The Pennsylvania State University

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# RAREFACTION WAVE INTERACTION OF PRESSURE-GRADIENT SYSTEM

A Thesis in

Mathematics

by

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# Abstract

The pressure-gradient system is a relatively new system of conservation laws. This system is a reduction from the two-dimensional(2-D) compressible Euler equations. Separating the pressure from the inertia in the flux of the Euler equations, we obtain the transport(or convective) system without pressure terms and the pressure-gradient system with pressure terms. Since the Cauchy problem for this system is still very difficult, We consider the Riemann problem, but it is also a complex open problem. According to the initial data, we have 12 main configurations that explain the interaction between various waves such as rarefaction waves, shocks and contact discontinuities. Here, we are interested in demonstrating the four rarefaction waves analytically. This problem is a mixed type nonlinear equation(elliptic, parabolic and hyperbolic). Yuxi Zheng proved that there exists a weak solution in the elliptic region, and we show that there exists a continuous and piecewise smooth solution in the hyperbolic region up to the domain of determinacy.

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This thesis is dedicated to my God who loves me forever, to my parents who have encouraged and supported me, to my lovely family, Mijung, Yeji and Yewon, who trust me and always make me happy, and finally, to teachers in my elementary, middle and high schools, and my college who guided me to the mathematical world.

# Chapter 1

# Introduction

The Euler Equations were essentially formed by Euler in 1776. Many Mathematicians have tried to prove the existence of solutions to the equations ever since 1776, but it is still open. Riemann initiated the Riemann problem for 1-D Euler equations, which is an initial value problem with a simple discontinuity at the origin in the 1840s. Solutions to a Riemann problem reveal the elementary waves(shock wave, rarefaction wave and contact discontinuity). Thereafter, the well-poseness of the 1-D hyperbolic system of conservation laws have been established for small data via a lot of mathematicians including von Neumann, Courant, Friedrichs, Oleinik, Lax, Glimm and Bressan. The existence of a solution to the 1-D compressible Euler equations with large data remains open.

What effort did mathematicians make in 2-D compressible Euler equations? Mach observed experimentally the regular reflection and Mach reflection of oblique shocks which are the most familiar examples of solutions in 2-D in the last century. Although several experiments and numerical simulations have been done on oblique shocks, there are few analytical demonstrations; in other words, we have a lot of work to do theoretically. It is difficult. Thus, we use the idea of flux splitting on the Euler equations to derive simpler models. Separating the pressure from the inertia in the flux of the Euler equations, we obtain the transport(or convective) system without pressure terms and the pressure-gradient system with pressure terms.

If we solve analytically the transport system and the pressure-gradient system, we can combine the two results and analyze the 2-D compressible Euler equations. Since the transport system has seen progress(see Li, Zhang and Yang's book [12]), we are interested in the pressure-gradient system. Since the Cauchy problem for this system is still very difficult, we consider the Riemann problem which is special Cauchy problem, but it is also a complex open problem. According to the initial data in the Riemann problem, we have 12 main configurations (see Li, Zhang and Yang's book [12])that explain the interaction between various waves such as rarefaction waves, shocks and contact discontinuities. We would like to resolve the four rarefaction waves analytically.

## 1.1 Pressure-Gradient System

We consider the two-dimensional compressible Euler equations

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uE + up \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vE + vp \end{pmatrix}_y = 0, \tag{1.1}$$

where $\rho$ is density, $p$ is pressure, $u$ and $v$ are velocities, $E = \frac{1}{2}(u^2 + v^2) + \frac{1}{\gamma-1}\frac{p}{\rho}$ is total energy per unit mass, and $\gamma > 1$ is the adiabatic index.

The initial value problem for Euler has been open since 1776. So we consider a simplified system from the full Euler equations; that is, the pressure-gradient system. Separating the pressure from the inertia in the flux of the Euler equations, we obtain two systems of equations

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 \\ \rho uv \\ \rho uE \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 \\ \rho vE \end{pmatrix}_y = 0, \tag{1.2}$$

and

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} 0 \\ p \\ 0 \\ up \end{pmatrix}_x + \begin{pmatrix} 0 \\ 0 \\ p \\ vp \end{pmatrix}_y = 0. \tag{1.3}$$

The first system is called the *transport* (or *convective*) system and the second system is called the *pressure-gradient* system. (See Agarwal and Halt [1], Peng Zhang, Ziequan Li, and Tong Zhang [18] , and Yuxi Zheng [22].) We are interested in the pressure-gradient system. From the first equation of system (1.3) we obtain

$$\rho_t = 0.$$

Thus $\rho$ is independent of time. Assuming that $\rho = 1$ for simplicity, we have

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ E_t + (up)_x + (vp)_y = 0, \end{cases} \tag{1.4}$$

where $E = (u^2 + v^2)/2 + p/(\gamma - 1)$. For smooth solutions or in regions where a solution is smooth, system (1.4) can be simplified to be

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ \frac{1}{\gamma-1} p_t + pu_x + pv_y = 0. \end{cases} \tag{1.5}$$

The transformation

$$\begin{cases} p = (\gamma - 1)P, \\ t = \frac{1}{\gamma-1} T \end{cases}$$

can change system (1.5) into

$$\begin{cases} u_T + P_x = 0, \\ v_T + P_y = 0, \\ P_T + Pu_x + Pv_y = 0. \end{cases} \tag{1.6}$$

From system (1.6) we can obtain one of the simplest two dimensional quasilinear wave equation

$$\left(\frac{P_T}{P}\right)_T = P_{xx} + P_{yy}. \tag{1.7}$$

Both Cauchy and Riemann problems for system (1.3), (1.4), or (1.7) are open. In the self-similar coordinates

$$\xi = \frac{x}{T}, \quad \eta = \frac{y}{T},$$

we have the second order nonlinear equation

$$(P-\xi^2)P_{\xi\xi} - 2\xi\eta P_{\xi\eta} + (P-\eta^2)P_{\eta\eta} + \frac{1}{P}(\xi P_\xi + \eta P_\eta)^2 - 2(\xi P_\xi + \eta P_\eta) = 0, \tag{1.8}$$

which has two eigenvalues,

$$\Lambda_\pm = \frac{\xi\eta \pm \sqrt{P(\xi^2+\eta^2-P)}}{\xi^2 - P}. \tag{1.9}$$

## 1.2 Four Rarefaction Wave Problem

We propose a four-constant initial value problem for system (1.6) so as to yield a continuous solution for $t > 0$. Consider the data

$$(u, v, P)|_{T=0} = (u_i, v_i, P_i) \tag{1.10}$$

for $(x, y)$ in the $i$-th quadrant, $i = 1, 2, 3, 4$, where $\{(u_i, v_i, P_i)\}_{i=1}^4$ are constants. Consider two states $(u_1, v_1, P_1)$ and $(u_2, v_2, P_2)$. Assume that $v_1 = v_2$ ( i.e., ignore $v$). From system (1.6) we obtain

$$\begin{cases} u_T + P_x = 0, \\ P_T + P u_x = 0. \end{cases} \tag{1.11}$$

The self-similar coordinates $\xi = \frac{x}{T}, \quad \eta = \frac{y}{T}$ can reduce variables of (1.11) as follows:

$$\begin{cases} -\xi u_\xi + P_\xi = 0, \\ -\xi P_\xi + P u_\xi = 0. \end{cases} \tag{1.12}$$

Thus we obtain the solution to (1.12),

$$\begin{cases} P = \xi^2, \\ u = 2\xi + k, \end{cases} \tag{1.13}$$

where $k$ is a constant.

We assume $P_2 < P_1$. The value of $(u, P)$ in the forward rarefaction wave is given by

$$\begin{cases} P = \xi^2, \\ u = 2(\xi - \sqrt{P_1}) + u_1, \quad \sqrt{P_2} < \xi < \sqrt{P_1}. \end{cases} \tag{1.14}$$

Thus, for the states $(u_1, v_1, P_1)$ and $(u_2, v_2, P_2)$ to be connected by a single forward rarefaction wave, we need the compatibility conditions

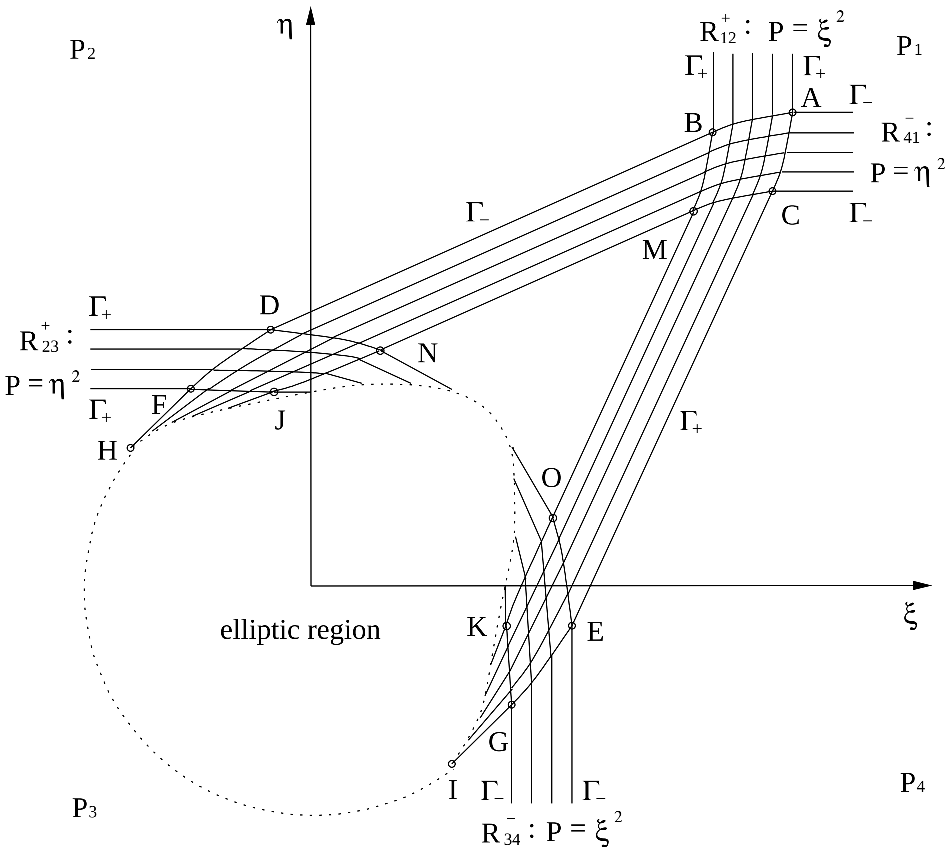


Figure 1.1. Four rarefaction waves

$$\begin{cases} v_2 = v_1, \\ u_2 - u_1 = 2(\sqrt{P_2} - \sqrt{P_1}). \end{cases} \tag{1.15}$$

Similarly the compatibility conditions between states 1 and 4, 2 and 3, and 3 and 4 in Figure 1.1 can be found as follows:

$$\begin{cases} u_4 = u_1, \\ v_4 - v_1 = 2(\sqrt{P_4} - \sqrt{P_1}), \end{cases} \qquad P_4 < P_1, \tag{1.16}$$

$$\begin{cases} u_3 = u_2 \left(= u_1 + 2(\sqrt{P_2} - \sqrt{P_1})\right), \\ v_3 - v_2 = 2(\sqrt{P_3} - \sqrt{P_2}), \end{cases} \qquad P_3 < P_2, \tag{1.17}$$

$$\begin{cases} v_3 = v_4 \left(= v_1 + 2(\sqrt{P_4} - \sqrt{P_1})\right), \\ u_3 - u_4 = 2(\sqrt{P_3} - \sqrt{P_4}), \end{cases} \qquad P_3 < P_4. \tag{1.18}$$

For (1.17) and (1.18) to be consistent, we need the necessary compatibility condition

$$\sqrt{P_1} - \sqrt{P_2} = \sqrt{P_4} - \sqrt{P_3} \tag{1.19}$$

which is also sufficient. Therefore for any $P_1$, $P_2$, $P_3$, $u_1$, $v_1$, we can have $P_4$, $u_i$, $v_i$ $(i = 2, 3, 4)$ from compatibility conditions.

$R_{12}^+(\xi)$ and $R_{41}^-(\eta)$ from infinity meet at $A = (\sqrt{P_1}, \sqrt{P_1})$. Then they will interact with each other. So the part of the boundary of the interaction region should be the extensions of characteristic lines $\Gamma_-$ and $\Gamma_+$ from $A$. Thus the equations of the arc $\widehat{AB}$ and the arc $\widehat{AC}$ satisfy, respectively,

$$\widehat{AB}: \quad \frac{d\eta}{d\xi} = \Lambda_-, \qquad P = \xi^2, \tag{1.20}$$

$$\widehat{AC}: \quad \frac{d\eta}{d\xi} = \Lambda_+, \qquad P = \eta^2. \tag{1.21}$$

$\Gamma_-$ penetrates $R_{12}^+(\xi)$ and ends at $B$, and then goes straight until it intersects $R_{23}^+(\eta)$ at $D$. Thus we can find the exact values of $B$ and $D$, and the equation of the straight segment $\overline{BD}$. This characteristic curve $\Gamma_-$ continues to pass through $R_{23}^+$ and ends at $F$. Thus the equation of the arc $\widehat{DF}$ satisfies

$$\frac{d\eta}{d\xi} = \Lambda_-, \qquad P = \eta^2, \tag{1.22}$$

$\Gamma_-$ goes straight again until it is tangent to the circular arc $\widehat{HI}$ that is a part of the circle centered at the origin with a radius $\sqrt{P_3}$. Therefore we can find the equation to the straight segment $\overline{FH}$.

Similarly we can find the equations of $\widehat{AC}$, $\overline{CE}$, $\widehat{EG}$, and $\overline{GI}$. The points and the equations are as follows:

$$A = (\sqrt{P_1}, \sqrt{P_1}),$$

$$B = (\sqrt{P_2}, \sqrt{2\sqrt{P_1 P_2} - P_2}),$$

$$C = (\sqrt{2\sqrt{P_1 P_4} - P_4}, \sqrt{P_4}),$$

$$D = \left( \frac{\sqrt{P_2}(\sqrt{2\sqrt{P_1P_2} - P_2} - \sqrt{P_1})}{\sqrt{P_1} - \sqrt{P_2}}, \sqrt{P_2} \right),$$

$$E = \left( \sqrt{P_4}, \frac{\sqrt{P_4}(\sqrt{2\sqrt{P_1P_4} - P_4} - \sqrt{P_1})}{\sqrt{P_1} - \sqrt{P_4}} \right),$$

$$F = \left( -\sqrt{2k_{12}\sqrt{P_3} - P_3}, \sqrt{P_3} \right),$$

$$G = \left( \sqrt{P_3}, -\sqrt{2k_{34}\sqrt{P_3} - P_3} \right),$$

$$H = \left( -\frac{\sqrt{P_3}\sqrt{2k_{12}\sqrt{P_3} - P_3}}{k_{12}}, \frac{\sqrt{P_3}(\sqrt{P_3} - k_{12})}{k_{12}} \right),$$

$$I = \left( \frac{\sqrt{P_3}(\sqrt{P_3} - k_{34})}{k_{34}}, -\frac{\sqrt{P_3}\sqrt{2k_{34}\sqrt{P_3} - P_3}}{k_{34}} \right),$$

$$\widehat{AB} \quad : \quad (\xi - \sqrt{P_1})^2 + \eta^2 = P_1,$$

$$\widehat{AC} \quad : \quad \xi^2 + (\eta - \sqrt{P_1})^2 = P_1,$$

$$\overline{BD} \quad : \quad \eta = \frac{\sqrt{P_1} - \sqrt{P_2}}{\sqrt{2\sqrt{P_1P_2} - P_2}}\xi + \frac{\sqrt{P_1P_2}}{\sqrt{2\sqrt{P_1P_2} - P_2}},$$

$$\overline{CE} \quad : \quad \eta = \frac{\sqrt{2\sqrt{P_1P_4} - P_4}}{\sqrt{P_1} - \sqrt{P_4}}\xi - \frac{\sqrt{P_1P_4}}{\sqrt{P_1} - \sqrt{P_4}},$$

$$\widehat{DF} \quad : \quad \xi^2 + (\eta - k_{12})^2 = k_{12}^2,$$

$$\widehat{EG} \quad : \quad (\xi - k_{34})^2 + \eta^2 = k_{34}^2,$$

$$\overline{FH} \quad : \quad \eta = \frac{\sqrt{2k_{12}\sqrt{P_3} - P_3}}{\sqrt{P_3} - k_{12}}\xi + \frac{k_{12}\sqrt{P_3}}{\sqrt{P_3} - k_{12}},$$

$$\overline{GI} \quad : \quad \eta = \frac{\sqrt{P_3} - k_{34}}{\sqrt{2k_{34}\sqrt{P_3} - P_3}}\xi - \frac{k_{34}\sqrt{P_3}}{\sqrt{2k_{34}\sqrt{P_3} - P_3}},$$

where

$$k_{12} = \frac{\sqrt{P_1 P_2}(\sqrt{P_1} - \sqrt{2\sqrt{P_1 P_2} - P_2})}{P_1 + P_2 - 2\sqrt{P_1 P_2}} > 0,$$

$$k_{34} = \frac{\sqrt{P_1 P_4}(\sqrt{P_1} - \sqrt{2\sqrt{P_1 P_4} - P_4})}{P_1 + P_4 - 2\sqrt{P_1 P_4}} > 0.$$

The waves between states 1 and 2, and 1 and 4 coming from infinity begin to interact at the point $A = (\sqrt{P_1}, \sqrt{P_1})$ in Figure 1.1. The two characteristics from the point $A$, and possibly also part of the sonic circle of the state 3, form a loop which separates the $(\xi, \eta)$-plane into two regions. The exterior region consists of the four rarefaction waves, that is, $R_{12}^+$, $R_{23}^+$, $R_{34}^-$, $R_{41}^-$, and constant states.

The interior region consists also of two region; one is elliptic (the region inside the dashed line near the origin in Figure 1.1.) and the other is hyperbolic where the four rarefaction waves interact. Characteristics in the hyperbolic region and on the parabolic curve may look like what is shown in Figure 1.1. The parabolic curve may be expected to be convex and smooth. Some part of the parabolic curve may be circular. On the whole curve there holds the relation

$$P = \xi^2 + \eta^2.$$

Yuxi Zheng proved the existence of a subsonic solution to (1.8) in the elliptic region which is shown in Figure 1.1, and Zihuan Dai and Tong Zhang proved the existence of the supersonic solution to (1.8) in a part of the hyperbolic region in Figure 1.1; i.e., the two rarefaction waves, $R^+_{12}$ and $R^-_{41}$, interact. See the book [22] by Yuxi Zheng, and the paper [5] by Zihuan Dai and Tong Zhang. There is a gap between the two works. We will cover the gap, in other words, we will prove that there exists a continuous and piecewise smooth solution in the part of the hyperbolic region up to the domain of determinacy in addition to the region bounded by the positive characteristic lines $\widehat{AC}$, $\widehat{BM}$ and the the negative characteristic lines $\widehat{AB}$, $\widehat{CM}$.

## 1.3 Three Full Rarefaction Wave Problem; Core Problem

In order to prove that there exists a smooth solution in the region where $R^+_{12}$ and $R^-_{34}$ intersect, i.e. $\Diamond EOKG$, we change the four rarefaction wave problem into the three full rarefaction wave problem.

