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MODELING OF INVERTED ANNULAR FILM BOILING USING AN INTEGRAL METHOD

A Thesis in

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by

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Abstract

In modeling Inverted Annular Film Boiling (IAFB), several important phenomena such as interaction between the liquid and the vapor phases and irregular nature of the interface, which greatly influence the momentum and heat transfer at the interface, need to be accounted for. However, due to the complexity of these phenomena, they were not modeled in previous studies. Since two-phase heat transfer equations and relationships rely heavily on experimental data, many closure relationships that were used in previous studies to solve the problem are empirical in nature. Also, in deriving the relationships, the experimental data were often extrapolated beyond the intended range of conditions, causing errors in predictions. In some cases, empirical correlations that were derived from situations other than IAFB, and whose applicability to IAFB was questionable, were used. Moreover, arbitrary constants were introduced in the model developed in previous studies to provide good fit to the experimental data. These constants have no physical basis, thereby leading to questionable accuracy in the model predictions.

In the present work, modeling of Inverted Annular Film Boiling (IAFB) is done using Integral Method. Two-dimensional formulation of IAFB is presented. Separate equations for the conservation of mass, momentum and energy are derived from first principles, for the vapor film and the liquid core. Turbulence is incorporated in the formulation. The system of second-order partial differential equations is integrated over the radial direction to obtain a system of integral differential equations. In order to solve the system of equations, second order polynomial profiles are used to describe the non-dimensional velocity and temperatures. The unknown coefficients in the profiles are functions of the axial direction alone. Using the boundary conditions that govern the physical problem, equations for the unknown coefficients are derived in terms of the primary dependent variables: wall shear stress, interfacial shear stress, film thickness, pressure, wall temperature and the mass transfer rate due to evaporation. A system of non-linear first order coupled ordinary differential equations is obtained.

Due to the inherent mathematical complexity of the system of equations, simplifying assumptions are made to obtain a numerical solution. The system of equations is solved numerically to obtain values of the unknown quantities at each subsequent axial location. Derived quantities like void fraction and heat transfer coefficient are calculated at each axial location. The calculation is terminated when the void fraction reaches a value of 0.6, the upper limit of IAFB. The results obtained agree with the experimental trends observed. Void fraction increases along the heated length, while the heat transfer coefficient drops due to the increased resistance of the vapor film as expected.

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Nomenclature

A	Cross-sectional flow area, m^2
a_0, a_1, a_2	Coefficients in the non-dimensional vapor equation
b_0, b_1, b_2	Coefficients in the non-dimensional liquid equation
c_0, c_1, c_2	Coefficients in the non-dimensional vapor temperature equation
d_0, d_1, d_2	Coefficients in the non-dimensional liquid temperature equation
c_p	Specific heat at constant pressure, J/kg-K
g	Acceleration due to gravity, m/s^2
h	Enthalpy, J/kg
h	Heat transfer coefficient, W/m^2-K
k	Thermal conductivity, W/m-K
P	Pressure, Pa
P	Perimeter, m
q	Heat transfer rate, W
Pr	Prandtl number, dimensionless
r	radial coordinate, m
R	Radius of the channel, m
Re	Reynolds number, dimensionless
T	Temperature, K
u	Velocity component in axial direction, m/s
v	Velocity component in radial direction, m/s
y	Radial coordinate defined from the wall inwards for a pipe, m
z	Axial coordinate, m

Greek Symbols

α	Void fraction
δ	Thickness of vapor film, m
ε_m	Eddy diffusivity of momentum, m^2/s
ε_H	Eddy diffusivity of heat, m^2/s
μ	Dynamic viscosity, Pa-s

η	non-dimensional length, $\left(y^{+}\right)^{1/7}$
ν	Kinematic viscosity, m^2/s
ρ	Density, kg/m^3
σ	Stefan-Boltzmann constant, $W/m^2\text{-}K^4$
τ	Shear stress, N/m^2

Subscripts

i	Interface
l	Liquid
rad	radiation
sat	Saturation
v	Vapor
w	Wall

Superscripts

$'$	Fluctuating component
$''$	Per unit area
t	Turbulent
–	(Overbar) Mean component

Other Symbols

u^{+}	Non-dimensional axial component of velocity
v^{+}	Non-dimensional radial component of velocity
u_v^{*}	Friction velocity, m/s
y^{+}	Non-dimensional y coordinate, $y^{+} = \dfrac{y u_v^{+}}{\nu_v}$
$\dot{m}_l''$	Rate of mass flux transferred by evaporation or condensation, $kg/m^2\text{-}s$

Acknowledgements

'gurulEka eTuvaNTi guNiki teliyaga pOdu'- shrI thyAgarAja

Translation: No one, however virtuous he may be, without the grace of a teacher will know how to cut through the forest of mental ills – says Sri. Thyaagaraaja (a 17th century saint-composer of Carnatic music).

My advisors, Dr. Hochreiter and Dr. Cheung deserve special thanks for their guidance, support and encouragement during the course of this work. In particular, I am grateful to Dr. Cheung for his suggestions and physical insights into the seemingly complicated mathematical problem, and for the timely feedback. I would not have started my Ph.D., if not for the discussions I had with Dr. Kulkarni during my early days at Penn State. His words of encouragement meant a lot to me.

My sincere thanks to my office colleagues, Ruwan, Jong and Sule, in particular, for merely being there at all times. Many thanks are also due to many of my friends from the State College group – *Raaga*, who have always been a family away from home during the last several years. The Saturday evening music get-togethers will be cherished forever.

My parents, though several thousand miles away, have been a constant source of strength and encouragement all through my academic life and more so during my stay at Penn State. But for their support and faith in my abilities, this may not have been possible. Lastly, I would like to convey my deep appreciation to my wife, Hema, who patiently supported me through the last but the most challenging year of my academic life at Penn State.

Chapter 1

Introduction

1.1 Background

Safety analysis is performed on nuclear reactor power plants to ensure safety from accidents that are postulated to occur. Accidents that may occur over the lifetime of the plant, as well as hypothetical accidents which are not expected to occur, are postulated to determine the mitigating failures of the particular reactor design.

Within the reactor design basis, the most challenging accident that is examined is a large-break Loss of Coolant Accident (LOCA). Analysis of this particular accident can result in limits in the reactor total core power as well as the allowable peak linear fuel rod power in the hottest rods. For this type of accident, the initial coolant in the reactor core is lost through the broken piping and the core cooling is dependant on the Engineered Safeguards Systems. The analysis carried out verifies whether the design of the Engineered Safeguards Systems will mitigate the accident. In a large-break LOCA, the fuel rod cladding is calculated to rupture at high temperature, and the primary piping is assumed to have failed. Without adequate core cooling, the reactor will continue to overheat and this can lead to the failure and release of fission products from the fuel.

Study of the equilibrium pool boiling curve indicates that film boiling occurs beyond the transition region. The lowest heat flux associated with film boiling is called the minimum film boiling heat flux. The temperature associated with this point is the minimum film boiling temperature. It is the temperature at which a stable continuous vapor blanket can first exist between the surface and the bulk liquid. Film boiling is a very inefficient heat transfer process, as large temperature differences are required to produce small heat transfer rates.

1.2 Inverted Annular Film Boiling

During cooling of nuclear reactors, film boiling is expected to occur during a transient induced by a postulated large break Loss of Coolant Accident (LOCA). After the coolant has been lost from the core, the fuel elements are surrounded by vapor providing relatively poor natural convection heat transfer environment. Since the fuel elements are still generating heat, the temperatures become very high. To cool the core and to limit the fuel element temperature, emergency core cooling coolant is flooded from the bottom of the vessel into the core. However, at this time, the temperature of the fuel cladding can be well above the minimum film boiling temperature. Therefore, initial cooling of the reactor is accomplished through film boiling heat transfer. Several factors like liquid velocity and liquid subcooling influence the heat transfer rate in film boiling. A good understanding of the heat transfer processes occurring during reflooding stage is of paramount importance for the determination of peak cladding temperature that is likely to occur.

At relatively high flooding rates, typically 6 inches/sec (0.15 m/sec), when the wall temperature is too high for the liquid to rewet the wall, and particularly if the liquid at the axially progressing quench front is *subcooled*, a liquid column is formed downstream from the quench front separated from the hot wall by a thin vapor film. This flow regime is known as Inverted Annular Film Boiling (IAFB).

Inverted annular film boiling is characterized by a vapor film covering the heating surface. This film separates the liquid core, at or below the saturation temperature at that pressure, from the heating surface. IAFB involves convective heat transfer from the hot wall to the vapor blanket and from the vapor to the liquid core. Heat may also be transferred from the hot rods to the liquid core by radiation. The heat transfer coefficient associated with this mode of film boiling is relatively small primarily because of the low thermal conductivity of the vapor.

In the IAFB regime, heat is transported by conduction and radiation through the vapor film to the interface from where it is conducted into the subcooled core. For a subcooled liquid, part of the heat is used for vaporization and part for reducing the subcooling by condensing the vapor film and keeping it thin. For a saturated liquid, the heat is used exclusively for vaporization, hence rapidly increasing the vapor film thickness. Modeling of IAFB depends critically upon the interfacial heat transfer between the vapor and the liquid core. The net interfacial heat transfer determines the rate of vapor generation and hence the film thickness. Rapid steam generation accelerates the low-viscosity, low density vapor more easily than the liquid core and produces a high steam velocity such that the liquid column gets sheared into liquid ligaments, thus drastically increasing the interfacial heat transfer surface area.

Experimental observations show that heat transfer in IAFB region increases rapidly with the liquid subcooling. Higher subcooling promotes heat transfer to the liquid in the core and reduces vapor generation and hence thickness of the vapor film, thus enhancing heat transfer. At high flow rates, a strong increase of the heat transfer coefficient with mass flux is observed.

IAFB regime will terminate due to the growth of the vapor film. As IAFB region progresses, the difference between the velocities of the liquid and vapor increases, eventually, the liquid core will break up into droplets. Typically the IAFB region can be more than 0.3 m (1 ft), but it depends on the flooding rate into the bundle. The void fraction will change from near zero to very high values as the flow regime transitions from IAFB to dispersed flow film boiling.

For low flooding rates, typically around 0.0254 m/sec (1 inch/sec), there may be no subcooled inverted annular film boiling region. Because of low injection flow rate, the liquid quickly reaches saturation and there is bulk boiling of fluid below the quench front. In the region between the quench front and above it, there is a froth region in which void fraction changes from a low value, just below the quench front and a very high value, close to unity in the dispersed flow regions above the quench front. The dominant flow regime for the low flooding rate case is a dispersed flow film boiling region in which the heat transfer rates are very small. The heat transfer in this region occurs between the heated wall and the superheated steam. The liquid droplets in the superheated steam evaporate reducing the steam temperature as well as increasing the steam flow rate. As a result, calculated peak cladding temperature usually occurs in this region. Figure 1-1 shows a schematic of low and high flooding rate situations. The figure to the right represents IAFB situation.

1.3 Scope and Objectives of the Proposed Study and the Current Approach

Full analytical solution of the conservation equations for IAFB are too complex, hence empirical correlations best describing the physics of the phenomenon are needed to bring closure to the problem of modeling of Inverted Annular Film Boiling. Current models for Inverted Annular Film Boiling involve use of empirical correlations that were derived from situations other than IAFB, and whose applicability to IAFB was questionable.

There are various complex physical phenomena that contribute to enhancement of interfacial momentum and heat transfers in IAFB. The effect of various complex physical processes, such as interfacial waves, oscillations of the liquid core, droplet entrainment and redeposition and turbulence in the vapor film may significantly improve interfacial transport. Not all these are accounted for in earlier studies. The recent work by Cachard [4] models these phenomena using empirical correlations based on experimental data.

In the present work, the physical problem is formulated in two dimensions. Separate equations for the conservation of mass, momentum and energy are derived from first principles, for the vapor film and the liquid core. Turbulence is incorporated in the formulation. The system of second-order partial differential equations is integrated over the radial direction to obtain a system of integral differential equations. In order to solve the system of equations, non-dimensional axial component of velocity and temperature profiles of the form, $A_0 + A_1\eta + A_2\eta^2$ are assumed, where $\eta = \left(y^+\right)^{1/7}$, is a definition commonly used in turbulence formulations. The unknown coefficients in the profiles for the velocity and temperature are functions of the axial direction alone. Equations for the coefficients are derived in terms of the primary dependant variables: wall shear stress, interfacial shear stress, film thickness, pressure, wall temperature and the mass transfer rate due to evaporation.

Due to the inherent complexity of the system, simplifying assumptions are made to obtain a numerical solution to the system of equations. At each axial location, the simplified system of equations is solved numerically to obtain values of the dependant variables. Derived quantities like void fraction, heat transfer coefficient are evaluated. This procedure is repeated for successive axial locations. IAFB regime is assumed to terminate when the void fraction reaches 0.6.

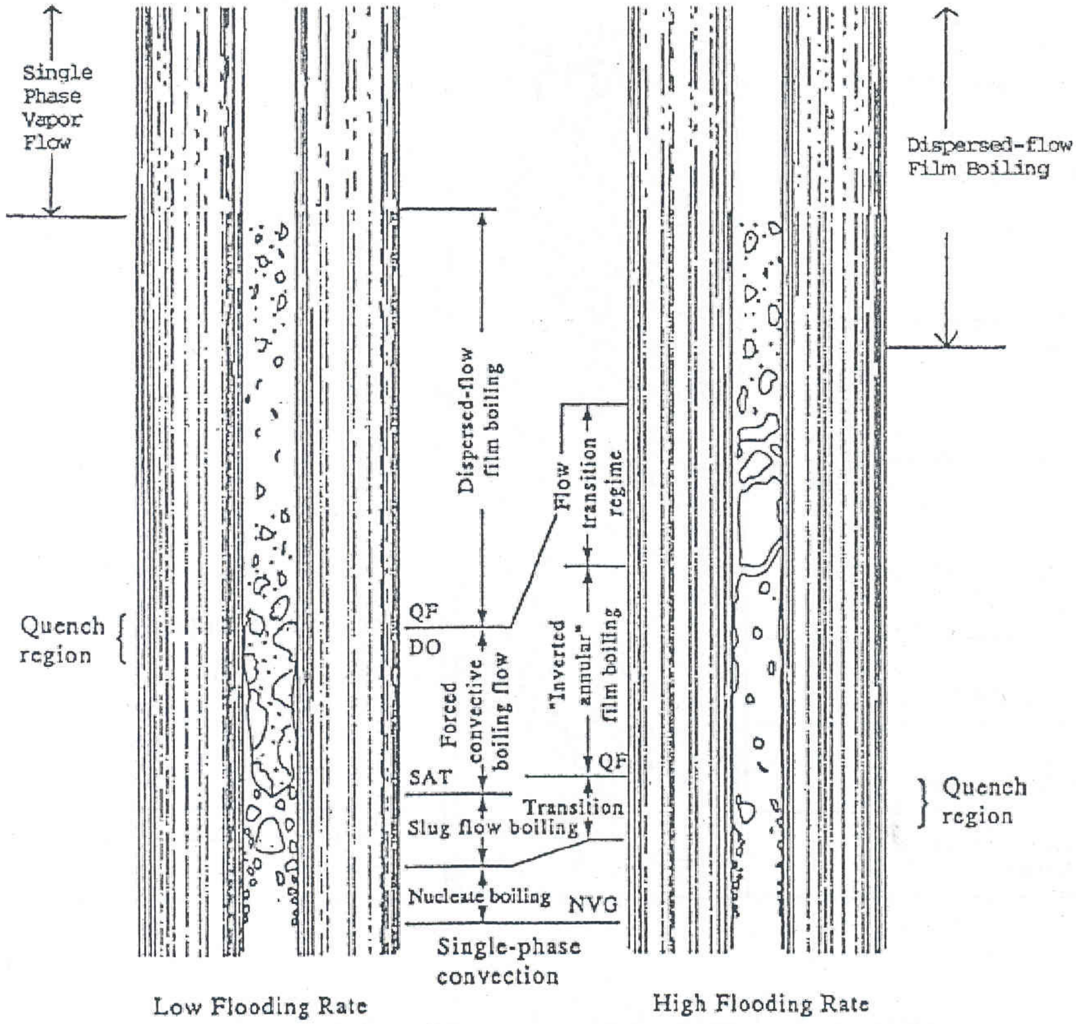

Figure 1-1: Reflood Flow Regimes for High and Low Flooding Rate [15]

Chapter 2

Literature Survey

2.1 Introduction

In order to understand the processes involved in Inverted Annular Film Boiling (IAFB), an extensive survey of available literature was conducted. Though film boiling has been studied for over fifty years, the information pertaining to IAFB is restricted to a few classical models. This is mainly due to the inherent complexity associated with the phenomenon of IAFB. This chapter presents a brief summary of the available information in the field of Film boiling in general and Inverted Annular Film Boiling in particular.

2.2 Classical Models in Film Boiling

Amongst the earliest references is the classical paper by Bromley [3]. Although the model is very restrictive because of several assumptions, this is nevertheless a primary source of reference in the field of film boiling.

Bromley studied laminar vertical film boiling following the approach of Nusselt for film-wise condensation.

The Bromley correlation is based on the following assumptions:

1. The wall temperature is constant.
2. The vapor flow is laminar and is controlled by the balance between shear and buoyancy forces only.
3. Heat transfer is by conduction only, and all of the heat supplied to the vapor goes into evaporation of the liquid phase.
4. The liquid-vapor interface is smooth and the vapor film is continuous.
5. The liquid is uniformly maintained at its saturation temperature,
6. The physical properties of the vapor can be evaluated at a temperature equal to the arithmetic average of the wall and liquid temperature.

The interfacial shear stress is evaluated considering two extreme cases:

1. The vapor flows between two stationary parallel planes.
2. The liquid moves in such a way that there is no interfacial shear.

These two assumptions alter the magnitude of the numerical coefficient in the expression obtained for the heat transfer coefficient.

The equation is developed based on experiments carried out on the boiling of liquid from the outside of horizontal tubes. The correlation for a vertical surface is obtained by applying the same theory as for the horizontal tube. Vapor film from outside the horizontal tube is in dynamic equilibrium as it rises under the action of the buoyant forces. Heat is supplied by conduction and radiation across the film.

The Bromley correlation for a vertical plate is given by

$$\bar{h} = 0.943\left[\frac{\tilde{h}_{fg}\, g \rho_g k_g^3 \left(\rho_f - \rho_g\right)}{\Delta T \mu_g L}\right]^{1/4} \tag{2-1}$$

where the value of the constant 0.943 is obtained from Y.Y. Hsu and Westwater [16].

ρ_f and ρ_g are the densities of the vapor and the liquid in ft^3/lbm.

k_g is the thermal conductivity of the vapor in Btu/hr-ft-F.

L is the length of the vertical surface in the direction of vapor flow, ft. (This is actually the distance from the location of the quench front to the point of interest in the IAFB region).

ΔT is the temperature difference between hot surface and the liquid at boiling point, ^oF.

$$\tilde{h}_{fg} = h_{fg}\left[1 + 0.34\frac{c_{p,v}\left(T_s - T_{sat}\right)}{h_{fg}}\right]^2 \tag{2-2}$$

Equation 2-2 represents the effective latent heat of vaporization accounting for the sensible heat of the vapor.

It is seen that the actual Bromley model incorporates a characteristic length term, which changes with the motion of the quench front. As the quench front advances, this value of L decreases, thereby indicating an increase in the heat transfer coefficient with advancing quench front.

The work by Hsu and Westwater [16] is another classical reference for film boiling studies based on Y.Y. Hsu's Ph.D. thesis [17]. Hsu argues that the Nusselt's model for condensation, which forms the basis for the Bromley model is not valid at the onset of turbulence because of the high liquid-vapor velocity difference that exists once turbulence begins. Hsu and Westwater postulate that Bromley's assumption of smooth, viscous film is incorrect, especially at and beyond a critical Reynolds number, when the flow becomes turbulent. The critical Reynolds number is given to be 100.

Their model considers vapor flow between a vertical solid heating surface and a body of liquid at its saturation temperature. The vapor flows upward because of buoyancy and flow is resisted by the drag at the hot solid and the vapor – liquid interface. The model states that in the lower portion of a vertical solid heating surface, the amount of vapor flow is small, so it is reasonable to assume that the flow is viscous, the film is thin and the film thickness changes with height as predicted by the Bromley model. This simple flow situation is seen to exist until a critical Reynolds number is reached after which a sharp transition to turbulence occurs. For this region, Hsu and Westwater used the universal velocity profile of Prandtl and Nikuradze. The buffer layer as assumed by Prandtl and Nikuradze is omitted, only the laminar sublayer and the turbulent core are considered. Most theoretical studies indicate that the laminar and turbulent regions show an intersection at $y^+ = 10$. So, Hsu assumed that the transition from the laminar sub-layer to turbulent core occurs at $y^+ = 10$. Thus, the critical Reynolds number is 100. The schematic used for the model development is shown in Figure 2-1.

The temperature in the turbulent core is assumed to be nearly constant and equal to the saturation temperature due to the good mixing achieved there. The velocity profile in the turbulent core is also flat. The laminar sublayer that exists next to the solid surface is assumed to provide the entire resistance to heat transfer. Based on a simple force balance and energy balance on a vapor element in the turbulent core of the vapor film, bounded by the laminar turbulent core interface on one side and the liquid-vapor interface on the other, an integral differential equation is obtained.

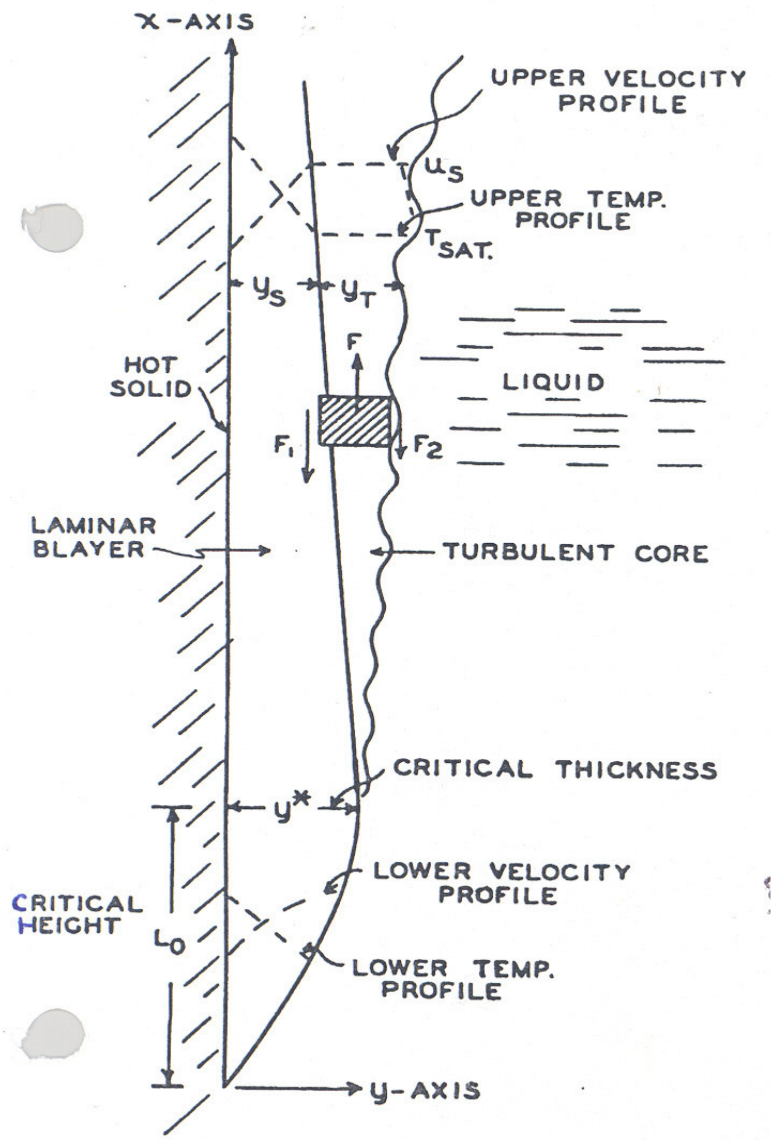

Figure 2-1: Schematic for Y.Y. Hsu's Model [16]

The analysis based on the above assumptions, after a lot of mathematical manipulations yield the following equations [16]:

$$\frac{\bar{h}L}{k_v} = \frac{2\lambda'\mu_v \operatorname{Re}^*}{3k_v\Delta T} + \frac{\left(B+\frac{1}{3}\right)}{A}\left[\left\{\frac{2}{3}\left(\frac{A}{B+\frac{1}{3}}\right)(L-L_o)+\left(\frac{1}{y^*}\right)^2\right\}^{3/2}+\left(\frac{1}{y^*}\right)^3\right] \tag{2-3}$$

where

$$A = \left[\frac{g(\rho_l - \rho_v)}{\rho_v}\right]\left(\frac{\bar{\rho}_v}{\mu_v \operatorname{Re}^*}\right)^2 \tag{2-4}$$

$$B = \frac{\mu_v + \frac{f\rho_v\mu_v \operatorname{Re}^*}{2\bar{\rho}_v} + \frac{k_v\Delta T}{\lambda}}{\frac{k_v\Delta T}{\lambda}} \tag{2-5}$$

$$y^* = \left[\frac{2\mu_v^2 \operatorname{Re}^*}{g\bar{\rho}_v(\rho_l - \bar{\rho}_v)}\right]^{1/3} \tag{2-6}$$

$$\operatorname{Re}^* = \frac{y^* u^* \bar{\rho}_v}{\mu_v} \tag{2-7}$$

$$L_o = \frac{\mu_v \operatorname{Re}^* \lambda' y^*}{2k_v\Delta T} \tag{2-8}$$

and $\lambda' = \lambda\left[1 + \frac{0.34c_v\Delta T}{\lambda}\right]^2$ (2-9)

In the above set of equations, λ refers to the latent heat of vaporization and λ' refers to the latent heat plus the sensible heat content of the vapor. L_o refers to the location of critical Reynolds number equal to 100. The other symbols are defined in the nomenclature section.

Tests were conducted using stainless steel and copper tubes of outside diameter 0.375" to 0.75", length between 2 to 6.3 inches. Five different liquids (methanol, benzene, carbon tetrachloride, nitrogen and air) were used for the experiments. It must be noted that water, with a high value of latent heat of vaporization was not tested.

This model predicts an increase in heat transfer coefficient with increase in length and increase in temperature difference. However the experimental data show the contrary. The equations predict heat transfer coefficients, which agree with experimental values within about ± 32%. This could be due to the fact that the model assumes the entire resistance to be in the laminar sublayer. In reality, the turbulent core also contributes to the resistance. The resistance to heat transfer increases along the axial direction, an aspect not considered by Hsu in his model.

The work by Dougall and Rohsenow [8] is another classical reference for film boiling studies. They developed a model to predict the local wall temperature and heat transfer coefficients for film boiling heat transfer inside a vertical tube with upward flow. The model was based on the following assumptions:

1. The wall heat flux is constant.
2. The entire heated length is in film boiling.
3. The liquid is saturated at entry into the heated portion of the tube.
4. Flow in the vapor film is turbulent except at entry.
5. Vapor flow is considered as flow between parallel plates (plane wall and liquid vapor interface).
6. Momentum forces are very much less than pressure and viscous forces.
7. Velocity distribution in vapor film is very similar to the universal velocity distribution profile. This is given as

Viscous sublayer: $$u^+ = y^+ ; \quad 0 \le y^+ \le 5 \qquad (2\text{-}10)$$

Buffer region: $$u^+ = -3.05 + 5.0 \ln y^+ ; \quad 5 \le y^+ \le 30 \qquad (2\text{-}11)$$

Turbulent core: $$u^+ = 5.5 + 2.5 \ln y^+ ; \quad 30 \le y^+ \le \frac{\delta^+}{2} \qquad (2\text{-}12)$$

This distribution is for half the vapor film thickness. Its mirror image is used for the other half.

The shear stress at the interface is postulated to have a value between zero and the wall shear stress. However, it is seen that for the case with zero shear stress at the interface and universal velocity profile that reached the maximum at the interface, heat transfer coefficients obtained were significantly higher than the experimental results. Therefore, it is assumed that the interface shear stress is the same as the wall shear stress.

Prandtl number for most vapors is about unity, hence the eddy diffusivities for momentum and heat transfer are assumed to be equal. The shear stress is assumed to be equal to the wall shear stress for the viscous sublayer and the buffer layer. The shear stress is assumed to be linear for the turbulent core region. It is given by

$$\tau = \tau_w \left(1 - \frac{y^+}{\delta^+ / 2} \right) \tag{2-13}$$

The distribution of eddy diffusivity for the vapor film is given by

Viscous sublayer: $\frac{\varepsilon}{\nu_v} = 0$; $0 \le y^+ \le 5$ (2-14)

Buffer region: $\frac{\varepsilon}{\nu_v} = \frac{y^+}{5} - 1$; $5 \le y^+ \le 30$ (2-15)

Turbulent core: $\frac{\varepsilon}{\nu_v} = \left(1 - \frac{y^+}{\delta^+ / 2} \right) \frac{y^+}{2.5}$; $30 \le y^+ \le \frac{\delta^+}{2}$ (2-16)

The thermal resistance in the other half of the vapor film, next to the liquid vapor interface is called the interface resistance. Dougall and Rohsenow considered three different expressions for the interface resistance.

1. A LBT interface theory where the interface resistance is equal to the complete resistance in the other half of the film.
2. A BT interface resistance theory where the interface resistance is equal to the sum of the turbulent and the buffer resistances in the other half of the film.
3. A T interface resistance theory where the interface resistance is equal to just the turbulent resistance in the other half of the film.

This theory, based on an annular flow model with turbulent vapor flow agreed well with experimental data at low qualities and indicated an asymptotic behavior towards the DFFB regime. Flow regimes, local wall temperatures and local heat transfer coefficient for film boiling inside a vertical tube with upward flow of saturated liquid (Freon-113) were studied by using a transparent test section made of electrically conducting glass tubing so that the flow could be visualized. They used two different diameter test sections: 0.408 inch I.D. and 0.180 inch I.D. The entire region of boiling at constant wall heat flux in the tube was by film boiling. The flow involved only low vapor qualities and with small liquid velocities. Visual observations indicated that the flow regime was inverted annular with a liquid core surrounded by vapor at the walls.

The work by Takenaka et al. [26] is another effort in the modeling of Inverted Annular Flow. The following are the salient features of the model:

1. One dimensional energy balance, neglecting axial heat conduction.
2. Neglecting effects of phase change, radial distributions of the velocity and temperature of both phases are calculated by a developed turbulent boundary layer model. Reynolds analogy is assumed.
3. Reichardt's equation for the turbulent diffusivity coefficient is employed.
4. Radiation heat transfer is neglected.
5. Void fraction is assumed small.
6. Physical properties are evaluated at saturation temperature.

Within the film, the shear stress is zero at the location of maximum velocity. The eddy diffusivity of momentum is also symmetrical about the location of maximum velocity. Unlike Dougall and Rohsenow's work this location is not at half the film thickness. The eddy diffusivity does not reach zero at the interface but is some finite value. This is the unique feature of this model and is similar to observed phenomenon.

The shear stress is related to the gradient of velocity as

$$\tau = \rho(\varepsilon + \nu)\frac{du}{dy}$$

Reichardt's equation near the wall is used for the vapor film from the wall to the location of maximum velocity. Reichardt's equation is again used for this region from the location of maximum velocity to the interface.

For the liquid core, eddy diffusivity obtained using Reichardt's equation for wall region is equated to the eddy diffusivity obtained from the equation for the center region. This gives the location up to which each equation can be used so that the eddy diffusivity is continuous. Using this limiting location, the velocity distribution is obtained for the liquid core.

Thus the entire region from the wall to the centerline is split into 4 regions:

1. Vapor Film split into two regions: One region of the vapor film from the wall to the location of maximum vapor velocity and the other region from the location of maximum velocity to the liquid-vapor interface.
2. Liquid Core split into two regions: One part from the liquid-vapor interface to the location where the eddy diffusivity is continuous, while the other part is from this location to the centerline.

Velocity profile is obtained for each region. Temperature distribution is calculated by Reynolds analogy.

Figure 2-2 shows the radial distributions of shear stress, velocity and temperature from the model and comparison of analytical heat transfer coefficients with experimental data.

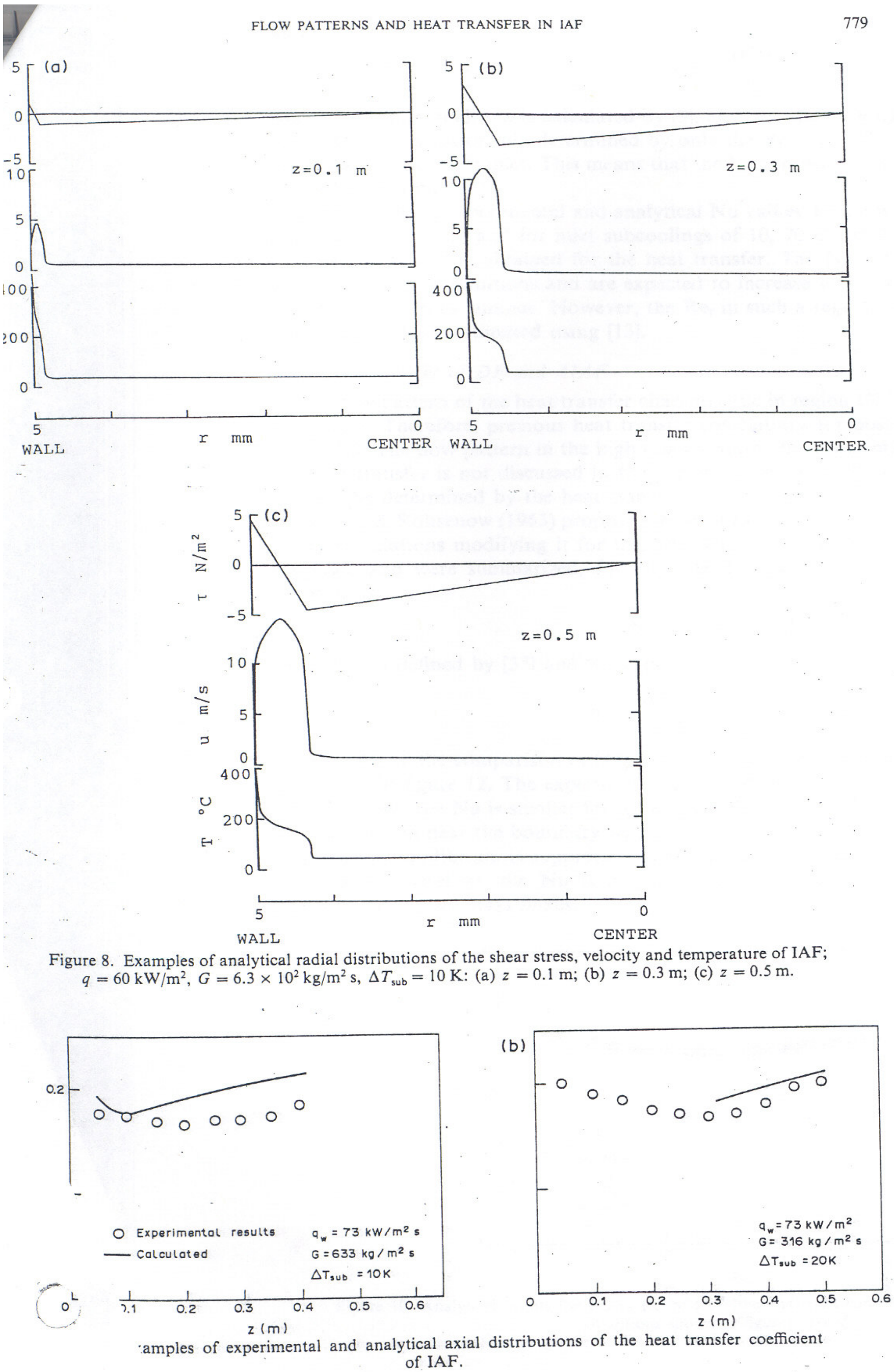

Figure 2-2: Radial distributions of shear stress, velocity and temperature from Takenaka's model and comparison of analytical heat transfer coefficients with experimental data.

Based on a two fluid formulation, a mechanistic model for the Inverted Annular Film Boiling region was developed by Analytis and Yadigaroglu [1].

Interfacial velocity is assumed to be the same as the liquid velocity. To account for the effect of non-smoothness of the interface on the interfacial exchanges, the model includes an enhancement factor given by Wallis [30]. This is given in the equation below:

$$\lambda = 1 + 150\left(\frac{\delta}{R}\right) \tag{2-17}$$

where δ denotes the film thickness and R the tube diameter. This factor is applied to both the vapor interface momentum and heat transfer equations. Also, this enhancement factor λ, is applied only to the vapor-interface heat flux and not to the wall-vapor heat flux.

The entrance length effect is approximately accounted by applying the following enhancement factor, to the interface-liquid heat transfer:

$$F = 1 + 1.4\left(\frac{R}{z}\right) \tag{2-18}$$

In the above equation, R denotes the tube radius and z denotes the distance from the quench front. In addition to this, another enhancement factor is needed (a constant value of 2.5 is used) in some cases, in order to get acceptable agreement between experimental data and computational work. This is attributed to “roughness” effect.

The predictions of the model have been successfully compared to heat transfer coefficients from experimental data from a series of single tube reflooding experiments. The model predictions agree with experimental data for a number of cases. However, it does not do a good job of predicting cases with high subcooling, high flooding rate and high wall temperature. In order to fit the data well, a factor of 2.5 is used along with the enhancement factor F, as discussed above. Samples of the results are shown in Figures 2-3 and 2-4.

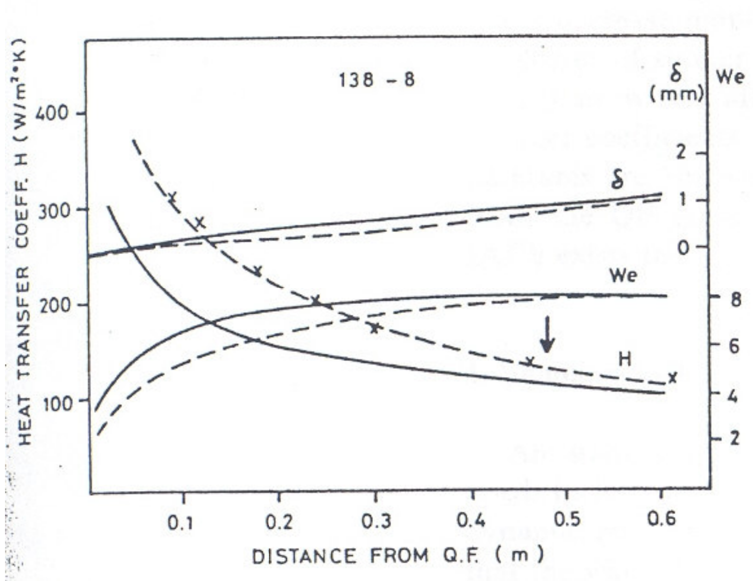

Figure 2-3: Analytis and Yadigaroglu Model - Intermediate subcooling, very high reflooding rate, very high wall temperature case (Run 138-8, 14-7-70).

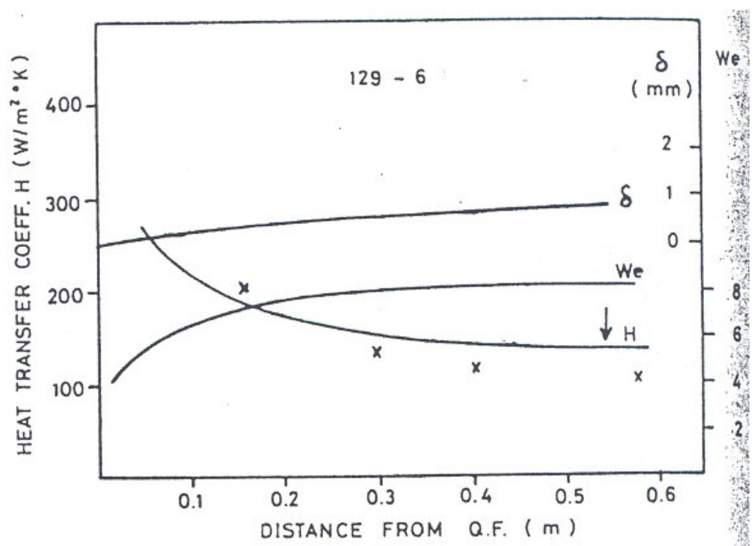

Figure 2-4: Analytis and Yadigaroglu Model - High subcooling, high reflooding rate, high wall temperature case (Run 129-6, 12-7-70).

The authors recommend detailed analysis around the quench front region and an additional enhancement factor for high flooding rates.

The work by Cachard [4] builds on the work by Yadigaroglu and Analytis [1]. This work is the most recent and complete work related to Inverted Annular Film Boiling. A six-equation model is implemented with closure laws specific to the IAFB regime. Closure laws applicable to both tubes and various bundle geometries have been proposed.

In the model proposed, convective liquid heating is related to the liquid velocity relative to the interface, and not to the absolute liquid velocity as in the previous models. This relative velocity is deduced from the interfacial shear stress, using the liquid - interface friction law. With this modification, the prediction of the experimental trends is greatly improved. The vapor is treated as flow between two parallel plates (wall and the liquid interface).

In the closure laws proposed, the effects of non-smoothness of the liquid vapor interface (waves, oscillations of the liquid core, droplet entrainment and redeposition, and turbulence in the vapor film) are also accounted by semi-empirical laws based on physical models. These phenomena significantly affect interfacial transfer.

The model introduces three enhancement factors, $\lambda_{fv}, \lambda_{hv}$ and λ_l which represent enhancement factors for vapor interface momentum transfer, vapor film heat transfer and interface liquid transfers respectively. These are the key points of the model by Cachard. A non-dimensional film thickness δ^* is used as a correlating parameter for the vapor-interface transfer enhancement during IAFB in reflooding situations. The enhancement factors are given by the following relations.

$$\lambda_{fv} = 0.0362\left(\delta^*\right)^{1.96} \tag{2-19}$$

$$\lambda_{hv} = 0.679\left(\delta^*\right)^{0.509} \tag{2-20}$$

$$\lambda_l = 4.18\left(1 + 0.7\frac{D_h}{z}\right) \tag{2-21}$$

$$\delta^* = \delta\left[\frac{\rho_v(\rho_l - \rho_v)g}{\mu_v^2}\right]^{1/3} \tag{2-22}$$

D_h represents the hydraulic diameter and z represents distance from the quench front.

The correlation obtained for λ_l (interface transfers) does not fit well with available experimental data. This is due to the lack of proper understanding of the physical phenomena. In addition to this, when z is small, the value of the enhancement factor is maximized to 10. This has no physical basis.

The work considers three possible geometries: flow in a tube and flow in square or triangular lattice rod bundles. Heat transfer within the liquid core is deduced from momentum transfer using the Chilton Colburn analogy. The model can be applied to steady state as well as reflooding conditions, with very different geometries, i.e. tubes and rod bundles with hydraulic diameters ranging from 4 to 14 mm, and for large parameter ranges, i.e. 3 to 50 cm/s in flooding rate, 0 to 30^{o}C in subcooling, 1 to 4 bar in pressure, and 300 to 1000^{o}C in wall temperature. Forced flow subcooled film boiling experimental results from four different sources were analyzed. The model was successfully assessed against 57 experiments corresponding to very different geometries and parameter ranges. IAFB was assumed to end when the void fraction reached 60%. Calculations were stopped arbitrarily at this point.

Thus, even the model provided by Cachard has a lot of empirical factors, some of which satisfy a wide range of experimental data. Here too, the factor accounting for the vapor-liquid interaction is questionable and needs further analysis based on physical phenomena. This work also recommends a thorough study of the region near the quench front to account for the physical phenomena involved in that region.

Elias and Chambre [9] studied IAFB from vertical surfaces. The study is based on the existence of two distinct regions of inverted annular film boiling. The first region is assumed to occur immediately following the quench front and is characterized by a smooth vapor film of constant thickness separating the bulk liquid from the hot wall. The vapor layer thickness depends upon the flow velocity and subcooling. The second region is considered to be the region where a wavy and unstable liquid vapor interface is observed.

The result of the study is a model based on the solution of the energy conservation equations of the liquid and the vapor regions downstream of the quench front. The model is developed *for the region of constant vapor thickness* only. The region of wavy liquid vapor interface is not considered in the analysis. The following assumptions are also made:

1. All the vapor is generated in the region at the quench front and the vapor is at the saturation temperature at this location.
2. The liquid core and the liquid-vapor interface are below the saturation temperature, so that no evaporation occurs from the liquid, hence there is no interfacial mass transfer.
3. A constant (uniform) velocity profile is assumed for the annular vapor film.

The model also uses two existing correlations for determining the quench front velocity that is needed to determine the vapor velocity.

The predictions of the model have been compared to some experimental data by choosing values of vapor film thickness. The value of vapor film thickness is not verified experimentally and hence could be a source of error. The region of applicability of the model is relatively small compared to the region where the interface begins to get wavy and eventually transitions to high void fraction DFFB region. Thus, the model is applicable to a very small portion of the IAFB region.

Denham [7] extended the Bromley model to account for the effects of flow rate and quality. Net rate of evaporation of water is determined by the net flux to the steam-vapor interface. This is given by:

$$\frac{d\Gamma_g}{dz} = \frac{(\phi_w + \phi_r - \phi_i)}{i'_{fg}} \tag{2-23}$$

where Γ_g is the mass flow rate of vapor per unit width of plate, ϕ_w is the heat flux from the wall by conduction to the vapor film, ϕ_r is the heat transferred by radiation and ϕ_i is the heat flux across the vapor film-liquid interface removed by conduction in to the subcooled liquid core. These heat fluxes are given by the following expressions:

$$\phi_w = \frac{k_g}{\delta}\Delta T \tag{2-24}$$

$$\phi_r = \frac{\sigma\left[T_w^4 - T_f^4(z)\right]}{\dfrac{1}{\varepsilon_f} + \dfrac{1}{\varepsilon_w} - 1} \tag{2-25}$$

$$\phi_i = -k_f \left(\frac{\partial T}{\partial r}\right)_i = k_f \frac{\left[T_{sat} - T_f(z)\right]}{\pi \alpha_f t} \frac{1}{1 + 3.75\left(\alpha_f t / r^2\right)} \tag{2-26}$$

where α_f is the thermal diffusivity of liquid, the thickness of the vapor film used in the Bromley equation is given by

$$\delta = \left[\frac{4\Gamma_g \mu_g}{g\rho_g\left(\rho_f - \rho_g\right)}\right]^{1/3} \tag{2-27}$$

and where r is the radius of the liquid core (approximately D/2) and the time t is that taken by the liquid core to rise from the start of the film boiling region (z = 0).

These equations form a system of first-order non-linear differential equations that are solved using different boundary conditions. Four different ‘options’ are considered for the initial flow rate and core thermal properties. Experimental measurements of heat flux and wall temperature just ahead of the quench front are obtained from the single tube reflooding experiments. Data from the single tube reflooding experiments compared well with the predictions when the constant in the equation for δ is changed from 4 to 5.20. Values of the emissivity for the wall and the film are 0.8 and 0.95 respectively. Inverted annular flow is said to break down when the core Weber number exceeds the order of 20. The result of the analysis is shown in Figure 2-5.

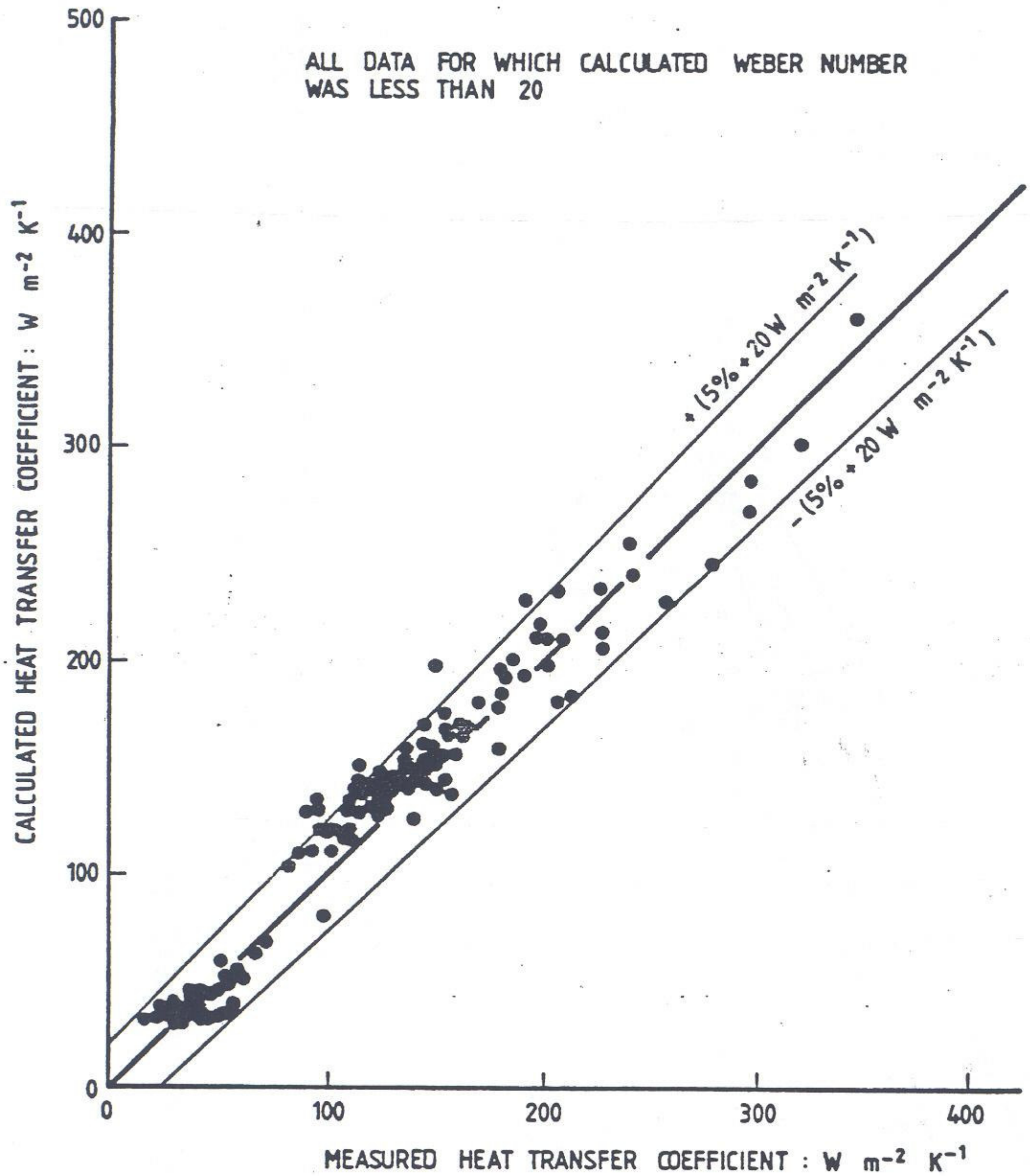

Figure 2-5: Comparison of measured and calculated heat transfer coefficients – Denham's model.

2.3 Other Relevant Works in Film Boiling

De Jarlais and Ishii [19] studied IAFB experimentally. They observed that Inverted annular flow occurs when low quality flow is coupled with wall surface temperature and heat flux values too high to allow liquid/wall contact. The resulting flow pattern is a liquid core surrounded by a blanketing annulus of vapor. Inverted annular flow occurs in light water reactor accident situations in which, after a Loss of Coolant Accident (LOCA) core reflood brings coolant into the confined regions between very hot rods. For inverted annular flow, the shape of the liquid/vapor interface, the stability of the liquid jet core and the disintegration of the liquid core must be understood and their predictive methods established, so that the modeling of this regime and the development of interfacial transfer correlations for mass, momentum and energy can be done.

A two-fluid model was developed by Hammouda et. al. [12] to predict the wall temperature of a *tube* during Inverted Annular Film Boiling (IAFB). The two-fluid model predictions of heat transfer in IAFB are based on the concept of reduction of the number of degrees of freedom of the system. The model is based on general constitutive relations, with a minimum of empirical coefficients. The mathematical modeling involved the following:

- conservation equations for mass, momentum, and energy for each phase;
- relevant constitutive relations for mass, momentum and energy transfer
- boundary and initial conditions.

The constitutive relations are different from those developed in earlier two-fluid models. The relations for the shear stress and heat transfer rates are the major components in this model. The model predictions were compared to experimental data from four fluids (water, Freon-12, Freon-22 and Freon-134a). Instead of developing closure relationships for each unknown quantity, Hammouda et. al. attempt to provide physically sound relationships between groups of the unknown quantities - various heat flux terms are related to each other; similarly terms relating to shear stress (wall, interface) are related to each other. This reduces the number of degrees of freedom. However, in doing so, they make certain assumptions that are questionable. For example, they assume that the heat transfer coefficient from the wall to the vapor and the vapor to the interface are of the same order of magnitude. This could be true, but not in all situations, especially in extreme situations of very high subcooling or mass flux.

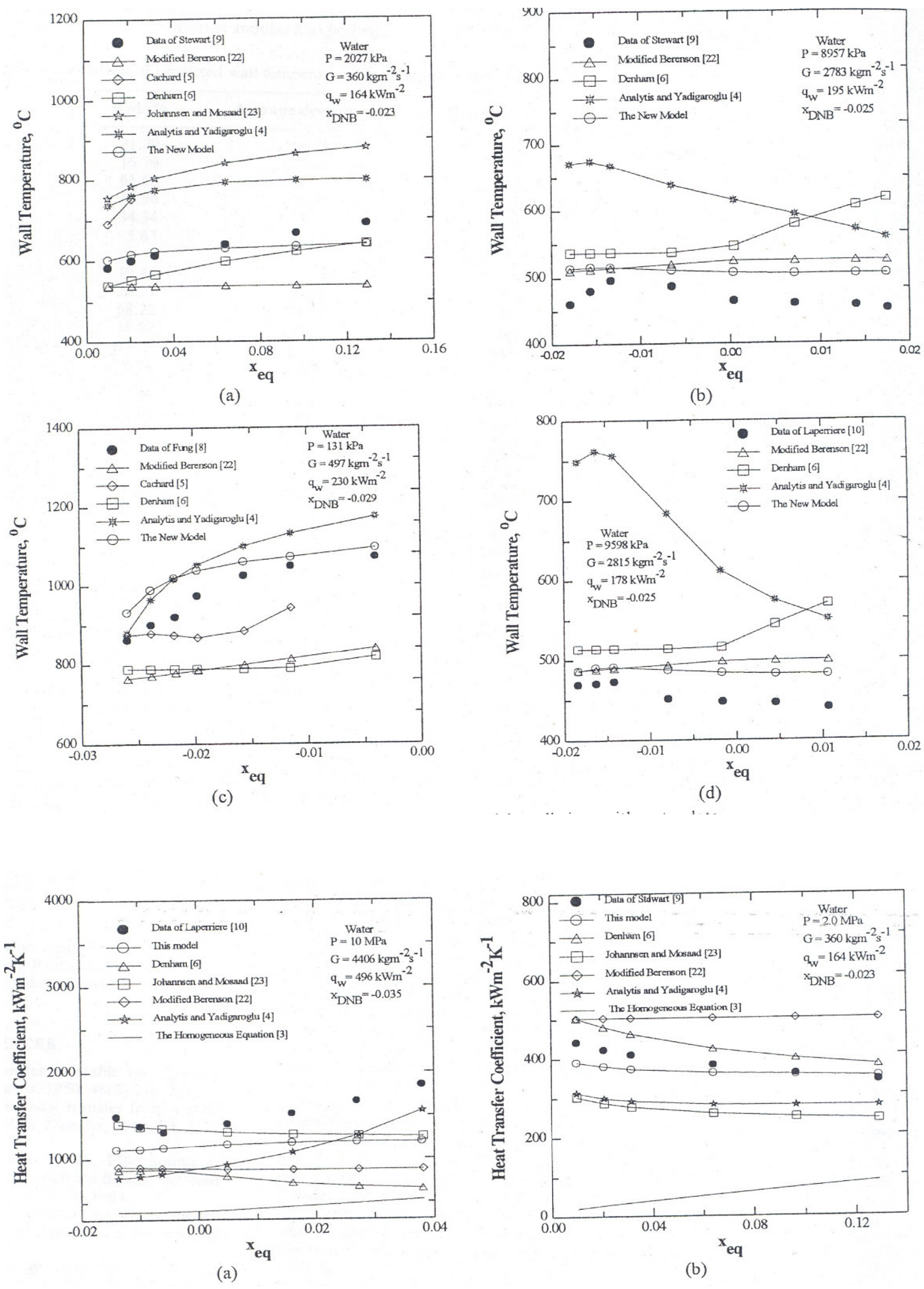

Figure 2-6: Comparison of results from Hammouda's model with other models and experimental data.

