MODELING OF PIEZOELECTRIC SMART STRUCTURES FOR
ACTIVE VIBRATION AND NOISE CONTROL APPLICATIONS

A Thesis in
Engineering Science and Mechanics
by
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ABSTRACT

Active vibration control and active structural acoustic control using piezoelectric sensors and actuators have recently emerged as a practical and promising technology. Efficient and accurate modeling of these structures bonded to or embedded with actuators and sensors is needed for efficient design of smart structures. This dissertation addresses the modeling of these structures and the associated control system design technique. Modeling of structures with both laminated and discrete type of actuators and sensors are addressed.

For piezoelectric laminates the governing equations of motion are derived using First Order Shear Deformation Theory (FSDT) and for the first time the dynamic response fields inside the laminate are obtained and compared with full elasticity solutions. This comparison brought out the effect of assumptions made with respect to the electric and mechanical fields using FSDT and Classical Laminate Theory (CLT) in previous work. It is expected that this analysis and the interior field estimations would help designers to understand the shortcomings of FSDT in modeling piezoelectric laminates, and help them to adopt this theory properly for use in FE or other numerical models.

For surface bonded discrete patch type actuators/sensors, the governing dynamic equation of motion for a plate is derived. The solution to this equation is obtained using a Fourier series method and the effect of passive stiffness and mass on the natural frequency is studied. The studies showed that ignoring the mass and the passive stiffness
of actuators/sensors leads to large errors in estimating the vibration characteristics of the smart plate. A Rayleigh-Ritz (RR) approach is then presented for studying the active vibration and transmitted noise control of a smart plate with discrete piezoelectric patches. Classical laminated plate theory is used to model the composite plate and electroelastic theory is used model the piezoelectric patches. The dynamic equations of motion for the coupled smart panel-cavity system are derived using Hamilton’s principle. Close agreement between the present approach and the finite element and experimental results confirmed the validity of the approach. The RR approach is thus presented as a simple, computationally inexpensive approach when compared to the finite element method. The RR method also proved to be powerful method for modeling the adjacent acoustic medium and for the associated control system design.

A finite element approach for the integrated design of a structure and its control system for suppressing vibration and the radiated noise are presented. A finite element model for a smart plate with surface bonded piezoelectric patches is developed using shell, brick and transition elements. The free and forced vibration characteristics of the plate are studied with and without closed loop feedback control. An optimal (multi-input multi-output) MIMO controller design for the vibration suppression of a clamped plate using the FE model is proposed. Numerical simulation showed that an optimal controller designed for controlling the smart plate vibration also reduces the transmitted noise to 20 dB for the first mode and to 40 dB for the second mode of plate.

The RR approach accurately models rigid walled acoustic cavities, but flexible
elastic boundaries or sound absorbing walls cannot be modeled using this approach. To model such acoustic domains, a novel hybrid Rayleigh-Ritz/Boundary Element solution method is proposed. This method would enable designers to model the panel with piezoelectric actuators and sensors and the adjacent acoustic medium with the presence of passive absorbers at the interface. The predicted sound pressure attenuation for three different thicknesses of passive absorber in the frequency range of 200 to 1200 Hz is calculated and an optimal thickness value of for the absorber for the smart panel is calculated. The attenuation in sound pressure levels due to an active control system in the presence of passive absorber is also computed. The system matrices resulting from this method are very smaller in size when compared to the FE models, which makes this approach most suitable for optimization studies. This new approach can be further extended to model the more complicated acoustic enclosures with complex interface conditions.
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Chapter 1

INTRODUCTION

1.1 Background

During the last two decades there has been an accelerating level of interest in the control of sound by active techniques. Many of the physical principles involved have long been established, but the technological means for the successful implementation of active noise control have only recently become feasible. Noise radiation and sound transmission from vibrating structures e.g., automobile bodies, aircraft fuselages and cabins, industrial machinery, etc., are common problems in noise control practices. Two such areas of application are the control of propeller noise in the passenger cabins of aircraft, and the control of low frequency engine-induced noise inside cars.

High acoustic levels are present within these enclosures (Von Flotow and Mercadel, 1995), typically caused by propellers or engine fan spool speed imbalances transmitted via the wing structure, or periodic pressures exerted on the cabin from passing propeller blades. Early noise control strategies for aircraft cabins were based on field cancellation techniques, employing microphone sensors, and speakers as acoustic actuators. These techniques are most effective at low frequencies and easier to implement for periodic sound fields than for random sound fields. For example the sound pressure level inside the passenger cabin of helicopters shows (Figure 1.1) (Wilby 1996) strong components at 15 to 20 Hz, which correspond to the Blade Passing Frequency (BPF) of
the main rotor. Although this noise component is below the lower frequency limit of hearing, prolonged exposure will decrease the comfort of the passengers.

Figure 1.1 The sound pressure level spectrum inside the helicopter cabin  
(G. Niesl, E. Laudien)

The primary contributions to the noise spectrum in a helicopter are the main rotor, tail rotor and turbines, which operate at frequencies ranging from 50 to 500 Hz and the gear mechanisms in the main transmission, which operate at frequencies above 500 Hz. Likewise, the sound pressure spectrum inside the passenger compartments of a propeller aircraft contains strong tonal components at harmonics of the BPF of the propellers, which are difficult to attenuate the using passive absorption (Metzger, 1981; Wilby et al., 1980). The noise cancellation technique typically demonstrated 10 to 15 dB reductions in
sound pressure levels over the first several harmonics of the propeller blade passage frequency. However, reductions were highly spatial, and required many sources (up to 32) for achieving control for small turboprop aircraft (Elliott, 1989). These numbers are required due to the spatial mismatch between the primary acoustic field and the field produced by the interior sources (speakers), creating an interior acoustic field that is more broadband in nature. More details can be obtained on the practical active control system operating in the passenger cabin from references Elliott et al (1989a) and Silcox (1990).

Apart from having to track the excitation frequency, the noise control task inside the car cabin is less extensive than in the aircraft case since the volume of the enclosure is smaller, and fewer loudspeakers and microphones need to be used. At the lowest frequency, only a single loudspeaker and microphone would be needed, which could control a single dominant acoustic mode (Oswald 1984).

1.2 Active Structural Acoustic Control

A more direct approach for noise control makes use of the fact that the acoustic field must pass through the structure. The control of radiated noise is achieved via controlling the vibration of flexible walls and other radiating structures. The methods for suppressing the vibration and the radiated noise from structural members or mechanical components are identified as “Structural Acoustic Control” (SAC), and are classified into two major categories, namely passive and active control methods. The underlying technique in these methods is to alter the vibration characteristics of the radiating structural members and thereby minimize the radiated sound field. Some examples of these radiating structural members are airplane fuselage panels, helicopter cabin panels,
and compressor housings. The first method involves altering the vibration characteristics of structures passively by tailoring their material properties. The second strategy involves altering the vibration characteristics by placing actuators at pre-selected points on the structures and driving these actuators to a predetermined force amplitude and phase. Traditional means for the control of undesired sound and vibration are often referred to as passive methods since no power source is required for the control system. The noise reduction is achieved by either insulating the enclosure walls or by covering them with porous sound absorbers. These passive control methods are quite effective at high frequencies or in narrow frequency bands. The thickness requirement for a passive absorber at lower frequencies is impractical. For example, at low frequency near 200 Hz, the wavelength of the sound is approximately 1.7 meters and efficiently designed passive absorbers for the enclosure wall would be more than a meter in thickness to absorb the sound energy. So the active control method is an attractive alternative to passive methods at these low frequencies.

The active control of vibration and structurally radiated noise also known as Active Structural Acoustic Control (ASAC) using active materials has emerged as a viable technology in recent years. The past decade has seen great advances towards integrating actuators and sensors with electro-active materials into devices and structures. The new design paradigms focus on constructing smaller, more precise systems with emphasis on improving performance and efficiency while meeting increasingly stringent weight, size, and power requirements. This has led to the development of various systems using piezoelectrics, electrostrictors, and shape memory ceramics, and other active
materials (Crawley 1987). A material, which can sense and respond to one or more external stimuli such as pressure, temperature, voltage, electric and magnetic fields, chemicals etc., can be called as an active material. Active materials (also sometimes called smart materials) and structures integrated with these materials have gained worldwide attention in the past few years because of their application in every branch of engineering. Actuators and sensors are becoming integral parts of the structures, which are making the host structures more ‘adaptive’ or ‘smart’. These piezoelectric transducers are used to control beam (Varadan V.K et al, 1990), truss (V. V. Varadan et al, 1993), plates (V.V. Varadan et al, 1991) and shell structures (Tzou et al 1993b). Varadan V. V. et al (1991) showed that the active control of vibration and radiated noise using piezoelectric sensors and actuators is a practical and feasible approach for plate structures. In this work authors have observed that when using an analog controller, the experimentally measured radiated noise showed a maximum reduction of 20 dB. In another experimental investigation of active control of transmitted sound through a plate into a cubic sound enclosure, Xiaoqi Bao et al (1995) showed that one actuator/sensor control system successfully reduced the sound level at the first three resonance frequencies by 15-22 dB. Vibration suppression of the fuselage structure has been studied using both point force devices (shakers) and in-plane bonded piezoceramics as actuators (Silcox et al., 1993; Fuller et al., 1992), illustrating reductions in interior noise levels similar to those obtained with speaker sources. More importantly, these reductions can be broadband in nature, are global throughout the structure, and can be achieved with far few actuator sources. Several full-scale demonstrations have also shown that
piezoceramics are feasible for this application (Fuller and Gibbs, 1994; Mathur and Tran, 1993). Only small out-of-plane structural displacements (on the order of 25 $\mu$m) are required to obtain significant control authority. Studies indicate that ‘extended’ actuator patches are more effective in reducing controller spillover, which is the limiting factor in control performance (Lester and Silcox, 1992). In a more general sense, it is advantageous to tailor the size, location and number of actuators in order to optimize the energy coupling between acoustic and structural responses (Silcox et al., 1993). Both of these conclusions point to the need for directional and (semi) continuous actuator materials. A brief overview of the actuation and sensing materials is presented in the next section.

1.3 Actuation And Sensing Materials

Various piezoelectric materials have been investigated for aerospace and other applications. PZT is a widely used piezoceramic material due to its high piezoelectric, dielectric and elasticity coefficients (Crawley, 1994). Actuation and sensing is applied through electrical signals, and their low field linear behavior has aided in modeling for transducer applications. Their high stiffness gives adequate energy densities, and their fast response times provide high bandwidth. In comparison (Table 1.1) (Bent 1997), other solid-state actuation materials are less well suited for the desired applications.

Piezo-polymer films (PVDF) are robust to damage, but lack high stiffness. Electrostrictive materials (PMN) have low hysteresis losses and high stiffness, but have poor temperature stability, and require high currents to operate due to their high material dielectric. Shape memory alloys (Nitinol) are capable of very high strains, but are limited
to ultra-low bandwidth applications (< 5 Hz) due to the time needed for thermal dissipation/heating. Finally, magnetostrictive actuators (Terfenol-D) have similar actuation energy density and bandwidth as piezoceramics, but are very heavy when the coils and flux path materials are accounted for. The actuators and sensors are incorporated into the host structures in many different forms depending upon the environmental and operating requirements of the host structure. Beams, truss structures, plate and shell-like structures are frequently used host structures for piezoelectric sensors and actuators for vibration and noise control applications. Several have been conceived experimentally such as for vibration control (for plates Bayer et al, 1991, Ghidella and Steven 1994; for beams—Bailey and Hubbard 1985), shape control (Koconis et al 1994), and buckling control (Thompson and Loughlan 1995). The actuators and sensors could either be surface bonded or embedded inside the layers in the form of lamina or fibers (Bent 1997) of the host laminate. A piezoelectric laminate with feedback control circuit is shown in Figure 1.1.

Figure 1.2 Piezoelectric laminate with Feedback circuit
Because of its very stiff and brittle nature, the fabrication of laminates containing PZT layers will be a challenge. Also PZT cannot be subjected to very high temperatures when fabricating the laminates, to prevent depoling and loss of piezoelectric properties. Again, for the surface bonded actuators and sensors the size and shape of these actuators/sensors vary depending on the application for which it is designed.

Table 1.1 Comparison of solid-state actuation materials that provide significant actuation

<table>
<thead>
<tr>
<th>Actuation Mechanism</th>
<th>PZT5H</th>
<th>PVDF</th>
<th>PMN</th>
<th>Terfenol D</th>
<th>Nitinol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max strain</td>
<td>0.13%</td>
<td>0.07%</td>
<td>0.1%</td>
<td>0.2%</td>
<td>2% -8%</td>
</tr>
<tr>
<td>Modulus (GPa)</td>
<td>60.6</td>
<td>2</td>
<td>64.5</td>
<td>29.7</td>
<td>75(hi-temp)-28(lo-temp)</td>
</tr>
<tr>
<td>Density (Kg/m3)</td>
<td>7500</td>
<td>1780</td>
<td>7800</td>
<td>9250</td>
<td>7100</td>
</tr>
<tr>
<td>Temp Range</td>
<td>-20 to 200oC</td>
<td>Low</td>
<td>0 to 40oC</td>
<td>High</td>
<td>-</td>
</tr>
<tr>
<td>Useful frequency range</td>
<td>0-100 MHz</td>
<td>0-10 MHz</td>
<td>100 kHz</td>
<td>&lt;10 kHz</td>
<td>&lt;5 Hz</td>
</tr>
<tr>
<td>Actuation Energy density (J/Kg)</td>
<td>6.83</td>
<td>0.28</td>
<td>4.13</td>
<td>6.42</td>
<td>252-4032</td>
</tr>
</tbody>
</table>

Rectangular or circular patches are the most common form for actuators and sensors in noise and vibration control applications. Development of simple and accurate models, analysis methods, and a reliable controller design for the structures bonded with the piezoelectric layers or patches are interesting areas of research.

This dissertation deals with modeling of both patch type and film type actuators/sensors for vibration and noise control purposes. There are numerous other aerospace and related field applications that can or do utilize active materials to achieve improved performance: structural health monitoring, damage mitigation, and de-icing, to name a few. The models and analysis approaches developed in the present work can be
used or extended without any difficulty for these applications.

1.4 Smart Structure Modeling Issues

In order to achieve low weight, smaller size and cost-effective design of smart structures using strain sensing and actuation properties of piezoelectric materials, we need accurate and reliable models for analysis and design. To develop such models, the interaction between the structure and the actuators/sensors must be well understood. There have been many theories and models proposed for analysis of laminated composite beams and plates containing active and passive piezoelectric layers. Early research in the design and analysis of smart structures for various applications used simplified theories and models for various types of structures. Classical (closed form solutions), numerical (such as finite difference), and physical modeling (such as finite element and boundary element) approaches have been extensively used to predict and analyze characteristics of the structures and machinery. However, the continuous demand for more precise systems has forced researchers and engineers to search for more accurate and efficient models to investigate the dynamics of structures with attached (or embedded) smart devices.

Some of the earliest models (Crawley and de Luis 1987, Bailey and Hubbard 1985, Robbins and Reddy 1991) used the induced strain by the piezoelectric actuators as an applied strain that contributed to the total strain of the non-active structure, similar to a thermal strain contribution. However, for complex intelligent structures with a significant amount of sensors and actuators distributed in the structure, the electro-mechanical coupling within the piezoelectric material as well as the coupling with the substrate must
be more fully integrated into the formulation. To find analytical solutions, electrical and mechanical equilibrium or governing field equations have to be solved for a set of boundary conditions (e.g. for plates—Heyliger 1997). The coupling of the electrical and mechanical constitutive equations will lead to the coupling of some boundary conditions. Hence, applying mechanical boundary conditions in the conventional way may not always be correct due to the coupling with electrical boundary conditions. In addition, analytical solutions often assume the solution with a certain series form and establish coefficients using the governing equations of mechanics. Since the assumed solution depends on the geometry of the structure, a set of solutions derived for rectangular plates would not be applicable to a circular plate structure or any other type of geometry. An alternative to solving exact analytical equations is to use finite element analysis (FEA). A truly sophisticated smart or intelligent structure is a composite of active and non-active materials and thus all coupling between actuators, substrate and sensors must be included in the model. This requirement is effectively facilitated by the Hamilton variational principle or energy methods. The variational principle based FE formulation is an effective method for complex structures because mechanical and electrical equilibrium equations do not need to be solved explicitly. The physics of the entire structure has been fully accounted for in the energy integrals and there is no need to derive equations based on forces and moments

1.5 Objectives Of the Present Research

As we have seen from the overview of the previous research carried out in this
area the modeling approach and analysis techniques differ considerably between the laminated type smart structures and smart structures bonded with the discrete type active materials. The objective of this thesis is to present a linear modeling approach and analysis method for analyzing smart structures. There are five major components to the work being presented here.

1. First order Shear deformation model for piezoelectric laminates.
2. Classical laminated plate model for smart plates incorporating patch type actuators/sensors.
3. Finite element modeling and demonstration of closed loop feedback control for vibration and radiated noise control application.
4. Demonstration of Rayleigh-Ritz analysis method using the model developed for smart structures.
5. A novel and efficient method of coupling the Rayleigh-Ritz approach for the smart panel with the Boundary Element formulation for the cavity.

The first two models would contribute to the smart structure design and analysis community to use more accurate models for analysis and better understanding of the behavior of these electromechanical materials when used as actuators or sensors in aerospace structures. The third and fourth models are aimed at the development of closed loop active radiated noise control analysis techniques using the finite element and Rayleigh Ritz approaches. These methods are applied for the first time for noise control applications using closed loop feedback techniques.
1.6 Scope Of Thesis

The objective of the present thesis work is to develop the linear modeling techniques for electromechanically coupled material to the extent which would permit analysis of structures incorporating active materials.

The review of the previous research related to this dissertation is presented in chapter 2. The various models and analysis methods developed for the analysis of smart structures for various applications are reviewed and their merits and drawbacks are discussed.

The development of the First Order Shear Deformation Theory (FSDT) for piezoelectric laminates is presented in Chapter 3. The electric and mechanical fields obtained using this FSDT model is compared with fields obtained using full electroelastic models for the laminate.

This is followed by the development of an electroelastic model for plates with surface bonded discrete piezoelectric patches presented in chapter 4. Two methods of solution approaches are attempted. Firstly, a Fourier series method is presented to solve the dynamic equations resulting from this model. Secondly, using the analytical model developed for the smart structure a vibroacoustic system with a smart panel is studied using the Rayleigh-Ritz approach and active control of noise transmitted into a rectangular enclosure is demonstrated.

Chapter 5 is dedicated to the formulation of the linear finite element model for the closed loop vibration and radiated noise. The smart structure is modeled using the flat shell and brick elements and the acoustic cavity is modeled using the rigid walled cavity
modes.

Chapter 6 presents an elegant approach for the active noise control problems by coupling the Rayleigh-Ritz model of the host structure and the dual reciprocity boundary element model of the acoustic cavity.

Chapter 7 closes the thesis with a summary of the contributions and recommendations for future work in the modeling of smart structures for active vibration and radiated noise control applications.
Chapter 2

RESEARCH BACKGROUND

2.1 Introduction

The references listed in this chapter are grouped under three major categories. Group I consists of references that are related to the previous models developed for analysis of piezoelectric laminates. Previous works on both analytical and finite element models are presented in this section. Group II references are related to the models developed for the smart structure attached with patch type actuators/sensor. In group III, previous research carried out on developing vibroacoustic models, which include smart materials and on the control system design for smart structures are discussed.

2.2 Models for Piezoelectric Laminates

The development of smart composites offers great potential for advanced aerospace structural applications. Piezopolymeric and piezoceramic are employed as both actuators and sensors in the development of these structures by taking advantage of direct and converse piezoelectric effects. The modeling of piezoelectric laminates can be enhanced in two ways; more accurate mechanics models to address the characteristics of composite laminates and more accurate electroelastic models which addresses the coupling effects between the electrical and mechanical fields inside the actuators and sensors.
The fundamental work of Tiersten (1969) gave much of the necessary theoretical development for the static and dynamic behavior of a single-layer piezoelectric plate. Lee and Moon (1989), and Lee (1990) used the assumptions of Classical (Kirchoff’s) Laminated Plate Theory (CLPT) to derive a simple theory for piezoelectric laminates, used primarily for the design of piezoelectric laminates for bending and torsional control. Many researchers (Lam et al 1999, Chandrashekhara et al 1993 and Hwang et al 1993) used this model and variations of this model for designing piezoelectric laminates for various applications. They have adopted the CLPT model in both analytical and numerical methods for analysis and design purposes. These models use simplifying approximations in characterizing the induced strain field and electric fields generated due to an external load or applied voltage. The kinematic approximations made on the mechanical fields by the laminate theories impose restrictions on choosing the electrical field variables, which affects the accuracy of estimating the electromechanical characteristics of the laminate structure. Therefore, it is necessary to understand the effect of applied load or voltage on the induced field distribution inside the layers to validate the range of these assumptions. No attempt has been made to study the effect of these CLPT assumptions or to study the electromechanical field distributions inside these laminates until now. The fields inside layers of a piezoelectric laminate are previously examined by Ray et al (1993), Roh et al (1996) and Heyliger (1997) using full elasticity theories, without any approximations and assumptions on the mechanical and electrical fields. The exact solution obtained using this exact elasticity theory indicated that the electric and elastic field distributions are often poorly modeled using simplified theories.
They showed that the electric field inside the sensor layers is not zero and both the electrical and mechanical field distributions are evidently affected by the relative values of the dielectric constants of the layers in a three layered cross-ply PVDF laminate. To increase the order of variation of electrical and mechanical fields inside the layers, higher order theories are to be employed in the laminate models. Bisegna et al (1996), showed that when the thickness to width ratio of the plate ($AR$) is less than or equal to $\frac{1}{5}$, First Order Shear Deformation Theory, (FSDT) provides results which differ from the exact solution by 20% for displacements, electric potential and the in-plane stress components. More recently Yang (1999) included higher order (quadratic) electric potential variation through the thickness of the actuators and obtained two-dimensional equations for the bending motion of elastic plates with partially electroded piezoelectric actuators attached to the top and bottom surfaces of a thick plate. Although negligible for thin actuators, this effect needs to be considered for thick actuators. Tiersten (1993) derived the approximate equations for extensional and flexural motion of a thin piezoelectric plate subjected to large electric fields. Up to cubic order terms are included in the expansion of electric potential across the plate thickness to describe the higher order electrical behavior, and showed that for a very thin plate, the quadratic and cubic terms in the expansion can be ignored. Using a variational formulation, Krommer and Irschik (1998) observed that for a Timoshenko type smart piezoelectric beam, the potential inside the smart beam could be expressed as a quadratic function in the thickness coordinate. The sensor signal derived using this expansion for closed electrode
conditions leads to the one obtained by Lee et al (1990).

Both two and three-dimensional models were developed to study the dynamic behavior of the piezoelectric laminate. Most of the two-dimensional FE analysis for plate and shell like smart structure are again based on the classical plate or laminated theories in which the in-plane displacement fields are linear through the thickness. More accurate theories, namely discrete layer theories (Mitchell and Reddy 1995), and layerwise theories (Saravanos et al 1997) are developed for the static and dynamic analysis of piezoelectric laminates. Again in most of the corresponding finite element modeling (Lam et al 1997, Hwang et al 1993, and Chandrashekhara et al 1993) using CLPT, the electric field inside the actuator is assumed to be constant and the sensor signal is obtained using the approach shown by Lee et al (1990).

In the case of layerwise theory (Saravanos et al 1997), for the piezoelectric laminates the mechanical displacements and the electric potential are assumed to be piecewise continuous across the thickness of the laminate. These theories provide a much more kinematically correct representation of cross sectional warping and capturing nonlinear variation of electric potential through the thickness associated with thick laminates. The developments of layerwise laminate theory for a laminate with embedded piezoelectric sensors and actuators are presented in the above-mentioned reference. Comparisons of the predicted free vibration results from the developed layerwise theory with the exact solutions for a simply supported piezoelectric laminate brings out the improved accuracy and robustness of the layerwise theory over the Classical (Kirchoff’s)
Laminated Plate Theory (CLT) or First Order Shear Deformation Theory (FSDT) (Gopinathan et al, 2000) for piezoelectric laminates.

Therefore, it is necessary to know the electrical and mechanical field distributions inside the piezoelectric laminates when modeled using CLPT to validate the assumptions of the CLPT on both fields. To the author’s knowledge no comparison has been attempted of the electromechanical fields inside the layers of a piezoelectric laminate modeled using the first order shear deformation theory (FSDT) and a more exact elasticity theory to bring out the effects and validity of the approximations made in FSDT. In the next chapter, using FSDT, the equations of motion for three-layered piezoelectric laminate are formulated and the electromechanical field response inside the layers due to an applied mechanical load or electric potential is estimated. The results obtained from this theory are compared with those obtained using the two-dimensional electroelasticity solution approach.

2.3 Models for Smart Structures with Discrete Piezoelectric Patches

Alternatively, discrete monolithic pieces of piezoelectric material can also be used for sensing and control purposes, instead of covering the entire surface in the form of layers due to weight and difficulty in fabrication considerations. The discrete patches can either be attached to the substrate surface as shown by Crawley and Lazarus (1991), Thomson and Loughlan (1995), Ha, Keilers and Chang (1992), Valey and Rao (1994), or embedded within the substrate as in the work of Crawley and deLuis (1987). In all these previous models the usual assumption is that unless an electric field is applied the
presence of the piezoelectric material on or in the substrate does not alter the overall structural properties significantly. It is also assumed that the thickness of the bonding adhesive layer is negligible and causes negligible property changes.

The analysis to determine the induced strain actuation for a composite laminated plate was developed by Wang and Rogers (1991) using CLPT. Induced strain due to discrete piezoelectric actuators contributed to the total strain and the distribution of these actuators throughout the laminate were accounted for by using the Heaviside function. Tzou et al (1994a,b) investigated the distributed sensing and controlling of a simply supported plate structure using discrete piezoelectric patches. In this work the modal sensing and control of the plate is derived using the modal expansion method and equivalent line control moments are derived in the modal domain. Authors used classical plate theory and mass and passive stiffness effects of the patch are neglected in estimating the dynamic characteristics of the plate. Many researchers have used this analytical model for various applications where piezoelectric patches are used for controlling beams, plates and shells.

For discrete piezoelectric patches embedded or bonded on beam and plate structures, researchers have mostly used numerical methods like the finite element method. Obtaining exact solutions is difficult for these types of structures so the finite element method is the best-suited method for the static and dynamic analysis of such structures. Ha and Keilers (1992) developed a three-dimensional brick element to study the dynamic as well as the static response of plates containing distributed piezoelectric ceramics. However, they adopted some special techniques to overcome the disadvantages
and inaccuracy of modeling a plate with three-dimensional elements. Kim et al (1997) developed a transition element to connect the three-dimensional solid elements in the piezoelectric region to the flat-shell elements used for the plate. This approach has merits in terms of accuracy in modeling the piezoelectric patches and computational economy for the plate structure. The use of shell elements is preferred for the structure since brick elements unless chosen properly, lead to unnatural stiffening of the plate and artificially high natural frequencies. Hwang and Park (1993) used a four-noded quadrilateral element and Chandrashekhara and Agarwal (1993) developed a nine-noded shear flexible finite element to study the dynamics of the laminated plate with actuators and sensors.