We extend the arc $\widehat{AB}$ and $\widehat{EG}$ to the origin in Figure 1.2 on the basis of [5], [10]. Thus the second and third quadrants have vacuum, that is, $P = 0$. The second order quasilinear equation (1.8) has the boundary value condition as follows:

$$P|_{l_+} = \xi^2, \quad P|_{l_-} = f_1(\xi, \eta) \tag{1.23}$$

where

$$l_+ : (\xi - k)^2 + \eta^2 = k^2, \tag{1.24}$$

$$l_- : \eta = g_1(\xi) \tag{1.25}$$

with

$$k = \frac{\sqrt{P_1 P_4}(\sqrt{P_1} - \sqrt{2\sqrt{P_1 P_4} - P_4})}{P_1 + P_4 - 2\sqrt{P_1 P_4}} > 0.$$

In the polar coordinate system $(\theta, r)(= (\arctan \eta/\xi, \sqrt{\xi^2 + \eta^2}))$, equation (1.8), the boundary value condition (1.23) and the function of the supports can be written as

$$(P - r^2)P_{rr} + \frac{P}{r^2}P_{\theta\theta} + \frac{P}{r}P_r + \frac{1}{P}(rP_r)^2 - 2rP_r = 0, \tag{1.26}$$

$$P|_{l'_+} = r^2 \cos^2 \theta, \quad P|_{l'_-} = f(r, \theta), \tag{1.27}$$

$$l'_+ : \theta = \theta_+(r) = \arccos\left(\frac{r}{2k}\right), \quad 0 \le r \le r_m, \quad -\theta_b \le \theta \le -\theta_m, \tag{1.28}$$

$$l'_- : \theta = \theta_-(r) = g(r), \qquad 0 \le r \le r_m, \quad -\theta_m \le \theta \le \theta_a, \tag{1.29}$$

where $P(r, \theta)$ denotes $P(r\cos\theta, r\cos\theta)$ for simplicity and $r_m$ is the length between the origin and $E$ in figure 1.2, that is, $r_m = \sqrt{2k\sqrt{P_4}}$, $\theta_m$ is the angle between $\xi$-axis and the straight line $\overline{EO}$, $\theta_b$ is the angle between $\xi$-axis and the tangent line to $l'_+$ at the

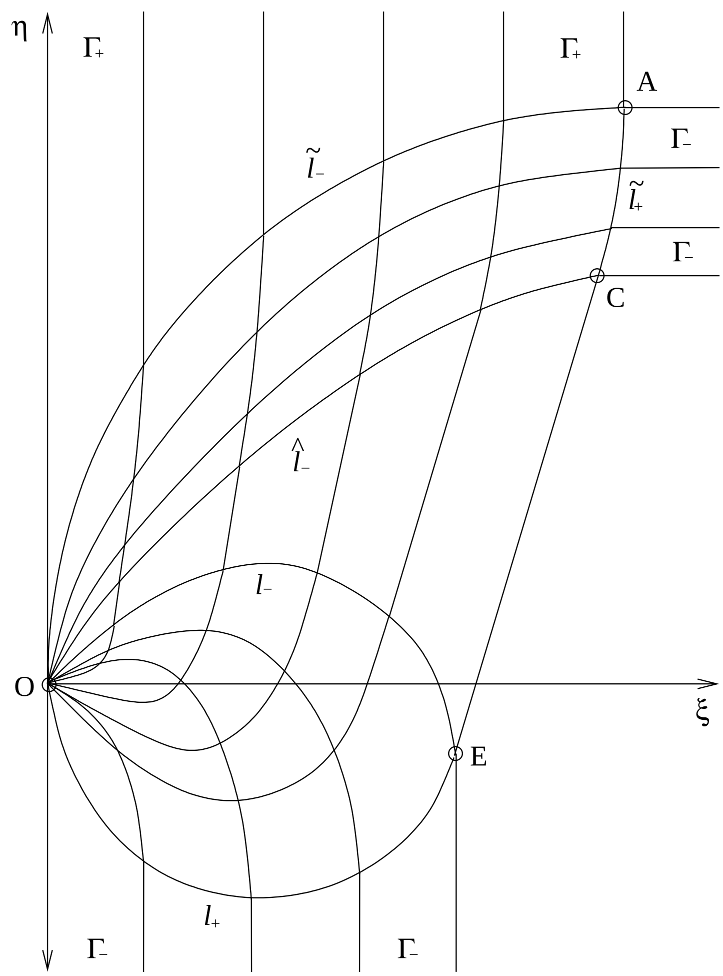


Figure 1.2. Three full rarefaction waves

origin, and $\theta_a$ is the angle between $\xi$-axis and the tangent line to $l'_-$ at the origin.

We transform (1.26) into a system. If we denote $P_r$, $P_\theta$ by $Q$, $R$, then (1.26) can be written as the system

$$\begin{pmatrix} P \\ Q \\ R \end{pmatrix}_r + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{-P}{r^2(r^2-P)} \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} P \\ Q \\ R \end{pmatrix}_\theta = \begin{pmatrix} Q \\ \bar{B} \\ 0 \end{pmatrix}, \tag{1.30}$$

where

$$\bar{B} = \frac{P^2Q + r^3Q^2 - 2r^2PQ}{rP(r^2 - P)}. \tag{1.31}$$

For convenience we introduce $W = (P, Q, R)^T$. System (1.30) has three eigenvalues

$$\lambda_0(\theta, r, W) = 0, \quad \lambda_\pm(\theta, r, W) = \pm\sqrt{\frac{P}{r^2(r^2 - P)}}, \tag{1.32}$$

and three left eigenvectors associated with them in turn

$$\vec{l_0}(\theta, r, W) = (1, 0, 0) \quad \vec{l_\pm}(\theta, r, W) = (0, 1, \lambda_\mp(\theta, r, W)). \tag{1.33}$$

Multiplying (1.30) by $\vec{l_i}(\theta, r, W)$, $i = 0, \pm$, we get the characteristic form of (1.30)

$$\vec{l_i}(\theta, r, W)(W_r + \lambda_i(\theta, r, W)W_\theta) = C_i(\theta, r, W), \quad i = 0, \pm, \tag{1.34}$$

where

$$C_0 = Q, \quad C_\pm = \bar{B}. \tag{1.35}$$

Now the Goursat problem (1.26) - (1.27) is transformed into a Goursat problem for (1.30) with the boundary value condition (1.27), but (1.26) - (1.27) is not equivalent to (1.30) with the boundary value condition (1.27) because a system is not necessarily equivalent to the second order equation from which it comes. So we need to give the boundary condition for $Q = P_r$ and $R = P_\theta$ that explains the relation of $P$, $Q$, and $R$ on the boundary because $P$, $Q$, $R$ are completely independent in the system. We use $\frac{dP}{dr} = P_r + \frac{d\theta}{dr}P_\theta$ that gives a notational consistency for equivalence between the second order equation (1.26) and system (1.30). As we give boundary values for $Q$, $R$, say, $Q|_{l'_\pm} = Q_\pm(r)$, $R|_{l'_\pm} = R_\pm(r)$, $Q_\pm(r)$, $R_\pm(r)$ satisfy two conditions

$$Q_+ + \frac{R_+}{\sqrt{4k^2-r^2}} = \frac{dP_+}{dr} = \frac{r^3}{k^2}, \tag{1.36}$$

$$Q_- + g'(r)R_- = \frac{dP_-}{dr} = f'\left(r, g\left(r\right)\right), \tag{1.37}$$

where

$$g'(r) = \frac{d\theta}{dr} = \lambda_- = -\frac{\sqrt{P}}{\sqrt{r^2(r^2-P)}} = -\frac{\sqrt{f}}{\sqrt{r^2(r^2-f)}}, \tag{1.38}$$

the second order equation (1.26) with (1.27) is equivalent to the Goursat problem for (1.30) with a new boundary value conditions:

$$W|_{l'_\pm} = W_\pm(r) =: (P_\pm(r), Q_\pm(r), R_\pm(r))^T, \tag{1.39}$$

where $P_\pm(r) = P|_{l'_\pm}$.

**Remarks 1.1.** *(i) These Goursat problems are unusual since at the ends of their support we have $P = 0$, which means the equations are degenerate hyperbolic. We call them degenerate Goursat problems.*

*(ii) By the result of Zihuan Dai and Tong Zhang's paper [5] and Zhen Lei and Yuxi Zheng's paper [10], we know that there exists a smooth solution in the region bounded by $\widehat{OA}$, $\widehat{OC}$, $\widehat{AC}$ in Figure 1.2, and vacuum occurs only at the origin.*

*(iii) We need to use directional derivatives in this thesis. So we use the following notations:*

$$\frac{d}{d_i\xi} = \frac{\partial}{\partial\xi} + \Lambda_i\frac{\partial}{\partial\eta}, \quad \frac{d}{d_i\eta} = \frac{\partial}{\partial\eta} + \Lambda_i^{-1}\frac{\partial}{\partial\xi}, \tag{1.40}$$

$$\frac{d}{d_i r} = \frac{\partial}{\partial r} + \lambda_i\frac{\partial}{\partial\theta}, \quad \frac{d}{d_i\theta} = \frac{\partial}{\partial\theta} + \lambda_i^{-1}\frac{\partial}{\partial r}, \tag{1.41}$$

*where $i = 0, \pm$, $\Lambda_0 = \eta/\xi$ and $\Lambda_\pm$, $\lambda_\pm$, $\lambda_0$ are defined in (1.9) and (1.32).*

From (1.34) and the notations in Remarks 1.1 we derive positive and negative characteristic forms.

$$\frac{dQ}{d_+ r} + \lambda_- \frac{dR}{d_+ r} = \bar{B} \tag{1.42}$$

$$\frac{dQ}{d_- r} + \lambda_+ \frac{dR}{d_- r} = \bar{B} \tag{1.43}$$

# Chapter 2

# Simple Waves

Now we figure out what happens in the simple wave region($\Omega_s$) bounded by two negative characteristic lines $\widehat{OC}(\hat{l}_-)$, $\widehat{OE(l_-)}$ and the straight line $\overline{CE}$ in Figure 1.2. What is a simple wave? For a system of hyperbolic conservation laws in one dimension, a centered rarefaction wave is a simple wave, in which one family of characteristics are straight lines and the dependent variables are constant along a characteristic. Simple waves for the pressure-gradient system in two dimensions are similar; i.e., one family of characteristics are straight and the pressure are constant along the characteristics. Note that the characteristic lines of simple waves may not be parallel to each other. In this chapter, we obtain three results. The first result is that positive characteristic lines do not intersect each other in the simple wave region. The second result is that $dP/d_-\xi > 0$ along $l_-$. The last result is that $l_-$ is concave with respect to the $\xi$-axis.

## 2.1 Preliminaries

We quote important equations from paper [5] for proving one of two main theorems in this paper. If (1.26) has a smooth solution in the hyperbolic region, it can be written as the following characteristic separating form:

$$\frac{d}{d_+\theta}\frac{dP}{d_-\theta}=\frac{r^4P_r\lambda_+}{2P^2}\frac{dP}{d_-\theta},\quad \frac{d}{d_-\theta}\frac{dP}{d_+\theta}=\frac{r^4P_r\lambda_-}{2P^2}\frac{dP}{d_+\theta}. \tag{2.1}$$

From these formula we see that $\Omega_s$ is a simple wave region(before characteristics intersect). Using the definition of the $\Lambda_\pm$, we have

$$\left(P-\xi^2\right)\Lambda_\pm^2+2\xi\eta\Lambda_\pm+\left(P-\eta^2\right)=0, \tag{2.2}$$

and therefore

$$P=\frac{(\xi\Lambda_+-\eta)^2}{\Lambda_+^2+1}=\frac{\left(\eta\Lambda_+^{-1}-\xi\right)^2}{\Lambda_+^{-2}+1},\quad P=\frac{(\xi\Lambda_--\eta)^2}{\Lambda_-^2+1}=\frac{\left(\eta\Lambda_-^{-1}-\xi\right)^2}{\Lambda_-^{-2}+1}. \tag{2.3}$$

Differentiating $P$ with respect to $\xi$ and $\eta$ along $\Gamma_+$ and $\Gamma_-$, respectively, we have

$$\frac{dP}{d_+\xi} = \frac{2\left(\xi\Lambda_+-\eta\right)\left(\xi+\eta\Lambda_+\right)}{\left(\Lambda_+^2+1\right)^2}\frac{d\Lambda_+}{d_+\xi}, \tag{2.4}$$

$$\frac{dP}{d_+\eta} = \frac{2\left(\eta\Lambda_+^{-1}-\xi\right)\left(\xi\Lambda_+^{-1}+\eta\right)}{\left(\Lambda_+^{-2}+1\right)^2}\frac{d\Lambda_+^{-1}}{d_+\eta}, \tag{2.5}$$

$$\frac{dP}{d_-\xi} = \frac{2\left(\xi\Lambda_--\eta\right)\left(\xi+\eta\Lambda_-\right)}{\left(\Lambda_-^2+1\right)^2}\frac{d\Lambda_-}{d_-\xi}, \tag{2.6}$$

$$\frac{dP}{d_-\eta} = \frac{2\left(\eta\Lambda_-^{-1}-\xi\right)\left(\xi\Lambda_-^{-1}+\eta\right)}{\left(\Lambda_-^{-2}+1\right)^2}\frac{d\Lambda_-^{-1}}{d_-\eta}. \tag{2.7}$$

## 2.2 Results in the Simple Wave Region

We prove the result that $P$ has a continuous solution in the simple wave region.

**Lemma 2.1.** *The positive characteristic lines in $\Omega_s$ do not intersect each other.*

**Proof .** *Suppose that $Z \in \Omega_s$ is the first intersection point of two positive characteristic lines $\overline{XZ}$, $\overline{YZ}$ with $X, Y \in \hat{l}_-$ such that*

$$\left.\frac{dP}{d_-\theta}\right|_Z = -\infty. \tag{2.8}$$

*Denote $X$, $Y$, $Z$ by $X = (\hat{\theta}_1, \hat{r}_1)$, $Y = (\hat{\theta}_2, \hat{r}_2)$ and $Z = (\mu, \nu)$ in the polar coordinates. By the definition of directional derivatives $dP/d_+\theta$ and $dP/d_-\theta$, we have*

$$P_r = \frac{1}{2}\lambda_+\left(\frac{dP}{d_+\theta} - \frac{dP}{d_-\theta}\right).$$

*Since $P$ is constant along the positive characteristic line $\overline{XZ}$, $dP/d_+\theta = 0$ on $\overline{XZ}$. Thus we have*

$$P_r = -\frac{1}{2}\lambda_+\frac{dP}{d_-\theta} \quad \text{on} \quad \overline{XZ}. \tag{2.9}$$

*Substituting (2.9) into (2.1), we have*

$$\frac{d}{d_+\theta}\frac{dP}{d\_\theta} = -\frac{r^4\lambda_+^2}{4P^2}\left(\frac{dP}{d\_\theta}\right)^2 \quad \text{on} \quad \overline{XZ}. \tag{2.10}$$

*Integrating (2.10) along $\overline{XZ}$, we have*

$$\frac{1}{\left(\frac{dP}{d\_\theta}\right)(\mu,\nu)} - \frac{1}{\left(\frac{dP}{d\_\theta}\right)(\hat{\theta}_1,\hat{r}_1)} = \int_{\hat{\theta}_1}^{\mu}\frac{r^2}{4P(r^2-P)}d_+\theta. \tag{2.11}$$

*From (2.8), (2.11), we have one equation*

$$\frac{1}{\left(\frac{dP}{d\_\theta}\right)(\hat{\theta}_1,\hat{r}_1)} = \int_{\mu}^{\hat{\theta}_1}\frac{r^2}{4P(r^2-P)}d_+\theta. \tag{2.12}$$

*From the fact that the left side of (2.12) is finite, $r^2 - P > 0$ on $\overline{XZ}$ and the right side of (2.12) is finite and positive. Thus we have*

$$\frac{dP}{d\_\theta}(\hat{\theta}_1,\hat{r}_1) > 0,$$

*which contradicts that $dP/d\_\theta(\hat{\theta}_1,\hat{r}_1) < 0$ by the Dai and Zhang's paper [5]. Therefore this lemma holds.* □

In order to obtain the second result, we need Lemma 2.2 as follows:

**Lemma 2.2.**

$$\frac{dP}{d\_\theta} < 0 \quad \text{along} \quad l'_- \tag{2.13}$$

**Proof .** *We choose any point $Y = (\theta, r)$ in the polar coordinate system on $l\_$ and draw the positive characteristic line($\Gamma_+^Y$) passing through $Y$. Denote the intersection point of $\Gamma_+^Y$ and $\hat{l_-}$ by $X = (\hat{\theta}, \hat{r})$ in the polar coordinate system. Integrating the first equation in (2.1) along the positive characteristic line $\Gamma_+^Y$ from $\hat{\theta}$ to $\theta$, we have*

$$\begin{aligned} \frac{dP}{d\_\theta}(\theta, r) &= \frac{dP}{d\_\theta}(\hat{\theta}, \hat{r}) \exp\left(\int_{\hat{\theta}}^{\theta} \frac{r^4 P_r \lambda_+}{2P^2} d_+\theta\right) \\ &= -\sqrt{\frac{\hat{r}^2(\hat{r}^2 - P(\hat{\theta}, \hat{r}))}{P(\hat{\theta}, \hat{r})}} \cdot \frac{dP}{d\_r}(\hat{\theta}, \hat{r}) \exp\left(\int_{\hat{\theta}}^{\theta} \frac{r^4 P_r \lambda_+}{2P^2} d_+\theta\right). \end{aligned} \tag{2.14}$$

*Since $dP/d\_r > 0$ and $r^2 - P > 0$ on $\hat{l}\_$, this lemma holds.* □

**Lemma 2.3.**

$$\frac{dP}{d\_\xi} > 0 \quad \text{along} \quad l\_ \quad \text{on} \quad \left(0, \sqrt{P_4}\right] \tag{2.15}$$

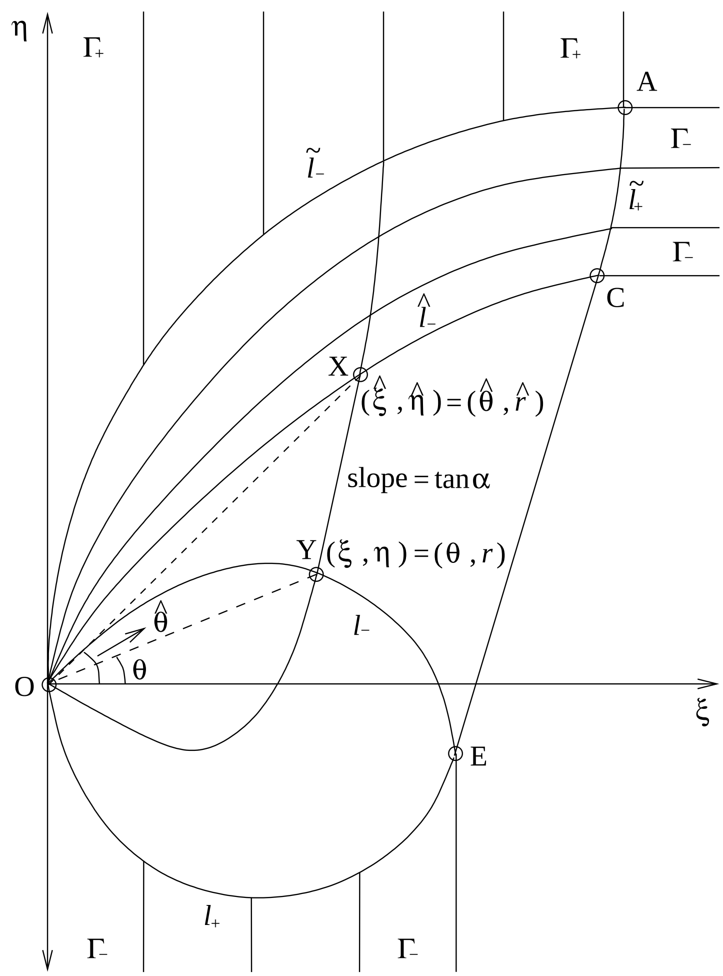


Figure 2.1. Positive characteristic lines

**Proof .** *We have an equation that explains the relation between* $\frac{dP}{d\_\xi}$ *and* $\frac{dP}{d\_\theta}$ *as follows:*

$$-\frac{dP}{d\_\theta} = \left( r\sin\theta + \cos\theta\sqrt{\frac{r^2(r^2-P)}{P}} \right)\frac{dP}{d\_\xi}.$$

*Since* $dP/d\_\theta < 0$ *by Lemma 2.2 and* $r\sin\theta + \cos\theta\sqrt{\frac{r^2(r^2-P)}{P}} > 0$ *for every* $\theta \in [0, \pi/2)$, *we only need to prove that*

$$r\sin\theta + \cos\theta\sqrt{\frac{r^2(r^2-P)}{P}} > 0 \quad \text{for} \quad \text{every} \quad \theta \in [-\theta_m, 0).$$

*For convenience, we replace* $\theta$ *by* $-\theta$. *Assume that there exists* $-\theta_1 \in [-\theta_m, 0)$ *such that*

$$r_1\sin(-\theta_1) + \cos(-\theta_1)\sqrt{\frac{r_1^2(r_1^2 - P(-\theta_1, r_1))}{P(-\theta_1, r_1)}} = 0$$

*which is equivalent to* $P(-\theta_1, r_1) = r_1^2\cos^2\theta_1 =: \xi_1^2$. *From the definition of* $\Lambda_+$, *we have* $\Lambda_+(\xi_1, \eta_1) = \infty$ *which is a contradiction by the fact that the slope of every positive characteristic line in* $\Omega_s$ *is finite. Therefore we have the inequality of this lemma.*□

Now I prove that $l\_$ is concave with respect to $\xi$.