The authors acknowledge that despite several simplifying assumptions, the model agrees well with experimental data. This could be due to compensating errors too. Also, for high subcooling cases, this model overpredicts the wall temperature by a significant margin. This suggests that the heat transfer relations used in the prediction methods are no longer applicable and that the film thickness is over predicted. They suggest that reliable equations for flow in very small gaps be used to overcome this deficiency.

Jensen and Holdredge [20] outlined a mechanism for inception of liquid entrainment and derived a mathematical model was derived for the prediction of the inception of the liquid entrainment. They concluded that the mechanism for liquid entrainment was the breakup of the liquid phase due to the instability of the vapor-liquid interface in the film-boiling regime. The liquid vapor interface was assumed sinusoidal and the equations were derived by the simultaneous solution of the continuity and momentum equations. Equations were derived to calculate the growth rate of the unstable interface and the length the wave must travel before liquid entrainment occurs. The model was compared to available data from Emergency Core Cooling experiments.

Liquid entrainment is defined as the disintegration of the liquid phase in the inverse annular flow regime due to the instability of the liquid-vapor interface.

Review of several high-speed moving pictures of the emergency core coolant injection suggested that liquid entrainment results from the instability of the liquid vapor interface and the subsequent breakdown of the continuous liquid phase into large droplets of slugs of water.

The liquid phase break up progresses in the following manner: At a given location and at some time, the amplitude of the unstable wavy interface formed in the inverted annular regime is not yet large enough for liquid phase break up to occur. After some time, at that location; break up of the liquid phase occurs and the resulting length of the liquid slug is some multiple of the wavelength. After some time, further disintegration of the slug occurs. Further disintegration of the droplets is controlled by droplet inertia and surface tension forces.

Assumptions made in the derivation of the mathematical model are as follows:

1. Rectangular coordinates.

2. Both phases are ideal frictionless fluids.

3. Irrotational flow, velocity is uniform across each phase.

4. Mass transfer at the interface is neglected.

5. Interfacial shear is neglected.

6. Change in velocity of each phase in axial direction is negligible.

7. Film thickness remains constant.

8. Incompressible.

9. Interface is of the form $b = B\cos(Ky - \sigma t)$.

where:

b is the location of the wave surface in the x-direction, in.

B is the wave amplitude, in.

K is the wave number $\left(2\pi/\lambda\right)$, in^{-1}.

t is time, sec.

σ is the wave frequency, sec^{-1}.

These assumptions are not entirely physically true but they do simplify the continuity and momentum equations to a great extent and eliminate the energy equation altogether. The interface was modeled by a cosine function.

Neglecting mass transfer at the interface implies that there is no phase change occurring at the interface. This implies that the energy equation that couples the two phases does not play a role in the analysis. Neglecting interfacial shear also simplifies the momentum equation greatly. The momentum equations for the liquid and vapor phases are decoupled from one another. Assumption of constant film thickness implies that the interface is not moving in the x-direction, thereby fixing the coordinates to obtain a boundary condition to solve the system of equations.

The continuity equations for the liquid and vapor phases respectively are:

$$\frac{\partial u_l}{\partial x} + \frac{\partial v_l}{\partial y} = 0 \text{ and } \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0 \qquad (2\text{-}28)$$

The simplified momentum equation for the x-direction is written as:

$$-\frac{1}{\rho_l}\frac{\partial p_l}{\partial x}=\frac{\partial u_l}{\partial t}+u_l\frac{\partial u_l}{\partial x}+v_l\frac{\partial u_l}{\partial y}\ \text{(Liquid phase)} \tag{2-29}$$

$$-\frac{1}{\rho_g}\frac{\partial p_g}{\partial x}=\frac{\partial u_g}{\partial t}+u_g\frac{\partial u_g}{\partial x}+v_g\frac{\partial u_g}{\partial y}\ \text{(Gas phase)} \tag{2-30}$$

Because of the simplifying assumption of no phase change at the interface, there is no energy equation involved in the analysis.

These equations were then solved simultaneously to determine the growth rate of the interface waves. Based on some previous experimental work and basic wave propagation equations, they derived an expression for the entrainment length (length required to initiate entrainment). The dimensionless growth rate was found to be

$$\Pi=2\pi R\left[\frac{\bar{\rho}_l\,\bar{\rho}_g}{\bar{\rho}^2}-\frac{2\pi R}{We}\left(\frac{\rho_l}{\bar{\rho}}\right)\right]^{1/2} \tag{2-31}$$

where the overbars indicate that the properties are void fraction weighted. Thus, the growth rate of the interface waves as a function of the system parameters: phase velocity, void fraction, pressure and wavelength.

The dimensionless Weber number is given as :

$$We=\frac{\rho_l d V_{gl}^2}{T} \tag{2-32}$$

where

T is the surface tension

$d=\delta_l+\delta_g$ (δ refer to the thickness of the appropriate phase)

$$V_{gl} = V_g - V_l$$

$$R = \frac{d}{\lambda}$$

They also performed a parametric study of the growth rate. The effects of pressure, wavelengths, void fractions and Weber number were studied. It showed the existence of a maximum growth rate and a corresponding wavelength

Jensen and Holdredge also obtained an equation which related the wave growth rate to the length which a wave travels before the wave amplitude required to initiate liquid entrainment is achieved. This is called the entrainment length, which is also a function of the wave velocity and wave growth rate.

The trend of the predictions by the model agreed with the experimental results. However, the entrainment length predicted by the model was much smaller than the actual entrainment lengths observed. Since interfacial shear was neglected in the model development, they found that the wave velocity was nearly independent of the vapor velocity and was approximately equal to the liquid velocity. This is quite different from the actual observations in experiments.

This study by Jensen and Holdredge is quite useful in understanding the mathematical complexity of the problem. An analytical solution was obtained by making several assumptions, many of which cannot be made in modeling of reflood heat transfer. For example, neglecting interfacial mass transfer amounts to no interaction between the two phases and no phase change. This also means that the vapor film thickness would remain constant as assumed. Thus, one assumption may actually imply several other assumptions in the analysis. This model is a good simplified starting step in the modeling of the onset of entrainment. A more realistic model should consider the mass transfer at the interface, phase change at the interface and interfacial shear. This would bring in the energy equation and couple it with a more complex momentum equation. Analytical solution of such a system of equations may not be possible.

Barnea et. al. [2] used stability analysis to develop a criterion for transition from a steady inverted annular flow (IAF) to inverted slug flow during the rewetting of a heated vertical surface surrounded by an annular channel. The analysis was based on the Kelvin-Helmholtz instability using small perturbations on the main velocities due to the waviness of the liquid-vapor interface. Vapor velocity was calculated assuming that evaporation takes place only at the quench front. The analysis indicated that the stable wavelength depends upon velocities and film thickness of the two phases. The stable wavelength was calculated using an empirical parameter that was used to account for the heat transfer between the phases. The model predictions compared favorably to experimental results.

The work by Fung [10] is a complete theoretical and experimental work on inverted annular film boiling. Unlike previous experimental works, water was used as the working fluid. Fung obtained steady state, post dryout heat transfer data for vertical flow of water inside electrically heated tubes of about 12 mm inside diameter and 800 mm length. The data was obtained using the hot patch technique, in which an indirectly heated copper block brazed onto the inlet of the test section supplied the critical heat flux. The wall temperatures were measured at ten different locations and the heat transfer rates derived from these measurements taking into account axial conduction and heat losses. Gamma densitometer was used to measure the void fraction at five locations. The outcome of the experimental work was subcooled film boiling data for water under forced flow conditions inside a tube. The data covered a wide range of mass flux and inlet subcooling similar to those expected to exist in a nuclear fuel channel under accident conditions. Most previous data were using cryogenic fluids and not with water.

A theoretical model developed used a one-dimensional integral technique similar to the analysis of single phase boundary layers. The model predicts the surface temperature of a tube during Inverted Annular Film Boiling. The model considers each phase to flow separately without any entrainment of the other phase. It incorporates the effects of mass flux, inlet subcooling, pressure and hydraulic diameter. The model considers initial stage of the vapor film to be in laminar flow and as the film thickens, it changes to turbulent flow. A subcooled, rather than a saturated liquid core is considered for the analysis. The model assumed a uniform velocity in the liquid core, and the liquid core to be turbulent. In other words, the velocity boundary layer of the liquid core at the liquid-vapor interface is neglected. The eddy diffusivities of heat and momentum have been assumed to be equal in the analysis. An empirical relation from Wallis [30] is used for the interfacial friction factor. The predictions from the model have been compared to experimental results.

2.4 Models in COBRA-TF for Inverted Annular Film Boiling [28]

Heat transfer in film boiling is assumed to result from one of the two mechanisms: inverted annular film boiling (IAFB) or dispersed flow film boiling (DFFB).

Dispersed flow film boiling differs from IAFB in that there is no continuous liquid flow present in the flow field. Rather, the liquid phase is in the form of droplets that are dispersed in a continuous vapor phase. The void fraction is greater than 0.9. DFFB is modeled using a two-step method where the dominant heat transfer mode is forced convection to superheated steam. Heat fluxes due to wall to droplet radiation and droplet impingement are super-imposed on the vapor convective heat flux. Thus, the total heat flux is

$$q''_{DFFB} = q''_{FC} + q''_{R} + q''_{W-D} \tag{2-33}$$

where

q''_{FC} = Vapor convection heat flux

q''_{R} = Radiation heat flux

q''_{W-D} = Drop impingement heat flux

Vapor convection heat flux:

The superheat for the calculation of the vapor convection heat flux is determined by the interfacial heat transfer rate to the entrained droplets as a part of the hydrodynamic solution.
The vapor convection heat flux is given by:

$$q''_{FC} = \psi\left[h_{SPV}\left(T_w - T_v\right)\right] \tag{2-34}$$

The heat transfer coefficient h_{SPV} is calculated using the maximum of the values obtained by the following equations:

Dittus Boelter (Steam)

$$htc = 0.023\frac{k_v}{D_h}\left(\frac{G_v D_h}{\mu_v}\right)^{0..8}\left(\Pr_v\right)^{0.4} \tag{2-35}$$

FLECHT SEASET 161-rod bundle

$$htc = 0.0797\frac{k_v}{D_h}\left(\frac{G_v D_h}{\mu_v}\right)^{0..6774}(\Pr_v)^{0.333} \qquad (2\text{-}36)$$

Laminar Flow

$htc = Nu\left(\frac{k_v}{D_H}\right)$, where a value of 10 is assigned to the Nusselt number.

Some DFFB studies have shown that interfacial shear between dispersed particles and a continuous phase increases the turbulence levels and enhances the convective heat transfer. The two-phase enhancement factor for dispersed flow, ψ, is approximated by an extension of the analogy between wall shear stress and heat transfer, wherein the turbulent convection heat transfer coefficient is, to a first approximation, proportional to the square root of the shear stress.

$$h_{SPV} = (\tau_w)^{0.5} \qquad (2\text{-}37)$$

Then,

$$\psi = \frac{h_{2\phi}}{h_{SPV}} = \sqrt{\frac{\tau_{2\phi}}{\tau_w}} \qquad (2\text{-}38)$$

Heat transfer due to droplets striking the wall, q''_{W-D}, is evaluated using the drop deposition models of Ganic and Rohsenow [11].

$$q''_{WET} = S''_{DE} h_{fg} \eta \qquad (2\text{-}39)$$

where S''_{DE} is the drop migration rate towards wall and is calculated as:

$$S''_{DE} = C.k_D \qquad (2\text{-}40)$$

where

the deposition coefficient, $k_D = 0.102\left(\frac{\mu_v}{D_H \sigma \rho_l}\right)^{1/2}\left(\frac{f}{2}\right)^{1/2}\left(\frac{G_v}{\rho_v}\right)$ (2-41)

and the droplet concentration, $C = \left(\frac{G_D}{G_v}\right)\rho_v$

G_D and G_v are the droplet and vapor mass flux values respectively.

The drop evaporation efficiency, η, is approximated by

$$\eta = \exp\left[1 - \left(\frac{T_w}{T_f}\right)^2\right] \tag{2-42}$$

The radiative heat transfer, q''_R, is calculated using the subchannel based model.

Inverted annular film boiling is assumed to occur if void fraction is less than 0.4. The heat flux for the IAFB regime is computed from the larger of either the value calculated for dispersed flow film boiling or the value from the modified Bromley correlation:

$$q''_{BROM} = 0.62\left(\frac{D_H}{\lambda_c}\right)^{0.172}\left[\frac{k_g^3 g h'_{fg}\left(\rho_f - \rho_g\right)\rho_g}{D_H \mu_g \left(T_w - T_f\right)}\right]^{1/4}\left(T_w - T_f\right) \tag{2-43}$$

where

$$h'_{fg} = h_{fg}\left[1 + \frac{0.4 c_{pv}\left(T_w - T_f\right)}{h_{fg}}\right] \tag{2-44}$$

and

$$\lambda_c = 2\pi\left[\frac{\sigma g_c}{g\left(\rho_f - \rho_g\right)}\right]^{1/2} \tag{2-45}$$

The work by Momoki et. al. [23] involved prediction of the heat transfer around a finite length vertical cylinder using a separate model for each of the top, bottom and the circumferential surfaces of the cylinder. The results were also compared to experiments conducted using cylinders of six different materials and known aspect ratio. A simple explicit finite difference scheme was employed for predicting the temperature distribution inside the cylinder. The convective boundary condition was obtained from a modified Bromley equation with the effect of the vapor from the bottom surface for the vertical surface around the lower corner and another model for the region when the liquid vapor interface becomes wavy. The numerical results obtained from the predictions agreed with the conducted experiments quite well.

2.5 Models for Eddy Diffusivity of Momentum

Measured velocity profiles for fully developed turbulent flows in smooth circular ducts are well fitted by relations of the type

$$u^+ = C\left(y^+\right)^{1/n}$$

as pointed out by Schlichting. C and n are somewhat dependant on the Reynolds number $\mathrm{Re}_D = \frac{U_{av} D}{\nu}$. For $\mathrm{Re}_D \approx 10^5$, C = 8.74 and n = 7. For turbulent pipe flows, the following definitions are commonly used

$$y = R - r\ ,\ \eta = \frac{r}{R}$$

where R is the radius of the pipe. ‘y’ is the distance from the wall. Dimensionless quantities are formed by defining the following

$$u^* = \sqrt{\frac{\tau_w}{\rho}}$$, called friction velocity.

$$u^+ = \frac{\bar{u}}{u^*}\ ,\ y^+ = \frac{y u^*}{\nu}$$

Prandtl and Taylor introduced the concept of a transition between the laminar sublayer (a layer which allows turbulence effects to exist), where viscous effects prevail, and a fully turbulent core where the turbulent forces are dominant.

In turbulent flow formulations, time averaging results in terms that include product of fluctuating quantities. As is the convention, the product of the fluctuating components of the velocity is related to the turbulent shear stress as $-\rho\overline{u'v'} = \mu^t \frac{\partial \overline{u}}{\partial r}$ where the $\mu^t = \rho\varepsilon_m$. ε_m is called the eddy diffusivity of momentum, which is an unknown quantity.

Several researchers have developed empirical relations for eddy viscosity and consequently, the velocity distribution. The basic requirements of such models are:

1. The eddy viscosity must be represented by a smooth, continuous curve.
2. The eddy viscosity vanishes with the third power of the distance from the wall.
3. The gradient of the mean velocity distribution vanishes at the center of the pipe.
4. The continuity equation is satisfied.
5. The ratio of the bulk velocity to the centerline velocity must agree with the experimental data.

In 1877, Boussinesq postulated that the eddy viscosity of momentum is a constant. This was probably guided by the fact that molecular viscosity is a constant. But in reality, the eddy diffusivity of momentum is not a constant. Particularly near solid surfaces, it is dependant on velocity and geometry. Nevertheless, a constant ε_m is used for convenience. Later on, Prandtl devised the Prandtl mixing length theory based on ideas from kinetic theory of gases. It is given by:

$\varepsilon_m = l^2 \left| \frac{d\overline{u}}{dy} \right|$, where $l = K_1 y$ is the mixing length. $K_1 = 0.36$. (2-46)

Later on von Karman gave the mixing length as

$$l = K_3 \left| \frac{d\overline{u}/dy}{d^2\overline{u}/dy^2} \right| \text{ with } K_3 = 0.4 \qquad (2\text{-}47)$$

Even with Prandtl's mixing length theory, there is an unwarrantedly large discrepancy between data and prediction observed in the region very close to the wall.

von Karman introduced the concept of buffer region between the viscous sublayer and the turbulent core. Using the velocity distribution data of Nikuradse, the following regions were defined:

Viscous sublayer: $u^+ = y^+$; $0 \le y^+ \le 5$ (2-48)

Buffer region: $u^+ = -3.05 + 5.0 \ln y^+$; $5 \le y^+ \le 30$ (2-49)

Turbulent core: $u^+ = 5.5 + 2.5 \ln y^+$; $y^+ \ge 30$ (2-50)

This three-region profile suffers from a discontinuity in slope at $y^+ = 30$; moreover, the velocity gradient does not vanish at the center of the pipe. Thus, the logarithmic velocity distribution, although useful and often used for approximate predictions of turbulent velocity profiles, violates a number of physical conditions. Therefore, it cannot form a completely accurate description of the flow field.

After von karman, Deissler proposed that the eddy viscosity approaches zero as the square of the distance from the wall. His expression for the eddy viscosity in the neighborhood of the wall is:

$$\frac{\varepsilon}{\nu} = m u^+ y^+ \qquad (2\text{-}51)$$

where m was determined as 0.0119. Deissler's two region velocity profile is given by:

$$y^+ = \sqrt{\frac{\pi}{2m}} \exp\left(\frac{m u^{+}}{2}\right) erf\left(u^+ \sqrt{\frac{m}{2}}\right); \qquad y^+ \le 26 \qquad (2\text{-}51)$$

$$u^+ = 3.8 + 2.78 \ln y^+; \qquad y^+ > 26 \qquad (2\text{-}52)$$

This profile also suffers from the same disadvantages as the standard logarithmic velocity profile given by von karman.

Reichardt assumed the following relations:

$$\left.\frac{\varepsilon_m}{\nu}\right|_w = 0.4\left[y^+ - 11\tanh\left(\frac{y^+}{11}\right)\right]$$, for the region near the wall and (2-53)

$$\left.\frac{\varepsilon_m}{\nu}\right|_c = \frac{k}{6}\cdot\frac{\mathrm{Re}^*}{2}\left(1-\eta^2\right)\left(1+2\eta^2\right)$$, for the turbulent core. (2-54)

These equations satisfy the basic requirements given above, however the momentum equation cannot be solved in a closed form. Values of eddy viscosity rise from the wall to a maximum value approximately midway between the wall and the centerline. It then decreases at the centerline to a value that is generally 50% or more of the maximum value in the pipe. Reichardt's profile gives nearly uniform $\frac{\varepsilon_m}{\nu}$ in the central region. The high value of $\frac{\varepsilon_m}{\nu}$ in the central region testifies the effectiveness of vigorous turbulent mixing in increasing the effective viscosity there. Since turbulent viscosity is about two orders of magnitude more than the molecular value, the central core very nearly behaves as a solid slug.

Travis [29] proposed a modification of Reichardt's two region eddy viscosity model to describe mean velocity and eddy viscosity distributions in steady, fully developed isothermal pipe flows $\left(4\times10^3 < \mathrm{Re}_D < 5\times10^6\right)$. Travis used Reichardt's two-region empirical equations without the numerical constants as:

$$\left.\frac{\varepsilon_m}{\nu}\right|_w = k\left[y^+ - A\tanh\left(\frac{y^+}{A}\right)\right]$$ for the wall region and, (2-55)

$$\left.\frac{\varepsilon_m}{\nu}\right|_c = \frac{k}{6}\cdot\frac{\mathrm{Re}^*}{2}\left[1-(F\eta)^2\right]\left[\frac{2}{3}+2(F\eta)^2\right]$$ for the turbulent core. (2-56)

The factor F in the above equation is greater than 1. It has the effect of reducing the radius over which the central equation applies from $\eta = 1$ to $\eta = \frac{1}{F}$. This is done so that the two equations for eddy diffusivity may be equal at some distance away from the wall and that they join in a smooth fashion (i.e.. equal slopes). The ratio of the eddy viscosity of the centerline to the maximum value is chosen to be 0.75, because it agrees better with the shape of the velocity profiles. The factor 2/3 in the equation for the turbulent core is introduced for this purpose.

In addition to the above equations for eddy diffusivity of momentum, continuity equation

$$1 = \frac{\mathrm{Re}^{*2}}{\mathrm{Re}} \int_0^1 \eta d\eta \int_0^1 \frac{\eta}{(\varepsilon_m/\nu)+1} d\eta \tag{2-57}$$

and the equation for the centerline velocity

$$1 = \left(\frac{u_b}{\bar{u}_{\max}}\right) \frac{\mathrm{Re}^{*2}}{2 \cdot \mathrm{Re}} \int_0^1 \frac{\eta}{(\varepsilon_m/\nu)+1} d\eta \tag{2-58}$$

require to be satisfied. Thus, four non-linear equations should be solved for the four unknowns: k, A, F and the non-dimensional η. Travis obtained plots of k, A, F, η as a function of Reynolds number.

The model used by Travis provides excellent agreement with Laufer's experimental data and a better mean velocity distribution than Van Driest's analytical investigation.

2.6 Concluding Remarks

The study of literature in the field of IAFB suggests that modeling of IAFB still has several empirically dependent models and correlations, some of which should not be used due to lack of physical explanation or extrapolation beyond the range of parameters for which they were developed.

The work done by Bromley [3], Dougall and Rohsenow [8] represent classical studies in film boiling while those of Analytis [1] and Cachard [4] are the more recent and complete works in modeling IAFB. Despite the efforts put in, all the phenomena of interest are not modeled accurately.

The main challenge in implementing IAFB models into two-fluid codes is the proper choice of the interfacial heat and momentum transfer relationships. Interfacial heat exchange enhancements may be due to turbulence in the vapor film, violent vaporization at the quench front, liquid contacts with the wall at the quench front. Also, there is a large amount of vapor generated at the quench front (release of heat stored in the wall due to quenching). There is a great deal of uncertainty about this, since the vapor at the quench front is generated violently under highly non-equilibrium conditions. In general there are several adjustable parameters and assumptions influencing the results (introducing degrees of freedom into the model).

Chapter 3

Problem Formulation

3.1 Introduction

The physical phenomenon of Inverted Annular Film Boiling (IAFB) has been explained in Chapter 1. Models from the earliest, the Bromley model to the most recent ones are reviewed in Chapter 2. In addition to this, various models for eddy diffusivity of momentum have been described in the previous chapter.

A schematic for Inverted Annular Film Boiling is shown in Figure 3-1. In this chapter, the differential equations that describe the physical phenomena are derived from fundamental principles. The liquid core and the vapor film are considered separately. Governing equations of mass, momentum and energy conservation are derived for the liquid and vapor considering a differential control volume in each region. The effect of turbulence is introduced in the governing equations by defining the velocities, temperature, shear stress and pressure as the sum of a mean component and fluctuating component. Time averaging of the governing equations result in terms that can be re-defined in terms of eddy diffusivity of momentum and eddy diffusivity of heat. By definition, some of the terms are zero, thereby simplifying the governing equations.

The resulting set of governing equations is a system of eight partial differential equations in r and z.

3.2 Derivation of Governing Equations

Consider a control volume in the form of an elemental cylindrical ring of radius r, thickness dr and height dz. A schematic of the control volume chosen is shown in Figure 3-2. It must be noted that both the liquid and vapor velocities and the temperatures are all functions of both r and z.

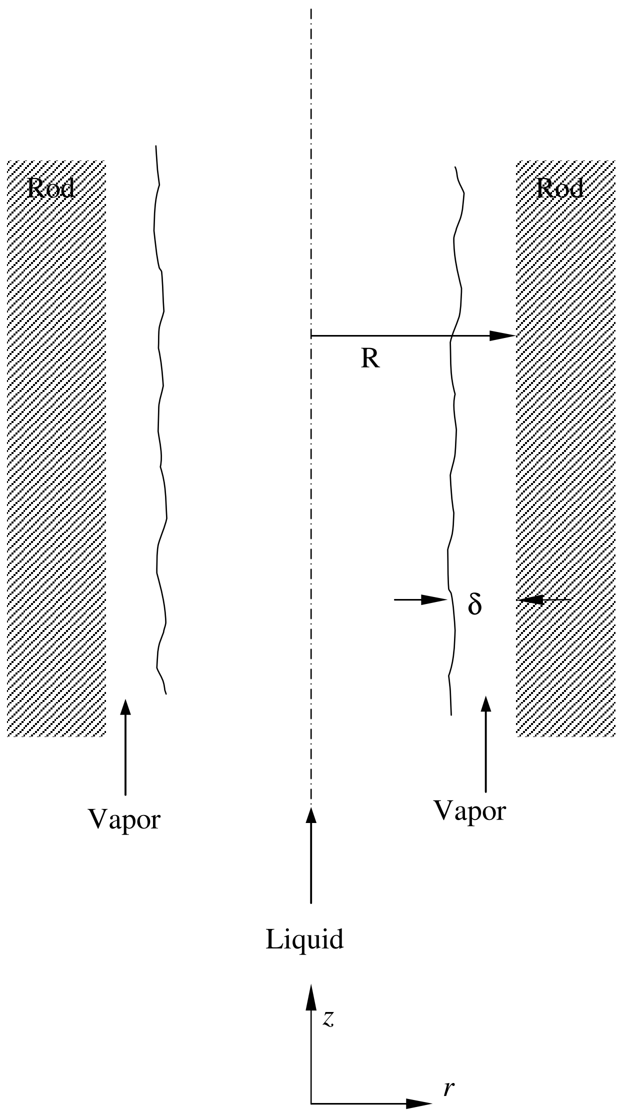

Figure 3-1: Schematic showing the flow configuration and coordinate axes for IAFB.

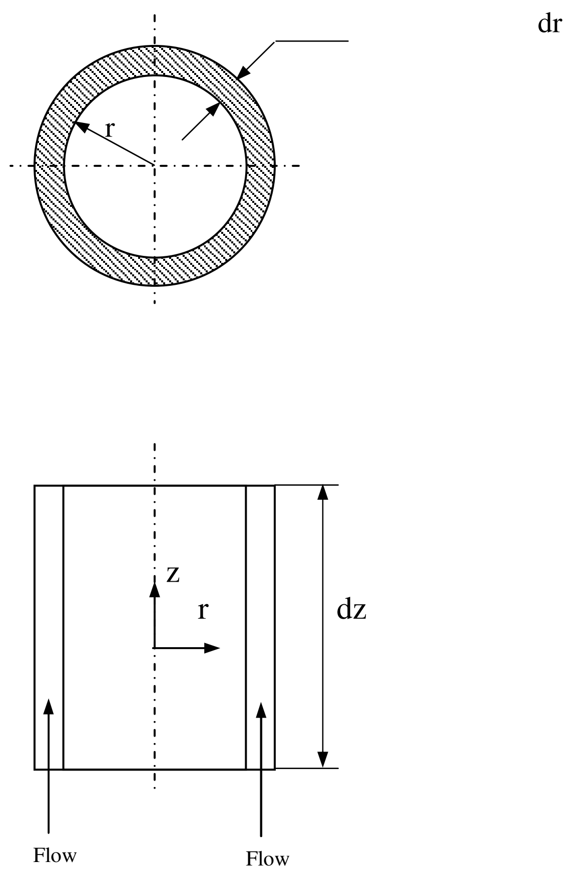

Figure 3-2: Control Volume for Analysis

3.2.1 Assumptions

1. The flow is quasi-steady. In the reflooding process, the quench front progresses slowly along the heated rods. Thus the phenomena change slowly relative to the frame of reference moving with the quench front.
2. The flow is two dimensional and axisymmetric. This is usually the case for channels in the center of the heated bundle.
3. $\rho_l = \overline{\rho}_l = \text{constant}$
4. Constant thermal conductivities of liquid and vapor phases.

Liquid region: $0 \le r \le R - \delta$
Vapor region: $R - \delta \le r \le R$

3.2.2 Turbulence Fundamentals

While describing a turbulent flow in mathematical terms, it is convenient to separate it into mean and fluctuating component. Denoting the time-average of the x-component of the velocity by $\overline{u}$ and its fluctuating component by u', the instantaneous velocity is given by:

$$u = \overline{u} + u'$$

Time averages are formed at a fixed point in space and are given by:

$$\overline{u} = \frac{1}{\Delta t} \int_{t}^{t+\Delta t} u dt$$

where Δt is a time interval large enough relative to the turbulent fluctuations, so that the mean values are completely independent of time. The time averages of all fluctuating quantities are equal to zero.

3.2.3 Conservation of Mass

Based on the assumptions and for the control volume chosen, the conservation of mass for the liquid and vapor phases based on the *mean* velocities are given below:

Liquid Phase: $\frac{\partial \bar{u}_l}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{v}_l\right) = 0$ (3-1)

Vapor Phase: $\frac{\partial\left(\rho_v \bar{u}_v\right)}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \bar{v}_v\right) = 0$ (3-2)

3.2.4 Conservation of Momentum

Since both the liquid and vapor velocities are functions of both r and z, the momentum equations will have two components for each phase.

3.2.4.1 Z direction Liquid Phase Equation

Rate of change of momentum: $\frac{\partial}{\partial z}\left[\rho_l \left(2\pi r dr\right) u_l u_l\right]dz + \frac{\partial}{\partial r}\left[\rho_l\left(2\pi r dz\right) u_l v_l\right]dr$ (3-3)

The above expression can be re-written as:

$$\left(2\pi r dr dz\right)\left[\frac{\partial}{\partial z}\left(\rho_l u_l u_l\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_l u_l v_l\right)\right] \quad \text{(3-3a)}$$

Forces acting on the control volume:

Pressure: $P\left(2\pi r\right)dr - \left(P + \frac{\partial P}{\partial z}dz\right)\left(2\pi r\right)dr = -\frac{\partial P}{\partial z}\left(2\pi r dr\right)dz$ (3-4)

Gravity: $-\rho_l g\left(2\pi r dr dz\right)$ (3-5)

Shear: $-\tau_{rz,l}\left(2\pi r dz\right) + \left\{\tau_{rz,l}\left(2\pi r dz\right) + \frac{\partial}{\partial r}\left[\tau_{rz,l}\left(2\pi r dz\right)\right]dr\right\} = \frac{\partial}{\partial r}\left[\tau_{rz,l}\left(2\pi r dz\right)\right]dr$ (3-6)

From Newton's second Law, the rate of change of momentum is equal to the sum of the forces. Combining terms given by 3-3a, 3-4, 3-5 and 3-6, we get:

$$\left[\frac{\partial}{\partial z}(\rho_l u_l u_l)+\frac{1}{r}\frac{\partial}{\partial r}(r\rho_l u_l v_l)\right]=-\rho_l g-\frac{\partial P}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz,l}) \tag{3-7}$$

Equation 3-7 represents the z direction momentum equation for the liquid phase.

3.2.4.2 R direction Liquid Phase Equation

With no body forces in the 'r' direction, the only forces of interest in the radial direction are the pressure forces and shear forces.

The resultant pressure force in the radial direction: $-\frac{\partial P}{\partial r}(2\pi r dz)dr$ (3-8)

Shear Forces: $\left\{-\tau_{zr,l}+\tau_{zr,l}+\frac{\partial}{\partial z}\tau_{zr,l}dz\right\}2\pi r dr=\frac{\partial}{\partial z}(\tau_{zr,l})2\pi r dr dz$ (3-9)

Rate of change of momentum: $\frac{\partial}{\partial z}[\rho_l(2\pi r dr)u_l v_l]dz+\frac{\partial}{\partial r}[\rho_l(2\pi r dz)v_l v_l]dr$ (3-10)

The above expression can be re-written as:

$$(2\pi r dr dz)\left\{\frac{\partial}{\partial z}(\rho_l u_l v_l)+\frac{1}{r}\frac{\partial}{\partial r}(r\rho_l v_l v_l)\right\} \tag{3-10a}$$

Combining 3-8, 3-9 and 3-10a, the momentum equation for the liquid phase in r direction can be written as:

$$\left[\frac{\partial}{\partial z}(\rho_l u_l v_l)+\frac{1}{r}\frac{\partial}{\partial r}(r\rho_l v_l v_l)\right]=-\frac{\partial P}{\partial r}+\frac{\partial}{\partial z}(\tau_{zr,l}) \tag{3-11}$$

Equations 3-7 and 3-11 represent the liquid momentum equations in the z and r directions respectively.

3.2.5 Turbulent Flow Formulation

3.2.5.1 Z direction momentum Equation

Expressing Equation 3-7 in terms of mean and fluctuating quantities, we can write:

$$\frac{\partial}{\partial z}\left[\rho_l(\bar{u}_l+u_l')(\bar{u}_l+u_l')\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\rho_l(\bar{u}_l+u_l')(\bar{v}_l+v_l')\right]=-\frac{\partial}{\partial z}\left(\bar{P}+P'\right)-\rho_l g+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\bar{\tau}_{rz,l}+\tau'_{rz,l}\right)\right] \qquad (3\text{-}12)$$

Statistically averaging, and invoking $\overline{\bar{u}_l u_l'}=\overline{u_l'\bar{v}_l}=\overline{\bar{u}_l v_l'}=0$, $\overline{\tau'}_{rz,l}=0$ and $\overline{P'}=0$, we get:

$$\frac{\partial}{\partial z}\left[\rho_l\left(\bar{u}_l\bar{u}_l+\overline{u_l'u_l'}\right)\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\rho_l\left(\bar{u}_l\bar{v}_l+\overline{u_l'v_l'}\right)\right]=-\frac{\partial\bar{P}}{\partial z}-\rho_l g+\frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{\tau}_{rz,l}\right) \qquad (3\text{-}13)$$

Re-arranging, we get:

$$\frac{\partial}{\partial z}\left(\rho_l\bar{u}_l\bar{u}_l\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_l\bar{u}_l\bar{v}_l\right)=-\frac{\partial\bar{P}}{\partial z}-\rho_l g+\frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{\tau}_{rz,l}\right)-\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_l\overline{u_l'v_l'}\right)-\frac{\partial}{\partial z}\left(\rho_l\overline{u_l'u_l'}\right) \qquad (3\text{-}14)$$

As is the convention, the product of the fluctuating components of the velocity is related to the turbulent shear stress as $-\rho_l\overline{u_l'v_l'}=\mu_l^t\frac{\partial\bar{u}_l}{\partial r}$ and using $\bar{\tau}_{rz,l}=\mu_l\frac{\partial\bar{u}_l}{\partial r}$, the two terms can be combined to give:

$$\bar{\tau}_{rz,l}-\rho_l\overline{u_l'v_l'}=\mu_l\frac{\partial\bar{u}_l}{\partial r}+\mu_l^t\frac{\partial\bar{u}_l}{\partial r}=\rho_l\left(\nu_l+\varepsilon_{m,l}\right)\frac{\partial\bar{u}_l}{\partial r} \qquad (3\text{-}15)$$

Using the above simplification, Equation 3-14 can be re-written as:

$$\frac{\partial}{\partial z}(\rho_l \bar{u}_l \bar{u}_l) + \frac{1}{r}\frac{\partial}{\partial r}(r\rho_l \bar{u}_l \bar{v}_l) = -\frac{\partial \bar{P}}{\partial z} - \rho_l g + \frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\rho_l(\nu_l + \varepsilon_{m,l})\frac{\partial \bar{u}_l}{\partial r}\right]\right\} - \frac{\partial}{\partial z}\left(\rho_l \overline{u_l' u_l'}\right) \tag{3-16}$$

The last term in Equation 3-16 can be written as:

$$-\rho_l \overline{u_l' u_l'} = \mu_l^t \frac{\partial \bar{u}_l}{\partial z} = \rho_l \varepsilon_{m,l} \frac{\partial \bar{u}_l}{\partial z}$$

Thus the z direction momentum equation for the liquid phase is given by:

$$\frac{\partial}{\partial z}(\rho_l \bar{u}_l \bar{u}_l) + \frac{1}{r}\frac{\partial}{\partial r}(r\rho_l \bar{u}_l \bar{v}_l) = -\frac{\partial \bar{P}}{\partial z} - \rho_l g + \frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\rho_l(\nu_l + \varepsilon_{m,l})\frac{\partial \bar{u}_l}{\partial r}\right]\right\} + \frac{\partial}{\partial z}\left(\rho_l \varepsilon_{m,l} \frac{\partial \bar{u}_l}{\partial z}\right) \tag{3-17}$$

3.2.5.2 'R' direction Momentum Equation

Equation 3-11, expanded in terms of mean and fluctuating components can be written as:

$$\frac{\partial}{\partial z}\left[\rho_l(\bar{u}_l + u_l')(\bar{v}_l + v_l')\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[r\rho_l(\bar{v}_l + v_l')(\bar{v}_l + v_l')\right] = -\frac{\partial}{\partial r}\left(\bar{P} + P'\right) + \frac{\partial}{\partial z}\left(\bar{\tau}_{zr,l} + \tau_{zr,l}'\right) \tag{3-18}$$

Statistically averaging, and invoking $\overline{\bar{v}_l v_l'} = \overline{u_l' \bar{v}_l} = \overline{\bar{u}_l v_l'} = 0$, $\overline{\tau'}_{zr,l} = 0$ and $\overline{P'} = 0$, we get:

$$\frac{\partial}{\partial z}\left[\rho_l\left(\bar{u}_l \bar{v}_l + \overline{u_l' v_l'}\right)\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[r\rho_l\left(\bar{v}_l \bar{v}_l + \overline{v_l' v_l'}\right)\right] = -\frac{\partial \bar{P}}{\partial r} + \frac{\partial}{\partial z}\left(\bar{\tau}_{zr,l}\right) \tag{3-19}$$

Re-arranging, we get:

$$\frac{\partial}{\partial z}(\rho_l \bar{u}_l \bar{v}_l) + \frac{1}{r}\frac{\partial}{\partial r}(r\rho_l \bar{v}_l \bar{v}_l) = -\frac{\partial \bar{P}}{\partial r} + \frac{\partial}{\partial z}\left(\bar{\tau}_{zr,l}\right) - \frac{\partial}{\partial z}\left(\rho_l \overline{u_l' v_l'}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_l \overline{v_l' v_l'}\right) \tag{3-20}$$

As is the convention, the product of the fluctuating components of the velocity is related to the turbulent shear stress as $-\rho_l \overline{u_l' v_l'} = \mu_l^t \frac{\partial \bar{v}_l}{\partial z}$ and using $\tau_{zr,l} = \mu_l \frac{\partial \bar{v}_l}{\partial z}$, the two terms can be combined to give:

$$\bar{\tau}_{zr,l} - \rho_l \overline{u_l' v_l'} = \mu_l \frac{\partial \bar{v}_l}{\partial z} + \mu_l^t \frac{\partial \bar{v}_l}{\partial z} = \rho_l \left(\nu_l + \varepsilon_{m,l}\right) \frac{\partial \bar{v}_l}{\partial z} \tag{3-21}$$

Using the above simplification, Equation 3-20 can be re-written as:

$$\frac{\partial}{\partial z}\left(\rho_l \bar{u}_l \bar{v}_l\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_l \bar{v}_l \bar{v}_l\right) = -\frac{\partial \bar{P}}{\partial r} + \frac{\partial}{\partial z}\left[\rho_l \left(\nu_l + \varepsilon_{m,l}\right)\frac{\partial \bar{v}_l}{\partial z}\right] - \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_l \overline{v_l' v_l'}\right) \tag{3-22}$$

The last term in Equation 3-22 can be written as $-\rho_l \overline{v_l' v_l'} = \mu_l^t \frac{\partial \bar{v}_l}{\partial r} = \rho_l \varepsilon_{m,l} \frac{\partial \bar{v}_l}{\partial r}$

Thus the R-direction momentum equation for the liquid phase is given by:

$$\frac{\partial}{\partial z}\left(\rho_l \bar{u}_l \bar{v}_l\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_l \bar{v}_l \bar{v}_l\right) = -\frac{\partial \bar{P}}{\partial r} + \frac{\partial}{\partial z}\left[\rho_l \left(\nu_l + \varepsilon_{m,l}\right)\frac{\partial \bar{v}_l}{\partial z}\right] + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_l \varepsilon_{m,l} \frac{\partial \bar{v}_l}{\partial r}\right) \tag{3-23}$$

Thus, Equations 3-17 and 3-23 represent the liquid phase momentum equations in z and r directions respectively. The equations for the vapor phases can be derived in a similar manner and have the same form, except that the velocities and properties are for the vapor phase.

$$\frac{\partial}{\partial z}\left(\rho_v \bar{u}_v \bar{u}_v\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \bar{u}_v \bar{v}_v\right) = -\frac{\partial \bar{P}}{\partial z} - \rho_v g + \frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\rho_v \left(\nu_v + \varepsilon_{m,v}\right)\frac{\partial \bar{u}_v}{\partial r}\right]\right\} + \frac{\partial}{\partial z}\left(\rho_v \varepsilon_{m,v} \frac{\partial \bar{u}_v}{\partial z}\right) \tag{3-24}$$

$$\frac{\partial}{\partial z}\left(\rho_v \bar{u}_v \bar{v}_v\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \bar{v}_v \bar{v}_v\right) = -\frac{\partial \bar{P}}{\partial r} + \frac{\partial}{\partial z}\left[\rho_v \left(\nu_v + \varepsilon_{m,v}\right)\frac{\partial \bar{v}_v}{\partial z}\right] + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \varepsilon_{m,v} \frac{\partial \bar{v}_v}{\partial r}\right) \tag{3-25}$$

Equations 3-24 and 3-25 represent the vapor phase momentum equations in the z and r directions respectively.

3.2.6 Energy Equations

3.2.6.1 Liquid Phase Energy Equation

The energy equation can be obtained from a simple energy balance of a fluid element in the form of a ring. It is given in Equation 3-26 below.

$$\frac{\partial}{\partial z}(q_{cond})dz+\frac{\partial}{\partial r}(q_{cond})dr+\frac{\partial}{\partial z}\left[\rho_l(2\pi r dr)u_l c_{p,l}T_l\right]dz+\frac{\partial}{\partial r}\left[\rho_l(2\pi r dr)v_l c_{p,l}T_l\right]dr=0 \tag{3-26}$$

Applying Fourier's Law, Equation 3-26 can be written as:

$$\frac{\partial}{\partial z}\left[k_l(2\pi r dr)\frac{\partial T_l}{\partial z}\right]dz+\frac{\partial}{\partial r}\left[k_l(2\pi r dz)\frac{\partial T_l}{\partial r}\right]dr=\frac{\partial}{\partial z}\left[\rho_l(2\pi r dr)u_l c_{p,l}T_l\right]dz+\frac{\partial}{\partial r}\left[\rho_l(2\pi r dr)v_l c_{p,l}T_l\right]dr \tag{3-27}$$

Dividing throughout by $2\pi r dr dz$, the above equation becomes

$$\frac{\partial}{\partial z}\left[k_l\frac{\partial T_l}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[k_l r\frac{\partial T_l}{\partial r}\right]=\frac{\partial}{\partial z}\left[\rho_l u_l c_{p,l}T_l\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[\rho_l r v_l c_{p,l}T_l\right] \tag{3-28}$$

Assuming constant properties, Equation 3-28 becomes:

$$k_l\left\{\frac{\partial}{\partial z}\left(\frac{\partial T_l}{\partial z}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_l}{\partial r}\right)\right\}=\rho_l c_{p,l}\left\{\frac{\partial}{\partial z}\left[u_l T_l\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r v_l T_l\right]\right\} \tag{3-29}$$

3.2.6.2 Turbulent Flow Formulation

In terms of mean and fluctuating quantities, the liquid phase energy equation, 3-29, can be written

$$k_l\left\{\frac{\partial}{\partial z}\left[\frac{\partial(\overline{T}_l+T_l')}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial(\overline{T}_l+T_l')}{\partial r}\right]\right\}=\rho_l c_{p,l}\left\{\frac{\partial}{\partial z}\left[(\overline{u}_l+u_l')(\overline{T}_l+T_l')\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r(\overline{v}_l+v_l')(\overline{T}_l+T_l')\right]\right\} \tag{3-30}$$

Statistically averaging, and invoking $\overline{u_l'\overline{T}_l}=\overline{v_l'\overline{T}_l}=\overline{\overline{u}_l T_l'}=\overline{\overline{v}_l T_l'}=0$ we get

$$k_l\left\{\frac{\partial}{\partial z}\left(\frac{\partial \overline{T}_l}{\partial z}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \overline{T}_l}{\partial r}\right)\right\}=\rho_l c_{p,l}\left\{\frac{\partial}{\partial z}\left(\overline{u}_l\overline{T}_l+\overline{u_l'T_l'}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\overline{v}_l\overline{T}_l+\overline{v_l'T_l'}\right)\right]\right\} \tag{3-31}$$

Re-arranging

$$\rho_l c_{p,l}\left[\frac{\partial}{\partial z}\left(\overline{u}_l\overline{T}_l\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{v}_l\overline{T}_l\right)\right]=\frac{\partial}{\partial z}\left[k_l\left(\frac{\partial \overline{T}_l}{\partial z}\right)-\rho_l c_{p,l}\left(\overline{u_l'T'}\right)\right]+\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left[\left(k_l\frac{\partial \overline{T}_l}{\partial r}\right)-\rho_l c_{p,l}\left(\overline{v_l'T_l'}\right)\right]\right\} \tag{3-32}$$

As is the convention, the product of the fluctuating components of the velocity and temperature is related as follows:

$$-\rho_l c_{p,l}\overline{u_l'T_l'}=k_l^t\frac{\partial \overline{T}_l}{\partial z} \text{ and } -\rho_l c_{p,l}\overline{v_l'T_l'}=k_l^t\frac{\partial \overline{T}_l}{\partial r} \tag{3-33}$$

Using the above definitions, Equation 3-32 becomes:

$$\rho_l c_{p,l}\left[\frac{\partial}{\partial z}\left(\overline{u}_l\overline{T}_l\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{v}_l\overline{T}_l\right)\right]=\frac{\partial}{\partial z}\left[\left(k_l+k_l^t\right)\frac{\partial \overline{T}_l}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(k_l+k_l^t\right)\frac{\partial \overline{T}_l}{\partial r}\right] \tag{3-34}$$

Invoking the definition of thermal diffusivity $\alpha_l=\frac{k_l}{\rho_l c_{p,l}}$, Prandtl number $\mathrm{Pr}_l=\frac{\nu_l}{\alpha_l}$ and defining $\frac{k_l^t}{\rho_l c_{p,l}}=\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}$ where Pr_l^t is the turbulent Prandtl number for liquid phase and $\varepsilon_{m,l}$ is the eddy diffusivity of momentum for liquid phase, Equation 3-34 can be written as:

$$\frac{\partial}{\partial z}\left(\overline{u}_l\overline{T}_l\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{v}_l\overline{T}_l\right)=\frac{\partial}{\partial z}\left[\left(\frac{\nu_l}{\mathrm{Pr}_l}+\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\right)\frac{\partial \overline{T}_l}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\nu_l}{\mathrm{Pr}_l}+\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\right)\frac{\partial \overline{T}_l}{\partial r}\right] \tag{3-35}$$

Equation 3-35 represents the energy equation for the liquid phase. The energy equation for the vapor phase is identical in form, except that properties and temperatures correspond to those for. vapor phase. The specific heat of vapor can be treated as constant, hence can be factored out as in the liquid energy equation, however, the density of vapor cannot be treated as a constant. It needs to remain inside the differential sign.

$$\frac{\partial}{\partial z}\left(\rho_v \overline{u}_v \overline{T}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \overline{v}_v \overline{T}_v\right)=\frac{\partial}{\partial z}\left[\left(\frac{\nu_v}{\mathrm{Pr}_v}+\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\right)\rho_v \frac{\partial \overline{T}_v}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\nu_v}{\mathrm{Pr}_v}+\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\right)\rho_v \frac{\partial \overline{T}_v}{\partial r}\right] \quad (3\text{-}36)$$

Equation 3-36 represents the vapor phase energy equation.

3.2.7 Unknowns in the Governing Equations

Equations 3-1, 3-2, 3-17, 3-23, 3-24, 3-25, 3-35 and 3-36 represent the governing mass, momentum and energy conservation equations. There are 11 unknowns in the above system of equations. These are:

- Velocity distribution of liquid and vapor phases, $\overline{u}_l, \overline{v}_l, \overline{u}_v$ and $\overline{v}_v$.
- Temperature distribution of liquid and vapor phases, $\overline{T}_l$ and $\overline{T}_v$.
- Eddy diffusivities of momentum for liquid and vapor phases, $\varepsilon_{m,l}$ and $\varepsilon_{m,v}$.
- Turbulent Prandtl number of liquid and vapor phases, Pr_l^t and Pr_v^t.
- Pressure, $\overline{P}$.

The boundary conditions needed for the solution of the set of above equations are presented in Appendices B through G. It should be noted that the actual boundary conditions used are suitably non-dimensionalized to reflect the modified nature of the set of equations. The modification of the governing equations and the boundary conditions are presented in the succeeding chapters and Appendices B through G. As mentioned earlier, the liquid phase extends from the centerline to $r = R - \delta$. This radial location represents the liquid-vapor interface. The phenomena occurring at the interface serve as boundary conditions to the liquid region. The vapor phase extends from $r = R - \delta$, the liquid-vapor interface to $r = R$, which is the surface of the rod.

Chapter 4

Integral Formulation

4.1 Introduction

In order to account for the dominating effects near the heated surface and away from it, the vapor film is split into a wall region where viscous effects dominate and a turbulent core where the effects of turbulence are predominant. The governing equations derived in Chapter 3 are recast to account for the dominant effects. These are integrated over the radial location at a particular axial location. It may be recalled, that the governing equations derived in Chapter 3 are partial differential equations in both r and z. On integration over the radial direction, these are transformed to ordinary differential equations with the axial location, z, as the independent variable.

4.2 Simplification of Governing Equations

The continuity equations derived earlier remain the same.

$$\frac{\partial \bar{u}_l}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{v}_l\right) = 0 \qquad (3\text{-}1)$$

$$\frac{\partial\left(\rho_v \bar{u}_v\right)}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v\bar{v}_v\right) = 0 \qquad (3\text{-}2)$$

The momentum and energy equations can be simplified to account for the effects of the dominating terms in the liquid core and the vapor film.

4.2.1 Equations for Liquid Core

Liquid occupies the central core in IAFB, thus it is far from the solid surfaces, where viscous effects are seen to dominate. Since the flow is fully turbulent and since $\varepsilon_{m,l} >> \nu_l$, the liquid momentum equations can be simplified by neglecting ν_l as compared to $\varepsilon_{m,l}$. Density of liquid can be treated as a constant and hence, it is factored out. The simplified forms of the momentum equation are given below.

Z-Momentum: $$\frac{\partial}{\partial z}\left(\overline{u}_l\overline{u}_l\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{u}_l\overline{v}_l\right)=-\frac{1}{\rho_l}\frac{\partial\overline{P}}{\partial z}-g+\frac{1}{r}\frac{\partial}{\partial r}\left[r\varepsilon_{m,l}\frac{\partial\overline{u}_l}{\partial r}\right]+\frac{\partial}{\partial z}\left(\varepsilon_{m,l}\frac{\partial\overline{u}_l}{\partial z}\right) \quad (4\text{-}1)$$

R-Momentum: $$\frac{\partial}{\partial z}\left(\overline{u}_l\overline{v}_l\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{v}_l\overline{v}_l\right)=-\frac{1}{\rho_l}\frac{\partial\overline{P}}{\partial r}+\frac{\partial}{\partial z}\left[\varepsilon_{m,l}\frac{\partial\overline{v}_l}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left(r\varepsilon_{m,l}\frac{\partial\overline{v}_l}{\partial r}\right) \quad (4\text{-}2)$$

The energy equation for liquid simplifies as follows

$$\frac{\partial}{\partial z}\left(\overline{u}_l\overline{T}_l\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{v}_l\overline{T}_l\right)=\frac{\partial}{\partial z}\left[\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\frac{\partial\overline{T}_l}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\frac{\partial\overline{T}_l}{\partial r}\right] \quad (4\text{-}3)$$

4.2.2 Equations for Vapor Film

The vapor region, which extends from $r = R - \delta$ to $r = R$ is split into two regions, the turbulent vapor core extending from $r = R - \delta$ to a location corresponding to $y_v^+ = 5$ and a wall region extending from the location of $y_v^+ = 5$ to $r = R$.

For the turbulent core of the vapor film that exists beyond $y^+ = 5$, viscous effects can be treated as negligible compared to the effects of turbulence. Thus, the vapor momentum equations simplify as shown below.

Z-Momentum:

$$\frac{\partial}{\partial z}\left(\rho_v\overline{u}_v\overline{u}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v\overline{u}_v\overline{v}_v\right)=-\frac{\partial\overline{P}}{\partial z}-\rho_v g+\frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\rho_v\varepsilon_{m,v}\frac{\partial\overline{u}_v}{\partial r}\right]\right\}+\frac{\partial}{\partial z}\left(\rho_v\varepsilon_{m,v}\frac{\partial\overline{u}_v}{\partial z}\right) \quad (4\text{-}4)$$

R-Momentum:

$$\frac{\partial}{\partial z}\left(\rho_v \bar{u}_v \bar{v}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \bar{v}_v \bar{v}_v\right)=-\frac{\partial \bar{P}}{\partial r}+\frac{\partial}{\partial z}\left[\rho_v \varepsilon_{m,v}\frac{\partial \bar{v}_v}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \varepsilon_{m,v}\frac{\partial \bar{v}_v}{\partial r}\right) \tag{4-5}$$

The simplified energy equation for the turbulent core of the vapor film is given below.

$$\frac{\partial}{\partial z}\left(\rho_v \bar{u}_v \bar{T}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \bar{v}_v \bar{T}_v\right)=\frac{\partial}{\partial z}\left[\frac{\varepsilon_{m,v}}{\Pr_v^t}\rho_v \frac{\partial \bar{T}_v}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\rho_v \frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial \bar{T}_v}{\partial r}\right] \tag{4-6}$$

The region near the wall is where viscous effects dominate compared to the effects of turbulence. Hence for the region near the wall $y^+ < 5$, the vapor momentum equations simplify as shown below.

Z-Momentum:

$$\frac{\partial}{\partial z}\left(\rho_v \bar{u}_v \bar{u}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \bar{u}_v \bar{v}_v\right)=-\frac{\partial \bar{P}}{\partial z}-\rho_v g+\frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\rho_v \nu_v \frac{\partial \bar{u}_v}{\partial r}\right]\right\} \tag{4-7}$$

R-Momentum:

$$\frac{\partial}{\partial z}\left(\rho_v \bar{u}_v \bar{v}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \bar{v}_v \bar{v}_v\right)=-\frac{\partial \bar{P}}{\partial r}+\frac{\partial}{\partial z}\left[\rho_v \nu_v \frac{\partial \bar{v}_v}{\partial z}\right] \tag{4-8}$$

The energy equation for the wall region of vapor film is as given below

$$\frac{\partial}{\partial z}\left(\rho_v \bar{u}_v \bar{T}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \bar{v}_v \bar{T}_v\right)=\frac{\partial}{\partial z}\left[\frac{\nu_v}{\Pr_v}\rho_v \frac{\partial \bar{T}_v}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\rho_v \frac{\nu_v}{\Pr_v}\frac{\partial \bar{T}_v}{\partial r}\right] \tag{4-9}$$

The governing equations 4-1 through 4-9 are partial differential equations in r and z. These equations are converted to a system of ordinary differential equations in z, by integrating the equations over the appropriate radial region of interest. The detailed derivation is shown in Appendix A, while the complete set of integrated governing equations is shown below.

