VECTOR MESON PRODUCTION AND THE DIPOLE PICTURE IN PHOTON-HADRON SCATTERING

A Thesis in
Physics
by
Ted C. Rogers

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Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

August 2006
The thesis of Ted C. Rogers was reviewed and approved* by the following:

Mark Strikman  
Professor of Physics  
Thesis Adviser  
Chair of Committee  

John Collins  
Distinguished Professor of Physics  

Richard W. Robinett  
Professor of Physics  

Yuri Zarhin  
Professor of Mathematics  

Jayanth Banavar  
Professor of Physics  
Head of the Department of Physics  

*Signatures are on file in the Graduate School.
Abstract

This thesis extends and combines traditional methods of hadronic physics, quantum chromodynamics, and nuclear physics so that they may be consistently applied to particular new and interesting regimes of photon(lepton)-hadron scattering. High energy photon(lepton)-production from the deuteron offers a unique opportunity to experimentally test qualitative aspects of the Strong interaction, and to investigate the possibility of applying perturbative quantum chromodynamics (QCD) methods to nuclear physics. New experiments planned at Jefferson National Laboratory and Brookhaven National Laboratory may soon make this possible. Furthermore, there has been a steady increase in the amount of theoretical work being done in nonperturbative QCD, and in extending perturbative QCD methods to new kinematical regimes. As such, it is becoming increasingly important to understand the interplay of perturbative and nonperturbative effects in hadronic interactions, and to establish the boundary of the kinematic regions where traditional pQCD methods can be applied. This thesis elaborates on methods, namely the color dipole picture and the vector meson dominance model, for describing photon(lepton)-proton scattering, uses them to determine the regions of applicability of perturbative QCD. We give extensions of this analysis to the case of ultra-high energy photons which has possible applications in particle astrophysics. Also, we discuss the traditional Glauber-Gribov method used in deuteron scattering, and attempt to place it on a more formal footing. With the widespread use of the Glauber optical method in nuclear physics, it is important to establish where Glauber Theory is applicable and what modifications, if any, need to be made in order for it to be used in current applications. The long-term goal of this work is to combine a picture of photon(lepton)-nucleon scattering based on the Color Dipole Picture and hadronic fluctuations with the Glauber-Gribov model of deuteron scattering to obtain a complete description of photon(lepton)-nucleus interactions.

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Acknowledgments

I cannot thank my adviser, Mark Strikman, enough for his patience and kindness. It has been truly fascinating to work with him.

I thank the members of my committee, John Collins, Richard Robinett, and Yuri Zarhin for their useful comments regarding my thesis and for their guidance.

I have had many exciting and educational interactions with my colleagues, Vadim Guzey, Misak Sargsian, and Xiaomin Zu. I thank them for adding to my over-all learning experience. I am also grateful for the warm hospitality I received on two stays at Florida International University.

I owe a great deal to the many people with whom I have benefitted from interesting and inspirational conversations. These include especially, John Collins, Sudarshan Fernando, Leonid Frankfurt, Tsutomu Mibe, Sean McReynolds, Isaac Mognet, and Christian Weiss.

I give a special thanks to Umberto Gutierrez, and Un Jeong Kim for their support and friendship. I will not forget the many weekends and vacations that we spent together.

Finally, I thank my family for their constant support.
dedicated to my parents
Chapter 1

Historical Introduction

1.1 Hadronic Physics Before QCD

Before the advent of Quantum Chromodynamics (QCD), strong interaction physics was based almost entirely on phenomenological models whose theoretical bases lay only with the fundamental principles of analyticity, unitarity, Lorentz invariance, and crossing symmetry in the S-matrix. It was speculated at the time that a complete theory of strong interaction physics would one day be formulated entirely by enumerating the analytic properties of the strong interaction S-matrix. Although this program failed, the pre-QCD period produced a number of very general and useful observations about hadronic interactions including Regge theory and the vector meson dominance model (VMD). The picture of hadron-hadron scattering that emerged from the 1950s described hadrons scattering at high energies and small angles with large cross sections that grow slowly with energy. It was recognized early on that high energy, low angle hadron scattering could be successfully described by quantum mechanical diffraction. Throughout the 1960s, a number of general properties of the Strong interaction S-matrix were proved. For example, the Froissart bound placed a limit on the absolute size and rate of growth of total cross sections, and the Pomeranchuk theorem established that only vacuum quantum numbers are exchanged between hadrons scattering in the high energy, small angle limit (for a review, see, e.g., ). Hadron phenomenology was conveniently organized according to poles of the S-matrix in the complex angular momentum plane (Regge theory). It was pointed out by Gribov in 1961 that if the hadron-hadron cross section grows at the rate prescribed by diffractive scattering, then there will be a violation of unitarity unless the radius of the interaction is allowed to grow slowly with the center of mass energy. The vector meson dominance model provided a relationship between photon and hadron interactions. Indeed, it was observed that the photon behaves, in many ways, like a hadron at high energies, allowing much of the theoretical framework of hadron-hadron scattering to be carried over to vector meson photo-production. The combination of the these and other early observations of hadron-hadron and photon-hadron scattering led to a very general spacetime picture: A photon/hadron scatters from the hadron target at high energies with a small angle and a large cross section that grows slowly with energy, and a radius of interaction that also grows...
slowly with energy.

In parallel with the development of hadron phenomenology, during the 1950s and 1960s a basic theory of hadron-nucleus scattering was developed by Glauber and others using purely non-relativistic quantum mechanical techniques [6]. The description of high energy/low angle scattering of a light projectile probe from a nucleus is described in a simple way in non-relativistic quantum mechanics when the nucleus is assumed to be a loosely bound system of nearly on-shell nucleons, and the plane wave eikonal approximation is assumed (to be discussed more in later sections). Gribov noted that a modified spacetime picture is needed to describe hadron-nucleus scattering at very high energies which takes into account the cross section fluctuations of the incident particle [7]. We refer to the combined work of Glauber and Gribov as the Glauber-Gribov theory.

The behavior in hadron-nucleus scattering can be extended to the case of incident photons by using the vector meson dominance model (VMD). The VMD hypothesis of high energy photon-hadron interactions successfully describes low angle photo-production of vector mesons from hadrons. In this approach, it is found that the amplitude for vector meson photo-production is roughly proportional to the amplitude for vector meson scattering, with a proportionality constant related to the virtuality of the incident photon, $Q^2$. Using the VMD approach, Kolbig and Margolis made a simple extension of the Glauber method for hadron-nucleus scattering to the case of vector meson photo-production from nuclei [8].

Although many of the pre-QCD techniques discussed above are largely phenomenological, they have a high success rate in describing experimental data. Today, the technical complexity of QCD ensures that, in order to obtain a complete description of many hadronic processes, it is necessary to combine QCD calculations with phenomenological descriptions in a consistent way. The combination of S-matrix theory, Regge theory, Glauber-Gribov theory, vector meson dominance, and cross section fluctuations form the framework with which we must now approach studies of the transition between different regimes in strong interaction physics including the break-down of leading twist perturbative QCD in DIS and the transition between nucleon and parton degrees of freedom in high energy nuclear scattering. In the next few sections, we will discuss several examples of such situations.

1.2 Small-x Deep Inelastic Scattering and the Black Disk Limit

In discussions of Deep Inelastic Scattering (DIS) we will use the usual notational conventions. In $\gamma^* N$ scattering, the photon has 4-momentum $q$ and the photon virtuality is denoted
by $Q^2 = -q^2$. The target proton has 4-momentum, $P$. The Bjorken scaling variable is,

$$x = \frac{Q^2}{2P \cdot q}. \quad (1.1)$$

DIS refers to the kinematic limit of fixed center of mass energy and large $Q^2$. Small-$x$ DIS involves a unique interplay between soft physics (describable by Regge theory and pre-QCD phenomenological models) and hard physics (involving a perturbatively small QCD coupling constant). Let us start by further describing the pre-QCD picture of hadron(photon)-hadron scattering.

In the case of a heavy nuclear target, the density of the target is large enough that scattering occurs with nearly the maximum value allowed by the unitarity constraint for all values of impact parameters in the radius of the target. Gribov considered the high energy limit, where the incident photon (real or virtual) fluctuates into a hadronic configuration a long distance before scattering from the target nucleus. He noted [9] that calculations of nuclear structure functions become very simple if the incident hadron is assumed to scatter with unit probability so long as it is incident upon the disk of area containing the target. We will refer to such scattering processes as taking place in “black disk limit” (BDL). (This approach neglects the contribution from large mass fluctuations in the photon.)

Before QCD was placed on a firm footing, models of the general spacetime picture of high energy interactions such as that used by Gribov to describe high energy nuclear scattering were used to describe experimental data and were a first step toward understanding dynamics. Indeed, Bjorken scaling and the formulation of the parton model by Feynman and Bjorken [10] ultimately led to an understanding of asymptotic freedom and to the formulation of QCD. Bjorken and Kogut recognized, however, that the parton model, if taken literally, was in gross contradiction to the behavior ascribed to hadrons and nuclei by Gribov in earlier studies [11]. The spacetime picture of DIS in the center of mass system, successfully described by the parton model is, on its surface, very different from the spacetime picture that had previously been used to describe high energy photo(lepto)-production of vector mesons in which the vector meson is produced a large distance before interacting with the target nucleon and then scatters from the target with a large cross section. It was noted by Bjorken in [11] that the black disk behavior mentioned by Gribov leads to a gross violations of scaling. He believed that there should be a “correspondence” between the parton model behavior and the general behavior observed in previous high energy scattering experiments, i.e., the spacetime picture of the parton model should reduce smoothly to the spacetime pictures of other strong interaction regimes. Bjorken believed that this correspondence principle, in the spirit of Bohr’s principle, could be used as a guide toward understanding new physics. In order to avoid what was perceived as a paradox in the opposing descriptions of high energy photon(lepton)-hadron scattering, Bjorken introduced
the aligned jet model (AJM). In the AJM, a $q\bar{q}$ that is produced by the incident photon interacts with a large probability ($\sim 2\pi r^2 A$) according to the Gribov logic (and violates Bjorken scaling), while the parton model would predict that a $q\bar{q}$ pair with relative momentum, $k_\perp^2 \sim Q^2$, would interact with very small probability. Bjorken, therefore, hypothesized that only a fraction $\sim k_0^2/Q^2$ of the $q\bar{q}$ states may interact\footnote{Bjorken did not consider diopes, but rather pairs of general hadronic jets. The argument, though, is the same.} for some maximum transverse momentum $k_0$. Hence, Bjorken scaling is restored in the large $Q^2$ limit. Since the allowed $q\bar{q}$ configurations have momentum aligned with the incident virtual photon, it was called the aligned jet model. The AJM was applied again well after QCD had been established in order to resolve controversy regarding the degree of shadowing in nuclei for hard interactions; the relationship between the smallness of the interaction with nucleons, the small size of the incident configuration (color transparency), and Bjorken scaling was emphasized in\cite{12} where it was demonstrated that small and large size configurations give comparable contributions to $F_2(x,Q^2)$ at small $x$ and $Q^2 \sim \text{few GeV}^2$ - the QCD aligned jet model.

During the 1970s a fundamental theory of strong interactions (QCD), based on a principle of non-Abelian SU(3) gauge symmetry, was finally developed and successfully applied to explain such phenomena as $e^+e^-$ annihilation, inclusive DIS, and the Drell-Yan process. Throughout the late 1970’s and early 1980’s, perturbative methods in QCD were developed and placed upon a rigorous mathematical footing (see, e.g.,\cite{13}). Within inclusive DIS, the complicated, non-perturbative information about the strong interactions is contained in the universal integrated parton distribution functions (PDFs) which have since been rigorously defined non-perturbatively in terms of operator products (see\cite{14} for an overview). Factorization theorems, which establish the factorizability of short range contributions to scattering cross sections from the long range contributions in the calculation of observable quantities were proved for a number of processes that involve at least one hard scale. Renormalization group methods of quantum field theory were applied to PDFs to describe their evolution with the hard scale. The renormalization group leads to the evolution of the parton distribution functions with the hard scale ($Q^2$ in inclusive DIS) through the well-known DGLAP (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) equation. A factorization theorem for a particular process states that, within a certain kinematical regime, for that process, the cross section may be expanded in powers of the inverse of the hard scale where the first term is in the factorized form (see, for example,\cite{15}). The factorized form is the “leading twist” term in the language of the operator product expansion, and therefore we will refer to this standard approach to pQCD as leading twist pQCD (LT pQCD).

Perturbative QCD has been remarkably successful at making predictions within its limited range of applicability. Inclusive DIS in the Bjorken limit of large $Q^2$ and fixed $x$ is generally an ideal application of leading twist perturbative QCD. The fact that QCD
is a renormalizable gauge theory, however, leads to the general observation that the high transverse momentum contributions to the scattering cross section increase quickly with energy. For extremely small $x$, an expansion in a finite number of terms clearly becomes inappropriate, and the usual methods of LT pQCD breakdown. The problems that develop at small Bjorken-$x$ are particularly interesting because they involve a connection with the earlier work on Regge theory and the black disk behavior of hadrons. A result of pQCD is that $(\alpha_s \ln 1/x)^n$ terms appear at the $n$th order of a perturbative calculation. At small-$x$, these terms become large and it is clear that a new approach needs to be taken. One approach is to re-sum the series of terms proportional to $\alpha_s \ln 1/x$. This leads to another evolution equation (the BFKL equation) [16]. The terms of order $\alpha_s \ln 1/x$ lead to a very rapid growth of the gluon distribution function at small-$x$. Assume that as $x$ decreases, the gluon distribution rises faster than,

$$g(x, \mu^2) \sim x^{-1},$$

(1.2)

at small-$x$. Since the gluon distribution dominates, and $F_2(x) \sim x g(x, \mu^2)$ in pQCD, then this yields the prediction that at small $x$, the gluon distribution increases at small $x$ faster than,

$$F_2(x) \sim x^{-\lambda},$$

(1.3)

where $\lambda \gtrsim 0$. This behavior will eventually lead to a violation of the pre-QCD Froissart bound and hence to a violation of unitarity. It is instructive to approach the problem from the perspective of Regge theory. At small-$x$, one approaches the Regge limit, and it is tempting to use a more Regge-like description of the distribution over $x$. In that case, the structure functions should be expressible as a sum over Regge trajectories with the leading behavior established by a Pomeron-like trajectory:

$$F_2(x) \approx F_{IP} \left( \frac{s}{s_0} \right)^{\alpha_{IP}(t) - 1}$$

$$\approx F_{IP} \left( \frac{x}{x_0} \right)^{1 - \alpha_{IP}(t)}.$$  

(1.4)

where $F_2(x)$ is the usual structure function of DIS, $s$ is the usual Mandelstam variable for the $\gamma^*\text{-proton}$ interaction, and $F_{IP}$ is the residue of the Pomeron trajectory. The problem that arises in description of Eq. (1.3) is, therefore, reminiscent of the unitarity problem that arises in Regge theory when $\alpha_0 > 1.0$.

Thus, we may associate the breakdown of LT pQCD that occurs at small-$x$ with the rapid increase in the distribution of gluons that take part in the interaction at high energies. While the structure function, $F_2(x, Q^2)$, grows without bound at small-$x$, the QCD coupling constant is still perturbatively small if $Q^2$ is sufficiently large. One of the signif-
icant observations that came from HERA during the early 1990s was that the increase in the structure functions is very rapid at small-$x$. It was found that $\lambda \approx .3$ in Eq. (1.3) for $Q^2 \gtrsim 10$ GeV (see e.g. [17]). A desire to connect formal QCD with preQCD methods such as Regge theory and $S$-matrix theory has led to very active research in the area of small-$x$ DIS. Furthermore, as the understanding of quantum field theory has progressed, properties of different phases in QCD has become an interesting and accessible topic (see, e.g. [18]). For example, attempts are now being made to describe scattering at low-$x$ as a “color glass condensate”.

All of this leads to a complex interplay between well established perturbative calculations and non-perturbative models. The need for a consistent framework with which to describe photon-hadron interactions in various kinematical regimes, and which contains all desired perturbative and non-perturbative effects led to the development of several versions of the color dipole picture. Furthermore, in studying the properties of QCD phase transitions, it is necessary to know the kinematical boundaries of LT pQCD, and the regime where the nucleon behaves more like the black disk described by Gribov. The dipole model allows for a smooth transition between pQCD and non-perturbative models in the spirit of Bjorken and Kogut’s correspondence principle.

In recent years, the dipole picture has found a new venue of application in the realm of high energy particle astrophysics where it is applied to the case of an ultra-high energy real photon (of the order of $10^{20}$ electron volts). Clearly, the direct application of pQCD is inappropriate at these energies. However, there is a possibility to use very general assumptions along with low energy matching results to obtain constraints on the behavior of UHE photon cross-sections. It turns out that at ultra-high energies, the density of gluons becomes large enough that there is a possibility of a large contribution from highly virtual fluctuations of the photon. These issues will be discussed in later chapters.

1.3 Vector Meson Production and Photon-Nucleus Scattering

The vector meson dominance hypothesis arose for phenomenological reasons in the description of small angle vector meson photo-production and is supported by the many observed similarities between the photon-hadron and hadron-hadron cross sections. (For an overview of the similarities between photon and hadron cross sections, see [19].) The basic assumption of the VMD hypothesis is that the photon state may be expanded in terms of a “bare” photon state and a set of hadronic states that undergo ordinary hadronic interactions with the target hadron or nucleus:

$$|\gamma\rangle \approx |\gamma_0\rangle + \sqrt{\alpha_{\text{e.m.}}} |h_n\rangle.$$  (1.5)
Assuming that the vector meson production cross section is dominated by resonances corresponding to the $\rho^0$, $\omega$, and $\phi$ meson leads to the statement of the VMD model:

$$\sqrt{\alpha_{e.m.}} \langle h_n \rangle \approx \sum_n \frac{e}{f_n m_n^2} \frac{m_n^2}{m_n^2 + Q^2} |n\rangle. \quad (1.6)$$

A heuristic derivation of this formula can be given within the framework of time-independent perturbation theory. The VMD model may be generalized somewhat by including contributions from states other than the main vector meson resonances (generalized VMD). Given the VMD hypothesis, the relationship between photon-hadron and hadron-hadron interactions becomes very simple. For example, The $\gamma N \rightarrow V N$ cross section is directly proportional to the $V N \rightarrow V N$ cross section:

$$\sigma_{\gamma N \rightarrow V N} = \frac{e^2}{f_V^2} \sigma_{V N \rightarrow V N}, \quad (1.7)$$

where $f_V$ are coupling constants that can be extracted from $e^+ e^-$ decay widths. Hence, the VMD hypothesis allows one to study the interaction of vector mesons with other hadrons via the interactions of photons with hadrons. It is important for a general understanding of strong interactions, then, to improve understanding of the VMD behavior as well as its limitations. We are particularly interested in the application of the vector meson dominance model photon-nucleus scattering.

The basic framework for the scattering of a projectile probe from a nuclear target that persists to this day is based on the early non-relativistic quantum mechanical work of Glauber [6, 20] in the context of hadron-nucleus scattering. The purpose of this early work was to quantify the suppression of a nuclear cross section relative to the sum of the cross sections of the constituent nucleons (nuclear shadowing). An important result that is still used today is the a formula for relating the total nucleon-nucleon cross section to the $A$-dependence of the nucleon-nucleus scattering cross section. All of the amplitudes in the Glauber formalism are elastic, and the incident projectile may scatter at most once from each of the nucleons in the target nucleus. The multiple scattering interactions lead to a reduction in the total cross section relative to the sum of the cross sections for scattering from the individual nucleons. The incorporation of very high energy dynamics effects in multiple scattering was achieved by Gribov in 1969 [7]. The resulting spacetime picture, which requires relativistic considerations, deviates significantly from the non-relativistic picture discussed by Glauber. Heuristically, the incident projectile is actually a linear combination of a bare particle state plus quantum mechanical fluctuations into multi-particle states as one would expect in a relativistic quantum field theory. At very high energies, the spatial duration of these fluctuations ($\sim 2M_N^2/E_\gamma$) becomes larger than the radius of the
struck nucleus. Consequently, when the incident particle interacts with the first nucleon, there is not enough time for the debris to reform the initial particle state. As a result, inelastic scattering begins to become more significant at very high energies. The relationship between nuclear scattering and diffractive scattering from nucleons in the Glauber-Gribov formalism introduces the potential to use scattering from nuclei as a way to study diffraction of hadrons on nuclei. It was found that including the Gribov effects increases the amount of nuclear shadowing significantly and thus gives a measurable physical relationship between nuclear shadowing and diffraction dissociation. The Glauber-Gribov framework is naturally extendible to vector meson production from nuclei; the incident photon is treated as a linear combination of three hadronic states. Kolbig and Margolis extended the basic non-relativistic Glauber formalism to the case of vector meson production in 1968 using VMD ideas. As the modeling of hadronic scattering in terms of field theoretic pictures progressed, it became desirable to formulate the Glauber-Gribov theory in diagrammatic language while continuing to use the nonrelativistic wavefunctions of nuclear physics (see Appendix C for an elaboration of the steps).

Since the A-dependence of the $\gamma(\gamma^*)$-A cross sections is related to the basic nucleon cross sections via the Glauber-Gribov formalism, multiple scattering from the nucleus introduces another way to analyze strong interaction physics under various conditions. The appearance of color transparency (the sharp drop in the cross section as the hard part of the interaction becomes dominant) can be used to study the transition between quark and nucleon degrees of freedom in the nucleus. In the context of current interests, the production of vector mesons through multiple scattering with nuclei is used to probe interesting and poorly understood regimes of strong interactions and to test the limits of applicability of the VMD hypothesis. In [21], for example, the authors use coherent vector meson lepto-production in the hard regime to determine the on-set of color transparency. In the vector meson dominance model, the basic $\gamma(\gamma^*) \to VN$ cross sections are nearly independent of energy. The transition from the VMD regime to the color transparency regime will be signaled by a substantial drop in the double scattering. Furthermore, by choosing different scattering angles for the produced vector meson, one can control the relative distances between the nucleons at which rescattering occurs and thereby study the spacetime picture of the production process.

Finally, a long-standing goal of modern nuclear physics is to understand hadronic interactions in terms of an underlying field theory, and even to relate that effective field theory to QCD. Efforts to formulate a field theoretical description of vector meson production from nuclei have been made in more recent years by (see, e.g. [22, 23]), and experiments are now being conducted at SPring/LEPS and Jefferson Lab to confirm some of their more unusual predictions. As it becomes more desirable to focus on the details of vector meson production interactions, it becomes necessary to reconsider the effects of those details on
the original Glauber-Gribov formulation.

1.4 In This Thesis

Deep Inelastic Scattering on the proton has remained one of the main tools with which the details of the strong interaction and the standard model are investigated, and it continues to lead to a better understand of strong interaction dynamics. DIS on nuclei has lead to the verification that nuclear shadowing persists in the Bjorken limit (the EMC effect [24]). However, as we discussed in the previous two sections, many interesting aspects of strong interaction physics lie outside the range of applicability of perturbative QCD, and many interactions still depend upon tentative phenomenological models. This is true, for example, for the vector meson dominance property. On the basis of the VMD, for example, recent studies of the production of $\phi$-mesons at SPring-LEPS have reported the strange observation that the $\phi$-nucleon cross section in $\phi$-nucleus scattering seems to be larger by at least a factor of three than for $\phi$ scattering from a free nucleon at moderate photon energies. In another extreme case, upper limits have been placed on the growth of ultra-high energy cosmic ray photon cross sections on the basis of old-fashioned Pomeron physics and QCD-inspired models.

It becomes necessary to establish a consistent basic framework for applying the phenomenological models to nuclei and/or combining them with predictions from perturbative QCD. This is the main purpose of this thesis. The thesis can be considered as consisting of two main parts; the first part consists of Chapters 2 through 4 and is concerned with very high energy photon(lepton) interactions with proton and nuclear targets, while the second part consists of Chapters 5 through 6 and is concerned with intermediate energy photo-production of vector mesons off the deuteron. In the Chapter 2, we will give a general discussion of the dipole picture of DIS at small Bjorken-$x$. The particular version of the dipole picture that we will apply was formulated by McDermott, Frankfurt, Guzey, and Strikman (MFGS) though we will include comparisons with other models. In Chapter 3 we will discuss its use to determine the proximity of small Bjorken-$x$ totally inclusive DIS to the Black Disk Limit (BDL). Original work presented here is the application of the QCD-improved dipole picture to estimate the proximity of small-$x$ DIS to the BDL, and a analysis of the importance of the quark mass in the dipole picture. In Chapter 4, we will then discuss the extension of the MFGS model to the case of Ultra High Energy photons (UHE) and the resulting possible application to cosmic ray astrophysics. Original work here is the application of unitarity to the individual $q\bar{q}$ configuration cross sections to extract upper limits on the growth of the cross section for high energy real photons.

In Chapter 5, we will discuss another regime of photon scattering. This is the regime of photo-production of vector mesons from the deuteron when the energy of the photon is not
large enough for either the application of perturbative QCD or the usual Glauber-Gribov approach to multiple scattering in nuclei. We will deconstruct the commonly used applied Glauber formulae for vector meson production with multiple scattering from the deuteron and set up a framework for taking into account effects that are normally neglected in the Glauber-Gribov multiple scattering approach. In Chapter 6, we will provide examples of where these effects become important in the context of recent interest in $\phi$-meson production from the deuteron. These are the first calculations that we know of wherein the energy dependence of the basic photon-nucleon amplitude is considered in the integral for coherent vector meson photo-production from a nucleus.
Chapter 2

The Dipole Picture

The color dipole picture is closely related to the concept of cross section fluctuations (see, e.g. [25]) which predates QCD. It is natural, then, that it provides a useful framework with which one can relate basic properties of hadrons known well before the advent of QCD with the results of pQCD predictions. It will therefore be necessary to give an overview of the dipole picture and its various applications in the current approach to small-$x$ DIS.

2.1 Background and Basic Formulation

Early approaches to high energy particle diffraction are similar in spirit to the current form of the color dipole picture. Good and Walker [26], following the earlier work of Feinberg and Pomeranchuk, in 1960 [2] gave a heuristic description of high energy particle diffraction dissociation in which the incident beam (of hadrons, in their case) was regarded as a linear combination of quantum mechanical fluctuations near the on-shell free “bare” hadron state. While the initial application of this formalism was to diffractive hadron-hadron scattering, the basic idea appeared in the description of photon scattering though in a different guise when, in 1971, Kogut, Soper and Bjorken [27] calculated the dimuon production cross section from an external field where the photon transforms into a $\mu^+\mu^-$ dipole before scattering. Soon after the formulation of QCD, searches began for a QCD mechanism to describe the diffractive behavior observed in high energy, low angle hadronic scattering. Low and Nussinov [28, 29] independently demonstrated that the basic properties of hadron scattering could be recovered in a two-gluon exchange model (two gluons because the exchanged object must carry vacuum quantum numbers in diffractive scattering). Miettenin and Pumplin [30] later suggested that the gluons in the target should be associated with the partonic states of perturbative QCD. The application of the dipole picture to the scattering of a $q\bar{q}$ using the eikonal model at fixed impact parameter was used by Nikolaev and Zakahrov [31] in the early 1990s to describe color transparency in deep inelastic scattering from nuclei. Ultimately, we want to use the dipole picture to study the amplitude in impact parameter space in the spirit of the early work by Cheng and Wu [32]. It was pointed out in [25] that the cross section of the dipole nucleon interaction within the leading log approximation is actually proportional to the gluon density at high virtuality and small $x$, resulting in a fast growth
of the total cross section. The resulting dipole-nucleon cross sections were used to take into account finite $Q^2$ corrections to exclusive vector meson production in DIS \cite{33}. Splitting of the dipoles into systems of dipoles was studied within the BFKL approximation in \cite{34}. The basic kinematical idea behind the dipole picture is as follows:

Gluon induced reactions dominate at small-$x$ where the gluon distribution becomes large. A typical Feynman diagram contribution (the leading order gluon-induced graph) is shown in Fig. 2.1. It is convenient to write the momentum of the target in a frame where there is no transverse component for both the incident virtual photon and the target nucleon in light-cone coordinates as,

$$ P = \left( P^+ , \frac{M_N^2}{2P^+} , 0_\perp \right), $$

(2.1)

where $M_N$ is the mass of the nucleon. Then, the virtual photon 4-momentum can be written as,

$$ q = \left( -xP^+ , \frac{Q^2}{2xP^+} , 0_\perp \right). $$

(2.2)

In the rest frame of the target,

$$ q_z \approx \frac{Q^2}{2M_N x}. $$

(2.3)
If we write the momenta of the produced quark and anti-quark respectively as,

\[ k_1 = \left( zq^+, \frac{\perp^2}{2zq^+}, \perp \right) \]
\[ k_2 = \left( (1 - z)q^+, \frac{\perp^2}{2(1 - z)q^+}, -\perp \right) , \tag{2.4} \]

then the energy dominator for the transition from photon to \( q\bar{q} \) pair is,

\[ \Delta E = \frac{M_{q\bar{q}}^2 + Q^2}{2q_z}, \tag{2.6} \]

where \( M_{q\bar{q}}^2 = \frac{\perp^2}{z(1 - z)} \) is the invariant energy squared of the \( q\bar{q} \) system. In the limit of large \( Q^2 \) (and \( x \) small but not too small), standard LT pQCD is applicable, and the interaction may be described as the scattering of a \( q\bar{q} \) pair with very large relative transverse momentum, \( \perp^2 \sim Q^2 \) with the target. If \( M_{q\bar{q}}^2 \approx Q^2 \), then the time-scale, \( \Delta t \), over which the dynamics of the \( q\bar{q} \) pair takes place can then be estimated from the Heisenberg uncertainty principle,

\[ \Delta E \sim \frac{1}{\Delta t} \sim \frac{q_z}{Q^2} \sim \frac{1}{2M_Nx}. \tag{2.7} \]

That is to say, the \( q\bar{q} \) pair evolves very slowly in the rest frame of the target. Alternatively, if the \( q\bar{q} \) pair has high enough energy that it is moving at near the speed of light, then the distance over which the interaction takes place is,

\[ l_C \sim \frac{1}{2M_Nx} \tag{2.8} \]

in natural units. We call \( l_C \) the coherence length. If the coherence length is very large, then the \( q\bar{q} \) pair evolves very slowly over the longitudinal dimension of the target nucleon, and the photon fluctuates into the \( q\bar{q} \) configuration a long distance before interacting with the target. Therefore, the \( q\bar{q} \) pair may be considered to be “frozen” at a certain size, \( d \), as it passes through the target. This is illustrated in the cartoon picture shown in Fig. 2.2. An appealing property of this picture is that it is very similar to the general picture of a photon or a hadronic projectile that fluctuates into a general hadronic state a long distance before interacting with the target - a property that suggests that it may be used to smoothly related perturbative and non-perturbative effects in strong interaction physics. The phenomenological convenience in this approach is its similarity to the concept of hadronic fluctuations. The basic idea of the cross section fluctuation description of hadron scattering is that the incident hadron may be expanded in a basis of cross section eigenstates, and that a calculation of the total cross section can be obtained from an understanding of the individual cross section eigenstates. For example, unitarity constraints cannot be placed on
the total $\gamma^* N$ cross section because of the non-normalizability of the photon wave function. However, unitarity constraints can be applied to the individual hadronic Fock states of the incident photon.

In the Bjorken limit, it is possible to directly obtain an equation for the cross section in terms of the interaction of a $q\bar{q}$ pair with a nucleon target in LT pQCD. One must calculate the contribution from graphs like that in Fig. 2.1. The result is a convolution of a photon wavefunction calculated within pQCD with the cross section for the interaction of a $q\bar{q}$ of a particular transverse size, $d$. \footnote{b was used to denote the hadronic size in Ref. \cite{35} and other texts. In this work we use the variable, $d$, for “dipole” to avoid confusion with the impact parameter.}

$$\sigma_{L,T}^{\gamma^*N}(Q^2, x) = \int_0^1 dz \int d^2d |\psi_{L,T}(z, d)|^2 \hat{\sigma}_{tot}(d, x'), \quad (2.9)$$

The photon wavefunctions for transversely and longitudinally polarized photons are, respec-
tively [31],
\[
|\psi_T(z,d)|^2 = \frac{3}{2\pi^2} \alpha_{e.m.} \sum_{q=1}^{n_f} e_q^2 [(z^2 + (1 - z)^2)e^2K_1^2(\epsilon d) + m_q^2 K_0^2(\epsilon d)],
\]
(2.10)
\[
|\psi_L(z,d)|^2 = \frac{6}{\pi^2} \alpha_{e.m.} \sum_{q=1}^{n_f} e_q^2 [z^2 (1 - z)^2 K_0^2(\epsilon d)],
\]
(2.11)

where \( e^2 = Q^2 z (1 - z) + m_q^2 \), and the cross section for the interaction of a \( q\bar{q} \) pair with a nucleon target is given in leading order pQCD by,
\[
\hat{\sigma}_{pQCD}(d,x) = \frac{\pi^2}{3} d^2 \alpha_s(Q^2) x' g(x',\bar{Q}^2).
\]
(2.12)

A derivation of this formula is given in [36].