**Lemma 2.4.** *The arc* $\widehat{EO}(l\_)$ *is concave with respect to* $\xi$, *in other words,*

$$\frac{d^2\eta}{d\xi^2} = \frac{d\Lambda_-}{d_-\xi} < 0 \ \text{ along } \ \widehat{EO}(l_-). \tag{2.16}$$

**Proof .** *By (2.6) and Lemma 2.3, this lemma holds.* □

# Chapter 3

# Local Smooth Solutions

Before we consider global solutions in three full rarefaction wave problem, we prove that there exists a local smooth solution to the three Goursat problems (1.8) (1.23), (1.26) (1.27) and (1.30) (1.39) near $E$ in Figure 1.2 and to an initial boundary value problem. Several schemes have been used to prove that there exist local smooth solutions to these kinds of problems. Here I use the compatability conditions of [17].

## 3.1 The Existence of Local Solutions to the Goursat Problem

Let $R(\delta)$ be the closed domain in the polar coordinate system bounded by $l_+^{'}$, $l_-^{'}$, and $l_0^{'} : r = r_m - \delta$, $\delta > 0$. Now we prove the following theorem.

**Theorem 3.1.** *There exist appropriate* $Q_\pm$*,* $R_\pm$*,* $\delta_0$ *and* $c_0$*,* $c_1$ *such that* $Q_\pm$*,* $P_\pm$ *satisfy (1.36), (1.37) and the Goursat problem (1.30), (1.39) has a solution* $W \in C^1(R(\delta_0))$ *which satisfies*

$$||W||_{C^1(R(\delta_0))} \leq c_0, \quad c_1 \leq P \leq r^2 - c_1, \tag{3.1}$$

*where* $\delta_o$*,* $c_0$*, and* $c_1$ *depends on* $P_1$ *and* $P_4$*.*

**Proof .** *we know that the Goursat problem (1.30), (1.39) has a solution if and only if the following compatibility conditions are satisfied:*

$$W_+(r_m) = W_-(r_m) =: W_0 =: (P_0, Q_0, R_0)^T , \tag{3.2}$$

$$\frac{\vec{l_0}^{(0)} W'_+(r_m) - C_0^{(0)}}{\lambda_+^{(0)} - \lambda_0^{(0)}} = \frac{\vec{l_0}^{(0)} W'_-(r_m) - C_0^{(0)}}{\lambda_-^{(0)} - \lambda_0^{(0)}} \tag{3.3}$$

*where* $f^{(0)} = f(-\theta_m, r_m, W_0)$. *From (1.42), (1.43), we have*

$$\frac{dQ_+}{dr} - \sqrt{\frac{P_+}{r^2(r^2 - P_+)}} \cdot \frac{dR_+}{dr} = \frac{P_+^2 Q_+ + r^3 Q_+^2 - 2r^2 P_+ Q_+}{r P_+ (r^2 - P_+)} \tag{3.4}$$

$$\frac{dQ_-}{dr} + \sqrt{\frac{P_-}{r^2(r^2 - P_-)}} \cdot \frac{dR_-}{dr} = \frac{P_-^2 Q_- + r^3 Q_-^2 - 2r^2 P_- Q_-}{r P_- (r^2 - P_-)} \tag{3.5}$$

*From (3.3) we obtain*

$$Q_0 = \frac{1}{2} \left( \frac{r_m^3}{k^2} + f'(r_m, g(r_m)) \right) . \tag{3.6}$$

*Applying (3.2) to (3.6), the initial values for* $Q_\pm$ *are*

$$Q_\pm(r_m) = Q_0 = \frac{r_m^3}{2k^2} + \frac{f'(r_m, g(r_m))}{2}. \tag{3.7}$$

*Next we obtain the following two ordinary differential equations from (1.36), (1.37), (3.4), (3.5):*

$$\frac{dQ_+}{dr} = \frac{6k^2r^2 - 2r^4}{k^2(4k^2 - r^2)} - \frac{1}{r}Q_+ + \frac{8k^4}{r^4(4k^2 - r^2)}Q_+^2, \tag{3.8}$$

$$= a_1(r) + b_1(r)Q_+ + c_1(r)Q_+^2,$$

$$\frac{dQ_-}{dr} = \frac{-2f^2f' + 4r^2ff' - r^3(f')^2}{4rf(r^2 - f)} + \frac{f''}{2}$$

$$+\frac{4f^2 - 8r^2f + r^3f'}{4rf(r^2 - f)}Q_- + \frac{r^2}{2f(r^2 - f)}Q_-^2 \tag{3.9}$$

$$= a_2(r) + b_2(r)Q_- + c_2(r)Q_-^2.$$

*which are Riccati equations and have special solutions, that is,* $\frac{r^3}{2k^2}$ *and* $\frac{f'}{2}$ *respectively. By using the special solutions we have the unique solutions to the initial value problems for (3.8) and (3.9) with the initial value condition (3.7) as follows:*

$$Q_+(r) = \left[\frac{2k^2 - r^2}{2r^3} + \frac{(r^2 - 4k^2)[2\ln r - \ln(4k^2 - r^2) - 8k^2\hat{c}]}{8k^2r}\right]^{-1}$$

$$+\frac{r^3}{2k^2}, \tag{3.10}$$

$$Q_-(r) = \left\{r\left(\frac{r^2 - f}{f}\right)^{3/4}\exp\left(-\frac{1}{2}\int\frac{r}{r^2 - f}dr\right)\right.$$

$$\cdot\int\left[\frac{-r}{2f^{1/4}(r^2 - f)^{7/4}}\exp\left(\frac{1}{2}\int\frac{r}{r^2 - f}dr\right)\right]dr$$

$$+\tilde{c}\cdot r\left(\frac{r^2-f}{f}\right)^{3/4}\exp\left(-\frac{1}{2}\int\frac{r}{r^2-f}dr\right)\Bigg\}^{-1}+\frac{f'}{2}, \tag{3.11}$$

*where*

$$\begin{aligned}\hat{c} &= \frac{f'(r_m)\left(2k(k-\sqrt{P_4})+\sqrt{P_4}(\sqrt{P_4}-2k)\ln\frac{\sqrt{P_4}}{2k-\sqrt{P_4}}\right)}{8k^2\sqrt{P_4}(\sqrt{P_4}-2k)f'(r_m)}\\ &\quad -\frac{8k\sqrt{P_4}\sqrt{2k\sqrt{P_4}}}{8k^2\sqrt{P_4}(\sqrt{P_4}-2k)f'(r_m)}, \qquad (3.12)\\ \tilde{c} &= \frac{2k^2}{r_m^4}\left(\frac{f(r_m)}{r_m^2-f(r_m)}\right)^{3/4}\exp\left(\frac{1}{2}F(r_m)\right)-G(r_m), \qquad (3.13)\\ F(r) &= \int\frac{r}{r^2-f}dr,\\ G(r) &= \int\left[\frac{-r}{2f^{1/4}(r^2-f)^{7/4}}\exp\left(\frac{1}{2}\int\frac{r}{r^2-f}dr\right)\right]dr.\end{aligned}$$

*From (1.36) and (1.37), we have*

$$R_+(r)=\sqrt{4k^2-r^2}\left(\frac{r^3}{k^2}-Q_+(r)\right), \tag{3.14}$$

$$R_-(r)=-\frac{r^2(r^2-f)}{f}\left(f'-Q_-(r)\right). \tag{3.15}$$

*The boundedness of* $||W||_{C^1(R(\delta_0))}$ *follows from (3.10), (3.11), (3.14), (3.15).* □

**Remarks 3.1.** *(i) By (1.36), (1.37), (3.8), (3.9), we have*

$$\frac{dR_+}{dr} = \frac{2r^2(3k^2-r^2)}{k^2\sqrt{4k^2-r^2}} + \frac{4k^2}{r\sqrt{4k^2-r^2}}Q_+ - \frac{8k^4}{r^4\sqrt{4k^2-r^2}}Q_+^2, \tag{3.16}$$

$$\begin{aligned}\frac{dR_-}{dr} &= \frac{2f^2f' - 4r^2ff' + r^3(f')^2}{4f\sqrt{f}\sqrt{r^2-f}} - \frac{f''r\sqrt{r^2-f}}{2\sqrt{f}} \\ &\quad - \frac{r^3f'}{4f\sqrt{f}\sqrt{r^2-f}}Q_- + \frac{r^3}{2f\sqrt{f}\sqrt{r^2-f}}Q_-^2 \qquad (3.17) \\ &= \hat{a}_2(r) + \hat{b}_2(r)Q_- + \hat{c}_2(r)Q_-^2.\end{aligned}$$

*(ii) Differentiating* $\frac{dQ_\pm}{dr}$, $\frac{dR_\pm}{dr}$ *with respect to* $r$*, we have*

$$\begin{aligned}\frac{d^2Q_+}{dr^2} &= \frac{4r(12k^4-8k^2r^2+r^4)}{k^2(4k^2-r^2)^2} + \frac{1}{r^2}Q_+ - \frac{1}{r}\frac{dQ_+}{dr} \\ &\quad - \frac{16k^4(8k^2-3r^2)}{r^5(4k^2-r^2)}Q_+^2 + \frac{16k^4}{r^4(4k^2-r^2)}Q_+\frac{dQ_+}{dr},\end{aligned} \tag{3.18}$$

$$\begin{aligned}\frac{d^2Q_-}{dr^2} &= a_2'(r) + b_2'(r)Q_- + b_2(r)\frac{dQ_-}{dr} + c_2'(r)Q_-^2 + 2c_2(r)Q_-\frac{dQ_-}{dr} \\ &= a_2'(r) + [b_2(r) + 2c_2(r)Q_-]\frac{dQ_-}{dr} + b_2'(r)Q_- + c_2'(r)Q_-^2,\end{aligned} \tag{3.19}$$

$$\begin{aligned}\frac{d^2R_+}{dr^2} &= \frac{r(48k^4-38k^2r^2+6r^4)}{k^2(4k^2-r^2)\sqrt{4k^2-r^2}} - \frac{8k^2(2k^2-r^2)}{r^2(4k^2-r^2)\sqrt{4k^2-r^2}}Q_+ \\ &\quad + \frac{4k^2}{r\sqrt{4k^2-r^2}}\frac{dQ_+}{dr} + \frac{8k^4(16k^2-5r^2)}{r^5(4k^2-r^2)\sqrt{4k^2-r^2}}Q_+^2 \\ &\quad - \frac{16k^4}{r^4\sqrt{4k^2-r^2}}Q_+\frac{dQ_+}{dr},\end{aligned} \tag{3.20}$$

$$\frac{d^2R_-}{dr^2} = \hat{a}_2'(r) + \hat{b}_2'(r)Q_- + \hat{b}_2(r)\frac{dQ_-}{dr} + \hat{c}_2'(r)Q_-^2 + 2\hat{c}_2(r)Q_-\frac{dQ_-}{dr}$$
$$= \hat{a}_2'(r) + [\hat{b}_2(r) + 2\hat{c}_2(r)Q_-]\frac{dQ_-}{dr} + \hat{b}_2'(r)Q_- + \hat{c}_2'(r)Q_-^2. \tag{3.21}$$

Using Theorem 3.1 and the equivalence of the three Goursat problems, we can obtain the following theorem.

**Theorem 3.2.** *There must exist $\delta_0$, $c_0$, $c_1$ depending only on $P_1$ and $P_4$ such that the Goursat problem (1.26), (1.27) has a solution $P \in C^2(R(\delta_0))$ which satisfies*

$$||P||_{C^2(R(\delta_0))} \leq c_0, \quad c_1 \leq P \leq r^2 - c_1, \quad P_r|_{l'_\pm} = Q_\pm, \quad P_\theta|_{l'_\pm} = R_\pm. \tag{3.22}$$

*Thus the Goursat problem (1.8), (1.23) has a solution $P \in C^2(R'(\delta_0))$ which satisfies*

$$||P||_{C^2(R'(\delta_0))} \leq c_0, \quad c_1 \leq P \leq \xi^2 + \eta^2 - c_1, \tag{3.23}$$

*where $R'(\delta_0)$ is the domain in the $(\xi, \eta)$ plane corresponding to $R(\delta_0)$.*

## 3.2 Local Solution to the Initial Boundary Value Problem

Now consider an initial boundary value problem for the system (1.30). Denote the points on the $l'_+$, $l'_-$ by $X = (\theta_l, 2k\cos\theta_l)$, $Y = (\theta_r, g^{-1}(\theta_r))$ respectively in this

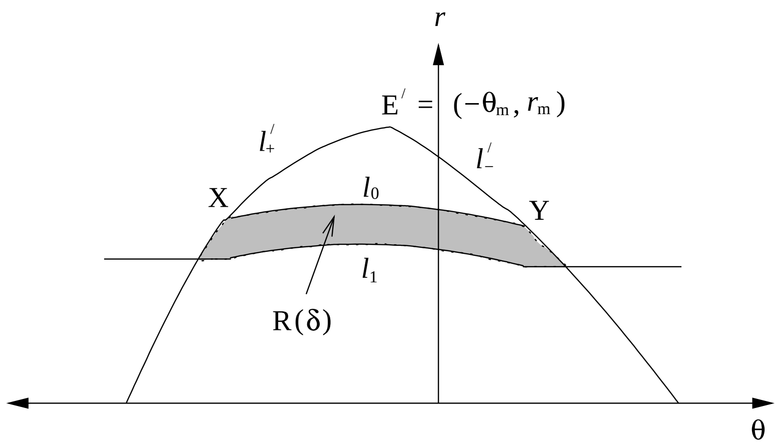


Figure 3.1. Region and boundary in the polar coordinate system

section. Let $l_0 : r = r_0(\theta)$ be a line which stays in the domain bounded by $l'_{\pm}$, the $\theta$ axis and meet with $l'_+$, $l'_-$ at $X$, $Y$ respectively in Figure 3.1. We also suppose that $r_0(\theta) \in C^1([\theta_l, \theta_r])$. Let $W_0(\theta) = (P_L(\theta), Q_0(\theta), R_0(\theta)) \in C^1([\theta_l, \theta_r])$ such that:

(1) there exist a positive constant $P_L$ such that

$$P_L < P_L(\theta) < \min\{r_0^2(\theta)\cos^2\theta, r_0^2(\theta)\cos^2(\theta+\alpha)\}, \quad \theta_l \le \theta \le \theta_r, \tag{3.24}$$

$$\alpha = \tan^{-1}\left(\max\left\{\frac{\xi\eta + \sqrt{f_1(\xi^2+\eta^2-f_1)}}{\xi^2 - f_1} : \xi = r\cos\theta, \eta = r\sin\theta, (\theta, r) \in \widehat{E'X}\right\}\right).$$

(2) The $l_0$ with the data $W_0(\theta)$ has nowhere a characteristic direction, i.e.,

$$\lambda_-^{-1}(\theta, r_0(\theta), W_0(\theta)) < r_0'(\theta) < \lambda_+^{-1}(\theta, r_0(\theta), W_0(\theta)), \quad \theta_l \le \theta \le \theta_r. \tag{3.25}$$

(3) It satisfies the compatibility conditions at $X$ and $Y$:

$$W_0(\theta_l) = W_+(2k\cos\theta_l) =: W_X, \tag{3.26}$$

$$W_0(\theta_r) = W_-(g^{-1}(\theta_r)) =: W_Y, \tag{3.27}$$

$$\frac{\bar{l}_i^{(X)} W_0'(\theta_l) - C_i^{(X)} r_0'(\theta_l)}{1-\lambda_i^{(X)} r_0'(\theta_l)} = \frac{\bar{l}_i^{(X)} W_+'(2k\cos\theta_l) - C_i^{(X)}}{\lambda_+^{(X)} - \lambda_i^{(X)}}, \tag{3.28}$$

$$\frac{\bar{l}_i^{(Y)} W_0'(\theta_r) - C_i^{(Y)} r_0'(\theta_r)}{1-\lambda_i^{(Y)} r_0'(\theta_r)} = \frac{\bar{l}_i^{(Y)} W_-'(g^{-1}(\theta_r)) - C_i^{(Y)}}{\lambda_-^{(Y)} - \lambda_i^{(Y)}}, \tag{3.29}$$

where $h^{(X)} = h(\theta_l, 2k\cos\theta_l, W_X)$ and $h^{(Y)} = f(\theta_r, g^{-1}(\theta_r), W_Y)$ for any function $h(\theta, r, W)$ and $i = 0, -$ in (3.28), $i = +, 0$ in (3.29).

Let $l_1 : r = r_1(\theta)$ be a line defined as follows:

$$l_1 : r = r_1(\theta) = \begin{cases} 2k\cos\theta_l - \delta, & \theta \in (-\infty, \theta_l], \\ r_0(\theta) - \delta, & \theta \in [\theta_l, \theta_r], \\ g^{-1}(\theta_r) - \delta, & \theta \in [\theta_r, \infty). \end{cases} \tag{3.30}$$

Let $R(\delta)$ denote the closed domain bounded by $l'_\pm$, $l_0$, $l_1$. Now we prove the following theorem.