Vapor Film Turbulent Core

(V-C) $$\frac{d}{dz}\left[\int_{r=R-\delta}^{r=R-l}(r\rho_v\bar{u}_v)dr\right]+(r\rho_v\bar{u}_v)\big|_{R-l}\frac{dl}{dz}-(r\rho_v\bar{u}_v)\big|_{R-\delta}\frac{d\delta}{dz}+(r\rho_v\bar{v}_v)\big|_{R-l}-(r\rho_v\bar{v}_v)\big|_{R-\delta}=0$$

(V-Z) $$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\rho_v\bar{u}_v\bar{u}_v)dr\right]+(r\rho_v\bar{u}_v\bar{u}_v)\big|_{R-l}\frac{dl}{dz}-(r\rho_v\bar{u}_v\bar{u}_v)\big|_{R-\delta}\frac{d\delta}{dz}+\left\{(r\rho_v\bar{u}_v\bar{v}_v)\big|_{R-l}-(r\rho_v\bar{u}_v\bar{v}_v)\big|_{R-\delta}\right\}=$$
$$-\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\bar{P})dr\right]-(r\bar{P})\big|_{R-l}\frac{dl}{dz}+(r\bar{P})\big|_{R-\delta}\frac{d\delta}{dz}+\left\{\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial r}\right)\Big|_{R-l}-\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial r}\right)\Big|_{R-\delta}\right\}$$
$$-\frac{g\rho_v}{2}\left[(R-l)^2-(R-\delta)^2\right]+\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial z}\right)dr\right]+\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial z}\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial z}\right)\Big|_{R-\delta}\frac{d\delta}{dz}$$

(V-R) $$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\rho_v\bar{u}_v\bar{v}_v)dr\right]+(r\rho_v\bar{u}_v\bar{v}_v)\big|_{R-l}\frac{dl}{dz}-(r\rho_v\bar{u}_v\bar{v}_v)\big|_{R-\delta}\frac{d\delta}{dz}+\left\{(r\rho_v\bar{v}_v\bar{v}_v)\big|_{R-l}-(r\rho_v\bar{v}_v\bar{v}_v)\big|_{R-\delta}\right\}=$$
$$-\left[\int_{R-\delta}^{R-l}\left(r\frac{\partial\bar{P}}{\partial r}\right)dr\right]+\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial z}\right)dr\right]+\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial z}\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial z}\right)\Big|_{R-\delta}\frac{d\delta}{dz}$$
$$+\left\{\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial r}\right)\Big|_{R-l}-\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial r}\right)\Big|_{R-\delta}\right\}$$

(V-E) $$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\rho_v\bar{u}_v\bar{T}_v)dr\right]+(r\rho_v\bar{u}_v\bar{T}_v)\big|_{R-l}\frac{dl}{dz}-(r\rho_v\bar{u}_v\bar{T}_v)\big|_{R-\delta}\frac{d\delta}{dz}+\left\{(r\rho_v\bar{v}_v\bar{T}_v)\big|_{R-l}-(r\rho_v\bar{v}_v\bar{T}_v)\big|_{R-\delta}\right\}=$$
$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial\bar{T}_v}{\partial z}\right)dr\right]+\left(r\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial\bar{T}_v}{\partial z}\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial\bar{T}_v}{\partial z}\right)\Big|_{R-\delta}\frac{d\delta}{dz}$$
$$+\left\{\left(r\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial\bar{T}_v}{\partial r}\right)\Big|_{R-l}-\left(r\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial\bar{T}_v}{\partial r}\right)\Big|_{R-\delta}\right\}$$

Vapor Film Wall Region

(V-C) $$\frac{d}{dz}\left[\int_{r=R-l}^{r=R}(r\rho_v\bar{u}_v)dr\right]-\left(r\rho_v\bar{u}_v\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\bar{v}_v\right)\Big|_{R-l}=0$$

(V-Z) $$\frac{d}{dz}\left[\int_{R-l}^{R}(r\rho_v\bar{u}_v\bar{u}_v)dr\right]-\left(r\rho_v\bar{u}_v\bar{u}_v\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\bar{u}_v\bar{v}_v\right)\Big|_{R-l}=$$
$$-\frac{d}{dz}\left[\int_{R-l}^{R}(r\bar{P})dr\right]+\left(r\bar{P}\right)\Big|_{R-l}\frac{dl}{dz}-\frac{g\rho_v}{2}\left[2Rl-l^2\right]+\left\{\left(r\rho_v\nu_v\frac{\partial\bar{u}_v}{\partial r}\right)\Big|_{R}-\left(r\rho_v\nu_v\frac{\partial\bar{u}_v}{\partial r}\right)\Big|_{R-l}\right\}$$

(V-R) $$\frac{d}{dz}\left[\int_{R-l}^{R}(r\rho_v\bar{u}_v\bar{v}_v)dr\right]-\left(r\rho_v\bar{u}_v\bar{v}_v\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\bar{v}_v\bar{v}_v\right)\Big|_{R-l}=$$
$$-\int_{R-l}^{R}\left(r\frac{\partial\bar{P}}{\partial r}\right)dr+\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\nu_v\frac{\partial\bar{v}_v}{\partial z}\right)dr\right]-\left(r\rho_v\nu_v\frac{\partial\bar{v}_v}{\partial z}\right)\Big|_{R-l}\frac{dl}{dz}$$

(V-E) $$\frac{d}{dz}\left[\int_{R-l}^{R}(r\rho_v\bar{u}_v\bar{T}_v)dr\right]-\left(r\rho_v\bar{u}_v\bar{T}_v\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\bar{v}_v\bar{T}_v\right)\Big|_{R-l}=$$
$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\frac{\nu_v}{\mathrm{Pr}_v}\frac{\partial\bar{T}_v}{\partial z}\right)dr\right]-\left(r\rho_v\frac{\nu_v}{\mathrm{Pr}_v}\frac{\partial\bar{T}_v}{\partial z}\right)\Big|_{R-l}\frac{dl}{dz}+\left\{\left(r\rho_v\frac{\nu_v}{\mathrm{Pr}_v}\frac{\partial\bar{T}_v}{\partial r}\right)\Big|_{R}-\left(r\rho_v\frac{\nu_v}{\mathrm{Pr}_v}\frac{\partial\bar{T}_v}{\partial r}\right)\Big|_{R-l}\right\}$$

Liquid Core

(L-C) $$\frac{d}{dz}\left[\int_{r=0}^{r=R-\delta}(r\bar{u}_l)dr\right]+\left(r\bar{u}_l\right)\Big|_{r=R-\delta}\frac{d\delta}{dz}+\left(r\bar{v}_l\right)\Big|_{r=R-\delta}=0$$

(L-Z) $$\frac{d}{dz}\left[\int_{0}^{R-\delta}(r\bar{u}_l\bar{u}_l)dr\right]+\left(r\bar{u}_l\bar{u}_l\right)\Big|_{R-\delta}\frac{d\delta}{dz}+\left(r\bar{u}_l\bar{v}_l\right)\Big|_{R-\delta}=-\frac{1}{\rho_l}\frac{d}{dz}\left[\int_{0}^{R-\delta}\left(r\bar{P}\right)dr\right]-\frac{\left(r\bar{P}\right)}{\rho_l}\Bigg|_{R-\delta}\frac{d\delta}{dz}-\frac{g(R-\delta)^2}{2}$$
$$+\left[r\varepsilon_{m,l}\frac{\partial\bar{u}_l}{\partial r}\right]_{R-\delta}+\frac{d}{dz}\left[\int_{0}^{R-\delta}\left(r\varepsilon_{m,l}\frac{\partial\bar{u}_l}{\partial z}\right)dr\right]+\left[r\varepsilon_{m,l}\frac{\partial\bar{u}_l}{\partial z}\right]_{R-\delta}\frac{d\delta}{dz}$$

(L-R) $$\frac{d}{dz}\left[\int_{0}^{R-\delta}(r\bar{u}_l\bar{v}_l)dr\right]+\left(r\bar{u}_l\bar{v}_l\right)\Big|_{R-\delta}\frac{d\delta}{dz}+\left(r\bar{v}_l\bar{v}_l\right)\Big|_{R-\delta}=-\frac{1}{\rho_l}\left[\int_{0}^{R-\delta}\left(r\frac{\partial\bar{P}}{\partial r}\right)dr\right]+$$
$$\frac{d}{dz}\left[\int_{0}^{R-\delta}\left(r\varepsilon_{m,l}\frac{\partial\bar{v}_l}{\partial z}\right)dr\right]+\left[r\varepsilon_{m,l}\frac{\partial\bar{v}_l}{\partial z}\right]_{R-\delta}\frac{d\delta}{dz}+\left[r\varepsilon_{m,l}\frac{\partial\bar{v}_l}{\partial r}\right]_{R-\delta}$$

(L-E) $$\frac{d}{dz}\left[\int_{0}^{R-\delta}(r\bar{u}_l\bar{T}_l)dr\right]+\left(r\bar{u}_l\bar{T}_l\right)\Big|_{R-\delta}\frac{d\delta}{dz}+\left(r\bar{v}_l\bar{T}_l\right)\Big|_{R-\delta}=\frac{d}{dz}\left[\int_{0}^{R-\delta}\left(r\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\frac{\partial\bar{T}_l}{\partial z}\right)dr\right]+$$
$$\left[r\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\frac{\partial\bar{T}_l}{\partial z}\right]_{R-\delta}\frac{d\delta}{dz}+\left[r\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\frac{\partial\bar{T}_l}{\partial r}\right]_{R-\delta}$$

Chapter 5

Simplification and Numerical Solution

5.1 Introduction

In order to solve the system of integral differential governing equations derived in Appendix A using an Integral Method, suitable second order polynomial profiles of the form $A_0 + A_1\eta + A_2\eta^2$ is assumed for the non-dimensional axial component of the vapor and liquid velocities, the liquid and vapor temperatures. $\eta = \left(y^+\right)^{1/7}$ is a definition commonly used in turbulence formulations. Detailed derivation of the profiles is presented in Appendices B through G. Based on the physical phenomena occurring and the boundary conditions the coefficients occurring in the non-dimensional profiles are expressed in terms of the variables: $\tau_w, \tau_i, \delta, \overline{P}, T_w, m_l''$. The equations for the liquid and vapor velocities and temperature are substituted in the integrated governing equations to obtain a system of non-linear first order ordinary differential equations. These are shown in Appendices H through K.

5.2 Simplification of the Integrated Governing Equations

In order to solve the system of integrated governing equations numerically, some assumptions need to be made to simplify the same. Since the axial direction is the predominant direction compared to the radial direction, an order of magnitude analysis shows that the second derivatives with respect to z can be neglected in comparison to the second derivatives with respect to r. Based on this assumption, the governing equations derived in Chapter 4 simplify as shown below.

Equations 5-1 through 5-3 represent the simplified equations for the turbulent core of the vapor film.

Z-Momentum:
$$\frac{\partial}{\partial z}\left(\rho_v \overline{u}_v \overline{u}_v\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \overline{u}_v \overline{v}_v\right) = -\frac{\partial \overline{P}}{\partial z} - \rho_v g + \frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\rho_v \varepsilon_{m,v}\frac{\partial \overline{u}_v}{\partial r}\right]\right\} \qquad (5\text{-}1)$$

R-Momentum:
$$\frac{\partial}{\partial z}\left(\rho_v \overline{u}_v \overline{v}_v\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \overline{v}_v \overline{v}_v\right) = -\frac{\partial \overline{P}}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \varepsilon_{m,v}\frac{\partial \overline{v}_v}{\partial r}\right) \qquad (5\text{-}2)$$

Energy: $$\frac{\partial}{\partial z}\left(\rho_v \overline{u}_v \overline{T}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \overline{v}_v \overline{T}_v\right)=\frac{1}{r}\frac{\partial}{\partial r}\left[\rho_v r\frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial \overline{T}_v}{\partial r}\right] \tag{5-3}$$

For the wall region of the vapor film, the equations simplify as shown in Equations 5-4 through 5-6.

Z-Momentum: $$\frac{\partial}{\partial z}\left(\rho_v \overline{u}_v \overline{u}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \overline{u}_v \overline{v}_v\right)=-\frac{\partial \overline{P}}{\partial z}-\rho_v g+\frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\rho_v \nu_v \frac{\partial \overline{u}_v}{\partial r}\right]\right\} \tag{5-4}$$

R-Momentum: $$\frac{\partial}{\partial z}\left(\rho_v \overline{u}_v \overline{v}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \overline{v}_v \overline{v}_v\right)=-\frac{\partial \overline{P}}{\partial r} \tag{5-5}$$

Energy: $$\frac{\partial}{\partial z}\left(\rho_v \overline{u}_v \overline{T}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \overline{v}_v \overline{T}_v\right)=\frac{1}{r}\frac{\partial}{\partial r}\left[r\rho_v \frac{\nu_v}{\Pr_v}\frac{\partial \overline{T}_v}{\partial r}\right] \tag{5-6}$$

The simplified liquid phase equations are given below.

Z-Momentum: $$\frac{\partial}{\partial z}\left(\rho_l \overline{u}_l \overline{u}_l\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_l \overline{u}_l \overline{v}_l\right)=-\frac{\partial \overline{P}}{\partial z}-\rho_l g+\frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\rho_l\left(\nu_l+\varepsilon_{m,l}\right)\frac{\partial \overline{u}_l}{\partial r}\right]\right\} \tag{5-7}$$

R-Momentum: $$\frac{\partial}{\partial z}\left(\rho_l \overline{u}_l \overline{v}_l\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_l \overline{v}_l \overline{v}_l\right)=-\frac{\partial \overline{P}}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_l \varepsilon_{m,l}\frac{\partial \overline{v}_l}{\partial r}\right) \tag{5-8}$$

Energy: $$\frac{\partial}{\partial z}\left(\overline{u}_l \overline{T}_l\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{v}_l \overline{T}_l\right)=\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\nu_l}{\Pr_l}+\frac{\varepsilon_{m,l}}{\Pr_l^t}\right)\frac{\partial \overline{T}_l}{\partial r}\right] \tag{5-9}$$

Assuming that the velocity in the radial direction v is negligible, the r-direction momentum equations show that pressure is only a function of the axial direction. This assumption is made so that the system of integrated differential equations is mathematically solvable.

Based on these assumptions, the system of integral differential equations derived in Chapter 4 can be simplified considerably.

5.2.1 Vapor Film Turbulent Core

The simplified governing equations for the turbulent core of the vapor film are presented below:

(V-C) $$\frac{d}{dz}\left[\int_{r=R-\delta}^{r=R-l}(r\rho_v\overline{u}_v)dr\right]+\left(r\rho_v\overline{u}_v\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\overline{u}_v\right)\Big|_{R-\delta}\frac{d\delta}{dz}=0 \tag{5-10}$$

(V-Z) $$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\rho_v\overline{u}_v\overline{u}_v)dr\right]+\left(r\rho_v\overline{u}_v\overline{u}_v\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\overline{u}_v\overline{u}_v\right)\Big|_{R-\delta}\frac{d\delta}{dz}=-\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\overline{P})dr\right]-\left(r\overline{P}\right)\Big|_{R-l}\frac{dl}{dz}+$$
$$\left(r\overline{P}\right)\Big|_{R-\delta}\frac{d\delta}{dz}+\left\{\left(r\rho_v\varepsilon_{m,v}\frac{\partial\overline{u}_v}{\partial r}\right)\Bigg|_{R-l}-\left(r\rho_v\varepsilon_{m,v}\frac{\partial\overline{u}_v}{\partial r}\right)\Bigg|_{R-\delta}\right\}-\frac{g\rho_v}{2}\left[(R-l)^2-(R-\delta)^2\right] \tag{5-11}$$

(V-E) $$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\rho_v\overline{u}_v\overline{T}_v)dr\right]+\left(r\rho_v\overline{u}_v\overline{T}_v\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\overline{u}_v\overline{T}_v\right)\Big|_{R-\delta}\frac{d\delta}{dz}=$$
$$\left\{\left(r\rho_v\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\frac{\partial\overline{T}_v}{\partial r}\right)\Bigg|_{R-l}-\left(r\rho_v\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\frac{\partial\overline{T}_v}{\partial r}\right)\Bigg|_{R-\delta}\right\} \tag{5-12}$$

Substituting Equations H-4 through H-17 into Equation 5-11 and simplifying yields:

$$\left\{\begin{array}{l}\frac{a_0^2}{2}\left[(R-l)^2-(R-\delta)^2\right]+14a_0a_1\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right]\\+7\left(a_1^2+2a_0a_2\right)\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right]\\+14a_1a_2\left[(5)^{3/7}(l)\left(\frac{l}{17}-\frac{R}{10}\right)-\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)\right]+\\7a_2^2\left[(5)^{4/7}(l)\left(\frac{l}{18}-\frac{R}{11}\right)-\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)\right]\end{array}\right\}\frac{d\tau_w}{dz}+$$

$$\tau_w\left\{\begin{array}{l}\left[(R-l)^2-(R-\delta)^2\right]a_0\frac{da_0}{dz}+a_0^2\left[(R-\delta)\frac{d\delta}{dz}-(R-l)\frac{dl}{dz}\right]\\+14\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right]\left(a_0\frac{da_1}{dz}+a_1\frac{da_0}{dz}\right)+\\14a_0a_1\left[5^{1/7}\left(\frac{2l}{15}-\frac{R}{8}\right)\frac{dl}{dz}-\left(\delta^+\right)^{1/7}\left(\frac{2\delta}{15}-\frac{R}{8}\right)\frac{d\delta}{dz}-\frac{1}{7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}\right]\\+14\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right]\left(a_1\frac{da_1}{dz}+a_2\frac{da_0}{dz}+a_0\frac{da_2}{dz}\right)+\\7\left(a_1^2+2a_0a_2\right)\left[(5)^{2/7}\left(\frac{2l}{16}-\frac{R}{9}\right)\frac{dl}{dz}-\left(\delta^+\right)^{2/7}\left(\frac{2\delta}{16}-\frac{R}{9}\right)\frac{d\delta}{dz}-\frac{2}{7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}\right]\\+14\left[(5)^{3/7}(l)\left(\frac{l}{17}-\frac{R}{10}\right)-\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)\right]\left(a_2\frac{da_1}{dz}+a_1\frac{da_2}{dz}\right)+\\14a_1a_2\left[(5)^{3/7}\left(\frac{2l}{17}-\frac{R}{10}\right)\frac{dl}{dz}-\left(\delta^+\right)^{3/7}\left(\frac{2\delta}{17}-\frac{R}{10}\right)\frac{d\delta}{dz}-\frac{3}{7}\left(\frac{\delta^2}{17}-\frac{R\delta}{10}\right)\left(\delta^+\right)^{-4/7}\frac{d\delta^+}{dz}\right]\\+14\left[(5)^{4/7}(l)\left(\frac{l}{18}-\frac{R}{11}\right)-\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)\right]\left(a_2\frac{da_2}{dz}\right)+\\7a_2^2\left[(5)^{4/7}\left(\frac{2l}{18}-\frac{R}{11}\right)\frac{dl}{dz}-\left(\delta^+\right)^{4/7}\left(\frac{2\delta}{18}-\frac{R}{11}\right)\frac{d\delta}{dz}-\frac{4}{7}\left(\frac{\delta^2}{18}-\frac{R\delta}{11}\right)\left(\delta^+\right)^{-3/7}\frac{d\delta^+}{dz}\right]\end{array}\right\}$$

$$-\frac{1}{2}(R-l)l\left[a_0^2+2a_0a_1(5)^{1/7}+\left(a_1^2+2a_0a_2\right)(5)^{2/7}+2a_1a_2(5)^{3/7}+a_2^2(5)^{4/7}\right]\frac{d\tau_w}{dz}$$

$$-\tau_w(R-\delta)\left[a_0^2+2a_0a_1\left(\delta^+\right)^{1/7}+\left(a_1^2+2a_0a_2\right)\left(\delta^+\right)^{2/7}+2a_1a_2\left(\delta^+\right)^{3/7}+a_2^2\left(\delta^+\right)^{4/7}\right]\frac{d\delta}{dz} \qquad (5\text{-}13)$$

$$
= -\left[\frac{(R-l)^2-(R-\delta)^2}{2}\right]\frac{d\overline{P}}{dz} - \frac{\rho_v u_v^* \varepsilon_{m,v}}{7l}(R-l)\left[a_1(5)^{1/7} + 2a_2(5)^{2/7}\right] +
$$

$$
\frac{\rho_v u_v^* \varepsilon_{m,v}(R-\delta)}{7\delta}\left[a_1\left(\delta^+\right)^{1/7} + 2a_2\left(\delta^+\right)^{2/7}\right] - \frac{g\rho_v}{2}\left[(R-l)^2-(R-\delta)^2\right]
$$

Substituting Equations J-3, J-4, J-5, J-11 and J-12 in Equation 5-12 yields:

$$\rho_v \left\{ \begin{array}{l} \frac{a_0}{2}\left[(R-l)^2-(R-\delta)^2\right]+7a_1\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right] \\ +7a_2\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right] \end{array} \right\} \left[u_v^*\frac{dT_w}{dz}+\frac{T_w}{2\sqrt{\rho_v\tau_w}}\frac{d\tau_w}{dz}\right]+$$

$$\rho_v T_w u_v^* \left\{ \begin{array}{l} \frac{\left[(R-l)^2-(R-\delta)^2\right]}{2}\frac{da_0}{dz}+7\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right]\frac{da_1}{dz} \\ +7\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right]\frac{da_2}{dz}+ \\ a_0\left[(R-\delta)\frac{d\delta}{dz}-(R-l)\frac{dl}{dz}\right]+ \\ 7a_1\left[5^{1/7}\left(\frac{2l}{15}-\frac{R}{8}\right)\frac{dl}{dz}-\left(\delta^+\right)^{1/7}\left(\frac{2\delta}{15}-\frac{R}{8}\right)\frac{d\delta}{dz}-\frac{1}{7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}\right]+ \\ 7a_2\left[5^{2/7}\left(\frac{2l}{16}-\frac{R}{9}\right)\frac{dl}{dz}-\left(\delta^+\right)^{2/7}\left(\frac{2\delta}{16}-\frac{R}{9}\right)\frac{d\delta}{dz}-\frac{2}{7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}\right] \end{array} \right\}+$$

$$\frac{1}{c_{p,v}} \left\{ \begin{array}{l} \frac{a_0c_0}{2}\left[(R-l)^2-(R-\delta)^2\right]+7\left(a_1c_0+c_1a_0\right)\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right] \\ +7\left(a_2c_0+a_1c_1+a_0c_2\right)\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right] \\ +7\left(a_1c_2+a_2c_1\right)\left[(5)^{3/7}(l)\left(\frac{l}{17}-\frac{R}{10}\right)-\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)\right] \\ +7a_2c_2\left[(5)^{4/7}(l)\left(\frac{l}{18}-\frac{R}{11}\right)-\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)\right] \end{array} \right\}\frac{dq''_w}{dz}+$$

$$
\frac{q''_w}{c_{p,v}}\left\{
\begin{aligned}
&\frac{\left[(R-l)^2-(R-\delta)^2\right]}{2}\left(a_0\frac{dc_0}{dz}+c_0\frac{da_0}{dz}\right)+a_0c_0\left[(R-\delta)\frac{d\delta}{dz}-(R-l)\frac{dl}{dz}\right]\\
&+7\left(a_1\frac{dc_0}{dz}+c_0\frac{da_1}{dz}+a_0\frac{dc_1}{dz}+c_1\frac{da_0}{dz}\right)\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right]\\
&+7\left(a_2\frac{dc_0}{dz}+c_0\frac{da_2}{dz}+a_0\frac{dc_2}{dz}+c_2\frac{da_0}{dz}+a_1\frac{dc_1}{dz}+c_1\frac{da_1}{dz}\right)\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right]\\
&+7\left(a_1\frac{dc_2}{dz}+c_2\frac{da_1}{dz}+a_2\frac{dc_1}{dz}+c_1\frac{da_2}{dz}\right)\left[(5)^{3/7}(l)\left(\frac{l}{17}-\frac{R}{10}\right)-\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)\right]\\
&+7\left(a_2\frac{dc_2}{dz}+c_2\frac{da_2}{dz}\right)\left[(5)^{4/7}(l)\left(\frac{l}{18}-\frac{R}{11}\right)-\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)\right]\\
&+7(a_1c_0+c_1a_0)\left[5^{1/7}\left(\frac{2l}{15}-\frac{R}{8}\right)\frac{dl}{dz}-\left(\delta^+\right)^{1/7}\left(\frac{2\delta}{15}-\frac{R}{8}\right)\frac{d\delta}{dz}-\frac{1}{7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}\right]\\
&+7(a_2c_0+a_1c_1+a_0c_2)\left[5^{2/7}\left(\frac{2l}{16}-\frac{R}{9}\right)\frac{dl}{dz}-\left(\delta^+\right)^{2/7}\left(\frac{2\delta}{16}-\frac{R}{9}\right)\frac{d\delta}{dz}-\frac{2}{7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}\right]\\
&+7(a_1c_2+a_2c_1)\left[5^{3/7}\left(\frac{2l}{17}-\frac{R}{10}\right)\frac{dl}{dz}-\left(\delta^+\right)^{3/7}\left(\frac{2\delta}{17}-\frac{R}{10}\right)\frac{d\delta}{dz}-\frac{3}{7}\left(\frac{\delta^2}{17}-\frac{R\delta}{10}\right)\left(\delta^+\right)^{-4/7}\frac{d\delta^+}{dz}\right]\\
&+7a_2c_2\left[5^{4/7}\left(\frac{2l}{18}-\frac{R}{11}\right)\frac{dl}{dz}-\left(\delta^+\right)^{4/7}\left(\frac{2\delta}{18}-\frac{R}{11}\right)\frac{d\delta}{dz}-\frac{4}{7}\left(\frac{\delta^2}{18}-\frac{R\delta}{11}\right)\left(\delta^+\right)^{-3/7}\frac{d\delta^+}{dz}\right]
\end{aligned}
\right\}
$$

$$
-(R-l)l\frac{d\tau_w}{dz}\left\{\frac{\rho_v T_w u_v^*}{2\tau_w}\left[a_0+a_1(5)^{1/7}+a_2(5)^{2/7}\right]+\frac{q''_w}{2\tau_w c_{p,v}}\left[\begin{aligned}&a_0c_0+(a_1c_0+a_0c_1)(5)^{1/7}\\&+(a_2c_0+a_1c_1+a_0c_2)(5)^{2/7}+\\&(a_1c_2+a_2c_1)(5)^{3/7}+a_2c_2(5)^{4/7}\end{aligned}\right]\right\}
$$

$$
-\left\{\rho_v u_v^* T_w\left[a_0+a_1\left(\delta^+\right)^{1/7}+a_2\left(\delta^+\right)^{2/7}\right]+\frac{q''_w}{c_{p,v}}\left[\begin{aligned}&a_0c_0+(a_1c_0+a_0c_1)\left(\delta^+\right)^{1/7}+\\&(a_2c_0+a_1c_1+a_0c_2)\left(\delta^+\right)^{2/7}+\\&(a_1c_2+a_2c_1)\left(\delta^+\right)^{3/7}+a_2c_2\left(\delta^+\right)^{4/7}\end{aligned}\right]\right\}(R-\delta)\frac{d\delta}{dz}
$$

$$
=\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\left(\frac{q''_w}{\rho_v c_{p,v}u_v^*}\right)\left\{\frac{(R-\delta)\left[c_1\left(\delta^+\right)^{1/7}+2c_2\left(\delta^+\right)^{2/7}\right]}{7\delta}-\frac{\left[c_1(5)^{1/7}+2c_2(5)^{2/7}\right](R-l)}{7l}\right\} \tag{5-14}
$$

5.2.2 Vapor Film Wall Region

The simplified governing equations for the turbulent core of the vapor film are presented below.

(V-C) $$\frac{d}{dz}\left[\int_{r=R-l}^{r=R}\left(r\rho_v\bar{u}_v\right)dr\right]-\left(r\rho_v\bar{u}_v\right)\Big|_{R-l}\frac{dl}{dz}=0 \tag{5-15}$$

(V-Z) $$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\bar{u}_v\bar{u}_v\right)dr\right]-\left(r\rho_v\bar{u}_v\bar{u}_v\right)\Big|_{R-l}\frac{dl}{dz}=-\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\bar{P}\right)dr\right]+ \left(r\bar{P}\right)\Big|_{R-l}\frac{dl}{dz}-\frac{g\rho_v}{2}\left[2Rl-l^2\right]+\left\{\left(r\rho_v\nu_v\frac{\partial\bar{u}_v}{\partial r}\right)\Bigg|_{R}-\left(r\rho_v\nu_v\frac{\partial\bar{u}_v}{\partial r}\right)\Bigg|_{R-l}\right\} \tag{5-16}$$

(V-E) $$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\bar{u}_v\bar{T}_v\right)dr\right]-\left(r\rho_v\bar{u}_v\bar{T}_v\right)\Big|_{R-l}\frac{dl}{dz}=\left\{\left(r\rho_v\frac{\nu_v}{\Pr_v}\frac{\partial\bar{T}_v}{\partial r}\right)\Bigg|_{R}-\left(r\rho_v\frac{\nu_v}{\Pr_v}\frac{\partial\bar{T}_v}{\partial r}\right)\Bigg|_{R-l}\right\} \tag{5-17}$$

Substituting Equations H-20 through H-27 in Equation 5-16, the simplified z-momentum equation for the vapor wall region becomes:

$$\frac{25}{2}\left(\frac{5\nu_v}{u_v^*}-\frac{R}{3}\right)\frac{d\tau_w}{dz}+\left[R-\left(\frac{5\nu_v}{2u_v^*}\right)\right]\frac{d\bar{P}}{dz}+g\rho_v\left[R-\frac{5\nu_v}{2u_v^*}\right]+\tau_w=0 \tag{5-18}$$

Substituting Equations J-15, J-16, J-20 and J-21 in Equation 5-17, the simplified wall region vapor energy equation becomes:

$$5\rho_v\left(\frac{R}{2}-\frac{5\nu_v}{3u_v^*}\right)\frac{dT_w}{dz}+\frac{25\Pr_v}{c_{p,v}u_v^*}\left(\frac{R}{3}-\frac{5\nu_v}{4u_v^*}\right)\frac{dq''_w}{dz}+ \left(\frac{5\rho_vRT_w}{2\tau_w}-\frac{25}{3}\frac{\rho_v\nu_vT_w}{\tau_wu_v^*}-\frac{125}{4}\frac{\Pr_v\nu_vq''_w}{c_{p,v}\tau_w\left(u_v^*\right)^2}+\frac{25}{3}\frac{R\Pr_vq''_w}{\tau_wu_v^*c_{p,v}}\right)\frac{d\tau_w}{dz}=-\frac{q''_w}{c_{p,v}}\left(\frac{1}{u_v^*}\right) \tag{5-19}$$

5.2.3 Liquid Core

The simplified governing equations for the turbulent core of the vapor film are presented below:

(L-C) $$\frac{d}{dz}\left[\int_{r=0}^{r=R-\delta}\left(r\overline{u}_l\right)dr\right]+\left.\left(r\overline{u}_l\right)\right|_{r=R-\delta}\frac{d\delta}{dz}=0 \quad (5\text{-}20)$$

(L-Z) $$\begin{aligned}&\frac{d}{dz}\left[\int_{0}^{R-\delta}\left(r\overline{u}_l\overline{u}_l\right)dr\right]+\left.\left(r\overline{u}_l\overline{u}_l\right)\right|_{R-\delta}\frac{d\delta}{dz}=\\&-\frac{1}{\rho_l}\frac{d}{dz}\left[\int_{0}^{R-\delta}\left(r\overline{P}\right)dr\right]-\left.\frac{\left(r\overline{P}\right)}{\rho_l}\right|_{R-\delta}\frac{d\delta}{dz}-\frac{g\left(R-\delta\right)^2}{2}+\left[r\varepsilon_{m,l}\frac{\partial\overline{u}_l}{\partial r}\right]_{R-\delta}\end{aligned} \quad (5\text{-}21)$$

(L-E) $$\frac{d}{dz}\left[\int_{0}^{R-\delta}\left(r\overline{u}_l\overline{T}_l\right)dr\right]+\left.\left(r\overline{u}_l\overline{T}_l\right)\right|_{R-\delta}\frac{d\delta}{dz}=\left[r\frac{\varepsilon_{m,l}}{\Pr_l^t}\frac{\partial\overline{T}_l}{\partial r}\right]_{R-\delta} \quad (5\text{-}22)$$

Substituting Equations I-4 through I-9 in Equation 5-21, the simplified z-momentum equation for the vapor wall region becomes:

$$
\left\{\begin{array}{l}
\frac{b_0^2}{2}(R-\delta)^2+14b_0b_1\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right] \\
+7\left(b_1^2+2b_0b_2\right)\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]+ \\
14b_1b_2\left[\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)+\frac{7R^2}{170}\left(R^+\right)^{3/7}\right]+7b_2^2\left[\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)+\frac{7R^2}{198}\left(R^+\right)^{4/7}\right]
\end{array}\right\}\frac{1}{\rho_v}\frac{d\tau_w}{dz}+
$$

$$
\left(u_v^*\right)^2\left\{\begin{array}{l}
(R-\delta)^2 b_0\frac{db_0}{dz}-b_0^2(R-\delta)\frac{d\delta}{dz}+14\left(b_0\frac{db_1}{dz}+b_1\frac{db_0}{dz}\right)\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right] \\
+14\left(b_1\frac{db_1}{dz}+b_0\frac{db_2}{dz}+b_2\frac{db_0}{dz}\right)\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right] \\
+14\left(b_2\frac{db_1}{dz}+b_1\frac{db_2}{dz}\right)\left[\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)+\frac{7R^2}{170}\left(R^+\right)^{3/7}\right] \\
+14b_2\frac{db_2}{dz}\left[\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)+\frac{7R^2}{198}\left(R^+\right)^{4/7}\right] \\
+14b_0b_1\left[\left(\delta^+\right)^{1/7}\left(\frac{2\delta}{15}-\frac{R}{8}\right)\frac{d\delta}{dz}+\frac{1}{7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}+\frac{R^2}{120}\left(R^+\right)^{-6/7}\frac{dR^+}{dz}\right] \\
+7\left(b_1^2+2b_0b_2\right)\left[\left(\delta^+\right)^{2/7}\left(\frac{2\delta}{16}-\frac{R}{9}\right)\frac{d\delta}{dz}+\frac{2}{7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}+\frac{R^2}{72}\left(R^+\right)^{-5/7}\frac{dR^+}{dz}\right] \\
+14b_1b_2\left[\left(\delta^+\right)^{3/7}\left(\frac{2\delta}{17}-\frac{R}{10}\right)\frac{d\delta}{dz}+\frac{3}{7}\left(\frac{\delta^2}{17}-\frac{R\delta}{10}\right)\left(\delta^+\right)^{-4/7}\frac{d\delta^+}{dz}+\frac{3R^2}{170}\left(R^+\right)^{-4/7}\frac{dR^+}{dz}\right] \\
+7b_2^2\left[\left(\delta^+\right)^{4/7}\left(\frac{2\delta}{18}-\frac{R}{11}\right)\frac{d\delta}{dz}+\frac{4}{7}\left(\frac{\delta^2}{18}-\frac{R\delta}{11}\right)\left(\delta^+\right)^{-3/7}\frac{d\delta^+}{dz}+\frac{2R^2}{99}\left(R^+\right)^{-3/7}\frac{dR^+}{dz}\right]
\end{array}\right\}
$$

$$
+\left(u_v^*\right)^2(R-\delta)\left\{b_0^2+2b_0b_1\left(\delta^+\right)^{1/7}+\left(b_1^2+2b_0b_2\right)\left(\delta^+\right)^{2/7}+2b_1b_2\left(\delta^+\right)^{3/7}+b_2^2\left(\delta^+\right)^{4/7}\right\}\frac{d\delta}{dz}
$$

$$
=-\frac{(R-\delta)^2}{2\rho_l}\frac{d\overline{P}}{dz}-\frac{g(R-\delta)^2}{2}-\frac{(R-\delta)\varepsilon_{m,l}u_v^*\left[b_1\left(\delta^+\right)^{1/7}+2b_1\left(\delta^+\right)^{2/7}\right]}{7\delta} \qquad (5\text{-}23)
$$

Substituting Equations K-3, K-4 and K-8 in Equation 5-22 yields the liquid core energy equation

$$
\begin{aligned}
&\left\{\begin{aligned}
&\frac{b_0}{2}(R-\delta)^2+7b_1\left[\left(\delta^+\right)^{1/7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]\\
&+7b_2\left[\left(\delta^+\right)^{2/7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]
\end{aligned}\right\}\left(u_v^*\frac{dT_w}{dz}+\frac{T_w}{2\sqrt{\rho_v\tau_w}}\frac{d\tau_w}{dz}\right)\\
&+\left(u_v^*T_w\right)\left\{\begin{aligned}
&\frac{(R-\delta)^2}{2}\frac{db_0}{dz}-b_0(R-\delta)\frac{d\delta}{dz}+7\left[\left(\delta^+\right)^{1/7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]\frac{db_1}{dz}+\\
&7b_1\left[\left(\delta^+\right)^{1/7}\left(\frac{2\delta}{15}-\frac{R}{8}\right)\frac{d\delta}{dz}+\frac{1}{7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}+\frac{R^2}{120}\left(R^+\right)^{-6/7}\frac{dR^+}{dz}\right]+\\
&7\left[\left(\delta^+\right)^{2/7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]\frac{db_2}{dz}+\\
&7b_2\left[\left(\delta^+\right)^{2/7}\left(\frac{2\delta}{16}-\frac{R}{9}\right)\frac{d\delta}{dz}+\frac{2}{7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}+\frac{R^2}{72}\left(R^+\right)^{-5/7}\frac{dR^+}{dz}\right]
\end{aligned}\right\}\\
&\frac{q''_w}{\rho_v c_{p,v}}\left\{\begin{aligned}
&\frac{(R-\delta)^2}{2}\left(d_0\frac{db_0}{dz}+b_0\frac{dd_0}{dz}\right)-b_0d_0(R-\delta)\frac{d\delta}{dz}+\\
&7\left(b_1\frac{dd_0}{dz}+d_1\frac{db_0}{dz}+b_0\frac{dd_1}{dz}+d_0\frac{db_1}{dz}\right)\left[\left(\delta^+\right)^{1/7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]+\\
&7b_1d_0\left[\left(\delta^+\right)^{1/7}\left(\frac{2\delta}{15}-\frac{R}{8}\right)\frac{d\delta}{dz}+\frac{1}{7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}+\frac{R^2}{120}\left(R^+\right)^{-6/7}\frac{dR^+}{dz}\right]+\\
&7\left(b_2\frac{dd_0}{dz}+b_1\frac{dd_1}{dz}+b_0\frac{dd_2}{dz}+d_0\frac{db_2}{dz}+d_1\frac{db_1}{dz}+d_2\frac{db_0}{dz}\right)\left[\left(\delta^+\right)^{2/7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]+\\
&7\left(b_2d_0+b_1d_1+b_0d_2\right)\left[\left(\delta^+\right)^{2/7}\left(\frac{2\delta}{16}-\frac{R}{9}\right)\frac{d\delta}{dz}+\frac{2}{7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}+\frac{R^2}{72}\left(R^+\right)^{-5/7}\frac{dR^+}{dz}\right]+\\
&7\left(b_2\frac{dd_1}{dz}+b_1\frac{dd_2}{dz}+b_2\frac{dd_1}{dz}+d_1\frac{db_2}{dz}\right)\left[\left(\delta^+\right)^{3/7}\left(\frac{\delta^2}{17}-\frac{R\delta}{10}\right)+\frac{7R^2}{170}\left(R^+\right)^{3/7}\right]+\\
&7\left(b_1d_2+b_2d_1\right)\left[\left(\delta^+\right)^{3/7}\left(\frac{2\delta}{17}-\frac{R}{10}\right)\frac{d\delta}{dz}+\frac{3}{7}\left(\frac{\delta^2}{17}-\frac{R\delta}{10}\right)\left(\delta^+\right)^{-4/7}\frac{d\delta^+}{dz}+\frac{3R^2}{170}\left(R^+\right)^{-4/7}\frac{dR^+}{dz}\right]+\\
&7\left(d_2\frac{db_2}{dz}+b_2\frac{dd_2}{dz}\right)\left[\left(\delta^+\right)^{4/7}\left(\frac{\delta^2}{18}-\frac{R\delta}{11}\right)+\frac{7R^2}{198}\left(R^+\right)^{4/7}\right]+\\
&7b_2d_2\left[\left(\delta^+\right)^{4/7}\left(\frac{2\delta}{18}-\frac{R}{11}\right)\frac{d\delta}{dz}+\frac{4}{7}\left(\frac{\delta^2}{18}-\frac{R\delta}{11}\right)\left(\delta^+\right)^{-3/7}\frac{d\delta^+}{dz}+\frac{2R^2}{99}\left(R^+\right)^{-3/7}\frac{dR^+}{dz}\right]
\end{aligned}\right\}+
\end{aligned}
$$

$$\frac{1}{\rho_v c_{p,v}}\left\{\begin{aligned}&\frac{b_0 d_0}{2}(R-\delta)^2+7(b_1 d_0+d_1 b_0)\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]\\&+7(b_2 d_0+b_1 d_1+b_0 d_2)\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]+\\&7(b_1 d_2+b_2 d_1)\left[\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)+\frac{7R^2}{170}\left(R^+\right)^{3/7}\right]\\&+7b_2 d_2\left[\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)+\frac{7R^2}{198}\left(R^+\right)^{4/7}\right]\end{aligned}\right\}\frac{dq_w''}{dz}$$

$$+(R-\delta)(u_v^*)\left[b_0+b_1\left(\delta^+\right)^{1/7}+b_2\left(\delta^+\right)^{2/7}\right]\left\{T_w+\left(\frac{q_w''}{\rho_v c_{p,v}u_v^*}\right)\left[d_0+d_1\left(\delta^+\right)^{1/7}+d_2\left(\delta^+\right)^{2/7}\right]\right\}\frac{d\delta}{dz} \tag{5-24}$$

$$=-(R-\delta)\left(\frac{\varepsilon_{m,l}}{\Pr_l^t}\right)\left(\frac{q_w''}{\rho_v c_{p,v}u_v^*}\right)\frac{\left[d_1\left(\delta^+\right)^{1/7}+2d_2\left(\delta^+\right)^{2/7}\right]}{7\delta}$$

5.3 Solution Methodology

Equations 5-13, 5-14, 5-18, 5-19, 5-23 and 5-24 represent the system of six non-linear first order ordinary differential equations. Substituting for the coefficients $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, d_0, d_1, d_2$ and their derivatives along with the expressions for the derivatives of δ^+ and R^+ in the above equations results in a system with $\tau_w, \tau_i, \delta, \overline{P}, T_w, m_l''$ as the dependent variables (unknowns) and z as the independent variable. These need to be solved numerically.

In order to solve the system of equations, they are cast in the form

$$[A][X]=[C] \tag{5-25}$$

where the coefficient matrix, $[A]$ is a non-singular 6×6 square matrix with known coefficients. The matrices $[X]$ and $[C]$ are single column matrices with 6 rows. These are symbolically shown below,

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{16} \\ a_{21} & a_{22} & . & . & . & a_{26} \\ a_{31} & . & . & . & . & a_{36} \\ a_{41} & . & . & . & . & a_{46} \\ a_{51} & . & . & . & . & a_{56} \\ a_{61} & a_{62} & . & . & . & a_{66} \end{bmatrix} \quad X \equiv \begin{bmatrix} \frac{d\tau_w}{dz} \\ \frac{d\tau_i}{dz} \\ \frac{d\delta}{dz} \\ \frac{dP}{dz} \\ \frac{dT_w}{dz} \\ \frac{d\dot{m}_l''}{dz} \end{bmatrix} \quad C \equiv \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} \tag{5-26}$$

The matrix $[X]$ represents the solution that needs to be obtained, is called the solution vector, while the right hand side matrix $[C]$ is the known constants that appear in the governing equations. The solution vector $[X]$ is obtained by converting coefficient matrix, $[A]$, into an upper triangular matrix by Gaussian elimination. The solution vector can then be obtained using back substitution. The value of the dependent variable can be evaluated at a subsequent z location by a simple first-order method. Thus,

$$\tau_w\big|_{z+\Delta z} = \tau_w\big|_z + \left(\frac{d\tau_w}{dz}\right)\Delta z$$

where

$\tau_w\big|_{z+\Delta z}$ is the value of the wall shear stress at the location $z+\Delta z$.

$\tau_w\big|_z$ is the value of the wall shear stress at the location z.

$\left(\frac{d\tau_w}{dz}\right)$ is the value of the gradient calculated from the matrix inversion procedure.

Δz is the step size.

The other variables can also be calculated in a similar manner. Once the values of the dependent variables are calculated, quantities like void fraction, heat transfer coefficient can be calculated at the new location. The procedure is repeated until IAFB is terminated when the void fraction reaches 0.6.

In order to start the calculation, initial values of the dependant variables $\tau_w, \tau_i, \delta, \overline{P}, T_w, m_l''$ need to be specified. Also, the wall heat flux, q_w'' and the gradient of the heat flux $\frac{dq_w''}{dz}$ need to be specified. It must be noted that the initial value of the interfacial shear stress is calculated using the fact that interfacial shear is a fraction of the wall shear at the same axial location. This is an input by the user. The initial values of the film thickness and the mass flux due to evaporation should be a value higher than zero. The value of initial wall shear stress is calculated using single-phase liquid flow and is multiplied by a two-phase multiplier to account for the two-phase flow situation. The initial wall temperature is chosen to be 922 K (1200 deg F) or 1144 K (1600 deg F). The eddy diffusivity of momentum of the liquid and vapor are assumed to be identical to each other. These are evaluated using Travis' model. It was found that Travis' model gives realistic predictions of the eddy diffusivity.

Once the calculation is started, it proceeds until the value of void fraction reaches 0.6 at which the IAFB region is assumed to terminate. The convective heat transfer coefficient is estimated by the following relation:

$$h = \frac{k_v}{\delta} \tag{5-27}$$

where δ is the vapor film thickness and k_v is the thermal conductivity of vapor. The effect of heat transfer due to radiation is added by evaluating a radiative heat transfer coefficient, which is given by:

$$h_r = \frac{\sigma\left(T_w^4 - T_{sat}^4\right)}{\left[\frac{1}{\varepsilon_w} + \frac{P_w}{P_i}\left(\frac{1}{\varepsilon_l} - 1\right)\right]\left(T_w - T_{sat}\right)} \tag{5-28}$$

The total heat transfer coefficient is the sum of the convective and the radiative components.

Chapter 6

Results and Discussion

6.1 Introduction

Inverted annular film boiling is characterized by a vapor film covering the heating surface. This film separates the liquid core, at or below the saturation temperature at that pressure, from the heating surface. IAFB involves convective heat transfer from the hot wall to the vapor blanket and from the vapor to the liquid core. Heat is also transferred from the hot rods to the liquid core by radiation. The heat transfer coefficient associated with this mode of film boiling is very small primarily because of the low thermal conductivity of the vapor.

6.2 Results and Discussion

The solution methodology outlined in Chapter 5 was used and the model was executed for a couple of test cases. Due to the highly coupled nature of the equations, numerical stability issues were observed in some cases.

For a set of input conditions given in Table 6-1, two test cases with wall temperatures of 922 K (1200 deg F) and 1144 K (1600 deg F) were considered. Plots of void fraction and vapor film thickness as a function of the axial direction are made. These are shown in Figure 6-1 and 6-2 for the case with wall temperature of 922 K (1200 deg F). Figure 6-3 shows the plot of heat transfer coefficient as a function of void fraction for the same case. Figures 6-4 and 6-5 show the zoomed in void fraction and heat transfer coefficient plots for the same case.

Table 6-1 provides a listing of the conditions for the runs. The values of wall shear stress that is needed as an input quantity is calculated using the relation given by Equation 6-1.

$$\tau_w = C_f \frac{\rho V^2}{2} \qquad (6\text{-}1)$$

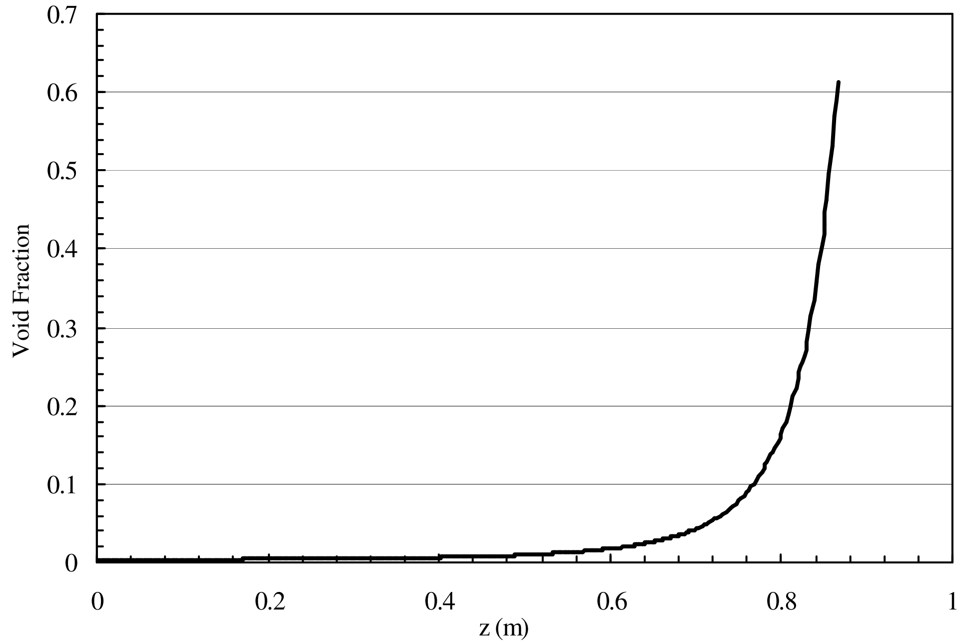

Figure 6-1: Void Fraction as function of axial distance, $T_w = 922K(1200 \deg F)$

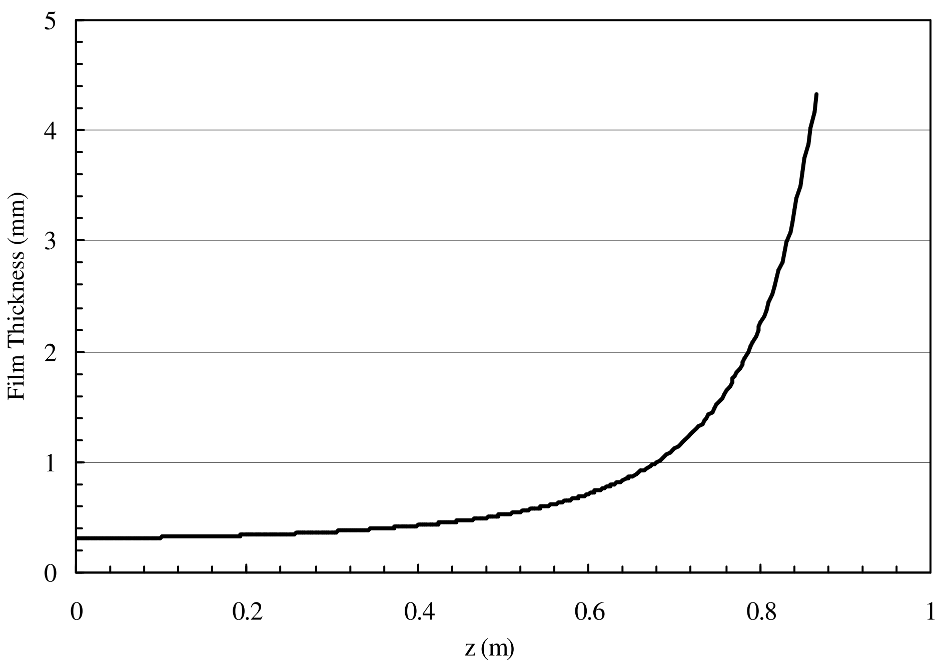

Figure 6-2: Film Thickness as function of axial distance, $T_w = 922K(1200 \deg F)$

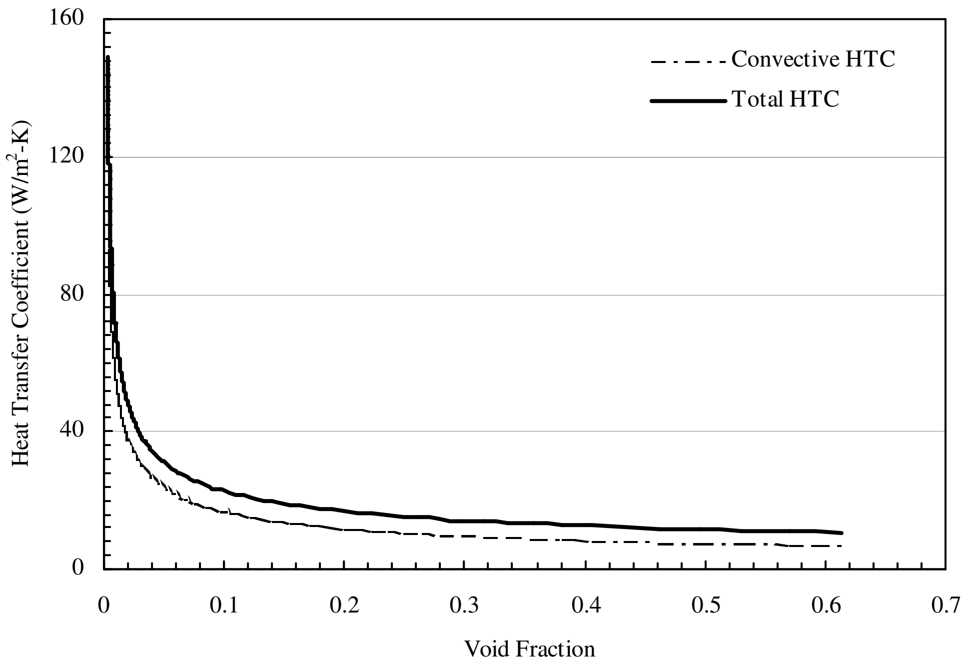

Figure 6-3: Heat Transfer Coefficient Vs. Void Fraction, $T_w = 922K(1200 \deg F)$

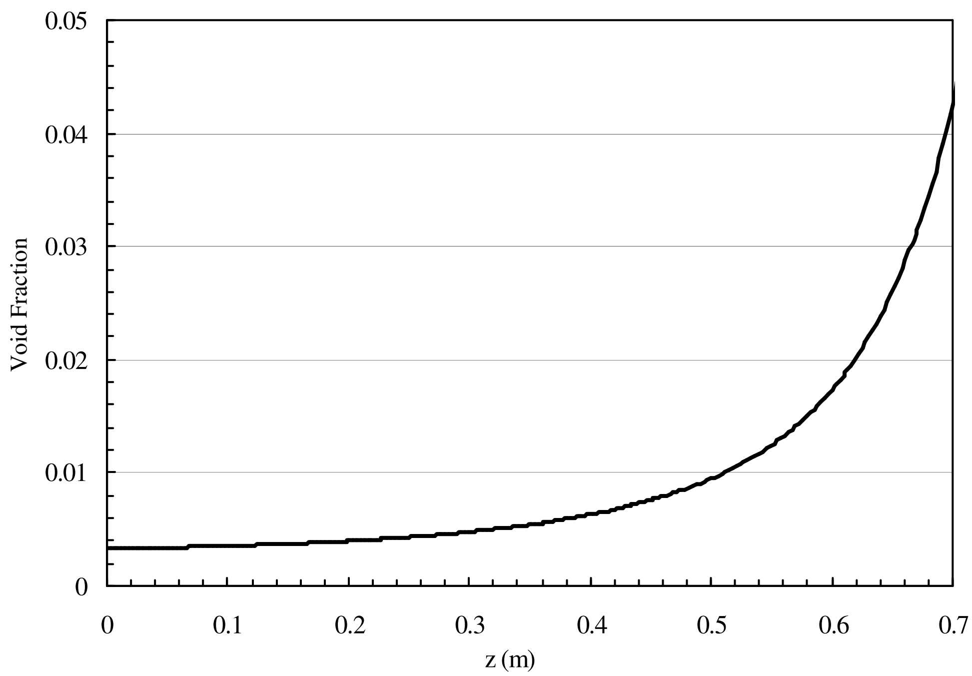

Figure 6-4: Void Fraction as a function of axial distance - detail, $T_w = 922K(1200 \deg F)$

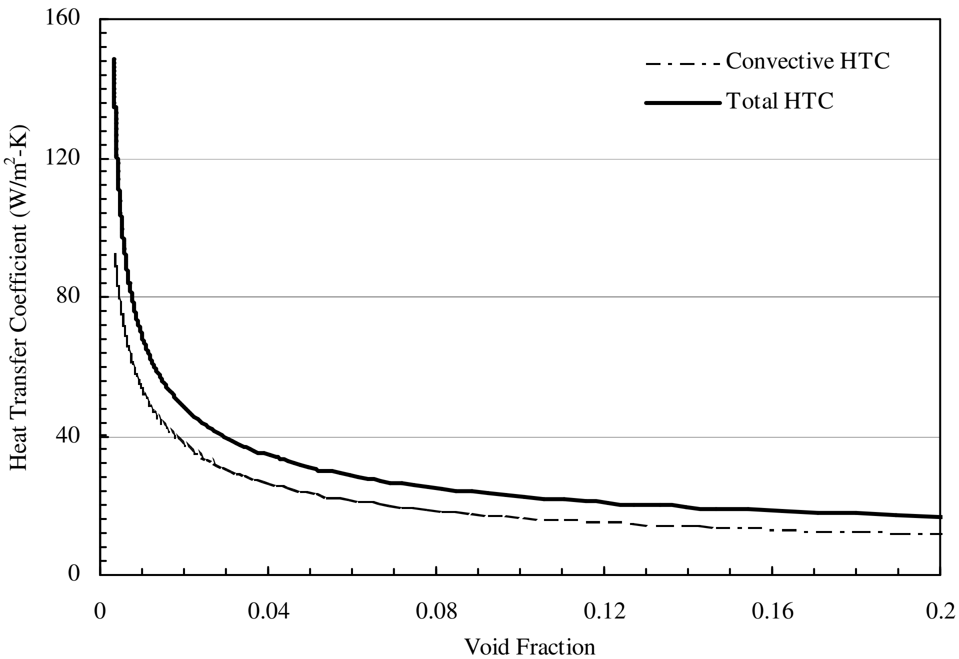

Figure 6-5: Heat Transfer Coefficient Vs. Void Fraction - detail, $T_w = 922K(1200\deg F)$

The value of C_f is obtained from the Equation 6-2.

$$C_f = \frac{0.184}{\mathrm{Re}_D^{0.2}} \tag{6-2}$$

The Reynolds number is defined by

$$\mathrm{Re}_D = \frac{\rho_f D_h V}{\mu_f} \tag{6-3}$$

The wall heat flux needed as input is calculated from FLECHT-SEASET experiment 31701. This is a typical high flooding rate experiment with conditions given in Table 6-2.