The parameters that enter into Eq. (2.12) are the virtuality of the \( q\bar{q} \) system, \( \bar{Q}^2 \), given by \( \bar{Q}^2 = \lambda d^2 \) where \( \lambda \) is a parameter that can be fixed by data, and the gluon momentum fraction, \( x' \). (We will discuss the relationship of \( x' \) to external kinematics when we discuss particular versions of the dipole model.) The interpretation of Eq. (2.9) is simple; the total cross section for the interaction of the virtual photon with a nucleon is the sum of the probabilities for a dipole of a particular size to interact with a nucleon, weighted by the probability for a virtual photon to fluctuate into a \( q\bar{q} \) pair. The total \( \gamma^*N \) cross section is the mean of the cross sections of the individual \( q\bar{q} \) components. This formulation can be shown to be formally equivalent to the standard result of leading twist pQCD (see Appendix A) in the Bjorken limit where the distribution in the sizes of the \( q\bar{q} \) pairs becomes sharply peaked around a particular high virtuality for the longitudinal structure function.

For small size configurations, Eq. (2.12) demonstrates the well-known property of color transparency which appears at small-\( x \) as a sharp drop in the cross section. This behavior leads to a reduction in the amount of shadowing seen in low-\( x \) DIS scattering from nuclei less than what is normally expected in nuclear physics where cross sections are large. As one considers softer interactions, contributions from larger sizes become more important and the cross section grows. At a certain point, the interaction becomes too soft for the perturbative QCD expression to be applicable. However, Eq. (2.9) may be smoothly extended into the region where the squared wavefunction represents the probability that the photon fluctuates into a constituent \( q\bar{q} \) pair (or a more general hadronic state) which scatters from the nucleon with a cross section typical for a hadronic state as in the VMD picture. Thus, the dipole picture is ideal for probing the transition between quark and hadronic degrees of freedom.

Another situation that occurs at low-\( x \) is that terms involving \( \ln x \) represented by diagrams with many radiated gluons being emitted become important. If such diagrams are not taken into account, then pushing the LT pQCD result to extremely small-\( x \) leads to a large rise in
the gluon density and hence a large rise in structure functions. This qualitative behavior, predicted by LT pQCD, is evident experimentally in the observed rapid rise in structure functions at low-$x$. However, if such a rapid rise as is predicted by LT pQCD continues down to smaller $x$, then s-channel unitarity constraints are strongly violated. Hence, non-leading twist effects must come into play are expected to cause the rise in the cross section with decreasing $x$ to level off at some hadronic-size value. Therefore, the region of very small-$x$ is qualitatively very different from the regime of moderate $x$. For very small-$x$, cross sections become large but, unlike in the soft limit, the QCD coupling constant remains small. This has lead to many attempts to study the small-$x$ limit as a QCD phase transition with small coupling techniques. Another interesting question is how this small-$x$ behavior affects nuclear shadowing and color transparency.

Toward this end, it becomes desirable to model the transition between LT pQCD physics, where cross sections are small but rapidly increasing with energy, and soft physics (or small-$x$ physics) where the cross section becomes large and is almost constant with energy. On the basis of unitarity conditions, it was generally assumed early on that DIS at HERA kinematics reaches the saturation phase at small $x$. This was discussed in Ref. [33] based on integral quantities, $\sigma_{\text{elastic}}$ and $\sigma_{\text{tot}}$ where the hope was expressed that saturation-like behavior would be relevant for HERA kinematics. The purpose of the dipole picture is to continue to use Eq. (2.9) beyond the range of applicability of LT pQCD by modeling the behavior of the basic cross section and imposing conditions that ensure that the cross section is tamed at sufficiently high energies. The first and simplest of these approaches was introduced by Golec-Biernat and Wüsthoff [37] and will be referred to as the saturation model (or the GBW model) following the standard convention. The purpose of the GBW model was to justify quantitatively the assumption of what was referred to as “saturation at HERA.” The saturation model simply uses exponential behavior for the growth with $d^2$:

$$\hat{\sigma}(x, d) = \sigma_0 \{1 - e^{-4R_0(x)d^2}\}.$$  \hspace{1cm} (2.13)

where,

$$R_0(x) = \frac{1}{GeV^2} \left(\frac{x}{x_0}\right)^{\frac{\lambda}{2}}.$$  \hspace{1cm} (2.14)

The parameters, $\sigma_0$, $x_0$, and $\lambda$ may be adjusted to fit the model with data. The strategy of the saturation model is simply to restrain the growth of the cross section at a constant value for kinematics that deviate significantly from the Bjorken limit. For large $d$, Eq. (2.13) becomes a simple constant, $\hat{\sigma} = \sigma_0 \approx 23 \text{ mb}$, of the order of a typical light hadronic cross
section (e.g. pion). In the limit of small $d$,

$$\hat{\sigma} = \frac{d^2}{4} \left( \frac{x}{x_0} \right)^{\tilde{\lambda}}.$$  

(2.15)

Thus, for $\tilde{\lambda} > 0$, the saturation model exhibits the general behavior of color transparency and rapid rise in energy in the large $s$, large $Q^2$ regime. The saturation model $q\bar{q}$ cross section exhibits color transparency and a rise at small $x$, but has a number of significant limitations. One is that by having fixed the shape of the function that relates large and small size dipoles, one limits the ability to model the large and small size dipoles independently. Another is that, while the size of the total dipole cross section has been fixed at the pion cross section in the large size or small-$x$ limit, it is insufficient to conclude that it is dipole cross sections of this size that are in the “saturation” phase, or that their interactions resemble hadronic interactions which are characterized by the proximity to the unitarity limit. This is because a size of the total cross section is not a good measure of the strength of the interaction unless the physical transverse size of the target is specified. For instance, the same value of total cross section that would seem large if it were associated with the proton would seem tiny if associated with a lead nucleus. Depending upon the $t$-dependence of the interaction, the radius of the target would become very broad at low-$x$ so that the cross section could become very large without violating $s$-channel unitarity, even if the interaction strength is small (a direct illustration of how this could happen will be given later in the discussion of ultra-high energy photons). In the next section, we will justify the existence of saturation at HERA on the basis of a more realistic model that takes these effects into account.

## 2.2 The MFGS Model

A formulation of the dipole picture that directly incorporates pQCD by using Eq. (2.12) was formulated in 1999 by McDermott, Frankfurt, Guzey, and Strikman (MFGS) [35]. The small size configurations behave like pQCD $q\bar{q}$ pairs whereas the large sizes behave like typical hadronic states (e.g. pions). A complete description of the basic $q\bar{q}$-N cross section is then obtained by simply interpolating smoothly between these two regions. A major advantage of this approach over the GBW model is that the behavior of large and small size configurations may be modified independently from one another. The shape of the cross section is not dependent upon the use of the exponential function to interpolate between large and small size behavior. Of course, as with any dipole model, the behavior of the intermediate size dipoles between the pQCD region and the hadronic region is poorly understood, and the best hope for making physical calculations is that the result of a calculation is not sensitive to a particular method for interpolating between the soft and
hard regions.

We start with the formulation of the total DIS longitudinal and transverse cross sections written as the convolution product of Eq. (2.9). The cross section for a particular hadronic fluctuation of the virtual photon is denoted by $\hat{\sigma}_{\text{tot}}(d,x')$. Although, strictly speaking, configurations away from the Bjorken limit are not dipoles, to simplify the discussion we will often refer to this quantity as the dipole cross section (DCS).

According to the pQCD factorization theorem for the longitudinal cross section, at large $Q^2$ the main contribution to the total cross section comes from highly virtual configurations of the $q\bar{q}$ pair, $Q^2 = \lambda/d^2$ where $\lambda$ is a matching parameter to be determined from data. The MFGS model of $\hat{\sigma}_{\text{tot}}$ should therefore reproduce this behavior in the large $Q^2$ fixed energy limit. At this point, an ansatz is required to relate the size of the dipole to the virtuality that is to be used in the gluon density function away from the Bjorken limit. A general observation of the bound states of two particles is that the invariant energy of the system is inversely proportional to the size squared. Therefore, we decide to fix $\lambda$ at its value obtained in the Bjorken limit and define $\bar{Q}^2$ in all kinematic limits to be,

$$\bar{Q}^2 \equiv \frac{\lambda}{d^2}. \quad (2.16)$$

The use of this ansatz ensures that in the Bjorken limit, there is a suppressed contribution from $d \gtrsim \sqrt{\lambda}/Q$. In the limit of large-$Q^2$, the photon wavefunction ensures that most of the contribution to the total cross section comes from small sizes. The large size behavior is modeled phenomenologically so that Eq. (2.16) does not enter. In the limit of small-$x$, the $\ln x$ contributions to the small-size behavior that we have been neglecting become large, and surely lead to taming of the growth, since otherwise the unitarity limit would be violated as we will discuss later. For this kinematical region, we approximate the cross section as simply being black. The fact that, as we will show, data is reasonably described down to low $Q^2$ and low-$x$ is evidence that the ansatz provides an adequate parameterization. The constant, $\lambda$, can be determined from a study of $F_L$ and is found to be of order $10^{-38}$.

From data for $J/\psi$ production in DIS, it is found that the value of $\lambda \approx 4$ provides a better interpolation to larger sizes so this is the value that we will use. However, for the description of DIS data, a variation of $\lambda$ between 4 and 10 has a small effect. In the Bjorken limit of large $Q^2$ and fixed $x$, Eq. (2.16) yields the relation, $Q^2 \approx \bar{Q}^2$.

In order to determine how to sample values of $x'$ in the gluon distribution we quote the standard LO pQCD formula for the longitudinal structure function $F_L$,

$$F_L(x,Q^2) = \frac{4\alpha_s(Q^2)T_R}{2\pi} \sum_{q=1}^{2n_f} e_q^2 \int_x^1 d'x' \ g(x',Q^2) \ \frac{x^2}{x'^3} \ (1 - \frac{x'}{x}). \quad (2.17)$$

Here, $T_R$, is the usual group theoretical factor and $n_f$ is the number of active flavors. From
Figure 2.3: Distribution of $x'/x$ values in the integrand of the LO longitudinal structure function calculation. This graph is borrowed from Ref. [35]. The lower dashed curve corresponds to $Q^2 = 4$ GeV$^2$, and the upper dashed curve corresponds to $Q^2 = 10$ GeV$^2$.

This it is possible to check, as is done in [35], that the distribution in values of $x'$ are peaked at around $1.3 \times x$ for a wide range of $x$ and $Q^2$ (see Fig. 2.3). Because of the skewed shape of the distribution, the value of $x'$ used in the sampling of the gluon distribution for $\bar{Q}^2 = Q^2$ is chosen to be the value of $x'$ up to which one must integrate in order to obtain half of the integral. This turns out to be $x' \approx 1.75 x$. (In most other versions of the dipole picture, it is assumed that $x' \approx x$ for the whole range of kinematics.) If $x'$ is simply proportional to $x$, then the $q\bar{q}$ cross section in Eq. (2.12) is completely independent of external variables and is in this sense a universal parameter for DIS reactions.
Away from the Bjorken limit there is a broad distribution in relevant sizes in the calculation of the longitudinal cross section, and the assumptions needed to justify the universality of the $q\bar{q}$ cross section are absent. Away from $Q^2 = Q^2$, we can determine if and how $x'$ should depend on $d$. This can extracted from basic kinematics (see Fig. 2.1). The invariant energy squared, $M_{q\bar{q}}^2$ of the $q\bar{q}$ system is,

$$M_{q\bar{q}}^2 = \frac{m_q^2 + k_{\perp}^2}{z(1-z)}.$$  \hspace{1cm} (2.18)

It will be very convenient if we can perform the integral over $z$ only once. If we assume suppression of aligned jet configurations ($z \rightarrow 1, 0$) then we can make the approximation, $z \approx 1/2$. (aligned jet contributions are suppressed by factors of $1/Q^2$ for longitudinal structure functions due to the properties of the longitudinal component of the light-cone wavefunction.) Then we have,

$$M_{q\bar{q}}^2 \approx 4(m_q^2 + k_{\perp}^2).$$  \hspace{1cm} (2.19)

The exact expression for $x'$ in terms of the relative transverse momenta of the quark and antiquark is given by,

$$x' = \frac{Q^2 + M_{q\bar{q}}^2}{s + Q^2}.$$  \hspace{1cm} (2.20)

Using Eq. (2.19), this becomes,

$$x' = x \left(1 + \frac{4m_q^2}{Q^2} \right) \left(1 + \frac{k_{\perp}^2}{(Q^2 + 4m_q^2)} \right).$$  \hspace{1cm} (2.21)

The typical numerical value $k_{\perp}^2$ contributing to the integral should be similar to the average $\langle k_{\perp}^2 \rangle \sim \lambda/d^2$ that we use in sampling the gluon distribution. In order to reproduce the $x' \sim 1.75x$ behavior in the Bjorken limit, we use $k_{\perp}^2 = 0.75\lambda/d^2$. So,

$$x' = x \left(1 + \frac{4m_q^2}{Q^2} \right) \left(1 + \frac{0.75\lambda}{d^2(Q^2 + 4m_q^2)} \right).$$  \hspace{1cm} (2.22)

Now let us discuss the procedure for modeling the $d$ and $x'$ dependence of the basic cross section. As we mentioned before, this is done in three steps: describing the small size behavior, describing the large size behavior and interpolating between the two regions. For the small sizes, we use the pQCD result of Eq. (2.12) with the prescription given above for relating $d$ and $x'$ to external kinematics. For large sizes, we take a cross section which mimics the behavior of the pion-nucleon total cross section:

$$\sigma_{\pi N}(d, x) = 23.78 \left(\frac{x_0}{x} \right)^{0.08} \text{mb.}$$  \hspace{1cm} (2.23)
A simple interpolation is used to connect Eq. (2.12) with Eq. (2.23). If the magnitude of the perturbative calculation at the border of the non-perturbative region is $\hat{\sigma}(x = 10^{-2}, d^2_{Q_0}) \approx 6 \text{ mb}$, where $Q_0^2 \equiv 1/d^2_{Q_0}$, then we may use an interpolation of the form

$$\hat{\sigma}_I(x, d^2) = (\hat{\sigma}_{\pi N}(x, d^2) - \hat{\sigma}_{pqcd}(x, d^2_{Q_0})) \left( \frac{d^2 - d^2_{Q_0}}{d^2_{\pi} - d^2_{Q_0}} \right) + \hat{\sigma}_{pqcd}(d^2_{Q_0}). \quad (2.24)$$

If one wishes to make a smoother interpolation, one may use a cubic interpolation (see Fig. 2.5) or an exponential interpolation as was done in Ref. [35]. However, the use of different interpolations results in a small numerical variation, while there is no good theoretical justification to use one over the other, so we will continue to use the linear interpolation. As $x$ decreases, the cross section as predicted by the LO gluon distribution begins to grow very large away from $d = 0$. If the size of the configuration becomes larger than about one-half of the pion size, then using the LO pQCD result becomes highly unrealistic. Therefore, at very small $x$ we reset the interpolation region to begin where the pQCD regime would predict a cross section that is one-half of the pion-nucleon cross section:

$$\hat{\sigma}(x, d_{crit}) = \frac{\pi^2}{3} d^2 \alpha_s(\bar{Q}^2_{crit}) x'(x', \bar{Q}^2_{crit}) = \frac{\hat{\sigma}(x, d_x)}{2} \quad (2.25)$$
\[ Q^2 = 3.0 \text{ GeV}^2 \]

Figure 2.4: A sampling of the behavior in the MFGS model of the basic \( q\bar{q} \) cross section.

The result is the model of the basic \( \gamma - N \) cross section which appears in Fig. 2.4. In summary, we have modeled the interaction of a virtual photon in such a way that we can now study the effect of different hadronic Fock state contributions. In order to study the interaction in the impact parameter representation, we must first model the \( t \)-dependence of the basic cross sections. This will be the purpose of the next section.

### 2.3 Modeling the \( t \)-dependence

We will discuss a model of the \( t \)-dependence in the dipole picture that was formulated and applied in [40]. Starting with the expression for the total cross section in Eq. (2.12), we devise a model for the scattering amplitude by writing it in the form,

\[ A_{hN}(s, t) = is\hat{\sigma}_{tot}f(s, t), \tag{2.26} \]

We assume that the amplitude is purely imaginary, and the amplitude is normalized so that

\[ \frac{1}{s}\Im A_{hN}(s, t = 0) = \hat{\sigma}_{tot}, \]

where \( f(s, t) \) accounts for the \( t \)-dependence of the interaction, and \( \hat{\sigma}_{tot} \) is determined from the QCD improved dipole picture (Eq. (2.12)). The “hat” on
Figure 2.5: The MFGS model with a cubic interpolation and $Q^2 = 3$ GeV. (The $x = .0001$ case is poorly fit with a cubic interpolation.)
\( \sigma_{\text{tot}} \) is meant to distinguish the total cross section for the scattering of one component of the photon wave function from the total \( \gamma^* N \) cross section which we consider in Sect. 3.2. Applying the optical theorem in the large \( s \) limit reproduces \( \sigma_{\text{tot}} \). We will return to the question of a real part of the amplitude at the end of Sect. 3.1.

No procedure exists as yet to calculate the \( t \)-dependence from first principles so we must make a model of the form of \( f(s, t) \) which will take into account the nonzero size of the \( q\bar{q} \) dipole and which will smoothly interpolate between perturbative and nonperturbative regimes using the hadronic size, \( d \), as a parameter. As we did for the total cross section in the previous section, we start by modeling the \( t \)-dependence of the amplitude in the soft and hard regimes (separately) and by using the dipole size, \( d \), as a parameter to build a smooth interpolation. The three steps: building a model for the small dipole region, building a model for the large wave-packet region, and interpolating between the two regions are outlined in the next three paragraphs.

We start by writing the general structure of the amplitude. The \( t \)-dependence, \( f(t, x, d) \), is written as the product of three functions

\[
f(t, x, d) = F_N(t, d)F_h(t, d)F_{IP}(t, x, d).
\] (2.27)

Here and in the rest of this section, the dependence of \( f \) upon \( s \) is replaced by dependence upon \( x \) and \( d \). \( F_N(t, d) \) describes the \( t \)-dependence of the nucleon target, \( F_h(t, d) \) describes the \( t \)-dependence of the hadronic projectile, and \( F_{IP}(t, x, d) \) accounts for Gribov diffusion. This method of separating the \( t \)-dependence into three factors corresponding to different sources of \( t \)-dependence is similar to what is used in Ref. [41].

The next task is to model the small dipole size \( t \)-dependence. Both the soft Pomeron exchange factor and the hadronic form factor must approach unity as the size, \( d \), shrinks to zero. One of the key predictions of the QCD factorization theorem for vector meson production is that the \( t \)-dependence becomes universal, i.e., independent of the flavor of the quarks in the incident dipole in the large \( Q^2 \) limit. Because of the large mass of the charm quarks, the \( J/\psi \) meson has a natural hard scale associated with it [42]. Therefore, the \( J/\psi \)-meson is produced in a small-size configuration, thus contributing very little to the \( t \)-dependence. This was pointed out in [43, 44, 38] and is confirmed in experimental observations (see summary in Ref. [45]) which have shown that the \( t \)-dependence for \( J/\psi \)-production changes very little over a large range of \( Q^2 \). (This can be contrasted with the case of \( \rho \)-production, where the \( t \)-dependence changes drastically with changes in \( Q^2 \)). Furthermore, the contribution to the \( t \)-dependence from Gribov diffusion should become suppressed at moderate \( x \) for the case of \( J/\psi \)-production. Thus, the \( t \)-dependence extracted from \( J/\psi \)-production provides a natural estimate for the two-gluon form factor. It was shown by Frankfurt and Strikman in [39] that data from \( J/\psi \) production can be parameterized if one assumes a two-gluon
form factor of the form,
\[ F_N(t, d \to 0) = F_1(t) \sim \frac{1}{(1-t/m_1^2)^2}. \] (2.28)

The subscript, 1, labels the two-gluon form factor and \( m_1^2 \) is a measurable parameter in the two-gluon form factor. The value, \( m_1^2 \approx 1.1 \text{ GeV}^2 \) is extracted from data in Refs. [46, 47, 48]. It was found that the dipole form factor contributes only about 0.3 GeV\(^{-2} \) to the slope of the \( t \)-dependence. The assumption that only the gluon form factor is relevant for \( J/\psi \) production has been successfully tested against data in Refs. [49, 50, 51].

Hence, in the limit of small dipole sizes,
\[ f(t, x, d \to 0) = F_1(t). \] (2.29)

Next we construct a model for the large size dipole behavior. When the hadronic state has a large size, the \( t \)-dependence receives contributions from sources other than the two-gluon form factor. We rewrite Eq. (2.27) in the form,
\[ f(t, x, d) = F_{e.m.}^N(t, d) F_h(t, d) F_{IP}(t, x, d). \] (2.30)

This method of writing the product of form factors is reminiscent of the Regge pole representation of amplitudes. Now, \( F_{e.m.}^N(t) \) is the electromagnetic form factor of the nucleon which is known phenomenologically to take the form,
\[ F_{e.m.}^N(t) \sim \frac{1}{(1-t/m_0^2)^2}, \] (2.31)

where \( m_0^2 \approx 0.7 \text{ GeV}^2 \). Large size hadronic configurations can be reasonably expected to have \( t \)-dependence similar to the pion electromagnetic form factor. Thus, for the hadronic form factor we use the well-known form of the pion form factor,
\[ F_h(t, d \to d_\pi) \sim \frac{1}{1-t/m_2^2}, \] (2.32)

with \( m_2^2 \approx 0.6 \text{ GeV}^2 \). Here, \( d_\pi \) is the characteristic size of the pion, and takes on a value of approximately 0.65 fm. This value for the pion size is consistent with what is used in the matching ansatz of Ref. [35] and agrees well with data for the \( \pi N \) cross section in Ref. [52]. For low-\( x \) soft scattering there is also a factor that arises from Gribov diffusion effects:
\[ F_{IP}(t, x, d \to d_\pi) \sim e^{-\alpha' \ln \frac{m_0}{x}}. \] (2.33)

The factor, \( F_{IP}(t, x, d) \), describes the exchange of a soft Pomeron with a Regge slope \( \alpha' \approx \)
0.25 GeV$^{-2}$ and $x_0 = 0.01$. The value of $x_0$ is determined by the boundary of the region where Gribov diffusion effects become significant.

Finally, we must find a reasonable way to interpolate between the hard and the soft regions. We use the $t$-dependence discussed in the previous two paragraphs to guess the following form for the hadronic configuration-nucleon amplitude:

$$A_{hN}(s, t) = i s \hat{\sigma}_{\text{tot}} \frac{1}{(1 - t/M^2(d^2))^2} \frac{1}{1 - td^2/d_\pi^2 m_2^2} e^{-\frac{d^2}{d_\pi^2} \ln \frac{x_0}{x}}. \quad (2.34)$$

To interpolate between the nucleon and the two-gluon form factors, we have defined the function,

$$M^2(d^2) = \begin{cases} m_1^2 - (m_1^2 - m_0^2) \frac{d^2}{d_\pi^2} & , \quad d \leq d_\pi \\ m_0^2 & , \quad \text{otherwise} \end{cases} \quad (2.35)$$

Note that when $d$ equals $d_\pi$, $A_{hN}$ is the product of Eqs. (2.31), (2.32), and (2.33). In the small $d$ limit, the dipole form factor and the Pomeron form factor approach unity, $M^2(d^2) \to m_1^2$, and the limit in Eq. (2.29) is recovered. Varying $d^2$ interpolates smoothly between the hard and soft regions. Note that we neglect a possible small $x$ dependence of $F_N(t, d)$ at $x \lesssim 0.01$. (See the discussion in Ref. [53].) However, our model is adjusted to reproduce the observed $x$-dependence of the slope for photoproduction of $J/\psi$ mesons.
Chapter 3

Unitarity and The Black Disk Limit

In this chapter we discuss unitarity constraints and we investigate the approach to the unitarity limit as predicted by the MFGS model, with the model of \( t \)-dependence discussed in Chapter 2. We will argue that a small but significant fraction of the total cross section in HERA data arises from interaction near the BDL.

### 3.1 Impact Parameter Analysis

Having obtained Eq. (2.34), the next step is to transform to the impact parameter representation where the profile function is defined in the high energy limit by the relation,

\[
A_{h,N}(s,t) = 2i s \int d^2b e^{-iq \cdot b} \Gamma_h(s,b),
\]

(3.1)

where \( t = -q^2 \). The subscript, \( h \), indicates that we are considering the profile function for the scattering of a single hadronic component of the photon wave function from the proton. We get the profile function by inverting Eq. (3.1),

\[
\Gamma_h(s,b) = \frac{1}{2is(2\pi)^2} \int d^2q e^{iq \cdot b} A_{h,N}(s,t).
\]

(3.2)

For an imaginary amplitude, the BDL is reached when \( \Gamma_h(s,b) = 1 \) and the elastic and inelastic cross sections are equal. If a dipole consists of color octet representations of SU(3) (as would be the case for gluons in a higher order calculation), then Eq. (2.12) has an extra factor of 9/4. Actually carrying out a NLO calculation is very complicated, but we note that a calculation of \( F_2 \) in this case would require the presence of both \( q\bar{q} \) pairs and configurations with a gluon, while the over-all magnitude of \( F_2 \) should be the same. Although a \( q\bar{q}g \) configuration’s contribution to \( F_2 \) would be smaller than the \( q\bar{q} \) contribution calculated above for LO, such configurations typically come with an extra factor of 9/4 which would put them closer to the unitarity limit. That is, \( \Gamma_h \gtrsim 1/2 \) for the \( q\bar{q} \) situation would roughly correspond to a unitarity violation for octet dipoles.

In the rest of this section, we will suppress explicit reference to the argument, \( s \), in the profile function. We have plotted the function \( \Gamma_h(b) \) for different values of the dipole size
Figure 3.1: The hadronic configuration-nucleon profile function for different $x$ values. The large $\Gamma_h(b)$ region ($\Gamma_h \gtrsim 1/2$) is reached for intermediate hadronic sizes. (See figure 3.3.) Here, $Q^2$ is taken to be 2 GeV$^2$.

and $x$ in Fig. 3.1. We have used gluon distributions from CTEQ5L in the perturbative calculation of $\hat{\sigma}_{tot}$ [54]. Our model of the $t$-dependence requires that we specify the external photon virtuality. Since we are interested in the possibility of reaching the BDL at a few GeV$^2$, we have set $Q^2 = 2$ GeV$^2$ in Fig. 3.1. We note that it is somewhat strange that when we shift from $x = 10^{-2}$ to $x = 10^{-3}$ at large $d$, the profile function actually decreases at $b = 0$. This is probably a symptom of the simplifications of the model. The rate of growth of the basic cross section may not be large enough, or the slope of the large size $t$-dependence may be too small.

A Gaussian ansatz is commonly used in experiments to extrapolate the $t$-dependence to large values. Let us, therefore, compare the behavior of our model to that of a simple Gaussian. We start with the form

$$A_{hN}(s, t) = is\hat{\sigma}_{tot}e^{Ct/2},$$  \hspace{1cm} (3.3)
Figure 3.2: Comparison of the $b$ behavior for our model with that of a Gaussian model. Our model falls off more slowly with $b$. The slope of the Gaussian used here is 0.17 fm$^2$.

which is then transformed into impact parameter space giving,

$$\Gamma_{hN}(b) = \hat{\sigma}_{tot} \frac{4}{\pi C} e^{-\frac{b^2}{2C}}.$$  \hspace{1cm} (3.4)

The slope of the Gaussian, $C$, is chosen so that it yields the same standard deviation in $\Gamma_h(b)$ as our model. One danger in using a Gaussian model is that it neglects the importance of interactions in peripheral regions. The approach that we described in Sec. 2.3 attempts to fix this problem by spreading out the distribution in $t$. (Note in Fig. 3.2 that our model fall off more slowly with $b$.)

Now let us estimate the contribution of large values of $\Gamma_h(b)$ to the total hadronic cross section. The total cross section follows from the optical theorem,

$$\hat{\sigma}_{tot} = 2 \int d^2 b R e \Gamma(s, b).$$  \hspace{1cm} (3.5)

We have made a numerical evaluation of the fraction of the total $hN$ cross section obtained by setting different upper limits on the $b$-integral in Eq. (3.5). Looking at Fig. 3.2, we see that putting such a limit, $b_{max}$, on the range of impact parameters restricts contributions from the profile function of any hadronic configuration to values greater than $\Gamma_h(b_{max})$. In Fig. 3.3, one can see that no more than about thirty percent of the total hadronic cross section is due to values of $\Gamma_h \gtrsim 1/2$. Moreover, contributions from large values of $\Gamma_h(b)$
occur for hadronic sizes close to the pion size, \( d \approx 0.6 \) fm. Averaging over the photon wave function will lead to a suppression of contributions from larger size hadronic configurations, so there will indeed be a small contribution to the total DIS cross section due to large values of \( \Gamma_h(b) \) (see Fig. 3.4). The goal of Sect. 3.2 will be to determine whether the contribution to the \( \gamma^*N \) cross section from large values of \( \Gamma_h(b) \) is nevertheless significant enough that we may expect to see black disk behavior within HERA kinematics.

To summarize, Fig. 3.1 demonstrates that large values of \( \Gamma_h(b) \) are approached for hadronic \( hN \) scattering at central impact parameters, \( b \lesssim 0.5 \) fm. In Fig. 3.1 it is seen that this is particularly true for hadronic sizes around \( d \approx 0.6 \) fm. Figure 3.3 shows that for \( d \sim 0.6 \) fm, a maximum of about 1/3 of the total hadron-nucleon cross section comes from values of \( \Gamma_h(b) \) that approach the black limit. When \( d \lesssim 0.2 \) fm, a very small fraction of the total hadronic configuration-nucleon cross section comes from large values of \( \Gamma_h(b) \).
The only contribution from $\Gamma_h \gtrsim 1/2$ to the total cross section for $d \lesssim .2$ fm occurs at very small $x$ ($x \lesssim 10^{-4}$).

Most of the model dependence in this calculation comes from uncertainty in the large $-t$ behavior of the amplitude. Fig. 3.5 shows the numerical effect on how our model changes if we remove contributions from large $t$. Notice that simply removing the contribution from $-t \gtrsim 3.3$ GeV$^2$ leads to an error of less than ten percent. Thus, we do not expect the uncertainty in the large $-t$ behavior to have a drastic effect.

We should also point out that we have considered only the non-spin-flip interactions. Corrections which account for the spin-flip amplitude would result in a smaller non-spin-flip amplitude than what we consider here. Experimental results in Ref. [55] demonstrate that the polarization, $P$, is less than 0.2 for the range of $t$ we are discussing. From the formula relating $P$ to the spin-flip amplitude,

$$P = -\frac{2Im(A_{++}A_{+-}^*)}{|A_{++}|^2 + |A_{+-}|^2} \approx -\frac{2|A_{+-}|}{|A_{++}|}, \quad (3.6)$$

we find the fraction, $\frac{|A_{+-}|}{|A_{++}|} \lesssim 0.1$. Here, $A_{++}$ represents the amplitude with no spin-flip whereas $A_{+-}$ represents the amplitude with spin-flip.