**Theorem 3.3.** *If $l_0$, $W_0$ satisfy the conditions listed above, then there exist appropriate $\delta_0$, $c_0$, and $c_1$ such that the initial boundary value problem for (1.30) with the initial value condition and boundary value condition*

$$W|_{l'_\pm} = W_\pm(r), \quad W|_{l_0} = W_0(\theta) \tag{3.31}$$

*has a solution* $W \in C^1(R(\delta_0))$ *which satisfies*

$$||W||_{C^1(R(\delta_0))} \leq c_0, \quad c_1 \leq P \leq r^2 - c_1, \tag{3.32}$$

*where* $\delta_0$*,* $c_0$*,* $c_1$ *depend only on* $P_1$*,* $P_4$*, and* $P_L$*.*

**Proof .** *By(3.25) there must exist* $\theta_X, \theta_Y \in (-\pi/2, \pi/2)$ *such that*

$$r_0'(\theta_l) < \tan\theta_X < \lambda_+^{-1}(\theta_l, r_0(\theta_l), W_0(\theta_l)), \tag{3.33}$$

$$r_0'(\theta_r) > \tan\theta_Y > \lambda_-^{-1}(\theta_r, r_0(\theta_r), W_0(\theta_r)). \tag{3.34}$$

*We define* $l_X$*,* $l_Y$*,* $l'_X$*,* $l'_Y$ *as follows:*

$$l_X : r - r_0(\theta_l) = (\theta - \theta_l)\tan\theta_X; \quad l'_X || l_X, \quad \mathrm{dist}(\mathrm{l'_X}, \mathrm{l_X}) = \delta;$$

$$l_Y : r - r_0(\theta_r) = (\theta - \theta_r)\tan\theta_Y; \quad l'_Y || l_Y, \quad \mathrm{dist}(\mathrm{l'_Y}, \mathrm{l_Y}) = \delta,$$

*where* $\mathrm{dist}(,)$ *denotes the distance between two lines. Let* $R_X(\delta)$ *(or* $R_Y(\delta)$*) denote the closed domain bounded by* $l'_+$*,* $l_0$*,* $l'_X$ *(or,* $l'_-$*,* $l_0$*,* $l'_Y$*).*

*From (3.33) (3.34) there exist* $\theta'_X$*,* $\theta'_Y$ *such that*

$$\lambda_-^{-1}(\theta_l, r_0(\theta_l), W_0(\theta_l)) < \tan\theta'_X < r'_0(\theta_l),$$

$$r'_0(\theta_r) < tan\theta'_Y < \lambda_+^{-1}(\theta_r, r_0(\theta_r), W_0(\theta_r)).$$

*We define* $l''_X$, $l''_Y$ *as*

$$l''_X : r - r_0(\theta l) = (\theta - \theta_l)\tan\theta'_X,$$

$$l''_Y : r - r_0(\theta r) = (\theta - \theta_r)\tan\theta'_Y.$$

*Let* $R_{XY}(\delta)$ *denote the closed domain bounded by* $l''_X$, $l''_Y$, $l_0$, $l_1$. *By the results in [17] and (3.25) - (3.29) we can prove the following three results.*

*(1) There exist appropriate* $\delta_0$, $c_0$ *and* $c_1$ *depending only on* $P_1$, $P_4$, *and* $P_L$ *such that the boundary value problem for (1.39) with the boundary value condition*

$$W|_{l'_+} = W_+(r), \quad W|_{l_0} = W_0(\theta) \tag{3.35}$$

*has a solution* $W_X \in C^1(R_X(\delta_0))$.

*(2) There exist appropriate* $\delta_0$, $c_0$ *and* $c_1$ *depending only on* $P_1$, $P_4$, *and* $P_L$ *such that the boundary value problem for (1.39) with the boundary value condition*

$$W|_{l'_-} = W_-(r), \quad W|_{l_0} = W_0(\theta) \tag{3.36}$$

*has a solution* $W_Y \in C^1(R_Y(\delta_0))$.

*(3) There exist appropriate* $\delta_0$, $c_0$ *and* $c_1$ *depending only on* $P_1$, $P_4$, *and* $P_L$ *such that the boundary value problem for (1.39) with the initial value condition*

$$W|_{l_0} = W_0(\theta) \tag{3.37}$$

*has a solution* $W_{XY} \in C^1(R_{XY}(\delta_0))$.

*By the uniqueness of solution in [17], we have*

$$W_X = W_{XY}, \quad \text{for} \quad \text{every} \quad (\theta, r) \in R_X(\delta_0) \cap R_{XY}(\delta_0) \tag{3.38}$$

$$W_Y = W_{XY}, \quad \text{for} \quad \text{every} \quad (\theta, r) \in R_Y(\delta_0) \cap R_{XY}(\delta_0) \tag{3.39}$$

*Since* $R(\delta) \subset R_X(\delta_0) \cup R_{XY}(\delta_0) \cup R_Y(\delta_0)$ *when* $\delta$ *is small enough, this theorem is valid.*

□

# Chapter 4

# Some *a priori* Estimates

So far, we treated the existence of local solutions to the three full rarefaction wave problem. In this chapter, we obtain some estimates on $P$ by calculating directional derivatives and the norm in $C^1(\Omega)$ that play an important role in proving the existence of global solutions. In order to obtain these results, we have two definitions as follows:

**Definition 4.1.** *Let $\Omega'$ be a closed domain bounded by $l'_+$, $l'_-$ and $l'_0$, where $l'_0$ is a line which intersects $l'_+$, $l'_-$ and stays in the domain bounded by $l'_+$, $l'_-$ and the $\theta$ axis. Let $P(\theta, r) \in C(\Omega')$, and $0 < P < r^2$. We call $\Omega'$ a determinate domain of $P$ provided that for all $(\theta_0, r_0) \in \Omega'$ the three curves defined by*

$$\frac{d\theta}{dr} = \lambda_i, \quad \theta(r_0) = \theta_0, \quad r \geq r_0, \quad i = 0, \pm \tag{4.1}$$

*intersect only with $l'_+$, $l'_-$, where $\lambda_i$ are given by (1.9).*

**Definition 4.2.** *Let $\Omega$ be a closed domain bounded by $l_+$, $l_-$ and $l_0$, where $l_0$ is a line which intersects $l_+$, $l_-$ and stays in the domain bounded by $l_+$, $l_-$. Let $P(\xi, \eta) \in C(\Omega)$, and $0 < P < \xi^2 + \eta^2$. We call $\Omega$ a determinate domain of $P$ provided that for all $(\xi_0, \eta_0) \in \Omega$ the three curves defined by*

$$\frac{d\eta}{d\xi} = \Lambda_i, \quad \eta(\xi_0) = \eta_0, \quad \xi \geq \xi_0, \quad i = 0, \pm \tag{4.2}$$

*intersect only with $l_+$, $l_-$, where $\Lambda_\pm$ are given by (1.32) and*

$$\Lambda_0 = \frac{\eta}{\xi}.$$

## 4.1 Derivatives and Boundedness of Pressure

Now we can obtain results of directional derivatives for $P$ in Lemma 4.1 and Lemma 4.2.

**Lemma 4.1.** *Let $\Omega'$ be a domain described in Definition 4.1. Let $P \in C^2(\Omega')$ be a solution to the Goursat problems (1.26), (1.27) such that $0 < P < r^2$ and $\Omega'$ is a determinate domain of $P$. Then*

$$\frac{dP}{d_i r} > 0 \quad i = +, 0, -. \tag{4.3}$$

**Proof .** *Let the positive characteristic line be* $\Gamma_+ : r = r_+(\theta)$ *and the negative characteristic line be* $\Gamma_- : r = r_-(\theta)$ *which pass through any point* $(\mu, \nu)$ *in the domain* $\Omega'$*. Since* $\Omega'$ *is a determinate domain of* $P$*,* $\Gamma_+$ *must intersect with* $l'_-$ *at* $(\theta_-, r_-)$ *and* $\Gamma_-$ *must intersect with* $l'_+$ *at* $(\theta_+, r_+)$*. Making use of (2.1) and (1.27) we obtain*

$$\frac{dP}{d_-\theta}(\mu,\nu) = \frac{d}{d\theta} f(g^{-1}(\theta_-), \theta_-) \exp\left(\int_{\theta-}^{\mu} \frac{r^4 P_r \lambda_+}{2P^2} d_+\theta\right) \tag{4.4}$$

$$\frac{dP}{d_+\theta}(\mu,\nu) = -16k^2 \cos^3\theta_+ \sin\theta_+ \exp\left(\int_{\theta+}^{\mu} \frac{r^4 P_r \lambda_-}{2P^2} d_-\theta\right) \tag{4.5}$$

*where* $\int h(\theta, r) d_\pm\theta$ *means* $\int h(\theta, r_\pm(\theta)) d\theta$*. So we have*

$$\frac{dP}{d_-\theta}(\mu,\nu) < 0, \quad \frac{dP}{d_+\theta}(\mu,\nu) > 0 \tag{4.6}$$

*and*

$$P_r = \frac{\lambda_+}{2}\left(\frac{dP}{d_+\theta} - \frac{dP}{d_-\theta}\right) > 0. \tag{4.7}$$

*The definition of* $\lambda_i$ *gives* $\frac{dP}{d_i r} > 0$*,* $i = +, 0, -.\square$

By Lemma 4.1, we can prove that the directional derivatives for $P$ along the characteristic lines in Cartesian coordinates are positive to the Goursat problem (1.8), (1.23) as follows:

**Lemma 4.2.** *Let $P \in C^2(\Omega)$ be a hyperbolic solution to the Goursat problem (1.8), (1.23), that is, $0 < P < \xi^2 + \eta^2$, and $\Omega$ is a determinate domain of $P$. Then*

$$\frac{dP}{d_i\xi} > 0, \quad i = \pm, 0. \tag{4.8}$$

**Proof .** *Suppose that $\Omega$ and $P(\xi,\eta)$ are transformed as $\Omega'$ and $\bar{P}(\theta,r) = P(r\cos\theta, r\sin\theta)$ in the polar coordinates system $(\theta,r)$. Then $\bar{P}(\theta,r)$ is a solution to the Goursat problem (1.26), (1.27) and $\Omega'$ is a determinate domain of $\bar{P}(\theta,r)$. By Lemma 4.1, we have*

$$\bar{P}_\theta + \bar{\Lambda}\bar{P}_r > 0, \quad \bar{P}_r > 0, \quad \bar{P}_\theta - \bar{\Lambda}\bar{P}_r < 0, \tag{4.9}$$

*where $\bar{\Lambda} = \sqrt{r^2(r^2 - \bar{P})/\bar{P}}$. Thus we can get*

$$P_\eta(r\cos\theta + \bar{\Lambda}\sin\theta) + P_\xi(-r\sin\theta + \bar{\Lambda}\cos\theta) > 0, \tag{4.10}$$

$$P_\xi\cos\theta + P_\eta\sin\theta > 0, \tag{4.11}$$

$$P_\xi(r\sin\theta + \bar{\Lambda}\cos\theta) + P_\eta(-r\cos\theta + \bar{\Lambda}\sin\theta) > 0. \tag{4.12}$$

*From (4.11) we have*

$$\frac{dP}{d_0\xi} = P_\xi + \frac{\sin\theta}{\cos\theta} P_\eta > 0 \quad \text{for} \quad \text{every} \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{4.13}$$

*Next we will prove that* $dP/d_+\xi > 0$*. From the left hand sides of (4.9) and (4.10) we have*

$$\frac{dP}{d_+r} = \left(\cos\theta - \sin\theta \frac{\sqrt{P}}{\sqrt{r^2 - P}}\right) \frac{dP}{d_+\xi}. \tag{4.14}$$

*Since* $dP/d_+\xi > 0$ *for every* $\theta \in (-\pi/2, 0]$ *by Lemma 4.1 and (4.14), it suffices to show that* $\cos\theta - \sin\theta\sqrt{P}/\sqrt{r^2 - P} > 0$ *for every* $\theta \in (0, \pi/2)$*. Assume that there exists* $\theta_1 \in (0, \pi/2)$ *such that*

$$\cos\theta_1 - \sin\theta_1 \frac{\sqrt{P(\theta_1, r_1)}}{\sqrt{r_1^2 - P(\theta_1, r_1)}} = 0.$$

*Thus we have* $P(\theta_1, r_1) = r_1^2 \cos^2\theta_1 =: \xi_1^2$ *and* $d\xi/d\eta(\xi_1, \eta_1) = \Lambda_+^{-1}(\xi_1, \eta_1) = 0$*. Making use of nonnegativeness of* $d^2\xi/d\eta^2(\xi_1, \eta_1) = d\Lambda_+^{-1}/d_+\eta(\xi_1, \eta_1)$ *and (2.5), we have*

$$\frac{dP}{d_+\eta}(\xi_1, \eta_1) \leq 0$$

*which contradicts that* $dP/d_+\eta = P_\eta + (-r\sin\theta + \bar{\Lambda}\cos\theta)P_\xi/(r\cos\theta + \bar{\Lambda}\sin\theta) > 0$ *for every* $\theta \in (0, \pi/2)$ *by (4.10). Therefore, we have*

$$\frac{dP}{d_+\xi} > 0. \tag{4.15}$$

*Now we will prove that* $dP/d\_\xi > 0$. *From the left hand sides of (4.9) and (4.12) we have*

$$\frac{dP}{d\_r} = \left(\cos\theta + \sin\theta\frac{\sqrt{P}}{\sqrt{r^2 - P}}\right)\frac{dP}{d_+\xi}. \tag{4.16}$$

*Since* $dP/d\_\xi > 0$ *for every* $\theta \in [0, \pi/2]$ *by Lemma 4.1 and (4.16), it suffices to show that* $\cos\theta + \sin\theta\sqrt{P}/\sqrt{r^2 - P} > 0$ *for every* $\theta \in (-\pi/2, 0)$. *Assume that there exists* $\theta_2 \in (-\pi/2, 0)$ *such that*

$$\cos\theta_2 + \sin\theta_2\frac{\sqrt{P(\theta_2, r_2)}}{\sqrt{r_2^2 - P(\theta_2, r_2)}} = 0.$$

*Thus we have* $P(\theta_2, r_2) = r_2^2\cos^2\theta_2 =: \xi_2^2$ *and* $d\xi/d\eta(\xi_2, \eta_2) = \Lambda_-^{-1}(\xi_2, \eta_2) = 0$. *By (2.7) and (4.12), we have*

$$\frac{d^2\xi}{d\eta^2}(\xi_2, \eta_2) = \frac{d\Lambda_-^{-1}}{d\_\eta}(\xi_2, \eta_2) < 0.$$

*Thus the negative characteristic line* $\Gamma_-$ *has a local maximum at* $(\xi_2, \eta_2)$ *with respect to* $\eta$ *and there exists a point* $(\xi_3, \eta_3)$ *on* $\Gamma_-$ *such that*

$$\frac{d^2\xi}{d\eta^2}(\xi_3, \eta_3) = \frac{\Lambda_-^{-1}}{d_-\eta}(\xi_3, \eta_3) = 0, \quad \xi_3 < 0, \quad \eta_3 < \eta_2.$$

*From (2.7), we have*

$$\frac{dP}{d_-\eta}(\xi_3, \eta_3) = 0$$

*which contradicts that* $dP/d_-\eta = P_\eta + (r\sin\theta + \bar{\Lambda}\cos\theta)P_\xi/(-r\cos\theta + \bar{\Lambda}\sin\theta) < 0$ *for* $\theta \in (-\pi/2, 0)$ *by (4.12). Therefore, we have*

$$\frac{dP}{d_-\xi} > 0. \tag{4.17}$$

□

We can get the estimate on $P$ from Lemma 4.2

**Lemma 4.3.** *If* $\Omega$ *and* $P$ *satisfy the conditions listed in Lemma 4.2, then the characteristic lines* $\Gamma_+$, $\Gamma_-$ *in the domain* $\Omega$ *corresponding to* $P$ *are convex and concave, respectively, and*

$$P \leq \min\{\xi^2, (\xi \cos\alpha - \eta \sin\alpha)^2\}. \tag{4.18}$$

*If* $\Omega'$ *and* $P$ *satisfy the conditions listed in Lemma 4.1, then*

$$P \leq \min\{r^2 \cos^2\theta, r^2 \cos^2(\alpha + \theta)\}. \tag{4.19}$$

**Proof .** *We show the convexity for positive characteristic lines and the concavity for negative characteristic lines. Since* $d\Lambda_+/d_+\xi = k^2/((2k\xi - \xi^2)\sqrt{2k\xi - \xi^2})$ *along* $l_+$, $l_+$ *is convex, i.e.,*

$$\frac{d\Lambda_+}{d_+\xi} > 0 \quad \text{along} \quad l_+. \tag{4.20}$$

*We proved that* $l_-$ *is concave in Lemma 2.4, i.e.,*

$$\frac{d\Lambda_-}{d_-\xi} < 0 \quad \text{along} \quad l_-. \tag{4.21}$$

*By (2.4), (2.6), (4.20), (4.21) we can assert that*

$$\Phi_+ =: \frac{d\Lambda_+}{d_+\xi} > 0, \quad \Phi_- =: \frac{d\Lambda_-}{d_-\xi} < 0. \tag{4.22}$$

*are valid on* $\Omega$*. We only prove that* $\Phi_+ > 0$ *on* $\Omega$*. By the similar method we can prove that* $\Phi_- < 0$ *on* $\Omega$*. Suppose that there exists a point* $X$ *such that* $\Phi_+|_X \leq 0$*. The negative characteristic line* $\Gamma_-$ *through* $X$ *must intersect with* $l_+$ *at some point* $Y$*. By (4.20) we have* $\Phi_+|_Y > 0$*. Thus there must exist a point* $Z$ *on* $\Gamma_-$ *such that* $\Phi_+|_Z = 0$*. By (1.36) we have* $dP/d_+\xi = 0$ *at* $Z$*, which contradicts (4.15). Thus* $\Phi_+ > 0$ *on* $\Omega$*.*

*Since* $\Lambda_+ = h(\xi)$ *on* $l_-(\widehat{EX})$ *and* $\Lambda_- = -\infty$ *on* $l_+$ *where* $h(\xi)$ *is the slopes of the positive characteristic lines along* $l_-(\widehat{EX})$*, (4.22) gives*

$$\Lambda_+ \leq \max_{(\xi,\eta)\in\widehat{EX}} h(\xi) = \cot\alpha, \quad \Lambda_- > -\infty, \quad \text{in} \quad \Omega. \tag{4.23}$$

*By (2.2) we have*

$$\frac{\partial\Lambda_+}{\partial P} = \frac{\Lambda_+^2 + 1}{2\sqrt{P\left(\xi^2+\eta^2-P\right)}} > 0, \tag{4.24}$$

$$\frac{\partial\Lambda_-}{\partial P} = -\frac{\Lambda_-^2 + 1}{2\sqrt{P\left(\xi^2+\eta^2-P\right)}} < 0.$$

*Thus for any fixed point* $(\xi,\eta)$*, the function* $\Lambda_+(P;\xi,\eta)$ *is increasing and* $\Lambda_-(P;\xi,\eta)$ *is decreasing in the interval* $\left[0,\xi^2+\eta^2\right]$*. Since* $\Lambda_+\left((\xi\cos\alpha-\eta\sin\alpha)^2;\xi,\eta\right) = \cot\alpha$ *and* $\Lambda_+ < \cot\alpha$ *in* $\Omega$*,* $P \leq (\xi\cos\alpha-\eta\sin\alpha)^2$*. Since* $\Lambda_-\left(\xi^2;\xi,\eta\right) = -\infty$ *and* $\Lambda_- > -\infty$*,* $P \leq \xi^2$*.*

*We can easily obtain the second assertion of this lemma from the first one.* □

**Remarks 4.1.** *(i) By (4.19), we have*

$$P \leq c_1 r^2 \Leftrightarrow r \geq \frac{\sqrt{P}}{\sqrt{c_1}} \tag{4.25}$$

*where $c_1$ is a constant such that $c_1 < 1$ and depends only on $P_1$ and $P_4$.*

*(ii) For simplicity, we will use $c_0$ to denote any positive constant depending only on $P_1$, $P_4$, $c_1$ and let $P_* = \min\{P(\theta, r)|(\theta, r) \in \Omega'\}$.*

*(iii) We also have an inequality for $\lambda_+$ by using (i) as follows:*

$$\frac{\sqrt{P_*}}{r^2} \leq \frac{\sqrt{P}}{r\sqrt{r^2}} \leq \lambda_+ = \sqrt{\frac{P}{r^2(r^2 - P)}} \leq \sqrt{\frac{c_1 r^2}{r^2(r^2 - c_1 r^2)}} \leq \sqrt{\frac{c_1}{1 - c_1}} \cdot \frac{1}{r}. \tag{4.26}$$

*(iv) We can prove that $|Q_\pm|$, $|R_\pm|$, $|Q'_\pm|$, $|R'_\pm|$, $|Q''_\pm|$, $|R''_\pm|$ are bounded by using the equation for $Q_\pm$, $R_\pm$ from Theorem 3.1, Remarks 3.1 and (i), (ii), (iii) of Remarks 4.1 as follows:*

$$\begin{gathered} |Q_+|,\ |Q'_+|,\ |Q''_+|,\ |R_+|,\ |R'_+|,\ |R''_+| \leq c_0, \\ |Q_-| \leq c_0,\ |Q'_-| \leq c_0 P_*^{-1},\ |Q''_-| \leq c_0 P_*^{-12}, \\ |R_-| \leq c_0 P_*^{-1/2},\ |R'_-| \leq c_0 P_*^{-3/2},\ |R'_-| \leq c_0 P_*^{-5/2} \end{gathered} \tag{4.27}$$

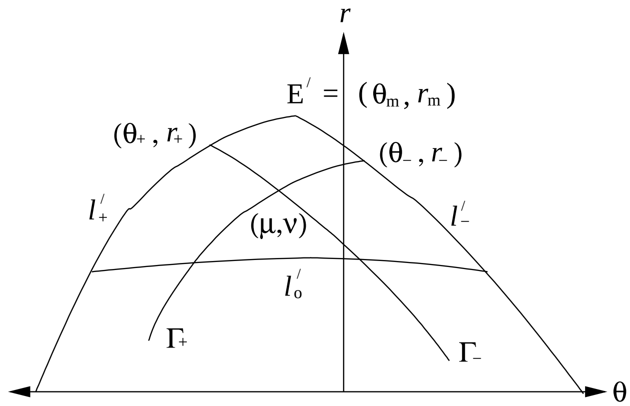


Figure 4.1. Characteristic lines in $\Omega'$

## 4.2 The Boundedness of the Norm in $C^1(\Omega)$ and $C^{1,1}(\Omega)$

Making use of the characteristic separating forms and Remarks 4.1, we can estimate $P_r$, $P_\theta$, $||W||_{C^1(\Omega')}$, $|W|_{C^{1,1}(\Omega')}$ in Lemma 4.4-4.6.