Table 6-1: Summary of Initial Conditions

Quantity	SI Units	British Units
Pressure, P	275.786 kPa	40 psia
Wall shear stress, τ_w	441.788 Pa	9.227 lbf/ft^2
Fraction, $\frac{\tau_i}{\tau_w}$	0.1	0.1
Initial Film thickness, δ	0.3048 mm	0.001 ft
Initial mass flux rate, $\dot{m}_l''$	0.00488 kg/sec-m^2	0.001 lbm/sec-ft^2
Wall temperature, T_w	922 K	1200 deg F
Wall heat flux, q_w''	139.89 kW/m^2	12.322 Btu/sec-ft^2
$\frac{dq_w''}{dz}$	-740 kW/m^3	-19.87 Btu/sec-ft^3

From Figures 6-1 and 6-4, it is seen that void fraction increases along the length as expected. This is due to the vaporization of liquid as it moves along the length of the bundle. The vapor film thickness increases as seen from Figure 6-2. The plot of heat transfer coefficient Vs. void fraction shows that at low void fractions, heat transfer coefficient is quite high and it decreases quite rapidly as we move along the length of the heated bundle. This is due to the fact that at the beginning, the vapor film is thin, and hence the resistance due to the film thickness is smaller. Also, as seen from Figure 6-6, the heat transfer due to radiation is very significant at the lower elevations. At higher elevations, due to the heat removal and hence reduction in the wall temperature, radiation contribution becomes much smaller. This is seen in Figure 6-6.

Table 6-2: Conditions for FLECHT-SEASET Experiment 31701

Upper Plenum Pressure	0.28 MPa (40 psia)
Rod Peak Power	2.3 kW/m (0.7 kW/ft)
Flow Rate	155 mm/sec (6.1 inch/sec)
Coolant Temperature	53^{o}C (127 deg F)

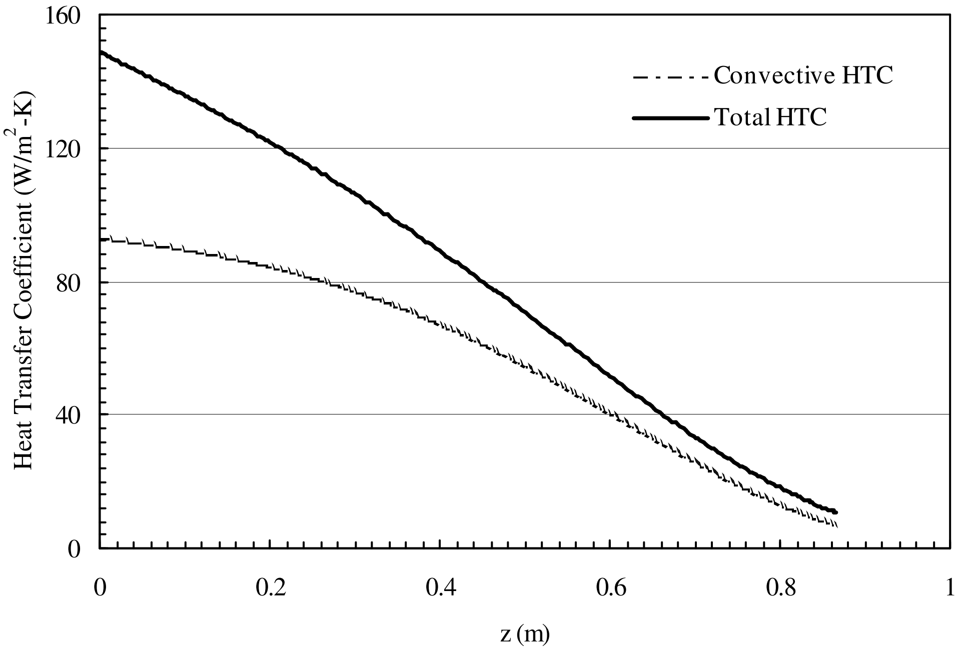

Figure 6-6: Comparison of convective and radiative heat transfer coefficient components, $T_w = 922K(1200 \deg F)$

Figures 6-7 through 6-12 show the plots for case with the same conditions as given in Table 6-1 except that the initial wall temperature was assumed to be 1144 K (1600 deg F)

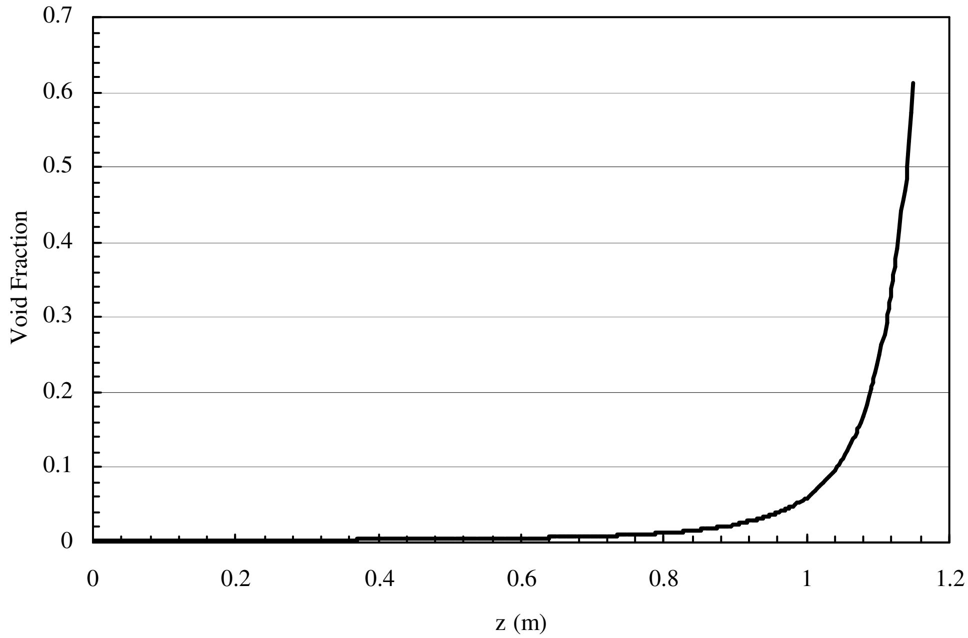

Figure 6-7: Void Fraction as function of axial distance, $T_w = 1144K(1600\deg F)$

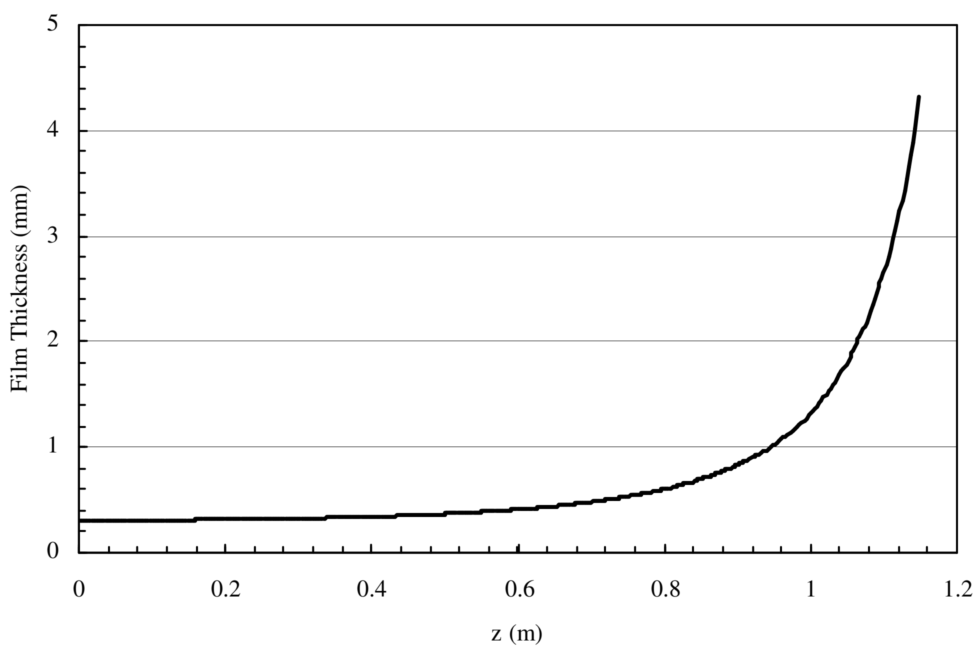

Figure 6-8: Film Thickness as function of axial distance, $T_w = 1144K(1600\deg F)$

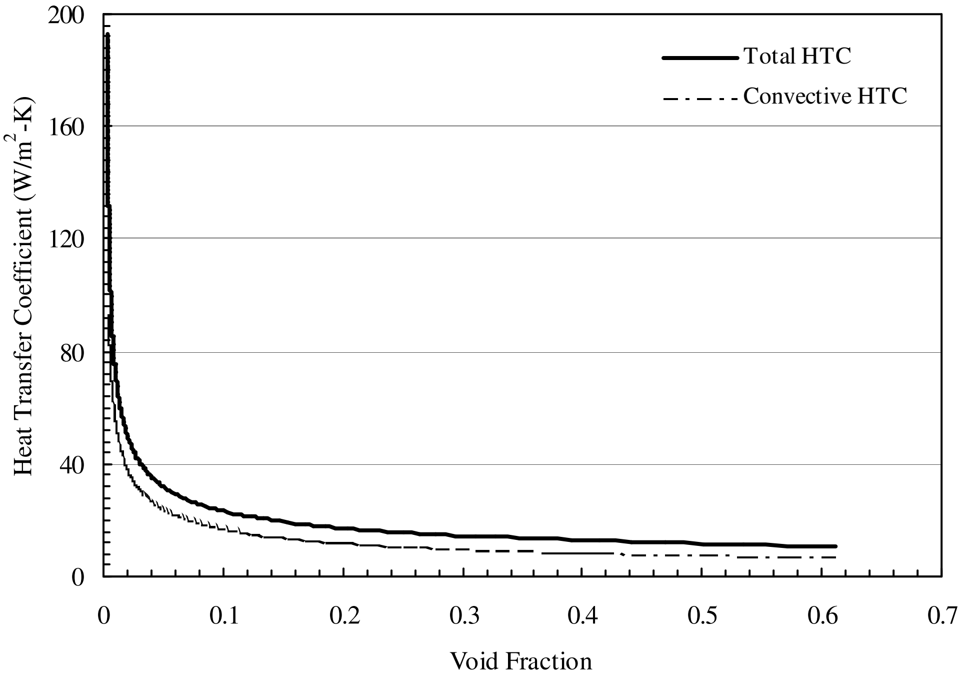

Figure 6-9: Heat Transfer Coefficient Vs. Void Fraction, $T_w = 1144K(1600 \deg F)$

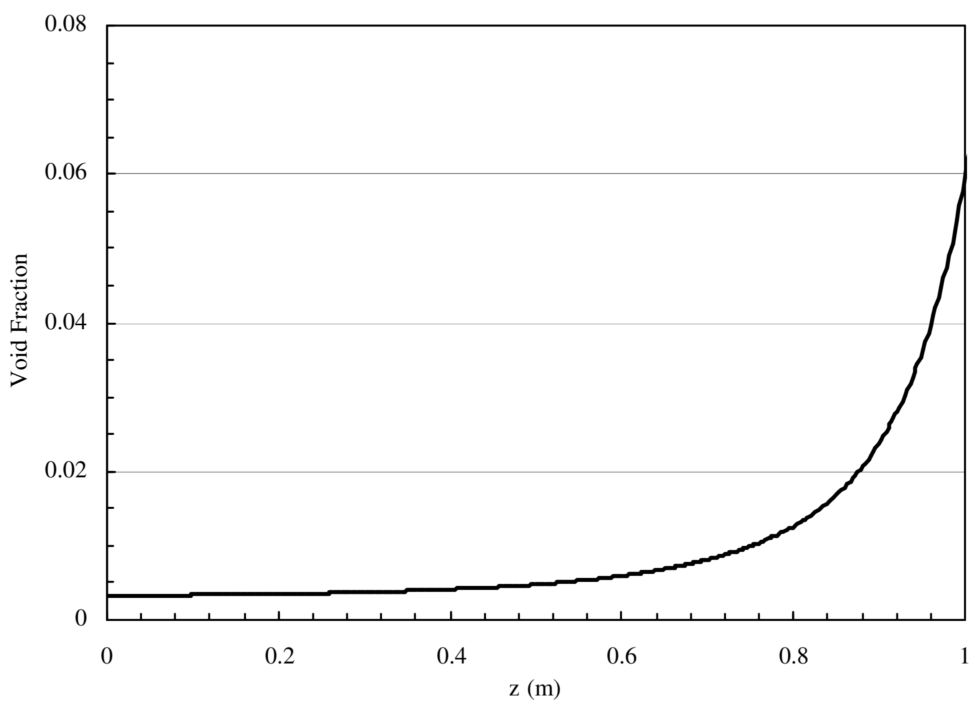

Figure 6-10: Void Fraction as a function of axial distance - detail, $T_w = 1144K(1600 \deg F)$

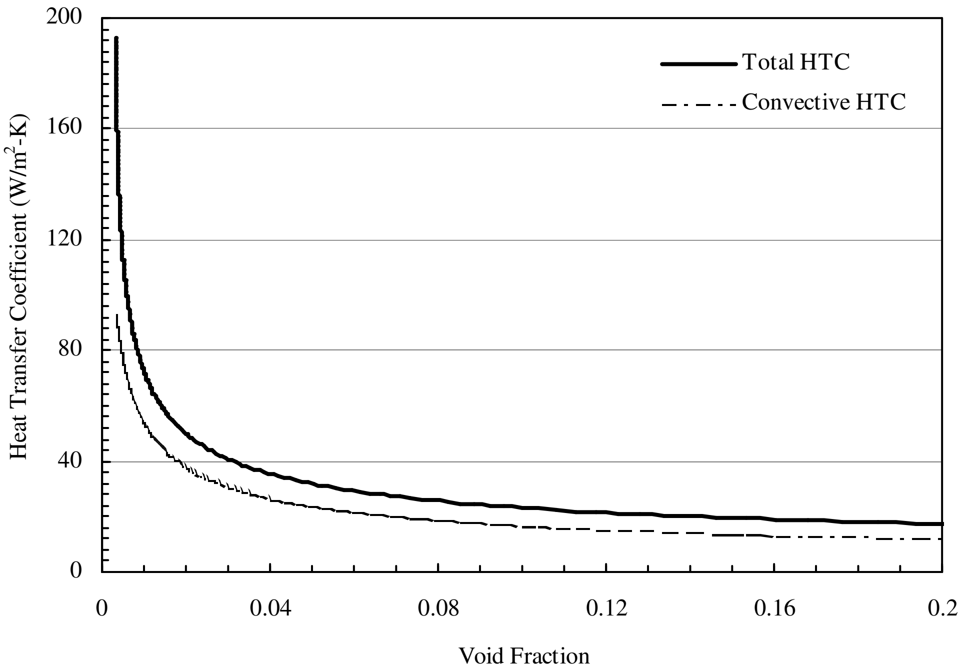

Figure 6-11: Heat Transfer Coefficient Vs. Void Fraction - detail, $T_w = 1144K(1600 \deg F)$

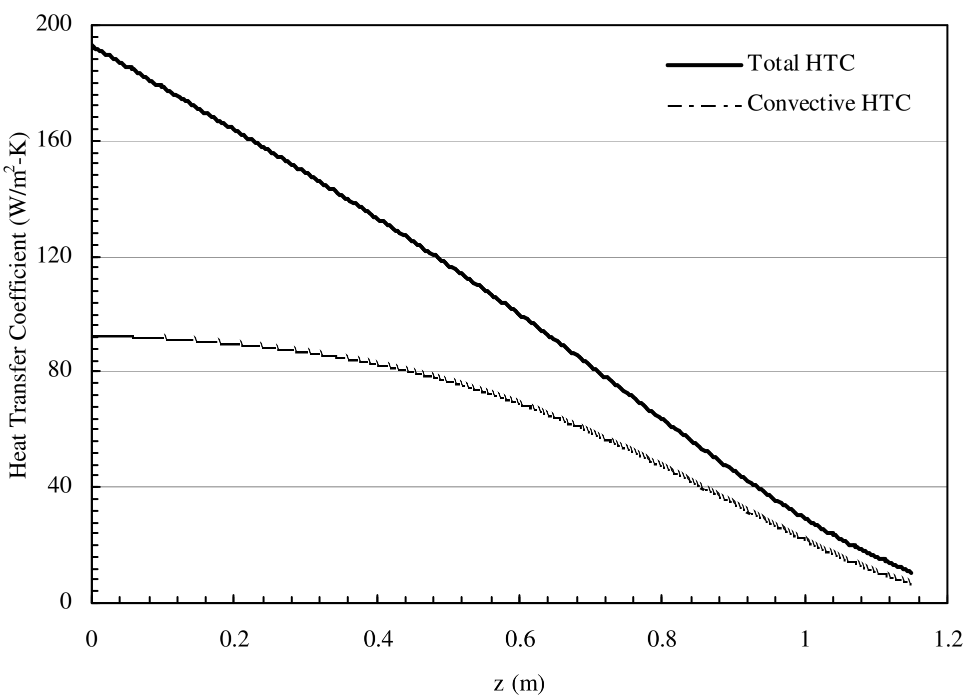

Figure 6-12: Comparison of convective and radiative heat transfer coefficient components, $T_w = 1144K(1600 \deg F)$

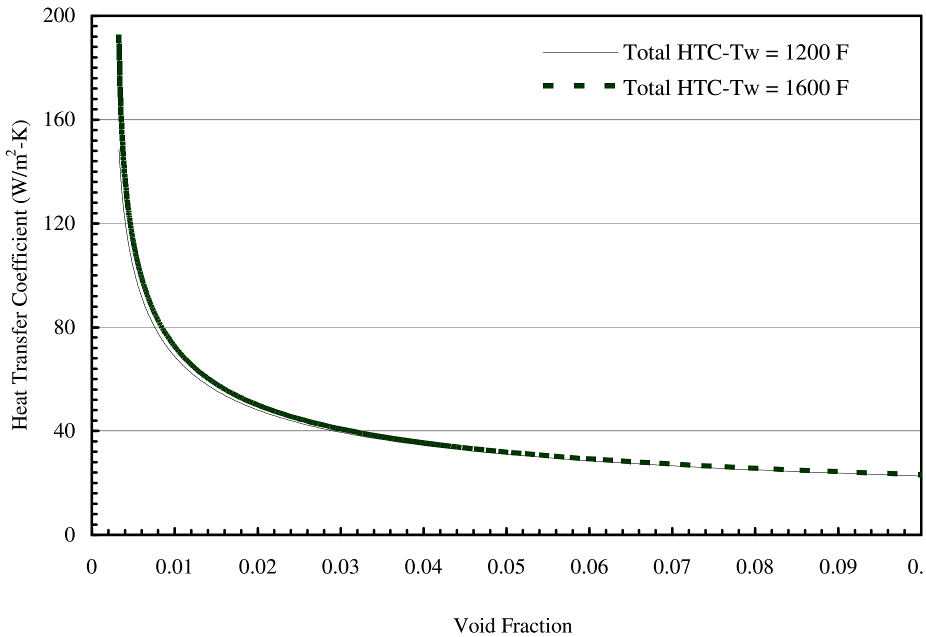

Figure 6-13: Comparison of Heat transfer coefficients for cases with different initial wall temperature.

6.3 Effect of Variation of Input Parameters

Comparison of the cases, with different initial wall temperature reveals that, for the case with higher initial wall temperature, IAFB regime exists for a greater distance as compared to the case with lower wall temperature. The heat transfer coefficient for the case with $T_w = 1144K(1600\deg F)$ starts off at a higher value than the case with $T_w = 922K(1200\deg F)$. This is mainly due to the radiation component, which is higher due to the higher initial wall temperature. The heat transfer coefficient then drops off in the same manner for both cases. This is shown in Figure 6-13.

The interfacial shear is taken to be a fraction of the wall shear stress. The initial value of the interfacial shear can be varied by changing the value of this fraction parameter. As the fraction is made larger, it is found that the IAFB regime gets shorter, this can be explained by the fact that as the initial interfacial shear is made larger, the destabilizing force that leads to the break up of the liquid core is made larger. This effect propagates in the axial direction, thereby causing the IAFB regime to terminate earlier.

As expected, the IAFB region is shorter for a case with higher wall heat flux, with all other input parameters remaining the same due to the fact that more heat is available for evaporation of the liquid.

The results are not very significant to changes in the step size. The accuracy of the results is significantly poorer with a coarser step size. However, making the step size finer than 0.01 ft, does not increase the accuracy of the results. Hence, calculations are done with a step size of 0.01 ft.

6.4 Comparison with Experimental Data

The model predictions have been compared to experimental data from two Penn State – NRC Rod Bundle Heat Transfer (RBHT) experiments. These are

Experiment 1223: 40 psia, 6 inch/sec, 1600 deg F initial temperature and 20 deg F subcooling.
Experiment 1285: 40 psia, 6 inch/sec, 1600 deg F initial temperature and 150 deg F subcooling.

Of the several experiments performed in the RBHT series, these experiments come closest to the conditions wherein IAFB could exist.

The plots of heat transfer coefficient as a function of the axial distance, z, are shown in Figures 6-14 through 6-16.

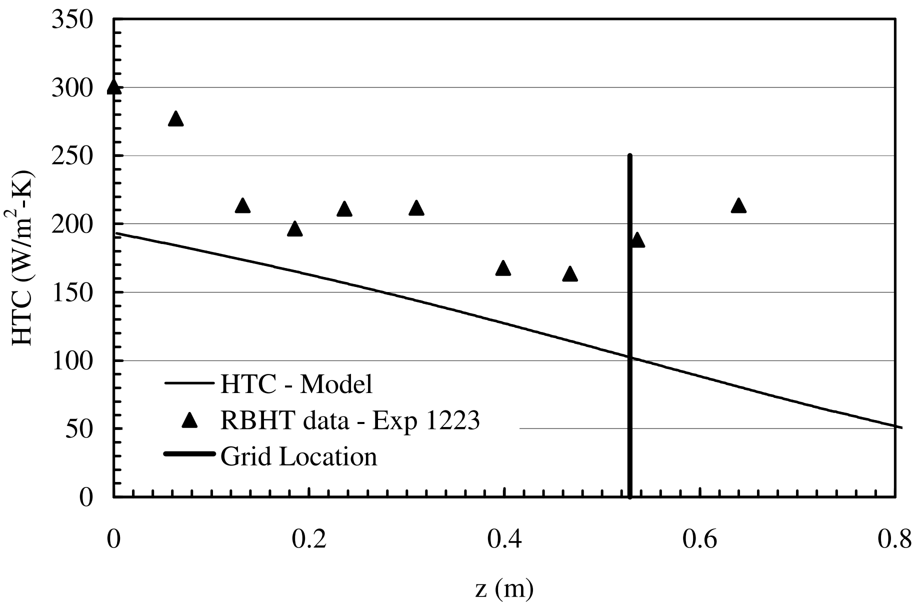

Figure 6-14: Comparison of heat transfer coefficients: RBHT data (Expt. 1223) with present model.

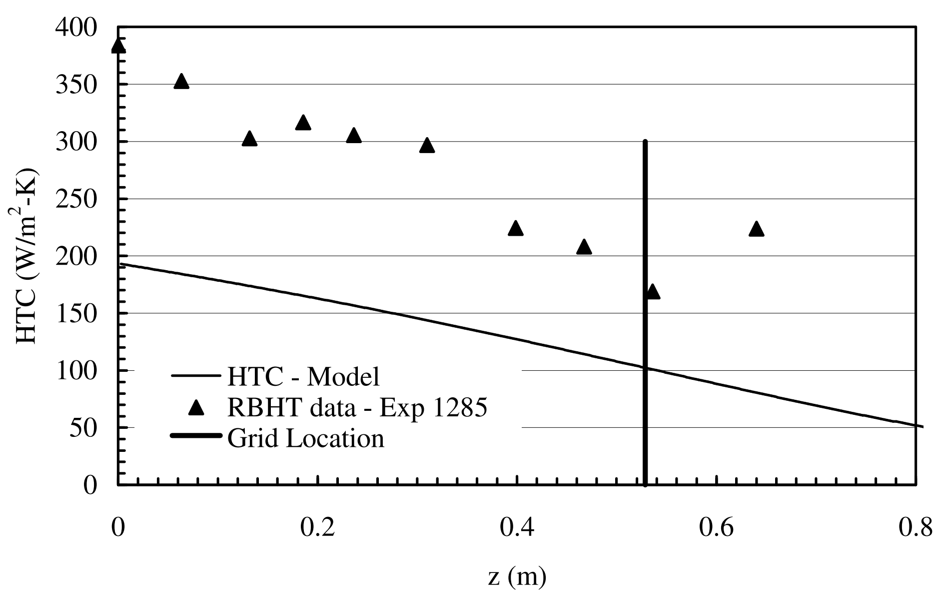

Figure 6-15: Comparison of heat transfer coefficients: RBHT data (Expt. 1285) with present model.

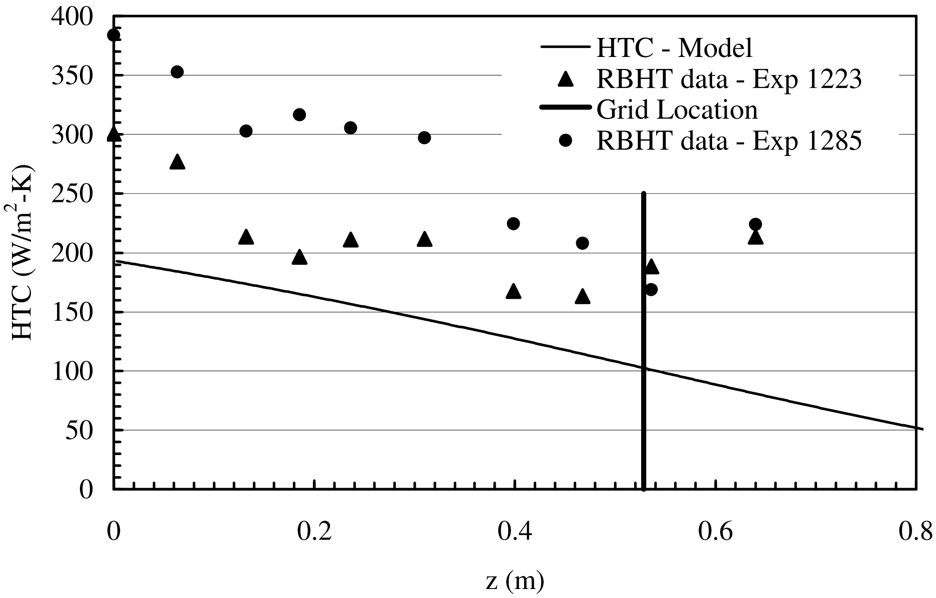

Figure 6-16: Comparison of heat transfer coefficients: RBHT data (Expt. 1223 and 1285) with present model.

As seen from the above figures, comparison of model prediction with RBHT data is conservative. It should be noted that the experiments chosen were the ones in which IAFB would have existed, if at all, though it was not clear if the experiment actually simulated IAFB. The present model does not account for the liquid subcooling, where as both RBHT experiments chosen were with initial subcooling. As seen from the graphs, higher the subcooling, greater is the difference between data and predictions. Presence of spacer grids enhance turbulence and increase heat transfer due to good mixing. The current model does not account for spacer grid effects whereas the RBHT experimental setup had spacer grids for better mixing. The increase in the heat transfer coefficient downstream of the spacer grid shown on the plots is not accounted for by the model. Also, wall temperature predicted by model is lower than what is observed, thereby reducing heat transfer coefficients. The liquid-vapor interface in IAFB is typically wavy. This causes an increased heat transfer, due to the larger surface area. The effect of waviness of the interface is not considered in the model, thereby making the predictions conservative.

Chapter 7

Conclusions and Recommendations for Future Work

7.1 Conclusions

Models for Inverted Annular Film Boiling have been developed over the last few decades. Due to the inherent complexity of the problem, none of the models have been comprehensive. The current work employs a slightly different approach to modeling the phenomenon. The derivation of the governing equations and using an Integral Method technique for the solution is the unique to this work.

Inverted Annular Film Boiling phenomenon has been modeled from first principles. The governing equations were derived in two dimensions assuming axisymmetric behavior. Turbulence was incorporated in the formulation by use of eddy diffusivity of momentum and energy. Second order partial differential equations for the conservation of mass, momentum and energy were derived for both the liquid core and the vapor film. The vapor film was split into two regions – the wall region near the heated surface where the effects of viscosity dominate and a turbulent core away from the wall where the effect of turbulence is predominant. These equations were then integrated over the radial direction to obtain a system of first order integral differential equations.

In order to solve this system of equations, the axial component of the non-dimensional liquid and vapor velocity, non-dimensional liquid and vapor temperature was assumed to be represented by a second order polynomial of the form $A_0 + A_1\eta + A_2\eta^2$ where $\eta = \left(y^+\right)^{1/7}$ and y^+ represents the dimensionless distance from the wall. Based on the physical phenomena and boundary conditions, expressions for the coefficients that appear in the assumed polynomial profiles were derived in terms of the six dependent variables: $\tau_w, \tau_i, \delta, \overline{P}, T_w, m_l''$. Using the assumed profiles, the system of integral differential equations was converted into a system of non-linear first order ordinary differential equations.

An order of magnitude analysis ensured that the terms of the form $\frac{\partial^2}{\partial z^2}$ were negligible compared to terms of the form $\frac{\partial^2}{\partial r^2}$. Although the assumption of one-dimensional flow is not physically realistic, it was necessary to obtain a set of solvable non-linear first order ordinary differential equations from the original set of equations, which contained terms with products of derivatives of the dependent variable making it a very complicated set of equations mathematically.

A simple numerical solution of the now simplified governing system of equations resulted in reasonable results. Plots of void fraction as a function of the distance from the quench front show an increase as expected, with the void fraction increasing slowly at the beginning, and then more rapidly later on. The IAFB region was assumed to end when the void fraction reached a value of 0.6 at which the calculations were terminated. The vapor film thickness also increased as the distance from the quench front increases. This is due to continuous evaporation of the liquid, which is at saturation temperature into vapor, thereby increasing the film thickness. It must be noted that in the present analysis there was no provision for dealing with liquid subcooling effects. The plot of heat transfer coefficient as a function of the void fraction shows a sharp decrease at low void fractions and a slow decrease later on. Nevertheless the heat transfer coefficient continually decreased with increasing void fraction. This trend is along expected lines, as the resistance to heat transfer is significantly increased as the vapor film gets thicker due to the poor thermal conductivity of the vapor. For cases with higher initial wall temperature, the heat transfer coefficient starts higher due to the increased effect of radiation. IAFB region extends to a greater length along the bundle.

7.2 Recommendations for Future Work

Modeling of Inverted Annular Film Boiling is a very challenging task due to the complex nature of the phenomena involved. The current modeling effort is quite unique when compared with existing models in this area. This is, by no means a complete work in itself and there are several aspects that can be improved upon. Subcooling of the liquid is an important parameter in the analysis of IAFB. The current modeling approach does not account for the inlet subcooling effects. In order to make the system of equations mathematically tractable, the radial component of the velocity is assumed to be negligible. This is not true, especially near the interface where the liquid that is vaporized will have radial component of velocity as it enters the vapor film. For simplicity of derivation, the eddy diffusivities were treated as a constant. This is also not realistic. The variation of the eddy diffusivity can be incorporated in the integral equations. The current model does not account for the waviness of the liquid-vapor interface. Waviness of the interface contributes to enhanced heat transfer due to higher interfacial area. If the waviness can be modeled and incorporated, the predictions of the model can be improved. There are studies in the literature on interfacial structure in vertical upwards annular flow. These can be used as a starting point for such an analysis. Modeling of IAFB using Integral Method has not been accomplished in this level of detail so far. This can be used as a starting point for further improvements future work in this area.

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Appendix A

Integration of Simplified Governing Equations

A.1 Introduction

The governing equations for mass, momentum and energy conservation derived earlier in Chapter 3 are partial differential equations with r and z as the independent variables. Integrating these over the radial direction at a given elevation yield terms which are regular derivatives with respect to z or terms which involve both integration and differentiation.

The liquid region extends from $r = 0$ to $r = R - \delta$. The vapor region, which extends from $r = R - \delta$ to $r = R$ is split into two regions, the turbulent vapor core extending from $r = R - \delta$ to a location corresponding to $y_v^+ = 5$ and a wall region extending from the location of $y_v^+ = 5$ to $r = R$.

The location corresponding to $y_v^+ = 5$ needs to be written in terms of the radial coordinate. Using the definition for $y_v^+ = \dfrac{y u_v^*}{\nu_v}$ and $y = R - r$, $y_v^+ = 5$ can be transformed to $r = R - \left(\dfrac{5\nu_v}{u_v^*}\right)$. It must be noted that the location of $y_v^+ = 5$ is a function of the axial location, z. As the thickness of the vapor film changes, the location of $y_v^+ = 5$ also changes. For convenience, let us define $l = \left(\dfrac{5\nu_v}{u_v^*}\right)$, to represent the demarcation between the vapor wall region and the turbulent core of the vapor film.

The integration is carried out by multiplying the governing equations by $dA = 2\pi r dr$ and applying the appropriate limits for the independent variable, r, as described above. The integration is with respect to the independent variable r, while the limits of integration are also functions of the other independent variable, z. Thus, one needs to employ Leibnitz formula to carry out the integration of some of the terms in the governing equations.

Leibnitz's formula is given as follows:

$$\frac{d}{dz}\left[\int_{x=\alpha(z)}^{x=\beta(z)} f(r,z)dr\right] = \int_{x=\alpha(z)}^{x=\beta(z)} \frac{\partial f(r,z)}{\partial z}dr + f[\beta(z),z]\frac{d\beta(z)}{dz} - f[\alpha(z),z]\frac{d\alpha(z)}{dz} \qquad \text{(A-1)}$$

A.2 Continuity Equations

The continuity equations for the liquid core and vapor film are integrated over the radial direction. As discussed, the vapor film is split into the wall region and the fully turbulent region.

A.2.1 Liquid Core

The liquid continuity equation is given by:

$$\frac{\partial \bar{u}_l}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{v}_l\right)=0 \tag{3-1}$$

Integrating the above, within specified limits over the radial direction

$$\int_{r=0}^{r=R-\delta}\left[\frac{\partial \bar{u}_l}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{v}_l\right)\right]2\pi r dr=0$$

$$\int_{r=0}^{r=R-\delta}\left(\frac{\partial \bar{u}_l}{\partial z}\right)2\pi r dr+\int_{r=0}^{r=R-\delta}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{v}_l\right)\right]2\pi r dr=0 \tag{A-2}$$

Canceling out 2π, the above equation can be integrated term by term. The second term is relatively straightforward, as it simplifies to the difference between the values of $\left(r\bar{v}_l\right)$ at the upper and the lower limits.

$$\int_{r=0}^{r=R-\delta}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{v}_l\right)\right]rdr=\int_{r=0}^{r=R-\delta}\frac{\partial}{\partial r}\left(r\bar{v}_l\right)dr=\left(r\bar{v}_l\right)\Big|_{r=0}^{r=R-\delta}=\left(r\bar{v}_l\right)\Big|_{r=R-\delta}-\left(r\bar{v}_l\right)\Big|_{r=0}=\left(r\bar{v}_l\right)\Big|_{r=R-\delta} \tag{A-3}$$

The integration of the first term involves use of Leibnitz formula.

$$\int_{r=0}^{r=R-\delta}\left(\frac{\partial \bar{u}_l}{\partial z}\right)rdr=\int_{r=0}^{r=R-\delta}\frac{\partial\left(r\bar{u}_l\right)}{\partial z}dr=\frac{d}{dz}\left[\int_{r=0}^{r=R-\delta}\left(r\bar{u}_l\right)dr\right]-\left(r\bar{u}_l\right)\Big|_{R-\delta}\frac{d}{dz}\left(R-\delta\right)+\left(r\bar{u}_l\right)\Big|_{0}\frac{d}{dz}(0)$$

Since R is a constant, the above equation simplifies to:

$$\int_{r=0}^{r=R-\delta} \frac{\partial(r\bar{u}_l)}{\partial z} dr = \frac{d}{dz}\left[\int_{r=0}^{r=R-\delta} (r\bar{u}_l)dr\right] + (r\bar{u}_l)\Big|_{R-\delta} \frac{d\delta}{dz} \tag{A-4}$$

Thus, combining Equations A-3 and A-4, the integrated form of the liquid continuity equation is:

$$\frac{d}{dz}\left[\int_{r=0}^{r=R-\delta} (r\bar{u}_l)dr\right] + (r\bar{u}_l)\Big|_{r=R-\delta} \frac{d\delta}{dz} + (r\bar{v}_l)\Big|_{r=R-\delta} = 0 \tag{A-5}$$

A.2.2 Vapor Phase

The vapor continuity equation is given by:

$$\frac{\partial(\rho_v \bar{u}_v)}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho_v \bar{v}_v) = 0 \tag{A-6}$$

The vapor equation needs to be integrated in two parts: the turbulent core from $r = R - \delta$ to $r = R - l$ and the wall region from $r = R - l$ to $r = R$, over the radial direction.

For the fully turbulent core region, the vapor continuity equation can be integrated as shown below.

$$\int_{R-\delta}^{R-l} \left[\frac{\partial(\rho_v \bar{u}_v)}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho_v \bar{v}_v)\right] 2\pi r dr = 0$$

Splitting into two the two terms, canceling 2π , the equation becomes:

$$\int_{R-\delta}^{R-l} \frac{\partial(\rho_v \bar{u}_v)}{\partial z} r dr + \int_{R-\delta}^{R-l} \frac{1}{r}\frac{\partial}{\partial r}(r\rho_v \bar{v}_v) r dr = 0 \tag{A-7}$$

The second term is relatively straightforward, as it simplifies to the difference between the values of $(r\rho_v \bar{v}_v)$ at the upper and the lower limits.

$$\int_{r=R-\delta}^{r=R-l}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\rho_v\bar{v}_v)\right]rdr=\int_{r=R-\delta}^{r=R-l}\frac{\partial}{\partial r}(r\rho_v\bar{v}_v)dr=(r\rho_v\bar{v}_v)\Big|_{r=R-\delta}^{r=R-l}=(r\rho_v\bar{v}_v)\Big|_{r=R-l}-(r\rho_v\bar{v}_v)\Big|_{r=R-\delta} \tag{A-8}$$

The integration of the first term involves use of Leibnitz formula.

$$\int_{r=R-\delta}^{r=R-l}\frac{\partial}{\partial z}(\rho_v\bar{u}_v)rdr=\int_{r=R-\delta}^{r=R-l}\frac{\partial(r\rho_v\bar{u}_v)}{\partial z}dr=\frac{d}{dz}\left[\int_{r=R-\delta}^{r=R-l}(r\rho_v\bar{u}_v)dr\right]-(r\rho_v\bar{u}_v)\Big|_{R-l}\frac{d}{dz}(R-l)+(r\rho_v\bar{u}_v)\Big|_{R-\delta}\frac{d}{dz}(R-\delta)$$

Since R is a constant, the above equation simplifies to

$$\int_{r=R-\delta}^{r=R-l}\frac{\partial}{\partial z}(\rho_v\bar{u}_v)rdr=\frac{d}{dz}\left[\int_{r=R-\delta}^{r=R-l}(r\rho_v\bar{u}_v)dr\right]+(r\rho_v\bar{u}_v)\Big|_{R-l}\frac{dl}{dz}-(r\rho_v\bar{u}_v)\Big|_{R-\delta}\frac{d\delta}{dz} \tag{A-9}$$

Using the same explanation as given in the derivation of the integral continuity equation for the liquid phase, the term for mass addition due to generation of vapor needs to appear on the right hand side of the vapor continuity equation. This is of the same form as the term in the integrated liquid continuity equation, except that the sign is reversed, indicating a mass addition due to vapor generation. If condensation of vapor occurred due to the presence of a subcooled liquid, the sign would change to negative to indicate depletion of vapor.

Combining Equations A-8 and A-9, the integrated form of the vapor continuity equation for the turbulent core is

$$\frac{d}{dz}\left[\int_{r=R-\delta}^{r=R-l}(r\rho_v\bar{u}_v)dr\right]+(r\rho_v\bar{u}_v)\Big|_{R-l}\frac{dl}{dz}-(r\rho_v\bar{u}_v)\Big|_{R-\delta}\frac{d\delta}{dz}+(r\rho_v\bar{v}_v)\Big|_{R-l}-(r\rho_v\bar{v}_v)\Big|_{R-\delta}=\dot{m}_l''P_i \tag{A-10}$$

For the wall region of the vapor film, the integration is performed in the same manner. In this case, the limits of integration are different from that for the turbulent core.

$$\int_{R-l}^{R}\left[\frac{\partial(\rho_v\bar{u}_v)}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho_v\bar{v}_v)\right]2\pi rdr=0 \tag{A-11}$$

Thus, the integrated form of the vapor continuity equation for the wall region is of the same form as that for the turbulent core with the limits being different.

$$\frac{d}{dz}\left[\int_{r=R-l}^{r=R}(r\rho_v\bar{u}_v)dr\right]-(r\rho_v\bar{u}_v)\big|_R\frac{dR}{dz}+(r\rho_v\bar{u}_v)\big|_{R-l}\frac{d(R-l)}{dz}+(r\rho_v\bar{v}_v)\big|_R-(r\rho_v\bar{v}_v)\big|_{R-l}=0$$

Also, the derivative of R with respect to z is zero. No slip condition at the wall implies that vapor velocity at $r=R$ is zero. Thus, the above equation simplifies to:

$$\frac{d}{dz}\left[\int_{r=R-l}^{r=R}(r\rho_v\bar{u}_v)dr\right]-(r\rho_v\bar{u}_v)\big|_{R-l}\frac{dl}{dz}-(r\rho_v\bar{v}_v)\big|_{R-l}=0 \qquad \text{(A-12)}$$

Equation A-12 represents the integrated form of the vapor continuity equation for the wall region.

A.3 Momentum and Energy Equations

The momentum and energy equations for the liquid and vapor phase can be integrated in a similar manner. The vapor region is split into two regions – the wall and the turbulent core regions as before. Only a few intermediate steps and the final result are presented for each equation.

A.3.1 Liquid Phase

For the liquid core which is assumed to be fully turbulent, $\varepsilon_{m,l} >> \nu_l$, also the liquid density can be assumed to be a constant and pulled out of the differential sign.

The z-direction liquid momentum equation is given by:

$$\frac{\partial}{\partial z}(\bar{u}_l\bar{u}_l)+\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}_l\bar{v}_l)=-\frac{1}{\rho_l}\frac{\partial\bar{P}}{\partial z}-g+\frac{1}{r}\frac{\partial}{\partial r}\left[r\varepsilon_{m,l}\frac{\partial\bar{u}_l}{\partial r}\right]+\frac{\partial}{\partial z}\left(\varepsilon_{m,l}\frac{\partial\bar{u}_l}{\partial z}\right) \qquad \text{(4-1)}$$

Integrating the above, within specified limits, $r=0$ to $r=R-\delta$ over the radial direction gives

$$\int_{r=0}^{r=R-\delta}\left[\frac{\partial}{\partial z}(\overline{u}_l\overline{u}_l)+\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u}_l\overline{v}_l)\right]2\pi r dr=\int_{r=0}^{r=R-\delta}\left[-\frac{1}{\rho_l}\frac{\partial\overline{P}}{\partial z}-g+\frac{1}{r}\frac{\partial}{\partial r}\left[r\varepsilon_{m,l}\frac{\partial\overline{u}_l}{\partial r}\right]+\frac{\partial}{\partial z}\left(\varepsilon_{m,l}\frac{\partial\overline{u}_l}{\partial z}\right)\right]2\pi r dr$$

Upon integration, and employing Leibnitz formula to simplify terms with $\frac{\partial}{\partial z}$, the z direction momentum equation integrates to

$$\frac{d}{dz}\left[\int_0^{R-\delta}(r\overline{u}_l\overline{u}_l)dr\right]+(r\overline{u}_l\overline{u}_l)\Big|_{R-\delta}\frac{d\delta}{dz}+(r\overline{u}_l\overline{v}_l)\Big|_{R-\delta}=-\frac{1}{\rho_l}\frac{d}{dz}\left[\int_0^{R-\delta}(r\overline{P})dr\right]-\frac{(r\overline{P})}{\rho_l}\Bigg|_{R-\delta}\frac{d\delta}{dz}-\frac{g(R-\delta)^2}{2}$$
$$+\left[r\varepsilon_{m,l}\frac{\partial\overline{u}_l}{\partial r}\right]_{R-\delta}+\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\varepsilon_{m,l}\frac{\partial\overline{u}_l}{\partial z}\right)dr\right]+\left[r\varepsilon_{m,l}\frac{\partial\overline{u}_l}{\partial z}\right]_{R-\delta}\frac{d\delta}{dz} \tag{A-13}$$

The r-direction liquid momentum equation is given by

$$\frac{\partial}{\partial z}(\overline{u}_l\overline{v}_l)+\frac{1}{r}\frac{\partial}{\partial r}(r\overline{v}_l\overline{v}_l)=-\frac{1}{\rho_l}\frac{\partial\overline{P}}{\partial r}+\frac{\partial}{\partial z}\left[\varepsilon_{m,l}\frac{\partial\overline{v}_l}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left(r\varepsilon_{m,l}\frac{\partial\overline{v}_l}{\partial r}\right) \tag{4-2}$$

Integrating the above, within specified limits, $r=0$ to $r=R-\delta$ over the radial direction gives

$$\int_{r=0}^{r=R-\delta}\left[\frac{\partial}{\partial z}(\overline{u}_l\overline{v}_l)+\frac{1}{r}\frac{\partial}{\partial r}(r\overline{v}_l\overline{v}_l)\right]2\pi r dr=\int_{r=0}^{r=R-\delta}\left[-\frac{1}{\rho_l}\frac{\partial\overline{P}}{\partial r}+\frac{\partial}{\partial z}\left[\varepsilon_{m,l}\frac{\partial\overline{v}_l}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left(r\varepsilon_{m,l}\frac{\partial\overline{v}_l}{\partial r}\right)\right]2\pi r dr$$

The final integrated form of the r-direction liquid momentum equation is

$$\frac{d}{dz}\left[\int_0^{R-\delta}(r\overline{u}_l\overline{v}_l)dr\right]+(r\overline{u}_l\overline{v}_l)\Big|_{R-\delta}\frac{d\delta}{dz}+(r\overline{v}_l\overline{v}_l)\Big|_{R-\delta}=-\frac{1}{\rho_l}\left[\int_0^{R-\delta}\left(r\frac{\partial\overline{P}}{\partial r}\right)dr\right]+$$
$$\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\varepsilon_{m,l}\frac{\partial\overline{v}_l}{\partial z}\right)dr\right]+\left[r\varepsilon_{m,l}\frac{\partial\overline{v}_l}{\partial z}\right]_{R-\delta}\frac{d\delta}{dz}+\left[r\varepsilon_{m,l}\frac{\partial\overline{v}_l}{\partial r}\right]_{R-\delta} \tag{A-14}$$

Energy Equation for Liquid Phase

$$\frac{\partial}{\partial z}\left(\overline{u}_l\overline{T}_l\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{v}_l\overline{T}_l\right)=\frac{\partial}{\partial z}\left[\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\frac{\partial\overline{T}_l}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\frac{\partial\overline{T}_l}{\partial r}\right] \tag{4-3}$$

Integrating the above, within specified limits, $r=0$ to $r=R-\delta$ over the radial direction gives

$$\int_{r=0}^{r=R-\delta}\left[\frac{\partial}{\partial z}\left(\overline{u}_l\overline{T}_l\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{v}_l\overline{T}_l\right)\right]2\pi r dr=\int_{r=0}^{r=R-\delta}\left\{\frac{\partial}{\partial z}\left[\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\frac{\partial\overline{T}_l}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\frac{\partial\overline{T}_l}{\partial r}\right]\right\}2\pi r dr$$

Applying Leibnitz theorem, the integrated form of the liquid energy equation is

$$\begin{aligned}&\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\overline{u}_l\overline{T}_l\right)dr\right]+\left(r\overline{u}_l\overline{T}_l\right)\Big|_{R-\delta}\frac{d\delta}{dz}+\left(r\overline{v}_l\overline{T}_l\right)\Big|_{R-\delta}=\\&\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\frac{\partial\overline{T}_l}{\partial z}\right)dr\right]+\left[r\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\frac{\partial\overline{T}_l}{\partial z}\right]_{R-\delta}\frac{d\delta}{dz}+\left[r\frac{\varepsilon_{m,l}}{\mathrm{Pr}_l^t}\frac{\partial\overline{T}_l}{\partial r}\right]_{R-\delta}\end{aligned} \tag{A-15}$$

A.3.2 Vapor Film

The turbulent core of the vapor film is now considered for integration. In this region, the effects of turbulence are dominating as compared to the viscous effects.

A.3.2.1 Turbulent Core

The z-direction vapor momentum equation is given by

$$\frac{\partial}{\partial z}\left(\rho_v\overline{u}_v\overline{u}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v\overline{u}_v\overline{v}_v\right)=-\frac{\partial\overline{P}}{\partial z}-\rho_v g+\frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\rho_v\varepsilon_{m,v}\frac{\partial\overline{u}_v}{\partial r}\right]\right\}+\frac{\partial}{\partial z}\left(\rho_v\varepsilon_{m,v}\frac{\partial\overline{u}_v}{\partial z}\right) \tag{4-4}$$

The turbulent core equations are integrated over the radial direction from $r=R-\delta$ to $r=R-l$ where $l=\left(\frac{5\nu_v}{u_v^*}\right)$, as described earlier in this chapter.

$$\int_{r=R-\delta}^{r=R-l}\left[\frac{\partial}{\partial z}\left(\rho_v\bar{u}_v\bar{u}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v\bar{u}_v\bar{v}_v\right)\right]2\pi r dr =$$
$$\int_{r=R-\delta}^{r=R-l}\left[-\frac{\partial\bar{P}}{\partial z}-\rho_v g+\frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial r}\right]\right\}+\frac{\partial}{\partial z}\left(\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial z}\right)\right]2\pi r dr \tag{A-16}$$

Integrating and using Leibnitz formula, the z direction momentum equation for the turbulent core of the vapor film becomes

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\bar{u}_v\bar{u}_v\right)dr\right]+\left(r\rho_v\bar{u}_v\bar{u}_v\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\bar{u}_v\bar{u}_v\right)\Big|_{R-\delta}\frac{d\delta}{dz}+\left\{\left(r\rho_v\bar{u}_v\bar{v}_v\right)\Big|_{R-l}-\left(r\rho_v\bar{u}_v\bar{v}_v\right)\Big|_{R-\delta}\right\}=$$
$$-\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\bar{P}\right)dr\right]-\left(r\bar{P}\right)\Big|_{R-l}\frac{dl}{dz}+\left(r\bar{P}\right)\Big|_{R-\delta}\frac{d\delta}{dz}+\left\{\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial r}\right)\Big|_{R-l}-\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial r}\right)\Big|_{R-\delta}\right\} \tag{A-17}$$
$$-\frac{g\rho_v}{2}\left[(R-l)^2-(R-\delta)^2\right]+\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial z}\right)dr\right]+\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial z}\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial z}\right)\Big|_{R-\delta}\frac{d\delta}{dz}$$

The r-direction vapor momentum equation is given by

$$\frac{\partial}{\partial z}\left(\rho_v\bar{u}_v\bar{v}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v\bar{v}_v\bar{v}_v\right)=-\frac{\partial\bar{P}}{\partial r}+\frac{\partial}{\partial z}\left[\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial r}\right) \tag{4-5}$$

Integrating this in the radial direction between limits $r=R-\delta$ to $r=R-l$

$$\int_{r=R-\delta}^{r=R-l}\left[\frac{\partial}{\partial z}\left(\rho_v\bar{u}_v\bar{v}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v\bar{v}_v\bar{v}_v\right)\right]2\pi r dr =$$
$$\int_{r=R-\delta}^{r=R-l}\left\{-\frac{\partial\bar{P}}{\partial r}+\frac{\partial}{\partial z}\left[\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial r}\right)\right\}2\pi r dr \tag{A-18}$$

Integrating and using Leibnitz formula, the r-direction momentum equation for the turbulent core of the vapor film becomes

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\rho_v\bar{u}_v\bar{v}_v)dr\right]+(r\rho_v\bar{u}_v\bar{v}_v)\Big|_{R-l}\frac{dl}{dz}-(r\rho_v\bar{u}_v\bar{v}_v)\Big|_{R-\delta}\frac{d\delta}{dz}+\left\{(r\rho_v\bar{v}_v\bar{v}_v)\Big|_{R-l}-(r\rho_v\bar{v}_v\bar{v}_v)\Big|_{R-\delta}\right\}=$$

$$-\left[\int_{R-\delta}^{R-l}\left(r\frac{\partial\bar{P}}{\partial r}\right)dr\right]+\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial z}\right)dr\right]+\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial z}\right)\Bigg|_{R-l}\frac{dl}{dz}-\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial z}\right)\Bigg|_{R-\delta}\frac{d\delta}{dz} \quad \text{(A-19)}$$

$$+\left\{\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial r}\right)\Bigg|_{R-l}-\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial r}\right)\Bigg|_{R-\delta}\right\}$$

The turbulent core vapor energy equation is given by

$$\frac{\partial}{\partial z}(\rho_v\bar{u}_v\bar{T}_v)+\frac{1}{r}\frac{\partial}{\partial r}(r\rho_v\bar{v}_v\bar{T}_v)=\frac{\partial}{\partial z}\left[\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\rho_v\frac{\partial\bar{T}_v}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\rho_v\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\frac{\partial\bar{T}_v}{\partial r}\right] \quad \text{(4-6)}$$

Integrating this in the radial direction between limits $r=R-\delta$ to $r=R-l$

$$\int_{r=R-\delta}^{r=R-l}\left[\frac{\partial}{\partial z}(\rho_v\bar{u}_v\bar{T}_v)+\frac{1}{r}\frac{\partial}{\partial r}(r\rho_v\bar{v}_v\bar{T}_v)\right]2\pi r dr=\int_{r=R-\delta}^{r=R-l}\left\{\frac{\partial}{\partial z}\left[\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\rho_v\frac{\partial\bar{T}_v}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\rho_v\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\frac{\partial\bar{T}_v}{\partial r}\right]\right\}2\pi r dr \quad \text{(A-20)}$$

Integrating and using Leibnitz formula, the energy equation for the turbulent core of the vapor film becomes

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\rho_v\bar{u}_v\bar{T}_v)dr\right]+(r\rho_v\bar{u}_v\bar{T}_v)\Big|_{R-l}\frac{dl}{dz}-(r\rho_v\bar{u}_v\bar{T}_v)\Big|_{R-\delta}\frac{d\delta}{dz}+\left\{(r\rho_v\bar{v}_v\bar{T}_v)\Big|_{R-l}-(r\rho_v\bar{v}_v\bar{T}_v)\Big|_{R-\delta}\right\}=$$

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\frac{\partial\bar{T}_v}{\partial z}\right)dr\right]+\left(r\rho_v\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\frac{\partial\bar{T}_v}{\partial z}\right)\Bigg|_{R-l}\frac{dl}{dz}-\left(r\rho_v\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\frac{\partial\bar{T}_v}{\partial z}\right)\Bigg|_{R-\delta}\frac{d\delta}{dz} \quad \text{(A-21)}$$

$$+\left\{\left(r\rho_v\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\frac{\partial\bar{T}_v}{\partial r}\right)\Bigg|_{R-l}-\left(r\rho_v\frac{\varepsilon_{m,v}}{\mathrm{Pr}_v^t}\frac{\partial\bar{T}_v}{\partial r}\right)\Bigg|_{R-\delta}\right\}$$

A.3.2.2 Wall Region of Vapor Film

The turbulent core equations are integrated over the radial direction from $r = R - l$ where $l = \left(\frac{5\nu_v}{u_v^*}\right)$ to $r = R$. It should be noted that R is a constant, hence $\frac{dR}{dz} = 0$. Also, no slip condition at $r = R$, implies that the velocities at that location are zero. These conditions will be used to simplify the integrated equations.

