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BISTABLE DEVICES FOR MORPHING ROTOR BLADES

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by

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ABSTRACT

This dissertation presents two bistable concepts for morphing rotor blades. These concepts are simple and are composed of bistable devices that act as coupling structures between an actuator and the rotor blade. Bistable or “snap-through” mechanisms have two stable equilibrium states and are a novel way to achieve large actuation output stroke at relatively modest effort for gross rotor morphing applications. This is because in addition to the large actuation stroke associated with the snap-through (relative to conventional actuator/amplification systems) coming at relatively low actuation effort, no locking is required in either equilibrium state (since they are both stable). The first concept that is presented in this dissertation is a that is composed of a bistable twisting device that twists the tip of helicopter rotor blades. This work examines the performance of the presented bistable twisting device for rotor morphing, specifically, blade tip twist under an aerodynamic lift load. The device is analyzed using finite element analysis to predict its load carrying capability and bistable behavior.

The second concept that is presented is a concept that is composed of a bistable arch for rotor blade chord extension. The bistable arch is coupled to a thin flat plate that is supported by rollers. Increasing the chord of the rotor blade is expected to generate more lift-load and improve helicopter performance. In this work, a methodology is presented to design the bistable arches for chord morphing using the finite element analysis and pseudo-rigid body model method. This work also examines the effect of different arches, arch hinge size and shape, inertial loads and rigidity on arch performance. Finally, this work shows results from an experiment that was conducted to validate the developed numerical model and demonstrates how the arch can be actuated using a Nitinol Shape Memory Alloy (SMA) wire to extend the chord of a helicopter rotor blade.
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Chapter 1

Introduction and Literature Review

Objectives and Organization of the Dissertation

The theme of this work is bistable concepts for helicopter rotor blade morphing applications. These concepts are composed of an actuator and bistable device that act as a coupling structure between the actuator and a movable part within the system such as a rotor blade or flat plate. These concepts act to change the shape of the rotor blade while the helicopter is in flight. Studies have shown (as discussed in chapters 3 and 4) that changing the shape of the helicopter’s rotor blade while in flight can potentially increase the maximum speed and flight altitude of the helicopter, decrease the power required to maintain flight, and increase the helicopter’s payload carrying capability. The objectives of this dissertation is to (1) present two bistable concept for helicopter rotor blade morphing, (2) investigative the feasibility of these concepts for helicopter applications, and (3) demonstrate to the concepts.

The first concept that is presented (Chapter 3) is the use of a bistable twisting device for rotor blade tip twist morphing. This concept is unique because it enables large tip twist of a helicopter rotor blade, it does not require an additional on-board locking mechanism to maintain the twisted shape at the blade’s tip and its performance is examined under external load. These contributions are further explored in Chapter 3.

The second concept that is presented (Chapter 4) is a bistable arch for chord extension. This concept is unique because it is composed of a lightweight coupling device that is used for chord extension and the coupling device is a bistable compliant mechanism used for helicopter morphing applications that considers external load. These contributions are further explored in Chapter 4.

The remainder of this chapter defines aircraft morphing, and surveys previous work in the areas of morphing concepts for rotor blade twist tip twist and chord extension. In addition, a literature survey is
conducted on bistable compliant mechanisms and bistable devices developed for aircraft morphing concepts.

**Introduction**

Conventional aircraft are designed for a specific mission and/or set of performance requirements. For example, a long endurance fixed-winged aircraft, such as the Global Hawk, cannot fly at high speeds like the Bell X-1. In order for a conventional fixed-winged aircraft to fly diversified missions with sufficient performance, it must be able to make large configuration changes in an efficient reversible manner. Over the past decade, organizations such as NASA, DARPA, AFRL and NextGen have been interested in technologies and concepts that enable large shape change of the wings of fixed-wing aircrafts. These large shape changes include a 200% change in aspect ratio, 50% change in wing area, 50% change in wing twist and a 20 degree change in wing sweep. These changes are implemented by new actuation concepts that use smart materials and adaptive structures. Such large shape change in fixed-wing aircraft are commonly referred to as aircraft morphing (Figure 1.1). Jha and Kudva (2004a) define a morphing aircraft as an aircraft that changes its configuration (i.e., shape or arrangement of parts) at radically different flight conditions. These configuration changes can take place in any part of the aircraft, e.g., fuselage, wing, engine, and tail. Anna McGowan of NASA (Levine, 2003) describes morphing as a visible change of form or structure, and it includes small and large changes using structures and fluids for control. Overall, the fundamental goal of morphing is to enable an aircraft to execute multiple missions (i.e., takeoff, climb, cruise, acceleration, etc.) at optimal performance.
Current research in aircraft morphing has expanded to include rotary-wing aircraft or rotorcraft. A rotorcraft is an aircraft that uses lift generated by wings that revolve around a mast for flight. These wings are called rotor blades. The collection of rotor blades and hub assembly is called the rotor. The main rotor provides thrust and propulsive force to the rest of the rotorcraft. It is also the source of the forces and moments on the rotorcraft that control the rotorcraft’s position, altitude and velocity. The rotorcraft of interest in this work is the helicopter. Helicopter morphing can be classified as shown in Figure 1.2. While it is feasible to consider that morphing can occur in any section of the helicopter, a large part of morphing research is focused on the rotor system, in particular the rotor blades.
The morphing of rotor blades can be categorized into three broad groups: discrete shape change, continuous shape change, and area change. A discrete shape change is a noticeable transformation in blade configuration by a distinct part such as a plate, flap or blade section. A discrete shape change of the rotor blade can occur over the entire blade, pre-selected segments of the rotor blade or to the blade’s airfoil profile. Examples of morphing due to whole or segmented rotation and airfoil profile are blade pitch, Gurney Flaps (e.g., Thiel, 2006), folding of the blade tips (e.g., Testa, 2005), trailing edge flaps along the blade (e.g., Centolanza, 2002a) and a rigid body rotation of blade tips (e.g., Bernhard and Chopra, 1997a, 2002d). These techniques are useful in increasing blade angle of attack, increasing and redistributing blade lift and reducing Blade-Vortex Interaction (BVI) noise. BVI noise is impulsive loading noise that occurs when a rotor blade passes within close proximity of the shed tip vortices of a previous blade (Hardin, 1987). However, these concepts are often complex and require sizable actuation systems. In addition, discrete shape change of blade sections may adversely affect blade aerodynamics.

The second type of morphing that occurs is continuous shape change (Figure 1.2). A continuous shape change is a noticeable transformation in blade configuration by a blade deformation. Continuous shape change includes blade/tip twist and smooth cambering (Nissly, 2005; Tieck, 2004; Anusonti-Inthra, 2005; Cadogan, 2006; Gandhi, 2008; Kota, 2008, Maucher, 2007). Variable twist allows for the redistribution of lift in different flight regimes. For example, in hover flight, negatively twisted blades redistribute the lift inboard and reduce induce power. In forward flight, highly twisted blades can cause a loss of thrust and propulsive performance due to negative lift on the advancing blade tip. In smooth cambering, the camber of the airfoil is modified by an internal structure for the purpose of increased lift and performance. The conformable airfoil can also act as an alternative to trailing-edge flaps used for active helicopter vibration reduction.

The third morphing rotor blade category is area change. Area change is the stretching or sliding of the surface of the planform to increase blade area. The morphing types that make up this category are span extension (e.g., Fenny, 2005; Prabhakar, 2007; Scott, 2007) and chord extension. Variable diameter rotors are effective because they allow a variation in rotor size when desired. Large rotors are desirable because
they have good autorotation and hover performance, produce reduced power requirement due to high tip speeds and increase rotor thrust. Smaller rotors are advantageous in reducing static droop of blades and are good for storage. Chord extension can be used to increase blade lift coefficient as angle of attack is increased (e.g., Liu, 2007) and reduce noise (e.g., Kobiki, 2003, 2004). The two morphing techniques that are of interest in this work are variable twist and chord extension.

**Morphing Concepts for Blade/Blade Tip Twist**

The work done on a rotor to generate lift leads to a gain in kinetic energy in the air in the wake of the rotor. This unavoidable energy loss is called induced power and it consumes a majority of the total power required of a rotor in hover and low speed forward flight. Reducing induced power can lead to an increase in payload carrying capability in hover and low speed flight. A high negative (nose down) linear twist results in a downwash and lift distribution over the blade that reduces induced power in hover and low speed flight. In high speed forward flight, on the other hand, a large negative twist together with cyclic pitch can result in negative lift in the vicinity of the advancing blade tip. As a result, most rotor designs incorporate a negative linear twist (between 8° and 15°) that is a compromise between maximizing rotor performance in hover while simultaneously limiting detrimental effects in high speed forward flight. Some blade designs include a nonlinear twist or double linear blade twist to give better forward flight performance while retaining most of the hover performance (Keys et al., 1987; Leishman, 2006). Another possible solution to improve flight performance is to incorporate a blade design that allows variable span-wise or blade tip twist while in flight. This type of blade would allow near optimum rotor performance in different flight regimes.

As an alternative to bearing the blade twist, a segmented rotor blade can be used to improve flight performance. In a segmented rotor blade, the pitch of the outboard segment of the blade is varied independent of the inboard region of the blade to obtain a better lift distribution. Stroub (1980) introduced a rotor blade composed of two sections, an inboard section similar to a conventional blade, and a small outboard segment that pitches freely about the spar. Stroub showed that the free-tip rotor is effective in
reducing the total power requirement of the rotor in forward flight compared to the conventional rotor blade. This improvement resulted from a change in the rotor blade’s span-wise lift distribution that decreased both profile and induced power in the entire rotor. Stroub also showed that the free tip has the potential to resist negative lift on the advancing blade tip that is commonly experienced by conventional rotor blades. This resulted in a reduction in drag experienced by the rotor and a reduced power requirement in forward flight.

Current research in rotor twist is conducted (largely) using smart materials for the purpose of helicopter noise and vibration reduction. Thakkar (2005) conducted a detailed survey of techniques used in rotor active rotor twist. Maucher et al (2007) describe several methods to directly and indirectly twist a rotor blade. An indirect method to twist the blade is conducted by the use of trailing edge flaps. Trailing edge flaps twist the blade by changing the blade’s aerodynamic pitching moment. Conversely, active twist concepts affect the spanwise lift distribution across the blade without affecting the aerodynamic pitching moment. This method directly twists the blade by structure-borne twist actuation. In the remainder of this section, several techniques are assessed to illustrate the state of the art in direct helicopter rotor blade twist. These concepts can be categorized as piezo-electric (pzt) based and non piezo-electric based actuation concepts.

An example of blade twist using conventional pzt patches is found in Chopra and Chen (1996a, 1997b, 1997c). Chopra and Chen (1996a, 1997b, 1997c) developed a Froude scale rotor blade with embedded monolithic piezo-ceramic (PZT-5H) actuators positioned at +/- 45 degrees on the top and bottom surfaces of the rotor blade to control blade vibrations. Experiments showed that the maximum tip twist achieved was 0.6 degrees. Additional studies showed that actuator geometric parameters affect twist performance of the rotor blades.

An alternate approach to direct twist is the use of AFCs (Active Fiber Composites) as an actuator. In this approach, AFCs are integrated within a composite rotor blade to induce a distributed twisting moment. Rodgers and Hagood (1997a, 1998b) used anisotropic active plies, positioned about the quarter-chord, that are embedded within the composite spar of the blade to induce shear stress that creates twist. A
bench test demonstrated that a maximum twist of 1 to 1.6 degrees peak-to-peak could be achieved. In general, active fiber composites are less fragile than monolithic piezoelectric patches and are easier to implement into practical structures. Wilbur et al (2002a, 2002b) and Shin et al (2008), (NASA/ARMY/MIT Active Twist Rotor Project) demonstrated rotor blade twist using AFC actuators for helicopters in forward flight. Results showed that a rotor blade twist of 1.1 degrees is achieved at an actuation frequency of three times the rotor rotational frequency, 3P, and 1.5 degrees at 5P when actuated at voltage amplitudes of 1000 Volts. Yet, difficulties encountered with the reliability and reproducibility of piezo fiber based actuators lead to the development Macro Fiber Composite (MFC). Wierach et al (2005) developed a model that combined distributed pzt MFC actuators with an orthotropic rotor blade skin to generate twist within the blade. Results showed that improved twist performance could be achieved if the rotor blade skin contained on outer active layer.

It is known that active twist control concepts require high actuation power for twisting the entire blade compared to the actuation of (smaller) finite span flaps. This disadvantage can be alleviated by the use of single-crystal material actuators for active twist. Pawar et al (2008a, 2008b) developed elastically coupled (extension-torsion) composite rotor blades using a newly developed single-crystal material for shear actuation. One of the advantages of using the single-crystal material is that it compares well with finite span flaps in terms of power usage. In slow speed flight, the blades generated as much as 2.5 degrees of positive twist. In high speed flight, the blades generated as much as 1.1 degrees of positive twist. Park and Kim (2008) developed the advanced active twist rotor (AATR) blade that incorporated single crystal MFC in the blade’s skin for the purpose of the vibration and noise reduction in helicopters. The designed AATR blade achieved a twist actuation of $5.58 \text{ deg/m at 1000 V}_{\text{peak-to-peak}}$, whereas the twist actuation of the active tip rotor (ATR) prototype blade produced $4.52 \text{ deg/m at 4000 V}_{\text{peak-to-peak}}$. The input voltage to the AATR blade is lower and it can produce much higher twist actuation than the ATR prototype blade.

Additional concepts in pzt actuation are the induced shear pzt tube and bending-torsion coupled composite beam. Smith and Centolanza (2002b) analytically evaluated an induced shear piezo-electric tube as an active blade twist actuator for both small and full-scale rotor blade applications. The analysis showed that a 48 inch long tube actuator generates a tip twist of $\pm 1.1$ degrees in a full scale blade, and an 18 inch
long actuator produces a tip twist of +/-1.50 degrees for a small scale blade. In comparison with AFC actuators, the shear tube actuator generates blade twist at a lower applied voltage level, but is very heavy. Bernhard and Chopra (1997a, 2002d) presented an active moving blade tip with torsional actuation via a piezo-induced bending-torsion coupled composite beam. Results showed that under zero loading, the actuator can generate +/- 1.25 degrees of pitch. If the blade tips are locked down, twist can be achieved along the span of the blade. In this case, the actuator beam functions as an active torsion spring in parallel with the blade.

A concept that does not include pzt actuation is the Reconfigurable Rotor blade (RRB) (Figure 1-3). Bushnell et al (2008) and Ruggeri et al (2008) developed the Reconfigurable Rotor blade (RRB) concept to twist the entire blade into a different configuration to provide greater lift during takeoff and landing. The rotor blade system is composed of an actuator located inside the spar near the root of the blade and a passive torque tube that transmits torque from the actuator location to the tip of the blade. The actuator mechanism consists of SMA torque tube actuators that apply torque to a spring mechanism that results in blade twist. The purpose of the spring mechanism is to increase the blade’s twist magnitude and end-state stability. This concept is able to produce about 2 degrees of blade twist. Another non-pzt actuation concept was developed by Mihir (2008) at The Pennsylvania State University. Mihir (2008) developed two concepts for the rotor blade twist. In the first concept, bimoments are applied to the tips of flanges in an I-beam spar to induced twist. More specifically, the axial loads applied to the beam’s cross-section warps the cross-section of the beam inducing beam twist. Analytical results demonstrated that is technique could produce approximately 12 degrees of spar twist for I-cross-sections with low flange and web thickness. The second concept is coined Skin Warp Induced Variable Twist. In this concept, a trailing edge actuator assembly is used to warp the skin. It was demonstrated experimentally that this concept could produce 15-20 degrees of twist when unloaded.
Introduction of New Concept: Bistable Tip Twist Device

In most of the previous works, only a small change in blade twist and blade tip pitch could be achieved (0.5 – 1.6 degrees). This amount of blade twist or blade tip pitch can be used to reduce vibration and blade vortex interaction noise, but not increase rotor performance. To increase rotor performance, a large amount of blade twist is needed and a new actuation concept must be considered. A concept that may be useful in achieving a large amount of blade twist is Schultz’s bistable twisting structure (Schultz, 2007). This bistable twisting structure (Figure 1.4) moves from one equilibrium state to the next by snap-through. Snap-Through is characterized by a visible and sudden jump from one equilibrium state to the next (Simitses and Hodges, 1976).
The device consists of curved shells that are pinched by riveting the short ends together. When pinched the structure assumes one of the equilibrium configurations as shown in Figure 1.4. If the root end is constrained and a torque is applied at some span-wise location, the structure can be made to snap through to its second stable equilibrium condition. If this structure constituted the outer section of a rotor blade, a large change in the twist of the section could be realized through the application of a torque. The advantages to this type of structure include: 1. power to the actuation system is required only when transforming the device form one shape to another, 2. the design is mechanically simple, and 3. large shape change can be accomplished without a complicated system of multiple actuators.

A twisting device, similar to Schultz’s device, may be used to induce blade tip twist in rotorcraft. One method is to place the device within the rotor blade and allow the blade section to take the shape of the device (Figure 1.5). In this case, the device must support the aerodynamic loads and the blade must be covered with a flexible skin in order to allow a span-wise twist along the blade. As a result, an evaluation of the device’s load carrying capability must be conducted before it can be implemented within a rotor. One of the objectives of this dissertation is to examine the performance of the bistable twisting device under aerodynamic load with geometric and material properties appropriate for rotorcraft application.

![Figure 1.5. Tip Twist Rotor blade and Actuator Model](image)

To further define the scope of the twisting device, trade studies were conducted that considered direct twist high frequency actuation concepts (Table 1-1) direct twist quasi-static actuation concepts
(Table 1-2) and bistable devices for morphing applications (Table 1-6). A description of the categories listed in each table is shown below:

**Description of Categories in Table 1-1 and Table 1-2.**

1. **Group**: Institution of first author
2. **Objective**: Goal intended to be attained by the author(s)
3. **Actuation Type**: Mode of actuation
4. **Quasi-Static Free Stroke Displacement**: Free stroke of actuators that undergo quasi-loading
5. **Blade Twist Magnitude**: Maximum amount of twist that can be attained by the actuator when coupled with a rotor blade.
6. **Aerodynamic Load Considered**.
7. **Percentage of Blade Twisted**: Only considered in Table 1-2
8. **Weight**: This is an approximation of the weight the actuation system adds to a blade based on qualitative analysis. The analysis is based on (a.) the number of parts and (b.) the approximate weight of each part. Light concepts add an addition weight to the blade that is 0% - 5% of the weight of a blade. Medium concepts add an addition weight to the blade that is 6% - 15% of the weight of a blade. Heavy concepts add an addition weight to the blade that is 16% - 100% of the weight of a blade.
9. **System Type**: This is based on (a) actuation system fabrication and (b) number of system components. An actuation system is complex if it requires 3 or more manufacturing steps or system components.
10. **Application to Date**.

**Description of Categories in Table 1-6.**

1. **Group**.
2. **External Load Considered**: This category considers the devices ability to support a non-actuation external load without the assistance of an applied external force or support.
3. **Application to Date**.
4. **Application**.
5. **Device Deformation Type**.

As demonstrated in Table 1-1 and Table 1-2, a majority of the previous concepts developed for direct blade twist was for the purpose of vibration and noise reduction. Therefore, it is necessary that these devices are actuated at a high frequency and low blade twist magnitude (0.75 - 5 degrees). As shown in
Table 1-1, more than half of the direct twist high frequency actuation concepts are light. In addition, most of these concepts are complex in that they require many parts. Concepts that require many moving parts may be challenging to repair once they are installed within the rotor system.

Only recently (within the past three years) have direct twist concepts been developed for the purpose of increased flight performance. As shown in Table 1-2, these concepts twist the blade quasi-statically to induce large twist. In addition, most of these concepts add a medium amount of weight to the blade and do not consider the affect of aerodynamic load. The concept presented in this dissertation generates 20 - 40 degrees of quasi-static free stroke displacement, it is designed to twist the 20 % outboard section of the blade, add a light to medium amount of weight to the blade in that these devices only add an additional 3 % to 10 % of the blade’s weight to the blade. In addition, these systems are also simple. The analysis is solely analytical and aerodynamic load is considered. The full details of this concept are given in Chapter 3.
Table 1-1. Trade Study Comparison of Direct Twist High Frequency Actuation Concepts

<table>
<thead>
<tr>
<th></th>
<th>PZT-5H Patches (Chen and Chopra, 1996a, 1997b, 1997c)</th>
<th>**Active Fiber Composite (Rodgers and Hagood, 1997a, 1998b)</th>
<th>**Macro Fiber Composite (Wierach and Riemenschneider, 2005)</th>
<th>**Single Crystal (Park and Kim, 2008)</th>
<th>Induced-Shear Tube (Centolanza and Smith, 2002b)</th>
<th>PZT B-T Composite Beam (Bernhard and Chopra, 1997a, 2002b)</th>
<th>SABT (Bernhard and Chopra, 2002b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Groups</td>
<td>Univ. of Maryland</td>
<td>NASA/MIT/Army</td>
<td>German Aero. Centre</td>
<td>Seoul Nat. Univ.</td>
<td>Penn State</td>
<td>Univ. of Maryland</td>
<td>Univ. of Maryland</td>
</tr>
<tr>
<td>Objective</td>
<td>Vib./Noise Reduction</td>
<td>Vib./Noise Reduction</td>
<td>Vib./Noise Reduction</td>
<td>Vib./Noise Reduction</td>
<td>Vib./Noise Reduction</td>
<td>Vib./Noise Reduction</td>
<td>Vib./Noise Reduction</td>
</tr>
<tr>
<td>Blade Response to Actuation</td>
<td>Dynamic</td>
<td>Dynamic</td>
<td>Dynamic</td>
<td>Dynamic</td>
<td>Dynamic</td>
<td>Dynamic</td>
<td>Dynamic</td>
</tr>
<tr>
<td>Quasi-Static Free Stroke Displacement</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>Blade Twist Magnitude</td>
<td>2.0 degrees</td>
<td>0.75 degrees</td>
<td>1.3 degrees</td>
<td>5.58 degrees</td>
<td>1.45 degrees</td>
<td>0.85 degrees</td>
<td>2 – 5 degrees</td>
</tr>
<tr>
<td>Aerodynamic Load Considered</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Weight</td>
<td>Light</td>
<td>Light</td>
<td>Light</td>
<td>Light</td>
<td>Light/ Medium</td>
<td>Heavy†</td>
<td>Heavy†</td>
</tr>
<tr>
<td>System Type</td>
<td>Complex</td>
<td>Complex</td>
<td>Complex</td>
<td>Simple</td>
<td>Complex†</td>
<td>Complex†</td>
<td>Complex†</td>
</tr>
<tr>
<td>Application to Date</td>
<td>Froude-Scale Rotor blade</td>
<td>Mach-Scale Rotor blade</td>
<td>Active Twist Box Beam</td>
<td>Analytical</td>
<td>Analytical</td>
<td>Mach-Scale Rotor</td>
<td>Froude and Mach-Scale</td>
</tr>
</tbody>
</table>

**Additional work done in this area not shown in the table

A. Non-rotating twist response at 100 V_{rms} excitation and resonance. Rotating twist response at 100 V_{rms} excitation, 150 rpm and resonance is 1 degree. Rotating twist response at 100 V_{rms} excitation, 900 rpm (Froude-scale operating speed) and resonance is 0.1 degrees.
B. 5.58 degrees per meter at 1000 V_{pp}
C. 1.3 degrees per 2 kV. Six actuator patches were used. Three bonded to the top and bottom layers of the blade.
D. Contains four components
E. Contains Aluminum or Composites pinned thin shells w/ activating mechanism.
F. 22% of the BO-105 blade span. Full Scale: maximum actuator weight is 5% rotor blade weight. Small Scale: maximum actuator weight adds 2.5 lbs to blade.
G. Up to 12 degrees of twist when a curved flange I-beam is considered.
H. The composite actuation system has a mass comprising 28% of the total blade mass.
I. Contain four components.
Table 1-2. Trade Study Comparison of Direct Twist Quasi-Static Actuation Concepts

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Groups</td>
<td>Univ. of Maryland</td>
<td>Boeing</td>
<td>Penn State</td>
<td>Penn State</td>
</tr>
<tr>
<td>Objective</td>
<td>Increase Performance</td>
<td>Increase Performance</td>
<td>Increase Performance</td>
<td>Increase Performance</td>
</tr>
<tr>
<td>Blade Response to Actuation</td>
<td>Static</td>
<td>Static</td>
<td>Static</td>
<td>Static</td>
</tr>
<tr>
<td>Quasi-Static Free Stroke Displacement</td>
<td>9 – 11 degrees</td>
<td>16° degrees</td>
<td>12° degrees</td>
<td>--</td>
</tr>
<tr>
<td>Blade Twist Magnitude</td>
<td>--</td>
<td>2 degrees</td>
<td>--</td>
<td>15 – 20° degrees</td>
</tr>
<tr>
<td>Aerodynamic Load Considered</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Percentage of Blade Twisted</td>
<td>--</td>
<td>100 %</td>
<td>Outboard 12-35%</td>
<td>100 %</td>
</tr>
<tr>
<td>Weight</td>
<td>--</td>
<td>Light</td>
<td>Medium</td>
<td>--</td>
</tr>
<tr>
<td>System Type</td>
<td>--</td>
<td>Complex</td>
<td>Simple</td>
<td>Simple</td>
</tr>
<tr>
<td>Application to Date</td>
<td>Analytical</td>
<td>Quarter-Scale V-22 Blades</td>
<td>Analytical</td>
<td>Model Blade</td>
</tr>
</tbody>
</table>

**Additional work done in this area not shown in the table**

A. Mechanical stops were placed at +/- 16 degrees of PPT angle
B. Up to 12 degrees of twist when a curved flange I-beam is considered.
C. Includes device, activation device and activation device box.
D. 15-20 degrees both loaded and unloaded.
E. Contain aluminum or composites pinned thin shells w/ actuating mechanism.
F. Includes device, activation device and activation device box.
G. SMA actuator produced more than 60 in-lb of torque and 30 in-lb of energy.
Morphing Concepts for Chord Extension

In helicopters, chord extension may prevent stall on the retreating side of the disk in high speed forward flight by providing additional lift where needed; and reduce disk loading (the ratio of rotor thrust to rotor disk area) which is significant to reducing power loading (ratio of rotor power required to rotor thrust) in hover flight. An early example of chord extension is the Fowler flap used on fixed-wing aircraft. The Fowler flap slides back from the wing and rotates down creating a slot between it and the wing (Figure 1.6). The flap acts to increase wing area and provide additional lift to the aircraft in comparison to a traditional flap.

![Figure 1.6. Fowler Flap (Day, 2008)](image1)

Current research in chording morphing is focused on developing innovative actuation concepts to induce area change of wing or rotor blade planform. JAXA and Kawanda Industires Inc. (Kobiki, 2003, 2004) developed the Active Tab (Figure 1.7) as a concept to reduce helicopter noise. The Active Tab is located in the aft position of the airfoil and driven back and forth dynamically (with anhedral) by an electric motor to reduce BVI noise and vibration by blade lift control. When used statically, the Active Tab can be used to increase blade lift so that rotor speed can be reduced for the purpose of noise reduction in climb flight. Reed et al (2005) developed a chord morphing wing that is composed of a flexible-skin made out of DMC (Dynamic Modulus Composites) and SMP (Shape Memory Polymer). The internal structure of the wing consists of sliding rods and a motor and lead screw assembly to drive the leading and trailing edge sections of the planform when actuated. A flexible honeycomb-like structural system is used to support the
skin, maintain airfoil shape, and allow morphing in the chord. Diaconu et al (2008) developed a chord morphing concept using composite plates. The proposed assembly for chord length change is shown in Figure 1.8. A rectangular, bistable, cross-ply non-symmetric laminate is inserted into the airfoil in a vertical position along the main spar. The plate is connected at its corners to the spar. The airfoil is composed of two separate sections, a leading edge part and a trailing edge part. When an actuating transverse load is applied at the center of the plate, the chord length of the airfoil is changed between two stable states (Figure 1.8 b and c). In the second stable state, the plate acts as an internal stiffener for the airfoil section because half of it fibers are perpendicular to the chord and in the direction of the aerodynamic forces transverse to the airfoil.

![Figure 1.8. (a.) Assembly for Chord Length Change. (b) Isometric Shape View for Chord Length Change First Stable Shape (c) Second Stable Shape (Diaconu 2008).](image)

Introduction of New Concept: Chord Extension using Bistable Arch

A new actuation concept for chord morphing presented in this dissertation is the use of a bistable arch for chord extension. The concept presented in this dissertation is a discrete chord morphing concept that purports to increase flight performance by quasi-statically actuating a thin plate aft of the trailing edge of the airfoil. As shown in Figure 1-9, the concept is composed of an arch, flat plate and roller track where all components of the system are initially rest within the blade. This configuration is called the first stable state. When activated, the arch snaps to a second position where the flat plate is positioned outside of the blade, aft of the trailing edge. The roller track within the blade supports the flat plate as it slides from the interior to aft trailing edge of the blade. This configuration is called the second stable state. This simple concept proposes to increase the chord by as much as 20% while resisting inertial loads. A bistable arch is
useful for this application because it can hold the plate in a desired position without the need for additional force input, and it is a simple system that does not require many moving parts. This concept is described in more detail in Chapter 4.