We now return to the issue of a real part of the amplitude which we ignored in the previous chapter. In the considered kinematic region ($t \lesssim -2$ GeV$^2$), the ratio of the real to imaginary part of the amplitude, $\eta$, is rather small. Indeed, if we adopt power law

![Figure 3.4: The distribution of the integrand in Eq. (2.9) over hadronic sizes for both the transverse and longitudinal cross sections.](image)
Figure 3.5: A demonstration of the rapid convergence of the profile function. Here, the profile function is plotted for different values of the upper limit, $U$, on the integral over $t$ ($-t = U^2$).

behavior for the total cross section, $\hat{\sigma}_{\text{tot}} \sim s^\rho$, we can estimate the value of $\eta(0)$ using the following formula which follows from the Gribov-Migdal result [56] at high energies in the near forward direction,

$$\eta(t) = \frac{\text{Re} A_{hN}(s,t)}{\text{Im} A_{hN}(s,t)} = \frac{\pi}{2} \left( \frac{\partial \ln \hat{\sigma}_{\text{tot}}}{\partial \ln s} \right) = \pi \frac{\rho}{2}. \quad (3.7)$$

The amplitude can be rewritten as,

$$A_{hN}(s,t) \rightarrow s(i + \eta(t))f(s,t), \quad (3.8)$$

where the function, $f(s,t)$, is assumed to be strictly real. For soft kinematics, the total cross section has the approximate $s$-behavior of the $\pi N$ cross section as in Ref. [35], consistent with the behavior of a Donnachie-Landshoff soft Pomeron [57]. In that case, $\rho \approx 0.08$, and Eq. (3.7) gives $\eta \approx 0.1$. The second term in Eq. (3.8) appears squared in the calculation of the cross section, so the correction to the cross section is approximately one percent. For the high $Q^2$, low-$x$ region, the total cross section experiences rapid growth and $\rho \approx 0.25$, or, by Eq. (3.7), $\eta \approx 0.35$. The correction to the squared amplitude is therefore approximately ten percent near the forward direction. Away from the forward direction, one must account for the small variation of $\eta$ with $t$. The effect can be estimated by considering the signature
factor in the general form of the Reggeon amplitude. For $-t \leq 2.0 \text{ GeV}^2$, $\eta(t)$ continues to contribute a negligible amount to the amplitude.

The elastic scattering cross section associated with Eq. (3.8) for one hadronic component of the photon is found by integrating the profile function over impact parameters,

$$\hat{\sigma}_{el} = \int d^2 b |\Gamma_h(s, b)|^2.$$  \hspace{1cm} (3.9)

The total cross section is found using Eq. (3.5). Therefore, the inelastic cross section is,

$$\hat{\sigma}_{inel} = \int d^2 b (2Re\Gamma_h(s, b) - |\Gamma_h(s, b)|^2),$$  \hspace{1cm} (3.10)

with the unitarity constraint,

$$2Re\Gamma_h(s, b) - |\Gamma_h(s, b)|^2 \leq 1.$$  \hspace{1cm} (3.11)

If the amplitude is purely imaginary, then $\hat{\sigma}_{el} \leq \hat{\sigma}_{inel}$ and the unitarity constraint is that $\Gamma_h \leq 1$. Note that by considering only the imaginary part of the amplitude, we have considered only the real part of the profile function. If the amplitude is given a real part correction, then the profile function will obtain an imaginary part, the elastic cross section will increase, and the inelastic cross section will decrease. The correction to the unitarity constraint on $Re\Gamma$ is, from Eq. (3.11), $-(\eta Re(\Gamma(s, b))^2$. In the region of large $Q^2$, the effect of a real part in the amplitude would clearly be noticeable. By Eq. (3.11), the unitarity limit on the real part of the profile function for $\eta \approx 0.35$ would be,

$$Re\Gamma \leq \frac{1}{1 + \eta^2} \sim 0.9.$$  \hspace{1cm} (3.12)

Thus, the unitarity limit on the real part of the profile function may be less than unity by as much as ten percent. At this point, we should point out that both the contribution from the real part of the amplitude and the contribution from inelastic diffraction will tend to raise the boundary in impact parameter space where the BDL is reached. We neglect both effects in our model. As a consequence, when our model predicts that the BDL has been reached below a certain impact parameter, we can be confident that the same would be true in a model that incorporates inelastic diffraction and the effects of a real component of the amplitude. On the other hand, if our model predicts that the BDL has not been reached, we must keep in mind that corrections due to inelastic diffraction and a real part of the amplitude may be important. In other words, the BDL may already be approached at larger values of $b$ than what our model predicts. This observation is relevant for determining the effect of a real part on the taming - it predicts that the taming starts earlier.

There is also a range of uncertainty in the model due to the matching that is needed
Figure 3.6: Comparison of the profile function for different values of $\lambda$ in the case of an intermediate hadronic size equal to 0.4 fm. $\Gamma_h$ changes by about fifteen percent at $b = 0$ if $\lambda$ is changed from 4 to 10.

between the soft and hard kinematical regions of $\tilde{\sigma}$ as shown in Fig. 2.4. The uncertainty in the matching region is expressed by the uncertainty in the parameter, $\lambda$. However, values of the order of 4 to 10 seem to work well and, as shown in Fig. 3.6, there is a variation of only about fifteen percent at small impact parameters when we vary $\lambda$ from 10 to 4. Note that this is done for a hadronic size of 0.4 fm which is in the region where the dependence upon $\lambda$ should be at its greatest. However, there remains another subtlety related to the matching of kinematic regions. First, recall the distinction between the energy scale, $Q^2$, denoting the virtuality of the photon in a particular scattering process, and the scale, $\bar{Q}^2$, which is the energy scale related to the hadronic size, $d$, through the scaling ansatz of Ref. [35]. These two scales are nearly equal as long as we consider hadronic sizes in the vicinity of the average hadronic size for $F_2$. In determining how to relate the value of $x'$ to external kinematics, we found in Eq. (2.22) that the universality of the $q\bar{q}$ cross section breaks down away from large $Q^2$ and large $d^2$. The value of $x'$ used in a calculation of the hadronic cross section will be significantly larger than $x$ for small hadronic sizes and fixed $Q^2$, leading to a suppression of the cross section in the small size region. In particular, in Fig. 3.1, the approach to the BDL at small $d$ is slowed due to the large values of $x$ needed to push the small size configuration on shell. In investigating the hadronic profile function, it may also be reasonable to determine $Q^2$ by letting it equal $\bar{Q}^2$ so that the value of $d$ always corresponds to a typical component of the virtual photon. We have done this in
Figure 3.7: These graphs are identical to those in Fig. 3.1 except that the value of $Q^2$ used to make each graph is calculated from the hadronic size. Note the larger values of the profile function at small $d$ compared with Fig. 3.1.
Figure 3.8: Demonstration of reasonable agreement between the color dipole model and recent HERA data [58] for $F_2$ at low $Q^2$. The different curves correspond to the different parton distributions CTEQ6L, CTEQ5L, and MRST98 [59, 54, 60]. In our calculations we used CTEQ5L parton distributions because this yields optimal agreement between the dipole model and current data.
Figure 3.9: Demonstration of reasonable comparison between the color dipole model and preliminary HERA results [58] for $F_L$ at low $Q^2$ and low $x$. The two points in the upper graph correspond to different methods of taking data. The different curves correspond to different parton distributions, CTEQ6L, CTEQ5L, and MRST98 [59, 54, 60].
Fig. 3.7 and we can see that at small $d$, $\Gamma_h(b)$ is substantially larger, especially at small $x$. Comparing Figs. 3.1 and 3.7, we see that at $d = .1$ fm this effect is significant while at intermediate hadronic sizes the effect is very small. For $d \gtrsim .5$ fm there is no discernible difference between the two cases. The physical interpretation of this effect is that the profile function for a small size configuration approaches the BDL more slowly if it is far off shell for a given $Q^2$. Note that once we begin to calculate the total cross section, an external value for $Q^2$ is explicit, and we no longer have this ambiguity.

Furthermore, there is some uncertainty in the gluon distribution used to calculate $\sigma_{tot}$. This is demonstrated in Figs. 3.8 and 3.9 where we compare results for the structure functions using CTEQ5L [54], CTEQ6L [59], and MRST98 [60] leading order gluon parton distributions. The dependence upon the parton distribution is seen to be small, but we used CTEQ5L parton distributions for all other calculations because they seem to yield optimal consistency with data for $J/\psi$ production.

As we mentioned in the last chapter, the value of $d_\pi$ that we used is consistent with the slope of the $\pi N$ cross section as measured in Ref. [52] and with the matching ansatz used in Ref. [35]. In the model of the $t$-dependence, $d_\pi$ determines where soft Pomeron behavior becomes important, and one may well ask whether a different value of $d_\pi$ is appropriate. For models with a larger value of $d_\pi$, the suppression of the profile function due to the Pomeron form factor, $F_P$, does not occur until one considers larger hadronic configurations. Therefore, for intermediate hadronic sizes, the profile function rapidly approaches the unitarity limit as $x$ decreases when $d_\pi$ is large. This can be seen in Fig. 3.10 where we have repeated the calculation of Fig. 3.1, this time using $d_\pi = .8$ fm. Note the large values of $\Gamma_h(b)$ at small $b$ for $d \sim .5$ fm. Therefore, by choosing a smaller value for $d_\pi$, we are making a conservative estimate of the approach to the BDL. Note that using the larger value of pion size leads to a somewhat more natural behavior for the large size profile function than what was observed in Fig. 3.1.

### 3.2 Estimating the Proximity of the Total $\gamma^*N$ Cross Section to the BDL

To properly study the proximity of DIS to the unitarity limit, we must evaluate the degree to which the different hadronic components contribute to the $\gamma^*N$ cross section, $\sigma_{L,T}^{\gamma* N}$. $T$ and $L$ refer, respectively, to the transverse and longitudinal cross sections. Using Eq. (2.9) we have calculated $\sigma_{L,T}^{\gamma* N}$ using $\hat{\sigma}(d,x)$ with $t$-dependence determined in section 2.3. Plots of $F_2$ and $F_L$ are shown in Fig. 3.8 and Fig. 3.9. The structure functions $F_2$ and $F_L$ are
Figure 3.10: The hadronic configuration-nucleon profile function for different $x$ values. Here we have used $d_\pi = .8$ fm. Compare with Fig. 3.1.
defined as,

\[ F_L(x, Q^2) = \frac{Q^2}{4\pi^2\alpha_{\text{e.m.}}} \sigma_{L}^{\gamma^*N}, \]

\[ F_2(x, Q^2) = \frac{Q^2}{4\pi^2\alpha_{\text{e.m.}}} (\sigma_{L}^{\gamma^*N} + \sigma_T). \]  

(3.13)

Note that the \(t\)-dependence is not needed for calculations of \(F_2\) and \(F_L\). For more plots of the total cross section calculated within the QCD improved dipole model see Ref. [35]. Figure 3.4 shows the distribution of the integrand in Eq. (2.9) over total hadronic sizes and demonstrates the suppression of large size hadronic configurations.

We would like to study the contribution of intervals of \(\Gamma(b)\) to the \(\gamma^*N\) cross section. However, the profile function for \(\gamma^*N\) scattering by itself is not useful because the photon wave function is not normalizable and because it depends on \(\alpha_{\text{e.m.}}\). Thus, in order to look for the proximity to the BDL, we have plotted the fraction of the total \(\gamma^*N\) cross section due to different regions of the hadronic profile function. Plots with different values of \(x\) are shown in Fig. 3.11. The vertical axis denotes the fraction of the longitudinal (transverse) cross section with contributions from \(\Gamma_h(b)\) greater than the corresponding value on the \(x\)-axis. Note that when \(Q^2 = 2\) GeV\(^2\) and \(x \sim 10^{-4}\), about 1/4 of the longitudinal cross sections are due to dipole configurations corresponding to \(\Gamma_h(b) > 1/2\).

At \(Q^2 = 20\) GeV\(^2\), Fig. 3.11 demonstrates the recovery of the leading twist behavior, especially for \(\sigma_L^{\gamma^*N}\) at \(x = .01\), where less than one-tenth of \(\sigma_L^{\gamma^*N}\) is due to hadronic configurations with \(\Gamma_h(b) > 1/2\). It is clear from Fig. 3.11 that, for central impact parameters and low enough \(Q^2\), a significant portion of the total cross section is due to \(hN\) interactions that are close to the unitarity limit. In high energy \(\gamma^*A\) scattering, where the effects of the BDL are enhanced, we may be able to use DIS to probe the BDL. This possibility is discussed within the context of the QCD improved dipole model in Sect. 3.4.

### 3.3 Comparison with Results of Other Studies

The reasonableness of our model is demonstrated in Fig. 3.8 where the dipole model is seen to be consistent with recent HERA data for \(F_2\) at low values of \(Q^2\) and \(x\). Furthermore, in Fig. 3.9, our model is seen to be consistent with preliminary results from HERA for \(F_L\) [58]. Other studies of the impact parameter picture of hadronic interactions with nucleons were done in [61], where \(\rho\) production data was used to extract the \(t\)-dependence. The analysis in [61] used the S-matrix convention, \(S(b) = 1 - \Gamma(b)\), in place of the profile function. In Fig. 3.12 we have plotted our prediction of the S-matrix profile for central impact parameters along with earlier result from Munier et al. [61]. In their analysis, the authors were restricted to using \(\rho\) production data to model the \(t\)-dependence. Data for \(\rho\) production is limited.
Figure 3.11: The fraction of $\sigma_{L,T}^{\gamma^*N}$ with contributions coming from values of $\Gamma_h$ greater than the corresponding values listed on the $x$-axis. When $Q^2 = 2$ GeV$^2$ and $x = 10^{-4}$, about $1/4$ of $\sigma_{L,T}^{\gamma^*N}$ comes from hadronic components scattering with $\Gamma_h > 1/2$. 
Figure 3.12: Comparison of the S-matrix calculated using Eq. (2.34) with results obtained in [61]. The bold line shows our result, while the dashed and dotted lines show results taken from Ref. [61] using \( \rho \) production with three different interpolations for the \( t \)-dependence.

to kinematics where \(-t \lesssim 0.6 \text{ GeV}^2 (b \gtrsim 0.3 \text{ fm})\) so the accuracy of their results is limited to moderate impact parameters. Further complication is introduced into their analysis by the need to model the \( \rho \)-meson wavefunction. Our model uses \( J/\psi \) production data and is therefore valid at small values of \( b \). As mentioned in section 2.3, production of \( J/\psi \) depends only on the two-gluon form factor, and can therefore be extended down to very small impact parameters. It is natural to compare our results for \( S(b) \) with the median value of the dipole size as it is used in the calculation of the amplitude estimated in Ref. [43]. In our case, the value of \( d \) that corresponds to \( S(b) \) evaluated at \( Q^2 = 7 \text{ GeV}^2 \) is about \( d \approx 0.32 \text{ fm} \). With this assumption, Figure 3.12 shows that our model has very good agreement with the results of Ref. [61] at moderate values of \( b \) while there is an expected deviation between the two models for low values of \( b^2 \). Since our model will have \( \sim t^{-4} \) behavior at large values of \(-t\), then even in the small \( b \) region, our model deviates from the results of Ref. [61] by no more than about twenty-five percent. Reference [61] used a simple exponential or power ansatz to interpolate to larger values of \( t \) as indicated in the figure. In Fig. 3.12, it is seen that the model used in [61] has a high degree of uncertainty at small values of \( b \) because of the necessity to guess the form of the function that interpolates to large \( t \). In contrast, our

\footnote{We thank A. Mueller for discussions on how to optimally compare results of the two analyses.}

\footnote{This is consistent with the observed \( Q^2 \) variation of the \( \rho \)-meson slope from ZEUS02 [63] with predictions in Ref. [43].}
model uses information about $J/\psi$ production and DIS to model the small $b$ behavior.

After preliminary results of our study were presented, there appeared an experimental analysis with improved data on inclusive cross sections and vector meson production at HERA by Kowalski and Teaney [62]. They carry out an analysis similar to that used in [61]. Therefore, it differs from our analysis in that it does not include information about large $t$ behavior of the two-gluon form factor. Also, the model of Ref. [62] is sensitive to the $\rho$-meson wavefunction, and it uses an unrealistic value for the quark mass (see discussion in Sec. 3.5).

3.4 Scattering off a Heavy Nuclear Target

It is interesting to examine how the profile function $\Gamma_h(s, b, d)$ changes when the free proton target is substituted by a heavy nuclear target such as the nucleus of $^{208}$Pb. (Recall that the BDL was originally applied by Gribov to calculate the structure functions for scattering from a nuclear target [9].) In the heavy nucleus case, the procedure for obtaining $\Gamma_h(s, b, d)$ differs from the one in the nucleon case. A method for computing nuclear parton distribution functions using LT pQCD is described in [64] by using the Glauber-Gribov formula at a particular fixed value of $Q^2$ (using a quasi-eikonal approximation), and then evolving to different scales using leading twist DGLAP evolution. This method for determining the degree of nuclear shadowing is called leading twist (LT) nuclear shadowing to be contrasted with the eikonal approximation. First, for dipoles of small transverse sizes, $d < 0.2$ fm, the inelastic scattering cross section at a given impact parameter $b$ is given by the perturbative QCD expression involving the impact parameter dependent nuclear gluon distribution, $g_A(x, Q^2, b)$ (compare to Eq. (2.12)). Using the fact that $\hat{\sigma}_{\text{inel}} \approx \hat{\sigma}_{\text{tot}}$, we find,

$$\Gamma_{\text{inel}}^\text{pQCD}(d, x, b) \approx \frac{\pi^2}{3} d^2 \alpha_s(Q^2) x g_A(x', Q^2, b),$$

(3.14)

where $x'$ is given by Eq. (2.22). The gluon distribution $g_A(x, Q^2, b)$, normalized such that $\int d^2 b g_A(x, Q^2, b) = g_A(x, Q^2)$, was evaluated in [64] using the theory of leading twist nuclear shadowing. The profile function $\Gamma_h(s, b, d)$ can be found from Eq. (3.10) (see also [65]),

$$2 \text{Re} \Gamma_h(s, b, d) - |\Gamma_h(s, b, d)|^2 = \Gamma_{\text{inel}}^\text{pQCD}(d, x, b),$$

(3.15)

Ignoring the small imaginary part of $\Gamma_h(s, b, d)$, which is even smaller in the heavy nucleus case than in the free proton case because of the effect of nuclear shadowing, Eq. (3.15) gives

$$\Gamma_h(s, b, d) \approx 1 - \sqrt{1 - \Gamma_{\text{inel}}^\text{pQCD}(d, x, b)},$$

(3.16)
Figure 3.13: The hadronic configuration-nucleus \((^{208}\text{Pb})\) profile function. The upper solid curve correspond to \(x = 10^{-5}\), and the lower solid curve corresponds to \(x = 10^{-4}\); the dashed curve correspond to \(x = 10^{-3}\); the dot-dashed curve correspond to \(x = 0.01\).

which is valid for \(d < 0.2\) fm. The approximation in Eq. 3.16 is sufficient for our purposes because it gives the correct value when \(\Gamma\) is small, and is close to one when \(\Gamma\) is large.

Second, for dipoles of a larger size, \(d_0 = 0.2 < d < d_\pi = 0.65\) fm, the profile function is found by interpolating between the pQCD expression of Eq. (3.16) and the profile function calculated at \(d = d_\pi\) [35]

\[
\Gamma_h(s, b, d) = (\Gamma_h(s, b, d_\pi) - \Gamma_h(s, b, d_0)) \frac{d^2 - d_0^2}{d_\pi^2 - d_0^2} + \Gamma_h(s, b, d_0).
\]  

(3.17)

The profile function \(\Gamma_h(s, b, d_\pi)\) is calculated using the Glauber multiple scattering formalism [20]

\[
\Gamma_h(s, b, d_\pi) = 1 - e^{A_T N(s) T(b)/2},
\]

(3.18)
Figure 3.14: These plots are the analogue of those appearing in Fig. 3.3. They correspond to the profile functions for the nuclear target in Fig. 3.13. In these plots, $x'$ corresponds to $Q^2 = 2 \text{ GeV}^2$. 
Figure 3.15: These plots are the analogue of those appearing in Fig 3.11. The fraction of $\sigma^{\gamma^*A}$ due to values of the hadronic profile function larger than $\Gamma_h$ is plotted versus $\Gamma_h$. 
where \( \sigma_{\pi N}(s) \) is the energy-dependent pion-nucleon total scattering cross section, \( \sigma_{\pi N}(s) = 23.78(s/s_0)^{0.08} \text{mb} \); \( s_0 = 200 \text{GeV} \); \( T(b) \) is the nuclear optical density normalized such that \( \int d^2 b T(b) = 1 \). \( A \) is the number of nucleons in the target. A standard Fermi step fit is used for the nuclear optical density.

Third, for the dipoles with the size \( d > d_\pi \), the profile function is given by Eq. (3.18), where the pion-nucleon cross section is allowed to slowly grow as

\[
\sigma_{\pi N}(s, d) = \sigma_{\pi N}(s) \left( \frac{1.5 d^2}{d^2 + d_\pi^2/2} \right).
\]

The results for the profile function \( \Gamma_h(s, b, d) \) for the nucleus of \(^{208}\text{Pb}\) are presented in Fig. 3.13 by the two solid curves (\( x = 10^{-4} \) and \( x = 10^{-5} \)), dashed (\( x = 10^{-3} \)) and dot-dashed (\( x = 0.01 \)) curves.

The profile function for the nuclear target shows some similarity with the profile function for the proton target. The main differences are that the BDL is approached over a larger range of impact parameters than in the case of a proton target. This is not surprising because of the larger thickness of the nuclear target. The plots in Fig. 3.14 show the fraction of the hadronic cross section due to large values of \( \Gamma_h \).

Large leading twist gluon shadowing tames the growth of the interaction of hadronic components of the photon with the nucleus so that the unitarity constraint is satisfied for \( x \gtrsim 10^{-4} \) while the BDL may be reached for a large range of impact parameters. For smaller \( x \), unitarity starts to break down at central impact parameters. For large \( d \), unitarity is automatically satisfied since the Glauber model for large total cross sections leads to a \( \Gamma_h \) that approaches unity.

Finally, we have included plots in Fig. 3.15 for the nuclear target showing the fraction of \( \sigma^{\gamma^* A} \) due to large values of \( \Gamma_h \) analogous to those in Fig. 3.11. In Fig. 3.15 we see that the BDL is approached for nearly all values of \( x \) at \( Q^2 = 2 \text{GeV}^2 \) and \( Q^2 = 20 \text{GeV}^2 \). (Note the recovery of leading twist behavior for the longitudinal cross section at large \( x \) and \( Q^2 = 20 \text{GeV}^2 \).) Notice also that the fraction of \( \sigma^{\gamma^* A} \) due to large values of \( \Gamma_h \) for \( x = .01 \) is actually larger in some cases than for the case, \( x = .001 \). This effect can be explained qualitatively by inspection of Fig. 3.13. For the case of the nuclear target, the main contributions to \( \sigma^{\gamma^* A} \) come from smaller values of \( d \) (\( d \approx .2 \text{fm} \)). The growth of the profile function with decreasing \( x \) at small \( d \) is slower for smaller values of \( x \) (\( x \approx .01 \)) than for larger values. Thus, the tail of profile function at large impact parameter may become significant in these regions.
3.5 Comparison with the Saturation Model

Having set up a procedure for determining the proximity of the particular hadronic Fock states to the unitarity limit within the MFGS model, it is now interesting to compare this with what would be predicted by the simpler saturation model. The approach of the saturation model is to consider the absolute magnitude of the total $q\bar{q} - N$ cross section. At large $Q^2$, the saturation model presumes that the $\gamma^*N$ interaction has reached a new phase at the point where the $\hat{\sigma}_{tot}$ is near to the pion cross section, $\hat{\sigma}_0 \sim \hat{\sigma}_\pi \sim 23$ mb where the gluon density has become large enough that the cross section is of hadronic size, and new dynamics prevent it from growing further. The rate of growth of the small size dipole-nucleon cross section is given by a hard Pomeron-like growth that does not vary with $Q^2$. The cross section for large size configurations is constant with energy in the saturation model. These conditions, however, puts rather severe constraints on how large the cross section may grow. The approach of the MFGS model is instead to consider the proximity to the s-channel unitarity limit as an indicator of where saturation-like behavior must occur. The growth of the cross section is then probably over-estimated, but the criterion for measuring the approach to saturation is more conservative in that there is no model dependent taming effect. That is, the cross section is allowed to continue to grow at the most rapid rate predicted by the pQCD in the small size regime, and the large size regime includes the growth characteristic of soft Pomeron dynamics. Thus, when the profile function approaches the black disk limit, the gluon density must begin to undergo saturation effects. The MFGS model therefore provides a truly conservative way of estimating the approach of the total DIS cross section to the BDL.

Another problem that emerges within the saturation model is the issue of choosing the correct quark mass [66]. If the cross section is dominated by the hard regime where $Q^2 \gg m_q^2$, then the mass of the quarks has little effect on the cross section. The mass of the quarks only appears in the light-cone wavefunctions of Eqs. (2.10) and (2.11). A large quark mass, however, leads to a suppression of the higher mass components in the total DIS cross section. If the quark mass chosen is too small, then, as one considers lower $Q^2$ one may begin to over-estimate the total DIS cross section to a very large degree. In fact, as is clear from Eqs. (2.10) and (2.11), the total DIS cross section becomes ill-defined in the photo-production limit if the quarks have zero mass. This can be seen graphically in Figures 3.16 and 3.17. We note that a light quark mass of around 300 MeV is consistent with what is found in current nonperturbative models [67]. Figure 3.16 shows the mean squared size of the $q\bar{q}$ configurations as a function of $Q^2$, and demonstrate that the mean size of the configuration diverges in the case of a zero quark mass. This will lead to an extremely rapid variation of the $t$-dependence of the in the form factors for exclusive processes like vector meson production in the small $Q^2$ region, so that using a small quark mass may
lead to large quantitative errors for such processes. The issue of quark mass is not merely a numerical problem. Taking the mass of the quark to zero amounts to including low transverse momentum in the hard scattering part of the LT pQCD calculation. However, the hard scattering part of the interaction should contain only high transverse momenta to be consistent with formal pQCD. Therefore, the saturation approach with a very small quark mass is formally incorrect with regards to LT pQCD.

The problem of taking the photo-production limit may seem irrelevant given that the interaction is overwhelmingly dominated by non-perturbative, soft components of the photon wave function in the photo-production limit. However, we will see in the next chapter that, for extremely high energies, even the real photon may gain a significant contribution from small-size configurations. Furthermore, an attempt was made to include pQCD in the saturation model at large $Q^2$, and it was found that small quark masses were needed to fit to DIS data [68]. The problems with quark mass, of course, extends to this “QCD improved” saturation model as is shown in Fig. 3.17. The basic idea for formulating a pQCD improved version of the saturation model is to write an exponential factor that reduces to Eq. (2.12) in the limit of small sizes. In fact, it was pointed out in Ref. [68] that a quark mass of zero gives the best fit to DIS data for the pQCD evolved saturation model. In Figure 3.17 we have therefore plotted the QCD improved saturation model prediction of the distribution in sizes (the integrand of the $d$-integral in Eq. 2.9), and we see that the case of the pQCD improved saturation model yields the largest tail of large $d$ values for the case of a light quark mass of $10^{-4} \text{ GeV}$.

\footnote{We should note that we have used a different parameterization for the parton distribution function from what was used in Ref. [68]. We use the CTEQ5LO gluon distribution function in order to make a more direct comparison with the MFGS model.}
Figure 3.16: The mean squared dipole size (in $fm^2$) as a function of $Q^2$ calculated in the MFGS model for $x = 10^{-4}$ compared with the result calculated in the GBW model.
Figure 3.17: Comparison of the MFGS model prediction for the distribution of hadronic sizes compared with the GBW prediction for various values of the light quark mass.
Chapter 4

Ultra High Energy Photon Scattering from Protons and Light Nuclei

4.1 Status of Ultra-High Energy Cosmic Ray Photon Astrophysics

In recent years, as experiments sensitive to rare, ultra-high energy cosmic ray particles that are incident upon the earth have become feasible, another venue for the application of the dipole picture has arisen. In this chapter, we will investigate the limit of the dipole picture where the incident photon is real and has extremely large energies at the level relevant to cosmic ray particle physics \[69\]. This investigation serves both to check the consistency of the MFGS model in the kinematical extremes as well as to provide limits for experimental searches.

There is currently an interest in the types of showers induced by ultra-high energy (UHE) cosmic ray neutrinos \[70\] which will be relevant to the Auger and Icecube experiments. The resulting showers are presumed to be initiated by the Bremsstrahlung photons radiated from the electron produced in the initial reaction,

\[
\nu_e + A \rightarrow e + X, \tag{4.1}
\]

where $A$ is a nucleus in the target medium, and $X$ is a hadronic jet produced in the initial reaction. Due to the Landau-Pomeranchuk-Migdal (LPM) \[71\] effect, at UHE soft electromagnetic radiation is suppressed and most of the energy of the electron is transferred directly to the photon. Furthermore, the cross section for $e^+e^-$ pair production drops at UHE and the hadronic interaction between the photon and the target nucleus may dominate while the electromagnetic interaction becomes negligible. The suppression due to the LPM effect becomes stronger depending on the density of the target medium. For a general overview of the LPM effect and electromagnetic suppression various media, see Ref.\[72\]. Furthermore, the shapes of showers may depend upon whether they are dominantly electromagnetic or hadronic \[73\]. Along these lines, it also important to determine what fraction of showers are due to charmed particles. This is important for IceCube and MACRO
because an increase in the number of charmed mesons in the initial reaction will increase
the number of high energy muons seen in experiments, but the contribution of high energy
muons is one factor used to determine the composition of cosmic rays. In addition, charmed
particles contribute to the flux of atmospheric neutrinos. Thus, experiments will need to
take this contribution by charm quarks into account in searches for diffuse astrophysical
neutrinos.

Another possible source of UHE cosmic photons is the decay of extremely massive ex-
otic particles, topological defects, and Z-burst models [74]. Particles with masses as high
as $10^{26}$ eV are speculated to explain the observation of super-GZK energy cosmic rays [75].
Indeed, the calculations in Ref. [75] have shown, using both standard QCD and super-
symmetric QCD, that a large part of the spectrum in the decay of super-massive particles
consists of photons. A characteristic of these “top down” models is the existence of a large
photon flux in cosmic rays. The ratio of protons to photons in the primary interaction for
the production of showers can be used to distinguish between various top-down scenarios.
It is necessary, in order to address the issues above, to place upper limits on the growth
of the real photon cross section. Upper limits have been placed on the the fraction of pri-
mary photons at 26% [76] by analysis at the Auger observatory, but those upper limits are
sensitive to the photon-nucleus interaction which we wish to discuss here.

In this chapter, we will consider the MFGS dipole model in the limit that the incident
photon is real. It was found in Ref. [69] that one can apply unitarity constraints and
phenomenological expectations to estimate the actual growth of the real photon cross section
with target nucleons/nuclei at UHE. We allow the cross section to grow with energy as fast
as possible under the constraints of s-channel unitarity, and thereby at the very least place
upper limits on the growth of photon-nucleus cross sections. It is worth emphasizing here
that one cannot simply use a smooth extrapolation of the cross section to higher energies by
assuming (as it is often done for the case of hadron-hadron scattering) a parameterization
of the cross section inspired by the Froissart bound of the form, $\sigma_{\gamma N}^{tot} = a + b \ln^2(s/s_0)$. 
Asymptotically, the photon - hadron cross section can grow faster than the rate of growth
supplied by the Froissart bound due to the the fact that the photon wave function is non-
normalizable. In fact, as we will discuss in section 4.5, the rate of growth with energy of
the photon-hadron cross section may be as fast as $\ln^3 E_\gamma$ [77].