**Lemma 4.4.** *If $\Omega'$, $P$ satisfy the conditions listed in Lemma 4.1, then there must exist a positive constant $c_0$ depending only on $P_1$, $P_4$ such that*

$$c_0 P_* \exp\left(-c_0 P_*^{-2}\right) \leq P_r \leq \frac{c_0}{r} P_*^{-\frac{1}{2}}, \quad |P_\theta| \quad \leq c_0 P_*^{-\frac{1}{2}}. \tag{4.28}$$

**Proof .** *Before we prove the inequalities in this lemma, we introduce two inequalities. Since $P_r > 0$ by (4.7), we have*

$$\frac{r^4 P_r \lambda_+}{2P^2} > 0, \quad \frac{r^4 P_r \lambda_-}{2P^2} < 0.$$

*Thus we have*

$$\exp\left(\int_{\theta-}^{\mu} \frac{r^4 P_r \lambda_+}{2P^2} d_+\theta\right) \leq 1, \quad \exp\left(\int_{\theta+}^{\mu} \frac{r^4 P_r \lambda_-}{2P^2} d_-\theta\right) \leq 1. \tag{4.29}$$

*From (4.4), (4.5), (4.29), Lemma 4.1-4.3 and Remarks 4.1, for all* $(\mu, \nu) \in \Omega'$*, we have*

$$\begin{aligned} |P_\theta(\mu,\nu)| &\leq -\frac{1}{2}\frac{d}{d\theta} f(g^{-1}(\theta_-), \theta_-) \exp\left(\int_{\theta-}^{\mu} \frac{r^4 P_r \lambda_+}{2P^2} d_+\theta\right) \\ &\quad -8k^2 \cos^3\theta_+ \sin\theta_+ \exp\left(\int_{\theta+}^{\mu} \frac{r^4 P_r \lambda_-}{2P^2} d_-\theta\right) \\ &\leq -\frac{1}{2}\frac{d}{d\theta} f(g^{-1}(\theta_-), \theta_-) - 8k^2 \cos^3\theta_+ \sin\theta_+ \\ &\leq c_0 P_*^{-\frac{1}{2}}. \end{aligned} \tag{4.30}$$

*From (4.32), we have*

$$P_r(\mu,\nu) = \lambda_+ |P_\theta(\mu,\nu)| \leq \sqrt{\frac{c_1}{1-c_1}} \cdot \frac{1}{\nu} \cdot c_0 P_*^{-\frac{1}{2}} := \frac{c_0}{\nu} P_*^{-\frac{1}{2}}. \tag{4.31}$$

*By (4.4), (4.5), (4.32) and Remark 4.1, we have*

$$
\begin{aligned}
P_r(\mu,\nu) &= -8\lambda_+ k^2 \cos^3\theta_+ \sin\theta_+ \exp\left(\int_{\theta+}^{\mu} \frac{r^4 P_r \lambda_-}{2P^2} d\_\theta\right) \\
&\quad -\frac{\lambda_+}{2}\frac{d}{d\theta}\tilde{f}(\tilde{g}^{-1}(\theta_-),\theta_-)\exp\left(\int_{\theta-}^{\mu} \frac{r^4 P_r \lambda_+}{2P^2} d_+\theta\right) \\
&\geq \lambda_+ \left[-8k^2 \cos^3\theta_+ \sin\theta_+ \exp\left(\int_{\theta+}^{\mu} \frac{r^4 P_r \lambda_-}{2P^2} d\_\theta\right)\right] \\
&\geq c_0 P_* \left[\exp\left(\int_{\theta+}^{\mu} \frac{r^4 P_r \lambda_-}{2P^2} d\_\theta\right)\right] \\
&\geq c_0 P_* \exp\left(-c_0(P_*)^{-2}\right). \qquad (4.32)
\end{aligned}
$$

□

**Lemma 4.5.** *Let $\Omega'$ be a domain described as in Definition 4.1. Let $W = (P, Q, R) \in C^1(\Omega')$ be a supersonic to the Goursat problem (1.30), (1.39), i.e., $0 < P < r^2$ and $\Omega'$ is a determinate domain of $P$. Then*

$$
||W||_{C^1(\Omega')} \leq c_0(P_*)^{-5} \exp\left(c_0(P_*^{-9/2}\right). \qquad (4.33)
$$

**Proof .** *By the equivalence of the three Goursat problems, we know that $P$ is a hyperbolic $C^2$ solution to the Goursat problem (1.26), (1.27) and $Q = P_r$, $R = P_\theta$. By Using Lemma 4.4 we have*

$$0 < Q \leq \frac{c_0}{r} P_*^{-1/2}, \quad |R| \leq c_0 P_*^{-1/2}. \tag{4.34}$$

*By (1.32)-(1.35) and the definition of* $W$, $C_i$, $B$, *we have*

$$(1, \lambda_{\mp}) \left( \begin{bmatrix} Q_r \\ R_r \end{bmatrix} + \lambda_{\pm} \begin{bmatrix} Q_\theta \\ R_\theta \end{bmatrix} \right) = \bar{B}. \tag{4.35}$$

*Since* $R_r = Q_\theta$, *differentiating (4.35) with respect to* $\theta$ *gives*

$$\begin{aligned} \frac{d}{d_-r}\frac{dR}{d_+r} = G_+ =: \quad & \frac{\partial \bar{B}}{\partial P} R + \frac{\partial \bar{B}}{\partial Q} \frac{1}{2} \left( \frac{dR}{d_+r} + \frac{dR}{d_-r} \right) \\ & + \frac{1}{2\lambda_+} \frac{d\lambda_+}{d_-r} \left( \frac{dR}{d_+r} + \frac{dR}{d_-r} \right), \end{aligned} \tag{4.36}$$

$$\begin{aligned} \frac{d}{d_+r}\frac{dR}{d_-r} = G_- =: \quad & \frac{\partial \bar{B}}{\partial P} R + \frac{\partial \bar{B}}{\partial Q} \frac{1}{2} \left( \frac{dR}{d_+r} + \frac{dR}{d_-r} \right) \\ & + \frac{1}{2\lambda_+} \frac{d\lambda_-}{d_+r} \left( \frac{dR}{d_+r} + \frac{dR}{d_-r} \right). \end{aligned} \tag{4.37}$$

*Through any point* $(\widetilde{\theta}, \widetilde{r}) \in \Omega'$, *we draw the positive(or negative) characteristic line* $\Gamma_+ : \theta = \theta_+(r; \widetilde{\theta}, \widetilde{r})$*(resp.,* $\Gamma_- : \theta = \theta_-(r; \widetilde{\theta}, \widetilde{r})$*) which intersects with* $l'_-$ *(resp.,* $l'_+$*) at* $(\widetilde{\theta}_-, \widetilde{r}_-)$*(resp.,* $(\widetilde{\theta}_+, \widetilde{r}_+)$*). Integrating (4.36), (4.37) along* $\Gamma_+$, $\Gamma_-$ *gives*

$$\frac{dR}{d_+r}(\widetilde{\theta},\widetilde{r}) = \frac{dR}{d_+r}(\widetilde{\theta}_+,\widetilde{r}_+) + \int_{\widetilde{r}_+}^{\widetilde{r}} G_+(\theta,r)d_-r, \tag{4.38}$$

$$\frac{dR}{d_-r}(\widetilde{\theta},\widetilde{r}) = \frac{dR}{d_-r}(\widetilde{\theta}_-,\widetilde{r}_-) + \int_{\widetilde{r}_-}^{\widetilde{r}} G_-(\theta,r)d_+r, \tag{4.39}$$

*where* $\int f(\theta,r)d_\pm r = \int f(\theta_\pm(r;\widetilde{\theta},\widetilde{r}),r)dr$*. By the definition of* $(\widetilde{\theta}_\pm,\widetilde{r}_\pm)$ *we know that*

$$\frac{dR}{d_\pm r}(\widetilde{\theta}_\pm,\widetilde{r}_\pm) = \frac{dR_\pm}{dr}(\widetilde{r}_\pm). \tag{4.40}$$

*Thus we have* $|\frac{dR}{d_\pm r}(\widetilde{\theta}_\pm,\widetilde{r}_\pm)| \le c_0 P_*^{-3/2}$ *by Remarks 4.1(iv). From the definition of* $\bar{B}$*, (4.34) and Remarks 4.1, we have*

$$\begin{aligned}
\left|\frac{\partial\bar{B}}{\partial P}R\right| &\le |R|\left|\frac{\partial\bar{B}}{\partial P}\right| \le c_0 P_*^{-1/2}\left|\frac{rQ(2rPQ-r^3Q-P^2)}{\left[P\left(r^2-P\right)\right]^2}\right| \\
&\le \frac{c_0}{(P_*)^{9/2}}, \qquad (4.41)\\
\left|\frac{\partial\bar{B}}{\partial Q}\right| &= \left|\frac{P^2+2r^3Q-2r^2P}{rP(r^2-P)}\right| \le \frac{c_0}{(P_*)^2}, \qquad (4.42)\\
\left|\frac{1}{\lambda_+}\frac{d\lambda_\pm}{d_\mp r}\right| &\le \left|\frac{1}{\lambda_+}\frac{\partial\lambda_+}{\partial r}\right| + \left|\frac{\partial\lambda_+}{\partial\theta}\right| \\
&= \frac{1}{2}\left|\frac{r^3Q-4r^2P+2P^2}{rP(r^2-P)}\right| + \frac{1}{2}\frac{|R|r\sqrt{r^2-P}}{\sqrt{P}(r^2-P)^2} \\
&\le \frac{c_0}{(P_*)^{7/2}}. \qquad (4.43)
\end{aligned}$$

*Thus from (4.36) - (4.40) we have*

$$\begin{aligned}
\left|\frac{dR}{d_+r}(\widetilde{\theta},\widetilde{r})\right| &\leq \left|\frac{dR}{d_+r}(\widetilde{\theta}_+,\widetilde{r}_+)\right| + \int_{\widetilde{r}}^{\widetilde{r}_+} \left|G_+(\theta,r)\right| d_-r \\
&\leq c_0 P_*^{-3/2} + \int_{\widetilde{r}}^{\widetilde{r}_+} \left|\frac{\partial \bar{B}}{\partial P} R\right| d_-r + \frac{1}{2}\int_{\widetilde{r}}^{\widetilde{r}_+} \left|\frac{\partial \bar{B}}{\partial Q}\right| \left|\frac{dR}{d_+r} + \frac{dR}{d_-r}\right| d_-r \\
&\quad + \frac{1}{2}\int_{\widetilde{r}}^{\widetilde{r}_+} \left|\frac{1}{\lambda_+}\frac{d\lambda_+}{d_-r}\right| \left|\frac{dR}{d_+r} + \frac{dR}{d_-r}\right| d_-r \\
&\leq c_0 (P_*)^{-9/2} + c_0 (P_*)^{-9/2} \int_{\widetilde{r}}^{\widetilde{r}_+} \left(\left|\frac{dR}{d_+r}\right| + \left|\frac{dR}{d_-r}\right|\right) d_-r, \qquad (4.44) \\
\left|\frac{dR}{d_-r}(\widetilde{\theta},\widetilde{r})\right| &\leq c_0 (P_*)^{-9/2} + c_0 (P_*)^{-9/2} \int_{\widetilde{r}}^{\widetilde{r}_-} \left(\left|\frac{dR}{d_+r}\right| + \left|\frac{dR}{d_-r}\right|\right) d_+r. \qquad (4.45)
\end{aligned}$$

*Define $\Omega'(\tau)$, $r^*$, and $V(\tau)$ as follows:*

$$\begin{aligned}
\Omega'(\tau) &= \Omega' \cap \{r = \tau\}, \qquad (4.46) \\
r^* &= \min\left\{ r | (\theta, r) \in \Omega' \right\}, \qquad (4.47) \\
V(\tau) &= \max\left\{ \left\|\frac{dR}{d_+r}\right\|_{C^0(\Omega'(\tau))}, \left\|\frac{dR}{d_-r}\right\|_{C^0(\Omega'(\tau))} \right\}. \qquad (4.48)
\end{aligned}$$

*Let $(\widetilde{\theta},\widetilde{r}) \in \Omega'$ be given. From (4.44) and (4.45), we have*

$$\left|\frac{dR}{d_+r}(\widetilde{\theta},\widetilde{r})\right| \leq c_0 (P_*)^{-9/2} + c_0 (P_*)^{-9/2} \int_{\widetilde{r}}^{\widetilde{r}_+} \left(\left|\frac{dR}{d_+r}\right| + \left|\frac{dR}{d_-r}\right|\right) d_-r$$

$$
\begin{aligned}
&\leq c_0 (P_*)^{-9/2} \\
&\quad + c_0 (P_*)^{-9/2} \int_{\widetilde{r}}^{\sqrt{2k\sqrt{P_4}}} \left( \left\| \frac{dR}{d_+r}(\tau) \right\|_{C^0(\Omega'(\tau))} + \left\| \frac{dR}{d_-r}(\tau) \right\|_{C^0(\Omega'(\tau))} \right) d\tau \\
&\leq c_0 (P_*)^{-9/2} + c_0 (P_*)^{-9/2} \int_{\widetilde{r}}^{\sqrt{2k\sqrt{P_4}}} V(\tau) d\tau, \\
\left| \frac{dR}{d_-r}(\widetilde{\theta}, \widetilde{r}) \right| &\leq c_0 (P_*)^{-9/2} + c_0 (P_*)^{-9/2} \int_{\widetilde{r}}^{\sqrt{2k\sqrt{P_4}}} V(\tau) d\tau.
\end{aligned}
$$

*Since $(\widetilde{\theta}, \widetilde{r})$ is arbitrary in $\Omega'$, we have*

$$
V(r) \leq c_0 (P_*)^{-9/2} + c_0 (P_*)^{-9/2} \int_{\widetilde{r}}^{\sqrt{2k\sqrt{P_4}}} V(\tau) d\tau, \quad r^* \leq r \leq \sqrt{2k\sqrt{P_4}}. \tag{4.49}
$$

*By an appropriate transformation and the Gronwall inequality we get*

$$
\begin{aligned}
V(r) &\leq c_0 (P_*)^{-9/2} \exp\left( \int_{\widetilde{r}}^{\sqrt{2k\sqrt{P_4}}} c_0 (P_*)^{-9/2} d\tau \right) \\
&\leq c_0 (P_*)^{-9/2} \exp\left( c_0 (P_*)^{-9/2} \right), \quad r^* \leq r \leq \sqrt{2k\sqrt{P_4}}.
\end{aligned} \tag{4.50}
$$

*Now we claim that*

$$
\|R\|_{C^1(\Omega')} \leq c_0 (P_*)^{-5} \exp\left( c_0 (P_*)^{-9/2} \right), \tag{4.51}
$$

$$
\|Q\|_{C^1(\Omega')} \leq c_0 (P_*)^{-5} \exp\left( c_0 (P_*)^{-9/2} \right). \tag{4.52}
$$

*From the definition of $V(r)$ and (4.48), we have*

$$|R_r| = \frac{1}{2}\left|\frac{dR}{d_+r} + \frac{dR}{d_-r}\right| \leq \frac{1}{2}\left(\left|\frac{dR}{d_+r}\right| + \left|\frac{dR}{d_-r}\right|\right) \leq V(r), \tag{4.53}$$

$$|R_\theta| = \frac{1}{2\lambda_+}\left|\frac{dR}{d_+r} - \frac{dR}{d_-r}\right| \leq \frac{1}{2\lambda_+}\left(\left|\frac{dR}{d_+r}\right| + \left|\frac{dR}{d_-r}\right|\right) \leq \frac{V(r)}{\lambda_+}. \tag{4.54}$$

*Making use of (4.50), (4.53), and (4.54), we have*

$$\begin{aligned} \sqrt{R_r^2 + \frac{1}{r^2}R_\theta^2} &\leq \sqrt{c_0(P_*)^{-9}[\exp c_0(P_*)^{-9/2}]^2 + c_0(P_*)^{-10}[\exp c_0(P_*)^{-9/2}]^2} \\ &\leq c_0\,(P_*)^{-5}\exp\left(c_0\,(P_*)^{-9/2}\right) \end{aligned} \tag{4.55}$$

*Thus we have*

$$\begin{aligned} \|R\|_{C^1(\Omega')} &= \sup|R| + \sup\sqrt{R_r^2 + \frac{1}{r^2}R_\theta^2} \\ &\leq c_0 P_*^{-1/2} + c_0\,(P_*)^{-5}\exp\left(c_0\,(P_*)^{-9/2}\right) \\ &\leq c_0\,(P_*)^{-5}\exp\left(c_0\,(P_*)^{-9/2}\right). \end{aligned} \tag{4.56}$$

*Now we calculate $||Q||_{C^1(\Omega')}$. By (4.35), $Q_\theta = R_r$, we have*

$$\|Q\|_{C^1(\Omega')} = \sup|Q| + \sup\sqrt{Q_r^2 + \frac{1}{r^2}Q_\theta^2}$$

$$= \sup|Q| + \sup\sqrt{(\lambda_+^2 R_\theta + \bar{B})^2 + \frac{1}{r^2}R_r^2}$$

$$= \sup|Q| + \sup\sqrt{\frac{1}{r^2}R_r^2 + \lambda_+^4 R_\theta^2 + 2\lambda_+^2 \bar{B} R_\theta + \bar{B}^2}.$$