Actuation concepts found in literature for chord extension are shown in Table 1-3. The scope of chord extension concepts in aircraft is defined in Table 1-3. A description of the categories listed in this table is shown in the next section. Table 1-3 shows that the plate response to actuation for most of the cases shown is static. This means that the plate is held fixed when it is moved aft of the trailing edge of the blade. In addition, most of these concepts do not require a stretching skin and have medium to heavy weight. In contrast, the concept proposed in this work is light weight.

In general, the arch moves from one equilibrium state to the next by applying a point load or moment at a suitable location along the arch. At the critical load, the arch moves from one equilibrium state to the next by symmetric or asymmetric snap-through. Arches have a symmetric geometry where its shape can be split into two-half arches. Symmetric snap-through occurs when the half arches have the same deformation pattern from one equilibrium state to the next. Asymmetric snap-through occurs when the half arches have dissimilar deformation patterns from one stable equilibrium state to the next. Shallow arches are expected to undergo symmetric snap-through, whereas high arches can undergo symmetric or asymmetric snap-through (Abaqus, 2007).

Figure 1.9. Chord Extension Concept: (left) First Stable Shape and (right) Second Stable Shape.
Description of Categories in Table 1-3:

2. Objective.
3. Actuation Type.
4. Amount of Extension: The increase in chord that results from actuation. The quantity is measured in terms of the chord of the un-extended airfoil.
5. Requires Stretching Skin.
6. Aerodynamic Load Considered.
7. Weight.
8. System Type.
9. Application to Date.

The bistable arch that is purposed to be used for chord extension is a fully compliant mechanism because its motion is obtained from the deflection of its compliant members. The scope of multi-stable mechanisms for non-morphing concepts and bistable devices for morphing concepts is defined in Table 1-4, Table 1-5 and Table 1-6, respectively. An evaluation of Table 1-4 and Table 1-5 demonstrate that previous multi-stable mechanism designs have not considered non-actuation based external load and are typically developed for non-morphing applications. Therefore, the chord extension concept and design process presented in this dissertation contain unique features in relation to previous devices that have been developed.
### Table 1-3. Trade Study Comparison of Chord Extension Actuation Concepts

<table>
<thead>
<tr>
<th>Groups</th>
<th>Objective</th>
<th>Plate Response to Actuation</th>
<th>Amount of Extension</th>
<th>Requires Stretching Skin</th>
<th>Aerodynamic Load Considered</th>
<th>Weight</th>
<th>System Type</th>
<th>Application to Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>JAXA</td>
<td>Increase Performance/Vib./Noise Reduction</td>
<td>Static/Dynamic</td>
<td>20 % of chord</td>
<td>No</td>
<td>Yes</td>
<td>Medium</td>
<td>Simple</td>
<td>Reduced Size One-Bladed Rotor System</td>
</tr>
<tr>
<td>CRG</td>
<td>Improved (fixed-wing) Maneuverability</td>
<td>Static</td>
<td>80 % of chord</td>
<td>Yes</td>
<td>No</td>
<td>Medium</td>
<td>Simple</td>
<td>Qualitative Study/CAD Model</td>
</tr>
<tr>
<td>University of Bristol</td>
<td>Increase Performance</td>
<td>Static</td>
<td>10 - 14 % of chord</td>
<td>Yes</td>
<td>No</td>
<td>Medium</td>
<td>Simple</td>
<td>Analytical /Model Built</td>
</tr>
<tr>
<td>Penn State</td>
<td>Increase Performance</td>
<td>Static</td>
<td>10 - 30 % of chord</td>
<td>No</td>
<td>No</td>
<td>Heavy</td>
<td>Simple</td>
<td>Analytical</td>
</tr>
<tr>
<td>Penn State</td>
<td>Increase Performance</td>
<td>Static</td>
<td>20 - 40 % of chord</td>
<td>Yes</td>
<td>Yes</td>
<td>Light</td>
<td>Simple</td>
<td></td>
</tr>
</tbody>
</table>

A. Includes plate, activation device, and activation device box. This is an assumption, system not defined in detail.
B. Includes tab, activation device, and activation device box. This is an assumption, system not defined in detail.
C. Tab extension is also designed to deflect angularly by 10 degrees.
D. Objective is to add no additional weight is added to the initial wing design.
E. Includes sliding ribs, miniature motor, and a lead screw.
Table 1-4. Trade Study Comparison of Multi-stable Compliant Mechanisms and Bistable Devices, Non Morphing

<table>
<thead>
<tr>
<th></th>
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<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>External Load Considered</td>
<td>BYU</td>
<td>BYU</td>
<td>BYU</td>
<td>Sandia and BYU</td>
<td>BYU</td>
<td>Andong Nat. Univ</td>
<td>Univ. of Cambridge</td>
<td>BYU</td>
</tr>
<tr>
<td>Application to Date</td>
<td>Micro-Mechanism Prototype</td>
<td>Micro-Mechanism Prototype</td>
<td>Prototype</td>
<td>Bistable CD Ejection Actuator</td>
<td>Micro-Relay Prototype</td>
<td>Millimeter-Scale Brass Mechanism Prototype</td>
<td>Analytical</td>
<td>Bistable Micro-Mechanism Prototypes</td>
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</tbody>
</table>

Table 1-5. Trade Study Comparison of Multi-stable Compliant Mechanisms and Bistable Devices, Non Morphing

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>External Load Considered</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>Application to Date</td>
<td>Analytical</td>
<td>Analytical</td>
<td>Analytical</td>
<td>Analytical</td>
<td>Analytical</td>
<td>Analytical</td>
<td>Analytical</td>
<td>Prototype</td>
<td>Prototype Manufactured Laminate</td>
<td>Prototype</td>
<td></td>
</tr>
<tr>
<td>Group</td>
<td>Michigan, BYU</td>
<td>Michigan, BYU, Sandia</td>
<td>BYU</td>
<td>Michigan State</td>
<td>Univ. of Cambridge</td>
<td>Univ. of Maryland Baltimore County</td>
<td>University of Bath</td>
<td>Universidade do Porto/ Univ. of Bristol</td>
<td>Univ. of Bristol</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 1-6. Trade Study Comparison of Bistable Devices Wing/Rotor blade, Morphing

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Group</td>
<td>Bristol Univ.</td>
<td>Bristol Univ.</td>
<td>Univ. of Bristol</td>
<td>Univ. of Bristol</td>
<td>Univ. of Bristol</td>
<td>Penn State</td>
<td>Penn State</td>
<td></td>
</tr>
<tr>
<td>External Load Considered</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>Application</td>
<td>Variable Sweep</td>
<td>Winglet</td>
<td>Variable Camber</td>
<td>Variable Camber</td>
<td>Variable Chord</td>
<td>Variable Chord</td>
<td>Variable Twist</td>
<td>Variable Chord</td>
</tr>
<tr>
<td>Application to Date</td>
<td>Wingbox Prototype</td>
<td>Prototype</td>
<td>Prototype, Variable Camber, Trailing Edge</td>
<td>Analytical</td>
<td>Analytical</td>
<td>Analytical</td>
<td>Analytical</td>
<td>Analytical</td>
</tr>
<tr>
<td>Device Deformation Type</td>
<td>Bending</td>
<td>Bending</td>
<td>__</td>
<td>Bending</td>
<td>Bending</td>
<td>Linear</td>
<td>Twist</td>
<td>Linear</td>
</tr>
</tbody>
</table>
Bistable Devices

The final literature survey presented in this dissertation is a survey of bistable compliant mechanisms and bistable devices designed for morphing aircraft. The bistable arch presented in this dissertation is a bistable compliant mechanism. The bistable twisting device is designed for morphing helicopters.

Many of the bistable compliant mechanisms found in literature where developed by the Brigham Young Compliant Mechanism Research Group directed by Dr. Larry Howell. Examples of bistable compliant mechanisms are as follows: Jensen et al (1999, 2001, and 2004) developed bistable compliant mechanisms using three separate solution techniques: 1. Pseudo-Rigid-Body Model (PRBM), 2. Finite Element Method, and 3. a theory that provides prior knowledge of mechanism configurations that guarantee bistable behavior. Baker and Howell (2002) developed a compliant bistable mechanism which can be snapped from one-position to the next by an on-chip thermal actuation. In this work, the simulated annealing algorithm in conjunction with the pseudo-rigid-body model analysis method is used to develop compliant mechanisms that possess bistability and satisfy actuation constraints of a linear thermal actuator used to actuate the mechanism. Masters and Howell (2003) developed a fully compliant (i.e., they are monolithic and get their motion from the deflection of elastic components) bistable mechanism with added self-retraction for low-power switching applications. The self-retracting fully compliant bistable micromechanism was modeled using PRBM to find the strain energy of the mechanism. Gomm et al (2002) investigated the linear displacement bistable mechanism (LDBM) for use in microrelays. Chen et al (2009) presented a new class of tristable mechanisms called double tensural tristable mechanisms (DTTMs) which are fully compliant and can be fabricated at the micro scale. A pseudo-rigid-body model (PRBM) for the DTTM was developed. The results demonstrate tristable behavior of the mechanism and show that the PRBM can be used to identify tristable configurations and predict their performance. Oh and Kota (2009) present a mathematical approach to synthesize multistable compliant mechanisms by combining multiple bistable equilibrium mechanisms. More specifically, they identified and categorized various types of bistabilities by characterizing the essential elements of their complicated deformation pattern.
Other flexible devices found in literature are as follows, Su and McCarthy (2007) developed a formulation to design unistable and multi-stable compliant mechanisms with specified equilibrium configurations using polynomial homotopy. Polynomial homotopy is a globally convergent method for finding all the isolated solutions to systems of polynomial equations. Prasad and Diaz (2006) developed a formulation for the automatic Synthesis of 2-D bistable, compliant periodic structures. The term bistable periodic structure refers to the periodic arrangement on interconnected bistable structures that tile a plane. Han et al (2007) developed a quadstable monolithic mechanism (QsMM), which provides four stable equilibrium positions within its planar range of operation. The QsMM is composed of two pairs of curved snapping beams to achieve X and Y directional bistability. Santer and Pellegrino (2004) designed and constructed a fully actuated bistable monolithic structure that is capable of doing useful work over the transition between its stable states (the structure can jump approximately 100 mm from a datum surface). Iqbal and Pellegrino (2000) and Pellegrino (2005) studied the behavior of a bistable structure using ABAQUS. The structure is in stable equilibrium when it is either straight and extended or rolled-up. Iqbal and Pellegrino showed that there are two strain energy minimums of the shell, therefore the shell is bistable.

Bistable devices used in aerospace applications have been developed for morphing applications such as blade twist/tip twist, adaptive trailing edge, winglet, adaptive airfoil camber, chord, variable sweep, and space deployable structures. Variable blade twist is used to improve the flight performance of a helicopter. For example, a twisted blade redistributes the induced inflow over the blade which reduces the power requirement. An example of a device that can be used to induce blade twist is Schultz’s (2007) bistable twisting device (Figure 1-10). Each laminate of the device is composed to uni-directional graphite/epoxy where one layer has a 0 degree fiber angle and the other a 90 degree fiber angle. The device is made active by bonding MFC actuators to the top and bottom surfaces of the device. When the actuators are operated at opposing voltages, the actuators act to twist the device.
Trailing edge flaps are actuated in order to reduce the stalling speed of the aircraft. A proposed method for the deflection of trailing edge flaps using bistable composites is the Composite Trailing Edge Box (Figure 1.11, Diaconu, 2008). To represent typical flap geometry, the structure should rotate with an angle of 10 degrees between stable states around an axis which is located at 15% of the chord from the trailing edge. The proposed trailing edge box concept (Figure 1.11) is completely made out of composite material. The stable region (shown in Figure 1.11) is the web, rear spar, and top/bottom surfaces connected to the spar. The composites in this region are made from symmetric or anti-symmetric laminates. Composites in the bistable region (located near the trailing edge) are made from unsymmetric laminates. Since the device is passive, a load is applied to the tip of the trailing edge to snap it between the first and second stable states (Figure 1.12).
Winglets are used to improve performance by reducing lift-induced drag. A proposed concept for tip morphing is to mount a bistable panel on to the tip of a traditional wing (Mattioni, 2008). The outboard section (Figure 1.13) of the panel is composed of a bistable unsymmetric laminate $[0_4/90_4]_T$ whereas, the inboard is composed of a symmetric laminate $[0_2/90_2/0_2]_T$. When the panel is extended, it is predicted to generate more lift. However, when it is deployed, it is predicted to increase performance. The winglets are snapped between both stable configurations by the aerodynamic loads as the speed of the aircraft is increased.

Another type of airfoil morphing is adaptive camber. The camberline is the curve located halfway between the upper and lower surfaces of the airfoil. Camber is added to the airfoil to increase lift and/or reduce critical angle of attack. The proposed assembly of the camber concept is shown in Figure 1-14 (Diaconu, 2008). A bistable plate is inserted into the airfoil section to induce that change in camber. The plate is approximately 234 mm x 234 mm in size. The plate is clamped to the spar and hinged to the web. In addition, the web is hinged to the airfoil surface to allow relative movement between the hinge and skin.

The spar, web, and leading edge are made $[45/-452 /45/90/02 /90]_{AS}$ for high stiffness. The trailing edge skin is made of $[45/-452 /45]_{AS}$ to ensure flexibility.
Another concept of aircraft morphing is chord extension. Chord extension is expected to increase lift and improve flight performance. The proposed assembly of chord length change is shown in Figure 1.15 (Diaconu, 2008). A rectangular bistable plate is inserted into the airfoil in a vertical position along the main spar. The plate is connected at its corners to the spar. The airfoil is composed of two separate sections, a leading edge part and a trailing edge part. The leading edge is fixed to the center of the plate. When an actuating transverse load is applied at the center of the plate, the chord length of the airfoil is changed between two stable states (Figure 1.15). In the second stable state, the plate acts as an internal stiffener for the airfoil section because half of it fibers are perpendicular to the chord and in the direction of the aerodynamic forces transverse to the airfoil. Diaconu et al (2009) also conducted a dynamic analysis of the snap-through phenomena of rectangular bi-stable laminated composite plates. The model was used to evaluate the initial displacements for the stable states and also to investigate the static and dynamic transitions from one stable state to another.
Another concept of aircraft morphing is variable sweep angle. The benefits of variable sweep angle are a delayed rise in drag at Mach numbers near unity and reduced bending moment at root by redistribution of loads. The disadvantages of variable sweep angle are that a large metal pivot is needed to sweep wing and there is an increase maintenance cost and a decrease fuel performance. Therefore, new sweep wing concepts should be considered. Mattioni et al (2008) developed a variable wing sweep structure that is composed of two spars with a interconnected truss-rib structure (Figure 1.16). The two spars are composite bistable shells that allow the wing to snap into multiple stable positions. The ribs do not constrain the spars during sweep and a compatible skin must allow transition between the two configurations. At low forward speeds, the spars are designed not to snap; at high speeds, the wings sweep back to an equilibrium position.

Figure 1.16. Wingbox Structure for the Variable Sweep Wing (Mattioni et al, 2008)

Composites are also used in space applications such as deployable structures. For example, Lockheed Martin rolled solar array (Figure 1.17) is a possible application for composite tape springs. The benefits of composite tape springs (Murphey, 2004) are as follows, (1) small force is required to roll and unroll, (2) can be actuated with low force actuators (e.g. SMA, pzt film), (3) mass efficient (w.r.t stiffness and strength), and (4) neutrally stable (i.e. force not needed to hold). The tape springs are constructed of fiber reinforced cross-ply curve lamina where two lamina are bonded together into a flat configuration (Figure 1-18). The resulting tape springs are stable when completely rolled.
Overall, bistable composites are used in solutions for aircraft morphing. Applications include twist, adaptive trailing edge and airfoil, winglet, variable sweep and deployable space structure. Bistable structures are obtained by inducing a residual stress field into the structure through pre-stress or thermal stresses. This survey is limited to thermally induced bistable unsymmetric laminates. Residual stress in a traditional symmetric laminate will not induce multi-stability because all the plies are symmetrically loaded. For non-bistable composites, previous work has shown that bistability may be possible by adding unidirectional reinforcement along the edges of the composite.

The reminder of this dissertation will cover the following topics. Chapter 2 will give a brief review of the elastic stability of structures. In Chapter 3, the bistable twisting device concept previously presented for morphing rotors will be examined under aerodynamic load. In Chapter 4, a novel bistable
concept for extending the chord of rotor blades is presented. In chapters 3 and 4, methods are introduced for the analysis and design of the bistable twisting device and the bistable arch, respectively. These methods were developed with a focus on helicopter rotor blade application.
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Chapter 2
A Review of the Concepts of Stability

Introduction

As outlined in the previous chapter, the attribute that is common to both aircraft morphing coupling devices that are presented in this work is the ability of the device to reside in two stable equilibrium states. The term that is commonly used to describe this attribute is bistability. Up to the point, the term bistability has been used to describe structures used in several different engineering applications but no discussion has been given on the definition of stability. Therefore, this chapter presents a definition of stability and outlines methods to determine the presence and characteristics of multiple stable states. The terminology and methods presented in this section will be useful in understanding how stability was determined for the twisting device (Chapter 3) and the arch (Chapter 4) coupling devices, respectively.

The Concept of Stability

A system can be defined as a collection of components that act together to perform a certain task. All existing systems change with time, but when the rates of change are not significant, the systems are referred to as static systems. When structural components of static systems are subjected to external causes (load and temperature changes), the stiffness and geometry of the components may change. A key aspect of the analysis of structural components is defining the character of the response of the structure. The character of the response is often associated directly with the load-carrying capability of structural components and systems. The character of the response is defined by three states of equilibrium: stable, unstable and neutral.

The concept of stability can be illustrated in many different ways (Simitses, 1986; Chen, 1987); here it is demonstrated by the ball-on-hill analogy (Figure 2.1). In the figure, the ball can be located a
positions A, B, or C. In position A, the ball sits near the base of the hill. If a small disturbance is applied to the ball in position A, the ball will oscillate about position A then come to a state of rest at position A. Therefore, position A is called a stable equilibrium position. When the ball is located at position B, it sits at the top of the hill. If a small disturbance is applied to the ball at position B, then the ball will roll down the hill and never return to position B. Therefore, position B is called an unstable equilibrium position. Lastly, when the ball is located at position C, it is also located near the base of the hill. However, unlike position A, position C is located on a flat surface. If a small disturbance is applied to the ball when it’s located at position C, the ball will move away from position C and immediately come to rest in a different position on the flat surface. Therefore, position C is called a neutral equilibrium system.

In the ball-on-hill analogy, the ball represents a simple mechanical system and the positions A, B and C represent different states of the system. The state of the system refers to system properties such as the level of external load applied to the system, the system’s stiffness and configuration. When the system is in state A, then the system is in a stable state. When the state of the system is in state B, then the system is in an unstable state. When the state of the system is in state C, the system is in a neutrally stable state. If the system is initially in a stable state, the level of external load applied at which the state of the system changes to an unstable or neutrally stable state is called the critical load. In practice, the lowest critical load is called the buckling load.

The concept of stability can also be interpreted by means of the concept of minimum total potential energy. In nature, elastic systems tend to reside in a state where the total potential energy is at a minimum. A system is in a stable equilibrium state if any deviation from that state causes an increase in the total potential energy of the system. A system is in an unstable equilibrium state if any deviation from
that state results in a decrease in the total potential energy of the system. A system is in neutral equilibrium if any deviation from that state results in neither an increase nor a decrease in the total potential energy of the system. Using the total potential energy of the system, an energy concept has been developed to find the buckling load of an elastic system (LaGrange, 1788).

**Bifurcation Points**

Consider the rigid rod system and force displacement plot shown in Figure 2.2. One end of the rod is supported by an elastic hinge and the opposite end is loaded by a vertical force $P$ that remains vertical as the rod is deflected by an angle $\psi$. Before loading, the rod axis is exactly vertical. The ordinate of the force displacement plot corresponds to the initial vertical state of the rod and the curve that intersects the ordinate at point A corresponds to the deflected state of the rod. The initial vertical state of the rod is stable and the rod under the load remains vertical up until load reaches point A on the equilibrium path of the force displacement plot. After the applied load ($P$) is increased beyond point A, the initial vertical equilibrium state becomes unstable, and any arbitrarily small disturbance shifts the rod out of the vertical state. However, at point A, the system may remain stable if force $P$ is nonlinearly increased as angular displacement, $\psi$, is increased. The deflected states at point A represent the solutions to a nonlinear formulation of the system. Such points, where a solution (e.g. initial vertical state of rigid rod) splits into several branches, are called bifurcation points. Branches in the vicinity of a bifurcation point are not restricted as stable equilibrium branches; these branches can also be unstable.
Methods of Analyses in Stability

The objective of stability analyses is to determine the critical conditions of the system. The three most commonly used methods to establish the critical loads and system mode shapes are the classical approach, the energy approach and the dynamic approach.

Classical Approach

The classical approach can be described as a search for the load where the system has a nontrivial equilibrium configuration. The load at which the system has a nontrivial equilibrium configuration is called the critical load (bifurcation load if branching occurs) because it’s the load that corresponds to the lost of the stable equilibrium state of the system. In this approach, the equations of equilibrium can be determined by setting the first variation of the total potential energy equal to zero $\delta^I \pi = 0$ or by considering the free body diagram of the structure. The equation, $\delta^I \pi = 0$, implies that the total potential does not change at the equilibrium configuration and it is often referred to as the Principal of Stationary Potential. The critical loads are found by performing an eigenvalue analysis on the system. The steps to determine the critical load of a system using this approach are as follows:

1. Determine the equations of equilibrium
2. Determine the general solution to the equations
3. Apply boundary conditions to the general solution
4. Conduct eigenvalue analysis on the resulting equations to determine critical load.

For simple systems, the eigenvalue analysis can be represented by equations 2-1 and 2-2. In Equation 2-1, matrix $A$ represents the system’s stiffness matrix, $P$ represents the external load and $x$ represents the generalized displacements of the system. Equation 2-1 represents a homogenous linear system. By Cramer’s theorem (Kreysig, 2006), it has a nontrivial solution, $x \neq 0$, if and only if its coefficient determinant is zero (Equation 2-2). It is important to note that the eigenvalue approach is an idealized mathematical approach to determine the critical loads and mode shapes of a geometrically perfect system. If geometric imperfections are present, deflections will start at the beginning of loading, and therefore, the problem becomes a load-deflection problem. These problems can be characterized by the question: What is the value of the external load for which the static displacements of the imperfect system become excessive or infinite? For a load-deflection problem, the classical approach cannot be applied.

$$([A] - P[I])\{x\} = 0$$  \hspace{1cm} \text{Equation 2-1}

$$\text{det}([A] - P[I]) = 0$$  \hspace{1cm} \text{Equation 2-2}

**Energy Approach**

Another way to determine the critical load of a system is by the energy approach. In this approach, the critical load is determined by evaluating the stability of the system about a point of interest (e.g. initial position of the system) using the second variation of total potential energy. Table 2-1 shows how the second variation of potential energy is used to determine stability. As shown in the table, if the second variation of the total potential energy is positive definite, then the equilibrium is stable. If the second variation of the total potential energy ceases to be positive definite, then the equilibrium is neutrally stable or unstable. The load at which the system ceases to be stable is called the critical load (Shames and Dym, 1991). For an elastic system subjected to conservative forces, the total potential energy can be
expressed in terms of generalized coordinates and external force. The conservative forces are those forces whose potential is dependent on the initial and final values of deflection and not on the path of deflection.

Table 2-1. Energy Approach: System Stability

<table>
<thead>
<tr>
<th>Behavior of the Second Variation of Potential Energy</th>
<th>Equilibrium Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta^2 \pi &gt; 0$</td>
<td>Stable</td>
</tr>
<tr>
<td>$\delta^2 \pi = 0$</td>
<td>Neutrally Stable</td>
</tr>
<tr>
<td>$\delta^2 \pi &lt; 0$</td>
<td>Unstable</td>
</tr>
</tbody>
</table>

**Dynamic Approach**

In this approach, a system of equations of motion governing the small free vibrations of the system must be written as a function of generalized displacements and the external applied force. A critical load is reached if a slight disturbance causes the motion of the system to become unbounded. The equilibrium is stable if a small disturbance causes a small deviation from the equilibrium configuration of system and the small deviation decreases as the magnitude of the disturbance decreases. The equilibrium is unstable if the magnitude of the motion increase without bound when the system is subjected to a slight disturbance.

**Basic Approaches of Stability Analysis**

In this section, three example problems are investigated, in the context of small and large deflection analysis, by using the classical, energy and dynamic approach. In the first example problem, the critical load for a pinned-ended column is determined using the equilibrium and energy approaches. In the second example problem, the critical load of a one degree of freedom problem is examined using the dynamic approach. In the third and final problem, the critical load of a system that experiences snap-through is examined using the energy approach.
Example Problem 1: Pinned-Ended Column – Small Deflection Analysis

A column can be defined as a member that sustains an axial load only. A beam-column is a member that is required to sustain both axial and lateral loads. This example only considers a column. The following assumptions regarding the geometry, kinematics, and material of the column are used (Chen, 1987a):

1. The column is initially perfectly straight.
2. The axial load is applied along the centroidal axis of the column.
3. Plane sections before deformation remain plane after deformation.
4. Deflection of the member is due to axial and bending shortening. The effect of traverse shear is negligible.
5. The material of the element is homogeneous and isotropic.
6. The material obeys Hooke’s Law

With the above assumptions, the governing differential equation of the column is derived using the calculus of variation in conjunction with the principle of stationary total potential energy. These techniques are used to determine the conditions that must be satisfied if the column is to be in equilibrium in this slightly deformed configuration.

Figure 2.3 shows a column that has been slightly deformed by axial load P at the instance of buckling. In order to develop the total potential energy of the column, the strain energy U and potential energy V of the column must be derived separately. A complete derivation of the equation of equilibrium and boundary conditions using the calculus of variation in conjunction with the principle of stationary total potential energy is given in the Appendix (Sections A.1.1 – A.1.4).
Equilibrium Approach

The equilibrium approach is used to find the critical load of the system. The derived governing differential equation of the Pinned-Pinned Column shown in Figure 2.3 is

$$EI \frac{d^4v}{dx^4} + P \frac{d^2v}{dx^2} = 0 \quad \text{or} \quad \frac{d^4v}{dx^4} + k^2 \frac{d^2v}{dx^2} = 0$$

Equation 2-3

where $k^2$ is equal to $P / EI$. The solution that satisfies the condition of Equation 2-3 is

$$v = A \sin kx + B \cos kx + Cx + D$$

Equation 2-4

where $x$ is the coordinate of a point in the $x$-direction of Figure 2.3. The term $v$ is displacement in the $y$ direction of Figure 2.3.

To determine the critical load, four boundary conditions must be specified because Equation 2-3 is a fourth order differential equation. At each boundary, $x = 0$ and $x = L$, a kinematic and natural boundary condition will be specified. From the kinematic boundary conditions (Section A.1.4), it was
determined that $\delta v = 0$ at $x = 0$ and $x = L$. Therefore, $v(0)$ and $v(L) = 0$. In addition, it is also shown that the bending moment at the ends of the column is equivalent to zero. Therefore,

$$\left( \frac{d^2v}{dx^2} \right)_{x=L} = 0 \quad \text{and} \quad \left( \frac{d^2v}{dx^2} \right)_{x=0} = 0 \quad \text{Equation 2-5}$$

Using the conditions $v(0) = 0$ and $\left( \frac{d^2v}{dx^2} \right)_{x=0} = 0$, we obtain $B = D = 0$ from Equation 2-5.