2.4 Models for Vibroacoustic Systems with Smart Structures

Application of discrete sensors and actuators mounted to structures for active radiated noise control via controlling the panel (ASAC) vibration is demonstrated by many researchers. A theoretical analysis by Fuller et al (1990) showed the feasibility of actively controlling the sound transmission/radiation with a few actuators applying forces to the plate. Hong et al (1993) showed that a few discretely located small actuators can be effectively used to control the vibrations and radiated sound from relatively large structures. In Hong’s work, in an experiment considering an automobile fuel tank as an acoustic enclosure they have achieved a considerable reduction in the noise levels inside the tank just by using two disc shaped piezoelectric actuators mounted on the tank wall. In an experiment using a cubic enclosure, Xiaoqi Bao et al (1995) also showed that the
one pair of discrete piezoelectric patches with closed loop feedback control showed significant reductions of 15-22 dB in the enclosure pressure field. Howarth et al (1991) used a 1-3 piezocomposite for absorbing the incident acoustic energy, thereby acting as a non-reflective surface coating. The absorbed acoustic energy is dissipated as heat from electrical networks to which this actuator is connected. A model for a two-dimensional acoustic cavity with a flexible boundary (beam) controlled via piezoceramic patches producing bending moments in the beam is considered in a work carried out by Banks et al (1991). Feedback control of these actuators demonstrated the noise reduction inside the cavity. In a work carried out by Lester et al (1993), analytical models for piezoelectric actuators, adapted from flat plate concepts, are developed for noise and vibration control applications associated with vibrating circular cylinders. The loadings applied to the cylinder by the piezoelectric actuators for the bending and in-plane force models are approximated by line moment and line force distributions, respectively, acting on the perimeter of the actuator patch area. Coupling between the cylinder and interior acoustic cavity is examined by studying the modal spectra, particularly for the low-order cylinder modes that couple efficiently with the cavity at low frequencies. Using a similar type of analytical model for the actuator/sensor mounted panel, in a recent study by Balachandran et al (1996), local noise reductions of 30 to 40 dB were achieved in a three-dimensional enclosure using microphone sensors and piezoelectric actuators. As far as the acoustic cavity is considered, there are many possible theoretical models of enclosed sound fields, using, for example, modes, images or rays, or numerical methods using, for example, finite element and boundary element methods. Many researchers prefer the
modal models, which are simple and can be used for rectangular and cylindrical type cavities without any difficulty.

Though the simple analytical models serve the purpose of modeling the smart walls and panels of simple rectangular and circular shapes, more powerful numerical models like finite element models are needed for modeling complex shaped smart panels or boundary conditions other than the simply supported case. Using a detailed finite element model, Jaehwan Kim et al (1995) carried out the optimization of the geometry and the excitation voltage applied to a piezoelectric actuator bonded on a plate to reduce the sound radiation from the host plate. In chapter 4 a closed loop feedback system with a P-D controller is employed to control the vibration amplitudes of the wall with collocated actuators/sensors. A detailed finite element model was used in modeling the panels with actuators/sensors and a simple modal model was used to represent the pressure field inside the cavity. The pressure at any point inside the enclosure is expressed as the sum of acoustic mode shapes of the rigid walled cavity.

2.5 Closure

In this chapter, many relevant earlier research works were reviewed and their merits and drawbacks are discussed. The models derived in forthcoming chapters for piezoelectric laminates and laminates with the discrete piezoelectric patches for vibration and radiated noise control applications have better approximations over these models discussed in this chapter and a more detailed discussions on the improvements are discussed during the development of the models.
Chapter 3

FIRST ORDER SHEAR DEFORMATION THEORY FOR PIEZOELECTRIC LAMINATES

3.1 Introduction

There are many finite element models developed for laminates using CLPT with various assumptions and approximations on the electrical and mechanical fields. While the consequences of approximations made on mechanical fields are understood no study has been carried out until now to understand the effect of approximations made on the electrical fields. No comparison has been attempted of the behavior of the electromechanical fields inside the layers of a piezoelectric laminate using the CLT or FSDT and a more exact elasticity theory to delineate the effects and validity of the approximations made in CLPT or FSDT. In this chapter, using FSDT, the equations of motion for three-layered piezoelectric laminate is formulated and the electromechanical response field inside the layers due to an applied mechanical load or electric potential are estimated. The results obtained from this theory are compared with those obtained using the two-dimensional electroelasticity solution approach.

3.2 First order Shear Deformation Laminate Theory

In this section the response of a piezoelectric laminated beam is estimated using first order shear deformation laminated plate theory (FSDT). The frequencies and the field distributions are estimated for the sensor and actuator type piezoelectric
laminates. The fields estimated from this theory are then compared with those obtained from the exact solution method described in the next section. The governing equations of fully coupled linear piezoelectricity and the constitutive relations for a thickness polarized (poled in z-direction) transversely isotropic (with 6mm symmetry) are given in Appendix A.

In classical lamination theory, the displacements for the bending vibrations of a thin beam are assumed as

\[ w = w(y)e^{i\alpha} \]
\[ v = (v_0(y) - z\psi(y))e^{i\alpha} \]  \hspace{1cm} (3.1)

where \( v \) and \( w \) are the axial and transverse displacements of the beam and \( \psi \) is the shear deformation angle (Fig. 3.1).

![Figure 3.1 Undeformed and deformed configuration of the piezoelectric laminate](image)

(a) Initial undeformed configuration (b) Deformed configuration

The shear deformation angle \( \psi \) is assumed to be independent of the transverse displacement \( w \). We assume that the displacements, stresses, electric potential and electric displacement are continuous across the layer interface. Also from the plate theory
assumption the transverse normal stress $\sigma_{zz}$ is zero. The equations of motion are

$$\frac{\partial \sigma_{y}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = \rho \ddot{y}, \quad \text{and} \quad \frac{\partial \sigma_{z}}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} = \rho \ddot{w}$$

(3.2)

where $\sigma_y$ is the bending stress, $\tau_{yz}$ is the transverse shear stress and $\rho$ is the density of the layer material. Neglecting the $y$ component of electric displacement ($D_y$), the charge equation (refer equation A.1 of appendix A) reduces to

$$\frac{dD_y}{dz} = 0.$$  

(3.3)

The assumption $D_y = 0$ imposes a condition which relates the $y$ component of the electric field $E_y$ and the shear strain $\gamma_{yz}$. Using this and the constitutive equation (refer to equation A.4) for the $y$ component of electric displacement, we find that the shear strain inside a piezoelectric layer is related to $E_y$ as

$$E_y = \frac{e_{24}}{\varepsilon_{22}} \gamma_{yz}.$$  

(3.4)

Alternatively one can assume the $y$ component of the electric field to be zero ($E_y = 0$) instead. In this case, the charge equation reduces to

$$e_{24} \frac{\partial \gamma_{yz}}{\partial y} + e_{32} \frac{\partial \varepsilon_{yz}}{\partial z} = \varepsilon_{33} \frac{dE_z}{dz} = 0.$$  

(3.5)

Firstly we consider the $D_y = 0$ assumption and the $E_y = 0$ assumption is considered later in this chapter.

The first order approximations to the strains are
\[ \varepsilon_y = \frac{\partial v}{\partial y} = \frac{dv_0}{dy} - z \frac{d\psi}{dy} = \varepsilon_y^0 - z\varepsilon_y^{(1)}, \quad \text{and} \]
\[ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{dw}{dy} - \psi, \]

where \( \varepsilon_y^0 = \frac{dv_0}{dy} \) and \( \varepsilon_y^{(1)} = \frac{d\psi}{dy} \).

The constitutive equations for each layer are
\[ \sigma_y = C_{22}\varepsilon_y + e_{32}E_z, \]
\[ \tau_{yz} = \left\{ C_{44} + \left(\frac{e_{24}}{\varepsilon_{22}}\right)^2 \right\} \gamma_{yz}, \quad \text{and} \]
\[ D_z = e_{32}\varepsilon_y - \varepsilon_{33}E_z. \]

From the charge equation we see that the assumption of linear variation of the electric potential inside the piezoelectric layers would result in the bending strain \( \varepsilon_y^{(1)} \) being zero everywhere inside the layers. This is not physically possible, therefore to avoid this the potential inside the upper and lower piezoelectric layers is assumed to be a quadratic function of \( z \).

### 3.2.1 Sensor Problem

A three-layered laminate shown in figure 3.2 is considered for the present numerical study. The superscripts \( u, m \) and \( l \) denote the parameters corresponding to top, middle and bottom layers. The potential distributions inside the upper and lower piezoelectric layers are assumed to be
\[ \phi^u = \phi_0^u + z\phi_1^u + z^2\phi_2^u \quad \text{for layer } 3, \quad \text{and} \]
\[ \phi^l = \phi_0^l + z\phi_1^l + z^2\phi_2^l \quad \text{for layer } 1. \]
Now using the charge equation we get

\[ \phi_z^u = \frac{e_{32}^u}{2 \varepsilon_{33}^u} e_y^{(1)}. \]  \hspace{1cm} (3.9)

Here the top and bottom layers are modeled as sensor layers. The electrodes at the interfaces are assumed to be at zero potential. Substituting the assumed displacements and the potential in the constitutive equation, and using the boundary condition \( D_z \mid_{z=z_3} = -D_z \mid_{z=z_3} \) for the top sensor layer we obtain

\[ \phi^u = \phi_0^u + \frac{z e_{32}^u}{\varepsilon_{33}^u} e_y^{(0)} + z^2 \frac{e_{32}^u}{2 \varepsilon_{33}^u} e_y^{(1)} \hspace{0.5cm} \text{for} \hspace{0.5cm} z > z_2. \]  \hspace{1cm} (3.10)

Similarly for the bottom layer;

\[ \phi' = \phi_0^l + \frac{z e_{32}^l}{\varepsilon_{33}^l} e_y^{(0)} + z^2 \frac{e_{32}^l}{2 \varepsilon_{33}^l} e_y^{(1)} \hspace{0.5cm} \text{for} \hspace{0.5cm} z < z_1. \]  \hspace{1cm} (3.11)

The electric field inside the piezoelectric layers are given by
\[ E_z^u = \frac{\varepsilon_{32}^u \varepsilon_{y}^{(0)} + z \varepsilon_{32}^u \varepsilon_{y}^{(1)}}{\varepsilon_{33}^u} \text{ for } z > z_2, \text{ and} \]

\[ E_z^l = \frac{\varepsilon_{32}^l \varepsilon_{y}^{(0)} + z \varepsilon_{32}^l \varepsilon_{y}^{(1)}}{2 \varepsilon_{33}^l} \text{ for } z < z_1. \]

To satisfy the reduced form of the charge equation the electric displacement field should not vary in the thickness direction. This condition, together with the electrical boundary condition on the surface of the piezoelectric layers forces the \( z \) component of the electric displacement to be zero everywhere inside the piezoelectric layer. Thus the CLT and the FSDT force the electric displacement to be zero everywhere inside the sensor layers. If higher order terms are included in the assumed expansion for the potential, then it is possible to estimate the electric displacement field inside the sensor layers. This is possible only if higher order terms are also present in the assumed expansion for the axial displacement component. In other words, a higher order laminate theory would predict a better distribution of electric displacement inside the sensor layers. The potential difference between the electrodes of the piezoelectric layers can be found from eqns. (3.10) and (3.11) and are given by

\[ V_0^u = \phi^u |_{z = z_2} - \phi^u |_{z = z_3} = \frac{h^u \varepsilon_{32}^u \varepsilon_{y}^{(0)} + (z_3 + z_4)}{2 \varepsilon_{33}^u} \]

for the top layer, and

\[ V_0^l = \phi^l |_{z = z_0} - \phi^l |_{z = z_1} = \frac{h^l \varepsilon_{32}^l \varepsilon_{y}^{(0)} + (z_4 + z_0)}{2 \varepsilon_{33}^l} \]

for the bottom layer.

If the piezoelectric sensor is viewed as a capacitor, the equivalent charges that will appear on the electrodes corresponding to the voltage generated on each layer are given by
We observe that the above equations are similar to the corresponding charge equations derived in (Lee et al, 1990) for laminated plates, where $E_z$ is forced to be zero by external circuitry. This external enforcement is not required and the effect of the $z$ component electric field is already considered in equation (3.14). Now the bending stresses and the electric displacement vector inside the layers are found to be

\[
\sigma_{y}^{u} = \left( C_{22}^{u} + \frac{\varepsilon_{33}^{u}}{\varepsilon_{33}} \right) \epsilon_{y}^{(0)} + z \left( C_{22}^{u} + \frac{\varepsilon_{32}^{u}}{\varepsilon_{33}} \right) \epsilon_{y}^{(1)} \quad \text{for the top layer}
\]

\[
\sigma_{y}^{m} = C_{22}^{m} \left( \epsilon_{y}^{(0)} + z \epsilon_{y}^{(1)} \right) \quad \text{for the middle layer},
\]

and

\[
\sigma_{y}^{l} = \left( C_{22}^{l} + \frac{\varepsilon_{33}^{l}}{\varepsilon_{33}} \right) \epsilon_{y}^{(0)} + z \left( C_{22}^{l} + \frac{\varepsilon_{32}^{l}}{\varepsilon_{33}} \right) \epsilon_{y}^{(1)} \quad \text{for the bottom layer}.
\]

Multiplying the first equation of (3.2) by $z$ and integrating over the thickness of the laminate with respect to $z$ we get

\[
\frac{dM}{dy} = Q = \rho_1 \omega^2 v_0 + \rho_2 \omega^2 \epsilon_{y}^{(1)}, \quad (3.16)
\]

where $M = \int_{z_0}^{z_1} \sigma_y z dz$ is the bending moment and $Q = \int_{z_0}^{z_1} \tau_{xy} dz$ is the shear force along the length of the beam. Substituting the values for the axial stress and shear stress in the definition of the bending moment and shear force, we get the expressions for these as
where

\[
Y_1 = \frac{(z_i^2 - z_0^2)\beta^l}{2} + \frac{(z_i^2 - z_0^2)\beta^m}{2} + \frac{(z_i^2 - z_0^2)\beta^u}{2},
\]

\[
Y_2 = \frac{(z_i^3 - z_0^3)\beta^l}{3} + \frac{(z_i^3 - z_0^3)\beta^m}{3} + \frac{(z_i^3 - z_0^3)\beta^u}{3},
\]

\[
Y_3 = h\kappa(\alpha + \alpha^m + \alpha''),
\]

and

\[
\beta^u = \left( C_{22}^u + \frac{(e_{12}^u)^2}{\varepsilon_{33}^u} \right), \quad \alpha^u = \left( C_{44}^u + \frac{(e_{24}^u)^2}{\varepsilon_{22}^u} \right),
\]

\[
\beta^l = \left( C_{22}^l + \frac{(e_{12}^l)^2}{\varepsilon_{33}^l} \right), \quad \alpha^l = \left( C_{44}^l + \frac{(e_{24}^l)^2}{\varepsilon_{22}^l} \right),
\]

\[
\beta^m = C_{22}^m, \quad \alpha^m = C_{44}^m.
\]

In the expression for the shear force the parameter \( h\kappa \) is called the reduced section and it is computed from classical beam theory. The factor \( \kappa \) is equal to \( \frac{5}{6} \) for beams of rectangular cross section and \( \frac{1}{1.175} \) for beams of circular cross section. From the above equations we observe that for the estimation of bending and shear rigidities, the stiffness coefficients \( C_{22} \) and \( C_{44} \) for piezoelectric layers are replaced by the parameters \( \beta \) and \( \alpha \) of the layers (Tiersten 93). Integrating the second equation of (3.2) with respect to \( z \) we obtain
\[
\frac{dQ}{dy} - q(y) = \rho_3 \omega^2 w .
\] (3.20)

In the above expressions the density parameters \(\rho_1\), \(\rho_2\) and \(\rho_3\) are given by

\[
\rho_1 = \frac{1}{2} \left[ (z_1^3 - z_0^2) \rho_i + (z_2^2 - z_1^2) \rho_m + (z_3^2 - z_0^2) \rho_a \right],
\]

\[
\rho_2 = \frac{1}{3} \left[ (z_1^3 - z_0^3) \rho_i + (z_2^3 - z_1^3) \rho_m + (z_3^3 - z_0^3) \rho_a \right], \text{ and}
\]

\[
\rho_3 = \left[ h_i \rho_i + h_m \rho_m + h_a \rho_a \right].
\] (3.21)

Combining eqns. (3.16) and (3.20), we obtain the final equation of motion for the piezoelectric laminate:

\[
\frac{d^4 w}{dy^4} + \left( \frac{\rho_1 + \rho_2}{Y_3} \right) \omega^2 \frac{d^2 w}{dy^2} - \frac{Y_1}{Y_2} \frac{d^2 v_0}{dy^2} = \frac{q(y)}{Y_2} - \frac{\rho_1}{Y_2} \omega^2 v_0 - \left( \frac{\rho_3 \omega^4}{Y_3 Y_2} - \frac{\rho_i \omega^2}{Y_3} \right) w .
\] (3.22)

The boundary conditions for a simply supported beam are \(w(0) = w(l) = M(0) = M(l) = 0\). The bending angle \(\psi\) can be derived from eqn. (3.17) as

\[
\psi = \frac{1}{(Y_3 - \rho_i \omega^2)} \left\{ Y_2 \frac{d^3 w}{dy^3} + \left( \frac{\rho_i \omega^2}{Y_3} + Y_3 \right) \frac{dw}{dy} + \frac{dq}{dy} \right\} .
\] (3.23)

The above equation of motion is solved for \(v\) and \(w\) and the potential distribution inside the sensor layer can be determined from eqns. (3.10) and (3.11).

3.2.2 Actuator Problem

The three-layered laminate shown in Fig. 3.2 is also considered for this study. The assumed fields for the displacements and the potential are the same as in the above case. The potentials at the top and bottom surfaces of the laminate are equal to the externally applied voltages and the electrodes on the interfaces are assumed to be at zero
potential, so the boundary conditions are:

\[
\phi = +V_0(y)e^{i\alpha} \text{ at } z=z_3 ,
\]
\[
= -V_0(y)e^{i\alpha} \text{ at } z=z_0 , \text{ and }
\]
\[
= 0 \text{ at } z=z_2 \text{ and } z=z_1.
\]  

(3.24)

The potential distributions inside the upper and lower piezoelectric layer are assumed as

\[
\phi'' = \phi''_0 + z\phi''_1 + z^2\phi''_2 \text{ for the layer 3, and }
\]
\[
\phi' = \phi'_0 + z\phi'_1 + z^2\phi'_2 \text{ for the layer 1.}
\]  

(3.25)

Applying the electric field boundary conditions and using the charge equation, the potential distributions for the top and bottom layers can be written as

\[
\phi'' = \frac{(z-z_2)V_0}{h} + [(z_2-z)(z_3-z)]\frac{\epsilon''_{32}}{2\epsilon_{33}}\epsilon^{(1)}_y \text{ for the layer 3, and }
\]
\[
\phi' = \frac{(z-z_1)V_0}{h} + [(z_1-z)(z_0-z)]\frac{\epsilon'^{1}}{2\epsilon_{33}}\epsilon^{(1)}_y \text{ for the layer 1.}
\]  

(3.26)

Now using the constitutive equation we get bending stresses inside the layers as

\[
\sigma''_y = \left(C''_{22} + \frac{(\epsilon''_{32})^2}{\epsilon_{33}}\right)e^{(0)}_y + z\left(C''_{22} + \frac{(\epsilon''_{32})^2}{\epsilon_{33}}\right)e^{(1)}_y - \frac{(z_2+z_3)}{2\epsilon_{33}}e^{(2)}_y + \frac{\epsilon''_{32}V_0}{h} \text{ for the top layer,}
\]
\[
\sigma'^{m}_y = C''_{22}(e^{(0)}_y + ze^{(1)}_y) \text{ for the middle layer, and}
\]
\[
\sigma'^{l}_y = \left(C'^{l}_{22} + \frac{(\epsilon'^{l}_{32})^2}{\epsilon_{33}}\right)e^{(0)}_y + z\left(C'^{l}_{22} + \frac{(\epsilon'^{l}_{32})^2}{\epsilon_{33}}\right)e^{(1)}_y - \frac{(z_0+z_1)}{2\epsilon_{33}}e^{(2)}_y - \frac{\epsilon'^{'l}_{32}V_0}{h} \text{ for the bottom layer}
\]  

(3.27)

The bending moment and the shear force are derived from the axial stress resultants and are given by
\[ M = Y_2 \varepsilon_y^{(0)} + Y_3 \varepsilon_y^{(1)} + Y_4 V_0, \text{ and} \]
\[ Q = Y_3 \left( \frac{dw}{dy} - \psi \right) \]  
(3.28)

where

\[ Y_2' = \frac{(z_3^3 - z_0^3)\beta^l}{3} + \frac{(z_2^3 - z_1^3)\beta^m}{3} + \frac{(z_3^3 - z_2^3)\beta^n}{3} \left\{ \frac{h_y(z_0 + z_1)^2(e_{32}^l)^2}{2\varepsilon_{33}^l} + \frac{h_u(z_2 + z_3)^2(e_{32}^u)^2}{2\varepsilon_{33}^u} \right\}, \]

and

\[ Y_4 = \frac{(z_0 + z_1)e_{32}^l}{2} - \frac{(z_3 + z_2)e_{32}^u}{2}. \]  
(3.29)

The electric field and the charge density inside the piezoelectric layers are given by the following equations:

\[ E_z^u = \frac{V_0}{h} + \left[ 2z - (z_2 + z_1) \right] \frac{e_{32}^u}{2\varepsilon_{33}^u} \varepsilon_y^{(1)} \text{ for } z > z_u, \]

\[ E_z^l = \frac{V_0}{h} + \left[ 2z - (z_1 + z_0) \right] \frac{e_{32}^l}{2\varepsilon_{33}^l} \varepsilon_y^{(1)} \text{ for } z < z_l, \]  
(3.30)

\[ D_z^u = -e_{32}^u \varepsilon_y^{(0)} + \left( \frac{z_2 + z_1}{2} \right) e_{32}^u \varepsilon_y^{(1)} \frac{\varepsilon_{33}^u V_0}{h} \text{ for } z > z_u, \text{ and} \]

\[ D_z^l = -e_{32}^l \varepsilon_y^{(0)} + \left( \frac{z_1 + z_0}{2} \right) e_{32}^l \varepsilon_y^{(1)} + \frac{\varepsilon_{33}^l V_0}{h} \text{ for } z < z_l. \]

From the expressions obtained for the electric fields inside the piezoelectric layers we can see that the electric field inside the layers are not constant, but vary linearly through the thickness of the actuator layer. The electric field inside the layers can be approximated to \( \frac{V_0}{h} \) if \( \frac{he_{32}^l \varepsilon_y^{(1)}}{2\varepsilon_{33}} \) is small when compared to the term \( \frac{V_0}{h} \). For thin actuators the induced strain field inside the actuators is small when compared to the \( \frac{V_0}{h} \).
term and the second term can be ignored. But when the actuator layer is subjected to a large deformation or large strain field, this term becomes comparable to the first one. As in the sensor case, we see that the reduced form of the charge equation forces the electric displacement to be constant across the thickness of the piezoelectric layer. The electric displacement is non-zero for this case and is proportional to the sum of the mid-plane bending and membrane strains and the applied voltage on the electrode surface of the layer. Once again, as in the sensor case, the governing equation of motion for the laminate is obtained from the equations of equilibrium. The equation of motion for the laminate, which includes actuator layers, is

$$\frac{d^4 w}{dy^4} + \left( \frac{\rho_3}{Y_3} + \frac{\rho_2}{Y_2} \right) \phi^2 \frac{d^2 w}{dy^2} - \frac{Y_1}{Y_2} \frac{d^2 V_0}{dy^2} \frac{Y_4}{Y_2} \frac{d^2 (V_0)}{dy^2} = \frac{q(y)}{Y_2} \frac{\rho_1}{Y_2} \omega^2 V_0 + \left( \frac{\rho_3 \rho_2 \omega^2}{Y_3 Y_2} - \frac{\rho_3 \omega^2}{Y_2} \right) w.$$  

(3.31)

The boundary conditions are $$w(0) = w(l) = M(0) = M(l) = 0.$$ Using eqn. (3.28), the boundary condition on the moment further reduces to $$Y_4 V_0(y) = Y_2 \epsilon^{(1)}_y - Y_1 \epsilon^{(0)}_y$$ at $$y = 0$$ and $$y = 1.$$ The above equation is solved for $$v$$ and $$w$$ from which the variation of bending stress and strain and the potential inside the layers are determined.

### 3.3 Method Of Solution

The solution for the equations of motion is obtained for a laminated beam undergoing pure bending vibration when there are no axial vibration modes present. In this case, eqns. (3.22) and (3.31) contain only $$w$$ displacement and the neutral layer axial displacement $$v_0$$ is taken to be zero. Also for free vibration, the external surface load $$q(y)$$
3.3.1 Sensor Solution

The solution can be obtained by assuming the displacement to be

\[ w(y, t) = a \exp \left( \lambda y / l \right) \exp (i \omega t) \]  \hspace{1cm} (3.32)

Substituting this assumed displacement in the eqn. (3.31) yields the eigenvalue problem

\[ \lambda^2 + \Omega(\eta + \chi) \lambda - \Omega + \Omega^2 \eta \chi = 0 \]  \hspace{1cm} (3.33)

where \( \Omega = \frac{\omega^2 \rho_s l^4}{Y_2} \), \( \chi = \frac{\rho_2}{\rho s l^2} \), and \( \eta = \frac{Y_3}{Y_2 l^2} \). The roots of this quadratic equation in \( \lambda^2 \) are

\[ \lambda^2 = \frac{-\Omega(\eta + \chi) \pm \sqrt{\Omega^2 (\eta - \chi)^2 + 4\Omega}}{2} \]  \hspace{1cm} (3.34)

The eigenvalue eqn. (3.33) has one positive and one negative root under the condition \( \Omega \eta \chi < 1 \) or \( \rho_2 \omega^2 < Y_3 \). So, for low frequencies where the contributions from the shear and rotary inertia are small, the roots may be put in the form \( \pm i \lambda_1, \pm \lambda_2 \) which are functions of \( \omega \) and the general solutions takes the form

\[ w(y) = a_1 \sin \left( \frac{\lambda_1 y}{l} \right) + a_2 \cos \left( \frac{\lambda_1 y}{l} \right) + a_3 \sinh \left( \frac{\lambda_2 y}{l} \right) + a_4 \cosh \left( \frac{\lambda_2 y}{l} \right) \]  \hspace{1cm} (3.35)

Applying the boundary conditions at \( y = 0 \) and \( l \) for the simply supported beam, we get for a non-trivial solution \( \sin \lambda = 0 \). The roots of this equation are \( \lambda = n \pi, n = 1, 2, \ldots \infty \). Substituting this in eqn. (3.33) we obtain the characteristic equation

\[ \Omega^2 \eta \chi - \Omega[(\eta + \chi)n^2 \pi^2 + 1] + n^4 \pi^4 = 0 \]  \hspace{1cm} (3.36)

Solution of the above equation yields the frequency parameter \( \Omega \) from which the
natural frequency of the laminate beam $\omega_n$ can be determined. If rotary inertia and shear effects are neglected (i.e. $\chi = \eta = 0$), the above solution reduces to that of the Euler-Bernoulli beam theory. After estimating the natural frequencies, the response produced by a distributed harmonic force applied on the laminate surface can be estimated using the normal mode method (Timoshenko 1990).