4.2 The Photo-Production Limit in the Dipole Picture

Recall that the total DIS cross section in the dipole picture is the convolution of the basic
cross section for the interaction of individual $q\bar{q}$ pairs with the squared light-cone wave

\footnote{In the case of scattering off heavy nuclei the non-normalizability of the photon wave function results in $\sigma_{\gamma A} \propto \ln(s/s_0)$ behavior [9].}
functions, Eqs. 2.10 and 2.11, of the photon. Of course, the cross section for scattering of longitudinal polarizations vanishes in the limit that the photon is real, $Q^2 \to 0$, and the transverse component becomes a unique function of the quark momentum fraction, $z$, and the hadronic size, $d$. The transversely polarized photon’s light-cone wave function is,

$$|\psi_T(z,d)|^2 = \frac{3}{2\pi^2} \alpha_{\text{e.m.}} \sum_{q=1}^{n_f} e_q^2(z^2+(1-z)^2)\epsilon^2 K_1^2(\epsilon d) + m_q^2 K_0^2(\epsilon d),$$

(4.2)

where the sum is over $n_f$ active quark flavors, the $K$’s are the modified Bessel functions of the second kind, and $\epsilon^2 = m_q^2$. (Here, it is relevant to recall the discussion of quark masses at the end of section 3.5) For UHE photons, Eq. (4.2) will include a term for the light quarks ($m_q \approx .3$ GeV) as well as a term for charm dipoles ($m_q \approx 1.5$ GeV). More massive quarks are strongly suppressed by the light-cone wave function and are neglected in the present analysis (but see Sec. 4.3 for more discussion of the heavier quarks). The energy dependence in Eq. (2.9) enters through the Bjorken-$x$ variable. In the original formulation of the MFGS model as presented in chapter 2, $x$ and $Q^2$ are taken as input to the calculation of a particular cross section. Recall that in Eq. (2.9) the value of Bjorken-$x$, $x'$, used in the basic dipole-nucleon cross section is not the same as the external value of Bjorken-$x$:

$$x' = \frac{Q^2}{2P \cdot q} \left(1 + \frac{4m_q^2}{Q^2}\right) \left(1 + \frac{0.75\lambda}{d^2(Q^2+4m_q^2)}\right).$$

(4.3)

This prescription for relating the external value of $x$ to $x'$ was originally meant to reproduce realistic behavior in the medium $x$ Bjorken limit while ensuring that the result is well defined in the region of low-$x$ and small $Q^2$. The model is well-defined (finite) in the photo-production limit, $Q^2 \to 0$; but the external value of $x$ vanishes for all energies when $Q^2 = 0$ and is clearly not appropriate as input to the basic cross section. Rather, we would like to fix $Q^2 = 0$ and specify $E_\gamma$. To this end, we note how the original value of $x'$ used in Ref. [35] behaves for $Q^2 \to 0$:

$$x'_{Q^2 \to 0} = \frac{4m_q^2}{2P \cdot q} \left(1 + \frac{0.75\lambda}{4d^2m_q^2}\right).$$

(4.4)

The method of calculating $x'$ used in Eq. (4.3) is derived for the small size components of the photon where the photon fluctuates into a $q\bar{q}$ pair which then exchanges a single gluon with the target. Note that, even when $x \to 0$, the effective value, $x'$, is large when $d \to 0$.

In the original formulation of the MFGS model, the large size configurations were characterized by growth with energy that mimicked the $\ln^2(W_0^2/W_0^2)$ growth of the pion-nucleon total cross section with $W_0^2 = 400$ GeV$^2$ (or $x_0 \approx 0.01$ for DIS kinematics). That is, it was
assumed in Ref. [35] that for large sizes,

\[ \sigma_{d\sim d_\pi}(x', Q^2) \approx \sigma_{\pi N}(x', Q^2) = 23.78 \text{ mb} \left( \frac{x_0}{x} \right)^{0.08}. \quad (4.5) \]

This behavior for the pion cross section was extracted from data in Ref. [52]. Furthermore, soft Pomeron exchange leads to a factor of \( e^{\alpha' \frac{d}{d\ln(x)}} \) with \( \alpha' = .25 \text{ GeV}^{-2} \) and \( d_{\pi} = .65 \text{ fm} \) in the scattering amplitude. In the case of the virtual photon discussed in chapter 2 and Refs. [35, 40], the external value of \( x \) was used for the large size behavior which makes sense for large \( Q^2 \). Clearly, this is inappropriate in the photo-production limit where \( x = 0 \).

In fact, we will now argue that the appropriate value of \( x \) to use for the large size configurations is the same \( x' \) (Eq. (4.3)) that was used for the small size behavior in Ref. [35]. To see this, consider the large \( W^2 \), fixed \( q^2 \), limit of the pion-nucleon scattering cross-section (considering, for the moment, \( q \) to be the pion 4-momentum):

\[
\begin{align*}
W^2 &= (q + P)^2 \\
&= M^2 + 2P \cdot q + q^2 \\
\lim_{x \to 0} &= 2P \cdot q.
\end{align*}
\]  

Then,

\[
\ln \left( \frac{W^2}{W_0^2} \right) = \ln \left( \frac{2P \cdot q}{2P \cdot q_0} \right) = \ln \left( \frac{x_0}{x} \right) + \ln \left( \frac{Q^2}{Q_0^2} \right), \quad (4.7)
\]

where \( Q_0 \) is defined so that \( x_0 \equiv Q_0^2/(2P \cdot q_0) = .01 \). If we regard the pion mass as a particular value for the photon virtuality, then we see that Eq. (4.7) generalizes to any small photon virtuality \( Q^2 \) and small Bjorken scaling variable, \( x \). On the other hand, in the limit of small \( Q^2 \) and non-vanishing size, \( d \), Eqs. 4.3, 4.4 and 4.7 yield,

\[
\ln \left( \frac{x'_0}{x'} \right) = \ln \left( \frac{2P \cdot q}{2P \cdot q_0} \right) = \ln \left( \frac{W^2}{W_0^2} \right), \quad (4.8)
\]

where \( x'_0 \) corresponds to taking \( Q \rightarrow Q_0 \). In other words, to truly mimic the behavior of the pion-nucleon cross section, one should use the effective Bjorken-x that was used in sampling the gluon distribution. Comparing Eqs. 4.7 and 4.9, we also have

\[
\ln \left( \frac{x'_0}{x'} \right) - \ln \left( \frac{x_0}{x} \right) = \ln \left( \frac{Q^2}{Q_0^2} \right). \quad (4.10)
\]
Thus, if one uses the external values of $x$ in Eq. (4.5) then one over-estimates the large size cross section by a factor of $(Q_0^2/Q^2)^{0.08}$ which diverges in the limit of a real photon. Finally, we can determine the value of $x'_0$ by noting that $W_0^2 = 400 \text{ GeV}^2$ implies $Q_0^2 = 4 \text{ GeV}^2$. Therefore, from Eq. (4.4), we have,

$$x'_0 = x_0(1 + m_q^2) \left(1 + \frac{75\lambda}{4d^2(1 + m_q^2)}\right). \tag{4.11}$$

The second term, $m_q^2$, in parentheses in Eq. (4.11) is implicitly divided by 1 GeV$^2$ so that it is unitless. Notice that the difference between $x_0$ and $x'_0$ is only significant for small $d$ or large $m_q$. The conclusion of this section is that it is appropriate to use the effective $x'$ given in Eq. (4.3) for all values of $Q^2$ and that Eq. (4.3) must be used in the photo-production limit. Furthermore, $x'$ is calculated unambiguously from Eq. (4.4) and $E_\gamma$ and with the condition that the target nucleon is at rest so that $P \cdot q = M E_\gamma$. For the rest of this section, we will assume that the target nucleon is at rest and we will specify $E_\gamma$ as input for the dipole model of the real photon.

### 4.3 Growth of the Cross-Section at Very High Energies

When one considers the profile function as in Ref. [40], or the S-matrix for DIS as is done in Ref. [61], one finds that the unitarity constraint is usually violated at extremely low values of Bjorken-$x$ (Ultra-High Energy). As discussed previously, this behavior is direct evidence for the breaking of the leading twist approximation and indicates the onset of qualitatively new QCD phenomena. However, the small size configurations of the UHE real photon wildly violate the unitarity constraint due to the rapid growth of the perturbative expression,

$$\tilde{\sigma}_{pQCD}(d, x') = \frac{\pi^2}{3} d^2 \alpha_s(Q^2)x' g_N(x', Q^2), \tag{4.12}$$

at small $x'$ even when the increase of the radius of the interaction with energy is taken into account. Therefore, it becomes necessary to introduce some new assumptions about the unitarity violating components of the total cross section in order to make some sense of the UHE behavior. We choose to adhere to the usual assumption that at UHE, in the region where small size perturbative methods begin to break down, the cross section is at (or near) the limit set by unitarity. This assumption is supported by our studies of the amplitudes of the dipole - nucleon interaction at HERA energies [40] and is described in chapter 2. Very large size configurations tend not to violate the unitarity constraint in a gross way - the total cross section for large size configurations increases slowly with energy, but there is suppression of the impact parameter at small $b$ with increasing energy due to diffusion consistent with the Donnachie-Landshoff soft Pomeron [3]. In fact, the fit of
[3] does violate S-channel unitarity for LHC energies and above. However, this leads to a very small effect on the total cross sections for the energies we discuss here. We can enforce the unitarity assumption about the high energy behavior if we re-calculate the total photo-production cross section by using the basic cross section of the MFGS model, with a linear interpolation everywhere, except where the unitarity limit is violated. When the unitarity limit is violated, we set the cross section equal to the maximum value allowed by unitarity. More specifically, we first decompose the the basic cross section in terms of the impact parameter representation of the amplitude (the profile function) using the optical theorem. Let $F_{hN}(E_{\gamma}, l)$ be the hadron-nucleon amplitude and let $l$ be the 4-momentum exchanged in the subprocess. Then, we calculate,

$$\sigma_{\gamma N}^\gamma(E_{\gamma}) = 2 \int_0^1 dz \int d^2d |\psi_T(z, d)|^2 \int d^2b \Gamma_h(E_{\gamma}, b, d),$$  \hspace{1cm} (4.13)$$

where $\Gamma(E_{\gamma}, b, d)$ is the profile function for a configuration of size $d$. When we calculate the hadronic profile function, we model the amplitude using the MFGS model and the model of the $t$-dependence of Eq. (2.34) from chapter 2 so long as it is less than one. If it exceeds unity, then we explicitly reset $\Gamma(E_{\gamma}, b, d) = 1.0$. In symbols,

$$\Gamma_h(E_{\gamma}, b, d) = \left\{ \begin{array}{ll} \frac{1}{2s(2\pi)^2} \int d^2q e^{ib\cdot q} A_{hN}(q) & , \frac{1}{2s(2\pi)^2} \int d^2q e^{ib\cdot q} A_{hN}(q) < 1 \\ 1 & , \text{otherwise} \end{array} \right. \hspace{1cm} (4.14)$$

Note that in the limit of $d \rightarrow 0$, both $x'$ and the photon wavefunction become mass independent. As we shall see, the distribution in hadronic sizes involved in the interaction becomes more and more sharply peaked around small sizes as $E_{\gamma} \rightarrow \infty$. Thus, in the UHE limit, the sum in Eq. (2.9) will contain significant contributions from all of the more massive strongly interacting particles. The energies that we consider in this paper are still not high enough to include all heavy particles, but, as we will discuss, there is a significant charm contribution. We include the charm pair contribution in the calculation of the cross section and note that the resulting hadronic showers will likely consist of a significant number of open charmed particles like D-mesons. The contribution from bottom quark pairs will be suppressed relative to the charm due to its larger mass and smaller electromagnetic coupling. We neglect the contribution from bottom and all heavier particles.

In our calculations, we use CTEQ5L gluon distributions [54]. We note that, since significant sections of the matching region will violate the unitarity constraint, then trying to achieve an extremely smooth interpolation is an arbitrary modification to the model which achieves no genuine improvement. For simplicity, therefore, we use the linear interpolation in the MFGS model. The plots in Figs. 4.1 and 4.2 demonstrate the resulting distribution of hadronic sizes in the photon. Naturally, the peak at small sizes becomes sharper in the
Figure 4.1: The distribution of the integrand in Eq. (2.9) over hadronic sizes. Here, the unitarity constraint, $\Gamma \leq 1$, is explicitly enforced. Nevertheless, the distribution becomes sharply peaked around small hadronic sizes for UHE photons.
Figure 4.2: This is the distribution of the charm contribution normalized to one. Comparing with Fig. 4.1, we see the suppression of massive quark contributions. Comparing this with Fig. 4.1, we see that the distribution for heavier quarks is more sharply peaked around small transverse sizes relative to lighter quarks.
UHE limit.

Before leaving the subject of \(t\)-dependence in the UHE photon, we recall that in the original MFGS model of the \(t\)-dependence, discussed in Ref. [40], the diffusion of the small size \(q\bar{q}\) pairs was neglected. This was reasonable in the energy range of DIS. However, in a recent overview [78] of the behavior of hadronic cross sections at UHE, the maximum diffusion of small size configurations at UHE was considered\(^2\). In order to correct for the small size diffusion, we slightly modify the diffusion factor in the amplitude of Ref. [40] to the form,

\[
F_{IP}(t, x) = e^{\alpha'(d) \frac{1}{2} \ln(x_0/x)},
\]

(4.15)

where \(\alpha'(d) = 0.25(1 - 0.5e^{-18d^4}) \text{ GeV}^{-2}\). For sizes greater than the pion size \((d_\pi \approx .65 \text{ fm})\), the value of \(\alpha'\) quickly approaches the usual slope \((\alpha' = 0.25 \text{ GeV}^{-2})\) in accordance with the Donnachie-Landshoff soft Pomeron as in Ref. [40]. However, unlike the Regge slope in Ref. [40] which vanishes at small sizes, the value of \(\alpha'\) approaches \(0.125 \text{ GeV}^{-2}\) for \(d \lesssim .2 \text{ fm}\), which is the diffusion rate for small size dipoles determined recently in Ref. [78]. We want the transition from between the small and the large size slope to be fast but continuous, so we use an exponential function to interpolate between the soft and hard region. The slope of 18 ensures that \(\alpha'\) approaches \(0.25 \text{ GeV}^{-2}\) rapidly for \(d \gtrsim .65 \text{ fm}\) and \(0.125 \text{ GeV}^{-2}\) for \(d \lesssim .2 \text{ fm}\). Note that at very high energies we should take into account that the Fourier transform of \(F_{IP}(t, x)\) should contain a tail \(\propto \exp(-\mu b)\) for some mass scale, \(\mu\). However, in the energy range discussed in this paper it is a small effect and hence we neglect it.

Samples of the profile function obtained when we use the above \(t\)-dependence and the above procedure for taming the unitarity violations are shown in Fig. 4.3. Note the extremely rapid growth with energy at sizes of \(d = .1 \text{ fm}\).

The procedure described above may be regarded as placing an upper limit on the growth of the cross section since we have taken the maximal rate of growth that does not violate unitarity. The resulting nucleon cross section has been plotted in Fig. 4.4. To test the numerical sensitivity to a variation in the upper limit of the profile function, we have included the result of placing the upper limit of the profile function at 0.8 rather than at 1.0. In the studies [33, 35] the matching parameter, \(\lambda\) was estimated based on the analysis of the expressions for \(\sigma_L(x, Q^2)\) to be of order 10. A later analysis of the \(J/\psi\) production [38] suggested that a better description of the cross section for the intermediate \(0.5 \geq d \geq 0.3 \text{ fm}\) is given by \(\lambda \sim 4\) while the cross section in the perturbative region depends very weakly on

\(^2\)The experimental status of \(\alpha'\) is a bit confusing at the moment. The H1 on photo-production of \(\rho\)-mesons presented at DIS06 finds a value for \(\alpha'\) that is factor of two smaller than the Donnachie-Landshoff value and is consistent with previous ZEUS data. However the may also be interpreted as nonlinearity in the effective Pomeron trajectory with \(\alpha'\) close to the Donnachie-Landshoff value for \(-t > .2 \text{ GeV}\) and going to zero at larger \(-t\). In view of these uncertainties we have stuck to our original parameterization in the previous section, but we note that in order to obtain an upper limit in this section, it is safe to use the larger value of \(\alpha'\).
Figure 4.3: Samples of the profile function for the $hN$ interaction for real photon energies of $10^4$ GeV (dashed line), $10^6$ GeV (dotted line), $10^8$ (dot-dashed), and $10^{11}$ GeV (solid line) for a range of hadronic sizes.
Figure 4.4: Growth of the total photon nucleon cross section for the range of energies from $E_\gamma = 10^3$ GeV to $E_\gamma = 10^{12}$ GeV. On the x-axis of the bottom panel, $\ln^3\left(\frac{E_\gamma}{E_0}\right)$ ($E_0 = 1.0$ GeV) is plotted to allow for easy comparison with the expected energy dependence at UHE (see section 4.5). The lowest three curves in each panel are the result of using the MFGS model with variations in model dependent parameters to demonstrate numerical sensitivity (see text). For comparison, the upper curve shows the two Pomeron model of Donnachie and Landshoff Ref. [79] and the dotted curve shows the model of Shoshi et. al. Ref. [80].
\( \lambda \). We therefore include in Fig. 4.4 the result of using \( \lambda = 10 \) to test the sensitivity to the matching ansatz. It is evident that variations in these parameters yield a small reduction in the upper limit. For the rest of this paper, we assume that \( \lambda = 4 \) and that the unitarity constraint implies, \( \Gamma_h \leq 1 \).

For comparison, the Donnachie-Landshoff two-Pomeron model [79] with no unitarity constraint is also shown in Fig. 4.4. The two-Pomeron is a fit obtained by using a linear superposition of two Pomeron trajectories; one with a large value of \( \alpha_0 \) characteristic of hard scattering and one with a small value of \( \alpha_0 \) characteristic of soft scattering. (The hard Pomeron dominates at very high energies.) (This was used recently to estimate the possible contribution of UHE photons to atmospheric showers.) Without any unitarity constraint, the growth is, of course, extremely rapid and disagrees strongly with our result. Yet another approach was used for calculations in Ref. [80]. The result of this calculation is shown as the dotted curve in Fig. 4.4. In the model of Ref. [80], the large size components are modeled by non-perturbative QCD techniques, but the small size components use a Donnachie-Landshoff two Pomeron model with parameters somewhat different from the original model. While the method used in Ref. [80] imposes impact parameter space unitarity in the dipole picture, it does not account for the non-normalizability of the photon wavefunction. The diverging integral over \( d \) is arbitrarily cut off from below to obtain a normalization for the photon wavefunction. A Unitarity constraint is then imposed upon the \( S \)-matrix for \( \gamma \)-proton scattering, and is thus less restrictive than our approach which applies unitarity constraints to the profile function for individual hadronic configurations. Our approach thus restricts the growth of the cross section more than either of the two above scenarios as can be seen in Fig. 4.4.

It has been observed that agreement with data in a standard QCD analysis is improved if one uses \( \overline{\text{MS}} \) NLO pdfs in very low-\( x \) DIS experiments. Our main concern is that the rate of increase described by the interpolation that we obtained in Eqs. B.1-B.3 of the appendix is not drastically modified by the inclusion of higher order effects. In particular, the rate of growth, and the order of magnitude of higher order corrections should not be drastically altered from what one obtains at leading order. To argue that this is the case, we notice that the LO and NLO parton densities grow at nearly the same rate at very high energies. This is demonstrated for a typical small configuration size in Fig. 4.5 where we compare the LO and NLO CTEQ5 parton densities lowest values of \( x \) where parameterizations exist. The main difference between the two parton densities is that, in the high energy limit, the leading order gluon density is nearly a constant factor larger than the NLO pdf. (This makes sense when one considers the color factor 9/4 difference between quark-antiquark dipoles, and gluon dipoles and the need to include \( q\bar{q}g \) dipoles in a consistent NLO formulation of the dipole model.) Indeed, above approximately \( E_\gamma = 20000 \) GeV in the plot of Fig. 4.5, the LO curve is shifted by a constant factor of about 9/4 upward from the NLO curve, and
Figure 4.5: Plot comparing the the growth of LO and NLO pdfs at high energy for the small dipole size, \( d = 0.05 \) fm.

The energy dependence used (see Eq. (B.2) of Appendix B) describes both curves with an accuracy of \( \leq 10\% \) for the energy range \( E_\gamma > 20,000 \) GeV. In the spirit of obtaining an upper limit on the growth of the cross section, we continue to used the LO pQCD result. Since we checked that our LO inspired parameterization of \( \sigma(d, x') \) describes the data well down to \( x \sim 10^{-3} \), the difference in the normalization is likely to be partially absorbed in the definition of the cross section. Note also that the recent studies of the small \( x \) behavior of the the gluon densities indicate that the NLO approximation is close to the resumed result down to \( x \sim 10^{-7} \), see a review in [81].

Before leaving the discussion of the nucleon target, we note several advantages and disadvantages of the current approach: First, Fig. 4.4 is consistent with the rate growth of the cross section for a real photon at UHE which we find in Sec. 4.5 to be \( \sim \ln^3(E_\gamma/E_0) \), where \( E_0 \) is of order 1 GeV (see e.g. [3] for a review of the Froissart bound for the hadron - hadron scattering cross section, See Sec. 4.5 for a generalized discussion of energy dependence for real photons.) This is one consistency check for the MFGS model.

Second, we are also able to analyze the contribution from different flavors separately. In Fig. 4.6, we see that the fraction of the total cross section due to charmed particles rises to nearly 25% at \( 10^{12} \) GeV\(^3\). This indicates that a large fraction of showers initiated by charm could, of course, be much larger if there is some mechanism that suppresses the light quark contribution but not the charm contribution.

\(^3\text{This calculation of the fractional contribution of charm assumes that the cross section for the scattering of each quark flavor individually grows at the maximum rate allowed by unitarity. The fractional contribution of charm could, of course, be much larger if there is some mechanism that suppresses the light quark contribution but not the charm contribution.}\)
UHE photons should contain a pair of leading charmed particles. It is worth investigating whether such showers have a substantially longer penetration depth in the atmosphere and could be separated by the Auger detector.

Furthermore, by recalling the relationship between the diffractive component of the basic cross section, $\hat{\sigma}_{\text{diff}}$ and the hadronic profile function,

$$\hat{\sigma}_{\text{diff}} = \int d^2 b |\Gamma_h(b)|^2,$$

we may also separate the fraction of the total hadronic profile function due to diffractive scattering. This is shown in Fig. 4.7, which gives an integrated measure of the proximity to the black disk limit. In the black disk limit, diffractive scattering accounts for exactly half of the total cross section. Therefore, the fact that, as Fig. 4.7 shows, the fraction of the cross section due to diffractive scattering is around .35 at UHE indicates that diffractive scattering plays a significant role and that there will be large shadowing in nuclei.

Since the cross section grows extremely quickly at UHE, the unitarity limit is saturated even at values of dipole size around $d = 0.1$ fm. Figure 4.1 demonstrates that the largest contributions to the total photo-production cross section come from regions around $d \approx .1$ fm and from the transition region. Sizes smaller than this contribute very little to the total cross section. Thus, pQCD provide very little detailed information, since any model of the basic cross section (at small sizes) which violates the unitarity constraint at $d \approx 0.1$ fm will give very similar results. Furthermore, since the gluon distribution rises very sharply
between $d = 0$ fm and $d = 0.1$ fm at very high energies, the calculation in Eq. (2.9) becomes more sensitive to how the gluon distribution is sampled. Thus, there is more sensitivity to the parameter, $\lambda$, used to relate the hadronic size to the hardness of the interaction. As seen in Fig. 4.4, if $\lambda = 10$, the cross section is suppressed by around ten percent from its value when $\lambda = 4$.

The largest source of theoretical uncertainty is the region of large hadronic sizes. Though the behavior of large configurations can be reasonably expected to follow pion behavior at accelerator energies, we do not have any experimental data for $\gtrsim 10^6$ GeV hadrons with which to model these extremely high energy Fock states. Moreover, we have so far been associating each hadronic Fock state with a particular size. It may be that as the energy of the photon increases, a large number of hadronic Fock states (perhaps multiple pion states) may be associated with a single size. Moreover, contributions from large impact parameters become significant for extremely energetic photons. Thus, predictions become sensitive to how the model handles the $t$-dependence of the amplitude. The current model is based on the assumption that the typical $t$-dependence for low energies continues into the UHE regime. Note, however, that in the case of $pp$ scattering the analysis of [78] indicates that, though the black disk limit leads to the slope $B \propto \ln^2 s$, this is a very small correction for the energies discussed here. Finally, we emphasize that we allow the impact parameter amplitude to approach $\Gamma = 1$ without a slowdown at $\Gamma \sim 0.5$ as happens in many other
dipole models where eikonal type parameterizations of the dipole-nucleon cross section are used (see, for example, [68]). Therefore, this approach indeed yields a very conservative upper bound.

### 4.4 Nuclear Targets

Since we are interested in the interaction of UHE photons with atmospheric nuclear targets, we will now go on to investigate the growth of the UHE cross section for the case of a real photon scattering from a nuclear target. The steps apply to any light nuclei constituting the atmosphere, but we use $^{12}$C for the purpose of demonstration since the $^{12}$C nucleus has the approximate number of nucleons for a typical atmospheric nucleus.

At first glance it seems natural to repeat the procedure we followed for the proton case in Sec. 4.3 with the profile function given by a nuclear shape, and cross section for the interaction of the small dipoles as given by Eq. (2.9). However we found that if we follow this procedure we end up with the obviously wrong result that in the UHE limit, this approach quickly leads to the situation that $\sigma_{\gamma A} \geq A\sigma_{\gamma N}$ which is physically unreasonable. The reason for this paradox becomes clear if we visualize the relationship between the proton PDF and the nuclear PDF. The unitarity condition is not sensitive to effects of transverse correlations of the partons. The unitarity constraint would tame the dynamics if one could assume that the nucleus is a perfectly homogeneous distribution of nuclear matter. At high energies, the disk of nuclear matter “seen” by the incident hadronic configuration blackens as represented schematically in Fig. 4.8. Any inhomogeneity in the distribution of nucleons is accounted for at low energies by the grayness of the nuclear disk without yielding a quantitative difference in the total cross section. However, the actual distribution of nuclear matter in light nuclei “seen” by the incident dipole probably looks more like that of Fig. 4.9 - a collection concentrated regions of nuclear matter which individually grow black in the high energy limit but far apart from each other transversely in impact parameter space. If the nuclear system is dilute and nucleons do not overlap transversely, the use of the $\Gamma(b) < 1$ condition becomes insufficient. Thus, we will certainly find that $\sigma_{\gamma A} > A\sigma_{\gamma N}$ if we assume that both the disk of the individual nucleon and the disk of the nucleus grow black in the high energy limit. That is, the cross section resulting from Fig. 4.9(b) is certainly larger than the sum of the cross sections from each of the blackened nucleons seen in Fig. 4.8(b). The simplest illustrative example would be to consider scattering off the deuteron - in this case neglect of the cluster structure of the system would grossly overestimate the maximal cross section for the interaction of this system with a small dipole. It is worth emphasizing that all these considerations are valid for the light nuclei and are likely to be a small correction for the scattering off sufficiently heavy nuclei.

A more meaningful upper estimate of the cross section of scattering off light nuclei is,
Figure 4.8: The nuclear disk as “seen” by the incident hadronic configuration in the simple homogeneous model of the unintegrated nuclear PDFs. The level of absorption by the disk is indicated by the level of grayness. At low energies, (a), the disk is weakly absorbing and homogeneous. At very high energies in (b) the disk becomes black and is thus totally absorbing.
Figure 4.9: A more accurate way to visualize what the incident hadronic configuration probably actually “sees”. The nucleus consists of nucleons separated over a large distance with concentrated nuclear matter in (a). At very high energies (b), each of the nucleons becomes totally absorbing (black).
therefore, to take into account first the taming of the elementary cross sections and next the Glauber - Gribov theory of nuclear shadowing due to diffraction [6, 7] which does not rely on the twist decomposition of the cross section. Since we observed that diffraction constitutes a large fraction of the total cross section we expect that a large shadowing effect will emerge with growing energy. Consequently, our result will automatically be consistent with S-channel unitarity.

Hence, we take the usual approach to nuclear scattering when the product of the nuclear optical density with the cross section is small: \( A\sigma_N T(b) < 1 \). Following in the spirit of the treatment of hadronic fluctuations in the Good and Walker picture as presented in e.g. [25], we consider the states of the incident photon to be a linear combination of nearly “frozen” hadronic states that do not mix with one another during their passage through the target. Each of these states scatters from the target with a cross section \( \langle \hat{\sigma}_h \rangle = \hat{\sigma}_h \). Let us write the total photon-nucleus cross section as,

\[
\sigma^{\gamma A} = A\hat{\sigma}_h - \Delta\sigma. \tag{4.17}
\]

A standard result of the Glauber-Gribov theory in the language of hadronic fluctuations is that the full shadowing correction, \( \Delta\sigma \), can be written as,

\[
\Delta\sigma = \frac{A}{4} \int d^2b T^2(b) \langle \hat{\sigma}_h^2 \rangle e^{-\frac{1}{2}A(\hat{\sigma}_h)T(b)}, \tag{4.18}
\]

where \( T(b) \) is the nuclear optical density normalized to unity. Equation (4.18) is an approximate formula valid for the case of small fluctuations or small nuclear thickness (the later is true in our case). We will find a convenient expression for \( A_{eff}/A \) if we expand this expression using the argument of the exponent as a small parameter:

\[
\sigma^{\gamma A} = A\hat{\sigma}_h - \frac{A\langle \hat{\sigma}_h^2 \rangle}{4} \int d^2b T^2(b) + \cdots \tag{4.19}
\]

If \( A_{eff} \) is defined by the relation, \( \sigma^{\gamma A} = A_{eff}\sigma^{\gamma N} \), then dividing Eq. 4.19 by \( A\sigma^{\gamma N} \) gives,

\[
\frac{A_{eff}}{A} = 1 - \frac{\langle \hat{\sigma}_h^2 \rangle}{4\sigma^{\gamma N}} \int d^2b T^2(b) + \cdots. \tag{4.20}
\]

Define an effective cross section,

\[
\sigma_{eff} = \frac{\langle \hat{\sigma}_h^2 \rangle}{\sigma^{\gamma N}} = \frac{16\pi}{\sigma^{\gamma N}} \left. \frac{d\sigma_{diff}}{dt} \right|_{t=0}, \tag{4.21}
\]

where by definition,

\[
\langle \hat{\sigma}_h^2 \rangle = \int_0^1 dz \int d^2d |\psi_T(z, d)|^2 \hat{\sigma}_h^2(d, d'). \tag{4.22}
\]
Then we can write,

\[
\frac{A_{\text{eff}}}{A} = \frac{\sigma_{\text{eff}} - \frac{\sigma_{\text{eff}}^2}{4} \int d^2b T^2(b) + \cdots}{\sigma_{\text{eff}}} = \frac{\int d^2b \left( T(b)\sigma_{\text{eff}} - \frac{\sigma_{\text{eff}}^2}{4} T^2(b) + \cdots \right)}{\sigma_{\text{eff}}}. \tag{4.23}
\]

Considering the first two terms in Eq. 4.23 as the first terms in a power series expansion\(^4\), we may write,

\[
\frac{A_{\text{eff}}}{A} = \frac{2 \int d^2b \left( 1 - e^{-\frac{1}{2}\sigma_{\text{eff}} T(b)} \right)}{\sigma_{\text{eff}}}, \tag{4.24}
\]

If we identify the effective profile function as,

\[
\Gamma_{\text{eff}}(b) \equiv 1 - e^{-\frac{1}{2}\sigma_{\text{eff}} T(b)}, \tag{4.25}
\]

then we see that both the effective profile function and the shadowing ratio are less than unity by construction. Furthermore, because \(A_{\text{eff}}\) must be less than \(A\) (for scattering in any range of impact parameters), then we may regard \(\Gamma_{\text{eff}}(b) < 1\) as our unitarity condition for the nucleus.