Therefore, the deflection function, (Equation 2-4) reduces to

$$v = A \sin kx + C x \quad \text{Equation 2-6}$$

Using the conditions $v(L) = 0$ and $\left( \frac{d^2v}{dx^2} \right)_{x=L} = 0$, we obtain

$$A \sin kL + C L = 0 \quad \text{Equation 2-7}$$

and

$$-A k^2 \sin kL = 0 \quad \text{Equation 2-8}$$

In matrix form

$$\begin{bmatrix} \sin kL & L \\ -k^2 \sin kL & 0 \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{Equation 2-9}$$

This is the eigenvalue problem and the nontrivial solution can be obtained by specifying that the determinant of the coefficient matrix equal zero. Therefore,

$$\det \begin{bmatrix} \sin kL & L \\ -k^2 \sin kL & 0 \end{bmatrix} = 0 \quad \text{or} \quad k^2L \sin kL = 0 \quad \text{Equation 2-10}$$

Since $k^2$ is not equal to zero, then $\sin kL$ must equal zero. Therefore, $kL$ must equal $n\pi$. The critical load can be obtained by substituting in $P / EI$ for $k^2$ and by setting $n = 1$ to give

$$P_{cr} = \frac{\pi^2EI}{L^2} \quad \text{Equation 2-11}$$
The critical load can also be found by using the Energy Approach. The $\delta^{(2)} \pi$ ceases to be positive definite at the critical load. This criteria can also be rewritten as $D_n = 0$ where $D_n$ is

$$D_n = \det \begin{bmatrix} \frac{\partial^2 U_1}{\partial q_1 \partial q_1} & \frac{\partial^2 U_1}{\partial q_1 \partial q_2} & \ldots & \frac{\partial^2 U_1}{\partial q_1 \partial q_n} \\ \frac{\partial^2 U_2}{\partial q_1 \partial q_1} & \frac{\partial^2 U_2}{\partial q_1 \partial q_2} & \ldots & \frac{\partial^2 U_2}{\partial q_1 \partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 U_n}{\partial q_1 \partial q_1} & \frac{\partial^2 U_n}{\partial q_1 \partial q_2} & \ldots & \frac{\partial^2 U_n}{\partial q_1 \partial q_n} \end{bmatrix}$$

Equation 2-12

With this background, the pin-ended column is reexamined using the energy approach. The general solutions to the axial and bending deformation of the column are

$$u = \sum_{n=1}^{\infty} B_n x$$

Equation 2-13

$$v = \sum_{n=1}^{\infty} A_n \sin \frac{n \pi x}{L}$$

Equation 2-14

respectively.

The total potential energy is

$$\pi = \frac{1}{2} \int_0^L E A \left( \frac{d^2 u}{dx^2} \right)^2 dx + \frac{1}{2} \int_0^L E I \left( \frac{d^2 v}{dx^2} \right)^2 dx - \frac{P L}{2} \int_0^L \frac{d u}{dx} dx - \frac{P L}{2} \int_0^L \left( \frac{d v}{dx} \right)^2 dx$$

Equation 2-15

The general solutions must be applied to the total potential energy equation. A detailed breakdown of the strain energy and force potential with general solutions applied are listed in the Appendix section A.1.5.

$$\pi = \frac{1}{2} E A \sum_{n=1}^{\infty} n^2 B_n^2 \left( \frac{L}{2} - \frac{L \sin[2 \pi n]}{4 \pi n} \right) + \frac{1}{2} E I \sum_{n=1}^{\infty} n^4 A_n^2 \left( \frac{L}{2} - \frac{L \sin[2 \pi n]}{4 \pi n} \right) - \frac{P}{2} \sum_{n=1}^{\infty} L^n B_n$$

Equation 2-16

The second variation of the total potential energy of the column is

$$\delta^2 \pi = \frac{\partial^2 \pi}{\partial A \partial A} \delta A \delta A + \frac{\partial^2 \pi}{\partial A \partial B} \delta A \delta B + \frac{\partial^2 \pi}{\partial B \partial A} \delta B \delta A + \frac{\partial^2 \pi}{\partial B \partial B} \delta B \delta B$$

Equation 2-17

More specifically,

$$\delta^2 \pi = \sum_{n=1}^{\infty} \left( \frac{E I n^6 \pi^4}{L^4} - \frac{P n^2 \pi^2 L}{L^2} + \frac{L \sin[2 \pi n]}{4 \pi n} \right) \frac{\delta A_n^2}{1+2 \pi} + \sum_{n=1}^{\infty} \frac{A E L^{-1+2 \pi} n^2}{-1+2 \pi} \frac{\delta B_n^2}{-1+2 \pi}$$

Equation 2-18

Here, the critical load can be determined by evaluating the neutral equilibrium condition (i.e. $\delta^{(2)} \pi = 0$) because it is the state between which $\delta^{(2)} \pi$ transitions between being $> 0$ and $< 0$. In Equation 2-18,
δA_n^2 is an arbitrary value and the quantity \( \frac{AE\ell^{1+2n}n^2}{-1+2n} \) is a positive quantity because all of the variables are positive and constant. Therefore, \( \delta B_n^2 \) must equal zero in order to satisfy the critical condition. Since \( \delta B_n^2 \) is a kinematically admissible quantity, this demonstrates that axial deflection does not have a role in determining the critical load or stability of the column in this problem. Therefore, the critical load is determined by equation

\[
\frac{EI_n^4}{L^4} \left( \frac{L}{2} - \frac{L \sin(2n \pi)}{4n \pi} \right) - \frac{P_n^2 \pi^2}{L^2} \left( \frac{L}{2} + \frac{L \sin(2n \pi)}{4n \pi} \right) = 0
\]

Equation 2-19

For \( n = 1 \), the critical load and the buckling load are

\[
P_{cr} = \frac{\pi^2 EI}{L^2}
\]

Equation 2-20

The critical load determined is identical to the critical load calculated is using the equilibrium approach.

**Example Problem 2: One Degree of Freedom Model – Large Deflection Analysis**

In this section, a one degree of freedom model (Figure 2-4) is used to demonstrate the dynamic approach to resolving the stability of a system. In Figure 2-4, the rigid bar has length \( l \). It is hinged at one end and free at the other. It is also supported through a frictionless ring connected to a spring that can move only horizontally. The free end is loaded with a force \( P \) in the direction of the bar and it is assumed that the direction of the force remains unchanged throughout the motion of the system.
Dynamic Approach

The objective of this approach is to determine the value of the load at which the free motion of the system, in the vicinity of its settled position, ceases to be bounded. A full derivation of the equation of motion of the system using Lagrange’s equation is given in the Appendix Sections A.2.1 and A.2.2. The equation of motion of the system is shown below:

\[ l \ddot{\theta} + a^2 k \sec^2[\theta] \tan[\theta] = l P \sin[\theta] \quad \text{or} \quad l \ddot{\phi} - l P \sin[\theta] + \frac{a^2 k \sin[\theta]}{\cos^2[\theta]} = 0 \quad \text{Equation 2-21} \]

This approach is concerned with the behavior of the system’s response to a small disturbance. Therefore, the system can be linearized to analyze the response of the system in the small. The linearization of \( \theta, \sin \theta, \) and \( \cos^3 \theta \) is shown in Section A.2.2.

The resulting equation of motion is

\[ l \ddot{\phi} - l P \sin(\theta_{eq}) - \varphi l P \cos(\theta_{eq}) + \frac{a^2 k (\sin(\theta_{eq}) + \varphi \cos(\theta_{eq}))}{\cos^2(\theta_{eq}) - \varphi 3 \cos^2(\theta_{eq}) \sin(\theta_{eq})} \quad \text{Equation 2-22} \]

The equilibrium of the system is located at \( \theta_{eq} = 0 \) and Equation 2-22 can be re-written as

\[ l \ddot{\phi} - \varphi l P + a^2 k \varphi = 0 \quad \text{or} \quad l \ddot{\phi} + ( -l P + a^2 k ) \varphi = 0 \quad \text{Equation 2-23} \]

The solution to Equation 2-23 is dependent on the sign of \( (k a^2 - P l) \).
Case 1: When \((ka^2 - P) > 0\), the solution can be expressed as

\[
\varphi(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t
\]

Equation 2-24

where \(\omega_n = (ka^2 - P)^{1/2}\)

In this case, the solution is demonstrated to be stable for a small initial perturbation represented by \(A_1\) and \(A_2\).

Case 2: When \((ka^2 - P) < 0\), the solution can be expressed as

\[
\varphi(t) = B_1 e^{\alpha t} + B_2 e^{-\alpha t}
\]

Equation 2-25

where \(\alpha = (P - ka^2)^{1/2}\)

For initial conditions \(\theta(t = 0) = \theta_o\) and \(\dot{\theta}(t = 0) = \dot{\theta}_o\), \(f(t)\) becomes,

\[
\varphi(t) = \frac{1}{2\alpha} \left[ (\alpha \theta_0 + \dot{\theta}_0) e^{\alpha t} + (\alpha \theta_0 - \dot{\theta}_0) e^{-\alpha t} \right]
\]

Equation 2-26

This shows that \(f(t)\) increases exponentially with time; hence the motion is unstable.

Case 3: When \((ka^2 - P) = 0\), the governing equation reduces to \(\ddot{\theta} = 0\) and the solution can be obtained directly by integrating twice as

\[
\varphi(t) = C_1 t + C_2
\]

Equation 2-27

For initial conditions \(\theta(t = 0) = \theta_o\) and \(\dot{\theta}(t = 0) = \dot{\theta}_o\), \(f(t)\) becomes,

\[
\varphi(t) = \theta_0 t + \theta_o
\]

Equation 2-28

This equation shows that when \(\dot{\theta} = 0\) and \(\theta_0 \neq 0\), then \(\varphi(t)\) is constant for a small initial perturbation. In this case, \(P = \frac{k a^2}{l}\) and is a point of bifurcation and stable equilibrium. When \(\dot{\theta}_0 \neq 0\) and \(\theta_0 \neq 0\), then \(\varphi(t)\) is will grow linearly for a small initial perturbation. In this case, the equilibrium is unstable. Overall, \(P = \frac{k a^2}{l}\) is the critical load because it is the greatest load at which the system is no longer stable.
Example Problem 3: A Snap-Through Model

This analysis demonstrates the type of buckling known as snap-through. Snap-through is characterized by a visible and sudden jump from one equilibrium state to the next (Simitses and Hodges, 1976). As an example consider the mechanism shown below if Figure 2.5.

![Figure 2.5. Snap-Through Model (Simitses and Hodges, 2006, pg 32)](image)

The mechanism is composed of three rigid bars of length $l$ that are pinned together. The left end of the rigid bar mechanism is pinned to an immovable support; the right end is pinned to a block and linear horizontal spring system. The initial configuration of the mechanism is denoted by the angle $\alpha$ where the horizontal spring is unstretched and unloaded. Then, the mechanism is loaded by a load $P$. As the load is increased quasistatically from zero, the spring will be compressed and the two bars make an angle $\theta$ with the horizontal.

In this problem, the objective is to determine the critical load of the system. This will also be the load that will cause the system to snap-through to its second stable state where $\theta$ has negative values. The critical load of the system will be determined using the energy approach.

The strain energy of the system due to the spring is

$$U_{spring} = \frac{1}{2} k x^2 = \frac{1}{2} k \left(2l \cos \alpha - 2l \cos \theta\right)^2$$

Equation 2-29

The potential due to the applied external load is

$$V_{load} = -P \left(l \sin \alpha - l \sin \theta\right)$$

Equation 2-30
The total potential is
\[ \pi = U_{spring} + V_{load} = \frac{1}{2} k (2 l \cos \alpha - 2 l \cos \theta)^2 - P (l \sin \alpha - l \sin \theta) \]  
Equation 2-31

The second variation of the total potential is
\[ \delta^{(2)} \pi = -2 k l \cos \theta (-2 l \cos \alpha + 2 l \cos \theta) - l P \sin \theta + 4 k l^2 \sin^2 \theta \]  
Equation 2-32

If the second variation of the total potential energy is stable, then
\[ \delta^{(2)} \pi = -2 k l \cos \theta (-2 l \cos \alpha + 2 l \cos \theta) - l P \sin \theta + 4 k l^2 \sin^2 \theta > 0 \]  
Equation 2-33

or
\[ P < 4kl \left( \cos \alpha \cot \theta - \cos 2 \theta \csc \theta \right) \quad \text{or} \quad -P > 4kl \left( -\cos \alpha \cot \theta + \cos 2 \theta \csc \theta \right) \]  
Equation 2-34

Since it is known that the critical load represents the load amid the stable and unstable equilibrium states, the critical load is
\[ P = 4kl \left( \cos \alpha \cot \theta - \cos 2 \theta \csc \theta \right) \]  
Equation 2-35

To further explore snap-through, numerical values are given to the parameters in Figure 2.5, \( k = 10 \text{ lbs/in} \), \( l = 10 \text{ in} \), and \( \alpha = 0.611 \text{ rad} \).

Using Equation 2-35, the force-displacement results are shown in Table 2-2 and Figure 2.6. Table 2-2 shows that the system is stable between points 1 and 7 and between points 25 and 32 where the second variation of the total potential energy is positive. Table 2-2 also shows that the system is unstable between points 8 and 24 where the second variation of the total potential energy is negative. The intermediate point between the stable and un-stable equilibrium states is represented by points 7 and a point (not shown) between points 24 and 25. These points are neutral equilibrium states and they define the critical loads if the system. Physically, after the system moves from point 7, it becomes unstable. Then, the system jumps from its current equilibrium configuration shown in Figure 2.6 to its second stable equilibrium position.
In this chapter, stability and the aspects of stability have been defined. In this chapter, the objective of stability analysis was to determine the buckling load of the system. It was shown that the stability of a system can be analyzed in many different ways. The three methods that were explored for the stability analysis of structures were the Classical, Energy and Dynamic approaches. Although the stability of a system changes with respect to time, the stability of the system may be analyzed by using the Classical or Energy method if the changes with respect to time are negligible (e.g. when a load is applied quasi-statically). In the column and one degree of freedom model, it was demonstrated that the buckling load can be defined by using the Classical, Energy and Dynamic methods. It was also demonstrated that the systems may be in a stable or unstable equilibrium state while they are in their initial configuration. Although, in the snap-through problem, the system jumps from its 1st stable configuration to its 2nd stable configuration to maintain stable equilibrium.
References


LaGrange, J. L. 1788. Méchanique analitique, Paris: Desaint


Appendix

Pinned-Ended Column – Small Deflection Analysis

Strain Energy

For a linear elastic system, the strain energy can be expressed as

\[ U = \frac{1}{2} \int \sigma_{ij} \epsilon_{ij} \, dV = \frac{1}{2} \int \left( \sigma_{xx} \epsilon_{xx} + \sigma_{xy} \epsilon_{xy} + \sigma_{yx} \epsilon_{yx} + \sigma_{yy} \epsilon_{yy} + \sigma_{xz} \epsilon_{xz} + \sigma_{yz} \epsilon_{yz} + \sigma_{zx} \epsilon_{zx} + \sigma_{zy} \epsilon_{zy} + \sigma_{zz} \epsilon_{zz} \right) \, dV \]  

Equation 2-36

The deformations that are expected from the applied load on the column are axial shortening and bending. The displacement at a point in the column located at \( x \) due to axial shortening is \( u_1(x) \). According to the assumption that plane sections remain plane, deformation due to bending can be defined by a translation in the \( y \) direction (\( v(x) \)) and a rotation of the plane section (\( dv/dx \)) (Shames and Dym, 1991, pg 187). Therefore, the total displacement at a point on the column due to the applied loading is

\[ u_1 = (u_1)_s - y \frac{dv(x)}{dx} \]  

Equation 2-37

and the corresponding strain is then

\[ \epsilon_{xx} = \frac{d[u_1(x)]_s}{dx} - y \frac{d^2v(x)}{dx^2} \]  

Equation 2-38

The displacement of a point in the \( y \) direction is \( v(x) \). Since plane sections are assumed to act like rigid surfaces and neglecting horizontal moment of a point due to rotations of the plane sections, strain in the \( y \)-direction is

\[ \epsilon_{yy} = 0 \]  

Equation 2-39

Finally, there is no relative motion in the \( z \) direction at time for points in the cross section of the column. Therefore,

\[ \epsilon_{zz} = 0 \]  

Equation 2-40

Assuming that the effect of shear is negligible and that Poisson’s ratio is equal to zero, the strain energy of the column is
The equation can be expanded and shown to be equivalent to
\[
\frac{1}{2} \int_{0}^{L} E A \left( \frac{dv}{dx} \right)^2 dx + \frac{1}{2} \int_{0}^{L} E I \left( \frac{d^2v}{dx^2} \right)^2 dx
\]  
Equation 2-42

**Potential Energy**

The work done by the external load that acts on the column can be represented as a potential energy (Equation 5-8).

\[
W_{ext} = P \Delta = -V
\]  
Equation 2-43

In Equation 5-8, \( \Delta \) represents the shortening of the column and the negative sign on \( V \) denotes that the external force decreases the potential energy of the system. It is important to note that the factor one-half does not appear in Equation 5-8 because the external load is constant as the column passes from the straight to the slightly bent configuration. The shortening of the column, \( \Delta \), consist of two parts: axial shortening, \( \Delta_a \), and bending shortening, \( \Delta_b \).

The axial shortening, \( \Delta_a \), can be obtained by integrating the axial strain \( \epsilon_s = \frac{d[u(x)]}{dx} \) over the length of the column, that is
\[
\Delta_a = \int_{0}^{L} \epsilon_a \ dx = \int_{0}^{L} \frac{d[u]}{dx} \ dx
\]  
Equation 2-44

The bending shortening, \( \Delta_b \), can be represented by Equation 2-45 and the derivation is shown below.

\[
\Delta_b = \int_{0}^{L} d \Delta_b = \frac{1}{2} \int_{0}^{L} \left( \frac{d[v]}{dx} \right)^2 \ dx
\]  
Equation 2-45

The differential shortening due to rotation \( \Theta \) is \( d \Delta_b = dx (1 - \cos \Theta) \) where \( \Theta \) is the angle between the un-deformed and deformed elements. Since a small amount of deflection is assumed, \( \cos \Theta \) can be represented as \( \cos \Theta = 1 - \frac{1}{2} \Theta^2 \) (neglecting higher order terms). Therefore, \( d \Delta_b \) can be represented as, \( d \Delta_b = \frac{1}{2} (\Theta^2) \ dx \). For small bending deformation, \( \Theta \) can be represented by \( dv/dx \). Therefore, the total
shortening in the column can be represented by \( \Delta_a + \Delta_b \). The potential energy of the column in a slightly bent configuration, considering both axial and bending shortening is,

\[
V = -P \Delta = -P (\Delta_a + \Delta_b) = -P \int_0^L \left( \frac{d^2 u}{dx^2} \right) dx - \frac{P}{2} \int_0^L \left( \frac{d^2 v}{dx^2} \right)^2 dx \quad \text{Equation 2-46}
\]

**Total Potential Energy of the Column**

The total potential energy of the column can be expressed as

\[
\pi = U + V = \frac{1}{2} \int_0^L E_A \left( \frac{d u}{dx} \right)^2 dx + \frac{1}{2} \int_0^L E_I \left( \frac{d^2 v}{dx^2} \right)^2 dx - P \int_0^L \left( \frac{d u}{dx} \right) dx - \frac{P}{2} \int_0^L \left( \frac{d^2 v}{dx^2} \right)^2 dx \quad \text{Equation 2-47}
\]

For equilibrium, according to the Principal of Stationary Potential, the first variation of the total potential energy of the column must vanish. Therefore, the condition for equilibrium can be written as

\[
\delta \pi = E_A \int_0^L \left( \frac{d u}{dx} \right) \left( \frac{d \delta u}{dx} \right) dx + E_I \int_0^L \left( \frac{d^2 v}{dx^2} \right) \left( \frac{d^2 \delta v}{dx^2} \right) dx - P \int_0^L \left( \frac{d u}{dx} \right) \delta u dx - \frac{P}{2} \int_0^L \left( \frac{d^2 v}{dx^2} \right)^2 \delta v dx = 0 \quad \text{Equation 2-48}
\]

It must be pointed out that the variational operator \( \delta \) and the differential operator \( d \) are commutative. To transform the terms involving derivatives of \( \delta u \) or \( \delta v \) so that a common factor of \( \delta u \) or \( \delta v \) appear in Equation 5-13, integration by parts. For the first term,

\[
E_A \int_0^L \left( \frac{d u}{dx} \right) \left( \frac{d \delta u}{dx} \right) dx = E_A \frac{d u}{dx} \delta u \bigg|_0^L - E_A \int_0^L \left( \frac{d^2 u}{dx^2} \right) \delta u dx \quad \text{Equation 2-49}
\]

For the second term,

\[
E_I \int_0^L \left( \frac{d^2 v}{dx^2} \right) \left( \frac{d^2 \delta v}{dx^2} \right) dx = E_I \frac{d^2 v}{dx^2} \delta v \bigg|_0^L - E_I \int_0^L \left( \frac{d^3 v}{dx^3} \right) \delta v dx + \frac{E_I}{2} \int_0^L \left( \frac{d^4 v}{dx^4} \right) \delta v dx \quad \text{Equation 2-50}
\]

In Equation 5-15, \( \delta v = 0 \) at \( x = 0 \) and \( x = L \), so the second term on the right side of the equation vanishes.

Using integration by parts, the third term in Equation 5-13 can be written as,

\[
- P \int_0^L \left( \frac{d \delta u}{dx} \right) dx = - P \delta u \bigg|_0^L \quad \text{Equation 2-51}
\]
Finally, the last term in Equation 5-13 can be written as,
\[-P \int_0^L \left(\frac{d^2v}{dx^2}\right) \frac{d\delta v}{dx} \, dx = P \int_0^L \left(\frac{d^2v}{dx^2}\right)^3 \, dx \delta v \, dx\]

Equation 2-52

Substituting Equation 5-14, 5-15, 5-16 and 5-17 into 5-13 and rearranging,
\[
(EA \frac{du}{dx} - P) \delta u \bigg|_0^L - EA \int_0^L \frac{d^2u}{dx^2} \, dx \delta u + EI \int_0^L \left(\frac{d^4v}{dx^4} + P \frac{d^2v}{dx^2}\right) \, dx \delta v = 0
\]

\[
\delta \pi =
\]

Equation 2-53

In order for Equation 5-18 to be satisfied, each term must equal zero. Therefore, from the first term
\[
(EA \frac{du}{dx} - P) \bigg|_{x=L} = 0 \text{ or } \delta u \bigg|_{x=L} = 0 \quad \text{and} \quad (EA \frac{du}{dx} - P) \bigg|_{x=0} = 0 \text{ or } \delta u \bigg|_{x=0} = 0
\]

Equation 2-54

and from the second term
\[
EA \int_0^L \frac{d^2u}{dx^2} \, dx = 0 \quad \text{or} \quad EA \frac{d^2u}{dx^2} = 0
\]

Equation 2-55

and from the third term
\[
EI \left(\frac{d^2v}{dx^2}\right) \bigg|_{x=L} = 0 \quad \text{or} \quad \delta \left(\frac{d^2v}{dx}\right) \bigg|_{x=L} = 0
\]

Equation 2-56

and
\[
EI \left(\frac{d^2v}{dx^2}\right) \bigg|_{x=0} = 0 \quad \text{or} \quad \delta \left(\frac{d^2v}{dx}\right) \bigg|_{x=0} = 0
\]

Equation 2-57

and from the fourth term
\[
\int_0^L \left(\frac{4}{d^4x^4} + P \frac{d^2v}{dx^2}\right) \, dx = 0 \quad \text{or} \quad EI \frac{d^4v}{dx^4} + P \frac{d^2v}{dx^2} = 0
\]

Equation 2-58

**Boundary Conditions**

Note that equations 5-19 and 5-20 pertain to the axial shortening of the column that occurs before buckling, while equations 5-21 through 5-23 pertain to bending shortening that occurs at buckling. In
equation 5-18, the natural boundary conditions are denoted by \( (E A \frac{d^2 u}{d x^2} - P) \), whereas, the kinematic boundary conditions are denoted by \( \delta u \). For the column shown in Figure 2.3, it can be seen that the axial displacement at the end \( x = L \) is not zero, therefore \( \delta u \neq 0 \) at \( x = L \). As a result, from Equation 5-19,

\[
(E A \frac{d u}{d x}) \bigg|_{x=L} = P \quad \text{Equation 2-59}
\]

At \( x = 0 \), it is obvious that

\[
\delta u \big|_{x=0} = 0 \quad \text{or} \quad u \big|_{x=0} = 0 \quad \text{Equation 2-60}
\]

This demonstrates that the axial force at the top end of the column is equal to \( P \) and the axial displacement at the bottom end of the column is equal to zero. Equation 5-20 is simply the derivative of natural boundary condition in 5-19; therefore, no further work is required here.

In Equations 5-21 and 5-22, the natural boundary conditions are denoted by \( (E I \frac{d^2 v}{d x^2}) \) whereas, the kinematic boundary conditions are denoted by \( \delta \left( \frac{d v}{d x} \right) \). For the column shown in Figure 2.3, it can be seen that the end slopes are not zero, therefore \( \delta \left( \frac{d v}{d x} \right) \neq 0 \) at \( x = 0 \) and \( x = L \). Therefore, from Equations 5-21 and 5-22,

\[
E I \left( \frac{d^2 v}{d x^2} \right) \bigg|_{x=L} = 0 \quad \text{and} \quad E I \left( \frac{d^2 v}{d x^2} \right) \bigg|_{x=0} = 0 \quad \text{Equation 2-61}
\]

Since the bending moment is represented by \( M = E I \left( \frac{d^2 v}{d x^2} \right) \), these equations indicate that there is no bending moment at the ends of the column. The differential equation in Equation 5-23 is the governing equation of the column due to bending. This fourth order differential equation is a general formulation and it is applicable to all columns with boundary conditions. The equation shows that the axial deformation that occurs before buckling has no bearing on the evaluation of the critical loads of the column.
Energy Approach

The strain energy due to axial deflection becomes

\[ U_{axial} = \frac{1}{2} \int_0^L E A \left( \frac{d^2 u}{dx^2} \right)^2 \, dx = \frac{1}{2} E A n^2 B_n^2 \int_0^L x^{-2+2n} \, dx = \frac{1}{2} E A n^2 B_n^2 \left[ \frac{x^{2n-1}}{2n-1} \right]_0^L = \frac{1}{2} E A n^2 B_n^2 \left[ \frac{L^{2n-1}}{2n-1} \right] \]

Equation 2-62

The strain energy due to bending deflection becomes

\[ U_{bending} = \frac{1}{2} \int_0^L E I \left( \frac{d^2 v}{dx^2} \right)^2 \, dx = \frac{1}{2} E I n^4 A_n^2 \int_0^L \sin^2 \left( \frac{n \pi x}{L} \right) \, dx = \frac{1}{2} E I n^4 A_n^2 \left( \frac{L^2 - \sin(2 n \pi)}{4 n \pi} \right) \]

Equation 2-63

The force potential due to axial deflection becomes

\[ V_{axial} = -P \int_0^L \left( \frac{d u}{dx} \right) \, dx = -P x^n B_n |_0^L = -P L^n B_n \]

Equation 2-64

The force potential due to bending deflection becomes

\[ V_{bending} = -\frac{P}{2} \int_0^L \left( \frac{d v}{dx} \right)^2 \, dx = -\frac{n^2 P}{2} \left( \frac{L^2 - \sin(2 n \pi)}{4 n \pi} \right) A_n^2 \]

Equation 2-65

One Degree of Freedom Model – Large Deflection Analysis

Derivation of Equation of Motion using Lagrange’s Equation

In this section, the equation of motion of the system is derived using Lagrange’s equation (Equation 5-31).

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q_i}} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial \dot{q_i}} = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0 \]

Equation 2-66

The kinetic energy of the mass is given by

\[ T = \frac{1}{2} I \dot{\theta}^2 \]

Equation 2-67

The potential of the spring and force are given by
\[ V_{spring} = \frac{1}{2} k (\alpha \tan \theta)^2 \quad \text{Equation 2-68} \]

\[ V_{force} = P (l - l \cos \theta) \quad \text{Equation 2-69} \]

The total potential of the system is given by

\[ V_{total} = V_{spring} - V_{force} = \frac{1}{2} k (\alpha \tan \theta)^2 - P (l - l \cos \theta) \quad \text{Equation 2-70} \]

Since the external load reduces the potential of the system, \( V_{force} \) has a minus sign in the equation. The components of Lagrange’s equation are calculated as follows:

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) = L \ddot{\theta} \quad \text{Equation 2-71} \]

\[ \frac{\partial T}{\partial \theta} = 0 \quad \text{Equation 2-72} \]

\[ \frac{\partial V}{\partial \theta} = a^2k \sec^2[\theta] \tan[\theta] - lP \sin[\theta] \quad \text{Equation 2-73} \]

These terms are substituted into Equation 5-31 to give

\[ L \ddot{\theta} + a^2k \sec^2[\theta] \tan[\theta] = lP \sin[\theta] \quad \text{Equation 2-74} \]

**Linearization of \( \theta, \sin \theta, \text{and } \cos^3 \theta **

The linearization of \( \Theta \) is as follows,

\[ \dot{\theta} = \dot{\theta}_{eq} + \ddot{\theta}_{eq} \theta_{eq} = \ddot{\theta} \quad \text{Equation 2-75} \]

The linearization of \( \sin \theta \) is as follows,

\[ \sin \theta \overset{\text{yields}}{\longrightarrow} \sin(\theta_{eq}) + \varphi \frac{\partial}{\partial \theta} (\sin \theta)|_{\theta_{eq}} = \sin(\theta_{eq}) + \varphi \cos(\theta_{eq}) \quad \text{Equation 2-76} \]

The linearization of \( \cos^3 \theta \) is as follows,

\[ \cos^3 \theta \overset{\text{yields}}{\longrightarrow} \cos^3(\theta_{eq}) + \varphi \frac{\partial}{\partial \theta} (\cos^3 \theta)|_{\theta_{eq}} = \cos^3(\theta_{eq}) + \varphi (-3 \cos^2 \theta \sin \theta)|_{\theta_{eq}} = \cos^3(\theta_{eq}) - \varphi (3 \cos^2 \theta_{eq} \sin \theta_{eq}) \quad \text{Equation 2-77} \]
Table 2-2. Force-Displacement Results

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<td>-10</td>
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<td>-45</td>
<td>-327.7</td>
<td>2634</td>
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</table>
Chapter 3

A Bistable Concept for Rotor Blade Tip Twist

Introduction

There are two objectives of this chapter. The first objective is to present a concept for twisting the tip of the rotor blade. The concept that is presented is the use of a bistable twisting coupling device that will twist the tip of the rotor blade when actuated. A possible means of actuating this device is presented in the Summary section of this chapter. The second objective of this chapter is to investigate the feasibility of the bistable twisting concept for helicopter applications. The method that is developed to accomplish this objective is as follows: First, a finite element model of the bistable twisting device is developed. Second, design constraints are developed for the purpose of helicopter rotor blade applications. Finally, the developed finite element model and design constraints are used to investigate the feasibility of the twisting device for helicopter application. The feasibility of these devices was investigated for the BO 105, R22 and SR200 helicopters. The outline of this chapter closely follows the presented objectives. In the next section of this chapter, the benefits of rotor blade twist are presented.