*The similar calculation for estimating* $||R||_{C^1(\Omega')}$ *gives the estimates for* $\frac{1}{r^2}R_r^2$, $\lambda_+^4 R_\theta^2$, $2\lambda_+^2 \bar{B} R_\theta$ *and* $\bar{B}^2$ *as follows:*

$$\frac{1}{r^2}R_r^2 \leq \frac{c}{P}V(r)^2 \leq c_0\,(P_*)^{-10}\left[\exp\left(c_0\,(P_*)^{-9/2}\right)\right]^2, \tag{4.57}$$

$$\begin{aligned} \lambda_+^4 R_\theta^2 &\leq \lambda_+^4 \cdot \frac{1}{\lambda_+^2}V(r)^2 = \lambda_+^2 V(r)^2 = \frac{P}{r^2(r^2-P)}V(r)^2 \\ &\leq c_0\,(P_*)^{-10}\left[\exp\left(c_0\,(P_*)^{-9/2}\right)\right]^2, \end{aligned} \tag{4.58}$$

$$\begin{aligned} \lambda_+^2 \bar{B} R_\theta &\leq \lambda_+^2|\bar{B}|\frac{1}{\lambda_+}V(r) = \lambda_+|\bar{B}|V(r) = \lambda_+\left|\frac{P^2Q + r^3Q^2 - 2r^2PQ}{rP(r^2-P)}\right|V(r) \\ &\leq c_0\,(P_*)^{-8}\left[\exp\left(c_0\,(P_*)^{-9/2}\right)\right]^2, \end{aligned} \tag{4.59}$$

$$\bar{B}^2 \leq \left(\frac{c_0}{P_*^2}\right)^2 \leq c_0\,(P_*)^{-4}\left[\exp\left(c_0\,(P_*)^{-9/2}\right)\right]^2. \tag{4.60}$$

*From (4.57) - (4.60), we have*

$$\begin{aligned}\|Q\|_{C^1(\Omega')} &\leq \frac{c_0}{r} P_*^{-1/2} + c_0 (P_*)^{-5} \exp\left(c_0 (P_*)^{-9/2}\right) \\ &\leq c_0 P_*^{-1} + c_0 (P_*)^{-5} \exp\left(c_0 (P_*)^{-9/2}\right) \\ &\leq c_0 (P_*)^{-5} \exp\left(c_0 (P_*)^{-9/2}\right). \qquad (4.61)\end{aligned}$$

*Next we estimate* $\|P\|_{C^1(\Omega')}$. *By Remarks 4.1 and Lemma 4.4, we have*

$$\begin{aligned}\|P\|_{C^1(\Omega')} &= \sup|P| + \sup\sqrt{P_r^2 + \frac{1}{r^2}P_\theta^2} \\ &\leq c_0 P_*^{-1} \leq c_0 (P_*)^{-1} \exp\left(c_0 (P_*)^{-9/2}\right). \qquad (4.62)\end{aligned}$$

*Therefore we have*

$$\begin{aligned}\|W\|_{C^1(\Omega')} &= \max\left\{\|P\|_{C^1(\Omega')}, \|Q\|_{C^1(\Omega')}, \|R\|_{C^1(\Omega')}\right\} \\ &\leq c_0 (P_*)^{-5} \exp\left(c_0 (P_*)^{-9/2}\right).\end{aligned}$$

□

**Lemma 4.6.** *If* $\Omega'$, $W$, $P$ *satisfy the conditions listed in Lemma 4.5, then*

$$|W|_{C^{1,1}(\Omega')} \leq c_0(P_*)^{-25/2} \exp\left(c_0(P_*)^{-9/2}\right) \exp\left(c_0(P_*)^{-5/2} \exp\left(c_0(P_*)^{-2}\right)\right) \quad (4.63)$$

**Proof .** *First we can prove that*

$$|P|_{C^{1,1}(\Omega')} \leq c_0(P_*)^{-5} \exp\left(c_0(P_*)^{-9/2}\right). \quad (4.64)$$

*by using the definition of* $|P|_{C^{1,1}(\Omega')}$, *the Mean Value Theorem, and Lemma 4.5.*

*Next We estimate* $|Q|_{C^{1,1}(\Omega')}$ *and* $|R|_{C^{1,1}(\Omega')}$. *We first give an estimate on* $|\frac{dR}{d_+r}|_{C^{0,1}(\Omega')}$. *Set*

$$D(\tau) = \Omega' \cap \{r \leq \tau\},$$
$$w(\tau) = \max\left\{\left|\frac{dR}{d_+r}\right|_{C^{0,1}(D(\tau))}, \left|\frac{dR}{d_-r}\right|_{C^{0,1}(D(\tau))}\right\}. \quad (4.65)$$

*From any two points* $(\tilde{\theta}, \tilde{r})$, $(\hat{\theta}, \hat{r}) \in D(\tau)$ *we draw the negative characteristic lines* $\tilde{\Gamma}_-$ : $\theta = \theta(r; \tilde{\theta}, \tilde{r})$, $\widehat{\Gamma}_- : \theta = \theta(r; \hat{\theta}, \hat{r})$. *The* $\tilde{\Gamma}_-$, $\widehat{\Gamma}_-$ *must intersect* $l'_+$ *at* $(\tilde{\theta}_+, \tilde{r}_+)$, $(\hat{\theta}_+, \hat{r}_+)$ *respectively. Set*

$$\rho\left((\tilde{\theta},\tilde{r}),(\hat{\theta},\hat{r})\right)=\sqrt{(\tilde{\theta}-\hat{\theta})^2+(\tilde{r}-\hat{r})^2}. \tag{4.66}$$

*Making use of (4.38) we have*

$$\begin{aligned}
&\left|\frac{dR}{d_+r}(\tilde{\theta},\tilde{r})-\frac{dR}{d_+r}(\hat{\theta},\hat{r})\right|\\
&\leq\left|\frac{dR}{d_+r}(\tilde{\theta}_+,\tilde{r}_+)-\frac{dR}{d_+r}(\hat{\theta}_+,\hat{r}_+)\right|\\
&\quad+\left|\int_{\tilde{r}_+}^{\tilde{r}}G_+\left(\theta(r;\tilde{\theta},\tilde{r}),r\right)dr-\int_{\hat{r}_+}^{\hat{r}}G_+\left(\theta(r;\hat{\theta},\hat{r}),r\right)dr\right|\\
&\leq\left|\int_{\hat{r}}^{\tilde{r}}G_+\left(\theta(r;\hat{\theta},\hat{r}),r\right)dr\right|+\left|\int_{\hat{r}_+}^{\tilde{r}_+}G_+\left(\theta(r;\hat{\theta},\hat{r}),r\right)dr\right|\\
&\quad+\left|\int_{\tilde{r}_+}^{\tilde{r}}\left\{G_+\left(\theta(r;\tilde{\theta},\tilde{r}),r\right)-G_+\left(\theta(r;\hat{\theta},\hat{r}),r\right)\right\}dr\right|\\
&\quad+\left|\frac{dR}{d_+r}(\tilde{\theta}_+,\tilde{r}_+)-\frac{dR}{d_+r}(\hat{\theta}_+,\hat{r}_+)\right|\\
&=:T_1+T_2+T_3+T_4,
\end{aligned} \tag{4.67}$$

*where, without loss of generality, we suppose* $\tilde{r}\geq\hat{r}$, $\tilde{r}_+\leq\hat{r}_+$.

*By the definition of* $\theta(r;\tilde{\theta},\tilde{r})$*, we have* $\frac{d\theta(r;\tilde{\theta},\tilde{r})}{dr}=\lambda_-(\theta(r;\tilde{\theta},\tilde{r}),r)$*. Differentiating it with respect to* $\tilde{\theta}$ *and integrating it, we have*

$$\frac{d}{dr}\frac{\partial\theta(r;\tilde{\theta},\tilde{r})}{\partial\tilde{\theta}} = \frac{\partial\lambda_-}{\partial\theta}(\theta(r;\tilde{\theta},\tilde{r}),r)\frac{\partial\theta(r;\tilde{\theta},\tilde{r})}{\partial\tilde{\theta}}, \tag{4.68}$$

$$\frac{\partial\theta(r;\tilde{\theta},\tilde{r})}{\partial\tilde{\theta}} = \exp\left(\int_{\tilde{r}}^{r}\frac{\partial\lambda_-}{\partial\theta}(\theta(r;\tilde{\theta},\tilde{r}),r)dr\right). \tag{4.69}$$

*From the definition of* $\theta(r;\tilde{\theta},\tilde{r})$*, we can also derive*

$$\frac{\partial\theta(r;\tilde{\theta},\tilde{r})}{\partial\tilde{r}} = -\lambda_-(\tilde{\theta},\tilde{r},w(\tilde{\theta},\tilde{r}))\exp\left(\int_{\tilde{r}}^{r}\frac{\partial\lambda_-}{\partial\theta}(\theta(r;\tilde{\theta},\tilde{r}),r)dr\right).$$

*Hence we have*

$$\max\left\{\left|\frac{\partial\theta(r;\tilde{\theta},\tilde{r})}{\partial\tilde{\theta}}\right|,\left|\frac{\partial\theta(r;\tilde{\theta},\tilde{r})}{\partial\tilde{r}}\right|\right\} \leq c_0P_*^{-1/2}\exp\left(c_0(P_*)^{-2}\right). \tag{4.70}$$

*When* $r=\tilde{r}_+$*, we get*

$$\max\left\{\left|\frac{\partial\tilde{\theta}_+}{\partial\tilde{\theta}}\right|,\left|\frac{\partial\tilde{\theta}_+}{\partial\tilde{r}}\right|\right\} \leq c_0P_*^{-1/2}\exp\left(c_0(P_*)^{-2}\right). \tag{4.71}$$

*By the Remarks 4.1(iv) and Lemma 4.5, we can obtain*

$$|R''_\pm(r)| \leq c_0, \quad |G_+| \leq c_0P_*^{-9}\exp\left(c_0(P_*)^{-9/2}\right), \tag{4.72}$$

$$|G_+(\theta_1, r) - G_+(\theta_2, r)|$$

$$\leq \left\{ c_0 P_*^{-12} \exp\left(c_0 (P_*)^{-9/2}\right) + c_0 P_*^{-2} w(r) \right\} \cdot |\theta_1 - \theta_2|. \tag{4.73}$$

*Now we will estimate $T_1$, $T_2$, $T_3$, and $T_4$. From (4.72), we have*

$$\begin{aligned} T_1 &\leq \left| \int_{\hat{r}}^{\tilde{r}} G_+(\theta(r; \hat{\theta}, \hat{r}), r) dr \right| \\ &\leq \int_{\hat{r}}^{\tilde{r}} \left| G_+(\theta(r; \hat{\theta}, \hat{r}), r) \right| dr \\ &\leq \int_{\hat{r}}^{\tilde{r}} c_0 P_*^{-9} \exp\left(c_0 (P_*)^{-9/2}\right) dr \\ &\leq c_0 P_*^{-9} \exp\left(c_0 (P_*)^{-9/2}\right) |\tilde{r} - \hat{r}| \\ &\leq c_0 P_*^{-9} \exp\left(c_0 (P_*)^{-9/2}\right) \rho\left((\tilde{\theta}, \tilde{r}), (\hat{\theta}, \hat{r})\right). \end{aligned} \tag{4.74}$$

*By making use of the definition of $l'_+ : r = 2k\cos\theta$, the Mean Value Theorem, (4.71) and (4.72), we have*

$$\begin{aligned} T_2 &\leq \int_{\hat{r}_+}^{\tilde{r}_+} \left| G_+\left(\theta(r; \hat{\theta}, \hat{r}), r\right) \right| dr \\ &\leq c_0 P_*^{-9} \exp\left(c_0 (P_*)^{-9/2}\right) |\hat{r}_+ - \tilde{r}_+| \\ &\leq c_0 P_*^{-9} \exp\left(c_0 (P_*)^{-9/2}\right) |\hat{\theta}_+ - \tilde{\theta}_+| \\ &\leq c_0 P_*^{-9} \exp\left(c_0 (P_*)^{-9/2}\right) \cdot c_0 P_*^{-1/2} \exp\left(c_0 (P_*)^{-2}\right) \rho\left((\tilde{\theta}, \tilde{r}), (\hat{\theta}, \hat{r})\right) \end{aligned}$$

$$\leq \quad c_0(P_*)^{-19/2}\exp\left(c_0(P_*)^{-9/2}\right)\rho\left((\tilde{\theta},\tilde{r}),(\hat{\theta},\hat{r})\right). \tag{4.75}$$

*From (4.70), (4.73), we have*

$$\begin{aligned}
T_3 &\leq \int_{\tilde{r}_+}^{\tilde{r}} \left|G_+\left(\theta(r;\tilde{\theta},\tilde{r}),r\right) - G_+\left(\theta(r;\hat{\theta},\hat{r}),r\right)\right| dr \\
&\leq \int_{\tilde{r}_+}^{\tilde{r}} \left\{c_0 P_*^{-12}\exp\left(c_0(P_*)^{-9/2}\right) + c_0 P_*^{-2} w(r)\right\} \cdot |\theta(r;\tilde{\theta},\tilde{r}) - \theta(r;\hat{\theta},\hat{r})| dr \\
&\leq c_0(P_*)^{-5/2}\exp\left(c_0(P_*)^{-2}\right)\rho\left((\tilde{\theta},\tilde{r}),(\hat{\theta},\hat{r})\right)\int_{\tilde{r}_+}^{\tilde{r}} w(r)dr \\
&\quad + c_0(P_*)^{-25/2}\exp\left(c_0(P_*)^{-9/2}\right)\rho\left((\tilde{\theta},\tilde{r}),(\hat{\theta},\hat{r})\right).
\end{aligned} \tag{4.76}$$

*By the definition of $R_\pm$, (4.71) and $|R''_+| \leq c_0 P_*^{-5/2}$, we have*

$$\begin{aligned}
T_4 &= \left|\frac{dR}{d_+ r}(\tilde{\theta}_+,\tilde{r}_+) - \frac{dR}{d_+ r}(\hat{\theta}_+,\hat{r}_+)\right| = |R'_+(\tilde{r}_+) - R'_+(\hat{r}_+)| \\
&\leq |R''_+||\tilde{r}_+ - \hat{r}_+| \leq c_0 P_*^{-5/2}|\tilde{r}_+ - \hat{r}_+| \leq c_0 P_*^{-5/2}|\tilde{\theta}_+ - \hat{\theta}_+| \\
&\leq c_0(P_*)^{-3}\exp\left(c_0(P_*)^{-2}\right)\rho\left((\tilde{\theta},\tilde{r}),(\hat{\theta},\hat{r})\right).
\end{aligned} \tag{4.77}$$

*Making use of (4.74)- (4.77) and (4.67), we obtain*

$$\left|\frac{dR}{d_+ r}\right|_{C^{0,1}(D(t))} \leq \sup \frac{\left|\frac{dR}{d_+ r}(\tilde{\theta},\tilde{r}) - \frac{dR}{d_+ r}(\hat{\theta},\hat{r})\right|}{\rho\left((\tilde{\theta},\tilde{r}),(\hat{\theta},\hat{r})\right)}$$

$$\leq \quad (T_1+T_2+T_3+T_4)\frac{1}{\rho\left((\tilde{\theta},\tilde{r}),(\hat{\theta},\hat{r})\right)}$$

$$\leq \quad c_0(P_*)^{-5/2}\exp\left(c_0(P_*)^{-2}\right)\int_t^{r_0} w(r)dr$$

$$+c_0(P_*)^{-25/2}\exp\left(c_0(P_*)^{-9/2}\right). \tag{4.78}$$

*By using the same calculation on* $\frac{dR}{d\_r}$*, we know that* $\frac{dR}{d\_r}$ *also satisfies (4.78). Hence*

$$\begin{aligned} w(t) &= \max\left\{\left|\frac{dR}{d_+r}\right|_{C^{0,1}(D(t))}, \left|\frac{dR}{d_-r}\right|_{C^{0,1}(D(t))}\right\} \\ &\leq c_0(P_*)^{-5/2}\exp\left(c_0(P_*)^{-2}\right)\int_t^{r_0} w(r)dr \\ &\quad +c_0(P_*)^{-25/2}\exp\left(c_0(P_*)^{-9/2}\right). \end{aligned} \tag{4.79}$$

*By the Gronwall inequality, we have*

$$w(r) \leq c_0(P_*)^{-25/2}\exp\left(c_0(P_*)^{-9/2}\right) exp\Big(c_0(P_*)^{-5/2}\exp\left(c_0(P_*)^{-2}\right)\Big),$$

$$r^* \leq r \leq r_0,$$

*which combined with (4.35) and the definition of* $w(r)$*, yield the results of this lemma.*

□

**Theorem 4.1.** *If* $\Omega'$*,* $W$*,* $P$ *satisfy the conditions listed in Lemma 4.5, then*

$$||W||_{C^{1,1}(\Omega')} \leq c_0(P_*)^{-25/2} \exp\left(c_0(P_*)^{-9/2}\right) \exp\left(c_0(P_*)^{-5/2} \exp\left(c_0(P_*)^{-2}\right)\right) \quad (4.80)$$

# Chapter 5

# Global Smooth Solutions for Three Full Rarefaction Waves

In order to prove the existence for global solutions for three full rarefaction wave problem, we first prove a lemma.

**Lemma 5.1.** *For every* $r : 0 < r \leq r_m = \sqrt{2k\sqrt{P_4}}$*, there is a smooth curve* $l_r : r = r(\theta) \in C^1$ *such that*

*(1) (1.30), (1.39) has a supersonic solution* $W = (P, Q, R) \in C^1(\Omega_r)$*, i.e.,* $0 < P < r^2$*,*

*(2)* $P$ *takes a fixed value on* $l_r$*,*

*(3)* $\Omega_r$ *is a determinate domain of* $P$*,*

*where* $l_r$ *stays in the domain bounded by* $l'_+$*,* $l'_-$ *at* $(\arccos \frac{r}{2k}, r)$*,* $(g(r), r)$ *respectively, and* $\Omega_r$ *is the closed domain bounded by* $l'_+$*,* $l'_-$*, and* $l_r$*.*

**Proof .** *Let* $S$ *be the set which consists of all numbers of* $(0, r_m]$ *which satisfy (1),(2),(3), listed in this lemma. By Lemma 4.1, we have* $P_r > 0$*, from which it follows that, if* $\tilde{r}_m \in S$*, then* $[\tilde{r}_m, r_m] \subset S$*. So we need only prove that* $S$ *is not empty set and* $\inf S = 0$*. By making use of Theorem 3.1,* $[r_m - \delta, r_m] \subset S$ *if* $\delta$ *is small enough. We need only to prove that* $\inf S = 0$*. We use the contradiction argument. Suppose that* $\inf S = \hat{r} > 0$*. The following argument is divided into two steps. In step 1 we will prove* $\hat{r} \in S$*. In step 2 we will prove that there exists a small* $\delta > 0$ *such that* $[\hat{r} - \delta, \hat{r}] \subset S$*.*

*Step 1. By the definition of infimum for* $\hat{r}$ *there exists a monotone decreasing sequence*

$\{r_i\}_{i=1}^{\infty} \subset S$ *satisfying* $\lim_{i\to\infty} r_i = \hat{r}$. *Then for every* $r_i$, *there exists a smooth curve* $l_i : r = r_i(\theta) \in C^1$ *such that*

*(1) (1.30), (1.39) has a supersonic solution* $W = (P, Q, R) \in C^1(\Omega_i)$, *i.e.,* $0 < P < r^2$,

*(2)* $P$ *takes a fixed value on* $l_i$,

*(3)* $\Omega_i$ *is a determinate domain of* $P$,

*where* $l_i$ *stays in the domain bounded by* $l'_+$, $l'_-$ *and the* $\theta$ *axis and intersects with* $l'_+$, $l'_-$ *at* $(\arccos \frac{r_i}{2k}, r_i)$, $(g(r_i), r_i)$ *respectively, and* $\Omega_i$ *is the closed domain bounded by* $l'_+$, $l'_-$, *and* $l_i$.

