**The Pennsylvania State University**
**The Graduate School**

# SMOOTH CONJUGACY IN HYPERBOLIC DYNAMICS

A Dissertation in
Mathematics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2009

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# Abstract

This thesis is devoted to the study of smooth conjugacy in hyperbolic dynamics.

Let $L$ be a hyperbolic automorphism of $\mathbb{T}^d$, $d \geq 3$. We study the smooth conjugacy problem in a small $C^1$-neighborhood $\mathcal{U}$ of $L$. Two dimensional case was fully understood by de la Llave, Marco and Moriyón in late eighties.

The main result establishes $C^{1+\nu}$ regularity of the conjugacy between two Anosov systems with the same periodic eigenvalue data. We assume that these systems are $C^1$-close to an irreducible linear hyperbolic automorphism $L$ with simple real spectrum and that they satisfy a natural transitivity assumption on certain intermediate foliations.

We elaborate on the example of de la Llave of two Anosov systems on $\mathbb{T}^4$ with the same constant periodic eigenvalue data that are only Hölder conjugate. We show that these examples exhaust all possible ways to perturb $C^{1+\nu}$ conjugacy class without changing periodic eigenvalue data. Also we generalize these examples to majority of reducible toral automorphisms as well as to certain diffeomorphisms of $\mathbb{T}^4$ $C^1$-close to the original example.

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# Acknowledgments

I would like to thank Anatole Katok, my advisor, for his support and encouragement during my years in graduate school. I am grateful for his suggestion to study the higher dimensional smooth conjugacy problem.

I also would like to thank Misha Guysinsky; his help was invaluable, especially at early stages. Part of this thesis is a joint work with him.

Chapter 1

# Introduction

Consider an Anosov diffeomorphism $f$ of a compact smooth manifold. Structural stability asserts that if a diffeomorphism $g$ is $C^1$ close to $f$ then $f$ and $g$ are topologically conjugate. The conjugacy $h$ is unique in the neighborhood of identity.

$$h \circ f = g \circ h$$

It is known that $h$ is Hölder continuous.

There are simple obstructions for $h$ to be smooth. Namely, let $x$ be a periodic point of $f$, $f^p(x) = x$ then $g^p(h(x)) = h(x)$ and if $h$ were differentiable then

$$Df^p(x) = (Dh(x))^{-1} Dg^p(h(x))Dh(x)$$

i. e. $Df^p(x)$ and $Dg^p(h(x))$ are conjugate. We see that every periodic point carries a modulus of smooth conjugacy.

Suppose that for every periodic point $x$, $f^p(x) = x$, differentials of return maps $Df^p(x)$ and $Dg^p(h(x))$ are conjugate then we say that *periodic data* (p. d.) of $f$ and $g$ coincide.

The main question that we will be concerned with is the following.

**Question 1.** *Suppose that p. d. coincide, is $h$ differentiable? If it is then how smooth is it?*

We describe situations when p. d. form full set of moduli of $C^1$ conjugacy.

The only surface that supports Anosov diffeomorphisms is two dimensional torus. For Anosov diffeomorphisms of $\mathbb{T}^2$ the complete answer was given by de la Llave, Marco and Moriyón.

**Theorem 1** ([LMM87], [L92])**.** *Let $f$ and $g$ be $C^r$, $r>1$, Anosov diffeomorphisms of $\mathbb{T}^2$ that are topologically conjugate,*

$$h\circ f = g\circ h.$$

*Suppose that p. d. coincide. Then $h$ is $C^{r-\varepsilon}$ where $\varepsilon>0$ is arbitrarily small.*

De la Llave [L92] also observed that the answer is negative for Anosov diffeomorphisms of $\mathbb{T}^d$, $d\geq 4$. He constructed two diffeomorphisms with the same p. d. which are only Hölder conjugate. We describe this example in Section 4.3. In Chapter 4 we analyze the example in detail and present a generalization for automorphisms of tori of higher dimension.

In dimension three the only manifold that supports Anosov diffeomorphisms is three dimensional torus. Moreover, all Anosov diffeomorphisms of $\mathbb{T}^3$ are topologically conjugate to the linear automorphisms of $\mathbb{T}^3$. Nevertheless the answer to the Question 1 is not known.

**Conjecture 1.** *Let $f$ and $g$ be $C^r$, $r>1$, Anosov diffeomorphisms of $\mathbb{T}^3$ that are topologically conjugate,*

$$h\circ f = g\circ h.$$

*Suppose that p. d. coincide. Then $h$ is at least $C^1$.*

In Chapter 2 we present a partial result in the direction of the conjecture above.

**Theorem 2.** *Let $L$ be a hyperbolic automorphism of $\mathbb{T}^3$ with real eigenvalues. Then there exists a $C^1$-neighborhood $\mathcal{U}$ of $L$ such that any $f$ and $g$ in $\mathcal{U}$ having the same p. d. are $C^{1+\nu}$ conjugate.*

Another partial result is the following

**Theorem 3** ([KS07])**.** *Let $L$ be a hyperbolic automorphism of $\mathbb{T}^3$ that has one real and two complex eigenvalues. Then any $f$ sufficiently $C^1$ close to $L$ that has the same p. d. as $L$ is $C^\infty$ conjugate to $L$.*

In higher dimensions not much is known. In recent years big progress has been made (see [L02, KS03, L04, F04, F07, S05, KS07]) in the case when stable and unstable foliations carry invariant conformal structures. To ensure existence of these conformal structures one has at least to assume that every periodic orbit has only one positive and one negative Lyapunov exponent. This is a very restrictive assumption on p. d.

In contrast to above we will study smooth conjugacy problem in proximity of a hyperbolic automorphism $L : \mathbb{T}^d \to \mathbb{T}^d$ with simple spectrum. Namely, we will assume that the eigenvalues of $L$ are real and have different absolute values. For the sake of notation we assume that the eigenvalues of $L$ are positive. This is not restrictive.

Let $l$ be the dimension of the stable subspace of $L$ and $k$ be the dimension of the unstable subspace of $L$, $k + l = d$. Consider $L$-invariant splitting

$$T\mathbb{T}^d = F_l \oplus F_{l-1} \oplus \ldots \oplus F_1 \oplus E_1 \oplus E_2 \oplus \ldots \oplus E_k$$

along the eigendirections with corresponding eigenvalues

$$\mu_l < \mu_{l-1} < \ldots < \mu_1 < 1 < \lambda_1 < \lambda_2 < \ldots < \lambda_k.$$

Let $\mathcal{U}$ be a $C^1$-neighborhood of $L$. Precise choice of $\mathcal{U}$ is described in Section 3.4.1. Theory of partially hyperbolic dynamical systems guarantees that for any $f$ in $\mathcal{U}$ the invariant splitting survives (e. g. see [Pes04])

$$T\mathbb{T}^d = F_l^f \oplus F_{l-1}^f \oplus \ldots \oplus F_1^f \oplus E_1^f \oplus E_2^f \oplus \ldots \oplus E_k^f.$$

We will see these one dimensional invariant distributions integrate uniquely to foliations $U_l^f$, $U_{l-1}^f$,... $U_1^f$, $V_1^f$, $V_2^f$,... $V_k^f$.

Given a foliation $\mathcal{F}$ on $\mathbb{T}^d$ and an open set $B$ define

$$\mathcal{F}(B) = \bigcup_{y \in B} \mathcal{F}(y).$$

We will be assuming the following property of $f$

**Property $\mathcal{A}$.** For every $x \in \mathbb{T}^d$ and every open ball $B \ni x$

$$\begin{aligned}\overline{U^f_{l-1}(B)} = \overline{U^f_{l-2}(B)} = \ldots = \overline{U^f_1(B)}\\ = \overline{V^f_1(B)} = \overline{V^f_2(B)} = \ldots = \overline{V^f_{k-1}(B)} = \mathbb{T}^d.\end{aligned} \qquad (\mathcal{A})$$

We discuss this property in Section 3.2.1.

The following is our main result which is a generalization of Theorem 2. The proof appears in Chapter 3.

**Theorem A.** *Let $L$ be a hyperbolic automorphism of $\mathbb{T}^d, d \geq 3$, with simple real spectrum. Assume that characteristic polynomial of $L$ is irreducible over $\mathbb{Z}$. There exists a $C^1$-neighborhood $\mathcal{U} \subset \mathrm{Diff}^r(\mathbb{T}^d)$, $r \geq 2$, of $L$ such that any $f \in \mathcal{U}$ satisfying $\mathcal{A}$ and any $g \in \mathcal{U}$ with the same p. d. are $C^{1+\nu}$ conjugate.*

*Remark.* We will see in Section 3.2.1 that irreducibility of characteristic polynomial of $L$ is necessary for $f$ to satisfy $\mathcal{A}$. Formally, we could have omitted the irreducibility assumption above. Theorem B below shows that irreducibility of $L$ is a necessary assumption for the conjugacy to be $C^1$. We believe that Theorem A holds when $L$ is irreducible without assuming that $f$ satisfies $\mathcal{A}$.

*Remark.* Number $\nu$ is a small positive number. It is possible to estimate $\nu$ from below in terms of eigenvalues of $L$ and the size of $\mathcal{U}$.

*Remark* (A remark on terminilogy)*.* Throughout the manuscript we will be proving that various maps are $C^{1+\nu}$-differentiable. This should be understood in the usual way: the map is $C^1$ differentiable and the derivative is Hölder continuous with some positive exponent $\nu$. Number $\nu$ is not the same in different statements.

When we say that a map is $C^{1+\nu}$-differentiable along foliation $\mathcal{F}$ we mean that restrictions of the map to the leaves of $\mathcal{F}$ are $C^{1+\nu}$-differentiable and the derivative is a Hölder continuous function on the manifold, not only on the leaf.

Chapter 2

# Smooth conjugacy of Anosov diffeomorphisms on $\mathbb{T}^3$

The results of these chapter were obtained jointly with Misha Guysinky. The main goal is to establish Theorem 2 from the introduction. The proof is a model for our main general higher-dimensional result, Theorem A.

## 2.1 Formulation of the result

Let $f$ be an Anosov diffeomorphism of $\mathbb{T}^d$. It is known [Mann74] that $f$ is topologically conjugate to a linear torus automorphism $L$. It is also known that Anosov diffeomorphisms of $\mathbb{T}^3$ are the only Anosov diffeomorphisms on three dimensional manifolds [Fr70], [N70].

Let $L$ be a hyperbolic automorphism of $\mathbb{T}^3$ with real eigenvalues. It is easy to show that absolute values of these eigenvalues are distinct. For the sake of notation we also assume that the eigenvalues are positive. This is not restrictive.

We will always assume that the Anosov diffeomorphisms that we are dealing with are at least $C^2$.

**Theorem 4** (Theorem 2)**.** *Given $L$ as above there exists a $C^1$-neighborhood $\mathcal{U}$ of $L$ such that any $f$ and $g$ in $\mathcal{U}$ having the same p. d. are $C^{1+\nu}$ conjugate, $\nu > 0$.*

*Remark.* The constant $\nu$ depends on the size of $\mathcal{U}$ and provided sufficient smoothness of $f$ and $g$ can be made as close as desired to $\log \lambda_3 / \log \lambda_2$ (see the definition in the next section) by shrinking the size of $\mathcal{U}$.

*Remark.* We don't know how to bootstrap regularity of $h$ to the regularity $f$ and $g$ like it was done in dimension two.

A result about integrability of central distribution [BI07] allows to show a stronger statement.

**Theorem 5.** *Let $f$ and $g$ be Anosov diffeomorphisms of $\mathbb{T}^3$ and*

$$h \circ f = g \circ h,$$

*where $h$ is a homeomorphism homotopic to identity. Suppose that p. d. coincide.*

*Also assume that $f$ and $g$ can be viewed as partially hyperbolic diffeomorphisms: there is an $f$-invariant splitting $T\mathbb{T}^3 = E_f^s \oplus E_f^{wu} \oplus E_f^{su}$ and constants $c > 0$, $0 < \alpha < 1 < \tilde{\beta} < \beta < \gamma$ such that for $n > 0$*

$$\begin{aligned}
&\|D(f^n)(x)(v)\| \le c\alpha^n \|v\|, \quad v \in E_f^s(x),\\
&\frac{1}{c}\tilde{\beta}^n \|v\| \le \|D(f^n)(x)(v)\| \le c\beta^n \|v\|, \quad v \in E_f^{wu}(x),\\
&\frac{1}{c}\gamma^n \|v\| \le \|D(f^n)(x)(v)\|, \quad v \in E_f^{su}(x)
\end{aligned} \tag{2.1}$$

*Analogous conditions with possibly different set of constants hold for a $g$-invariant splitting $T\mathbb{T}^3 = E_g^s \oplus E_g^{wu} \oplus E_g^{su}$.*

*Then the conjugacy $h$ is $C^{1+\nu}$, $\nu > 0$.*

*Remark.* Here and further in the paper we assume that the unstable distribution has dimension two. Obviously one can formulate the counterpart of Theorem 2 in the case when stable distribution has dimension two.

## 2.2 Scheme of the proof

Here we outline the proof of Theorem 4.

Let $\lambda_1$, $\lambda_2$ and $\lambda_3$ be the eigenvalues of the linear automorphism $L$, $0 < \lambda_1 < 1 < \lambda_2 < \lambda_3$. We choose $\mathcal{U}$ in such a way that every $f \in \mathcal{U}$ is partially hyperbolic, satifying (3.11) with constants $\alpha, \tilde{\beta}, \beta, \gamma$ independent on the choice of $f$, $0 < \lambda_1 <$

$\alpha < 1 < \tilde{\beta} < \lambda_2 < \beta < \gamma < \lambda_3$ and

$$\measuredangle(E_L^\sigma, E_f^\sigma) < k < \frac{\pi}{2}, \ \sigma = s, wu, su. \tag{2.2}$$

First we concentrate on a single diffeomorphism $f$ in $\mathcal{U}$. It is well known that distributions $E_f^s$, $E_f^u = E_f^{wu} \oplus E_f^{su}$ and $E_f^{su}$ integrate uniquely to stable, unstable and strong unstable foliations $W_f^s$, $W_f^u$ and $W_f^{su}$ respectively. We denote by $W_f^\sigma(x)$ the leaf of $W_f^\sigma$ passing through $x$, $\sigma = s, u, su$ and later $wu$. By $W_f^\sigma(x, R)$ we denote the local leaf of size $R$, i. e., a ball of radius $R$ inside of $W_f^\sigma(x)$ centered at $x$, $\sigma = s, u, wu, su$. Let $h_f$ be conjugacy between $f$ and $L$, $h_f \circ f = L \circ h_f$. Stable and unstable foliations can be characterized topologically, e.g.

$$W_f^s(x) = \{y : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty\}.$$

As a consequence we have that $h_f(W_f^s) = W_L^s$ and $h_f(W_f^u) = W_L^u$. In other words $h_f$ maps leaves of foliations for $f$ into leaves of corresponding foliations for $L$.

We prove two simple lemmas.

**Lemma 2.2.1.** *Let $f$ be in $\mathcal{U}$. Then the distribution $E_f^{wu}$ integrates uniquely to the foliation $W_f^{wu}$.*

**Lemma 2.2.2.** *Define $h_f$ as above. Then $h_f(W_f^{wu}) = W_L^{wu}$.*

Now let $f$ and $g$ be as in Theorem 4. For each of them we have the system of one dimensional invariant foliations. We know that $h(W_f^s) = W_g^s$. Also from Lemma 2.2.2 we have $h(W_f^{wu}) = W_g^{wu}$ since $h = h_g^{-1} \circ h_f$. Consider restrictions of $h$ to the leaves of $W_f^s$ and $W_f^{wu}$. These restrictions are one dimensional maps. We show that they are smooth.

**Lemma 2.2.3.** *The conjugacy $h$ is $C^{1+\nu}$ along $W_f^s$.*

Which means that $h$ is differentiable along the stable foliation and the derivative is a Hölder continuous function on $\mathbb{T}^3$ with exponent $\nu$

*Remark.* The general strategy of the proof of Theorem 4 is similar to de la Llave's strategy for Anosov diffeomorphisms of $\mathbb{T}^2$ [Ll92]. One proves smoothness of $h$ along one dimensional stable and unstable foliations. In particular proof of

Lemma 2.2.3 can be carried out in the same way as in dimension two. The hard part is showing smoothness of $h$ along two dimensional unstable foliation.

We would like to show the same for the foliation $W_f^{wu}$ but we split the proof into two steps.

**Lemma 2.2.4.** *The conjugacy $h$ is uniformly Lipschitz along $W_f^{wu}$.*

**Lemma 2.2.5.** *The conjugacy $h$ is $C^{1+\nu}$ along $W_f^{wu}$.*

After that we deal with the remaining foliation.

**Lemma 2.2.6.** $h(W_f^{su}) = W_g^{su}$.

*Remark.* We would like to remark that Lemma 2.2.6 requires only the coincidence of p. d. in the weak unstable direction. It is not true in general that strong unstable foliations match.

**Lemma 2.2.7.** *The conjugacy $h$ is $C^{1+\nu}$ along $W_f^{su}$.*

*Remark.* Proofs of smoothness along the foliations $W_f^s$ and $W_f^{su}$ are similar and use the coincidence of periodic data in corresponding directions. Showing smoothness along the weak unstable foliation is more subtle.

Now smoothness of $h$ is a simple consequence of a regularity result.

**Regularity Lemma.** *[J88] Let $M_j$ be a manifold and $W_j^s$, $W_j^u$ be continuous transverse foliations with uniformly smooth leaves, $j = 1, 2$. Suppose that $h : M_1 \to M_2$ is a homeomorphism that maps $W_1^s$ into $W_2^s$ and $W_1^u$ into $W_2^u$. Moreover assume that the restrictions of $h$ to the leaves of these foliations are uniformly $C^{r+\nu}, r \in \mathbb{N},\ 0 < \nu < 1$, then $h$ is $C^{r+\nu}$.*

First we apply the lemma on every unstable leaf of $W_f^u$ for the pair of foliations $W_f^{wu}, W_f^{su}$. After we know that $h$ is $C^{1+\nu}$ along $W_f^u$ we finish by applying the lemma to stable and unstable foliations.

The structure of the next section is the following. We prove Lemmas 2.2.1 and 2.2.2 in Section 2.3.1. Section 2.3.2 is devoted to the proof of Lemma 2.2.4. Sections 2.3.3 and 2.3.4 are the heart of our argument and contain proofs of Lemmas 2.2.5 and 2.2.6 correspondingly.

## 2.3 Proof of Theorems 4 and 5

First we prove Theorem 4.

### 2.3.1 Weak unstable foliation

In the proofs of Lemmas 2.2.1 and 2.2.2 we work with lifts of maps, distributions and foliations to $\mathbb{R}^3$. We use the same notation for the lifts as for the objects themselves.

Denote by $d(\cdot,\cdot)$ the usual distance in $\mathbb{R}^3$ and let $d_f^\sigma(\cdot,\cdot)$ be the distance in the leaves of $W_f^\sigma$ which is defined only for pairs of points lying in the same leaf of $W_f^\sigma$, $\sigma = s, u, su, wu$.

*Proof of Lemma 2.2.1.* Let us reason by contradiction. If $E_f^{wu}$ is not uniquely integrable then it must branch and we can find points $a, b, c \in \mathbb{R}^3$ such that

1. $a, b \in W_f^u(c)$,
2. there are smooth curves $\tau_{ca}, \tau_{cb} : [0,1] \to W_f^u(c)$ such that $\tau_{ca}(0) = \tau_{cb}(0) = c$, $\tau_{ca}(1) = a$, $\tau_{cb}(1) = b$, and $\{\dot\tau_{ca}, \dot\tau_{cb}\} \subset E_f^{wu}$,
3. $a \in W_f^{su}(b)$.

Then for $n \geq 1$

$$d(f^n(a), f^n(b)) \leq d(f^n(a), f^n(c)) + d(f^n(c), f^n(b)) \leq c_1\beta^n, \tag{2.3}$$

on the other hand

$$d_f^{su}(f^n(a), f^n(b)) \geq c_2\gamma^n. \tag{2.4}$$

For every $x \in \mathbb{R}^3$ consider a cone $Cone(x) = \{v \in T_x\mathbb{R}^3 : \measuredangle(v, E_L^{su}(x)) \leq k\}$. The assumption (3.10) tells us that $E_f^{su}(x) \subset Cone(x)$. Hence a leaf of $W_f^{su}$ can be considered as a graph of a Lipschitz function over $E_L^{su}$. The Lipschitz constant depends only on $k$. It follows that $W_f^{su}$ is quasi-isometric:

$$\exists c_3 > 0 \text{ such that for } x \in W_f^{su}(y) \;\; d_f^{su}(x,y) \leq c_3 d(x,y). \tag{2.5}$$

Inequalities (2.3), (2.4) and (2.5) sum up to a contradiction. □

*Proof of Lemma 2.2.2.* Suppose that there are two points $a$ and $b$, $a \in W_f^{wu}(b)$ such that $h_f(a) \notin W_L^{wu}(h_f(b))$ then we have

$$d(f^n(a), f^n(b)) \leq c_1\beta^n, \tag{2.6}$$

and since $h_f(a)$ and $h_f(b)$ lie in the same unstable leaf but not in the same weak unstable leaf we get

$$d(h_f(f^n(a)), h_f(f^n(b))) = d(L^n(h_f(a)), L^n(h_f(b))) \geq c_2\gamma^n. \tag{2.7}$$

Finally since

$$h_f(x + \bar{m}) = h_f(x) + \bar{m}, \quad \bar{m} \in \mathbb{Z}^3 \tag{2.8}$$

we have that $d(h_f(x), h_f(y)) \leq c(\varepsilon)d(x, y)$ for any $x$ and $y$ such that $d(x, y) \geq \varepsilon$. Hence

$$d(h_f(f^n(a)), h_f(f^n(b))) \leq c_3 d(f^n(a), f^n(b)), \tag{2.9}$$

where $c_3$ depends on $d(a, b)$. Inequalities (2.6), (2.7) and (2.9) sum up to a contradiction. □

### 2.3.2 Affine structure on the weak unstable foliation

Let $f$ be in $\mathcal{U}$. For any $x$ and $y$, $y \in W_f^{wu}(x)$ define the function

$$\rho_f(x, y) = \prod_{n \geq 1} \frac{D_f^{wu}(f^{-n}(y))}{D_f^{wu}(f^{-n}(x))}$$

where $D_f^{wu}(z) = \|D(f)\big|_{E_f^{wu}}(z)\|$. The following properties are easy to prove:

(P1) $\rho_f(x, \cdot)$ is well defined and Hölder continuous.

(P2) $\forall x, y \in W_f^{wu}(z) \;\; \rho_f(x, y)\rho_f(y, z) = \rho_f(x, z)$.

(P3) $\rho_f(f(x), f(y)) = \frac{D_f^{wu}(y)}{D_f^{wu}(x)}\rho_f(x, y)$.

(P4) The function $\rho(\cdot,\cdot)$ is the only continuous function satisfying $\rho_f(x,x) = 1$ and Property 3.

(P5) $\forall K > 0\ \exists C > 0$ such that $C > \rho_f(x,y) > \frac{1}{C}$ whenever $d^{wu}(x,y) < K$.

The goal is to show that $h$ is differentiable along $W_f^{wu}$ ($wu$-differentiable) and

$$\rho_g(h(x),h(y)) = \frac{D_h^{wu}(y)}{D_h^{wu}(x)}\rho_f(x,y), \tag{2.10}$$

*Proof of Lemma 2.2.4.* Fix an arbitrary point $p$. Let $h_p : W_f^{wu}(p) \to W_f^{wu}(h(p))$ be the restriction of $h$ to $W_f^{wu}(p)$. We would like to show that $h_p$ is Lipschitz with a constant that does not depend on $p$. Let $m$ be the induced volume on $W_f^{wu}(p)$. Consider the function $\tilde{d}_f$

$$\tilde{d}_f(x,y) = \int_x^y \frac{1}{\rho_f(x,z)}dm(z), \quad x,y \in W_f^{wu}(p),$$

we integrate along the leaf with respect to the measure $m$.

Function $\tilde{d}_f$ has the following properties which are simple corollories of the properties of $\rho_f$ and the definition of $\tilde{d}_f$.

(D1) $\tilde{d}_f(x,y) = d_f^{wu}(x,y) + o(d_f^{wu}(x,y))$,

(D2) $\tilde{d}_f(f(x),f(y)) = D_f^{wu}(x)\tilde{d}_f(x,y)$,

(D3) $\forall K > 0\ \exists C > 0$ such that

$$\frac{1}{C}\tilde{d}_f(x,y) \le d_f^{wu}(x,y) \le C\tilde{d}_f(x,y) \tag{2.11}$$

whenever $d_f^{wu}(x,y) < K$.

(D4) The function $\tilde{d}_f$ is continuous. To state this property precisely we consider lift of $\tilde{d}_f$. We speak about lifts of points and leaves.

$$\begin{aligned} &\forall \varepsilon > 0\ \exists \delta > 0 \quad \text{such that} \quad \forall x,y \in \mathbb{R}^3, y \in W_f^{wu}(x) \\ &\qquad \text{and} \quad \forall z,q \in \mathbb{R}^3, q \in W_f^{wu}(z), z \in B(x,\delta), q \in B(y,\delta) \\ &\qquad\qquad \text{we have} \quad |\tilde{d}_f(x,y) - \tilde{d}_f(z,q)| < \varepsilon. \end{aligned}$$

We will also need $\tilde{d}_g$ which is defined analogously on the leaves of $W_g^{wu}$ and has analogous properties.

The lift of the conjugacy $h$ satisfies the equation (2.8) which implies the following

$$\exists C > 0 : \forall x, y \quad d(h(x), h(y)) \leq C d(x, y) \ \text{ if } \ d(x, y) \geq 1.$$

Also we know that weak unstable foliation is quasi-isometric which gives us the same for the distance in weak unstable foliations

$$\exists C > 0 : \forall x, y \quad d_g^{wu}(h(x), h(y)) \leq C d_f^{wu}(x, y) \ \text{ if } \ d_f^{wu}(x, y) \geq 1. \tag{2.12}$$

This tells us that $h_p$ is Lipschitz for points that are far enough. So we need to estimate $d_g^{wu}(h(x), h(y))$ for $x$ and $y$ close. Note that (D3) allows us to use $\tilde{d}_g$ and $\tilde{d}_f$ in these estimates instead of $d_g^{wu}$ and $d_f^{wu}$.

Apply Livshits Theorem for $\varphi_1 = D_f^{wu}(\cdot)$ and $\varphi_2 = D_g^{wu}(h(\cdot))$. The condition of the Livshitz Theorem is satisfied because of the assumption on p. d. We have

$$\forall n > 0 \qquad \prod_{i=0}^{n-1} \frac{D_g^{wu}\left(h(f^i(x))\right)}{D_f^{wu}(f^i(x))} = \frac{P(x)}{P(f^n(x))}. \tag{2.13}$$

Choose points $x$ and $y$ close on the leaf $W_f^{wu}(p)$. Choose the smallest $N$ such that $d_f^{wu}(f^N(x), f^N(y)) \geq 1$. Then

$$\begin{aligned}\frac{\tilde{d}_g(h(x), h(y))}{\tilde{d}_f(x, y)} &= \prod_{i=0}^{N-1} \frac{D_g^{wu}\left(g^i(h(x))\right)}{D_f^{wu}(f^i(x))} \cdot \frac{\tilde{d}_g\left(g^N(h(x)), g^N(h(y))\right)}{\tilde{d}_f(f^N(x), f^N(y))} \\ &= \frac{P(x)}{P(f^N(x))} \cdot \frac{\tilde{d}_g\left(g^N(h(x)), g^N(h(y))\right)}{\tilde{d}_f(f^N(x), f^N(y))} \leq \frac{P(x)}{P(f^N(x))} \cdot constant.\end{aligned}$$

Here we used (2.12) and (D3) for $\tilde{d}_f$ and $\tilde{d}_g$. Function $P$ is bounded away from zero and infinity so we get that $h$ is uniformly Lipschitz along the weak unstable foliation. □

### 2.3.3 Transitive point argument and construction of a measure absolutely continuous with respect to weak unstable foliation

We divide the proof of Lemma 2.2.5 into several steps. The conjugacy $h$ is Lipschitz along $W_f^{wu}$ and hence $wu$-differentiable at almost every point with respect to Lebesgue measure on the leaves of $W_f^{wu}$. It is obvious that $wu$-differentiability of $h$ at $x$ implies $wu$-differentiability of $h$ at any point from the orbit $\{f^i(x), i \in \mathbb{Z}\}$. Moreover:

**Step 1.** *Suppose that $h$ is wu-differentiable at $x$ and $\overline{\{f^i(x), i \geq 0\}} = \mathbb{T}^3$ then $h$ is $C^{1+\nu}$ along $W_f^{wu}$ and (2.10).*

The problem now is to show existence of such a transitive point $x$. We know that almost every point is transitive with respect to a given ergodic measure with full support. On the other hand $h$ is $wu$-differentiable at almost every point with respect to Lebesgue measure on the leaves. Unfortunately it can happen that for natural ergodic "physical measures" these two "full measure" sets do not intersect. In other words weak unstable foliation is not absolutely continuous with respect to a "physical measure".

Let us explain this phenomenon in more detail. Consider a volume preserving $C^1$ small perturbation $\tilde{L}$ of $L$, $H \circ L = \tilde{L} \circ H$. The Lyapunov exponents of $\tilde{L}$ are defined on a full volume set of regular points $\mathcal{R}$ and are given by the formula

$$\chi^\sigma = \int_{\mathbb{T}^3} \log D_{\tilde{L}}^\sigma d\text{vol}, \quad \sigma = s, wu, su.$$

The perturbation $\tilde{L}$ can be chosen in such a way that $\chi^{wu} > \log \lambda_2$ (see [BB03], Proposition 0.3). It is easy to show that the weak unstable foliation of $\tilde{L}$ is not absolutely continuous. Namely, let $\Delta$ be a segment of a weak unstable leaf of $L$. Then by Lemma 2.2.2 $H(\Delta)$ is a piece of a weak unstable leaf of $\tilde{L}$. We show that Lebesgue measure of $\mathcal{R} \cap H(\Delta)$ is equal to zero. For any $n \geq 0$ $H(L^n(\Delta)) = \tilde{L}^n(H(\Delta))$ and (3.10) guarantees that $\tilde{L}^n(H(\Delta))$ can be viewed as a graph of a Lipschitz function over a leaf of the weak unstable foliation of $L$. Hence

$$\text{length}(\tilde{L}^n(H(\Delta)) \leq c_1 \cdot \text{length}(L^n(\Delta)) = \lambda_2^n \cdot \text{length}(\Delta), \;\; n \geq 0.$$

Suppose that $\text{Leb}(\mathcal{R} \cap H(\Delta)) > 0$ then

$$\text{length}(\tilde{L}^n(H(\Delta)) \geq c_2 e^{n(\chi^{wu}-\varepsilon)}, \ \ \varepsilon = \frac{1}{2}(\chi^{wu} - \log \lambda_2)$$

which contradicts the previous inequality.

This observation answers a question of Hirayama and Pesin [HP07] about existence of non-absolutely continuous foliations with non-compact leaves. For a generalization see [SX08]

To overcome this problem we do

**Step 2.** *Construction of a measure $\mu$ absolutely continuous with respect to $W_f^{wu}$.*

This construction follows the lines of Pesin-Sinai [PS83] construction of $u$-Gibbs measures. In our setup the construction is simpler so for the sake of completeness we present it here. Measure $\mu$ has full support. Thus ergodicity of $\mu$ would imply that almost every point is transitive and hence by Step 1 $h$ would be $wu$-differentiable. We do not know how to show ergodicity of $\mu$. Instead we do

**Step 3.** *Set of transitive points is a full measure $\mu$ set.*

Steps 2 and 3 guarantee existence of a transitive point needed in Step 1.

*Proof of Lemma 2.2.5.*
**Step 1.** Let us pick a point $y \in \mathbb{T}^3$ and show that $h$ is $wu$-differentiable at $y$ and moreover

$$D_h^{wu}(y) = \frac{P(y)}{P(x)} D_h^{wu}(x) \tag{2.14}$$

where $P$ is the same as in the proof of Lemma 4.

Choose $y' \in W_f^{wu}(y)$. Property (D1) of $\tilde{d}_f$, $\tilde{d}_g$ ensures that it is enough to show that

$$\frac{\tilde{d}_g(h(y), h(y'))}{\tilde{d}_f(y, y')} = \frac{P(y)}{P(x)} D_h^{wu}(x). \tag{2.15}$$

Fix an $\varepsilon > 0$ small compared to $\tilde{d}_f(y, y')$. Choose a small open ball $B$ centered at $y$ and define

$$B' = \{z' : \ \ \exists z \in B \ \text{ such that } \ \tilde{d}_f(z, z') = \tilde{d}_f(y, y')$$

and $(z, z')$ has the same orientation as $(y, y')\}$.

The condition about orientation ensures that $B'$ has only one connected component. The set $B'$ is a small neighborhood of $y'$ because of the continuity of $\tilde{d}_f$ (D4). The size of $B$ must be chosen in such a way that

1. $|P(z) - P(y)| < \varepsilon$ if $z \in B$,
2. $|\tilde{d}_g(h(z), h(z')) - \tilde{d}_g(h(y), h(y'))| < \varepsilon$ where $z$ and $z'$ are the same as in definition of $B'$.