Aerodynamic Significance of Blade Twist

Rotor blade twist is a common feature of helicopter rotor blades and it describes a configuration of the blade where the entire blade is fixed twisted from root to tip. Conventional rotor blades are twisted in a linear or nonlinear pattern with respect to the blade’s root and tip in order to improve performance. Most rotor designs incorporate a negative linear twist (between 8° and 15°) that is a compromise between maximizing rotor performance in hover while simultaneously limiting detrimental effects in high speed forward flight. In the following sections, the effects of blade twist on performance in hover and forward
flight are presented. The concept of blade tip pitch is also presented where the tip of a twisted blade is pitched for the purpose of increased flight performance.

**Blade Twist, Hover**

The work done on a rotor to generate lift leads to a gain in kinetic energy in the air in the wake of the rotor. This unavoidable energy loss is called induced power and it consumes a majority of the total power required of a rotor in hover and low speed forward flight. It has been demonstrated in the literature that a rotor’s performance in hover can be determined by evaluating the rotor’s total power requirement and Figure of Merit. The rotor’s total power requirement in hover is defined by the coefficient of power (Equation 3-1).

\[
C_p = \frac{\delta C_T^{3/2}}{\sqrt{2}} + \text{Profile Power Loss}
\]

Equation 3-1

The quantity \(\frac{\delta C_T^{3/2}}{\sqrt{2}}\) is the induced power. The quantity \(\delta\) is called an induced power factor. The power coefficient is derived from rotor and flight tests and it includes nonideal effects such as nonuniform inflow and tip losses. The quantity \(C_T\) is the coefficient of thrust. Here, profile power is the power that is necessary to overcome drag forces.

The Figure of Merit (FM) is a nondimensional measure of hovering thrust efficiency (Equation 3-2). The FM is the ratio of ideal power versus actual power required to hover. The ideal power does not include nonideal effects encountered in flight and it represents the minimum total power requirement from a hovering blade. Bagai and Leishman (1992) demonstrate that Equations 3-1 and 3-2 are sufficient in assessing the total power and FM of a hovering rotor, respectively.

\[
FM = \frac{\text{Ideal power required to hover}}{\text{Actual power required to hover}} = \frac{C_T^{3/2}}{\sqrt{2}} + \text{Profile Power Loss}
\]

Equation 3-2

Keys (1987) conducted a rotor test to determine the effect of twist on hover power required and FM (Figure 3.1). The rotor was ten feet in diameter and composed of four Mach-scale composite rotor blades. The expression “REF” in Figure 3.1 refers to the baseline level rotor blade where \(C_T/\sigma_T\) is 0.08 and
blade twist is 11.5 degrees. Results in Figure 3.1(a) demonstrates that increasing blade twist reduces the power requirement. Figure 3.1(b) indicates that the FM is also increased by 2.4% from the baseline rotor blade’s FM at a $C_L/\sigma_T$ of 0.08. This amount of improvement corresponds to a 160 lb increase in hover gross weight capability or a 5% increase in useful load for a 10,000 lb single rotor helicopter. Therefore, it is concluded that rotor blade twist is useful in hover flight to increase performance.

![Figure 3.1](image)

**Figure 3.1.** (a) Effect of Twist on Hover Power Required, (b) Effect of Twist on Hover Figure of Merit (Keys, 1987)

**Blade Twist, Forward Flight**

In forward flight, Keys (1987) demonstrates that increasing blade twist from 11.5 degrees to 17.3 degrees results in an increase in the power requirement (Figure 3.2(a)). At an advanced ratio of 0.434 (180 kt, 700 ft/sec tip speed) and $C_L/\sigma_T$ of 0.08, Figure 3.2(b) demonstrates a measured power increase of 5% when blade twist is increased. This effect is also complimented by degradation in rotor lift to effective drag ratio which is shown in Figure 3.2(b). Lastly, Keys (1987) also showed that an increased blade twist in forward flight also increases hub and blade vibratory loads. It has been demonstrated that blade twist required for optimal hovering flight can be detrimental to performance in forward flight.
Blade Tip Pitch

An additional method to improving rotor performance is by pitching the outboard tip segment independent of the remaining inboard section of a blade (Figure 3.3).

Figure 3.3. The Free-Tip Rotor Schematic. (Stroub, 1980)

Stroub (1980) conducted an analytical study on the effects of blade tip pitch on rotor performance. Comparisons were conducted between the conventional rotor and the Free Tip rotor (Figure 3.3) where each rotor generated the same thrust (13,700 lbs) and propulsive force. The conventional blade twist distribution was -8 degrees and the blade’s radius was 26 ft. The analytical model assumed a nonuniform
downwash distribution based on a prescribed wake. The blade’s lift, drag, and pitch-moment coefficients were calculated using blade-element theory and included compressibility effects, stall, and unsteady effects. A constant pitching moment was applied to the blade tip by a controller and the tip was allowed to continually adjust its own pitch. This resulted in an increase in lift at the tip of the rotor in forward flight and a net decrease in the rotor shaft power requirement. An analysis of the power requirement in forward flight (Figure 3.4 (a)) indicated that the Free Tip rotor required less power than the conventional rotor. In addition, around the azimuth (Figure 3.4 (b)), the Free Tip blade does not experience negative lift in the critical region around $\psi = 90$ degrees.

![Graph showing speed-power characteristics and lift coefficients](image)

Figure 3.4. (a) Comparison of speed-power characteristics at sea level, standard conditions, (b) Instantaneous lift coefficients at the 0.955 radial station for thrust = 13700 lbs. (Stroub,1980)

This concept demonstrates that performance improvements in forward flight can be attained by pitching the tip of a pre-twisted blade. In the next section, a morphing concept is proposed to twist the 20% outboard section of the blade, for the purpose of performance improvement.

**Introduction of Concept: Tip Twist using Bistable Twisting Device**

To increase rotor performance, a large amount of blade twist is needed and therefore new actuation concepts must be considered. A concept that may be useful in achieving a large amount of blade twist is Schultz’s bistable twisting structure (Schultz, 2007). The device consists of curved shells that are
pinched by riveting the short ends together. When pinched the structure assumes one of the equilibrium configurations as shown in Figure 3.5. A twisting device, similar to Schultz’s device, may be used to induce blade tip twist in rotorcraft. One possible method to induce blade tip twist using this device is to place the device within the rotor blade (at the blade’s tip) and allow the blade section to take the shape of the device (Figure 3.6). This concept is further discussed in Chapter 1.

As previously mentioned, the bistable twisting device is to replace the spar of the rotor blade only at the outboard tip of the blade. Therefore, the device must support aerodynamic and inertial load that is exerted at the tip of the blade. Since the device is placed at the tip of the rotor blade, the device will experience large aerodynamic and inertial load that may make the device ineffective in producing large twist at the tip of the rotor blade. Therefore, it is necessary that the performance of the device be examined while the device is loaded by external loads. The objective of this work is to investigate the feasibility of the proposed bistable tip twisting concept for helicopter rotor blade morphing applications. In this work, this is done in four steps. First, a finite element model of the device is developed. Second, the finite element model is validated by comparing the model to experimental results. Third, design constraints are introduced. Fourth, the performance of the device is examined while the device is loaded by external load.

Figure 3.5. Two stable configurations of the steel twisting device (Schultz, 2007)
To study the snap-through behavior of the device, the device is modeled using the dynamic finite-element code, ABAQUS/Explicit version 6.7. The shells are modeled using a 16 x 20 mesh of explicit, linear, shell elements (S4R). Enhanced hourglass control is also used to resist zero energy modes. The materials used to model the device are structural A36 steel or 6061-T6 aluminum. The material properties are shown in Table 3-1. In all examples below, the length of the device is 20 % of the length of the rotor blade of interest. The bistable device is modeled in ABAQUS using two basic steps: 1. Create and Align Shells, and 2. Pinch Shells Together. Each shell is created in ABAQUS by specifying shell radius of curvature, width and length as shown in Figure 3.7. Once both shells are created, the shells are aligned as shown in Figure 3.8.

<table>
<thead>
<tr>
<th>Material:</th>
<th>Steel</th>
<th>Aluminum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson's Ratio:</td>
<td>0.32</td>
<td>0.35</td>
</tr>
<tr>
<td>Density (kg/m)</td>
<td>7850</td>
<td>2720</td>
</tr>
<tr>
<td>Modulus</td>
<td>$207 \times 10^9$</td>
<td>$70 \times 10^9$</td>
</tr>
</tbody>
</table>
Next, the two shells are pinched together by specifying boundary conditions. The boundary conditions used to pinch the shells are shown in Table 3-2. In the table, the variable $\Delta$ represents the initial vertical distance between the central nodes of the top and bottom shells (points A and B) and can be
measured in ABAQUS. In the table, point A refers to the central node at the front end of the top shell. Point B refers to the central node at the front end of the bottom shell. Point C refers to the central node at the rear end of the top shell. Point D refers to the central node at the rear end of the bottom shell. Boundary condition UR3 in Table 3-2 refers to rotation about the 3 axis. Additional constraints that were applied were coupling constraints at nodes A and B in the 1 and 3 directions. These constraints caused the two shells to be connected once the shells were pinched together.

Contact conditions were imposed on this model to represent contact that may occur between the inner surfaces of the shells when the two ends are pinched together. The contact properties are shown in Table 3-3. The tangential behavior term in Table 3-3 refers to the tangential interaction between the two shells in the finite element model. The tangential interaction selection in this model was frictionless. The normal behavior term in Table 3-3 refers to the normal interaction between the two shells. The objective of the normal behavior function is to limit penetration between parts within a finite element model. The normal interaction behavior that worked best for the shells in this work was the linear function. In using the linear function, a pressure – overclosure slope value must be specified. The slope of linear function, k, selected in this work was between $1 \times 10^{10}$ and $5 \times 10^{10}$. Devices with shell thickness between 0.1 - 1 mm were given a k value of $1 \times 10^{10}$. Devices with shell thickness between 1 – 5 mm were given a k value of $5 \times 10^{10}$. The contact stiffness values given above were chosen because (1.) these values allow even contact of the two shells when they are pinched together in ABAQUS and (2.) the values produces a tip angle that is similar to experimental results. A detailed treatment of contact properties given in Table 3-3 can be found in Abaqus (2007).

<table>
<thead>
<tr>
<th>Location</th>
<th>Boundary Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$U_1 = 0, U_2 = \Delta$</td>
</tr>
<tr>
<td>C</td>
<td>$U_1, U_3 = 0, U_2 = \Delta$</td>
</tr>
<tr>
<td>B</td>
<td>$U_1, U_2 = 0$</td>
</tr>
<tr>
<td>D</td>
<td>$U_1, U_2, U_3, UR3 = 0$</td>
</tr>
</tbody>
</table>

Pinching the shells together was simulated using the ABAQUS Dynamic-Explicit solver. The Explicit solver was chosen because it is computationally efficient for analysis of large models with short
dynamic response times (ABAQUS, 2007). The solver uses the explicit central difference method to calculate displacements. The model also includes geometrically nonlinear effects because large displacement is expected from the device. The step time to pinch the shell together was set to 0.65 seconds.

Table 3-3. Contact Properties

<table>
<thead>
<tr>
<th>Type: General Contact Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangential behavior: Frictionless</td>
</tr>
<tr>
<td>Normal behavior: Linear</td>
</tr>
</tbody>
</table>

Once the shells are pinched in the numerical model (Figure 3.9), snap-through can be predicted by applying additional boundary conditions, a tip moment and viscous pressure. The boundary conditions used to induce snap-through are shown in Figure 3.10 and Table 3-4. Constraints are added in the U2 direction at points E, F, G, and H to model a fixture that grips the bistable device at the rear end while the device rest within the blade. The device must be allowed to move in the U1 and U3 direction at points E, F, G and H, in order for snap through to occur. Additional constraints are added at point I to keep the device from rotating about the 1 and 2 axes. Once boundary conditions from Table 3-4 are applied, a tip moment is applied, as shown in Figure 3.9 to achieve snapping. Since the model has no damping, viscous pressure is applied to the top and bottom surfaces of the device to reduce twisting oscillations in the system due to snap-through.

In the next section, the developed finite model is validated an experiment conducted by Schultz (2008).
Table 3-4. Boundary Conditions, Step: Snap-Through

<table>
<thead>
<tr>
<th>Location</th>
<th>Boundary Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>U1, U3 = 0, U2 = \Delta</td>
</tr>
<tr>
<td>D</td>
<td>U1, U2, U3, UR3 = 0</td>
</tr>
<tr>
<td>E</td>
<td>U2 = 0</td>
</tr>
<tr>
<td>G</td>
<td>U2 = 0</td>
</tr>
<tr>
<td>F</td>
<td>U2 = 0</td>
</tr>
<tr>
<td>H</td>
<td>U2 = 0</td>
</tr>
<tr>
<td>I</td>
<td>U1, U2 = 0</td>
</tr>
</tbody>
</table>
Comparison to Experimental Results

To validate the finite model developed, the tip angle of the model is compared to the tip angle of a twisting device developed by Schultz (2008). In his work, Schultz conducted an experiment that measured tip angle and central thickness of a bistable twisting device made of low carbon steel as shown Figure 3.5. Schultz defines the tip angle of the device as the angle between the root and tip end of the device. Central thickness is the thickness at the midpoint along the length of the device. This thickness is the maximum thickness along the length of the device. The device size used in Schultz’s experiment was 203 x 44.5 mm². The thickness of the individual shells of the device is 0.324 mm. Schultz measured the tip angle of the device to be 16 degrees and the central thickness to be 7.6 mm.

A comparison of the results obtained from Schultz’s experiment to results developed using the finite element model in this work was conducted. The results from the comparison are shown in Figure 3.11. The tip angle results are normalized by 16 degrees. The mid-point thickness results are normalized by 7.6 mm. The x-axis of Figure 3.11 plots the tip angle and midpoint thickness of the device. The y-axis of Figure 3.11 plots non dimensional value. The blue blocks represent the measurements form Schultz’s experiment. The red blocks represent results from the developed finite element model where the shells of the device have a curvature of 11.2 (1/m). The green blocks represent results from the developed finite element model where the shells of the device have a curvature of 13.2 (1/m). Schultz documented in his work that the initial curvature of the shells of the manufactured device was not uniform, and that the actual shell curvatures could not be accurately determined. Therefore, when Schultz compared the tip angle of his developed finite element model to the tip angle measured by experiment, two different devices models were used. Schultz presumes that the shell curvatures of the manufactured device are between 11 and 14 (1/m). Likewise, when comparing the results from the finite element model developed in this work to the experiment conducted by Schultz, two device models are used. The shell curvatures of the device models are 11.2 and 13.2 (1/m). Results show that the experimentally measured tip angle is within the two numerically calculated tip angles. Therefore, it is concluded that the finite element model developed in this work accurately models the behavior of the twisting devices.
The next step in investigating the feasibility of the twisting devices for helicopter rotor blade morphing is to present design constraints.

![Figure 3.11. Comparison of Shell Curvature and Tip Angle](image)

**Design Constraints**

The goal of this work is to evaluate the performance of a bistable twisting device under aerodynamic and inertial load. This work considers only the lift and centrifugal loads. Drag loads are not considered. In order to evaluate the performance of the twisting device, devices are designed for two helicopters: BO105 and the SR200. The geometric properties for these helicopters are shown in Table 3-5.

The devices are designed to fit inside of the rotor blades by subjecting the device design to a geometric constraint as shown in Figure 3.11. The geometric constraint is used to ensure that the device fits inside the cross-section of the rotor blade. In this work, the midpoint thickness of the device is constrained to be less than 80% of $t_{\text{max}}$ as denoted by the box in Figure 3.11. The midpoint thickness of the device is the maximum measured thickness of the device along the length of the device. This measurement is taken near the halfway point along the length of the device. The parameter, $t_{\text{max}}$ is defined as the maximum thickness of the airfoil. The horizontal boundary is denoted by $H_{\text{max}}$. $H_{\text{max}}$ constrains the
width of the device to remain within the airfoil. In this work, the geometry of the device is restricted such that the width of the pinched shells, $H$, is less than or equal to $H_{\text{max}}$. The length of the devices is 20% of the radius of the rotor blade. The axis of twist of the bistable device is placed at the quarter-chord. According to experimental results for the NACA 0012 shown in Anderson (2005), the quarter chord location is the location of the center of pressure. The center of pressure is the location on the airfoil where the summation of moments due to the pressure and shear stress distribution on the airfoil’s surface is equal to zero. Therefore, putting the device at this position alleviates the device from supporting a twisting moment.

Once it is determined that a particular device is satisfies the geometric constraint, then load carrying capability of the device is investigated. The lift load carrying capability of the device is examined first. Then, an investigation of the capability of the device to support centrifugal load is conducted.

![Figure 3.11. Airfoil with Design Domain](image)

As a first step of an analysis of the aerodynamic loads that act at the tip of the rotor blade, only the lift load during the hover flight condition is considered in this work. An idealized lift load profile over a helicopter rotor blade in hover flight (Leishman, 2006) is shown in Figure 3.12. The $y$-axis is normalized by the (coefficient of lift of the rotor blade) * (the density of air) * (the planform area of the blade) * (0.5) * (the angular velocity of the radial position)$^2$ * (radial position). For this calculation, the blade is un-twisted. The $x$-axis is normalized by the span of the rotor blade. This profile does not include tip loss over the outboard 6% section of the rotor blade. The bistable twisting device is located in the 20% outboard section of the radius of the rotor blade. By comparing the area under the Lift Load Profile curve shown in Figure 3.12 for blade radius region 0 to 0.8 to the area under the curve for blade radius region 0.8 to 1, it
was calculated that the bistable twisting device must support approximately 36% of the lift load that acts over the rotor blade.

In addition, it is also assumed that the bistable device should deflect no more than 10% of the length the device due to the lift load. This restriction has been established to prevent excessive tip deflection due to the lift load. Therefore, an additional displacement boundary condition that is equivalent to 10% of the device’s length is applied to the tip end of the pinched device. The 36% lift load constraint and the 10% tip deflection constraint are combined and modeled as a displacement constraint at the tip of the bistable twisting device. The displacement constraint is applied in the 2 direction as shown in Figure 3.8 at a tip node of the device. The reaction force at this tip node is calculated to determine the device’s lift load carrying capability. If the reaction load at the tip node is greater than the 36% lift load constraint, then the device satisfies the loading constraint. If the reaction load at the tip node is below the 36% lift load constraint, then the device does not satisfy the loading constraint. Table 3-5 shows the approximate amount of lift load the blade tips are required to carry for several helicopter type.

Figure 3.12. Idealized Lift Load Profile over a Blade in Hover Flight
Table 3-5. Lift Loads (Sekula, 2002, Rotomotion, 2010)

<table>
<thead>
<tr>
<th>Helicopter Type</th>
<th>BO105</th>
<th>SR200</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blade Chord (cm)</td>
<td>38.70</td>
<td>10.00</td>
</tr>
<tr>
<td>Weight (N)</td>
<td>21,224</td>
<td>76.60</td>
</tr>
<tr>
<td>Number of Blades</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Blade Radius (m)</td>
<td>4.93</td>
<td>1.5</td>
</tr>
<tr>
<td>Lift in each Blade (N)</td>
<td>5,306</td>
<td>38.00</td>
</tr>
<tr>
<td>Assumed Lift at Blade Tip (N)</td>
<td>$\textbf{1,910}$</td>
<td>$\textbf{14}$</td>
</tr>
</tbody>
</table>

In final constraint applied is the ability of the device to support centrifugal force. The centrifugal force that acts on the device is defined by the product of the mass of the device and the acceleration of the device. The centrifugal force vector is defined with respect to coordinate system $e_x, e_y, e_z$ as shown in Figure 3.13. The coordinate system $e_x, e_y, e_z$ is a moving reference frame and moves with the rotor blade.

The definitions of terms in Figure 3.13 are as follows:

$\theta =$ azimuth angle

$SH =$ spar height

$x_{blade} =$ the length in the $e_x$ direction that starts at the hub and stops at the rear end of the device boundary

$y_{blade} =$ the length in the $e_y$ direction that starts at the lead edge of the rotor blade and stops at the center of the twisting device

$x =$ $x$ coordinate of a point on the device from the origin of the $e_1, e_2, e_3$ axis

$y =$ $y$ coordinate of a point on the device from the origin of the $e_1, e_2, e_3$ axis

$z =$ $z$ coordinate of a point on the device from the origin of the $e_1, e_2, e_3$ axis

As a simplification, the blade is not allowed to flap. The position vector is given in Equation 1. The analysis only considers the rotor operating at normal operating speed in hover flight. The angular...
velocity ($\dot{\vartheta}$) of the rotor is equal to 40 radians/second. It does not consider rotor spin up or rotor spin down. Therefore, $\dot{\vartheta} = 0$. Coordinate system $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ shown in Figure 3.13 is centered at the midpoint node at the rear end of the device. The coordinates of Point A are defined by quantities $\overline{x}$, $\overline{y}$ and $\overline{z}$. The quantities $\overline{x}$, $\overline{y}$ and $\overline{z}$ are defined with respect to the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ coordinate system as shown in Figure 3.13. The acceleration vector is given in Equation 2.

$$\ddot{r} = (x_{\text{blade}} + \overline{x}) \mathbf{e}_x + (y_{\text{blade}} + \overline{y}) \mathbf{e}_y + \overline{z} \mathbf{e}_z$$  \hspace{1cm} \text{Equation (3)}$$

$$\ddot{r} = T_1 \mathbf{e}_x + T_2 \mathbf{e}_y$$  \hspace{1cm} \text{Equation (4)}$$

where

$$T_1 = -(x_{\text{blade}} + \overline{x}) \dot{\vartheta}^2$$  \hspace{1cm} \text{Equation (5)}$$

$$T_2 = \left(y_{\text{blade}} - \overline{y} - \frac{\sin \vartheta}{\overline{z}} \right) \dot{\vartheta}^2$$  \hspace{1cm} \text{Equation (6)}$$

The components ($T_1$ and $T_2$) of the acceleration equation are multiplied by the density of the device to generate body force equations. The developed body force equations are used in a dynamic analysis conducted using the developed finite element model in Abaqus to determine the effect of inertial load on device. In the Abaqus model, the prescribed body forces are defined with respect to coordinate system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ as shown in Figure 3.13.
The quantities that are measured to examine the capability of the device to support centrifugal load are the device tip angle and the maximum von Mises stress within the device. Measuring the tip angle of the device is a way to determine if the device is bistable. In observing the static equilibrium position of the pinched shells, the observer must determine if the pinched shells are twisted or untwisted (Figure 3.14 and Figure 3.15). If, when the shells are pinched together they remain untwisted, then the device is unistable. If, when the shells are pinched together they immediately twist, then the device has bistable configurations. Devices with bistable configurations are of interest in this work.

The von Mises stress is used as the yield criterion. If von Mises stresses within the twisting device exceeds the tensile strength (i.e. yield strength) of the material of the twisting device, then the device undergoes yielding. Yielding is the condition where a material experiences plastic deformation instead of elastic deformation while under load.
Results and Discussion

In this section, an examination of the performance of the device is conducted when the device is loaded by external load. The steps that are taken to conduct the study are as follows: First, devices are designed for a pre-selected helicopter and the geometric constraints are applied. Second, those devices that satisfy the geometric constraints are used to examine the performance of the device while the device is loaded by lift and centrifugal load. In this analysis, the lift and centrifugal loads are applied separately to the device. In the first example presented below, devices are designed for the BO105 helicopter. In the second example, devices are designed for the SR200 unmanned helicopter.

In the results and discussion given below, all of the parameters and calculated quantities are normalized by a force, length and mass scale. In order to apply this normalization, a material for the device must be pre-selected. The force scale = (material modulus) x (length of the device)$^2$. The length scale = (length of the device). The mass scale = (material density) x (length of the device)$^3$. The selected normalization scale definition simplifies the finite element model and generalizes the results. As a result of normalization, the results shown below are valid for any given homogenous isotropic material and device length. However, the lift constraint is dependent upon the material of the device because the lift load is (in part) normalized by the force scale as defined above. For example, the lift load requirement at the tip of the BO105 rotor blade is 1911 N. If aluminum is selected as the material of the bistable twisting device, then the normalized lift load constraint is $2.69 \times 10^{-8}$. However, if steel is selected as the material of the bistable twisting device, then the normalized lift load constraint is $9.49 \times 10^{-9}$. The material of the devices selected for the studies below is Aluminum 2014. Aluminum alloy is selected because of its superior strength to weight ratio in comparison to other aircraft materials such as steel. Aluminum alloy is widely used in aircraft applications and it is cheaper to manufacture than other widely used aircraft materials such as composites (Megson, 2007).
Example 1: BO105

In the first example, a set of bistable devices are examined for the BO105 rotor blade. It is assumed that the airfoil shape is NACA 0012. The imposed geometric constraints are \( t \leq 3.8 \times 10^{-2} \) and \( H \leq 2 \times 10^{-1} \). Designs with various values of shell curvature, thickness and width were evaluated. Shell thickness is the thickness of each individual shells of the device. The shell curvature range is \( 0.5 \leq c \leq 12 \); the shell thickness range is \( 6 \times 10^{-4} \leq t \leq 2 \times 10^{-2} \). These ranges of curvature and shell thickness were chosen based on the geometric constraint. The shell curvature and thickness were incremented, and the midpoint thicknesses of the devices were calculated. In this study, the parameters used to normalize the results are as follows: material modulus = \( 2.07 \times 10^{11} \) (N/m\(^2\)), density = 7850 (kg/l\(^3\)), length of the device = 0.986 m.

Figure 3.17 plots the non dimensional shell curvature versus non dimensional shell thickness for devices that were designed for the BO105 helicopter. The red square symbol represents designs that do not satisfy the geometric constraint. The blue diamond symbol represents designs that satisfy the geometric constraint. A list of the numerical data found in Figure 3.17–Figure 3.21 can also be found Johnson (2010). Results in Figure 3.17 show that designs that satisfy the geometric constraint have a curvature, \( c < 3 \), and a thickness, \( t < 4 \times 10^{-3} \). In addition, the results in Figure 3.17 show that both curvature and thickness have an effect on the midpoint thickness of the device. For example, a device that has shells with a curvature = 3 and thickness = \( 2 \times 10^{-3} \) satisfies the geometric constraints; whereas, a device that has shells with a curvature = 3 and thickness = \( 2 \times 10^{-3} \) does not satisfy the geometric constraints. In another example, a device that has a shell curvature = 3 and shell thickness = \( 3 \times 10^{-3} \) satisfies the geometric constraints; whereas, a device that has shells with a curvature = 3 and thickness = \( 4 \times 10^{-3} \) does not satisfy the geometric constraints.

Physically, the violation of the geometric constraint occurs when the shells of the device have a high bending stiffness. A shell with high bending stiffness can be obtained when the thickness or curvature of that shell is increased. When two shells are pressed together at their ends to form a twisting device, the pressing forces act to fully flatten both shells. If the shells that are being pressed together have a low bending stiffness, then the device’s midpoint thickness will be small as shown in Figure 3.16.
Alternatively, if the shells that are being pressed together have a high bending stiffness, then the device’s midpoint thickness is large as shown in Figure 3.16. Large device central thickness values occur because both shells are resistant to bending at the midpoint along the length of the device.

![Figure 3.16. Curved Shells](image)

Next, the lift load carrying ability of the devices that satisfy the geometric constraint is evaluated. A study of the effect of the geometric parameters of the shells on the lift load carrying capability of the device is shown in Figure 3.18. In this study, the widths of the shells are 0.193. The x axis plots the non dimensional shell thickness. The y-axis plots the non dimensional force. In the legend, the term ndc means non dimensional shell curvature. The lift load constraint is $2.69 \times 10^{-8}$. The results in Figure 3.18 show that for a constant shell width and curvature, the ability of the device to support a lift load is increased as the shell thickness is increased. This increase can be as much as 7000% as shown in Figure 3.18 when the lift load is compared in the cases where shell curvature is 0.986 and shell thickness is $1 \times 10^{-3}$ and $4 \times 10^{-3}$. However, these devices do not satisfy the lift load constraint as they can only support 12% of the required load.