(Note that the diffractive components have mass squared proportional to \(1/d^2\) and therefore correspond to rather large hadron multiplicities.) The result of evaluating Eq. (4.24) and solving for the \(\gamma A\) cross section is shown in Fig. 4.11. One can see that the energy dependence of the \(\gamma^{-12}C\) scattering cross section is substantially weaker than for the proton target due to nuclear shadowing, though the increase of the cross section as compared to the energies studied experimentally is still large. Since there is less shadowing for the case of incident charm dipoles, then our analysis indicates that there is a larger fraction of the total photon-\(12C\) cross section that arises due to charm dipoles than in the case of a proton target. This is shown in Fig. 4.12. In the case of a target \(12C\) nucleus, the fraction of the total cross section due to charm dipoles is around 30%. The large amount of shadowing that we find already has implications for energies around \(\sim 100\) TeV which are relevant to a number of current cosmic ray experiments [82]. In addition, the forthcoming studies of ultra-peripheral heavy ion collisions at the LHC would allow, to some extent, a check of our predictions by measuring shadowing for photon-heavy ion interactions for the range of values for \(\sqrt{s}\) from 1000 GeV to 2000 GeV. Within our model, we find that \(A_{\text{eff}}/A \approx .3\) for \(\sqrt{s} = 100\) GeV and \(A_{\text{eff}}/A \approx .2\) for \(\sqrt{s} = 2000\) GeV with \(A = 220\). In order to allow

\(^4\)In principle one can write a more accurate formula which would take into account deviations of \(\frac{\langle \hat{\sigma}_{n} \rangle}{\sigma_{\text{eff}}^n}\) from \(\sigma_{\text{eff}}^{-1}\) for \(n \geq 3\). However numerically these effects are small especially for the light nuclei where double scattering gives a dominant contribution.
for a simple extrapolation from the shadowing ratio, \( (A_{\text{eff}}/A)_C \), for Carbon to other nuclei with masses typical of atmospheric atoms we use the function,

\[
\frac{A_{\text{eff}}}{A} = \left( \frac{A_{\text{eff}}}{A} \right)_C \left( \frac{A}{12} \right)^n,
\]

where we then determine \( n \) for a set of fixed photon energies and for the range of atomic masses, \( 12 \leq A \leq 16 \). As a sample, we list the following: For \( E_\gamma = 10^{12} \text{ GeV} \), we find \( n = -0.41 \); for \( E_\gamma = 10^9 \text{ GeV} \), we find \( n = -0.35 \); and for \( E_\gamma = 10^6 \text{ GeV} \), we find \( n = -0.3 \).

### 4.5 Limiting behavior of Energy Dependence at Ultra-High Energies.

It is possible to obtain the general energy dependence, \( \sigma^{\gamma h} \sim \ln^3 E_\gamma \) for the \( \gamma \)-hadron cross section in the UHE limit within the dipole model. Here we give a general proof based only on the following assumptions about the UHE limit:

- The dipole model for a finite number of active quark flavors holds for the real photon in the UHE limit in the sense that at a fixed energy, the dipole cross section increases with \( d \) no faster than \( d^2 \).

- A given finite size hadronic Fock component of the real photon scatters with exactly
Figure 4.11: The upper panel shows the dependence of the $\gamma + ^{12}C$ cross section on the incident photon energy. The lower panel shows the dependence of the shadowing ration $\sigma_{\gamma A}/(A\sigma_{\gamma N})$. 
the maximum possible cross section allowed by the unitarity constraint when $E_\gamma \to \infty$, and the rate of increase of the cross section for each individual Fock component of size, $d$, is limited by the rate of growth in the Froissart limit.

- The very small size hadronic Fock components of the real photon scatter with a cross section whose rate of growth is no faster than a power of $x'$ (or $E_\gamma$).

The second bullet above requires some clarification. Usually, the Froissart bound is only applied to the interaction between two hadrons rather than to the interaction between a wave-packet and a hadron. However, since the derivation of the Froissart bound is based on analyticity in the $t$-channel which leads to the requirement that the amplitude in impact parameter space falls off at least as fast as $e^{-2m_\pi b}$ [3] then the argument works in our case as well.

We are only interested in the variation of the cross section with energy. Therefore we will leave out over-all factors in order to simplify the argument. Note that $\ln \frac{x'}{x'_0}$ can always be separated into a sum of $\ln(E_\gamma)$ and terms that only depend on $d$. For the rest of this section, we will always write $\ln \frac{x'}{x'_0}$ as $\ln(E_\gamma)$ since it is the leading powers of photon energy that will interest us. To be concise, the symbol $\sim$ will indicate how a cross section varies

\[ \frac{\gamma-C^{12}}{C^{12}} \text{ Due to Charm Dipoles} \]

Figure 4.12: The fraction of the total $\gamma$-C$^{12}$ cross section due to charm dipoles.

\[ \frac{\sigma_{\text{charm}}}{\sigma_{\text{tot}}} \]

\[ E_\gamma \text{(GeV)} \]

\[ 10^{-3} \quad 10^3 \quad 10^6 \quad 10^9 \quad 10^{12} \]

5Actually, in the MFGS model, the amplitude behaves as $\sim e^{a \ln \frac{x'}{x'_0}}$ in the UHE limit where $a$ is a positive constant. So, it falls off faster than $e^{-2m_\pi b}$ in impact parameter space and therefore the cross section should not increase at precisely the maximum rate allowed by unitarity. Therefore, the fact that the MFGS model is nearly linear in $\ln^3 \frac{E_\gamma}{E_0}$ in Fig. 4.4 indicates that pre-asymptotic affects are still significant in the considered energy range.
with photon energy, $E_\gamma$, whereas $\alpha$ will indicate how a cross section varies with hadronic size, $d$. The first of the above assumptions allows us to state that,

$$\sigma(E_\gamma) \sim \sum_{\text{flavor}} \int_0^\infty d \, d |\psi(d)|^2 \hat{\sigma}(d, x') \, .$$

(4.27)

The integral over momentum fraction from Eq. (2.9) is assumed to be implicit. Also, for the rest of this section, the sum over flavors in Eq. (4.27) will be understood and left out. Since the energy dependence of the integrand in Eq. (4.27) can be understood in the extreme limits of $d \to 0$ or $d \to \infty$, but is model dependent in the intermediate range of $d$, then let us separate Eq. (4.27) into the sum of three terms:

$$\sigma(E_\gamma) \sim \int_0^\epsilon d \, d |\psi(d)|^2 \hat{\sigma}(d, x') + \int_\epsilon^\Delta d \, d |\psi(d)|^2 \hat{\sigma}(d, x') + \int_\Delta^\infty d \, d |\psi(d)|^2 \hat{\sigma}(d, x') \, .$$

(4.28)

Call the terms in Eq. (4.28) regions 1, 2, and 3 respectively. For any given range of photon energies, one can choose sufficiently large $\Delta$ and sufficiently small $\epsilon$, that regions 1 and 3 must give a negligible contribution to the over-all cross section. We will justify this statement now.

First, in region 1, the cross section for the subprocess has the following behavior due to the first and last bulleted assumption above:

$$\hat{\sigma}_1(d, x') \sim d^2 (E_\gamma)^\alpha \, ,$$

(4.29)

where $\alpha$ is some positive real number. Furthermore, the light cone wavefunction of the photon gets its energy dependence from the leading behavior of the modified Bessel functions in Eq. (2.9). In the limit of $d << 1/m_q$, $|\psi(d)|^2 \propto 1/d^2$. Hence, in the limit defined by region 1, we have the following general energy dependence:

$$\text{region 1} \sim \int_0^\epsilon d \, d \frac{1}{d^2} d^2 (E_\gamma)^\alpha \sim c^2 (E_\gamma)^{\alpha+1} \cdot$$

(4.30)

Next we consider the other extreme: $d \to \infty$. Away from $d = 0$, Eq. (4.29) shows that the cross section, $\hat{\sigma}(d, x')$ for the subprocess rises very quickly to values that violate the unitarity constraint since we are considering UHE photons. At a certain value of $d$, the growth of $\hat{\sigma}(d, x')$ with $d$ must level out. Within the dipole model, the growth of $\hat{\sigma}(d, x')$ is flat with respect to variations in the transverse size of the hadronic component in the limit that $d$ is large. Call the upper limit of the basic cross section, $\hat{\sigma}_{\text{max}}$. Also, for $d \to \infty$, the Bessel functions give $|\psi(d)|^2 \propto e^{-2mqd}/d$. The energy dependence of the large size cross section can grow no faster than $\ln^2 E_\gamma$ due to the Froissart bound (the second bulleted assumption...
Thus, for region 3 we have,

\[
\text{region 3} \sim \int_{\Delta}^{\infty} d d e^{-2m_\gamma d} d \hat{\sigma}_{\text{max}} \ln^2(E_\gamma) \sim e^{-2m_\gamma \Delta \ln^2 E_\gamma}. \tag{4.31}
\]

For a particular range of photon energies, we may always choose \( \epsilon \) small enough, and \( \Delta \) large enough that regions 1 and 2 give a negligible contribution to the total over-all integral in Eq. (4.28). From now on, assume that \( \epsilon \) and \( \Delta \) are always chosen, in each energy range, so that regions 1 and regions 2 are defined to be negligibly small. Due to the general properties of the dipole model there is always a very small contribution from very large hadronic sizes \((d > \Delta)\) that grows slowly with energy \((\sim \ln^2 E_\gamma)\), and there is always a small contribution from very small hadronic sizes whose contribution may grow very quickly (as a power of \( E_\gamma \), according to the third bulleted assumption) due to the fact that there will always be a contribution from extremely small sizes whose value of \( \hat{\sigma}(d, x') \) has not yet reached the unitarity limit at a given photon energy. We will now consider the rate of growth of the cross section that results from assuming that the cross section attains the maximum value allowed by unitarity for the largest range of sizes possible within the general constraints of the dipole model. Regions 1 and 3 give negligible contributions to the total integral, as discussed above, and the values of \( \hat{\sigma}(d, x') \) will be assumed to saturate the unitarity constraint for all values of \( d \) outside of range of region 1. This means that for region 2, the basic cross section (denoted by a subscript 2) has reached the maximum allowed value, \( \hat{\sigma}_{\text{max}} \), in terms of its growth with \( d \), and the rate of growth with \( E_\gamma \) is the maximum allowed by the Froissart bound. The cross section appearing in the integrand of region 2 then becomes,

\[
\hat{\sigma}_2(d, x') \sim \hat{\sigma}_{\text{max}} \ln^2 E_\gamma. \tag{4.32}
\]

However, the requirement that region 2 contains all of the unitarity saturating contribution, and that region 1 contains a negligible contribution demands that we allow \( \epsilon \) to have some energy dependence. This is because, as Eq. (4.29) shows, the basic cross section at small sizes may have as much as a quadratic \( d \)-dependence and potentially very rapid energy dependence. Therefore, at a small but fixed value of \( d \), the basic cross section quickly rises from small values to unitarity violating values with increasing energy. However, region 1 by definition contains only the suppressed part of the integrand near \( d = 0 \), whereas the unitarity saturating region should be associated entirely with region 2. As the energy of the photon is increased, therefore, we must continuously redefine region 1 so that the integrand of region 1 is confined to a smaller and smaller region around \( d = 0 \). This sort of behavior does not exist at \( d \gtrsim \Delta \), because at such large values of hadronic size, the cross section only increases as \( \ln^2 E_\gamma \) and there is almost no variation with hadronic size. Therefore, \( \Delta \) is defined without any energy dependence (\( \Delta \) may be given weak energy dependence, but
that will only result in subleading powers of \( \ln E_\gamma \) in the final result.) Equation (4.29) tells us that the maximum rate at which \( \epsilon \) may decrease at small \( d \) is,

\[
\epsilon(E_\gamma) \sim E_\gamma^{-(1+\alpha)}.
\] (4.33)

We thus write region 2 as,

\[
\text{region 2} \sim \int_{\epsilon(E_\gamma)}^{\Delta} d \epsilon \frac{1}{d^2} \hat{\sigma}_{\text{max}} \ln^2 E_\gamma \sim \ln^3(E_\gamma)
\] (4.34)

Here we have continued to use \( 1/d^2 \) behavior for the squared photon wavefunction because this yields the fastest possible rate of divergence of the wavefunction at the lower end of the integral and thus yields the most conservative upper limit on the rate of growth. Notice that after having given \( \epsilon \) energy dependence, Eq. (4.30) becomes,

\[
\text{region 1} \sim \int_0^{\epsilon(E_\gamma)} d \epsilon \frac{1}{d^2} d^2 \ln^2(E_\gamma) \sim E_\gamma^{-(1+\alpha)}.
\] (4.35)

Thus we have established the behavior of each of the regions in Eq. (4.27). Regions 1 yields a vanishing contribution to the total integral as \( E_\gamma \to \infty \) and region 3 has energy dependence \( \lesssim \ln^2 E_\gamma \) whereas region 2 has energy dependence, \( \lesssim \ln^3(E_\gamma) \). Taking the leading behavior in Eq. (4.27) therefore gives,

\[
\sigma(\gamma \text{ hadron} \to \text{ hadrons}) \lesssim \text{Constant} \times \ln^3 E_\gamma.
\] (4.36)

Equation (4.36) applies for each active flavor individually and thus for the sum of flavors. A possible way that the rate of growth at ultra-high energies violates Eq. (4.36) in spite of the unitarity limit being saturated for each flavor would be for there to be a large proliferation of new active flavors at ultra high energies.
Chapter 5

Vector Meson Photo-Production from the Deuteron at Intermediate Energies

5.1 Introduction

Having just considered the basic consistency requirements for the scattering of ultra-high energy photons, we now consider another limit of photon scattering which also requires a review of the consistency of traditional approaches. In this chapter we consider vector meson photo-production off nuclear targets by real photons in the intermediate range of energies starting a few giga-electron volts above the threshold. These reactions have great potential for probing unusual phenomena such as non-diffractive, OZI violating mechanisms for vector meson production mesons, in-medium modifications of vector mesons, the importance of “non-ideal” $\omega - \phi$ mixing, and other new mechanisms for vector meson production (see Refs. [22, 23, 83]).

Furthermore, it would be interesting to discover whether the $\phi$-meson is produced with a small enough transverse size that quark degrees of freedom may become relevant, as in the case of $J/\psi$-production. Actually, in the case of $J/\psi$, the cross section of the $J/\psi - N$ interaction $\sigma_{J/\psi-N} \sim 3 \text{mb}$, [84] estimated based on the A-dependence of $J/\psi$ photo-production at energies $\sim 20$ GeV, is much larger than the estimate based on the VDM: $\lesssim 1 \text{mb}$. This is likely due to the color transparency phenomenon [84]. A natural question is whether a trace of this effect remains in the case of $\phi$-production. Jefferson Lab has produced data for $\phi$ production that is currently being analyzed.

The interest in intermediate energy reactions makes it necessary to re-evaluate the assumptions of the traditional Glauber series method, and to develop a new basic theoretical framework. This new framework was presented in [85]. This chapter addresses the issues one must face when considering photon energies large enough that the eikonal approximation is an appropriate description of hadronic re-interactions, but not large enough that it is appropriate to neglect vector meson masses in kinematical calculations or any non-trivial energy dependence of the amplitude for photo-production of vector mesons from the nucleon. (We will discuss precisely what range of energy to which this corresponds for the $\phi$-meson in the next section.) Furthermore, for small photon energies ($\lesssim 3$ GeV) the VMD hypothesis be-
comes suspect as a description of the $\gamma N \rightarrow VN$ amplitude. Therefore, we will not restrict ourselves to the VMD model of $\gamma N \rightarrow VN$ amplitude, considering instead the adequately parameterized form of the photon-nucleon amplitudes. We will argue that there may be a range of photon energies for which the eikonal approximation is valid, but where the usual Glauber theory assumptions of factorization and ultra-relativistic kinematics break down.

Although we retain the eikonal approximation, our approach is distinctly different from the usual Glauber-VMD approach. In particular, one of the basic assumptions used in the Glauber approach is that the basic $\gamma N \rightarrow VN$ and $VN \rightarrow VN$ cross sections are slowly varying functions of center of mass energy and that the small Fermi momentum of the nucleons can be neglected in the evaluation of the total center of mass energy of the $\gamma N$ and $VN$ systems. These assumptions result in the usual factorizability already discussed above. At intermediate energies, however, the photon energy is comparable to the vector meson mass, and the basic amplitude may gain non-trivial energy dependence due to the fact that Regge theory may be inadequate at intermediate photon energies. The usual smooth, slow rise in the total $\gamma N \rightarrow VN$ cross section characteristic of high energy diffractive scattering may be absent at intermediate energies. Fermi motion effects thereby destroy the factorizability of nuclear scattering into basic amplitudes and form factors. Also, the longitudinal momentum transferred (proportional to $M_V^2/E_\gamma$) plays an important role relative to reactions in the diffractive regime and further calls into question this factorization assumption. Earlier work (e.g. [8]) has considered the effect of longitudinal momentum transfers, but the breakdown of factorization has not been discussed.

To summarize, the particular reaction in which we are interested is the coherent photoproduction of vector mesons from the deuteron. However, the energy dependence of the $\gamma N \rightarrow VN$ will require that we account for Fermi motion effects which, in turn, will require that we account for non-factorization effects. In the derivation of the total $\gamma D \rightarrow VD$ amplitude we will use the eikonal approximation with effective Feynman diagram rules (see, e.g. [7, 86, 87, 88]). This is the approximation, valid at appropriately high energies, that allows us to derive the scattering amplitudes starting with corresponding effective Feynman diagrams while neglecting multiple scattering from the same nucleon. This is similar to the Generalized Eikonal Approximation (GEA) that has been applied to the $A(e,e'p)X$ reactions on the nucleus [88].

By maintaining the result in terms of momentum space integrals, within the eikonal approximation, transferred longitudinal momentum and Fermi motion effects may be explicitly and consistently taken into account. In our derivations, we keep only the corrections to the basic amplitudes that are of linear order in longitudinally exchanged momentum or nucleon momentum (neglecting order $k_N^2/m_N^2$ corrections, where $k_N$ is the bound state nucleon momentum). This allows us to relate the $D \rightarrow NN$ transition vertex to the nonrelativistic wavefunction of the deuteron. Since dynamical, model-dependent corrections related to the
N-N interaction are expected to be of quadratic or higher order in nucleon momentum, then linear order corrections arising from intermediate energy kinematics should be taken into account before any specific theory of the basic bound state amplitude that deviates from the nearly flat behavior of Regge theory is considered and used in the typical Glauber theory approach.

As it was explained above, we work in the kinematic regime in which diffractive behavior is not yet fully established but the momenta of the produced vector mesons are high enough that the eikonal approximation for the hadronic rescatterings is justified. As a result, a formalism should be maintained that allows the $\gamma N \rightarrow VN$ and the $VN \rightarrow VN$ amplitudes to be independently modeled. Fitting data to our modified form of the Glauber theory by using the $VN \rightarrow VN$ amplitude as a parameter allows one to infer a value for the $VN \rightarrow VN$ cross section. We emphasize that the main steps in sections 5.2 through 5.2.3 have been known for several decades; the Glauber theory in terms of effective Feynman diagrams was established in Ref. [7]. The effects of longitudinal momentum transfer in terms of phase shifts have also been studied [8]. However, as far as we are aware, there has never been a direct numerical study of the effect of the breakdown of factorization in Glauber theory as it applies to vector meson production. (The effects of factorization breakdown due to the electromagnetic current for multiple scattering in proton knock-out have been studied in Ref. [89].) We will also demonstrate that the breakdown of factorization persists even in the limit that off-shell effects in the bound state nucleon amplitudes are negligible.

### 5.2 Formulae for the Amplitudes

#### 5.2.1 Reaction and Kinematics

We study the coherent photo-production of vector mesons off the deuteron in the reaction:

$$\gamma + D \rightarrow V + D',$$  \hspace{1cm} (5.1)

where $P \equiv (M_D, 0)$ and $P' \equiv (E_D, P')$ define the initial and final four momenta of the deuteron. $q \equiv (E_\gamma, q)$ and $P_V \equiv (E_V, P_V)$ define the 4-momenta of the initial photon and the final state meson respectively. The three-momentum transferred in the reaction is defined as $l = q - P_V$.

In our calculations we concentrate on intermediate energy kinematics in which, although the photon energies are not high enough for the diffractive regime to be established for the photoproduction amplitude, the produced vector meson is sufficiently energetic that the eikonal approximation can be applied to the calculation of final state hadronic rescatterings. This require further elaboration: The eikonal approximation is the “straight line”
Figure 5.1: 3-momentum of a vector meson produced by a photon scattering off a nucleon target as a function of $-t$ for a given set of fixed $E_\gamma$. The solid lines correspond to $\phi$-meson production, whereas the dotted lines correspond to $\rho^0$-meson production. The incident photon energies in each case, going from the bottom curve to the top curve are 1.6, 1.8, 2.0, 2.2, 2.4 GeV. Details are discussed in the text.

approximation in that the incident particle follows a nearly straight line path through the nucleus. Clearly this must occur at high enough energies that higher partial waves than just the $s$-wave contribute. In order to establish the appropriate kinematical regime for our approach, we have plotted in Fig. 5.1 the lab frame 3-momentum of the final state vector meson as a function of $-t$ for a set of incident photon energies for the case of $\rho^0$ and $\phi$-meson production from a nucleon. Experience with the application of the Glauber model to the description of proton-nucleus scattering [90] as well as $A(e,e'p)X$ reactions [91] indicates that the eikonal approximation works roughly for $p_N/m_N \gtrsim 1$ ($p_N$ is the proton 3-momentum) and it works very well for $E_N \gtrsim 2m_N$. Since $m_\phi \approx m_N$, we expect to see the onset of the applicability of the eikonal approximation for a similar range of momenta for the case of $\phi$-meson production. By analogy with the proton case, we continue to use the criterion that $E_\gamma \gtrsim 2M_V$, and we find that the value of vector meson 3-momentum above which the eikonal approximation may certainly be applied is $P_V \gtrsim 1.8$ GeV for $\phi$-meson production. The values of 3-momentum, 1 GeV and 1.8 GeV have been indicated by horizontal dashed lines in Fig. 5.1. These dashed lines may be viewed as separating kinematic configurations where our approach may be applied to $\phi$-meson production from kinematic regions where both the current approach and the standard Glauber approach
should be abandoned entirely with regard to $\phi$-meson production. Below $P_V \approx 1$ GeV, both the current approach and the usual Glauber approach should be abandoned. Between 1 GeV and 1.8 GeV, the eikonal approximation may become rough, but above 1.8 GeV, the approach that we use (with the eikonal approach) is a very good approximation. For the $\rho^0$-meson, the eikonal regime begins at smaller values of momentum than for the $\phi$-meson due to its smaller mass, so to avoid confusion we do not include the corresponding range of applicability of the eikonal approach to $\rho^0$-meson production in Fig. 5.1.

When we discuss applications, we will consider the production of the $\phi$-meson at around 3 GeV, so the application of the eikonal approximation is quite safe. We will find that another problem arises at $t \approx t_{\text{min}}$, and this will be discussed in Sec. 5.4, but the above argument remains applicable as long as $-t$ is more than a few tens of MeVs larger than $-t_{\text{min}}$. We further assume that the non-relativistic model of the $N-N$ interaction can be represented by a $D \rightarrow NN$ vertex. In the case of the deuteron, there are only two relevant diagrams: the single scattering diagram (Born term) of Fig. 5.2 and the double scattering diagram of Fig. 5.3.

Figure 5.2: The impulse diagram for photo-production. The cross on the spectator nucleon line indicates that the spectator nucleon will be taken on shell in the non-relativistic approximation. (For all Feynman graphs we use Jaxodraw [92].)
5.2.2 The Born Amplitude

We start with the calculation of the amplitude corresponding to the Born term of Fig. 5.2. $F^0_{m,m'}(s,t)$ will denote the $\gamma D \rightarrow VD$ scattering amplitude for the Born term in which only one of the nucleons takes part in the interaction, whereas $\hat{F}(\hat{s},\hat{t})$ will denote the basic $\gamma N \rightarrow VN$ scattering amplitude. A hat on a variable indicates that it is associated with the $\gamma N \rightarrow VN$ subprocess rather than the process of Eq. (5.1). The superscript, 0, is meant to distinguish the Born term from the double scattering term. The initial and final polarizations of the deuteron are denoted by $m$ and $m'$ respectively. Because we consider only intermediate energies, the $\hat{F}(\hat{s},\hat{t})$ amplitude is not necessarily diffractive and we do not assume the validity of the VMD hypothesis. We neglect the spin-flip component of the basic amplitude (i.e. $\hat{F}$ and $\bar{F}$ are approximately diagonal in nucleon spin.) The $D \rightarrow NN$ vertex is denoted by $\Gamma_m$. All variables correspond to the labels in the Feynman diagram of Fig. 5.2 for the single scattering (Born) term. The free nucleon mass is denoted by $m_N$.

By applying effective Feynman rules to the graph in Fig. 5.2 we obtain the covariant
scattering amplitude,
\[
F^0_{m,m'}(s,t) = -\int \frac{d^4k}{i(2\pi)^4} \frac{\Gamma_{m'}^\dagger(P - k + l)\hat{F}(\hat{s},t)\Gamma_m(P - k)}{[(P - k + l)^2 - m_N^2 + i\epsilon] \left[(P - k)^2 - m_N^2 + i\epsilon\right] [k^2 - m_N^2 + i\epsilon]}
\]
\[+ (p \leftrightarrow n).\]

\[
(p \leftrightarrow n) \text{ refers to the term in which the neutron and proton are inverted. In the remainder}
\]
\[\text{of this text, Mandelstam variables that appear within an integral are understood to be}
\]
\[\text{functions of internal nucleon 4-momentum and the incident photon 4-momentum.}
\]

We proceed with the derivation by estimating the loop integral in Eq. (5.2) up to terms
\[\text{of order } \frac{k^2}{m_N^2}.\] This approximation allows us to evaluate the integral in Eq. (5.2) by keeping
\[\text{only the pole contribution which yields a positive energy for the spectator nucleon (see}
\]
\[\text{appendix C). We find,}
\]
\[
F^0_{m,m'}(s,t) = \int \frac{d^3k}{(2\pi)^3} \frac{\Gamma_{m'}^\dagger(P - k + l)\hat{F}(\hat{s},t)\Gamma_m(P - k)}{2k_0 \left[(P - k + l)^2 - m_N^2 + i\epsilon\right] \left[(P - k)^2 - m_N^2 + i\epsilon\right] [k^2 - m_N^2 + i\epsilon]}
\]
\[+ (n \leftrightarrow p).\]

(Note that \(k_0 = m_N\) up to correction terms of order \(k^2/m_N^2.\)) We now make use of the
\[\text{correspondence between the non-relativistic wave function and the vertex function,}
\]
\[
\tilde{\Psi}_m(k_{rel}) = \frac{-\Gamma_m(P - k)}{2\sqrt{k_0(2\pi)^3}D(P - k)}. \tag{5.4}
\]

The momentum dependence in Eq. (5.4) follows from the definition of the \(D \to NN\) vertex
\[\text{function and the over-all factors are established by assuming that the wavefunction is nor-}
\]
\[\text{malized to unity. Here, } -D(P - k) \text{ is the propagator denominator of the struck nucleon. We}
\]
\[\text{write } k_{rel} \text{ to indicate that the argument of the wave function is the relative 3-momentum}
\]
\[\text{of the two nucleons. Using Eq. (5.3) with Eq. (5.4) and using lab frame kinematics yields,}
\]
\[
F^0_{m,m'}(E_\gamma, l) = 2 \int d^3k \tilde{\Psi}_{m'}^\dagger(k - 1/2)\hat{F}(E_\gamma, k, l)\tilde{\Psi}_m(k) + (n \leftrightarrow p). \tag{5.5}
\]

We stress, at this point, that Eq. (5.5) does \textbf{not} yet coincide with the conventional VMD-
\[\text{Glauber theory because we have abandoned the usual assumptions that allow us to ignore}
\]
\[\text{the } k \text{-dependence in the basic amplitude, which would normally allow us to factor the basic}
\]
\[\text{amplitude out of the integral and leave us with the product of the basic amplitude with}
\]
\[\text{the non-relativistic form factor. For heavier vector mesons (like the } \phi \text{-meson), the vector}
\]
\[\text{meson mass may not be negligible, and the } \hat{s} \text{-dependence of the basic amplitude becomes}
\]
\[\text{non-trivial at intermediate photon energies.}
5.2.3 The Double Scattering Amplitude

Having obtained the Born term in Eq. (5.3), we move on to calculate the double scattering term of Fig. 5.3. Applying effective Feynman diagrammatic rules, we obtain

\[
F_{m,m'}^{1}(s,t) = - \int \frac{d^4 p_s}{i(2\pi)^4} \frac{d^4 p_s'}{i(2\pi)^4} \frac{\Gamma_{m'}^\dagger (P + l - p_s') \tilde{F}(\hat{s}, \hat{t}) \hat{F}(\hat{s}, \hat{t}) \Gamma_{m}(P - p_s)}{[p_s^2 - m_N^2 + ie] [p_s'^2 - m_N^2 + ie] [(P - p_s)^2 - m_N^2 + ie]} \times \left[ \frac{1}{[(P + l - p_s')^2 - m_N^2 + ie]} \right] \left[ \frac{1}{[(q - l + p_s' - p_s)^2 - M_V^2 + ie]} \right] \]  

(5.6)

Figure 5.3 and Eq. (5.6) express the following sequence of events: The incident photon scatters from a nucleon with center of mass energy, \( \sqrt{\tilde{s}} \), producing an intermediate state with invariant mass, \( M_V \). The intermediate state propagates through the deuteron before scattering from the other nucleon with center of mass energy \( \bar{s} \). (Bars over variables will indicate that they correspond to the secondary scattering.) We neglect fluctuations of the intermediate state for the present purposes. Now let us integrate over, \( p_s, 0 \) and \( p_s', 0 \). The integration over \( p_s, 0 \) is similar to the integration over \( k_0 \) for the Born term of Eq. (5.2). For the \( p_s', 0 \) integration one can choose one of the positive energy poles at \( M_D + l_0 - \sqrt{m_N^2 + (1 - p_s')^2 + ie} \) and \( \sqrt{m_N^2 + p_s'^2 - ie} \) at the upper and lower complex semiplane of \( p_s' \). Note that within the approximation in which \( p_s'^2/m_N^2, l^2/m_N^2 \) terms are consistently neglected, the integration over either pole will yield the same result. We choose the \( \sqrt{m_N^2 + p_s'^2 - ie} \) pole (the poles chosen for the integration are identified by the crosses shown in Fig. 5.3) because this choice reproduces the usual Glauber formula in the most direct way. Applying the definition in Eq. (5.4), we recover the formula quoted in [21],

\[
F_{m,m'}^{1}(E_\gamma, l) = - \int \frac{d^3 p_s'}{(2\pi)^3} \frac{d^3 p_s}{(2\pi)^3} \frac{\tilde{\Psi}_m^\dagger \left( \frac{1}{2} - p_s' \right) \tilde{F}(\hat{s}, \hat{t}) \hat{F}(\hat{s}, \hat{t}) \tilde{\Psi}_m(-p_s)}{\sqrt{p_s,0p_s',0} \left[ [(q - l + p_s' - p_s)^2 - M_V^2 + ie] \right]} .
\]

(5.7)

The \( (p \rightarrow n) \) term is implicit in these equations. Finally, we put this equation into a form that makes the next section slightly more manageable by transforming the variables of integration from \( p_s' \) and \( p_s \) to \( p \equiv (p_s + p_s')/2 \) and \( k \equiv p_s' - p_s \), and we make the redefinitions,
$k \rightarrow k + l/2$, and $p \rightarrow p + \frac{l}{4}$. The result of these changes is:

$$F_{m,m'}^1(E_\gamma, l)$$

$$= -\int \frac{d^3 p d^3 k}{(2\pi)^3} \frac{\tilde{\Psi}^\dagger_{m'}(p + \frac{k}{2}) \tilde{F}(\hat{s}, \hat{t}) \tilde{\Psi}_m(p - \frac{k}{2})}{m_N \left[ (q + k - \frac{l}{2})^2 - M_V^2 + i\epsilon \right]} \tag{5.8}$$

$$+ (p \leftrightarrow n).$$

In Eq. (5.8), we have given the amplitude a superscript, 1, to distinguish it from the Born term.