Since $x$ is transitive there is an arbitrarily large $N$ such that $f^N(x) \in B$. Choose $z$ on $W_f^{wu}(x)$ such that $\tilde{d}_f(f^N(x), f^N(z)) = \tilde{d}_f(y, y')$ so that $f^N(z) \in B'$ by the definition. We choose $N$ big enough so that

$$\left| \frac{\tilde{d}_g(h(x), h(z))}{\tilde{d}_f(x, z)} - D_h^{wu}(x) \right| < \varepsilon.$$

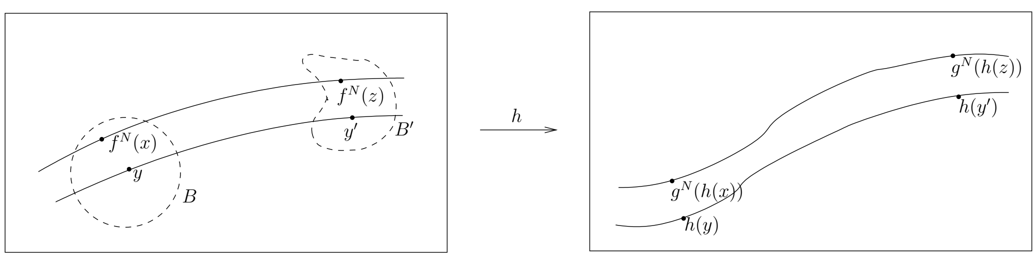


**Figure 2.1.** Differentiability of $h$ at the point $y$.

Now we are ready to do the estimates

$$\begin{aligned}
\tilde{d}_g(h(y), h(y')) &= \varepsilon_1 + \tilde{d}_g\left(h(f^N(x)), h(f^N(z))\right) \\
&= \varepsilon_1 + \tilde{d}_g\left(g^N(h(x)), g^N(h(z))\right) \\
&= \varepsilon_1 + \prod_{i=0}^{N-1} D_g^{wu}\left(g^i(h(x))\right) \cdot \tilde{d}_g\left(h(x), h(z)\right) \\
&= \varepsilon_1 + \prod_{i=0}^{N-1} D_g^{wu}\left(h(f^i(x))\right)(D_h^{wu}(x) + \varepsilon_2)\tilde{d}_f(x, z) \\
&= \varepsilon_1 + \frac{\prod_{i=0}^{N-1} D_g^{wu}\left(h(f^i(x))\right)}{\prod_{i=0}^{N-1} D_f^{wu}(f^i(x))}(D_h^{wu}(x) + \varepsilon_2)\tilde{d}_f(f^N(x), f^N(z))
\end{aligned}$$

$$= \varepsilon_1 + \frac{P(f^N(x))}{P(x)}(D_h^{wu}(x) + \varepsilon_2)\tilde{d}_f(y, y')$$
$$= \varepsilon_1 + \frac{P(y) + \varepsilon_3}{P(x)}(D_h^{wu}(x) + \varepsilon_2)\tilde{d}_f(y, y')$$

with $\max(|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3|) \le \varepsilon$. Now letting $\varepsilon$ go to 0 we get (2.15).

To show (2.10) define

$$\tilde{\rho}_g(h(x), h(y)) = \frac{D_h^{wu}(y)}{D_h^{wu}(x)}\rho_f(x, y).$$

Then

$$\tilde{\rho}_g\left(g(h(x)), g(h(y))\right) = \tilde{\rho}_g\left(h(f(x)), h(f(y))\right)$$
$$= \frac{D_h^{wu}(f(y))}{D_h^{wu}(f(x))}\rho_f(f(x), f(y)) = \frac{D_h^{wu}(f(y))}{D_h^{wu}(f(x))} \cdot \frac{D_f^{wu}(y)}{D_f^{wu}(x)}\rho_f(x, y)$$
$$= \frac{D_{h\circ f}^{wu}(y)}{D_{h\circ f}^{wu}(x)}\rho_f(x, y) = \frac{D_{g\circ h}^{wu}(y)}{D_{g\circ h}^{wu}(x)}\rho_f(x, y) = \frac{D_g^{wu}(h(y))}{D_g^{wu}(h(x))} \cdot \frac{D_h^{wu}(y)}{D_h^{wu}(x)}\rho_f(x, y) =$$
$$\frac{D_g^{wu}(h(y))}{D_g^{wu}(h(x))}\tilde{\rho}_g(h(x), h(y)).$$

This by $(P4)$ implies that $\tilde{\rho}_g = \rho_g$ which is equivalent to (2.10).

**Step 2.** Let $x_0$ be a fixed point for $f$ and let $V_0$ be an open bounded neighborhood of $x_0$ in $W_f^{wu}(x_0)$. Consider a probability measure $\eta^0$ supported on $V_0$ with density proportional to $\rho_f(x_0, \cdot)$. For $n > 0$ define

$$V_n = f^n(V_0),\ \eta^n = (f^n)_*\eta^0$$

so that $\eta^n$ is supported on $V_n$ and has density proportional to $\rho_f(x_0, \cdot)$ by $(P3)$.

Let $\mu^n = \frac{1}{n}\sum_{i=0}^{n-1}\eta^i$. By the Krylov-Bogoljubov theorem $\{\mu^n; n \ge 0\}$ is weakly compact and any of its limits is $f$-invariant. Let $\mu$ be a one of those limits along a subsequence $\{n_k; k \ge 1\}$. We would like to prove that $\mu$ has absolutely continuous conditional measures on the pieces of weak unstable foliation.

Let us be more precise. Consider a small open set $X \subset \mathbb{T}^3$ which can be decomposed in the following way

$$X = \bigcup_{y \in Y} W_f^{wu}(y, R_y).$$

Here $Y$ is a two dimensional transversal. To simplify the notation let $W(y) = W_f^{wu}(y, R_y)$. Denote by $\mu_T$ the transverse measure on $Y$: for $Y' \subset Y$ $\mu_T(Y') = \mu(\cup_{y \in Y'} W(y))$. Similary define $\eta_T^n$ and $\mu_T^n$. Obviously $\mu_T^{n_k} \to \mu_T$ weakly as $k \to \infty$. We show that for $\mu_T$ almost every $y$, $y \in Y$ the conditional measure $\mu_y$ on the local leaf $W(y)$ is absolutely continuous with respect to Lebesgue measure $m_y$ on $W(y)$.

The conditional measures are characterized by the following property

$$\forall F \in C(X) \quad \int_X F d\mu = \int_Y d\mu_T(y) \int_{W(y)} F(y, z) d\mu_y(z). \tag{2.16}$$

First we look at conditional measures of $\eta^n$. We fix $X$ and $Y$ as above and we assume that the end points of $V_n$ lie outside of $X$. Let $\{a_1, a_2, \ldots a_m\} = Y \cap V_n$. Then the formulas for the transverse measure and conditional measures are obvious:

$$\eta_T^n = \sum_{i=1}^{m} \left( \int_{W(a_i)} \rho_f(x_0, z) dm_{a_i}(z) \right) \delta(a_i),$$

$$d\eta_y^n(z) = \left( \int_{W(y)} \rho_f(y, z) dm_y(z) \right)^{-1} \rho_f(y, z) dm_y(z). \tag{2.17}$$

Notice that $\eta_y^n$ actually do not depend on $n$.

The goal now is to show that $d\mu_y = \left( \int_{W(y)} \rho_f(y, z) dm_y(z) \right)^{-1} \rho_f(y, \cdot)$ for almost every $y$. It could happen that the end points of $V_n$ lie inside of $X$. Support $S_n$ of $\eta_T^n$ consists of finitely many points. Some of these points correspond to the end points of $V_n$. Denote the set of these points by $B_n$, $|B_n| \in \{0, 1, 2\}$. Let $A_n = S_n \backslash B_n$ then there is a natural decomposition of the transverse measure $\eta_T^n$

$$\eta_T^n = \sum_{a \in A_n} \left( \int_{W(a)} \rho(x_0, z) dm_a(z) \right) \delta(a)$$

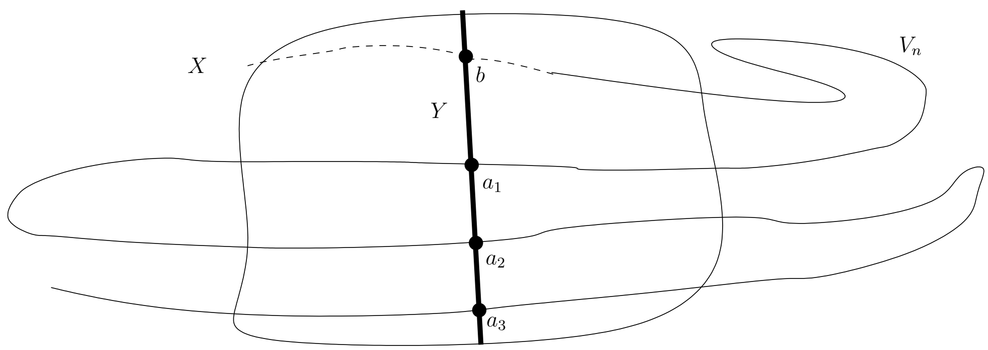


**Figure 2.2.** Decomposition of the transverse measure $\nu_T^n$.

$$+\sum_{b\in B_n}\left(\int_{W(b)\cap V_n}\rho(x_0,z)dm_b(z)\right)\delta(b)=\eta^n_{(T,A)}+\eta^n_{(T,B)}.$$

The conditional measures $\eta_y^n$ for $y\notin B_n$ are given by formula (2.17). Since $W_f^{wu}$ is uniformly expanding it is clear that $\eta_T^n(B_n)\to 0$ as $n\to\infty$. Hence

$$\frac{1}{n}\left(\sum_{i=0}^{n-1}\eta_T^i(B_i)\right)\to 0,\quad n\to\infty \tag{2.18}$$

and

$$\mu_T=\lim_{k\to\infty}\mu_T^{n_k}=\lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\eta^i_{(T,A)}. \tag{2.19}$$

Consider a continuous function $F$ on $X$.

$$\begin{aligned}\int_X Fd\mu=\lim_{k\to\infty}\int_X Fd\mu^{n_k}&=\lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\int_X Fd\eta^{n_k}\\ &=\lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\int_Y d\eta_T^{n_k}(y)\int_{W(y)}F(y,z)d\eta_k^{n_k}(z)\\ &=\lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\int_Y d\eta^{n_k}_{(T,A)}(y)\int_{W(y)}F(y,z)d\eta_y^{n_k}(z)\\ &\quad+\lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1}\int_Y d\eta^{n_k}_{(T,B)}(y)\int_{W(y)}F(y,z)d\eta_y^{n_k}(z).\end{aligned}$$

The function $F$ is bounded so it follows from (2.18) that the last limit is zero. So we get

$$\int_X F d\mu = \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \int_Y d\eta_{(T,A)}^{n_k}(y) \int_{W(y)} F(y,z) d\eta_y^{n_k}(z)$$
$$= \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \int_Y d\eta_{(T,A)}^{n_k}(y) \Big( \int_{W(y)} \rho_f(y,z) dm_y(z) \Big)^{-1} \int_{W(y)} F(y,z) \rho_f(y,z) dm_y(z).$$

Now notice that the function that we integrate with respect to $\eta_{(T,A)}^{n_k}$ is continuous and does not depend on $n_k$. Hence using (2.19) we get

$$\int_X F d\mu$$
$$= \lim_{k\to\infty} \int_Y d\mu_T^{n_k}(y) \left( \int_{W(y)} \rho_f(y,z) dm_y(z) \right)^{-1} \int_{W(y)} F(y,z) \rho_f(y,z) dm_y(z)$$
$$= \int_Y d\mu_T(y) \left( \int_{W(y)} \rho_f(y,z) dm_y(z) \right)^{-1} \int_{W(y)} F(y,z) \rho_f(y,z) dm_y(z)$$

and by (2.16) we see that up to normalization the density of the conditional measure on $W(y)$ is equal to $\rho_f(y,\cdot)$ for $\mu_T$ a. e. $y$.

The leaf $W_f^{wu}(x_0)$ is dense in $\mathbb{T}^3$ since $W_f^{wu}(x_0) = h_f^{-1}\left(W_L^{wu}(h_f^{-1}(x_0))\right)$ and $W_L^{wu}(h_f^{-1}(x_0))$ is a dense irrational line in $\mathbb{T}^3$. Hence the support $\mu$ is the whole torus.

**Step 3.** To prove that $\mu$ a. e. point is transitive we fix a ball in $\mathbb{T}^3$ and show that a. e. point visits the ball infinitely many times. Then to conclude transitivity we only need to cover $\mathbb{T}^3$ by a countable collection of balls such that every point is contained in an arbitrarily small ball.

So let us fix a ball $B'$ and a slightly smaller ball $B$, $B \subset B'$. Let $\psi$ be a non-negative continuous function supported on $B'$ and equal to 1 on $B$. By Birkhoff ergodic theorem

$$E(\psi|\mathcal{I}) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi \circ f^i \tag{2.20}$$

where $\mathcal{I}$ is $\sigma$-algebra of $f$-invariant sets.

Let $A = \{x : E(\psi|\mathcal{I})(x) = 0\}$. Then $\mu(A \cap B) = 0$ since

$$\int_A \psi d\mu = \int_A E(\psi|\mathcal{I}) d\mu = 0.$$

Hence

$$E(\psi|\mathcal{I})(x) > 0 \text{ for } \mu \text{ a.e. } x \in B.$$

Since $h_f(W_f^{wu}) = W_L^{wu}$ it is possible to find $R > 0$ such that $\cup_{b\in B} W(b, R) = \mathbb{T}^3$.

*Remark.* This observation also implies that $\mu$ has full support.

Applying the standard Hopf argument we get that for $\mu$ a. e. $x$ the function $E(\psi|\mathcal{I})$ is constant on $W(x, R)$. Now absolute continuity of $W_f^{wu}$ together with above observations shows that $E(\psi|\mathcal{I}) > 0$ for $\mu$ a. e. $x$ which means according to (4.24) that a. e. $x$ visits $B'$ infinitely many times. □

### 2.3.4 Strong unstable foliations match

Let us point out once again that in the proof of Lemma 2.2.6 we only use *wu*-differentiability of $h$ which as we showed is equivalent to coincidence of p. d. in the weak unstable direction.

*Proof of Lemma 2.2.6.* We will be working on two dimensional leaves of $W_f^u$. We know that each of these leaves is subfoliated by $W_f^{wu}$ as well as by $W_f^{su}$. The goal is to prove that $h(W_f^{su}) = W_g^{su}$ so we consider the foliation $U = h^{-1}(W_g^{su})$. As for usual foliations $U(x)$ stands for the leaf of $U$ passing through $x$ and $U(x, R)$ stands for the local leaf of size $R$. Obviously $U$ subfoliate $W_f^u$. A priori the leaves of $U$ are just Hölder continuous curves. Since weak unstable foliations match we see that a leaf $U(x)$ intersects each $W_f^{wu}(y)$, $y \in W_f^u(x)$ exactly once.

Let us prove several auxiliary claims.

**Claim 1.** *Consider a point $a \in \mathbb{T}^3$. Suppose that there is a point $b \neq a$, $b \in W_f^{su}(a) \cap U(a)$. Let $c \in W_f^{wu}(a)$ and $d = W_f^{wu}(b) \cap W_f^{su}(c)$, $e = W_f^{wu}(b) \cap U(c)$. Then $d = e$.*

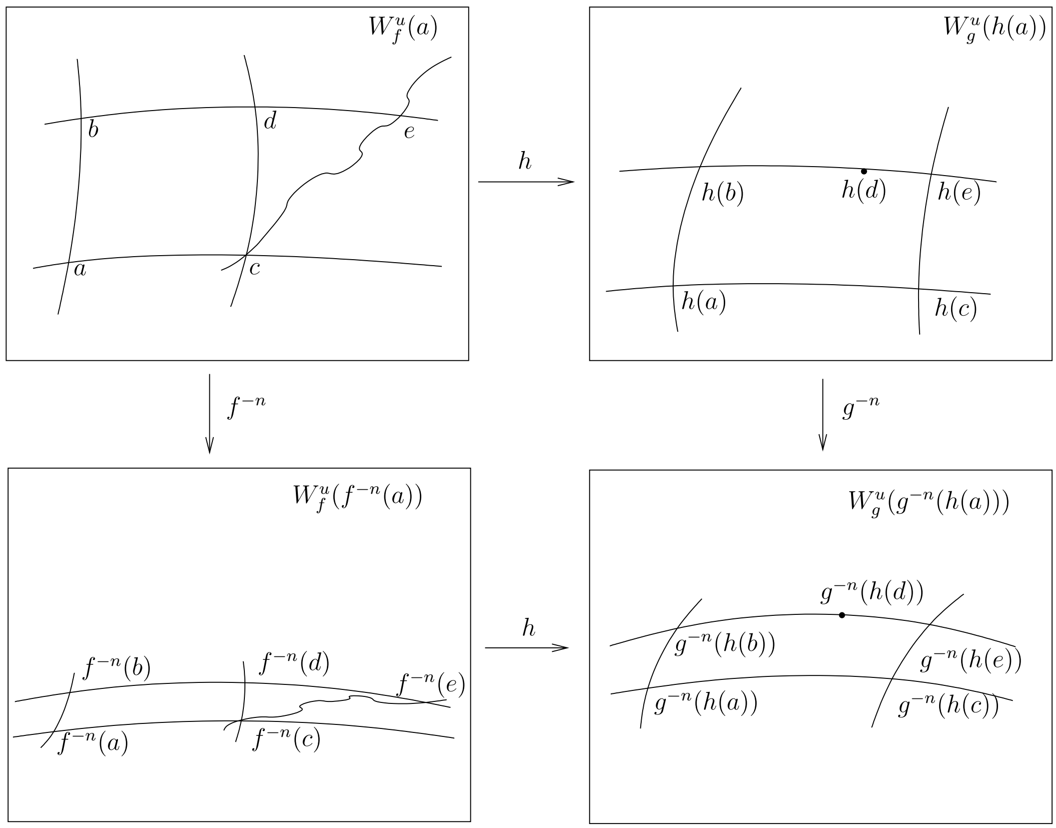


**Figure 2.3.** Illustration to the proof of the Claim 1. Notice that the actual size of the bottom pictures should be much smaller.

Assume that $d \neq e$. For the sake of concreteness we also assume that $d$ lies between $b$ and $e$. We look at configurations $\{a, b, c, d, e\} \in W^u_f(a)$, $\{h(a), h(b), h(c), h(d), h(e)\} \in W^u_g(h(a))$ and study their evolution under $f^{-n}, n > 0$ and $g^{-n}, n > 0$ respectively. Since under the action of $f^{-1}$ strong unstable leaves contract exponentially faster then weak unstable leaves we get that

$$\forall \varepsilon > 0 \;\; \exists n_0 : \; \forall n > n_0 \quad \left| \frac{d^{wu}_f(f^{-n}(a), f^{-n}(c))}{d^{wu}_f(f^{-n}(b), f^{-n}(d))} - 1 \right| < \varepsilon. \tag{2.21}$$

Analogously

$$\forall \varepsilon > 0 \;\; \exists n_1 : \; \forall n > n_1 \quad \left| \frac{d^{wu}_g\left(g^{-n}(h(a)), g^{-n}(h(c))\right)}{d^{wu}_g\left(g^{-n}(h(b)), g^{-n}(h(e))\right)} - 1 \right| < \varepsilon. \tag{2.22}$$

The next statement is a direct corollary of (D1) and (D2). There exists a $\delta > 0$ which depends on the initial configuration $\{a, b, c, d, e\}$ such that

$$\forall n > 0 \qquad \frac{d_f^{wu}(f^{-n}(b), f^{-n}(e))}{d_f^{wu}(f^{-n}(b), f^{-n}(d))} > 1 + \delta. \tag{2.23}$$

Combining (2.21) and (2.23) we get

$$\exists \delta' > 0 : \; \forall n > n_0 \qquad \frac{d_f^{wu}(f^{-n}(b), f^{-n}(e))}{d_f^{wu}(f^{-n}(a), f^{-n}(c))} > 1 + \delta'. \tag{2.24}$$

On the other hand we know that $h$ is continuously $wu$-differentiable, hence

$$\forall \varepsilon > 0 \;\; \exists n_2 : \; \forall n > n_2 \qquad \left| \frac{d_g^{wu}\left(g^{-n}(h(a)), g^{-n}(h(c))\right)}{d_f^{wu}(f^{-n}(a), f^{-n}(c))} - D_h^{wu}(f^{-n}(a)) \right| < \varepsilon$$

$$\text{and} \quad \left| \frac{d_g^{wu}\left(g^{-n}(h(b)), g^{-n}(h(e))\right)}{d_f^{wu}(f^{-n}(b), f^{-n}(e))} - D_h^{wu}(f^{-n}(a)) \right| < \varepsilon. \tag{2.25}$$

It is easy to see that (2.24) contradicts (2.25) and (2.22) so we are done.

**Claim 2.** *Consider a weak unstable leaf $W_f^{wu}(a)$ and $b \in W_f^{su}(a)$, $b \neq a$. For any $y \in W_f^{wu}(a)$ let $y' = W_f^{wu}(b) \cap W_f^{su}(y)$. Then $\exists c_1, c_2 > 0$ such that $\forall y \in W_f^{wu}(a)$ $c_1 > d_f^{su}(y, y') > c_2$.*

Recall that $h_f(W_f^{wu}) = W_L^{wu}$. The leaves $W_L^{wu}(h_f(a))$ and $W_L^{wu}(h_f(b))$ are parallel lines in $W_L^u(h_f(a))$ that are fixed distance apart. Hence the estimate from below is a direct consequence of uniform continuity of $h_f|_{W_f^u}(a)$ with respect to metrics $d_f^u$ and $d_L^u$.

Now we prove the estimate from above. We need to show that the strip between $W_f^{wu}(a)$ and $W_f^{wu}(b)$ cannot contain arbitrarily long pieces of strong unstable leaves. The reason for this is uniform transversality of weak unstable and strong unstable foliation.

For any positive number $\varepsilon$ we can choose a finite number of points $\{a_0 = a, a_1, a_2, \ldots a_m = b\}$ between $a$ and $b$ on $W_f^{su}(a)$ in such a way that $W_f^{wu}(a_i)$ is contained in $\varepsilon$-neighborhood of $W_f^{wu}(a_{i-1})$ and vice versa, $i = 1, \ldots m$. Again this is possible because $W_L^{wu}(h_f(a_i))$, $i = 0, \ldots m$ are parallel lines and $h_f^{-1}|_{W_L^u(h_f(a))}$ is uniformly continuous.

Let $s = \min_{x\in\mathbb{T}^3} \measuredangle(E_f^{wu}(x), E_f^{su}(x))$. Choose a small $\delta > 0$ such that in any ball $B$ of size $\delta$

$$\max_{x,y\in B} \max\{\measuredangle(E_f^{wu}(x), E_f^{wu}(y)), \measuredangle(E_f^{su}(x), E_f^{su}(y))\} < \frac{s}{10}.$$

In such a ball the direction of $E_f^{wu}$ is almost constant comparing to the angle between $E_f^{wu}$ and $E_f^{su}$. Clearly it is possible to choose a small $\varepsilon = \varepsilon(s, \delta)$ and correspondingly the points $\{a_0, a_1, \dots a_m\}$ as above such that any strong unstable leave crosses the strip between $W_f^{wu}(a_{i-1})$ and $W_f^{wu}(a_i)$ in a ball of size $\delta$, $i = 1, \dots m$. This gives us uniform estimates on the lengths of pieces of strong unstable leaves in the strips between $W_f^{wu}(a_{i-1})$ and $W_f^{wu}(a_i)$, $i = \overline{1,m}$. The sum of these estimates gives us the desired uniform estimate from above.

**Claim 3.** *Suppose $\exists a \in \mathbb{T}^3$ and $R > 0$ such that $W_f^{su}(a, R) = U(a, R)$ then $W_f^{su} = U$.*

Consider a point $c \in W_f^{wu}(a)$ then applying Claim 1 to the points $b \in W_f^{su}(a, R)$ we get that $\exists R_c > 0$ such that $W_f^{su}(c, R_c) = U(c, R_c)$. Moreover by Claim 2 numbers $R_c$, $c \in W_f^{wu}(a)$ are uniformly bounded away from zero. Now the statement follows from denseness of $W_f^{wu}(a)$ in $\mathbb{T}^3$.

We are ready to prove the lemma.

We say that $W_f^{su}(x)$ and $U(x)$ intersect transversally at $y$ if $y \in W_f^{su}(x) \cap U(x)$ and $\forall R > 0$ the local leaf $U(y, R)$ lies on both sides of $W_f^{su}(y)$.

We consider two cases.

**Case 1.** *At every periodic point $x_0$ the leaves $W_f^{su}(x_0)$ and $U(x_0)$ do not intersect transversally at a point different from $x_0$.*

Notice that the property of having a transverse intersection is stable — if $W_f^{su}(x)$ and $U(x)$ intersect transversally then there is a neighborhood $V$ of $x$ such that $\forall z \in V$ $W_f^{su}(z)$ and $U(z)$ intersect transversally. Periodic points are dense therefore absence of transverse intersections at periodic points leads to absence of transverse intersections at all points.

We assume that $W_f^{su} \neq U$. Then the above observation together with Claim 3 tell us that for any point $x$ the leaves $W_f^{su}(x)$ and $U(x)$ intersect only at $x$. Let $x_0$ to be a fixed point of $f$. For each $y \in U(x_0)$ the leaf $W_f^{su}(y)$ intersects $U(x_0)$

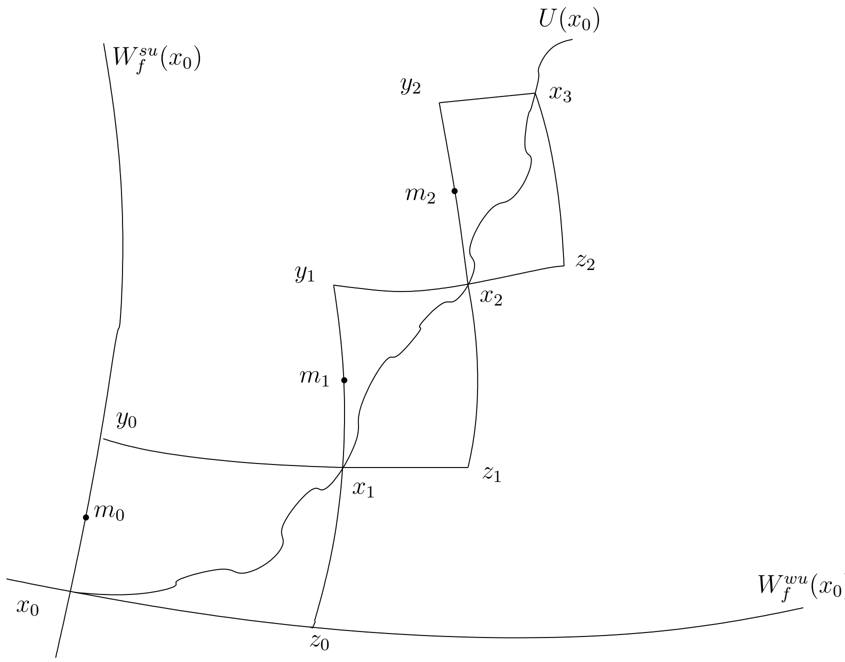


**Figure 2.4.** The ladder of rectangles.

only at $y$. Thus we are able to build a ladder of rectangles in $W_f^u(x_0)$ as shown on the Figure 2.4. The sides of the rectangles are pieces of weak unstable and strong unstable leaves. The rectangles are subject to condition

$$d_f^{su}(x_i, y_i) = 1, \ i \geq 0.$$

This guarantees that after the choice of $y_0$ (there are two choices) the sequence of rectangles is defined uniquely. Let $d_i = d_f^{wu}(y_i, x_{i+1}), \ i \geq 0$ and let $\{m_i; \ i \geq 0\}$ be midpoints on the sides of rectangles as shown on the picture.

Suppose that $\inf_{i\geq 0} d_i > 0$. Apply $f^{-n}, \ n > 0$ to the ladder of rectangles. The leaf $U(x_0)$ is invariant while the rectangles shrink and become flat as shown on Figure 2.5. Namely

$$\forall \varepsilon > 0 \ \exists n_0 : \ \forall n > n_0 \quad \text{and} \ \forall i \geq 0 \ \ \frac{d_f^{su}(f^{-n}(x_i), f^{-n}(y_i))}{d_f^{wu}(f^{-n}(y_i), f^{-n}(x_{i+1}))} < \varepsilon.$$

This means that in any fixed bounded neighborhood of $x_0$ the leaf $U(x_0)$ is arbitrarily close to $W_f^{wu}(x_0)$. In particular we have that $x_1$ is arbitrarily close to

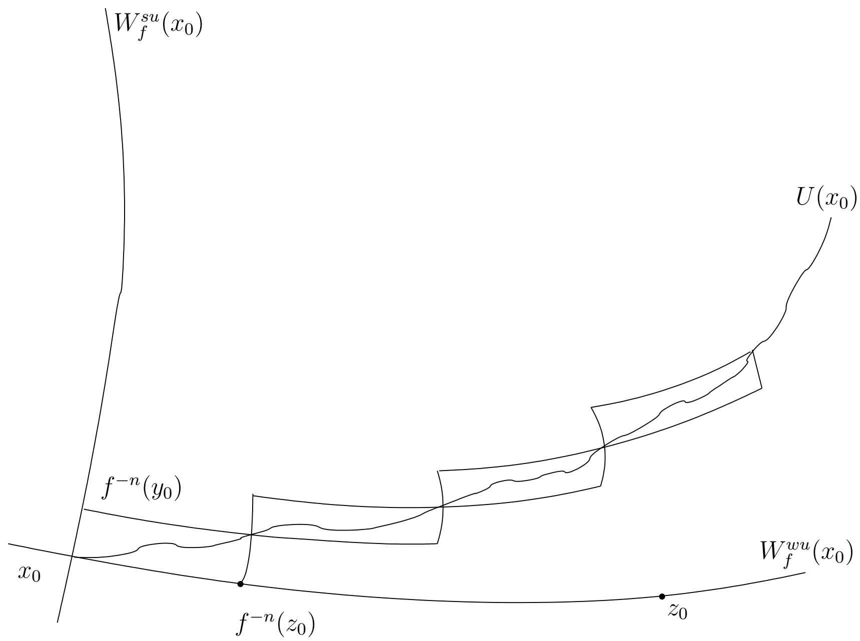


**Figure 2.5.** Ladder of rectangles after several iterations.

$z_0$ while we know that they are some fixed distance apart. To make this argument completely rigorous one needs carry out an estimate on the distance between $z_0$ and $x_1$ using regularity of holonomies along $W_f^{wu}$ and $W_f^{su}$ inside of the leaf $W_f^u(x_0)$. We conclude that $\inf_{i\geq 0} d_i = 0$.

Then choose a subsequence $\{m_{n_k}; k \geq 0\}$ such that corresponding rectangles have width going to zero as $k$ tend to infinity. Each of these rectangles contains a piece of $U(x_0)$ inside of it. Let $m$ be an accumulation point of $\{m_{n_k}; k \geq 0\}$ considered as a sequence of points in $\mathbb{T}^3$ rather than on $W_f^u(x_0)$. Since the width of the rectangles is shrinking and the foliations are continuous we get that $W_f^{su}(m, \frac{1}{2}) = U(m, \frac{1}{2})$. Hence $W_f^{su} = U$ by Claim 3 and we move on to the second case.