The increase in the lift load carrying ability of the device is due to the increase in the bending stiffness of the device. The bending stiffness of a shell of the device is increased by increasing the thickness or curvature of the shell (Calladine, 1983). In principal, it is possible to design a device that supports the lift load requirement by giving the shells of the device a large thickness, large curvature or some in between combination of large thickness and large curvature. Yet, the previous study demonstrates
that devices with large shell curvature and thickness typically do not satisfy the geometric constraints. Overall Figure 3.18 indicates that the lift load carrying capability of the device can be increased by increasing shell thickness and curvature while leaving shell width constant.

The final examination that is conducted for this example is an examination of the capability of the twisting devices to support centrifugal force. The quantities of interest are device tip angle and von Mises stress. The devices that are examined in this study are those devices that satisfy the geometric constraints. Figure 3.19 plots the tip angles of the devices before the centrifugal force is applied and after the centrifugal force is applied to the devices. The y-axis plots the tip angle of the devices in degrees. The x-axis plots the device cases that are being considered. The parameters of the shells of the devices that are considered in this study are shown in Table 3-6. In Figure 3.19, the blue blocks represent the instance before the centrifugal force is applied. The red blocks represent the instance after the centrifugal force is applied.

The results in Figure 3.19 show that cases A, B and D have a twist angle of approximately zero degrees before the centrifugal forces are applied. This shows that these devices are unistable. Case C has a tip angle of approximately 16 degrees. Therefore, this device is bistable. However, after the centrifugal force is applied to Case C, the tip angle of the device is approximately 0 degrees. This shows that the centrifugal force acts to untwist the device. Schultz (2008) states that the devices become bistable where there is a release of tensile and compressive axial stresses within the shells of the device. However, the centrifugal force reintroduces large tensile stress to the device, thus causing the device to untwist.

An examination of the von Mises stress within the devices is shown in Figure 3.20. The y-axis plots the non dimensional von Mises stress within the device. The x-axis plots the device cases that are being considered. The blue blocks represent device results before the centrifugal force is applied. The red blocks represent device results after the centrifugal force is applied. The results show that the von Mises stress within the device is increased after the centrifugal force is applied. The region of maximum von Mises stress within the device, before and after the application of the centrifugal load, is located at the pinched end near the section where the device is riveted. The region of maximum von Mises stress is shown below in Figure 3.21.
Figure 3.17. BO-105 Non Dimensional Shell Curvature vs. Non Dimensional Shell Thickness

Figure 3.18. BO-105 Curvature vs. Shell Thickness vs. Central Thickness, Lift Load

Lift Load Constraint

- ndc = 0.49
- ndc = 0.99
- ndc = 1.97
- ndc = 2.9
Table 3-6.

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<th>Non Dimensional Width</th>
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</thead>
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<td>1.57E-01</td>
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<tr>
<td>B</td>
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<td>6.67E-04</td>
<td>1.83E-01</td>
</tr>
<tr>
<td>C</td>
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<td>6.67E-04</td>
<td>1.83E-01</td>
</tr>
<tr>
<td>D</td>
<td>1.95</td>
<td>1.00E-03</td>
<td>1.57E-01</td>
</tr>
</tbody>
</table>

Figure 3.19. Tip Angle of the Cases A through D
Figure 3.20. Nondimensional von Mises Stress of the Cases A through D

Figure 3.21. Bistable Device with Region of Maximum von Mises Stress highlighted
**Example 3: SR200**

In this example, the SR200 unmanned air vehicle helicopter is used in the evaluation of lift and centrifugal force performance of the twisting devices. The SR200 was selected for this example because it is approximately 99.6% lighter than the BO105 helicopter as shown in Table 3.5. The imposed geometric constraints are $t \leq 3.2 \times 10^{-2}$ and $H \leq 2 \times 10^{-1}$ and the assumed airfoil shape is 0012. In this study, the parameters used to normalize the results are as follows: material modulus = $7.00 \times 10^{10}$ (N/m$^2$), density = 2720 (kg/l$^3$), length of the device = 0.3 m.

Figure 3.22 plots non-dimensional shell thickness versus non-dimensional shell curvature. The x-axis plots the non-dimensional shell thickness. The y-axis plots the non-dimensional curvature. The red square symbol represents designs that do not satisfy the geometric constraints. The blue diamond symbol represents designs that satisfy the geometric constraints. Similar to Figure 3.17 of the BO105 example, Figure 3.22 shows that designs that satisfy the geometric constraints appear in the lower left section of the plot. However, in this example, the designs that satisfy the geometric constraints have a shell thickness $< 3 \times 10^{-3}$ and shell curvature $< 2.5$. In addition, Figure 3.22 is very similar to Figure 3.17 because both plots contain many of the same data points. However, Figure 3.22 eliminates those data points that have a shell width value greater than $2 \times 10^{-1}$. A list of the data found in Figure 3.22 and Figure 3.23 can be found in Johnson (2010).

For those designs that satisfy the geometric constraints, the load carrying ability of the device is evaluated. The results in Figure 3.23 show lift load results for those designs that satisfy the geometric constraints. The x-axis plots the non-dimensional shell thickness. The y-axis plots the non-dimensional force. The red square symbol represents designs that have shell curvatures of 1.95. The blue diamond symbol represents designs that have shell curvatures of 2.4. The lift load constraint is $2.16 \times 10^{-9}$. The results in Figure 3.23 show that the device can support (at maximum) 6% of the lift load at the blade’s tip.

The final study that is conducted is an evaluation of the effect of centrifugal force on the device. Here, only one device that satisfies the geometric constraint is evaluated. The parameters for this device
are as follows: curvature = 1.95, thickness = $6.67 \times 10^{-4}$, width = $1.57 \times 10^1$. Figure 3.24 plots the tip angle of the device. Figure 3.25 plots the von Mises stress of the device. Figure 3.24 shows that the device is bistable before the centrifugal force is applied. However, the centrifugal force causes this device to become unistable. In addition, Figure 3.25 shows that the von Mises stress within the device is increased by as much as 10%. These results are consistent with the results presented in Figure 3.19 and Figure 3.20.

Figure 3.22. SR200 - Curvature vs. Shell Thickness vs. Central Thickness
Figure 3.23. SR200 - Curvature vs. Shell Thickness vs. Lift Load, Loading Constraint

Figure 3.24. Tip Angle
In this section, an investigation is conducted of how the geometry of the shells affect the tip angle of the device. Figure 3.26 plots the non dimensional shell thickness versus the tip angle of the device. The x-axis plots the non dimensional shell thickness. The y-axis plot the tip angle of the device as previous defined. The red blocks represent devices that have a shell curvature of 2.4 and a shell width of 0.183. The green triangles represent devices that have a shell curvature of 3.83 and a shell width of 0.21. The purple x points represent devices that have a shell curvature of 3.83 and a shell width of 0.25. The results in Figure 3.26 show that the tip angle is affected by the geometry of the device. In particular, for a given shell curvature and width, tip angle is increased as shell thickness is decreased. This demonstrates that decreasing the thickness of the shells decreases the torsional stiffness of the twisting device.

When Figure 3.18, Figure 3.23 and Figure 3.26 are compared, it is evident that there is a trade-off between the amount of tip angle a device possesses and the lift load carrying capability of the device. Figure 3.26 shows that a device must have thin shells in order to have large tip angle. In addition, the trend
of points in Figure 3.26 shows that as the thickness of the shell is increased, the tip angle of the device approaches zero degrees. This demonstrates that thin shells are also necessary to produce devices that are bistable. However, Figure 3.18 and Figure 3.23 shows that thick shells are required in order for the device to satisfy the lift load constraint. This analysis shows that the devices can be designed to possess large tip angle or to satisfy the lift load constraint. Yet, it is unlikely that devices can be designed the possess large tip angle and satisfy the lift load constraint.

![SR200 Twist Angle vs. Shell Width](image)

**Figure 3.26. SR200 Twist Angle vs. Shell Width**

**Actuation Energy**

Figure 3.27 plots the applied moment versus twist angle for a sample twisting device for the BO105 helicopter. The tip angle as defined by Schultz (2008) is the difference in angle between the root end and the tip end of the device. The twist angle (change in tip angle) in Figure 3.27 is the full angle between the two stable states. In Figure 3.27, the y-axis plots non dimensional tip moment. The x-axis
plots the twist angle in radians. In this study, the parameters used to normalize the results are as follows:

material modulus = $2.07 \times 10^{11} \text{ (N/m}^2\text{)},$ density = $7850 \text{ (kg/l}^3\text{)},$ length of the device = $0.766 \text{ m}$.

In the unloaded case, the device is initially twisted and resides in a stable equilibrium state, point A in Figure 3.27, where its twist angle is assigned a value of zero. A concentrated point moment is applied quasi-statically to tip end of the device to induce snapping. In Figure 3.27, snapping occurs when the device jumps from point B to C. After the loading duration is complete, the applied moment is removed and the device settles at point D. Noise in the data, from points A to D, occurs because there is no damping in the model. As a result, viscous pressure is applied to the top and bottom shells to achieve a steady-state solution at point D. The amount of actuation energy required to snap the device is calculated from the area under the curve from points A to B. To simplify the calculation of the area under the curve from points A to B, the curve between points A and B is approximated by a third order trend-line using Microsoft excel. The third order polynomial equation is given as 

$$y = (6 \times 10^{-9}) x^3 + (7 \times 10^{-9}) x^2 + (3 \times 10^{-9}) x - (5 \times 10^{-12})$$

and this equation is used to generate a smooth curve between points A and B (Figure 3.27). The area under the smooth curve is calculated using the trapezoidal rule and it gives an approximation of the amount of energy required to snap the device. Using the trapezoidal rule, the amount of energy required to snap the device is approximately $2.54 \times 10^{-9}$. This indicates that an actuation system must (at least) be able to generate $2.54 \times 10^{-9}$ to actuate the system. However, the amount of actuation energy may be increased or decreased if aerodynamic and inertial load act on the device. This device may be actuated by a rod-motor system where the rod is inserted through the root-end of the device and applies a moment to the tip-end of the device. For an actuation system of this type, studies must be conducted to determine the energy to weight ratio in order to determine its feasibility.
Figure 3.27. Applied Moment vs. Twist Angle

Summary

In this work, the performance of bistable devices for rotor blade tip twist is examined. In the BO105 example, the devices can carry 12% of the required lift load. Results showed that the lift load carrying ability of the devices can be increased by increasing shell thickness and curvature. In the SR200 example, the devices can carry 6% of the required lift load. A study of the effect of centrifugal force shows that the centrifugal force acts to untwist the device and increase the von Mises stress. A study of the effect of the device’s geometric properties on twist angle indicated the bistable devices have thin shells, but these thin shells prevent the device from supporting the required amount of lift load. Overall, the results presented in this work are a first step in evaluating the performance of bistable twisting devices for rotor blade twist and tip twist.


Table 3-7. Designs in Figure 3.17, BO105 Example

<table>
<thead>
<tr>
<th>Non Dim Cuvature</th>
<th>Non Dim Shell Thickness</th>
<th>Non Dim Central Thickness</th>
<th>Non Dim Device Width</th>
<th>Description</th>
</tr>
</thead>
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<td>1.08E-02</td>
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<td>1.93E-01</td>
<td>satisfies geometric constraint</td>
</tr>
<tr>
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</tr>
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</tr>
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Table 3-8. Designs in Figure 3.18, BO105 Example

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Table 3-9. Designs in Figure 3.22

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<th>Non Dim Central Thickness</th>
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<th>Description</th>
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<td>1.57E-01</td>
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<td>1.57E-01</td>
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Table 3-10. Designs in Figure 3.22

4.20E+00 1.33E-02 1.40E-01 1.57E-01
does not satisfy geometric constraint
Chapter 4
A Bistable Concept for Rotor Blade Chord Extension

Introduction

There are three objectives of this chapter. The first objective of this chapter is to present a new concept for helicopter rotor blade chord extension. The concept that is presented is a bistable arch that is coupled to a flat plate that is supported by rollers. The second objective of this chapter is to select a solution method to resolve the force –displacement relationship of the arch. The solution methods that are evaluated in this chapter are the Pseudo Rigid Body Method and the Finite Element Method. The final objective of this chapter is to investigate how the size and shape of the hinges of the arch affect the performance of the arch. This was done by parametric studies where the width, and length and shape of the hinges were varied.

In the next section of this chapter, the benefits of rotor blade chord extension are presented.

Aerodynamic Significance of Chord Extension

Retreating rotor blade stall is an area of concern in the high speed forward flight, very highly loaded rotors, at high altitudes and during maneuvers of helicopters. Retreating blade stall causes a loss in lift, and a sharp increase in drag and pitching moment. An approach to generate additional lift near stall is to extend the chord at the trailing edge of the blade’s airfoil. Chord extension may also reduce noise and vibration in forward flight. Studies were conducted by Lui et al (2007) and Leon, Hayden and Gandhi (2009) that demonstrate the aerodynamic effects of quasi-statically extending a trailing-edge discrete plate for both symmetric and non-symmetric airfoils, respectively.
Lui et al (2007) conducted experiments and presented the lift and drag coefficients of a baseline NACA0012 airfoil model (Figure 4.1(a) and Figure 4.1(b)). The model used to conduct these experiments had a chord and span of 10 and 12 inches, respectively. End plates were used on the finite blade section to reduce the three-dimensional flow effects. In the test, an aluminum sheet (0.216 mm thick) and a Mylar sheet (0.254 mm thick) was used, separately, to increase the chord of the airfoil. In the literature, these sheets are called static extended trailing edges (SETEs). The baseline rotor had a NACA0012 airfoil with no SETE extension. In the results shown in Figure 4.1(a) and Figure 4.1(b), the sheet is extended by 5 and 10 percent of the chord and deflected by as much as 14 degrees. The results in Figure 4.1(a) show that as the angle of attack is increased, the lift coefficient is increased as the SETE plate is extended by 5 and 10 percent of the airfoil’s chord with and without SETE plate deflection. The drag polar plot shows that the drag penalty is not increased with an increase in SETE extension and deflection.

Figure 4.1. NACA0012 airfoil model with the SETE at Rec = 4.74 x 10^5: a) lift coefficient as a function of the AOA and (b) drag polar. (Lui, 2007)

Leon, Hayden and Gandhi (2009) conducted numerical analysis for a non-symmetric airfoil. In this work, the baseline airfoil is the SC-1094R8 and it was assumed that the addition of a thin extended trailing edge does not significantly modify the aerodynamic coefficients compared to a clean SC-1094R8 airfoil with the same chord. Therefore, airfoil sections with an extended trailing edge were modeled by the baseline airfoil with an increase in chord. Studies were conducted to examine the effect of SETEs on
helicopter performance. In these studies, the trailing edge was extended at a length of 10%, 20% and 30% of the chord and located between the 70–90% radial positions. Results showed that for a given rotorcraft weight, installed power and altitude (high altitude), SETEs may supply an increase in the maximum speed and altitude of the vehicle (Figure 4.2 and Figure 4.3). In addition, for a given airspeed, altitude and within a limited gross weight range, the SETEs reduced the rotor power requirements (Figure 4.4). Likewise, for a given airspeed, altitude and power requirement, the SETEs increased the helicopter’s payload capacity. Finally, the change in power requirement with SETE size and location was examined. Results showed that SETEs located near the outboard section of the blade produced the highest reduction in blade power requirement.

Figure 4.2. UH-60A 18,000 lbs gross weight, altitude versus maximum speed envelope extension using SETEs (Leon, Hayden and Gandhi, 2009)
Figure 4.3. UH-60A 24,000 lbs gross weight, altitude versus maximum speed envelope extension using SETEs (Leon, Hayden and Gandhi, 2009)

Figure 4.4. Rotor power required as a function of gross weight, for 8,000 ft altitude, 112 knots airspeed (Leon, Hayden and Gandhi, 2009)

The previously discussed works showed that chord extension can increase performance in forward flight. The next section introduces a new chord extension concept using bistable mechanisms.
**Introduction of Concept: Bistable Mechanism for Chord Extension**

A new actuation concept for chord morphing presented in this work is the use of a bistable arch for chord extension. A bistable structure is a structure that has two stable energy states. The bistable arch concept as it would be applied to a helicopter rotor blade is pictured in Figure 4.5. As shown in Figure 4.5, the concept is composed of an arch, flat plate and roller track where all components of the system are initially at rest within the blade. This configuration is called the first stable state. When activated, the arch snaps to a second position where the flat plate is positioned outside of the blade, aft of the trailing edge. This configuration is called the second stable state. The arch’s function is to act as an actuator to the flat plate, and the arch is expected to withstand aerodynamic and inertial loads. The bistable arch and plate can increase the chord of the rotor blade (while in flight) by as much as 25% of the blade’s chord. A bistable arch is useful for this application because it can hold the plate in either of its equilibrium positions without the need for additional energy input, and it is a simple system that does not require many moving parts. In addition, the bistable arch was selected for this concept because it has the ability to supply large stroke while only requiring a moderate amount of force to produce this stroke.

![Figure 4.5. Chord Extension Concept in a Helicopter Rotor blade](image)

**Analysis**

When activated, the bistable arch snaps from its first to second stable state as shown in Figure 4.6 (a). The arch has living hinges (flexural pivots with minute stiffness) at its boundaries and the arch-living
hinges system forms a uniform structure. A living hinge can be modeled as a pin joint; hence living hinges in the arch models will be represented by pin joints. In this work, living hinges are used at the boundaries in order to maximize the arch’s stroke. Maximum stroke is desired because the amount of rotor blade chord extension attained is equivalent to the amount of arch stroke. For example, if the arch’s stroke is 10 cm, then the rotor blade’s chord is increased by 10 cm. Another type of device that is also considered is an arch with two segments joined by living hinges (Edwards, Jensen and Howell, 1999) as shown in Figure 4.6 (b). Similar to a one segment arch, the two segment arch is bistable. This type of arch is considered because it is expected to require less force to initiate snap-through than that of the continuous arch shown in Figure 4.6 (a).

For the purpose of analysis, the arch models are developed using two different solution methods: Pseudo Rigid Body Model (PRBM) and nonlinear Finite Element Analysis (FEA).

\[ \text{Figure 4.6. Bistable Arches: (a) Continuous Arch and (b) Two Continuous Arches Joined by Living Hinge (Edwards, Jensen and Howell, 1999).} \]

**Pseudo-Rigid-Body Model (PRBM)**

The aim of the PRBM method is to simplify the model of a system containing highly flexible members for the purpose of analysis and design. The pseudo-rigid-body model represents flexible members of a system as rigid links that are joined by pin joints (Howell, 2001a). The stiffness of a flexible member is modeled by placing a spring at the pin joint of the rigid links. The length of a rigid link in the
The quantity \( n \) can be positive or negative. The quantity \( \phi = \tan^{-1}(-1/n) \) is defined as, \[ \alpha^2 = \frac{P \sin(\phi - \Theta) l^2 (1 + n^2)^{1/2}}{EI} \] where \( P \) is the applied vertical load as shown in Figure 4.7. The quantity \( n \) can be positive or negative. The quantity \( l \) is the length of the segment and \( EI \) is rigidity. For initially curved segments, the characteristic radius is defined as \( pl \) instead of \( \gamma l \). Development of the PRBM method for initially curved segments has been done by Howell (2001a). A summary of the PRBM method is given in the Appendix section Pseudo Rigid Body Model Summary.

![Figure 4.7. PRBM](image)

In this work, it is assumed that the arches (Figure 4.6) will undergo symmetric deformation and therefore, only half of an arch is modeled. It is expected that the arches will undergo good symmetric deformation because the height of the arch is shallow. The PRBM of the half-arch is defined by a series of fixed-pinned segments as described above. For example, the half-arch shown in Figure 4.8(a) is modeled by three initially curved fixed-pinned segments as shown in Figure 4.8(c). The pinned end of Segment 1 in
Figure 4.8(c) is joined to the boundary. As the PRBM of the arch is progressed from its initial configuration to its final configuration, the reaction forces at the pinned end of segment 1 are changed in magnitude (see $P$ and $nP$ in Figure 4.7). To maintain accuracy, the location of the characteristic pivot (hence link lengths) and stiffness coefficient must be updated at every increment of the arch’s motion. This method is called the More Accurate Method. For models with many fixed-pinned segments, the More Accurate Method is implemented by imposing an initial displacement on a node on the link of the mechanism. Then, the mechanism’s motion (i.e. link angles, etc.), required force and reaction forces are calculated for each segment of the system. Next, the values of $n$, the characteristic pivots and stiffness coefficients for each segment are updated. Subsequently, the method can be repeated by prescribing an updated displacement value. A simplified version of this method is to take the average value of the characteristic radius factor, $\gamma$, and spring coefficient as a constant value for each segment of the mechanism throughout the mechanism’s motion. Average values of the characteristic radius factor and spring coefficient for initially straight and curved segments are reported in Howell (Howell, 2001b, 2001c; Edwards, Jensen and Howell, 1999).

The pinned ends of Segments 2 and 3 are joined to form Segment 4. As the half arch (Figure 4.8(a)) undergoes deflection from the first to the second stable state, it is expected that the arch will have an inflection point that travels along the length of the arch during deflection. This inflection point can be represented by the pinned condition in Segment 4 of Figure 4.8(c) because it does not support a moment. Therefore, segment 4 can be considered as a fixed-fixed segment with moving inflection point. There are many complex techniques to model fixed-fixed segments using PRBMs (Lyon, Howell and Roach, 2000; Saxena and Kramer, 1998). These methods are very accurate but they are complicated and difficult to implement in design. As a simplification, Howell (2001b) suggests modeling this type of fixed-fixed segment as a fixed-guided segment. This simplification is less accurate, but practical for modeling fixed-fixed segments. This involves replacing the two rigid links that have a pin joint connection in Figure 4.8(c) with a single rigid link. In this work, this method will be called The Simple Method.
Figure 4.8. (a) Half-Arch, (b) PRBM of Half-Arch, (c) PRBM with Segments Denoted.
In this work, PRBM method is used to approximate the required load needed to move the half-arch from its first to second stable state. Force equilibrium and kinematic constraint equations are determined for the system of links and springs shown in Figure 4.8(b). The unknown link angles and reaction forces are determined using Mathematica’s NMinimize function. NMinimize is a nonlinear constrained global optimizer that can be used to solve multi-variable nonlinear algebraic problems. The optimization problem can be stated as:

\[
\min \sum_{i=1}^{n} f(\theta_i)^2 \quad \text{Equation 4-2}
\]

\[
\text{s.t. } w - \sum_{i=1}^{n} L_i \cos \theta_i = 0, \quad n = 1 \text{ to } m \quad \text{Equation 4-3}
\]

where \(m\) is the maximum number or rigid links in the system. The term \(f\) in Equation 4-2 represents an equation of equilibrium. The quantities \(L_i\) and \(\theta_i\) in Equation 4-3 represent the length and angular position of a given link, respectively. The force equilibrium and constraint equations for the four-bar and three spring half-arch (Figure 4.8(b)) can be written as follows,

\[
f_a: \ P L_a \sin \theta_a - F L_a \cos \theta_a - k_a(\theta_a - \theta_{a, \text{initial}} - \theta_{a+1} - \theta_{(a+1), \text{initial}}) + k_{a-1}(\theta_{a-1} - \theta_{(a-1), \text{initial}}) - \theta_a + \theta_{a, \text{initial}} = 0 \quad a = 1 \text{ to } m - 1 \quad \text{Equation 4-4}
\]

\[
f_m: \ k_{(m-1)}(\theta_{(m-1)} - \theta_{(m-1), \text{initial}}) + M - F L_m = 0 \quad \text{Equation 4-5}
\]

The spring constants are represented by \(k_a\) and \(w\) represents the length of the base of the half-arch. \(P, F\) and \(M\) represent the reaction loads at the sliding boundary condition. In the analysis, the half-arch is moved from its first to second stable state by incrementally changing the values of a preselected \(\theta_i\) value and solving for the unknowns \(P, F, \theta_1\) and \(\theta_2\).

**Finite Element Analysis (FEA)**

The second solution method used to analyze the half arch is nonlinear FEA. In this work, Abaqus 6.7 is used to implement the FEA. The snap-through of the half-arch is modeled using static analysis.
Displacement control is used to simulate snapping the half-arch from its first to second stable state. The snap-through force (F) is calculated as a reaction load at the right boundary of the half-arch. The pinned-end of the half-arch at the left boundary is modeled by setting displacement in the horizontal and vertical direction equal to zero. For the continuous arch, the right boundary of the half-arch has constrained rotation and horizontal displacement (Figure 4.9). For the continuous arches joined by living hinges, the right boundary of the half-arch is modeled by only constraining horizontal displacement (Figure 4.10). In both cases, displacement in the vertical direction is prescribed. The half-arches are modeled in Abaqus by using 2-node linear interpolation in-plane beams with a hybrid formulation. The hybrid formulation option is recommended for beams that are expected to rotate significantly (Abaqus, 2007).

![Beam Elements](image)

**Figure 4.9. Continuous Arch with Boundary Conditions**

![Continuous Arches joined by Hinges Boundary Conditions](image)

**Figure 4.10. Continuous Arches joined by Hinges Boundary Conditions**

**Results**

The following sub-sections, the PRBM method and FEA solution methods are compared and one of these methods is chosen for further study. In addition, the continuous arch and continuous arches joined
by living hinges are also compared and one of these arch types is selected for further study. Following these comparisons, an addition study was conducted to determine the effect of the arch’s rigidity on the energy required to snap the arch.

**Solution Method Comparison**

There are many solution methods that may be used to determine reaction forces, displacements and strain energy of the half-arch. The solution methods used in this work are PRBM and FEA. The FEA method was chosen because of its ability to accurately solve nonlinear large deflection problems. In addition, the FEA method can also be implemented by benchmark solvers, such as Abaqus, that can handle complex geometries without simplifying the problem. The PRBM method was chosen because of its ability to simplify complex flexible systems. In addition, this method has been used in previous work (Edwards, Jensen and Howell, 1999) to accurately model the behavior of a bistable arch. In this study, the Simple Method is used in implementing the PRBM. The baseline material and cross-section shape chosen are Aluminum and rectangular, respectively. Table 4-2 gives the PRBM parameters. The quantity \( \kappa_o = \frac{l}{R} \) is a non-dimensionalized parameter that relates an arch segment’s initial curvature to the segment’s arch length. For the \( \kappa_o \) values given, the corresponding segment spring constants are also given in Table 4-2. The corresponding PRBMs for the cases shown in Table 4-2 are shown in Figure 4.12. The red circles represent springs and the black lines represent the links. A comparison of the force versus displacement and strain energy versus displacement using PRBM and FEA is shown in Figure 4.11. In Figure 4.11, force and strain energy are non-dimensionalized by the baseline arch snap-through force and strain energy as shown in Table 4-1. The x-axis in Figure 4.11 represents non-dimensional displacement where displacement is non-dimensionalized by half of the height of the half-arch (shown in Table 4-1). In the Force vs. Displacement plot (Figure 4.11, left); applied load is plotted versus output displacement. The applied load and output displacement are positive downward as shown in Figure 4.8 (b) and Figure 4.13(a).
Figure 4.11. (top) Force versus Displacement, (bottom) Strain Energy versus Displacement
Table 4-1. Baseline Continuous Arch Parameters

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<th>Arch Parameter</th>
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<td>Modulus (N/m²)</td>
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Table 4-2. PRBM Cases

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<tr>
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<td></td>
<td>0.25,</td>
<td>$L_1 = 0.1243, L_2 = 0.051, L_3 = 0.6027, L_4 = 0.1140$</td>
</tr>
<tr>
<td>Case B</td>
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<td>0.25,</td>
<td>$L_1 = 0.0255, L_2 = 0.2487, L_3 = 0.0510, L_4 = 0.2487, L_7 = 0.0255$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.25,</td>
<td>$L_1 = 4.7, L_2 = 4.7, L_3 = 4.7, L_4 = 4.7, L_5 = 4.7, L_6 = 4.7, L_7 = 4.7$</td>
</tr>
<tr>
<td>Case C</td>
<td>6</td>
<td>0.25,</td>
<td>$L_1 = 4.7, L_2 = 4.7, L_3 = 4.7, L_4 = 4.7, L_5 = 4.7, L_6 = 4.7, L_7 = 4.7$</td>
</tr>
</tbody>
</table>

Figure 4.12. PRBMs that correspond to cases in Table 4-2

The strain energy plot (Figure 4.11, top) shows that both FEA and PRBM illustrate the bistable nature of the arch because all curves have two local minima. In comparison to the FEA model, results
show that PRBM-Case A agrees more closely with the FEA solution than cases B and C. In comparing Case A to Case C, the difference in energy between the systems indicate that adding more segments and springs to the model using the PRBM approach results in an increase in the predicted energy within the system and may not increase the accuracy of results. This indicates that there may be an optimal number of segments and segment lengths that accurately model the force-displacement relationship of the arch. Howell (2001c) suggests that these types of models (models developed using the Simple Method), although inaccurate, are useful for visualizing the motion and predicting the behavior of the system. As suggested above, the accuracy of the PRBM approximations can be improved by allowing the values of $\gamma$ and spring coefficient to change throughout the mechanisms motion using the More Accurate Method.