### 3.3.2 Actuator Solution

In the closed (or short circuit) condition, the applied voltages on the top and bottom surfaces of the laminate are assumed to be zero. The frequency of vibration of the beam is estimated as described above. It is found that the eqn. (3.34) can also be used for estimating the frequencies of the laminate for closed or short circuit condition except that the parameter $\eta = \frac{Y_2}{Y_3 l^2}$. When the applied voltage on the laminate surfaces are not zero, the moment boundary conditions along with the $w = 0$ conditions can be written at $y = 0, l$ as

$$\frac{d^2w}{dy^2} = \frac{d^2w}{dy^2} + \frac{\rho_3 \omega^2 w}{Y_3} = \frac{d^2w(y = 0, l; t)}{dy^2} = -\frac{Y_4}{Y_2} V_0(y = 0, l; t) = M_{\text{end}}. \quad (3.37)$$

So we see that the boundary conditions for the actuator are equivalent to those of a piezoelectric laminate beam with a concentrated moment $M_{\text{end}}$ acting on the ends, where $M_{\text{end}}$ is proportional to the applied voltage (refer sec. 10.5 of Miu (1993)). For the actuator case, the response of the laminate due to this harmonic moment applied at the ends of the beam can be estimated using the normal mode method.

The effect of the other assumption ($E_y = 0$) poses some difficulty in obtaining the
solution for the vibration analysis of smart beams. The $E_y=0$ assumption imposes the potential inside the piezoelectric layers to be independent of $y$. So the coefficients $\phi_0, \phi_1$ and $\phi_2$ used in the expansion of the potential are constants and are not functions of $y$.

From the constitutive equations we can write $D_y = e_{24}\gamma_{yz}$ and $D_z = e_{32}\varepsilon_y - e_{33}\varepsilon_z$. Now substituting these constitutive equations in the charge equation we obtain the relation

$$e_{24} \frac{d^2w}{dy^2} - (e_{24} - e_{32}) \frac{d\psi}{dy} = 2e_{33} \phi_2.$$

The solution to the vibration of a piezoelectric sandwich beam can be obtained in a similar way to that used for the case of the $D_y=0$ assumption, and the displacement ($w$) and the bending angle ($\psi$) can be determined. The potential that will appear on the sensor electrodes can be determined using the above equation. So from the solution obtained for $w$ and $\psi$, we observe that the coefficient $\phi_2$ cannot be a constant and is a function of $y$, which violates the $E_y=0$ assumption. For the case of a thin beam or a beam with no external transverse load and no inertia force then we have the $\frac{dD_y}{dy} = 0$ condition. The solution for this case is the same as that obtained for the $D_y=0$ assumption case. Here the coefficient $\phi_2$ is found to be proportional to the strain $\varepsilon_y^{(1)}$ which is a function of $y$. This again violates the assumption of $E_y=0$. This discussion holds for both sensor and actuator layers, and in both cases the coefficient $\phi_2$ is a function of the derivatives of $w$ and $\psi$ (refer to equations (3.10), (3.11) and (3.26)). So finding a solution using the FSDT for the
forced vibration analysis of piezoelectric laminated beams with the assumption $E_y=0$ is not possible.

### 3.4 Elasticity Solution

The main objective here is to study the effect of the applied potential or normal load on the induced mechanical and electrical field distribution inside the layers when the laminate vibrates at resonance and off resonance conditions. Elasticity solutions for the composite laminates were first attempted by Pagano (1969) to determine the stress and displacement fields inside each layer. This approach was then extended to piezoelectric laminates by Ray et al (1993) and Heyliger (1997) to find the field distribution inside the layers either to estimate the fields for static loading condition or to estimate normal mode field distributions. No study has been done on estimating the response fields inside a piezoelectric laminate due to harmonic force or voltage excitation. In this section, using a three-dimensional electroelasticity approach, we investigate the nature of the electric and stress fields inside the layers due to an applied harmonic normal load or the potential on the surface of the laminate, which are then used for comparing with the FSDT solution.

The linearised field equation for the dynamic motion of each lamina is given in equation (3.2). The conservation of charge equation in terms of electric displacement components $D_i$ is given by $\frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0$. If each layer is orthotropic with the material symmetry axes coinciding with the coordinate axes, the linear constitutive equations for the stress $\sigma_{ij}$ and the electric displacement $D_i$, in terms of strain tensor components $\varepsilon_{ij}$, are
given by
\[
\sigma_i = C_{ij} e_j - e_{ik} E_k, \quad \text{and}
\]
\[
D_k = e_{kj} e_j + \epsilon_{km} E_m,
\]
where \(C_{ij}\) are the elastic stiffness coefficients, \(e_{ij}\) are the piezoelectric coefficients and \(E_k\) are the components of the electric field. The electric field strength \(E_i\) is the gradient of the electric potential \(\phi\),
\[
E_m = -\frac{\partial \phi}{\partial x_m}. \quad (3.39)
\]
The strain-displacement relations are given by
\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (3.40)
\]
where \(u_1 = u\), \(u_2 = v\) and \(u_3 = w\). Since we assume the plane deformations in the \(yz\) plane, the in-plane components of the displacements field, \(u_1, u_2\) and the electric potential \(\phi\), are functions of \(y\) and \(z\) only and the out of displacement field \(u_3\) is constant rigid body translation. Substituting these assumptions in the charge equation and strain displacement equations, we can see that \(E_x = 0\) and \(S_x = S_y = S_z = 0\). Substitution of these into the constitutive relations results in \(\sigma_{yx} = \sigma_{zx} = D_x = 0\). Substituting equations (3.38), (3.39), (3.40) in the dynamic equation of motion and charge equation, we obtain the equations of motion for a lamina in terms the displacement components and the electric potential as
\[
C_{22} \frac{\partial^2 v}{\partial y^2} + C_{23} \frac{\partial^2 w}{\partial y \partial z} + e_{32} \frac{\partial^2 \phi}{\partial y \partial z} + C_{44} \left( \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial y \partial z} \right) + e_{24} \frac{\partial^2 \phi}{\partial y \partial z} = \rho \frac{\partial^2 v}{\partial t^2},
\]
\[
C_{33} \frac{\partial^2 w}{\partial z^2} + C_{23} \frac{\partial^2 v}{\partial y \partial z} + e_{24} \frac{\partial^2 \phi}{\partial y \partial z} + C_{44} \left( \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial z} \right) + e_{33} \frac{\partial^2 \phi}{\partial z^2} = \rho \frac{\partial^2 w}{\partial t^2}, \text{ and (3.45)}
\]
\[
e_{33} \frac{\partial^2 w}{\partial z^2} + e_{32} \frac{\partial^2 v}{\partial y \partial z} - e_{22} \frac{\partial^2 \phi}{\partial y \partial z} + e_{24} \left( \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial z} \right) - e_{33} \frac{\partial^2 \phi}{\partial z^2} = 0.
\]

These are the governing equations for each layer inside the laminate and are associated with the mechanical and electric boundary conditions. For a laminated beam simply supported at both ends, the boundary conditions are given by \( \sigma_y(0,z) = \sigma_y(L,z) = 0 \), \( w(0,z) = w(L,z) = 0 \) and \( \phi(0,z) = \phi(L,z) = 0 \).

The displacement components, electric potential and the applied load can be expressed in the Fourier series form as
\[
v(y, z) = \sum_{n=1}^{\infty} V_n \cos \left( \frac{n\pi y}{l} \right) e^{imz} e^{j\omega t},
\]
\[
w(y, z) = \sum_{n=1}^{\infty} W_n \sin \left( \frac{n\pi y}{l} \right) e^{imz} e^{j\omega t}, \text{ and (3.46)}
\]
\[
\phi(y, z) = \sum_{n=1}^{\infty} \Phi_n \sin \left( \frac{n\pi y}{l} \right) e^{imz} e^{j\omega t},
\]

where \( j = \sqrt{-1} \), \( m \) is the unknown parameter and \( \omega \) is the frequency of the vibration.

Using the constitutive equations the stress and the electric displacement for any piezoelectric layer can now be expressed in a Fourier series using the assumed series for displacement and electric potential.

The Fourier series can be truncated at some order depending upon the number of terms to be included for the response study. Substitution of assumed displacement
components and the electric potential into the equation of motion and the conservation of charge equations results in a set of homogenous linear algebraic equations

\[
\begin{bmatrix}
C_{44}m^2 - C_{22}p^2 + \rho \omega^2 - j\omega \\
-pm(C_{44} + C_{23}) \\
-pm(e_{32} + e_{24})
\end{bmatrix}
\begin{bmatrix}
V_n \\
W_n \\
\Phi_n
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

(3.47)

for \( n = 1, 2, \ldots \infty \).

The necessary condition for a non-trivial solution for \( V_n, W_n \) and \( \Phi_n \) is that the determinant of the coefficient matrix in the above equation be zero for all values of \( 'n' \). Setting the determinant to be zero yields all six roots for the unknown parameter \( m \) in terms of \( \omega \). For orthotropic materials the above condition yields six distinct roots, with some roots being complex conjugate pairs. The six roots are the eigenvalues and each of these are associated with eigenvectors \( V^*_n, W^*_n, \) and \( \Phi^*_n \) for \( k = 1, 2, \ldots 6 \). The general solution for \( u, w, \) and \( \phi \) can be expressed in Fourier series form as

\[
v(y, z) = \sum_{n=1}^{\infty} \sum_{k=1}^{6} \alpha^k_n V^k_n \cos \left( \frac{n\pi y}{l} \right) e^{m_{ax} z} e^{i\alpha x},
\]

\[
w(y, z) = \sum_{n=1}^{m} \sum_{k=1}^{6} \alpha^k_n W^k_n \sin \left( \frac{n\pi y}{l} \right) e^{m_{ax} z} e^{i\alpha x}, \text{ and}
\]

\[
\phi(y, z) = \sum_{n=1}^{\infty} \sum_{k=1}^{6} \alpha^k_n \Phi^k_n \sin \left( \frac{n\pi y}{l} \right) e^{m_{ax} z} e^{i\alpha x},
\]

(3.48)

where \( \alpha^i_n \) are the scalar constants. It is to be noted that the roots \( m^*_n, V^*_n, W^*_n, \) and \( \Phi^*_n \), are functions of \( \omega \). The Fourier series can be truncated at some order depending upon the number of terms to be included for the response study. After obtaining the roots and the eigenvectors, the stress and electric displacement components are obtained using the
constitutive relation and the strain displacement equations for each layer and are:

\[
\sigma_y = \sum_{n=1}^{\infty} \sum_{k=1}^{6} (-pC_{22}\alpha_n^k V_n + C_{23}m_n \alpha_n^k W_n + e_{32}m_n \alpha_n^k \Phi_n )e^{j\omega y} \sin(py)e^{jvx},
\]

\[
\sigma_z = \sum_{n=1}^{\infty} \sum_{k=1}^{6} (-pC_{33}\alpha_n^k V_n + C_{32}m_n \alpha_n^k W_n + e_{33}m_n \alpha_n^k \Phi_n )e^{j\omega y} \sin(py)e^{jvx},
\]

\[
\tau_{yn} = \sum_{n=1}^{\infty} \sum_{k=1}^{6} \{C_{44}(m_n \alpha_n^k V_n + p\alpha_n^k W_n) + e_{24}p\alpha_n^k \Phi_n \}e^{j\omega y} \cos(py)e^{jvx}, \quad (3.49)
\]

\[
D_y = \sum_{n=1}^{\infty} \sum_{k=1}^{6} \{e_{24}(m_n \alpha_n^k V_n + p\alpha_n^k W_n) - \varepsilon_{24} p\alpha_n^k \Phi_n \}e^{j\omega y} \cos(py)e^{jvx},
\]

\[
D_z = \sum_{n=1}^{\infty} \sum_{k=1}^{6} (-pe_{32}\alpha_n^k V_n + e_{33}m_n \alpha_n^k W_n - \varepsilon_{33} m\alpha_n^k \Phi_n )e^{j\omega y} \sin(py)e^{jvx},
\]

where \( p = \frac{n\pi y}{l} \).

For a piezoelectric layer, the elastic and the electric fields decouple and the stress fields are expressed with the four roots for the elastic solution. There are six unknown constants \( (\alpha_n^k) \) in the case of piezoelectric layers and four unknowns in the case of nonpiezoelectric layers.

At the interfaces at each of the lamina the continuity conditions of displacement, stress, potential and the normal electric displacement must be enforced. If a layer is nonpiezoelectric, there are no interface conditions on the electric potential and the electric displacement. For the system shown in Figure (3.1), the interface conditions for any interface between \( n \) and \( n+1 \) piezoelectric layers are
\[
\begin{align*}
\sigma_z^{(n)}\left(y, \frac{-h_n}{2}\right) &= \sigma_z^{(n+1)}\left(y, \frac{h_{n+1}}{2}\right), \\
\tau_{yz}^{(n)}\left(y, \frac{-h_n}{2}\right) &= \tau_{yz}^{(n+1)}\left(y, \frac{h_{n+1}}{2}\right), \\
y^{(n)}\left(y, \frac{-h_n}{2}\right) &= y^{(n+1)}\left(y, \frac{h_{n+1}}{2}\right), \\
w^{(n)}\left(y, \frac{-h_n}{2}\right) &= w^{(n+1)}\left(y, \frac{h_{n+1}}{2}\right), \\
\phi^{(n)}\left(y, \frac{-h_n}{2}\right) &= \phi^{(n+1)}\left(y, \frac{h_{n+1}}{2}\right), \quad \text{and} \\
D_z^{(n)}\left(y, \frac{-h_n}{2}\right) &= D_z^{(n+1)}\left(y, \frac{h_{n+1}}{2}\right).
\end{align*}
\] (3.50)

If a layer is not piezoelectric the electric and elastic fields decouple and there are no interface conditions on the electric potential and the electric displacement. There are still six unknown constants for the piezoelectric layers and four unknown constants for non-piezoelectric layers left to be determined for each value of \(n\). The continuity conditions at the interfaces and the boundary conditions at the top and bottom surfaces of the laminate (at \(z = \pm \frac{h_{\text{lamine}}}{2}\)) are expressed in terms of the unknown coefficients \(\alpha_n\) in matrix form as shown below:

\[
\begin{bmatrix}[B_n] \{\alpha_n\} = \{P_n\} \quad \text{for} \quad n=1, \ldots, \infty, \]
\] (3.51)

where vector \(\alpha_n\) contains the unknown constants, and matrix \(B_n\) contains the coefficients \(m_n^k, V_n^k, W_n^k,\) and \(\Phi_n^k\) (which are functions of \(\omega\)). The right hand side vector \(P_n\) contains the applied force or the applied potential values on the surface of the laminate and zero for the free vibration case. The natural frequencies of the laminate can be found by
finding the value of $\omega_n$ for which the determinant of the $B$ matrix to zero. The mechanical and electrical fields inside the laminate can be computed once the coefficients $\alpha_n$ are determined.

The harmonic driving potential or the traction can be expressed as a Fourier series in $y$ as given below.

$$\phi_{z=\frac{b_{\text{num}}}{2}} = \sum_{n=1,3,5,7}^{\infty} \frac{4\phi_0}{n\pi} \sin\left(\frac{n\pi y}{l}\right)e^{j\omega t}, \text{and}$$

$$\sigma_{z=\frac{b_{\text{num}}}{2}} = \sum_{n=1,3,5,7}^{\infty} \frac{4\sigma_0}{n\pi} \sin\left(\frac{n\pi y}{l}\right)e^{j\omega t}. \tag{3.52}$$

To find the response of the laminate for applied traction or potential at some value close to its natural frequency $\omega_n$, the above system of equations is solved for the unknown constants $\alpha_n$ after substituting the value of $\omega_n$ in the coefficients of the $B_n$ matrix. The mechanical and electrical fields inside the laminate are completely determined once these coefficients are determined.

### 3.5 Results And Discussion

In this section, the application of FSDT to a three layered composite laminate is carried out and the electromechanical fields inside the layers of this laminate are estimated and compared with the fields obtained using the three-dimensional elasticity approach described in previous section. A composite orthotropic lamina is sandwiched between two piezoelectric layers. Two different aspect ratios of thickness to width ($l/h$) are considered for the numerical studies.

One value of the aspect ratio is 50 (very thin plate), which is well within the
assumptions of CLT and FSDT, and the second value of 20 (moderately thick plate) is at the limits of the assumptions of CLT and FSDT.

The thickness of the piezoelectric laminate is 10 mm. The properties of the laminate used for this analysis are given in Table 3.1.

<table>
<thead>
<tr>
<th>Property</th>
<th>Piezoelectric layers</th>
<th>Structural layer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{11}$ (GPa)</td>
<td>81.3</td>
<td>10.756</td>
</tr>
<tr>
<td>$C_{22}$</td>
<td>81.3</td>
<td>132.38</td>
</tr>
<tr>
<td>$C_{33}$</td>
<td>64.5</td>
<td>10.756</td>
</tr>
<tr>
<td>$C_{12}$</td>
<td>0.329</td>
<td>0.24</td>
</tr>
<tr>
<td>$C_{13}$</td>
<td>0.432</td>
<td>0.49</td>
</tr>
<tr>
<td>$C_{23}$</td>
<td>0.432</td>
<td>0.24</td>
</tr>
<tr>
<td>$C_{44}$</td>
<td>25.6</td>
<td>3.606</td>
</tr>
<tr>
<td>$C_{55}$</td>
<td>25.6</td>
<td>5.6537</td>
</tr>
<tr>
<td>$C_{66}$</td>
<td>30.6</td>
<td>5.6537</td>
</tr>
<tr>
<td>$e_{24}$ (C/m²)</td>
<td>12.72</td>
<td>0.0</td>
</tr>
<tr>
<td>$e_{31}$</td>
<td>-5.2</td>
<td>0.0</td>
</tr>
<tr>
<td>$e_{32}$</td>
<td>-5.2</td>
<td>0.0</td>
</tr>
<tr>
<td>$e_{33}$</td>
<td>15.08</td>
<td>0.0</td>
</tr>
<tr>
<td>$\varepsilon_{33}/\varepsilon_0$</td>
<td>1300</td>
<td>3.0</td>
</tr>
<tr>
<td>$\varepsilon_{33}/\varepsilon_0$</td>
<td>1475</td>
<td>3.5</td>
</tr>
<tr>
<td>$\rho$ (Kg/m³)</td>
<td>7500</td>
<td>2750</td>
</tr>
</tbody>
</table>

The piezoelectric laminate beam under study is shown in fig.3.2. Before solving for the response of the laminate, the free vibration study of the laminate is performed by making the applied load/potential vector zero and the first five resonance frequencies of
the laminate are determined using both the theories.

The first four bending mode frequencies are estimated using FSDT and the elasticity solution approach and are shown in Table 3.2. When compared to the exact solution, the FSDT estimates the frequencies with a maximum error of 3% for the first five modes and the error increases with increasing order of the mode number. Also from the Table 3.2 we see that the error in estimating the resonance frequencies using FSDT increases with decreasing aspect ratio of the laminate.

Table 3.2 Natural Frequencies for Single and Three Layered Laminate Beams

<table>
<thead>
<tr>
<th>Mode</th>
<th>Single Layer PZT (Aspect Ratio 20)</th>
<th>Three Layer Laminate (Aspect Ratio 50)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FSDT Solution(present) (Hz)</td>
<td>Elasticity Solution (Hz)</td>
</tr>
<tr>
<td>1</td>
<td>623.3</td>
<td>622.0</td>
</tr>
<tr>
<td>2</td>
<td>2470.3</td>
<td>2455.8</td>
</tr>
<tr>
<td>3</td>
<td>54790.8</td>
<td>54255.8</td>
</tr>
</tbody>
</table>

The displacement, potential, stress, strain, electric field, and electric displacement field distributions for the case of the distributed unit harmonic load applied on the surface \( z = \pm \frac{h_{\text{laminate}}}{2} \) (sensor case) are shown in figures 3.3-3.15 and for the case of unit potential applied on the electrodes (actuator case) shown in figures 3.16-3.28 for a laminate with (the length to width ratio) aspect ratio (AR) 20. The field distributions for
the actuator case and sensor case for the laminate with aspect ratio of 50 are shown in figures 3.29-3.54. The in-plane displacement, shear stress, shear strain, \( y \) component of electric field and the \( y \) component of electric displacement shown in the figures 3.3–3.54 correspond to the edges of the beam, while the remaining fields are estimated at the mid-span of the beam. The excitation frequency is 90\% of the first resonance frequency for both mechanical and electrical loading. The convergence of the Fourier series expansion is checked before estimating fields.

From figures 3.3 to 3.4 we see that for \( AR = 20 \), FSDT predicts the in-plane displacements and the transverse displacements with an error of 45 and 25 percent. The error in estimating the potential (Fig. 3.5) is nearly 42 percent. The variation of potential through the thickness is almost linear inside the sensor layer as estimated by the full electroelasticity theory. The error in estimating the bending stress is 45 percent. The FSDT estimates zero transverse normal stress, which is not true as shown by the elasticity solution in figure 3.7. From figure 3.8, we see that the shear stress is constant inside the layers in contrast to the nonlinear variation predicted by elasticity theory. The error in estimating the bending strains using FSDT is the same as that in estimating the in-plane displacement. Figure 3.12 and 3.13, shows the variation of \( y \) and \( z \) component of the electric fields inside the layers. There is a large difference in the \( y \) component electric field predicted by FSDT and elasticity theory. The zero \( y \) component electric displacement and the constant shear strain assumption in the FSDT leads to a large error in estimating the \( y \) component electric field inside the piezoelectric layers. Elasticity theory predicts almost a zero \( y \) component electric field inside the layers which one
would have obtained if the $E_y = 0$ assumption is used instead of the $D_y = 0$ assumption. There is a large error even in predicting the $z$ component electric field inside a sensor by FSDT for thick laminates, which is shown in figure 3.13. Both the $y$ and $z$ component of the $D$ field are not zero inside the sensor layers but have a very small value predicted by elasticity theory in contrast to the zero $y$ and $z$ component $D$ field predicted by FSDT.

Figures 3.16-3.28 show the electromechanical fields inside the laminate for $AR = 20$ when a unit harmonic potential is applied on the outer surface electrodes of the actuator layers. For this case also the fields estimated by the FSDT do not agree well with those predicted by elasticity theory except for the potential and $z$ component of the electric field. This is because the first term in eqns (3.26) and (3.30) for electric potential and electric fields inside the actuator layer is large when compared to the second term, which results in a linear variation of potential through the thickness and constant $E_z$ field inside the actuator layers. The difference between the two estimates of in-plane and $w$ displacement by FSDT and elasticity theory is approximately 45 percent. There is a large difference in the estimation of bending stress and bending strain values inside the layers between the two theories. Figure 3.20 shows that even in the absence of an external transverse load for the actuator case, the transverse normal stress inside the laminate is nonzero which cannot be predicted by FSDT. As in the sensor case, the FSDT predicts a nonzero $E_y$ field inside the piezoelectric layers in contrast to the zero $E_y$ value estimated by elasticity theory. Elasticity theory predicts a nonzero $D_y$ field inside the actuator layer.

For lower aspect ratio beams, $AR = 20$ or less, from figures 3.3 through 3.28, we see that FSDT cannot estimate the electromechanical fields accurately and the errors
involved in estimating the electrical fields are higher than the errors involved in the mechanical fields.

Figures 3.29 through 3.54 show the variation of electromechanical fields for the sensor and actuator cases inside the three layer laminate for $AR=50$. For slender beams the fields predicted by FSDT and elasticity theory agree well. In figures 3.31 and 3.39, FSDT estimates the potential and $E_z$ field inside the sensor layers with 30-35 percent error even for the case of very thin laminates. The difference between the two estimates of shear and transverse normal stresses and strains are due to the kinematic assumptions made in FSDT. As in the previous case, the estimation of $E_y$ by FSDT differs considerably (fig. 3.38) from the values given by elasticity theory. This is due to the constant shear strain assumption through the thickness of the laminate in FSDT and neglecting $D_y$. Both mechanical and electrical fields inside a thin laminate estimated by FSDT agree very well with that predicted by elasticity theory when a unit harmonic potential is applied on the outer electrode surfaces of the actuator layers except for the $E_y$ field. The field distributions for the laminate with $AR=50$ when a unit potential is applied on the laminate surfaces (actuator case) are shown in figures 3.42-3.54. Here also the FSDT is able to predict the fields inside the material and actuator layers accurately except for $E_y$, $D_y$, normal stress and strain fields due to assumptions made in FSDT as explained before. Unlike the sensor case, for the actuator case, the FSDT is able to predict the electrical potential and $E_z$ field more accurately. If the actuator or sensor layers are thick then we expect a quadratic variation of potential and linear variation of $E_z$ field inside the layers as suggested in (Yang, 1999).
3.6 Summary

The FSDT for the piezoelectric laminates is presented. The governing equations for a piezoelectric laminate are derived using first order shear deformation laminated beam theory and the solutions for the free and forced vibration problems of a simply supported piezoelectric laminated beam are obtained using the derived formulations. For the first time, the dynamic response fields inside the laminate obtained using FSDT are compared with the corresponding fields obtained using three-dimensional elasticity theory to bring out the effect of the kinematic and other assumptions made in FSDT. A Fourier series solution method is employed to calculate the field distributions inside the layers for the elasticity solution approach. For a thin laminate or a laminate with high aspect ratio ($AR=50$), the electromechanical fields predicted by FSDT agree well with the elasticity solution approach. But for the sensor case, the FSDT could not predict the potential distribution inside the sensor layers accurately though it predicts the mechanical fields with good accuracy. The assumption that $D_y = 0$, along with the constant shear strain approximation introduces a large error in estimating the $y$ component of the electric field inside the piezoelectric layers. It is observed that the FSDT estimates a constant electric displacement field inside the piezoelectric layer. This together with the reduced charge equation and sensor boundary condition on the sensor electrodes forces the $z$ component of the electric displacement field to be zero everywhere inside the sensor layer. For a better estimation of the electric displacement field, higher order laminated theories without approximations to the charge equation are needed. This is needed particularly when the strains induced in the piezoelectric layers are large. Though only a
three-layered symmetric laminate is considered for numerical calculations in the present study, the approach presented here can be applied to a nonsymmetrical laminate with multiple embedded piezoelectric layers.