**Case 2.** *There exist a periodic point $x_0$ and a point $y_0, y_0 \neq x_0$ such that $W_f^{su}(x_0)$ and $U(x_0)$ intersect at $y_0$ transversally.*

Without loss of generality we can assume that $x_0$ is a fixed point. We chose a sequence $\{x_i \in W_f^{wu}(x_0); i \geq 1\}$ such that $x_i \to y_0$, $i \to \infty$. Here and afterwards we speak about convergence on the torus, not in the leaf $W_f^u(x_0)$. By Claim 1 we know that for any $i$ the leaves $W_f^{wu}(y_0)$, $W_f^{su}(x_i)$ and $U(x_i)$ intersect at one point $z_i$. Up to the choice of a subsequence we have that $z_i \to y_1$, $i \to \infty$, where $y_1$ is some point on $W_f^{su}(y_0)$. Since the foliation $U$ is continuous we have that $y_1 \in U(y_0) = U(x_0)$ as well. The strong unstable foliation is orientable and the pairs $(x_0, y_0), (x_i, z_i), i \geq 1$ have the same orientation i. e. $y_0$ lies between $x_0$ and $y_1$. Now we would like to

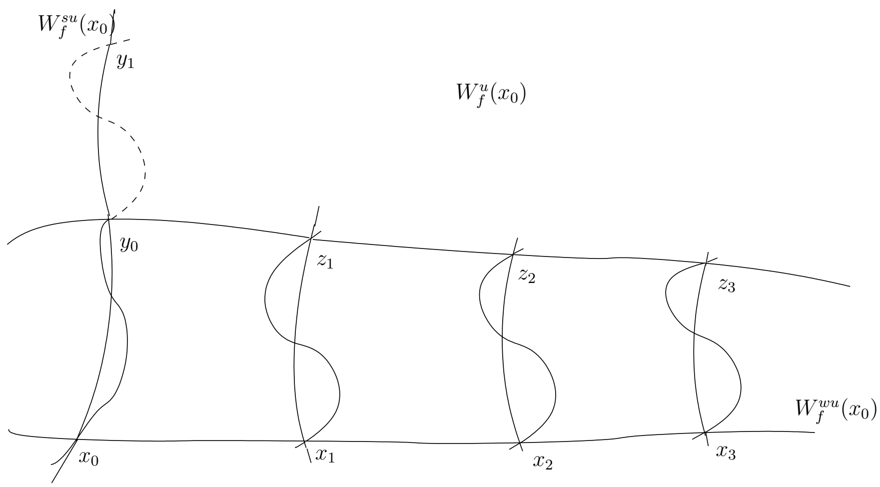


**Figure 2.6.** Curves $U(x_i)$ that pass through $x_i$ and $z_i$ are the preimages of the strong unstable manifolds. The leaf $W_f^u(x_0)$ is immersed into $\mathbb{T}^3$. In $\mathbb{T}^3$ curves $U(x_i)$ converge to the curve $U(y_0)$ (dashed curve in the picture). Hence $U(x_0)$ intersects $W_f^{su}(x_0)$ at $y_1$ with $d_f^{su}(x_0, y_0) \approx d_f^{su}(y_0, y_1)$.

repeat the procedure. Consider another sequence $\{\tilde{x}_i \in W_f^{wu}(x_0); i \geq 1\}$, $\tilde{x}_i \to y_1$ as $i \to \infty$ and corresponding sequence $\{\tilde{z}_i \in W_f^{wu}(y_0); i \geq 1\}$. Then $\tilde{z}_i \to y_2$ as $i \to \infty$, $y_2 \in W_f^{su}(x_0) \cap U(x_0)$. In this way by induction we obtain a sequence of points $\{y_i \in W_f^{su}(x_0) \cap U(x_0); i \geq 1\}$. These points are ordered on $W_f^{su}(x_0)$ — for any positive $i$ point $y_{i-1}$ lies between $x_0$ and $y_i$. By Claim 2 we know that there are constants $c_1$ and $c_2$ which depend only on the initial choice of $x_0$ and $y_0$ such that $\forall i \geq 0 \quad c_1 > d_f^{su}(y_i, y_{i+1}) > c_2$. This guarantees that the set $\{f^{-n}(y_i); n \geq 0, i \geq 0\} \subset W_f^{su}(x_0) \cap U(x_0)$ is dense and hence applying Claim 3 one more time we get that $W_f^{su} = U$. $\square$

### 2.3.5 Remarks

We did not discuss the proofs of Lemmas 2.2.3 and 2.2.7. They can be carried out in the same way as the proof of Lemma 2.2.5. The technical difficulty with constructing special measure is not present. One can use SRB measures instead (as a matter of fact the construction in Step 2 applied to $W_f^s$ and $W_f^{su}$ will produce SRB measures).

Notice that we used the assumption that $f, g \in \mathcal{U}$ only to prove Lemmas 2.2.1 and 2.2.2. So for Theorem 5 we only need to reprove these two lemmas in the new setting. We use a result from [BI07] that states the following.

**Theorem 6.** *Let $f$ be a partially hyperbolic diffeomorphism of $\mathbb{T}^3$. Then the lifts of stable and unstable foliations are quasi-isometric and the hence the central distribution is uniquely integrable.*

Thus Lemma 2.2.1 is automatic. Proof of Lemma 2.2.2 go through with minor differences since we know that $W_f^{su}$ is quasi-isometric.

The bootstrap of regularity of $h$ to the regularity of $f$ and $g$ cannot be done straightforwardly. The reason is the lack of smoothness of weak unstable foliation. Let $N = [\log \lambda_3 / \log \lambda_2]$. It is known [LW95] that given $f$ sufficiently $C^1$-close to $L$ the individual leaves of weak unstable foliation are $C^N$ immersed curves. In general the the leaves of weak untable foliation cannot be more than $C^N$ smooth. An example was constructed in [JPL95]. Hence our method cannot lead to smoothness higher than $C^N$.

Chapter 3

# Smooth conjugacy of Anosov diffeomorphisms on $\mathbb{T}^d$

## 3.1 Formulation of the results

The main objective of this Chapter is to prove Theorem A. Recall the statement.

**Theorem A.** *Let $L$ be a hyperbolic automorphism of $\mathbb{T}^d, d \geq 3$, with simple real spectrum. Assume that characteristic polynomial of $L$ is irreducible over $\mathbb{Z}$. There exists a $C^1$-neighborhood $\mathcal{U} \subset \mathrm{Diff}^r(\mathbb{T}^d)$, $r \geq 2$, of $L$ such that any $f \in \mathcal{U}$ satisfying $\mathcal{A}$ and any $g \in \mathcal{U}$ with the same p. d. are $C^{1+\nu}$ conjugate.*

*Remark.* Obviously analogous result holds on finite factors of tori. But we do not know how prove it on nilmanifolds. The problem is that for an algebraic Anosov automorphism of a nilmanifold various intermediate distributions may happen to be non-integrable.

Theorem A is a generalization of the three dimensional result of the previous chapter. Our method does not lead to higher regularity of the conjugacy (see the last section of the previous chapter for an explanation). Nevertheless we conjecture that the situation is the same as in dimension two.

**Conjecture 2.** *In the setup of Theorem A one can actually conclude that $f$ and $g$ are $C^{r-\varepsilon}$ conjugate, where $\varepsilon$ is an arbitrarily small positive number.*

Simple examples of diffeomorphisms that possess Property $\mathcal{A}$ include $f = L$ and any $f \in \mathcal{U}$ when $\max(k,l) \le 2$ (see Section 3.2.1). In addition we construct a $C^1$-open set of Anosov diffeomorphisms of $\mathbb{T}^5$ and $\mathbb{T}^6$ close to $L$ that have Property $\mathcal{A}$. It seems that this construction can be extended to arbitrary dimension.

We describe this open set when $l = 2$ and $k = 3$. Given $f \in \mathcal{U}$ denote by $D_f^{wu}$ the derivative of $f$ along $V_1^f$. Choose $f \in \mathcal{U}$ in such a way that

$$\forall x \neq x_0 \quad D_f^{wu}(x) > D_f^{wu}(x_0),$$

where $x_0$ is a fixed point of $f$. Then any diffeomorphism sufficiently $C^1$ close to $f$ possess Property $\mathcal{A}$.

In Sections 3.2 and 3.3 we discuss Property $\mathcal{A}$ and construct examples of diffeomorphisms that satisfy Property $\mathcal{A}$. These sections are self-contained.

Section 3.4 is devoted to the proof of our main result, Theorem A.

## 3.2 On the Property $\mathcal{A}$

### 3.2.1 Transitivity versus minimality

Here we discuss Property $\mathcal{A}$. Let $\mathcal{F}$ be a foliation of a compact manifold $M$. As usually $\mathcal{F}(x)$ stands for the leaf of $\mathcal{F}$ that contains $x$ and $\mathcal{F}(x, R)$ stands for the ball of radius $R$ centered at $x$ inside of $\mathcal{F}(x)$.

*Definition* 1. Foliation $\mathcal{F}$ is called minimal if every leaf of $\mathcal{F}$ is dense in $M$.

*Definition* 2. Foliation $\mathcal{F}$ is called transitive if these exists a leaf of $\mathcal{F}$ that is dense in $M$.

*Definition* 3. Foliation $\mathcal{F}$ is called tubularly minimal if for every $x$ and every open ball $B \ni x$

$$\overline{\bigcup_{y \in B} \mathcal{F}(y)} = M.$$

Property $\mathcal{A}$ simply requires foliations $U_{l-1}^f$, $U_{l-2}^f$,... $U_1^f$, $V_1^f$, $V_2^f$,... $V_{k-1}^f$ to be tubularly minimal.

**Property $\mathcal{A}'$.** Foliations $U_{l-1}^f$, $U_{l-2}^f$,... $U_1^f$, $V_1^f$, $V_2^f$,... $V_{k-1}^f$ are minimal. $(\mathcal{A}')$

**Proposition 7.** *Foliation $\mathcal{F}$ is transitive if and only if it is tubularly minimal.*[1]

*Proof.* Transitivity obviously implies tubular minimality.

Assume that $\mathcal{F}$ is tubularly minimal. Let $\{B_n, n \geq 1\}$ be a countable basis for the topology of $M$. By the definition of tubular minimality sets $\mathcal{F}(B_n)$ are open and dense in $M$. Hence by Baire category theorem we have the set

$$B = \bigcap_{n \geq 1} \mathcal{F}(B_n)$$

is non-empty. For every $x \in B$ the leaf $\mathcal{F}(x)$ is dense in $M$. □

*Remark.* We define Property $\mathcal{A}$ in terms of tubular minimality as opposed to transitivity because we need denseness of the tubes to carry out the proof of Theorem A.

A priori, transitivity is weaker than minimality. Hence, a priori, Property $\mathcal{A}$ is weaker than Property $\mathcal{A}'$.

If in Theorem A we require $f$ to satisfy $\mathcal{A}'$ instead of $\mathcal{A}$ then the induction procedure that we use (induction step 1) is much simpler. Proof of the induction step 1 assuming only Property $\mathcal{A}$ requires much more lengthy and delicate argument. It is not clear to us what is the relation between Properties $\mathcal{A}$ and $\mathcal{A}'$. They may happen to be equivalent. Thus first we provide a proof of Theorem A assuming that $f$ has Property $\mathcal{A}'$. Then we present a separate proof of induction step 1 (namely Lemma 3.4.6) that uses only Property $\mathcal{A}$.

Minimality of a foliation can be characterized similarly to tubular transitivity.

**Proposition 8.** *Foliation $\mathcal{F}$ is minimal if and only if for every $x$ and every open ball $B \ni x$*

$$\bigcup_{y \in B} \mathcal{F}(y) = M.$$

The proof is simple so we omit it. As a corollary we get that foliation $\mathcal{F}$ is minimal if and only if for every $x$ and every open ball $B \ni x$ there exists a number $R$ such that

$$\bigcup_{y \in B} \mathcal{F}(y, R) = M. \tag{3.1}$$

This is the property which we will actually use in the proof of the induction step 1.

[1]We would like to thank the referee of [G08] for pointing out this fact.

### 3.2.2 Examples of diffeomorphisms that satisfy Property $\mathcal{A}$

**Proposition 9.** *Assume that* $L$ *is irreducible. Then foliations* $U_j^L$, $V_i^L$, $j = 1 \dots l$, $i = 1 \dots k$ *are minimal.*

*Proof.* Denote by $\mathcal{F}$ one of the foliations under consideration. Since $\mathcal{F}$ is a foliation by straight lines the closure of a leaf $\mathcal{F}(x)$ is a subtorus of $\mathbb{T}^d$. This subtorus lifts to a rational invariant subspace of $\mathbb{R}^d$. The invariant subspace corresponds to a rational factor of the characteristic polynomial of $L$ while we have assumed that it is irreducible over $\mathbb{Q}$. Hence the invariant subspace is the whole $\mathbb{R}^d$ and the subtorus is the whole $\mathbb{T}^d$. □

Hence the conclusion of Theorem A holds at least for $f = L$.

We will see in Section 3.4.1 that for any $f \in \mathcal{U}$ foliations $U_1^f$ and $V_1^f$ are minimal. Hence the conclusion of Theorem A holds for any $f \in \mathcal{U}$ if $\max(k, l) \le 2$.

It is easy to construct $f \ne L$ that satisfies $\mathcal{A}$ when $k = 3$ and $l = 2$ since we only have to worry about the foliation $V_2^f$. We let $f = s \circ L$ where $s$ is any small shift along $V_2^f$. Clearly $V_2^f = V_2^L$ and hence $f$ satisfies $\mathcal{A}$.

Question about robust minimality of foliations $U_{l-1}^f$, $U_{l-2}^f$,... $U_1^f$, $V_1^f$, $V_2^f$,... $V_{k-1}^f$ arises naturally. Robust minimality of strong stable and strong unstable foliations of partially hyperbolic systems received some attention in the literature due to its intimate connection with robust transitivity. See [Ma78] and more recent papers [BDU02], [PS06], where robust minimality of the *full* expanding foliation is established under some assumptions. We do not have this luxury in our setting: expanding foliations that we are intrested in subfoliate full unstable foliation. A representative problem here is the following.

**Question 2.** *Let* $L : \mathbb{T}^3 \to \mathbb{T}^3$ *be a hyperbolic linear automorphism with real spectrum* $\lambda_1 < 1 < \lambda_2 < \lambda_3$*. Consider one dimensional strong unstable foliation. Is it true that this foliation is robustly minimal? In other words, is it true that for any* $f$ *sufficiently* $C^1$*-close to* $L$ *the strong unstable foliation of* $f$ *is minimal?*

In addition to the simple examples above we construct a $C^1$-open set of diffeomorphisms that possess Property $\mathcal{A}$ in the next section. The following statement can be obtained by applying the construction and the arguments of the next section in the setup of Question 2.

**Proposition 10.** *Let $L$ be as in Question 2. Then there exists a $C^1$-open set $\mathcal{U}$ $C^1$-close to $L$ such that for every $f \in \mathcal{U}$ the strong unstable foliation of $f$ is transitive.*

## 3.3 An example of an open neighborhood of diffeomorphisms that possess Property $\mathcal{A}$

Let $L : \mathbb{T}^5 \to \mathbb{T}^5$ be a hyperbolic automorphism as in Theorem A, $l = 2, k = 3$, and let $\mathcal{U}$ be a $C^1$-neighborhood of $L$ chosen as in Section 3.4.1.

Recall that $D_f^{wu}$ stands for the derivative of $f \in \mathcal{U}$ along $V_1^f$. Choose $f \in \mathcal{U}$ in such a way that

$$\forall x \neq x_0 \quad D_f^{wu}(x) > D_f^{wu}(x_0), \tag{3.2}$$

where $x_0$ is a fixed point of $f$.

**Proposition 11.** *There exists a $C^1$-neighborhood $\tilde{\mathcal{U}}$ of $f$ such that any diffeomorphism $g \in \tilde{\mathcal{U}}$ has Property $\mathcal{A}$.*

*Remark.* Similar example can be constructed on $\mathbb{T}^6$ with $l = 3$, $k = 3$. We only need to do the trick described below for both stable and unstable manifolds of the fixed point $x_0$.

Before proving the proposition let us briefly explain the idea behind the proof. We know that $U_1^g$ and $V_1^g$ are minimal. Hence we only need to show that foliation $V_2^g$ is tubularly minimal i. e. for every $x \in \mathbb{T}^5$ and every open ball $B \ni x$

$$\overline{\bigcup_{y \in B} V_2^g(y)} = \mathbb{T}^5. \tag{3.3}$$

To illustrate the idea we take $g = f$ and $x = x_0$. We work on the universal cover $\mathbb{R}^5$ with lifted foliations. Let

$$\mathcal{T} \stackrel{\text{def}}{=} \bigcup_{y \in B} V_2^f(y) \subset \mathbb{R}^5, \tag{3.4}$$

which is an open tube.

We show that $\mathcal{T}$ contains arbitrarily long connected pieces of the leaves of $V_1^f$ as shown on Figure 3.1. It would follow that $\mathcal{T}$ is dense in $\mathbb{T}^5$. Indeed, foliation $V_1^f$ is not just minimal but uniformly minimal: for any $\varepsilon > 0$ there exists $R > 0$ such that $\forall z \in \mathbb{T}^5$ $V_1^f(z, R)$ is $\varepsilon$-dense in $\mathbb{T}^5$. This property follows from the fact that $V_1^f$ is conjugate to the linear foliation $V_1^L$.

Pick $y_0 \in B \cap V_1^f(x_0)$ close to $x_0$. Let $x \in V_2^f(x_0)$ be a point far away in the tube $\mathcal{T}$ and $y = V_1^f(x) \cap V_2^f(y_0)$. To show that $\mathcal{T}$ contains arbitrarily long pieces of leaves of $V_1^f$ we prove that $d_1^f(x, y)$ (recall that $d_i^f$ is the Riemannian distance along $V_i^f$) is unbounded function of $x$.

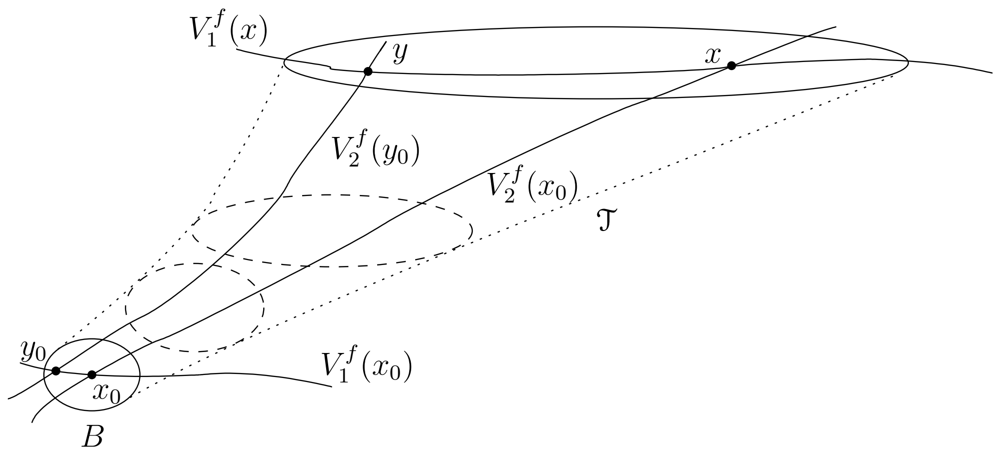


**Figure 3.1.** Tube $\mathcal{T}$ contains arbitrarily long pieces of leaves of $V_1^f$.

We make use of the affine structure on $V_1^f$. We refer to Chapter 2 for the definition of affine distance-like function $\tilde{d}_1$. Recall crucial properties of $\tilde{d}_1$

(D1) $\tilde{d}_1(x, y) = d_1^f(x, y) + o(d_1^f(x, y))$,

(D2) $\tilde{d}_1(f(x), f(y)) = D_f^{wu}(x)\tilde{d}_1(x, y)$,

(D3) $\forall K > 0\ \exists C > 0$ such that

$$\frac{1}{C}\tilde{d}_1(x, y) \le d_1^f(x, y) \le C\tilde{d}_1(x, y)$$

whenever $d_1(x, y) < K$.

By property (D3) it is enough to show that $\tilde{d}_1(x, y)$ is unbounded.

Given $x$ as above pick $N$ large so that the ratio

$$\tilde{d}_1(f^{-N}(x), f^{-N}(y))/\tilde{d}_1(x_0, f^{-N}(y_0))$$

is close to 1 as shown on the Figure 3.2. It is possible since $V_2^f$ contracts exponentially faster than $V_1^f$ under the action of $f^{-1}$.

It is not hard to see that given a large number $n$ we can pick $x$ (and $N$ correspondingly) far enough from $x_0$ so that at least $n$ points from the orbit $\{x, f^{-1}(x), \ldots f^{-N}(x)\}$ lie outside of $B$. For such a point $z = f^{-i}(x)$ that is not in $B$

$$D_f^{wu}(z) \geq D_f^{wu}(x_0) + \delta,$$

where $\delta > 0$ depends only on the size of $B$.

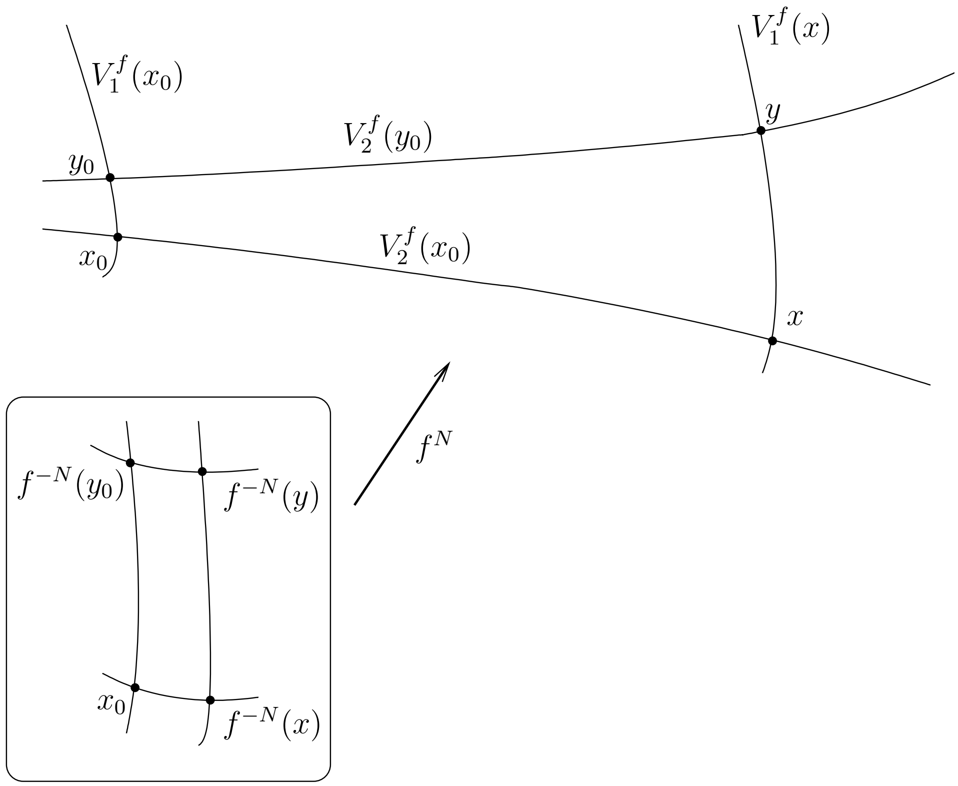


**Figure 3.2.** Illustration to the argument. Quadrilateral in the box is much smaller then the one outside.

Using (D2) we get

$$\frac{\tilde{d}_1(x,y)}{\tilde{d}_1(x_0,y_0)} = \prod_{i=1}^{N} \frac{D_f^{wu}(f^{-i}(x))}{D_f^{wu}(x_0)} \cdot \frac{\tilde{d}_1(f^{-N}(x), f^{-N}(y))}{\tilde{d}_1(x_0, f^{-N}(y_0))}$$
$$\geq \left(\frac{D_f^{wu}(x_0)+\delta}{D_f^{wu}(x_0)}\right)^n \cdot \frac{\tilde{d}_1(f^{-N}(x), f^{-N}(y))}{\tilde{d}_1(x_0, f^{-N}(y_0))} \quad (3.5)$$

which is an arbitrary large number. Hence $\tilde{d}_1(x,y)$ is arbitrarily large and we are done.

*Remark.* Although Proposition 11 deals with a pretty special situation we believe that the picture on Figure 3.1 is generic. To be more precise we think that for any $g \in \mathcal{U}$ the following alternative holds. Either $V_2^g$ is conjugate to the linear foliation $V_2^L$ or there exist a dense set $\Lambda$ such that for any $x \in \Lambda$ and any $B \ni x$ the tube

$$\bigcup_{y \in B} V_2^f(y) \subset \mathbb{R}^5$$

contains arbitrarily long connected pieces of the leaves of $V_1^g$.

*Proof of Proposition 11.* The argument is more delicate than the one presented above since we do not know that the minimum of the derivative is achieved at $x_0$.

Let $B_0$ be a small ball around $x_0$ and $B_1 \supset B_0$ be a bigger ball. Condition (3.2) guarantees that we can choose them in such a way that

$$m_0 < D_f^{wu}(x_0) < \sup_{x \in B_0} D_f^{wu}(x) < m_1 < M < \min_{x \notin B_1} D_f^{wu}(x)$$

with $m_0$, $m_1$ and $M$ satisfying

$$\frac{M m_0^{q-1}}{m_1^q} > 1, \quad (3.6)$$

where $q$ is an integer that depends only on the size of $\mathcal{U}$ and the size of $B_1$. After that we choose $\tilde{\mathcal{U}} \subset \mathcal{U}$ so the fixed point of $g$ (that corresponds to $x_0$) is inside of $B_0$ and the property above persists. Namely,

$$\forall g \in \tilde{\mathcal{U}} \quad m_0 < \inf_{x \in B_0} D_g^{wu}(x) < \sup_{x \in B_0} D_g^{wu}(x) < m_1 < M < \min_{x \notin B_1} D_f^{wu}(x). \quad (3.7)$$

Note that provided that $f$ is sufficiently $C^1$-close to $L$ and the ball $B_1$ is small enough any piece of a leaf of $V_2^g$ outside of $B_1$ that starts and ends on the boundary of $B_1$ cannot be homotoped into a point keeping the endpoints on the boundary. This is a minor technical detail that makes sure that the picture shown on Figure 3.3a does not occur. Thus there is a lower bound $R$ on the lengths of pieces of leaves of $V_2^g$ outside of $B_1$ with endpoints on the boundary of $B_1$. Obviously, there is also an upper bound $r$ on the lengths of pieces of leaves of $V_2^g$ inside $B_1$.

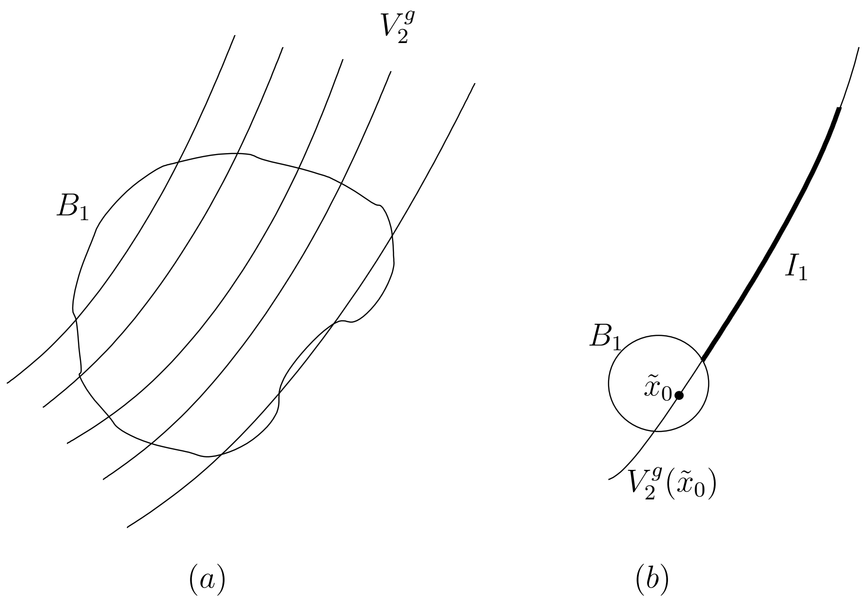


**Figure 3.3.** (a) does not occur if $B$ is sufficiently small; (b) choice of $I_1$.

It is enough to check (3.3) for a dense set $\Lambda$ of points $x \in \mathbb{T}^5$. We take $\Lambda$ to be a subset of the set of periodic points of $g$

$$\Lambda = \{p : \quad D^{wu}_{f^{n(p)}}(p) \le m_1^{n(p)}\}, \tag{3.8}$$

where $n(p)$ stands for the period of $p$. Set $\Lambda$ consists of periodic points that spend large but fixed percentage of time inside of $B_0$. It is fairly easy to show that $\Lambda$ is dense in $\mathbb{T}^5$. The proof is a trivial corollary of specification property (e. g. see [KH95]).

So we fix $\tilde{x}_0 \in \Lambda$, a small ball $B$ centered at $\tilde{x}_0$ and $y_0 \in B \cap V_1^g(x_0)$ close to $\tilde{x}_0$. Our goal now is to find $x \in V_2^g(\tilde{x}_0)$ far in the tube $\mathfrak{T}$ defined by (3.4) for which we can carry out estimates similar to (3.5).

We will be working with pieces of leaves of $V_2^g$. Given a piece $I$ with endpoints $z_1$ and $z_2$ let $|I| = d_2^g(z_1, z_2)$. Let $q$ be a number such that for any piece $I$, $|I| = R$, we have

$$|g^q(I)| > 2R + r. \tag{3.9}$$

Notice that $q$ can be chosen to be independent of $g$ and depends only on $\tilde{\beta}_2$, $R$ and $r$.

Pick $I_1 \subset V_2^g(\tilde{x}_0)$, $|I_1| = R$, $I_1 \cap B_1 = \emptyset$, as close to $\tilde{x}_0$ as possible if $\tilde{x}_0 \in B_1$ (see Figure 3.3b) or passing through $\tilde{x}_0$ if $\tilde{x}_0 \notin B_1$. Given $I_i$, $i \geq 1$ we choose $I_{i+1} \subset f^q(I_i)$, $|I_{i+1}| = R$, $I_{i+1} \cap B_1 = \emptyset$. Condition (3.9) guarantees that such choice is possible.

We fix $N$ large and take $x \in I_{Nq} \subset V_2^g(\tilde{x}_0)$. Let $y = V_1^g(x) \cap V_2^g(y_0)$ as before. Construction of the sequence $\{I_i, i \geq 1\}$ ensures that points $f^{-qi}(x)$, $i = 0, \ldots N-1$, are outside $B_1$. This fact together with (3.7) and (3.8) allows to carry out the following estimate

$$\frac{\tilde{d}_1(x,y)}{\tilde{d}_1(\tilde{x}_0,y_0)} = \prod_{i=1}^{Nq} \frac{D_g^{wu}(g^{-i}(x))}{D_g^{wu}(g^{-i}(\tilde{x}_0))} \cdot \frac{\tilde{d}_1(f^{-Nq}(x), f^{-Nq}(y))}{\tilde{d}_1(\tilde{x}_0, f^{-Nq}(y_0))}$$
$$\geq \frac{M^N m_0^{N(q-1)}}{m_1^{Nq}} \cdot \frac{\tilde{d}_1(f^{-Nq}(x), f^{-Nq}(y))}{\tilde{d}_1(\tilde{x}_0, f^{-Nq}(y_0))}.$$

The affine-like distance ratio on the right is bounded away from 0 independently of $N$ since $f^{-Nq}(x) \in I_1$ while the coefficient in front of it is arbitrarily large according to (3.6). Hence $\tilde{d}_1^g(x,y)$ is arbitrarily large and the projection of tube $\mathcal{T}$ is dense in $\mathbb{T}^5$. □

## 3.4 Proof of Theorem A

For reasons explained in Section 3.2 we first prove Theorem A assuming that $f$ has Property $\mathcal{A}'$. The only place where we use $\mathcal{A}'$ is the proof of Lemma 3.4.6. In Section 3.4.6 we give another proof of Lemma 3.4.6 that uses Property $\mathcal{A}$ only.

### 3.4.1 Scheme of the proof of Theorem A

Recall the notation from Chapter 1 for the $L$-invariant splitting

$$T\mathbb{T}^d = F_l \oplus F_{l-1} \oplus \ldots \oplus F_1 \oplus E_1 \oplus E_2 \oplus \ldots \oplus E_k$$

along the eigendirections with corresponding eigenvalues

$$\mu_l < \mu_{l-1} < \ldots < \mu_1 < 1 < \lambda_1 < \lambda_2 < \ldots < \lambda_k.$$

We choose neighborhood $\mathcal{U}$ in such a way that for any $f$ in $\mathcal{U}$ the invariant splitting survives

$$T\mathbb{T}^d = F_l^f \oplus F_{l-1}^f \oplus \ldots \oplus F_1^f \oplus E_1^f \oplus E_2^f \oplus \ldots \oplus E_k^f,$$

with

$$\measuredangle(F_i, F_i^f) < \frac{\pi}{2}, \ \measuredangle(E_j, E_j^f) < \frac{\pi}{2}, \ \ i = 1, \ldots l, \ j = 1, \ldots k \tag{3.10}$$

and $f$ is partially hyperbolic in the strongest sense: there exist $C > 0$ and constants

$$\alpha_l < \tilde{\alpha}_{l-1} < \alpha_{l-1} < \ldots < \tilde{\alpha}_1 < \alpha_1 < 1 < \tilde{\beta}_1 < \beta_1 < \ldots < \tilde{\beta}_k$$

independent of the choice of $f$ in $\mathcal{U}$ such that for $n > 0$

$$\begin{aligned}
&\|D(f^n)(x)(v)\| \le C\alpha_l^n \|v\|, \quad v \in F_l^f(x),\\
&\frac{1}{C}\tilde{\alpha}_{l-1}^n \|v\| \le \|D(f^n)(x)(v)\| \le C\alpha_{l-1}^n \|v\|, \quad v \in F_{l-1}^f(x),\\
&\qquad\qquad\qquad\qquad \ldots\\
&\frac{1}{C}\tilde{\alpha}_1^n \|v\| \le \|D(f^n)(x)(v)\| \le C\alpha_1^n \|v\|, \quad v \in F_1^f(x),\\
&\frac{1}{C}\tilde{\beta}_1^n \|v\| \le \|D(f^n)(x)(v)\| \le C\beta_1^n \|v\|, \quad v \in E_1^f(x),\\
&\qquad\qquad\qquad\qquad \ldots\\
&\frac{1}{C}\tilde{\beta}_k^n \|v\| \le \|D(f^n)(x)(v)\|, \quad v \in E_k^f(x).
\end{aligned} \tag{3.11}$$

Equivalently the Mather spectrum of $f$ does not contain 1 and has $d$ connected components.

We show that the choice of $\mathcal{U}$ guarantees unique integrability of intermediate distributions. From now on for the sake of concreteness we work with unstable distributions and foliations.

For a given $f \in \mathcal{U}$ let $E^f(i,j) = E_i^f \oplus E_{i+1}^f \oplus \ldots \oplus E_j^f$, $i \le j$.

**Lemma 3.4.1.** *For any $f$ in $\mathcal{U}$ distribution $E^f(1,1), E^f(1,2), \ldots E^f(1,k)$ are uniquely integrable.*

This is a direct corollary of Hirsch, Pugh and Shub theorem but we will present a direct proof.

Let $W_1^f \subset W_2^f \subset \ldots \subset W_k^f$ be the corresponding flag of weak unstable foliations. The last foliation in the flag is the unstable foliation $W^f = W_k^f$.