As shown in Figure 4.11, the PRBM Case B snaps at a lower output displacement than do cases A and C. Physically, this may be due to the location of the stiffer springs within the PRBM as they are nearly adjacent to one another and are located close to the arch’s pinned boundary. The location of the springs has a significant effect on the mechanism’s motion and snap-through displacement. The stiffness of the springs has a significant effect on the required load for snapping. For example, if the springs of the PRBM are rearranged and the spacing between the springs is increased (as in Case A), the snap-through displacement of the arch is increased.

**Arch Comparison**

In this work, Mathematica was used to obtain results for the PRBMs. Solutions generated using Mathematica for the half-arch problem were generated non-iteratively (which is a lengthy process). Solving the system of PRBM equations using a nonlinear iterative solver is complex to implement for this problem because there are many unknown variables. In contrast, the FEA method via Abaqus allows the user to obtain force, displacement and energy data all at once. The amount of data points recorded is specified by the user. Abaqus’s GUI allows the user to specify complex geometries, boundary conditions, mesh and loads without simplifying the problem. It is also able to solve nonlinear large displacements
problems which are studied in this work. Therefore, the remainder of the analysis is conducted using FEA via Abaqus instead of PRBM.

The two arch types that are evaluated using FEA are the continuous arch and the two continuous arches joined by a living hinge as shown in Figure 4.6. In addition, the two input load types considered are a concentrated moment and a concentrated force. The concentrated force was applied at the right boundary as shown in Figure 4.13 (a). The concentrated moment load type is applied to the left boundary as shown in Figure 4.13(b). The strain energy of the arch at each non dimensional displacement increment on the x-axis shown in Figure 4.14 is the sum of the strain energy within each individual element within the arch at each displacement increment.

A comparison of the energy required to snap the arch for different arch and load types is shown in Table 4-3. The results show that the arch and load type that requires the least amount of energy is the continuous arches joined by living hinges with a concentrated load. A comparison of the applied non-dimensional force versus non-dimensional displacement of both arch types is shown in Figure 4-14. Force and energy are normalized by the baseline snap through force and energy that are given in Table 4-1. Displacement is normalized by two times the height of the arch which in this case is 1.12 m. Results show that the continuous arch requires a larger force to snap the arch than do the continuous arches joined by a living hinges. Overall, the studies demonstrate that continuous arches joined by living hinges require the least amount of energy to activate the mechanism. Therefore, this arch type is used for further study in this work.
Figure 4.13. (a) Half-Arch with Concentrated Positive Load Applied. (b) Half-arch with Positive Moment Applied.

<table>
<thead>
<tr>
<th>Arch Type</th>
<th>Applied Load Type</th>
<th>Energy Required to Snap (N·m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CA</td>
<td>Concentrated Load</td>
<td>1.4</td>
</tr>
<tr>
<td>CA</td>
<td>Concentrated Moment</td>
<td>4.1</td>
</tr>
<tr>
<td>CAsLH</td>
<td>Concentrated Load</td>
<td>0.15</td>
</tr>
<tr>
<td>CAsLH</td>
<td>Concentrated Moment</td>
<td>3</td>
</tr>
</tbody>
</table>

CA = Continuous Arch  
CAsLH = Continuous Arches Joined by Living Hinges
Figure 4.14. (top) Applied Force and Strain Energy vs. Displacement, (bottom) Applied Force vs. Displacement
Figure 4.14 (bottom) compares the nondimensional force verse nondimensional displacement of the continuous arches joined by living hinges as curvature is increased. Force is normalized by the baseline snap through force and displacement is normalized by two times the height of the arch for each case. For curvatures 0.04, 0.63 and 1.7 (1/m), the values used in normalizing displacement are 0.14, 0.2323, and 1.12 (m), respectively. Results show that as curvature is increased, the snap-through force is increased.

An additional study is conducted to determine how changing the half-arch’s rigidity affects the energy required to snap (Figure 4.15). In this study, the arch’s rigidity (EI) was varied by changing the Elastic Modulus (E) and the cross-sectional geometry of the arch. Change in the cross-sectional geometry of the arch affects the arch’s area moment of inertia (I). Energy is normalized by the energy required to snap a continuous arch with living hinges as shown in Table 4-3. Results show that as the arch’s rigidity is increased, the energy required to snap is increased nearly linearly. This indicates that an arch with low rigidity is ideal because it requires the least amount of energy to move the arch.

Discussion

The results in Figure 4.14 showed that the continuous arches joined by living hinges require less force and energy to snap the arch than does the continuous arch. This occurs because the living hinges that are located at the input of the continuous arches joined by living hinges have low stiffness and reduce the
stiffness of the arch system. For load bearing applications, the continuous arch is preferred because more force and energy are needed to snap the arch.

In addition, the results from Figure 4.14 (bottom) show that the snap-through force of the arch is increased as the curvature of the arch is increased. In this study, the base of the arch is held fixed. Physically, the snap-through force of the arch increases as the curvature of the arch is increased because the arch’s bending stiffness is increased. This increase in bending stiffness is physically due to the addition of material to the arch system.

In Figure 4.15, the results showed that increasing the rigidity of the arch also increased the amount of energy required to snap the arch. Physically, this demonstrates that the energy to snap the arch can be reduced simply by changing the arch’s material or cross sectional shape. This further implies that the snap through force and displacement can be altered by changing these properties. No change to the arch’s base, height or thickness is required (Figure 4.16).

The previous studies were conducted using 1D beam-rod elements in Abaqus. Further studies are conducted using a 3D FEA model of the continuous arched joined by living hinges. As stated previously, the continuous arches joined by living hinges require less force and stroke than does the continuous arch. It also requires less energy to snap the arch when a concentrated force is applied to the arch’s input. The 3D models are generated in SolidWorks and imported into Abaqus. This allows for the analysis of an arch that has the exact geometry of an arch that can be manufactured for testing. The 3D models also enhances stress visualization and allows for locating regions of the arch where von Mises stresses exceed the material’s yield stress.

**Design of the Arch**

A fully compliant arch is a continuous structure (Figure 4.16). Previous work (Edwards, Jensen, Howell, 1999) has shown that adding living hinges at the arch’s input and boundary promotes bistability (Figure 4.17). The width and thickness of the arches developed in this work are approximately six times greater than the arch developed in the previous work (Edwards, Jensen, Howell, 1999). Therefore, hinges
of greater thickness and width (called flexible hinges) than the thickness and width used in Edwards, Jensen and Howell (1999) are used in this work at the arch’s input and boundaries as shown Figure 4.17. The following studies evaluate how the arch’s deformation, number of stable states, internal stress, and snap-through force are affected by the location of the flexible hinges. Three locations are considered: boundaries, input, and combined boundaries and input. Another important aspect of this work is the effect of hinge geometry on arch performance. It is expected that hinge geometry and shape greatly affects arch stability. The geometric hinge parameters that are studied in this work are hinge width and length as shown in Figure 4.18.
Figure 4.16 Continuous Arch

Figure 4.17 Continuous Arch with Flexible Hinges
In this work, arches are designed to extend the chord of the rotor blades of a BO-105 helicopter. The features of the BO-105 blade are as follows: blade span = 4.94 m, blade chord = 0.27 m (Jane’s, 2010); blade spar width is approximated to be 35% of the blade’s chord = 9.6 cm. This work assumes a 0012 airfoil for the BO-105 blade. The arch’s height is 3.8 cm. This value of arch height was selected in order to allow up to 30% of chord extension upon snap through. Based on the geometry of the 0012 airfoil, it was determined that the arch’s thickness must not exceed 2 cm. Arch thickness is measured in the z direction of the arch in Figure 4.16. The performance criteria for the arch are twofold: (1) the arch must have two stable states, and (2) the arch must undergo symmetric deformation.

Several different arches were considered. Analysis to determine the arch’s snap-through force and von Mises stress (Abaqus, 2007) was conducted using Abaqus Version 6.9-2, a Non-Linear Finite Element solver. In this work, the von Mises stress is used as the yield criterion. If von Mises stresses within the arch exceed the tensile strength (i.e. yield strength) of the material of the arch, then the arch undergoes yielding. Yielding is the condition where a material experiences plastic deformation instead of elastic deformation while under load. The von Mises stress is considered immediately after the arch snaps. Consider Figure 4.19 below which shows the applied non dimensional force versus non dimensional displacement of the arch. At point A, the arch is in its first bistable configuration. As a load is applied to the input of the arch (see Figure 4.17 for arch input location), the force-displacement behavior of the arch
approaches point B. When the load value of point B is exceeded, the arch jumps to point C. As the applied load is gradually removed from the arch’s input, the force-displacement behavior of the arch approaches point D. Point D is the second bistable configuration of the arch. In this work, the von Mises stress within the arches is considered at point C. The Von Mises stresses are considered at point C because the stresses within the arch are greatest at this point assuming the input of the arch does not exceed non dimensional displacement 2.2.

Figure 4.19. Non Dimensional Force vs. Non Dimensional Displacement for the Arch

The selected material for the arch is Du Pont Delrin II 150SA NC010. This material was selected because of its high fatigue strength and because it is readily available. The stress-strain data (Table 4-5) for Du Pont Delrin II 150SA NC010 is assumed to be the same as Delrin 100 (Du Pont, 2010). The yield stress for the Delrin 100 is between 69 N/mm$^2$ and 73 N/mm$^2$ depending on external factors (Du Pont, 2010). The material was modeled as a hyperelastic material. The Marlow form strain energy potential is used in the numerical simulation to fully approximate the stress-strain data for Delrin 100 in tension and compression. From this point forward, Delrin 100 will simply be referred to as Delrin.
The arches developed in this work were modeled and solved using the finite element method. Abaqus version 6.9-2 was used to implement the finite element method. The procedure used in Abaqus to solve the nonlinear model was the standard Newton-Raphson procedure (General, Static). Twenty-node brick elements (C3D20RH) with hybrid formulation and reduced integration were chosen as the elements to discretize the arch. The analysis included the nonlinear effects of large displacements. The left and right boundaries of the arch were constrained in translation in $x$, $y$, and $z$ and in rotation in $x$, $y$, and $z$. Displacement control was used to simulate the deformation of the arch between its first and second bistable configurations. A displacement constraint of 95 mm was applied at the center of the arch to simulate an actuator pushing the arch input downward. The numerical simulation was used to predict the arch’s mode of deformation, number of stable states, and force required to snap-through.

In this work, all parameters are nondimensionalized by a force scale and a displacement scale. The force scale $= YS * t^2$. Where $YS$ is the yield strength of Delrin and $t$ is the thickness of the arch. The calculated force scale in this work is $2.76 \times 10^7$ kg mm$^2$/s$^3$. The length scale $= \text{arch height}$. The arch height used in this work is 39 mm. Next, the developed model is used to investigate the effects of hinge location, size and shape on arch performance.

**Arch with Boundary Hinges**

This section evaluates how hinges located at the arch’s boundaries (as shown in Figure 4.17) affect arch performance. Detailed results from this study can be found in the Appendix in Table 4-6, Cases 2 - 14. A summary of the results from this study is shown in Figure 4.20. In Figure 4.20, non dimensional force is plotted on the vertical axis. The width and length of the hinges are nondimensionalized and expressed as a percentage. In this study, the thickness of the hinge is held constant. In addition, the width and length of the arch hinges are varied to determine their effect on arch performance. In each case, both boundary hinges on the arches are identical.

The results show that as the width of the boundary hinges are increased, the arch’s deformation shifts from non-symmetric to symmetric. Whereas, increasing the length of the hinge without changing the
width of the hinge results only in symmetric or non-symmetric deformation. A symmetric deformation means that the input of the arch follows a vertical straight-line path (Figure 4.22). A non-symmetric deformation means that the arch input does not follow a vertical straight-line (Figure 4.23). Non-symmetric deformation occurs when the stiffness of the boundary hinges is low. As the arch progresses in the direction of snapping, forces and moments are exerted on the boundary hinges. When the numerical simulation predicts that one hinge of the two boundary hinges allows more rotational deflection than the other, the arch input is pulled in the direction of the hinge that allows more rotational deflection. As shown in Figure 4.23, hinge A allows more deflection upon loading (in the counterclockwise direction) than hinge B. The resulting deformation is a move of the arch’s input toward the left. It is not certain how the numerical simulation chooses which direction to move the arch’s geometry about its mid-line. Thus, non-symmetric deformation as predicted in the simulation may be the result of numerical approximation. Non-symmetric deformation may occur in practice if there are imperfections in the arch’s geometry due to manufacturing.

Figure 4.20. Arches with Boundary Hinges only
Red: Symmetric Deformation
Gray: Non Symmetric Deformation
Outlined: 1 Stable State
Filled: 2 Stable States

Figure 4.21. Arches with Input Hinges only

Figure 4.22. Arch with Symmetric Deformation

Hinge A allows more counter clockwise deflection than Hinge B.

Figure 4.23. Arch with Non-symmetric Deformation
The results in Figure 4.20 also show that bistability can be achieved by using boundary hinges alone. In addition, Figure 4.20 shows that the arches are bistable when the hinge width is between 5 and 8%. This shows that there is an optimum geometric hinge condition for bistability. This also indicates that the bistable nature of the arches may not be greatly affected by an increase in hinge length as shown when the width of the hinge is 8%. Next, an evaluation of the snap-through force of the arch (Figure 4.20) shows that the snap-through force is generally increased by increasing the width and length of the boundary hinges. Overall, this study showed that it is not likely that an arch with boundary hinges only can have symmetric deformation and bistability.

**Discussion**

The results in Figure 4.20 of an arch with boundary hinges showed that soft boundary hinges are needed for bistability and that there is an optimal geometric hinge range where the hinges are bistable. Physically, this occurs because the hinges of the arch apply a force and moment to the arch. Large forces and moments applied at the boundaries of the arch physically keep the arch from releasing strain energy. Instead, strain energy is increased in these conditions. The ideal boundary conditions for releasing strain energy are pinned ends because arch’s ends are allowed to rotate freely which allows a relief in stress within the arch. Therefore, stiff hinges cause the arch to have only one stable state. Thus, the boundary hinges must be soft enough to allow bistability. In Figure 4.20, this occurs when the boundary hinge width is less than 8%.

Figure 4.20 also showed that an increase in arch force occurs when an increase in hinge length and width occurs. Physically, this occurs because the stiffness of the hinges is increasing, thereby causing an increase in the resistive moment applied to the arch at the arch boundaries. This condition results in an overall increase in the amount of force required to snap the arch.
Arch with Input Hinges

The next study conducted was an evaluation of input hinges on arch performance. Results from this study are shown in Figure 4-21. Detailed results from this study can be found in the Appendix in Table 4-6, Cases 15 - 21. In this study, boundary hinges are not present in the arch. The results show that as the width of the input hinge is increased, the deformation goes from non-symmetric to symmetric. In addition, Figure 4-21 shows that non-symmetric deformation occurs for the case when the stiffness of the hinge is low. In this case, the numerical simulation predicts that the deformation moves left despite that arch’s symmetric geometry about its mid-line. This non-symmetric solution may not be physically realizable. Results also show that increasing the width of the hinge does not change the arch’s stability condition and causes an increase in the snap-through force of the arch.

In the case where the length of the input hinges are increased and hinges widths held constant, the deformation type is symmetric and there is only one stable state in the arch. Results also show that snap-through force decreases in this case. In comparing Figure 4.20 to Figure 4.21, the results demonstrate that bistability can only be obtained when there are soft hinges at the boundaries. This likely occurs because there are large reaction moments at the boundaries that work to resist deformation. The incorporation of flexible hinges at the boundaries reduces stiffness in this region and produces a favorable condition for bistability.

Discussion

In Figure 4.21, the results showed that increasing the width of the hinge (i.e. increasing hinge stiffness) does not change that arch’s stability condition. As stated in the previous section, this condition occurs when there is high stiffness at the boundaries of the arch that prevent the arch from releasing strain energy. However, hinges at the arch’s input affects the snap-through force. The function of the input hinges of the arch is to transfer forcing to the each of the half arch sections that are adjacent to the input hinges. If the input hinges are stiff, the force is directly transferred to the half arch sections which require
high force to deform. However, if the input hinges are flexible, the hinges absorb deformation thereby requiring less deformation from the half arch section to deform the arch system.

**Arch with Boundary and Input Hinges**

Next, we examine arches with boundary and input hinges to study their effect on arch performance. Detailed results from this study can be found in the Appendix in Table 4-6, Cases 22 - 48. Figure 4-24 and Figure 4.25 compares an arch with no hinges to arches with hinges. Figure 4.24 compares the nondimensional snap-through force of the aches. Figure 4.25 compares the maximum von Mises stress with the arches. The maximum von Mises stress is located within the boundary hinges of the arch as shown in Figure 4.29. In the force and stress comparison, input hinges of 13 % width and 43 % length are added to an arch with boundary hinges that have a width of 13 % and length of 41 %. The comparison showed that all cases in Figure 4-24 and Figure 4.25 have symmetric deformation and one stable state. This indicates that adding input hinges to an arch does not necessarily have an effect on arch deformation and the number of stable states. In addition, the results in Figure 4-24 show that the snap-through force may be reduced but that the von Mises stress is not greatly affected by adding input hinges to an arch with boundary hinges.
Figure 4.24. Comparison of Arches, Snap Through Force

Figure 4.25. Comparison of Arches, Stress
To further explore the effects of adding hinges to the arch, input hinges are added to an arch that has boundary hinges. This study compares Case 26 to Case 46 in Table 4-6 of the Appendix. In this case, the widths of the boundary hinges are 8% and the lengths are 29%. The added input hinges have a width of 13% and 14% respectively. The length of both the input hinges is 14%. The arch with input hinge width = 13% and length = 14% has symmetric deformation and has two stable states. The results show that this arch can be made to have non-symmetric deformation by increasing its input hinge width to 15%. This demonstrates that the performance of the arch is very sensitive to changes in the geometry of the hinge.

Figure 4.26 shows a summary of results where the input hinge’s width are held constant at 13% while the input hinge’s lengths and boundary hinges geometry are varied. The x-axis shows the input hinge length. The y-axis plots the nondimensional force required the snap the arch. The arches are grouped by the geometry of their boundaries hinges as shown in the legend. The quantity BH denotes the width of the boundary hinges of the arch. The quantity LH denotes the length of the boundary hinges of the arch. The results show that all the arches in Figure 4.26 have symmetric deformation. The results also show that an
increase in the lengths of the input hinges results in an overall decrease in snap-through force. In addition, Figure 4.26 shows that softening the boundary hinges by increasing the length of the boundary hinges minimally affects snap-through force for a given input hinge.

An additional study was conducted where the arch base was increased from 14 to 15.4. The arch used in this study has the following geometric parameters: boundary hinges width = 8 %, length = 29 %, input hinges width = 13 %, length = 36 % and arch thickness = 0.5 (cases 52-54 in the Appendix). All other parameters are held constant. The results show that increasing the arch’s base geometry leads to a decrease in snap-through force and maximum Von Mises stress.

An additional study was conducted where the thickness of the arch is increased from 0.1 to 0.4 while all other geometric parameters are held constant (cases 22 to 25). The arch used in this study has the following geometric parameters: boundary hinges width = 8 %, length = 29 %, input hinges width = 13 %, and length = 32 %. When the arch thicknesses are 0.1 and 0.2, the numerical solution shows that the arches deform in the z-direction of Figure 4.16 and in the direction of snapping. This out-of-plane behavior is unusual given that the arch’s geometry is symmetric and the applied load is applied in the y-direction. This may not be a physically realizable solution. The results also show that as the thickness of the arch is increased, the deformation type becomes bistable and the snap-through force is increased. This increase in snap-through force is due to a general increase in the arch’s stiffness. Bistability occurs because the boundary hinges remain in the optimal stiffness region as discussed in section Arch with Boundary Hinges.

**Discussion**

The results in Figure 4.26 show that increasing the length of the input hinge decreases the force required to snap the arch to its second stable state. This is expected (as stated in the previous sections discussion) because increasing the lengths of the hinges reduces the stiffness of these hinges and makes them more flexible. These flexible hinges deform due to the application of load and reduce the amount of deformation required by the half arches. Reduced deformation from the half arches means that less load was required to deform the arch thereby resulting in a reduction in snap through force.
In addition, increasing the arch’s base length decreased the snap through force and maximum von Mises stress. This occurs because the half arch sections of the arch increases in length and therefore deforms less when a forcing is applied at the arch’s input. Another observation is that the highest level of von Mises stress in the arch occurs in the arch’s hinges. As the arch’s half-length section deforms less due to an increase in the arch’s base length, the boundary hinges also deform less which results in a reduction in stress found within the arch.

In summary, adding input hinges to an arch with boundary hinges acts to decrease snap-through load. Adding input hinges can also change the arch’s deformation type and number of stable states but this is heavily dependent on the input and boundary hinges geometry. A decrease in the maximum Von Mises stress in the arch can be obtained by increasing the base of the arch. A summary of the results from the previous studies is shown in Table 4-4.
Table 4-4. Summary of Results

<table>
<thead>
<tr>
<th>Objective</th>
<th>Results</th>
</tr>
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<tbody>
<tr>
<td>Obtain Bistability</td>
<td>Boundary Hinges Only: Increase Hinge Width(^A)</td>
</tr>
<tr>
<td></td>
<td>Input Hinges Only: Improbable</td>
</tr>
<tr>
<td></td>
<td>Boundary and Input Hinge: Dependent on Hinge Geometry</td>
</tr>
<tr>
<td>Obtain Symmetric Deformation</td>
<td>Boundary Hinges Only: Increase Hinge Width</td>
</tr>
<tr>
<td></td>
<td>Input Hinges Only: Increase Hinge Width</td>
</tr>
<tr>
<td></td>
<td>Boundary and Input Hinge: Dependent on Hinge Geometry</td>
</tr>
<tr>
<td>Obtain Bistability and Symmetric Deformation</td>
<td>Boundary Hinges Only: Improbable</td>
</tr>
<tr>
<td></td>
<td>Input Hinges Only: Improbable</td>
</tr>
<tr>
<td></td>
<td>Boundary and Input Hinge: Dependent on Hinge Geometry</td>
</tr>
<tr>
<td>Increase Snap-Through Force</td>
<td>Boundary Hinges Only: Increase Hinge Width</td>
</tr>
<tr>
<td></td>
<td>Input Hinges Only: Increase Hinge Width</td>
</tr>
<tr>
<td></td>
<td>Boundary and Input Hinge: Decrease Input Hinge Length</td>
</tr>
</tbody>
</table>

\(^A\): Study showed that bistability occurs when hinge width is between 2 to 3 mm.

Arch Hinge Shape

In this study, we evaluate how hinge shape affects deformation type, the number of stable states, and snap-through force. The arch used in this study has the following geometric parameters: boundary hinges width = 8 %, length = 29 %, input hinges width = 13 %, length = 36 %, arch thickness = 0.5 and arch base = 14. This arch was selected for further study because it provides a reasonable size for testing, has symmetric deformation and two stable states. Detailed results from this study can be found in the Appendix in Table 4-7. In this study, the thicknesses and length of the boundary and input hinges, and the arch’s thickness and base length are held constant.

Figure 4.27 shows the six different hinge shapes that were evaluated: Rectangular, Corner Filleted, Non-symmetric Corner Filleted, Non-symmetric Circular, Symmetric Circular and Taper. Lobontiu (2003) gives a comprehensive treatment of living hinges that undergo small deformation. In the present work, the hinges have nonlinear stiffness and undergo large deflection. The results from this study are summarized in Figure 4.28. In this figure, WH specifics the end width of the hinge as shown in Figure...
4.27. In this study, the width of the arch is 0.19. The hinges radius is specified by R. In Figure 4.28, results where WH is 5, 10, and 14% represent arches that have rectangular hinges with parameters 13% width and 36% length at its input. Results where WH is 5-5, 10-10, and 14 – 14% represent arches that have custom designed hinges (as shown in Figure 4.27) located at both the hinges input and boundaries. In the cases where the arch has symmetric and non-symmetric circular hinges respectively, these hinges are located at both the boundaries and input. The results in Figure 4.28 shows that increasing the hinges width and radius (R) promotes symmetric deformation except in the case where the arch has non-symmetric corner-filleted hinges at the boundaries and input. In this case, there is a region where the boundary and input hinges produce symmetric deformation. Results also show that bistability is not obtained when there are hinges at the boundaries and input for the cases when the hinge shape is symmetric circular and tapered.

Figure 4.27. Hinge Geometry (Lobontiu, 2003)
A second evaluation was conducted to determine the effect of arch shape on von Mises stress. As previously mentioned, the maximum amount of von Mises stress within the arch occurs in the arch hinges. Figure 4.29 shows the region of maximum von Mises stress within an arch that has rectangular hinges. The maximum von Mises occurs on the top surface of the hinge near the boundary and the half-length section of the arch. High levels of von Mises stress also occurs on the bottom surface of the hinge near the boundary of the arch. However, changing the shape of the hinges of the arch reduces the level of von Mises stress within the arch’s hinge. For example, Figure 4.30 shows a comparison of an arch with rectangular hinges to an arch with corner-filleted hinges. The arches in Figure 4.30 have hinges located at both the input and boundaries. The results show that the von Mises stress is reduced by changing the shape of the hinge, but the von Mises stress for the corner filleted hinged arch still exceeds the yield stress of the arch. This
reduction in von Mises stress is largely due to a reduction in the stiffness of the hinge. This reduction in stiffness is seen by comparing the snap-through force of the arches as shown in Figure 4.31. The snap through force of the arch with corner-filleted hinges is lower than the snap-through force of the arch with rectangular hinges.

A way to reduce the von Mises stress below the yield stress is to increase the thickness of the input hinges and reduce the width of the arch half sections to make the arch more flexible. Figure 4.32 shows a comparison of an arch with thick input hinges (15 %) with reduced arch width (.10) and the baseline arch with rectangular input hinges and thick arch width (.19). The comparison shows that the arch with thick input hinges has less bending in the boundary hinges of the arch than the arch with rectangular input hinges. Physically, the reduction in boundary hinge bending occurs because the arch’s mid-sections are very flexible and bends as the arch moves to its second bistable configuration. As a result of the high flexibility of the arch, the arch mid-section exerts less moment on the boundary hinges of the arch as the arch is deformed. This decrease in moment causes a reduction in hinge bending and results in less stress within the hinge.
Figure 4.29. Arch and Hinge that Shows the Region of Maximum Stress

Figure 4.30. Stress Comparison

- Non Dimensional von Mises Stress

Yield Stress, Rectangular Hinges, Corner-Filleted Hinges, Thick Input Hinges

Graph showing stress comparison for different hinge types.
Figure 4.31. Force Comparison

Figure 4.32. Arch comparison
Overall, the results from this study show that changing the hinge’s shape at the boundaries and input can result in a change in the magnitude of the snap-through force, a decrease in von Mises stress within the arch. As a result of this study, the arch where the boundaries and input hinges that are corner-filleted in shape and have a WH of 5-5% is selected for model validation because it is symmetric, bistable and requires moderate snap-through force.

Summary

A novel concept to achieve chord morphing using a bistable arch was presented. Two solution methods were presented to analyze the arch model: PRBM and FEA. A comparison of the methods showed that both methods were able to capture the bistable nature of the arch. It was demonstrated that using the Simple Method for PRBMs may result in an inaccurate representation of the arch. As a result, FEA was chosen as the solution method to analyze the arch. A comparison of arch types and load conditions showed that the continuous arches joined by living hinges required the least amount energy to snap the mechanism. A study of energy required to snap the arch versus the rigidity of the arch showed that as the arch’s rigidity is increased, the energy required to snap the arch is increased. In the next chapter, an experimental validation is conducted on the developed model and a demonstration of the arch-actuator system is presented. Additional studies were conducted using a 3D FEA model to determine the effects that hinges had on arch performance. Studies showed that bistability can only be obtained when the arch’s boundary hinges are soft. In addition, bistability and symmetric deformation can be achieved when there are flexible hinges at the arch’s boundaries and input.
References


[Jane's all the World's Aircraft], S. Low, Marston & Company, London (1909 - )


Appendix

Pseudo Rigid Body Model Summary

Introduction

The pseudo rigid body model (PRBM) method was developed by Howell (2001) as an efficient method to develop and improve initial designs of compliant mechanisms. The PRBM method provides a simple technique to analyze systems that are composed of flexible segments that undergo large nonlinear deflections. The segments are modeled using rigid-body components that predict the segment’s deflection path and force-deflection relationship. The deflection path of a flexible segment is modeled by two rigid links that are attached at a pin joint. The rigid link system is stiffened by adding a spring at the pin joint. The objective of the PRBM method is to determine the location of the pin joint and the value of the spring constant that best allows the rigid link system to replicate the deflection and force of the flexible segment.