We will summarize this section by cleaning up our notation and by writing out the correct expressions for the kinematic variables in terms of the integration variables, taking into account the variable transformations that were needed to get Eqs. (5.5) and (5.8). We explicitly expand each expression to linear order in nucleon momentum in the lab frame. Furthermore, we assume that nucleon 3-momentum and the exchanged 3-momentum are both small and are of the same order of magnitude relative to all masses involved. Subscripts $a$ denote Born amplitude quantities while subscripts $b$ denote double scattering quantities. The variables in each expression are established in the particular diagram under consideration. First, we have,

$$\hat{s}_a = ((P - k) + q)^2$$

$$= m_N^2 + 2E_\gamma m_N + 2E_\gamma k_z + \mathcal{O}(k^2). \tag{5.9}$$

Recalling the variable transformations we made in the double scattering term and noting that $p$, $k$ and $l$ are all of the same order of magnitude, we have,

$$\hat{s}_b = (q + P - ps)^2$$

$$= m_N^2 + 2E_\gamma m_N + 2E_\gamma \left( p_z - \frac{k_z}{2} \right) + \mathcal{O}(p^2). \tag{5.10}$$

Note that there is only dependence upon $k_z$ and that $k$ contributions come into play only at higher order in nucleon momentum. For the rescattering amplitude, we get,

$$\bar{s}_b = (k_V + ps)^2$$

$$= M_V^2 + m_N^2 + 2E_\gamma m_N - 2E_\gamma \left( p_z - \frac{k_z}{2} \right) + \mathcal{O}(p^2). \tag{5.11}$$

This last expression is obtained after the pole in $k_z$ is taken, giving the intermediate state
an invariant mass of $k_V^2 = M_V^2$. The values of $t$ to be used in each of these cases are,

$$\hat{t}_a = t$$  \hfill (5.12)

$$\hat{t}_b = \left( \frac{l}{2} - k \right)^2 = \left( \frac{l_0}{2} \right)^2 + \frac{l_z k_z}{2} - \left( \frac{l_\perp}{2} - k_\perp \right)^2 + O(k^2)$$  \hfill (5.13)

$$\tilde{t}_b = \left( \frac{l}{2} + k \right)^2 = \left( \frac{l_0}{2} \right)^2 - \frac{l_z k_z}{2} - \left( \frac{l_\perp}{2} + k_\perp \right)^2 + O(k^2).$$  \hfill (5.14)

In the usual VMD-Glauber theory expression for the double scattering term, one keeps only the perpendicular components of momentum in $\hat{t}$ and $\tilde{t}$. The terms proportional to $k_z$ are small and since they come with opposite sign, they tend to cancel if the $t$-dependence of the basic amplitude is nearly exponential. The terms with $l_0^2$ are proportional to $t^2/M_D^4$.

Thus, we continue to neglect both of the first two terms in Eqs. (5.13) and (5.14). Finally, we stress that $(P - k)^2 = (M_D - m_N)^2 + O(k^2)$ so that the struck nucleon may be treated kinematically as being on shell up to terms quadratic in the nucleon momentum.

By using the kinematic expressions of Eqs. (5.9) through (5.14) in Eqs. (5.5) and (5.8), we may ensure that the factors multiplying the deuteron wavefunction in each of the integrals are correct to linear order in nucleon 3-momentum (or exchanged 3-momentum).

5.3 Numerical Estimates and the Relationship with VMD-Glauber Theory

5.3.1 Differential Cross Section

Now that we have calculated the Born and double scattering amplitudes, let us set up notation that allows us to express the total differential cross section in terms of the basic amplitudes for $\gamma N$ and $VN$ scattering. We do not discuss any physics in this section, but simply formulate our notation to allow for convenient comparisons between the present approach and the standard Glauber-VMD approach.

For any exclusive two body reaction involving incoming particles of mass $m_1$ and $m_2$ and center of mass energy squared, $s$, the differential cross section may be represented as follows:

$$\frac{d\sigma^{m,m'}}{dt} = \frac{1}{16\pi\Phi(s,m_1,m_2)}|F_{m,m'}(s,t)|^2,$$  \hfill (5.15)

where,

$$\Phi(s,m_1,m_2) \equiv ((s - m_1^2)^2 + m_2^4 - 2sm_2^2 - 2m_1^2m_2^2).$$  \hfill (5.16)
In particular, the differential cross section for the reaction in Eq. (5.1) is,

$$\frac{d\sigma^{m,m'}}{dt} = \frac{1}{16\pi\Phi(s,0,m_N)} |F^0_{m,m'}(s,t) + F^1_{m,m'}(s,t)|^2. \quad (5.17)$$

It follows from Eqs. (5.2) and (5.8) that the numerical calculations of Eq. (5.17) will require as input the amplitudes for both the $\gamma N \rightarrow VN$ and the $VN \rightarrow VN$ interactions.

To proceed, we construct a parameterization of the photo-production differential cross section in a form that will provide a smooth transition to the VMD-Glauber regime by writing,

$$d\hat{\sigma}^\gamma_N \rightarrow VN(\hat{s},\hat{t}) = \hat{n}_0^2 \frac{1}{16\pi} \left( \frac{\hat{s}}{\hat{s}_0} \right)^{2(\hat{\alpha}(\hat{t})-1)} \hat{f}^2(\hat{t})\hat{g}^2(\hat{s},\hat{t})(i + \hat{\eta}), \quad (5.18)$$

for the basic $\gamma N \rightarrow VN$ interaction. In the high energy photon limit, the function $\hat{f}(\hat{t})$ will reduce, by construction, to the usual exponential dependence, $e^{\hat{B}\hat{t}/2}$, with the constant $\hat{B}$ that is typically used to parameterize experimental data as in, for example, Ref. [19]. The Regge trajectory is $\hat{\alpha}(t) = \hat{\alpha}'t + \hat{\alpha}_0$. The factor of $(\frac{\hat{s}}{\hat{s}_0})^{\hat{\alpha}(\hat{t})-1}$ is the Regge parameterization obtained in the VMD-Glauber regime and $\hat{g}(\hat{s},\hat{t})$ is a function which adjusts for other $s$ and $t$ dependence that may appear in the intermediate energy regime, but such that $\hat{g}(\hat{s},0)(\frac{s}{s_0})^{\hat{\alpha}(0)-1}$ reduces to 1 in the high energy photon limit. By substituting Eq. (5.18) into Eq. (5.15), we obtain,

$$\hat{F}^\gamma N \rightarrow VN(\hat{s},\hat{t}) = \hat{n}_0(s - m_N^2) \left( \frac{\hat{s}}{\hat{s}_0} \right)^{\hat{\alpha}(\hat{t})-1} \hat{f}(\hat{t})\hat{g}(\hat{s},\hat{t})(i + \hat{\eta}). \quad (5.19)$$

The overall normalization is labeled $\hat{n}_0$ and is not necessarily related to a total cross section as it would be in the VMD model. The variable, $\hat{\eta}$, is a possible real contribution to the amplitude. Because $P_V \gtrsim 1$ GeV for the kinematic regime under consideration (see Sec. 5.2.1), the parameterization we use for the $VN \rightarrow VN$ simply takes a nearly diffractive form,

$$\hat{F}^{VN \rightarrow VN}(\hat{s},\hat{t}) = \sigma_{VN}(\hat{s})(i + \hat{\eta})\sqrt{\Phi(\hat{s},m_N,M_V)}\hat{f}(\hat{s},\hat{t}). \quad (5.20)$$

The function, $\hat{f}(\hat{s},\hat{t})$ reduces by construction to a Regge parameterization, $(\frac{s}{s_0})^{\hat{\alpha}(\hat{t})-\hat{\alpha}(0)}e^{\hat{B}\hat{t}/2}$ in the VMD regime. By applying the optical theorem to Eq. (5.20), we see that $\sigma_{VN}(\hat{s})$ is, indeed, the total $VN$ cross section. The variable, $\hat{\eta}$ is a possible real part of the amplitude.

Our peculiar choice of notation is made so that we may smoothly recover the usual Regge parameterizations when we consider the VMD-Glauber approximation. Indeed, applying the VMD hypothesis in the appropriate kinematical regime allows us to assume that $\hat{F}(\hat{s},\hat{t}) \propto \hat{F}(\hat{s},\hat{t})$. Thus, applying the optical theorem would allow one to deduce the
VN → VN amplitude. With the standard high energy approximations, we have,

\[
\hat{F}(\hat{s}, \hat{t}) \xrightarrow{E_\gamma >> M_V} \hat{s}n_0(i + \hat{\eta}) s\hat{s}/i e^{\hat{B}t/2} \\
\hat{F}(\hat{s}, \hat{t}) \xrightarrow{E_\gamma >> M_V} \hat{s}\sigma_{VN}(i + \hat{\eta}) s\hat{s}/e^{\hat{B}t/2}.
\] (5.21)

Here, we have put \(\hat{s}_0 = 1\) GeV for convenience as is often done in parameterizations. In this way, we show how our parameterizations reduce smoothly to the expressions obtained within Regge theory and the VMD hypothesis.

One may fit all of the functions that define the expression for \(\hat{F}(\hat{s}, t)\) directly to data for \(d\sigma^{\gamma N \rightarrow VN}/dt\). The function, \(\hat{g}(\hat{s}, \hat{t})\) has been introduced to account for peaks in the energy dependence or other irregular energy dependence at intermediate energies. Without the VMD hypothesis, we can assume no relationship between \(\hat{F}(\hat{s}, \hat{t})\) and \(\bar{F}(\bar{s}, \bar{t})\). At intermediate energies, therefore, \(\bar{F}(\bar{s}, \bar{t})\) must be obtained from a theoretical model or by other experimental means. Conversely, one can use data for the reaction in Eq. (5.1) to extract \(\bar{F}(\bar{s}, \bar{t})\).

### 5.3.2 Corrections to Factorizability and an Effective Form Factor

We now define an effective form factor,

\[
S_{eff}^{m,m'}(E_\gamma, \frac{1}{2}) \equiv \int \frac{d^3k(\hat{s}_a - m_N^2)}{2E_\gamma m_N} \left(\frac{\hat{s}_a}{2E_\gamma m_N}\right)^{\alpha(t)-1} \hat{g}(\hat{s}, \hat{t}) \bar{\Psi}_{m'}^\dagger \left(k - \frac{1}{2}\right) \Psi_m (k),
\] (5.22)

for the Born term, and an effective basic amplitude,

\[
\hat{F}_{eff}(E_\gamma, t) \equiv 2E_\gamma m_N \hat{n}_0 (i + \hat{\eta}) \left(\frac{2E_\gamma m_N}{\hat{s}_0}\right)^{\alpha(t)-1} f(t).
\] (5.23)

If we substitute Eq. (5.19) into Eq. (5.5), then the Born amplitude for production from the deuteron is,

\[
F_{m,m'}^0(E_\gamma, l) = 2\hat{F}_{eff}(E_\gamma, l) S_{eff,a}^{m,m'}(E_\gamma, l/2) + (n \leftrightarrow p).
\] (5.24)

The definition in Eq. (5.23) takes the form of a general diffractive parameterization obtained when one makes the VMD hypothesis. However, Eq. (5.24) is exactly correct without any approximations. We have recovered the structure of the Born expression - the product of a diffractive basic amplitude with a form factor. The new feature in Eq. (5.24) is that our effective form factor depends on the energy of the photon. The definitions that we made in Eqs. (5.22) and (5.23) ensure that the effective form factor and the effective diffractive amplitude reduce to the usual non-relativistic form factor and the true diffractive basic
amplitude in the limit that \( E_γ \gg M_V \):

\[
S_{eff}^{m,m'}(E_γ,1/2) \xrightarrow{E_γ \gg M_V} S^{m,m'}(1/2),
\]

\[
\hat{F}_{eff}^{0}(E_γ,l) \xrightarrow{E_γ \gg M_V} \hat{F}_{VN \to VN}(s,t).
\] (5.25)

By following the usual methods of VMD-Glauber theory, one will extract the effective amplitude from the \( \gamma D \to V D \) cross section rather than the true amplitude. If, in the region of very small \(-t\) where the Born cross section dominates, the amplitude for the \( \gamma N \to VN \) scattering is inferred from data using the usual VMD-Glauber theory, then Eq. 5.24 can be used to obtain a corrected amplitude that accounts for non-factorizability.

### 5.3.3 Corrections to Factorizability in Double Scattering

The double scattering term is more complicated due to the fact that, in Eq. (5.8), the energy dependence cannot easily be factorized out of the integrand. We may rewrite Eq. (5.8) using Eqs. (5.19) and (5.20) as,

\[
F_{m,m'}^{1}(E_γ,t) = - \int dk_z \int d^2k_\perp \int \frac{d^3p}{(2\pi)^3} \frac{\hat{f}(\hat{t}_b)\hat{f}(\hat{s}_b,\hat{t}_b)\hat{\Psi}_{m'}^\dagger(p + \frac{k}{2})\hat{\Psi}_m(p - \frac{k}{2})}{m_N \left[ (q + k - \frac{1}{2})^2 - M_V^2 + i\epsilon \right]} \times \left[ \left( \frac{\hat{s}}{\hat{s}_0} \right)^\dagger(\hat{t}) - 1 \right] \frac{\sqrt{\Phi(\hat{s},m_N,0)\Phi(\hat{s},m_N,M_V)}}{\hat{g}(\hat{s},\hat{t})\hat{n}_0\sigma_{VN}(\hat{s})(i + \hat{n})(i + \hat{n})}.
\] (5.26)

The nonfactorizability of Eq. (5.26) near threshold comes from the fact that the basic amplitudes and the factors in braces have non-trivial dependence upon the integration variables. We determine that there is no simple reformulation of the integral in Eq. (5.26) which consistently accounts for corrections linear in momentum. Therefore, we conclude that a direct numerical evaluation is necessary. Note that, although we have set up the integral for a specific parameterization, the analysis applies to any smooth, slowly varying energy dependent basic amplitude. The \( k_z \) integral is determined by expanding the denominator in Eq. (5.26),

\[
\left( q + k - \frac{1}{2} \right)^2 - M_V^2 + i\epsilon \approx 2E_γ \left[ -k_z + \frac{k_z}{2} - \frac{M_V^2}{2E_γ} + (k - \frac{1}{2})_0 + i\epsilon \right] = 2E_γ \left[ -k_z - \Delta + i\epsilon \right].
\] (5.27)

The second line fixes the definition of \( \Delta \). Notice that by ignoring the term \( (k - \frac{1}{2})^2/2E_γ \) we have ignored the possibility of contributions from intermediate mesons which are far off shell and which correspond to nucleon 3-momenta that are strongly suppressed by the deuteron.
wavefunction. Furthermore, the pole value of $k_z$ in this approximation only depends on the external variables and is independent of the transverse motion of the nucleons. The resulting double scattering amplitude is then,

$$F_{1m,m'}(E_\gamma, t) = \int d^2 k_\perp \int \frac{d^3 p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \int \frac{d^3 \bar{p}}{(2\pi)^3} \tilde{f}(\bar{s}_b, \bar{t}_b) \bar{f}_{m'}^\dagger(p + \frac{k}{2}) \bar{f}_m(p - \frac{k}{2})$$

$$\times \left( \frac{\hat{s}}{\hat{s}_0} \right)^{\hat{A}(\hat{t})^{-1}} \sqrt{\phi(\hat{s}, m_N, 0) \phi(\hat{s}, M_V, M_V)} \hat{g}(\hat{s}, \hat{t}) \hat{n}_0 \sigma_{V_N}(\hat{s}_b) (i + \hat{\eta}) (i + \bar{\eta})$$

$$= \int d^2 k_\perp \int \frac{d^3 p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{I_{m,m'}(k_\perp, k_z, p, s)}{2E_\gamma m_N [k_z + \Delta - i\epsilon]}.$$  (5.28)

We have gathered all factors apart from the energy denominators in the integrand into a function, $I_{m,m'}(k_\perp, k_z, p, s)$. Assuming identical protons and neutrons, we get an identical term for the case where the roles of the neutron and proton are inverted. A convenient way to reorganize this formula so that it more closely resembles the standard non-relativistic quantum mechanical theory is to write the integrand in terms of its Fourier components in the following mixed representation:

$$I_{m,m'}(k_\perp, k_z, p, s) = 1 \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dz \tilde{I}_{m,m'}(k_\perp, z, p, E_\gamma) e^{-ik_zz}.$$  (5.29)

The vector meson propagator may be rewritten using the identity,

$$\frac{1}{p - i\epsilon} = \int_{-\infty}^{\infty} dz \Theta(-z) e^{i(p - i\epsilon)z}.$$  (5.30)

Summing the two terms for the neutron and the proton and using the fact that $\Theta(z) + \Theta(-z) = 1$ yields,

$$F_{1m,m'}(E_\gamma, t) = i \int \frac{d^3 p d^3 k_\perp}{2E_\gamma m_N (2\pi)^2} I_{m,m'}(k_\perp, -\Delta, p, E_\gamma)$$

$$- \frac{1}{\sqrt{2\pi}} \int \frac{d^3 p d^3 k_\perp}{2E_\gamma m_N (2\pi)^2} \int_{-\infty}^{\infty} dz \tilde{I}_{m,m'}(k_\perp, z, p, s) \sin(-\Delta z) \Theta(z).$$  (5.31)

In the VMD-Glauber approximation, $\Delta \to 0$ and $I_{m,m'}(k_\perp, 0, p, s)$ is the usual energy independent density matrix. Hence, the first term in Eq. (5.31) reduces to the traditional Glauber expression for double scattering and the second term vanishes in the limit where the usual VMD-Glauber assumptions are applicable. The second term is a correction discussed in Ref. [21] which arises from the non-zero phase shift in the vector meson wave function induced by longitudinal momentum transfer. In the phase shift term, the factor of $\sin(-\Delta z)$ is itself a correction of order $k_z$, so we neglect Fermi motion and energy dependent corrections to $I_{m,m'}(k_\perp, 0, p, s)$ in the phase shift term.
For a real photon, the double scattering term picks out the relative longitudinal nucleon momentum,

\[ \Delta = \frac{\ell_-}{2} + \frac{M_V^2}{2E_\gamma}. \]  

(5.32)

Furthermore, \( \ell_- = -\left( M_V^2 - t \right) / 2E_\gamma \), so

\[ \Delta = \frac{t + M_V^2}{4E_\gamma}, \]  

(5.33)

and we see that \( \Delta \) is indeed negligible at large center of mass energies and small \( t \). Corrections to the double scattering term, at linear order in momentum, arise from performing the integral in the first term of Eq. (5.31) numerically, and by retaining the phase shift term.

We end this section by noting that the breakdown in factorization comes simultaneously from the fact that longitudinal momentum transfer is non-negligible, and the fact that the longitudinal momentum of the bound nucleons is non-negligible; the contribution to the basic amplitudes from the longitudinal component of the bound nucleon momentum at linear order would vanish by symmetry in all of the integrals if the longitudinal momentum transfer were neglected in the wavefunctions.

5.4 Sample Calculations

5.4.1 Cross Section Calculation

It is usually the case that one calculates the charge and quadrapole form factors in the coordinate space formulation of the form factor. This method reduces the system of formulae to an extremely simple form and allows one to deal simply and directly with polarizations. For the purpose of modifying the basic amplitude, however, so that it has nucleon momentum dependence, we must maintain the momentum space formulation that results from a direct evaluation of the effective Feynman diagrams in Figs. (5.2, 4.1). Carrying this procedure out was the topic of the previous two sections. The calculation is straightforward, but becomes numerically cumbersome, and the longitudinal momentum exchanged leads to a breakdown of the orthogonality relations for spherical harmonics that usually lead to a very simple coordinate space formulation. However, dealing with the deuteron polarizations can still be simplified if one chooses the axis of quantization along the direction of momentum exchange [93, 6]. An overview of the non-relativistic deuteron wave function with polarizations are described in Appendix D.1.

In this section we provide some sample calculations by using simple models of the basic amplitudes. For this purpose, we restore the assumption of VMD and we use very simple parameterizations of the \( s \) and \( t \) dependence in the basic amplitudes. The purpose for do-
ing this is mainly to provide estimates of the sensitivity to non-factorizability rather than because VMD is thought to be appropriate at intermediate energies. We have extracted estimates of the parameters for production of the $\rho^0$ and $\phi$ vector mesons from the basic nucleon interaction cross section data appearing in Ref. [19], and we have made rough estimates of the parameterization of the $s$ and $t$-dependence of these amplitudes (see Sec. B.2 for a description of our parameterizations). This provides us with a reasonable model to work with, though we stress that refinements are ultimately needed. For all of our calculations we use the non-relativistic wavefunction obtained from the Paris N-N potential [94].

We are mainly interested in the $\phi$-production cross section which is dominated by natural parity exchange, even at energies close to threshold, due to the OZI rule. However, to demonstrate the consistency of our approach with traditional methods, we consider first the case of the photo-production of $\rho^0$-mesons which has been well understood for some time. The basic amplitude for $\rho^0$-production is dominated by soft Pomeron exchange at large energies, so that it is constant at high energies, but undergoes a relatively steep rise at energies near threshold due to meson exchanges. The parameterization we use is shown in appendix B.2. We use a typical exponential slope factor of 7.0 GeV$^{-2}$ for the $t$-dependence. Fig. (5.4) shows the cross section for $\rho^0$-production at the high energy of $E_\gamma = 12.0$ GeV. For comparison, we show data taken at 12.0 GeV from Ref. [19, 95]. The comparison with data is reasonable, as it is with the traditional Glauber approach.

Now we consider the more interesting case of $\phi$-meson photo-production. At high energy, we use the Regge dependence, $\alpha(t) = .27t + 1.14$ given in Ref.[19]. The parameterization that we used is described further in appendix B.2. As noted in Ref. [19], the energy dependence of the $\phi$-meson photo-production cross section is very weak, but the current state of experimental data is still ambiguous as to how much this energy dependence continues at lower energies. However, the large negative ratio of the real to imaginary part of the amplitude ($\eta = -.48$) [96] suggests that some mechanism other than soft Pomeron exchange is significant. This value of the ratio of the real to imaginary part of the $\phi$-meson cross section has large error bars and was calculated neglecting longitudinal momentum transfer. However, it is the only measurement we know of presently so we use it for the purpose of demonstration. At lower photon energies than what we consider here, the energy dependence of the basic cross section may become highly non-trivial as is suggested by data in Ref. [97] (see discussion in chapter 6). The results of the calculation done with each combination of initial and final deuteron polarizations are shown in the separate panels for a photon energy of $E_\gamma = 30.0$ GeV in Fig. 5.5 and for a photon energy of $E_\gamma = 3.0$ GeV in Fig. 5.6. A comparison of the results for each combination of initial and final deuteron polarizations is summarized in Figs. 5.7 and 5.8 which show the differential cross section for different polarizations along with the unpolarized cross section for photon energies of 30.0 GeV and 3.0 GeV respectively.
Figure 5.4: The unpolarized differential cross section for coherent $\rho^0$-meson production compared with the total cross section for different polarizations. The calculation is done with the large photon energy $E_\gamma = 12$ GeV, and the data for $E_\gamma = 12$ GeV is taken from Ref. [19].
Each of the curves in Figs. 5.5 and 5.6 separately represent the contribution to the total cross section from a term in the squared amplitude when we apply Eq. (5.17). The Born and double scattering terms are obtained from the square of Eq. (5.24) and square of the first term of Eq. 5.31, respectively. The phase shift term arises from the second term in Eq. (5.31). (The phase shift curve plotted in Figs. 5.5 and 5.6 includes both the square modulus of the second term in Eq. (5.31) and its interference with other terms.) We call it the phase shift term because, in the language of non-relativistic quantum mechanical wave-functions, it arises due to a phase difference between the incoming photon and the produced vector meson. The interference term arises from the interference between Eq. (5.24) and Eq. (5.31). Note that the interference term is negative, but it is plotted on the positive axis for demonstration purposes. Note also that there is no contribution from the Born term for the $m = +/−1$ to $m = −/+1$ transition, and therefore the total cross section for the spin-flip reaction has none of the large dips characteristic of the Born cross section.

An important feature that can be seen in Figs. 5.5 and 5.6 is that the double scattering term is suppressed in the intermediate energy case relative to the high energy case. We can see this most clearly by comparing the upper left panel of Fig. 5.5 with the upper left panel of Fig. 5.6. It is clear that the double scattering contribution is important in the $E_γ = 30$ GeV case at moderate values of $−t$, whereas for the $E_γ = 3$ GeV case the cross section is dominated by the Born term all the way up to $−t ≈ 0.4$ GeV$^2$. In the general case of multiple scattering from complex nuclei, it is the rescattering contributions which lead to the usual A-dependence (A is the number of nucleons) of Glauber theory. The fact that multiple scattering is suppressed in double scattering in the deuteron suggests that our method would yield a rather different A-dependence from that of usual Glauber theory if it were extended to complex nuclei. Extending our approach to complex nuclei will be the subject of future work.

Another problem begins to emerge at lower photon energies and extremely small $−t$ (at $t ≈ t_{min}$): A large fraction of the momentum integrals begins to violate relativistic kinematic constraints. It is likely that the basic amplitudes vary extremely rapidly with $s$ and $−t$ in these regions of the integral and that expanding in nucleon momentum is not valid (at least to linear order). In order to make progress, a precise understanding of the dynamics of off-shell amplitudes based on field theory may be necessary. Therefore, our approximation is only valid at $−t$ sufficiently large that the integrand does not contain significant contributions from kinematically forbidden nucleon configurations. We have tested the effect of this region in our calculations, and in performing our calculation, we find that there is virtually no contribution from kinematically forbidden regions for any situation that we consider as long as $−t + t_{min}$ is greater than a few tens of MeVs. We note that, even at relatively low photon energies, the data is consistent with a smooth exponential $−t$-dependence (see Ref. [97]) as long as $−t$ is not exactly $−t_{min}$. Note that this theoretical
problem of considering \( t \) at exactly \(-t_{\text{min}}\) exists at high energies as well, but that at high energies \(-t_{\text{min}}\) is generally too small for it to show up in plots. So that we may perform our calculations numerically at all values of \(-t\) greater than \(-t_{\text{min}}\), we choose to make the basic amplitude vanish in kinematically forbidden configurations (when \(-t \leq -\hat{t}_{\text{min}}\)). This results in a small dip just above \(-t_{\text{min}}\) in our plots. The small dip is, therefore, unphysical, and should not be regarded as a prediction. We leave it in our plots merely to illustrate a general failure of the Glauber theory approach at extremely small \(-t\) (see Fig. 5.8 at \(-t \lesssim .06\, \text{GeV}^2\)). In summary of the above, the small dip at extremely small \(-t\) denotes a kinematic region in which no known multiple scattering formalism works. Numerically, our calculation is only correct in the region of \(-t\) above the dip at small \(-t\); that is, when \(-\hat{t} + \hat{t}_{\text{min}}\) is greater than a few tens of MeVs.
Figure 5.5: The long-dashed, dotted, dashed, solid, and dot-dashed lines refer to the Born, double, interference, total, and phase shift terms respectively for a photon energy of $E_\gamma = 30.0$ GeV. The interference term is negative but is plotted for illustration on the positive axis. See text for detailed discussion.
Figure 5.6: The long-dashed, dotted, dashed, solid, and dot-dashed lines refer to the Born, double, interference, total, and phase shift terms respectively for a photon energy of, $E_\gamma = 3.0$ GeV. Note the different scale on the axis for the spin-flip contribution. The interference term is negative but is plotted for illustration on the positive axis. See text for detailed discussion.
Figure 5.7: The differential cross section for $\phi$-meson production for different polarizations for a photon energy of $E_\gamma = 30.0$ GeV. See text for detailed discussion.
Figure 5.8: The differential cross section for $\phi$-meson production for different polarizations for a photon energy of $E_\gamma = 3.0$ GeV. The deuteron spin flip term is negligible at these energies. See text for detailed discussion.
Next we consider the total unpolarized cross sections as a function of photon energy for a set of fixed values of $-t$. This allows us to compare the factorized and unfactorized calculations directly and to determine at approximately what value of energy the transition to the VMD-Glauber regime occurs. Recall that it is the motion of the nucleons in the deuteron (the Fermi motion) that leads to the non-factorizability of the basic amplitudes. Factorization refers the practice of ignoring the dependence of nucleon momentum inside the basic amplitudes when integrals over nucleon momentum are performed. The use of non-factorized amplitudes is the essential difference between our approach and the usual Glauber approach. The ratio of the cross section with the usual factorization assumption to the cross section which accounts for non-factorizability effects (Fermi motion) is shown in Figs. 5.9. The upper panel refers to the case of $\rho^0$ production, whereas the lower panel refers to $\phi$-production. The ratio is given for two small values of $-t$: $t = -.04$ GeV$^2$ and $t = -.14$ GeV$^2$.

The upper panel demonstrates that the effect of non-factorizability is small for the case of $\rho^0$ for the entire range of intermediate energies. This is in contrast to the case of $\phi$ production in the lower panel of Fig. 5.9. Note that we only plot the case of $t = -.04$ GeV down to $E_\gamma = 5$ GeV for the $\phi$-meson case. This is because, for photon energies lower than 5 GeV, $t = -.04$ GeV becomes too close to $t_{\text{min}}$. On the other hand, for the curve corresponding to $t = -.14$ GeV, there is nearly a 30 percent suppression of the factorized cross section relative to the unfactorized cross section at the lowest energy, $E_\gamma = 3$ GeV, shown in the lower panel of Fig. 5.9. We emphasize that this result is for a photon energy (3 GeV) that is well into the kinematic region where the eikonal approximation may be applied (see Sec. 5.2.1), and that $-t = .14$ GeV$^2$ is certainly large enough relative to $-t_{\text{min}} = .036$ GeV$^2$ that there are none of the problems discussed earlier related to nearness to $-t_{\text{min}}$. Therefore, our method of calculation is ideally suited to the kinematics of the dotted curve in Fig. 5.9, where a significant effect from the break down of the factorizability assumption is already seen.

Note from the general behavior in Fig. 5.9 that the cross section rises when the factorization assumption is removed. This effect is mainly due to the suppression of multiple scattering when non-factorizability is taken into account. To see this, observe that at $-t = .14$ GeV$^2$ the main effect of double scattering in the usual Glauber approach is to produce a large negative cross term that has a canceling effect. All terms apart from the Born term and the interference term are negligible at this value of $t$. (See, for example, the upper left panels of Figs. 5.5 and 5.6.) Therefore, if multiple scattering is suppressed, as it is in our approach, then the absolute value of the cross term becomes smaller, and the Born contribution is no longer suppressed by multiple scattering. Thus, the curve representing our approach in Fig. 5.9 is smaller than what is found in the standard Glauber calculation.

At high energies we expect the two methods to agree, and they do within the range of
Figure 5.9: The energy dependence of the ratio between the differential cross section calculated using the usual factorization assumption and the differential cross section calculated with factorization break-down taken into account.
experimental uncertainties of non-relativistic deuteron form factors. The fact that the two methods have slight disagreement at high energies is a reflection of the fact that, even at high energies, we have not calculated the form factor with exactly the same approximation as in the usual Glauber approach. In the usual non-relativistic form factor, any dependence on longitudinal transferred momentum is ignored. If one takes into account exact kinematics, one finds that there are two distinct effects which may cause this assumption to be violated. It is easiest to see this by writing out the exact expression for the transferred longitudinal momentum:

\[ l_z = \frac{-t}{2MD} + \frac{M_V^2 - t}{2E_\gamma}. \] (5.34)

From this we see that there are two approximations that are normally made in the Glauber approach that allow one to neglect \( l_z \). The first is the ultra-relativistic approximation for the incident vector meson, \( E_\gamma >> M_V \), and the second is the non-relativistic approximation for the exchanged 4-momentum, \( -t << MD \). If \( t \) is small relative to \( MD \) then there is still a significant contribution to \( l_z \) when \( M_V \) is non-negligible relative to \( E_\gamma \). This is the effect that interests us. It is safe to use the non-relativistic form factor because the transferred energy is,

\[ l_0 = -\frac{t}{2MD}, \] (5.35)

which is small at small \(-t\). On the other hand, as long as \( t \) is not exactly zero, there will be a component of \( l_z \) that does not die out with energy. This effect represents the error induced by ignoring relativistic recoil. In the future we plan to generalize the formalism to the case of light-cone wave functions so that it may be extended to higher \(-t\).
Chapter 6

\(\phi\)-Meson Production and Further Extensions

We emphasize that the work of the previous chapter is a first step in a set of refinements to the usual techniques applied to multiple scattering in vector meson production from the deuteron that are needed for various experimental applications. We plan to extend these refinements in the future to include, for example, light-cone kinematics in the treatment of the deuteron wavefunction, and spin-flip effects.