**Lemma 3.4.2.** *For any $f$ in $\mathcal{U}$ and $i \le j$ distribution $E(i,j)$ is uniquely integrable.*

Denote by $W^f(i,j)$, $i \le j$, the integral foliation of $E^f(i,j)$. Also recall that we denote by $V_1^f, V_2^f, \ldots V_k^f$ the integral foliations of $E_1^f, E_2^f, \ldots E_k^f$ correspondingly. Notice that $V_i^f = W^f(i,i)$ and $W_i^f = W^f(1,i)$, $i = 1, \ldots k$.

Now we consider $f$ and $g$ as in Theorem A, $h \circ f = g \circ h$. The conjugacy $h$ maps unstable (stable) foliation of $f$ into unstable (stable) foliation of $g$. Moreover, $h$ preserves the whole flag of weak unstable (stable) foliations.

**Lemma 3.4.3.** *Fix an $i = 1, \ldots k$. Then $h(W_i^f) = W_i^g$.*

*Remark.* Proof of this lemma does not use the assumption on p. d. We only need $f$ and $g$ to be in $\mathcal{U}$.

Lemmas 3.4.1, 3.4.2 and 3.4.3 can be proved under a milder assumption. Instead of requiring $f$ and $g$ to be in $\mathcal{U}$ we can require an

**Alternative assumption:** $f$ and $g$ are partially hyperbolic in the strongest sence (3.11) with the rate constants satisfying

$$\mu_l < \alpha_l < \tilde{\alpha}_{l-1} < \mu_{l-1} < \alpha_{l-1} < \ldots < \tilde{\beta}_{k-1} < \lambda_{k-1} < \beta_{k-1} < \tilde{\beta}_k < \lambda_k. \qquad (\star)$$

We think that $(\star)$ is actually automatic from (3.11).

*Remark.* To carry out proofs of Lemmas above under the Alternative assumption one needs to transfer the picture to the linear model by the conjugacy and use inequalities $(\star)$ for growth arguments. This way one uses quasi-isometric foliations by straight lines of the linear model instead of foliations of $f$ which are a priori not known to be quasi-isometric.

**Conjecture 3.** *Suppose that $f$ is homotopic to $L$ and partially hyperbolic in the strongest sense (3.11) then the rate constants satisfy $(\star)$.*

*Remark.* The proof of Lemmas 3.4.1, 3.4.2 and 3.4.3 is the only place where we really need $f$ and $g$ to be in $\mathcal{U}$. So in Theorem A the assumption that $f, g \in \mathcal{U}$ can be substituted by the alternative assumption.

**Lemma 3.4.4.** *A leaf $W_1^f(x)$ is dense in $\mathbb{T}^d$*

*Proof.* By Lemma 3.4.3 we have that the conjugacy between $L$ and $f$ takes the foliation $W_1^L$ into the foliation $W_1^f$. According to Proposition 9 leaves of $W_1^L$ are dense. Hence leaves of $W_1^f$ are dense. □

Next we describe the inductive procedure which leads to smoothness of $h$ along the unstable foliation.

*Induction base.* We know that $h$ takes $W_1^f$ into $W_1^g$.

**Lemma 3.4.5.** *Conjugacy $h$ is $C^{1+\nu}$-differentiable along $W_1^f$ i. e. restrictions of $h$ to the leaves of $W_1^f$ are differentiable and the derivative is $C^\nu$ function on $\mathbb{T}^d$.*

Provided that we have Lemma 3.4.4 the proof of Lemma 3.4.5 is the same as the proof of Lemma 2.2.5.

*Induction step.* The induction procedure is based on the following lemmas.

**Lemma 3.4.6.** *Assume that $h$ is $C^{1+\nu}$-differentiable along $W_{m-1}^f$ and $h(V_i^f) = h(V_i^g)$, $i = 1, \dots m-1$, $1 < m \le k$. Then $h(V_m^f) = V_m^g$.*

**Lemma 3.4.7.** *Assume that $h(V_m^f) = V_m^g$ for some $m = 1, \dots k$. Then $h$ is $C^{1+\nu}$-differentiable along $V_m^f$.*

Again we use a regularity result due to Journé. Recall the statement.

**Regularity Lemma** ([J88]). *Let $M_j$ be a manifold and $W_j^s$, $W_j^u$ be continuous transverse foliations with uniformly smooth leaves, $j = 1, 2$. Suppose that $h : M_1 \to M_2$ is a homeomorphism that maps $W_1^s$ into $W_2^s$ and $W_1^u$ into $W_2^u$. Moreover, assume that the restrictions of $h$ to the leaves of these foliations are uniformly $C^{r+\nu}, r \in \mathbb{N},\ 0 < \nu < 1$. Then $h$ is $C^{r+\nu}$.*

*Remark.* There are two more methods of proving analytical results of this flavor besides Journé's. One is due to de la Llave, Marco, Moriyón and the other one is due to Hurder and Katok (see [KN08] for a detailed discussion and proofs). We remark that we really need Journé's result since the alternative approaches require foliations to be absolutely continuous while we apply the Regularity Lemma to various foliations that do not have to be absolutely continuous.

Now the inductive scheme can be described as follows. Assume that $h$ is $C^{1+\nu}$ along $W^f_{m-1}$ for some $m \le k$ and $h(V_i^f) = h(V_i^g)$, $i = 1, \dots\ m-1$. By Lemma 3.4.6 we have that $h(V_m^f) = V_m^g$ and by Lemma 3.4.7 $h$ is $C^{1+\nu}$ along $V_m^f$. Fix a leaf $W_m^f(x)$. Leaves of $W^f_{m-1}$ and $V_m^f$ subfoliate $W_m^f(x)$ and it is clear that the Regularity Lemma can be applied for $h : W_m^f(x) \to W_m^g(h(x))$. Hence we get that $h$ is $C^{1+\nu}$ on every leaf of $W_m^f$. Hölder continuity of the derivative of $h$ in the direction transverse to $W_m^f$ is direct consequence of Hölder of the derivatives along $W^f_{m-1}$ and $V_m^f$. We conclude that $h$ is $C^{1+\nu}$-differentiable along $W_m^f$.

By induction we get that $h$ is $C^{1+\nu}$-differentiable along the unstable foliation and analogously along the stable foliation. We finish the proof of the Theorem A by applying the Regularity Lemma to stable and unstable foliations.

### 3.4.2 Proof of the integrability lemmas

In the proofs of Lemmas 3.4.1 and 3.4.2 we work with lifts of maps, distributions and foliations to $\mathbb{R}^d$. We use the same notation for lifts as for the objects themselves.

*Proof of Lemma 3.4.1.* Fix $i < k$. We assume that the distribution $E^f(1, i)$ is not integrable or it is integrable but not uniquely. In any case it follows that we can find distinct points $a_0, a_1, \dots a_m$ such that

1. $\{a_1, a_2, \dots a_m\} \subset W^f(a_0)$,

2. there are smooth curves $\tau_j : [0,1] \to W^f(a_0)$, $j = 1, \dots m$, such that $\tau_j(0) = a_{j-1}$, $\tau_j(1) = a_j$ and $\dot{\tau}_j \subset E^f_{p(j)}$, where $p(j) \le i$,

3. there is a smooth curve $\tau : [0,1] \to W^f(a_0)$ such that $\tau(0) = a_0$, $\tau(1) = a_m$ and $\dot{\tau}_j \subset E^f_q$ for some $q > i$.

Let $\tilde{\tau}$ be a piecewise smooth curve obtained by concatenating $\tau_1$, $\tau_2$,... $\tau_{m-1}$ and $\tau_m$. From the second property above and (3.11) we get the following rough estimate

$$\forall n \ge 0 \quad \text{length}(f^n(\tilde{\tau})) \le \beta_i^n \text{length}(\tilde{\tau}). \tag{3.12}$$

Similarly

$$\forall n \ge 0 \quad \text{length}(f^n(\tau)) \ge \tilde{\beta}_{i+1}^n \text{length}(\tau). \tag{3.13}$$

Denote by $d(\cdot,\cdot)$ the usual distance in $\mathbb{R}^d$. It follows from the assumption (3.10) that any curve $\gamma : [0,1] \to \mathbb{R}^d$ tangent to the distribution $E^f_q$ is quasi-isometric:

$$\exists c > 0 \text{ such that } \quad \text{length}(\gamma) \le c\, d(\gamma(0), \gamma(1)).$$

In particular

$$\forall n \ge 0 \quad d(f^n(a_0), f^n(a_m)) \ge \frac{1}{c} \text{length}(f^n(\tau)). \tag{3.14}$$

Inequalities (3.12), (3.13) and (3.14) sum up to a contradiction. □

*Proof of Lemma 3.4.2.* The theory of partial hyperbolicity guarantees that distributions $E^f(i,k)$, $i = 1, \dots k$, integrate uniquely to foliations $W^f(i,k)$. Let us fix $i$ and $j$, $i < j$, and define $W^f(i,j) = W^f(1,j) \cap W^f(i,k)$. Obviously $W^f(i,j)$ is an integral foliation for $E^f(i,j)$. Unique integrability of $E^f(i,j)$ is a direct consequence of the unique integrability of $E^f(1,j)$ and $E^f(i,k)$. □

### 3.4.3 Weak unstable flag is preserved.

*Proof.* We continue working on the universal cover. Pick two points $a$ and $b$, $a \in W^f_i(b)$. Since

$$h_f(x + \vec{m}) = h_f(x) + \vec{m}, \quad \vec{m} \in \mathbb{Z}^d \tag{3.15}$$

we have that $d(h(x),h(y)) \leq c_1 d(x,y)$ for any $x$ and $y$ such that $d(x,y) \geq 1$.

Hence for any $n > 0$

$$d(g^n(h(a)), g^n(h(b))) = d(h(f^n(a)), h(f^n(b))) \leq c_2 d(f^n(a), f^n(b)) \leq c_2 c_3 \beta_i^n,$$

where $c_2$ and $c_3$ depend on $d(a,b)$. The above inequality guarantees that $h(a) \in W_i^g(h(b))$. Since the choice of $a$ and $b$ was arbitrary we conclude that $h(W_i^f) = W_i^g$. □

### 3.4.4 Induction step 1: the conjugacy preserves foliation $V_m$

We prove Lemma 3.4.6 which is the key ingredient in the proof of Theorem A. The proof is based on the idea from Chapter 2 but we take a rather different approach in order to deal with high dimension of $W^f$.

The goal is to prove that $h(V_m^f) = V_m^g$. So we consider foliation $U = h^{-1}(V_m^g)$. As for usual foliations $U(x)$ stands for the leaf of $U$ passing through $x$ and $U(x,R)$ stands for the local leaf of size $R$. A priori, the leaves of $U$ are just Hölder continuous curves. Hence the local leaf needs to be defined with certain care. One way is to consider the lift of $U$ and define the lift of local leaf $U(x,R)$ as connected component of $x$ of the intersection $U(x) \cap B(x,R)$. We prove Lemma 3.4.6 by induction.

*Induction base.*

We will be working on $m$-dimensional leaves of $W_m^f$. By Lemma 3.4.3 $U$ subfoliates $W_m^f$. In other words for any $x \in \mathbb{T}^d$ $U(x) \subset W_m^f(x)$.

*Induction step.*

*Suppose that $U$ subfoliate $W^f(i,m)$ for some $i < m$. Then $U$ subfoliate $W^f(i+1,m)$.*

By induction we get that $U$ subfoliate $W^f(m,m) = V_m^f$. Hence $U = V_m$.

First let us prove several auxiliary claims. Note that all foliations that we are dealing with are oriented and the orientation is preserved under the dynamics.

Denote by $d_j^f$ and $d_j^g$ the induced distances on the leaves of $V_j^f$ and $V_j^g$ correspondingly, $j = 1, \dots k$.

**Lemma 3.4.8.** *Consider a point $a \in \mathbb{T}^d$. Pick a point $b \in U(a)$ and let $\tilde{b} = V_i^f(b) \cap W^f(i+1, m)(a)$. Assume that $\tilde{b} \neq b$. Pick a point $c \in V_i^f(a)$ and let $d = U(c) \cap W^f(i, m-1)(b)$, $\tilde{d} = V_i^f(d) \cap W^f(i+1, m)(c)$. Then $\tilde{d} \neq d$ and the orientations of the pairs $(b, \tilde{b})$ and $(d, \tilde{d})$ in $V_i^f$ are the same.*

The statement of the lemma when $i = 1$ and $m = 3$ is illustrated on Figure 3.4.

*Remark.* Since by the induction hypothesis $h(W^f(i, m-1)) = W^g(i, m-1)$ we see that the leaf $U(a)$ intersects each leaf $W^f(i, m-1)(x)$, $x \in W^f(i, m)(a)$ exactly once.

*Proof.* Let $e = V_i^f(b) \cap W^f(i+1, m)(d)$ and $\tilde{e} = V_i^f(b) \cap W^f(i+1, m)(\tilde{d})$. Obviously $(e, \tilde{e})$ has the same orientation as $(d, \tilde{d})$ and also has advantage of lying on the leaf $V_i^f(b)$. Therefore we forget about $(d, \tilde{d})$ and work with $(e, \tilde{e})$.

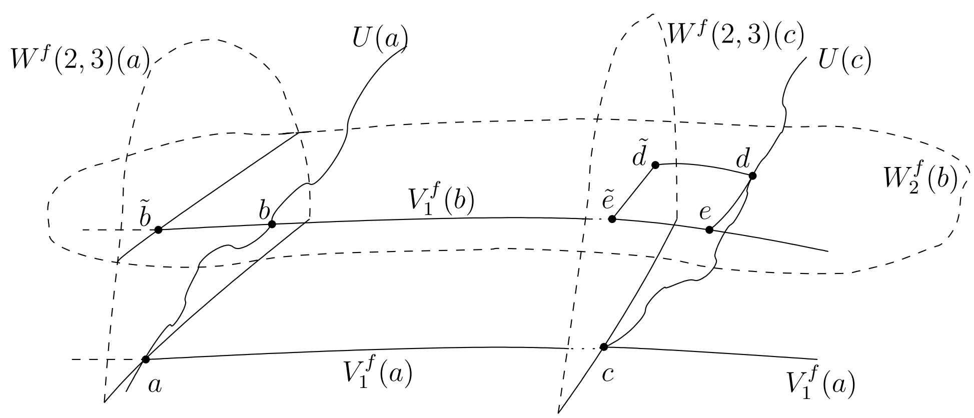


**Figure 3.4.** Illustration to Lemma 3.4.8 when $i = 1$ and $m = 3$.

We use affine structure on the expanding foliation $V_i^f$. Namely we work with affine distance-like function $\tilde{d}_i$. We refer to Chapter 2 for the definition. There we define affine distance-like function on weak unstable foliation. The definition for foliation $V_i^f$ is the same with obvious modifications. Recall crucial properties of $\tilde{d}_i$

(D1) $\tilde{d}_i(x, y) = d_i^f(x, y) + o(d_i^f(x, y))$,

(D2) $\tilde{d}_i(f(x), f(y)) = D_f^i(x)\tilde{d}_i(x, y)$, where $D_f^i$ is the derivative of $f$ along $V_i^f$.

(D3) $\forall K > 0\ \exists C > 0$ such that

$$\frac{1}{C}\tilde{d}_i(x,y) \le d_i^f(x,y) \le C\tilde{d}_i(x,y)$$

whenever $d_i(x,y) < K$.

Assume that $(e,\tilde{e})$ has orientation opposite to $(b,\tilde{b})$ or $e = \tilde{e}$. For the sake of concreteness we assume that these points lie on $V_i^f(b)$ in the order $b, \tilde{b}, \tilde{e}, e$. All other cases can be treated similarly. Then

$$\tilde{d}_i(b,e) \ge \tilde{d}_i(b,\tilde{e}) > \tilde{d}_i(b,\tilde{e}) - \tilde{d}_i(b,\tilde{b}).$$

*Remark.* Notice that $\tilde{d}_i(b,\tilde{e}) - \tilde{d}_i(b,\tilde{b}) \ne \tilde{d}_i(\tilde{b},\tilde{e})$ since $\tilde{d}_i$ is neither symmetric nor additive. Distance $\tilde{d}_i$ is given by an integral of a certain density with normalization defined by the first argument. As long as the first argument (point $b$ in the above inequality) is the same all natural inequalities hold.

Applying (D2) we get that

$$\forall n > 0 \quad \frac{\tilde{d}_i(f^{-n}(b), f^{-n}(e))}{\tilde{d}_i(f^{-n}(b), f^{-n}(\tilde{e})) - \tilde{d}_i(f^{-n}(b), f^{-n}(\tilde{b}))} = c_1 > 1$$

where $c_1$ does not depend on $n$. By property (D1) we can switch to the usual distance

$$\exists N:\ \forall n > N \quad \frac{d_i^f(f^{-n}(b), f^{-n}(e))}{d_i^f(f^{-n}(\tilde{b}), f^{-n}(\tilde{e}))} > c_2 > 1 \tag{3.16}$$

where $c_2$ does not depend on $n$.

Under the action of $f^{-1}$ strong unstable leaves of $W^f(i+1,m)$ contract exponentially faster then weak unstable leaves of $V_i^f$. Thus we get that

$$\forall \varepsilon > 0\ \exists N:\ \forall n > N \quad \left| \frac{d_i^f(f^{-n}(a), f^{-n}(c))}{d_i^f(f^{-n}(\tilde{b}), f^{-n}(\tilde{e}))} - 1 \right| < \varepsilon. \tag{3.17}$$

Point $h(e) \in W^g(i+1,m)(h(c))$. Indeed, notice that

$$e = V_i^f(b) \cap W^f(i+1,m)(d) = V_i^f(b) \cap W^f(i+1,m-1)(d)$$

(if $i = m-1$ than we have $e = d$). Thus

$$\begin{aligned} h(e) = h(V_i^f(b) \cap W^f(i+1, m-1)(d)) &= V_i^g(h(b)) \cap W^g(i+1, m-1)(h(d)) \\ = V_i^g(h(b)) \cap W^g(i+1, m)(h(d)) &= V_i^g(h(b)) \cap W^g(i+1, m)(h(c)), \end{aligned}$$

where the last equality is justified by the fact that $h(d) \in V_m^g(h(c))$. We know also that $h(b) \in W^g(i+1, m)(h(a))$. Hence, analogously to (3.17), we have

$$\forall \varepsilon > 0 \ \exists N : \ \forall n > N \quad \left| \frac{d_i^g\left(g^{-n}(h(a)), g^{-n}(h(c))\right)}{d_i^g\left(g^{-n}(h(b)), g^{-n}(h(e))\right)} - 1 \right| < \varepsilon. \tag{3.18}$$

On the other hand, we know that $h$ is continuously differentiable along $V_i^f$. Hence

$$\begin{aligned} \forall \varepsilon > 0 \ \exists N : \ \forall n > N \quad & \left| \frac{d_i^g\left(g^{-n}(h(a)), g^{-n}(h(c))\right)}{d_i^f(f^{-n}(a), f^{-n}(c))} - D_h^i(f^{-n}(a)) \right| < \varepsilon \\ \text{and} \quad & \left| \frac{d_i^g\left(g^{-n}(h(b)), g^{-n}(h(e))\right)}{d_i^g(f^{-n}(b), f^{-n}(e))} - D_h^i(f^{-n}(a)) \right| < \varepsilon. \end{aligned} \tag{3.19}$$

Therefore from (3.18) and (3.19) we have

$$\forall \varepsilon > 0 \ \exists N : \ \forall n > N \quad \left| \frac{d_i^f(f^{-n}(a), f^{-n}(c))}{d_i^f(f^{-n}(b), f^{-n}(e))} - 1 \right| < \varepsilon,$$

which we combine with (3.17) to get

$$\forall \varepsilon > 0 \ \exists N : \ \forall n > N \quad \left| \frac{d_i^f(f^{-n}(b), f^{-n}(e))}{d_i^f(f^{-n}(\tilde{b}), f^{-n}(\tilde{e}))} - 1 \right| < \varepsilon.$$

We have reached a contradiction with (3.16) □

*Remark.* By the same argument one can prove that if $b = \tilde{b}$ then $d = \tilde{d}$.

**Lemma 3.4.9.** *Consider a weak unstable leaf $W_{m-1}^f(a)$ and $b \in V_m^f(a)$, $b \neq a$. For any $y \in W_{m-1}^f(a)$ let $y' = W_{m-1}^f(b) \cap V_m^f(y)$. Then $\exists c_1, c_2 > 0$ such that $\forall y \in W_{m-1}^f(a) \quad c_1 > d_m^f(y, y') > c_2$.*

*Proof.* We will be working on the universal cover $\mathbb{R}^d$. We abuse the notation slightly by using the same notation for the lifted objects. Note that the leaves on $\mathbb{R}^d$ are connected components of preimages by the projection map of the leaves on $\mathbb{T}^d$.

Let $h_f$ be the conjugacy with the linear model, $h_f \circ f = L \circ h_f$. Lemma 3.4.3 holds for $h_f$: $h_f(W^f_{m-1}) = W^L_{m-1}$. Leaves $W^L_{m-1}(h_f(a))$ and $W^L_{m-1}(h_f(b))$ are parallel hyperplanes. Thus the lower bound follows from the uniform continuity of $h_f$.

It follows from (3.15) that $h_f^{-1} - Id$ is bounded. Hence we can find positive $R$ that depends only on size of $\mathcal{U}$ such that

$$W^f_{m-1}(a) \subset Tube_a \stackrel{\text{def}}{=} \cup_{x \in B(a,R)} W^L_{m-1}(x)$$

and

$$W^f_{m-1}(b) \subset Tube_b \stackrel{\text{def}}{=} \cup_{x \in B(b,R)} W^L_{m-1}(x).$$

Then, obviously,

$$d^f_m(y, y') \le \sup\{d^f_m(x, x') \mid x \in Tube_a, x' \in Tube_b \cap V^f_m(x)\}.$$

Assumption (3.10) guarantees that $E^f_m$ is uniformly transversal to $TW^L_{m-1} = E^L_1 \oplus E^L_2 \oplus \ldots \oplus E^L_{m-1}$. Thus the supremum above is finite. □

*Remark.* Given two points $a, b \in \mathbb{R}^d$ let

$$\hat{d}(a, b) = distance(W^L_{m-1}(h_f(a)), W^L_{m-1}(h_f(b))).$$

It is clear from the proof that constants $c_1$ and $c_2$ can be chosen in such a way that they depend only on $\hat{d}(a, b)$.

*Remark.* In the proof above we do not use the fact that both $W^f_{m-1}$ and $V^f_m$ are expanding. We only need them to be transversal. Thus, if we substitute weak unstable foliation $W^f_{m-1}$ by some weak stable foliation $\mathcal{F}$, the statement still holds.

*Remark.* As mentioned earlier the assumption (3.10) is crucial only for Lemmas 3.4.1, 3.4.2 and 3.4.3. We used this assumption in the proof above only for convenience. Slightly more delicate argument goes through without using assumption (3.10).

*Proof of the induction step.* We will be working inside of the leaves of $W^f(i,m)$. Assume that $U$ does not subfoliate $W^f(i+1,m)$. Then there exists a point $x_0$ and $x_1 \in U(x_0)$ close to $x_0$ such that $x_1 \notin W^f(i+1,m)(x_0)$.

We fix orientation $\mathcal{O}$ of $U$ and $V_i^f$ that is defined on pairs of points $(x,y)$, $y \in U(x)$ and $(x,y)$, $y \in V_i^f(x)$. Although we denote these orientations by the same symbol it will not cause any confusion since $U$ and $V_i^f$ are topologically transverse.

For every $(x,y)$, $y \in U(x)$ with $\mathcal{O}(x,y) = \mathcal{O}(x_0,x_1)$, define

$$[x,y] = W^f(i+1,m)(x) \cap V_i^f(y).$$

For instance in Lemma 3.4.8 $\tilde{b} = [a,b]$, $\tilde{d} = [c,d]$.

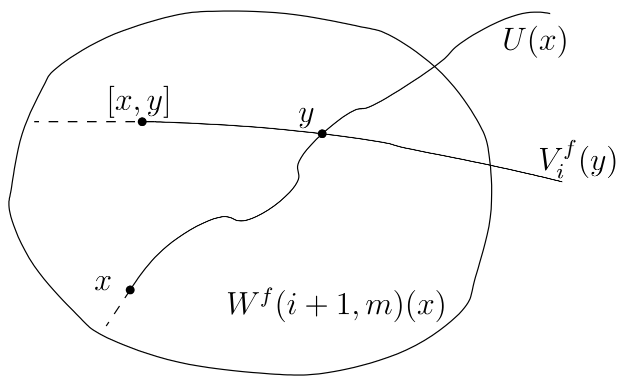


**Figure 3.5.** Definition of $[x,y]$.

**Lemma 3.4.10.** *For every $(x,y)$ as above either $[x,y] = y$ or $\mathcal{O}([x,y],y) = O^+ \stackrel{\text{def}}{=} \mathcal{O}([x_0,x_1],x_1)$.*

*Proof.* Let $a_0 = \hat{d}(x_0,x_1)$ (for definition of $\hat{d}$ see the remark after the proof of Lemma 3.4.9). Number $a_0$ is positive since $U(x)$ is transverse to $W_{m-1}^f$.

For any $y \in \mathbb{T}^d$ there is a unique point $sh(y) \in U(y)$ such that $\hat{d}(y, sh(y)) = a_0$ and $\mathcal{O}(y, sh(y)) = \mathcal{O}(x_0,x_1)$.

The leaves of all foliations that we consider depend continuously on the point. Therefore we can find a small ball $B$ centered at $x_0$ such that $\forall y \in B$ $[y, sh(y)] \neq sh(y)$ and $\mathcal{O}([y,sh(y)], sh(y)) = O^+$.

Next, let us fix $y \in B$ and choose any $z \in V_i^f(y)$. Apply Lemma 3.4.8 for $a = y$, $b = sh(y)$, $c = z$, $d = sh(z)$ to get that $[z, sh(z)] \neq sh(z)$ and $\mathcal{O}([z, sh(z)], sh(z)) = \mathcal{O}([y, sh(y)], sh(y)) = O^+$ as shown on the Figure 3.6.

By Property $\mathcal{A}$

$$\bigcup_{y \in B} V_i^f = \mathbb{T}^d.$$

Thus

$$\forall z \in \mathbb{T}^d \quad [z, sh(z)] \neq sh(z) \text{ and } \mathcal{O}([z, sh(z)], sh(z)) = O^+. \tag{3.20}$$

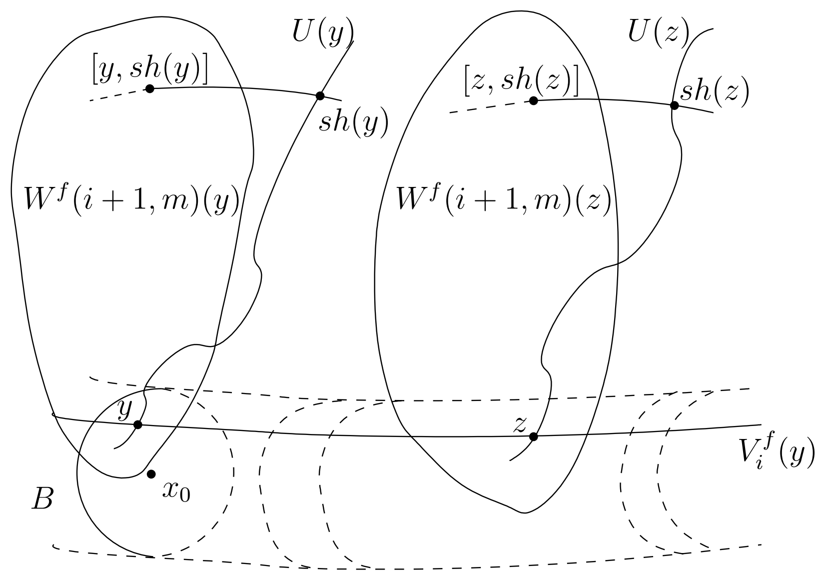


**Figure 3.6.** Orientation of $([z, sh(z)], sh(z))$ is positive for any $z$ in the $V_i^f$-tube through the ball $B$. Foliation $W^f(i+1, m)$ is two dimensional on the picture.

Now let us assume contrary to the statement of the lemma. Namely, assume that there exists $\tilde{x}_0$ and $\tilde{x}_1$, $\tilde{x}_1 \in U(\tilde{x}_0)$, $\mathcal{O}(\tilde{x}_0, \tilde{x}_1) = \mathcal{O}(x_0, x_1)$, such that $[\tilde{x}_0, \tilde{x}_1] \neq \tilde{x}_1$ and $\mathcal{O}([\tilde{x}_0, \tilde{x}_1], \tilde{x}_1) \stackrel{\text{def}}{=} O^- \neq O^+$. By tinkering $\tilde{x}_1$ infinitesimally along $U(\tilde{x}_0)$ we can ensure that $N_1 a_0 = N_2 \hat{d}(\tilde{x}_0, \tilde{x}_1)$, where $N_1$ and $N_2$ are some large integer numbers.

For any $y \in \mathbb{T}^d$ there is a unique point $\tilde{sh}(y) \in U(y)$ such that $\hat{d}(y, \tilde{sh}(y)) = \hat{d}(\tilde{x}_0, \tilde{x}_1)$ and $\mathcal{O}(y, \tilde{sh}(y)) = \mathcal{O}(\tilde{x}_0, \tilde{x}_1)$. Then by the same argument we show an analogue of (3.20):

$$\forall z \in \mathbb{T}^d \quad [z, \tilde{sh}(z)] \neq \tilde{sh}(z) \text{ and } \mathcal{O}([z, \tilde{sh}(z)], \tilde{sh}(z)) = O^-. \tag{3.21}$$

Pick a point $x \in \mathbb{T}^d$ and $y, z \in U(x)$, $\mathcal{O}(x,y) = \mathcal{O}(y,z)$. Assume that $\mathcal{O}([x,y],y) = \mathcal{O}([y,z],z)$. Then $\mathcal{O}([x,z],z) = \mathcal{O}([x,y],y)$. This obvious property allows us to "iterate" $sh$ and $\tilde{sh}$.

Choose any $z$ and "iterate" (3.20) and (3.21) $N_1$ and $N_2$ times correspondingly as shown on the Figure 3.7.

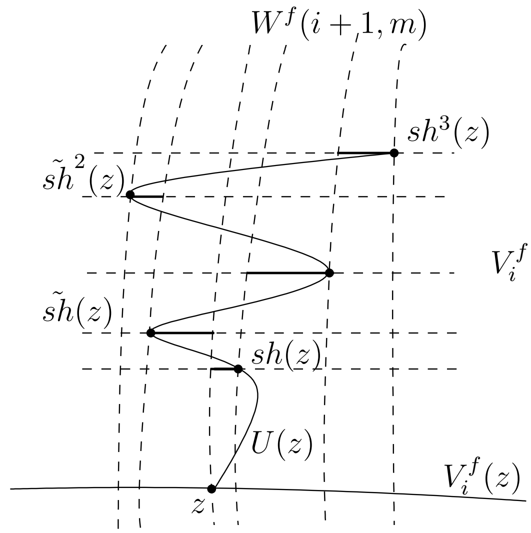


**Figure 3.7.** Illustration to the argument with shifts along the leaf $U(z)$. Foliation $W^f(i+1,m)$ is one dimensional here, $N_1 = 3$, $N_2 = 2$. Black segments of $V_i^f$ carry known information about orientation of $([\cdot, sh(\cdot)], sh(\cdot))$ and $([\cdot, \tilde{sh}(\cdot)], \tilde{sh}(\cdot))$. This picture is clearly impossible if $sh^{N_1} = \tilde{sh}^{N_2}$.

We get that

$$\mathcal{O}([z, sh^{N_1}(z)], sh^{N_1}(z)) = O^+ \text{ and } \mathcal{O}([z, \tilde{sh}^{N_2}(z)], \tilde{sh}^{N_2}(z)) = O^-.$$

To get a contradiction it remains to notice that $sh^{N_1} = \tilde{sh}^{N_2}$. Hence the lemma is proved. □

From (3.20) we see that for any $z \in \mathbb{T}^d$ $d_i^f([z, sh(z)], sh(z)) > 0$. Hence, due to compactness and continuity of function $d_i^f([\cdot, sh(\cdot)], sh(\cdot))$, we have $\delta <$

$d_i^f([z, sh(z)], sh(z)) < \Delta$ for some positive $\delta$ and $\Delta$. Lemma 3.4.10 guarantees even more,

$$\forall x \in \mathbb{T}^d \text{ and } y \in U(x), \mathcal{O}(x,y) = \mathcal{O}(x_0, x_1), \text{ such that } \hat{d}(x,y) \le a_0 \\ \text{we have } d_i^f([x,y],y) < \Delta. \quad (3.22)$$

From now on it is more convenient to work on the universal cover. Although formally we do not have to do it since we are working inside of the leaves of $W^f(i,m)$ which are isometric to their lifts.

Let $x_n = sh^n(x_0)$, $n > 0$. For every $n \ge 0$ $\mathcal{O}([x_n, x_{n+1}], x_{n+1}) = O^+$ and $d_i^f([x_n, x_{n+1}], x_{n+1}) > \delta$. Lemma 3.4.10 also tells us that $U$ is monotone with respect to $W^f(i+1,m)$. Namely, for any $x \in \mathbb{T}^d$ the intersection $U(x) \cap W^f(i+1,m)(x)$ is a connected piece of $U(x)$.