*We know that* $l_j$ *is below* $l_i$ *if* $i < j$ *by Lemma 4.1 because (1.30), (1.39) has the unique supersonic solution in* $\Omega_i$. *Thus* $\{r_i(\theta)\}_{i=1}^{\infty}$ *is a monotone decreasing sequence. Hence there exists a function* $\hat{r}(\theta)$ *such that*

$$\lim_{i\to\infty} r_i(\theta) = \hat{r}(\theta), \quad \text{for} \quad \text{every} \quad \theta \in \left[\arccos \frac{\hat{r}}{2k}, g(\hat{r})\right]. \tag{5.1}$$

*Denote by* $\hat{l}$ *the graph of* $r = \hat{r}(\theta)$ *and by* $\hat{\Omega}$ *the closed domain bounded by* $l'_+$, $l'_-$ *and* $\hat{l}$. *Then (1.30), (1.39) has a* $C^1$ *solution on* $\hat{\Omega} - \hat{l}$ *and* $\hat{\Omega} - \hat{l}$ *is a determinate domain of* $P$.

*By Lemma 4.1-4.6 for all* $i : 1 \le i < \infty$, *there exists a positive constant* $c_0$ *depending on* $P_1$ *and* $P_4$ *such that*

$$P_* = \frac{r_i^4}{4k^2} \quad \le \quad P \le \min\left\{r^2 \cos\theta, r^2 \cos(\alpha + \theta)\right\}$$

$$\begin{aligned}
c_0 r_i^4 \exp(-c_0 r_i^{-6}) &\leq P_r \leq c_0 r_i^{-3}, \\
P_\theta &\leq c_0 r_i^{-2}, \\
||W||_{C^{1,1}(\Omega_i)} &\leq c_0 r_i^{-50} \exp(-c_0 r_i^{-18}) \exp\Big(c_0 r_i^{-10} \exp(-c_0 r_i^{-8})\Big),
\end{aligned}$$

*hold on* $\Omega_i$.

*Thus there exist two positive constants* $c_0$ *and* $c_1$ *depending only on* $\hat{r}$, $P_1$, $P_4$ *such that*

$$\begin{aligned}
c_1 \leq P &\leq \min\Big\{r^2 \cos\theta, r^2 \cos(\alpha+\theta)\Big\} \\
c_1 \leq P_r &\leq c_0, \quad P_\theta \leq c_0, \\
||W||_{C^{1,1}(\Omega_i)} &\leq c_0 \qquad (5.2)
\end{aligned}$$

*hold on* $\hat{\Omega} - \hat{l}$.

*Since* $P$ *is constant along* $l_i$, *we have*

$$P_\theta + r_i'(\theta) P_r = 0 \quad \text{along} \quad l_i. \qquad (5.3)$$

*Thus by (5.2) and the Mean Value Theorem, there exists a positive constant* $c_0$ *depending only on* $\hat{r}$, $P_1$, $P_4$ *such that*

$$||r_i||_{C^{1,1}} \leq c_0. \tag{5.4}$$

*Hence we have* $\hat{r}(\theta) \in C^1$ *by the Arzela-Ascoli Theorem.*

*Define* $W_i(\theta) = W|_{l_i} := (P_i, Q_i, R_i)$. *Noting (5.2) and (5.4), there exists a positive constant* $c_0$ *depending on* $\hat{r}$, $P_1$, $P_4$ *such that*

$$||W_i||_{C^{1,1}} \leq c_0. \tag{5.5}$$

*By using the Arzela-Ascoli Theorem again, there exists a function* $\hat{W}(\theta) = (\hat{P}(\theta), \hat{Q}(\theta), \hat{R}(\theta)) \in C^1$ *such that*

$$\lim_{i\to\infty} W_i(\theta) = \hat{W}(\theta), \quad \hat{P}(\theta) = \frac{\hat{r}^4}{4k^2}. \tag{5.6}$$

*Now we extend* $W(\theta, r)$ *to* $\hat{\Omega}$ *by letting* $W|_{\hat{l}} = \hat{W}(\theta)$. *Thus the Goursat problem (1.30), (1.39) has a* $C^1$ *solution on* $\hat{\Omega}$. *Since* $P$ *takes a fixed value along* $\hat{l}$, *we have*

$$P_\theta + \hat{r}'(\theta)P_r = 0 \quad \text{along} \quad \hat{l}. \tag{5.7}$$

*Since* $|P_\theta + \lambda_\pm^{-1} P_r| > 0$ *holds on* $\hat{\Omega} - \hat{l}$ *where* $c_1$ *depends only on* $\hat{r}$, $P_1$, $P_4$ *by Theorem 2.2, (4.4) and (4.5), we can get*

$$P_\theta + \lambda_\pm^{-1} P_r \neq 0 \quad \text{along} \quad \hat{l}. \tag{5.8}$$

*So*

$$\hat{r}'(\theta) \neq \lambda_\pm^{-1} \quad \text{along} \quad \hat{l}. \tag{5.9}$$

*From the above results we can find* $\hat{r} \in S$.

*Step 2. Since* $\hat{r} \in S$, *by Theorem 3.3, the initial boundary value problem for (1.30) with the initial value condition and boundary condition*

$$W|_{l'_\pm} = W_\pm(r), \quad W|_{\hat{l}} = \hat{W}(\theta), \tag{5.10}$$

*has a local* $C^1$ *solution in a closed domain (we denote it by R). In fact, we can check that the compatibility conditions (3.26)- (3.29) and the conditions listed in Theorem 3.3 are satisfied, since* $\hat{r} \in S$. *Let us denote the limits of* $(P_r, Q_r, R_r, P_\theta, Q_\theta, R_\theta)$ *on the upper and the lower side of the line* $\hat{l}$ *by* $(P_r^u, Q_r^u, R_r^u, P_\theta^u, Q_\theta^u, R_\theta^u)(\theta)$ *and* $(P_r^d, Q_r^d, R_r^d, P_\theta^d, Q_\theta^d, R_\theta^d)(\theta)$ *respectively. Both of these vector functions are solutions to the system*

$$A(\theta)u(\theta) = B(\theta), \tag{5.11}$$

*where* $u(\theta) = ((u_1, u_2, u_3, u_4, u_5, u_6)(\theta))^T$ *is the unknown function,*

$$A(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{-\hat{P}}{\hat{r}^2(\theta)(\hat{r}^2(\theta)-\hat{P})} \\ 0 & 0 & 1 & 0 & -1 & 0 \\ \hat{r}'(\theta) & 0 & 0 & 1 & 0 & 0 \\ 0 & \hat{r}'(\theta) & 0 & 0 & 1 & 0 \\ 0 & 0 & \hat{r}'(\theta) & 0 & 0 & 1 \end{pmatrix}, B(\theta) = \begin{pmatrix} \hat{Q}(\theta) \\ \hat{B}(\theta) \\ 0 \\ 0 \\ \hat{Q}'(\theta) \\ \hat{R}'(\theta) \end{pmatrix}, \tag{5.12}$$

*and* $\hat{Q}(\theta) = Q(\theta, \hat{r}(\theta))$, $\hat{B}(\theta) = B(\theta, \hat{r}(\theta))$. *By (5.9) we have* $det(A(\theta)) \neq 0$ *along* $\hat{r}$. *Thus the system has only on solution and*

$$(P_r^u, Q_r^u, R_r^u, P_\theta^u, Q_\theta^u, R_\theta^u)(\theta) = (P_r^d, Q_r^d, R_r^d, P_\theta^d, Q_\theta^d, R_\theta^d)(\theta) \quad \text{along} \quad \hat{l}. \tag{5.13}$$

*Thus the Goursat problem (1.30), (1.39) has a* $C^1$ *solution on* $\hat{\Omega} \cup R$ *and* $\hat{\Omega} \cup R$ *is a determinate domain of* $P$. *By Lemma 4.1 we have* $P_r > 0$ *in* $\hat{\Omega} \cup R$. *By the Implicit Function Theorem, if* $\delta > 0$ *is small enough, then the equation* $P(\theta, r) = (\hat{r} - \delta)^4/(4k^2)$ *defines a* $C^1$ *function* $r = \tilde{r}(\theta)$ *whose graph (we denote it by* $\tilde{l}$*) lies in* $R$. *By the same*

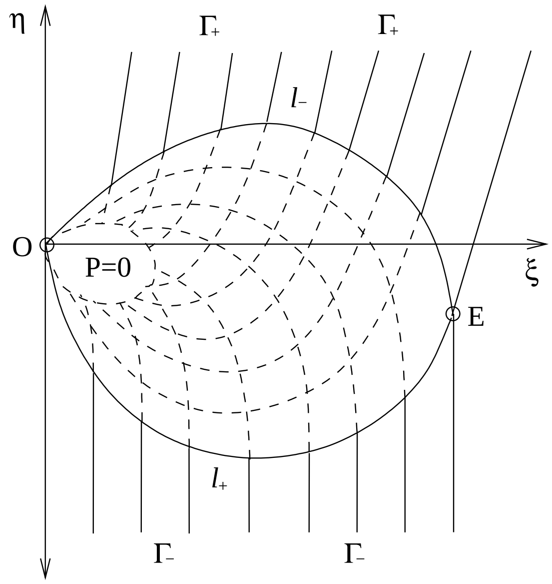


Figure 5.1. Solutions with a vacuum bubble

*method as for* $\hat{r}$*, we can prove that* $\hat{r} - \delta \in S$ *which is a contradiction. Thus* $\inf S = 0$ *and the lemma is valid.* □

From Lemma 5.1, we can prove Theorem 5.1 as follow:

**Theorem 5.1.** *The Goursat problem (1.8), (1.23) - (1.25) has a global solution* $P \in C^2(\Omega - l_*)$ *which satisfies*

$$P(\xi, \eta) \longrightarrow 0 \quad \text{as}(\xi, \eta) \longrightarrow \text{a point on } l_*, \tag{5.14}$$

*where* $l_*$ *is a curve which starts from the point* $(0,0)$ *ends at the point* $(0,0)$ *and stays in the domain bounded by* $l_+$*,* $l_-$*.*

**Remark 5.1.** *From Theorem 5.1 we know that this Goursat problem for three full rarefaction waves has smooth solutions, which take the value zero in a bubble near the origin, i.e., the domain bounded by the dashed line in Figure 5.1.*

# Chapter 6

# Complete Global Solutions for Three Full Rarefaction Waves

The global existence of a smooth solution was established in Theorem 5.1 up to the free boundary of vacuum. We prove that the vacuum bubble is trivial and the entire vacuum boundary is the trivial coordinate axes in the self-similar plane (see Figure 6.1).

## 6.1 Iterative Expressions by Integration along Characteristics

Before we prove the main theorem in this chapter, we need to find iterative formulas for $\partial_+ P$ and $\partial_- P$. We can easily know that the pressure gradient equation (1.8) is invariant under the following scaling transformation

$$(\xi, \eta, P) \longrightarrow \left(\frac{\xi}{k}, \frac{\eta}{k}, \frac{P}{k^2}\right)$$

where

$$k = \frac{\sqrt{P_1 P_4}\left(\sqrt{P_1} - \sqrt{2\sqrt{P_1 P_4} - P_4}\right)}{P_1 + P_4 - 2\sqrt{P_1 P_4}} > 0.$$

Thus, without loss of generality, the corresponding boundary condition (1.27), (1.28), (1.29) can be written as follows:

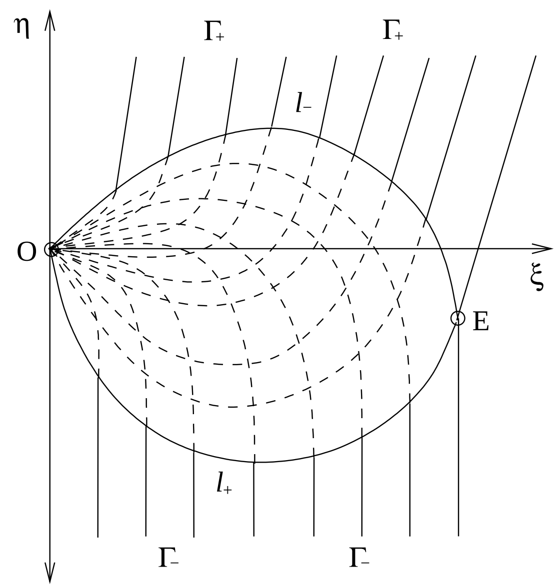


Figure 6.1. The global solution without vacuum bubble

$$P = r^2 \cos^2\theta = 4\cos^4\theta \quad \text{on} \quad r = 2\cos\theta, \quad -\theta_b \le \theta \le -\theta_m,$$

$$P = f(r,\theta) = f(g^{-1}(\theta),\theta) \quad \text{on} \quad r = g^{-1}(\theta), \quad -\theta_m \le \theta \le \theta_a, \tag{6.1}$$

where $g(r) := g(kr)$ for convenience.

For simplicity we will use the following notations:

$$\partial_\pm = \frac{\partial}{\partial_\theta} \pm \frac{1}{\lambda}\frac{\partial}{\partial_r}, \quad \lambda = \sqrt{\frac{P}{r^2(r^2-P)}}, \quad q = \frac{r^2}{4P(r^2-P)}$$

The characteristic separating form (2.1) has played an important role and was a powerful

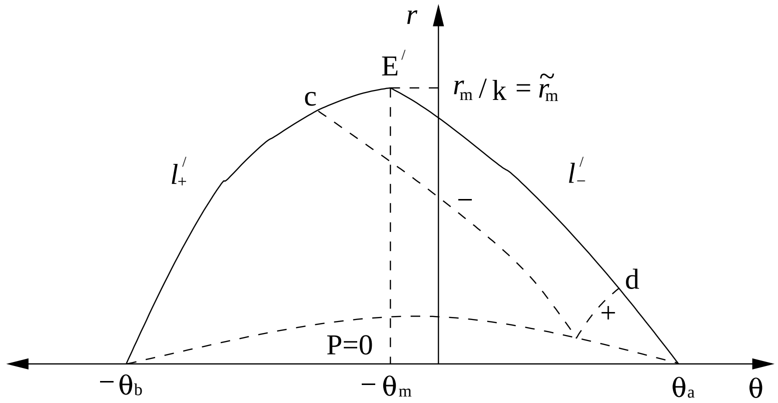


Figure 6.2. Vacuum in the polar coordinates

tool for proving that there is no vacuum bubble. For this purpose, we rewrite (2.1) as

$$\partial_+\partial_- P = q\partial_+ P\partial_- P - q(\partial_- P)^2,$$

$$\partial_-\partial_+ P = q\partial_+ P\partial_- P - q(\partial_+ P)^2. \tag{6.2}$$

Define the characteristic lines $r^c_-(\theta)$ and $r^d_+(\theta)$ by

$$\begin{cases} \dfrac{dr^c_-(\theta)}{d\theta} = -\dfrac{1}{\lambda(r^c_-(\theta),\theta)}, \\ r^c_-(\theta_c) = 2\cos\theta_c, \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{dr^d_+(\theta)}{d\theta} = \dfrac{1}{\lambda(r^d_+(\theta),\theta)}, \\ r^d_+(\theta_d) = g^{-1}(\theta_d). \end{cases} \tag{6.3}$$

We also use the notation $r^c_-(r,\theta)$ and $r^d_+(r,\theta)$ which represent the characteristic lines passing through the point $(r,\theta)$ and intersect $l'_+$ at point $c$ and $l'_-$ at point $d$, respectively in Figure 6.2.

Now we can change two equalities in (6.2) to the following equalities:

$$\partial_+ \left( \frac{1}{\partial_- P} \exp \int_{\theta_d}^{\theta} q \partial_+ P(r_+^d(\phi), \phi) d\phi \right) = q \exp \int_{\theta_d}^{\theta} q \partial_+ P(r_+^d(\phi), \phi) d\phi,$$

$$\partial_- \left( \frac{1}{\partial_+ P} \exp \int_{\theta_c}^{\theta} q \partial_- P(r_-^c(\phi), \phi) d\phi \right) = q \exp \int_{\theta_c}^{\theta} q \partial_- P(r_-^c(\phi), \phi) d\phi. \quad (6.4)$$

Integrating two rewritten equalities obtained from characteristic separating form in (6.4) along the positive and negative characteristics $r_+^d(\theta)$ and $r_-^c(\theta)$ from $\theta_d$ and $\theta_c$ to $\theta$, respectively, with respect to $\theta$, we obtain the iterative expressions of $\partial_+ P$ and $\partial_- P$ as follows:

$$\frac{1}{\partial_- P} \exp \int_{\theta_d}^{\theta} q \partial_+ P(r_+^d(\phi), \phi) d\phi = \frac{1}{\partial_- P}(g^{-1}(\theta_d), \theta_d)$$
$$+ \int_{\theta_d}^{\theta} q(r_+^d(\psi), \psi) \left\{ \exp \int_{\theta_d}^{\psi} q \partial_+ P(r_+^d(\phi), \phi) d\phi \right\} d\psi,$$
$$\frac{1}{\partial_+ P} \exp \int_{\theta_c}^{\theta} q \partial_- P(r_-^c(\phi), \phi) d\phi = \frac{1}{\partial_+ P}(2 \cos \theta_c, \theta_c)$$
$$+ \int_{\theta_c}^{\theta} q(r_-^c(\psi), \psi) \left\{ \exp \int_{\theta_c}^{\psi} q \partial_- P(r_-^c(\phi), \phi) d\phi \right\} d\psi. \quad (6.5)$$

Making use of the boundary condition (6.1), we have

$$\exp \int_{\theta_d}^{\theta} q \partial_+ P(r_+^d(\phi), \phi) d\phi$$

$$= \exp \frac{1}{4} \int_{\theta_d}^{\theta} \left( \frac{1}{P} + \frac{1}{r^2 - P} \right) \partial_+ P(r_+^d(\phi), \phi) d\phi$$

$$= \frac{P^{1/4}(r_+^d(\theta), \theta)}{f^{1/4}(g^{-1}(\theta_d), \theta_d)} \exp \frac{1}{4} \left\{ \int_{P(g^{-1}(\theta_d), \theta_d)}^{P(r_+^d(\theta), \theta)} \frac{1}{r^2(r_+^d) - P_0} dP_0 \right\}. \tag{6.6}$$

Similarly, we have

$$\exp \int_{\theta_c}^{\theta} q \partial_- P(r_-^c(\phi), \phi) d\phi$$

$$= \frac{P^{1/4}(r_-^c(\theta), \theta)}{\sqrt{2} \cos \theta_c} \exp \left\{ \frac{1}{4} \int_{P(2\cos\theta_c, \theta_c)}^{P(r_-^c(\theta), \theta)} \frac{1}{r^2(r_-^c) - P_0} dP_0 \right\}. \tag{6.7}$$

From (6.6) and (6.7) We also have

$$\int_{\theta_d}^{\theta} q(r_+^d(\psi), \psi) \left\{ \exp \int_{\theta_d}^{\psi} q \partial_+ P(r_+^d(\phi), \phi) d\phi \right\} d\psi$$

$$= \frac{1}{4 f^{1/4}(g^{-1}(\theta_d), \theta_d)} \int_{\theta_d}^{\theta} \frac{r^2 P^{-3/4}}{r^2 - P} (r_+^d(\psi), \psi)$$

$$\cdot \exp \left\{ \frac{1}{4} \int_{P(g^{-1}(\theta_d), \theta_d)}^{P(r_+^d(\psi), \psi)} \frac{1}{r^2(r_+^d) - P_0} dP_0 \right\} d\psi, \tag{6.8}$$

$$\int_{\theta_c}^{\theta} q(r_-^c(\psi), \psi) \left\{ \exp \int_{\theta_c}^{\psi} q \partial_- P(r_-^c(\phi), \phi) d\phi \right\} d\psi$$

$$= \frac{1}{4\sqrt{2} \cos \theta_c} \int_{\theta_c}^{\theta} \frac{r^2 P^{-3/4}}{r^2 - P} (r_-^c(\psi), \psi)$$

$$\cdot \exp \left\{ \frac{1}{4} \int_{P(2\cos\theta_c, \theta_c)}^{P(r_-^c(\psi), \psi)} \frac{1}{r^2(r_-^c) - P_0} dP_0 \right\} d\psi. \tag{6.9}$$