When using FSDT to model a structure, careful consideration must be given not only to the aspect ratio of the layers but also to the type of loads applied to the structure. In active vibration and noise control, it is important to control the dominant modes of vibration of the structure. It is seen from the present study that errors resulting from the use of FSDT are magnified at such frequencies. In addition, one must also consider the practicality of using piezoelectric layers that cover the entire structure. PVDF is the only polymeric piezoelectric material in common use. However, it is not very efficient as an actuator, particularly for the control of metallic plates. PZT is an efficient actuator material, however it is very stiff and brittle and fabrication of laminates containing PZT layers will be a challenge. Since both materials are poled, they cannot be subjected to very high temperatures when fabricating the laminates to prevent depoling and loss of piezoelectric properties. The originally designed structure will be changed drastically after embedding the piezoelectric layers with respect to resonance frequencies and dynamic response. For these reasons, laminates appear to be impractical for real structures. The many research publications on piezoelectric laminates are a natural extension of FSDT for composite laminates. However, the coupling of the electro-mechanical fields result in large errors if thin plate theories are used. There appears to be no simple analytical way to use FSDT for the case of discrete piezoelectric patches applied to plate structures except as proposed by Kim and Varadan (Kim et al 1996).
They have proposed a finite element approach with transition elements to connect the piezoelectric patch regions to the plate regions. It is hoped that the detailed analysis and plots of all components of the mechanical and electric fields in the sensor and actuator layers will be of use to researchers in this field.
Figure 3.3 Axial Displacement ($v$) for Sensor Case ($l/h = 20$)

Figure 3.4 Transverse Displacement ($w$) for Sensor Case ($l/h = 20$)
Figure 3.5 Electric Potential ($\phi$) for Sensor case ($l/h = 20$)

Figure 3.6 Bending Stress ($\sigma_y$) for Sensor case ($l/h = 20$)
Figure 3.7 Transverse Normal Stress for Sensor Case ($l/h = 20$)

Figure 3.8 Shear Stress for Sensor case ($l/h = 20$)
Figure 3.9 Bending strain for Sensor case ($l/h = 20$)

Figure 3.10 Transverse normal strain for Sensor case ($l/h = 20$)
Figure 3.11 Shear strain for Sensor case ($l/h = 20$)

Figure 3.12 $y$ Component of the Electric Field for Sensor Case ($l/h = 20$)
Figure 3.13 z Component Electric Field for Sensor Case ($l/h = 20$)

Figure 3.14 y Component Electric Displacement for Sensor Case ($l/h = 20$)
Figure 3.15 $z$ component Electric Displacement for Sensor Case ($l/h = 20$)
Figure 3.16 Axial Displacement for Actuator case ($l/h = 20$)

Figure 3.17 Transverse Displacement for Actuator Case ($l/h = 20$)
Figure 3.18 Electric Potential Actuator case ($l/h = 20$)

Figure 3.19 Bending Stress for Actuator case ($l/h = 20$)
Figure 3.20 Transverse Normal Stress Actuator Case ($l/h = 20$)

Figure 3.21 Shear Stress for Actuator case ($l/h = 20$)
Figure 3.22 Bending strain for Actuator case ($l/h = 20$)

Figure 3.23 Transverse normal strain for Actuator case ($l/h = 20$)
Figure 3.24 Shear strain for Actuator case ($l/h = 20$)

Figure 3.25 y Component Electric Field for Actuator case ($l/h = 20$)
Figure 3.26 $z$ Component Electric Field for Actuator case ($l/h = 20$)

Figure 3.27 $y$ component of the Electric Displacement for Actuator case ($l/h = 20$)
Figure 3.28 $z$ Component Electric Displacement for Actuator case ($l/h = 20$)

Figure 3.29 Axial Displacement for Sensor case ($l/h = 50$)
Figure 3.30 Transverse Displacement for Sensor Case ($l/h = 50$)
Figure 3.31 Electric Potential for Sensor case ($l/h = 50$)

Figure 3.32 Bending Stress for Sensor case ($l/h = 50$)
Figure 3.33 Bending strain for Sensor case ($l/h = 50$)

Figure 3.34 Shear Stress for Sensor case ($l/h = 50$)
Figure 3.35 Transverse Normal Stress for Sensor Case (\(l/h = 50\))

Figure 3.36 Transverse normal strain for Sensor case (\(l/h = 50\))
Figure 3.37 Shear strain for Sensor case ($l/h = 50$)

Figure 3.38 $y$ Component Electric Field for Sensor case ($l/h = 50$)
Figure 3.39 z Component Electric Field for Sensor case ($l/h = 50$)

Figure 3.40 y Component Electric Displacement for Sensor case ($l/h = 50$)
Figure 3.41 $z$ Component Electric Displacement for Sensor case ($l/h = 50$)

Figure 3.42 Axial Displacement for Actuator case ($l/h = 50$)
Figure 3.43 Transverse Displacement for Actuator Case ($l/h = 50$)

Figure 3.44 Electric Potential for Actuator case ($l/h = 20$)
Figure 3.45 Bending Stress for Actuator case ($l/h = 50$)

Figure 3.46 Transverse Normal Stress for Actuator Case ($l/h = 50$)
Figure 3.47 Shear Stress for Actuator case ($l/h = 50$)

Figure 3.48 Bending strain for Actuator case ($l/h = 50$)
Figure 3.49 Transverse normal strain for Actuator case ($l/h = 50$)

Figure 3.50 Shear strain for Actuator case ($l/h = 50$)
Figure 3.51 $y$ Component Electric Field for Actuator case ($l/h = 50$)

Figure 3.52 $z$ Component Electric Field for Actuator case ($l/h = 50$)
Figure 3.53 y component Electric Displacement for Actuator case ($l/h = 50$)

Figure 3.54 z Component Electric Displacement for Actuator case ($l/h = 50$)
Chapter 4

MODELING OF VIBRATION AND NOISE CONTROL USING DISCRETE PIEZOELECTRIC ACTUATORS/SENSORS

-a) Fourier series method b) Rayleigh-Ritz approach

4.1 Introduction

Surface bonded discrete piezoelectric patches for plates and shell structures have proven to be an efficient and simple approach to actively control vibration and radiated noise. By proper placement, distributed sensors can be used to sense membrane response or bending response or both. The sensing and actuation effects of distributed piezoelectric sensors and actuators depend on their shape, thickness, material properties, placement, spatial shaping and spatial distribution. When a single piece fully distributed piezoelectric sensor/actuator is used there are observability and controllability deficiencies in monitoring and control of plates and shells (Tzou 1993). Spatial shaping of distributed sensors and actuators, for example by segmenting them into a number of smaller pieces, can improve the controllability and observability. The effectiveness of distributed piezoelectric sensors and actuators, in the form of rectangular patches, bonded on simply supported rectangular thin composite plates and circular cylindrical shell panels have been investigated by Tzou and Fu (1994a,b). When the piezoelectric patches are surface bonded to the plate structure, the presence of a patch can affect the dynamic characteristics of the host structure in two ways (Banks et al 1995). The fact that the patch has a definite thickness and material properties, the contribution of the patch stiffness and mass to the host structures stiffness and mass changes the global dynamic
behavior of the smart plate. This leads to additional terms in the force and moment resultant of the host plate that must be estimated accurately in order to predict the dynamic characteristics of the smart plate like frequencies and mode shapes. The other contribution is due to the application of voltages (active stiffness) on the electrodes of the actuator, which generates the moments and forces which appear as forcing functions in the final dynamic equations of motion. In all earlier models the mass and stiffness contributions from sensor and actuator patches were neglected for estimating the natural frequencies of the smart plate. Also the thickness direction electric fields and strain fields inside the patches are assumed to be constant over the entire area of the patch. The validity of these assumptions depends on the size and relative stiffness of the patches and has not been investigated before. In this section, the Classical (Kirchoff’s) Plate Theory (CPT) is used to estimate the natural frequencies of a plate structure with surface bonded piezoelectric patches without the above-mentioned assumptions. A detailed modeling of the patches is developed by expressing the electric potential inside the patch as a quadratic function of the thickness coordinate. The equations of motion are derived for a generally isotropic plate with surface bonded discrete patches. The solution to the dynamic equations of motion is obtained using a Fourier series method for a plate with collocated piezoelectric actuator/sensor patches. The effect of the passive stiffness of the surface bonded actuator and sensor patches on the dynamic characteristics of host plate structure is investigated.

In the present formulation the piezoelectric patch model is refined using the electroelastic theory and by satisfying the charge equation. Also the contributions from
the discrete patches to a host plate are added and the effect of these contributions on the
dynamic characteristics of the smart plate is studied. Though the model development is
based on CPT this can be easily extended to FSDT and higher order plate theories
depending on geometric and material properties. This is one of the powerful features of
this formulation and also can be extended to shells.

Using the CPT model developed, a vibroacoustic model for studying the active
control of noise transmitted into a rectangular enclosure using discrete piezoelectric
patches is developed using the Rayleigh-Ritz (RR) method. The enclosure has five rigid
walls and a flexible smart plate, which is embedded with discrete piezoelectric actuators
and sensors. Eigenfunctions of a clamped-clamped beam are used as the Ritz functions
for the panel and the rigid-walled cavity modes are used to model the acoustic cavity. The
dynamic equations of motion for the coupled smart panel-cavity system are derived using
Hamilton’s principle. The forcing term due to the acoustic pressure in the cavity is
determined by using virtual work considerations. For the present study, five collocated
pairs of actuator/sensor are attached to the plate at predetermined positions. The results
obtained using the Rayleigh-Ritz procedure for a smart aluminum panel-cubic cavity is
compared with finite element (FE) and experimental results. Close agreement between
the present approach and the FE and experimental results confirms the validity of the
results. The RR method is suggested for the modeling of laboratory smart structures as an
alternative to the FE method because it provides the same accuracy with significant
reduction in computational size and time.
4.2 Piezoelectric Patches: Modeling And Formulation

Figure 4.1 shows a rectangular laminate on which piezoelectric patches are bonded. The length and width of the each patch are ‘a’ and ‘b’. $L_x$ and $L_y$ are the dimensions of the laminate.

![Figure 4.1 Configuration of surface bonded piezoelectric patch](image)

The piezoelectric patches are polarized in the thickness direction and the major surfaces are covered with electrodes of negligible thickness. The laminate is assumed to lie on the $z = 0$ plane. The classical laminated plate theory, which is an extension of Kirchoff’s (classical) plate theory to laminated composite plates assume the following displacement fields:
\[
\begin{align*}
    u(x, y, z, t) &= u_0(x, y, t) - z \frac{\partial w_0}{\partial x}, \\
    v(x, y, z, t) &= v_0(x, y, t) - z \frac{\partial w_0}{\partial y}, \text{ and} \\
    w(x, y, z, t) &= w_0(x, y, t),
\end{align*}
\] (4.1)

where \( u, v, w \) are the displacements along the \( x, y, z \) coordinate directions, respectively, of a point on the midplane (i.e., \( z = 0 \)). Kirchoff’s assumption amounts to neglecting both transverse shear and transverse normal effects, i.e., deformation is entirely due to bending and in-plane stretching. This leads to zero transverse strain components \( \gamma_{yz}, \gamma_{zx}, \gamma_{xy} \).

Thus for small strains and moderate rotations the strain-displacement relations take the form

\[
\begin{align*}
    \varepsilon_x &= \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2}, \\
    \varepsilon_y &= \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w_0}{\partial y^2}, \text{ and} \\
    \gamma_x &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w_0}{\partial x \partial y},
\end{align*}
\] (4.2)

or shortly \( \varepsilon = \varepsilon^{(0)} + z \varepsilon^{(1)} \),

where \( \varepsilon^{(0)} = \begin{bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{bmatrix} \) and \( \varepsilon^{(1)} = \begin{bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2 \frac{\partial^2 w_0}{\partial x \partial y} \end{bmatrix} \).

The linear constitutive relations for the host laminate in the principal material directions
are
\[
\begin{bmatrix}
\sigma_{xx}^{(k)} \\
\sigma_{yy}^{(k)} \\
\tau_{xy}^{(k)}
\end{bmatrix}
= \begin{bmatrix}
\overline{Q}_{11}^{(k)} & \overline{Q}_{12}^{(k)} & \overline{Q}_{16}^{(k)} \\
\overline{Q}_{21}^{(k)} & \overline{Q}_{22}^{(k)} & \overline{Q}_{26}^{(k)} \\
\overline{Q}_{16}^{(k)} & \overline{Q}_{26}^{(k)} & \overline{Q}_{66}^{(k)}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx}^{(k)} \\
\varepsilon_{yy}^{(k)} \\
\gamma_{xy}^{(k)}
\end{bmatrix},
\]
where \(\overline{Q}_{ij}^{(k)}\) are the transformed stiffness coefficients. The linear constitutive relations written in terms of nonzero stress and strain components for the attached piezoelectric patches are
\[
\begin{align*}
\begin{bmatrix}
\sigma_{xx}^{(a),(x)} \\
\sigma_{yy}^{(a),(x)} \\
\tau_{xy}^{(a),(x)}
\end{bmatrix}
= & \begin{bmatrix}
C_{11}^{(a),(x)} & C_{12}^{(a),(x)} & 0 \\
C_{12}^{(a),(x)} & C_{22}^{(a),(x)} & 0 \\
0 & 0 & C_{66}^{(a),(x)}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx}^{(a),(x)} \\
\varepsilon_{yy}^{(a),(x)} \\
\gamma_{xy}^{(a),(x)}
\end{bmatrix}
- \begin{bmatrix}
0 & 0 & \varepsilon_{13}^{(a),(x)} \\
0 & 0 & \varepsilon_{23}^{(a),(x)} \\
0 & 0 & \varepsilon_{33}^{(a),(x)}
\end{bmatrix}
\begin{bmatrix}
E_{x}^{(a),(x)} \\
E_{y}^{(a),(x)} \\
E_{z}^{(a),(x)}
\end{bmatrix},
\end{align*}
\]
and
\[
\begin{align*}
\begin{bmatrix}
D_{x}^{(a),(x)} \\
D_{y}^{(a),(x)} \\
D_{z}^{(a),(x)}
\end{bmatrix}
= & \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\varepsilon_{13}^{(a),(x)} & \varepsilon_{23}^{(a),(x)} & 0
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx}^{(a),(x)} \\
\varepsilon_{yy}^{(a),(x)} \\
\gamma_{xy}^{(a),(x)}
\end{bmatrix}
+ \begin{bmatrix}
\varepsilon_{11}^{(a),(x)} & 0 & 0 \\
0 & \varepsilon_{22}^{(a),(x)} & 0 \\
0 & 0 & \varepsilon_{33}^{(a),(x)}
\end{bmatrix}
\begin{bmatrix}
E_{x}^{(a),(x)} \\
E_{y}^{(a),(x)} \\
E_{z}^{(a),(x)}
\end{bmatrix},
\end{align*}
\]
where \(C_{ij}^{(a),(x)}, \varepsilon_{ij}^{(a),(x)}\) and \(\varepsilon_{ii}^{(a),(x)}\) are the stiffness coefficients at constant electric field, piezoelectric constants and dielectric permittivities at constant state of stress of the actuator and sensor patches. \(\sigma_{i}\) and \(D_{i}\) are the stress and electric displacement components. The superscripts \((a)\) and \((s)\) refer to the actuator and sensor patches. Apart from satisfying the equilibrium equations the piezoelectric patch regions should satisfy the charge equation \(\nabla \cdot \mathbf{D} = 0\). If the patches are thin, we can assume that the x and y component electric displacement fields are constant along x and y directions within the patch so that the charge equation reduces to \(\frac{dD_z}{dz} = 0\).
4.2.1 Actuator Patch Model

The electric potential inside the actuator patch ($\phi^a$) is assumed to vary quadratically along the thickness direction as:

$$\phi^a = \phi_0^a + z\phi_1^a + z^2\phi_2^a.$$  \hspace{1cm} (4.4)

It is assumed, that for the actuator patches the electrodes located at the laminate patch interface (i.e., $z=z_1$) are always grounded and a potential of $V^a$ is applied at the other electrode surface (i.e., $z=z_0$). Using these conditions in equation (4.4) we obtain

$$\phi_1^a = \frac{V^a}{h^a} - 2h_m^a \phi_2^a$$  \hspace{1cm} (4.5)

where $h_a = (z_0 - z_1)$ is the thickness of the actuator patch and $h_m^a = \frac{(z_0 + z_1)}{2}$ is the distance of the actuator middle surface from the laminate middle surface. Using the constitutive equation and the reduced charge equation we obtain

$$\phi_z^a = -\frac{\varepsilon_{31}}{2\varepsilon_{33}} \left( \varepsilon^{(1)}_{xx} + \varepsilon^{(1)}_{yy} \right)^{\gamma^a}. \hspace{1cm} (4.6)$$

From equations (4.4), (4.5) and (4.6) the electric potential and electric field inside the actuator patch is:

$$E_z^a = -\frac{d\phi^a}{dz} = -\frac{V^a}{h^a} + \varepsilon_{31} \left( \frac{z - h_m^a}{\varepsilon_{33}} \right) \left( \varepsilon^{(1)}_{xx} + \varepsilon^{(1)}_{yy} \right)^{\gamma^a}. \hspace{1cm} (4.7)$$

Substituting equation (4.7) into equation (4.3a) we obtain the stress-strain relation for the actuator patches.
\[ \sigma_{xx}^a = C_{11}^a \varepsilon_{xx}^a + C_{12}^a \varepsilon_{yy}^a + \frac{e_{31} V^a}{h^a} + \frac{e_{33}^2 \left( \frac{z - h_m^a}{h} \right)}{\varepsilon_{33}^a} \left( \varepsilon_{xx}^{(1)} + \varepsilon_{yy}^{(1)} \right)^a, \]

\[ \sigma_{yy}^a = C_{12}^a \varepsilon_{xx}^a + C_{22}^a \varepsilon_{yy}^a + \frac{e_{31} V^a}{h^a} + \frac{e_{33}^2 \left( \frac{z - h_m^a}{h} \right)}{\varepsilon_{33}^a} \left( \varepsilon_{xx}^{(1)} + \varepsilon_{yy}^{(1)} \right)^a, \text{ and} \]

\[ \tau_{xy} = C_{66}^a \tau_{xy}^a. \tag{4.8} \]

The above constitutive relations are valid only inside the actuator patches.

### 4.2.2 Sensor Patch Model

From the reduced charge equation, we observe that the \( z \) component electric displacement should be constant along the thickness (\( z \)) direction of the sensor patch. Also for the sensor there is no external supply of charge into the patch, so the total charge that appears on the sensor patch electrode surfaces should be zero. This condition for the present case can be written as

\[ D_z \big|_{z = z_a} = - D_z \big|_{z = z_a + h}. \]

This condition along with the reduced charge equation forces \( D_z \) to be zero everywhere inside the patch. So from equation (4.3b) we find

\[ E_z = - \frac{e_{31}}{\varepsilon_{33}^a} \left( \varepsilon_{xx}^a + \varepsilon_{yy}^a \right)^a z = - \frac{e_{31}}{\varepsilon_{33}^a} \left\{ \left( \varepsilon_{xx}^{(0)} + \varepsilon_{yy}^{(0)} \right)^a z + z \left( \varepsilon_{xx}^{(1)} + \varepsilon_{yy}^{(1)} \right)^a \right\}. \tag{4.9} \]

If the electric potential inside the sensor patch is also assumed to vary quadratically with \( z \) then we write:

\[ \phi^z = \phi_0^z + z \phi_1^z + z^2 \phi_2^z. \tag{4.10} \]

The electric field inside the patch is then obtained as

\[ E_z = - \frac{d \phi^z}{dz} = - \phi_1^z - 2z \phi_2^z. \tag{4.11} \]

Comparing equations (4.9) and (4.11) we obtain
\[ \phi_1' = -\frac{e_{31}}{e_{33}} (\varepsilon^{(0)}_{xx} + \varepsilon^{(0)}_{yy}) \phi_2' = -\frac{e_{31}}{2e_{33}} (\varepsilon^{(1)}_{xx} + \varepsilon^{(1)}_{yy}). \]

The average sensor potential (i.e., the voltage that appears between the sensor electrodes is

\[ V^s = \frac{1}{A_s} \int_A \left( \phi^s_{z=z_3} - \phi^s_{z=z_2} \right) dA = \frac{h' e_{31}}{A_s} \int_A \left\{ (\varepsilon^{(0)}_{xx} + \varepsilon^{(0)}_{yy}) - h_m (\varepsilon^{(1)}_{xx} + \varepsilon^{(1)}_{yy}) \right\} dA, \tag{4.12} \]

where \( h_s = (z_{n+1} - z_n) \) is the thickness of the sensor patch and \( h_m = \frac{(z_{n+1} + z_n)}{2} \) is the distance of the sensor patch middle surface from the laminate middle surface. Substituting equation (4.11) in (4.3a) the stress-strain relation for the sensor patch is obtained as:

\[ \sigma_{xx}^s = \left( C_{11} - \frac{e_{31}^2}{e_{33}} \right) \varepsilon_{xx}^s + \left( C_{12} - \frac{e_{31}^2}{e_{33}} \right) \varepsilon_{yy}^s, \]
\[ \sigma_{yy}^s = \left( C_{22} - \frac{e_{31}^2}{e_{33}} \right) \varepsilon_{xx}^s + \left( C_{22} - \frac{e_{31}^2}{e_{33}} \right) \varepsilon_{yy}^s, \text{ and} \]
\[ \tau_{xy}^s = C_{66} \gamma_{xy}^s. \tag{4.13} \]

### 4.3 Smart Plate Model

The equilibrium equations for the plate with actuator and sensor patches bonded on the surface are given as:
\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = \rho h \frac{\partial^2 u}{\partial t^2},
\]
\[
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = \rho h \frac{\partial^2 v}{\partial t^2}, \quad \text{and}
\]
\[
\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\rho h \frac{\partial^2 w}{\partial t^2} - q(x, y, t),
\]

where
\[
\begin{align*}
\{ N_x \} &= \int_{z_1}^{z_2} \{ \sigma_{xx}^E \} d\tau + \int_{z_1}^{z_2} \{ \sigma_{xx}^a \} d\tau + \int_{z_1}^{z_2} \{ \sigma_{xy}^a \} d\tau + \int_{z_1}^{z_2} \{ \tau_{xy}^a \} d\tau \\
\{ N_y \} &= \int_{z_1}^{z_2} \{ \sigma_{xy}^E \} d\tau + \int_{z_1}^{z_2} \{ \sigma_{xy}^a \} d\tau + \int_{z_1}^{z_2} \{ \tau_{xy}^a \} d\tau
\end{align*}
\]

are the membrane force resultants. Here \( \sigma_{ii}^E \) are the stress components for the host plate and \( R \) is the location function defined as
\[
R(x, y) = [H(x-a) - H(x+a)][H(y-a) - H(y+a)],
\]
\[
= 1, -a \leq x \leq a, -b \leq y \leq b, \quad \text{and}
\]
\[
= 0, \text{ elsewhere},
\]
in which \( H \) is the Heaviside function. The moment resultants \( M_i \) are defined as
\[
\begin{align*}
\{ M_x \} &= \int_{z_1}^{z_2} \{ \sigma_{xx}^E \} d\tau + \int_{z_1}^{z_2} \{ \sigma_{xx}^a \} d\tau + \int_{z_1}^{z_2} \{ \sigma_{xy}^a \} d\tau + \int_{z_1}^{z_2} \{ \tau_{xy}^a \} d\tau \\
\{ M_y \} &= \int_{z_1}^{z_2} \{ \sigma_{xy}^E \} d\tau + \int_{z_1}^{z_2} \{ \sigma_{xy}^a \} d\tau + \int_{z_1}^{z_2} \{ \tau_{xy}^a \} d\tau \\
\{ M_{xy} \} &= \int_{z_1}^{z_2} \{ \tau_{xy}^E \} d\tau + \int_{z_1}^{z_2} \{ \tau_{xy}^a \} d\tau + \int_{z_1}^{z_2} \{ \tau_{xy}^a \} d\tau
\end{align*}
\]

The local mass density function \( \bar{\rho} \) is defined as
\[
\bar{\rho} = \rho^E h^E + (\rho^{(a)} h^{(a)} + \rho^{(a)} h^{(a)}) R,
\]
where \( \rho^{(a)} \) and \( \rho^{(s)} \) are the mass densities of the sensor and actuator patch materials and \( \rho^E \) is the density of the host plate. Here \( q \) is the distributed harmonic load acting on the plate surface. Substituting the stress-strain relations for the plate, actuator and sensor
patches from equations (4.8), (4.13) and (4.15) into the force and moment resultant equations we obtain

\[
\{\mathbf{N}\} = \begin{bmatrix} \mathbf{A} & \mathbf{r}^{(0)} \\ \mathbf{B} & \mathbf{r}^{(1)} \end{bmatrix} + A^a \mathbf{r}_a^{(0)} + A^a \mathbf{r}_a^{(1)} + \begin{bmatrix} \mathbf{A}^t & \mathbf{r}^{(0)} \\ \mathbf{B}^t & \mathbf{r}^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{A}^t & \mathbf{r}^{(0)} \\ \mathbf{B}^t & \mathbf{r}^{(1)} \end{bmatrix} + [1] \varepsilon_{33} V^a ,
\]

\[
\{\mathbf{M}\} = \begin{bmatrix} \mathbf{B} & \mathbf{r}^{(0)} \\ \mathbf{D} & \mathbf{r}^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{B}^t & \mathbf{r}^{(0)} \\ \mathbf{D}^t & \mathbf{r}^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{B}^t & \mathbf{r}^{(0)} \\ \mathbf{D}^t & \mathbf{r}^{(1)} \end{bmatrix} + [1] \varepsilon_{33} V^a h_m^a ,
\]

where \( A_{ij} = \sum_{k=1}^{n} (\mathbf{Q}_{ij})_k (z^k_{ij} - z^k_{ij-1}); A^a_{ij} = C^a_{ij} h^a R ; A^a_{ij} = (C^a_{ij} + \varepsilon_{33}^2) h^a R , \)

\( B_{ij} = \frac{1}{2} \sum_{k=1}^{n} (\mathbf{Q}_{ij})_k (z^2_{ij} - z^2_{ij-1}); B^a_{ij} = C^a_{ij} h^a h^a R ; B^a_{ij} = (C^a_{ij} + \varepsilon_{33}^2) h^a h^a R , \)

\( D_{ij} = \sum_{k=1}^{n} (\mathbf{Q}_{ij})_k (z^3_{ij} - z^3_{ij-1}); D^a_{ij} = \frac{1}{3} C^a_{ij} (z^3_{ij} - z^3_{ij-1}) R ; D^a_{ij} = \frac{1}{3} (C^a_{ij} + \varepsilon_{33}^2) (z^3_{ij} - z^3_{ij-1}) R . \)

and \([\mathbf{I}]\) is the identity matrix. Substituting the force and moment resultants (4.18) into the plate equilibrium equation (4.14) we obtain the equation of motion for the smart plate.