The z direction vapor momentum equation for the wall region is given by

$$\frac{\partial}{\partial z}\left(\rho_v \overline{u}_v \overline{u}_v\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \overline{u}_v \overline{v}_v\right) = -\frac{\partial \overline{P}}{\partial z} - \rho_v g + \frac{1}{r}\left\{\frac{\partial}{\partial r}\left[r\rho_v \nu_v \frac{\partial \overline{u}_v}{\partial r}\right]\right\} \tag{4-7}$$

Integrating over the radial direction

$$\int_{r=R-l}^{r=R}\left[\frac{\partial}{\partial z}\left(\rho_v \overline{u}_v \overline{u}_v\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \overline{u}_v \overline{v}_v\right)\right]2\pi r dr = \int_{r=R-l}^{r=R}\left[-\frac{\partial \overline{P}}{\partial z} - \rho_v g + \frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r\rho_v \nu_v \frac{\partial \overline{u}_v}{\partial r}\right)\right\}\right]2\pi r dr \tag{A-22}$$

Using Leibnitz formula and the boundary conditions specified above, the equation upon integration becomes

$$\begin{aligned} &\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v \overline{u}_v \overline{u}_v\right)dr\right] - \left(r\rho_v \overline{u}_v \overline{u}_v\right)\Big|_{R-l}\frac{dl}{dz} - \left(r\rho_v \overline{u}_v \overline{v}_v\right)\Big|_{R-l} = \\ &-\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\overline{P}\right)dr\right] + \left(r\overline{P}\right)\Big|_{R-l}\frac{dl}{dz} - \frac{g\rho_v}{2}\left[2Rl - l^2\right] + \left\{\left(r\rho_v \nu_v \frac{\partial \overline{u}_v}{\partial r}\right)\Bigg|_{R} - \left(r\rho_v \nu_v \frac{\partial \overline{u}_v}{\partial r}\right)\Bigg|_{R-l}\right\} \end{aligned} \tag{A-23}$$

The r-direction vapor momentum equation for the wall region is given by

$$\frac{\partial}{\partial z}\left(\rho_v \overline{u}_v \overline{v}_v\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v \overline{v}_v \overline{v}_v\right) = -\frac{\partial \overline{P}}{\partial r} + \frac{\partial}{\partial z}\left[\rho_v \nu_v \frac{\partial \overline{v}_v}{\partial z}\right] \tag{4-8}$$

Integrating over the radial direction and using Leibnitz formula and the boundary conditions specified above, the equation becomes

$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\bar{u}_v\bar{v}_v\right)dr\right]-\left(r\rho_v\bar{u}_v\bar{v}_v\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\bar{v}_v\bar{v}_v\right)\Big|_{R-l}=$$
$$-\int_{R-l}^{R}\left(r\frac{\partial\bar{P}}{\partial r}\right)dr+\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\nu_v\frac{\partial\bar{v}_v}{\partial z}\right)dr\right]-\left(r\rho_v\nu_v\frac{\partial\bar{v}_v}{\partial z}\right)\bigg|_{R-l}\frac{dl}{dz} \tag{A-24}$$

The energy equation for the wall region of vapor film is given below

$$\frac{\partial}{\partial z}\left(\rho_v\bar{u}_v\bar{T}_v\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v\bar{v}_v\bar{T}_v\right)=\frac{\partial}{\partial z}\left[\frac{\nu_v}{\Pr_v}\rho_v\frac{\partial\bar{T}_v}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[r\rho_v\frac{\nu_v}{\Pr_v}\frac{\partial\bar{T}_v}{\partial r}\right] \tag{4-9}$$

Integrating over the radial direction and using Leibnitz formula and the boundary conditions specified above, the equation becomes

$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\bar{u}_v\bar{T}_v\right)dr\right]-\left(r\rho_v\bar{u}_v\bar{T}_v\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\bar{v}_v\bar{T}_v\right)\Big|_{R-l}=$$
$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\frac{\nu_v}{\Pr_v}\frac{\partial\bar{T}_v}{\partial z}\right)dr\right]-\left(r\rho_v\frac{\nu_v}{\Pr_v}\frac{\partial\bar{T}_v}{\partial z}\right)\bigg|_{R-l}\frac{dl}{dz}+\left\{\left(r\rho_v\frac{\nu_v}{\Pr_v}\frac{\partial\bar{T}_v}{\partial r}\right)\bigg|_{R}-\left(r\rho_v\frac{\nu_v}{\Pr_v}\frac{\partial\bar{T}_v}{\partial r}\right)\bigg|_{R-l}\right\} \tag{A-25}$$

Thus, a system of twelve ordinary differential equations has been obtained by integrating the original system of partial differential governing equations over the radial direction. The vapor film was split into two regions – the turbulent core, where the effects of turbulence dominate; and the wall region, where the viscous effects dominate.

Appendix B

Derivation of Vapor Phase Axial Velocity Profile

The following definitions typical for turbulent flow are used through this analysis.

$$u_v^* = \sqrt{\frac{\tau_w}{\rho_v}}, \ u_v^+ = \frac{\bar{u}_v}{u_v^*}, \ y^+ = \frac{y u_v^*}{\nu_v}, \ y = R - r \tag{B-1}$$

A non-dimensional variable η, is defined such that

$$\eta = \left(y^+\right)^{1/7} \tag{B-2}$$

Therefore $d\eta = \dfrac{dy^+}{7\eta^6}$ (B-3)

The vapor film being turbulent, the axial component of the velocity is assumed to be of the form given by Equations B-4 and B-5.

$$u_v^+ = y^+ \text{ (Wall Region)} \left(0 \le y^+ \le 5\right) \tag{B-4}$$

$$u_v^+ = a_0 + a_1\eta + a_2\eta^2 \text{ (Turbulent Core)} \left(5 \le y^+ \le \delta^+\right) \tag{B-5}$$

In order to evaluate the unknown coefficients a_0, a_1 and a_2, three conditions are needed on the velocity or its derivative. Differentiating Equation B-5, with respect to η, and invoking Equation B-3 yields:

$$\frac{du_v^+}{d\eta} = a_1 + 2a_2\eta$$

$$\frac{du_v^+}{dy^+} = \frac{1}{7\eta^6}\frac{du_v^+}{d\eta} = \frac{1}{7\eta^6}\left(a_1 + 2a_2\eta\right) \tag{B-6}$$

Condition 1:

The velocity is continuous at $y^+ = 5$. Using this condition in Equations B-4 and B-5 yields:

$$u_v^+ = 5 = a_0 + a_1(5)^{1/7} + a_2(5)^{2/7}$$

$$a_0 + a_1(5)^{1/7} + a_2(5)^{2/7} = 5 \tag{B-7}$$

Condition 2:

The derivative of the velocity is continuous at $y^+ = 5$. Using this condition in Equations B-4 and B-6 yields

From Equation B-4, $\left.\dfrac{du_v^+}{dy^+}\right|_{y^+=5} = 1$

From Equation B-6, $\left.\dfrac{du_v^+}{dy^+}\right|_{y^+=5} = \dfrac{1}{7\eta^6}\left.\dfrac{du_v^+}{d\eta}\right|_{\eta=5^{1/7}} = \left.\dfrac{(a_1 + 2a_2\eta)}{7\eta^6}\right|_{\eta=5^{1/7}}$

Combining these, Equation B-8 is obtained as below

$$a_1 + 2a_2(5)^{1/7} = 7(5)^{6/7} \tag{B-8}$$

Condition 3:

The third condition needed is obtained by evaluating the shear stress at the liquid-vapor interface.

Shear stress at $y^+ = \delta^+$ is τ_i, which is also an unknown.

$$\mu_v \left.\frac{\partial u_v}{\partial y}\right|_{y=\delta} = \tau_i$$

Using Equations B-2, B-3 and B-4 to non-dimensionalize, the above equation can be written as

$$\left.\frac{du_v^+}{dy^+}\right|_{y^+=\delta^+} = \frac{1}{7\eta^6}\left.\frac{du_v^+}{d\eta}\right|_{\eta=(\delta^+)^{1/7}} = \frac{\tau_i}{\tau_w} \tag{B-9}$$

Using Equation B-6, the above equation can be re-written as

$$a_1 + 2a_2\left(\delta^+\right)^{1/7} = 7\frac{\tau_i}{\tau_w}\left(\delta^+\right)^{6/7} \tag{B-10}$$

Equations B-7, B-8 and B-10 are the three equations for the vapor film that can be used to determine the unknown coefficients a_0, a_1 and a_2. Eliminating a_1 from equations B-8 and B-10 and re-arranging gives

- $$a_2 = \frac{7\left[\frac{\tau_i}{\tau_w}\left(\delta^+\right)^{6/7} - (5)^{6/7}\right]}{2\left[\left(\delta^+\right)^{1/7} - (5)^{1/7}\right]} \tag{B-11}$$

Using Equation B-8, a_1 can be obtained as

- $$a_1 = \frac{7\left[(5)^{6/7}\left(\delta^+\right)^{1/7} - (5)^{1/7}\left(\delta^+\right)^{6/7}\frac{\tau_i}{\tau_w}\right]}{\left[\left(\delta^+\right)^{1/7} - (5)^{1/7}\right]} \tag{B-12}$$

Substituting the values of a_1 and a_2 in Equation B-7, a_0 can be obtained as

- $$a_0 = \frac{\left[-60\left(\delta^+\right)^{1/7} + (5)^{15/7} + 7(5)^{2/7}\left(\delta^+\right)^{6/7}\frac{\tau_i}{\tau_w}\right]}{2\left[\left(\delta^+\right)^{1/7} - (5)^{1/7}\right]} \tag{B-13}$$

Equations B-11, B-12 and B-13 give the coefficients a_0, a_1 and a_2 respectively. These are not constants, but functions of the axial coordinate, z alone.

B.1 Gradients of the Vapor Velocity

Derivatives of the axial component of the vapor velocity, $\overline{u}_v$ with respect to z and r occur in the integrated form of the governing conservation equations.

B.1.1 Gradient with respect to the radial coordinate

The derivative of $\bar{u}_v$ with respect to r is given as

$$\frac{\partial \bar{u}_v}{\partial r} = \frac{\partial}{\partial r}\left[u_v^+ u_v^*\right] = u_v^+ \frac{\partial u_v^*}{\partial r} + u_v^* \frac{\partial u_v^+}{\partial r} \tag{B-14}$$

The friction velocity u_v^* is only a function of z, and independent of r, hence $\frac{\partial u_v^*}{\partial r} = 0$.

$$\frac{\partial \bar{u}_v}{\partial r} = \frac{\partial}{\partial r}\left[u_v^+ u_v^*\right] = u_v^* \frac{\partial u_v^+}{\partial r} \tag{B-15}$$

The turbulent core of the vapor film

Substituting for the dimensionless vapor velocity profile for the turbulent core (Equation B-5), Equation B-15 can be re-written as

$$\frac{\partial \bar{u}_v}{\partial r} = u_v^* \frac{\partial u_v^+}{\partial r} = u_v^* \frac{\partial}{\partial r}\left(a_0 + a_1\eta + a_2\eta^2\right) = u_v^*\left[\frac{\partial a_0}{\partial r} + \eta\frac{\partial a_1}{\partial r} + a_1\frac{\partial \eta}{\partial r} + 2a_2\eta\frac{\partial \eta}{\partial r} + \eta^2\frac{\partial a_2}{\partial r}\right] \tag{B-16}$$

The coefficients are not functions of r, hence their derivatives with respect to r is equal to zero. The other terms involve the derivative of η with respect to r. Equation B-16 simplifies to

$$\frac{\partial \bar{u}_v}{\partial r} = u_v^*\left[a_1 + 2a_2\eta\right]\frac{\partial \eta}{\partial r} \tag{B-17}$$

Using Equation B-2 and B-1, the non-dimensional variable η is defined as

$$\eta = \left(y^+\right)^{1/7} = \left[\frac{y}{\nu_v}\sqrt{\frac{\tau_w}{\rho_v}}\right]^{1/7} \tag{B-18}$$

In the above equation, all quantities, except the independent variable y are functions of z alone. Hence

- $$\frac{\partial \eta}{\partial y} = -\frac{\partial \eta}{\partial r} = \frac{\left(\tau_w / \rho_v \right)^{1/14}}{7 \nu_v^{1/7} y^{6/7}} = \frac{\eta}{7y} \qquad \text{(B-19)}$$

and

- $$\frac{\partial \eta}{\partial z} = \frac{\left(y / \nu_v \right)^{1/7}}{14} \left(\frac{1}{\tau_w^{13/14} \rho_v^{1/14}} \right) \frac{d\tau_w}{dz} = \frac{1}{14} \frac{\eta}{\tau_w} \frac{d\tau_w}{dz} \qquad \text{(B-20)}$$

In deriving Equations B-19 and B-20, the properties of vapor are treated as constant, although, strictly speaking, they are functions of temperature, which is a function of r and z.
Using Equation B-19 in Equation B-17 yields

- $$\frac{\partial \overline{u}_v}{\partial r} = -\frac{u_v^* \left[a_1 + 2a_2 \eta \right] \eta}{7y} \qquad \text{(B-21)}$$

The wall region of the vapor film

Using the velocity profile for the wall region from Equation B-4, and the definitions from Equation B-1

$$\frac{\partial u_v^+}{\partial y^+} = \frac{\nu_v}{u_v^*} \frac{\partial u_v^+}{\partial y} = -\frac{\nu_v}{u_v^*} \frac{\partial u_v^+}{\partial r} = 1 \qquad \text{(B-22)}$$

Using Equations B-15 and B-22

- $$\frac{\partial \overline{u}_v}{\partial r} = \frac{\partial}{\partial r} \left[u_v^+ u_v^* \right] = u_v^* \frac{\partial u_v^+}{\partial r} = -u_v^* \left(\frac{u_v^*}{\nu_v} \right) = -\frac{\left(u_v^* \right)^2}{\nu_v} \qquad \text{(B-23)}$$

B.1.2 Gradient with respect to the axial coordinate

The derivative of $\overline{u}_v$ with respect to z is given as

$$\frac{\partial \overline{u}_v}{\partial z} = \frac{\partial}{\partial z}\left[u_v^+ u_v^*\right] = u_v^+ \frac{\partial u_v^*}{\partial z} + u_v^* \frac{\partial u_v^+}{\partial z} \tag{B-24}$$

The turbulent core of the vapor film

Substituting for the dimensionless vapor velocity profile for the turbulent core, and using the fact that the friction velocity u_v^*, and the coefficients a_0, a_1 and a_2 are functions of z alone, Equation B-24 becomes

$$\frac{\partial \overline{u}_v}{\partial z} = \frac{\partial}{\partial z}\left[u_v^+ u_v^*\right] = \left(a_0 + a_1\eta + a_2\eta^2\right)\frac{du_v^*}{dz} + u_v^* \frac{\partial}{\partial z}\left(a_0 + a_1\eta + a_2\eta^2\right) \tag{B-25}$$

$$\frac{\partial \overline{u}_v}{\partial z} = \left(a_0 + a_1\eta + a_2\eta^2\right)\frac{du_v^*}{dz} + u_v^*\left[\frac{da_0}{dz} + \eta\frac{da_1}{dz} + a_1\frac{\partial \eta}{\partial z} + 2a_2\eta\frac{\partial \eta}{\partial z} + \eta^2\frac{da_2}{dz}\right]$$

$$\frac{\partial \overline{u}_v}{\partial z} = \left(a_0 + a_1\eta + a_2\eta^2\right)\frac{du_v^*}{dz} + u_v^*\left[\frac{da_0}{dz} + \eta\frac{da_1}{dz} + \eta^2\frac{da_2}{dz} + \left(a_1 + 2a_2\eta\right)\frac{\partial \eta}{\partial z}\right] \tag{B-26}$$

The friction velocity u_v^* is a function of z. Its derivative with respect to z is given as

$$\frac{du_v^*}{dz} = \frac{1}{2\sqrt{\tau_w \rho_v}}\frac{d\tau_w}{dz} = \frac{u_v^*}{2\tau_w}\frac{d\tau_w}{dz} \tag{B-27}$$

Equation B-26 requires the derivatives of the coefficients a_0, a_1 and a_2 with respect to z. Taking derivative of Equation B-11 with respect to z gives

$$\frac{da_2}{dz} = \left(da2tw\right)\frac{d\tau_w}{dz} + \left(da2ti\right)\frac{d\tau_i}{dz} + \left(da2d\right)\frac{d\delta}{dz} \tag{B-28}$$

where

$$da2tw = \frac{1}{\left[\left(\delta^+\right)^{1/7} - (5)^{1/7}\right]^2}\left[\frac{2\tau_i (5)^{1/7}\left(\delta^+\right)^{6/7}}{\tau_w^2} + \frac{(5)^{6/7}\left(\delta^+\right)^{1/7}}{4\tau_w} - \frac{9\tau_i \delta}{4\tau_w^2}\right] \quad \text{(B-28a)}$$

$$da2d = \frac{1}{\left[\left(\delta^+\right)^{1/7} - (5)^{1/7}\right]^2}\left[\frac{5\tau_i \delta^+}{2\delta\tau_w} - \frac{3(5)^{1/7}\left(\delta^+\right)^{6/7}\tau_i}{\tau_w \delta} + \frac{\tau_i\left(\delta^+\right)^{1/7}(5)^{6/7}}{2\tau_w \delta}\right] \quad \text{(B-28b)}$$

$$da2ti = \frac{7\left(\delta^+\right)^{6/7}}{2\tau_w\left[\left(\delta^+\right)^{1/7} - (5)^{1/7}\right]} \quad \text{(B-28c)}$$

Substituting Equations B-28a, B-28b and B-28c in Equation B-28 yields an expression for $\frac{da_2}{dz}$. In deriving Equation B-28, the derivative of the non-dimensional film thickness δ^+ is required. Using the definition of δ^+ from Equation B-1 and differentiating with respect to z and using Equation B-27 gives

$$\frac{d\delta^+}{dz} = \left(\frac{\delta^+}{2\tau_w}\right)\frac{d\tau_w}{dz} + \left(\frac{u_v^*}{\nu_v}\right)\frac{d\delta}{dz} \quad \text{(B-29)}$$

Differentiating Equation B-8 gives

$$\frac{da_1}{dz} + 2(5)^{1/7}\left(\frac{da_2}{dz}\right) = 0$$

Re-arranging

$$\frac{da_1}{dz} = -2(5)^{1/7}\left(\frac{da_2}{dz}\right) \quad \text{(B-30)}$$

Combining Equations B-28 and B-30 gives $\frac{da_1}{dz}$

Differentiating Equation B-7 gives

$$\frac{da_0}{dz} = -(5)^{1/7}\frac{da_1}{dz} - (5)^{2/7}\frac{da_2}{dz} \quad \text{(B-31)}$$

Using Equation B-30 into Equation B-31 gives

$$\frac{da_0}{dz} = (5)^{2/7}\frac{da_2}{dz} \quad \text{(B-32)}$$

Knowing $\frac{da_2}{dz}$ from Equation B-28 $\frac{da_0}{dz}$ can be obtained.

Substituting Equations B-19, B-27, B-28, B-30 and B-32 in Equation B-26 yields

$$\frac{\partial \overline{u}_v}{\partial z} = \frac{\left(a_0 + a_1\eta + a_2\eta^2\right)}{2\sqrt{\tau_w \rho_v}}\frac{d\tau_w}{dz} + u_v^*\left\{\frac{\left(a_1 + 2a_2\eta\right)}{14}\left(\frac{\eta}{\tau_w}\right)\frac{d\tau_w}{dz} + \left[\eta - (5)^{1/7}\right]^2\frac{da_2}{dz}\right\}$$

Using the definition for friction velocity and re-arranging gives

- $$\frac{\partial \overline{u}_v}{\partial z} = \frac{\left(a_0 + a_1\eta + a_2\eta^2\right)}{2\sqrt{\tau_w \rho_v}}\frac{d\tau_w}{dz} + \frac{\left(a_1\eta + 2a_2\eta^2\right)}{14\sqrt{\tau_w \rho_v}}\frac{d\tau_w}{dz} + u_v^*\left[\eta - (5)^{1/7}\right]^2\frac{da_2}{dz} \quad \text{(B-33)}$$

This is the gradient of the turbulent core vapor velocity in the axial direction.

The wall region of the vapor film

Using the velocity profile for the wall region from Equation B-4, we get:

$$u_v^+ = y^+ = \eta^7$$

Therefore, $\frac{\partial u_v^+}{\partial z} = 7\eta^6\frac{\partial \eta}{\partial z}$ (B-34)

Using Equation B-20, the above equation becomes

$$\frac{\partial u_v^+}{\partial z} = \frac{\eta^7}{2\tau_w}\frac{d\tau_w}{dz} \tag{B-35}$$

Substituting in Equation B-24 for $\frac{\partial u_v^+}{\partial z}$, u_v^+ and using the fact that the friction velocity u_v^*, is a function of z alone, gives

$$\frac{\partial \overline{u}_v}{\partial z} = \frac{\partial}{\partial z}\left[u_v^+ u_v^*\right] = \eta^7 \frac{\partial u_v^*}{\partial z} + u_v^*\left(\frac{\eta^7}{2\tau_w}\frac{d\tau_w}{dz}\right) \tag{B-36}$$

Using Equation B-27 in the above

- $$\frac{\partial \overline{u}_v}{\partial z} = \left(\frac{1}{\sqrt{\tau_w \rho_v}} + \frac{u_v^*}{\tau_w}\right)\frac{\eta^7}{2}\frac{d\tau_w}{dz} \tag{B-37}$$

Equations B-21 and B-23 give the gradient of the axial component of the vapor velocity with respect to the radial coordinate, r, while Equations B-33 and B-37 give the derivatives with respect to the axial coordinate, z.

Appendix C

Derivation of Liquid Phase Axial Velocity Profile

The liquid core being turbulent, the axial component of the velocity is assumed to be of the form given by equation C-1.

$$u_l^+ = \frac{\overline{u}_l}{u_v^*} = b_0 + b_1\eta + b_2\eta^2 \qquad \text{(C-1)}$$

The non-dimensional variable η is defined in the same manner

$$\eta = \left(y^+\right)^{1/7}$$

In order to evaluate the unknown coefficients b_0, b_1 and b_2, three conditions are needed on the liquid velocity or its derivative.

Condition 1:

The velocity gradient is zero at the center of the hydraulic channel. Mathematically, this can be written as given below

$$\text{At } r = 0 \text{ or } y^+ = R^+; \left.\frac{\partial \overline{u}_l}{\partial r}\right|_{r=0} = 0 \qquad \text{(C-2)}$$

Differentiating Equation C-1, with respect to η, and invoking Equation B-3 yields:

$$\left.\frac{du_l^+}{d\eta}\right|_{\left(R^+\right)^{1/7}} = 0$$

$$\left(b_1 + 2b_2\eta\right)\Big|_{\eta=\left(R^+\right)^{1/7}} = 0$$

$$b_1 + 2b_2\left(R^+\right)^{1/7} = 0 \qquad \text{(C-3)}$$

Condition 2:

The velocity is continuous at the liquid-vapor interface. Mathematically, it can be written as

At $y^+ = \delta^+$ $\qquad u_l^+\Big|_{y^+=\delta^+} = u_v^+\Big|_{y^+=\delta^+}$

Using the assumed velocity profiles, the above condition becomes

$$a_0 + a_1\left(\delta^+\right)^{1/7} + a_2\left(\delta^+\right)^{2/7} = b_0 + b_1\left(\delta^+\right)^{1/7} + b_2\left(\delta^+\right)^{2/7} \qquad \text{(C-4)}$$

Condition 3:

The third condition needed is obtained by evaluating the shear stress at the liquid-vapor interface. Shear stress at $y^+ = \delta^+$ is τ_i, which is also an unknown. Mathematically, it can be written as:

$$\mu_l \frac{\partial u_l}{\partial y}\bigg|_{y=\delta} = \tau_i$$

$$\mu_l \frac{\left(u_v^*\right)^2}{\nu_v}\frac{\partial u_l^+}{\partial y^+}\bigg|_{y^+=\delta^+} = \tau_i$$

$$\left(\frac{\mu_l}{\mu_v}\right)\mu_v \frac{\left(u_v^*\right)^2}{\nu_v}\frac{\partial u_l^+}{\partial y^+}\bigg|_{y^+=\delta^+} = \tau_i$$

The wall shear stress is given by $\tau_w = \rho_v\left(u_v^*\right)^2$, using Equation B-1. Using this, the above equation becomes

$$\tau_w \frac{\partial u_l^+}{\partial y^+}\bigg|_{y^+=\delta^+} = \frac{\tau_i \mu_v}{\mu_l}$$

From Equation B-3, the above condition can be re-written as

$$\left.\frac{\partial u_l^+}{\partial y^+}\right|_{y^+=\delta^+} = \frac{1}{7\eta^6}\left.\frac{du_l^+}{d\eta}\right|_{\eta=\left(\delta^+\right)^{1/7}} = \frac{\tau_i\mu_v}{\tau_w\mu_l}$$

Differentiating Equation C-1 and substituting in the above equation gives

$$\left.\frac{1}{7\eta^6}\left(b_1 + 2b_2\eta\right)\right|_{\left(\delta^+\right)^{1/7}} = \frac{\tau_i\mu_v}{\tau_w\mu_l}$$

$$\frac{b_1 + 2b_2\left(\delta^+\right)^{1/7}}{7\left(\delta^+\right)^{6/7}} = \frac{\tau_i\mu_v}{\tau_w\mu_l}$$

$$b_1 + 2b_2\left(\delta^+\right)^{1/7} = 7\left(\delta^+\right)^{6/7}\left(\frac{\tau_i\mu_v}{\tau_w\mu_l}\right) \qquad \text{(C-5)}$$

Equations C-3, C-4 and C-5 represent the three conditions that need to be satisfied. These can be solved to obtain the unknown coefficients.

Subtracting Equation C-3 from C-5 yields

$$2b_2\left(R^{+^{1/7}} - \delta^{+^{1/7}}\right) = -7\frac{\tau_i\mu_v}{\tau_w\mu_l}\left(\delta^{+^{1/7}}\right)$$

- $$b_2 = \frac{-7\left(\frac{\tau_i\mu_v}{\tau_w\mu_l}\right)\left(\delta^+\right)^{1/7}}{2\left(R^{+^{1/7}} - \delta^{+^{1/7}}\right)} \qquad \text{(C-6)}$$

From Equation C-3, $b_1 = -2b_2\left(R^+\right)^{1/7}$

$$b_1 = -2\left\{\frac{-7\left(\frac{\tau_i}{\tau_w}\frac{\mu_v}{\mu_l}\right)\delta^{+^{1/7}}}{2\left(R^{+^{1/7}} - \delta^{+^{1/7}}\right)}\right\}R^{+^{1/7}}$$

- $$b_1 = \frac{7\left(\frac{\tau_i}{\tau_w}\frac{\mu_v}{\mu_l}\right)\delta^{+^{1/7}} R^{+^{1/7}}}{\left(R^{+^{1/7}} - \delta^{+^{1/7}}\right)} \quad \text{(C-7)}$$

Substituting b_1 and b_2 in Equation C-4 along with values of $a_{0,}a_{1,}a_2$ gives b_0 in terms of $R^+, \delta^+, \tau_i, \tau_w, \mu_v, \mu_l$.

- $$b_0 = a_0 + (a_1 - b_1)(\delta^+)^{1/7} + (a_2 - b_2)(\delta^+)^{2/7} \quad \text{(C-8)}$$

Equations C-6, C-7 and C-8 give the coefficients b_0, b_1 and b_2 respectively. These are not constants, but functions of the axial coordinate, z alone.

C.1 Gradients of the Liquid Axial Velocity

Derivatives of the axial component of the liquid velocity, $\overline{u}_l$ with respect to z and r occur in the integrated form of the governing conservation equations.

$$\frac{\partial \overline{u}_l}{\partial r} = \frac{\partial}{\partial r}\left[u_l^+ u_v^*\right] = u_l^+ \frac{\partial u_v^*}{\partial r} + u_v^* \frac{\partial u_l^+}{\partial r} \quad \text{(C-9)}$$

The friction velocity u_v^* is only a function of z, and independent of r, hence $\frac{\partial u_v^*}{\partial r} = 0$. Using this, and substituting for the dimensionless vapor velocity profile, Equation C-9 can be re-written as

$$\frac{\partial \overline{u}_l}{\partial r} = u_v^* \frac{\partial u_l^+}{\partial r} = u_v^* \frac{\partial}{\partial r}\left(b_0 + b_1\eta + b_2\eta^2\right) = u_v^*\left[\frac{\partial b_0}{\partial r} + \eta\frac{\partial b_1}{\partial r} + b_1\frac{\partial \eta}{\partial r} + 2b_2\eta\frac{\partial \eta}{\partial r} + \eta^2\frac{\partial b_2}{\partial r}\right] \quad \text{(C-10)}$$

The coefficients are not functions of r, hence their derivatives with respect to r is equal to zero. The other terms involve the derivative of η with respect to r. Equation C-10 simplifies to

$$\frac{\partial \overline{u}_l}{\partial r} = u_v^*\left[b_1 + 2b_2\eta\right]\frac{\partial \eta}{\partial r} \quad \text{(C-11)}$$

Using Equation B-19 in Equation C-11 yields

- $$\frac{\partial \overline{u}_l}{\partial r} = -\frac{u_v^*[b_1 + 2b_2\eta]\eta}{7y} \qquad \text{(C-12)}$$

The derivative of $\overline{u}_l$ with respect to z is given as

$$\frac{\partial \overline{u}_l}{\partial z} = \frac{\partial}{\partial z}\left[u_l^+ u_v^*\right] = u_l^+ \frac{\partial u_v^*}{\partial z} + u_v^* \frac{\partial u_l^+}{\partial z} \qquad \text{(C-13)}$$

Substituting for the dimensionless vapor velocity profile, and using the fact that the friction velocity u_v^*, and the coefficients b_0, b_1 and b_2 are functions of z alone, Equation C-13 becomes

$$\frac{\partial \overline{u}_l}{\partial z} = \frac{\partial}{\partial z}\left[u_l^+ u_v^*\right] = \left(b_0 + b_1\eta + b_2\eta^2\right)\frac{du_v^*}{dz} + u_v^* \frac{\partial}{\partial z}\left(b_0 + b_1\eta + b_2\eta^2\right) \qquad \text{(C-14)}$$

$$\frac{\partial \overline{u}_l}{\partial z} = \left(b_0 + b_1\eta + b_2\eta^2\right)\frac{du_v^*}{dz} + u_v^*\left[\frac{db_0}{dz} + \eta\frac{db_1}{dz} + b_1\frac{\partial \eta}{\partial z} + 2\eta b_2\frac{\partial \eta}{\partial z} + \eta^2\frac{db_2}{dz}\right]$$

- $$\frac{\partial \overline{u}_l}{\partial z} = \left(b_0 + b_1\eta + b_2\eta^2\right)\frac{du_v^*}{dz} + u_v^*\left[\frac{db_0}{dz} + \eta\frac{db_1}{dz} + \eta^2\frac{db_2}{dz} + \left(b_1 + 2b_2\eta\right)\frac{\partial \eta}{\partial z}\right] \qquad \text{(C-15)}$$

Equation C-15 requires the derivatives of the coefficients b_0, b_1 and b_2 with respect to z. Taking derivative of Equation C-6 requires the derivative of the non-dimensional radius R^+ with respect to z. Using the definition $R^+ = \frac{Ru_v^*}{\nu_v}$ gives

$$\frac{dR^+}{dz} = \frac{R^+}{2\tau_w}\frac{d\tau_w}{dz} \qquad \text{(C-16)}$$

Using Equations B-29 and C-16 and the definition of b_2 yields

$$\frac{db_2}{dz} = \frac{b_2}{\tau_i}\left(\frac{d\tau_i}{dz}\right) - \frac{b_2}{\tau_w}\left(\frac{d\tau_w}{dz}\right) + \frac{b_2}{7\delta}\left[\frac{\left(R^+\right)^{1/7}}{\left(R^+\right)^{1/7} - \left(\delta^+\right)^{1/7}}\right]\left(\frac{d\delta}{dz}\right) \tag{C-17}$$

Using Equation C-3, $\frac{db_1}{dz}$ can be obtained as

$$\frac{db_1}{dz} = -2(R^+)^{1/7}\left(\frac{db_2}{dz}\right) - \frac{2}{7}(R^+)^{-6/7} b_2\left(\frac{dR^+}{dz}\right)$$

which can be simplified using Equations C-16 and C-17 to give

$$\frac{db_1}{dz} = \frac{b_1}{\tau_i}\left(\frac{d\tau_i}{dz}\right) - \frac{13}{14}\frac{b_1}{\tau_w}\left(\frac{d\tau_w}{dz}\right) + \frac{b_1}{7\delta}\left[\frac{\left(R^+\right)^{1/7}}{\left(R^+\right)^{1/7} - \left(\delta^+\right)^{1/7}}\right]\left(\frac{d\delta}{dz}\right) \tag{C-18}$$

Differentiating Equation C-8 yields

$$\frac{db_0}{dz} = \frac{da_0}{dz} + \frac{d}{dz}\left[\left(a_1 - b_1\right)\left(\delta^+\right)^{1/7}\right] + \frac{d}{dz}\left[\left(a_2 - b_2\right)\left(\delta^+\right)^{2/7}\right]$$

Simplifying, we get

$$\begin{aligned}\frac{db_0}{dz} &= \frac{da_0}{dz} + \left(\delta^+\right)^{1/7}\left(\frac{da_1}{dz} - \frac{db_1}{dz}\right) + \frac{\left(a_1 - b_1\right)}{7}\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz} \\ &+ \left(\delta^+\right)^{2/7}\left(\frac{da_2}{dz} - \frac{db_2}{dz}\right) + \frac{2\left(a_2 - b_2\right)}{7}\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}\end{aligned} \tag{C-19}$$

Using the derivatives of $a_{0,} a_1, a_2, b_1$ and b_2 with respect to z, which are already known, and substituting from Equation B-29, $\frac{db_0}{dz}$ can be evaluated from Equation C-19 as

$$\frac{db_0}{dz} = \left(db0tw\right)\frac{d\tau_w}{dz} + \left(db0ti\right)\frac{d\tau_i}{dz} + \left(db0d\right)\frac{d\delta}{dz} \tag{C-20}$$

where

$$db0tw = \frac{(a_1 + 12b_1)(\delta^+)^{1/7} + (2a_2 + 12b_2)(\delta^+)^{2/7}}{14\tau_w} + (5 - \delta^+)^2 (da2tw) \tag{C-20a}$$

$$db0ti = \left\{ (5 - \delta^+)^2 (da2ti) - \frac{b_1 (\delta^+)^{1/7}}{\tau_i} - \frac{b_2 (\delta^+)^{2/7}}{\tau_i} \right\} \tag{C-20b}$$

$$dbod = \left\{ \begin{array}{l} (5 - \delta^+)^2 (da2d) + \left[\dfrac{(a_1 - b_1)(\delta^+)^{1/7} + 2(a_2 - b_2)(\delta^+)^{2/7}}{7\delta} \right] \\ - \dfrac{(R^+)^{1/7} \left[b_1 (\delta^+)^{1/7} + b_2 (\delta^+)^{2/7} \right]}{7\delta \left[(R^+)^{1/7} - (\delta^+)^{1/7} \right]} \end{array} \right\} \tag{C-20c}$$

Using Equations B-20 for $\frac{\partial \eta}{\partial z}$, B-27 for $\frac{du_v^*}{dz}$, C-17, C-18 and C-20 in Equation C-15 yields an expression for $\frac{\partial \bar{u}_l}{\partial z}$.

Appendix D

Derivation of Vapor Phase Radial Velocity Profile

The continuity equation for the vapor phase is given by

$$\frac{\partial(\rho_v \bar{u}_v)}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho_v \bar{v}_v) = 0 \qquad \text{(D-1)}$$

The radial component of the vapor velocity, just like the axial component, is non-dimensionalized by the friction velocity. Thus,

$$v_v^+ = \frac{\bar{v}_v}{u_v^*} \qquad \text{(D-2)}$$

where the friction velocity and the other terms are defined in Equation B-1.

$$u_v^* = \sqrt{\frac{\tau_w}{\rho_v}}\,,\; u_v^+ = \frac{\bar{u}_v}{u_v^*}\,,\; y^+ = \frac{y u_v^*}{\nu_v}\,,\; y = R - r \qquad \text{(B-1)}$$

The vapor film being turbulent, the axial component of the velocity is assumed to be of the form given by equations B-4 and B-5.

$$u_v^+ = y^+ \quad \left(0 \le y^+ \le 5\right) \qquad \text{(B-4)}$$

$$u_v^+ = a_0 + a_1\eta + a_2\eta^2 \quad \left(5 \le y^+ \le \delta^+\right) \qquad \text{(B-5)}$$

Using the definitions in Equations B-1 and D-2, the vapor continuity equation becomes

$$\frac{\partial\left(\rho_v u_v^+ u_v^*\right)}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho_v v_v^+ u_v^*\right) = 0 \qquad \text{(D-3)}$$

In the above, the vapor density, though not a constant, can be factored out for simplicity. The friction velocity is a function of z alone, hence it can be treated as constant, when differentiating with respect to r. Employing the product rule of differentiation, Equation D-3 becomes

$$u_v^+ \frac{du_v^*}{dz} + u_v^* \frac{\partial u_v^+}{\partial z} + \frac{u_v^*}{r}\frac{\partial}{\partial r}\left(r v_v^+\right) = 0 \qquad \text{(D-4)}$$

Thus, the continuity equation becomes

$$\frac{\partial}{\partial r}\left(r v_v^+\right) = -\frac{r}{u_v^*}\left[u_v^+ \frac{du_v^*}{dz} + u_v^* \frac{\partial u_v^+}{\partial z}\right]$$

$$\frac{\partial}{\partial r}\left(r v_v^+\right) = -\frac{r u_v^+}{u_v^*}\frac{du_v^*}{dz} - r\frac{\partial u_v^+}{\partial z} \qquad \text{(D-5)}$$

Wall Region of Vapor Film

Using Equations B-4, B-27 and B-35 in Equation D-5 for $u_v^+, \frac{du_v^*}{dz}$ and $\frac{\partial u_v^+}{\partial z}$ respectively gives

$$\frac{\partial}{\partial r}\left(r v_v^+\right) = -\frac{r\eta^7}{u_v^*}\frac{1}{2\sqrt{\tau_w \rho_v}}\frac{d\tau_w}{dz} - \frac{r\eta^7}{2\tau_w}\frac{d\tau_w}{dz}$$

Using the definition of $u_v^* = \sqrt{\frac{\tau_w}{\rho_v}}$

$$\frac{\partial}{\partial r}\left(r v_v^+\right) = -\frac{r\eta^7}{\tau_w}\frac{d\tau_w}{dz} \qquad \text{(D-6)}$$

Integrating Equation D-6 with respect to r gives

$$v_v^+ = -\frac{r}{\nu_v}\frac{1}{\sqrt{\tau_w \rho_v}}\left(\frac{R}{2} - \frac{r}{3}\right)\frac{d\tau_w}{dz} + \frac{K_3(z)}{r} \text{ or } v_v^+ = -\frac{u_v^*}{\nu_v \tau_w}\frac{d\tau_w}{dz}\left(\frac{Rr}{2} - \frac{r^2}{3}\right) + \frac{K_3(z)}{r} \qquad \text{(D-7)}$$

where K_3 is a function of z, which can be evaluated by using the fact that at the wall, $r = R$, the velocity is zero. Thus,

$$K_3(z) = \frac{R^3}{6\nu_v\sqrt{\tau_w\rho_v}}\frac{d\tau_w}{dz} \text{ or } K_3(z) = \frac{R^3}{6}\frac{u_v^*}{\nu_v\tau_w}\frac{d\tau_w}{dz} \tag{D-8}$$

Substituting the value of $K_3(z)$ in Equation D-7 above gives

$$v_v^+ = -\frac{1}{\nu_v\sqrt{\tau_w\rho_v}}\left(\frac{Rr}{2} - \frac{r^2}{3}\right)\frac{d\tau_w}{dz} + \frac{R^3}{6r\nu_v\sqrt{\tau_w\rho_v}}\frac{d\tau_w}{dz} \tag{D-9}$$

An alternate form of the expression for v_v^+ is

$$v_v^+ = \frac{u_v^*}{\nu_v\tau_w}\frac{d\tau_w}{dz}\left(\frac{R^3}{6r} - \frac{Rr}{2} + \frac{r^2}{3}\right) \tag{D-10}$$

Equation D-9 or D-10 represents the radial component of the vapor velocity for the wall region.

Turbulent Core of Vapor Film

Equation D-4 can be re-arranged to become

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ru_v^*v_v^+\right) = -\frac{\partial \overline{u}_v}{\partial z}$$

The friction velocity, u_v^* is a function of z alone

$$\frac{\partial}{\partial r}\left(rv_v^+\right) = -\frac{r}{u_v^*}\frac{\partial \overline{u}_v}{\partial z} \tag{D-11}$$

From Equation B-33,

$$\frac{\partial \bar{u}_v}{\partial z} = \frac{\left(a_0 + a_1\eta + a_2\eta^2\right)}{2\sqrt{\tau_w \rho_v}}\frac{d\tau_w}{dz} + u_v^*\left\{\frac{(a_1 + 2a_2\eta)}{14}\left(\frac{\eta}{\tau_w}\right)\frac{d\tau_w}{dz} + \left[\eta - (5)^{1/7}\right]^2 \frac{da_2}{dz}\right\}$$

Substituting Equation B-33 in Equation D-11

$$\frac{\partial}{\partial r}\left(rv_v^+\right) = -\frac{r}{u_v^*}\left\{\frac{\left(a_0 + a_1\eta + a_2\eta^2\right)}{2\sqrt{\tau_w \rho_v}}\frac{d\tau_w}{dz} + u_v^*\left\{\frac{(a_1 + 2a_2\eta)}{14}\left(\frac{\eta}{\tau_w}\right)\frac{d\tau_w}{dz} + \left[\eta - (5)^{1/7}\right]^2 \frac{da_2}{dz}\right\}\right\} \qquad \text{(D-12)}$$

Substituting for the friction velocity

$$\frac{\partial}{\partial r}\left(rv_v^+\right) = -\frac{\left(a_0 + a_1\eta + a_2\eta^2\right)r}{2\tau_w}\frac{d\tau_w}{dz} - \frac{\left(a_1\eta + 2a_2\eta^2\right)r}{14\tau_w}\frac{d\tau_w}{dz} - \left[\eta^2 - 2\eta(5)^{1/7} + (5)^{2/7}\right]r\frac{da_2}{dz}$$

Integrating the above equation with respect to r and realizing that $\tau_w, \frac{d\tau_w}{dz}, \frac{da_2}{dz}$ are functions of z alone, yields

$$rv_v^+ = \left\{\begin{array}{l} -\frac{1}{2\tau_w}\frac{d\tau_w}{dz}\left[\frac{a_0 r^2}{2} + 7a_1\eta\left(\frac{y^2}{15} - \frac{Ry}{8}\right) + 7a_2\eta^2\left(\frac{y^2}{16} - \frac{Ry}{9}\right)\right] \\ -\frac{1}{2\tau_w}\frac{d\tau_w}{dz}\left[a_1\eta\left(\frac{y^2}{15} - \frac{Ry}{8}\right) + 2a_2\eta^2\left(\frac{y^2}{16} - \frac{Ry}{9}\right)\right] \\ -\frac{da_2}{dz}\left[7\eta^2\left(\frac{y^2}{16} - \frac{Ry}{9}\right) - 14(5)^{1/7}\eta\left(\frac{y^2}{15} - \frac{Ry}{8}\right) + (5)^{2/7}\left(\frac{r^2}{2}\right)\right] \end{array}\right\} + K_4(z)$$

Collecting like terms and dividing both sides by r gives

$$v_v^+ = \frac{1}{r}\left\{\begin{array}{l} -\frac{1}{2\tau_w}\frac{d\tau_w}{dz}\left[\frac{a_0 r^2}{2} + 8a_1\eta\left(\frac{y^2}{15} - \frac{Ry}{8}\right) + 9a_2\eta^2\left(\frac{y^2}{16} - \frac{Ry}{9}\right)\right] \\ -\frac{da_2}{dz}\left[7\eta^2\left(\frac{y^2}{16} - \frac{Ry}{9}\right) - 14(5)^{1/7}\eta\left(\frac{y^2}{15} - \frac{Ry}{8}\right) + (5)^{2/7}\left(\frac{r^2}{2}\right)\right] \end{array}\right\} + \frac{K_4(z)}{r} \qquad \text{(D-13)}$$

Equation D-13 represents the radial component of the vapor velocity for the turbulent core. $K_4(z)$ can be evaluated by equating the velocity of the vapor at the junction of the wall region and the turbulent core. In other words, the velocity obtained from Equation D-9 or D-10 at $y^+ = 5$ should equal the velocity obtained by Equation D-13. The location $y^+ = 5$ corresponds to $\eta = (5)^{1/7}$, $y = \frac{5\nu_v}{u_v^*}$. The corresponding radial coordinate is given by the definition from Equation B-1 as $r = R - \frac{5\nu_v}{u_v^*}$.

From Equation D-13 and Equation D-10, at $y^+ = 5$ or $r = R - \frac{5\nu_v}{u_v^*}$ or $y = \frac{5\nu_v}{u_v^*} = l$ or $\eta = (5)^{1/7}$.

Substituting the values of r, y, y^+ and η in the above equations gives

$$rv_v^+ = \left\{ \begin{array}{l} -\dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{a_0(R-l)^2}{2} + 8a_1(5)^{1/7} l\left(\dfrac{l}{15}-\dfrac{R}{8}\right) + 9a_2(5)^{2/7} l\left(\dfrac{l}{16}-\dfrac{R}{9}\right)\right] \\ -\dfrac{da_2}{dz}\left[7(5)^{2/7} l\left(\dfrac{l}{16}-\dfrac{R}{9}\right) - 14(5)^{2/7} l\left(\dfrac{l}{15}-\dfrac{R}{8}\right) + \dfrac{(5)^{2/7}(R-l)^2}{2}\right] \end{array} \right\} + K_4(z) = \frac{u_v^* l^2}{\nu_v \tau_w}\left(\frac{R}{2}-\frac{l}{3}\right)\frac{d\tau_w}{dz}$$

Thus,

$$K_4(z) = \frac{u_v^* l^2}{\nu_v \tau_w}\left(\frac{R}{2}-\frac{l}{3}\right)\frac{d\tau_w}{dz} + \left\{ \begin{array}{l} \dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\begin{array}{l} \dfrac{a_0(R-l)^2}{2} + 8a_1(5)^{1/7} l\left(\dfrac{l}{15}-\dfrac{R}{8}\right) \\ + 9a_2(5)^{2/7} l\left(\dfrac{l}{16}-\dfrac{R}{9}\right) \end{array}\right] \\ + \dfrac{da_2}{dz}\left[\begin{array}{l} 7(5)^{2/7} l\left(\dfrac{l}{16}-\dfrac{R}{9}\right) - 14(5)^{2/7} l\left(\dfrac{l}{15}-\dfrac{R}{8}\right) \\ + \dfrac{(5)^{2/7}(R-l)^2}{2} \end{array}\right] \end{array} \right\} \tag{D-14}$$

Substituting the value of $K_4(z)$ in Equation D-13 yields an expression for the non-dimensional radial component of the vapor velocity, v_v^+, as given below.

$$v_v^+ = \frac{1}{r}\left[\begin{array}{l} \left\{\begin{array}{l} -\dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{a_0 r^2}{2} + 8a_1\eta\left(\dfrac{y^2}{15} - \dfrac{Ry}{8}\right) + 9a_2\eta^2\left(\dfrac{y^2}{16} - \dfrac{Ry}{9}\right)\right] \\ -\dfrac{da_2}{dz}\left[7\eta^2\left(\dfrac{y^2}{16} - \dfrac{Ry}{9}\right) - 14(5)^{1/7}\eta\left(\dfrac{y^2}{15} - \dfrac{Ry}{8}\right) + (5)^{2/7}\left(\dfrac{r^2}{2}\right)\right] \end{array}\right\} + \dfrac{u_v^* l^2}{\nu_v \tau_w}\left(\dfrac{R}{2} - \dfrac{l}{3}\right)\dfrac{d\tau_w}{dz} \\ + \left\{\begin{array}{l} \dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{a_0(R-l)^2}{2} + 8a_1(5)^{1/7} l\left(\dfrac{l}{15} - \dfrac{R}{8}\right) + 9a_2(5)^{2/7} l\left(\dfrac{l}{16} - \dfrac{R}{9}\right)\right] \\ + \dfrac{da_2}{dz}\left[7(5)^{2/7} l\left(\dfrac{l}{16} - \dfrac{R}{9}\right) - 14(5)^{2/7} l\left(\dfrac{l}{15} - \dfrac{R}{8}\right) + \dfrac{(5)^{2/7}(R-l)^2}{2}\right] \end{array}\right\} \end{array}\right] +$$

Collecting similar terms together gives

$$v_v^+ = \frac{1}{r}\left[\begin{array}{l} \dfrac{u_v^* l^2}{\nu_v \tau_w}\left(\dfrac{R}{2} - \dfrac{l}{3}\right)\dfrac{d\tau_w}{dz} + \\ \left\{\begin{array}{l} \dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left\{\begin{array}{l} \dfrac{a_0}{2}\left[(R-l)^2 - r^2\right] + 8a_1\left[(5)^{1/7} l\left(\dfrac{l}{15} - \dfrac{R}{8}\right) - \eta\left(\dfrac{y^2}{15} - \dfrac{Ry}{8}\right)\right] \\ + 9a_2\left[(5)^{2/7} l\left(\dfrac{l}{16} - \dfrac{R}{9}\right) - \eta^2\left(\dfrac{y^2}{16} - \dfrac{Ry}{9}\right)\right] \end{array}\right\} \\ + \dfrac{da_2}{dz}\left\{\begin{array}{l} 7\left[(5)^{2/7} l\left(\dfrac{l}{16} - \dfrac{R}{9}\right) - \eta^2\left(\dfrac{y^2}{16} - \dfrac{Ry}{9}\right)\right] \\ -14\left[(5)^{2/7} l\left(\dfrac{l}{15} - \dfrac{R}{8}\right) - (5)^{1/7}\eta\left(\dfrac{y^2}{15} - \dfrac{Ry}{8}\right)\right] \\ + \dfrac{(5)^{2/7}}{2}\left[(R-l)^2 - r^2\right] \end{array}\right\} \end{array}\right\} \end{array}\right] \qquad \text{(D-15)}$$

Equation D-15 represents the non-dimensional radial component of the vapor velocity.

Appendix E

Derivation of Liquid Phase Radial Velocity Profile

The continuity equation for the liquid phase is given by

$$\frac{\partial \bar{u}_l}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{v}_l\right) = 0 \qquad \text{(E-1)}$$

The radial component of the liquid velocity, just like the axial component, is non-dimensionalized by the friction velocity. Thus,

$$v_l^+ = \frac{\bar{v}_l}{u_v^*} \qquad \text{(E-2)}$$

where the friction velocity and the other terms are defined in Equation B-1.

$$u_v^* = \sqrt{\frac{\tau_w}{\rho_v}}\text{, } u_v^+ = \frac{\bar{u}_v}{u_v^*}\text{, } y^+ = \frac{y u_v^*}{\nu_v}\text{, } y = R - r \qquad \text{(B-1)}$$

The vapor film being turbulent, the axial component of the velocity is assumed to be of the form given by equation C-1.

$$u_v^+ = b_0 + b_1\eta + b_2\eta^2 \qquad \text{(C-1)}$$

Using the definitions in Equations B-1 and E-2, the vapor continuity equation becomes

$$\frac{\partial \bar{u}_l}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r v_l^+ u_v^*\right) = 0 \qquad \text{(E-3)}$$

The friction velocity is a function of z alone, hence it can be treated as constant, when differentiating with respect to r. Employing the product rule of differentiation, and using $\frac{\partial \bar{u}_l}{\partial z}$ from Equation C-15, Equation E-3 becomes

$$\frac{u_v^*}{r}\frac{\partial}{\partial r}\left(rv_l^+\right) = -\frac{\partial \overline{u}_l}{\partial z} = -\left\{\left(b_0 + b_1\eta + b_2\eta^2\right)\frac{du_v^*}{dz} + u_v^*\left[\frac{db_0}{dz} + \eta\frac{db_1}{dz} + \eta^2\frac{db_2}{dz} + \left(b_1 + 2b_2\eta\right)\frac{\partial \eta}{\partial z}\right]\right\} \tag{E-4}$$

$$\frac{\partial}{\partial r}\left(rv_l^+\right) = -\frac{r}{u_v^*}\left\{\left(b_0 + b_1\eta + b_2\eta^2\right)\frac{du_v^*}{dz} + u_v^*\left[\frac{db_0}{dz} + \eta\frac{db_1}{dz} + \eta^2\frac{db_2}{dz} + \left(b_1 + 2b_2\eta\right)\frac{\partial \eta}{\partial z}\right]\right\}$$

Substituting Equations B-20 and B-27 in the above equation gives

$$\frac{\partial}{\partial r}\left(rv_l^+\right) = -\frac{\left(b_0 + b_1\eta + b_2\eta^2\right)r}{2u_v^*\sqrt{\tau_w\rho_v}}\frac{d\tau_w}{dz} - \left[\frac{db_0}{dz} + \eta\frac{db_1}{dz} + \eta^2\frac{db_2}{dz} + \frac{\left(b_1 + 2b_2\eta\right)}{14}\frac{\eta}{\tau_w}\frac{d\tau_w}{dz}\right]r$$

Substituting for the friction velocity

$$\frac{\partial}{\partial r}\left(rv_l^+\right) = -\frac{\left(b_0 + b_1\eta + b_2\eta^2\right)r}{2\tau_w}\frac{d\tau_w}{dz} - \left[\frac{db_0}{dz} + \eta\frac{db_1}{dz} + \eta^2\frac{db_2}{dz} + \frac{\left(b_1\eta + 2b_2\eta^2\right)}{14}\frac{1}{\tau_w}\frac{d\tau_w}{dz}\right]r \tag{E-5}$$

Integrating with respect to r, and noting that $\tau_w, \frac{d\tau_w}{dz}, \frac{db_0}{dz}, \frac{db_1}{dz}, \frac{db_2}{dz}, b_0, b_1$ and b_2 are functions of z alone, gives

$$rv_l^+ = -\left\{\begin{array}{l} \frac{1}{2\tau_w}\frac{d\tau_w}{dz}\left[\frac{b_0 r^2}{2} + 7b_1\eta\left(\frac{y^2}{15} - \frac{Ry}{8}\right) + 7b_2\eta^2\left(\frac{y^2}{16} - \frac{Ry}{9}\right)\right] + \\ \frac{1}{14\tau_w}\frac{d\tau_w}{dz}\left[7b_1\eta\left(\frac{y^2}{15} - \frac{Ry}{8}\right) + 14b_2\eta^2\left(\frac{y^2}{16} - \frac{Ry}{9}\right)\right] \\ + \left[\frac{db_0}{dz}\frac{r^2}{2} + \left(\frac{db_1}{dz}\right)7\eta\left(\frac{y^2}{15} - \frac{Ry}{8}\right) + \frac{db_2}{dz}7\eta^2\left(\frac{y^2}{16} - \frac{Ry}{9}\right)\right] \end{array}\right\} + K_1(z) \tag{E-6}$$

$$rv_l^+ = -\left\{\begin{array}{l} \frac{1}{2\tau_w}\frac{d\tau_w}{dz}\left[\frac{b_0 r^2}{2} + 8\eta b_1\left(\frac{y^2}{15} - \frac{Ry}{8}\right) + 9b_2\eta^2\left(\frac{y^2}{16} - \frac{Ry}{9}\right)\right] \\ + \left[\frac{db_0}{dz}\frac{r^2}{2} + 7\eta\frac{db_1}{dz}\left(\frac{y^2}{15} - \frac{Ry}{8}\right) + 7\eta^2\frac{db_2}{dz}\left(\frac{y^2}{16} - \frac{Ry}{9}\right)\right] \end{array}\right\} + K_1(z)$$

Therefore, the non-dimensional liquid velocity is

$$v_l^+ = -\left\{ \begin{array}{l} \dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{b_0 r}{2} + \dfrac{8\eta b_1}{r}\left(\dfrac{y^2}{15} - \dfrac{Ry}{8}\right) + \dfrac{9b_2\eta^2}{r}\left(\dfrac{y^2}{16} - \dfrac{Ry}{9}\right)\right] \\ + \left[\dfrac{db_0}{dz}\dfrac{r}{2} + \dfrac{7\eta}{r}\dfrac{db_1}{dz}\left(\dfrac{y^2}{15} - \dfrac{Ry}{8}\right) + \dfrac{7\eta^2}{r}\dfrac{db_2}{dz}\left(\dfrac{y^2}{16} - \dfrac{Ry}{9}\right)\right] \end{array} \right\} + \frac{K_1(z)}{r} \tag{E-7}$$

In order to obtain $K_1(z)$, the radial component of the non-dimensional liquid velocity, v_l^+ and the radial component of the non-dimensional vapor velocity v_v^+ at $y = \delta$ or $y^+ = \delta^+$; $\eta = (\delta^+)^{1/7}$ $r = R - \delta$ are equal.