6.1 Directions for Future Work

6.1.1 Parameterizations of Basic Amplitudes

Obtaining precise parameterizations of the \(s\) and \(t\) dependence is one of several steps needed for refinements in the calculation. We note that a peak in the energy dependence has been reported in Ref. [97] for photo-production of \(\phi\)-mesons from a proton target at \(E_\gamma = 2\) GeV and it is these data that we used in our parameterization (see Appendix B.2). We would like to point out, however, that the measurements in Ref. [97] are for the differential cross section at \(t = t_{\text{min}}\). Therefore, since the value of \(t_{\text{min}}\) varies significantly with energy in these near threshold measurements, then the reported measurements give the differential cross section at very different values of \(t\). We have indicated this in Fig. 6.1(A.). In order to infer the energy dependence at a fixed value of \(t\), one needs to assume a form for the \(t\)-dependence. The actual \(t\)-dependence at these low energies is not well known, but it is straightforward to see that even a simple exponential \(t\)-dependence will have an effect on the shape of the overall energy dependence of the cross section. As an example, we have plotted in Fig. 6.1(A.) the set of data as it was originally presented in Ref. [97] alongside Fig. 6.1(B.) where the data have been shifted to a fixed value of \(t\). We have used an exponential slope parameter of \(4\) GeV\(^2\) which gives reasonable agreement with the data. In the original form of the plot, Fig. 6.1 (A.), the data is shown at a different value of \(t\) at each energy. The highest value of \(-t_{\text{min}}\) occurs at the lowest energy plotted which is around 1.6 GeV. In Fig. 6.1(B.) we have re-plotted the energy dependence, but with the value of \(t\) for each data point fixed at \(-t_{\text{min}}\) for \(E_\gamma = 1.6\) GeV since this is the largest value of \(-t\) that is kinematically allowed for every point on the plot. Let \(t_{\text{min}}[1.6]\) represent the value
Figure 6.1: Plot of recent data from LEPS, taken from Ref. [97]. We indicate the significant variation of $t_{min}$ with photon energy. This may have an effect on the overall energy dependence of the cross section. (A.) shows how the set of data was originally presented: at a different value of $t$ for each energy. In (B.) we have shifted all of the data points to the same value of $t$ by assuming a constant slope parameter of $4 \text{ GeV}^{-2}$. Each point in (B.) corresponds to the differential cross section at the fixed value of $t$ corresponding to $t_{min}$ for a 1.6 GeV photon. Note the different scales on the axes in (B.).

We use this formula to obtain Fig. 6.1 (B.). We see that much of the peak-like behavior is removed. Without a fuller understanding of the $t$-dependence, therefore, it cannot be ruled out that the observed peak arises from purely kinematical effects. However, the fact that the cross section at fixed $t$ does increase at smaller $E_\gamma$ is evidence that OZI-violating meson exchange effects become important at these energies.

Recently, preliminary data were reported from SPring-8/LEPS [98] which measured the dependence of the $\phi$-meson production cross section with a proton target at $E_\gamma \sim 2 \text{ GeV}$ on the linear polarization of the photon. Significant polarization is observed which requires
the presence of a non-vacuum exchange like $\pi, \eta$ exchange. Such exchanges lead to spin flip in the nucleon vertex. These contributions for small $t$ are strongly suppressed for coherent production off the deuteron (pion exchange does not contribute in any case due to the zero isospin of the deuteron.) These effects are determined by the deuteron magnetic form factor which is much smaller than the electric form factor for small $-t$. Hence, the coherent production of the $\phi$-meson may be used as a spin analyzer of the elementary amplitude in the kinematics where double scattering is a small correction. This topic has already been discussed in Ref. [23]. In the spirit of the original Glauber approach, we have neglected spin effects for the sake of simplicity. Above $-t = 0.5$ GeV$^2$, the quadrupole form factor becomes important for non spin-flip interactions and the magnetic form factor may be significant. We intend to investigate this further in the future. Future work will involve generalizations of our method to the case of spin dependent basic amplitudes. However, if one fits a combination of Pomeron trajectory and Reggeon trajectory to the recent preliminary SPring-8/LEPS data, and then extrapolates to 3 GeV, it appears that less than 20% of the basic $\gamma D$ cross section is due to spin-flip, whereas the corrections found here due to non-factorizability are as large as 30% at 3 GeV [98].

Before ending our analysis, we mention that, because $V-N$ cross sections are extracted from the multiple scattering term, quantities sensitive to the deuteron polarization would be ideal for testing whether the $V-N$ cross section is unusually large. In order to emphasize this, the cross section for scattering from a polarized deuteron, from $m = +/−1$ to $m = +/−1$, is plotted in Fig. 6.2 where the result of using a typical value for the total $\phi N$ cross section (11 mb) is compared with the case when the $\phi N$ cross section is enhanced by a factor of three. (For clarity we have only plotted the sum of all the terms from the squared amplitude rather than each term separately.) The sharp dip that normally appears, is due to the sharp dip in the Born cross section. However, the double scattering cross section is nearly flat in $-t$. Therefore, in the summed cross section the double scattering term dominates in the region of the dip, and may even cause the dip to vanish entirely if it becomes very large. Figure 6.2 shows that, even with the suppression of the double scattering term that results from the non-factorizability, the dip in the cross section is observed to flatten out when the basic $\phi N$ cross section is abnormally large.

6.1.2 The Problem of Bound State Amplitudes

We have treated the struck nucleon as being on-shell which is consistent with the neglect of terms quadratic in nucleon momentum. However, immediately at the threshold for particle production, the $\gamma N \to VN$ amplitude has very unpredictable behavior which may be modified significantly when the nucleon is in a bound state. This is especially clear when we realize that, for a given photon energy, $-t_{min}$ is different for a deuteron and an on-shell nucleon target. We cannot predict the effects of the off-shellness of the bound nucleon with-
Figure 6.2: The energy dependence of the $m = +/−1$ to $m = +/−1$ differential cross section for $\phi$-meson photo-production with $E_\gamma = 3$ GeV. The dashed curve shows the result of increasing the typical basic $\phi N$ cross section by a factor of 3.
out a complete, relativistic understanding of the basic amplitude. However, we have made predictions in the region of kinematics where it is reasonable to assume that the bound state amplitude is the same as that of the free nucleon amplitude. If one includes dependence upon the nucleon virtuality in the basic amplitude, then one may write the amplitude as \( \hat{F}(\hat{s}, \hat{t}, k_N^2) \). For \( k_N^2 = m_N^2 \), the amplitude reduces to the free nucleon amplitude. As we have stated, \( k_N^2 = m_N^2 \) up to corrections of order \( k_N^2/m_N^2 \) or higher whereas \( \hat{s} \) has linear order corrections in nucleon momentum. Thus, if \( \hat{F}(\hat{s}, \hat{t}, k_N^2) \) is an analytic function of kinematic variables, then there will be linear order corrections in nucleon momentum due to \( \hat{s} \) whereas the lowest order corrections due to the virtuality of the bound nucleon are only of quadratic order in nucleon momentum. In other words, Fermi motion effects may be important even when it is appropriate to neglect the off-shellness of the bound state amplitude. Of course, all of this depends on the validity of using \( k_N/m_N \) as a small expansion parameter which is only true if the basic amplitude has relatively weak \( s \)-dependence. This is one reason why we emphasize that we are considering intermediate energies rather than low energies. One may also include the deuteron binding energy in the calculation of the mass of the bound nucleons, but the binding energy arises from the full consideration of relativistic binding and higher order terms in nucleon momentum, so considering the nucleon binding energy is not consistent with the neglect of higher order nucleon momentum terms or the use of a non-relativistic potential for the N-N interaction.

In this subsection we propose a rough way to test the validity of the on-shell amplitude approximation. We do this in the next few paragraphs by directly comparing the amplitude when it is evaluated at the value of \( t_{\text{min}} \) for the deuteron with the case when it is evaluated at \( t_{\text{min}} \) for a free nucleon with \( s \) given by the exact expression for \( \hat{s} \),

\[
\hat{s} = 2E_\gamma \left( M_D - \sqrt{m_N^2 + k^2 + k_z} \right) + \left( M_D - \sqrt{m_N^2 + k^2} \right)^2 - k^2. \tag{6.2}
\]

The value of the nucleon 3-momentum thus parameterizes the off-shellness of the bound nucleons. \( t'_{\text{min}} \) will denote the lower bound of \(-t\) for the free nucleon, whereas \( t_{\text{min}} \) is the lower bound of \(-t\) for the deuteron. The struck nucleon inside the deuteron for the unprimed case has \( \hat{s} \) given by Eq. (6.2). We will now consider the case of a free nucleon, with the same \( \hat{s} \) as for the bound nucleon, but with the nucleon on-shell (i.e. \( k^2 = m_N^2 \)) and with a fixed value for \( k_z \). So that the free nucleon energy corresponds to the bound nucleon energy, we will continue to use \( M_D - \sqrt{m_N^2 + k^2} \) for the energy of the struck nucleon. In short, we are comparing \( t_{\text{min}} \) for \( \gamma \) scattering off a deuteron at rest with \( t'_{\text{min}} \) for \( \gamma \) scattering off a free nucleon, with energy corresponding to that of the bound nucleon in both cases.

We expect the rate of variation of the basic amplitude with \( t \) to be very large near \( t_{\text{min}} \). If there is a significant contribution to the integral in Eq. (5.5) from regions near \( t_{\text{min}} \), then \( t_{\text{min}} \) should nearly equal \( t'_{\text{min}} \) in order to make the on-shell amplitude a valid approximation.
to the bound state amplitude. We can use the difference between these two values of $t_{\text{min}}$ to estimate the effect of the off-shellness on the amplitude.

In order to test the effect of the off-shellness of the basic amplitude, we may consider two extremes. First, the bound state basic amplitude could be evaluated at the physical value of $t$ for the photon-deuteron process. That is, we could calculate the amplitude, $\hat{F}(\hat{s}, t)$ at $t$ where $t$ is the physical value of $t$ for the photon deuteron process. In this case, since $t_{\text{min}}$ is smaller for the deuteron than $t'_{\text{min}}$ is for the nucleon, then we are probably overestimating the cross-section. On the other hand, we could evaluate the basic amplitude at $F(\hat{s}, t - (t_{\text{min}} - t'_{\text{min}})) = F(\hat{s}, t - \Delta t)$ where $t'_{\text{min}}$ is the minimum $t$ for the free, on-shell nucleon. With this second method for choosing which value of $t$ to use in the basic amplitude, the basic amplitude behaves like the free, on-shell nucleon amplitude in the region of $t$ close to $t_{\text{min}}$. Hence, with this method, we are probably underestimating the value of the basic amplitude. In the high energy limit, $\Delta t$ vanishes and the two amplitudes are equal, and the difference between the two provides an estimate of the off-shell effects. (Note that we must specify a value for $\mathbf{k}$ in order to make a comparison.) Any amplitude which has a relatively slow and smooth variation with $t$ will yield a small difference between $\hat{F}(\hat{s}, t)$ and $F(\hat{s}, t - \Delta t)$. We made this comparison for $\phi$-meson production with the parameterization in appendix B.2 and we find only a few percent deviation. We conclude that at a few GeV above threshold it is reasonable to continue using the on-shell amplitude of the nucleon.
Chapter 7

Conclusions

A central theme of this thesis has been the need for consistency in combining pre-QCD physics and non-perturbative models of photon-hadron interactions when they are extended beyond their usual range of applicability or when they are combined with leading twist perturbative QCD.

In chapter 2, we discussed how combining the pQCD dipole picture of very high energy interactions with unitarity considerations provides one with a method for probing the transition between the region of applicability of leading twist pQCD and black disk behavior at very high energies. We hope that in the future we may be able to exploit the novel properties of interactions in the BDL to study new phases of pQCD. In Ref. [77] the signatures of the BDL for DIS were discussed with the hopes that they would be seen in future experiments. It remained to be determined, however, in which kinematical regions one can expect to see black disk behavior. With the MFGS model discussed in chapter 2, we are now able to address this question.

In chapter 3, we apply the MFSG model to make rough estimates of the fraction of the hadronic interactions that exhibit the characteristic behavior of black disk interactions. More precisely, since we know that the effects that we have ignored so far – inelastic diffraction and the real part of $A_{hN}$ – will tend to increase the proximity of the interactions to the unitarity limit, then we can place lower limits on the values of $x$ and $b$ where a significant fraction of the events will occur at or near the BDL. Our results show that, within available HERA kinematics, a significant fraction of the total DIS cross section is due to interactions of the hadronic components with the proton that occur near the BDL. In particular, Fig. 3.11 shows that at $Q^2 \approx 2.0$ GeV$^2$ and $x \lesssim 10^{-4}$, about 1/5 of the longitudinal cross section is due to values of $\Gamma_h(s,b,d) \gtrsim 1/2$. The agreement of the MFGS model combined with the $t$-dependence of chapter 3 with preliminary HERA data and with previous models helps to strengthen this conclusion. An improved model, with corrections for inelastic diffraction, will likely predict a more rapid approach to the BDL at small $x$ and central impact parameters. The approach to the BDL as $x$ and $b$ decrease occurs much more rapidly for the case of a heavy nuclear target than for the case of a proton target. This can be seen by comparing Figs. 3.3 and 3.14. For example, at $d = 0.4$ fm and $x \approx 10^{-4}$, Fig. 3.3 shows that, for the proton, less than 1/2 of the total cross section is due to contributions from
\[ \Gamma_h(s, b, d) > 0.5 \] whereas with a \(^{208}\text{Pb}\) target, Fig. 3.14 shows that over seventy percent of the total cross section is due to contributions from \(\Gamma_h(s, b, d) > 0.5\). This suggests that nuclear targets are ideal for studies of the BDL as a phase of QCD as has been discussed before in Ref. [77]. Future work on this subject should incorporate inelastic effects. Also, a greater understanding of the large \(t\) behavior would lead to greater accuracy in the model.

In chapter 4, we take the limit of the MFGS model of chapter 2 where the photon goes on shell and acquires astronomically large energy. This simultaneously provides us with a consistency check on the MFGS model and reasonable upper limits on cross sections for experimentalists looking for cosmic ray photons with ultra high energies. We investigate the photon-proton/nucleus cross section in the range of energies from \(10^3\) GeV to \(10^{12}\) GeV. Figure 4.4 demonstrates that the total cross section rises by about a factor of 12, but that there is a significant amount of uncertainty involved when the details of the model are varied. This result, however, gives us a very reasonable upper limit on the cross section since we have consistently taken the maximum cross section allowed by unitarity, and constrains the cross section much more than previous models. The cross section varies approximately linearly with \(\ln^3(E_\gamma/E_0)\) (see section 4.5). (We use \(E_0 = 1.0\) GeV which is consistent with hadronic sizes; see, for example, page 18 of Ref. [3].)

The solid curve in Fig. 4.4 provides a reasonable estimate (or if we prefer to take a more cautious attitude, an upper limit) to the \(\gamma N\) cross-section at extremely high photon energies. We note that the dipole approach is consistent with the direct extrapolation of the photo-nuclear cross section [99] and with a model based on unitarity in the \(t\)-channel [100], but not with the model of [79] which uses a two-Pomeron approach and places no unitarity constraint on the growth of the cross section. There is also disagreement with the model of Ref. [80] which uses the Donnachie-Landshoff two-Pomeron approach, but applies a unitarity constraint to the \(S\)-matrix for \(\gamma\)-proton scattering rather than to the profile function of individual hadronic components.

We have shown in Fig. 4.6 that as much as 25\% of the cross section may be due to charmed mesons for the case of a target proton, and this fraction rises to 30\% when we consider a target \(^{12}\text{C}\) nucleus (but see the footnote on page 61). For the \(^{12}\text{C}\) target, we use a direct application of the usual Gribov-Glauber theory, and we find that there is a large amount of shadowing increasing with energy. It is worth noting here that although our results for the elementary \(\gamma p\) cross section are rather close to the results of [101] for \(\sqrt{s} \leq 10^3\) GeV considered in this paper using a generalized vector dominance model with a point-like component in the photon wave function, the model of Ref. [101] leads to a nuclear shadowing effect which is practically energy independent.

This is consistent with the dipole-proton interaction approach to the black disk limit. The resulting cross section is shown in Fig. 4.11 which indicates a rise in the cross section of about a factor of 7 when the energy increases from \(10^3\) GeV to \(10^{12}\) GeV. The relevance
of these observations is in the characterization of atmospheric showers induced by UHE neutrinos and super-GZK cosmic rays, where upper limits on the allowed growth of the photon-nucleus cross section are needed.

In chapter 5, we switched to an entirely different realm of strong interaction physics by considering intermediate to low energy real photo-production of vector mesons from the deuteron. We show by direct numerical calculation that the effect of factorization breakdown is significant for intermediate photon energies. The Glauber approach is, strictly speaking, only applicable for the case of very high photon energies. However, there are current attempts to apply the factorization assumption of Glauber theory to the \( \phi \)-meson production reaction at energies as low as 1.5 GeV in both experimental and theoretical research. Therefore, in order to salvage the situation in the energy range of a few giga-electron volts above threshold, we have outlined the steps one must follow in order to obtain corrections to leading order in the bound nucleon momentum and transferred momentum. The main steps are essentially those of the original diagrammatic formulation of Glauber theory in terms of momentum space integrals and its extension to vector meson production [7, 8, 86]; we have started with most of the original assumptions, but we have removed the assumptions of factorizability, ultra-relativistic kinematics or VMD for the basic amplitudes, and we have numerically evaluated all integrals directly without any factorization approximations. By using a simple model for the basic amplitude (we restore VMD for the simple model) based on a fit to old and recent data, we have shown that, away from \( t = t_{\text{min}} \), ignoring Fermi motion (and the resulting breakdown of factorizability) can lead to a significant error in basic cross sections extracted from \( \gamma D \rightarrow \phi D \) cross section data (see Fig. 5.9). This effect will certainly need to be taken into account in future searches for new production mechanisms at intermediate energies. The breakdown in factorizability arises as a consequence of both the non-negligible longitudinal momentum exchanged, and the non-negligible Fermi motion. An important point is that the source of the departure from factorizability is the inadequacy of assuming the nearly flat s-dependence predicted by Regge theory in the basic amplitudes. Therefore, models of the basic \( \gamma N \rightarrow VN \) amplitude or the \( VN \rightarrow VN \) amplitude which depart significantly from nearly flat s-dependence must include at least the linear order nucleon momentum corrections that we discuss in chapter 5 if they are to be used in calculations with a deuteron target. This correction arises purely from the fact that the bound nucleons have non-vanishing momentum and it must be included regardless of the details of a particular model of the basic amplitudes. For the case of \( \phi \)-meson production, we find that our approach is reasonable when we use our particular simple model of the basic amplitude and as long as the photon energy is around 3 GeV or higher and \( t \) is not too close to \( t_{\text{min}} \).

However, we stress that in a model of the basic \( \gamma N \rightarrow VN \) amplitude that predicts much wilder energy dependence at intermediate energies than what we have assumed, the linear
order corrections will not be sufficient, and a complete and precise understanding of the
$N-N$ interaction and the bound state nucleon amplitudes are necessary in order to make a
correct calculation. For the $\phi$-meson photo-production cross section ($M_\phi \approx 1.02$ GeV), the
amplitude may vary wildly with energy at $E_\gamma = 2$ GeV or lower because of the very close
proximity to threshold. For this reason, and because the eikonal approximation begins to
break down, basic cross sections for photo-production from the nucleon extracted from data
for photo-production from the deuteron are suspect for photon energies less than or equal
to 2 GeV for the $\phi$ production reaction.

In our sample calculation, we observe that the contribution from double scattering
becomes numerically suppressed relative to the Born approximation as the incident photon
energy decreases. However, the multiple scattering terms are what lead to the characteristic
A-dependence of the Glauber theory for complex nuclei, $\sigma_{tot} \sim A^{2/3}$. This suggests that an
extension of our methods to complex nuclei will yield a rather different A-dependence for the
cross section at intermediate energies from what is predicted at high energies. Hence, there
will need to be a revision in efforts to extract basic cross sections from nuclear data using
extrapolations in $A$. The extension to complex nuclei, however, requires much more work.

We note, however, that data given in Ref. [102] were interpreted as implying a very high $\phi N$
total cross section on the basis of a very traditional Glauber approach at energies of only a
few GeVs. Therefore, our next step will be to determine how the non-factorization effects
discussed in this chapter affect a general, incoherent Glauber series. Furthermore, since it
is apparent that spin effects will be important, then a generalization with spin-dependent
amplitudes will be needed.

In our study of photo-production from the deuteron, we have purposefully over-simplified
our analysis here for the purposes of demonstration. In particular, we have applied the VMD
hypothesis at energies where it is suspect and we have neglected fluctuations and $\omega - \phi$
mixing in the intermediate vector meson in the double scattering term. Further analysis
will need to include these effects. In order to make further numerical progress, we will
need firmer parameterizations of the basic cross sections for vector meson production from
ucleons. For theoretical work, it would be useful for the purposes of comparison to have
a widely agreed upon set of parameterizations. We also need to consider the calculation
in light-cone coordinates and the effects of spin-flip. We will pursue these issues in future
work.

Finally, we will need a complete understanding of the off-shell amplitudes if we are to
take into account the higher order momentum corrections that will be necessary just at the
threshold, though we have argued that for smoothly varying basic cross sections, the effect
of off-shellness in the amplitudes is small relative to the effect of linear order corrections in
nucleon momentum.
Appendix A

The perturbative QCD dipole cross section

In this appendix, we give a detailed derivation of the high energy $\gamma^* N$ cross section equation (2.9) in language of $q\bar{q}$ dipole scattering. Several sources quote this result in some form [31, 103, 36]. The purpose of this appendix is to give a derivation that avoids assuming some form of $k_\perp$-factorization which is known to fail for light quarks.

Let us first establish the kinematic variables appropriate for very high energy scattering and establish several approximations. The concept of the cross section for an incident photon is ill-defined because the flux factor is, strictly speaking, arbitrary. A convention must be chosen for the flux factor in order to yield a well defined concept, and we chose the “hund” convention which ensures that the expression is similar to that for on-shell particles. Squaring the amplitude and summing over final states yields the general expression for the total cross section:

$$\sigma_{\gamma^* N}(x, Q^2) = \frac{1}{2s} \int |\hat{A}\lambda|^2 d\Pi,$$

(A.1)

where $\lambda$ denotes the photon polarization, $|\hat{A}\lambda|^2$ is the squared amplitude summed over final spins for a photon of polarization, $\lambda$, to fluctuate into a $q\bar{q}$ pair of 4-momenta $k_1$ and $k_2$ respectively which then scatters from the target proton, and $d\Pi$ is the total phase volume element. At high energies, gluon induced processes dominate.

To leading order in pQCD, there are two Feynman graphs, and the squared amplitude (or, alternatively, the imaginary part of the forward Compton amplitude) is shown in Eq. (A.1). Our goal is to rewrite the integral as a convolution of a photon-gluon cross section with the probability for the nucleon to emit a gluon. To this end, we rewrite Eq. (A.1) in terms of two gauge invariant blocks:

$$\sigma_{\gamma^* N}(x, Q^2) = \frac{e^2 e_g^2 s^2}{2s} \int \frac{d^4 l}{(2\pi)^4} \mathcal{M}_{\lambda\lambda}^{\mu\nu}(q, l) G_{\lambda\lambda}(l, P) \frac{D_{\mu\bar{\nu}}D_{\nu\bar{\nu}}}{(l^2)^2},$$

(A.2)

Electromagnetic indices will be neglected for now. Our final goal for this section will be to obtain the longitudinal cross section, although we formally keep the polarization arbitrary for the present. The projection operator necessary to obtain $\sigma_{\gamma^* N}(x, Q^2)$ is assumed and is indicated by the $\lambda$ subscripted on $\mathcal{M}_{\lambda\lambda}^{\mu\nu}(q, l)$. The 4-momentum carried by the exchanged
Figure A.1: Forward virtual Compton scattering amplitudes.
The gluon is denoted by $l$. The individual blocks are written,
\[
\mathcal{M}^{\mu\nu}(q,l) = \int \frac{d^4k}{(2\pi)^2} M^{\mu}(q,k) M^{\nu\dagger}(q,k) \delta_+(k^2 - m_n^2) \delta_+(k^2 - m_n^2),
\] (A.3)
and,
\[
G^{\mu\nu}(q,l) = \int d\Pi_{X} G^{\mu}(q,l) G^{\nu\dagger}(q,l).
\] (A.4)

For LT pQCD to be applicable, the integral, $\mathcal{M}^{\mu\nu}(q,l)$, should only run over large values of transverse momentum for the case of longitudinal polarizations. We have kept the quark mass in the phase space $\delta$-functions because, as we will see, a large value of quark mass suppresses the contribution from small transverse momenta. In the perturbative regime, the value of the quark mass should not matter. The upper block may thus be loosely regarded as the squared amplitude for a virtual photon to scattering from a gluon target.

Although we are considering the cases where $Q^2$ is large enough that pQCD is applicable, we are also considering the limit where $s \gg Q^2$. Let the struck proton and the virtual photon have momentum in light-cone variables given respectively by
\[
p = (m_n^2/(2p^-), p^-, 0_\perp)
\] (A.5)
\[
q = (-Q^2/2q^-, q^-, 0_\perp).
\] (A.6)

We then have,
\[
s = 2p \cdot q = \frac{-p^- Q^2}{2q^-} + \frac{q^- m_n^2}{2p^-}.
\] (A.7)

Therefore, in order to have $s \gg Q^2, m_n^2$, we must have either $|p^-| \gg |q^-|$ or $|q^-| \gg |p^-|$. We choose our coordinate system to correspond to the former. In this limit, the $p$ and $q$ momenta are approximated by the light-like momenta, $q' = (-Q^2/2p^-, 0, 0_\perp)$ and $p' = (0, p^-, 0_\perp)$. Using these momenta, the struck gluon 4-momentum is written with the Sudakov parameterization:
\[
l = -\alpha q' + x' p' + l_\perp,
\] (A.8)
where $q'$ and $p'$ are related to the photon and nucleon momenta via,
\[
p = p' + \frac{p^2}{2p' \cdot q'} q'
\] (A.9)
\[
q = q' + \frac{q^2}{2p' \cdot q'} p'.
\] (A.10)

(This way of writing the momenta was first employed by Gribov and Lipatov in [104] for
the case of quantum electrodynamics.) Note that,

\[ p'^2 = q'^2 = 0 \]  \hfill (A.11)

\[ q' \cdot q = Q^2/2 \]  \hfill (A.12)

\[ p' \cdot p = m_n^2/2. \]  \hfill (A.13)

Also, \( p' \cdot q' \approx p \cdot q \) with a correction suppressed by \( q^-/p^- \sim Q^2/s \). Recall the measure of integration in the Sudakov representation,

\[ d^4 l = \frac{s}{2} d\alpha dx' d^2 l_\perp. \]  \hfill (A.14)

We will be considering the leading \( \alpha_s \ln \frac{Q^2}{\mu^2} \) (where \( \mu \) is the renormalization scale) limit which corresponds to the restriction that the 4-momentum carried by the gluon has low virtuality compared with the hard scale, \( l^2 << Q^2 \) since a gluon momentum of \( l^2 \sim Q^2 \) should be associated with the next order in \( \alpha_s \ln \frac{Q^2}{\mu^2} \). That is, the condition for the applicability of the high energy/leading-log approximation is,

\[ |l^2| = |\alpha x's + l_\perp^2| << |k^2| \sim Q^2 << s. \]  \hfill (A.15)

Thus, according to Eq. (A.15) we must have \( \alpha x' << 1 \). The squared center of mass energy of the \( q\bar{q} \) system is,

\[ M_{q\bar{q}}^2 = (l + q)^2 = -\alpha x's - k_\perp^2 - Q^2 + x's \approx x's - Q^2, \]  \hfill (A.16)

where the second approximation follows from the leading-log approximation in Eq. (A.15). Thus,

\[ x' \sim \frac{M_{q\bar{q}}^2 + Q^2}{s} << 1. \]  \hfill (A.17)

Therefore, \( x' \) is small in the high energy limit, but \( x's \) is always of the order of \( Q^2 \). Therefore, in order to remain in the \( \alpha_s \ln \frac{Q^2}{\mu^2} \) regime, Eq. (A.15) demands that we have \( \alpha << 1 \). To summarize, the leading-log approximation ensures that \( \alpha << 1 \) and the high energy approximation ensures that \( x' << 1 \). It will be important later to note that this implies that \( l \) is approximately orthogonal to both \( q' \) and \( p' \).

In order to obtain the desired simplification for the high energy behavior, one needs to use a theorem concerning high energy behavior first introduced by Gribov, Lipatov and Frolov (GLF) \[105\] in the context of QED. This starts by writing the gluon propagator as,

\[ \frac{D_{\mu\nu}(q,l)}{l^2}. \]  \hfill (A.18)
where in the light-cone gauge, and in the frame where $q'$ and $k$ lie along the same line,

$$D_{\mu\nu}(q, l) = -g_{\mu\nu} + \frac{q'_{\mu}l_{\nu} + l_{\mu}q'_{\nu}}{l \cdot q'}.$$  \hspace{1cm} (A.19)

According to the GLF theorem, in the high energy limit, we may replace the propagator by,

$$D_{\mu\nu} \approx \frac{p'_{\mu}q'_{\nu}}{p' \cdot q'}.$$  \hspace{1cm} (A.20)

Returning to Eq. (A.3) we have,

$$\sigma_{\gamma^* N}^{\gamma^* N}(x, Q^2) = \alpha_{\text{e.m.}} \frac{e^2 e_q^2 g_s^2}{s^2} \int \frac{dydzd^2k_{\perp}}{(2\pi)^4} M_{\alpha\beta}(q, l) \mathcal{G}_{\bar{\nu} \nu}(l, P) \frac{p'_{\mu}p'_{\mu}q'_{\mu}q'_{\nu}}{(l^2)^2}.$$  \hspace{1cm} (A.21)

The definition of the fully integrated parton distribution function is,

$$x' \alpha_s(k^2) g(x', k^2) = \frac{2}{s} \int \frac{d\alpha d^2l_{\perp}}{(2\pi)^4} \mathcal{M}_{\alpha\beta}(q, l) p'_{\mu}p'_{\mu}q'_{\mu}q'_{\nu} \frac{2\alpha_s}{(2\pi)^4} \frac{\mathcal{G}_{\bar{\nu} \nu}(l, P)}{P_{\perp}^2}.$$  \hspace{1cm} (A.22)

where in the leading-log approximation, we may replace $l^2 \approx l_{\perp}^2$. The integral over $l_{\perp}$ runs to infinity and leads to an ultraviolet divergence. Therefore, the gluon distribution must be renormalized with respect to a hard scale which we choose to be the virtuality of the incident $q\bar{q}$ pair, $k^2 \sim Q^2$, leading to the $k^2$ dependence in the gluon distribution.

$$\sigma_{\gamma^* N}^{\gamma^* N}(x, Q^2) = \alpha_{\text{e.m.}} \frac{e^2 e_q^2 g_s^2}{s^2} \int \frac{dx'd^2l_{\perp}}{(2\pi)^4} M_{\alpha\beta}(q, l, k) P_{\perp} \frac{2\alpha_s}{(2\pi)^4} \frac{\mathcal{G}_{\bar{\nu} \nu}(l, P)}{P_{\perp}^2} \delta_{++}((q - k)^2 - m_q^2) \delta_{++}((k + l)^2 - m_q^2).$$  \hspace{1cm} (A.23)

From now on we will absorb $\alpha_{\text{e.m.}} e_q^2$ and the factor of $\frac{1}{s}$ in Eq. (A.21) into the definition of $M_{\alpha\beta}(q, l)$. The renormalization scale of the gluon distribution should be of the order of $k^2_{\perp} \approx Q^2$. Now let us evaluate $M_{\alpha\beta}(q, l)$ which may be loosely interpreted as being proportional to the cross section for a $\gamma^*$ to scatter from a gluon. Since we wish to analyze the structure of this block, we will write it separately while keeping in mind that it has dependence on $x'$ and $l$ and must ultimately be substituted back into integral Eq. (A.23).