For a square isotropic plate with collocated actuator/sensor patches the general equation of motion reduces to

\[
-D V^4 w = \frac{\rho}{\rho} \frac{\partial^2 w}{\partial t^2} + \sum_{i=\{x,y,z\}} \left[ \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) R + 2 \left[ \left( \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) R_x + \left( \frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) R_y \right] + \left[ \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) R_{xx} + \left[ \nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] R_{yy} \right] + \left[ 2(1-\nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right) R_{xy} \right] \right] + \frac{G e_{1s} h_m^a h_m^a}{\varepsilon_{33}^2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) (R_{xx} + R_{yy} + 2 R_{xy}) (4.19)
\]
where \( R_i = \frac{\partial R}{\partial x} , R_j = \frac{\partial R}{\partial y} , R_{xx} = \frac{\partial^2 R}{\partial x^2} , R_{yy} = \frac{\partial^2 R}{\partial y^2} , \) and \( R_{xy} = \frac{\partial^2 R}{\partial x\partial y} . \) The first derivatives of \( R \) correspond to the delta functions \( ( R = \frac{\partial^2 R}{\partial x^2} = \delta_x ) \) in \( x \) and \( y \) directions and these exist inside the integrals.

It should be noted that while deriving the above equation of motion for the plate no assumptions are made on the strain fields other than the basic assumptions of CPT. The terms inside the curly braces on the RHS of the above equations are due to the passive stiffnesses of the actuator and sensor patches. If the strains are assumed to be constant inside the patches then all these terms will vanish (B T Wang et al 1991, Tzou et al 1994a, 1994b). In equation (4.19) the actuator voltage \( V_a \) is replaced by feedback voltage \( G V^{\phi} \), where \( G \) is the feedback gain and \( V^{\phi} \) is the sensor output signal that is fed back to the actuator.

4.4 Solution Method

To obtain the natural frequency of the smart plate the feedback gain is considered to be zero. The solution for the equation (4.19) can be obtained by expanding the displacement in a Fourier series. Assuming a double Fourier sine series for the lateral displacement \( \text{'}w\text{'} \) as \( w = \sum_m \sum_n A_{mn} \sin\left(\frac{m\pi x}{l_x}\right) \sin\left(\frac{n\pi y}{l_y}\right) e^{i\omega t} \) the equation of motion for the smart plate reduces to
\[- \sum \sum (I_1 + \sum \sum (I_2 R_x^i - I_2 R_y^i - I_2 R_y^i)) A_{\text{mn}} \sin \left( \frac{m \pi x}{l_x} \right) \sin \left( \frac{n \pi y}{l_y} \right) + 2 \sum \sum \sum (I_2 R_x^i \cos \left( \frac{m \pi x}{l_x} \right) A_{\text{mn}} \sin \left( \frac{n \pi y}{l_y} \right) + I_3 R_y^i A_{\text{mn}} \sin \left( \frac{m \pi x}{l_x} \right) \cos \left( \frac{n \pi y}{l_y} \right)) - \sum \sum \sum 2(1 - \nu) I_1 R_x^i A_{\text{mn}} \cos \left( \frac{m \pi x}{l_x} \right) \cos \left( \frac{n \pi y}{l_y} \right) = - \omega^2 \sum \sum \sum \left( \rho h + \sum \sum \rho h^i R^i \right) A_{\text{mn}} \sin \left( \frac{m \pi x}{l_x} \right) \sin \left( \frac{n \pi y}{l_y} \right); \]

(4.20)

where

\[ I = D \left[ \left( \frac{m \pi}{l_x} \right)^2 + \left( \frac{n \pi}{l_y} \right)^2 \right]^2, \quad I_1 = D \left[ \left( \frac{m \pi}{l_x} \right)^2 + \left( \frac{n \pi}{l_y} \right)^2 \right]^2, \quad I_2 = D \left[ \left( \frac{m \pi}{l_x} \right)^2 + \left( \frac{n \pi}{l_y} \right)^2 \right]^2, \quad I_3 = D \left[ \left( \frac{m \pi}{l_x} \right)^2 + \left( \frac{n \pi}{l_y} \right)^2 \right]^2, \quad I_4 = D \left[ \left( \frac{m \pi}{l_x} \right)^3 + \left( \frac{n \pi}{l_y} \right)^3 \right], \quad I_5 = D \left[ \left( \frac{m \pi}{l_x} \right)^3 + \left( \frac{n \pi}{l_y} \right)^3 \right], \quad I_6 = D \left[ \left( \frac{mn \pi}{l_x l_y} \right)^2 + \left( \frac{mn \pi}{l_x l_y} \right)^2 \right], \quad I_7 = D \left[ \left( \frac{mn \pi}{l_x l_y} \right)^3 + \left( \frac{mn \pi}{l_x l_y} \right)^3 \right], \quad I_8 = D \left[ \left( \frac{mn \pi}{l_x l_y} \right)^3 + \left( \frac{mn \pi}{l_x l_y} \right)^3 \right], \quad I_9 = D \left[ \left( \frac{mn \pi}{l_x l_y} \right)^3 + \left( \frac{mn \pi}{l_x l_y} \right)^3 \right] \]

The natural frequencies and the mode shapes of the plate cannot be obtained directly from the above equation. So the modal method is used to estimate the frequencies of the plate. Multiplying both sides of equation (4.20) by \( \sin \left( \frac{p \pi x}{l_x} \right) \sin \left( \frac{q \pi y}{l_y} \right) \) where \( p = 1, 2, \ldots, r \) and \( q = 1, 2, \ldots, s \) and integrating over the area of the plate we can generate a system of equations for the various values of \( p \) and \( q \). Truncating the series at some values of \( m \) and \( n \), we can write the system of equations in
matrix form as

\[
[K][A] = -\omega^2[M][A].
\] (4.21)

Solution to the above Eigenvalue problem gives the natural frequency \( \omega_{pq} \) and the coefficients \( A_{mn} \). The accuracy of the solution can be improved by considering more terms in the series.

4.5 Results And Discussion

An aluminum plate with one pair of collocated piezoelectric sensor/actuator patches (case I) and a Plexiglas plate with a one pair of PVDF patches (case II) are considered for the present study. The properties of the plate and the piezoelectric patches are given in Table 4.1.

<table>
<thead>
<tr>
<th>Properties</th>
<th>Host Plate</th>
<th>PZT</th>
<th>Host Plate</th>
<th>PVDF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Youngs Modulus (E) N/m²</td>
<td>12.6e10</td>
<td>12.6e10</td>
<td>3.10e3</td>
<td>2.0e9</td>
</tr>
<tr>
<td>Density (ρ) Kg/m³</td>
<td>2800</td>
<td>7500</td>
<td>1190</td>
<td>1800</td>
</tr>
<tr>
<td>Thickness (h) m</td>
<td>1.0e-3</td>
<td>0.8e-3</td>
<td>1.6e-3</td>
<td>0.5e-3</td>
</tr>
<tr>
<td>Poission’s Ratio (v)</td>
<td>0.3</td>
<td>0.2</td>
<td>0.35</td>
<td>0.2</td>
</tr>
<tr>
<td>Piezoelectric Constant (e31) N/Vm</td>
<td>-</td>
<td>-0.52</td>
<td>-</td>
<td>-0.13</td>
</tr>
<tr>
<td>Size m × m</td>
<td>0.305 × 0.305 Variable</td>
<td>0.2 × 0.2</td>
<td>Variable</td>
<td></td>
</tr>
</tbody>
</table>

The convergence of each natural frequency value is confirmed by considering more terms in the series. Figure 4.2 shows the convergence of the first four natural
frequencies for a single pair of actuator/sensor for case I (patch size = 0.1 × 0.1 m) with number of terms in the series. Table 4.2 shows the difference between the estimated frequencies of the host plate alone and the plate with the collocated actuator/sensor patches.

Table 4.2 Natural frequencies of the smart plate

<table>
<thead>
<tr>
<th>Mode</th>
<th>Case I (Hz)</th>
<th>Case II (Hz)</th>
<th>Difference</th>
<th>Case I (Hz)</th>
<th>Case II (Hz)</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>Plate</td>
<td>62.51</td>
<td>62.80</td>
<td>(0.5 %)</td>
<td>40.57</td>
<td>41.58</td>
</tr>
<tr>
<td>(1,2)</td>
<td>156.26</td>
<td>156.51</td>
<td>(0.3 %)</td>
<td>101.42</td>
<td>104.52</td>
<td>(3.9 %)</td>
</tr>
<tr>
<td>(1,3)</td>
<td>312.54</td>
<td>312.84</td>
<td>(0.1 %)</td>
<td>202.84</td>
<td>207.92</td>
<td>(2.5 %)</td>
</tr>
</tbody>
</table>

Figure 4.2 Convergence of the frequencies
Figure 4.3 Variation of the resonant frequencies with patch size for PVDF/Plexiglas/PVDF

Figure 4.4 Variation of the resonant frequencies with patch size for PZT/Al/PZT
The patch covers 10% of the host plate area. From Table 4.2, we observe that the error due to neglecting the passive stiffness and mass of the patch is more pronounced for a smart plate with PZT patch on aluminum plate than a PVDF patch on Plexiglas. The stiffness and mass ratios of the host plate to patch materials; piezoelectric coupling coefficient and size of the patches are the important factors, which decide the amount of shift in the resonant frequencies of the host plate.

The effect of the patch size on the resonant frequencies can be seen from figures 4.3 and 4.4. Solving equation (4.20) with the active stiffness (i.e. due to either voltage excitation or closed loop feedback) is extremely difficult, because of the non-orthogonal nature of the assumed solution in the patch region (-a<x<a, -b<y<b).

4.6 Active Enclosure Noise Control using Rayleigh-Ritz Method

From the previous section we noticed, though we could derive the equation of motion for surface bonded discrete actuators and sensors, it is nearly impossible to obtain a solution for the forced vibration or feedback case. So a more powerful numerical method should be considered for solving this problem.

A significant amount of scientific literature has been published on modeling the smart panel systems using analytical and finite element methods. Although these earlier FE models predict the behavior of the structural panel and the fluid-structure interaction (in the case of active noise control system designs) accurately at low frequencies, at high frequencies the size of the model increases resulting in very long computational time. Furthermore, optimal actuator/sensor placement studies involve repeated and time-
consuming FE remeshing during the iterations. Thus, a closed form solution approach is preferred over the FE method. In the following sections an energy approach based on the Rayleigh-Ritz approach is presented. The formulation is presented for a smart panel backed cavity to demonstrate the fact that this approach is not just for smart structures, but could also be used for vibroacoustic systems with smart walls.

4.7 Potential And Kinetic Energies Of The System

4.7.1 Composite Plate

Using classical thin plate theory, the bending strain energy for a symmetrically layered composite panel is given by

\[
U_w^{\text{plate}} = \frac{1}{2} \int_0^{\ell_x} \int_0^{\ell_y} \int_{-h/2}^{h/2} \mathbf{e} \mathbf{\sigma} \, dx \, dy \, dz .
\]  
(4.22)

Substituting the expressions for the stresses and strains we obtain the strain energy for the plate as

\[
U_w^{\text{plate}} = \int_0^{\ell_x} \int_0^{\ell_y} D_{11} \left( \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x^2} \right) + D_{12} \left( \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial x^2} \right) + D_{22} \left( \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial y^2} \right) + 4D_{66} \left( \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right) \, dx \, dy ,
\]  
(4.23)

where \( D_j = \frac{1}{3} \sum_{k=1}^{n} (Q_{j,k})_k (z_k^3 - z_{k-1}^3) \).

The kinetic energy for the plate is given by

\[
T_w^{\text{plate}} = \frac{1}{2} \int_0^{\ell_x} \int_0^{\ell_y} \rho h \left( \frac{\partial w}{\partial t} \right)^2 \, dx \, dy .
\]  
(4.24)
4.7.2 Piezoelectric Patches

The strain energy stored in a piezoelectric patch is estimated in the same way and is given by

\[
U_{w_{pc}}^w = \frac{1}{2} \sum_{i=1}^{Na+Ns} \int_{x_{1i}}^{x_{2i}} \int_{y_{1i}}^{y_{2i}} \int_{z_{2}}^{z_{2}} \mathbf{e} : \mathbf{e}^T \sigma \, dx \, dy \, dz,
\]

where \( N_a \) and \( N_s \) are the total number of actuator and sensor patches bonded on the surface and \( x_{1i}, x_{2i}, y_{1i}, y_{2i} \) are the positions of the edges of the patches from origin.

Substituting the expressions for stresses and strains we obtain

\[
U_{w_{pc}}^{piezo} = \sum_{i=1}^{Na+Ns} \int_{x_{1i}}^{x_{2i}} \int_{y_{1i}}^{y_{2i}} D_{ij} \left( \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \frac{\partial^2 w}{\partial x^2} \right)
+ D_{ij} \left( \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + 4D_{ij} \left( \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial dy \partial dx} \right) \right) \, dy \, dx,
\]

where \( D_{ij}^{act} = \frac{1}{3} C_{ij}^a (z_3^3 - z_2^3) \) and \( D_{ij}^{elec} = \frac{1}{3} (C_{ij}^{elec} - \frac{\varepsilon_3^3}{\varepsilon_3^3}) (z_3^3 - z_0^3) \).

Apart from strain energy, the potential energy stored in the piezoelectric patches has one more component called electrical energy due to the presence of electric and electric displacement fields in the piezoelectric material.

The electric energy stored in a piezoelectric material is expressed as

\[
U_{\Phi_{piezo}} = \frac{1}{2} \sum_{i=1}^{Na} \int_{x_{1i}}^{x_{2i}} \int_{y_{1i}}^{y_{2i}} \int_{z_{2}}^{z_{2}} \mathbf{E} \cdot \mathbf{D} \, dx \, dy \, dz
= \frac{1}{2} \sum_{i=1}^{Na} \int_{x_{1i}}^{x_{2i}} \int_{y_{1i}}^{y_{2i}} \int_{z_{2}}^{z_{2}} D_{ij} E_j \, dx \, dy \, dz.
\]

Due to the assumption that the \( D \) field is zero inside the sensors there is no electric energy stored in the sensors. Substituting the expressions for the electric
displacement and electric field from eqns, (4.3b) and (4.7) we obtain the electric energy stored in the actuators as

\[
U_\Phi^{\text{piezo}} = \frac{1}{2} \sum_{i=1}^{N_a} \int_{y_{li}}^{y_{2i}} \int_{x_{li}}^{x_{2i}} \int_{z_2}^{z_3} \left\{ e_{31i}^a (\epsilon_{xx}^a + \epsilon_{yy}^a) \right. \\
\left. - \epsilon_{33}^a \frac{V^a}{h^a} + e_{31i}^a (z - h_m^a) (\epsilon_{xx}^{(1)} + \epsilon_{yy}^{(1)})^a \right\} dx \, dy \, dz
\] (4.28)

\[
= \frac{1}{2} \sum_{i=1}^{N_a} \int_{x_{li}}^{x_{2i}} \int_{y_{li}}^{y_{2i}} \int_{z_2}^{z_3} \left\{ - \frac{V^a}{h^a} + e_{31i}^a (z - h_m^a) (\epsilon_{xx}^{(1)} + \epsilon_{yy}^{(1)})^a \right\} dx \, dy \, dz
\] (4.28)

The total kinetic energy for the patches is

\[
T_w^{\text{piezo}} = \frac{1}{2} \sum_{i=1}^{N_a} \int_{x_{li}}^{x_{2i}} \int_{y_{li}}^{y_{2i}} \rho^i h^i \left( \frac{\partial w}{\partial t} \right)^2 \, dx \, dy \text{.} \tag{4.29}
\]

### 4.7.3 Acoustic Cavity

The potential energy associated with the acoustic cavity is given by

\[
U_{aco} = \frac{1}{2} \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \nabla p . \nabla p - \frac{1}{c^2} \frac{\partial^2 \tilde{p}}{\partial t^2} \, dx \, dy \, dz \tag{4.30}
\]

and the work done by the flexible walls is given by

\[
Q_{pr} = - \int_{x_0}^{x_1} \int_{y_0}^{y_1} p . \frac{\partial \tilde{p}}{\partial n} \, dx \, dy = - \int_{x_0}^{x_1} \int_{y_0}^{y_1} p . \rho_0 \tilde{w} \, dx \, dy \text{, where } \rho_0 \text{ is the density of air in the cavity and } c \text{ is the sound velocity.}
\]

### 4.8 Dynamic Equation of Motion: Smart Panel-Cavity System

After estimating all the energies associated in the host composite plate and attached piezoelectric patches, the dynamic equations of motion are obtained using Hamilton’s principle. Hamilton’s principle for the smart plate can be stated as
\[ \delta \int_{t_i}^{t_f} (T - U + W) \, dt = 0 , \quad (4.31) \]

where \( T \) and \( U \) are the total kinetic and strain energies for the plate and patches and \( W \) is the work done by the external forces on the smart plate. The total kinetic and strain energies for the present smart plate are given by \( T = T_p + T_{\text{piezo}} \) and \( U = U_p + U_{\text{piezo}} + U^e \). Before applying Hamilton’s principle we assume a series solution for the transverse displacement \( w \) containing approximation functions \( f_{ij}(x,y) \) multiplied by the modal coordinates \( W_{ij}(t) \). In view of rectangular geometry and clamped boundary conditions of the panel, the eigenfunctions of a beam with clamped ends is a good choice for \( f_{ij}(x,y) \). The transverse displacement \( w \) can be written as

\[
w = \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij}(x, y) W_{ij}(t) = \sum_{i=1}^{m} \sum_{j=1}^{n} X_i(x) Y_j(y) W_{ij}(t) = \mathbf{F}^T \mathbf{W} \quad (4.32)\]

where \( X_i(x) = \sin \lambda_x x - \sinh \lambda_x x + \alpha_i (\cosh \lambda_x x - \cosh \lambda_x x) \) for \( i = 1, 2, \ldots, m \), and \( Y_j(y) = \sin \beta_j y - \sinh \beta_j y + \alpha_j (\cosh \beta_j y - \cosh \beta_j y) \) for \( j = 1, 2, \ldots, n \)

in which \( \lambda_i \) and \( \beta_j \) are the roots of the characteristic equations

\[
\cos \lambda_x l_x \cos \lambda_y l_y + 1 = 0 \quad \text{and} \quad \cos \beta_x l_x \cos \beta_y l_y + 1 = 0 \quad \text{and} \quad \alpha_i = \frac{\sinh \lambda_x l_x + \sin \lambda_x l_x}{\cosh \lambda_x l_x + \cos \lambda_x l_x} .
\]

Substituting the expressions derived for the kinetic and strain energies in Hamilton’s equation and taking the variation with respect to the generalized coordinates \( W_{mn} \), we obtain the equation of motion for the smart plate as

\[
(M_w + M_{\text{piezo}}) \ddot{W} + (K_w + K_{\text{piezo}}) W + (K_{w\Phi})^T \Phi^a = Q_p + Q_m , \quad (4.33)
\]
where

\[
M_w = \int_0^{x_1} \int_0^{y_1} \rho h \mathbf{F} \mathbf{F}^T \, dx \, dy, \quad M_w^{p_c} = \sum_{i=1}^{N_x} \int_{y_{i-1}}^{y_i} \int_{x_{i-1}}^{x_i} \rho h \mathbf{F} \mathbf{F}^T \, dx \, dy + \sum_{j=1}^{N_y} \int_{x_{j-1}}^{x_j} \int_{y_{j-1}}^{y_j} \rho h \mathbf{F} \mathbf{F}^T \, dx \, dy,
\]

\[
K_w = \int_0^{x_1} \int_0^{y_1} D_{11} \left( \frac{\partial^2 \mathbf{F}}{\partial x^2} \frac{\partial^2 \mathbf{F}^T}{\partial x^2} \right) + 2D_{12} \left( \frac{\partial^2 \mathbf{F}}{\partial x^2} \frac{\partial^2 \mathbf{F}^T}{\partial y^2} \right) + D_{22} \left( \frac{\partial^2 \mathbf{F}}{\partial y^2} \frac{\partial^2 \mathbf{F}^T}{\partial y^2} \right) + 4D_{60} \left( \frac{\partial^2 \mathbf{F}}{\partial x \partial y} \frac{\partial^2 \mathbf{F}^T}{\partial x \partial y} \right) \, dx \, dy,
\]

\[
K_w^{p_c} = \sum_{i=1}^{N_x} \int_{y_{i-1}}^{y_i} \int_{x_{i-1}}^{x_i} D_{11} \left( \frac{\partial^2 \mathbf{F}}{\partial x^2} \frac{\partial^2 \mathbf{F}^T}{\partial x^2} \right) + 2D_{12} \left( \frac{\partial^2 \mathbf{F}}{\partial x^2} \frac{\partial^2 \mathbf{F}^T}{\partial y^2} \right) + D_{22} \left( \frac{\partial^2 \mathbf{F}}{\partial y^2} \frac{\partial^2 \mathbf{F}^T}{\partial y^2} \right) + 4D_{60} \left( \frac{\partial^2 \mathbf{F}}{\partial x \partial y} \frac{\partial^2 \mathbf{F}^T}{\partial x \partial y} \right) \, dx \, dy,
\]

\[
Q_{pr} = \int_0^{x_1} \int_0^{y_1} p \mathbf{F} \, dx \, dy, \quad \text{and} \quad Q_f = \int_0^{x_1} \int_0^{y_1} f^T \mathbf{F} \, dx \, dy.
\]

where \(D_{11}^{(a)} = \frac{1}{3} C_{ij} (z_{i3}^3 - z_{i2}^3), D_{11}^{(a)} = \frac{1}{3} C_{ij} (z_{i3}^3 - z_{i0}^3), \) and \(D_{11}^{(a)} = \sum_{k=1}^{n} (Q_{ij})_k (z_{i3}^k - z_{i0}^k).\)

\(Q_{pr}\) is the work done by the cavity acoustic pressure on the vibrating panel and \(p\) is the cavity pressure and \(f^T\) are the external forces (i.e., point forces or distributed forces like incident noise etc.,) acting on the top surface of the panel. The sensor voltage can be obtained from equation (4.33) and the matrix form of this equation is \(V' = K_w^{p_c} W',\)

where

\[
K_w^{p_c} = -\sum_{i=1}^{N_x} h_i h_{i+1} e_{31} \sum_{l=1}^{N_y} \int_{x_{l-1}}^{x_l} \int_{y_{l-1}}^{y_l} \left( \frac{\partial \mathbf{F}}{\partial x^2} \frac{\partial \mathbf{F}^T}{\partial y^2} \right),
\]

\(4.34\)

For the acoustic cavity, the cavity pressure is expressed as a sum of a finite number of rigid walled cavity modes

\[
p^c = \sum_{i=1}^{N_x} \sum_{m=1}^{N_y} \sum_{n=1}^{N_z} \psi_{i m n}(x, y, z) P_{i m n}(t) = \Psi^T \mathbf{P},
\]

\(4.35\)
where \( \psi_{lmn} (x, y, z) = \cos \left( \frac{l \pi x}{l_x} \right) \cos \left( \frac{m \pi y}{l_y} \right) \cos \left( \frac{n \pi z}{l_z} \right) \) are the rigid walled cavity modes.

The plate cavity coupling occurs via the force vector \( Q_{pw} \). Substituting the function expression assumed for the cavity pressure in the force vector term we obtain

\[
Q_{pw} = K_{pw} P, \quad \text{where} \quad K_{pw} = \int_0^{l_x} \int_0^{l_y} \Psi(x, y, 0) F^T dx dy.
\]

Substituting the assumed function for the pressure in the energy expression for the cavity and applying Hamilton’s equation and taking the variation with respect to the generalized pressure \( P_{lmn}(t) \), we obtain the governing equation for the panel backed cavity as

\[
M_p \ddot{P} + K_p P + K_{pw} \dot{W} = 0, \quad (4.36)
\]

where

\[
M_p = \frac{1}{c^2} \int_0^{l_x} \int_0^{l_y} \Psi^T \Psi dx dy dz,
\]

and

\[
K_p = \int_0^{l_x} \int_0^{l_y} \int_0^{l_z} \frac{\partial \Psi}{\partial x} \frac{\partial \Psi^T}{\partial x} + \frac{\partial \Psi}{\partial y} \frac{\partial \Psi^T}{\partial y} + \frac{\partial \Psi}{\partial z} \frac{\partial \Psi^T}{\partial z} dx dy dz.
\]

Combining eqns. (4.33) and (4.36), we obtain the coupled equations for the smart panel backed cavity as

\[
\begin{bmatrix}
M_w + M_w^{pc} & 0 \\
K_{pw} & M_p
\end{bmatrix}
\begin{bmatrix}
\dddot{W} \\
\dddot{P}
\end{bmatrix}
+ \begin{bmatrix}
K_w + K_w^{pc} & K_{pw}^T \\
0 & K_p
\end{bmatrix}
\begin{bmatrix}
W \\
\dot{P}
\end{bmatrix}
= \begin{bmatrix}
K_{w\Phi}^a \Phi^a \\
0
\end{bmatrix}
+ \begin{bmatrix}
Q^{\alpha a}
\end{bmatrix}
\quad (4.37)
\]

with the sensor equation \( \Phi^s = K_{w\Phi}^s W \).

4.9 Results and Discussion

To validate the RR method, the resonant frequencies of an aluminum panel structure with five sensor/actuator patches bonded to its surface are estimated using
NASTRAN and the Center for the Engineering of Electronic and Acoustic Materials (CEEAM) developed FE analysis code for smart structures. The details of the CEEAM FE model are given in chapter 5. The cavity with the smart panel and the acoustic excitation are shown in Fig. 4.5. The geometric and material properties of this smart panel are given in Table 4.3 (also refer Fig. 4.1).