Thus, $rv_v^+ = rv_l^+$ at the liquid-vapor interface.

$$-\left\{ \begin{array}{l} \dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{b_0 r^2}{2} + 8\eta b_1\left(\dfrac{y^2}{15} - \dfrac{Ry}{8}\right) + 9b_2\eta^2\left(\dfrac{y^2}{16} - \dfrac{Ry}{9}\right)\right] \\ + \left[\dfrac{db_0}{dz}\dfrac{r^2}{2} + 7\eta\dfrac{db_1}{dz}\left(\dfrac{y^2}{15} - \dfrac{Ry}{8}\right) + 7\eta^2\dfrac{db_2}{dz}\left(\dfrac{y^2}{16} - \dfrac{Ry}{9}\right)\right] \end{array} \right\} + K_1(z)$$

$$= -\left\{ \begin{array}{l} \dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{a_0 r^2}{2} + 8\eta a_1\left(\dfrac{y^2}{15} - \dfrac{Ry}{8}\right) + 9\eta^2 a_2\left(\dfrac{y^2}{16} - \dfrac{Ry}{8}\right)\right] \\ + \left[\dfrac{da_0}{dz}\dfrac{r^2}{2} + 7\eta\dfrac{da_1}{dz}\left(\dfrac{y^2}{15} - \dfrac{Ry}{8}\right) + 7\eta^2\dfrac{da_2}{dz}\left(\dfrac{y^2}{16} - \dfrac{Ry}{9}\right)\right] \end{array} \right\} + K_4(z)$$

Rearranging,

$$K_1(z) = \left\{ \begin{array}{l} K_4(z) - \dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{(a_0 - b_0)r^2}{2} + 8\eta\left(\dfrac{y^2}{15} - \dfrac{Ry}{8}\right)(a_1 - b_1) + 9\eta^2\left(\dfrac{y^2}{16} - \dfrac{Ry}{9}\right)(a_2 - b_2)\right] \\ -\left[\dfrac{r^2}{2}\left(\dfrac{da_0}{dz} - \dfrac{db_0}{dz}\right) + 7\eta\left(\dfrac{y^2}{15} - \dfrac{Ry}{8}\right)\left(\dfrac{da_1}{dz} - \dfrac{db_2}{dz}\right) + 7\eta^2\left(\dfrac{y^2}{16} - \dfrac{Ry}{9}\right)\left(\dfrac{da_2}{dz} - \dfrac{db_2}{dz}\right)\right] \end{array} \right\}$$

Substitute for $y = \delta, \eta = (\delta^+)^{1/7}, r = R - \delta$

$$K_1(z)=\left\{\begin{array}{l} K_4(z)-\dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\begin{array}{l}\left[\left(\dfrac{a_0-b_0}{2}\right)(R-\delta)^2\right]+\left[8\left(\delta^+\right)^{1/7}\left(\dfrac{\delta^2}{15}-\dfrac{R\delta}{8}\right)(a_1-b_1)\right]\\ +\left[9\left(\delta^+\right)^{2/7}\left(\dfrac{\delta^2}{16}-\dfrac{R\delta}{9}\right)(a_2-b_2)\right]\end{array}\right]\\ -\left[\begin{array}{l}\left[\dfrac{(R-\delta)^2}{2}\left(\dfrac{da_0}{dz}-\dfrac{db_0}{dz}\right)\right]+\left[7\left(\delta^+\right)^{1/7}\left(\dfrac{\delta^2}{15}-\dfrac{R\delta}{8}\right)\left(\dfrac{da_1}{dz}-\dfrac{db_1}{dz}\right)\right]+\\ \left[7\left(\delta^+\right)^{2/7}\left(\dfrac{\delta^2}{16}-\dfrac{R\delta}{9}\right)\left(\dfrac{da_2}{dz}-\dfrac{db_2}{dz}\right)\right]\end{array}\right]\end{array}\right\} \tag{E-8}$$

$K_4(z)$ is defined in Equation D-14. Substituting Equation E-8 in Equation E-7 gives an expression for the non-dimensional liquid radial velocity component.

Appendix F

Derivation of Vapor Temperature Profile

In addition to the definitions in Equation B-1, the vapor temperature is non-dimensionalized as

$$T_v^+ = \left(\overline{T}_v - T_w\right)\frac{\rho_v c_{p,v} u_v^*}{q_w''} \tag{F-1}$$

The radial heat flux is given by

$$q_r = \rho_v c_{p,v}\left(\frac{\varepsilon_{m,v}}{\mathrm{Pr}_{t,v}} + \frac{\nu_v}{\mathrm{Pr}_v}\right)\frac{\partial \overline{T}_v}{\partial y}$$

For the wall region $\left(0 \le y^+ \le 5\right)$, $\varepsilon_{m,v} << \nu_v$ and assuming Reynolds analogy to be valid, the non-dimensional vapor temperature in the wall region is given by

$$T_v^+ = \left(\mathrm{Pr}_v\right)y^+ \quad \left(0 \le y^+ \le 5\right) \tag{F-2}$$

For the turbulent vapor core, the non-dimensional temperature profile is given by Equation F-3.

$$T_v^+ = c_0 + c_1\eta + c_2\eta^2 \quad \left(5 \le y^+ \le \delta^+\right) \tag{F-3}$$

In the above equation, the coefficients c_0, c_1 and c_2 are unknown and need to be determined.

For the turbulent core, the derivative of T_v^+ with respect to η is given by

$$\frac{dT_v^+}{d\eta} = c_1 + 2c_2\eta \tag{F-4}$$

Using Equation B-3 and F-4

$$\frac{dT_v^+}{dy^+} = \frac{1}{7\eta^6}\frac{dT_v^+}{d\eta} = \frac{1}{7\eta^6}\left(c_1 + 2c_2\eta\right) \tag{F-5}$$

In order to evaluate the unknown coefficients c_0, c_1 and c_2, three conditions are needed on the vapor temperature or its derivative.

Condition 1:

The temperature is continuous at $y^+ = 5$

From Equations F-2 and F-3, at $y^+ = 5$ or $\eta = (5)^{1/7}$,

$$T_v^+ = 5\,\mathrm{Pr}_v = c_0 + c_1(5)^{1/7} + c_2(5)^{2/7} \tag{F-6}$$

Condition 2:

The gradient of the temperature is continuous at $y^+ = 5$.

From Equation F-2, $\left.\dfrac{dT_v^+}{dy^+}\right|_{y^+=5} = \mathrm{Pr}_v$

From Equation F-5, $\left.\dfrac{dT_v^+}{dy^+}\right|_{y^+=5} = \left.\dfrac{1}{7\eta^6}\dfrac{dT_v^+}{d\eta}\right|_{\eta=5^{1/7}} = \left.\dfrac{(c_1 + 2c_2\eta)}{7\eta^6}\right|_{\eta=5^{1/7}}$

Combining these, Equation F-7 is obtained as below

$$c_1 + 2c_2(5)^{1/7} = 7(5)^{6/7}\,\mathrm{Pr}_v \tag{F-7}$$

Condition 3:

The third condition is obtained by setting the interface temperature to the saturation temperature. At $y^+ = \delta^+$, $T_v = T_{sat}$

$$T_{sat}^{+} = \left(T_{sat} - T_w\right)\frac{\rho_v c_{p,v} u_v^{*}}{q_w''} = c_0 + c_1\left(\delta^{+}\right)^{1/7} + c_2\left(\delta^{+}\right)^{2/7} \tag{F-8}$$

Equations F-6, F-7 and F-8 represent the three conditions that need to be satisfied to obtain the coefficients c_0, c_1 and c_2.

Equation F-6 subtracted from Equation F-8 yields

$$c_1\left[\left(\delta^{+}\right)^{1/7} - 5^{1/7}\right] + c_2\left[\left(\delta^{+}\right)^{2/7} - 5^{2/7}\right] = T_{sat}^{+} - 5\,\mathrm{Pr}_v \tag{F-9}$$

Solving Equations F-7 and F-9 simultaneously gives

$$c_2 = \frac{T_{sat}^{+} + 30\,\mathrm{Pr}_v - 7\,\mathrm{Pr}_v (5)^{6/7}\left(\delta^{+}\right)^{1/7}}{\left[\left(\delta^{+}\right)^{1/7} - (5)^{1/7}\right]^2} \tag{F-10}$$

Using F-7, $c_1 = -2c_2(5)^{1/7} + 7(5)^{6/7}\,\mathrm{Pr}_v$

$$c_1 = \frac{\left\{7(5)^{6/7}\left(\delta^{+}\right)^{2/7}\mathrm{Pr}_v - 2(5)^{1/7} T_{sat}^{+} - 25(5)^{1/7}\,\mathrm{Pr}_v\right\}}{\left[\left(\delta^{+}\right)^{1/7} - 5^{1/7}\right]^2} \tag{F-11}$$

Using Equations F-6 and F-7 c_0 can be obtained as

$$c_0 = c_2(5)^{2/7} - 30\,\mathrm{Pr}_v$$

$$c_0 = \frac{\left\{25\,\mathrm{Pr}_v\left(\delta^{+}\right)^{1/7}(5)^{1/7} + (5)^{2/7} T_{sat}^{+} - 30\,\mathrm{Pr}_v\left(\delta^{+}\right)^{2/7}\right\}}{\left[\left(\delta^{+}\right)^{1/7} - 5^{1/7}\right]^2} \tag{F-12}$$

Equations F-10, F-11 and F-12 give the coefficients c_0, c_1 and c_2 respectively. These are not constants, but functions of the axial coordinate, z alone. Derivatives of the axial component of the vapor velocity, $\overline{T}_v$ with respect to z and r occur in the integrated form of the governing conservation equations.

F.1 Gradient in the Radial Direction

Rearranging Equation F-1 and differentiating with respect to r gives

$$\frac{\partial \overline{T}_v}{\partial r} = \frac{\partial T_w}{\partial r} + \frac{\partial}{\partial r}\left[\left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right) T_v^+\right] \qquad \text{(F-13)}$$

F.1.1 Vapor Film Turbulent Core

Wall temperature, wall heat flux, friction velocity are not functions of y (or r), hence they can be treated as constants.

For the turbulent core of the vapor film, using Equation F-3, the above relation becomes

$$\frac{\partial T_v}{\partial r} = \left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right) \frac{\partial}{\partial r}\left(c_0 + c_1\eta + c_2\eta^2\right)$$

The coefficients c_0, c_1 and c_2 are functions of z alone, hence

$$\frac{\partial \overline{T}_v}{\partial r} = \left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right)\left(c_1 \frac{\partial \eta}{\partial r} + 2c_2\eta \frac{\partial \eta}{\partial r}\right) \qquad \text{(F-14)}$$

Substituting for $\frac{\partial \eta}{\partial r}$ from Equation B-19, we get

- $$\frac{\partial \overline{T}_v}{\partial r} = -\left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right)(c_1 + 2c_2\eta)\left(\frac{\eta}{7y}\right) \tag{F-15}$$

F.1.2 Vapor Film Wall Region

For the wall region of the vapor film, differentiating Equation F-2 with respect to y^+ gives

$$\frac{\partial T_v^+}{\partial y^+} = \mathrm{Pr}_v$$

Using definitions from Equation B-1, we get

$$\frac{\partial T_v^+}{\partial r} = -\left(\frac{u_v^*}{\nu_v}\right)\mathrm{Pr}_v \tag{F-16}$$

Substituting Equation F-16 in Equation F-13 and knowing that the wall temperature, wall heat flux, friction velocity are not functions of y (or r), we get

$$\frac{\partial \overline{T}_v}{\partial r} = -\left(\frac{q''_w}{\rho_v c_{p,v} \nu_v}\right)\mathrm{Pr}_v$$

Using the definition of Prandtl number and the thermal diffusivity, we can write $\mathrm{Pr}_v = \frac{\nu_v}{\alpha_v} = \frac{\nu_v \rho_v c_{p,v}}{k_v}$

Thus, substituting in the above equation gives

- $$\frac{\partial \overline{T}_v}{\partial r} = -\frac{q''_w}{k_v} \tag{F-17}$$

Equations F-15 and F-17 represent the gradients of the vapor temperature with respect to the radial direction.

F.2 Gradient in the Axial Direction

Rearranging Equation F-1 and differentiating with respect to z gives

$$\frac{\partial \overline{T}_v}{\partial z} = \frac{\partial T_w}{\partial z} + \frac{\partial}{\partial z}\left[\left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right) T_v^+\right] \tag{F-18}$$

The wall temperature, friction velocity and the wall heat flux are all functions of z alone. Thus Equation F-18 becomes

$$\frac{\partial \overline{T}_v}{\partial z} = \frac{dT_w}{dz} + \left(\frac{1}{\rho_v c_{p,v}}\right)\left[\frac{q''_w}{u_v^*}\frac{\partial T_v^+}{\partial z} + \frac{T_v^+}{u_v^*}\frac{dq''_w}{dz} - \frac{T_v^+ q''_w}{\left(u_v^*\right)^2}\frac{du_v^*}{dz}\right] \tag{F-19}$$

F.2.1 Vapor Film Turbulent Core

Substituting for the non-dimensional temperature using Equation F-3, the above equation becomes

$$\frac{\partial \overline{T}_v}{\partial z} = \frac{dT_w}{dz} + \left(\frac{1}{\rho_v c_{p,v}}\right)\left[\frac{q''_w}{u_v^*}\frac{\partial}{\partial z}\left(c_0 + c_1\eta + c_2\eta^2\right) + \frac{T_v^+}{u_v^*}\frac{dq''_w}{dz} - \frac{T_v^+ q''_w}{\left(u_v^*\right)^2}\frac{du_v^*}{dz}\right] \tag{F-20}$$

Using the fact that the coefficients c_0, c_1 and c_2 are functions of z alone, the above equation becomes

$$\frac{\partial \overline{T}_v}{\partial z} = \frac{dT_w}{dz} + \left(\frac{1}{\rho_v c_{p,v}}\right)\left\{\begin{array}{l} \dfrac{q''_w}{u_v^*}\left[\dfrac{dc_0}{dz} + \eta\dfrac{dc_1}{dz} + \eta^2\dfrac{dc_2}{dz} + c_1\dfrac{\partial \eta}{\partial z} + 2c_2\eta\dfrac{\partial \eta}{\partial z}\right] \\ + \dfrac{T_v^+}{u_v^*}\dfrac{dq''_w}{dz} - \dfrac{T_v^+ q''_w}{\left(u_v^*\right)^2}\dfrac{du_v^*}{dz} \end{array}\right\} \tag{F-21}$$

Equation F-21 requires the derivatives of coefficients c_0, c_1 and c_2 with respect to z. These are evaluated below.

Using Equation F-8 for the non-dimensional saturation temperature, T_{sat}^{+} and evaluating the derivative of T_{sat}^{+} invoking Equation B-27 yields

$$\frac{dT_{sat}^{+}}{dz}=\frac{T_{sat}^{+}}{2\tau_{w}}\frac{d\tau_{w}}{dz}-\frac{T_{sat}^{+}}{q_{w}''}\frac{dq_{w}''}{dz}-\left(\frac{\rho_{v}c_{p,v}u_{v}^{*}}{q''}\right)\frac{dT_{w}}{dz} \tag{F-22}$$

Using Equation F-10 for the definition of c_2, and Equations B-29 and F-22 gives

$$\frac{dc_{2}}{dz}=(dc2tw)\frac{d\tau_{w}}{dz}+(dc2d)\frac{d\delta}{dz}+(dc2wt)\frac{dT_{w}}{dz}+(dc2qw)\frac{dq_{w}''}{dz} \tag{F-23}$$

$$dc2tw=\left\{\frac{T_{sat}^{+}-(5)^{6/7}(\delta^{+})^{1/7}\,\mathrm{Pr}_{v}-\frac{2}{7}c_{2}(\delta^{+})^{1/7}\left[(\delta^{+})^{1/7}-(5)^{1/7}\right]}{2\tau_{w}\left[(\delta^{+})^{1/7}-(5)^{1/7}\right]^{2}}\right\} \tag{F-23a}$$

$$dc2d=-\left\{\frac{(5)^{6/7}(\delta^{+})^{1/7}\,\mathrm{Pr}_{v}+\frac{2}{7}c_{2}(\delta^{+})^{1/7}\left[(\delta^{+})^{1/7}-(5)^{1/7}\right]}{\delta\left[(\delta^{+})^{1/7}-(5)^{1/7}\right]^{2}}\right\} \tag{F-23b}$$

$$dc2wt=-\frac{\rho_{v}c_{p,v}u_{v}^{*}}{q_{w}''\left[(\delta^{+})^{1/7}-(5^{+})^{1/7}\right]^{2}} \tag{F-23c}$$

$$dc2qw=-\frac{T_{sat}^{+}}{q_{w}''\left[(\delta^{+})^{1/7}-(5^{+})^{1/7}\right]^{2}} \tag{F-23d}$$

Using Equation F-7

$$\frac{dc_1}{dz} = -2(5)^{1/7}\frac{dc_2}{dz} \tag{F-24}$$

Combining Equations F-6 and F-7 to express c_0 in terms of c_2 and differentiating with respect to z yields

$$\frac{dc_0}{dz} = (5)^{2/7}\frac{dc_2}{dz} \tag{F-25}$$

Using Equations B-20, B-27, F-23, F-24 and F-25 in Equation F-21, $\frac{\partial \overline{T}_v}{\partial z}$ can be obtained for the turbulent core of the vapor film.

F.2.2 Vapor Film Wall Region

Equation F-2 can be re-written as

$$T_v^+ = (\mathrm{Pr}_v)y^+ = \mathrm{Pr}_v(\eta^7)$$

Therefore

$$\frac{\partial T_v^+}{\partial z} = 7\eta^6\,\mathrm{Pr}_v\frac{\partial \eta}{\partial z} \tag{F-26}$$

Substituting from Equation B-20 for $\frac{\partial \eta}{\partial z}$ yields

$$\frac{\partial T_v^+}{\partial z} = \left(\frac{\eta^7\,\mathrm{Pr}_v}{2\tau_w}\right)\frac{d\tau_w}{dz} \tag{F-27}$$

Substituting Equations F-27, F-2 in F-19 yields $\frac{\partial \overline{T}_v}{\partial z}$ for the wall region in the vapor film as

$$\frac{\partial \overline{T}_v}{\partial z} = \frac{dT_w}{dz} + \left(\frac{\eta^7 \Pr_v}{\rho_v c_{p,v} u_v^*} \right) \left[\frac{q_w''}{2\tau_w} \frac{\partial \tau_w}{\partial z} + \frac{dq_w''}{dz} - \frac{q_w''}{u_v^*} \frac{du_v^*}{dz} \right]$$

Using Equation B-27 for $\frac{\partial u_v^*}{\partial z}$ and the definition of friction velocity in the above and simplifying yields

$$\frac{\partial \overline{T}_v}{\partial z} = \frac{dT_w}{dz} + \left(\frac{\eta^7 \Pr_v}{\rho_v c_{p,v} u_v^*} \right) \frac{dq_w''}{dz} \tag{F-28}$$

Using the definitions from B-1 and B-2 $\eta^7 = y^+ = \frac{y u_v^*}{\nu_v}$ and the definition of $\Pr_v = \frac{\mu_v c_{p,v}}{k_v} = \frac{\rho_v \nu_v c_{p,v}}{k_v}$ in Equation F-28 gives

$$\frac{\partial \overline{T}_v}{\partial z} = \frac{dT_w}{dz} + \left(\frac{y}{k_v} \right) \frac{dq_w''}{dz} \tag{F-29}$$

Equation F-29 represents the gradient of the vapor temperature in the axial direction for the wall region.

Appendix G

Derivation of Liquid Temperature Profile

The temperature of the liquid core is non-dimensionalized in the same manner as the vapor temperature. Thus,

$$T_l^+ = \left(\overline{T_l} - T_w\right)\frac{\rho_v c_{p,v} u_v^*}{q_w''} = d_0 + d_1\eta + d_2\eta^2 \tag{G-1}$$

The derivative of T_l^+ with respect to η is given by

$$\frac{dT_l^+}{d\eta} = d_1 + 2d_2\eta \tag{G-2}$$

Rearranging Equation G-1 and differentiating with respect to y^+ gives

$$\frac{\partial \overline{T_l}}{\partial y^+} = \frac{\partial T_w}{\partial y^+} + \left(\frac{q_w''}{\rho_v c_{p,v} u_v^*}\right)\frac{\partial T_l^+}{\partial y^+} \tag{G-3}$$

Wall temperature, wall heat flux, friction velocity are not functions of y, hence they can be treated as constants. Using the definition of y^+, Equation B-1 and Equation B-3, the above equation becomes

$$\frac{\partial \overline{T_l}}{\partial y} = -\frac{\partial \overline{T_l}}{\partial r} = \frac{u_v^*}{\nu_v}\frac{\partial \overline{T_l}}{\partial y^+} = \frac{u_v^*}{\nu_v}\left(\frac{q_w''}{\rho_v c_{p,v} u_v^*}\right)\frac{\partial T_l^+}{\partial y^+} = \frac{1}{7\eta^6}\left(\frac{q_w''}{\rho_v c_{p,v} \nu_v}\right)\frac{dT_l^+}{d\eta} \tag{G-4}$$

Coefficients d_0, d_1 and d_2 are unknowns and need to be determined. Three conditions are needed to obtain these coefficients.

Condition 1:

The interface temperature to the saturation temperature. At $y^+ = \delta^+$, $T_l = T_{sat}$

$$T_{sat}^{+} = \left(T_{sat} - T_{w}\right)\frac{\rho_{v} c_{p,v} u_{v}^{*}}{q_{w}''} = d_0 + d_1\left(\delta^{+}\right)^{1/7} + d_2\left(\delta^{+}\right)^{2/7} \tag{G-5}$$

Condition 2:

The gradient of the temperature is equal to zero at the centerline. At $y^{+} = R^{+}$, $\frac{\partial T_l}{\partial r} = 0$

Using the definitions in Equation B-1, B-2 and B-3, this condition can be written as

$$\frac{1}{7\eta^6}\frac{dT_l^{+}}{d\eta}\bigg|_{\eta=\left(R^{+}\right)^{1/7}} = 0$$

This can be written using Equation G-2 as

$$d_1 + 2d_2\left(R^{+}\right)^{1/7} = 0 \tag{G-6}$$

Condition 3:

Performing an energy balance at the interface, this condition can be given as

$$q_{rad}'' + k_v \frac{\partial \overline{T}_v}{\partial r}\bigg|_{r=R-\delta} = k_l \frac{\partial \overline{T}_l}{\partial r}\bigg|_{r=R-\delta} + \dot{m}_l'' h_{fg} \tag{G-7}$$

The left hand side of the above equation refers to the heat coming into the interface from the wall by radiation and from the vapor by conduction. The right hand side refers to the heat transfer from the interface to the liquid core by conduction and the heat transfer due to mass transfer. This mass transfer term needs to be evaluated. Writing Equation G-7 in terms of the variable y gives

$$q_{rad}'' - k_v \frac{\partial \overline{T}_v}{\partial y}\bigg|_{y=\delta} + k_l \frac{\partial \overline{T}_l}{\partial y}\bigg|_{y=\delta} = \dot{m}_l'' h_{fg} \tag{G-8}$$

Substituting Equation G-4 in Equation G-8 yields

$$q''_{rad} - \left[\frac{k_v}{7\eta^6} \frac{q''_w}{\rho_v c_{p,v} \nu_v} \frac{dT_v^+}{d\eta} \right]_{\eta=(\delta^+)^{1/7}} + \left[\frac{k_l}{7\eta^6} \frac{q''_w}{\rho_v c_{p,v} \nu_v} \frac{dT_l^+}{d\eta} \right]_{\eta=(\delta^+)^{1/7}} = \dot{m}''_l h_{fg} \tag{G-9}$$

Using the definitions of thermal diffusivity, $\alpha_v = \dfrac{k_v}{\rho_v c_{p,v}}$ and $\Pr_v = \dfrac{\nu_v}{\alpha_v}$ in Equation G-9 yields

$$q''_{rad} - \left\{ \left(\frac{q''_w}{7\eta^6} \frac{1}{\Pr_v} \right) \left[\frac{dT_v^+}{d\eta} - \left(\frac{k_l}{k_v} \right) \frac{dT_l^+}{d\eta} \right] \right\}_{\eta=(\delta^+)^{1/7}} = \dot{m}''_l h_{fg} \tag{G-10}$$

Invoking Equations F-4 and G-2 for $\dfrac{dT_v^+}{d\eta}$ and $\dfrac{dT_l^+}{d\eta}$ respectively, Equation G-10 becomes

$$q''_{rad} - \left\{ \left(\frac{q''_w}{7\eta^6} \frac{1}{\Pr_v} \right) \left[(c_1 + 2c_2\eta) - \left(\frac{k_l}{k_v} \right) (d_1 + 2d_2\eta) \right] \right\}_{\eta=(\delta^+)^{1/7}} = \dot{m}''_l h_{fg}$$

Rearranging

$$\left[c_1 + 2c_2 (\delta^+)^{1/7} \right] - \left(\frac{k_l}{k_v} \right) \left[d_1 + 2d_2 (\delta^+)^{1/7} \right] = \frac{7 (\delta^+)^{6/7} \Pr_v \left(q''_{rad} - \dot{m}''_l h_{fg} \right)}{q''_w} \tag{G-11}$$

Equations G-5, G-6 and G-11 represent the system of equations that need to be solved to obtained d_0, d_1 and d_2.

Using Equations F-10, F-11 and G-6, d_2 can be obtained as

$$d_2 = \frac{\left(k_v / k_l \right)}{2\left[(\delta^+)^{1/7} - (R^+)^{1/7} \right]} \left\{ \frac{2T_{sat}^+ + 25\Pr_v - 7\Pr_v (\delta^+)^{1/7} (5)^{6/7}}{\left[(\delta^+)^{1/7} - (5)^{1/7} \right]} + \frac{7\Pr_v (\delta^+)^{6/7}}{q''_w} \left[\dot{m}''_l h_{fg} - q''_{rad} \right] \right\} \tag{G-12}$$

Using Equation G-6, d_1 can be obtained as

$$d_1 = -2d_2\left(R^+\right)^{1/7}$$

$$d_1 = \frac{-\left(R^+\right)^{1/7}\left(k_v/k_l\right)}{\left[\left(\delta^+\right)^{1/7} - \left(R^+\right)^{1/7}\right]}\left\{\frac{2T_{sat}^+ + 25\,\mathrm{Pr}_v - 7\,\mathrm{Pr}_v\left(\delta^+\right)^{1/7}(5)^{6/7}}{\left[\left(\delta^+\right)^{1/7} - (5)^{1/7}\right]} + \frac{7\,\mathrm{Pr}\left(\delta^+\right)^{6/7}}{q''_w}\left[\dot{m}''_l h_{fg} - q''_{rad}\right]\right\} \quad \text{(G-13)}$$

Solving for d_0 using Equation G-5 yields

$$d_0 = T_{sat}^+ - \left(\delta^+\right)^{1/7}\left(\frac{k_v}{k_l}\right)\left\{\frac{2T_{sat}^+ + 25\,\mathrm{Pr}_v - 7\,\mathrm{Pr}_v\left(\delta^+\right)^{1/7}(5)^{6/7}}{\left[\left(\delta^+\right)^{1/7} - (5)^{1/7}\right]} + \frac{7\,\mathrm{Pr}\left(\delta^+\right)^{6/7}}{q''_w}\left[\dot{m}''_l h_{fg} - q''_{rad}\right]\right\} \quad \text{(G-14)}$$

Equations G-12, G-13 and G-14 give the coefficients d_0, d_1 and d_2 respectively. These are not constants, but functions of the axial coordinate, z alone. Derivatives of the axial component of the vapor velocity, $\overline{T_l}$ with respect to z and r occur in the integrated form of the governing conservation equations.

In Equations G-7 to G-14, term q''_{rad} represents the radiation heat flux from the wall to the interface. For a rod bundle, the radiation heat transfer rate is given by

$$q''_{rad} = \frac{\sigma\left(T_w^4 - T_{sat}^4\right)}{\frac{1}{\varepsilon_w} + \frac{P_w}{P_i}\left(\frac{1}{\varepsilon_l} - 1\right)} \quad \text{(G-15)}$$

The radiation absorption in the vapor film is neglected. A value of $\varepsilon_l = 0.96$ is used for the water emittance (assumed to be equal to the absorbance), according to Eckert's recommendation for 0.1 mm or more thick water (in Rohsenow and Hartnett, 1973 pp. 15-23). Wall emittance of Inconel 600 is given by the following relation:

$$\varepsilon_w = 1.979\times10^{-4}T_w + 0.5735 \quad \text{(G-16)}$$

where T_w is in deg K. It is a linear approximation between 580 K and 1260 K, of the values recommended by Kawaji (1984, pp. 31), based on a report from *The International Nickel Company, Inc.*

Differentiating Equation G-15 with respect to z and noting that the interfacial perimeter is also a function of z, as it is a function of the vapor film thickness, δ yields

$$\frac{dq''_{rad}}{dz} = \frac{4\sigma T_w^3}{\left[\frac{1}{\varepsilon_w} + \frac{P_w}{P_i}\left(\frac{1}{\varepsilon_l} - 1\right)\right]} \frac{dT_w}{dz} - \frac{q''_{rad}}{\left[\frac{1}{\varepsilon_w} + \frac{P_w}{P_i}\left(\frac{1}{\varepsilon_l} - 1\right)\right]} \frac{R}{(R-\delta)^2} \left(\frac{1}{\varepsilon_l} - 1\right) \frac{d\delta}{dz} \tag{G-17}$$

G.1 Gradient in the Radial Direction

Rearranging Equation G-1 and differentiating with respect to r gives

$$\frac{\partial \overline{T}_l}{\partial r} = \frac{\partial T_w}{\partial r} + \left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right) \frac{\partial T_l^+}{\partial r} \tag{G-18}$$

Wall temperature, wall heat flux, friction velocity are not functions of y (or r), hence they can be treated as constants. Using Equation G-1, the above relation becomes

$$\frac{\partial \overline{T}_l}{\partial r} = \left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right) \frac{\partial}{\partial r}\left(d_0 + d_1\eta + d_2\eta^2\right)$$

The coefficients d_0, d_1 and d_2 are functions of z alone, hence

$$\frac{\partial \overline{T}_l}{\partial r} = \left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right)\left(d_1 \frac{\partial \eta}{\partial r} + 2d_2\eta \frac{\partial \eta}{\partial r}\right) \tag{G-19}$$

Substituting for $\frac{\partial \eta}{\partial r}$ from Equation B-19, we get

- $$\frac{\partial \overline{T}_l}{\partial r} = -\left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right)(d_1 + 2d_2\eta)\left(\frac{\eta}{7y}\right) \qquad \text{(G-20)}$$

G.2 Gradient in the Axial Direction

Rearranging Equation G-1 and differentiating with respect to z gives

$$\frac{\partial \overline{T}_l}{\partial z} = \frac{\partial T_w}{\partial z} + \frac{\partial}{\partial z}\left[\left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right) T_l^+\right] \qquad \text{(G-21)}$$

Substituting for the non-dimensional temperature profile from Equation G-1 and proceeding in the same manner as was done for the turbulent core of the vapor film, $\frac{\partial \overline{T}_l}{\partial z}$ can be obtained as

$$\frac{\partial \overline{T}_l}{\partial z} = \frac{dT_w}{dz} + \left(\frac{1}{\rho_v c_{p,v}}\right)\left\{\begin{aligned} &\frac{q''_w}{u_v^*}\left[\frac{dd_0}{dz} + \eta\frac{dd_1}{dz} + \eta^2\frac{dd_2}{dz} + c_1\frac{\partial \eta}{\partial z} + 2c_2\eta\frac{\partial \eta}{\partial z}\right] \\ &+ \frac{T_l^+}{u_v^*}\frac{dq''_w}{dz} - \frac{T_l^+ q''_w}{\left(u_v^*\right)^2}\frac{du_v^*}{dz} \end{aligned}\right\} \qquad \text{(G-22)}$$

The above equation requires derivatives of the coefficients d_0, d_1 and d_2 with respect to z. Using Equations B-20, B-27 and the derivatives of the coefficients d_0, d_1 and d_2 with respect to z, $\frac{\partial \overline{T}_l}{\partial z}$ can be obtained.

The above equation requires derivatives of the coefficients d_0, d_1 and d_2 with respect to z. Using Equation F-21 and other definitions, these are evaluated below.

$$\frac{dd_2}{dz}=\frac{k_v}{2k_l}\left\{\begin{array}{l}\left[\begin{array}{l}\dfrac{5\Pr_v(\delta^+)^{6/7}(\dot{m}_l''h_{fg}-q_{rad}'')}{2\tau_w(q_w'')\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]}-\dfrac{\left[2T_{sat}^+ +25\Pr_v-7\Pr_v(\delta^+)^{1/7}(5)^{6/7}\right]}{14\tau_w\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]\left[(\delta^+)^{1/7}-(5)^{1/7}\right]}\\ -\dfrac{\left[2T_{sat}^+ +25\Pr_v-7\Pr_v(\delta^+)^{1/7}(5)^{6/7}\right](\delta^+)^{1/7}}{14\tau_w\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]\left[(\delta^+)^{1/7}-(5)^{1/7}\right]^2}\\ +\left[\dfrac{T_{sat}^+}{\tau_w}-\dfrac{(5)^{6/7}(\delta^+)^{1/7}\Pr_v}{2\tau_w}\right]\dfrac{1}{\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]\left[(\delta^+)^{1/7}-(5)^{1/7}\right]}\end{array}\right]\dfrac{d\tau_w}{dz}\\ +\left[\begin{array}{l}\dfrac{6\Pr_v(\delta^+)^{6/7}(\dot{m}_l''h_{fg}-q_{rad}'')}{\delta q_w''\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]}-\dfrac{(5)^{6/7}(\delta^+)^{1/7}\Pr_v}{\delta\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]\left[(\delta^+)^{1/7}-(5)^{1/7}\right]}\\ -\left(\dfrac{2k_l}{k_v}\right)\dfrac{d_2(\delta^+)^{1/7}}{7\delta\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]}-\dfrac{\left[2T_{sat}^+ +25\Pr_v-7\Pr_v(\delta^+)^{1/7}(5)^{6/7}\right](\delta^+)^{1/7}}{7\delta\left[(\delta^+)^{1/7}-(5)^{1/7}\right]\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]^2}\end{array}\right]\dfrac{d\delta}{dz}\\ -\left[\dfrac{2\rho_v c_{p,v}u_v^*}{q_w''\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]\left[(\delta^+)^{1/7}-(5)^{1/7}\right]}\right]\dfrac{dT_w}{dz}+\left[\dfrac{7\Pr_v(\delta^+)^{6/7}h_{fg}}{\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]q_w''}\right]\dfrac{d\dot{m}_l''}{dz}\\ -\left[\dfrac{7\Pr_v(\delta^+)^{6/7}}{q_w''\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]}\right]\dfrac{dq_{rad}''}{dz}\\ -\left[\dfrac{7\Pr_v(\delta^+)^{6/7}(\dot{m}_l''h_{fg}-q_{rad}'')}{(q_w'')^2\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]}+\dfrac{2T_{sat}^+}{q_w''\left[(\delta^+)^{1/7}-(R^+)^{1/7}\right]\left[(\delta^+)^{1/7}-(5)^{1/7}\right]}\right]\dfrac{dq_w''}{dz}\end{array}\right\} \qquad \text{(G-23)}$$

Using Equation G-17 for the derivative of the radiation heat flux in the above equation yields the expression for $\frac{dd_2}{dz}$

Differentiating Equation G-6 gives

$$\frac{dd_1}{dz} = -2\left(R^+\right)^{1/7}\frac{dd_2}{dz} - \frac{2}{7}d_2\left(R^+\right)^{-6/7}\frac{dR^+}{dz}$$

which on simplification using Equation C-16 becomes

$$\frac{dd_1}{dz} = \left(\frac{d_1}{14\tau_w}\right)\frac{d\tau_w}{dz} - 2\left(R^+\right)^{1/7}\frac{dd_2}{dz} \tag{G-24}$$

Substituting from Equation G-24 for $\frac{dd_2}{dz}$ gives an equation for $\frac{dd_1}{dz}$

Differentiating Equation D-5 gives

$$\frac{dd_0}{dz} = \left\{ \begin{aligned} &\frac{d\left(T_{sat}^+\right)}{dz} + 2\left(\delta^+\right)^{1/7}\left[\left(R^+\right)^{1/7} - \left(\delta^+\right)^{1/7}\right]\frac{dd_2}{dz} \\ &+ \frac{2\left(\delta^+\right)^{1/7}d_2}{7}\left[\left(R^+\right)^{-6/7}\frac{dR^+}{dz} - \left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}\right] + \frac{2\left[\left(R^+\right)^{1/7} - \left(\delta^+\right)^{1/7}\right]d_2}{7}\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz} \end{aligned} \right\} \tag{G-25}$$

Using Equations B-20, B-27, B-29, F-22, G-17 and G-23 in Equation G-22, $\frac{\partial \overline{T}_l}{\partial z}$ can be obtained.

Appendix H

Integrated Vapor Momentum Equations

There are a total of four momentum equations for the vapor film, which is split into wall region and the fully turbulent core. The wall region extends from $y = 0 \left(y^+ = 0\right)$ to $y^+ = 5$. The turbulent core is assumed to extend from $y^+ = 5$ to $y = \delta \left(y^+ = \delta^+\right)$.

$l = \left(\dfrac{5\nu_v}{u_v^*}\right)$ has been defined in Appendix A.

Assuming constant properties, using Equation B-27 and the definition of friction velocity from Equation B-1 gives

$$\frac{dl}{dz} = -\left(\frac{5\nu_v}{u_v^*}\right)\frac{1}{2\tau_w}\frac{d\tau_w}{dz} \tag{H-1}$$

The four equations derived in Appendix A are considered separately here.

H.1 Z direction Momentum Equations

The Z-direction momentum equation for the turbulent core and the wall region of the vapor film are now considered. Using the definitions of the non-dimensional velocities, each term in the integrated equation is written in terms of the primary independent variables.

H.1.1 Turbulent Core

The integrated form of the Z-direction vapor momentum equation for the turbulent core is given by

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\rho_v\bar{u}_v\bar{u}_v)dr\right]+(r\rho_v\bar{u}_v\bar{u}_v)\Big|_{R-l}\frac{dl}{dz}-(r\rho_v\bar{u}_v\bar{u}_v)\Big|_{R-\delta}\frac{d\delta}{dz}+\left\{(r\rho_v\bar{u}_v\bar{v}_v)\Big|_{R-l}-(r\rho_v\bar{u}_v\bar{v}_v)\Big|_{R-\delta}\right\}=$$

Term1 Term2 Term3 Term4 Term5

$$-\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\bar{P})dr\right]-(r\bar{P})\Big|_{R-l}\frac{dl}{dz}+(r\bar{P})\Big|_{R-\delta}\frac{d\delta}{dz}+\left\{\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial r}\right)\Bigg|_{R-l}-\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial r}\right)\Bigg|_{R-\delta}\right\} \quad \text{(A-17)}$$

Term6 Term7 Term8 Term9 Term10

$$-\frac{g\rho_v}{2}\left[(R-l)^2-(R-\delta)^2\right]+\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial z}\right)dr\right]+\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial z}\right)\Bigg|_{R-l}\frac{dl}{dz}-\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{u}_v}{\partial z}\right)\Bigg|_{R-\delta}\frac{d\delta}{dz}$$

Term11 Term12 Term13 Term14

The turbulent core extends from

$y^+ = 5$, which corresponds to $y = l = \frac{5\nu_v}{u_v^*}, r = R - \frac{5\nu_v}{u_v^*}, \eta = (5)^{1/7}$ (H-2)

$y = \delta$, which corresponds to $y^+ = \delta^+, r = R - \delta, \eta = (\delta^+)^{1/7}$ (H-3)

Term 1:

Using Equations B-1 and B-5 gives

$$\bar{u}_v = (u_v^+)(u_v^*) = (u_v^*)(a_0 + a_1\eta + a_2\eta^2)$$

Thus, $\bar{u}_v\bar{u}_v = (u_v^*)^2\left(a_0^2 + 2a_0a_1\eta + a_1^2\eta^2 + 2a_0a_2\eta^2 + 2a_1a_2\eta^3 + a_2^2\eta^4\right)$

Substituting the above in the expression for Term 1, using the definition for friction velocity and assuming constant properties and integrating using Equations H-2 and H-3 for the limits gives

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\rho_v\bar{u}_v\bar{u}_v)dr\right]=\frac{d}{dz}\left[\tau_w\left\{\begin{array}{l}\frac{a_0^2}{2}\left[(R-l)^2-(R-\delta)^2\right]\\+14a_0a_1\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-(\delta^+)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right]\\+7(a_1^2+2a_0a_2)\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-(\delta^+)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right]\\+14a_1a_2\left[(5)^{3/7}(l)\left(\frac{l}{17}-\frac{R}{10}\right)-(\delta^+)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)\right]\\+7a_2^2\left[(5)^{4/7}(l)\left(\frac{l}{18}-\frac{R}{11}\right)-(\delta^+)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)\right]\end{array}\right\}\right]$$

In the above expression, the coefficients a_0, a_1, a_2, the friction velocity and the vapor film thickness are all functions of z. Differentiating the above expression gives an equation for Term 1. This is given in Equation H-4.

$$
\left\{\begin{array}{l}
\frac{a_0^2}{2}\left[(R-l)^2-(R-\delta)^2\right]+14a_0a_1\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right] \\
+7\left(a_1^2+2a_0a_2\right)\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right] \\
+14a_1a_2\left[(5)^{3/7}(l)\left(\frac{l}{17}-\frac{R}{10}\right)-\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)\right]+ \\
7a_2^2\left[(5)^{4/7}(l)\left(\frac{l}{18}-\frac{R}{11}\right)-\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)\right]
\end{array}\right\}\frac{d\tau_w}{dz}+
$$

$$
\tau_w\left\{\begin{array}{l}
\left[(R-l)^2-(R-\delta)^2\right]a_0\frac{da_0}{dz}+a_0^2\left[(R-\delta)\frac{d\delta}{dz}-(R-l)\frac{dl}{dz}\right] \\
+14\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right]\left(a_0\frac{da_1}{dz}+a_1\frac{da_0}{dz}\right)+ \\
14a_0a_1\left[5^{1/7}\left(\frac{2l}{15}-\frac{R}{8}\right)\frac{dl}{dz}-\left(\delta^+\right)^{1/7}\left(\frac{2\delta}{15}-\frac{R}{8}\right)\frac{d\delta}{dz}-\frac{1}{7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}\right] \\
+14\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right]\left(a_1\frac{da_1}{dz}+a_2\frac{da_0}{dz}+a_0\frac{da_2}{dz}\right)+ \\
7\left(a_1^2+2a_0a_2\right)\left[(5)^{2/7}\left(\frac{2l}{16}-\frac{R}{9}\right)\frac{dl}{dz}-\left(\delta^+\right)^{2/7}\left(\frac{2\delta}{16}-\frac{R}{9}\right)\frac{d\delta}{dz}-\frac{2}{7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}\right] \\
+14\left[(5)^{3/7}(l)\left(\frac{l}{17}-\frac{R}{10}\right)-\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)\right]\left(a_2\frac{da_1}{dz}+a_1\frac{da_2}{dz}\right)+ \\
14a_1a_2\left[(5)^{3/7}\left(\frac{2l}{17}-\frac{R}{10}\right)\frac{dl}{dz}-\left(\delta^+\right)^{3/7}\left(\frac{2\delta}{17}-\frac{R}{10}\right)\frac{d\delta}{dz}-\frac{3}{7}\left(\frac{\delta^2}{17}-\frac{R\delta}{10}\right)\left(\delta^+\right)^{-4/7}\frac{d\delta^+}{dz}\right] \\
+14\left[(5)^{4/7}(l)\left(\frac{l}{18}-\frac{R}{11}\right)-\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)\right]\left(a_2\frac{da_2}{dz}\right)+ \\
7a_2^2\left[(5)^{4/7}\left(\frac{2l}{18}-\frac{R}{11}\right)\frac{dl}{dz}-\left(\delta^+\right)^{4/7}\left(\frac{2\delta}{18}-\frac{R}{11}\right)\frac{d\delta}{dz}-\frac{4}{7}\left(\frac{\delta^2}{18}-\frac{R\delta}{11}\right)\left(\delta^+\right)^{-3/7}\frac{d\delta^+}{dz}\right]
\end{array}\right\} \quad \text{(H-4)}
$$

Term 2:

Using Equations B-1 and B-5 gives

$$\overline{u}_v=\left(u_v^+\right)\left(u_v^*\right)=\left(u_v^*\right)\left(a_0+a_1\eta+a_2\eta^2\right)$$

Evaluating the above at $y = l = \frac{5\nu_v}{u_v^*}, r = R - \frac{5\nu_v}{u_v^*}, \eta = (5)^{1/7}$ and invoking the definition for friction velocity, Term 2 becomes

$$\left(r\rho_v \bar{u}_v \bar{u}_v\right)\Big|_{R-l} \frac{dl}{dz} = \left(R - \frac{5\nu_v}{u_v^*}\right)\tau_w \left[a_0 + a_1 (5)^{1/7} + a_2 (5)^{2/7}\right]^2 \frac{dl}{dz}$$

Using Equation H-1 in the above gives Term 2 as

$$\left(r\rho_v \bar{u}_v \bar{u}_v\right)\Big|_{R-l} \frac{dl}{dz} = -\frac{1}{2}\left(R - \frac{5\nu_v}{u_v^*}\right)\left(\frac{5\nu_v}{u_v^*}\right)\left[\begin{array}{l} a_0^2 + 2a_0 a_1 (5)^{1/7} + \\ \left(a_1^2 + 2a_0 a_2\right)(5)^{2/7} + 2a_1 a_2 (5)^{3/7} + a_2^2 (5)^{4/7} \end{array}\right]\frac{d\tau_w}{dz} \tag{H-5}$$

Term 3:

Using Equations B-1 and B-5 gives

$$\bar{u}_v = \left(u_v^+\right)\left(u_v^*\right) = \left(u_v^*\right)\left(a_0 + a_1\eta + a_2\eta^2\right)$$

Evaluating the above at $y^+ = \delta^+, r = R - \delta, \eta = \left(\delta^+\right)^{1/7}$ and invoking the definition for friction velocity, Term 3 becomes

$$\tau_w (R - \delta)\left[a_0^2 + 2a_0 a_1 \left(\delta^+\right)^{1/7} + \left(a_1^2 + 2a_0 a_2\right)\left(\delta^+\right)^{2/7} + 2a_1 a_2 \left(\delta^+\right)^{3/7} + a_2^2 \left(\delta^+\right)^{4/7}\right]\frac{d\delta}{dz} \tag{H-6}$$

Term 4:

Using the definition of axial component of vapor velocity from Equation B-5 and the radial component of the vapor velocity from Equation D-2, along with the definitions from Equation B-1 yields

$$\left(r\rho_v \bar{u}_v \bar{v}_v\right)\Big|_{R-l} = \rho_v \left(u_v^*\right)^2 \left[u_v^+ \left(r v_v^+\right)\right]_{R-l} = \tau_w \left[\left(a_0 + a_1\eta + a_2\eta^2\right)\left(r v_v^+\right)\right]_{R-l}$$

The radial component of the vapor velocity was derived in Equation D-13. Substituting in the above and using Equation H-2 yields

$$\left(r\rho_v\bar{u}_v\bar{v}_v\right)\Big|_{R-l} =$$
$$\tau_w\left[a_0 + a_1(5)^{1/7} + a_2(5)^{2/7}\right]$$
$$\left\{\begin{array}{l} K_4(z) - \dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{a_0}{2}\left(R - \dfrac{5\nu_v}{u_v^*}\right)^2 + 8(5)^{1/7}a_1\left(\dfrac{5\nu_v}{u_v^*}\right)\left(\dfrac{5\nu_v}{15u_v^*} - \dfrac{R}{8}\right) + 9a_2(5)^{2/7}\left(\dfrac{5\nu_v}{u_v^*}\right)\left(\dfrac{5\nu_v}{16u_v^*} - \dfrac{R}{9}\right)\right] \\ -(5)^{2/7}\dfrac{da_2}{dz}\left[7\left(\dfrac{5\nu_v}{u_v^*}\right)\left(\dfrac{5\nu_v}{16u_v^*} - \dfrac{R}{9}\right) - 14\left(\dfrac{5\nu_v}{u_v^*}\right)\left(\dfrac{5\nu_v}{15u_v^*} - \dfrac{R}{8}\right) + \dfrac{1}{2}\left(R - \dfrac{5\nu_v}{u_v^*}\right)^2\right] \end{array}\right\}$$

Substituting for $K_4(z)$ from Equation D-14 and simplifying yields

$$\left(r\rho_v\bar{u}_v\bar{v}_v\right)\Big|_{R-l} = \left[a_0 + a_1(5)^{1/7} + a_2(5)^{2/7}\right]\frac{u_v^* l^2}{\nu_v}\left(\frac{R}{2} - \frac{l}{3}\right)\frac{d\tau_w}{dz} \tag{H-7}$$

Term 5:

Using the definition of axial component of vapor velocity from Equation B-5 and the radial component of the vapor velocity from Equation D-2, along with the definitions from Equation B-1 yields

$$\left(r\rho_v\bar{u}_v\bar{v}_v\right)\Big|_{R-\delta} = \rho_v\left(u_v^*\right)^2\left[u_v^+\left(rv_v^+\right)\right]\Big|_{R-\delta} = \tau_w\left[\left(a_0 + a_1\eta + a_2\eta^2\right)\left(rv_v^+\right)\right]\Big|_{R-\delta}$$

The radial component of the vapor velocity was derived in Equation D-13. Substituting in the above and using Equation H-3 yields

$$\left(r\rho_v \overline{u}_v \overline{v}_v\right)\Big|_{R-\delta} =$$

$$\tau_w \left[a_0 + a_1 \left(\delta^+\right)^{1/7} + a_2 \left(\delta^+\right)^{2/7} \right] \tag{H-8}$$

$$\left\{ \begin{array}{l} K_4(z) - \dfrac{1}{2\tau_w} \dfrac{d\tau_w}{dz} \left[\dfrac{a_0}{2}(R-\delta)^2 + 8a_1\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15} - \dfrac{R}{8}\right) + 9a_2\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16} - \dfrac{R}{9}\right) \right] \\ -\dfrac{da_2}{dz}\left[7\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16} - \dfrac{R}{9}\right) - 14\left(\delta^+\right)^{1/7}(5)^{1/7}(\delta)\left(\dfrac{\delta}{15} - \dfrac{R}{8}\right) + \dfrac{(5)^{2/7}}{2}(R-\delta)^2 \right] \end{array} \right\}$$

Substituting for $K_4(z)$ from Equation D-14 in the above yields an expression for Term 5.

Term 6:

Assuming pressure to be a weak function of the radial direction, performing integration and using Equations H-2 and H-3 gives

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l} \left(r\overline{P}\right) dr \right] = \frac{d}{dz}\left[\overline{P} \int_{R-\delta}^{R-l} (r) dr \right] = \frac{d}{dz}\left[\overline{P} \, \frac{(R-l)^2 - (R-\delta)^2}{2} \right]$$

Differentiating with respect to z gives

$$\left[\frac{(R-l)^2 - (R-\delta)^2}{2} \right] \frac{d\overline{P}}{dz} + \overline{P}\left[(R-\delta)\frac{d\delta}{dz} - (R-l)\frac{dl}{dz} \right]$$

Substituting from Equation H-1, Term 6 becomes

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l} \left(r\overline{P}\right) dr \right] = \left[\frac{(R-l)^2 - (R-\delta)^2}{2} \right] \frac{d\overline{P}}{dz} + \overline{P}\left[(R-\delta)\frac{d\delta}{dz} + (R-l)\frac{l}{2\tau_w}\frac{d\tau_w}{dz} \right] \tag{H-9}$$

Term 7:

Using Equations H-1 and H-2 yields

$$\left(r\overline{P}\right)_{R-l}\frac{dl}{dz}=-\left(R-\frac{5\nu_v}{u_v^*}\right)\left(\frac{5\nu_v}{u_v^*}\right)\frac{\overline{P}}{2\tau_w}\frac{d\tau_w}{dz} \tag{H-10}$$

Term 8:

Using Equation H-3 yields

$$\left(r\overline{P}\right)_{R-\delta}\frac{d\delta}{dz}=\left(R-\delta\right)\overline{P}\frac{d\delta}{dz} \tag{H-11}$$

Term 9:

Using Equation B-21 for the gradient of the vapor axial velocity with respect to r along with Equation H-2, Term 9 becomes

$$\left(r\rho_v\varepsilon_{m,v}\frac{\partial\overline{u}_v}{\partial r}\right)\Bigg|_{R-l}=-\frac{\rho_v u_v^*\varepsilon_{m,v}}{7\left(\frac{5\nu_v}{u_v^*}\right)}\left(R-\frac{5\nu_v}{u_v^*}\right)\left[a_1(5)^{1/7}+2a_2(5)^{2/7}\right] \tag{H-12}$$

Term 10

Using Equation B-21 for the gradient of the vapor axial velocity with respect to r along with Equation H-3, Term 10 becomes

$$\left(r\rho_v\varepsilon_{m,v}\frac{\partial\overline{u}_v}{\partial r}\right)\Bigg|_{R-\delta}=-\frac{\rho_v u_v^*\varepsilon_{m,v}\left(R-\delta\right)}{7\delta}\left[a_1\left(\delta^+\right)^{1/7}+2a_2\left(\delta^+\right)^{2/7}\right] \tag{H-13}$$

Term 11:

$$\frac{g\rho_v}{2}\left[\left(R-l\right)^2-\left(R-\delta\right)^2\right] \tag{H-14}$$

Term 12:

Equation B-33 gives the gradient of $\overline{u}_v$ in the z direction. Assuming constant properties, a constant value of the eddy diffusivity of momentum and integrating using Equation H-2 and H-3 for the limits gives

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\varepsilon_{m,v}\frac{\partial\overline{u}_v}{\partial z}\right)dr\right]=$$

$$\frac{d}{dz}\left[\rho_v\varepsilon_{m,v}\left\{\begin{array}{l}\dfrac{1}{2\sqrt{\rho_v\tau_w}}\dfrac{d\tau_w}{dz}\left\{\begin{array}{l}\left[\dfrac{(R-l)^2-(R-\delta)^2}{2}\right]a_0+\\ 7a_1\left[(5)^{1/7}(l)\left(\dfrac{l}{15}-\dfrac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)\right]+\\ 7a_2\left[(5)^{2/7}(l)\left(\dfrac{l}{16}-\dfrac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\end{array}\right\}\\ +\dfrac{1}{14\sqrt{\rho_v\tau_w}}\dfrac{d\tau_w}{dz}\left\{\begin{array}{l}7a_1\left[(5)^{1/7}(l)\left(\dfrac{l}{15}-\dfrac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)\right]+\\ 14a_2\left[(5)^{2/7}(l)\left(\dfrac{l}{16}-\dfrac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\end{array}\right\}\\ +u_v^*\dfrac{da_2}{dz}\left\{\begin{array}{l}(5)^{2/7}\left[\dfrac{(R-l)^2-(R-\delta)^2}{2}\right]\\ +7\left[(5)^{2/7}(l)\left(\dfrac{l}{16}-\dfrac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\\ -14(5)^{1/7}\left[(5)^{1/7}(l)\left(\dfrac{l}{15}-\dfrac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)\right]\end{array}\right\}\end{array}\right\}\right]$$

Simplifying gives

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\varepsilon_{m,v}\frac{\partial\overline{u}_v}{\partial z}\right)dr\right]=\frac{d}{dz}\left[\rho_v\varepsilon_{m,v}\left\{\begin{array}{l}\dfrac{1}{2\sqrt{\rho_v\tau_w}}\dfrac{d\tau_w}{dz}\left\{\begin{array}{l}\left[\dfrac{(R-l)^2-(R-\delta)^2}{2}\right]a_0+\\ 8a_1\left[(5)^{1/7}(l)\left(\dfrac{l}{15}-\dfrac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)\right]+\\ 9a_2\left[(5)^{2/7}(l)\left(\dfrac{l}{16}-\dfrac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\end{array}\right\}\\ +u_v^*\dfrac{da_2}{dz}\left\{\begin{array}{l}(5)^{2/7}\left[\dfrac{(R-l)^2-(R-\delta)^2}{2}\right]\\ +7\left[(5)^{2/7}(l)\left(\dfrac{l}{16}-\dfrac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\\ -14(5)^{1/7}\left[(5)^{1/7}(l)\left(\dfrac{l}{15}-\dfrac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)\right]\end{array}\right\}\end{array}\right\}\right]$$

In the above expression, the coefficients a_0, a_1, a_2, the friction velocity, wall shear stress and the vapor film thickness are all functions of z. Differentiating the above expression gives an equation for Term 12.