Denote by $\overline{x_n, x_{n+1}}$ the piece of $U(x_0)$ that lies between $x_n$ and $x_{n+1}$. We know that for any $n \ge 0$ $\overline{x_n, x_{n+1}}$ is confined between $W^f(i,m-1)(x_n)$ and $W^f(i,m-1)(x_{n+1})$. Lemma 3.4.10 guarantees that $\overline{x_n, x_{n+1}}$ is also confined between $W^f(i+1,m)(x_n)$ and $W^f(i+1,m)(x_{n+1})$ as shown on Figure 3.8. Thus, it makes sense to measure two different "dimensions" of $\overline{x_n, x_{n+1}}$. Namely, let $a_n = \hat{d}(x_n, x_{n+1})$ and $b_n = d_i^f([x_n, x_{n+1}], x_{n+1})$. As we have remarked earlier $b_n > \delta > 0$ and $a_n = a_0$ by the definition of $\hat{d}$ and $sh$.

This "dimensions" behave nicely under the dynamics. Namely,

$$\forall N > 0 \ \ (f_*)^{-N}(b_n) \stackrel{\text{def}}{=} d_i^f([f^{-N}(x_n), f^{-N}(x_{n+1})], f^{-N}(x_{n+1})) \ge \delta \beta_i^{-N}$$

and

$$\forall N > 0 \ \ (f_*)^{-N}(a_n) \stackrel{\text{def}}{=} \hat{d}(f^{-N}(x_n), f^{-N}(x_{n+1})) = a_0 \lambda_m^{-N}.$$

Recall that $\lambda_m > \beta_i$.

The idea now is to show that the leaf $U(f^{-N}(x_0))$ is "too close" to $W^f(i,m-1)(x_0)$ for $N$ large, which would lead to a contradiction.

Take $N$ large and let $M = \lfloor \lambda_m^N \rfloor$. Then

$$\hat{d}(f^{-N}(x_0), f^{-N}(x_M)) = \sum_{j=0}^{M-1} \hat{d}(f^{-N}(x_j), f^{-N}(x_{j+1}))$$

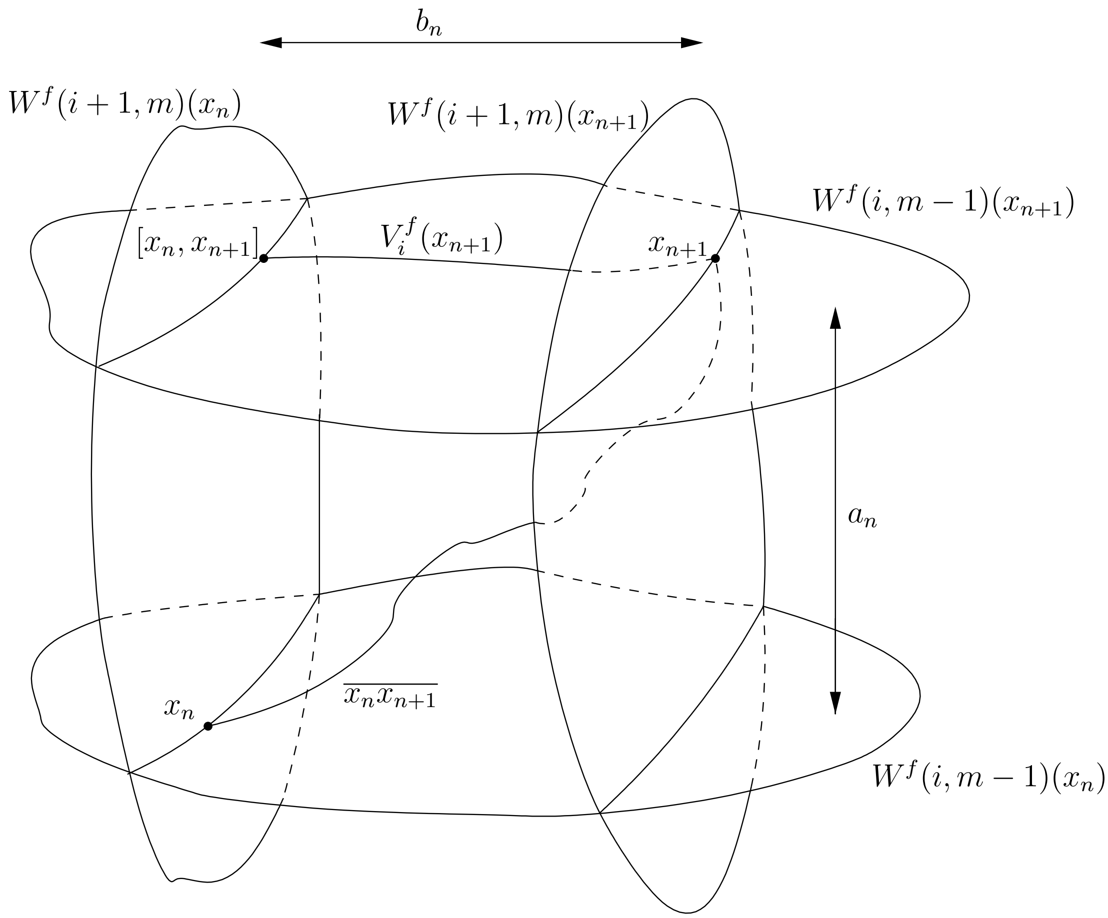


**Figure 3.8.** Piece $\overline{x_n x_{n+1}}$ is "monotone" with respect to foliation $W^f(i, m-1)$. By Lemma 3.4.10 $\overline{x_n x_{n+1}}$ is also "monotone" with respect to $W^f(i+1, m)$: the intersections of $\overline{x_n x_{n+1}}$ with local leaves of $W^f(i+1, m)$ are points or connected components of $\overline{x_n x_{n+1}}$. On this picture foliations $W^f(i, m-1)$ and $W^f(i+1, m)$ are two dimensional.

$$= \sum_{j=0}^{M-1} (f_*)^{-N}(a_j) = M a_0 \lambda_m^{-N} \le a_0. \quad (3.23)$$

The first equality holds since the holonomy along $W^f(i, m-1)$ is isometric with respect to $\hat{d}$.

To estimate $d_i^f([f^{-N}(x_0), f^{-N}(x_M)], f^{-N}(x_M))$ in the similar way we need to have control over holonomies along $W^f(i+1, m)$.

Fix two small one dimensional transversals $T(x) \subset V_i^f(x)$ and $T(y) \subset V_i^f(y)$, $y \in U(x)$ with $\hat{d}(x, y) \le a_0$. This condition ensures that the distance between $x$ and $y$ along $W^f(i, m)(x)$ is uniformly bounded from above. To see this we only need to bound the distance between $h(x)$ and $h(y)$ along $W^g(i,m)(h(x))$. This, in turn, is a direct consequence of Lemma 3.4.9 applied for $g$ since $h(y) \in V_m^g(h(x))$.

Consider holonomy map along $W^f(i+1,m)$ $H : T(x) \to T(y)$. This holonomy can be viewed as holonomy along $W^f(i+1,k)$. Recall that $W^f(i+1,k)$ is the fast unstable foliation. Since $f$ is at least $C^2$-differentiable $W^f(i+1,k)$ is Lipschitz inside of $W^f(i,k)$. Moreover, since the distance between $x$ and $y$ is bounded from above, the Lipschitz constant $C_{Hol}$ of $H$ is uniform in $x$ and $y$. For proof see [LY85], Section 4.2. They proof that the unstable foliation is Lipschitz within center-unstable leaves but the proof goes through for $W^f(i+1,k)$ within the leaves of $W^f(i,k)$.

Let $\tilde{x}_j = W^f(i+1,m)(f^{-N}(x_j)) \cap V_i^f(f^{-N}(x_M))$, $j = 1, \dots M$. Then

$$\begin{aligned} d_i^f([f^{-N}(x_0), f^{-N}(x_M)], f^{-N}(x_M)) &= \sum_{j=0}^{M-1} d_i^f(\tilde{x}_j, \tilde{x}_{j+1}) \\ \geq C_{Hol} \sum_{j=0}^{M-1} d_i^f([f^{-N}(x_j), f^{-N}(x_{j+1})], f^{-N}(x_{j+1})) &= C_{Hol} \sum_{j=0}^{M-1} (f_*)^{-N}(b_j) \\ &\geq C_{Hol} M \delta \beta_i^{-N}. \end{aligned}$$

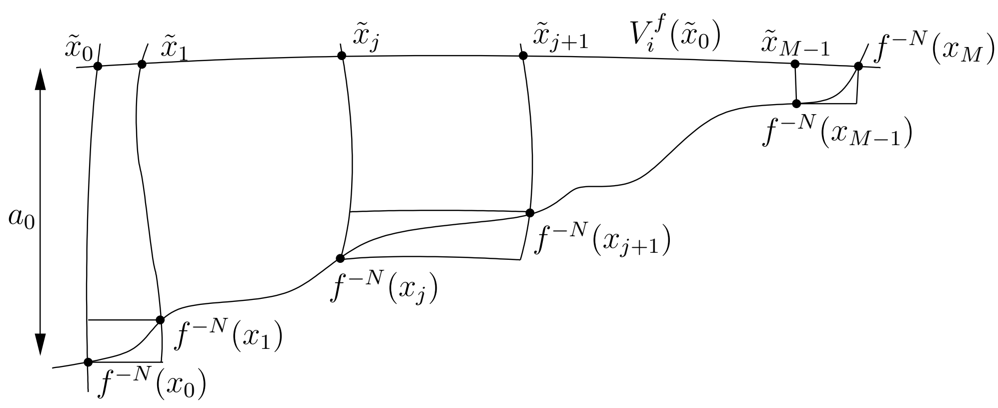


**Figure 3.9.** Small rectangles along leaf $U(f^{-N}(x_0))$ are very "flat" according to the estimates on $(f_*)^{-N}(b_n)$ and $(f_*)^{-N}(a_n)$. Together with Lipschitz property of foliation $W^f(i+1,m)$ this provides an estimate from below on the horizontal size $d_i^f(\tilde{x}_0, f^{-N}(x_M))$.

The holonomy constant is uniform since

$$\hat{d}(f^{-N}(x_j), \tilde{x}_j) \le \hat{d}(f^{-N}(x_0), \tilde{x}_j) = \hat{d}(f^{-N}(x_0), f^{-N}(x_M)) \le a_0$$

by (3.23).

Notice that $C_{Hol}M\delta\beta_i^{-N} \to \infty$ when $N \to \infty$, while $d(f^{-N}(x_0), f^{-N}(x_M)) \le a_0$ which contradicts to (3.22). Hence the induction step is established. □

### 3.4.5 Induction step 2: proof of Lemma 3.4.7 by transitive point argument

The proof of Lemma 3.4.7 is carried out in a way similar to the proofs of Lemmas 2.2.4 and 2.2.5 from Chapter 2. Here we overview the scheme and deal with complications that arise due to higher dimension.

First using the assumption on p. d. we argue that $h$ is uniformly Lipschitz along $V_m^f$, i. e., for any point $x$ the restriction $h\big|_{V_m^f(x)} : V_m^f(x) \to V_m^g(x)$ is a Lipschitz map with a Lipschitz constant that does not depend on $x$. At this step the assumption on p. d. along $V_m^f$ is used.

Lipschitz property implies differentiability at almost every point with respect to the Lebesgue measure on the leaves of $V_m^f$. The second step is to show that differentiability of $h$ along $V_m^f$ at a transitive point $x$ implies $C^{1+\nu}$-differentiability along $V_m^f$. This is done by a direct approximation argument (see Step 1 in Section 2.3.3). Transitive point $x$ "spreads differentiability" all over the torus.

Last but not the least, we need to find such a transitive point $x$. For that we would like to find an ergodic measure $\mu$ with full support such that the foliation $V_m^f$ is absolutely continuous with respect to $\mu$. Then by the Birkhoff ergodic theorem almost every point is transitive. And since $V_m^f$ is absolutely continuous we would have that almost every point with respect to the Lebesgue measure on the leaves is transitive. Hence we would find a full measure set of points that we are looking for.

Unfortunately we cannot carry out the scenario described above. The problem is that the foliation $V_m^f$ is not absolutely continuous with respect to natural ergodic measures (see Section 2.3.3 for detailed discussion and [SX08] for in-depth analysis of this phenomenon). Instead we construct a measure $\mu$ such that almost every point is transitive and $V_m^f$ is absolutely continuous with respect to $\mu$. This is clearly sufficient.

The construction follows the lines of Pesin-Sinai [PS83] construction of $u$-Gibbs measures. Given a partially hyperbolic diffeomorphism they construct a measure such that the unstable foliation is absolutely continuous with respect to the measure. In fact this construction works well for any expanding foliation. We apply this construction to $m$-dimensional foliation $W_m^f$.

Construction is described as follows. Let $x_0$ be a fixed point of $f$. For any $y \in W_m^f(x_0)$ define

$$\rho(y) = \prod_{n \geq 0} \frac{J_m^f(f^{-n}(y))}{J_m^f(x_0)},$$

where $J_m^f = Jacobian(f\big|_{W_m^f})$.

Let $\mathcal{V}_0$ be an open bounded neighborhood of $x_0$ in $W_m^f(x_0)$. Consider a probability measure $\eta_0$ supported on $\mathcal{V}_0$ with density proportional to $\rho(\cdot)$. For $n > 0$ define

$$\mathcal{V}_n = f^n(\mathcal{V}_0),\ \eta_n = (f^n)_* \eta_0.$$

Let

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \eta_i.$$

By the Krylov-Bogoljubov theorem $\{\mu_n; n \geq 0\}$ is weakly compact and any of its limits is $f$-invariant. Let $\mu$ be an accumulation point of $\{\mu_n; n \geq 0\}$. This is the measure that we are looking for.

Foliation $W_m^f$ is absolutely continuous with respect to $\mu$. Proof from Chapter 2 requires only minimal modifications that are due to higher dimension of $W_m^f$.

Since foliation $W_m^f$ is conjugate to the linear foliation $W_m^L$ we have that for any open ball $B$

$$\exists R > 0 \quad \bigcup_{y \in B} W_m^f(y, R) = \mathbb{T}^d,$$

where $W_m^f(y, R)$ is a ball of radius $R$ inside of the leaf $W_m^f(y)$. Together with absolute continuity this guarantees that $\mu$ almost every point is transitive. See, Section 2.3.3, Step 3 for the proof. We stress that we do not need to know that $\mu$ has full support in that argument.

It is left to show that the conjugacy $h$ is $C^{1+\nu}$-differentiable in the direction of $V^f_m$ at $\mu$ almost every point. For this we need to argue that $V^f_m$ is absolutely continuous with respect to $\mu$.

Foliation $W^f(m,k)$ is Lipschitz inside of a leaf of $W^f$ (we refer to [LY85], Section 4.2). Hence $V^f_m = W^f(m,k) \cap W^f_m$ is Lipschitz inside of a leaf of $W^f_m = W^f \cap W^f_m$. So we have that $V^f_m$ is absolutely continuous with respect to the Lebesgue measure on a leaf of $W^f_m$ while $W^f_m$ is absolutely continuous with respect to $\mu$. Therefore $V^f_m$ is absolutely continuous with respect to $\mu$.

### 3.4.6 Induction step 1 revisited

To carry out proof of Lemma 3.4.6 assuming Property $\mathcal{A}$ only we shrink neighborhood $\mathcal{U}$ even more. In addition to (3.10) and (3.11) we require $f \in \mathcal{U}$ to have narrow spectrum. Namely,

$$\forall m,\ 1 < m \le k \quad \frac{\log \tilde{\beta}_m}{\log \beta_m} > \frac{\log \beta_{m-1}}{\log \tilde{\beta}_m}$$

and the analogous condition on the contraction rates $\alpha_j$, $\tilde{\alpha}_j$. The following condition that we will actually use is obviously a consequence of the above one.

$$\forall i < k \quad \text{and} \quad \forall m,\ i < m \le k \quad \rho \overset{\text{def}}{=} \frac{\log \tilde{\beta}_m}{\log \beta_m} > \frac{\log \beta_i}{\log \tilde{\beta}_m}. \tag{3.24}$$

This inequality can be achieved by shrinking the size of $\mathcal{U}$ since $\beta_j$ and $\tilde{\beta}_j$ get arbitrarily close to $\lambda_j$, $j = 1, \dots k$.

*Remark.* Condition (3.24) greatly simplifies the proof of Lemma 3.4.6. We have yet another longer proof (but based on the same idea) of Lemma 3.4.6 that works for any $f$ with Property $\mathcal{A}$ in $\mathcal{U}$ as defined in Section 3.4.1. It will not appear here.

We start the proof as in Section 3.4.4. The first place where we use Property $\mathcal{A}'$ is the proof of Lemma 3.4.10. So we reprove induction step 1 with Property $\mathcal{A}$ only assuming that we have got everything that preceded Lemma 3.4.10. With Property $\mathcal{A}$ the proof of Lemma 3.4.10 still goes through. Although instead of (3.20) we get

$$\forall z \in \mathbb{T}^d \quad \text{either} \quad [z, sh(z)] = sh(z) \ \text{ or } \ \mathcal{O}([z, sh(z)], sh(z)) = O^+.$$

Thus we still have Lemma 3.4.10 and the upper bound (3.22) but not the lower bound $d_i^f([z, sh(z)], sh(z)) > \delta$. This is the reason why we cannot proceed with the proof of the induction step as at the end of Section 3.4.4.

*Proof of the induction step.* As before we need to show that $U$ subfoliate $W^f(i+1, m)$.

Fix orientation $\mathcal{O}$ on $V_m^f$ and $V_i^f$. Given $x \in \mathbb{T}^d$ and $\varepsilon > 0$ choose $\bar{x} \in V_m^f(x)$ such that $d_m^f(x, \bar{x}) = \varepsilon$ and $\mathcal{O}(x, \bar{x}) = O^+$. Let $\bar{y} = U(x) \cap W^f(i, m-1)(\bar{x})$ and $y = V_i^f(x) \cap W^f(i+1, m)(\bar{y})$. This way we define an $\varepsilon$-"rectangle" $\mathcal{R} = \mathcal{R}(x, \bar{x}, y, \bar{y})$ with the base point $x$, vertical size $d_m^f(x, \bar{x}) = \varepsilon$ and horizontal size $d_i^f(x, y) = \bar{\varepsilon}$.

*Remark.* Notice that we measure vertical size in a way different from one in 3.4.4.

It is clear that "rectangle" is uniquely defined by its "diagonal" $(x, \bar{y})$ (Figure 3.8 is the picture of "rectangle" with diagonal $(x_n, x_{n+1})$. Sometimes we will use notation $\mathcal{R}(x, \bar{y})$. Note that by Lemma 3.4.10 $\mathcal{O}(x, y)$ does not depend on $x$ and $\varepsilon$. Also it guarantees that the piece of $U(x)$ between $x$ and $\bar{y}$ lies "inside" of $\mathcal{R}(x, \bar{y})$. The horizontal size $\bar{\varepsilon}$ might happen to be equal to zero.

Next we define a set of base points $X_\varepsilon$ such that $U(x)$, $x \in X_\varepsilon$, has big Hölder slope inside of corresponding $\varepsilon$-rectangle.

$$X_\varepsilon = \{x \in \mathbb{T}^d : \ \ \bar{\varepsilon} \le \varepsilon^\delta\}$$

with some $\delta$ satisfying inequality $\rho > \delta > \log \beta_i / \log \tilde{\beta}_m$.

Let $\mu$ be the measure constructed in Section 3.4.5. Recall that $\mu$ almost every point is transitive. Foliation $W^f(i, m)$ is absolutely continuous with respect to $\mu$. The latter can be shown in the same way as absolute continuity of $V_m^f$ is shown in Section 3.4.5.

We consider two cases.

**Case 1.** $\overline{\lim_{\varepsilon \to 0}} \mu(X_\varepsilon) > 0$.

Then choose $\{X_{\varepsilon_n}, n \ge 1\}$, $\varepsilon_n \to 0$ as $n \to \infty$, such that $\overline{\lim}_{n\to\infty} \mu(X_{\varepsilon_n}) > 0$.

The idea now is to iterate a rectangle with base point in $X_{\varepsilon_n}$ and vertical size $\varepsilon_n$ until the vertical size is approximately 1. Since the Hölder slope of initial rectangle was big it will turn out that the horizontal size of the iterated rectangle is extremely small. This argument will show that for a set of base points of positive measure the horizontal size of rectangles is equal to zero. Hence the leaves of $U$ lie inside of the leaves of $W^f(i+1,m)$.

Given $n$ let $N=N(n)$ be the largest number such that $\frac{1}{C}\tilde{\beta}_m^N\varepsilon_n<1$ (constant $C$ here is from definition (3.11)). Take $x\in X_{\varepsilon_n}$ and corresponding $\varepsilon_n$-rectangle $\mathcal{R}(x,y,\bar{x},\bar{y})$ and consider its image $\mathcal{R}(f^N(x),f^N(y),f^N(\bar{x}),f^N(\bar{y}))$. Choice of $N$ provides lower bound on the vertical size

$$VS\left(\mathcal{R}(f^N(x),f^N(y),f^N(\bar{x}),f^N(\bar{y}))\right)=d_m^f(f^N(x),f^N(\bar{x}))\geq\frac{1}{\beta_m}.$$

While the horizontal size can be estimated as follows

$$\begin{aligned}HS\left(\mathcal{R}(f^N(x),f^N(y),f^N(\bar{x}),f^N(\bar{y}))\right)&=d_i^f(f^N(x),f^N(y))\\&\leq C\beta_i^N\bar{\varepsilon}\leq C\beta_i^N\varepsilon^\delta\leq C\beta_i^N\left(\frac{C}{\tilde{\beta}_m^N}\right)^\delta=C^{1+\delta}\left(\frac{\beta_i}{\tilde{\beta}_m^\delta}\right)^N.\end{aligned}$$

Instead of looking at rectangle $\mathcal{R}(f^N(x),f^N(y),f^N(\bar{x}),f^N(\bar{y}))$ let us look at the rectangle $\tilde{\mathcal{R}}(f^N(x))$ with base point $f^N(x)$ and fixed vertical size $1/\beta_m$. Then Lemma 3.4.10 together with the estimate above on the vertical size of the rectangle $\mathcal{R}(f^N(x),f^N(y),f^N(\bar{x}),f^N(\bar{y}))$ guarantees that horizontal size of $\tilde{\mathcal{R}}(f^N(x))$ is less than $C^{1+\delta}\left(\beta_i/\tilde{\beta}_m^\delta\right)^N$ as well.

Thus for every $x\in f^N(X_{\varepsilon_n})$ the horizontal size of $\tilde{\mathcal{R}}(x)=\tilde{\mathcal{R}}(x,z,\tilde{x},\tilde{z})$ is less than $C^{1+\delta}\left(\beta_i/\tilde{\beta}_m^\delta\right)^N$. Note that $\left(\beta_i/\tilde{\beta}_m^\delta\right)^N\to 0$ as $n\to\infty$ since $\beta_i/\tilde{\beta}_m^\delta<1$ and $N\to\infty$ as $n\to\infty$.

Let $X=\overline{\lim}_{n\to\infty}f^N(X_{\varepsilon_n})$. Since any $x\in X$ also belong to $f^N(X_{\varepsilon_n})$ with arbitrarily large $N$ we conclude that $\tilde{\mathcal{R}}(x)$ has zero horizontal size i. e. $x=z$. Hence by Lemma 3.4.10 we conclude that the piece of $U(x)$ from $x$ to $\tilde{z}$ lies inside of $W^f(i+1,m)(x)$.

It is a simple exercise in measure theory to show that

$$\mu(X)\geq\overline{\lim_{n\to\infty}}\,\mu(f^N(X_{\varepsilon_n}))=\overline{\lim_{n\to\infty}}\,\mu(X_{\varepsilon_n})>0.$$

Finally recall that $\mu$ almost every point is transitive ($\overline{\{f^j(x),\ j\geq 1\}}=\mathbb{T}^d$). Hence by taking a transitive point $x\in X$ and applying straightforward approximation argument we get that $\forall y\;\;U(y)\subset W^f(i+1,m)(y)$.

**Case 2.** $\overline{\lim_{\varepsilon\to 0}}\,\mu(X_\varepsilon)=0.$

In this case the idea is to use the assumption above to find a leaf $U(x)$ which is "flat" i. e. arbitrarily close to $W^f(i,m-1)(x)$. Since the leaf $U(x)$ has to "feel" measure $\mu$ we need to take it together with a small neighborhood. Choice of this neighborhood is done by multiple application of pigeonhole principle.

Given a point $\bar{y}\in U(x)$ denote by $U_{x\bar{y}}$ the piece of $U(x)$ between $x$ and $\bar{y}$. As before by $\mathcal{R}(x,\bar{y})$ we denote the rectangle spanned by $x$ and $\bar{y}$. Recall that $HS(\mathcal{R}(x,\bar{y}))$ and $VS(\mathcal{R}(x,\bar{y}))$ stand for horizontal and vertical sizes of $\mathcal{R}(x,\bar{y})$. Also we will need to measure sizes of $U_{x\bar{y}}$. Simply let $HS(U_{x\bar{y}})=HS(\mathcal{R}(x,\bar{y}))$ and $VS(U_{x\bar{y}})=VS(\mathcal{R}(x,\bar{y}))$.

**Iterating Pigeonhole Principle.** Divide $\mathbb{T}^d$ into a finite number of tubes $\mathcal{T}_1,\mathcal{T}_2,\dots\mathcal{T}_q$ foliated by $U$ such that any connected component of $U(x)\cap\mathcal{T}_j$, $j=1,\dots q$, has vertical size between $S_0$ and $S_1$. Numbers $S_0$ and $S_1$ are fixed, $0<S_0<S_1$. We also require every tube $\mathcal{T}_j$ to be $W^f(i,m-1)$-foliated so that it can be represented as

$$\mathcal{T}_j=\bigcup_{y\in Transv} Plaque(y),$$

where $Transv$ is a plaque of $U$ and $Plaque(y)$ are plaques of $W^f(i,m-1)$.

Given a small number $\tau>0$ we can find an $\varepsilon>0$ such that $\mu(X_\varepsilon)<\tau$. Then by the pigeonhole principle we can choose a tube $\mathcal{T}_j$ such that $\mu(\mathcal{T}_j)\neq 0$ and

$$\frac{\mu(\mathcal{T}_j\cap X_\varepsilon)}{\mu(\mathcal{T}_j)}<\tau.$$

Tube $\mathcal{T}_j$ can be represented as $\mathcal{T}_j=\bigcup_{z\in\hat{\mathcal{T}}_j}W(z)$, where $\hat{\mathcal{T}}_j$ is a transversal to $W^f(i,m)$ and $W(z)$, $z\in\hat{\mathcal{T}}_j$, are connected plaques of $W^f(i,m)$. By absolute continuity

$$\mu(\mathcal{T}_j)=\int_{\hat{\mathcal{T}}_j} d\hat{\mu}(z)\int_{W(z)} d\mu_{W(z)},$$

where $\hat{\mu}$ is the factor measure on $\hat{\mathcal{T}}_j$ and $\mu_{W(z)}$ is the conditional measure on $W(z)$.

Apply pigeonhole principle again to choose $W=W(z)$ such that

$$\mu_W(W\cap X_\varepsilon)<\tau.$$

Recall that $\mu_W(W) = 1$ by the definition of conditional measure and $\mu_W$ is equivalent to the induced Riemannian volume on $W$ by absolute continuity of $W^f(i, m)$.

Plaque $W$ is subfoliated by plaques of $U$ of sizes between $S_0$ and $S_1$. Unfortunately we do not know if $U$ is absolutely continuous with respect to $\mu_W$. So we construct a finite partition of $W$ into smaller plaques of $W^f(i, m)$ which are very thin $U$-foliated tubes.

To construct this partition we switch to $h(W)$ which is a plaque of $W^g(i, m)$ subfoliated by the plaques of $h(U) = V^g_m$. The partition $\{\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots \tilde{\mathcal{T}}_p\}$ will consist of $V^g_m$-tubes inside of $h(W)$ that can be represented as

$$\tilde{\mathcal{T}}_j = \bigcup_{z \in \hat{\mathcal{T}}_j} V(z), \quad j = 1, \dots p,$$

where $\hat{\mathcal{T}}_j$ is a transversal to $V^g_m$ inside of $h(W)$ and $V(z)$ are plaques of $V^g_m$. For every $j = 1, \dots p$ choose $z_j \in \hat{\mathcal{T}}$. Then the tube $\tilde{\mathcal{T}}_j$ can also be represented as

$$\tilde{\mathcal{T}}_j = \bigcup_{y \in V(z_j)} \tilde{P}_j(y),$$

where $\tilde{P}_j(y) \subset W^g(i, m-1)(y)$ are connected plaques.

Recall that $V^g_m$ is Lipschitz inside of $W^g(i, m)$. Hence for any $\xi > 0$ it is possible to find a partition $\{\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots \tilde{\mathcal{T}}_p\}$, $p = p(\xi)$, such that

$$\begin{aligned} &\forall j = 1, \dots p \;\; \forall y \in V(z_j) \quad \exists B_j(\tilde{C}_1\xi), B_j(\tilde{C}_2\xi) \subset W^g(i, m-1)(y) \\ &\qquad\qquad \text{such that} \quad B_j(\tilde{C}_1\xi) \subset \tilde{P}_j(y) \subset B_j(\tilde{C}_2\xi), \end{aligned} \tag{3.25}$$

where $B_j(\tilde{C}_1\xi)$ and $B_j(\tilde{C}_2\xi)$ are balls inside of $(W^g(i, m-1)(y)$, induced Riemannian distance) of radii $\tilde{C}_1\xi$ and $\tilde{C}_2\xi$ respectively. Constants $\tilde{C}_1$ and $\tilde{C}_2$ are independent of $\xi$. Since we are working in a bounded plaque $h(W)$ they also do not depend on any other choices but $S_1$.

In the sequel we will need to take $\xi$ to be much smaller than $\varepsilon$.

Now we pool this partition back into a partition of $W$.

$$\{\mathcal{T}_1, \mathcal{T}_2, \dots \mathcal{T}_p\} = \{h^{-1}(\tilde{\mathcal{T}}_1), h^{-1}(\tilde{\mathcal{T}}_2), \dots h^{-1}(\tilde{\mathcal{T}}_p)\}.$$

Although we use the same notation for this partition it is clearly different from the initial partition of $\mathbb{T}^d$.

Each tube $\mathcal{T}_j$ can be represented as

$$\mathcal{T}_j = \bigcup_{y \in U(h^{-1}(z_j))} P_j(y), \tag{3.26}$$

where $P_j(y) = h^{-1}(\tilde{P}_j(y)) \subset W^f(i, m-1)(y)$.

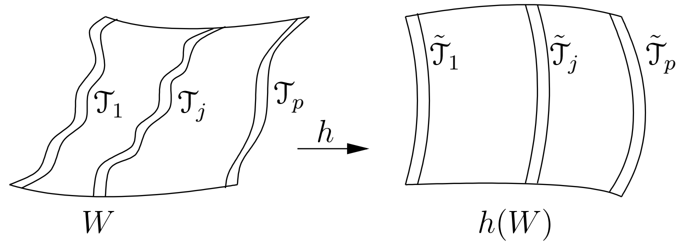


**Figure 3.10.** We construct partition $\{\mathcal{T}_1, \mathcal{T}_2, \dots \mathcal{T}_p\}$ as a pullback of partition of $h(W)$ by $V^g_m$-tubes. Foliation $V^g_m$ is Lipschitz and $h$ is continuously differentiable along $W^f(i, m-1)$. This guarantees that the "width" of a tube $\mathcal{T}_j$ is of the same order as we move along $\mathcal{T}_j$ (3.27). Hence $\mu_W$ is "uniformly distributed" along $\mathcal{T}_j$.

By Lemma 3.4.7 $h$ is continuously differentiable along $W^f(i, m-1)$. Moreover, the derivative depend continuously on the point in $W$. Hence property (3.25) persists

$$\begin{aligned} \forall j = 1, \dots p \;\; \forall y \in U(h^{-1}(z_j)) \quad & \exists B_j(C_1\xi), B_j(C_2\xi) \subset W^f(i, m-1)(y) \\ & \text{such that} \quad B_j(C_1\xi) \subset P_j(y) \subset B_j(C_2\xi). \end{aligned} \tag{3.27}$$

Constants $C_1$ and $C_2$ differ from $\tilde{C}_1$ and $\tilde{C}_2$ by a finite factor due to the bounded distortion along $W^f(i, m-1)$ by the differential of $h$.

Apply pigeonhole principle for the last time to find $\mathcal{T} \in \{\mathcal{T}_1, \mathcal{T}_2, \dots \mathcal{T}_p\}$ such that

$$\frac{\mu_W(\mathcal{T} \cap X_\varepsilon)}{\mu_W(\mathcal{T})} < \tau. \tag{3.28}$$

Take a plaque $U_{x\bar{y}}$ inside of $\mathcal{T}$. By the construction

$$S_0 < VS(U_{x\bar{y}}) < S_1.$$

**Estimating horizontal size of $U_{x\bar{y}}$ from below.** We have constructed $U_{x\bar{y}}$ so that a lot of points in the neighborhood of $U_{x\bar{y}}$ $\mathcal{T}$ lie outside of $X_\varepsilon$. Corresponding $\varepsilon$-rectangles $\mathcal{R}(x)$ have vertical size greater than $\varepsilon^\delta$. It is clear that we can use this fact to show that $VS(U_{x\bar{y}})$ is large.

Choose a sequence $\{x_0 = x, x_1, \dots x_N\} \subset U_{x\bar{y}}$ such that

$$VS(\mathcal{R}(x_0, x_N)) \geq S_0 \quad \text{and} \quad VS(\mathcal{R}(x_j, x_{j+1})) = \varepsilon,\ j = 0, \dots N-1.$$

First we estimate the number of rectangles $N$.