<table>
<thead>
<tr>
<th>Properties</th>
<th>Host Plate (Al)</th>
<th>PZT</th>
<th>AIR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus ‘E’ (N/m²)</td>
<td>68.0e+09</td>
<td>12.6e10</td>
<td></td>
</tr>
<tr>
<td>Density ‘ρ’ (Kg/m³)</td>
<td>2800</td>
<td>7500</td>
<td>1.293</td>
</tr>
<tr>
<td>Thickness ‘h’ (m)</td>
<td>0.8e-3</td>
<td>1.0e-3</td>
<td></td>
</tr>
<tr>
<td>Poisson’s Ratio ‘ν’</td>
<td>0.32</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>Piezoelectric Constant ‘e31’ (N/Vm)</td>
<td>-</td>
<td>-6.5</td>
<td></td>
</tr>
<tr>
<td>Size (m × m)</td>
<td>0.305 × 0.305</td>
<td>variable</td>
<td></td>
</tr>
<tr>
<td>Velocity of sound ‘C’ (m/sec)</td>
<td></td>
<td></td>
<td>330.0</td>
</tr>
</tbody>
</table>

The uncoupled resonance frequencies of the smart plate and cubic cavity estimated using the FE method and the RR method are shown in Table 4.4. Once the structural model for the panel is developed then it is coupled with the cavity and the coupled frequencies of the system are determined using equation (4.37). In the NASTRAN FE analysis, 36 shell elements and 512 acoustic elements are used to model the panel-cavity system. The actuators and sensors are modeled with the 10 non-piezoelectric brick elements.
Table 4.4 Uncoupled resonant frequencies (RR method) and comparison with FE solution

<table>
<thead>
<tr>
<th>Type</th>
<th>Cavity (Hz)</th>
<th>Smart Panel (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R-R Method</td>
<td>NASTRAN FE</td>
</tr>
<tr>
<td>(0,0,0)</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>(0,0,1)</td>
<td>540.9</td>
<td>544.0</td>
</tr>
<tr>
<td>(0,1,1)</td>
<td>765.0</td>
<td>770.1</td>
</tr>
<tr>
<td>(1,1,1)</td>
<td>937.0</td>
<td>951.0</td>
</tr>
<tr>
<td>(0,1,2)</td>
<td>1081.1</td>
<td>1178.0</td>
</tr>
</tbody>
</table>

The cavity is modeled using acoustic brick (CHEXA) elements. For the RR method, the first 7 mode shapes in each direction of the plate are included for the displacement and the first 6 rigid-walled cavity modes in each direction of the cavity are used in the series describes the cavity pressure. The coupled frequencies of the panel-backed cavity obtained from NASTRAN FE analysis and RR method are given in Table 4.5. From the results we note that the computationally easier RR approach is capable of
predicting the dynamics of the smart panel-cavity system accurately when compared with
the FE results.

<table>
<thead>
<tr>
<th>R-R Method (Hz)</th>
<th>NASTRAN/FE (Hz)</th>
<th>Experiment (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>84.5</td>
<td>85.3</td>
<td>85.7</td>
</tr>
<tr>
<td>155.7</td>
<td>151.9</td>
<td>-</td>
</tr>
<tr>
<td>252.8</td>
<td>249.8</td>
<td>256.0</td>
</tr>
<tr>
<td>315.4</td>
<td>304.3</td>
<td>-</td>
</tr>
</tbody>
</table>

The size of the system matrices resulting from the RR method for the present case is
265×265 and the FE system matrices have size of 800×800. This difference will
considerably increase for higher frequencies. For noise control studies, a uniform
harmonic pressure load of 2.0 Pa, which excites the panel at its first two resonant
frequencies, is applied on the panel surface. The pressure at the center of the cavity
\[
\left( \frac{l_x}{2}, \frac{l_y}{2}, \frac{l_z}{2} \right)
\]
is computed with and without velocity feedback control and is given in

<table>
<thead>
<tr>
<th>Excitation Frequency (Hz)</th>
<th>Pressure at the center of the Cavity (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Without Feedback</td>
</tr>
<tr>
<td>84.5</td>
<td>108</td>
</tr>
<tr>
<td>252.8</td>
<td>92</td>
</tr>
</tbody>
</table>
A simple velocity feedback control law $\mathbf{V}_a = -gain \times \mathbf{I} \times \dot{\mathbf{V}}_a$ is used in which $gain$ is the feedback gain value and $\mathbf{I}$ (5x5) is an identity matrix. From Table 4.6, we observe that the velocity feedback control using discrete sensor/actuator patches reduces the transmitted pressure by 14 dB for the first mode excitation and by 19 dB for the second mode excitation. We also see from equation (4.33), that the voltages induced in the sensors and the actuation forces supplied by the actuators are multiplied by the term

$$\int \int \left( \frac{\partial \mathbf{F}^T}{\partial x^2} + \frac{\partial \mathbf{F}^T}{\partial y^2} \right) dx dy,$$

which is a function of the displacement of the plate. This particular function, hereafter called amplification function, is plotted over the area of the plate in Fig.4.6 to Fig. 4.9 for the first four resonant modes of the plate. From the figures we observe that the peak values of the amplification function for any specific mode indicates the optimal positions of the sensors and actuators for controlling that particular mode. The long white dash dot lines represent the nodal lines (zero magnitude lines); over which if the sensor/actuators were placed would become ineffective. These amplification functions mapped, using either the mode shapes or the total response, can be used for rectangular plates to identify the optimal positions of the actuators and sensors.
Figure 4.6 Amplification function for mode 1

Figure 4.7 Amplification function for mode 2
Figure 4.8 Amplification function for mode 3

Figure 4.9 Amplification function for mode 4
Figure 4.10 Experimental Setup For Noise Control Measurement

Figure 4.11 The pressure (@cavity center) and the amplitude (@center of the plate) measured by the accelerometer and the microphone
Figure 4.10 shows the schematic diagram of the experimental setup used to determine the resonant frequencies for validating the RR model. The accelerometer senses the vibration levels at the center of the panel and a microphone inside the cabin senses the sound pressure levels. A B&K signal generator generates a chirp signal with a frequency range of 40 Hz to 500 Hz. This signal is then fed to a speaker, which generates a pressure field on the top surface of the smart panel thus simulating an external disturbance for the cavity-panel system. Figure 4.11 shows the frequency response of the accelerometer bonded at the center of the plate and the microphone placed at the center of the cavity. From the plot we observe that the responses are maximum at the resonant frequencies of the coupled system, which are shown in Table 4.5. The accelerometer and the microphone sensed only the symmetric modes because of the nature of the noise loading and their position on the plate and inside the cavity.

4.10 Summary

The analytical model for a smart plate bonded with discrete piezoelectric patches is developed without any assumptions on the strains inside the patches. The classical plate theory and the electroelastic theory are used in deriving the dynamic equations of motion for the smart plate. Though the examples used here includes only one pair of collocated sensor/actuator patches the solution can be obtained for any number of sensor/actuator pairs bonded to the host plate. The effect of the passive stiffnesses on the dynamic behavior of a host isotropic plate structure is studied. We observed that the presence of the patch changes the natural frequency considerably for of a smart plate with
surface bonded PZT piezoelectric patches. So the effects of passive stiffness cannot be neglected particularly for the Micro Electro Mechanical Systems (MEMS) applications where precise estimation of the frequencies is mandatory. For such applications the present analytical model can be used to estimate the dynamic characteristics of the smart plate. The effect of active stiffness and damping provided by the patch due to a closed loop system using the present modeling approach is presently difficult to obtain and should be attempted in the future. Alternatively, the Rayleigh-Ritz method is developed for modeling the plates with surface bonded discrete type actuators and sensors and extended for an active noise control system using this smart panel. A smart aluminum plate backed cavity is studied using this method and the results are compared with FE analysis and experimental results. From the results we conclude that the RR approach is quite accurate in predicting the dynamic behavior of the vibroacoustic system with an active noise control system. Results of studies showed that the RR model, which can predict the first few frequencies at the same level of accuracy, could execute four times faster than the corresponding FE model. At high frequencies, the FE method requires larger matrices, which are computationally expensive. Controller design and optimal placement of actuator/sensor pairs using the RR approach is preferred over the FE method due its simplicity and smaller computational size. For problems involving irregular boundaries and complicated boundary conditions, it is not possible to define a global shape function. In such cases, the RR approach loses some of its advantages over the FE method. Though the method developed here is used to study an isotropic panel, the approach is applicable to a general orthotropic smart panel-cavity system.
CHAPTER 5

FINITE ELEMENT SIMULATION OF SMART PANELS FOR VIBRATION AND NOISE CONTROL APPLICATIONS

5.1 Introduction

The design of efficient structural systems is of fundamental interest to both structural and control engineers. Systematic methodologies for both structural and active control system syntheses are well known and are receiving increased application in the design environment. Control system design for large flexible structures is an important and difficult problem because of the interaction between a structure and a control system in active vibration control. A number of important practical considerations such as the discretization of continuum models, total number of actuators and sensors used and their placement, and controller design must be taken into account before real implementation of active control can be put into action. The modeling and analysis of adaptive piezoelectric structures represents a high level of sophistication and complexity. Thus, the development of admissible mechanics for the numerical model as well as a sophisticated control algorithm for smart structures poses a great challenge. For given piezoelectric materials, structure, and control domain, an optimum procedure should be investigated for controlling the vibrations of the system.

There are two major experimental approaches to structural vibration control. One
uses distributed sensors and actuators in the form of piezoelectric polymer films (PVDF) that cover the entire structure and the electrodes deposited on these films are often shaped to shape the electric field to achieve the desired control. These sensors and actuators are called mode shape sensors and actuators because they respond and control only particular modes of vibration. The difficulty with this approach is that it is impractical and expensive to cover the whole structure or a large part of the structure with piezoelectric films. This changes the structural response of the system completely and there are major manufacturing difficulties. The optimal size, thickness and location of a single piezoelectric actuator on a flat plate for reducing the radiated sound are studied by Kim et al (1995), who have observed that for an optimal configuration of the patch type actuator, the voltage requirement is low.

Like the previous one this chapter also focuses on piezoelectric actuator and sensor patches that occupy a relatively small area of structures. It is unreasonable to use one sensor to obtain the vibration information of all points of the structure. Also, it is difficult to use one single actuator to control the vibration of structure. Most structures need multiple sensors and actuators to accurately observe and to effectively control the fundamental modes that dominate the structural vibration. The system thus becomes a multi-input-multi-output (MIMO) structure. Piezoelectric transducers made of PZT are also relatively thicker and smaller in size compared to the structure and are spatially distributed in many practical applications. Therefore, discrete sensors and actuators are preferred over distributed piezoelectric films for the multimode control of structures.

The fundamental problem of actively controlling flexible systems is to control a
very large dimensional system with a much smaller dimensional controller. The selection of appropriate feedback gains thus presents the difficulties for large-scale structures that require an infinite number of vibration modes to describe their dynamic behavior. Moreover, because of economical reasons and the limitations of on-board computer capability, the number of sensors and actuators available are restricted. To overcome these difficulties, output feedback controllers are preferred. An output feedback system applies control gains directly to sensor outputs, and these signals are fed back to the control actuators. Thus, this method is favorable for the application of discrete piezoelectric sensors and actuators to control the large-scale structures. This control hierarchy provides many advantages over the existing state feedback control. No state estimator is involved in this approach and, consequently, this greatly simplifies the internal complexity of the controller and reduces the on-board computer requirement.

Since large-scale structures involve a large number of degrees of freedom, modal reduction techniques are normally used to reduce the system model for controller designs instead of physical coordinates. This can be achieved by eigenvector assignment using output feedback control. Consequently, active control is reduced to a few low fundamental modes in control of structures in which the high frequency residual modes are truncated. The optimal control simulation of piezoelectric smart structures has been successfully investigated in modal space by using the finite element method (Rao and Sunar 1993, Chen et al 1997). Many earlier researchers adopted the state feedback control and assume a perfect knowledge about the space, even though the state has to be reconstructed from the sensor signals by the observer in the actual situation. In most
applications, particularly in those involving flexible large structures, the plate state (modal coordinates) cannot be physically measured or resolved without employing elaborate and model dependent state estimators. Under these conditions, the implementation of the control algorithm may encounter significant difficulties. Also, this approach cannot provide a natural and logical mechanism for implementing the discrete multiple-input-multiple-output control methodologies.

To overcome the drawbacks of state feedback, the optimal control objective to be achieved by modal output feedback computes the gain matrix to increase the modal damping ratios and natural frequencies of the selected fundamental vibration modes. However, it is not always possible to design the controller to be as stable as necessary since structural vibration reduction is not always guaranteed. This limitation usually comes when the number of sensors and actuators are a lot smaller than the number of vibration modes to be controlled. If there are enough pairs of sensors and actuators, then the output feedback control will show performance similar to state feedback control. The optimal output feedback problem is thus a significant extension of the optimal full state feedback problem.

In this chapter, five collocated sensors and actuators are used to effectively control ten modes of the flexible smart panel at different locations. It is well known that collocated sensors and actuators are advantageous from the viewpoint of stability. Also, controllability and observability matrices are full rank, so it is possible to assign virtually arbitrary stability to the controller. This present work demonstrates a detail description of the integrated finite element capability using output feedback for a controller based on
Linear Quadratic optimal control. For a numerical example, a hybrid approach using a finite element formulation for the radiating walls of an enclosure is also presented. Interior noise control in a cabin enclosure using active vibration control of the walls of the enclosure with discrete modeling technique is demonstrated. Though the control forces depend only on the state of the system, namely displacement and velocity, the reduction in the sound pressure level is achieved through actively controlling the modes of vibration, which contribute to the radiated pressure inside the enclosure.

5.2 Formulation

The finite element formulation of the structure is considered first. The modeling approach for the acoustic cavity is shown in subsequent sections. Finally the coupling of both models and the design of an optimal controller are discussed.

5.2.1 Finite Element Model Of The Panel Structure

The variational principle can be used to derive the finite element equations for coupled electro-elastic continuum. The approach used here to derive the dynamic equations of motion is based on the work carried out by Allik and Hughes (1970). The linear constitutive equations relating the elastic field and electric field can be written as

\[
\begin{align*}
T &= C^e S - h^T E, \\
D &= h S + b^T E,
\end{align*}
\] (5.1)

where

\[T = \text{Stress Tensor}, \quad D = \text{Electric Displacement},\]
\[S = \text{Strain tensor}, \quad h = \text{Piezoelectric coupling coefficient},\]
\(E = \) Electric Field, \(b^s = \) Dielectric constant at constant strain, and \(C^E = \) Elastic stiffness tensor evaluated at constant E field.

Dynamic equations of a piezoelectric continuum can be derived using Hamilton’s principle,

\[
\int_{t_1}^{t_2} \delta L \, dt = 0, \tag{5.2}
\]

where the operator \(\delta\) denotes the first order variation and \(t_1\) and \(t_2\) define the time interval, and all variations must vanish at \(t = t_1\) and \(t = t_2\). The Lagrangian \(L\) is determined by the energies available in the piezoelectric medium:

\[
L = E_{\text{kin}} - E_s + W, \tag{5.3}
\]

where the kinetic energy \(E_{\text{kin}}\) is given by \(E_{\text{kin}} = \frac{1}{2} \int \rho \dot{u}^2 \, dV\) and the elastic and dielectric energy \(E_s = \frac{1}{2} \int (S^T T - E^T D) \, dV\). Here \(\rho\) is the mass density and \(\dot{u}\) is the velocity vector and \(V\) is the volume of the piezoelectric medium. The energy \(W\) stored in the medium due to the external mechanical and electrical excitation is defined as

\[
W = \int_V u^T f_b \, dV + \int_{S_1} u^T f_s \, dS + \sum_{S_2} u^T f_c - \int_{S_1} \phi q_s \, dS_2 + \sum_{S_2} \phi q_c, \tag{5.4}
\]

where \(f_b\) is the body force, \(f_s\) is the surface force, \(f_c\) is the concentrated force and \(q_s\) is the surface charge, \(q_c\) is the point charge and \(S_1\) is the area where mechanical forces are applied and \(S_2\) is the area where electrical charges are applied.
Substituting the equations (5.3) and (5.4) into equation (5.2) and taking the first variation on $L$, and integrating by parts, we get

$$
\int \{ \delta S^T C^E S - \delta S^T h^T E - \delta E^T h S - \delta E^T b^T E + \rho \delta u^T \bar{u} - \delta u^T f_s \} dV
$$

$$
- \int \delta u^T f_c dS_1 + \int \delta \phi_q dS_2 - \delta u^T f_c + \delta \phi_q c = 0 .
$$

(5.5)

Now the electroelastic continuum is discretized with finite elements and the displacement $u$ and the electric potential $\phi$ are expressed in terms of nodal values via interpolation functions

$$
u = N_u u_i ,$$

$$\phi = N_\phi \Phi_i ,$$

(5.6)

where $N_u$ and $N_\phi$ are the interpolation functions for the field variables for $u$ and $\phi$ and $u_i$ and $\phi_i$ are the nodal point values.
It is to be noted that the linear shape functions for $\phi$ are identical to those for the displacement vector $u$. The shape functions $N_\phi$ and their derivatives are defined for the conventional isoparametric element.

The mechanical strains $S$ are related to the nodal displacements through the derivative of the shape function $B_u$

$$ S = B_u u_i, \quad (5.7) $$

where $B_u$ is the strain-displacement matrix. Similarly the electric field $E$ is related to the nodal potential by

$$ E = \nabla \phi = -\nabla N_\phi \phi = -B_\phi \Phi_i, \quad (5.8) $$

where $B_\phi$ is the electric field-displacement matrix.

Substituting equations (5.6)-(5.8) into equation (5.5) results in, for all values of virtual displacements, the two equilibrium equations:

$$
\begin{align*}
\{ M_{uu} \} \{ \dot{u} \} + \{ K_{uu} \} \{ u \} + \{ K_{\phi \phi} \} \{ \phi \} &= \{ F \}, \text{ and} \\
\{ K_{\phi u}^T \} \{ \dot{u} \} + \{ K_{\phi \phi} \} \{ \phi \} &= \{ Q \},
\end{align*}
$$

where

$$
\begin{align*}
\{ M_{uu} \} &= \int \rho N_u^T N_u \, dV \quad \text{is the mass matrix,} \\
\{ K_{uu} \} &= \int B_u^T C B_u \, dV \quad \text{is the elastic stiffness matrix,} \\
\{ K_{\phi u} \} &= \int B_u^T h^T B_\phi \, dV \quad \text{is the piezoelectric coupling matrix,} \\
\{ K_{\phi \phi} \} &= -\int B_\phi^T b^T B_\phi \, dV \quad \text{is the dielectric stiffness matrix,}
\end{align*}
$$
\{F\} = \int_V N_b^T f_b dV + \int_{S_1} N_{s_1}^T f_s dS_1 + N_u^T f_c \text{ is the mechanical force, and}

\{Q\} = -\int_{S_2} N_{s_2}^T q_{s_2} dS_2 - N_v^T q_c \text{ is the electrical charge.}

The above matrix equations are written in partitioned form to reflect coupling between the elastic and electric fields. The structural linear viscous damping term \( [C_{uu}] \mathbf{u} \) can be added in the first matrix equation (5.9) where \( [C_{uu}] = \eta [M_{uu}] + \lambda [K_{uu}] \) is the system damping matrix in which \( \eta \) and \( \lambda \) are called Rayleigh coefficients.

The electric field boundary condition requires that the electrode surface is an equipotential one and the summation of the nodal electric charges on it should be zero on the sensor electrodes since they are in ‘open circuit’ conditions:

\[ \Phi_i = \Phi_{i+1} = \ldots = \text{Constant} \quad \text{and} \quad \sum_{i} Q_i \big|_{\text{sensor}} = 0. \quad (5.10) \]

### 5.2.2 Condensation of System Matrices

The external charge applied on the actuator surface needs to be properly provided to produce the desired control forces. The external charge applied on the actuator surface needs to be properly modeled and cannot be neglected in order to produce the desired control forces. Equation (5.9) can be condensed to write the sensor potential in terms of the sensor displacement as

\[
\begin{bmatrix}
\mathbf{u} \\
\mathbf{\phi}
\end{bmatrix} = \begin{bmatrix}
I \\
- [K_{\phi\phi}]^{-1} [K_{\phi u}]
\end{bmatrix} \{u\}. \quad (5.11)
\]

After assembling all element matrices, and substituting the foregoing transformation matrix into eqn. (5.9), the system dynamic equation is written as
\[ [M_{uu}] \ddot{u} + [K^*] u = \{F_{ext}\} + \{F_{ac}\} + \{F_e\}, \quad (5.12) \]

Where
\[
[K^*] = [K_{uu}] - [K_{uφ}] [K_{φφ}]^{-1} [K_{φφ}^T], \quad \text{and} \\
\{F_e\} = [K_{uφ}] [K_{φφ}]^{-1} \{Q\}.
\]

\{F_{ext}\} and \{F_e\} are the external mechanical force acting on the structure and the electrical force, respectively. The electric force is contributed directly by the external applied electric charge \{Q\}. Ha (1992) considers that the voltage is the only form of electrical input to the actuator. He did not consider the effect of charge. However, external electric charge will play a very important role in vibration control applications. The force vector \{F_{ac}\} is due to the pressure loads acting on the structure when the structure is in contact with the fluid in the cavity.

The electrical potential can be recovered from
\[
\{Φ\} = [K_{φφ}]^{-1} \{Q\} - [K_{uφ}^T] u. \quad (5.13)
\]

Note that \{Q\} is zero in the sensor. Thus, the sensor equation is
\[
\{Φ_s\} = [K_{φφ}]^{-1} \{−[K_{uφ}^T] u\}. \quad (5.14)
\]

5.3 Control Mechanism

The sensor generates voltage outputs when the structure is oscillating. The signal can be amplified and fed back into the actuator using an appropriate controller. The feedback signal to the actuator is represented as
\[
\{Φ_a\} = G_a \{Φ_s\} + G_v \{Φ_s\}, \quad (5.15)
\]
where the subscript $a$ is for the actuator and $s$ is for the sensor. The quantities $\{\Phi_s\}$ and $\{\Phi_a\}$ are obtained from the sensor equation (5.14). $G_d$ and $G_v$ represent respectively the displacement feedback control gain and the velocity feedback gain. As the gains are constant with respect to time and frequency, this controller is known as a constant gain feedback controller. The direct effect in the piezoelectric actuator is negligible because the feedback voltage is much higher than the self-generated voltage.

5.3.1 Direct Charge Approach

When piezoelectric materials are used for actuation, an electric charge $\{Q\}$ is usually applied to the distributed piezoelectric actuator, and this charge is responsible for the potential difference that appears between two electrodes of the actuator. The electrical charge $\{Q\}$ and electric potential $\{\Phi\}$ can be related by

$$\{Q\} = [C] \{\Phi_a\}, \quad (5.16)$$

where $[C]$ is the capacitance of the piezoelectric actuator, and $\{\Phi_a\}$ is an external voltage applied on the piezoelectric actuator. The voltage output resulting from the distributed sensors is amplified and fed back into the distributed actuators as feedback voltages. Thus multiplying equation (5.15) by the capacitance of the actuator, one can derive the feedback charge injected into the actuator as

$$\{Q\} = C (G_d \{\Phi_s\} + G_v \{\Phi_s\}) \quad (5.16)$$

The piezoelectric capacitance of the actuator modeled as a parallel plate capacitor can be defined as $C=\varepsilon_{33}A/t_{pet}$, where $\varepsilon_{33}$ is the $z$-directional permittivity of
piezoceramics, \( A \) is the actuator area and \( t_{\text{act}} \) is the actuator thickness.

Substituting the feedback charge into equation (5.9) and utilizing the sensor output signal, the feedback force can be reformulated in matrix form as follows:

\[
\begin{bmatrix}
\{ F \}_c
\end{bmatrix} = -\begin{bmatrix}
\mathbf{K}_{\text{act}}
\end{bmatrix}^{-1} \begin{bmatrix}
\mathbf{G}_d
\end{bmatrix}^{\mathbf{T}} \mathbf{K}_{\text{act}}^{-1} \begin{bmatrix}
\{ \mathbf{u} \}
\end{bmatrix} - \begin{bmatrix}
\mathbf{K}_{\text{act}}
\end{bmatrix}^{-1} \begin{bmatrix}
\mathbf{G}_d
\end{bmatrix}^{\mathbf{T}} \mathbf{K}_{\text{act}}^{-1} \begin{bmatrix}
\{ \mathbf{u} \}
\end{bmatrix} \dot{\mathbf{u}}
\]

(5.17)

where \( \begin{bmatrix}
\mathbf{C}_{\text{ctrl}}
\end{bmatrix} = \begin{bmatrix}
\mathbf{K}_{\text{act}}
\end{bmatrix}^{-1} \begin{bmatrix}
\mathbf{G}_d
\end{bmatrix}^{\mathbf{T}} \mathbf{K}_{\text{act}}^{-1} \begin{bmatrix}
\mathbf{K}_{\text{act}}
\end{bmatrix} \) is the active damping matrix for charge control and \( \begin{bmatrix}
\mathbf{K}_{\text{ctrl}}
\end{bmatrix} = \begin{bmatrix}
\mathbf{K}_{\text{act}}
\end{bmatrix}^{-1} \begin{bmatrix}
\mathbf{G}_d
\end{bmatrix} \mathbf{K}_{\text{act}}^{-1} \begin{bmatrix}
\mathbf{K}_{\text{act}}
\end{bmatrix} \) is the active stiffness matrix for charge control, \( \begin{bmatrix}
\mathbf{G}_d
\end{bmatrix} \) and \( \begin{bmatrix}
\mathbf{G}_v
\end{bmatrix} \) are the displacement feedback gain matrix and the velocity feedback matrix, respectively. \( \begin{bmatrix}
\mathbf{G}_{d,ij}
\end{bmatrix} \) and \( \begin{bmatrix}
\mathbf{G}_{v,ij}
\end{bmatrix} \) have the meaning that sensed voltage of the \( j \)th piezoelectric element is fed back to control the \( i \)th piezoelectric element. The feedback control forces should be applied to piezoelectric actuators depending upon the information of velocity and displacement gain components.