Referring to the upper blocks in Fig. A.1, we write the produced quark’s momentum in the Sudakov representation:

$$k = -yp' + zq' + k_{\perp}$$  \hspace{1cm} (A.24)

Eq. (A.3) becomes,

$$M_{\alpha\beta}(q, l) = \alpha_{\text{e.m.}} \frac{e_q^2}{2} \int \frac{dydzd^2k_{\perp}}{(2\pi)^2} M_{\alpha}(q, l, k) M_{\beta\gamma}(q, l, k) \delta_{++}((q - k)^2 - m_q^2) \delta_{++}((k + l)^2 - m_q^2).$$  \hspace{1cm} (A.25)

Let us analyze the structure of this integral now and return later to the address the fact
that the gluon distribution function, which we have notably left out of Eq. (A.25), has $k$-dependence that should be taken into consideration. It is straightforward to show that Eq. (A.25) reduces to,

$$
\mathcal{M}^{\mu\nu}_{\alpha\beta}(q,l) = \frac{\alpha_{e.m.}\epsilon_q^2}{8\pi^2s^2} \int \frac{dz d^2k}{z(1-z)} \frac{d^2l}{l(1-z)} \frac{M^\mu_{\alpha}(q,l,k)M^{\nu\dagger}_{\beta}(q,l,k)}{z(1-z)s^2} \left( x' - \frac{z\mathbf{k}_\perp^2 + (1-z)(\mathbf{k}_\perp^2 + \mathbf{l}_\perp^2) + \epsilon^2}{z(1-z)s} \right),
$$

(A.26)

where $\epsilon^2 \equiv Q^2z(1-z) + m_q^2$. When the $x'$ and $y$ integrals are taken, the delta functions force the final state quark and antiquark on-shell. The value of $x'$ is fixed by the delta function, but it may be approximated as described in the main body of the text and an average value of $x'$ can be used in the evaluation of $G^{\mu\nu}$. In order to identify the photon wavefunction factors, the labeling of the momenta in the squared amplitude becomes very important. We anticipate that in the $s >> t$ limit, we will be able to identify the light-cone photon wavefunction. In anticipation of the space-time picture that emerges in the $s >> t$ regime, we label the momenta so that the quark and antiquark both have positive momentum going into the electromagnetic vertices in both sides of the squared amplitude. This is shown in Fig. A.1 which shows the squared amplitude (or, alternatively, the imaginary part of the forward Compton scattering amplitude). In order to calculate the traces, it is simplest to calculate for specific photon polarizations, say, the longitudinal and transverse polarizations. For this reason, we have restored the electromagnetic indices, $\alpha$ and $\beta$ in Eq. (A.25). Let us start with the longitudinal polarization. The longitudinal polarization projection operator may be written,

$$
P^\alpha_\beta = \frac{4Q^2}{s^2} p^\alpha p^\beta + \frac{2}{s^2} (p^\alpha q^\beta + p^\beta q^\alpha) + \frac{q^\alpha q^\beta}{Q^2}.
$$

(A.27)

In contracting with this with $\mathcal{M}^{\mu\nu}(q,l)$, we will get a vanishing result from all terms proportional to $q^\alpha$ or $q^\beta$ because of gauge invariance. Therefore let us only contract with,

$$
P^{\alpha\beta}_L = \frac{4Q^2}{s^2} p^\alpha p^\beta.
$$

(A.28)

Recalling the approximations discussed earlier, in the traces we may approximate $k \approx zq'$ and $l \cdot p \approx l \cdot q \approx 0$ because the other terms will lead to factor suppressed by $Q^2/s$ or a factor of $\alpha << 1$. Therefore, for graph (a) we have a numerator factor of,

$$
-\frac{4Q^2}{s^2} \text{Tr} \left[ (q - \mathbb{k}) p \mathbb{k}' (\mathbb{k} + l) p \mathbb{k}' \right] \approx -8z^3(1-z)s^2Q^2.
$$

(A.29)

The denominator from graph (a) (keeping the large quark mass to remove the low-\mathbf{k}_\perp
contribution) is found to be,
\[
\left( \frac{k^2 + \epsilon^2}{1 - z} \right)^2,
\]  
so that the integrand of graph (a) (apart from the $1/(8\pi^2)$ factor) may be written as,
\[
-8z^2(1 - z)^2Q^2 \frac{(k^2 + \epsilon^2)^2}{(k^2 + \epsilon^2)^2}.
\]  
(A.31)

Following the same steps, one can easily verify that each of the graphs in Fig. A.1 will have an over-all factor of,
\[
Num(z, Q^2) \equiv -8z^2(1 - z)^2Q^2,
\]  
(A.32)
apart from the sign reversal for graphs (c) and (d) mentioned above. Also, for each photon vertex with outgoing momenta $r$ and $q - r$, there is a denominator,
\[
Den(z, Q^2, r_\perp) \equiv (r^2 + \epsilon^2).
\]  
(A.33)

In addition to these factors, there is a factor of $T_F = Tr\{t^at^b\}$ where the $t^a$ are the generators of the color SU(3) Lie group in the fundamental representation. This factor arises when we sum over the final state color of the $q\bar{q}$ pair. We define the light-cone wave-function of the photon as,
\[
\psi_{\rho}(Q^2, z, r_\perp) \equiv \sqrt{\frac{\alpha_{e.m.}e^2 q N_c}{\alpha}} \bar{u}(r)\gamma_{\rho} u(r - q) \frac{r^2 + \epsilon^2}{(z(1-z))}.
\]  
(A.34)

The number of colors, $N_c$, is included as an over-all factor for reasons that will be convenient later. With the simplifications to the numerator factor made above, for the longitudinally polarized photon we may replace the wavefunction with,
\[
\psi_L(Q^2, z, r_\perp) \rightarrow \frac{\sqrt{\alpha_{e.m.}e^2 q N_c Num}}{(r^2 + \epsilon^2)}.
\]  
(A.35)

In this form, it is clear that the large quark mass suppresses the contribution from low-$k_\perp$ Only relative transverse momentum of the $q\bar{q}$ pair of the order of $\Lambda_{QCD}$ should be included in the calculation for a perturbative calculation to be valid. Therefore, the constituent quark mass of 300 MeV that we use in the main text is reasonable. Keeping in mind the minus sign associated with graphs (c) and (d), and using Eqs. (A.32) and (A.35), it follows directly that Eq. (A.26) may be rewritten as,
\[
\frac{p'_\mu p'_\nu}{8\pi^2 N_c}M^{\mu\nu}_{L}(q, l) = \frac{T_F}{8\pi^2 N_c} \int dz d^2k_\perp \psi^\dagger_{L}(Q^2, z, k_\perp)
\times (2\psi_L(Q^2, z, k_\perp) - \psi_L(Q^2, z, k_\perp - l_\perp) - \psi_L(Q^2, z, k_\perp + l_\perp)).
\]  
(A.36)
That is, the integrand is simply the product of light-cone wavefunctions. The intermediate quark propagators led only to an over-all factor of $s^2$ which canceled with the factors of $s$ from the gluon propagators. At very high energies, the main effect of the exchanged momentum, $l$, is to introduce a mismatch in the denominator between the initial and final photon wave functions in the forward virtual Compton amplitude. Now we are in a position where we may rewrite the photon wave functions in terms of their Fourier transforms:

$$\psi_{\mu}(Q^2, z, k) = \int d^2 d \tilde{\psi}_{\mu}(Q^2, z, d)e^{i d \cdot k_{\perp}}.$$ (A.37)

Substituting into Eq. (A.36), we get,

$$p'_{\mu} p'_{\nu} \mathcal{M}_L^{\mu \nu}(q, l) = \frac{T_F}{8\pi^2 N_c} \int dz d^2 k_{\perp} d^2 d' d^2 d'' |\tilde{\psi}_L(Q^2, z, d)|^2 \times (2e^{i k \cdot (d - d')} - e^{i d_{\perp} \cdot d} e^{i k \cdot (d - d')} - e^{-i d_{\perp} \cdot d} e^{i k \cdot (d' - d)}).$$ (A.38)

When Eq. (A.38) is included in Eq. (A.23), we must take into account the fact that the gluon distribution has $k$-dependence since we have used $k^2$ as the renormalization scale. In the full expression for $\sigma_{\gamma p}$ in Eq. (A.2), the gluon distribution must be included in the integral of Eq. (A.38). However, in the region of $k_{\perp} \sim Q$, the evolution of the gluon distribution is determined by the leading $\ln \frac{k_{\perp}^2}{\mu^2}$ term which varies much more slowly with $k_{\perp}$ that the other factors in the integrand of Eq. (A.38). In fact, in the hard limit, the integrand in Eq. (A.38) is sharply distributed around a particular value of $k_{\perp}$, which, as we discuss in the main text is around $k^2 \approx \lambda / d^2$. Therefore, for the purpose of performing the $k_{\perp}$ integral, we substitute $k^2$ in the gluon distribution by this average value, which will, nevertheless, vary with the hardness of the interaction since the typical value of $k^2$ varies as $d$ varies. That is we make the substitution,

$$\alpha_s(k^2) g(x, k^2) \rightarrow \alpha_s(\lambda / d^2) g(x, \lambda / d^2).$$ (A.39)

The integral over $k_{\perp}$ in Eq. (A.38) then yields a Dirac delta function (times $(2\pi)^2$) that sets $d = d'$ when we perform the integral over $d'$. After these integrals are performed, we are left with

$$p'_{\mu} p'_{\nu} \mathcal{M}_L^{\mu \nu}(q, l) = \frac{1}{8\pi^2} \int dz d^2 d |\tilde{\psi}_L(Q^2, z, d)|^2 (2 - e^{i l_{\perp} \cdot d} - e^{-i l_{\perp} \cdot d}) \frac{T_F(2\pi)^2}{N_c}.$$ (A.40)

Expanding the exponential function in terms of $l_{\perp}$, we find that the lowest order non-vanishing term in the parenthesis of Eq. (A.40) is $(l_{\perp} \cdot d)^2$. Since the photon wavefunction depends only on the magnitude of $d$, then we may make the replacement, $((l_{\perp} \cdot d \cos \phi)^2 \rightarrow$
Putting $M_{L}^{\mu
u}(q,l)$ back into Eq. (A.23), we have
\[
\sigma_{L}^{\gamma N}(Q^{2},x) = \int_{0}^{1} dz \int d^{2}d|\tilde{\psi}_{L}(Q^{2},z,d)|^{2}\{\alpha_{s}(\lambda/d^{2}) T_{F}/2N_{c}q^{2}x'g(x',\lambda/d^{2})(2\pi^{2})}\}. \tag{A.41}
\]
Let us identify everything in the braces with the $q\bar{q}$ cross section,
\[
\hat{\sigma}_{q\bar{q}}(x',Q^{2}) = \frac{\pi^{2}}{3} \alpha_{s}(\lambda/d^{2}) d^{2}x'g(x',\lambda/d^{2}), \tag{A.42}
\]
where we use $T_{F} = 1/2$ for SU(3). The longitudinally polarized wavefunction can be calculated using Eq. (A.35) and the identity [106],
\[
\int_{0}^{\infty} dt \frac{t^{\nu+1}J_{\nu}(at)}{t^{2}+y^{2}} = y^{\nu}K_{\nu}(ay). \tag{A.43}
\]
Where $K_{\nu}(x)$ is the $\nu$th order modified Bessel function of the first kind. The Fourier transform wavefunction is,
\[
\tilde{\psi}_{L}(Q^{2},d,z) = \sqrt{\alpha_{e.m.}e_{q}^{2}N_{c}8Q^{2}z^{2}(1-z)^{2}} \int_{0}^{\infty} dk_{\perp} 2\pi k_{\perp} \frac{J_{0}(k_{\perp}d)}{k_{\perp}^{2}+\epsilon^{2}}. \tag{A.44}
\]
This yields,
\[
|\tilde{\psi}_{L}(Q^{2},d,z)|^{2} = \frac{6}{\pi^{2}} \alpha_{e.m.} \sum q^{2}Q^{2}z^{2}(1-z)^{2}K_{0}^{2}(\epsilon d). \tag{A.45}
\]
We have used $N_{c} = 3$ and recall that $\epsilon^{2} = Q^{2}z(1-z) + m_{q}^{2}$. Thus, we recover the dipole formula for the case of a longitudinally polarized photon,
\[
\sigma_{L}^{\gamma N}(x,Q^{2}) = \int_{0}^{1} dz \int d^{2}d|\psi_{L}(Q^{2},z,d)|^{2}\hat{\sigma}_{q\bar{q}}(x',\lambda/d^{2}). \tag{A.46}
\]
With some abuse of notation, we have removed the tilde on the Fourier transformed wavefunction in order to coincide with earlier texts. The calculation for the transverse cross section follows parallel steps.
Appendix B

Parameterizations

In this appendix, we list the various parameterizations that we used for the purposes of sample calculations.

B.1 Gluon Distribution

In chapter 4 we needed to interpolate the gluon distribution to extremely small values of $x$. In order to further extrapolate the basic, small size, cross section to extremely small values of $x$, we make a fit to the CTEQ5L gluon distribution in the region of lowest $x$ ($10^{-5} > x > 10^{-4}$) where the parameterizations exist. We find that the following interpolation agrees to within a few percent over the range of small-$x$ and for $0.01 \text{ fm} > d > 0.2 \text{ fm}$:

$$x g_N(x, d) = a(d)x^{c(d)}$$  \hspace{1cm} (B.1)

$$c(d) = \frac{-0.28}{d^{-1.11}}$$ \hspace{1cm} (B.2)

$$a(d) = 3.9 - 13.9d + 20.1d^2 + 0.5 \ln d. \hspace{1cm} \text{(B.3)}$$

B.2 Sample Parameterizations of Energy Dependence in Vector Meson Production Amplitudes

Here we describe the fits of the basic cross sections that we used for our sample calculations. The object here is not necessarily to produce very accurate parameterizations, but rather to devise parameterizations that demonstrate the effects of Glauber factorization while being consistent with recent and established experimental results.

First we consider the $\gamma N \rightarrow \rho^0 N$ differential cross section. For this we use a simple exponential $t$-dependence with a typical exponential slope of $B = 7.0 \text{ GeV}^{-2}$ and an over all normalization of $105 \mu b/\text{GeV}^2$. (See, e.g. Ref. [19].) It is known that at low energies the normalization undergoes a steep rise. We take this into account in our calculation by including a factor of $(1 + \frac{Q^2}{E_t^2})$ in the overall normalization and then doing a least squares
fit to obtain the parameter, $a$. We find that $a \approx 32.7$. The cross section is thus,

$$\frac{d\sigma}{dt} = 105 \frac{\mu b}{GeV^2} \left( 1 + \frac{32.7 GeV}{E^4_\gamma} \right) e^{7.0 GeV^{-2} t}$$  \hspace{1cm} (B.4)

The result is shown in Fig. B.1. As is seen in the main part of the text, the variation is too weak to introduce a very large effect on the final $\gamma D \rightarrow \rho^0 D$ cross section from Fermi motion.

The case of the $\phi$-meson is more complicated due to the irregular behavior near threshold. The main point is to interpolate smoothly between recent low energy data and the standard higher energy parameterization. The normalization of the low energy data, taken from recent experimental work in Ref. [97], is obtained from an effective Pomeron and pseudo-scalar exchange model [107] as it was presented in Ref. [97]. We continue to use this so that our model will be consistent with current work. The high energy parameterization was obtained in Ref. [19] by fitting a diffraction-like cross section to a large set of experimental data. We want to interpolate quickly but smoothly between the low energy and high energy data. There is an exponential factor, $e^{Bt}$, associated with both the high and the low energy behavior, but the slope, $B$, is around $3.4 \text{ GeV}^{-2}$ for the low energy behavior ($E_\gamma \lesssim 4.0 \text{ GeV}$) while it is around $4.8 \text{ GeV}^{-2}$ for the high energy behavior. Thus, for the exponential slope we use,

$$B(E_\gamma) = \left( 4.8 - (4.8 - 3.4)e^{-0.001 \text{GeV}^{-4} E^4_\gamma} \right) \text{GeV}^{-2}. \hspace{1cm} (B.5)$$

Next, for the low energy region, there is no Regge slope. That is, $\alpha' = 0$ in the factor, $s^{\alpha' t}$. But, in the high energy region, $\alpha' = .27 \text{ GeV}^{-2}$. Thus, we use,

$$\alpha'(E_\gamma) = .27(1 - e^{-0.001 \text{GeV}^{-4} E^4_\gamma}) \text{GeV}^{-2}. \hspace{1cm} (B.6)$$

Now we consider the behavior of $\frac{d\sigma}{dt}|_{t=0}$ for photo-production of the $\phi$-meson from a proton target. The high energy parameterization in Ref. [19] is

$$\frac{d\sigma}{dt} \bigg|_{t=0} = 1.34 s^{28}, \hspace{1cm} (B.7)$$

and, as in Ref. [19], over-all units will be understood to be $\frac{\mu b}{GeV^2}$. In order to match to the data of Ref. [97] we want a peak to appear at around $E_\gamma = 2.0 \text{ GeV}$. Therefore, we adjust the parameterization to,

$$\frac{d\sigma}{dt} \bigg|_{t=0} = 1.34 s^{28}(1 + ae^{-b(E_\gamma - c)^2}). \hspace{1cm} (B.8)$$
We use Eq. (B.8) to fit to the low energy data of Ref. [97] while assuring that the high energy parameterization of Eq. (B.7) is reproduced at high energies. We find: $a = 0.71$, $b = 16.5$ GeV$^{-2}$, and $c = 2$ GeV$^{-2}$. Finally, we note that the low energy data is actually given for $t = t_{\text{min}}$ rather than $t = 0$. Therefore, we must be sure to include a factor of $e^{B(E_{\gamma}) t_{\text{min}}}$ in the final result for $\frac{d\sigma}{dt}\bigg|_{\tilde{t}=0}$. The result of our parameterization for $\frac{d\sigma}{dt}\bigg|_{\tilde{t}=0}$ for the $\phi$-meson is shown in Fig. B.2. We point out that in the intermediate energy range at around $E_{\gamma} = 3$ GeV, the energy dependence is not completely flat, but it is smooth, and slow enough that its effect may be treated as a small correction.
Figure B.1: We obtain this fit using data from Ref. [108, 109, 110] (listed in Ref. [19].) We use an inverse-fourth function at low energies and apply a least-squares fit. The peak in the parameterization yields a small effect from Fermi motion (at a few GeVs) because of the small mass of the $\rho^0$. 
Figure B.2: The low energy data here is from Ref. [97]. The curve at high energies was taken from Ref. [19]. The dashed curve shows its extension to lower energies. The high energy data is taken from Ref. [111] and is presented to establish the consistency of the high energy parameterization. The curve at low energies has been fit to the low energy data of Ref. [97] using a least-squares fit.
Appendix C

Multiple scattering in the Non-Relativistic Approximation.

In this appendix, we review some of the basic steps in the derivation of the Glauber-VMD formalism. The purpose is to verify the consistency of using a diagrammatic formalism analogous to that used in a relativistic quantum field theory in the non-relativistic limit.

In this appendix, we discuss in detail the steps that lead to Eq. (5.2). Since we want to examine how the usual factorization of the total Born amplitude into a Glauber series breaks down near threshold, we will use the Feynman amplitude, normalized as an ordinary quantum field theory amplitude. This is to be distinguished from the Regge amplitude used in, for example, Ref. [21] where it was convenient for the amplitude to be normalized so that \( \Re \hat{f}^{\gamma^* N - VN}(s, t = 0) = \sigma_{tot} \). Furthermore, we simplify the analysis by ignoring all factors related to spin into the vertex factors and the basic amplitude so that we are essentially treating the nucleon propagators as scalar field theory propagators.

C.1 Pole Structure

Let us write out all of the poles that appear in the complex \( k_0 \) plane in the integrand of Eq. (5.2):

\[
\begin{align*}
k_0^{\pm a} &= \pm \sqrt{k^2 + m_N^2} \mp i\epsilon \\
k_0^{\pm b} &= M_D \mp \sqrt{k^2 + m_N^2} \pm i\epsilon \\
k_0^{\pm c} &= M_D + l_0 \mp \sqrt{(1 - k)^2 + m_N^2} \pm i\epsilon.
\end{align*}
\]

where we assume \( P_0 = M_D \) (lab frame kinematics). The superscript on \( k_0 \) enumerates the different poles. The pole structure of the amplitude for Eq. (5.2) in the \( k_0 \) plane is shown in Fig. C.1.

If we close the contour of integration below, then we have a sum of three terms corresponding to each of the three poles lying in the lower half plane. For non-relativistic scattering it is reasonable to exclude the appearance of negative energy nucleons. In other words,
Figure C.1: The poles in the integrand of Eq. (5.2) shown schematically in the $k_0$-plane. The pole at $k_0 = k_0^{\pm a}$ gives the only physically meaningful contribution to the amplitude.

We assume that the vertex factor suppresses contributions from any pole where $k_0$ deviates significantly from $m_N$, or where $\frac{k^2}{m_N^2}$ is much greater than zero. As such, we notice that the two poles corresponding to $k_0 = M_D + \sqrt{k^2 + m_N^2}$ and $k_0 = M_D + l_0 + \sqrt{(1-k)^2 + m_N^2}$ correspond to negative energies for the struck nucleon (and to $k_0 \approx 3m_N$), and we neglect the terms in the amplitude that result from taking these two poles. We are left with the term corresponding to the pole at $k_0 = \sqrt{k^2 + m_N^2}$ which establishes that the spectator nucleon is on shell. To summarize, we take only the term which corresponds to all nucleons carrying positive energy, and the spectator nucleon being placed on shell. Of course, it is also necessary to assume proper convergence of the integral when the contour is closed below. The empirical success of the two-nucleon approximation, and hence the above effective Feynman rules supports this assumption. Note that we may also have closed the contour about and achieved the same result within the non-relativistic approximation. In that case, all three poles would need to be kept, but two of them would cancel. The contribution from the remaining term is equal to that obtained by closing the contour below up to corrections of order $\frac{-t}{M_D}$ and $\frac{(k-1)^2}{2m_N}$. This method of taking the pole of the spectator nucleon is denoted in Fig. 5.2 by a cross placed on the spectator nucleon line, and is usually referred to as the "virtual nucleon" method. (Note that we are forced to use a rather ad hoc method for dealing with the off-shellness of the bound nucleon. But, we argue in Sec 6.1.2, the off-shellness is a small effect for our purposes.)
Now consider the double scattering term of Fig. 4.1. The nucleon lines shown with crosses indicate that we keep the poles corresponding to the situation that the spectator nucleon is on-shell before and after the secondary scattering vertex. In order to verify that we are justified in taking only these poles, let us consider the integrations over $p_{s,0}$ and $p'_{s,0}$ individually. Fixing, for the moment, $p'_{s,0}$ and considering the poles in the $p_{s,0}$ plane, we find,

\[
p_{s,0} = \pm \sqrt{p_{s}^2 + m_{N}^2} \mp i\epsilon \quad \text{(C.4)}
\]

\[
p_{s,0} = M_D \mp \sqrt{p_{s}^2 + m_{N}^2} \pm i\epsilon \quad \text{(C.5)}
\]

\[
p_{s,0} = (q - l + p'_{s})_0 \mp \sqrt{(q - l + p'_{s} - p_{s})^2 + M_V^2} \pm i\epsilon. \quad \text{(C.6)}
\]

The situation here is very similar to what appeared in the Born term. Closing the contour down, we pick up three poles corresponding to three terms in the scattering amplitude.

Of the poles with negative imaginary parts, only the pole in Eq. (C.4) is significant for the same reasons discussed from the Born term. In particular, if we take the pole corresponding to Eq. (C.5) with a negative imaginary part, then the struck nucleon will have a negative energy and $p_{s,0} \approx 3m_N$. To see what happens when we take the pole corresponding to Eq. (C.6), we can calculate the energy of the struck nucleon after rescattering:

\[
p'_{s,0} = p_{s,0} - (q - l)_0 - \sqrt{(q - l + p'_{s} - p_{s})^2 + M_V^2}. \quad \text{(C.7)}
\]

The deuteron wave function constrains the energy of the spectator to be $p_{s,0} \sim m_N$ up to terms quadratic in nucleon momentum. However, since $(q - l)_0 \gtrsim 0$ and,

\[
\sqrt{(q - l + p'_{s} - p_{s})^2 + M_V^2} \gtrsim M_V^2,
\]

then, for non-zero $M_V$, we find that the calculated value of $p'_{s,0}$ is either negative or is much less than the nucleon mass. Either case is strongly suppressed by the deuteron wavefunction, so the pole in Eq. (C.6) is also neglected. Thus, we take the single remaining pole which corresponds to the condition that the spectator nucleon before rescattering is on-shell. A symmetric argument can then be constructed to establish that when taking the $p'_{s,0}$ integral, the significant pole is the one corresponding to the condition that the spectator nucleon after rescattering is on-shell.
C.2 Recovery of the Conventional Formalism in the VMD-Glauber Approximation.

Now let us verify that Eq. (5.5) reduces to the quantum mechanical Glauber formula when we use the formalism of non-relativistic quantum mechanics. First, we review the basic assumptions and approximations of the traditional Glauber model:

The Regge limit, \( s \gg t \) with fixed \( t \), is assumed, but relativistic effects in the deuteron wave function are neglected. In considering the high energy limit for the incident photon, one usually ignores Fermi movement completely and approximates \( s \approx 2E_\gamma M_d \approx 4E_\gamma m_N \) and \( \hat{s} \approx 2E_\gamma m_N \). Note that deuteron binding energy is neglected as well in this approximation. In the high energy limit, the Feynman amplitude is normalized so that the optical theorem reads,

\[
\frac{1}{\hat{s}} \Im \hat{F}(\hat{s}, t = 0) = \hat{\sigma}_{\text{tot}}^{\gamma N}.
\] (C.9)

To see explicitly what we neglect, we write the center of mass energy as,

\[
\hat{s} = 2E_\gamma m_N \left( 1 + \frac{k_z}{m_N} + \frac{m_N}{2E_\gamma} + \mathcal{O}\left(\frac{k^2}{m_N^2}\right) \right).
\] (C.10)

As long as we do not consider processes that are sensitive to large values of \( k \), then we can note that the deuteron wave function constrains \( k^2 \ll m_N^2 \). The mass term is neglected at large photon energies. Also, it is usually assumed that \( \frac{k_z}{m_N} \) is small and also is neglected. This term is actually non-negligible for precise calculations and it does not vanish in the high energy limit. It is usually argued that the term linear in \( k_z \) vanishes in the high energy limit because the product of wave functions becomes symmetric around \( k_z \). We note in the main text, however, that for non-zero \( t \), \( l_z \) is a finite value even in the high energy limit, and thus the wave function never becomes precisely symmetric around zero momentum. Neglecting \( k_z/m_N \) is required, however, in order to factor the basic amplitude out of the integral in Eq. (5.3) and form the usual first terms in a Glauber series. Note that if there is any significant dependence of \( \hat{s} \) upon the internal nucleon momentum, then the factorizability becomes problematic.

The simplifications and assumptions we are making now are not valid in the situation discussed in the main body of chapter 5. Indeed, near threshold we expect the amplitude to have non-trivial dependence upon center of mass energy (sensitive to variations in nucleon momentum) so that the Glauber factorization approximation is not precisely valid. In our intermediate energy calculations, we will continue to ignore terms of order \( \frac{|k|^2}{m_N^3} \), but we will keep all terms of order \( \frac{k_z}{m_N} \) in order to correct for Fermi movement. Furthermore, near threshold, we are not in the extreme Regge limit, so we will need to keep the exact form of the flux factor appearing in Eq. (5.16).
Let us apply the above assumptions to the effective Feynman diagram calculation and verify that we can reproduce the usual conventional Glauber-VMD expression. If \( f_m(s, t) \) denotes the non-relativistic scattering amplitude, then the optical theorem is,

\[
\frac{4\pi}{|q|} \Im f(s, t = 0) = \sigma_n^{hn}. \quad (C.11)
\]

This gives us the following rules relating the normalizations of the Feynman amplitudes with the non-relativistic amplitudes:

\[
\hat{F}(\hat{s}, t) = \frac{4\pi}{|q|} \hat{f}(\hat{s}, t) \approx \frac{8\pi E\gamma m_N}{|q|} \hat{f}(\hat{s}, t) \quad (C.12)
\]

\[
F_{m,m'}(s, t) = \frac{4\pi s}{|q|} f_{m,m'}(s, t) \approx \frac{16\pi E\gamma m_N}{|q|} f_{m,m'}(s, t). \quad (C.13)
\]

Using Eq. (C.13) to replace Feynman amplitudes with non-relativistic amplitudes in Eqs. (5.5), and factorizing the basic amplitude out of the integral by applying the approximations listed in the previous paragraph yields,

\[
f_{\gamma^dVd}(E\gamma, l) \approx \hat{f}(E\gamma, l) \int d^3k \tilde{\Psi}_{m'}^\dagger (k - \frac{l}{2}) \tilde{\Psi}_m(k) + (n \leftrightarrow p). \quad (C.14)
\]

Now, shifting the integration variable by \( k \rightarrow k + \frac{1}{4} \), we have

\[
f_{\gamma^dVd}(E\gamma, l) \approx \hat{f}(E\gamma, l) \int d^3k \tilde{\Psi}_{m'}^\dagger \left( k - \frac{1}{4} \right) \tilde{\Psi}_m \left( k + \frac{1}{4} \right) + (n \leftrightarrow p) \quad (C.15)
\]

where we have used the definition of the non-relativistic deuteron form factor,

\[
S^{m,m'}(l) = \int d^3k \tilde{\Psi}_{m'}^\dagger \left( k - \frac{1}{2} \right) \tilde{\Psi}_m \left( k + \frac{1}{2} \right). \quad (C.16)
\]

Thus, we have recovered the first two Born terms in the usual non-relativistic Glauber expansion. This is also the form of the Born term appearing in, e.g. [21], where relativistic Regge amplitudes are used.
Appendix D

The Non-Relativistic Deuteron Wave Function

D.1 Deuteron Polarization

In this appendix, we give an overview of the treatment of deuteron spin as it is presented in [93]. In order to evaluate the cross section, we must determine how the operator, $\bar{\Psi}_m^\dagger(k - \frac{1}{2})\bar{\Psi}_m(k)$ acts on the spin-1 ground state of the deuteron. The non-relativistic deuteron wave function in momentum space is written in terms of S and D states via the formula,

$$\bar{\Psi}_m(k) = \left[\bar{u}(k) - 8^{-1/2}\bar{w}(k)\hat{S}_{12}\right]|\hat{q}, m\rangle$$ \hspace{1cm} (D.1)

where,

$$\bar{u}(k) \equiv \frac{1}{\sqrt{2\pi}} \int_0^\infty rdrj_0(kr)u(r)$$

$$\bar{w}(k) \equiv \frac{1}{\sqrt{2\pi}} \int_0^\infty rdrj_2(kr)w(r).$$ \hspace{1cm} (D.2)

The real functions, $u(r)$ and $w(r)$, are taken from any realistic model of the deuteron wave function, and in our computations we use the Paris potential [94]. The functions, $j_0$ and $j_2$, are the zeroth and second order spherical Bessel functions. In Eq. (D.1), $|\hat{q}, m\rangle$ is a spin-one spinor representing the total angular momentum of the deuteron, and $\hat{q}$ is the quantization axis. The tensor operator, $\hat{S}_{12}$ acts upon the total angular momentum state to produce a sum over total spin states. In terms of the spins of the nucleons, it is given by:

$$\hat{S}_{12} = \frac{3(\sigma_1 \cdot r)(\sigma_2 \cdot r) - \sigma_1 \cdot \sigma_2}{r^2}.$$ \hspace{1cm} (D.3)
Here, $\sigma_1$ and $\sigma_2$ are the spin operators (divided by two) for the proton and neutron respectively. The projection onto total spin states is,

\[
\hat{S}|0, \vec{q}\rangle = \sqrt{\frac{48\pi}{5}} Y_2^1(\theta, \phi)|1\rangle - \sqrt{\frac{64\pi}{5}} Y_2^0(\theta, \phi)|0\rangle + \sqrt{\frac{48\pi}{5}} Y_2^{-1}(\theta, \phi)|1\rangle
\]

\[
\hat{S}|1, \vec{q}\rangle = \sqrt{\frac{16\pi}{5}} Y_2^0(\theta, \phi)|1\rangle - \sqrt{\frac{48\pi}{5}} Y_2^{-1}(\theta, \phi)|0\rangle + \sqrt{\frac{96\pi}{5}} Y_2^{-2}(\theta, \phi)|1\rangle
\]

\[
\hat{S}|1, \vec{q}\rangle = \sqrt{\frac{96\pi}{5}} Y_2^2(\theta, \phi)|1\rangle - \sqrt{\frac{48\pi}{5}} Y_2^1(\theta, \phi)|0\rangle + \sqrt{\frac{16\pi}{5}} Y_2^0(\theta, \phi)|1\rangle.
\]

The functions, $Y$, are the usual spherical harmonic functions. With these equations, we can calculate the effective form factor for each polarization, and then sum and average over final/initial deuteron polarizations.
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Vita

Born in Columbus, Ohio, Ted Rogers received the Bachelor of Arts degree in Physics and Mathematics from Kenyon College in May of 2000. He enrolled in the Ph.D. program at the Pennsylvania State University in August 2000.