**Lemma 3.4.11.** *The holonomy map $Hol : T(a) \to T(b)$, $b \in W^f(i,m)(a)$, $T(a) \subset V_m^f(a)$, $T(b) \subset V_m^f(b)$, along $W^f(i, m-1)$ is Hölder continuous with exponent*

$$\rho \stackrel{\text{def}}{=} \frac{\log \tilde{\beta}_m}{\log \beta_m}.$$

We postpone the proof until the end of the current section.

Let $\tilde{x}_j = W^f(i, m-1)(x_j) \cap V_m^f(x_0)$, $j = 0, \dots N$. Then according to the lemma above

$$d_m^f(\tilde{x}_{j-1}, \tilde{x}_j) \leq C_{Hol} VS(\mathcal{R}(x_{j-1}, x_j))^\rho = C_{Hol}\varepsilon^\rho, \quad j = 1, \dots N,$$

which allows to estimate $N$

$$S_0 \leq VS(\mathcal{R}(x_0, x_N)) = \sum_{j=1}^{N} d_m^f(\tilde{x}_{j-1}, \tilde{x}_j) \leq N C_{Hol} \varepsilon^\rho.$$

Hence

$$N \geq \frac{S_0}{C_{Hol}\varepsilon^\rho}. \tag{3.29}$$

Along with the rectangles $\mathcal{R}(x_j, x_{j+1})$ let us consider sets $A(x_j, x_{j+1}) \subset \mathcal{T}$, $j = 0, \dots N-1$ given by the formula

$$A(x_j, x_{j+1}) = \bigcup_{y \in U_{x_j x_{j+1}}} P(y),$$

where $P(y)$ are plaques of $W^f(i, m-1)$ from representation (3.26) for $\mathcal{T}$. Sets

$A(x_j, x_{j+1})$ have the same vertical size. The following property of these sets is a direct consequence of (3.27) and the fact that $\mu_W$ is equivalent to the Riemannian volume on $W$.

$$\exists C_{univ} \quad \text{such that} \quad \forall j, \tilde{j} = 1, \dots N-1 \quad \frac{1}{C_{univ}} < \frac{\mu_W(A(x_j, x_{j+1}))}{\mu_W(A(x_{\tilde{j}}, x_{\tilde{j}+1}))} < C_{univ}. \tag{3.30}$$

Constant $C_{univ}$ depends on $C_1$, $C_2$ and size of $W$, but independent of $\varepsilon$ and $\xi$.

Let

$$A_1 = \bigcup_{\substack{j=1 \\ j \text{ is odd}}}^{N-1} A(x_j, x_{j+1}) \quad \text{and} \quad A_2 = \bigcup_{\substack{j=1 \\ j \text{ is even}}}^{N-1} A(x_j, x_{j+1})$$

It follows from (3.28) that we have that either

$$\frac{\mu_W(A_1 \cap X_\varepsilon)}{\mu_W(A_1)} < \tau \quad \text{or} \quad \frac{\mu_W(A_2 \cap X_\varepsilon)}{\mu_W(A_2)} < \tau.$$

For concreteness assume that the first holds.

Bounds (3.30) allow to estimate the number $N_1$ of sets $A(x_j, x_{j+1}) \subset A_1$ that have a point $q_j \in A(x_j, x_{j+1})$ such that $q_j \notin X_\varepsilon$.

$$N_1 \geq \left\lfloor \frac{N}{2} \right\rfloor - \lfloor C_{univ} \tau N \rfloor.$$

Here $\lfloor N/2 \rfloor$ is the total number of sets $A(x_j, x_{j+1})$ in $A_1$ and $\lfloor C_{univ}\tau N \rfloor$ is the maximal possible number of sets $A(x_j, x_{j+1})$ in $A_1 \cap X_\varepsilon$. Clearly we can choose $\tau$ and $\varepsilon$ accordingly such that $N_1 \geq N/3$.

For every $A(x_j, x_{j+1})$ as above fix $q_j \in A(x_j, x_{j+1})$, $q_j \notin X_\varepsilon$, and consider rectangle $\mathcal{R}(q_j)$ of vertical size $\varepsilon$. Then

$$HS(\mathcal{R}(q_j)) \geq \varepsilon^\delta.$$

Consider two rectangles $\mathcal{R}(q_j)$ and $\mathcal{R}(q_{\tilde{j}})$ as above. Since $|j - \tilde{j}| \geq 2$ they do not "overlap" vertically if $\xi$ is sufficiently small (although this is not important to us). They might happen to "overlap" horizontally as shown on the Figure 3.11 but the size of the overlap cannot exceed the diameter of the tube $\mathcal{T}$ which, according to (3.27), is bounded by $C_2\xi$.

Above considerations result in the following estimate

$$
\begin{aligned}
HS(U_{x\bar{y}}) \geq HS(U_{x_0x_N}) &\geq \frac{1}{C_H}\sum_{j=1}^{N_1} HS(\mathcal{R}(q_j)) - C_H N_1 C_2 \xi \\
&\geq \frac{1}{C_H} N_1 \varepsilon^{\delta} - C_H N C_2 \xi \geq \frac{N}{3C_H}\varepsilon^{\delta} - NC_H C_2 \xi \geq \frac{S_0}{3C_H C_{Hol}}\varepsilon^{\delta-\rho} - NC_H C_2\xi,
\end{aligned}
\tag{3.31}
$$

where $C_H$ is the Lipschitz constant of the holonomy map along $W^f(i+1,m)$. We used estimate on $N_1$ and estimate (3.29) on $N$.

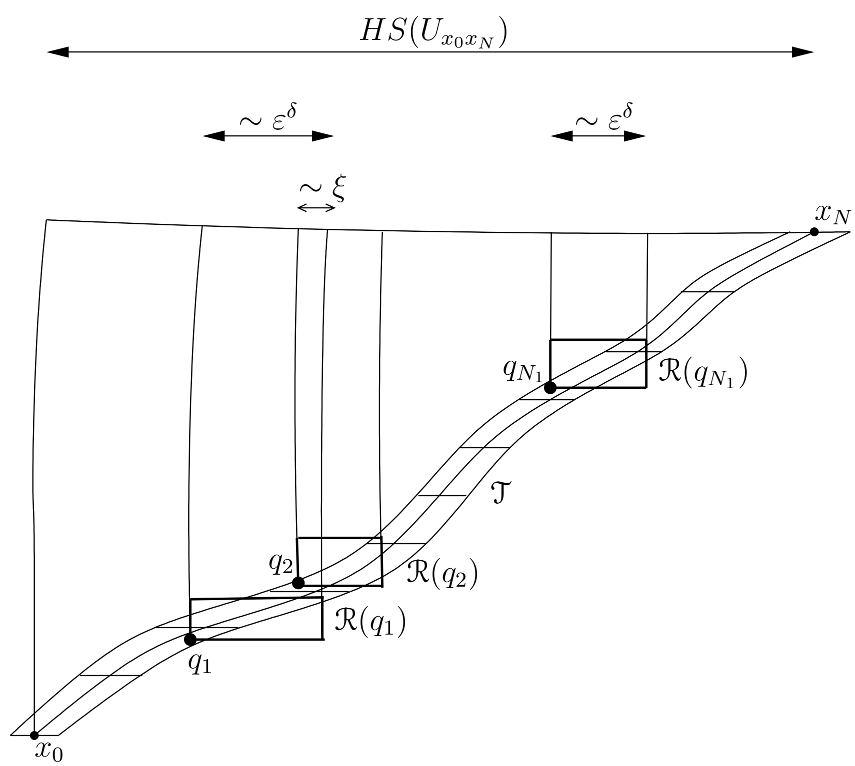


**Figure 3.11.** This picture illustrates the key estimate (3.31). Since the holonomy along $W^f(i+1,m)$ is Lipschitz the horizontal size of $U_{x_0x_N}$ can be estimated from below by the sum of horizontal sizes of "flat" rectangles with base points $q_j \in A_1 \subset \mathcal{T}$, $j = 1, \dots N_1$. They might overlap horizontally as shown but the overlap is of order $\xi \lll \varepsilon$.

Finally recall that $\delta - \rho < 0$ while $\xi$ can be chosen arbitrarily small independently of $\varepsilon$ (and hence $N$). Hence by choosing $\varepsilon$ small we can find $U_{x\bar{y}}$ with arbitrarily big horizontal size that contradicts to the uniform upper bound (3.22) that follows from compactness. Hence Case 2 is impossible. □

*Remark.* Note that we do not need to take $\tau$ arbitrarily small. Constant $\tau$ just need to be small enough to provide the estimate on $N_1$.

*Proof of Lemma 3.4.11.* Take points $x$ and $y \in V^f_m(x)$ such that

$$1 \leq d^f_m(x, y) \leq C\beta_m \tag{3.32}$$

By Lemma 3.4.9 there exist constants $c_1$ and $c_2$ such that

$$\forall \tilde{x}, \tilde{y}, \;\; \tilde{y} \in V^f_m(\tilde{x}), \;\; \tilde{x} \in W^f(i, m-1)(x), \;\; \tilde{y} \in W^f(i, m-1)(y)$$
$$c_1 < d^f_m(\tilde{x}, \tilde{y}) < c_2. \tag{3.33}$$

Moreover, since $c_1$ and $c_2$ depend only on $\hat{d}(x, y)$ (see remark after the proof of Lemma 3.4.9) they can be chosen independently of $x$ and $y$ as long as $x$ and $y$ satisfy (3.32).

Take $x, y \in T(a)$ close to each other. Let $N$ be the smallest integer such that $d^f_m(f^N(x), f^N(y)) \geq 1$. Then

$$d^f_m(f^N(x), f^N(y)) \geq \frac{1}{C}\tilde{\beta}^N_m d^f_m(x, y) \tag{3.34}$$

and, obviously,

$$d^f_m(f^N(x), f^N(y)) \leq C\beta_m. \tag{3.35}$$

Hence by taking in (3.33) $\tilde{x} = f^N(Hol(x))$ and $\tilde{y} = f^N(Hol(y))$ we get

$$d^f_m\left(f^N(Hol(x)), f^N(Hol(y))\right) > c_1. \tag{3.36}$$

On the other hand

$$d^f_m\left(f^N(Hol(x)), f^N(Hol(y))\right) \leq C\beta^N_m d^f_m(Hol(x), Hol(y)). \tag{3.37}$$

Combining (3.34), (3.35), (3.36) and (3.37) we finish the proof

$$d^f_m(x, y) \leq \frac{C}{\tilde{\beta}^N_m} d^f_m(f^N(x), f^N(y)) \leq \frac{C^2\beta_m}{c_1^\rho \tilde{\beta}^N_m} \cdot c_1^\rho$$

$$< \frac{C^2\beta_m}{c_1^\rho} \cdot \frac{1}{\tilde{\beta}_m^N} d_m^f \left(f^N(Hol(x)), f^N(Hol(y))\right)^\rho \leq C_{Hol} \frac{\beta_m^{\rho N}}{\tilde{\beta}_m^N} d_m^f(Hol(x), Hol(y))^\rho$$
$$= C_{Hol} d_m^f(Hol(x), Hol(y))^\rho.$$

We used (3.24) for the last equality. □

Chapter 4

# Around de la Llave's counterexample

## 4.1 When the coincidence of periodic data is not sufficient

First let us describe the counterexample of de la Llave.

Let $L:\mathbb{T}^4\to\mathbb{T}^4$ be an automorphism of the product type

$$L(x,y)=(Ax,By),\ \ (x,y)\in\mathbb{T}^2\times\mathbb{T}^2, \tag{4.1}$$

where $A$ and $B$ are Anosov automorphisms. Let $\lambda$, $\lambda^{-1}$ be the eigenvalues of $A$ and $\mu$, $\mu^{-1}$ be the eigenvalues of $B$. We assume that $\mu>\lambda>1$. Consider perturbations of the form

$$\tilde{L}=(Ax+\vec{\varphi}(y),By), \tag{4.2}$$

where $\vec{\varphi}:\mathbb{T}^2\to\mathbb{R}^2$ is a $C^1$-small $C^r$, $r>1$, function. Obviously p. d. of $L$ and $\tilde{L}$ coincide. We will see in Section 4.3 that majority of perturbations (A.2) are only Hölder conjugate to $L$. The following theorem is a simple generalization of this counterexample.

**Theorem B.** *Let $L:\mathbb{T}^d\to\mathbb{T}^d$ be a hyperbolic automorphism. Assume that characteristic polynomial of $L$ factors over $\mathbb{Q}$. Then there exist $C^\infty$ diffeomorphisms $\tilde{L}:\mathbb{T}^d\to\mathbb{T}^d$ and $\hat{L}:\mathbb{T}^d\to\mathbb{T}^d$ arbitrarily $C^1$-close to $L$ with the same p. d. such that the conjugacy between $\tilde{L}$ and $\hat{L}$ is not Lipschitz.*

*Remark.* In majority of cases one can take $\hat{L} = L$. The need to take $\tilde{L}$ and $\hat{L}$ both to be different from $L$ appears, for instance, when $L(x, y) = (Ax, Ay)$. It was shown in [L02] that p. d. form complete set of moduli for smooth conjugacy problem to $L$. This is a remarkable phenomenon due to invariance of conformal structures on stable and unstable foliations. Nevertheless we still have a counterexample if we go a little bit away from $L$.

Next we study smooth conjugacy problem in the neighborhood of (A.1) assuming that $\mu > \lambda > 1$. We show that perturbations (A.2) exhaust all possibilities. Before formulating the result precisely let us move to a slightly more general setting. Let $A$ and $B$ be as in (A.1) with $\mu > \lambda > 1$. Consider Anosov diffeomorphism

$$L(x, y) = (Ax, g(y)), \quad (x, y) \in \mathbb{T}^2 \times \mathbb{T}^2, \tag{4.3}$$

where $g$ is an Anosov diffeomorphism sufficiently $C^1$-close to $B$ so that $L$ can be treated as a partially hyperbolic diffeomorphism with automorphism $A$ acting in the central direction. Consider perturbations of the form

$$\tilde{L} = (Ax + \vec{\varphi}(y), g(y)). \tag{4.4}$$

As before, it is obvious that p. d. of $L$ and $\tilde{L}$ coincide. In Section 4.6 we will see that $L$ and $\tilde{L}$ with non-linear $g$ also provide a counterexample to Question 1.

**Theorem C.** *Given $L$ as in (4.3) with $\mu > \lambda > 1$ there exists a $C^1$-neighborhood $\mathcal{U} \subset \mathrm{Diff}^r(\mathbb{T}^4)$, $r \geq 2$, of $L$ such that any $f \in \mathcal{U}$ that has the same p. d. as $L$ is $C^{1+\nu}$, $\nu > 0$, conjugate to a diffeomorphism $\tilde{L}$ of type (4.4).*

## 4.2 Additional moduli of $C^1$ conjugacy in the neighborhood of the counterexample of de la Llave

Let $L$ be given by (A.1) with $\mu > \lambda > 1$ and let $\mathcal{U}$ be a small $C^1$-neighborhood of $L$. It is fruitful to think of diffeomorphisms from $\mathcal{U}$ as of partially hyperbolic diffeomorphisms with two dimensional central foliations. Consider $f, g \in \mathcal{U}$, $h \circ f =$

$g \circ h$. According to the celebrated theorem of Hirsch, Pugh and Shub [HPS77] the conjugacy $h$ maps the central foliation of $f$ into the central foliation of $g$.

Assume that p. d. of $f$ and $g$ are the same. Then we show that $h$ is $C^{1+\nu}$ along the central foliation. As described above it can still happen that $h$ is not a $C^1$-diffeomorphism. This means that the conjugacy is not differentiable in the direction transverse to the central foliation. The geometric reason for this is mismatch between strong stable (unstable) foliations of $f$ and $g$ — the conjugacy $h$ does not map strong stable (unstable) foliation of $f$ into strong stable (unstable) foliation of $g$.

Motivated by this observation we introduce additional moduli of continuously differentiable conjugacy. Roughly speaking these moduli measure the tilt of strong stable (unstable) leaves when compared to the model (A.1).

We define these moduli precisely. Let $W_L^{ss}$, $W_L^{ws}$, $W_L^{wu}$ and $W_L^{su}$ be the foliations by straight lines along the eigendirections with eigenvalues $\mu^{-1}$, $\lambda^{-1}$, $\lambda$ and $\mu$ respectively. For any $f \in \mathcal{U}$ these invariant foliations survive. We denote them by $W_f^{ss}$, $W_f^{ws}$, $W_f^{wu}$ and $W_f^{su}$. Also we write $W_f^s$ and $W_f^u$ for two dimensional stable and unstable foliations.

Let $h_f$ be the conjugacy to the linear model, $h_f \circ f = L \circ h_f$. Then

$$h_f(W_f^\sigma) = W_L^\sigma, \quad \sigma = s, u, ws, wu. \tag{4.5}$$

Fix orientation of $W_L^\sigma$, $\sigma = ss, ws, wu, su$. Then for every $x \in \mathbb{T}^4$ there exists a unique orientation preserving isometry $\mathcal{I}^\sigma(x) : W_L^\sigma(x) \to \mathbb{R}$, $\mathcal{I}^\sigma(x) = 0$, $\sigma = ss, ws, wu, su$.

Define $\Phi_f^u : \mathbb{T}^4 \times \mathbb{R} \to \mathbb{R}$ by the formula

$$\Phi_f^u(x,t) = \mathcal{I}^{wu}\big(\mathcal{I}^{su}(x)^{-1}(t)\big)\big(h_f(W_f^{su}(h_f^{-1}(x))) \cap W_L^{wu}(\mathcal{I}^{su}(x)^{-1}(t))\big).$$

The geometric meaning is transparent and illustrated on Figure 4.1. Image of strong unstable manifold $h_f(W_f^{su}(x))$ can be viewed as a graph of function $\Phi_f^u(x,\cdot)$ over $W_L^{su}(x)$. Analogously we define $\Phi_f^s : \mathbb{T}^4 \times \mathbb{R} \to \mathbb{R}$.

Clearly $\Phi_f^{s/u}$ are moduli of $C^1$ conjugacy. Indeed, assume that $f$ and $g$ are $C^1$ conjugate by $h$. Then $h(W_f^{su}) = h(W_g^{su})$ and $h(W_f^{ss}) = h(W_g^{ss})$ since strong stable and unstable foliations are characterized by the speed of convergence which

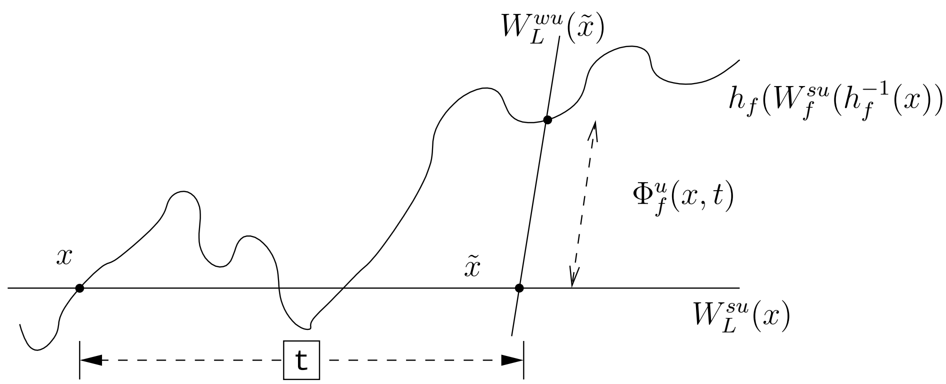


**Figure 4.1.** Geometric meaning of $\Phi^u_f$. Here $\tilde{x} = \mathcal{I}^{su}(x)^{-1}(t)$.

is preserved by $C^1$ conjugacy. Hence $\Phi^{s/u}_f = \Phi^{s/u}_g$.

It is possible to choose a subfamily of these moduli in an efficient way. We say that $f$ and $g$ from $\mathcal{U}$ have the same *strong unstable foliation moduli* if

$$\exists t \neq 0 \quad \text{such that} \quad \forall x \in \mathbb{T}^4, \quad \Phi^u_f(x,t) = \Phi^u_g(x,t) \tag{4.6}$$

or

$$\exists x \in \mathbb{T}^4 \quad \text{and} \quad \exists I = (a,b) \subset \mathbb{R} \quad \text{such that} \quad \forall t \in I \quad \Phi^u_f(x,t) = \Phi^u_g(x,t). \tag{4.7}$$

Definition of *strong stable foliation moduli* is analogous.

**Theorem D.** *Given $L$ as in (A.1) with $\mu > \lambda > 1$ there exists a $C^1$-neighborhood $\mathcal{U} \subset \mathrm{Diff}^r(\mathbb{T}^4)$, $r \geq 2$ of $L$ such that if $f, g \in \mathcal{U}$ have the same p. d. and the same strong unstable and strong stable foliation moduli. Then $f$ and $g$ are $C^{1+\nu}$ conjugate.*

*Remark.* In this case $C^{1+\nu}$-differentiability is in fact the optimal regularity.

In Section 4.3 we describe the counterexample of de la Llave in a way that allows us to generalize it to Theorem B in Section 4.4. Sections 4.3 and 4.4 are self contained and independent from the rest of the manuscript.

Theorem C is proved in Section 4.5. It is independent of the rest of the manuscript with an exception of a reference to Proposition 16.

Proof of Theorem D appears in Section 4.6 and relies on some technical results from Chapter 2.

## 4.3 The counterexample on $\mathbb{T}^4$

Here we describe the example of de la Llave of two Anosov diffeomorphisms of $\mathbb{T}^4$ with the same p. d. that are only Hölder conjugate. Understanding of the example is important for the proof of Theorem B.

Recall that we start with an automorphism $L : \mathbb{T}^4 \to \mathbb{T}^4$

$$L(x, y) = (Ax, By), \ \ (x, y) \in \mathbb{T}^2 \times \mathbb{T}^2,$$

where $A$ and $B$ are Anosov automorphisms, $Av = \lambda v$, $A\tilde{v} = \lambda^{-1}\tilde{v}$, $Bu = \mu u$, $B\tilde{u} = \mu^{-1}\tilde{u}$. We assume that $\mu \geq \lambda > 1$.

To simplify computations we consider a special perturbation of the form

$$\tilde{L} = (Ax + \varphi(y)v, By).$$

We look for the conjugacy $h$ of the form

$$h(x, y) = (x + \psi(y)v, y). \tag{4.8}$$

The conjugacy equation $h \circ \tilde{L} = L \circ h$ transforms into a cohomological equation on $\psi$

$$\varphi(y) + \psi(By) = \lambda\psi(y). \tag{4.9}$$

Let us solve for $\psi$ using the recurrent formula

$$\psi(y) = \lambda^{-1}\varphi(y) + \lambda^{-1}\psi(By).$$

We get a continuous solution to (A.2.2.1)

$$\psi(y) = \lambda^{-1} \sum_{k \geq 0} \lambda^{-k}\varphi(B^k y). \tag{4.10}$$

Hence the conjugacy is indeed given by the formula (A.3).

In the following proposition we denote by subscript $u$ the partial derivative in the direction of $u$.

**Proposition 12.** *Assume that $\mu > \lambda > 1$. Then function $\psi$ is Lipschitz in the direction of $u$ if and only if*

$$\sum_{k\in\mathbb{Z}} \left(\frac{\mu}{\lambda}\right)^k \varphi_u(B^k y) = 0, \tag{4.11}$$

*i. e. the series on the left converge in the sense of distribution convergence and the limit is equal to zero.*

*Proof.* First assume (4.11). Let us consider series (4.10) as series of distributions that converge to $\psi$. Then as a distribution $\psi_u$ is obtained by differentiating (4.10) termwise.

$$\psi_u = \lambda^{-1} \sum_{k\geq 0} \lambda^{-k} \mu^k \varphi_u(B^k). \tag{4.12}$$

Applying (4.11) we get

$$\psi_u = \lambda^{-1} \sum_{k<0} \lambda^{-k} \mu^k \varphi_u(B^k).$$

Since $\mu > \lambda$ the above series converge and the distribution is regular. Hence $\psi$ is differentiable in the direction of $u$.

Now assume that $\psi$ is $u$-Lipschitz. By differentiating (A.2.2.1) we get cohomological equation on $\psi_u$

$$\varphi_u(x) + \mu\psi_u(By) = \lambda\psi_u(y)$$

that is satisfied on a $B$-invariant set of full measure. We solve it using the recurrent formula

$$\psi_u(y) = -\frac{1}{\mu}\varphi_u(B^{-1}y) + \frac{\lambda}{\mu}\psi_u(B^{-1}y).$$

Hence

$$\psi_u = \lambda^{-1} \sum_{k<0} \lambda^{-k} \mu^k \varphi_u(B^k). \tag{4.13}$$

On the other hand we know that as a distribution $\psi_u$ is given by (4.12). Combining (4.12) and (4.13) we get the desired equality (4.11). □

If $\mu = \lambda$ then the argument above works only in one direction. We will see that in this case $L$ and $\tilde{L}$ do not provide a counterexample since p. d. are different.

**Proposition 13.** *Assume that $\mu = \lambda$. Then (4.11) is a necessary assumption for $\psi$ to be Lipschitz in the direction of $u$.*

*Proof.* As in the proof of Proposition 1, viewed as distribution, $\psi_u$ is given by

$$\psi_u = \lambda^{-1} \sum_{k \geq 0} \varphi_u(B^k). \tag{4.14}$$

Assume that $\psi$ is $u$-Lipschitz then analogously to (4.13) we get

$$\psi_u = \lambda^{-1} \sum_{-N \leq k < 0} \varphi_u(B^k) + \psi(B^N). \tag{4.15}$$

Note that in the sense of distributions $\psi(B^N) \to 0$ as $N \to \infty$ since $B$ is mixing. Hence, as a distibution, $\psi_u$ is given by

$$\psi_u = \lambda^{-1} \sum_{k < 0} \varphi_u(B^k). \tag{4.16}$$

Combining (4.14) and (4.16) we get (4.11). □

By rewriting condition (4.11) in terms of Fourier coefficients of $\varphi$ one can see that it is an infinite codimension condition. Moreover, one can easily construct functions that do not satisfy (4.11). One only need to make sure that some Fourier coefficients of the sum (4.11) are non-zero. For instance, for any $\varepsilon > 0$ and positive integer $p$ function

$$\varphi(y) = \varphi(y_1, y_2) = \varepsilon \sin(p\pi y_1) \tag{4.17}$$

will serve the purpose. Thus corresponding $\tilde{L}$ is not $C^1$ conjugate to $L$. Note that $\tilde{L}$ maybe chosen arbitrarily close to $L$.

*Remark.* Perturbations of the general type (A.2) can be treated analogously by decomposing $\vec{\phi} = \phi_1 v + \phi_2 \tilde{v}$.

*Remark.* Notice that the assumption $\mu \geq \lambda > 1$ is crucial in this construction.

*Remark.* By choosing appropriate $\lambda$ and $\mu$ one can get any desired regularity of the conjugacy (see [L92] for details). For example, if $\mu^2 > \lambda > \mu > 1$ then the conjugacy is $C^1$ but not $C^2$.

From now on let us assume that $\mu = \lambda$. As we have remarked in the introduction $L$ and $\tilde{L}$ do not provide a counterexample. Indeed, the derivative of $\tilde{L}$ in the basis $\{v, u, \tilde{v}, \tilde{u}\}$ is

$$\begin{pmatrix} \lambda & \varphi_u & 0 & \varphi_{\tilde{u}} \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}.$$

Let $x$ be a periodic point, $\tilde{L}^p(x) = x$. Then the derivative of the returm map at $x$ is

$$\begin{pmatrix} \lambda^p & \lambda^{p-1} \sum_{y \in \mathcal{O}(x)} \varphi_u(y) & 0 & * \\ 0 & \lambda^p & 0 & 0 \\ 0 & 0 & \lambda^{-p} & 0 \\ 0 & 0 & 0 & \lambda^{-p} \end{pmatrix}. \tag{4.18}$$

We see that it is likely to have a Jordan block while $L$ is diagolizable. Hence $L$ and $\tilde{L}$ have different p. d.

It is still easy to cook up a counterexample in the neighborhood of $L$. Let

$$\hat{L} = (Ax + \xi(y)v, By)$$

an let

$$h(x, y) = (x + \psi(y)v, y)$$

be the conjugacy between $\tilde{L}$ and $\hat{L}$

**Proposition 14.** *Condition*

$$\sum_{k \in \mathbb{Z}} (\xi - \varphi)_u(B^k y) = 0,$$

*is necessary for $\psi$ to be Lipschitz in the direction of $u$.*

The proof is exactly the same as the one of Proposition 13.

Take $\varphi$ that does not satisfy (4.11) as before and take $\xi = 2\varphi$. Then obviously the condition of Proposition 14 is not satisfied. Hence $h$ is not Lipschitz. By looking at (4.18) it is obvious that our choice of $\xi$ guarantees that Jordan normal forms of the derivatives of the return maps at periodic points of $\tilde{L}$ and $\hat{L}$ are the same.

*Remark.* Due to the special choice of $\xi$ it was easy to ensure that p. d. of $\tilde{L}$ and $\hat{L}$ are the same. We could have taken a different and somewhat more general approach. It is possible to show that for many choices of $\varphi$ the sum that appears over the diagonal in (4.18) is non-zero for every periodic point $x$. All correspoding diffeomorphism will have the same p. d. with a Jordan block at every periodic point.

## 4.4 Proof of Theorem B

Here we consider $L : \mathbb{T}^d \to \mathbb{T}^d$ with reducible characteristic polynomial. We show how to construct $\tilde{L}$ and $\hat{L}$ with the same p. d. which are not Lipschitz conjugate.

Assume that all real eigenvalues of $L$ are positive. Otherwise we would consider $L^2$. Let $M : \mathbb{R}^d \to \mathbb{R}^d$ be the lift of $L$. And let $\{e_1, e_2, \dots e_d\}$ be the canonical basis so that $\mathbb{T}^d = \mathbb{R}^d / \text{span}_{\mathbb{Z}}\{e_1, e_2, \dots e_d\}$.

It is well known that characteristic polynomial of $M$ factors over $\mathbb{Z}$ into the product of polynomials irreducible over $\mathbb{Q}$.

$$P(x) = P_1(x)P_2(x)\dots P_r(x), \quad r \geq 2.$$

Let $\lambda$ be the eigenvalue of $M$ with the smallest absolute value which is greater than one. Without loss of generality we assume that $P_1(\lambda) = 0$.

Let $V_i$ be the invariant subspace that corresponds to the roots of $P_i$. Then $\dim V_i = \deg P_i$ and it is easy to show that

$$V_i = \text{Ker}(P_i(M)).$$

Matrices of $P_i(M)$ have integer entries. Hence there is a basis $\{\tilde{e}_1, \tilde{e}_2, \dots \tilde{e}_d\}$, $\tilde{e}_i \in \text{span}_{\mathbb{Z}}\{e_1, e_2, \dots e_d\}, i = 1, \dots d$, such that matrix of $M$ in this basis has integer entries and is of a block diagonal form with blocks corresponding to invariant subspaces $V_i, i = 1, \dots r$.

We consider projection of $M$ to $\tilde{\mathbb{T}}^d = \mathbb{R}^d / \mathrm{span}_{\mathbb{Z}}\{\tilde{e}_1, \tilde{e}_2, \dots \tilde{e}_d\}$. Denote by $N$ the induced map on $\tilde{\mathbb{T}}^d$. We have the following commutative diagram where $\pi$ is a finite-to-one projection.

$$\begin{array}{ccc}
\mathbb{R}^d & \xrightarrow{M} & \mathbb{R}^d \\
\downarrow & & \downarrow \\
\tilde{\mathbb{T}}^d & \xrightarrow{N} & \tilde{\mathbb{T}}^d \\
\pi\downarrow & & \pi\downarrow \\
\mathbb{T}^d & \xrightarrow{L} & \mathbb{T}^d
\end{array}$$

Notice that $N$ has the form $N(x, y) = (Ax, By)$, $(x, y) \in \mathbb{T}^{\deg P_1} \times \mathbb{T}^{d - \deg P_1}$. Let $\mu$ be an eigenvalue of $B$. By construction $\lambda$, $|\lambda| \le |\mu|$, is an eigenvalue of $A$.

With certain care the construction of Section 4.3 can be applied to $N$. We have to distinguish the following cases.

1. $\lambda$ and $\mu$ are real.

2. $\lambda$ is real and $\mu$ is complex.

3. $\lambda$ is complex and $\mu$ is real.

4. $\lambda$ and $\mu$ are complex.

Assume that $|\lambda| < |\mu|$. Then we take $\hat{L} = L$.

In the first case construction of Section 4.3 applies straightforwardly. We use function of the type (4.17) to produce $\tilde{N}$. Now we only need to make sure that $\tilde{N}$ can be projected to a map $\tilde{L} : \mathbb{T}^d \to \mathbb{T}^d$. Since $\pi$ is a finite-to-one covering map this can be achieved by choosing suitable $p$ in (4.17).

Other cases require heavier calculations but follow the same scheme of Proposition 1. We outline the construction in the case 4 that can appear, for instance, if $A$ and $B$ are hyperbolic automorphisms of four dimensional tori without real eigenvalues.

Let $V_A = \text{span}\{v_1, v_2\}$ be the two dimensional $A$-invariant subspace corresponding to $\lambda$ and $V_B = \text{span}\{u_1, u_2\}$ be the two dimensional $B$-invariant subspace corresponding to $\mu$. Then $A$ acts on $V_A$ by multiplication by $|\lambda| R_A$ and $B$ acts on $V_B$ by multiplication by $|\mu| R_B$, where $R_A$ and $R_B$ are rotation matrices expressed in bases $\{v_1, v_2\}$ and $\{u_1, u_2\}$ respectively.