Substituting this feedback force into the system equation (5.15) gives the governing equation of motion as

\[
\begin{bmatrix}
\mathbf{M}_{\text{act}}
\end{bmatrix} \mathbf{u} + \{ \begin{bmatrix}
\mathbf{C}_{\text{act}}
\end{bmatrix} + \begin{bmatrix}
\mathbf{K}_{\text{act}}
\end{bmatrix}^{-1} \begin{bmatrix}
\mathbf{G}_d
\end{bmatrix}^{\mathbf{T}} \mathbf{K}_{\text{act}}^{-1} \begin{bmatrix}
\mathbf{K}_{\text{act}}
\end{bmatrix} \} \dot{\mathbf{u}} + \{ \begin{bmatrix}
\mathbf{K}
\end{bmatrix} + \begin{bmatrix}
\mathbf{K}_{\text{act}}
\end{bmatrix}^{-1} \begin{bmatrix}
\mathbf{G}_d
\end{bmatrix} \mathbf{K}_{\text{act}}^{-1} \begin{bmatrix}
\mathbf{K}_{\text{act}}
\end{bmatrix} \} \dot{\mathbf{u}} = \{ \mathbf{F} \},
\]

(5.18)

5.3.2 Direct Voltage Approach

If voltage is applied directly on the electrodes of the actuators then in equation (5.15) we substitute directly the voltage magnified by gain into the actuator voltage \( \{ \mathbf{V}_a \} \) degrees of freedom, equation (5.9) can be rewritten as
where the electrical potential vector \( \{ \Phi \} \) can be divided into two groups, \( \{ \Phi_a \} \) the potential vector at the actuator nodes and \( \{ \Phi_s \} \) the potential vector at the sensor nodes. \( Q \) contains the unknown nodal forces related to the actuator electrical potential, which may be derived from sensor equation (5.14). Therefore, charge can be found later after substitution of unknown degrees of freedom, and we can directly solve the above equation.

### 5.4 Eigenvalue Analysis

Before solving the equation of motion of a structure with the piezoelectric devices, the mode shapes and the resonant frequencies of the undamped system are obtained using Eigenvalue analysis. Solution of the Eigenvalue problems may use the full assembled matrices or the reduced matrix system obtained by matrix condensation of structural and potential degrees of freedom. Free vibration implies that \( \{ F \} = 0 \) and \( \{ Q \} = 0 \) in equation (5.9). Unspecified potentials are then condensed out of the matrices reducing the equations (5.9) to

\[
[K^*] - \omega^2 [M_{uu}] U = \{ 0 \},
\]

where \( \omega \) is the natural frequency. The new stiffness matrix \( [K^*] \) indicates that the structure is electromechanically stiffened.
5.4.1 Modal Analysis

The system equation of motion shown in equation (5.18) is not suitable for system analysis because the order of degrees of freedom of the system $u$ is usually too large in finite element method. Therefore, the equation has to be transformed into a set of properly chosen modal coordinates with much smaller and manageable degrees of freedom to supply great computer efficiency. The modal analysis is based on the orthogonality of natural modes and expansion theorem.

The modal extended displacement $u$ is represented by

$$u = [U]q,$$  \hspace{1cm} (5.21)

where $[U]$ is the modal matrix, and $q$ are referred to as modal coordinates.

The modal matrix $[U]$ is simply a square matrix in which the columns corresponding to eigenvectors of the system, satisfying an eigenvalue problem shown in equation (5.20). Substituting equation (5.21) into equation (5.18) and multiplying two sides of equation (5.18) by $[U^T]$ gives


Considering the orthogonality property, the above equation can be written as

$$[I]q + [U^T]C_{ctrl} [U]q + (-[I] \omega^2 + [U^T]K_{ctrl} [U])q = [U^T]F\} \hspace{1cm} (5.23)$$

Equation (5.23) is recognized as a closed-loop equation of motion. The process of using the above equation to control the system is known as modal control. Note that the rate feedback modifies the damping matrix of the system, and the displacement feedback modifies the stiffness matrix of the system. Before the behavior of the closed-loop system
can be established, it is necessary to determine the control gain.

### 5.5 Output Feedback Optimal Control

The Linear Quadratic Regulator (LQR) controller based on the modal space model is proposed and the explicit solutions are used for the corresponding Riccati equations of the output feedback optimal controller. In the most general case some of the piezoelectric patches act as actuators and others as sensors, and it is therefore necessary to partition the electrical potential $\Phi$ to extract actuator and sensor contributions. By using transformation matrices $T_a$ and $T_s$ respectively, for the vectors of the voltages at the actuators and at the sensors, the equation of motion (5.23) and the sensor output equation (5.14) in terms of a modal coordinate can be expressed as

\[
\begin{bmatrix}
I_1 \hat{q} + \left[2\zeta \omega \right] \hat{q} + \left[\omega^2 \right] q
= \left[U \right]^T \left[F \right] - \left[U \right]^T \left[K_{u\phi} \right] \left[K_{\phi\phi} \right]^{-1} \left[T_a \right] \Phi_a,
\end{bmatrix}
\]

\[
\Phi_s = -\left[T_s \right] \left[K_{u\phi} \right] \left[K_{\phi\phi} \right]^{-1} \left[U \right] \hat{q},
\]

$T_a$ is an $m \times n$ actuator location matrix for the actuator degrees of freedom and $T_s$ is an $n \times m$ sensor location matrix for the sensor degrees of freedom.

Here $m$ is the number of total electrical potential degrees of freedom and $n$ is the number of sensors and actuators. The columns of $T_a$ matrix respectively consist of the piezoelectric capacitance values of actuators corresponding to the actuator degrees of freedom, where the rows of the $T_s$ matrix contain unit values at the sensor degrees of freedom. The control forces are provided by $n$ number of actuators and the sensor voltages are also measured by $n$ number of sensors.

The governing equations can be written in modal state space form to provide a
standard mathematical basis for control studies. The linear time-invariant equations of motion which include the effects of piezoelectric control forces and external disturbance force are expressed as follows

\[
\begin{bmatrix}
\dot{q} \\
\dot{q}
\end{bmatrix} = A\mathbf{x} + B_d\Phi + B_a F, \quad \Phi = C\mathbf{x},
\]

where

\[
A = \begin{bmatrix}
0 & I \\
-\omega^2 & 2\zeta\omega
\end{bmatrix}, \quad B_a = -[U]^T[K_{ud}K_{op}^{-1}[T_a]^T],
\]

\[
C = [0 -[T_s[K_{op}^{-1}[K_{up}^T][U]]], \quad B_d = \begin{bmatrix}
0 \\
[U]^T
\end{bmatrix},
\]

where \(A, B_u, B_a\) and \(C\) are the corresponding state, disturbance, control and sensor matrices. The transfer functions matrices in terms of these state-space matrices are

\[
P_a(s) = C(sI - A)^{-1}B_a, \quad \text{and}
\]

\[
P_d(s) = C(sI - A)^{-1}B_d.
\]

In direct output feedback, control forces are calculated directly from the multiplication of output measurements by constant feedback gains, expressed as

\[
\{Q\} = C\left(G_a\{\Phi_d\} + G_v\{\Phi_v\}\right),
\]

where \(G\) is the time-invariant output feedback gain matrix to be determined by the design procedure. Control forces are just some combinations of output measurements, which in turn are combinations of system states. Based on this representation, the optimal controller can be formulated by calculating the applied voltage to minimize a linear quadratic performance.
\[
J = \int_0^\infty (x^T Q x + \Phi_2^T R \Phi_2) \, dt ,
\]

(5.28)

where \( Q \) (positive semi definite) and \( R \) (positive definite) are weighting matrices to be selected by the designer. The optimal solution is found from the procedure of (Levine and Athans 1970) that minimizes the expected value of performance index

\[
J = \text{tr}(PX) \quad \text{where} \quad X = E\{x(0)x^T(0)\} ,
\]

(5.29)

where \( P \) satisfies the Ricatti equation

\[
A_c^T P + PA_c + C^T G^T R G C + Q = 0 .
\]

(5.30)

By applying optimality conditions for the above problem, the optimal control gain matrix \( G \) can be obtained by solving the following coupled nonlinear algebraic matrix equations simultaneously with equation (5.30):

\[
A_c S + S A_c^T + X = 0 , \quad \text{and}
\]

(5.31)

\[
G = R^{-1} B^T P S C^T (C S C^T)^{-1} .
\]

(5.32)

Note that \( P, S \) and \( G \) are matrices to be determined from the above three equations. \( A_c \) denotes the closed loop system matrix

\[
A_c = A - B G C .
\]

(5.33)

There exist several numerical approaches to solving equations (5.30-5.32). In this study, the effective iterative solution algorithm proposed by Moerder and Calise (1985) is adopted. The weighting matrix \( Q \) for the modal space quadratic performance index is usually assumed to be diagonal and takes the form (Meirovitch et al 1983)

\[
Q = \text{diag}(\omega_1^2, \ldots, \omega_n^2, 1 \ldots 1) .
\]

(5.34)
However, there are no general guidelines for the choice of $R$, the weighting matrix for control forces. A diagonal $R$, as assumed in most applications, may be written as

$$R = \text{diag}(r_1, r_2, \ldots, r_n).$$

(5.35)

### 5.6 Radiated Sound Field Inside a Rectangular Enclosure

Although there are many possible theoretical models for the enclosed sound fields, the modal model, which is more appropriate for low frequency applications, is used here. In the low frequency range the modal model of an enclosed sound field requires relatively few parameters for an adequate description of the spatial and frequency dependency of the field. In this model the pressure at any point inside the enclosure is expressed as the sum of acoustic mode shapes of the cavity. These mode shapes correspond to the eigenfunctions of the homogenous Helmholtz equation with the rigid wall boundary condition.

The modal representation of the acoustic pressure can be written as

$$p = \sum A_{mn} \psi_{mn} = \sum A_{mn} \left(\cos\left(\frac{l\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi z}{c}\right)\right),$$

(5.36)

where $A_{mn}$ are the unknown modal weighting coefficients and $l, m, n$ are the cavity mode numbers.

The inhomogeneous Helmholtz equation for the cavity pressure field, generated due to the vibration of walls is given as

$$\nabla^2 p + \left(\frac{\omega^2}{c^2}\right)p = \rho_0 \omega^2 u_n,$$

(5.37)

where $u_n$ is the normal structural displacement and can be written in terms of the shape
functions \((N_s)\) and the nodal displacements \((u_s)\) of the structural finite element as
\[
    u_n = N_s u_s .
\] (5.38)

Substituting the assumed pressure field in equation (5.37) and using the orthogonality property of the modes, we find modal weights \(A_{l mn}\) as
\[
    A_{l mn} = \frac{\int \left( -\omega^2 \rho_0 N_s u_s \right) \psi_{l mn} ds}{\left[ \left( \frac{\omega}{c} \right)^2 - \left( \frac{l\pi}{a} \right)^2 + \left( \frac{m\pi}{b} \right)^2 + \left( \frac{n\pi}{c} \right)^2 \right]} \Lambda_{l mn},
\] (5.39)
where
\[
    \Lambda_{l mn} = \int_0^a \int_0^b \int_0^c \cos^2 \left( \frac{l\pi y}{a} \right) \cos^2 \left( \frac{m\pi y}{b} \right) \cos^2 \left( \frac{n\pi z}{c} \right) dxdydz.
\]

The force exerted on the plate structure by the pressure field inside the cavity is calculated as
\[
    F_{\text{fluid}} = \int p N_s ds = \sum \int A_{l mn} \psi_{l mn} N_s ds = -\omega^2 \mathbf{H} u_s ,
\] (5.40)
where
\[
    \mathbf{H} = \sum \int \left[ \left( \frac{\omega}{c} \right)^2 - \left( \frac{l\pi}{a} \right)^2 + \left( \frac{m\pi}{b} \right)^2 + \left( \frac{n\pi}{c} \right)^2 \right] \psi_{l mn} N_s dS.
\]

The finite element equation of motion for the smart plate structure is rewritten by including this force term along with the other external forces acting on the plate structure. The final coupled fluid-structure finite element equation of motion is written as
\[
    \left[ -\omega^2 \mathbf{M}_{uu} - \mathbf{H} \right] + i\omega \left[ \mathbf{C}_{ctrl} \right] + \left[ \mathbf{K}^t + \mathbf{K}_{ctrl} \right] [\mathbf{U}] = \{ \mathbf{F} \}.
\] (5.41)

The above equation can be solved for the displacements and weighting coefficients \(A_{l mn}\), from which the pressure at any point inside the cavity can be
determined from the equation (5.37).

5.7 Numerical Simulations of the Clamped Plate

To utilize the advantages of piezoelectric transducers, it is necessary to select appropriate positions of the transducers and to select the sensor signals that are to be fed back to the actuators. The problem of selecting the locations of transducers is a complete problem in itself. In this research, the actuator/sensor pairs are pre selected and held at a fixed location during an optimization search for control gain. The modal order and the corresponding mode shape function provide a significant insight of the best actuator locations on vibration control. Effective sensing and control depend on the locations of sensors and actuators. The actuator locations are studied in order to maximize actuator effectiveness. The positions of the plate at which the mechanical strain is highest are the best locations for sensors and actuators. Therefore, the objective of the multimode control is to place all of the actuators in regions of high average strain and away from areas of zero strain (anti-nodal line). One advantage of using this prediction guarantees the fewest number of sensors and actuators, the least power and the least coverage of structural area. This method is derived from a structural mechanics standpoint and is relatively simple. The analysis of active vibration control is performed for the clamped plate. The size of the aluminum plate is $0.305m \times 0.305m$ and the thickness is $0.8 \text{ mm}$. The five piezoelectric patches made of PZT-5H are used as collocated sensors and actuators over
Case I

Harmonic force excitation: \( F = 0.1 e^{-i\omega t} \) N

Unit step excitation:

\[
\begin{align*}
F (\text{N}) & = -0.1 \\
t (\text{sec}) & = 0
\end{align*}
\]

Figure 5.2 Aluminum panel with five surface bonded collocated piezoelectric actuators/sensors and schematic diagram of two loading cases

Harmonic force excitation for arbitrary points:

\[
\begin{align*}
F_c &= -0.1 \sin(50t) \\
F_d &= 0.3 \cos(40t) \\
F_e &= 0.5 \sin(60t)
\end{align*}
\]

Case II

Figure 5.3 Modeling of a geometrical transition with several types of finite element.
the upper and lower surfaces for multimode control of vibration of the plates (Fig. 5.2).

Figure 5.3 and 5.4 show the elements used and the finite element mesh for the smart plate.

Twenty-four 20-node solid elements are used for modeling the solid piezoelectric patches and the part of the plate beneath the piezoelectric patch. Around the piezoelectric patch, twenty-four 13-node transition elements are used to connect the solid elements to the remaining part of the plate that is modeled by 9 node flat-shell elements. A total of 136 nine node flat-shell elements are used to model the plate. The upper piezoelectric elements are used as actuators and the lower ones as sensors. The first 10 modes of the proposed finite element model are calculated using eigenvalue analysis. The mode shapes upon activation of the embedded collocated piezoelectric patches for the pure plate are shown in Fig. 5.5.

From previous chapters we found that the center of the plate is an appropriate
location for placing the actuator/sensor pair for controlling the first mode. At this location, the nodal lines corresponding to modes 2, 3, 5, and 6 intersect the piezoelectric patch.

These modes may be uncontrollable. On the other hand, the nodal lines corresponding to modes 1 and 4 do not intersect the patch. Therefore, it is anticipated that the symmetric modes 1 and 4 will be completely controlled. It can be predicted that the control effect to modes 2, 3 and 5 can be improved, if 4 more actuators are arranged along the diagonals of the plate (Fig. 5.2). In this case, the nodal lines corresponding to modes 2, 3, and 4 do not intersect the patches; these patches should impose some control effort over these modes. For the given base model, the loci of the closed loop system are plotted in Fig. 5.6. Several trial values of the weighting factor $\mathbf{R} = \text{diag}(r_1, r_2, \ldots, r_n)$ are
used to determine the optimal gain.

The eigenvalues of the open loop system (without control) have zero real parts, because it is assumed that the system has no inherent damping, and the only damping available to the plate is through the closed loop feedback system. When lower control cost weights are used, the poles of the closed loop are moved further to the left, indicating additional damping, and this is accomplished with the use of more control input to the structure. The use of a lower $R = \text{diag}(r_1, r_2 \ldots r_n)$ may extend the applicability of the controller to provide high damping factors. However, further decrease of $r_i$ may lead to instability of the system in the closed loop (Lin et al 1996) and problems of solving the Riccati equation.
Figure 5.7 Frequency spectrum at point A when harmonic excitation is applied at point A. (with optimal feedback gain for case I)

Figure 5.8 Time history plot at point A when a unit step excitation is applied at point A.
The weighting value for the optimal control is therefore chosen to be $10^{10}$.

All structural modes are not controlled in applying the output feedback optimal control. The control input is directed to modes that dominate the response. The active damping values for the closed loop system are also listed in Figure 5.6. The first four modes are significantly damped. However, for the fifth mode, the damping ratio has hardly been affected by this control strategy because the nodal lines corresponding to this mode intersect all piezoelectric patches as seen in Fig. 5.5.

Figure 5.9 Response of the clamped plate at point F due to three arbitrary harmonic forces (No control)

The weighting value for the optimal control is therefore chosen to be $1.0 \times 10^{-10}$.

All structural modes are not controlled in applying the output feedback optimal control. The control input is directed to modes that dominate the response. The active damping values for the closed loop system are also listed in Figure 5.6. The first four modes are significantly damped. However, for the fifth mode, the damping ratio has hardly been affected by this control strategy because the nodal lines corresponding to this mode intersect all piezoelectric patches as seen in Fig. 5.5.

Figure 5.2 also shows the disturbance source and the response measurement
position for two analysis cases. A harmonic force $F = 0.1 e^{i\omega} N$ is applied at point $A$ for the frequency response analysis. The response due to this force is estimated without feedback control (i.e., uncontrolled) and with closed loop optimal feedback control. Time domain response is also analyzed when a unit step force $0.1 N$ is applied at point $A$. Structural damping is not taken into account to show clearly the effects of active damping. All of the piezoelectric patches are used at the same time to allow the possibility of controlling all structural modes simultaneously. The optimal control gain values are simultaneously applied to five co-located piezoelectric patches under the same harmonic and unit step force excitation.

For Case I, figure 5.7 shows the frequency spectrum measured at point $A$ due to a harmonic force applied at point $A$ with and without active control system.
Using only five piezoelectric actuators, the amplitude reduction is achieved for all modes with the except for mode 5. The result also shows that the largest vibration reduction can be achieved with optimal control. From above results, it can be concluded that the use of an array of appropriately positioned actuators is necessary to provide good controllability of the structure. This means only spatially discrete actuators are needed to implement multimode control.

This cannot be done if the actuator is distributed over the entire surface of the plate. This increased effectiveness is attributed to the actuators being in the best possible position for control of all structural modes. It is shown in Fig. 5.8 that transient noise phenomenon affected by the superposition of antisymmetric modes at an initial time interval disappears as time increases. It is possible to estimate that the four control channels that are arranged along the diagonals of the plate have significant practical advantage for noise control. This is due to the fact that these actuator arrangements were tailored for antisymmetric modes. Therefore the higher overall damping and noise suppression efficiency are obtained by optimal feedback control.
Figure 5.11 Acoustic field in (vertical plane of the cavity) enclosure when first mode of panel is excited.

Figure 5.12 Acoustic field in vertical plane of the cavity enclosure when fourth mode is excited.
Figure 5.9 depicts the uncontrolled plate response in Case II when three different harmonic forces are respectively excited at arbitrary locations. Figure 5.10 shows the closed-loop (controlled) system with optimal gain. The plate vibration is significantly attenuated when the control circuit is on.

The cavity used in the previous chapter for the RR method (Figure 4.3) is now considered to show the effects of control on the cavity pressures. A uniform harmonic pressure load, which excites the panel at its resonance frequencies, is applied on the panel surface. The pressure inside the cavity due to this excitation is computed by solving eqns. (5.41), (5.39) and (5.36).

The two-dimensional pressure field in the horizontal and vertical planes of the cavity with and without control is plotted in Figures (5.11) and (5.12). From the distribution we observe that, when optimal control gain values are used for the vibration control of the panel the reduction in the radiated sound pressure field is nearly 20 dB for the first mode and 40 dB for the second mode. The reduction in the radiated sound pressure level for the fourth mode is more than the first mode, because of the fact that the fourth mode has the highest amplitude reduction among the first five modes which is shown in Fig. (5.7).

5.8 Summary

This chapter is concerned with the numerical modeling of discrete piezoelectric sensors and actuators for active modal control of a flexible clamped square plate structure. To avoid difficulty in modeling plate structures with piezoelectric transducers,
a three-dimensional solid element is used for piezoelectric materials and some parts of the structure with the combination of flat shell and transition elements. Various algorithms for dynamic simulation and active controls are studied. For a large structural system, it is suggested that the model superposition method be used, together with the $\alpha$-method for time integration, to improve the computational efficiency.

The actuator/sensor arrangement used for this simulation is geared towards control of the first six-modes control. In order to find out the capabilities of this kind of discrete control, the possibility of multimode control is carefully explored. From numerical results, it is possible to surmise that the active control effect can be achieved using a few discrete actuators at carefully selected positions separated from each other over the area of clamped plate. It is shown through finite element simulation that employing the proposed control schemes significantly reduces the undesirable vibration due to external disturbance. As a result, it has implications for material savings and reduced power requirements in applications as compared to truly distributed methods. The study also shows that the radiated sound pressure levels inside an enclosure generated by the vibration of a part of the enclosure wall can be reduced by optimally controlling the modes of the plate.

The problem of selecting appropriate feedback gains is equally important for selecting transducer positions. Therefore, further study of optimal design methods is needed to select optimal gains and placements of actuators by using output feedback control methods. Consideration of other design variables such as the sizes of the sensors, actuators and the number of piezoelectric patches are still need to be studied.
Chapter 6

DEVELOPMENT OF A RAYLEIGH-RITZ/BOUNDARY ELEMENT MODELING APPROACH FOR ACTIVE/PASSIVE CONTROL

6.1 Introduction

From previous chapters, we observed that successful noise reductions could be achieved at low frequencies by employing the active structural acoustic control methods. To expand the noise suppression capability at off-resonance and broadband frequencies, particularly higher frequencies one has to opt for the passive methods of noise reduction (Guigou and Fuller 1998). Sound absorbing materials such as sponges and glass fibers have been widely used in noise control engineering. Foamed aluminum has been being used recently because of its robustness against dispersion and moisture and its excellent performance even in high temperature compared to sponges and glass fibers. More recently the hybrid approaches that use both active and passive methods of controlling transmitted noise have been successfully used in the control of interior cabin noises in aircrafts and helicopters (Gentry et al, 1997, Fuller at al, 1994a and Kim et al, 1999). Gentry et al (1997) used smart foams (or adaptive foams), which use polyurethane foam for high frequency noise reduction and PVDF actuators with a feedback control mechanism for low frequency noise suppression. However, radiation control of more complex structures using these foams and increasing the control authority in such active foams is still under research. In this thesis a new approach for the active-passive method of using the discrete piezoelectric patches with feedback control and a passive absorber covering the interior of the panel is proposed. The primary contributions of this proposed
work are the formulation of the problem using effective numerical methods for the smart panel structure with sound absorbing foam and for the acoustic cavity. For the smart panel considered in the previous chapter we noticed that the Rayleigh-Ritz solution approach is more efficient than the FE methods because of its high degree of accuracy and smaller size of the system matrices. For the acoustic cavity, the RR method has a serious drawback in that the rigid walled cavity modes cannot be used for more complicated boundary conditions like the sound absorbing boundaries or pressure release ($p = 0$) conditions. So in this chapter a novel method of combining the RR method for the smart panel is combined with the boundary element (BE) formulation of the acoustic cavity. This approach enables us to design an active-passive noise control system for enclosures with any type of boundary conditions. For acoustic domains the boundary element approach is preferred over the finite element approach since this method reduces the size of the problem by a large magnitude. In the BE approach only the surface of the acoustic enclosure or the cavity needs to be meshed with boundary elements as compared to the FE method where the whole domain needs to be subdivided into elements. The number of finite elements required to model the acoustic domain increases with decreasing wavelength (or increasing frequency), so at high frequencies BE is the practical solution approach to acoustic analysis. In this chapter first the dual reciprocity method boundary element formulation is presented for the acoustic cavity then the coupling of the RR model of the smart panel with the BE model is presented in subsequent sections.
6.2 Boundary Element Model For Acoustic Enclosure

The Finite Difference (FD) and Finite Element (FE) methods fall within the group of the so-called domain methods. In these methods, the discretization of the whole domain is necessary. On the other hand, the Boundary Element (BE) method expresses the solution of the problem in terms of the integral equation and these are integrals are to be integrated only along the radiating surface. Only the boundary surface needs to be discretized and the radiated field into the enclosure or exterior domain is obtained using the solution obtained at the boundary points (or nodes). The main disadvantage of the finite element method is handling large quantities of data that results due to the discretization of full domain, particularly for three-dimensional problems.

6.2.1 Dual Reciprocity Method

The direct application for the BE method to many engineering problems which are governed by the Poisson’s and wave equation frequently results in formulations with integral equations containing domain as well as the boundary integrals. The domain integral results from the inertia force present in these governing differential equations. Because of the presence of the domain integral, the discretization is no longer restricted to the boundary alone and the domain also to be discretized (Brebbia et al 1979). The domain integral appearing in these equations can be transformed into its equivalent boundary form using any one of the following methods.

**Analytical integration of the domain integrals:** This method produces accurate results but it is applicable to a limited number of cases where the analytical integration of
the domain integral is possible.

**Fourier Expansion method:** This method is not straightforward and the calculation of matrix coefficients becomes cumbersome.

**Multiple reciprocity method:** This technique uses the higher order fundamental solution recursively and transforms the domain integrals into a series of boundary integrals. Though this method is efficient for eigenvalue analysis, it is not suited for time domain analysis.

**Dual Reciprocity Method:** This is a generalized way of constructing the particular integrals and most suitable for the coupled problem. The boundary element equation resulting from this method is in the same form as finite element equations for the structure. Also the formulated BE equations can be used for both time domain and frequency domain analysis.

The dual reciprocity method is a technique in which the domain integrals resulting from the inertia forces are transformed into a boundary integral form first proposed by Nardini and Brebbia (1982).

Consider the transient scalar wave equation

$$\nabla^2 p = \frac{1}{c^2} \ddot{p}.$$  \hspace{1cm} (6.1)

The solution of the wave equation can be decomposed into two parts:

$$p(x,t) = p^c(x,t) + p^f(x,t),$$  \hspace{1cm} (6.2)

where $p^c(x,t)$ is the complimentary function satisfying the homogenous equation

$$\nabla^2 p^c = 0$$  \hspace{1cm} (6.3)
and \( p^I(x, t) \) is the particular integral which satisfies the following equation

\[
\nabla^2 p^I = \frac{1}{c^2} \ddot{p} .
\]

(6.4)

The integral form of the Laplace equation \((\nabla^2 p = 0)\) is shown in appendix B. After the standard boundary element discretization (refer to appendix B) the equation (6.3) can be reformulated into the boundary element equation as

\[
\mathbf{H}p^I - \mathbf{G}q = 0 .
\]

(6.5)

Substituting the eqn. (6.2) into the above equation results in

\[
\mathbf{H}p^I - \mathbf{G}q - (\mathbf{H}q^I - \mathbf{G}q^I) = 0 .
\]

(6.6)

The dual reciprocity method uses a series of particular solutions \( \phi_j \) instead of a single particular solution. The number of terms in the series is equal to the total number of nodes in the problem. Then the following approximation for the inertia force is proposed

\[
p = \sum_{i=1}^{n} \lambda_i(x) \alpha_i(t) ,
\]

(6.7)

in which \( \lambda_i \) are a set of known shape functions, and \( \alpha_i \) is a set of time dependent functions to be determined. Thus, the solution of eqn (6.4) may be written as

\[
p^I = \sum_{i=1}^{n} \phi_i(x) \ddot{x}_i(t) , \text{ and} \]

(6.8)

\[q^I = \sum_{i=1}^{n} \psi_i(x) \dot{x}_i(t) ,\]

where \( q = \frac{\partial p}{\partial n} \) and \( \phi_j \) satisfy the Poisson’s equation
\[ \nabla^2 \phi_i = \frac{1}{c^2} \lambda_i, \]  

(6.9)

and \( \psi_i = \frac{\partial \phi_i}{\partial n} \). There is no restriction imposed on choosing the functions \( f_i \) and the following type of functions are usually suggested:

- polynomials constructed from the Pascal triangle,
- trigonometric functions, or
- the distance function \( r \) used in the definition of the fundamental solution.