A Description of PRBM Method

Cantilevered beams are used as the foundation for the development of the PRBM method. Cantilevered beams with large deflections require elliptic integral solutions or some other technique to solve the equations, but these methods can be cumbersome in the early design phases of flexible mechanisms. Therefore, it is assumed that the nearly circular path of a cantilevered beam’s end can be accurately modeled by two rigid links that are joined by a pivot along the beam. The pivot is called the characteristic pivot and it also locates the spring that adds stiffness to the rigid link system. The radius of the circular deflection path traversed by the end of the pseudo-rigid-body link is called the characteristic radius ($\gamma l$); $\gamma$ is the characteristic radius factor (Figure 4.33). The pseudo-rigid-body angle is the angle between the pinned pseudo-rigid body link and the horizontal.
Parametric Approximation of the Beam’s Deflection Path

An acceptable value for the characteristic radius factor, $\gamma$, and hence the characteristic radius may be found by determining the maximum acceptable percentage error in the pinned pseudo-rigid body link end deflection. The objective is to determine the value of $\gamma$ that allows the maximum pseudo-rigid-body angle while satisfying a prescribed maximum error constraint. The error is the relative deflection error (Howell, 2001, pgs 147-148). The characteristic radius factor is dependent on the amount of horizontal load applied to the cantilever beam model because as the applied load at the pinned end changes, the deflection path is also changed. The deflection path range of accuracy of the approximations of the characteristic radius factor is limited by the deflection calculated using the elliptic-integral approach. Table 5.1 (Howell, 2001, pg 151) list numerical values for $\gamma$ for various values of $n$. Here, only the kinematics is modeled; a spring was not added to the model. The quantity $n$ is a multiplication factor on the horizontal applied load $P$ as shown in Figure 4.33.

The characteristic radius factor does not vary much over a large range of force angles and may be roughly approximated by determining its average value over a specified range of $n$. This approximation is useful for a wide range of force angles and is helpful in making rough calculations.

![Figure 4.33. Fixed-Pinned PRBM](image)
Stiffness Coefficient

The beam’s resistance to bending is modeled by a stiffness coefficient. The stiffness coefficient and geometric and material properties are used to determine the value of the spring constant for the beam’s pseudo-rigid-body model. The force that will cause the link of the pseudo-rigid-body to deflect is tangent to the path of the beam’s endpoint ($F_t$). This load can be nondimensionalized as the nondimensionalized transverse load index (Howell, 2001, pg 153, Equation 5.57).

The force displacement relationship in terms of nondimensionalized pseudo-rigid-body parameters is written as shown in Howell (2001, pg 154, Equation 5.59). The relationship in Equation 5.59 may not be accurate for the entire range of the kinematic model. The limit of this model is expressed in terms of the maximum pseudo-rigid-body model angle ($Q_{max}$) allowed before exceeding the limits. An average stiffness coefficient value can also be calculated by Equation 5.64 (Howell, 2001, pg 156). The stiffness coefficient is calculated for a rigid link system that has already been kinematically optimized. The stiffness coefficient is directly proportional to the torsion spring constant as shown in Equation 5.72 (Howell, 2001, pg 157).

PRB Model Flexible Segment Types

Howell presents four flexible segment types in his book: fixed-pinned, initially curved fixed-pinned, fixed-guided and fixed-fixed (Howell, 2001, pgs 166-170, Table 5.2). The fixed end of a segment is defined as an end that is coupled to a rigid (e.g. ground) or flexible segment (Howell, 2001, pg 160). Fixed-guided flexible segments are initially straight and the segment’s ends are required to maintain a fixed angle. The pseudo-rigid-body model of the fixed-guided segment may be derived by combining the two fixed-pinned PBR segments. The pinned ends of the two half segments are approximated as one rigid link (Howell, 2001, pg 163-164).

Fixed-Fixed segments are segments that are loaded with a force and moment at the segments fixed ends. Fixed-Fixed segments that have a force and moment applied in the same direction at the beam ends (called load case 1) can be modeled by an initially curved fixed-pinned beam. This can be done because the applied moment creates a curvature in the beam that is continuous throughout the entire beam. The
Continuous curvature is the same as a beam that is initially curved (Howell, 2001, pg. 176). This means that the segment has the same behavior as an initially curved beam with radius and a force applied at its end. A second load type (called load case 2) for fixed-fixed segments is a force and moment loading in opposite directions with an inflection located along the beam. In this case, the beam can be modeled as two cantilevered fixed-pinned PRBM segments connected at the inflection point (Howell, 2001, pg 177).

The PRBMs for fixed-fixed segments are accurate but are too complicated to be practical in compliant mechanism design. For example from load case 1, the moment loading caused a constant curvature throughout the length of the beam. This is similar to the constant curvature from an initially curved beam; therefore the PRBM for an initially curved beam may be used for a fixed-fixed beam with case 1 loading. When the force loads at the pins and the moment loads that induce curvature are known, it is a simple and powerful method. However, in many cases the loads are unknown and this makes it difficult to know what curvature to use in the pseudo-rigid body model.

A third load case can be modeled by combining two fixed-pinned PRBMs that are connected at the point of inflection by a pin joint. In this case, the end angle is important in determining the overall displacement. As the displacement of the mechanism changes, the inflection point also changes. Therefore, the two PRBMs must be changed continuously to reflect the actual flexible segment. Because the inflection point is moving and its location is usually unknown at the beginning of the analysis, an iterative method is required to solve for the unknowns. A PRBM for fixed-fixed segments was also developed by Saxena and Kramer.

Simplified Pseudo-Rigid Body Models for Fixed-Fixed Segments

A fixed-guided segment is a special type of fixed-fixed segment. A fixed-fixed segment is accurately modeled by a pseudo-rigid-body model that has two pin joints located such that they are each located at the same distance from their respective end. Although, not as accurate as for fixed-guided segments, this model can be used to model fixed-fixed segments. The only difference in the model is that
the ends are not constrained to stay at the same angle and are allowed to move as needed. This model is easy to implement into design and is useful in the initial design phase.

In problems where the reaction forces in the pinned-fixed system are changed and yield different values of \( n \), the variation in \( n \) causes changes in the location of the characteristic pivot and stiffness coefficient and this change may be accounted for in two ways: 1. Simplest Method: use the averages of the characteristic radius factor and the stiffness coefficient as constant values. 2. Update the changing values of \( \gamma \) and \( K_0 \) at every increment of motion using the equations that relate \( n \) to \( \gamma \), and \( n \) to \( K_0 \). If the value of \( \gamma \) is held constant and \( K_0 \) is allowed to vary, the displacement and required force are calculated by imposing an initial displacement at the node of a link within the system, calculating the resulting mechanism motion, and determining the reaction forces. The values of \( n \) for each segment are then updated, and the mechanism is incremented to the next displacement.

The accuracy of the simplest method can be improved by allowing the values of \( \gamma \) and \( K_0 \) to change throughout the mechanism motion and combining these values with the kinematic equations of the mechanism that allows arbitrary link lengths. These simplified models are useful in visualizing the motion of large-deflection systems and predicting their behavior. They are also valuable in the initial design phase, allowing many different designs to be efficiently investigated. It is important to note that it is convenient to use a constant value of \( \gamma \) for various values of \( n \). Recommended values of \( \gamma \), rho, and \( K_0 \) for various values of \( \kappa_0 \) are listed in Table 5.2 (Howell, 2001m pg 170).
The PRB Method Applied to Half-Arch

The half-arch (Figure 4.34) can be modeled by a series of straight fixed-fixed segment. According to Howell, the fixed-fixed segments can be modeled using PRBM fixed-pinned segments (Figure 4.35) that have initial curvature. Therefore, the PRB model of the arch is as shown in Figure 4.36. This simplification can be used if the forces and moments are known. However, the forces and moments are unknown which makes it difficult to know what non-dimensional constant \((\kappa_0)\) to use in the PRBM. In addition, this discretization requires the generation of a large set of nonlinear force displacement equations which may be difficult to solve using commercial solvers.

In order to simplify the PRBM, the number of segments in the model is reduced (Figure 4.37) so that the half arch can be modeled with a fixed-pinned and fixed-fixed initially curved segment.
Howell does not present a method to model fixed-fixed segments that are initially curved. However, Howell does suggest a simplified method to model straight fixed-fixed segments as discussed above, and this technique is applied to this problem to model the PRBM that are composed of initially curved segments. Therefore, the initially curved fixed segment is modeled by a simplified PRBM for a straight fixed-fixed segment. The two pin joints of the fixed-fixed model are located such that they each have the same distance from the fixed ends.

Figure 4.37. Reduced PRBM
### Stress-Strain Data for Du Pont Delrin II 150SA NC010

Table 4-5. Stress-Strain Data

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### Tables of Arch Hinge Studies

Table 4-6. Results from Study of Arch with Rectangular Hinges

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AB: Arch Base  
AT: Arch Thickness  
DT: Deformation Type  
LH: Length of hinge (% of half length)  
NS: Non-symmetric  
OP: Out of Plane  
S: Symmetric, **symmetric deformation occurs when x-def is < 2mm  
SF: Snap-Through Force  
SS: Stable States  
WH: Width of Hinge  
1-S: One Stable State  
2-S: Two Stable State
Table 4-7. Results from Study of Arch with Various Hinge Shapes

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<th>Input Hinge WH (%)</th>
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* - denotes that only one end of hinge is at this height

DT: Deformation Type
HT: Hinge Type
MVMS: Maximum Von Mises Stress at Snap-Through
NS: Non Symmetric
R: Radius of Circular Hinge
S: Symmetric
SF: Snap-Through Force
SS: Stable States
WH: End Width of Hinge
1-S: One Stable State
2-S: Two Stable State
Chapter 5

A Experimental Validation and Feasibility Study of Bistable Arches

Introduction

This chapter is based on the analysis conducted in Chapter 4. In Chapter 4, the arch and plate system is presented as a concept to extend the chord of the rotor blades. In addition, solution method to solve the arch problem was presented. A 3D finite element model of the arch was presented and the model was used to investigate how the size and shape of the hinges of the arch affect the performance of the arch. The objectives of this chapter are to (1) validate the developed 3D finite element model presented in Chapter 4., (2) Investigate the feasibility of the arch and plate system for helicopter applications, and (3) to demonstrate the concept.

This chapter is organized as follows: first, results are shown from an experiment that was conducted in order to validate the developed finite element model. Second, the feasibility of the arch is investigated for helicopter application by simulating the performance or the arch under inertial load. Finally, the arch and plate system chord extension concept is demonstrated, where an SMA wire is attached to the arch and made to snap the arch to its second bistable configuration.

In the next section of this chapter, an experimental validation of the arch is presented.

Experimental Validation of the Arch

An experimental validation is conducted to verify the accuracy of the developed numerical simulation. A picture of the experimental setup is shown in Figure 5.1. In Figure 5.1, the arch is supported at its left and right boundary by three upright aluminum bars. The aluminum bars are coupled to the arch by a support plate. In addition, the aluminum bars are constrained to a reinforced base plate by L-joints. The Delrin arch is attached to an Instron machine Model 4206 with a 1KN load cell. The Delrin arch was
fabricated using a Waterjet machine where the arch’s shape was cut-out from a 0.75 in. thick sheet of Delrin II 150SA NC010 material. In this work, the Instron machine is used to obtain force-displacement data in both push-down and pull-up directions. The arch is loaded at a rate of 5 mm/min. The maximum push-down displacement is set to 75mm. The arch used in this study has the following geometric parameters: hinge shape = Symmetric Corner Filleted, boundary hinges end width = 5 %, and input hinges end width = 5 %. The force-displacement results from testing are shown in Figure 5.2. The results are further analyzed in Table 5-1. In this work, all parameters are nondimensionalized by a force scale and a displacement scale. The force scale = YS * t^2. Where YS is the yield strength of Delrin and t is the thickness of the arch. The calculated force scale in this work is $2.76 \times 10^7$ kg mm/s^2. The length scale = arch height. The arch height used in this work is 39 mm.

In Figure 5.2, push-down and pull-up are represented by blue and red curves, respectively. Push-down is downward forcing of the arch’s input in the vertical direction. Pull-up is upward forcing of the arch’s input in the vertical direction. The predicted force-displacement data is represented by purple x points. The predicted pull-up and push-down force-displacement data from the numerical simulation are identical. The results from Figure 5.2 show that the results from the numerical simulation and the experiment have very similar shapes. Both sets of data show that the arch is bistable. This is seen by the
second crossing of the force-displacement plot with the x-axis. As shown in Table 5-1, the difference in snap-through force and displacement in the push-down direction is 12 % and 14 % respectively. In the pull-up direction, the differences in snap-through force and displacement are 4% and 3% respectively. This indicates that there is good agreement between the numerical simulation model and the experiment.

![Figure 5.2. Experiment versus Model](image_url)

**Table 5-1. Summary of Results in Figure 5.2**

<table>
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<tr>
<th>LT</th>
<th>STF</th>
<th>STD</th>
<th>% DSTF</th>
<th>% DSTD</th>
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<td>PU</td>
<td>-7.2E-04</td>
<td>1.56</td>
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</table>

E - Experiment  
PD - Push Down  
LT - Loading Type  
AM - Abaqus Model  
PU - Push Up  
STF - Snap-Through Force  
STD - Snap-Through Displacement
DSTF - Difference in Snap-Through Force
DSTD - Difference in Snap-Through Displacement
The difference in force and displacement between the predicted and measured values may be due to the stress-strain data used in the numerical model. The stress-strain data for this material was assumed to be that of Delrin 100 (Du Pont, 2010); the stress-strain data for Delrin II 150SA NC010 was not available. In addition the numerical simulation allows the specification of stress-strain data in tension or compression, but not in both tension and compression. This affects the model because at high stress levels, the strain in compression is less than it is in tension (Du Pont, 2010). In addition, the stiffness properties of Delrin are affected by resin composition, rate of loading, temperature and humidity. For example, the Delrin Design Guide (Du Pont, 2010); shows that the yield stress for Delrin at 23°C is approximately 71 MPa, whereas, the yield stress at 66°C is approximately 65 MPa. The temperature and humidity of the arches while in storage and during testing were not directly controlled.

Overall, the numerical simulation model shows good agreement with the experiment. Therefore, this numerical model will be used to evaluate the feasibility of bistable arches for the chord extension of helicopter rotor blades.

**Application to Helicopter Rotor blades**

A preliminary study was conducted to determine the feasibility of the proposed concept for helicopter rotor blade chord extension. The arch is designed to extend the chord of a BO-105 rotor blade. The BO-105 rotor blade is assumed to have an angular velocity of 40 rad/sec (382 rev/min) at normal operating speed. Inertial effects exerted on the arch at this speed are expected to deform the arch in-plane and out-of-plane and affect the arch’s number of stable states and snap-through force. Therefore, it is necessary to evaluate the effect of inertial load on the performance of the arch. The inertial load that acts on the arch is defined by the product of the mass of the arch and the acceleration of the arch. The acceleration vector is defined with respect to coordinate system \( e_x, e_y, \) and \( e_z \) as shown in Figure 5.3 and Figure 5.4. In Figure 5.3, the coordinate system IJK is the fixed reference frame. The coordinate system \( e_x, e_y, \) and \( e_z \) is a moving reference frame and moves with the rotor blade. As shown in Figure 5.4, the arch lies in the \( e_x, e_y \) plane. The thickness of the arch as shown in Figure 4.16 is out-of-plane.
As a simplification, the blade is hinged about the hub and is only allowed to flap. The blade flaps with angle $\beta$ about the $e_y$ axis. The position vector is given in Equation 5-1. The velocity vector is given in Equation 5-2. The acceleration vector is given in Equation 5-3.

$$\begin{align*}
\mathbf{r} &= (x_{blade} + \bar{x})\mathbf{e}_x + (y_{blade} + \bar{y})\mathbf{e}_y + \bar{z}\mathbf{e}_z \\
\mathbf{v} &= (-y_{blade} + \bar{y})(-\dot{\theta} \cos \beta) \mathbf{e}_x + (\bar{x} \dot{\theta} \sin \beta + (x_{blade} + \bar{x}) \dot{\theta} \cos \beta) \mathbf{e}_y - (-y_{blade} + \bar{y}) \dot{\theta} \sin \beta \mathbf{e}_z
\end{align*}$$

Equation 5-1

$$\ddot{\mathbf{r}} = T_1\mathbf{e}_x + T_2\mathbf{e}_y + T_3\mathbf{e}_z$$

Equation 5-3

where

$$T_1 = (-y_{blade} + \bar{y} + \frac{SH}{2}) \sin \beta \dot{\theta} - (\cos \beta)^2(x_{blade} + \bar{x})\dot{\theta}^2 - (\cos \beta)\bar{z}(\sin \beta)\dot{\theta}^2 - (\cos \beta) \left( -y_{blade} + \bar{y} + \frac{SH}{2} \right) \dot{\theta}$$

Equation 5-4

$$T_2 = (\bar{x} \cos \beta - (x_{blade} + \bar{x}) \sin \beta) \dot{\theta} + (y_{blade} - \bar{y} - \frac{SH}{2}) \dot{\theta}^2 + ((x_{blade} + \bar{x}) \cos \beta + \bar{z} \sin \beta) \dot{\theta}$$

Equation 5-5

$$T_3 = (y_{blade} - \bar{y} - \frac{SH}{2}) \sin \beta \dot{\theta} - \sin \beta \dot{\theta}^2(\beta \bar{z} + (x_{blade} + \bar{x}) \cos \beta) + (y_{blade} - \bar{y} - \frac{SH}{2}) \cos \beta \dot{\theta} \dot{\theta}$$

Equation 5-6

The definitions of terms in the equations above are as follows:

$\theta$ = azimuth angle  
$\dot{\theta}$ = rotor angular velocity  
$\ddot{\theta}$ = rotor angular acceleration  
$\beta$ = flap angle  
$\dot{\beta}$ = blade flapping angular velocity  
$SH$ = spar height  
$x_{blade}$ = the length in the $e_x$ direction that starts at the hub and stops at the bottom left edge of the arch boundary as shown in Figure 5.4.  
$y_{blade}$ = the length in the $e_y$ direction that starts at the lead edge of the rotor blade and stops at the bottom left edge of the arch boundary as shown in Figure 5.4.  
$\bar{x}$ = $x$ coordinate of a point on the arch from the origin of the $e_1$, $e_2$, $e_3$ axis  
$\bar{y}$ = $y$ coordinate of a point on the arch from the origin of the $e_1$, $e_2$, $e_3$ axis  
$\bar{z}$ = $z$ coordinate of a point on the arch from the origin of the $e_1$, $e_2$, $e_3$ axis

The analysis only considers the rotor operating at normal operating speed in hover flight. It does not consider rotor spin up or rotor spin down. Therefore, $\ddot{\theta} = 0$. The analysis does not consider rotor blade flapping. Therefore, $\dot{\beta} = 0$. However, the blades flap up at an angle $\beta$ between $3^\circ$ to $6^\circ$ and remain in this position as the rotor operates at normal operating speed. Point A in Figure 5.3 represents a point on the
arch. Point A in Figure 5.3 is equivalent to Point A in Figure 5.4. Coordinate system e_x, e_y, e_z shown in Figure 5.4 is flush with the bottom face of the arch. The coordinates of Point A are defined by quantities \( \bar{x}, \bar{y} \) and \( \bar{z} \). The quantities \( \bar{x}, \bar{y} \) and \( \bar{z} \) are defined with respect to the \( e_1, e_2, e_3 \) coordinate system as shown in Figure 5.4. Figure 5.3 shows point A in 3D space. Figure 5.4 shows point A within the rotor blade. The rotor blade in Figure 5.4 is positioned in angle \( \beta \).

The small angle assumption is also used to simplify the equations. The simplified equations for \( T_1, T_2, \) and \( T_3 \) are shown below.

\[
T_1 = -(x_{blade} + \bar{x}) \dot{\theta}^2 - \bar{z} (\beta \dot{\theta})^2 \quad \text{Equation 5-7}
\]

\[
T_2 = (y_{blade} - \bar{y} - \frac{SH}{2}) \dot{\theta}^2 \quad \text{Equation 5-8}
\]

\[
T_3 = - (\beta \bar{z} + (x_{blade} + \bar{x}) \beta \dot{\theta}^2 \quad \text{Equation 5-9}
\]

The quantities assigned to the parameters of \( T_1, T_2, \) and \( T_3 \) for hover flight are shown below:

\[
\dot{\theta} = 40 \text{ rad/sec}
\]

\( \beta = 0.0698 \text{ radians} \)

\( SH = 95.5 \text{ mm} \)

\( x_{blade} \) is defined below.

\( y_{blade} = 37 \% \) of the chord of the rotor blade

The components (\( T_1, T_2, \) and \( T_3 \)) of the acceleration equation are multiplied by the density of the arch to generate body force equations. The developed body force equations are used in a static analysis conducted using the developed finite element model in Abaqus to determine the effect of inertial load on arch performance. In the Abaqus model, the prescribed body forces are defined with respect to coordinate system \( e_1, e_2, e_3 \) as shown in Figure 5.4.

The following studies examine the performance of the arch while loaded by inertial loads while the BO-105 helicopter operates in hover flight. The arch used in these studies has rectangular hinges at the input and boundaries of the arch. The input hinge geometric parameters are hinge width = 13 \% and hinge length = 36 \%. The boundary hinge geometric parameters are boundary hinge width = 8 \% and hinge length = 29 \%. The thickness of the arch = 0.5 and the length of the base of the arch = 14.
Figure 5.3. Point A in Coordinate System IJK

Figure 5.4. Rotor Blade and Arch with Point A
**Inertial Force Examination**

As the blade rotates about the hub of the helicopter, the inertial loads on the arch deform the arch and affect the performance of the arch. In this section, an examination of the affect of the inertial loads on the arch is conducted. The inertial loads are applied separately to the arch in the \( e_x \), \( e_y \), and \( e_z \) directions.

The results in Figure 5.6 show the non dimensional force versus non dimensional displacement for the arch. In this study, the arch is located at the spanwise position of 80 % of the radius of the blade and the 37 % chordwise position of the rotor blade as shown in Figure 5.5. The 80 % spanwise position was selected because this is within the region where stall occurs in forward flight. The objective of rotor blade chord extension is to increase lift within the stall region of the rotor. The 37 % chordwise position was selected because it allows the arch ample space to extend the chord of the rotor blade without striking the airfoil. The maximum chordwise location for the input of the arch before the arch strikes the airfoil while moving to its second bistable configuration is 39 % of the chord. Therefore, in the following studies, the arch is located at 37 % of chord.

![Figure 5.5. Arch and Plate within Rotor Blade](image)

The \( x \) data points in Figure 5.6 represent the case where there is No inertial load applied to the arch. The diamond points represent the case where inertial load is only applied in the \( e_x \) direction (see Figure 5.4). The square points represent the case where inertial load is only applied in the \( e_y \) direction. The triangle points represent the case where the inertial force only applied in the \( e_z \) direction. There are two directions for forcing shown in the force-displacement plots. The push down and pull up directions. The
term push down refers to the downward motion of the arch in Figure 5.1. The term pull up refers to the upward motion of the arch in Figure 5.1.

In comparing the cases No Load, $e_y$ direction forces and $e_z$ direction forces, the results show that the snap-through load of the arch for these cases is nearly similar when the arch is pushed down. However, the snap-through force is 19% lower and the snap-through displacement is 12% higher in the push down direction for the case when inertial load is only applied in the $e_x$ direction. The results also show that the arch becomes unistable when inertial force is applied in the $e_x$ direction of the arch. This occurs because of the large $e_x$ direction deflections of the arch that are caused by large $e_x$ direction inertial load applied to the arch as shown in Figure 5.7. Physically, in this highly deflected state the stiffness of the arch is effectively reduced and the amount of force required to snap the arch is decreased. The increase in the snap-through displacement is caused by $e_y$ deflection of the arch’s input that occurs as the $e_x$ direction inertial loads are applied to the arch.

In the pull-up direction of the arch, the snap-through load is lower than the snap-through load in the push-down direction. Physically this occurs because of the strain energy present within the arch when the arch is in its second bistable configuration. When the arch is in its second bistable configuration, the top surface of the arch is in compression and the bottom surface is in tension. As a result, there is strain energy within the arch. The presence of stain energy with the arch decreases the energy needed to snap the arch back to its first bistable configuration. A second possible reason for this reduction in snap-through force is the resistive moment applied to the arch by the arch’s flexible hinges. As the arch is deformed, these hinges deform in bending and provide resistive moments to the arch. These resistive moments exert additional forcing on the arch in the pull-up direction and act to decrease the snap-through force.
Figure 5.6. Non Dimensional Force vs. Non Dimensional Displacement of Arch. Inertial Forces Considered Separately

As the arch is moved along the span of the rotor blade, the magnitude of the inertial force on the arch is increased. This increase in the inertial force on the arch may affect the performance of the arch as the arch is moved from its first to its second bistable configuration. In this section, inertial forces in the $e_x$, $e_y$, and $e_z$ directions are considered separately.
$e_x$, $e_y$, and $e_z$ directions are applied to the arch and the performance of the arch is examined as the arch is moved from its first to its second bistable configuration.

Figure 5.8 plots the non dimensional force versus non dimensional displacement for an arch. The triangle points represent the case where there is no external inertial load applied to the arch. The diamond points represent the case when the arch is located at the 16 % blade radius position. The square points represent the case when the arch is at the 80 % blade radius position. The results show that the snap-through force in the push down direction is reduced when inertial load acts upon the arch. This occurs because the inertial forces deform the arch in the $e_x$, $e_y$, and $e_z$ directions. These deformations reduce the effective stiffness of the arch and cause a reduction in snap through force.

A comparison between the arch at the 16 % and 80 % radial positions along the blade (see Figure 5.8) shows that the input of the arch at the 80% radial position is offset from its initial position nonloaded position. At the initial nonloaded position of the arch, the nondimensional displacement is zero. This offset from the initial position also causes an offset in the snap-through displacement. This offset is due to an increase in $e_x$ direction inertial forces that act on the arch as the arch is moved along the blade’s span. The results also show that the stability of the arch is affected by the position of the arch along the span of the blade. In Figure 5.8, a comparison of the No Load case to the cases where the arch is at the 16 % and 80 % blade radius locations show that the arch loses its bistable nature. This effect is caused predominately by the increase in $e_x$ direction inertial force upon the arch. As shown in Figure 5.6, $e_x$ direction inertial force acts to change the stability of the arch from bistable to unistable.
Figure 5.8. Non Dimensional Force vs. Non Dimensional Displacement of the Arch, Spanwise Location Comparison

**Arch Input Constraint and Applied Plate Load Study**

The previous studies have shown that inertial forces affect the performance of the arch and the component the inertial force that has the greatest effect on the arch is the $e_x$ component. In this section, the performance of the arch is examined when the input of the arch is constrained in the $e_x$ and $e_z$ directions. In addition, the bistable chord extension concept proposed in this work is composed of a flat plate. As the rotor blades rotate about the hub of the rotor, inertial force acts on the plate as shown in Figure 5.11. This inertial force exerts a reaction force in the $e_y$ direction of the arch which acts to move the arch in the push down direction. This reaction may affect the performance of the arch. Therefore, a second study conducted in this section is the examination of the performance of the arch while it is acted upon by inertial forces and a plate load.

Figure 5.9 plots the non dimensional force versus non dimensional displacement for an arch. The x points represents the case where there is no external inertial load applied to the arch. The diamond points
represent the case when the arch is located at the 80% blade radius position. The square points represent the case when the arch is at the 80% blade radius position with the input of the arch constrained in the $e_x$ and $e_z$ directions. A comparison of the x points and the square points show that the two curves are nearly coincident. This shows that constraining the arch in the $e_x$ and $e_z$ directions act to significantly negate the effects of the inertial force that act on the arch. The difference between the snap-through force in the pull-up direction between the x points and square points may be due to residual $e_x$ and $e_z$ direction deformations within the arch that have not been canceled out by the constraint at the input of the arch. This effect is probably not caused by the inertial force component in the $e_y$ direction as this component of force acts to increase the snap through force in the pull up direction as shown in Figure 5.6.

Figure 5.9. Non Dimensional Force vs. Non Dimensional Displacement of Arch with Constrained Input

A second study was conducted that considered the effects of the plate load on arch performance. As shown in Figure 5.11, a centrifugal force acts on the plate as the rotor blades rotate about the hub of the rotor. The centrifugal force quantity that acts on the plate is calculated by the equation $m \cdot \theta^2 \cdot r$. Here, $m$ is the mass of the plate, $\theta$ is the angular velocity of the blade and $r$ is the radial distance from the hub to
the center of mass of the plate. In this study, a 1 mm thick plate is assumed for chord extension. For a 1 mm thick plate, the reaction force on the arch \( F_y \) is calculated as 31 N.