We are following the construction from the previous section. Let

$$\tilde{N}(x, y) = (Ax + \vec{\varphi}(y)\vec{v}, By) \stackrel{\text{def}}{=} (Ax + \varphi_1(y)v_1 + \varphi_2(y)v_2, By).$$

Then we look for the conjugacy in the form

$$h(x, y) = (x + \vec{\psi}(y)\vec{v}, y) \stackrel{\text{def}}{=} (x + \psi_1(y)v_1 + \psi_2(y)v_2, y).$$

The conjugacy equation $h \circ \tilde{N} = N \circ h$ transforms into

$$\vec{\varphi}(y)\vec{v} + \vec{\psi}(By)\vec{v} = |\lambda| R_A \vec{\psi}(y). \tag{4.19}$$

Solving for $\vec{\psi}$ gives

$$\vec{\psi}(y) = \sum_{k \geq 0} |\lambda|^{-k-1} R_A^{-k-1} \vec{\varphi}(B^k y),$$

which we would like to differentiate in the directions $u_1$ and $u_2$. We use the formula

$$\vec{\varphi}(By)_{\vec{u}} = \begin{pmatrix} \varphi_1(By)_{u_1} & \varphi_1(By)_{u_2} \\ \varphi_2(By)_{u_1} & \varphi_2(By)_{u_2} \end{pmatrix} = |\mu| \begin{pmatrix} (\varphi_1)_{u_1} & (\varphi_1)_{u_2} \\ (\varphi_2)_{u_1} & (\varphi_2)_{u_2} \end{pmatrix} (By) R_B = \vec{\varphi}_{\vec{u}}(By) R_B$$

to get that as a distribution

$$\vec{\psi}_{\vec{u}} = \sum_{k \geq 0} |\lambda|^{-k-1} |\mu|^k R_A^{-k-1} \vec{\varphi}_{\vec{u}}(B^k) R_B^k.$$

Now we assume that $\vec{\psi}$ is Lipschitz and we differentiate (4.19) in the directions $u_1$ and $u_2$

$$\vec{\varphi}_{\vec{u}}(y) + |\mu| \vec{\psi}_{\vec{u}}(By) R_B = |\lambda| R_A \vec{\psi}_{\vec{u}}(y).$$

Hence by the recurrent formula

$$\vec{\psi}_{\vec{u}} = \sum_{k<0} |\lambda|^{-k-1}|\mu|^k R_A^{-k-1}\vec{\varphi}_{\vec{u}}(B^k)R_B^k.$$

Combining the expressions for $\vec{\psi}_{\vec{u}}$ we get

$$\sum_{k\in\mathbb{Z}} |\lambda|^{-k}|\mu|^k R_A^{-k}\vec{\varphi}_{\vec{u}}(B^k)R_B^k = 0.$$

Using Fourier decomposition one can find functions $\vec{\varphi}$ that do not satisfy the condition above. One also needs to make sure that the choice of $\vec{\varphi}$ allows to project $\tilde{N}$ down to $\tilde{L}$. We omit this analysis since it is routine.

This is a contradiction and therefore $\vec{\psi}$ (and hence $h$) is not Lipschitz.

If $|\lambda| = |\mu|$ but $\lambda \neq \mu$ then the scheme above still works. Obviously extra Jordan blocks do not appear in the normal forms at periodic points of $\tilde{L}$.

Finally, the case $\lambda = \mu$ must be treated separately. We use the same trick as in previous section to find $\tilde{L}$ and $\hat{L}$ with the same p. d. that are only Hölder conjugate. The trick works well in the case of complex eigenvalues as well. We omit the details.

## 4.5 Proof of Theorem C

### 4.5.1 Scheme of the proof of Theorem C

The way to choose neighborhood $\mathcal{U}$ is the same as in Theorem A. We look at the $L$-invariant splitting

$$T\mathbb{T}^4 = E_L^{ss} \oplus E_L^{ws} \oplus E_L^{wu} \oplus E_L^{su},$$

where $E_L^{ws}$, $E_L^{wu}$ are eigendirections with eigenvalues $\lambda^{-1} < \lambda$ and $E_L^{ss} \oplus E_L^{su}$ is the Anosov splitting of $g$. We choose $\mathcal{U}$ in such a way that for any $f \in \mathcal{U}$ the invariant splitting survives

$$T\mathbb{T}^4 = E_f^{ss} \oplus E_f^{ws} \oplus E_f^{wu} \oplus E_f^{su} \tag{4.20}$$

with

$$\max_{x\in\mathbb{T}^4,\sigma=ss,ws,wu,su} \left(\measuredangle(E_f^\sigma(x), E_L^\sigma(x))\right) < \frac{\pi}{2} \tag{4.21}$$

and $f$ is partially hyperbolic in the strongest sense (3.11) with respect to the splitting (4.20).

Lemma 3.4.1 works for $f \in \mathcal{U}$. Hence the distributions $E_f^{ss}$, $E_f^{ws}$, $E_f^{wu}$ and $E_f^{su}$ integrate uniquely to foliations $W_f^{ss}$, $W_f^{ws}$, $W_f^{wu}$ and $W_f^{su}$. Also, as usually, $W_f^s$ and $W_f^u$ stand for two dimensional stable and unstable foliations.

Fix $f \in \mathcal{U}$ and let $H$ be the conjugacy with the model, $H \circ f = L \circ H$. Distribution $E_L^{ws} \oplus E_L^{wu}$ obviously integrate to foliation $W_L^c$ which is subfoliated by $W_L^{ws}$ and $W_L^{wu}$. Applying Lemma 3.4.3 to the weak foliations we get that $H(W_f^{ws}) = W_L^{ws}$ and $H(W_f^{wu}) = W_L^{wu}$. Hence distribution $E_f^{ws} \oplus E_f^{wu}$ integrates to foliation $W_f^c$ which is subfoliated by $W_f^{ws}$ and $W_f^{wu}$.

Note that the leaves of $W_f^c$ are embedded two dimensional tori.

**Lemma 4.5.1.** *Conjugacy $H$ is $C^{1+\nu}$along $W_f^{ws}$ and $W_f^{wu}$. Hence, by the Regularity Lemma, $H$ is $C^{1+\nu}$ along $W_f^c$.*

Proposition 16 is a more general statement which we prove in Section 4.6. So we omit the proof of Lemma 4.5.1 here.

We establish smoothness of central holonomies.

**Lemma 4.5.2.** *Let $T_1$ and $T_2$ be open $C^{1+\nu}$-disks transversal to $W_f^c$. Then the holonomy map along $W_f^c$, $H_f^c : T_1 \to T_2$, is $C^{1+\nu}$-differentiable.*

Next we introduce distance on the leaves of $W_f^{ws}$ and $W_f^{wu}$ by simply letting $d^\sigma(x,y) = d^\sigma(H(x), H(y))$, $y \in W_f^\sigma(x)$, $\sigma = ws, wu$. Notice that by Lemma 4.5.1 $d^{ws}$ and $d^{wu}$ are induced by a Hölder continuous Riemannian metric — the pullback by $DH^{-1}|_{W_L^c}$ of the Riemannian metric on $W_L^c$.

Let $x_0$ be the fixed point of $f$ and let $S_0$ be the two dimensional torus passing through $x_0$ and tangent to $E_L^{ss} \oplus E_L^{su}$. Assumption (4.21) guarantees that $S_0$ is transversal to $W_f^c$.

Now we construct foliation $S$ that is transversal to $W_f^c$. For any point $x \in \mathbb{T}^4$ let $x_1 = W_f^c(x) \cap S_0$ and $x_2$ be some point of intersection of $W_f^{ws}(x_1)$ and $W_f^{wu}(x)$. Fix $\tilde{x} \in \mathbb{T}^4$ and define

$$S(\tilde{x}) = \{x : \text{such that}$$
$$(x_1, x_2) \ \text{ and } \ (\tilde{x}_1, \tilde{x}_2) \ \text{ have the same orientation in } \ W_f^{ws};$$

$(x_2, x)$ and $(\tilde{x}_2, x)$ have the same orientation in $W_f^{wu}$;

$$d^{ws}(x_1, x_2) = d^{ws}(\tilde{x}_1, \tilde{x}_2);\ \ d^{wu}(x_2, x) = d^{wu}(\tilde{x}_2, \tilde{x})\}.$$

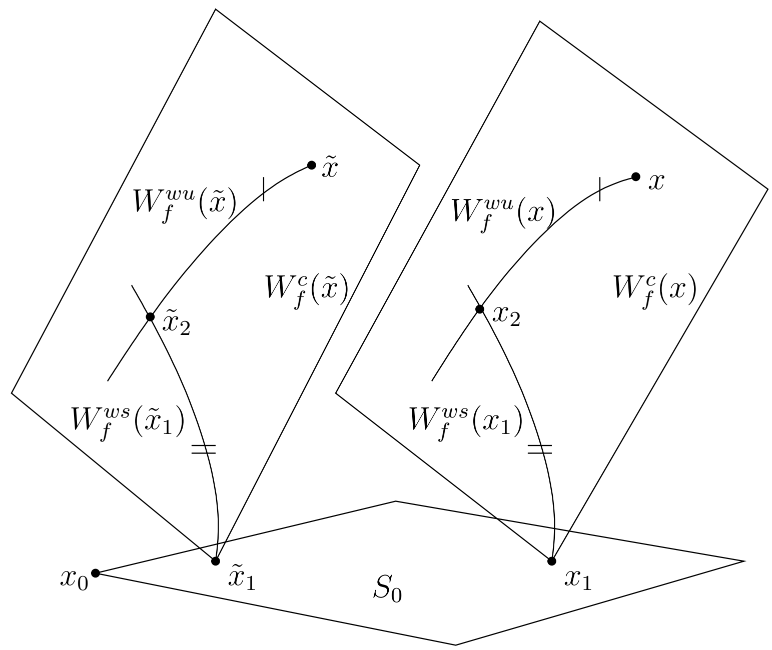


**Figure 4.2.** Definition of $S$. Point $x \in S(\tilde{x})$.

According to this definition $S(\tilde{x})$ intersects each leaf of $W_f^c$ exactly once. Also note that since the distances came from the model $L$ the definition above does not depend on the choice of $\tilde{x}_2$. It is clear that $S$ is a topological foliation into topological two dimensional tori. We show that these tori are in fact regular.

**Lemma 4.5.3.** *Leaves of $S$ are $C^{1+\nu}$ embedded two dimensional tori.*

Let $f_0 : S_0 \to S_0$ be the factor map of $f$, $f_0(x) = W_f^c(f(x)) \cap S_0$. Lemma 4.5.2 guarantees that $f_0$ is a $C^{1+\nu}$-diffeomorphism. Every periodic point of $f_0$ lifts to a periodic point of $f$. Applying Lemma 4.5.2 again we see that p. d. of $f_0$ are the same as strong stable and unstable p. d. of $f$ which is the same as p. d. of $g$. Hence there is a $C^{1+\nu}$-diffeomorphism $h_0$ homotopic to identity such that $h_0 \circ f_0 = g \circ h_0$.

Let $f_c : W_f^c(x_0) \to W_f^c(x_0)$ be the restriction of $f$ to $W_f^c(x_0)$. Obviously p. d. of $f_c$ and $A$ are the same. Hence there is a $C^{1+\nu}$ diffeomorphism $h_c$ homotopic to identity such that $h_c \circ f_c = A \circ h_c$.

We are ready to construct the conjugacy $h : \mathbb{T}^4 \to \mathbb{T}^2 \times \mathbb{T}^2$

$$h(x) = \left(h_c(S(x) \cap W_f^c(x_0)), h_0(W_f^c(x) \cap S_0)\right).$$

Homeomorphism $h$ maps central foliation into vertical foliation and foliation $S$ into horizontal foliation.

*Remark.* Notice that at this point we do not know if $h$ is $C^{1+\nu}$ diffeomorphism although $h_c$ and $h_0$ are $C^{1+\nu}$ differentiable.

**Lemma 4.5.4.** *Homeomorphism $h$ is $C^{1+\nu}$-differentiable along $W_f^c$.*

*Proof.* The projection $x \mapsto S(x) \cap W_f^c(x_0) \stackrel{\text{def}}{=} pr(x)$ projects weak stable leaf $W_f^{ws}(x)$ into $W_f^{ws}(pr(x))$. Moreover, it is clear from the definition of $S$ that the restriction of this projection to $W_f^{ws}(x)$ is an isometry with respect to distance $d^{ws}$. According to the formula for the first component of $h$ we compose this projection with $h_c$ which is an isometry when restricted to the leaf $W_f^{ws}(pr(x))$ by the definition of $d^{ws}$. Diffeomorphism $h_c$ straightens weak stable foliation into foliation by straight lines $W_L^{ws}$. Hence $h(W_f^{ws}) = W_L^{ws}$ and $h$ is an isometry as a map $(W_f^{ws}(x), d^{ws}) \mapsto (W_L^{ws}(h(x)),$ Riemannian metric$)$. Thus $h$ is $C^{1+\nu}$ along $W_f^{ws}$.

Everything above can be repeated for weak unstable foliation. Applying the Regularity Lemma we get the desired statement. □

**Lemma 4.5.5.** *Homeomorphism $h$ is $C^{1+\nu}$-differentiable along $S$.*

*Proof.* Restriction of $h$ to $S_0$ is just $h_0$. Restriction of $h$ to some other leaf $S(x)$ can be viewed as composition of holonomy $H_f^c$, $h_0$ and holonomy $H_L^c$. Hence this restriction is $C^{1+\nu}$-differentiable as well. We need to make sure that the derivative of $h$ along $S$ is Hölder continuous on $\mathbb{T}^4$. For this we only need to show that derivative of $H_f^c : S(x) \to S_0$ depends Hölder continuously on $x$. This assertion will become clear in the proof of Lemma 4.5.3. □

Hence, by the Regularity Lemma, we conclude that $h$ is $C^{1+\nu}$ diffeomorphism.

Let $\tilde{L} = h \circ f \circ h^{-1}$. Clearly foliations $W_L^{ws}$ and $W_L^{wu}$ are $\tilde{L}$-invariant. By construction $h$ and $h^{-1}$ are isometries when restricted to the leaves of weak foliations. Recall that $f$ stretches by factor $\lambda$ distance $d^{wu}$ on $W_f^{wu}$ and contracts by factor

$\lambda^{-1}$ distance $d^{ws}$ on $W_f^{ws}$. Hence if we consider restriction of $\tilde{L}$ on a fixed vertical two torus $W_L^c(x) \mapsto W_L^c(\tilde{L}(x))$ then it acts by hyperbolic automorphism $A$.

Also it is obvious from the construction of $h$ that the factor map of $\tilde{L}$ on a horizontal two torus is $g$. These observations show that $\tilde{L}$ is of the form

$$\tilde{L} = (Ax + \vec{\varphi}(y), g(y)). \tag{4}$$

Note that we do not have to argue additionally that $\vec{\varphi}$ is smooth since we know that $\tilde{L}$ is $C^{1+\nu}$-diffeomorphism.

*Remark.* An observant reader would notice that our choice of $h$ and hence $\tilde{L}$ is far from being unique. The starting point of the construction of $h$ is the torus $S_0$. Although we have chosen a concrete $S_0$, in fact, the only thing we need from $S_0$ is transversality to $W_f^c$. This is not surprising. Many diffeomorphisms of type (4.4) are $C^1$-conjugate to each other. In the linear case this is controlled by invariants (4.11).

In the rest of this section we prove Lemmas 4.5.2 and 4.5.3.

### 4.5.2 A technical Lemma

Before we proceed with proofs of Lemmas 4.5.2 and 4.5.3 we establish a crucial technical lemma which is a corollary of Lemma 4.5.1.

Let $U^\sigma = H(W_f^\sigma)$, $\sigma = ss, su$. These are foliations by Hölder continuous curves.

**Lemma 4.5.6.** *Fix $x \in \mathbb{T}^4$ and $y \in W_L^c(x)$. Let $\vec{v}$ be a vector connecting $x$ and $y$ inside of $W_L^c(x)$. Then*

$$U^\sigma(y) = U^\sigma(x) + \vec{v}.$$

*In other words foliation $U^\sigma$ is invariant under translations along $W_L^c$, $\sigma = ss, su$.*

*Proof.* For concreteness we take $\sigma = ss$. The proof in case $\sigma = su$ is the same.

First let us assume that $y \in W_L^{ws}(x)$. This allows to restrict our attention to the stable leaf $W_L^s(x)$ since $U^{ss}(x)$ and $U^{ss}(y)$ lie inside of $W_L^s(x)$. Pick a point $z \in U^{ss}(x)$ and let $\tilde{z} = W_L^{ws}(z) \cap U^{ss}(y)$. We only need to show that $d(x, y) = d(z, \tilde{z})$, where $d$ is the Riemannian distance along weak stable leaves.

Simple idea of the proof of Claim 1 from [GG08] works here. We briefly outline the argument.

Let $c = d(z, \tilde{z})/d(x, y)$. Obviously

$$\forall n \qquad \frac{d(L^n(z), L^n(\tilde{z}))}{d(L^n(x), L^n(y))} = c. \tag{4.22}$$

Since $H^{-1}(z) \in W_f^{ss}(x)$, $H^{-1}(\tilde{z}) \in W_f^{ss}(y)$ and strong stable leaves contract exponentially faster than weak stable leaves we have

$$\forall \varepsilon > 0 \quad \exists N : \forall n > N : \quad \left| \frac{d\left(H^{-1}(L^n(z)), H^{-1}(L^n(\tilde{z}))\right)}{d\left(H^{-1}(L^n(x)), H^{-1}(L^n(y))\right)} - 1 \right| \\ = \left| \frac{d\left(f^n(H^{-1}(z)), f^n(H^{-1}(\tilde{z}))\right)}{d\left(f^n(H^{-1}(x)), f^n(H^{-1}(y))\right)} - 1 \right| < \varepsilon. \tag{4.23}$$

On the other hand, since derivative of $H$ along $W_f^{ws}$ is continuous, the ratios

$$\frac{d(L^n(z), L^n(\tilde{z}))}{d\left(H^{-1}(L^n(z)), H^{-1}(L^n(\tilde{z}))\right)} \quad \text{and} \quad \frac{d(L^n(x), L^n(y))}{d\left(H^{-1}(L^n(x)), H^{-1}(L^n(y))\right)}$$

are arbitrarily close when $n \to +\infty$. Together with (4.23) this shows that constant $c$ from (4.22) is arbitrarily close to 1. Hence $c = 1$.

Finally, recall that for any $x$ leaf $W_L^{ws}(x)$ is dense in $W_L^c(x)$. Hence by continuity we get the statement of the lemma for any $y \in W_L^c(x)$. □

Lemma 4.5.6 leads to some non-trivial structural information about $f$ which is of interest on its own.

**Proposition 15.** *Distributions $E_f^{wu} \oplus E_f^{ss}$ and $E_f^{ws} \oplus E_f^{su}$ are integrable.*

*Proof.* It follows from Lemma 4.5.6 that foliations $W_L^{wu}$ and $U^{ss}$ integrate together. Thus foliations $W_f^{wu}$ and $W_f^{ss}$ integrate to a foliation with tangent distribution $E_f^{wu} \oplus E_f^{ss}$. □

### 4.5.3 Smoothness of central holonomies

We assume that the holonomy map $H_f^c : T_1 \to T_2$ is a bijection. It can be represented as a composition of holonomies along $W_f^{ws}$ and $W_f^{wu}$. Indeed, let us work on the universal cover and consider two open three dimensional submanifolds of $\mathbb{R}^4$ $M_1 = \bigcup_{x\in T_1} W_f^{wu}(x)$ and $M_2 = \bigcup_{x\in T_2} W_f^{ws}(x)$. Let $T_3 = M_1 \cap M_2$. Obviously $T_3$ is a smooth two dimensional open submanifold. Also it is easy to see that $T_3$ is connected since we are working on the universal cover. Then $H_f^c : T_1 \to T_2$ is the composition of $H_f^{wu} : T_1 \to T_3$ and $H_f^{ws} : T_3 \to T_2$.

So, it is sufficient to study holonomy map along $W_f^{wu}$ $H_f^{wu} : T_1 \to T_2$. The study of holonomies along $W_f^{ws}$ is the same.

First we make a reduction that allows to work with one dimensional transversals instead of two dimensional transversals. Let $\tilde{W}_f$ and $\tilde{W}_L$ be the integral foliations of $E_f^{ws} \oplus E_f^{wu} \oplus E_f^{su}$ and $E_L^{ws} \oplus E_L^{wu} \oplus E_L^{su}$ respectively. Also let $\bar{W}_f$ and $\bar{W}_L$ be the integral foliations of $E_f^{ss} \oplus E_f^{ws} \oplus E_f^{wu}$ and $E_L^{ss} \oplus E_L^{ws} \oplus E_L^{wu}$ respectively.

Any transversal $T$ to $W_f^c$ can be foliated by connected components of intersections with leaves of $\tilde{W}_f$. Call this foliation $\tilde{T}$. This is a well-defined one dimensional foliation since $T$ is two dimensional while the leaves of $\tilde{W}_f$ are three dimensional and both $T$ and $\tilde{W}_f$ are transversal to $W_f^{ss}$. The holonomy map $H_f^{wu} : T_1 \to T_2$ maps $\tilde{T}_1$ into $\tilde{T}_2$ since $W_f^{wu}$ subfoliate $\tilde{W}_f$.

Analogously any transversal $T$ can be foliated by connected components of intersections with leaves of $\bar{W}_f$. Call this foliation $\bar{T}$. Then $H_f^{wu}(\bar{T}_1) = \bar{T}_2$ since $W_f^{wu}$ subfoliate $\bar{W}_f$.

Hence we can consider restrictions of $H_f^{wu}$ to the leaves of $\tilde{T}_1$ and $\bar{T}_1$.

**Lemma 4.5.7.** *Restriction of holonomy $H_f^{wu}$ to a leaf of $\tilde{T}_1$, $H_f^{wu} : \tilde{T}_1(x) \to \tilde{T}_2(H_f^{wu}(x))$ is $C^{1+\nu}$-differentiable.*

**Lemma 4.5.8.** *Restriction of holonomy $H_f^{wu}$ to a leaf of $\bar{T}_1$, $H_f^{wu} : \bar{T}_1(x) \to \bar{T}_2(H_f^{wu}(x))$ is $C^{1+\nu}$-differentiable.*

Note that $\tilde{T}_i$ and $\bar{T}_i$ are transverse since $T_i$ is transverse to $W_f^c$, $i = 1, 2$. Hence, by the Regularity Lemma, the holonomy $H_f^{wu} : T_1 \to T_2$ is $C^{1+\nu}$-differentiable.

To prove Lemmas 4.5.7 and 4.5.8 we need to establish regularity of $H$ in strong unstable direction.

Given $x \in \mathbb{T}^4$ define $H_x : W_f^{su}(x) \to W_L^{su}(H(x))$ by the following composition.

$$W_f^{su}(x) \xrightarrow{H} U^{su}(H(x)) \xrightarrow{H_L^{wu}} W_L^{su}(H(x)).$$

First we map $W_f^{su}(x)$ into a Hölder continuous curve $U^{su}(H(x)) \subset W_L^u(H(x))$ and then we project it on $W_L^{su}(H(x))$ along the linear foliation $W_L^{wu}$ as shown on the Figure 4.3.

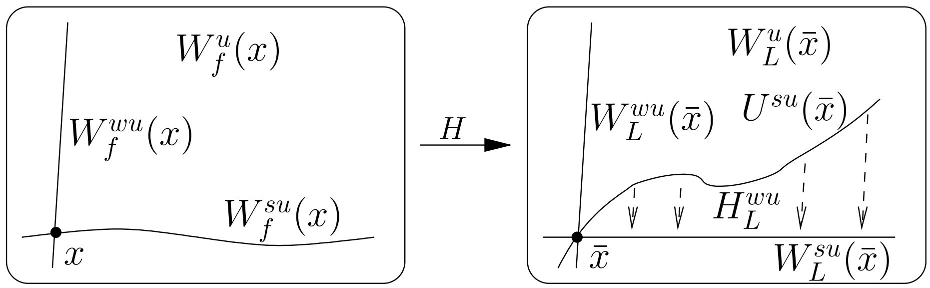


**Figure 4.3.** Definition of $H_x$. Here $\bar{x} \stackrel{def}{=} H(x)$.

**Lemma 4.5.9.** *For any $x \in \mathbb{T}^4$ the map $H_x$ is $C^{1+\nu}$-differentiable.*

*Proof.* Let us show first that $H_x$ is uniformly Lipshitz with a constant that does not depend on $x$. Denote by $d$, $d_f^{su}$, $d_L^u$ and $d_L^{su}$ Riemannian distances on the universal cover $\mathbb{R}^4$, along the leaves of $W_f^{su}$, along the leaves of $W_L^u$ and along the leaves of $W_L^{su}$ respectively. First we show that $H_x$ is Lipshitz if the points are far enough. Assume that $y, z \in W_f^{su}(x)$ and $d_f^{su}(y,z) \geq 1$. Then on the universal cover

$$
\begin{aligned}
&d_L^{su}(H_x(y), H_x(z)) \stackrel{1}{\leq} c_1 d_L^u(H_x(y), H_x(z)) \\
&\stackrel{2}{\leq} c_1 c_2 \inf\{d_L^u(\tilde{y}, \tilde{z}) : \tilde{y} \in W_L^{wu}(H_x(y)), \tilde{z} \in W_L^{wu}(H_x(z))\} \stackrel{3}{\leq} c_1 c_2 d_L^u(H(x), H(y)) \\
&\qquad\qquad \stackrel{4}{\leq} c_1 c_2 c_3 d(H(x), H(y)) \stackrel{5}{\leq} c_1 c_2 c_3 c_4 d(y,z) \stackrel{6}{\leq} c_1 c_2 c_3 c_4 d_f^{su}(y,z).
\end{aligned}
$$

First and fourth inequality hold since $W_L^{su}$ and $W_L^u$ are quasi-isometric. Second inequality holds with universal constant $c_2$ due to uniform transversality of $W_L^{wu}$ and $W_f^{su}$. Inequalities 3 and 6 are obvious. Fifth inequality holds since $d_f^{su}(y,z) \geq 1$ and the lift of the conjugacy satisfies

$$H(x + \vec{m}) = H(x) + \vec{m}, \quad x \in \mathbb{R}^4, \quad \vec{m} \in \mathbb{Z}^4.$$

Here we slightly abuse notation by denoting the lift and the map itself by the same letter.

Now we need to show that $H_x$ is Lipschitz if $y$ and $z$ are close on the leaf. Notice that $H_x$ is composition of $H_y$ and holonomy $H_L^{wu}: W_L^{su}(H(y)) \to W_L^{su}(H(x))$ which is just a translation. Hence to show that $H_x$ is Lipschitz at $y$ we only need to show that $H_y$ is Lipshitz at $y$.

So we fix $x$ and $y$ on $W_L^{su}(x)$ close to $x$ and show that $d_L^{su}(H_x(x), H_x(y)) \leq c\, d_f^{su}(x, y)$. The argument here is an adapted argument from the proof of Lemma 4 from [GG08]. Two major tools here are the Livshitz theorem and affine distance-like functions $\tilde{d}_f^{su}$ and $\tilde{d}_L^{su}$ on $W_f^{su}$ and $W_L^{su}$ respectively. We used the same distance like function on foliation $V_i^f$ in the proof of Lemma 3.4.8. Recall properties of $\tilde{d}_f^{su}$

(D1) $\tilde{d}_f^{su}(x, y) = d_f^{su}(x, y) + o(d_f^{su}(x, y))$,

(D2) $\tilde{d}_f^{su}(f(x), f(y)) = D_f^{su}(x)\tilde{d}_f^{su}(x, y)$,

(D3) $\forall K > 0\ \exists C > 0$ such that

$$\frac{1}{C}\tilde{d}_f^{su}(x, y) \leq d_f^{su}(x, y) \leq C\tilde{d}_f^{su}(x, y)$$

whenever $d_f^{su}(x, y) < K$.

Consider Hölder continuous functions $D_f^{su}(\cdot)$ and $D_L^{su}(H(\cdot))$. The assumption on p. d. of $f$ and $L$ guarantee that the products of these derivatives along periodic orbits coincide. Thus we can apply Livshitz theorem and get the Hölder continuous positive transfer function $P$ such that

$$\forall n > 0 \qquad \prod_{i=0}^{n-1} \frac{D_L^{su}\left(H(f^i(x))\right)}{D_f^{su}(f^i(x))} = \frac{P(x)}{P(f^n(x))}.$$

Choose the smallest $N$ such that $d_f^{su}(f^N(x), f^N(y)) \geq 1$. Then

$$\begin{aligned}\frac{\tilde{d}_L^{su}(H_x(x), H_x(y))}{\tilde{d}_f^{su}(x, y)} &= \prod_{i=0}^{N-1} \frac{D_L^{su}\left(L^i(H_x(x))\right)}{D_f^{wu}(f^i(x))} \cdot \frac{\tilde{d}_L^{su}\left(L^N(H_x(x)), L^N(H_x(y))\right)}{\tilde{d}_f^{su}(f^N(x), f^N(y))} \\ &= \frac{P(x)}{P(f^N(x))} \cdot \frac{\tilde{d}_L^{su}\left(H_{f^N(x)}(f^N(x)), H_{f^N(x)}(f^N(y))\right)}{\tilde{d}_f^{su}(f^N(x), f^N(y))} \leq \frac{P(x)}{P(f^N(x))} \cdot c_1c_2c_3c_4.\end{aligned}$$

Function $P$ is uniformly bounded away from zero and infinity. Hence, together with

(D3) this shows that $H_x$ is Lipschitz at $x$ uniformly in $x$ and hence is uniformly Lipschitz.

Next we apply the transitive point argument. Consider SRB measure $\mu^u$ which is the equilibrium state for the potential minus the logarithm of the unstable jacobian of $f$. It is well known that $W_f^u$ is absolutely continuous with respect to $\mu^u$. On a fixed leaf of $W_f^u$ foliation $W_f^{su}$ is absolutely continuous with respect to the Lebesgue measure on the leaf (for proof see [LY85], Section 4.2, they proof that the unstable foliation is Lipschitz with center-unstable leaves, but the proof goes through for strong unstable foliation within unstable leaves). Hence $W_f^{su}$ is absolutely continuous with respect to $\mu^u$.

We know that $H_x$ is Lipschitz and hence almost everywhere differentiable on $W_f^{su}(x)$. It is clear from the definition that $H_x$ is differentiable at $y$ if and only if $H_y$ is differentiable at $y$. Thus it does make sense to speak about differentiability at a point on strong unstable leaf without referring to a particular map $H_x$. Absolute continuity of $W_f^{su}$ allows to conclude that $H_x$ is differentiable at $x$ for $\mu^u$ almost every $x$.

Since $\mu^u$ is ergodic and has full support we can consider a transitive point $\bar{x}$ such that $H_{\bar{x}}$ is differentiable at $\bar{x}$. Now $C^1$-differentiability of $H_x$ for any $x \in \mathbb{T}^4$ can be shown by an approximation argument: we approximate the target point by iterates of $\bar{x}$. The argument is the same as the proof of Step 1, Lemma 5 from [GG08] with minimal modifications. So we omit it. This argument shows even more. Namely,

$$D(H_x)(x) = \frac{P(x)}{P(\bar{x})} D(H_{\bar{x}})(\bar{x}).$$

Note that $D(H_x)(y) = D(H_y)(y)$. Hence $H_x$ maps Lebesgue measure on the leaf $W_f^{su}(x)$ into absolutely continuous measure $dy \mapsto \frac{P(y)}{P(\bar{x})} d\mathrm{Leb}$. Recall that $P$ is Hölder continuous. Hence $H_x$ is $C^{1+\nu}$-differentiable. □

*Proof of Lemma 4.5.7.* We work in a ball $B$ inside of the leaf $\tilde{W}_f(x)$ that contains $\tilde{T}_1(x)$ and $\tilde{T}_2(H_f^{wu}(x))$. Recall that $B$ is subfoliated by $W_f^c$ and $W_f^{su}$. We apply the conjugacy map $H$ to the ball $B$. It maps $W_f^{su}$ and $W_f^c$ into $U^{su}$ and $W_L^c$ respectively. We construct a shift map $sh : H(B) \to \tilde{W}_L(H(x))$ in such a way that for any $z$ the leaf $W_L^c(z)$ is $sh$-invariant and the action of $sh$ on the leaf is a rigid translation.

Given a point $z \in H(B)$ let $y(z) = W_L^c(H(x)) \cap U^{su}(z)$. Define

$$sh(z) = W_L^{su}(y(z)) \cap W_L^{wu}(z).$$

Clearly $sh(U^{su}(H(x))) = W_L^{su}(H(x))$. Moreover, by Lemma 4.5.6 $sh(U^{su}) = W_L^{su}$.