Term 13:

Equation B-33 gives the gradient of $\overline{u}_v$ in the z direction. Substituting Equations B-33 and H-1 in Term 13, and evaluating using Equation H-2 yields

$$\left.\left(r\rho_v\varepsilon_{m,v}\frac{\partial\overline{u}_v}{\partial z}\right)\right|_{R-l}\frac{dl}{dz}=-\rho_v\varepsilon_{m,v}\left(\frac{5\nu_v}{u_v^*}\right)\left(R-\frac{5\nu_v}{u_v^*}\right)\left(\frac{1}{2\tau_w}\right)\left(\frac{d\tau_w}{dz}\right)^2\left\{\begin{array}{l}\dfrac{a_0+a_1(5)^{1/7}+a_2(5)^{2/7}}{2\sqrt{\tau_w\rho_v}}\\ +\dfrac{a_1(5)^{1/7}+2a_2(5)^{2/7}}{14\sqrt{\tau_w\rho_v}}\end{array}\right\} \qquad \text{(H-16)}$$

Term 14:

Using Equation B-33 in Term 14 and evaluating using Equation H-3 yields

$$\left.\left(r\rho_v\varepsilon_{m,v}\frac{\partial\overline{u}_v}{\partial z}\right)\right|_{R-\delta}\frac{d\delta}{dz}=\rho_v\varepsilon_{m,v}\left(R-\delta\right)\frac{d\delta}{dz}\left\{\begin{matrix}\left[u_v^*\left[\left(\delta^+\right)^{1/7}-\left(5\right)^{1/7}\right]^2\dfrac{da_2}{dz}\right]+\\\left[\dfrac{a_0+a_1\left(\delta^+\right)^{1/7}+a_2\left(\delta^+\right)^{2/7}}{2\sqrt{\tau_w\rho_v}}+\dfrac{a_1\left(\delta^+\right)^{1/7}+2a_2\left(\delta^+\right)^{2/7}}{14\sqrt{\tau_w\rho_v}}\right]\dfrac{d\tau_w}{dz}\end{matrix}\right\} \quad \text{(H-17)}$$

H.1.2 Wall Region

The integrated form of the Z-direction vapor momentum equation for the wall region is given by

$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\bar{u}_v\bar{u}_v\right)dr\right]-\left(r\rho_v\bar{u}_v\bar{u}_v\right)\Big|_{R-l}\frac{dl}{dz}-\left(r\rho_v\bar{u}_v\bar{v}_v\right)\Big|_{R-l}=$$

Term 1 Term 2 Term 3

$$-\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\bar{P}\right)dr\right]+\left(r\bar{P}\right)\Big|_{R-l}\frac{dl}{dz}-\frac{g\rho_v}{2}\left[2Rl-l^2\right]+\left\{\left(r\rho_v\nu_v\frac{\partial\bar{u}_v}{\partial r}\right)\Big|_R-\left(r\rho_v\nu_v\frac{\partial\bar{u}_v}{\partial r}\right)\Big|_{R-l}\right\} \tag{A-23}$$

Term 4 Term 5 Term 6 Term 7 Term 8

The wall region extends from

$$y=0 \text{ which corresponds to } y^+=0, r=R, \eta=0 \tag{H-18}$$

$$y^+=5 \text{ which corresponds to } y=l=\frac{5\nu_v}{u_v^*}, r=R-\frac{5\nu_v}{u_v^*}, \eta=(5)^{1/7} \tag{H-19}$$

Term 1:

Using the definitions from Equation B-1 and the dimensionless vapor velocity profile for the wall region from Equation B-4, gives

$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\bar{u}_v\bar{u}_v\right)dr\right]=\frac{d}{dz}\left[\tau_w\int_{R-l}^{R}\left(y^+\right)^2 rdr\right]=\frac{d}{dz}\left[\tau_w\left(\frac{u_v^*}{\nu_v}\right)^2\int_{R-l}^{R}(y)^2 rdr\right]$$

Integrating the above expression and applying the limits using Equations H-18 and H-19 yields

$$\frac{d}{dz}\left[\tau_w\left(\frac{u_v^*}{\nu_v}\right)^2\int_{R-l}^{R}(y)^2 rdr\right]=\frac{d}{dz}\left[\frac{125R\tau_w\nu_v}{3u_v^*}-\frac{625\tau_w}{4}\left(\frac{\nu_v}{u_v^*}\right)^2\right]$$

Using the definition of friction velocity, assuming constant properties and differentiating yields

$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\bar{u}_v\bar{u}_v\right)dr\right]=\frac{125}{6}\left(\frac{R\nu_v}{u_v^*}\right)\frac{d\tau_w}{dz} \tag{H-20}$$

Term 2:

Using the definitions from Equation B-1 and the dimensionless axial velocity profile for the wall region from Equation B-4 gives

$$\left(r\rho_v\bar{u}_v\bar{u}_v\right)\Big|_{R-l}\frac{dl}{dz}=\tau_w\left(ry^+y^+\right)\Big|_{R-l}\frac{dl}{dz}$$

Using Equation H-1 and evaluating the above using H-19 yields

$$\left(r\rho_v\bar{u}_v\bar{u}_v\right)\Big|_{R-l}\frac{dl}{dz}=-\frac{25}{2}\left(\frac{5\nu_v}{u_v^*}\right)\left(R-\frac{5\nu_v}{u_v^*}\right)\frac{d\tau_w}{dz} \tag{H-21}$$

Term 3:

Using the definitions from Equation B-1 and D-2 gives

$$\left(r\rho_v\bar{u}_v\bar{v}_v\right)\Big|_{R-l}=\tau_w\left(ru_v^+v_v^+\right)\Big|_{R-l}$$

Using the dimensionless axial velocity profile for the wall region from Equation B-4, the dimensionless radial velocity profile from Equation D-10 and evaluating the above using Equation H-19 gives

$$\left(r\rho_v\bar{u}_v\bar{v}_v\right)\Big|_{R-l}=25\left(\frac{5\nu_v}{u_v^*}\right)\left(\frac{R}{2}-\frac{5\nu_v}{3u_v^*}\right)\frac{d\tau_w}{dz} \tag{H-22}$$

Term 4:

Assuming pressure to be a weak function of the radial direction and performing integration and using Equations H-18 and H-19 gives

$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\overline{P}\right)dr\right]=\frac{d}{dz}\left[\overline{P}\int_{R-l}^{R}rdr\right]=\frac{d}{dz}\left\{\overline{P}\left[\frac{R^2-\left(R-l\right)^2}{2}\right]\right\}$$

Differentiating with respect to z gives

$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\overline{P}\right)dr\right]=\left[\frac{R^2-\left(R-l\right)^2}{2}\right]\frac{d\overline{P}}{dz}+\overline{P}\left(R-l\right)\frac{dl}{dz}$$

Using Equation H-1 gives

$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\overline{P}\right)dr\right]=\left[\frac{R^2-\left(R-l\right)^2}{2}\right]\frac{d\overline{P}}{dz}-\overline{P}\left(R-l\right)\left(\frac{5\nu_v}{u_v^*}\right)\frac{1}{2\tau_w}\frac{d\tau_w}{dz} \qquad \text{(H-23)}$$

Term 5:

Using Equations H-1 and H-2 yields

$$\left(r\overline{P}\right)_{R-l}\frac{dl}{dz}=-\left(R-\frac{5\nu_v}{u_v^*}\right)\left(\frac{5\nu_v}{u_v^*}\right)\frac{\overline{P}}{2\tau_w}\frac{d\tau_w}{dz} \qquad \text{(H-24)}$$

Term 6:

Evaluating the term using the definition in Equation H-19 gives

$$\frac{g\rho_v}{2}\left(2Rl-l^2\right)=g\rho_v\left(\frac{5\nu_v}{u_v^*}\right)\left[R-\frac{5\nu_v}{2u_v^*}\right] \qquad \text{(H-25)}$$

Term 7:

Using Equation B-23 for $\frac{\partial \overline{u}_v}{\partial r}$, simplifying and using the definition of friction velocity and Equation H-18 yields

$$\left. \left(r\rho_v \nu_v \frac{\partial \overline{u}_v}{\partial r} \right) \right|_R = -R\tau_w \qquad \text{(H-26)}$$

Term 8:

Using Equation B-23 for $\frac{\partial \overline{u}_v}{\partial r}$, simplifying and using the definition of friction velocity and Equation H-19 yields

$$\left. \left(r\rho_v \nu_v \frac{\partial \overline{u}_v}{\partial r} \right) \right|_{R-l} = -\tau_w \left(R - \frac{5\nu_v}{u_v^*} \right) \qquad \text{(H-27)}$$

H.2 R direction Momentum Equations

H.2.1 Turbulent Core

The integrated form of the r-direction vapor momentum equation for the turbulent core is given by

$$\underbrace{\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\rho_v\bar{u}_v\bar{v}_v)dr\right]}_{\text{Term 1}}+\underbrace{(r\rho_v\bar{u}_v\bar{v}_v)\Big|_{R-l}\frac{dl}{dz}}_{\text{Term 2}}-\underbrace{(r\rho_v\bar{u}_v\bar{v}_v)\Big|_{R-\delta}\frac{d\delta}{dz}}_{\text{Term 3}}+\left\{\underbrace{(r\rho_v\bar{v}_v\bar{v}_v)\Big|_{R-l}}_{\text{Term 4}}-\underbrace{(r\rho_v\bar{v}_v\bar{v}_v)\Big|_{R-\delta}}_{\text{Term 5}}\right\}=$$

$$-\underbrace{\left[\int_{R-\delta}^{R-l}\left(r\frac{\partial\bar{P}}{\partial r}\right)dr\right]}_{\text{Term 6}}+\underbrace{\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial z}\right)dr\right]}_{\text{Term 7}}+\underbrace{\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial z}\right)\Big|_{R-l}\frac{dl}{dz}}_{\text{Term 8}}-\underbrace{\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial z}\right)\Big|_{R-\delta}\frac{d\delta}{dz}}_{\text{Term 9}} \qquad \text{(A-19)}$$

$$+\left\{\underbrace{\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial r}\right)\Big|_{R-l}}_{\text{Term 10}}-\underbrace{\left(r\rho_v\varepsilon_{m,v}\frac{\partial\bar{v}_v}{\partial r}\right)\Big|_{R-\delta}}_{\text{Term 11}}\right\}$$

Terms 7, 8 and 9 can be neglected by order of magnitude approximation.

Term 1:

Using the definitions from Equation B-1 and D-2, Term 1 becomes

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}(r\rho_v\bar{u}_v\bar{v}_v)dr\right]=\frac{d}{dz}\left\{\int_{R-\delta}^{R-l}\left[\rho_v\left(u_v^*\right)^2\left(u_v^+.rv_v^+\right)\right]dr\right\}=\frac{d}{dz}\left\{\tau_w\int_{R-\delta}^{R-l}\left(u_v^+.rv_v^+\right)dr\right\}$$

Substituting for the non-dimensional velocities using Equations B-5 and D-13 gives

$$\frac{d}{dz}\left\{\tau_w\int_{R-\delta}^{R-l}\left(u_v^+.rv_v^+\right)dr\right\}=$$

$$\frac{d}{dz}\left[\tau_w\int_{R-\delta}^{R-l}\left(a_0+a_1\eta+a_2\eta^2\right)\left\{\begin{aligned}&K_4(z)-\frac{1}{2\tau_w}\frac{d\tau_w}{dz}\left[\frac{a_0r^2}{2}+8a_1\eta\left(\frac{y^2}{15}-\frac{Ry}{8}\right)+9a_2\eta^2\left(\frac{y^2}{16}-\frac{Ry}{9}\right)\right]\\&-\frac{da_2}{dz}\left[7\eta^2\left(\frac{y^2}{16}-\frac{Ry}{9}\right)-14(5)^{1/7}\eta\left(\frac{y^2}{15}-\frac{Ry}{8}\right)+(5)^{2/7}\left(\frac{r^2}{2}\right)\right]\end{aligned}\right\}dr\right]$$

Evaluating the product and then integrating, applying the limits using Equations H-2 and H-3 gives

$$\frac{d}{dz}\left[\left[K_4(z)\right]\tau_w\left\{\begin{aligned}&a_0\left[(R-l)-(R-\delta)\right]-\frac{7}{8}a_1\left[5^{1/7}(l)-\left(\delta^+\right)^{1/7}(\delta)\right]-\\&\frac{7}{9}a_2\left[5^{2/7}(l)-\left(\delta^+\right)^{2/7}(\delta)\right]\end{aligned}\right\}\right]$$

+

$$\frac{d}{dz}\left[\begin{aligned}&-\frac{1}{2}\frac{d\tau_w}{dz}\left\{\begin{aligned}&\frac{a_0^2}{6}\left[(R-l)^3-(R-\delta)^3\right]\\&+\frac{7a_0a_1}{2}\left[5^{1/7}(l)\left(\frac{2Rl}{15}-\frac{R^2}{8}-\frac{l^2}{22}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{2R\delta}{15}-\frac{R^2}{8}-\frac{\delta^2}{22}\right)\right]\\&+\frac{7a_0a_2}{2}\left[5^{2/7}(l)\left(\frac{2Rl}{16}-\frac{R^2}{9}-\frac{l^2}{23}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{2R\delta}{16}-\frac{R^2}{9}-\frac{\delta^2}{23}\right)\right]\end{aligned}\right\}\\&-\frac{56}{2}\frac{d\tau_w}{dz}\left\{\begin{aligned}&a_0a_1\left[5^{1/7}(l)^2\left(\frac{R}{120}-\frac{l}{330}\right)-\left(\delta^+\right)^{1/7}(\delta)^2\left(\frac{R}{120}-\frac{\delta}{330}\right)\right]\\&+a_1^2\left[5^{2/7}(l)^2\left(\frac{R}{128}-\frac{l}{345}\right)-\left(\delta^+\right)^{2/7}(\delta)^2\left(\frac{R}{128}-\frac{\delta}{345}\right)\right]\\&+a_2a_1\left[5^{3/7}(l)^2\left(\frac{R}{136}-\frac{l}{360}\right)-\left(\delta^+\right)^{3/7}(\delta)^2\left(\frac{R}{136}-\frac{\delta}{360}\right)\right]\end{aligned}\right\}\\&-\frac{63}{2}\frac{d\tau_w}{dz}\left\{\begin{aligned}&a_0a_2\left[5^{2/7}(l)^2\left(\frac{R}{144}-\frac{l}{368}\right)-\left(\delta^+\right)^{2/7}(\delta)^2\left(\frac{R}{144}-\frac{\delta}{368}\right)\right]\\&+a_1a_2\left[5^{3/7}(l)^2\left(\frac{R}{153}-\frac{l}{384}\right)-\left(\delta^+\right)^{3/7}(\delta)^2\left(\frac{R}{153}-\frac{\delta}{384}\right)\right]\\&+a_2^2\left[5^{4/7}(l)^2\left(\frac{R}{162}-\frac{l}{400}\right)-\left(\delta^+\right)^{4/7}(\delta)^2\left(\frac{R}{162}-\frac{\delta}{400}\right)\right]\end{aligned}\right\}\end{aligned}\right]$$

+

$$
-\frac{d}{dz}\tau_w\left[\begin{array}{l}
\frac{5^{2/7}}{2}\frac{da_2}{dz}\left\{\begin{array}{l}
\frac{a_0}{3}\left[(R-l)^3-(R-\delta)^3\right] \\
+7a_1\left[5^{1/7}(l)\left(\frac{2Rl}{15}-\frac{R^2}{8}-\frac{l^2}{22}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{2R\delta}{15}-\frac{R^2}{8}-\frac{\delta^2}{22}\right)\right] \\
+7a_2\left[5^{2/7}(l)\left(\frac{2Rl}{16}-\frac{R^2}{9}-\frac{l^2}{23}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{2R\delta}{16}-\frac{R^2}{9}-\frac{\delta^2}{23}\right)\right]
\end{array}\right\} \\
-98(5)^{1/7}\frac{da_2}{dz}\left\{\begin{array}{l}
a_0\left[5^{1/7}(l)^2\left(\frac{R}{120}-\frac{l}{330}\right)-\left(\delta^+\right)^{1/7}(\delta)^2\left(\frac{R}{120}-\frac{\delta}{330}\right)\right] \\
+a_1\left[5^{2/7}(l)^2\left(\frac{R}{128}-\frac{l}{345}\right)-\left(\delta^+\right)^{2/7}(\delta)^2\left(\frac{R}{128}-\frac{\delta}{345}\right)\right] \\
+a_2\left[5^{3/7}(l)^2\left(\frac{R}{136}-\frac{l}{360}\right)-\left(\delta^+\right)^{3/7}(\delta)^2\left(\frac{R}{136}-\frac{\delta}{360}\right)\right]
\end{array}\right\} \\
+49\frac{da_2}{dz}\left\{\begin{array}{l}
a_0\left[5^{2/7}(l)^2\left(\frac{R}{144}-\frac{l}{368}\right)-\left(\delta^+\right)^{2/7}(\delta)^2\left(\frac{R}{144}-\frac{\delta}{368}\right)\right] \\
+a_1\left[5^{3/7}(l)^2\left(\frac{R}{153}-\frac{l}{384}\right)-\left(\delta^+\right)^{3/7}(\delta)^2\left(\frac{R}{153}-\frac{\delta}{384}\right)\right] \\
+a_2\left[5^{4/7}(l)^2\left(\frac{R}{162}-\frac{l}{400}\right)-\left(\delta^+\right)^{4/7}(\delta)^2\left(\frac{R}{162}-\frac{\delta}{400}\right)\right]
\end{array}\right\}
\end{array}\right]
$$

In the above expression, the coefficients a_0, a_1, a_2, the friction velocity and the vapor film thickness, wall shear stress are all functions of z. Differentiating the above expression gives an equation for Term 1.

Term 2:

Using the definitions from Equations B-1 and D-2, Term 2 becomes

$$
\left(r\rho_v\bar{u}_v\bar{v}_v\right)\Big|_{R-l}\frac{dl}{dz}=\left[\rho_v\left(u_v^*\right)^2\left(u_v^+.rv_v^+\right)\right]\Big|_{R\text{-}l}\frac{dl}{dz}=\tau_w\left(u_v^+.rv_v^+\right)_{R\text{-}l}\frac{dl}{dz}
$$

Substituting for the non-dimensional velocity profiles from Equations B-5, D-13, using Equation H-1 and evaluating these using Equation H-2 gives Term 2 as

$$\left(r\rho_v \bar{u}_v \bar{v}_v\right)\Big|_{R-l} \frac{dl}{dz} =$$

$$-\left(\frac{5\nu_v}{2u_v^*}\right)\left[a_0 + a_1(5)^{1/7} + a_2(5)^{2/7}\right]$$

$$\left\{\begin{array}{l} K_4(z) - \dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{a_0}{2}(R-l)^2 + 8a_1(5)^{1/7}(l)\left(\dfrac{l}{15}-\dfrac{R}{8}\right) + 9a_2(5)^{2/7}(l)\left(\dfrac{l}{16}-\dfrac{R}{9}\right)\right] \\ -(5)^{2/7}\dfrac{da_2}{dz}\left[7(l)\left(\dfrac{l}{16}-\dfrac{R}{9}\right) - 14(l)\left(\dfrac{l}{15}-\dfrac{R}{8}\right) + \dfrac{(R-l)^2}{2}\right] \end{array}\right\}\frac{d\tau_w}{dz}$$

Substituting for $K_4(z)$ from Equation D-14, Term 2 can be simplified as

$$\left(r\rho_v \bar{u}_v \bar{v}_v\right)\Big|_{R-l} \frac{dl}{dz} = -\left(\frac{l}{2}\right)\left[a_0 + a_1(5)^{1/7} + a_2(5)^{2/7}\right]\frac{u_v^* l^2}{\nu_v \tau_w}\left(\frac{R}{2}-\frac{l}{3}\right)\left(\frac{d\tau_w}{dz}\right)^2 \qquad \text{(H-29)}$$

Term 3:

Using the definitions from Equations B-1 and D-2, Term 3 becomes

$$\left(r\rho_v \bar{u}_v \bar{v}_v\right)\Big|_{R-\delta} \frac{d\delta}{dz} = \left[\rho_v \left(u_v^*\right)^2\left(u_v^+ . r v_v^+\right)\right]\Big|_{R\text{-}\delta} \frac{d\delta}{dz} = \tau_w \left(u_v^+ . r v_v^+\right)_{R\text{-}\delta} \frac{d\delta}{dz}$$

Substituting for the non-dimensional velocity profiles from Equation B-5 and D-13 and evaluating these using Equation H-3 gives Term 3 as

$$\left(r\rho_v \bar{u}_v \bar{v}_v\right)\Big|_{R-\delta} \frac{d\delta}{dz} =$$

$$\tau_w\left[a_0 + a_1\left(\delta^+\right)^{1/7} + a_2\left(\delta^+\right)^{2/7}\right] \qquad \text{(H-30)}$$

$$\left\{\begin{array}{l} K_4(z) - \dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{a_0}{2}(R-\delta)^2 + 8a_1\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right) + 9a_2\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right] \\ -\dfrac{da_2}{dz}\left[7\left(\delta^+\right)^{2/7}\delta\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right) - 14\left(\delta^+\right)^{1/7}(5)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right) + \dfrac{5^{2/7}(R-\delta)^2}{2}\right] \end{array}\right\}\frac{d\delta}{dz}$$

Substituting for $K_4(z)$ from Equation D-14, Term 3 can be obtained.

Term 4:

Using Equations D-2 and D-13 for the non-dimensional vapor velocity and evaluating Term 4 using Equation H-2 gives

$$\left(r\rho_v\bar{v}_v\bar{v}_v\right)\Big|_{R-l} = \tau_w(R-l)\left\{\begin{array}{l} -\dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{a_0}{2}(R-l)+\dfrac{8a_1(5)^{1/7}(l)}{(R-l)}\left(\dfrac{l}{15}-\dfrac{R}{8}\right)+\dfrac{9a_2(5)^{2/7}(l)}{(R-l)}\left(\dfrac{l}{16}-\dfrac{R}{9}\right)\right]\Bigg]^2 \\ -\dfrac{da_2}{dz}\left[\dfrac{5^{2/7}}{2}(R-l)+\dfrac{7(5)^{2/7}(l)}{(R-l)}\left(\dfrac{5R}{36}-\dfrac{17\nu_v}{48u_v^*}\right)\right]+\dfrac{K_4(z)}{(R-l)} \end{array}\right\} \qquad \text{(H-31)}$$

Substituting for $K_4(z)$ from Equation D-14, Term 4 can be obtained.

Term 5:

Using Equations D-2 and D-13 for the non-dimensional vapor velocity along with Equation D-14 for $K_4(z)$ and evaluating Term 5 using Equation H-3 gives

$$\left(r\rho_v\bar{v}_v\bar{v}_v\right)\Big|_{R-\delta} = \tau_w(R-\delta)\left\{\begin{array}{l} -\dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{a_0}{2}(R-\delta)+\dfrac{8a_1(\delta^+)^{1/7}}{(R-\delta)}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)+\dfrac{9a_2(\delta^+)^{2/7}}{(R-\delta)}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right] \\ -\dfrac{da_2}{dz}\left[\dfrac{7(\delta^+)^{2/7}}{(R-\delta)}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)-\dfrac{14(\delta^+)^{1/7}5^{1/7}}{(R-\delta)}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)+\dfrac{5^{2/7}(R-\delta)}{2}\right]+\dfrac{K_4(z)}{(R-\delta)} \end{array}\right\}^2 \qquad \text{(H-32)}$$

Term 6:

Since pressure was assumed to be a weak function of the radial direction, Term 6 becomes equal to zero.

Term 10:

Terms 10 and 11 require the gradient of the radial component of the vapor velocity in the radial direction, $\frac{\partial \overline{v}_v}{\partial r}$. Instead of taking the derivative of the radial component of the vapor velocity with respect to r, an alternate approach can be used.

Assuming density to be a constant, the vapor continuity equation can be re-written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{v}_v\right)=-\frac{\partial \overline{u}_v}{\partial z}$$

Employing the product rule of differentiation, the above equation can be recast as

$$\frac{\partial \overline{v}_v}{\partial r}+\frac{\overline{v}_v}{r}=-\frac{\partial \overline{u}_v}{\partial z}$$

Rearranging and using Equation D-2

$$\frac{\partial \overline{v}_v}{\partial r}=-\frac{\partial \overline{u}_v}{\partial z}-\frac{\overline{v}_v}{r}=-\frac{\partial \overline{u}_v}{\partial z}-\frac{u_v^* v_v^+}{r} \qquad \text{(H-33)}$$

Substituting Equation H-33, Term 10 becomes

$$\left(r\rho_v\varepsilon_{m,v}\frac{\partial \overline{v}_v}{\partial r}\right)\Bigg|_{R-l}=-\left[r\rho_v\varepsilon_{m,v}\left(\frac{\partial \overline{u}_v}{\partial z}+\frac{\overline{v}_v}{r}\right)\right]\Bigg|_{R-l}=\left[-r\rho_v\varepsilon_{m,v}\frac{\partial \overline{u}_v}{\partial z}-\rho_v\varepsilon_{m,v}\left(u_v^* v_v^+\right)\right]_{R-l}$$

Using Equation B-32 for $\frac{\partial \overline{u}_v}{\partial z}$, Equation D-13 for the non-dimensional radial component of vapor velocity and evaluating using Equation H-2, Term 10 can be written as

$$\left. \left(r\rho_v \varepsilon_{m,v} \frac{\partial \bar{v}_v}{\partial r} \right) \right|_{R-l} =$$

$$- \rho_v \varepsilon_{m,v} u_v^* \left\{ \begin{array}{l} -\dfrac{1}{2\tau_w} \dfrac{d\tau_w}{dz} \left[\dfrac{a_0}{2}(R-l) + \dfrac{8a_1 (5)^{1/7} (l)}{(R-l)} \left(\dfrac{l}{15} - \dfrac{R}{8} \right) + \dfrac{9a_2 (5)^{2/7} (l)}{(R-l)} \left(\dfrac{l}{16} - \dfrac{R}{9} \right) \right] \\ -\dfrac{da_2}{dz} \left[\dfrac{5^{2/7}}{2}(R-l) + \dfrac{7(5)^{2/7}(l)}{(R-l)} \left(\dfrac{5R}{36} - \dfrac{17\nu_v}{48u_v^*} \right) \right] + \dfrac{K_4(z)}{(R-l)} \end{array} \right\}$$

$$- \frac{\rho_v \varepsilon_{m,v} (R-l)}{\sqrt{\tau_w \rho_v}} \frac{d\tau_w}{dz} \left\{ \frac{a_0 + a_1 (5)^{1/7} + a_2 (5)^{2/7}}{2} + \frac{a_1 (5)^{1/7} + 2a_2 (5)^{2/7}}{14} \right\} \tag{H-34}$$

Term 11:

Following the same approach as done for Term 10, and evaluating using Equation H-3, Term 11 becomes

$$\left. \left(r\rho_v \varepsilon_{m,v} \frac{\partial \bar{v}_v}{\partial r} \right) \right|_{R-\delta} =$$

$$- \rho_v \varepsilon_{m,v} u_v^* (R-\delta) \left[\left(\delta^+\right)^{1/7} - 5^{1/7} \right]^2 \frac{da_2}{dz}$$

$$- \frac{\rho_v \varepsilon_{m,v} (R-\delta)}{\sqrt{\tau_w \rho_v}} \left[\frac{a_0 + a_1 \left(\delta^+\right)^{1/7} + a_2 \left(\delta^+\right)^{2/7}}{2} + \frac{a_1 \left(\delta^+\right)^{1/7} + 2a_2 \left(\delta^+\right)^{2/7}}{14} \right] \frac{d\tau_w}{dz} \tag{H-35}$$

$$- \rho_v \varepsilon_{m,v} u_v^* \left\{ \begin{array}{l} -\dfrac{1}{2\tau_w} \dfrac{d\tau_w}{dz} \left[\dfrac{a_0}{2}(R-\delta) + \dfrac{8a_1 \left(\delta^+\right)^{1/7}}{(R-\delta)} (\delta) \left(\dfrac{\delta}{15} - \dfrac{R}{8} \right) + \dfrac{9a_2 \left(\delta^+\right)^{2/7}}{(R-\delta)} (\delta) \left(\dfrac{\delta}{16} - \dfrac{R}{9} \right) \right] \\ -\dfrac{da_2}{dz} \left[\dfrac{7\left(\delta^+\right)^{2/7}}{R-\delta} (\delta) \left(\dfrac{\delta}{16} - \dfrac{R}{9} \right) - \dfrac{14\left(\delta^+\right)^{1/7} 5^{1/7}}{(R-\delta)} (\delta) \left(\dfrac{\delta}{15} - \dfrac{R}{8} \right) + \dfrac{5^{2/7}(R-\delta)}{2} \right] + \dfrac{K_4(z)}{(R-\delta)} \end{array} \right\}$$

$K_4(z)$ can be substituted from Equation D-14.

H.2.2 Wall Region

The integrated form of the r-direction vapor momentum equation for the wall region is given by

$$\underbrace{\frac{d}{dz}\left[\int_{R-l}^{R}(r\rho_v\bar{u}_v\bar{v}_v)dr\right]}_{\text{Term 1}}-\underbrace{(r\rho_v\bar{u}_v\bar{v}_v)\big|_{R-l}\frac{dl}{dz}}_{\text{Term 2}}-\underbrace{(r\rho_v\bar{v}_v\bar{v}_v)\big|_{R-l}}_{\text{Term 3}}=$$
$$-\underbrace{\int_{R-l}^{R}\left(r\frac{\partial\bar{P}}{\partial r}\right)dr}_{\text{Term 4}}+\underbrace{\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\nu_v\frac{\partial\bar{v}_v}{\partial z}\right)dr\right]}_{\text{Term 5}}-\underbrace{\left(r\rho_v\nu_v\frac{\partial\bar{v}_v}{\partial z}\right)\bigg|_{R-l}\frac{dl}{dz}}_{\text{Term 6}} \tag{A-24}$$

Term 1:

Using the definitions from Equation B-1 and D-2, Term 1 becomes

$$\frac{d}{dz}\left[\int_{R-l}^{R}(r\rho_v\bar{u}_v\bar{v}_v)dr\right]=\frac{d}{dz}\left\{\int_{R-l}^{R}\left[\rho_v\left(u_v^*\right)^2\left(u_v^+.rv_v^+\right)\right]dr\right\}=\frac{d}{dz}\left\{\tau_w\int_{R-l}^{R}\left(u_v^+.rv_v^+\right)dr\right\}$$

Using Equations B-4 and D-10 for the non-dimensional velocities, and carrying out the integration using Equations H-18 and H-19 and then differentiating with respect to z gives

$$\frac{l^4}{\rho_v\nu_v^2}\left[\tau_w\left(\frac{R}{8}-\frac{l}{15}\right)\frac{d^2\tau_w}{dz^2}-\left(\frac{R}{8}-\frac{l}{10}\right)\left(\frac{d\tau_w}{dz}\right)^2\right] \tag{H-36}$$

Term 2:

Using the definitions from Equations B-1 and D-2, Term 2 becomes

$$(r\rho_v\bar{u}_v\bar{v}_v)\big|_{R-l}\frac{dl}{dz}=\left[\rho_v\left(u_v^*\right)^2\left(u_v^+.rv_v^+\right)\right]\Big|_{R\text{-}l}\frac{dl}{dz}=\tau_w\left(u_v^+.rv_v^+\right)_{R\text{-}l}\frac{dl}{dz}$$

Substituting for the non-dimensional velocity profiles from Equation B-4 and D-10, using Equation H-1, evaluating these using Equation H-19 and simplifying gives Term 2 as

$$\left(r\rho_v \overline{u}_v \overline{v}_v\right)\Big|_{R-l} \frac{dl}{dz} = \frac{25l^2}{2\tau_w}\left(\frac{l}{3}-\frac{R}{2}\right)\left(\frac{d\tau_w}{dz}\right)^2 \qquad \text{(H-37)}$$

Term 3:

Using definitions from Equations B-1 and D-2 gives

$$\left(r\rho_v \overline{v}_v \overline{v}_v\right)\Big|_{R-l} = r\rho_v \left(u_v^* v_v^+\right)\left(u_v^* v_v^+\right)\Big|_{R-l} = \tau_w \left(r v_v^+ v_v^+\right)\Big|_{R-l}$$

Using Equation D-10 for the radial component of the vapor velocity in the above, evaluating using Equation H-19 and simplifying gives

$$\left(r\rho_v \overline{v}_v \overline{v}_v\right)\Big|_{R-l} = \frac{l^4}{\rho_v \nu_v^2 (R-l)}\left(\frac{R}{2}-\frac{l}{3}\right)^2\left(\frac{d\tau_w}{dz}\right)^2 \qquad \text{(H-38)}$$

Term 4:

Since pressure was assumed to be a weak function of the radial direction, Term 6 becomes equal to zero.

Terms 5 and 6 are neglected using an order of magnitude approach.

Appendix I

Integrated Liquid Momentum Equations

The equations for the liquid core derived in Appendix A are considered separately here. The liquid core extends from $y = \delta \left(y^+ = \delta^+\right)$ to $y = R\left(y^+ = R^+\right)$.

I.1 Z direction Liquid Momentum Equation

The Z-direction momentum equation is now considered. Using the definitions of the non-dimensional velocities, each term in the integrated equation is written in terms of the primary independent variables.

The integrated form of the Z-direction liquid momentum equation is given by

$$\underbrace{\frac{d}{dz}\left[\int_0^{R-\delta}(r\bar{u}_l\bar{u}_l)dr\right]}_{\text{Term 1}} + \underbrace{(r\bar{u}_l\bar{u}_l)\Big|_{R-\delta}\frac{d\delta}{dz}}_{\text{Term 2}} + \underbrace{(r\bar{u}_l\bar{v}_l)\Big|_{R-\delta}}_{\text{Term 3}} = \underbrace{-\frac{1}{\rho_l}\frac{d}{dz}\left[\int_0^{R-\delta}(r\bar{P})dr\right]}_{\text{Term 4}} - \underbrace{\frac{(r\bar{P})}{\rho_l}\Big|_{R-\delta}\frac{d\delta}{dz}}_{\text{Term 5}} - \underbrace{\frac{g(R-\delta)^2}{2}}_{\text{Term 6}}$$

$$+\underbrace{\left[r\varepsilon_{m,l}\frac{\partial\bar{u}_l}{\partial r}\right]_{R-\delta}}_{\text{Term 7}} + \underbrace{\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\varepsilon_{m,l}\frac{\partial\bar{u}_l}{\partial z}\right)dr\right]}_{\text{Term 8}} + \underbrace{\left[r\varepsilon_{m,l}\frac{\partial\bar{u}_l}{\partial z}\right]_{R-\delta}\frac{d\delta}{dz}}_{\text{Term 9}} \qquad \text{(A-13)}$$

In the integrated form of the liquid energy equation,

$r = 0$ corresponds to $y = R, y^+ = R^+, \eta = \left(R^+\right)^{1/7}$ and (I-1)

$r = R-\delta$ corresponds to $y = \delta, y^+ = \delta^+, \eta = \left(\delta^+\right)^{1/7}$ (I-2)

Term 1:

The product $\bar{u}_l\bar{u}_l$ is evaluated using the definition of u_l^+ and using Equation B-1 and C-1 gives

$$\bar{u}_l\bar{u}_l = \left(u_v^* u_l^+\right)\left(u_v^* u_l^+\right) = \left(u_v^*\right)^2\left(b_0 + b_1\eta + b_2\eta^2\right)^2$$

$$\overline{u}_l\overline{u}_l = \left(u_v^*\right)^2\left(b_0^2 + 2b_0b_1\eta + b_1^2\eta^2 + 2b_0b_2\eta^2 + 2b_1b_2\eta^3 + b_2^2\eta^4\right) \tag{I-3}$$

Substituting the above in the expression for Term 1, integrating using Equations I-1 and I-2 for the limits gives

$$\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\rho_v\overline{u}_l\overline{u}_l\right)dr\right] = \frac{d}{dz}\left[\left(u_v^*\right)^2\left\{\begin{array}{l} \frac{b_0^2}{2}(R-\delta)^2 \\ +14b_0b_1\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right] \\ +7\left(b_1^2+2b_0b_2\right)\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right] \\ +14b_1b_2\left[\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)+\frac{7R^2}{170}\left(R^+\right)^{3/7}\right] \\ +7b_2^2\left[\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)+\frac{7R^2}{198}\left(R^+\right)^{4/7}\right] \end{array}\right\}\right]$$

In the above expression, the coefficients b_0, b_1, b_2, friction velocity and the vapor film thickness are all functions of z. Differentiating the above with respect to z yields an expression for Term 1.

$$\frac{d}{dz}\left[\int_0^{R-\delta}(r\rho_v\bar{u}_l\bar{u}_l)dr\right]=$$

$$\left\{\begin{aligned}&\frac{b_0^2}{2}(R-\delta)^2+14b_0b_1\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]\\&+7\left(b_1^2+2b_0b_2\right)\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]+\\&14b_1b_2\left[\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)+\frac{7R^2}{170}\left(R^+\right)^{3/7}\right]+7b_2^2\left[\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)+\frac{7R^2}{198}\left(R^+\right)^{4/7}\right]\end{aligned}\right\}\frac{1}{\rho_v}\frac{d\tau_w}{dz}+$$

$$\left(u_v^*\right)^2\left\{\begin{aligned}&(R-\delta)^2b_0\frac{db_0}{dz}-b_0^2(R-\delta)\frac{d\delta}{dz}+14\left(b_0\frac{db_1}{dz}+b_1\frac{db_0}{dz}\right)\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]\\&+14\left(b_1\frac{db_1}{dz}+b_0\frac{db_2}{dz}+b_2\frac{db_0}{dz}\right)\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]\\&+14\left(b_2\frac{db_1}{dz}+b_1\frac{db_2}{dz}\right)\left[\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)+\frac{7R^2}{170}\left(R^+\right)^{3/7}\right]\\&+14b_2\frac{db_2}{dz}\left[\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)+\frac{7R^2}{198}\left(R^+\right)^{4/7}\right]\\&+14b_0b_1\left[\left(\delta^+\right)^{1/7}\left(\frac{2\delta}{15}-\frac{R}{8}\right)\frac{d\delta}{dz}+\frac{1}{7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}+\frac{R^2}{120}\left(R^+\right)^{-6/7}\frac{dR^+}{dz}\right]\\&+7\left(b_1^2+2b_0b_2\right)\left[\left(\delta^+\right)^{2/7}\left(\frac{2\delta}{16}-\frac{R}{9}\right)\frac{d\delta}{dz}+\frac{2}{7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}+\frac{R^2}{72}\left(R^+\right)^{-5/7}\frac{dR^+}{dz}\right]\\&+14b_1b_2\left[\left(\delta^+\right)^{3/7}\left(\frac{2\delta}{17}-\frac{R}{10}\right)\frac{d\delta}{dz}+\frac{3}{7}\left(\frac{\delta^2}{17}-\frac{R\delta}{10}\right)\left(\delta^+\right)^{-4/7}\frac{d\delta^+}{dz}+\frac{3R^2}{170}\left(R^+\right)^{-4/7}\frac{dR^+}{dz}\right]\\&+7b_2^2\left[\left(\delta^+\right)^{4/7}\left(\frac{2\delta}{18}-\frac{R}{11}\right)\frac{d\delta}{dz}+\frac{4}{7}\left(\frac{\delta^2}{18}-\frac{R\delta}{11}\right)\left(\delta^+\right)^{-3/7}\frac{d\delta^+}{dz}+\frac{2R^2}{99}\left(R^+\right)^{-3/7}\frac{dR^+}{dz}\right]\end{aligned}\right\} \quad \text{(I-4)}$$

Term 2:

The product $\bar{u}_l\bar{u}_l$ is evaluated using the definition of u_l^+ and using Equations B-1 and C-1 is given in Equation I-3. Evaluating Term 2 using Equation I-2 yields

$$\left(r\bar{u}_l\bar{u}_l\right)\Big|_{R-\delta}\frac{d\delta}{dz}=\left(u_v^*\right)^2(R-\delta)\left\{b_0^2+2b_0b_1\left(\delta^+\right)^{1/7}+\left(b_1^2+2b_0b_2\right)\left(\delta^+\right)^{2/7}+2b_1b_2\left(\delta^+\right)^{3/7}+b_2^2\left(\delta^+\right)^{4/7}\right\}\frac{d\delta}{dz} \quad \text{(I-5)}$$

Term 3:

Using the definition of axial component of liquid velocity from Equation C-1 along with the definitions from Equation B-1 yields

$$\left(r\bar{u}_l\bar{v}_l\right)\Big|_{R-\delta} = \left(u_v^*\right)^2\left[u_l^+\left(rv_l^+\right)\right]\Big|_{R-\delta} = \left(u_v^*\right)^2\left[\left(b_0 + b_1\eta + b_2\eta^2\right)\left(rv_l^+\right)\right]\Big|_{R-\delta}$$

The radial component of the liquid velocity was derived in Equation E-7. Substituting in the above and using Equation I-2 gives

$$\begin{aligned}&\left(r\bar{u}_l\bar{v}_l\right)\Big|_{R-\delta} = \left(u_v^*\right)^2\left[b_0 + b_1\left(\delta^+\right)^{1/7} + b_2\left(\delta^+\right)^{2/7}\right]\\&\left\{\begin{aligned}&K_1(z) - \frac{1}{2\tau_w}\frac{d\tau_w}{dz}\left[\frac{b_0}{2}(R-\delta)^2 + 8b_1\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right) + 9b_2\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right]\\&-\left[7\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\frac{db_2}{dz} + 7\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\frac{db_1}{dz} + \frac{(R-\delta)^2}{2}\frac{db_0}{dz}\right]\end{aligned}\right\}\end{aligned} \tag{I-6}$$

$K_1(z)$ is defined in Equation E-8.

Term 4:

Assuming pressure to be a constant over the radial direction gives

$$\frac{1}{\rho_l}\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\bar{P}\right)dr\right] = \frac{1}{\rho_l}\frac{d}{dz}\left[\frac{\bar{P}(R-\delta)^2}{2}\right]$$

Differentiating with respect to z gives Term 4 as

$$\frac{1}{\rho_l}\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\bar{P}\right)dr\right] = \frac{(R-\delta)^2}{2\rho_l}\frac{d\bar{P}}{dz} - \frac{(R-\delta)\bar{P}}{\rho_l}\frac{d\delta}{dz} \tag{I-7}$$

Term 5:

Term 5 can be simplified as

$$\left.\frac{(r\overline{P})}{\rho_l}\right|_{R-\delta}\frac{d\delta}{dz}=\frac{(R-\delta)\overline{P}}{\rho_l}\frac{d\delta}{dz} \tag{I-8}$$

Term 7:

Using Equation C-12 for the gradient of the axial component of the liquid velocity, $\overline{u}_l$ with respect to r and evaluating Term 7 using Equation I-2 yields

$$\left[r\varepsilon_{m,l}\frac{\partial\overline{u}_l}{\partial r}\right]_{R-\delta}=-\frac{(R-\delta)\varepsilon_{m,l}u_v^*\left[b_1\left(\delta^+\right)^{1/7}+2b_2\left(\delta^+\right)^{2/7}\right]}{7\delta} \tag{I-9}$$

Term 8:

Using Equation C-15 for the gradient of the axial component of the liquid velocity $\overline{u}_l$ with respect to z. Assuming a constant value of the eddy diffusivity of momentum and integrating using Equation I-1 and I-2 for the limits gives

$$\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\varepsilon_{m,l}\frac{\partial \bar{u}_l}{\partial z}\right)dr\right]=$$

$$\frac{d}{dz}\left[\varepsilon_{m,l}\left\{\begin{aligned}&\frac{1}{2\sqrt{\rho_v\tau_w}}\frac{d\tau_w}{dz}\left\{\begin{aligned}&b_0\frac{(R-\delta)^2}{2}+\\&7b_1\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]+\\&7b_2\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]\end{aligned}\right\}\\&+\frac{1}{14\sqrt{\rho_v\tau_w}}\frac{d\tau_w}{dz}\left\{\begin{aligned}&7b_1\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]\\&14b_2\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]\end{aligned}\right\}\\&+u_v^*\left\{\begin{aligned}&\frac{(R-\delta)^2}{2}\frac{db_0}{dz}\\&+7\frac{db_1}{dz}\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]\\&+7\frac{db_2}{dz}\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]\end{aligned}\right\}\end{aligned}\right\}\right]$$

Simplifying the above gives

$$\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\varepsilon_{m,l}\frac{\partial\overline{u}_l}{\partial z}\right)dr\right]=$$

$$\frac{d}{dz}\left[\varepsilon_{m,l}\left\{\begin{array}{l}\dfrac{1}{2\sqrt{\rho_v\tau_w}}\dfrac{d\tau_w}{dz}\left\{\begin{array}{l}b_0\dfrac{(R-\delta)^2}{2}+\\ 8b_1\left[\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)+\dfrac{7R^2}{120}\left(R^+\right)^{1/7}\right]+\\ 9b_2\left[\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)+\dfrac{7R^2}{144}\left(R^+\right)^{2/7}\right]\end{array}\right\}\\ +u_v^*\left\{\begin{array}{l}\dfrac{(R-\delta)^2}{2}\dfrac{db_0}{dz}\\ +7\dfrac{db_1}{dz}\left[\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)+\dfrac{7R^2}{120}\left(R^+\right)^{1/7}\right]\\ +7\dfrac{db_2}{dz}\left[\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)+\dfrac{7R^2}{144}\left(R^+\right)^{2/7}\right]\end{array}\right\}\end{array}\right\}\right]$$

In the above expression, the coefficients b_0, b_1, b_2, the friction velocity, wall shear stress and the vapor film thickness are all functions of z. Differentiating the above expression gives an equation for Term 8.

Term 9:

Using Equation C-15 for the gradient of the axial component of the liquid velocity $\overline{u}_l$ with respect to z along with Equation B-20, assuming constant eddy diffusivity of momentum and using Equation I-2 yields

$$\left[r\varepsilon_{m,l}\frac{\partial\overline{u}_l}{\partial z}\right]_{R-\delta}\frac{d\delta}{dz}=\varepsilon_{m,l}(R-\delta)\left\{\begin{array}{l}u_v^*\left[\dfrac{db_0}{dz}+\dfrac{db_1}{dz}\left(\delta^+\right)^{1/7}+\dfrac{db_2}{dz}\left(\delta^+\right)^{2/7}\right]+\\ \left[\dfrac{b_0+b_1\left(\delta^+\right)^{1/7}+b_2\left(\delta^+\right)^{2/7}}{2\sqrt{\tau_w\rho_v}}+\dfrac{b_1\left(\delta^+\right)^{1/7}+2b_2\left(\delta^+\right)^{2/7}}{14\sqrt{\tau_w\rho_v}}\right]\dfrac{d\tau_w}{dz}\end{array}\right\}\frac{d\delta}{dz} \qquad \text{(I-11)}$$

I.2 R direction Liquid Momentum Equation

The final integrated form of the r-direction liquid momentum equation is

$$\underbrace{\frac{d}{dz}\left[\int_0^{R-\delta}(r\overline{u}_l\overline{v}_l)dr\right]}_{\text{Term 1}}+\underbrace{(r\overline{u}_l\overline{v}_l)\Big|_{R-\delta}\frac{d\delta}{dz}}_{\text{Term 2}}+\underbrace{(r\overline{v}_l\overline{v}_l)\Big|_{R-\delta}}_{\text{Term 3}}=\underbrace{-\frac{1}{\rho_l}\left[\int_0^{R-\delta}\left(r\frac{\partial\overline{P}}{\partial r}\right)dr\right]}_{\text{Term 4}}+$$
$$\underbrace{\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\varepsilon_{m,l}\frac{\partial\overline{v}_l}{\partial z}\right)dr\right]}_{\text{Term 5}}+\underbrace{\left[r\varepsilon_{m,l}\frac{\partial\overline{v}_l}{\partial z}\right]_{R-\delta}\frac{d\delta}{dz}}_{\text{Term 6}}+\underbrace{\left[r\varepsilon_{m,l}\frac{\partial\overline{v}_l}{\partial r}\right]_{R-\delta}}_{\text{Term 7}} \qquad \text{(A-14)}$$

Terms 5 and 6 have been neglected by order of magnitude analysis. Using the definitions of the non-dimensional velocities, each term in the integrated equation is written in terms of the primary independent variables.

Term 1:

Using the definition of axial component of liquid velocity from Equation C-1 along with the definitions from Equation B-1 yields

$$\frac{d}{dz}\left[\int_0^{R-\delta}(r\overline{u}_l\overline{v}_l)dr\right]=\frac{d}{dz}\left[\int_0^{R-\delta}\left[\left(u_v^*\right)^2u_l^+rv_l^+\right]dr\right]$$

Substituting for the axial component of the liquid velocity from Equation C-1 and radial component of the liquid velocity from Equation E-7 gives

$$\frac{d}{dz}\left[\int_0^{R-\delta}\left[\left(u_v^*\right)^2u_l^+rv_l^+\right]dr\right]=$$
$$\frac{d}{dz}\left[\left(u_v^*\right)^2\int_0^{R-\delta}\left(b_0+b_1\eta+b_2\eta^2\right)\left\{\begin{array}{l}K_1(z)-\frac{1}{2\tau_w}\frac{d\tau_w}{dz}\left[\frac{b_0r^2}{2}+8\eta b_1\left(\frac{y^2}{15}-\frac{Ry}{8}\right)+9b_2\eta^2\left(\frac{y^2}{16}-\frac{Ry}{9}\right)\right]\\ \left[\frac{db_0}{dz}\frac{r^2}{2}+7\eta\frac{db_1}{dz}\left(\frac{y^2}{15}-\frac{Ry}{8}\right)+7\eta^2\frac{db_2}{dz}\left(\frac{y^2}{16}-\frac{Ry}{9}\right)\right]\end{array}\right\}dr\right]$$

Integrating and applying limits from Equations I-1 and I-2 gives

$$\frac{d}{dz}\left[\left(u_v^*\right)^2 K_1(z)\left\{b_0(R-\delta)-\frac{7}{8}b_1\left[\left(\delta^+\right)^{1/7}(\delta)-R\left(R^+\right)^{1/7}\right]-\frac{7}{9}b_2\left[\left(\delta^+\right)^{2/7}(\delta)-R\left(R^+\right)^{2/7}\right]\right\}\right]$$

+

$$-\frac{d}{dz}\left[\begin{array}{l}\dfrac{\left(u_v^*\right)^2}{2\tau_w}\dfrac{d\tau_w}{dz}\left\{\begin{array}{l}\dfrac{b_0^2}{6}(R-\delta)^3\\ +\dfrac{7b_0b_1}{2}\left[\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{2R\delta}{15}-\dfrac{R^2}{8}-\dfrac{\delta^2}{22}\right)+\dfrac{49R^3}{1320}\left(R^+\right)^{1/7}\right]\\ +\dfrac{7b_0b_2}{2}\left[\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{2R\delta}{16}-\dfrac{R^2}{9}-\dfrac{\delta^2}{23}\right)+\dfrac{49R^3}{1656}\left(R^+\right)^{2/7}\right]\end{array}\right\}\\ +\dfrac{56\left(u_v^*\right)^2}{2\tau_w}\dfrac{d\tau_w}{dz}\left\{\begin{array}{l}b_0b_1\left[\left(\delta^+\right)^{1/7}(\delta)^2\left(\dfrac{R}{120}-\dfrac{\delta}{330}\right)-\dfrac{7R^3}{1320}\left(R^+\right)^{1/7}\right]\\ +b_1^2\left[\left(\delta^+\right)^{2/7}(\delta)^2\left(\dfrac{R}{128}-\dfrac{\delta}{345}\right)-\dfrac{217R^3}{44160}\left(R^+\right)^{2/7}\right]\\ +b_2b_1\left[\left(\delta^+\right)^{3/7}(\delta)^2\left(\dfrac{R}{136}-\dfrac{\delta}{360}\right)-\dfrac{7R^3}{1530}\left(R^+\right)^{3/7}\right]\end{array}\right\}\\ +\dfrac{63\left(u_v^*\right)^2}{2\tau_w}\dfrac{d\tau_w}{dz}\left\{\begin{array}{l}b_0b_2\left[\left(\delta^+\right)^{2/7}(\delta)^2\left(\dfrac{R}{144}-\dfrac{\delta}{368}\right)-\dfrac{7R^3}{1656}\left(R^+\right)^{2/7}\right]\\ +b_1b_2\left[\left(\delta^+\right)^{3/7}(\delta)^2\left(\dfrac{R}{153}-\dfrac{\delta}{384}\right)-\dfrac{77R^3}{19584}\left(R^+\right)^{3/7}\right]\\ +b_2^2\left[\left(\delta^+\right)^{4/7}(\delta)^2\left(\dfrac{R}{162}-\dfrac{\delta}{400}\right)-\dfrac{119R^3}{32400}\left(R^+\right)^{4/7}\right]\end{array}\right\}\end{array}\right]$$

+

$$
-\frac{d}{dz}\left[\begin{array}{l}
\left(u_v^*\right)^2\frac{db_0}{dz}\left\{\begin{array}{l}
\frac{b_0}{6}(R-\delta)^3\\
+\frac{7b_1}{2}\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{2R\delta}{15}-\frac{R^2}{8}-\frac{\delta^2}{22}\right)+\frac{49R^3}{1320}\left(R^+\right)^{1/7}\right]\\
+\frac{7b_2}{2}\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{2R\delta}{16}-\frac{R^2}{9}-\frac{\delta^2}{23}\right)+\frac{49R^3}{1656}\left(R^+\right)^{2/7}\right]
\end{array}\right\}\\
+49\left(u_v^*\right)^2\frac{db_1}{dz}\left\{\begin{array}{l}
b_0\left[\left(\delta^+\right)^{1/7}(\delta)^2\left(\frac{R}{120}-\frac{\delta}{330}\right)-\frac{7R^3}{1320}\left(R^+\right)^{1/7}\right]\\
+b_1\left[\left(\delta^+\right)^{2/7}(\delta)^2\left(\frac{R}{128}-\frac{\delta}{345}\right)-\frac{217R^3}{44160}\left(R^+\right)^{2/7}\right]\\
+b_2\left[\left(\delta^+\right)^{3/7}(\delta)^2\left(\frac{R}{136}-\frac{\delta}{360}\right)-\frac{7R^3}{1530}\left(R^+\right)^{3/7}\right]
\end{array}\right\}\\
+49\left(u_v^*\right)^2\frac{db_2}{dz}\left\{\begin{array}{l}
b_0\left[\left(\delta^+\right)^{2/7}(\delta)^2\left(\frac{R}{144}-\frac{\delta}{368}\right)-\frac{7R^3}{1656}\left(R^+\right)^{2/7}\right]\\
+b_1\left[\left(\delta^+\right)^{3/7}(\delta)^2\left(\frac{R}{153}-\frac{\delta}{384}\right)-\frac{77R^3}{19584}\left(R^+\right)^{3/7}\right]\\
+b_2\left[\left(\delta^+\right)^{4/7}(\delta)^2\left(\frac{R}{162}-\frac{\delta}{400}\right)-\frac{119R^3}{32400}\left(R^+\right)^{4/7}\right]
\end{array}\right\}
\end{array}\right]
$$

In the above expression, the coefficients b_0, b_1, b_2, the friction velocity and the vapor film thickness, wall shear stress are all functions of z. Differentiating the above expression gives an equation for Term 1.