From eqns (6.7), (6.8) and (6.9), we have

\[ p = \Lambda \alpha, \]
\[ p' = \Phi \Lambda^{-1} \dot{p}, \text{ and} \]
\[ q' = \Psi \Lambda^{-1} \dot{p}. \]

(6.10)

where

\[ p = (p_i)_n, \]
\[ q = (q_i)_n, \]
\[ \Lambda = (\Lambda_{ij})_{nm} = (\lambda_i(x^j))_{nm}, \text{ and} \]
\[ \Phi = (\Phi_{ij})_{nm} = (\phi_j(x^i, x^j))_{nm}. \]

Therefore, eqn (6.6) can be rewritten in the form

\[ M\ddot{p} + Hp - GQ = 0, \]

(6.11)

where

\[ M = (-H\Phi' + G\Psi')\Lambda^{-1}, \]

(6.12)

which is similar to the mass matrix in the FE equation.
6.3 Fluid-Structure Coupling

The boundary element equations of the acoustic domain can be rewritten as

$$\begin{bmatrix} M_I & M_{IB} \\ M_{BI} & M_{BB} \end{bmatrix} \begin{bmatrix} \dot{P}_I \\ \dot{P}_B \end{bmatrix} + \begin{bmatrix} H_I & H_{IB} \\ H_{BI} & H_{BB} \end{bmatrix} \begin{bmatrix} P_I \\ P_B \end{bmatrix} + \begin{bmatrix} -G_I & -G_{IB} \\ -G_{BI} & -G_{BB} \end{bmatrix} \begin{bmatrix} Q_I \\ Q_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (6.13)$$

The subscript $I$ denotes the quantities on the structure-fluid interface, which should be coupled with the finite element equation of the structure. The subscript $B$ denotes the functions at all other boundary locations.

Impedance boundary conditions can be applied on the fluid-absorber-structure boundary. The specific acoustic impedance is defined as the ratio of the transformed acoustic pressure, $p$, to the transformed particle velocity, in the direction (refer to fig. 6.2) and is given by
where \( \dot{\mathbf{u}}_r \) is the relative fluid particle velocity with respect to the structural surface and can be written as

\[
\dot{\mathbf{u}}_r = \dot{\mathbf{u}}_a - \dot{\mathbf{u}},
\]

where \( \dot{\mathbf{u}}_a \) is the true particle velocity and \( \dot{\mathbf{u}} \) is the normal vibration velocity of the structure. The relation between the true particle velocity and the fluid pressure at the boundary can be written as

\[
\frac{\partial p}{\partial n} = -\rho_0 \frac{\partial \dot{u}_a}{\partial t}.
\]

Combining eqns (6.14), (6.15) and (6.16), we get the relation between the pressure and the structural vibration velocity at the boundary as
\[
\frac{\partial \mathbf{P}}{\partial n} = -\rho_0 \left( \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{Z} \frac{\partial \mathbf{P}}{\partial t} \right).
\] (6.17)

The impedance of the absorber $Z$ is generally a complex function whose value changes with frequency and the geometry of the absorber. In accordance with current terminology, the surface with large acoustic impedance ($|Z| \to \infty$) is identified as acoustically hard, in which case $\frac{\partial \mathbf{P}}{\partial n} = -\rho_0 \frac{\partial^2 u}{\partial t^2}$, and the surface with small impedance ($|Z| \to 0$) as acoustically soft, in which case $\mathbf{p} = \mathbf{0}$ (“pressure release”). On the structurally rigid boundaries where no absorber material is used $\frac{\partial \mathbf{P}}{\partial n} = 0$.

### 6.4 Plate-Cavity Interface

Coupling the acoustic boundary element model with the Rayleigh Ritz model of the smart panel at the interface is easier in BE/RR method than the classical BE/FE approach. Since in FE/BE methods generally the structure is meshed in such a way that the finite element nodes exactly match with the acoustic nodes of the boundary acoustic elements. If these two meshes do not match then complicated interpolation schemes are to be used. In the case of the RR method we do not have this constraint because it is possible to obtain a displacement value at any point on the surface of the panel and we can always choose those points according to the boundary element mesh. For such a meshing scheme eqn. (6.17) can be written in terms of the acoustic pressure and flux vectors and the structural displacement vectors as

\[
\{Q_i\} = -[R_0]\{u_i\} + [Y]\{p_i\},
\] (6.18)
where $[Y]$ is the admittance matrix.

The force vector on the right hand side of eqn (4.33) contains the pressure loading due to the acoustic pressure inside the cavity. This force can be related to the nodal pressure values of the boundary elements by using force vector defined in eqn (4.33) and using eqn (A.7). Using these two equations we obtain the acoustic pressure-loading vector as

$$\{f_{ac}\}_i = \int_{s_3} F_{s_3}^T N_p^T [p]_i dS_3.$$  \hspace{1cm} (6.19)

As mentioned earlier, the vector $F_{s3}$ contains the characteristic functions estimated at the boundary element node positions. Estimating this force matrix for all the boundary elements at the interface and properly assembling these matrices we obtain the total force vector at the interface as

$$\{F_{ac}\} = \sum \{f_{ac}\}_i = \sum \left( \int_{s_3} F_{s_3}^T N_p^T dS_3 \right) [p]_i = [K_p] [P]_i.$$ \hspace{1cm} (6.20)

Using the above equation for acoustic loading on the structure and combining the boundary element eqn (6.13) with the Rayleigh-Ritz equation of the smart plate structure given by eqn (4.33), we have a coupled RR/BE equation for the smart paneled cavity as

$$
\begin{bmatrix}
M_{\text{str}} & 0 & 0 & \{\bar{u}\} & \{0\} & \{0\} & \{\bar{u}\} \\
\{G_I\} & M_I & M_{IB} & \{\bar{P}_I\} & 0 & -[G_I] [Y] & 0 & \{\bar{P}_I\} \\
\{G_{IB}\} & M_{BI} & M_B & \{\bar{P}_B\} & 0 & -[G_{IB}] [Y] & 0 & \{\bar{P}_B\} \\
K^* & K_p & 0 & \{u\} & \{F_{\text{ext}}\} & \{G_{IB}Q_I\} + \{K_p^{pc}\}_w \Phi_a \\
0 & H_I & H_{IB} & \{P_I\} & {G_{IB}Q_I} & \{0\} & \{0\} \\
0 & H_{BI} & H_B & \{P_B\} & \{G_{BB}Q_B\} & \{0\} & \{0\}
\end{bmatrix}
$$ \hspace{1cm} (6.21)
where $M_{ww} = (M_w + M_w^{pc})$ and $K^* = (K_w + K_w^{pc})$.

### 6.5 Results and Discussion

The BE formulation should be validated before coupling the cavity with the smart panel. For this validation we consider the same cubic cavity (0.305$m\times$0.305$m\times$0.305$m$) considered before. The resonance frequencies of the cavity for the rigid wall boundary conditions ($\frac{\partial p}{\partial n} = 0$) are estimated using eqn. (6.11). These values are then compared with the exact solutions in Table 6.1. The BE model consists of 96 quadratic elements with 386 surface nodes and the mesh is shown in Figure 6.1.

<table>
<thead>
<tr>
<th>Type</th>
<th>Dual reciprocity BE Method</th>
<th>Exact Solution</th>
<th>NASTRAN (FE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0,0)</td>
<td>$6.2\times10^{-12}$</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>(0,0,1)</td>
<td>546.1</td>
<td>540.9</td>
<td>544.0</td>
</tr>
<tr>
<td>(0,1,1)</td>
<td>769.3</td>
<td>765.0</td>
<td>770.1</td>
</tr>
<tr>
<td>(1,1,1)</td>
<td>938.4</td>
<td>937.0</td>
<td>951.0</td>
</tr>
<tr>
<td>(0,1,2)</td>
<td>1094.5</td>
<td>1081.1</td>
<td>1178.0</td>
</tr>
</tbody>
</table>

Figure 6.3 Boundary element mesh for the cavity (96 elements, 386 nodes)
The estimated frequencies agree well with the exact and FE solutions. In our next example a $6.0m \times 6.0m \times 6.0m$ acoustic domain is excited by a uniform harmonic flux of $50 \text{ N/m}^3$ on the upper face. The lower face has a prescribed pressure $p = 0$ and lateral faces have prescribed flux $\frac{\partial p}{\partial n} = 0$. The time history of the excitation flux is shown in fig. (6.4). The wave propagation velocity inside the cavity is $c = 100 \text{ m/s}$. The response flux at the bottom face of the cavity due to this exciting flux is calculated by solving the BE dynamic equation (6.11) and is plotted in fig. (6.5). From fig. (6.5) we observe that the response is converging to the static solution of $8.33 \text{ N/m}^3$.

![Figure 6.4 Excitation flux on the top surface of the cavity](image)

To demonstrate the effect of passive control using the RR/BE approach, the top surface of a cubic cavity $(0.305m \times 0.305m \times 0.305m)$ is assumed to have an aluminum
foam absorber at the cavity/plate interface. The complex admittance values for the aluminum foam are obtained from reference Jae-Eung OH et al (1999) for different frequencies and thickness of the absorber.

Figure 6.5 Time history of the flux at midpoint of the bottom face

The aluminum foam shows the best absorption performance in the range of 500 Hz to 1300 Hz. Since the acoustic enclosure considered throughout this study has the coupled system natural frequencies below this range, it is expected that the foamed aluminum absorber should provide the best absorption performances in the frequency range of interest. Figure 6.6 shows the real part of the admittance curves for the foam for the thickness of 20 mm, 24 mm and 30 mm. The optimal thickness of the absorber can be
determined by using the complex admittance values. The peak value of the admittance curve increases and moves towards lower frequencies as thickness increases. The predicted sound attenuation at different frequencies of the cavity is computed by solving eqn. (6.21). Before solving eqn. (6.21) with the admittance values the coupled system frequencies are estimated and shown in Table 6.2. The coupled system frequencies match very well with the earlier predictions using the NASTRAN/FE and RR/RR methods.

![Figure 6.6 Real part value curves of the admittance for the different thickness of foamed aluminum](image-url)
Table 6.2 Coupled resonant frequencies of the plate-cavity system

<table>
<thead>
<tr>
<th></th>
<th>RR/BE Method (Hz)</th>
<th>R-R Method (Hz)</th>
<th>NASTRAN/FE (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>83.5</td>
<td>84.5</td>
<td>85.3</td>
<td></td>
</tr>
<tr>
<td>152.9</td>
<td>155.7</td>
<td>151.9</td>
<td></td>
</tr>
<tr>
<td>243.6</td>
<td>252.8</td>
<td>249.8</td>
<td></td>
</tr>
<tr>
<td>295.7</td>
<td>315.4</td>
<td></td>
<td>304.3</td>
</tr>
</tbody>
</table>

The predicted sound pressure attenuation for three different thickness of absorber in the frequency range of 200 to 1200 Hz is calculated and displayed in fig. 6.7. The 12 mm thick absorber has the effective performance around 1300 Hz, the 20 mm absorber at 1000 Hz, and 24 mm absorber at 700 Hz.

Figure 6.7 Predicted sound pressure attenuation for different thickness of foam.
From the Figure 6.7 we observe that for each frequency there exists an optimal thickness for which foam provides best possible passive attenuation. The dotted line in Figure 6.7 represents the attenuation due to active and passive control. Here the feedback gain is the optimal value determined using the procedure shown in Chapter 5. The attenuation due to optimal feedback control is not significant in this frequency range when compared to the attenuation due to passive absorbers.

6.6 Summary

The major drawback of using the RR approach is its inability to model the complicated interface conditions that exist at cavity-panel interface. Particularly, when the walls are covered with some absorber materials then it is difficult to find a function to express the cavity pressure that satisfies this interface condition. Also it is very difficult to model arbitrarily shaped cavities with the RR method. To avoid these difficulties the cavity is modeled using boundary elements and is coupled with the RR model of the smart panel. This coupled system has smaller system matrices when compared to FE models for the vibroacoustic systems with smart walls. The present approach can be extended to cavities with complicated geometries. The challenge then would be finding a suitable Ritz functions for the enclosure walls.
Chapter 7

CONCLUSIONS AND FUTURE WORK

7.1 Introduction

The goal of this thesis is twofold. First, to understand the mechanics and develop simple and powerful new modeling methods for structures bonded with piezoelectric materials and secondly to develop solution methods, which enable us to efficiently analyze and design these structures for vibration and radiated noise control purposes. These models will be helpful for designers and researchers who work on the real aerospace structures in designing the active vibration and noise control systems. In this thesis the numerical models were developed and the results were obtained to prove the theory and to validate the models. Though the modeling of real life structures has not been attempted here, modeling these structures should be feasible and straightforward.

7.2 Conclusions And Future Work

In chapter 3 the FSDT model for piezoelectric laminate is developed in such a way that the formulation facilitates the determination of fields inside the layers of the laminate. Also for the first time the response of a piezoelectric laminate is estimated using a Fourier series approach for uniformly distributed mechanical load and potential. The results obtained using FSDT are then compared with the more accurate elasticity solution with no assumptions on the electric and mechanical fields. This comparison
showed that though FSDT is capable of predicting the frequencies of the laminate, it fails to estimate fields inside the layers, particularly electrical fields. It is concluded in this study that to obtain more accurate fields one needs to use higher order plate theories. But at the same time the use of higher order theories pose computational difficulties. The future research in this area would be in formulating the higher order theories for the piezoelectric laminates and solving the vibration problem analytically. This would really give researchers a powerful and accurate alternate to the large sized finite element (FE) formulation for the smart laminates.

Chapter 4 presents the Rayleigh-Ritz (RR) solution method for the acoustic enclosures surrounded by walls bonded with piezoelectric actuators and sensors. This is a powerful and simple analytical method for enclosure noise control studies using piezoelectric materials when compared to the finite element method. The resulting system matrices that arise from this method are much smaller than those of the FE method and this method is more suitable for optimization studies for the active vibration/noise control systems. Also for noise control studies for an enclosure at high frequencies where the modal density is high, the RR method is a better solution approach than any other presently available numerical methods. Future research in the RR method can be pursued in two major directions. Firstly, the possibilities of using simple polynomials or functions, which makes the computation of system matrices easier and faster, should be explored. Secondly, extending the approach to the other kinds of enclosures, like cylinders (for modeling the airplane fuselage) and walls with the absorbers should be attempted. The challenge in formulating this model would be finding the suitable
functions for the cavity and the surrounding shell structure.

Chapter 5 presented the finite element/analytical approach for the enclosure noise control problem. The smart panel is modeled with the shell elements and the piezoelectric actuators and sensors are modeled using the three dimensional brick elements. The brick elements are then connected to the base shell structure using transition elements. An analytical model was developed using rigid walled cavity modes then combined with the FE model for estimating the pressure inside the cavity. An optimal controller design for the smart panel is also presented. The reduction in the transmitted pressure due to the active control of the panel using this optimal controller inside the cavity is calculated for the first time using the elaborate model for the smart panel.

A powerful but simple model is developed for the smart panel-cavity system in chapter 6. The Rayleigh-Ritz approach is used to model the panel with surface bonded discrete piezoelectric patches and the acoustic cavity. Since the Rayleigh-Ritz approach works for any conservative system this approach can be extended to more arbitrary shaped panels. The challenge then would be finding the proper Ritz functions for more complicated shapes of the panel. As far as the acoustic cavity is considered if we are able to find rigid wall cavity functions for arbitrary shaped cavities then the cavity model can be easily interfaced with the smart panel. This method can also easily be extended to curved geometries like cylindrical and spherical cavities by proper choice of Ritz functions.

Even though the RR approach can be used for modeling cavities with different geometries, it is nearly impossible to model a cavity in this approach if we have more
complicated boundary conditions. One of the examples for such boundary conditions are impedance boundary conditions. In the case of passive noise control systems a sound absorber material is placed at the flexible panel/cavity interface to absorb the transmitted sound radiation through the panel. On such boundaries or interfaces the impedance boundary conditions are to be imposed which is not possible if one intend to use the Ritz approach using rigid wall cavity functions. A need for a better model for the acoustic cavity is developed using a dual reciprocity boundary element method. The boundary element (BE) method is a more suitable method for modeling acoustic domains due to the smaller system matrices when compared to available FE methods for the acoustics. Also this model can be used for more arbitrary shaped cavities. A novel method of combining the RR model of the panel with the BE model of the cavity is presented in Chapter 6. This approach yields excellent accuracy and incredibly small system matrices when compared to FE/FE type approaches. This model can be used efficiently for designing active noise control systems for acoustic enclosures of various shapes and having any kind of boundary conditions in a short time, particularly at high frequencies.
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Appendix A
Constitutive Relations For A Piezoelectric Medium

For linear piezoelastic materials the governing equations and the boundary conditions can be derived using the Hamilton’s principle (Tiersten H. F., 1969) and are given below.

Governing Equations in the absence of body forces and body charges,

\[ \sigma_{ij} = \rho u_j \text{, and } D_{i,i} = 0, \]  \hspace{1cm} (A.1)

associated with the boundary conditions

either \( \sigma_{ij} n_j = T_i \) or \( u_i \) is prescribed and

either \( D_{i,n_i} = q_i \) or \( \phi_i \) is prescribed  \hspace{1cm} (A.2)

The constitutive relations for piezoceramics poled in thickness direction are

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{32} \\
\sigma_{31} \\
\sigma_{12}
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{32} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
e_{15} \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
D_1 \\
D_2 \\
D_3
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & e_{15} & 0 \\
0 & 0 & 0 & e_{24} & 0 \\
e_{31} & e_{32} & e_{33} & 0 & 0
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{32} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix} = \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{11} \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}, \hspace{1cm} (A.3)
\]

\[
\begin{bmatrix}
D_1 \\
D_2 \\
D_3
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & e_{15} & 0 \\
0 & 0 & 0 & e_{24} & 0 \\
e_{31} & e_{32} & e_{33} & 0 & 0
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{32} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix} = \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{11} \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}, \hspace{1cm} (A.4)
\]
where $C_{ijkl}$ are the stiffness constants measured in constant electric field, $\varepsilon_{ij}$ are the dielectric constants measured at constant strain, $e_{ij}$ are the piezoelectric constants, $\varepsilon_{ij}$ and $\gamma_{ij}$ are the strain tensor, $\sigma_{ij}$ is the stress tensor $D_i$ is the electric displacement vector and $n_i$ is the outgoing unit normal vector to the boundary $S$. 
Appendix B
BE Formulation for the Acoustic Domain

BE equations can be obtained by multiplying the governing equation by a weighting function and using the reciprocity theorem. The fundamental solution \( g_L \) and its derivative along the outward normal to the boundary can be chosen as a weighting function. The fundamental solution (Green’s function) is usually interpreted as a potential field for an unbounded domain due to a unit point source. The BE equations can be derived by making use of reciprocity theorem (Brebbia 1978) also known as Green’s identity.

The reciprocity theorem for any two potential fields \( p \) and \( g_L \) is given by

\[
\int_{\Omega} (p \nabla^2 g_L - g_L \nabla^2 p) d\Omega = \int_{\Gamma} \left( p \frac{\partial g_L}{\partial n} - g_L \frac{\partial p}{\partial n} \right) d\Gamma,
\]

where \( n \) is the outward normal to the boundary \( \Gamma \) of the domain \( \Omega \). Generally we choose \( u \) as the unknown potential for the problem and the \( g_L \) as the fundamental solution, which satisfies the following differential equation:

\[
\nabla^2 g_L = \delta(\vec{r}_i, \vec{\xi}_i).
\]

In above eqn \( \delta(\vec{r}_i, \vec{\xi}_i) \) is a concentrated unit source at the position vector \( r_i \) and \( \xi_i \) is the observation point. Mathematically it represents a Dirac delta function, whose magnitude goes to infinity at the point where \( r_i \) equals to \( \xi_i \) and is zero elsewhere.

Differentiating potentials \( u \) and \( g_L \) along the outward normal the boundary fluxes \( q = \frac{\partial p}{\partial n} \)
and $q^* = \frac{\partial g_L}{\partial n}$ are obtained.

Let us suppose that the scalar-valued potential field $p$ defined in the bounded domain $\Omega$ is governed by the Laplace equation

$$\nabla^2 p = 0,$$

(B.3)

with the boundary conditions $p = 0$ on $\Gamma_1$ and $\frac{\partial p}{\partial n} = 0$ on $\Gamma_2$

where $\nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i}$. From the Green’s identity and eqn (B.3) we can write

$$\int_\Omega g_L \nabla^2 p d\Omega = \int_\Omega p \nabla^2 g_L d\Omega - \int_\Gamma \left( p \frac{\partial g_L}{\partial n} - g_L \frac{\partial p}{\partial n} \right) d\Gamma = 0. \quad (B.4)$$

Using the property of the fundamental solution for the Laplace equation, the above integral equation reduces to

$$\int_\Omega p(\xi_i) \delta(\vec{r}_i, \xi_i) d\Omega(\xi_i) - \int_\Gamma \left[ p(\xi_i) \frac{\partial g_L(\vec{r}_i, \xi_i)}{\partial n(\xi_i)} - g_L(\vec{r}_i, \xi_i) \frac{\partial p(\xi_i)}{\partial n(\xi_i)} \right] d\Gamma(\xi_i) = 0. \quad (B.5)$$

The integration and the differentiation is performed with respect to the coordinates of the observation point $\xi_i$. Using the property of Dirac delta function and performing some algebraic manipulation the following integral equation is obtained for the Laplace equation:

$$p(\vec{r}_i); \vec{r}_i \in \Omega \begin{cases} 0; \vec{r}_i \notin \Omega \\frac{1}{2} p(\vec{r}_i); \vec{r}_i \in \Gamma \end{cases} + \int_\Gamma q^*(\vec{r}_i, \xi_i) p(\xi_i) d\Gamma(\xi_i) = \int_\Gamma g_L(\vec{r}_i, \xi_i) q(\xi_i) d\Omega(\xi_i). \quad (B.6)$$

Equation (B.4) is the basis of the boundary element method wherein spatial
integration is performed numerically using the BE method. The boundary $\Gamma$ is divided into $N$ boundary elements denoted by $\Gamma_n$, and in each element, the variables defined on the boundary can be expressed in terms of their nodal values. Interpolation (also called shape functions) functions are assumed inside each element, which relate the variables at any point inside an element to the nodal values of the element as shown in equation below.

$$ p = \left[N_p^T\right]\{p_j\} \text{ and } q = \left[N_q^T\right]\{q_j\} $$ \hspace{1cm} (B.7)

In the derivation of generalized coordinate boundary elements, a local element coordinate system called natural coordinate system is often used. Depending on the order of the polynomial chosen for the shape functions, the boundary elements can be classified into three major categories.

*Subparametric elements* - In this type of element the order of the polynomial used for the shape function is lower than the order of the polynomial used in expressing the local coordinate system. The potential and flux are assumed to be constant over the entire element.

*Isoparametric element* - for which the order of the shape function polynomial is equal to the order of the polynomial used for expressing the element coordinate system. In these type of elements the potential and flux varies linearly between their values at the nodes.

*Superparametric elements* - Here the shape function polynomial order is higher than order of polynomial used to express the coordinate system. The variation of potential and flux are quadratic inside the element.
The potential field modeled using constant boundary elements cannot be continuous and higher order elements are capable of representing continuous variations of potential across the elements. But the continuity of fluxes is not guaranteed even in the higher order elements because the flux, a derivative of the potential along the outward normal, is not continuous at any corner.

Discretization of the boundary $\Gamma$ into a number of boundary elements replaces the boundary integral by a summation of integrals, each one over the boundary element $\Gamma_n$. Equation (B.4) after discretizing the boundary with $N$ elements becomes

$$
\sum \int_{\Gamma_n} \frac{\partial g^{(m)}_L}{\partial n} p d\Gamma = \sum_{j=1}^{N} \int_{\Gamma_j} g^{(m)}_L q d\Gamma .
$$

(B.8)

Substituting the assumed shape functions for the potential and the flux from eqn. (B.7) into (B.8) we obtain
Integrals over the elements are estimated numerically in the local coordinate system. There are no limitations regarding the location of source point $i$. Usually it is placed at every boundary node, which in turn generates $N$ equations. Since the potential field is continuous, this system of equations is assembled with respect to the nodal potentials. The fluxes are generally continuous, so they are not assembled and the discontinuity at the node point is still preserved. The size of the global $p$ vector is $N$ and for global $q$ vector is $2N$. The total number of columns in the potential influence matrix $H$ is $N$ and that of the flux influence matrix $G$ is $2N$ exactly matches with the size of vector $Q$. After assembling the matrices, the discrete form of the integral equation is

$$HP = GQ.$$  \hspace{1cm} (B.10)

The integrals over the boundary elements are of the form given by equation (B.9) and in order to do those integrals the boundary elements are subdivided into triangular regions with the a corner point at the collocation node. Once the element is subdivided a new system of coordinates $s_1, s_2$ is defined in such a way that the Jacobian becomes zero at the collocation point. The transformed domain in the $s_1, s_2$ system of coordinates is a square. Then the integrals of equation (B.9) can be done in the $s_1, s_2$ domain using the standard Gaussian quadrature. This strategy for integration was presented by Lacht (1975). Other versions were presented by Han et al (1985), and Cerrolaza and Alarcón (1989).
Vita

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In 1997 the author commenced his doctoral program in the Engineering Science and Mechanics department at the Pennsylvania State University. The focus of his doctoral research was on the active and passive methods of noise control using piezoelectric type smart structures for aerospace applications. His research involved in developing solution techniques for designing the smart structures for vibration and noise reduction applications for helicopter and aircraft cabins. The author has published his work in various conference proceedings and journals. His future interests involve in the development of efficient numerical methods for the analysis, design and development of active vibration and noise control systems.