Figure 5.10 shows the non dimensional force verses non dimensional displacement data for an arch. The x points represents the case where there is no external inertial load applied to the arch. The square points represent the case when the arch is located at the 80\% radial position and a plate load is exerted on the arch in the push down direction. The diamond points represent the case when the arch is at the 80 \% blade radius position, a plate load is exerted on the arch in the push down direction and the input of the arch is constrained in the e_x and e_z directions.

A comparison of the x points to the square points show that adding a plate load to the arch acts to offset the input of the arch in the push down direction. In addition, a comparison of the square points in Figure 5.10 to the diamond points in Figure 5.9 shows that the plate load increases the force required to snap the arch in the pull up direction. A comparison of the square points and the diamond points in Figure 5.10 shows that constraining the input of the arch in the e_x and e_z directions increase the snap through force in the push down and pull up forcing directions. However, unlike the square points in Figure 5.9, the diamond points in Figure 5.10 are not coincident with the x points (No Load case). This occurs because the plate load decreases the snap through load in the push down direction and increases the snap through load in the pull up direction.
In summary, an investigation of the feasibility of a bistable arch for helicopter chord extension was conducted. This examination only considered first order inertial loading effects and therefore considers the performance of the arch in the hove flight condition. The results showed that the inertial force component in the e_x direction greatly affects the performance of the arch. The e_x direction inertial force component is the largest centrifugal force component that acts on the arch. In addition, the magnitude
of the centrifugal forces on the arch is smaller when the arch is located inboard than when the arch is located outboard within the rotor blade. The large magnitude of centrifugal force that acts on the arch when the arch is located outboard within the rotor blade causes the arch to become unistable.

The results also showed that constraining the input of the arch acts to negate the effects of inertial load. Finally, adding a plate to the arch offsets the arch’s input. The plate load causes a decrease in the snap-through force in the push down direction and an increase in the snap-through force in the pull up direction. Overall, it is concluded that a Delrin bistable arch system is suitable for helicopter rotor blade hover flight applications. To extend this study to forward flight applications, periodic blade flapping and lead-lag must be considered.

**Shape Memory Alloy Wire Experiment**

A major area of concern for morphing concepts is actuation. In this work, the actuator that is selected to snap the arch is a Nitinol Shape Memory Alloy (SMA) wire. This actuation method was chosen because a Nitinol Shape Memory Alloy wires have the ability to produce large strain (up to 5%) and force (Dynalloy, 2010) once heated to its Austenite final temperature. Depending upon the wire’s configuration, the wire can produce up to 110 % strain and 120 degrees of stroke (Dynalloy, 2010). This actuation method is also ideal because it is light and can be heated using several different methods. The heating method of choice in this work is heating by an electrical current. In this work, the Nitinol Shape Memory Alloy wire is used to snap the arch from its first to its second stable state as shown in Figure 5.12 and Figure 5.13. The actuation concept operates by pulling the wire in tension while the wire sits in the arch handle. Then, the SMA wire is heated using an electric current. Once heated, the wire will shrink and produce a downward force on the arch handle. Once the snap-through force is exceeded, the arch will jump to its second stable state (Figure 5.13) where the wire will stretch out. There, the wire can be reheated to shift arch back to its first stable state.
An experiment was conducted to characterize the wire’s applied force to strain relationship. The wire selected for this experiment was a SM495 Nitinol, 0.0985 in. diameter, straight, black wire. The wire is 1.5 meters in length and was received from Nitinol Device Components (www.nitinol.com). This wire size was selected because it can exert up to 114 lbs in force. The experimental set-up is shown in Figure 5.14. The components of the experiment are listed in Table 5-2 shown below. The experiment was conducted in The Fluid Dynamics/ Water Channel facility. As shown in Figure 5.14, the wire was clamped at both ends and held in an upright position.
Figure 5.14. Wire Experiment Set-up
Table 5-2. Component of Wire Experiment

<table>
<thead>
<tr>
<th>Components</th>
<th>Description</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wire</td>
<td>8 gauge wire, Max current rating: 40 Amps</td>
<td>purchased from Lowes</td>
</tr>
<tr>
<td>Alligator Clips</td>
<td>Heavy Duty w/ crimp connection, 3” length, steel</td>
<td>McMaster part number: 7236K84</td>
</tr>
<tr>
<td>Resistor</td>
<td>Thick Film Power Resistor; Resistance:1ohm; Power Rating:600W;</td>
<td>Newark part number: 55R4168</td>
</tr>
<tr>
<td>LVDT</td>
<td>Linear Variable Differential Transformer, a common type of electromechanical transducer</td>
<td>Property of The Aero. Engineering Department</td>
</tr>
<tr>
<td>Weights</td>
<td></td>
<td>Property of The Aero. Engineering Department</td>
</tr>
</tbody>
</table>

The length of interest along the wire was 0.5513 m. The measured resistance of the wire at this length is 0.0840 ohms. This low level of resistance creates a short circuit condition. In order to alleviate the short circuit condition, a 1 ohm resistor was added to the circuit. The resistor, shown in Figure 5.14, was connected to a 2 x 2 x 0.5 in. aluminum block which acted as a heat sink to the resistor. Using the voltage to current relationship, \( V = IR \), the current in the circuit can be calculated. The current in the circuit when the power supply is at maximum voltage (15 V) is 13.84 A. Therefore, the power in the Nitinol wire when the current is 13.84 A is \( (\text{Power} = I^2 R) \) 16.1 Watts.

According to specifications (Nitinol Device Components, 2010) listed for the Nitinol SM495 wire, the Austenite final temperature is 60° C (140° F). From room temperature, the wire is required to undergo a 36° C change in temperature in order to reach its Austenite final temperature. The time required to heat the wire to its Austenite final temperature is inversely proportional to the electoral power provided to the wire. A calculation of the power required to heat the wire was calculated using power equations from Watlow’s Heating Catalog (Watlow, 2010). The power equation is as follows,
Power Required to Heat the Wire = A + C + \frac{2}{3} Loss + Safety Factor

where

\[ A = \frac{\text{Wire Mass} \times \text{Specific Heat of Material} \times \text{Temperature Rise}}{\text{Required Time}} \]

\[ C = \frac{\text{Wire Mass} \times \text{Heat of Fusion}}{\text{Required Time}} \]

\[ Loss = \frac{\text{Thermal Conductivity of material} \times \text{Surface Area} \times \text{Temp. Diff. to Ambient}}{\text{Thickness of material}} \]

The parameters used in these equations are as follows,

Wire Length: 0.5513 m

Wire Mass: 0.0179 kg

Specific Heat of Material: 620 J/kg °C (Huang, 2002; Duerig, 1994)

Temperature Rise: 36 °C

Heat of Fusion: 32,000 J/kg (Huang, 2002; Duerig, 1994)

Loss: 0

Safety Factor: 2 Watts

Power Required to Heat the Wire: 16.1 Watts (as previously calculated)

Therefore, the time required to raise the temperature of the wire by 36 °C is approximately 68 seconds.

An experiment was conducted where force and strain data was recorded for the wire. The force recorded is the weight that is applied to the wire as shown in Figure 5.14. The amount of shrinkage of the wire was recorded by an electronic LVDT. The shrinkage value was used to calculate engineering strain. The basic steps for conducting the experiment are as follows: first, weight was applied to the wire system as shown in Figure 5.14. Second, the wire was heated using an electric current. Third, after two minutes, the shrinkage read-out value from the electronic DVDT was recorded. Last, the wire was cooled by a fan for two minutes. This method was repeated several times to produce the results shown in Figure 5.15.

The results from the wire experiment are shown in Figure 5.15. In each case shown in Figure 5.15, the measured amount of temperature rise is between 33 and 42 °C after two minutes. This is
approximately double the time that was calculated using the heat energy equations. This error may be due to heat losses that occur in the circuit and inaccurate values approximated for the specific heat of material and heat of fusion parameters. The results show that the strain in the wire increases as the amount of weight applied to the wire is increased. The results also show that the maximum amount of strain that the wire can produce at 13,000 psi is approximately 1% of strain.

![Graph showing weight applied vs. strain](image)

Figure 5.15. Results from SMA Wire Experiment

The first step in actuating the arch using a Nitinol SMA wire is to determine the wire’s ability to move the arch to its second stable state. In order to complete this task, a second test on the Delrin arch was conducted where the input of the arch was not fully constrained. In this test, the input of the arch was allowed to move freely in the x and z directions but the input was constrained to move in the y direction. As the arch was pushed down, force and displacement data was recorded by an Instron machine. The goal here was to determine experimentally when the arch jumps to its second stable state. Previously it was expected that the arch would jump to its second stable state at point A shown in Figure 5.16. The force and displacement values at this point are called the snap-through force and snap-through displacement, respectively. Experimentally it was shown that the arch jumps to its second stable state at point B in Figure 5.16. This behavior may be due to the arch’s material which is Delrin. Delrin is a nonlinear material and its material properties change based on many factors such as temperature, humidity and strain rate (Du Pont, 2010). In addition, this behavior may also be due to yielding in the arch’s boundary hinges due to over-
loading during testing. The arch used in this experiment underwent many loading cycles before the arch was coupled to the SMA wire.

![Figure 5.16. Force vs. Displacement](image)

The amount of strain needed from the wire to snap the arch and the force in the wire at the corresponding strain value is shown in Figure 5.18. The strain equation used was,

$$
\varepsilon = \frac{\text{final length} - \text{current length}}{\text{current length}}
$$

Equation 5-10

where the final length of the wire was calculated based on the predicted final orientation of the wire. The current length of the wire was calculated based on the wire’s initial length and the current amount of arch input vertical displacement from its initial configuration. The force in the wire was calculated using equation,

$$
F_{\text{wire}} = \frac{F_{\text{Arch}}}{2 \cdot \sin \theta}
$$

Equation 5-11

where $F_{\text{Arch}}$ is the force applied to the wire by the arch. The parameter $\Theta$ represents the angular configuration of the wire as shown in Figure 5.17.
Push down means that the input of the arch was pushed downward. Pull up means that the input of the arch was pulled upward. In addition, the force in the wire is dependent on the force required to move the arch up or down and the wire’s orientation within the system. The strain needed to snap the arch is dependent on the length of the wire. For example, the data (in Figure 5.18) for push down shows that when the strain needed to snap the arch is approximately one percent, the force in the wire is 47 lbs. In contrast, the results from the wire experiment show that the amount of force required in the wire to achieve the desired one percent of strain in 100 lbs (as shown in Figure 5.18). Therefore, this demonstrates that the wire cannot actuate the arch to the point of snapping (point B in Figure 5.16) because there is only 47 lbs in the wire. The wire experiment showed that 100 lbs is needed within the wire to snap one percent.

Region P highlighted in Figure 5.18 demonstrates that there are wire and arch configurations along the arch’s pull up and push down path where it is possible for the wire to snap the arch. This occurs because the amount of force that is needed in the wire to generate the required level of strain to snap the arch is much lower than the current force in the wire. At these force values, it is predicted that the wire will strain more than what is needed to snap the arch.

Finally, the configuration of the wire must be different in the pull up and push down directions. That is, one configuration of the wire cannot be used to actuate the arch in the pull-up and pull-down direction if the wire is pre-tensioned and the length of the wire is held fixed. This is due to the location of the snap-through displacement of the arch in the push down direction (point B in Figure 5.16). This issue is further discussed in the Summary section.
Demonstration of the SMA Wire Actuating the Arch

A demonstration of the SMA wire snapping the arch to its second stable state was conducted. The demonstration was conducted using an arch that has rectangular hinges at its input and boundaries. The force-displacement behavior of this arch is similar to the behavior shown in Figure 5.16. The arch in its initial position is shown in Figure 5.19. In this configuration, the wire is below the arch in an inverted V configuration. The arch in its final position is shown in Figure 5.20. In this configuration, the SMA wire snapped the arch. In Figure 5.19, the arch is offset from its unloaded position (as shown in Figure 5.1) so that the wire could snap the arch. When the arch is held offset from its initial position, the arch provides tensile forcing to the wire (Figure 5.17). The wire requires tensile loading in order to produce a shrinkage strain when heated by an electric current. This offset configuration exists in Region P in Figure 5.18.

From its initial configuration, the wire moved the arch approximately 5 mm in the vertical direction to reach the point of snapping (point B, Figure 5.16). The total amount of stroke by the arch-wire system in the pull down direction is approximately 2.2 cm. The wire strain required to move the arch to the point of snapping was 0.03%. From the initial heat-up of the wire at room temperature, the arch took approximately 30 seconds to snap the arch to its second stable state.
A second demonstration of the arch was conducted in the pull-up direction as shown in Figure 5.21 and Figure 5.22. The snap through displacement of the arch in the pull-down direction is approximately seven times greater than the snap-through displacement of the arch in the pull-up direction as shown in Figure 5.16. Figure 5.23 compares the force in the wire to the amount of strain needed to snap the arch. The case Arch, IF = 0 N is the case where there is no additional load added to the arch prior to snapping. The case Arch, IF = 29 N indicates that a vertical force of 29 N was added to the arch prior to snapping. The case Wire Experiment is results from an experiment conducted on the wire alone. The experiment is explained in section Shape Memory Alloy Experiment and the results are also shown in Figure 5.15 and Figure 5.18. The results in Figure 5.23 show that case Arch, IF = 0 N does not cross the Wire experiment curve. This demonstrates that the arch cannot be snapped by the wire when the arch is unloaded. Therefore, additional preload was added to the wire as shown in Figure 5.21. This increased the load in the wire which was necessary to achieve the required amount of strain in the wire to snap the arch. Figure 5.23 also shows a Region P. Region P demonstrates that the wire can snap the arch by a strain only within the strain range marked by the region. Here, the maximum wire strain marked by Region P is approximately 0.22 % strain.

In Figure 5.21, the arch was offset from its initial position so that the wire could snap the arch. The wire moved the arch approximately 4 mm in the vertical direction to reach the point of snapping. The total amount of stroke by the arch-wire system in the pull down direction is approximately 8 cm. The strain
required to move the arch 4 mm to the point of snapping was 0.382 \%. From the initial heat-up of the wire at room temperature, the arch took approximately 30 seconds to snap the arch to its second stable state.

Figure 5.21. Pull Up, Initial Configuration

Figure 5.22. Pull Up, Final Configuration

Figure 5.23. Force vs. Strain

**Summary and Discussion**

In this work, an experiment was conducted to validate the developed 3D finite element model. Once validated, the finite element model was used to determine the feasibility of the proposed concept for helicopter morphing applications. The results showed that, in principal, a Delrin bistable arch system is suitable for helicopter rotor blade hover and forward flight applications. It was also proposed that a single wire could be used to actuate the arch. This concept is dependent on the tensile load in the wire. As stated previously, 13,000 psi of tensile stress is required in the wire in order to achieve approximately 1\% of
strain. Pre-tensioning the wire causes the input of the arch to drop down in the vertical direction as shown in the Figure 5.24. The length of the wire in this configuration is less than the length of the wire that is needed to allow the arch to reach its second stable state. Therefore, adding pre-tension to the wire and fixing the length of the wire in the system is not an effective means of snapping the arch to its second stable state.

Figure 5.24. Arch with Single Nitinol Wire

A component that can be added to the system to allow a single wire to snap the arch is a rotary and locking device (Figure 5.25) that allows the wire to increase, decrease and lock its length as needed. This concept could work by setting the rotary device to pretension the wire. Next, the wire is locked in this pretensioned configuration. Once heated, the wire will shrink and cause the arch to jump to its second stable state. As the arch moves to its second stable state, the wire’s length is increased. When the arch is in its second stable state, the wire is pretensioned by the rotary device and locked in this configuration. Then, the wire can be reheated to allow the arch to jump back to its first stable state.

Figure 5.25. Arch with Rotary Devices
References


Watlow. “Watlow, Revolutionizing the Heating Industry Heater Catalog”
Chapter 6
Conclusions and Recommendations

In this chapter, a summary of the research presented in Chapters 3-5 is given. The major contributions of each chapter are also given. Finally, recommendations are given for further research to further develop the proposed concepts for application on a helicopter platform.

Summary of Research and Conclusions

Helicopter rotor blade large shape change (morphing) is beneficial because it has the potential to improve the performance of the aircraft in different flight regimes. Several concepts to induce large shape change have been proposed previously, but most of these efforts do not examine the performance of the proposed concept while operating under external loads such as aerodynamic or inertial loads. Examining the performance of a concept while the concept it performs under external load is important because harsh external load can render the concept infeasible for morphing applications. Therefore, in this work, two concepts for helicopter rotor blade morphing were presented. The feasibility of each concept for helicopter rotor blade morphing was investigated in four steps: (1) development of a finite element model, (2) validation of the model by experiment, (3) introduction of design constraints, (4) examination of the performance of the device while external loads are exerted on it.

The first bistable concept for helicopter rotor blade morphing that was presented was a bistable twisting device to linearly twist the outboard 20% span section of the rotor blade (see Chapter 3). The bistable twisting device is to replace the spar of the rotor blade at the blade tip and provide twist to the outboard section of the rotor blade. However, since the twisting device is to replace the spar, the device must support aerodynamic loads at the tip of the rotor blade. Therefore, the objective of this work was to investigate the feasibility of these devices in an aerodynamic loaded environment.
In this work, only lift load was considered. The results showed that the bistable twisting devices can support some lift load at the tip of the blade. However, this lift load is below the assumed lift load requirement. At best, the devices can carry up to 12% of the required lift load. Therefore, these devices are not able to act as the sole load carrying structure at the tip of the rotor blade.

The second concept presented for helicopter rotor blade morphing is the use of a bistable arch and thin flat plate system. This system is to extend the chord of the rotor blade. The purpose of rotor blade chord extension is to provide additional lift to the rotor blade. This additional lift can be used in the region of stall of a helicopter rotor blade. The arch within this concept sits within the rotor blade and it is subjected to large inertial loads. Therefore, the feasibility of the arch to actuate the flat plate in this highly loaded inertial environment was examined. In this work, the only inertial loads that were considered were loads that are present in hover flight. The results showed that inertial loads generally decreases snap through force and displacement of the arch. The largest inertial load on the arch is the $e_x$ direction load as shown in Figure 5.4. Adding a plate load to the arch decreases the snap through force. Overall, it was shown that a bistable arch is feasible to extend the chord of a rotor blade in the hover flight condition. It was also demonstrated that a single Nitinol Shape Memory Alloy wire can be used to snap the arch from its first to its second bistable configuration.

**Contributions**

In this section the contributions of this work are briefly summarized. The contributions are stated according to the presented concept.

**Bistable Tip Twist**

The bistable tip twist concept provides unique contributions in the areas of direct twist actuation and bistable devices used in morphing. The first contribution of this concept is that it contains a bistable coupling device for direct blade tip twist. This concept is the first of its kind to be proposed for the
purpose of direct tip twist of helicopter rotor blades. Bushnell et al (2008) and Ruggeri et al (2008) developed a bistable coupling device to twist the entire blade. The device proposed in this work is designed only to twist the tip. In general, bistable devices have a self-locking attribute that make them ideal for rotor blade morphing concepts.

The second contribution of this concept is that it considers actuation under external load. Actuation/morphing of bistable devices while resisting aerodynamic loads has not been considered thus far in the literature. The device proposed in the work considers morphing while resisting lift and centrifugal loads.

The third contribution of the concept is that the bistable coupling device used in the concept does not require a locking mechanism. All other concepts for large angle blade twist in the literature require an additional on-board locking mechanism to keep the blade twisted. The concept proposed in this chapter does not require an additional on-board locking mechanism to keep the blade’s tip twisted because the bistable device self-locks after it is actuated.

Therefore, it is concluded that this concept provides unique contributions to the areas of direct blade twist and bistable devices for rotor blade morphing.

**Bistable Chord Extension**

This concept provides unique contributions in the areas of chord extension concepts and bistable devices used in morphing. The first contribution of this concept is that it is a lightweight device used for chord extension. Previous devices considered for chord extension appear to add a substantial amount of extra weight to the blade. The proposed concept is only composed of a lightweight bistable arch, SMA wires for actuation, a thin plate and support roller for the plate. This concept is expected to add little weight to the blade.

The second unique contribution of this concept is that it presents a bistable compliant mechanism that considers external load. The literature shows that bistable compliant mechanisms developed for
various applications have not considered external non-actuating loads. The bistable compliant mechanism presented in this work considers inertial load that act on the chord morphing system.

The third unique contribution of this concept is that it presents a bistable compliant mechanism developed for the purpose of helicopter rotor blade morphing. There has been no bistable compliant mechanism concept previously developed for the purpose of helicopter rotor blade morphing.

**Recommendations for Future Work**

The research presented in Chapters 3 and 4 investigated the feasibility of the presented bistable concepts for helicopter rotor blade morphing. In Chapter 5, an actuation method was presented for the bistable chord extension concept. In this section, recommendations are proposed for the purpose of further development of these concepts for use in helicopter platforms. For each proposed improvement, a problem statement, objective statement and remarks on how to satisfy the objectives are given.

**Replace Metal Bistable Twisting Device with a Composite Device**

**Problem Statement**

The numerical results given in Chapter 3 showed that devices that satisfy the geometric constraint and that possess bistability have thin shells. Whereas, devices that satisfy the loading constraint have thick shells. Shell curvature also has an effect on the ability of the device to satisfy the geometric, bistability and loading constraint. The ability to satisfy the loading, geometric and stability constraint is directly related to the stiffness of the shells. However, if the stiffness of the shells is tailored in the latitude and longitudinal directions, then it may be possible to develop devices that satisfy the geometric, loading and bistability constraints simultaneously. The shell must have high stiffness in the longitudinal direction of resist bending. The shells must have low stiffness in the latitudinal direction in order to allow the device to satisfy the geometric constraint.
Objective

Therefore, the objective is to optimize shell stiffness in the latitudinal and longitudinal directions. It is proposed that a composite laminate with multiple layers of fibers be used to produce shells that have optimal directional stiffness properties.

Remarks

Schultz (2008) in his work used steel and a composite laminate with two layers of unidirectional graphite/epoxy to develop twisting devices. However, the properties of the composite device were not optimized for aircraft morphing applications. Here, it is proposed that the properties of the composite device be optimized for aircraft morphing applications.

Steps to optimize the stiffness properties of the shells are as follows: (1) material selection, (2) develop a finite element model of the device, (3) validate the finite element model, (4) introduce design constraints, and (5) develop an optimization scheme to design the stiffness of the shells. A decision must be made on fiber type, direction of the fibers and number of plies. The second step is to develop a finite model of the device. This model will be used to predict the tip angle of the midpoint thickness of the device. This model must also predict the snap-through behavior of the device. In Chapter 3, a finite element model is presented for the bistable twisting device. Schultz (2008) presented a composite based finite element model for the device. Schultz presented model can be used as a starting point to develop a composite model of the bistable twisting device. The fourth step is to introduce design constraints. Careful consideration must be given to how the constraints are chosen as these constraints will be used as constraints in the optimization of the shells. Three possible constraints are a geometric, loading and stability constraints. Finally, an optimization scheme must be developed and used to optimize properties of the shells of the device. A possible objective function is to increase global stiffness in the longitudinal directions while decreasing global stiffness in the latitudinal direction. A possible method to complete the optimization is to couple a MatLab algorithm with the finite element model. MatLab has robust...
optimization functions. The developed composite device finite element model in Schultz (2008) is an accurate model and should be coupled with the optimization function.

Use Bistable Twisting Device as a Twisting Structure Only

Problem Statement

The bistable twisting device presented in Chapter 3 has been shown to possess bistability and support some lift load, but these devices cannot support the required lift load. Therefore, the bistable twisting devices may not be feasible for helicopter rotor blade morphing applications. However, these devices produce large angles of twist and therefore may possess the ability to induce large angles of twist at the tip of a rotor blade. Therefore, a concept must be developed that will allow the bistable twisting device to induce large angles of twist at the tip of the blade without the need to support the aerodynamic loads.

One possible concept that may work with this device to induce large tip twist angles is Berhard and Chopra’s Smart Active Blade Tip (SBAT) rotor blade concept (Bernhard and Chopra, 1997a, 2001b, 2002c, 2002d) (Figure 6.1). Bernhard and Chopra developed a piezo-induced bending-torsion coupled composite beam to obtain small rigid-body twist of the blade tip. By using the SABT twisting concept with the bistable concept presented in Chapter 3, large rigid-body twist angles may be obtained at the blade tip. A schematic of the bistable device and blade assembly is shown if Figure 6.1. The section of the blade where the bistable device resides has been made transparent. In this assembly, the bistable twisting device is surrounded by the spar of the rotor blade. The spar carries the aerodynamic loads while the device executes blade tip twist. In this model, the spar is inboard of the segmented blade tip. The bistable twisting device rest within the spar and provides a torque to the outboard blade tip.

In Figure 6.1, the twisting device is connected to the blade tip by a blade tip bearing and shaft assembly. The bearing fixture transfers lift, drag and associated bending moments of the blade tip to the main blade structure. A flanged sleeve bearing is added to the bearing fixture to resist centrifugal loads
exerted on the device. However, a moment must be applied to the twisting device to get the device to shift from its first to its second bistable configuration. Therefore, an actuator must be designed or selected for this purpose.

Figure 6.1. Bistable Device and Blade Assembly and Blade Tip Bearing and Shaft Assembly

Objective

The objective is to develop a concept similar to Chopra’s device that decouples the twist of the rotor blade from the aerodynamic load carrying requirement. The first step in satisfying this objective is the development of an actuator to twist the device.

Remarks

The actuator that is to twist the device has several constraints. The actuator must not add additional weight to the rotor blade, the actuator must not overheat, and the actuator must be able to apply the required moment and stroke to the device in order to get the device to jump to its second bistable configuration. A simple method to apply a moment to the device is to insert a rod through the root of the
device and attach the rod to the inner mid-section of the device. The rod must be powered by a motor in order to provide rotation and moment to the twisting device. To implement this concept, the root of the device must be un-screwed and there must be clearance between the two shells. Schultz (2007) showed that increasing clearance between the two shells does not necessarily result in a uni-stable device. An actuator that may provide a moment to the twisting device is Boeing’s twisting actuator (Bushnell et al, 2008; Ruggeri et al, 2008) as present in Chapter 1, Figure 1.3. This concept does not add additional weight to the rotor blade and can provide a large moment to the device.

Actuation of the Arch using Multiple Nitinol Shape Memory Alloy Wire Bundles

Problem Statement

In Chapter 5, a method was presented to actuate the arch using a Shape Memory Alloy wire. This method is beneficial because it is simple. However, the wire must be pre-tensioned in order to actuate the arch more than once in the pull-up and push-down directions. In certain cases, pre-tensioning the wire limits the pull-down motion of the arch and keeps the arch from reaching its second bistable configuration. In addition, a one wire concept requires a thick Shape Memory Alloy wire to provide large actuation force and stroke to the arch. Once heated beyond its austenite temperature, a thick wire requires a lengthy cooling time to reach its martensite phase which is necessary for actuation. Therefore, a modified actuation concept must be considered.

A Shape Memory Alloy based actuation system that will work for this concept is the use of multiple wires. These wires can be configured many different ways. One possible configuration is shown Figure 6.2. In this configuration, wire 1 is slack and wire 2 is taut when the arch is in its first stable state. The arch is represented by the first set of dashed lines as shown in Figure 6.2. The second set of dashed lines represents a connector between the arch and plate. When wire 2 is heated to its final austenite temperature, it shrinks and causes the arch to snap to its second bistable configuration. When the arch is in
its second bistable configuration, wire 2 is slack and wire 1 is taut. The arch can be returned to its first bistable configuration by heating wire 1. This system, in principle, demonstrates that two wires can be used to snap the arch.

In addition, instead of one thick Nitinol Shape Memory Alloy wire, a bundle of smaller diameter Nitinol Shape Memory Alloy wires can be used with convection cooling to decrease the cool down time of the wires. Therefore, to further develop the presented concept, the orientation of the wires must be determined, the wire bundle must be designed and a simple convection cooling method must be designed.

**First Stable State**

**Second Stable State**

Figure 6.2. Arch in Airfoil with Additional Wires
Objective

Therefore, the objectives of this work is as follows: (1) determine the number of wires and the thickness of each individual wire within the wire bundle that will replace the thick wire, (2) determine the orientation of the wire bundles that allow the arch to be shifted between its first and second bistable configurations, (3) develop a convection cooling system.

Remarks

Important parameters that must be known to determine the orientation of the wire bundles within the rotor blade are the snap through displacement of the arch, the size and geometry of the airfoil of the rotor blade, and the size of the segments along the span of the rotor blade that contain the chord extension system. Knowing these parameters will also help to determine the length and the diameters of the wires within the bundles. An additional parameter that is important in determining the diameter of the wires within the bundles is the snap through force of the arch. If the snap-through force of the arch is large, then the diameter of the Shape Memory Alloy wire must be large in order to provide enough force to shift the arch to its snap through displacement. A convection cooling system must also be developed in order to allow the wire to cool after being heated. One possible method is to increase the plate opening at the trailing edge of the rotor blade that is needed to extend the flat plate aft of the trailing edge of the rotor blade.
References


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