Finally, by substituting (6.7) and (6.9) ((6.6) and (6.8), respectively) into the second (first, respectively) equality of (6.5), we obtain two new iterative expressions for $\partial_+ P$ ($\partial_- P$, respectively) as follows:

$$\partial_+ P(r,\theta) = \frac{4P^{1/4}(r,\theta)\exp\left\{\frac{1}{4}\int_{P(2\cos\theta_c,\theta_c)}^{P(r_-^c(\theta),\theta)} \frac{1}{r^2(r_-^c)-P_0}dP_0\right\}}{F(r,\theta)}, \tag{6.10}$$

$$\partial_- P(r,\theta) = \frac{4P^{1/4}(r,\theta)\exp\frac{1}{4}\left\{\int_{P(g^{-1}(\theta_d),\theta_d)}^{P(r_+^d(\theta),\theta)} \frac{1}{r^2(r_+^d)-P_0}dP_0\right\}}{G(r,\theta)}, \tag{6.11}$$

where

$$F(r,\theta) = -\frac{1}{2\sqrt{2}\cos^2\theta_c\sin\theta_c} + \int_{\theta_c}^{\theta} \frac{r^2P^{-3/4}}{r^2-P}(r_-^c(\psi),\psi)\exp\left\{\frac{1}{4}\int_{P(2\cos\theta_c,\theta_c)}^{P(r_-^c(\psi),\psi)} \frac{1}{r^2(r_-^c)-P_0}dP_0\right\}d\psi,$$

$$G(r,\theta) = \frac{4f^{1/4}(g^{-1}(\theta_d),\theta_d)}{\frac{df}{d\theta}(g^{-1}(\theta_d),\theta_d)} + \int_{\theta_d}^{\theta} \frac{r^2P^{-3/4}}{r^2-P}(r_+^d(\psi),\psi)\exp\left\{\frac{1}{4}\int_{P(g^{-1}(\theta_d),\theta_d)}^{P(r_+^d(\psi),\psi)} \frac{1}{r^2(r_+^d)-P_0}dP_0\right\}d\psi.$$

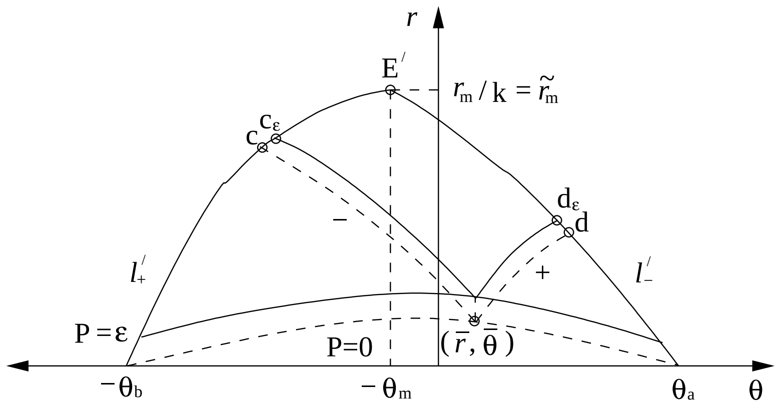


Figure 6.3. Bounded domain and notations

## 6.2 The Boundary of the Vacuum Bubble

In this section, we prove that the vacuum bubble is the trivial origin in the self-similar plane by using the iterative formulas (6.10) and (6.11). We employ the method of contradiction. Assume that there is a bubble with boundary $r_0(\theta) \geq 0$ for all $\theta \in (-\theta_b, \theta_a)$, and $r_0(\theta) > 0$ for some $\theta \in (-\theta_b, \theta_a)$. That means $P(r_0(\theta), \theta)$ for every $\theta \in (-\theta_b, \theta_a)$, and the solution is smooth in the domain bounded by the bubble $r_0(\theta)$ and the characteristic boundaries $l'_-$, $l'_+$. With the symmetry of system (6.2), let us restrict our arguments on $\theta \in (-\theta_b, -\theta_m)$. Fix a point $(\bar{r}, \bar{\theta})$ on the bubble boundary. Let us denote $\mathbb{D}(\bar{r}, \bar{\theta})$ the domain bounded by the positive and negative characteristic lines starting from $(\bar{r}, \bar{\theta})$ and $(\tilde{r}_m, -\theta_m)$ in Figure 6.3

For $0 < \varepsilon < \sqrt{P_4}/\sqrt{k}$, define a curve $r_\varepsilon$, $\theta \in (-\theta_b, -\theta_m)$, by

$$P(r_\varepsilon(\theta), \theta) = \varepsilon. \tag{6.12}$$

From (6.10) and (6.11), we have

$$\partial_+ P(r,\theta) > 0, \quad -\partial_- P(r,\theta) > 0 \ \text{ for } \ \theta \in (-\theta_b, -\theta_m). \tag{6.13}$$

By making use of the definition of $\partial_\pm$ and (6.13), we have

$$P_r(r,\theta) > 0, \tag{6.14}$$

which implies that the curve $r_\varepsilon$ defined in $(-\theta_b, \theta_a)$ from (6.12) is smooth if $0 < \varepsilon < \sqrt{P_4}/\sqrt{k}$. Here we still denote the characteristic lines passing through $(r_\varepsilon(\theta), \theta)$ by $r_-^c(\theta)$ and $r_+^d(\theta)$ although both $c = c_\varepsilon$ and $d = d_\varepsilon$ depend on $\varepsilon$ and $\theta$. Furthermore, we still denote the characteristic lines passing through $(r,\theta)$ by $r_-^c(\theta)$ and $r_+^d(\theta)$, where $c$ and $d$ depend on $(r,\theta)$.

Let us fix an $\varepsilon_0 \in (0, \sqrt{P_4}/\sqrt{k})$. Thus the curve $r_{\varepsilon_0}(\theta)$ exists. Define

$$M_1 = \max_{(r,\theta)\in\mathbb{S}} \{P^{-\frac{1}{2}}\partial_+ P, -P^{-\frac{1}{2}}\partial_- P\}, \tag{6.15}$$

where

$$\mathbb{S} := \{(r,\theta) | \theta \in (-\theta_b, \theta_a), r_{\varepsilon_0}(\theta) \le r \le \tilde{r}_m\}.$$

Then we have

$$\partial_+ P \le M_1 P^{\frac{1}{2}}, \quad -\partial_- P \le M_1 P^{\frac{1}{2}}, \tag{6.16}$$

for every $(r, \theta)$ with $r \ge r_{\varepsilon_0}(\theta)$. We know that $M_1$ depends on $\varepsilon_0$ and does not depend on $(\bar{r}, \bar{\theta})$.

For $(r, \theta) \in \mathbb{D}(\bar{r}, \bar{\theta})$, let $c$ be the intersection of the negative characteristic line passing through $(r, \theta)$ with $l'_+$, and $d$ the corresponding counterpart in Figure 6.3. Next we define $A(r, \theta)$ and $B(r, \theta)$ for convenience by

$$\begin{aligned}
A(r,\theta) &= \frac{\exp\left\{-\frac{1}{4}\int_{P(2\cos\theta_c,\theta_c)}^{P(r^c_-(\theta),\theta)} \frac{1}{r^2(r^c_-)-P_0} dP_0\right\}}{-2\sqrt{2}\cos^2\theta_c, \sin\theta_c} \\
B(r,\theta) &= \frac{4f^{\frac{1}{4}}(g^{-1}(\theta_d),\theta_d)\exp\left\{-\frac{1}{4}\int_{P(g^{-1}(\theta_d),\theta_d)}^{P(r^d_+(\theta),\theta)} \frac{1}{r^2(r^d_+)-P_0} dP_0\right\}}{-\frac{df}{d\theta}(g^{-1}(\theta_d),\theta_d)}.
\end{aligned} \tag{6.17}$$

Then, let

$$M_2 = \max\left\{ \max_{(r,\theta)\in\mathbb{D}(\bar{r},\bar{\theta})} \frac{4P^{-\frac{1}{4}}(2\cos\theta_c,\theta_c)}{A(r,\theta)}, \max_{(r,\theta)\in\mathbb{D}(\bar{r},\bar{\theta})} \frac{4P^{-\frac{1}{4}}(g^{-1}(\theta_d),\theta_d)}{B(r,\theta)} \right\}, \tag{6.18}$$

and

$$M3 = \max\{M_1, M_2 + 1\}. \tag{6.19}$$

We intend to prove that inequalities (6.16) are still valid for every $(r, \theta) \in \mathbb{D}(\bar{r}, \bar{\theta})$ with $M_1$ being replaced by $M_3$, that is,

$$\partial_+ P \leq M_3 P^{\frac{1}{2}}, \quad -\partial_- P \leq M_3 P^{\frac{1}{2}}, \tag{6.20}$$

for every $(r, \theta) \in \mathbb{D}(\bar{r}, \bar{\theta})$. Note that the positive constant $M_3$ $(\geq M_1)$ is independent of $(r, \theta) \in \mathbb{D}(\bar{r}, \bar{\theta})$, but depends on the fixed $(\bar{r}, \bar{\theta})$.

We will prove (6.20) for every $(r, \theta) \in \mathbb{D}(\bar{r}, \bar{\theta})$. Suppose that (6.20) is correct up to $r_\varepsilon(\theta) \cap \mathbb{D}(\bar{r}, \bar{\theta})$. As we improve (6.20) to strict inequalities on $r_\varepsilon(\theta) \cap \mathbb{D}(\bar{r}, \bar{\theta})$, we derive a contradiction. For $(r, \theta) \in \mathbb{D}(\bar{r}, \bar{\theta})$ with $r \leq r_{\varepsilon_0}(\theta)$, let us compute the inequality for $\partial_+ P(r, \theta)$ as follows:

$$\begin{aligned} & \partial_+ P(r, \theta) \\ = \; & 4P^{\frac{1}{4}}(r, \theta) \Bigg/ \Bigg\{ A(r, \theta) + \int_{\theta_c}^{\theta} \frac{r^2 P^{-\frac{3}{4}}}{r^2 - P}(r_-^c(\psi), \psi) \\ & \cdot \exp\left[ \frac{1}{4} \int_{P(r_-^c(\phi), \phi)}^{P(r_-^c(\psi), \psi)} \frac{1}{r^2 (r_-^c - P_0)} dP_0 \right] d\psi \Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= 4P^{\frac{1}{4}}(r,\theta) \Bigg/ \Bigg\{ A(r,\theta) + \int_{\theta_c}^{\theta} \frac{(-\partial_- P) r^2 P^{-\frac{3}{4}}}{(-\partial_- P)(r^2 - P)}(r_-^c(\psi),\psi) \\
&\quad \cdot \exp\left[\frac{1}{4}\int_{P(r_-^c(\phi),\phi)}^{P(r_-^c(\psi),\psi)} \frac{1}{r^2(r_-^c - P_0)} dP_0\right] d\psi \Bigg\} \\
&= M_3 P^{\frac{1}{4}}(r,\theta) \Bigg/ \Bigg\{ \frac{M_3 A(r,\theta)}{4} + \int_{P(2\cos\theta_c,\theta_c)}^{P(r_-^c(\theta),\theta)} \frac{M_3}{-\partial_- P} \cdot \frac{-\frac{1}{4} r^2 P^{-\frac{3}{4}}}{r^2 - P}(r_-^c(\psi),\psi) \\
&\quad \cdot \exp\left[\frac{1}{4}\int_{P(r_-^c(\phi),\phi)}^{P(r_-^c(\psi),\psi)} \frac{1}{r^2(r_-^c - P_0)} dP_0\right] dP \Bigg\} \\
&\le M_3 P^{\frac{1}{4}}(r,\theta) \Bigg/ \Bigg\{ \frac{M_3 A(r,\theta)}{4} + \int_{P(2\cos\theta_c,\theta_c)}^{P(r_-^c(\theta),\theta)} \frac{-\frac{1}{4} r^2 P^{-\frac{5}{4}}}{r^2 - P}(r_-^c(\psi),\psi) \\
&\quad \cdot \exp\left[\frac{1}{4}\int_{P(r_-^c(\phi),\phi)}^{P(r_-^c(\psi),\psi)} \frac{1}{r^2(r_-^c - P_0)} dP_0\right] dP \Bigg\} \\
&= \frac{M_3 P^{\frac{1}{4}}(r,\theta)}{\frac{M_3 A(r,\theta)}{4} + h_1(\theta^-) \int_{P(2\cos\theta_c,\theta_c)}^{P(r_-^c(\theta),\theta)} -\frac{1}{4} P^{-\frac{5}{4}} dP} \\
&= \frac{M_3 P^{\frac{1}{4}}(r,\theta)}{\frac{M_3 A(r,\theta)}{4} + h_1(\theta^-)\left\{ P^{-\frac{1}{4}}(r,\theta) - P^{-\frac{1}{4}}(2\cos\theta_c,\theta_c)\right\}}, \qquad (6.21)
\end{aligned}
$$

where

$$
\begin{aligned}
h_1(\theta^-) &= \frac{r^2}{r^2 - P}(r_-^c(\theta^-),\theta^-) \exp\left\{\frac{1}{4}\int_{P(r_-^c(\theta),\theta)}^{P(r_-^c(\theta^-),\theta^-)} \frac{1}{r^2(r_-^c) - P_0} dP_0\right\} \\
&> \exp\left\{\frac{1}{4}\int_{P(r_-^c(\theta),\theta)}^{P(r_-^c(\theta^-),\theta^-)} \frac{1}{\tilde{r}_m^2} dP_0\right\}
\end{aligned}
$$

$$= \exp\left\{\frac{P(r_-^c(\theta^-), \theta^-) - P(r_-^c(\theta), \theta)}{4\tilde{r}_m^2}\right\} > 1 \tag{6.22}$$

with some $\theta^- \in (\theta_c, \theta)$. Thus, from (6.18) (6.19), (6.21) and (6.22), we have

$$\begin{aligned}\partial_+ P(r,\theta) &< \frac{M_3 P^{\frac{1}{4}}(r,\theta)}{\frac{M_3 A(r,\theta)}{4} + \left\{P^{-\frac{1}{4}}(r,\theta) - P^{-\frac{1}{4}}(2\cos\theta_c, \theta_c)\right\}} \\ &\leq M_3 P^{\frac{1}{2}}(r,\theta) \end{aligned} \tag{6.23}$$

By making use of similar calculation in (6.21), we have an inequality for $\partial_- P$ as follows:

$$-\partial_- P(r,\theta) \leq \frac{M_3 P^{\frac{1}{4}}(r,\theta)}{\frac{M_3 B(r,\theta)}{4} + h_2(\theta^+)\left\{P^{-\frac{1}{4}}(r,\theta) - P^{-\frac{1}{4}}(g^{-1}(\theta_d), \theta_d)\right\}} \tag{6.24}$$

where

$$\begin{aligned} h_2(\theta^+) &= \frac{r^2}{r^2 - P}(r_+^d(\theta^+), \theta^+) \exp\left\{\frac{1}{4}\int_{P(r_+^d(\theta),\theta)}^{P(r_+^d(\theta^+),\theta^+)} \frac{1}{r^2(r_+^d) - P_0} dP_0\right\} \\ &> \exp\left\{\frac{P(r_+^d(\theta^+), \theta^+) - P(r_+^d(\theta), \theta)}{4\tilde{r}_m^2}\right\} > 1 \end{aligned} \tag{6.25}$$

with some $\theta^+ \in (\theta, \theta_d)$. Thus, from (6.18) (6.19), (6.24) and (6.25), we have

$$-\partial_{-}P(r,\theta) < M_3 P^{\frac{1}{2}}(r,\theta). \tag{6.26}$$

By (4.25), (6.23) and (6.26), we have

$$P_r = \frac{\lambda}{2}(\partial_{+}P - \partial_{-}P) \leq \frac{M_3}{\sqrt{1-c_1} \cdot r^2} P \leq \frac{1}{\sqrt{1-c_1}} \cdot \frac{M_3}{\bar{r}^2} P, \tag{6.27}$$

for every $(r,\theta)$ in a sufficiently small neighborhood of $(\bar{r},\bar{\theta})$ in $\mathbb{D}(\bar{r},\bar{\theta})$ with $c_1 < 1$. Thus, integrating (6.27) with respect to $r$ form $\bar{r}$ to $r$, we have

$$P(\bar{r},\bar{\theta}) \geq P(r,\bar{\theta}) \exp\left\{-\frac{1}{\sqrt{1-c_1}} \cdot \frac{M_3}{\bar{r}^2}(r-\bar{r})\right\} \quad \text{for} \quad \bar{r} < r. \tag{6.28}$$

Since $P(r,\bar{\theta}) > 0$ for every $(r,\theta)$ with $r > r_0(\theta)$ by (6.14) and the fact that $P(r_0(\theta),\theta) = 0$, (6.28) yields $P(\bar{r},\bar{\theta}) > 0$ which contradicts that $(\bar{r},\bar{\theta})$ is a point on the bubble. Therefore, we proved the following theorem.

**Theorem 6.1.** *The Goursat problem (1.26) with the boundary condition, (1.27), in full three rarefaction waves has a unique smooth solution. The pressure of the solution is strictly positive in* $\{(\xi,\eta)|\xi > 0\} \cap \Omega$ *of the self-similar plane where* $\Omega$ *is the region bounded by* $l_{+}$ *and* $l_{-}$*.*

# Chapter 7

# Existence of Solutions in the Hyperbolic Region for Four Rarefaction Wave Problem

Now we return back to the four rarefaction wave problems. We know that there exists a smooth solution $P$ in the region bounded by two positive characteristic lines, $\widehat{EG}$ and $\widehat{OK}$, and two negative characteristic lines, $\widehat{EO}$ and $\widehat{GK}$ in Figure 1.1 by Theorem 5.1. Now I prove that every positive characteristic line passing through a point on $\widehat{GK}$ does not intersect up to sonic line, that is, there is no shock in Lemma 7.1 below. Similarly, every negative characteristic line passing through a point on $\widehat{OK}$ does not intersect up to the sonic line.

**Lemma 7.1.** *Every positive characteristic line passing through a point on $\widehat{GK}$ does not intersect before it reaches to a sonic line and every negative characteristic line passing through a point on $\widehat{OK}$ does not intersect before it reaches to a sonic line.*

**Proof .** *Suppose that two positive characteristic lines, which pass through two points, $X$ and $Y$ on $\widehat{GK}$, meet a point before the lines reach to the sonic lines. Denote the first intersection point by $Z$. Thus we have*

$$\left.\frac{dP}{d_{-}\theta}\right|_{Z} = -\infty. \tag{7.1}$$

*Making use of the same proof for Lemma 2.1, we can derive a contradiction.*

*Similarly, we can prove the second assertion. Therefore this lemma holds.* □

Combining the existence of a smooth solution in the region bounded by $\widehat{EG}$, $\widehat{OK}$, $\widehat{EO}$ and $\widehat{GK}$ in Figure 1.1 with Lemma 7.1, we have the second main theorem as follows:

**Theorem 7.1.** *There exists a continuous and piecewise smooth solution to the extent of the domain of determinacy of the hyperbolic data from infinity(i.e., all real plane except the elliptic region bounded by the dotted line in Figure 1.1).*

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# Vita

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