Term 2:

Using the definition of axial component of liquid velocity from Equation C-1 along with the definitions from Equation B-1 yields

$$
\left(r\bar{u}_l\bar{v}_l\right)\Big|_{R-\delta}\frac{d\delta}{dz}=\left[\left(u_v^*\right)^2\left(u_l^+.rv_l^+\right)\right]\Big|_{R\text{-}\delta}\frac{d\delta}{dz}=\left(u_v^*\right)^2\left(u_l^+.rv_l^+\right)_{R\text{-}\delta}\frac{d\delta}{dz}
$$

The radial component of the liquid velocity was derived in Equation E-7. Substituting in the above and using Equation I-2 yields

$$\left.\left(r\bar{u}_l\bar{v}_l\right)\right|_{R-\delta}\frac{d\delta}{dz}=$$

$$\left[\begin{array}{l}\left(u_v^*\right)^2\left[b_0+b_1\left(\delta^+\right)^{1/7}+b_2\left(\delta^+\right)^{2/7}\right]\\ \left\{\begin{array}{l}K_1(z)-\dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{b_0}{2}(R-\delta)^2+8b_1\delta\left(\delta^+\right)^{1/7}\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)+9b_2\delta\left(\delta^+\right)^{2/7}\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\\ -\left[\dfrac{db_0}{dz}\dfrac{(R-\delta)^2}{2}+7\dfrac{db_1}{dz}\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)+7\dfrac{db_2}{dz}\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\end{array}\right\}\end{array}\right]\frac{d\delta}{dz} \tag{I-13}$$

Term 3:

Using the definition of axial component of liquid velocity from Equation C-1 gives

$$\left.\left(r\bar{v}_l\bar{v}_l\right)\right|_{R-\delta}=\left.\left[\left(u_v^*\right)^2\left(rv_l^+v_l^+\right)\right]\right|_{R\text{-}\delta}$$

Substituting from Equation E-7 for the radial component of the liquid velocity and using Equation I-2 gives Term 3 as

$$(R-\delta)\left(u_v^*\right)^2\left\{\begin{array}{l}\dfrac{K_1(z)}{(R-\delta)}-\dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{b_0}{2}(R-\delta)+\dfrac{8b_1(\delta)\left(\delta^+\right)^{1/7}}{(R-\delta)}\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)+\dfrac{9b_2(\delta)\left(\delta^+\right)^{2/7}}{(R-\delta)}\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\\ -\left[\dfrac{db_0}{dz}\left(\dfrac{R-\delta}{2}\right)+\dfrac{7\delta\left(\delta^+\right)^{1/7}}{(R-\delta)}\dfrac{db_1}{dz}\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)+\dfrac{7\delta\left(\delta^+\right)^{2/7}}{(R-\delta)}\dfrac{db_2}{dz}\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\end{array}\right\}^2 \tag{I-14}$$

Term 4:

Since pressure was assumed to be a weak function of the radial direction, Term 4 becomes equal to zero.

Term 7:

Using an approach similar to that used to obtain Equation H-33,

$$\frac{\partial \bar{v}_l}{\partial r} = -\frac{\partial \bar{u}_l}{\partial z} - \frac{\bar{v}_l}{r} = -\frac{\partial \bar{u}_l}{\partial z} - \frac{u_v^* v_l^+}{r}$$

Using the above equation in Term 7 yields

$$-\left[r\varepsilon_{m,l} \frac{\partial \bar{u}_l}{\partial z} + \varepsilon_{m,l} u_v^* v_l^+ \right]_{R-\delta}$$

Using Equation C-15 for $\frac{\partial \bar{u}_l}{\partial z}$ in the above equation, substituting for the non-dimensional radial velocity of liquid from Equation E-7, using Equation I-2 and assuming the eddy diffusivity of momentum to be a constant gives Term 7 as

$$\begin{aligned}
&-(R-\delta)\varepsilon_{m,l} u_v^* \left[\frac{db_0}{dz} + \left(\delta^+\right)^{1/7} \frac{db_1}{dz} + \left(\delta^+\right)^{2/7} \frac{db_2}{dz} \right] \\
&-\left[\frac{b_0 + b_1\left(\delta^+\right)^{1/7} + b_2\left(\delta^+\right)^{2/7}}{2} + \frac{b_1\left(\delta^+\right)^{1/7} + 2b_2\left(\delta^+\right)^{2/7}}{14} \right] \frac{(R-\delta)\varepsilon_{m,l}}{\sqrt{\tau_w \rho_v}} \frac{d\tau_w}{dz} \\
&-\varepsilon_{m,l} u_v^* \left\{ \begin{aligned} &\frac{K_1(z)}{R-\delta} - \frac{1}{2\tau_w} \frac{d\tau_w}{dz} \left[\frac{b_0(R-\delta)}{2} + \frac{8b_1\left(\delta^+\right)^{1/7}}{(R-\delta)} \delta\left(\frac{\delta}{15} - \frac{R}{8} \right) + \frac{9b_2\left(\delta^+\right)^{2/7}}{(R-\delta)} \delta\left(\frac{\delta}{16} - \frac{R}{9} \right) \right] \\ &-\left[\frac{db_0}{dz}\left(\frac{R-\delta}{2} \right) + \frac{7\left(\delta^+\right)^{1/7}}{(R-\delta)} \frac{db_1}{dz} (\delta)\left(\frac{\delta}{15} - \frac{R}{8} \right) + \frac{7\left(\delta^+\right)^{2/7}}{(R-\delta)} \frac{db_2}{dz} (\delta)\left(\frac{\delta}{16} - \frac{R}{9} \right) \right] \end{aligned} \right\}
\end{aligned} \qquad \text{(I-15)}$$

In all the above equations, the expression for $K_1(z)$ can be substituted from Equation E-8.

Appendix J

Integrated Vapor Energy Equations

The integrated form of the vapor energy equation for the turbulent core is given by

$$\underbrace{\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\overline{u}_v\overline{T}_v\right)dr\right]}_{\text{Term 1}}+\underbrace{\left(r\rho_v\overline{u}_v\overline{T}_v\right)\Big|_{R-l}\frac{dl}{dz}}_{\text{Term 2}}-\underbrace{\left(r\rho_v\overline{u}_v\overline{T}_v\right)\Big|_{R-\delta}\frac{d\delta}{dz}}_{\text{Term 3}}+\left\{\underbrace{\left(r\rho_v\overline{v}_v\overline{T}_v\right)\Big|_{R-l}}_{\text{Term 4}}-\underbrace{\left(r\rho_v\overline{v}_v\overline{T}_v\right)\Big|_{R-\delta}}_{\text{Term 5}}\right\}=$$

$$\underbrace{\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial\overline{T}_v}{\partial z}\right)dr\right]}_{\text{Term 6}}+\underbrace{\left(r\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial\overline{T}_v}{\partial z}\right)\Bigg|_{R-l}\frac{dl}{dz}}_{\text{Term 7}}-\underbrace{\left(r\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial\overline{T}_v}{\partial z}\right)\Bigg|_{R-\delta}\frac{d\delta}{dz}}_{\text{Term 8}} \qquad \text{(A-21)}$$

$$+\left\{\underbrace{\left(r\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial\overline{T}_v}{\partial r}\right)\Bigg|_{R-l}}_{\text{Term 9}}-\underbrace{\left(r\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial\overline{T}_v}{\partial r}\right)\Bigg|_{R-\delta}}_{\text{Term 10}}\right\}$$

The turbulent core extends from

$y^+=5$, which corresponds to $y=l=\dfrac{5\nu_v}{u_v^*}, r=R-\dfrac{5\nu_v}{u_v^*}, \eta=(5)^{1/7}$ (J-1)

$y=\delta$, which corresponds to $y^+=\delta^+, r=R-\delta, \eta=\left(\delta^+\right)^{1/7}$ (J-2)

Re-arranging Equations B-1 and F-1

$$\overline{u}_v=\left(u_v^+\right)\left(u_v^*\right)=\left(u_v^*\right)\left(a_0+a_1\eta+a_2\eta^2\right)$$

$$\overline{T}_v=T_w+\left(\frac{q_w''}{\rho_v c_{p,v}u_v^*}\right)T_v^+$$

$$\overline{T}_v=T_w+\left(\frac{q_w''}{\rho_v c_{p,v}u_v^*}\right)\left(c_0+c_1\eta+c_2\eta^2\right)$$

Term 1:

Using the definitions for the vapor temperature from Equation F-1 and the axial component of the vapor velocity from Equation B-1, assuming constant vapor density, Term 1 becomes

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(\rho_v r \bar{u}_v \overline{T}_v\right)dr\right]=\frac{d}{dz}\left\{\rho_v\int_{R-\delta}^{R-l}\left[\begin{array}{l}\left(T_w u_v^*\right)\left(a_0 r+a_1\eta r+a_2\eta^2 r\right)+\\ \dfrac{q_w''}{\rho_v c_{p,v}}\left[\begin{array}{l}a_0c_0 r+\left(a_1c_0+c_1a_0\right)\eta r+\left(a_2c_0+a_1c_1+a_0c_2\right)\eta^2 r+\\ \left(a_1c_2+a_2c_1\right)\eta^3 r+a_2c_2\eta^4 r\end{array}\right]\end{array}\right]dr\right\}$$

Integrating the above with respect to r and applying limits using Equations J-1 and J-2 yields

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(\rho_v r \bar{u}_v \overline{T}_v\right)dr\right]=\rho_v\frac{d}{dz}\left\{\begin{array}{l}\left(T_w u_v^*\right)\left\{\begin{array}{l}\dfrac{a_0}{2}\left[(R-l)^2-(R-\delta)^2\right]\\ +7a_1\left[5^{1/7}(l)\left(\dfrac{l}{15}-\dfrac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)\right]\\ +7a_2\left[(5)^{2/7}(l)\left(\dfrac{l}{16}-\dfrac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\end{array}\right\}+\\ \dfrac{q_w''}{\rho_v c_{p,v}}\left\{\begin{array}{l}\dfrac{a_0c_0}{2}\left[(R-l)^2-(R-\delta)^2\right]\\ +7\left(a_1c_0+c_1a_0\right)\left[5^{1/7}(l)\left(\dfrac{l}{15}-\dfrac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)\right]\\ +7\left(a_2c_0+a_1c_1+a_0c_2\right)\left[(5)^{2/7}(l)\left(\dfrac{l}{16}-\dfrac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\\ +7\left(a_1c_2+a_2c_1\right)\left[(5)^{3/7}(l)\left(\dfrac{l}{17}-\dfrac{R}{10}\right)-\left(\delta^+\right)^{3/7}(\delta)\left(\dfrac{\delta}{17}-\dfrac{R}{10}\right)\right]\\ +7a_2c_2\left[(5)^{4/7}(l)\left(\dfrac{l}{18}-\dfrac{R}{11}\right)-\left(\delta^+\right)^{4/7}(\delta)\left(\dfrac{\delta}{18}-\dfrac{R}{11}\right)\right]\end{array}\right\}\end{array}\right\}$$

In the above expression, the coefficients a_0, a_1, a_2 and c_0, c_1, c_2, the wall heat flux, friction velocity and the wall temperature are all functions of z. Differentiating the above expression gives an equation for Term 1. This is shown in Equation J-3.

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(\rho_v r\bar{u}_v\bar{T}_v\right)dr\right]=$$

$$\rho_v\left\{\begin{array}{l}\frac{a_0}{2}\left[(R-l)^2-(R-\delta)^2\right]+7a_1\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right]\\+7a_2\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right]\end{array}\right\}\left[u_v^*\frac{dT_w}{dz}+\frac{T_w}{2\sqrt{\rho_v\tau_w}}\frac{d\tau_w}{dz}\right]+$$

$$\rho_v T_w u_v^*\left\{\begin{array}{l}\frac{\left[(R-l)^2-(R-\delta)^2\right]}{2}\frac{da_0}{dz}+7\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right]\frac{da_1}{dz}\\+7\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right]\frac{da_2}{dz}+\\a_0\left[(R-\delta)\frac{d\delta}{dz}-(R-l)\frac{dl}{dz}\right]+\\7a_1\left[5^{1/7}\left(\frac{2l}{15}-\frac{R}{8}\right)\frac{dl}{dz}-\left(\delta^+\right)^{1/7}\left(\frac{2\delta}{15}-\frac{R}{8}\right)\frac{d\delta}{dz}-\frac{1}{7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}\right]+\\7a_2\left[5^{2/7}\left(\frac{2l}{16}-\frac{R}{9}\right)\frac{dl}{dz}-\left(\delta^+\right)^{2/7}\left(\frac{2\delta}{16}-\frac{R}{9}\right)\frac{d\delta}{dz}-\frac{2}{7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}\right]\end{array}\right\}+$$

$$\frac{1}{c_{p,v}}\left\{\begin{array}{l}\frac{a_0c_0}{2}\left[(R-l)^2-(R-\delta)^2\right]+7\left(a_1c_0+c_1a_0\right)\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right]\\+7\left(a_2c_0+a_1c_1+a_0c_2\right)\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right]\\+7\left(a_1c_2+a_2c_1\right)\left[(5)^{3/7}(l)\left(\frac{l}{17}-\frac{R}{10}\right)-\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)\right]\\+7a_2c_2\left[(5)^{4/7}(l)\left(\frac{l}{18}-\frac{R}{11}\right)-\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)\right]\end{array}\right\}\frac{dq''_w}{dz}+ \qquad \text{(J-3)}$$

$$
\frac{q''_w}{c_{p,v}}\left\{\begin{aligned}
&\frac{\left[(R-l)^2-(R-\delta)^2\right]}{2}\left(a_0\frac{dc_0}{dz}+c_0\frac{da_0}{dz}\right)+a_0c_0\left[(R-\delta)\frac{d\delta}{dz}-(R-l)\frac{dl}{dz}\right]\\
&+7\left(a_1\frac{dc_0}{dz}+c_0\frac{da_1}{dz}+a_0\frac{dc_1}{dz}+c_1\frac{da_0}{dz}\right)\left[5^{1/7}(l)\left(\frac{l}{15}-\frac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)\right]\\
&+7\left(a_2\frac{dc_0}{dz}+c_0\frac{da_2}{dz}+a_0\frac{dc_2}{dz}+c_2\frac{da_0}{dz}+a_1\frac{dc_1}{dz}+c_1\frac{da_1}{dz}\right)\left[(5)^{2/7}(l)\left(\frac{l}{16}-\frac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)\right]\\
&+7\left(a_1\frac{dc_2}{dz}+c_2\frac{da_1}{dz}+a_2\frac{dc_1}{dz}+c_1\frac{da_2}{dz}\right)\left[(5)^{3/7}(l)\left(\frac{l}{17}-\frac{R}{10}\right)-\left(\delta^+\right)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)\right]\\
&+7\left(a_2\frac{dc_2}{dz}+c_2\frac{da_2}{dz}\right)\left[(5)^{4/7}(l)\left(\frac{l}{18}-\frac{R}{11}\right)-\left(\delta^+\right)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)\right]\\
&+7\left(a_1c_0+c_1a_0\right)\left[5^{1/7}\left(\frac{2l}{15}-\frac{R}{8}\right)\frac{dl}{dz}-\left(\delta^+\right)^{1/7}\left(\frac{2\delta}{15}-\frac{R}{8}\right)\frac{d\delta}{dz}-\frac{1}{7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}\right]\\
&+7\left(a_2c_0+a_1c_1+a_0c_2\right)\left[5^{2/7}\left(\frac{2l}{16}-\frac{R}{9}\right)\frac{dl}{dz}-\left(\delta^+\right)^{2/7}\left(\frac{2\delta}{16}-\frac{R}{9}\right)\frac{d\delta}{dz}-\frac{2}{7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}\right]\\
&+7\left(a_1c_2+a_2c_1\right)\left[5^{3/7}\left(\frac{2l}{17}-\frac{R}{10}\right)\frac{dl}{dz}-\left(\delta^+\right)^{3/7}\left(\frac{2\delta}{17}-\frac{R}{10}\right)\frac{d\delta}{dz}-\frac{3}{7}\left(\frac{\delta^2}{17}-\frac{R\delta}{10}\right)\left(\delta^+\right)^{-4/7}\frac{d\delta^+}{dz}\right]\\
&+7a_2c_2\left[5^{4/7}\left(\frac{2l}{18}-\frac{R}{11}\right)\frac{dl}{dz}-\left(\delta^+\right)^{4/7}\left(\frac{2\delta}{18}-\frac{R}{11}\right)\frac{d\delta}{dz}-\frac{4}{7}\left(\frac{\delta^2}{18}-\frac{R\delta}{11}\right)\left(\delta^+\right)^{-3/7}\frac{d\delta^+}{dz}\right]
\end{aligned}\right\}
$$

Term 2:

Multiplying re-arranged form of Equations B-1 and F-1 yields

$$
\left(r\rho_v\bar{u}_v\bar{T}_v\right)\Big|_{R-l}\frac{dl}{dz}=\left\{r\rho_v\left(u_v^*\right)\left(a_0+a_1\eta+a_2\eta^2\right)\left[T_w+\left(\frac{q''_w}{\rho_v c_{p,v}u_v^*}\right)\left(c_0+c_1\eta+c_2\eta^2\right)\right]\right\}_{R-l}\frac{dl}{dz}
$$

Evaluating the above using Equation J-1, and substituting for $\frac{dl}{dz}$ using Equation H-1 gives

$$\left(r\rho_v\bar{u}_v\bar{T}_v\right)\Big|_{R-l}\frac{dl}{dz}=-\left(R-\frac{5\nu_v}{u_v^*}\right)\left(\frac{5\nu_v}{u_v^*}\right)\frac{d\tau_w}{dz}\left\{\begin{array}{l}\frac{\rho_v T_w u_v^*}{2\tau_w}\left[a_0+a_1(5)^{1/7}+a_2(5)^{2/7}\right]+\\ \frac{q''_w}{2\tau_w c_{p,v}}\left[\begin{array}{l}a_0c_0+(a_1c_0+a_0c_1)(5)^{1/7}+(a_1c_2+a_2c_1)(5)^{3/7}\\ +(a_2c_0+a_1c_1+a_0c_2)(5)^{2/7}+a_2c_2(5)^{4/7}\end{array}\right]\end{array}\right\} \quad \text{(J-4)}$$

Term 3:

Multiplying re-arranged form of Equations B-1 and F-1 yields

$$\left(r\rho_v\bar{u}_v\bar{T}_v\right)\Big|_{R-\delta}\frac{d\delta}{dz}=\left\{r\rho_v\left(u_v^*\right)\left(a_0+a_1\eta+a_2\eta^2\right)\left[T_w+\left(\frac{q''_w}{\rho_v c_{p,v}u_v^*}\right)\left(c_0+c_1\eta+c_2\eta^2\right)\right]\right\}_{R-\delta}\frac{d\delta}{dz}$$

Evaluating the above using Equation J-2 gives

$$\left(r\rho_v\bar{u}_v\bar{T}_v\right)\Big|_{R-\delta}\frac{d\delta}{dz}=\left\{\begin{array}{l}\rho_v u_v^* T_w\left[a_0+a_1\left(\delta^+\right)^{1/7}+a_2\left(\delta^+\right)^{2/7}\right]+\\ \frac{q''_w}{c_{p,v}}\left[\begin{array}{l}a_0c_0+(a_1c_0+a_0c_1)\left(\delta^+\right)^{1/7}+(a_1c_2+a_2c_1)\left(\delta^+\right)^{3/7}+\\ (a_2c_0+a_1c_1+a_0c_2)\left(\delta^+\right)^{2/7}+a_2c_2\left(\delta^+\right)^{4/7}\end{array}\right]\end{array}\right\}(R-\delta)\frac{d\delta}{dz} \quad \text{(J-5)}$$

Term 4:

Using the re-arranged form of Equation F-1 for the vapor temperature, the expression for the radial component of the vapor velocity from Equation D-13 and evaluating them using Equation J-1 yields

$$\left(r\rho_v\bar{v}_v\bar{T}_v\right)\Big|_{R-l}=\frac{1}{\nu_v}\left(\frac{R}{2}-\frac{5\nu_v}{3u_v^*}\right)\left(\frac{5\nu_v}{u_v^*}\right)^2\left\{T_w+\frac{q''_w}{\rho_v c_{p,v}u_v^*}\left[c_0+c_1(5)^{1/7}+c_2(5)^{2/7}\right]\right\}\frac{d\tau_w}{dz} \quad \text{(J-6)}$$

Term 5:

Using the re-arranged form of Equation F-1 for the vapor temperature, the expression for the radial component of the vapor velocity from Equation D-13 and evaluating them using Equation J-2 yields

$$
\left(\rho_v u_v^*\right)\left\{T_w + \frac{q_w''}{\rho_v c_{pv} u_v^*}\left[c_0 + c_1\left(\delta^+\right)^{1/7} + c_2\left(\delta^+\right)^{2/7}\right]\right\}
$$

$$
\left\{\begin{array}{l} K_4(z) - \dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\dfrac{a_0}{2}(R-\delta)^2 + 8a_1\left(\delta^+\right)^{1/7}\left(\dfrac{\delta^2}{15} - \dfrac{R\delta}{8}\right) + 9a_2\left(\delta^+\right)^{2/7}\left(\dfrac{\delta^2}{16} - \dfrac{R\delta}{9}\right)\right] \\ -\dfrac{da_2}{dz}\left[7\left(\delta^+\right)^{2/7}\left(\dfrac{\delta^2}{16} - \dfrac{R\delta}{9}\right) - 14\left(\delta^+\right)^{1/7}(5)^{1/7}\left(\dfrac{\delta^2}{15} - \dfrac{R\delta}{8}\right) + \dfrac{5^{2/7}(R-\delta)^2}{2}\right] \end{array}\right\}
$$

Substituting for $K_4(z)$ from Equation D-14 yields

$$
\left[\begin{array}{l} \left(\rho_v u_v^*\right)\left\{T_w + \dfrac{q_w''}{\rho_v c_{pv} u_v^*}\left[c_0 + c_1\left(\delta^+\right)^{1/7} + c_2\left(\delta^+\right)^{2/7}\right]\right\} \\ \left\{\begin{array}{l} \dfrac{u_v^* l^2}{\nu_v \tau_w}\left(\dfrac{R}{2} - \dfrac{l}{3}\right)\dfrac{d\tau_w}{dz} + \dfrac{1}{2\tau_w}\dfrac{d\tau_w}{dz}\left[\begin{array}{l} \dfrac{a_0}{2}\left[(R-l)^2 - (R-\delta)^2\right] + 8a_1\left[(5)^{1/7}\left(\dfrac{l^2}{15} - \dfrac{Rl}{8}\right) - \left(\delta^+\right)^{1/7}\left(\dfrac{\delta^2}{15} - \dfrac{R\delta}{8}\right)\right] \\ +9a_2\left[(5)^{2/7}\left(\dfrac{l^2}{16} - \dfrac{Rl}{9}\right) - \left(\delta^+\right)^{2/7}\left(\dfrac{\delta^2}{16} - \dfrac{R\delta}{9}\right)\right] \end{array}\right] \\ +\dfrac{da_2}{dz}\left[\begin{array}{l} 7\left[(5)^{2/7}\left(\dfrac{l^2}{16} - \dfrac{Rl}{9}\right) - \left(\delta^+\right)^{2/7}\left(\dfrac{\delta^2}{16} - \dfrac{R\delta}{9}\right)\right] - 14(5)^{1/7}\left[(5)^{1/7}\left(\dfrac{l^2}{15} - \dfrac{Rl}{8}\right) - \left(\delta^+\right)^{1/7}\left(\dfrac{\delta^2}{15} - \dfrac{R\delta}{8}\right)\right] \\ +\dfrac{(5)^{2/7}}{2}\left[(R-l)^2 - (R-\delta)^2\right] \end{array}\right] \end{array}\right\} \end{array}\right] \qquad \text{(J-7)}
$$

Term 6:

Using Equation F-21 for the gradient of the vapor temperature in the axial direction, assuming constant properties and eddy diffusivity of momentum for the vapor phase and performing integration using Equations J-1 and J-2 gives

$$\frac{d}{dz}\left[\int_{R-\delta}^{R-l}\left(r\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\frac{\partial\overline{T}_v}{\partial z}\right)dr\right]=$$

$$\left(\rho_v\frac{\varepsilon_{m,v}}{\Pr_v^t}\right)\frac{d}{dz}\left\{\begin{array}{l}\dfrac{dT_w}{dz}\left[\dfrac{(R-l)^2-(R-\delta)^2}{2}\right]+\\ \dfrac{q''_w}{\rho_v c_{p,v}u_v^*}\dfrac{dc_2}{dz}\left\{\begin{array}{l}(5)^{2/7}\left[\dfrac{(R-l)^2-(R-\delta)^2}{2}\right]\\ +7\left[(5)^{2/7}(l)\left(\dfrac{l}{16}-\dfrac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\\ -14(5)^{1/7}\left[(5)^{1/7}(l)\left(\dfrac{l}{15}-\dfrac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)\right]\end{array}\right\}+\\ \dfrac{1}{\rho_v c_{p,v}u_v^*}\dfrac{q''_w}{14\tau_w}\dfrac{d\tau_w}{dz}\left\{\begin{array}{l}7c_1\left[(5)^{1/7}(l)\left(\dfrac{l}{15}-\dfrac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)\right]+\\ 14c_2\left[(5)^{2/7}(l)\left(\dfrac{l}{16}-\dfrac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\end{array}\right\}+\\ \dfrac{1}{\rho_v c_{p,v}u_v^*}\left(\dfrac{dq''_w}{dz}-\dfrac{q''_w}{2\tau_w}\dfrac{d\tau_w}{dz}\right)\left\{\begin{array}{l}\left[\dfrac{(R-l)^2-(R-\delta)^2}{2}\right]c_0+\\ 7c_1\left[(5)^{1/7}(l)\left(\dfrac{l}{15}-\dfrac{R}{8}\right)-\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)\right]+\\ 7c_2\left[(5)^{2/7}(l)\left(\dfrac{l}{16}-\dfrac{R}{9}\right)-\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)\right]\end{array}\right\}\end{array}\right\} \quad \text{(J-8)}$$

Differentiating the above with respect to z gives an expression for Term 6. In the above equation, the vapor film thickness, friction velocity, wall heat flux, wall temperature, the wall shear stress and the coefficients c_0, c_1, c_2 are all functions of the axial variable z.

Term 7:

Using Equation F-21 for the gradient of the vapor temperature in the axial direction, evaluating Term 7 using Equation J-1 and substituting for $\frac{dl}{dz}$ using Equation H-1 gives

$$\left(r\rho_v \frac{\varepsilon_{m,v}}{\Pr_v^t} \frac{\partial \overline{T}_v}{\partial z} \right)\Bigg|_{R-l} \frac{dl}{dz} =$$

$$-\left(\frac{5\nu_v}{u_v^*}\right)\left(R - \frac{5\nu_v}{u_v^*}\right)\frac{\rho_v \varepsilon_{m,v}}{2\tau_w \Pr_v^t}\left\{\frac{dT_w}{dz} + \frac{1}{\rho_v c_{p,v} u_v^*}\left[\begin{array}{l}\left(c_0 + c_1(5)^{1/7} + c_2(5)^{2/7}\right)\frac{dq_w''}{dz} + \\ \left(c_1(5)^{1/7} + 2c_2(5)^{2/7}\right)\frac{q_w''}{14\tau_w}\frac{d\tau_w}{dz} - \\ \left(c_0 + c_1(5)^{1/7} + c_2(5)^{2/7}\right)\frac{q_w''}{2\tau_w}\frac{d\tau_w}{dz}\end{array}\right]\right\}\frac{d\tau_w}{dz} \tag{J-9}$$

Term 8:

Using Equation F-21 for the gradient of the vapor temperature in the axial direction, evaluating Term 8 using Equation J-2 gives

$$(R-\delta)\frac{\rho_v \varepsilon_{m,v}}{\Pr_v^t}\left\{\frac{dT_w}{dz} + \frac{1}{\rho_v c_{p,v} u_v^*}\left[\begin{array}{l} q_w''\left[\left(\delta^+\right)^{1/7} - (5)^{1/7}\right]\frac{dc_2}{dz} + \\ \left(c_0 + c_1\left(\delta^+\right)^{1/7} + c_2\left(\delta^+\right)^{2/7}\right)\frac{dq_w''}{dz} + \\ \left(c_1\left(\delta^+\right)^{1/7} + 2c_2\left(\delta^+\right)^{2/7}\right)\frac{q_w''}{14\tau_w}\frac{d\tau_w}{dz} - \\ \left(c_0 + c_1\left(\delta^+\right)^{1/7} + c_2\left(\delta^+\right)^{2/7}\right)\frac{q_w''}{2\tau_w}\frac{d\tau_w}{dz}\end{array}\right]\right\}\frac{d\delta}{dz} \tag{J-10}$$

Term 9:

Using Equation F-15 for $\frac{\partial \overline{T}_v}{\partial r}$ and evaluating Term 9 using Equation J-1 gives

$$-(R-l)\rho_v \frac{\varepsilon_{m,v}}{\Pr_v^t}\left(\frac{q_w''}{\rho_v c_{p,v} u_v^*}\right)\frac{\left\lfloor c_1(5)^{1/7} + 2c_2(5)^{2/7}\right\rfloor}{7\left(\frac{5\nu_v}{u_v^*}\right)} \tag{J-11}$$

Term 10:

Using Equation F-15 for $\frac{\partial \overline{T_v}}{\partial r}$ and evaluating Term 10 using Equation J-2 gives

$$-(R-\delta)\rho_v \frac{\varepsilon_{m,v}}{\Pr_v^t}\left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right)\frac{\left[c_1\left(\delta^+\right)^{1/7}+2c_2\left(\delta^+\right)^{2/7}\right]}{7\delta} \qquad \text{(J-12)}$$

Wall Region

The integrated form of the energy equation for the **wall region** of vapor film

$$\underbrace{\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\bar{u}_v\bar{T}_v\right)dr\right]}_{\text{Term 1}}-\underbrace{\left(r\rho_v\bar{u}_v\bar{T}_v\right)\Big|_{R-l}\frac{dl}{dz}}_{\text{Term 2}}-\underbrace{\left(r\rho_v\bar{v}_v\bar{T}_v\right)\Big|_{R-l}}_{\text{Term 3}}=$$

$$\underbrace{\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\frac{\nu_v}{\mathrm{Pr}_v}\frac{\partial\bar{T}_v}{\partial z}\right)dr\right]}_{\text{Term 4}}-\underbrace{\left(r\rho_v\frac{\nu_v}{\mathrm{Pr}_v}\frac{\partial\bar{T}_v}{\partial z}\right)\Bigg|_{R-l}\frac{dl}{dz}}_{\text{Term 5}}+\left\{\underbrace{\left(r\rho_v\frac{\nu_v}{\mathrm{Pr}_v}\frac{\partial\bar{T}_v}{\partial r}\right)\Bigg|_{R}}_{\text{Term 6}}-\underbrace{\left(r\rho_v\frac{\nu_v}{\mathrm{Pr}_v}\frac{\partial\bar{T}_v}{\partial r}\right)\Bigg|_{R-l}}_{\text{Term 7}}\right\} \quad \text{(A-25)}$$

$r=R$ corresponds to $y=0$ and (J-13)

$r=R-l$ corresponds to $y=l=\dfrac{5\nu_v}{u_v^*},\ y^+=5, \eta=(5)^{1/7}$ (J-14)

Term 1:

Using Equations B-4 and F-2 for the vapor velocity and temperature profiles for the wall region yields

$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\bar{u}_v\bar{T}_v\right)dr\right]=\frac{d}{dz}\left\{\int_{R-l}^{R}\left[r\rho_v\left(u_v^*y^+\right)\left(T_w+\frac{q_w''}{\rho_v c_{p,v}u_v^*}\left(\mathrm{Pr}_v\ y^+\right)\right)\right]dr\right\}$$

Using definitions from Equation B-1, integrating using specified limits and Equations J-13 and J-14, then differentiating with respect to z and simplifying gives

$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v\bar{u}_v\bar{T}_v\right)dr\right]=25\left\{\begin{array}{l}\rho_v\nu_v\left(\dfrac{R}{2}-\dfrac{5\nu_v}{3u_v^*}\right)\dfrac{dT_w}{dz}+\dfrac{5\nu_v\,\mathrm{Pr}_v}{c_{p,v}u_v^*}\left(\dfrac{R}{3}-\dfrac{5\nu_v}{4u_v^*}\right)\dfrac{dq_w''}{dz}+\\ \dfrac{5\nu_v}{2\tau_w u_v^*}\left(\dfrac{\rho_v\nu_v T_w}{3}+\dfrac{5}{2}\dfrac{\nu_v}{u_v^*}\dfrac{\mathrm{Pr}_v}{c_{p,v}}q_w''-\dfrac{1}{3}\dfrac{R\,\mathrm{Pr}_v\,q_w''}{c_{p,v}}\right)\dfrac{d\tau_w}{dz}\end{array}\right\} \quad \text{(J-15)}$$

Term 2:

Using Equations B-4 and F-2 for the vapor velocity and temperature profiles for the wall region in conjunction with Equation J-14 and H-1 gives

$$\left(r\rho_v \bar{u}_v \bar{T}_v\right)_{R-l} \frac{dl}{dz} = -\frac{25\nu_v \rho_v}{2\tau_w}\left(R - \frac{5\nu_v}{u_v^*}\right)\left(T_w + \frac{5q_w'' \Pr_v}{\rho_v c_{p,v} u_v^*}\right)\frac{d\tau_w}{dz} \tag{J-16}$$

Term 3:

Substituting for the radial velocity profile for the wall region from Equation D-10 and using Equation F-2 for the non-dimensional wall temperature distribution, and evaluating Term 3 using Equation J-14 gives

$$\left(r\rho_v \bar{v}_v \bar{T}_v\right)\Big|_{R-l} = \frac{1}{\nu_v}\left(\frac{5\nu_v}{u_v^*}\right)^2\left(\frac{R}{2} - \frac{5\nu_v}{3u_v^*}\right)\left(T_w + \frac{5q_w'' \Pr_v}{\rho_v c_{p,v} u_v^*}\right)\frac{d\tau_w}{dz} \tag{J-17}$$

Term 4:

Using Equation F-29 for $\frac{\partial \bar{T}_v}{\partial z}$, Term 4 becomes

$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v \frac{\nu_v}{\Pr_v}\frac{\partial \bar{T}_v}{\partial z}\right)dr\right] = \frac{d}{dz}\left\{\int_{R-l}^{R}\left[r\rho_v \frac{\nu_v}{\Pr_v}\left(\frac{dT_w}{dz} + \left(\frac{y}{k_v}\right)\frac{dq_w''}{dz}\right)\right]dr\right\}$$

Using the definitions from Equation B-1 and Equations J-13 and J-14 gives

$$\frac{d}{dz}\left[\int_{R-l}^{R}\left(r\rho_v \frac{\nu_v}{\Pr_v}\frac{\partial \bar{T}_v}{\partial z}\right)dr\right] = \left\{\begin{aligned} &\frac{l^2}{c_{p,v}}\left(\frac{R}{2} - \frac{l}{3}\right)\left(\frac{d^2 q_w''}{dz^2}\right) - \frac{(R-l)l^2}{2\tau_w c_{p,v}}\left(\frac{d\tau_w}{dz}\right)\left(\frac{dq_w''}{dz}\right) \\ &+\left(\frac{\rho_v \nu_v l}{\Pr_v}\right)\left(\frac{R-l}{2}\right)\left(\frac{d^2 T_w}{dz^2}\right) - \left(\frac{\rho_v \nu_v l}{\tau_w \Pr_v}\right)\left(\frac{R-2l}{4}\right)\left(\frac{dT_w}{dz}\right)\left(\frac{d\tau_w}{dz}\right)\end{aligned}\right\} \tag{J-18}$$

Term 5:

Using Equation F-29 and evaluating the gradient at $r = R - l$

$$\left.\frac{\partial \overline{T}_v}{\partial z}\right|_{R-l} = \frac{dT_w}{dz} + \left(\frac{5\nu_v}{u_v^* k_v}\right)\frac{dq_w''}{dz}$$

Substituting Equation H-1 along with the above in Term 5 yields

$$\left.\left(r\rho_v \frac{\nu_v}{\Pr_v}\frac{\partial \overline{T}_v}{\partial z}\right)\right|_{R-l}\frac{dl}{dz} = \left(-\frac{5\nu_v}{u_v^*}\right)\left(R - \frac{5\nu_v}{u_v^*}\right)\left(\frac{\rho_v \nu_v}{\Pr_v}\right)\left(\frac{1}{2\tau_w}\right)\left[\frac{dT_w}{dz} + \left(\frac{5\nu_v}{u_v^* k_v}\right)\frac{dq_w''}{dz}\right]\frac{d\tau_w}{dz} \tag{J-19}$$

Term 6:

Substituting for $\frac{\partial \overline{T}_v}{\partial r}$ using Equation F-17 and using the definition of Prandtl number, Term 6 simplifies to

$$\left.\left(r\rho_v \frac{\nu_v}{\Pr_v}\frac{\partial \overline{T}_v}{\partial r}\right)\right|_{R} = -\frac{q_w'' R}{c_{p,v}} \tag{J-20}$$

Term 7:

Term 7 is similar to Term 6 except that it is evaluated at a different radial location. Thus, Term 7 simplifies to

$$\left.\left(r\rho_v \frac{\nu_v}{\Pr_v}\frac{\partial \overline{T}_v}{\partial r}\right)\right|_{R-l} = -\frac{q_w''}{c_{p,v}}\left[R - \left(\frac{5\nu_v}{u_v^*}\right)\right] \tag{J-21}$$

Appendix K

Integrated Liquid Energy Equation

The integrated form of the liquid energy equation is given by

$$\begin{aligned}
&\underbrace{\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\bar{u}_l\bar{T}_l\right)dr\right]}_{\text{Term 1}}+\underbrace{\left(r\bar{u}_l\bar{T}_l\right)\Big|_{R-\delta}\frac{d\delta}{dz}}_{\text{Term 2}}+\underbrace{\left(r\bar{v}_l\bar{T}_l\right)\Big|_{R-\delta}}_{\text{Term 3}}= \\
&\underbrace{\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\frac{\varepsilon_{m,l}}{\Pr_l^t}\frac{\partial\bar{T}_l}{\partial z}\right)dr\right]}_{\text{Term 4}}+\underbrace{\left[r\frac{\varepsilon_{m,l}}{\Pr_l^t}\frac{\partial\bar{T}_l}{\partial z}\right]_{R-\delta}\frac{d\delta}{dz}}_{\text{Term 5}}+\underbrace{\left[r\frac{\varepsilon_{m,l}}{\Pr_l^t}\frac{\partial\bar{T}_l}{\partial r}\right]_{R-\delta}}_{\text{Term 6}}
\end{aligned} \tag{A-15}$$

In the integrated form of the liquid energy equation,

$$r=0 \text{ corresponds to } y=R,\, y^+=R^+,\eta=\left(R^+\right)^{1/7} \text{ and} \tag{K-1}$$

$$r=R-\delta \text{ corresponds to } y=\delta,\, y^+=\delta^+,\eta=\left(\delta^+\right)^{1/7} \tag{K-2}$$

Re-arranging C-1 and G-1

$$\bar{u}_l=\left(u_l^+\right)\left(u_v^*\right)=\left(u_v^*\right)\left(b_0+b_1\eta+b_2\eta^2\right)$$

$$\bar{T}_l=T_w+\left(\frac{q_w''}{\rho_v c_{p,v}u_v^*}\right)T_l^+$$

$$\bar{T}_l=T_w+\left(\frac{q_w''}{\rho_v c_{p,v}u_v^*}\right)\left(d_0+d_1\eta+d_2\eta^2\right)$$

Equation A-15 is now considered term-by-term.

Term 1:

Substituting from Equations C-1 and G-1 and evaluating the product gives

$$\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\bar{u}_l\overline{T}_l\right)dr\right]=\frac{d}{dz}\left\{\int_0^{R-\delta}\left[\begin{array}{l}\left(T_w u_v^*\right)\left(b_0 r+b_1\eta r+b_2\eta^2 r\right)+\\ \dfrac{q_w''}{\rho_v c_{p,v}}\left[\begin{array}{l}b_0 d_0 r+\left(b_1 d_0+d_1 b_0\right)\eta r+\left(b_2 d_0+b_1 d_1+b_0 d_2\right)\eta^2 r+\\ \left(b_1 d_2+b_2 d_1\right)\eta^3 r+b_2 d_2\eta^4 r\end{array}\right]\end{array}\right]dr\right\}$$

In the above expression, the coefficients b_0, b_1, b_2 and d_0, d_1, d_2, the wall heat flux, friction velocity and the wall temperature are all functions of z alone. Also, the variable η is a function of r. Integrating the above with respect to r and applying limits using Equations K-1 and K-2 yield

$$\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\bar{u}_l\overline{T}_l\right)dr\right]=\frac{d}{dz}\left\{\begin{array}{l}\left(T_w u_v^*\right)\left\{\begin{array}{l}\dfrac{b_0}{2}(R-\delta)^2+7b_1\left[\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)+\dfrac{7R^2}{120}\left(R^+\right)^{1/7}\right]\\ +7b_2\left[\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)+\dfrac{7R^2}{144}\left(R^+\right)^{2/7}\right]\end{array}\right\}+\\ \dfrac{q_w''}{\rho_v c_{p,v}}\left\{\begin{array}{l}\dfrac{b_0 d_0}{2}(R-\delta)^2+7\left(b_1 d_0+d_1 b_0\right)\left[\left(\delta^+\right)^{1/7}(\delta)\left(\dfrac{\delta}{15}-\dfrac{R}{8}\right)+\dfrac{7R^2}{120}\left(R^+\right)^{1/7}\right]\\ +7\left(b_2 d_0+b_1 d_1+b_0 d_2\right)\left[\left(\delta^+\right)^{2/7}(\delta)\left(\dfrac{\delta}{16}-\dfrac{R}{9}\right)+\dfrac{7R^2}{144}\left(R^+\right)^{2/7}\right]+\\ 7\left(b_1 d_2+b_2 d_1\right)\left[\left(\delta^+\right)^{3/7}(\delta)\left(\dfrac{\delta}{17}-\dfrac{R}{10}\right)+\dfrac{7R^2}{170}\left(R^+\right)^{3/7}\right]+\\ 7b_2 d_2\left[\left(\delta^+\right)^{4/7}(\delta)\left(\dfrac{\delta}{18}-\dfrac{R}{11}\right)+\dfrac{7R^2}{198}\left(R^+\right)^{4/7}\right]\end{array}\right\}\end{array}\right\}$$

Differentiating the above expression with respect to z yields an expression for Term 1. This is given in Equation K-3.

$$\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\overline{u}_l\overline{T}_l\right)dr\right]=$$

$$\left\{\begin{aligned}&\frac{b_0}{2}(R-\delta)^2+7b_1\left[\left(\delta^+\right)^{1/7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]\\&+7b_2\left[\left(\delta^+\right)^{2/7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]\end{aligned}\right\}\left(u_v^*\frac{dT_w}{dz}+\frac{T_w}{2\sqrt{\rho_v\tau_w}}\frac{d\tau_w}{dz}\right)$$

$$+\left(u_v^*T_w\right)\left\{\begin{aligned}&\frac{(R-\delta)^2}{2}\frac{db_0}{dz}-b_0(R-\delta)\frac{d\delta}{dz}+7\left[\left(\delta^+\right)^{1/7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]\frac{db_1}{dz}+\\&7b_1\left[\left(\delta^+\right)^{1/7}\left(\frac{2\delta}{15}-\frac{R}{8}\right)\frac{d\delta}{dz}+\frac{1}{7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}+\frac{R^2}{120}\left(R^+\right)^{-6/7}\frac{dR^+}{dz}\right]+\\&7\left[\left(\delta^+\right)^{2/7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]\frac{db_2}{dz}+\\&7b_2\left[\left(\delta^+\right)^{2/7}\left(\frac{2\delta}{16}-\frac{R}{9}\right)\frac{d\delta}{dz}+\frac{2}{7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}+\frac{R^2}{72}\left(R^+\right)^{-5/7}\frac{dR^+}{dz}\right]\end{aligned}\right\}+$$

$$\frac{q_w''}{\rho_v c_{p,v}}\left\{\begin{aligned}&\frac{(R-\delta)^2}{2}\left(d_0\frac{db_0}{dz}+b_0\frac{dd_0}{dz}\right)-b_0d_0(R-\delta)\frac{d\delta}{dz}+\\&7\left(b_1\frac{dd_0}{dz}+d_1\frac{db_0}{dz}+b_0\frac{dd_1}{dz}+d_0\frac{db_1}{dz}\right)\left[\left(\delta^+\right)^{1/7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]+\\&7b_1d_0\left[\left(\delta^+\right)^{1/7}\left(\frac{2\delta}{15}-\frac{R}{8}\right)\frac{d\delta}{dz}+\frac{1}{7}\left(\frac{\delta^2}{15}-\frac{R\delta}{8}\right)\left(\delta^+\right)^{-6/7}\frac{d\delta^+}{dz}+\frac{R^2}{120}\left(R^+\right)^{-6/7}\frac{dR^+}{dz}\right]+\\&7\left(b_2\frac{dd_0}{dz}+b_1\frac{dd_1}{dz}+b_0\frac{dd_2}{dz}+d_0\frac{db_2}{dz}+d_1\frac{db_1}{dz}+d_2\frac{db_0}{dz}\right)\left[\left(\delta^+\right)^{2/7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right]+\\&7\left(b_2d_0+b_1d_1+b_0d_2\right)\left[\left(\delta^+\right)^{2/7}\left(\frac{2\delta}{16}-\frac{R}{9}\right)\frac{d\delta}{dz}+\frac{2}{7}\left(\frac{\delta^2}{16}-\frac{R\delta}{9}\right)\left(\delta^+\right)^{-5/7}\frac{d\delta^+}{dz}+\frac{R^2}{72}\left(R^+\right)^{-5/7}\frac{dR^+}{dz}\right]+\\&7\left(b_2\frac{dd_1}{dz}+b_1\frac{dd_2}{dz}+b_2\frac{dd_1}{dz}+d_1\frac{db_2}{dz}\right)\left[\left(\delta^+\right)^{3/7}\left(\frac{\delta^2}{17}-\frac{R\delta}{10}\right)+\frac{7R^2}{170}\left(R^+\right)^{3/7}\right]+\\&7\left(b_1d_2+b_2d_1\right)\left[\left(\delta^+\right)^{3/7}\left(\frac{2\delta}{17}-\frac{R}{10}\right)\frac{d\delta}{dz}+\frac{3}{7}\left(\frac{\delta^2}{17}-\frac{R\delta}{10}\right)\left(\delta^+\right)^{-4/7}\frac{d\delta^+}{dz}+\frac{3R^2}{170}\left(R^+\right)^{-4/7}\frac{dR^+}{dz}\right]+\\&7\left(d_2\frac{db_2}{dz}+b_2\frac{dd_2}{dz}\right)\left[\left(\delta^+\right)^{4/7}\left(\frac{\delta^2}{18}-\frac{R\delta}{11}\right)+\frac{7R^2}{198}\left(R^+\right)^{4/7}\right]+\\&7b_2d_2\left[\left(\delta^+\right)^{4/7}\left(\frac{2\delta}{18}-\frac{R}{11}\right)\frac{d\delta}{dz}+\frac{4}{7}\left(\frac{\delta^2}{18}-\frac{R\delta}{11}\right)\left(\delta^+\right)^{-3/7}\frac{d\delta^+}{dz}+\frac{2R^2}{99}\left(R^+\right)^{-3/7}\frac{dR^+}{dz}\right]\end{aligned}\right\}$$

$$+\frac{1}{\rho_v c_{p,v}}\left\{\begin{array}{l}\frac{b_0 d_0}{2}(R-\delta)^2+7(b_1 d_0+d_1 b_0)\left[(\delta^+)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}(R^+)^{1/7}\right]\\ +7(b_2 d_0+b_1 d_1+b_0 d_2)\left[(\delta^+)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}(R^+)^{2/7}\right]+\\ 7(b_1 d_2+b_2 d_1)\left[(\delta^+)^{3/7}(\delta)\left(\frac{\delta}{17}-\frac{R}{10}\right)+\frac{7R^2}{170}(R^+)^{3/7}\right]\\ +7b_2 d_2\left[(\delta^+)^{4/7}(\delta)\left(\frac{\delta}{18}-\frac{R}{11}\right)+\frac{7R^2}{198}(R^+)^{4/7}\right]\end{array}\right\}\frac{dq''_w}{dz} \qquad \text{(K-3)}$$

Term 2:

Substituting from Equations C-1 and G-1 and evaluating Term 2 using Equation K-2 yields

$$(R-\delta)(u_v^*)\left[b_0+b_1(\delta^+)^{1/7}+b_2(\delta^+)^{2/7}\right]\left\{T_w+\left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right)\left[d_0+d_1(\delta^+)^{1/7}+d_2(\delta^+)^{2/7}\right]\right\}\frac{d\delta}{dz} \qquad \text{(K-4)}$$

Term 3:

Using definitions from Equation G-1 and E-2, Term 3 can be re-arranged as

$$\left(r\bar{v}_l\bar{T}_l\right)\Big|_{R-\delta}=\left\{r\left(u_v^* v_l^+\right)\left[T_w+T_l^+\left(\frac{q''_w}{\rho_v c_{p,v} u_v^*}\right)\right]\right\}_{R-\delta}$$

Substituting for the non-dimensional temperature distribution from Equation G-1 and the dimensionless radial component of the liquid velocity from Equation E-6 and evaluating using Equation K-2 gives

$$\left(r\bar{v}_l\bar{T}_l\right)\Big|_{R-\delta}=\left[\begin{array}{l} u_v^*\left\{T_w+\left[d_0+d_1\left(\delta^+\right)^{1/7}+d_2\left(\delta^+\right)^{2/7}\right]\frac{q_w''}{\rho_v c_{p,v}u_v^*}\right\} \\ \left\{\begin{array}{l} K_1(z)-\frac{1}{2\tau_w}\left[\frac{b_0(R-\delta)^2}{2}+8b_1(\delta)\left(\delta^+\right)^{1/7}\left(\frac{\delta}{15}-\frac{R}{8}\right)+9b_2(\delta)\left(\delta^+\right)^{2/7}\left(\frac{\delta}{16}-\frac{R}{9}\right)\right] \\ -\left[\frac{(R-\delta)^2}{2}\frac{db_0}{dz}+7\frac{db_2}{dz}\left[(\delta)\left(\delta^+\right)^{2/7}\left(\frac{\delta}{16}-\frac{R}{9}\right)-2\left(R^+\right)^{1/7}(\delta)\left(\delta^+\right)^{1/7}\left(\frac{\delta}{15}-\frac{R}{8}\right)\right]\right] \end{array}\right\} \end{array}\right] \qquad \text{(K-5)}$$

Substituting for $K_1(z)$ from Equation E-8 gives an expression for Term 3.

Term 4:

Using Equation G-22 for $\frac{\partial \bar{T}_l}{\partial z}$, assuming constant values for the turbulent Prandtl number for liquid and the eddy diffusivity of momentum for liquid phase, and performing the integration using Equations K-1 and K-2 gives

$$\frac{d}{dz}\left[\int_0^{R-\delta}\left(r\frac{\varepsilon_{m,l}}{\Pr_l^t}\frac{\partial \bar{T}_l}{\partial z}\right)dr\right]=\frac{\varepsilon_{m,l}}{\Pr_l^t}\frac{d}{dz}\left\{\begin{array}{l} \frac{dT_w}{dz}\frac{(R-\delta)^2}{2}+ \\ \frac{q_w''}{\rho_v c_{p,v}u_v^*}\left\{\begin{array}{l} \frac{(R-\delta)^2}{2}\frac{dd_0}{dz}+7\frac{dd_2}{dz}\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right] \\ -14\left(R^+\right)^{1/7}\frac{dd_2}{dz}\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right] \end{array}\right\}+ \\ \frac{1}{\rho_v c_{p,v}u_v^*}\frac{q_w''}{2\tau_w}\frac{d\tau_w}{dz}\left\{\begin{array}{l} d_1\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]+ \\ 2d_2\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right] \end{array}\right\}+ \\ \frac{1}{\rho_v c_{p,v}u_v^*}\left(\frac{dq_w''}{dz}-\frac{q_w''}{2\tau_w}\frac{d\tau_w}{dz}\right)\left\{\begin{array}{l} \frac{(R-\delta)^2}{2}d_0+ \\ 7d_1\left[\left(\delta^+\right)^{1/7}(\delta)\left(\frac{\delta}{15}-\frac{R}{8}\right)+\frac{7R^2}{120}\left(R^+\right)^{1/7}\right]+ \\ 7d_2\left[\left(\delta^+\right)^{2/7}(\delta)\left(\frac{\delta}{16}-\frac{R}{9}\right)+\frac{7R^2}{144}\left(R^+\right)^{2/7}\right] \end{array}\right\} \end{array}\right\}$$

Differentiating the above with respect to z gives an expression for Term 4. In the above equation, the vapor film thickness, friction velocity, wall heat flux, wall temperature, the wall shear stress and the coefficients d_0, d_1, d_2 are all functions of the axial variable z.

Term 5:

Using Equation G-22 for $\frac{\partial \overline{T}_l}{\partial z}$ along with Equations B-20 and B-27 and evaluating Term 5 using Equation K-2 yields

$$(R-\delta)\frac{\varepsilon_{m,l}}{\Pr_l^t}\left\{\frac{dT_w}{dz}+\frac{1}{\rho_v c_{p,v} u_v^*}\left[\begin{array}{l} q_w''\left(\frac{dd_0}{dz}+\left(\delta^+\right)^{2/7}\frac{dd_2}{dz}-2\left(R^+\delta^+\right)^{1/7}\frac{dd_2}{dz}\right)+ \\ \frac{q''}{14\tau_w}\left(d_1\left(\delta^+\right)^{1/7}+2d_2\left(\delta^+\right)^{2/7}\right)\frac{d\tau_w}{dz}+ \\ \left(d_0+d_1\left(\delta^+\right)^{1/7}+2d_2\left(\delta^+\right)^{2/7}\right)\left(\frac{dq_w''}{dz}-\frac{q_w''}{2\tau_w}\frac{d\tau_w}{dz}\right) \end{array}\right]\right\}\frac{d\delta}{dz} \qquad \text{(K-7)}$$

Term 6:

Using Equation G-20, $\frac{\partial \overline{T}_l}{\partial r}$ can be evaluated at $r = R - \delta$ as

$$\left.\frac{\partial \overline{T}_l}{\partial r}\right|_{R-\delta} = -\left(\frac{q_w''}{\rho_v c_{p,v} u_v^*}\right)\frac{\left[d_1\left(\delta^+\right)^{1/7}+2d_2\left(\delta^+\right)^{2/7}\right]}{7\delta}$$

Therefore, Term 6 becomes

$$-(R-\delta)\left(\frac{\varepsilon_{m,l}}{\Pr_l^t}\right)\left(\frac{q_w''}{\rho_v c_{p,v} u_v^*}\right)\frac{\left[d_1\left(\delta^+\right)^{1/7}+2d_2\left(\delta^+\right)^{2/7}\right]}{7\delta} \qquad \text{(K-8)}$$

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