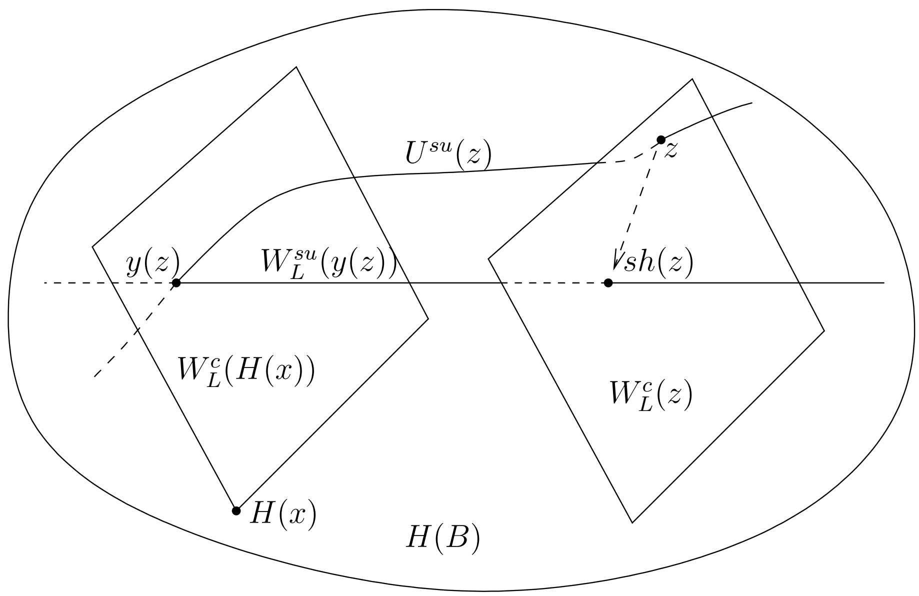


**Figure 4.4.** Definition of the shift.

The shift $sh$ is designed so that the composition $sh \circ H$ maps foliation $W_f^c$ into $W_L^c$ and foliation $W_f^{su}$ into $W_L^{su}$. According to Lemma 4.5.1 $sh \circ H$ is $C^{1+\nu}$-differentiable along $W_f^c$. Also notice that the restriction of $sh \circ H$ to a strong unstable leaf $W_f^{su}$ is nothing but $H_y$ composed with constant parallel transport along $W_L^{wu}$. Recall that $H_y$ is $C^{1+\nu}$-differentiable by Lemma 4.5.9. Hence, by the Regularity Lemma, we conclude that $sh \circ H$ is $C^{1+\nu}$-diffeomorphism.

Therefore $\hat{T}_1 = sh \circ H(\tilde{T}_1(x))$ and $\hat{T}_2 = sh \circ H(\tilde{T}_2(H_f^{wu}(x)))$ are smooth curves inside of $H(B)$ and the holonomy map $H_f^{wu}$ can be represented as a composition as shown on the commutative diagram

$$\begin{array}{ccc}
\tilde{T}_1(x) & \xrightarrow{H_f^{wu}} & \tilde{T}_2(H_f^{wu}(x)) \\
{\scriptstyle sh\circ H}\downarrow & & {\scriptstyle sh\circ H}\downarrow \\
\hat{T}_1 & \xrightarrow{H_L^{wu}} & \hat{T}_2
\end{array}$$

Holonomy $H_L^{wu}$ is smooth since $W_L^{wu}$ is a foliation by straight lines. Hence $H_f^{wu}$ is $C^{1+\nu}$-differentiable. □

*Remark.* Notice that this argument completely avoids dealing with geometry of transversals i. e. their relative position to the foliations.

*Proof of Lemma 4.5.8.* We use exactly the same argument as in the previous proof. Notice that the picture is not completely symmetric compared to the picture in Lemma 4.5.7 since we are dealing with weak unstable holonomy. Nevertheless the argument goes through by looking at transversals $\bar{T}_1(x)$ and $\bar{T}_2(H_f^{wu}(x))$ on the leaf of $\bar{W}_f$. The shift map must be constructed in such a way that it maps $U^{ss}$ into $W_L^{ss}$. □

*Proof of Lemma 4.5.3.* In this proof we exploit the same idea of composing $H$ with some shift map. We fix $S_1 = S(x_1) \in S$ which is, a priori, just an embedded topological torus. We assume that $x_1 \in W_f^{wu}(x_0)$. It is easy to see that this is not restrictive.

Foliate $S_0$ and $S_1$ by $\tilde{T}_0$, $\bar{T}_0$ and $\tilde{T}_1$, $\bar{T}_1$ respectively by taking intersections with leaves of $\tilde{W}_f$ and $\bar{W}_f$. To prove the lemma we only have to show that the leaves of $\tilde{T}_1$ and $\bar{T}_1$ are $C^{1+\nu}$-differentiable curves.

We restrict our attention to a leaf of $\tilde{W}_f$. Construct the shift map $sh$ in the same way as in Lemma 4.5.7. Fix an $x \in S_0$ and let $\hat{T}_0 = sh \circ H(\tilde{T}_0(x))$, $\hat{T}_1 = sh \circ H(\tilde{T}_1(H_f^{wu}(x)))$.

$\hat{T}_0$ is a $C^{1+\nu}$-curve since $sh \circ H$ is $C^{1+\nu}$-diffeomorphism. By the definition of $S_1$

$$\forall y \in \tilde{T}_0 \quad d^{wu}(y, H_f^{wu}(y)) = d^{wu}(x, H_L^{wu}(x)).$$

Recall the definition of $d^{wu}$ to see that conjugacy $H$ acts as an isometry on a weak unstable leaf. Obviously $sh$ is an isometry when restricted to a weak unstable leaf as well. Therefore

$$\forall y \in \hat{T}_0 \quad d(y, H_f^{wu}(y)) = d(sh \circ H(x), H_L^{wu}(sh \circ H(x))),$$

where $d$ is the Riemannian distance along $W_L^{wu}$.

Hence $\hat{T}_1$ is smooth being a parallel translation of $\hat{T}_0$. We conclude that $\tilde{T}_1(H_f^{wu}(x)) = (sh \circ H)^{-1}(\hat{T}_1)$ is $C^{1+\nu}$-curve.

Repeating the argument for $\bar{T}_0(x)$ and $\bar{T}_1(H_f^{wu}(x))$ we show that $\bar{T}_1(H_f^{wu}(x))$ is $C^{1+\nu}$-curve. Hence the lemma is proved. □

# 4.6 Proof of Theorem D

## 4.6.1 Scheme of the proof of Theorem D

We choose $\mathcal{U}$ in the same way as in 4.5.1. The only difference is that $L$ is given by (A.1) not by (4.3).

Given $f \in \mathcal{U}$ we denote by $W_f^c$ two dimensional central foliation. Take $f$ and $g$ in $\mathcal{U}$. Then they are conjugate, $h \circ f = g \circ h$.

**Proposition 16.** *Assume that $f$ and $g$ have the same p. d. Then $h(W_f^c) = W_g^c$ and the conjugacy $h$ is $C^{1+\nu}$-differentiable along $W_f^c$.*

*Remark.* In the proof we only need coincidence of p. d. in the central direction.

After we have differentiability along the central foliation strong stable and unstable foliation moduli come into the picture.

**Lemma 4.6.1.** *Assume that $f$ and $g$ have the same p. d. and the same strong unstable foliation moduli. Then $h(W_f^{su}) = W_g^{su}$.*

Now the proof of Theorem D follows immediately. Coincidence of p. d. in strong unstable direction guarantees $C^{1+\nu}$-differentiability of $h$ along $W_f^{su}$. This can be done by transitive point argument with SRB-measure in the same way as the proof of Lemma 3.4.7. Then we repeat everything for strong stable foliation. After this we apply Journé Regularity Lemma twice to conclude that $h$ is $C^{1+\nu}$-differentiable.

In particular this argument shows that in the counterexample of de la Llave strong stable and unstable foliations are not preserved by the conjugacy. We can make use of this fact by extending the counterexample for the diffeomorphisms of the form $(x, y) \mapsto (Ax + \vec{\varphi}(y), g(y))$. Namely, take $L = (Ax, By)$ and $\tilde{L} = (Ax + \vec{\varphi}(y), By)$ as in (A.1) and (A.2) respectively. We know that strong foliations of $L$ and $\tilde{L}$ do not match. Strong foliations depend continuously on the diffeomorphism in $C^1$ topology. Thus if we consider diffeomorphisms $L'(x, y) = (Ax, g(y))$ and

$\tilde{L}'(x,y) = (Ax + \vec{\varphi}(y), g(y))$ with $g$ being sufficiently $C^1$ close to $B$ then strong foliations of $L'$ and $\tilde{L}'$ do not much as well. Therefore $L'$ and $\tilde{L}'$ are not $C^1$ conjugate.

We do not know how to show that the counterexample extends to the whole neighborhood $\mathcal{U}$.

**Conjecture 4.** *For any $f \in \mathcal{U}$ there exists $g \in \mathcal{U}$ with the same p. d. which is not $C^1$ conjugate to $f$.*

*Proof of Lemma 4.6.1.* Let $U = h^{-1}(W_g^{su})$. We need to show that $U = W_f^{su}$. The main tool is the following statement

**Lemma 4.6.2.** *Consider a point $a \in \mathbb{T}^4$. Suppose that there is a point $b \neq a$, $b \in W_f^{su}(a) \cap U(a)$. Let $c \in W_f^{wu}(a)$ and $d = W_f^{wu}(b) \cap W_f^{su}(c)$, $e = W_f^{wu}(b) \cap U(c)$. Then $d = e$.*

This means that the "intersection structure" of $U$ and $W_f^{su}$ is invariant under the shifts along $W_f^{wu}$. See Chapter 2 for the proof. Claim 1 in Chapter 2 is exactly the same statement in the context of $\mathbb{T}^3$. The proof uses Proposition 16.

According to the definition of strong unstable foliation moduli we have to distinguish two cases.

First assume (4.7). It follows that there is a curve $\mathcal{C} \subset W_f^{su}(x)$ that corresponds to the interval $I$ such that $\mathcal{C} \subset U$ as well. Let

$$\mathcal{S} = \bigcup_{a \in \mathcal{C}} W_f^{wu}(a).$$

Obviously $\mathcal{S} \subset W_f^u(x)$. It follows from Lemma 4.6.2 that $W_f^{su} = U$ when restricted to $\mathcal{S}$. Then $W_f^{su} = U$ when restricted to $f^n(\mathcal{S})$, $n > 0$ as well. It remains to notice that $\cup_{n>0} f^n(\mathcal{S})$ is dense in $\mathbb{T}^4$ since $length(f^n(\mathcal{C})) \to \infty$ as $n \to \infty$. Hence $W_f^{su} = U$.

Now let us consider the second case. Namely, assume (4.6). Let $x_0$ be a fixed point. Define $x_1 = \mathcal{J}^{su}(x_0)^{-1}(t)$. Then by (4.6) we have that $x_1 \in W_f^{su}(x_0) \cap U(x_0)$. We continue to define a sequence $\{x_k; k \geq 0\}$ inductively. Given $x_k$ define $x_{k+1} = \mathcal{J}^{su}(x_k)^{-1}(t)$. Then for any $k$ $x_{k+1} \in W_f^{su}(x_k) \cap U(x_k) = W_f^{su}(x_0) \cap U(x_0)$. Obviously $f^{-n}(x_k) \in W_f^{su}(x_0) \cap U(x_0)$ as well.

Map $\mathcal{I}^{su}(x_0)$ is an isometry, hence $d_f^{su}(x_k, x_{k+1})$ does not depend on $k$. Therefore the set $\{f^{-n}(x_k); n \geq 0, k \geq 0\}$ is dense in $W_f^{su}(x_0)$ which guarantees that $W_f^{su}(x_0) = U(x_0)$. We can proceed as in the first case now to conclude that $W_f^{su} = U$. □

### 4.6.2 Smoothness along the central foliation

We apply the transitive point argument as in the proof of Lemma 3.4.7. The technical difficulty that we have to deal with is that the leaves of $W^c$ are not dense in $\mathbb{T}^4$.

Conjugacy $h$ preserves weak stable and unstable foliations. By the Regularity Lemma we only need to show $C^{1+\nu}$-differentiability of $h$ along these one dimensional foliations. For concreteness we work with weak unstable foliation $W_f^{wu}$.

For the transitive point argument to work we have to find an invariant measure $\mu$ such that $\mu$ a. e. point is transitive ($\overline{\{f^n(x); n \geq 0\}} = \mathbb{T}^4$) and $W_f^{wu}$ is absolutely continuous with respect to $\mu$. Provided that we have such a measure $\mu$ $C^{1+\nu}$-differentiability of $h$ along $W_f^{wu}$ is proved in the same way as Lemma 2.2.5.

We modify the construction from the proof of Lemma 3.4.7. Consider the space $\mathcal{T}$ of the leaves of $W_f^c$. Clearly this is a topological space homeomorphic to a two torus. Let $\tilde{f} : \mathcal{T} \to \mathcal{T}$ be the factor dynamics of $f$. Since the conjugacy to the linear model $L$ maps the central leaves to the central leaves, $\tilde{f}$ is conjugate to the automorphism $B : \mathbb{T}^2 \to \mathbb{T}^2$, $\tilde{h} \circ B = \tilde{f} \circ \tilde{h}$. Then the measure $\tilde{\mu} = \tilde{h}_*(\text{Lebesgue})$ is $\tilde{f}$-invariant and ergodic.

Pick a point $x_0$ on a $\tilde{\mu}$ typical central leaf. Let $\mathcal{V}_0$ be an open bounded neighborhood of $x_0$ in $W_f^{wu}(x_0)$. Given $x$ and $y \in W_f^{wu}(x)$ let

$$\rho(x, y) = \prod_{n \geq 0} \frac{D_f^{wu}(f^{-n}(y))}{D_f^{wu}(f^{-n}(x))}.$$

Consider a probability measure $\eta_0$ supported on $\mathcal{V}_0$ with density proportional to $\rho(x_0, \cdot)$. For $n > 0$ define

$$\mathcal{V}_n = f^n(\mathcal{V}_0),\ \eta_n = (f^n)_* \eta_0.$$

Let

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}\eta_i.$$

An accumulation point of $\{\mu_n; n \geq 0\}$ is the measure $\mu$ that we are looking for.

By the choice of $x_0$ the projection of $\mu$ to $\mathcal{T}$ is $\tilde{\mu}$.

Foliation $W_f^{wu}$ is absolutely continuous with respect to $\mu$. We refer Chapter 2 for the proof. The context is slightly different but the proof goes through with only minor changes. For instance, in Chapter 2 $x_0$ is a fixed point but we do not use it in the proof of absolute continuity.

Now we have to argue that $\mu$ a. e. point is transitive. We fix a ball in $\mathbb{T}^4$ and we show that a. e. point visits the ball infinitely many times. Then to conclude transitivity we only need to cover $\mathbb{T}^4$ by a countable collection of balls such that every point is contained in an arbitrarily small ball.

So let us fix a ball $B'$ and a slightly smaller ball $B$, $B \subset B'$. Let $\psi$ be a non-negative continuous function supported on $B'$ and equal to 1 on $B$. By Birkhoff ergodic theorem

$$E(\psi|\mathcal{I}) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\psi\circ f^i \tag{4.24}$$

where $\mathcal{I}$ is the $\sigma$-algebra of $f$-invariant sets.

Let $A = \{x : E(\psi|\mathcal{I})(x) = 0\}$. Then $\mu(A\cap B) = 0$ since

$$\int_A \psi d\mu = \int_A E(\psi|\mathcal{I})d\mu = 0.$$

Hence

$$E(\psi|\mathcal{I})(x) > 0 \text{ for } \mu \text{ a. e. } x \in B.$$

Let $\tilde{B} \subset B$ be a slightly smaller ball and let $W^c(\tilde{B}) = \cup_{x\in\tilde{B}} W_f^c(x)$. Since weak unstable leaves are dense in corresponding central leaves it is possible to find $R > 0$ such that

$$W^c(\tilde{B}) \subset \bigcup_{x\in B} W_f^{wu}(x, R).$$

Applying the standard Hopf argument we get that for $\mu$ a. e. $x$ the function $E(\psi|\mathcal{I})$ is constant on $W(x, R)$. Now absolute continuity of $W_f^{wu}$ together with above observations show that

$$E(\psi|\mathcal{I})(x) > 0 \ \text{ for } \ \mu \text{ a. e. } \ x \in W^c(\tilde{B}).$$

Obviously

$$\forall n \quad E(\psi|\mathcal{I})(x) > 0 \ \text{ for } \ \mu \text{ a. e. } \ x \in f^n(B).$$

Repeat the same argument to get

$$\forall n \quad E(\psi|\mathcal{I})(x) > 0 \ \text{ for } \ \mu \text{ a. e. } \ x \in W^c(f^n(\tilde{B})).$$

Let $\mathcal{O}(\tilde{B}) = \cup_{n\in\mathbb{Z}} f^n(\tilde{B})$ and $W^c(\mathcal{O}(\tilde{B})) = \cap_{x\in\mathcal{O}(\tilde{B})} W^c_f(x)$. Then

$$E(\psi|\mathcal{I})(x) > 0 \ \text{ for } \ \mu \text{ a. e. } \ x \in W^c(\mathcal{O}(\tilde{B})).$$

Set $W^c(\mathcal{O}(\tilde{B}))$ is $W^c_f$-saturated. Hence $\mu(W^c(\mathcal{O}(\tilde{B})))$ is equal to $\tilde{\mu}$ measure of its projection $proj(W^c(\mathcal{O}(\tilde{B}))) = proj(\mathcal{O}(\tilde{B}))$ on $\mathcal{T}$. Set $proj(\mathcal{O}(\tilde{B}))$ is an open $\tilde{f}$-invariant set. By ergodicity of $\tilde{f}$ it has full measure. Hence $\mu(W^c(\mathcal{O}(\tilde{B}))) = 1$ and

$$E(\psi|\mathcal{I})(x) > 0 \ \text{ for } \ \mu \text{ a. e. } \ x \in \mathbb{T}^4.$$

According to (4.24) this means that $\mu$ a. e. $x$ visits $B'$ infinitely many times.

Appendix A

# Preliminaries on hyperbolic and partially hyperbolic dynamics

Here we describe some well-known results in hyperbolic and partially hyperbolic dynamics that are important for our work.

## A.1 Hyperbolic Dynamics

We briefly go through definitions and some results in hyperbolic dynamics. For a comprehensive introduction see Section 6 of [KH95]. Also see [Hass02] for a recent survey.

Let $M$ be a Riemannian manifold. Let $f\colon M \to M$ be a diffeomorphism.

*Definition* 4. A compact $f$-invariant set $\Lambda \subset M$ is a *hyperbolic set* for $f$ if there exist constants $\lambda < 1$ and $C > 0$, and a continuous splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^u$ invariant under the derivative $Df$ such that for all $n > 0$

$$\|Df^n v\| \le C\lambda^n \|v\|,\ v \in E^s \quad \text{and} \quad \|Df^{-n} v\| \le C\lambda^n \|v\|,\ v \in E^u.$$

We also will be dealing with flows $\varphi\colon \mathbb{R} \to \mathrm{Diff}(M)$.

*Definition* 5. A compact invariant set $\Lambda \subset M$ is a *hyperbolic set* for $\varphi$ if there exist constants $\lambda < 1$, $C > 0$ and a continuous splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^c \oplus E^u$, $E^c = \dot{\varphi}$, invariant under the derivative $D\varphi$ and satisfying for all $t > 0$

$$\|D\varphi^t v\| \le C\lambda^t \|v\|,\ v \in E^s \quad \text{and} \quad \|D\varphi^{-t} v\| \le C\lambda^t \|v\|,\ v \in E^u.$$

*Definition* 6. If hyperbolic set is the whole manifold $M$ then the diffeomorphism or a flow is called *Anosov.*

We will be mostly concerned with Anosov systems.

The following characterization of hyperbolicity is a powerful tool that allows to check that given system is Anosov (or hyperbolicity of an invariant set).

**Theorem 17** (Cone Criterium, Alekseev)**.** *Diffeomorphism $f$ is Anosov if and only if there exists a Riemannian metric, numbers $\lambda < 1$ and $\varepsilon > 0$, and a splitting $TM = E^s \oplus E^u$ (not necessarily invariant) such that the cone fields*

$$\mathcal{C}^u(x) = \{(v^u, v^s) \in E^u \oplus E^s : \|v^s\| \le \varepsilon \|v^u\|\},$$
$$\mathcal{C}^s(x) = \{(v^u, v^s) \in E^u \oplus E^s : \|v^u\| \le \varepsilon \|v^s\|\}$$

*satisfy*

$$Df(x)(\mathcal{C}^u(x)) \subset \operatorname{Int} \mathcal{C}^u(f(x)) \quad \textit{and} \quad Df^{-1}(x)(\mathcal{C}^s(x)) \subset \operatorname{Int} \mathcal{C}^s(f^{-1}(x))$$

*and*

$$\|Dfv\| \le \lambda \|v\|, v \in \mathcal{C}^s \quad \textit{and} \quad \|Df^{-1}v\| \le \lambda \|v\|, v \in \mathcal{C}^u.$$

**Corollary 18.** *The set of Anosov diffeomorphisms is $C^1$-open in* $\operatorname{Diff}(M)$*.*

Another important characterization of Anosov systems was discovered by John Mather [Math68]. Denote by $\Gamma(TM)$ the set of continuous (or bounded) one dimensional sections of $TM$ with supremum norm. Given a diffeomorphism $f \colon M \to M$ define $f_* \colon \Gamma(TM) \to \Gamma(TM)$

$$f_* v(\cdot) = Df\left(v(f^{-1}(\cdot))\right).$$

The specrum $Q_f$ of the complexification of $f_*$ is called *Mather spectrum* of $f$.

**Theorem 19.** *If non-periodic points of $f$ are dense then any connected component of $Q_f$ is an annulus centered at* 0*. Diffeomorphism $f$ is Anosov if and only if* $1 \notin Q_f$.

We refer to [Pes04] for an exposition on Mather spectrum.

The closest relatives of Anosov systems are expanding maps.

*Definition* 7. A $C^r$-differentiable map, $r \geq 1$, $f\colon M \to M$ of a compact manifold $M$ is *expanding* if in some Riemannian metric $f$ stretches every tangent vector.

### A.1.1 Examples of Anosov diffeomorphisms

Take an integer matrix $L$ with determinant $\pm 1$ and eigenvalues lying in the complement to the unit circle. Linear map $L\colon \mathbb{R}^d \to \mathbb{R}^d$ preserves $\mathbb{Z}^d$ and hence induces an automorphism of $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. We abuse notation denoting by $L$ the induced map as well. The choice of splitting $E^s \oplus E^u$ is obvious. It does not depend on the point and satisfies conditions of Definition 4. Hence $L$ is Anosov.

This can be generalized to a wider family of automorphisms. Let $G$ be a Lie group and $\Gamma$ a cocompact lattice. Just as above we can hope to find a hyperbolic automorphism of $G$ that preserves $\Gamma$ and hence projects to an Anosov automorphism of $G/\Gamma$. First example of this kind was presented by Smale in [Sm67] and is attributed to Armand Borel. We describe it briefly, see Section 17.3 of [KH95] for detailed exposition. Take $G$ to be equal to the product of two copies of Heisenberg groups $H_1 \times H_2$.

$$H_1 = H_2 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL(3, \mathbb{R}).$$

Let $\{X_i, Y_i, Z_i\}$, $[X_i, Y_i] = Z_i$ be the standard bases of Lie algebras $\mathcal{L}(H_i)$, $i = 1, 2$. Introduce coordinates on $\mathcal{L}(G)$ by identifying $(\zeta_1, \eta_1, \xi_1, \xi_2, \eta_2, \zeta_2)$ with $\xi_1 X_1 + \eta_1 Y_1 + \zeta_1 Z_1 + \xi_2 X_2 + \eta_2 Y_2 + \zeta_2 Z_2$. Let $\lambda < 1$ be an eigenvalue of a hyperbolic matrix with integer coefficients. Then the map

$$(\zeta_1, \eta_1, \xi_1, \xi_2, \eta_2, \zeta_2) \mapsto (\lambda^3 \zeta_1, \lambda^2 \eta_1, \lambda \xi_1, \lambda^{-1} \xi_2, \lambda^{-2} \eta_2, \lambda^{-3} \zeta_2)$$

is a hyperbolic automorphism of $\mathcal{L}(G)$ that preserves a lattice that projects by the exponential map to a cocompact lattice $\Gamma \subset G$. Topologically $G/\Gamma$ is a non-trivial $\mathbb{T}^2$ bundle over $\mathbb{T}^4$.

This example and its modifications have served as a source of inspiration in partially hyperbolic dynamics. It can be viewed as partially hyperbolic diffeomorphism from several points of view, see [BW08].

### A.1.2 Structural Stability

Informally a dynamical system is structurally stable if its orbit structure does not change qualitatively if one perturbs the dynamics.

*Definition* 8. A diffeomorphism $f$ is *structurally stable* if any diffeomorphism $g$ sufficiently $C^1$ close to $f$ is topologically conjugate to $f$.

*Definition* 9. A flow $\varphi$ is *structurally stable* if any flow $\psi$ sufficiently $C^1$ close to $\varphi$ is topologically orbit equivalent to $\varphi$.

**Theorem 20** ([An69])**.** *Anosov systems are structurally stable. The conjugacy $h$ lies in a small $C^0$ neighborhood of identity and unique in this neighborhood.*

Moreover, it is easy to show that the conjugacy is Hölder continuous.

Related to structural stability is the following result of John Franks and Anthony Manning.

**Theorem 21** ([Fr69],[Mann74])**.** *Any Anosov diffeomorphism of an infranilmanifold is topologically conjugate to a hyperbolic infranilmanifold automorphism. The conjugacy is homotopic to identity.*

### A.1.3 On classification

In general, construction described in Section A.1.1 is possible only if $G$ is nilpotent ([GS99]). All known examples of Anosov diffeomorphisms live on factor spaces $G/\Gamma$ called *nilmanifolds* and their finite factors (*infranilmanifolds*). It is a big open question whether or not these are the only (up to a homeomorphism) manifolds that support Anosov diffeomorphisms.

There are results in this direction.

**Theorem 22** ([Fr70], [N70])**.** *Assume that Anosov diffeomorphism $f\colon M \to M$ has one dimensional stable or one dimensional unstable distribution. Then $M$ is homeomorphic to a torus.*

Theorems 21, 22 and classification of nilmanifolds in dimension three result in the following.

**Corollary 23.** *Any Anosov diffeomorphism of a three manifold is topologically conjugate to a hyperbolic automorphism of $\mathbb{T}^3$.*

According to Theorem 19 Mather spectrum of an Anosov diffeomorphism is contained in two annuli with radii $\lambda_1, \lambda_2$, $0 < \lambda_1 < \lambda_2 < 1$ and $\mu_1, \mu_2$, $1 < \mu_2 < \mu_1$.

The following result of Misha Brin and Anthony Manning is based on earlier works of Brin about Anosov diffeomorphisms with pinched spectrum and work of Gromov on groups with polynomial growth[1].

**Theorem 24** ([BM79])**.** *Suppose the numbers $\lambda_1, \lambda_2, \mu_2$ and $\mu_1$ associated with an Anosov diffeomorphism $f\colon M \to M$ satisfy either*

$$1 + \frac{\log \mu_2}{\log \mu_1} > \frac{\log \lambda_1}{\log \lambda_2}$$

*or*

$$1 + \frac{\log \lambda_2}{\log \lambda_1} > \frac{\log \mu_1}{\log \mu_2}.$$

*Then $M$ is homeomorphic to a infranilmanifold.*

*Remark.* In the last two theorems the smooth structure on the infranilmanifold can be different from the standard algebraic structure. Existence of Anosov diffeomorphisms on exotic tori was established by Farrell and Jones [FJ78].

An analog of classification conjecture for a special class of "very regular" Anosov systems was solved by Benoist and Labourie.

**Theorem 25** ([BL93])**.** *Let $f$ be an Anosov diffeomorphism of a $C^\infty$ manifold $M$. Suppose that the stable and unstable foliations are $C^\infty$ and that $f$ preserves a $C^\infty$ connection, or, in particular, a $C^\infty$ symplectic structure. Then $M$ is an infranilmanifold, and $f$ is $C^\infty$ conjugate to a hyperbolic automorphism of $M$.*

[1] I would like to thank Federico Rodriguez Hertz for explaining this to me and providing the reference

The zoo of Anosov flows is much bigger and has non-algebraic and even non-transitive animals (see [HT80] and [FW79]). Still restrictions on topology of $M$ must apply. For example the fundamental group of $M$ is believed to have exponential growth.

Finally let us mention that for expanding maps classification problem is solved by Shub, Franks and Gromov.

**Theorem 26** ([S69],[Fr70],[Gr81])**.** *If a compact manifold $M$ supports an expanding map, then $M$ is homeomorphic to an infranilmanifold.*

Shub showed that the universal cover of $M$ is diffeomorphic to an open ball. Then Franks proved that $\pi_1(M)$ has polynomial growth and $M$ is homeomorphic to an infranilmanifold provided that $\pi_1(M)$ is virtually solvable. Gromov showed that any group of polynomial growth is virtually nilpotent and thus completed the proof.

### A.1.4 Ergodicity of Anosov systems

Ergodicity of Anosov systems was established in [An69]. Anosov managed to apply an argument used earlier by Hopf for proving ergodicity of geodesic flow on compact surfaces of non-constant negative curvature with respect to Liouville measure.

We deal with volume preserving Anosov diffeomorphism $f\colon M \to M$. Stable and unstable distribution integrate uniquely to foliations $W^s$ and $W^u$. We need to show that $f$ is ergodic i. e. sigma algebra $\mathcal{A}$ of $f$-invariant sets is trivial.

Let $\varphi\colon M \to \mathbb{R}$ be a continuous function. Consider functions

$$\varphi^+ = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ f^i$$

$$\varphi^- = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ f^{-i}.$$

Denote by $A^+$ the set where $\varphi^+$ is well-defined and by $A^-$ the set where $\varphi^-$ is well defined. By Birkhoff Ergodic Theorem $A^+$ and $A^-$ have full volume and $\varphi^+(x) = \varphi^-(x)$ for $x \in A^+ \cap A^-$.

If $y \in W^s(x)$, $x \in A^+$ then, obviously, $y \in A^+$ and $\varphi^+(y) = \varphi^+(x)$. Analogously, if $y \in W^u(x)$, $x \in A^-$ then $y \in A^-$ and $\varphi^-(y) = \varphi^-(x)$. We would like to conclude that $\varphi^+(x) = const$ for a. e. $x$. For that we need some compatibility of invariant foliations and volume.

*Definition* 10. A open connected set $B \subset M$ is a *foliation box* if it can be represented as

$$B = \bigcup_{x \in T} \mathcal{G}(x),$$

where $T$ is a transversal to $\mathcal{F}$ and $\mathcal{G}(x) \subset \mathcal{F}(x)$ are foliation plaques.

Fix a measure $\mu$ on $M$ (e. g. volume) and a foliation box $B$. Denote by $\hat{\mu}$ projection of $\mu$ on $T$.

*Definition* 11. A *system of conditional measures* of $\mu$ relative to partition into plaques $\mathcal{G}(x)$, $x \in T$, is a family $\{(\mu_x, \mathcal{G}(x)) : x \in T\}$ of probability measures such that for any bounded measurable function $\psi \colon B \to \mathbb{R}$ the function $x \mapsto \int_{\mathcal{G}(x)} \psi d\mu_x$ is measurable and

$$\int_B \psi d\mu = \int_T d\hat{\mu} \int_{\mathcal{G}(x)} \psi d\mu_x.$$

The following result is a particular case of Rokhlin's theorem.

**Theorem 27** (Disintegration into conditional measures, [Roh62])**.** *Given a measure $\mu$ and a foliation box $B$ there exists a system of conditional measures of $\mu$ relative to partition into plaques $\mathcal{G}(x)$, $x \in T$*

We recommend [Via] for a modern exposition.

*Definition* 12. A foliation $\mathcal{F}$ is *absolutely continuous* with respect to a measure $\mu$ if for any foliation box $B$ the conditional measures and transverse measure $\hat{\mu}$ are absolutely continuous with respect to induced Riemannian volumes on $\mathcal{G}(x)$, $x \in T$, and $T$ respectively with continuous densities uniformly bounded away from 0 and infinity.

*Remark.* There are several different non-equivalent definitions of absolute continuity, see [HP07] for various definitions and relations between them. Our definition is fairly common.

**Theorem 28.** *Stable and unstable foliations of a volume preserving Anosov diffeomorphism are absolutely continuous.*

After this checking that $\varphi^+ = \varphi^- = const$ on a full volume subset of $A^+ \cap A^-$ becomes a simple exercise.

### A.1.5 SRB measure

Any expanding map $f$ preserves an absolutely continuous measure. Moreover the following result holds.

**Theorem 29** (Sacksteder, Krzyżewski, Szlenk)**.** *Let $f$ be a $C^r$ expanding maps of $M$ for $r = 2, \dots \infty, \omega$. Then there is a $C^{r-1}$ invariant probability measure for $f$.*

This measure is ergodic which can be shown by a classical argument with Lebesgue density point and distortion estimates.

Formally, the situation in Anosov case differs drastically.

**Theorem 30.** *An Anosov diffeomorphism $f$ preserves an absolutely continuous measure if and only if for every periodic $x$ with period $p$*

$$|\det(Df^p(x)| = 1.$$

It follows that set of Anosov diffeomorphisms that do not preserve volume is $C^1$ open and dense in the set of Anosov diffeomorphisms.

But actually the picture is completely parallel to the one for expanding maps.

SRB measures are the invariant measures most compatible with volume when invariant volume is not present.

**Theorem 31.** *Given an Anosov diffeomorphism $f$ there is a unique $f$-invariant ergodic probability measure $\mu^+$ that is characterized by each of the following equivalent conditions:*

1. *measure $\mu^+$ has absolutely continuous conditional measures on unstable leaves;*

2. *there is a set $A$ of full volume such that for every continuous function $\varphi\colon M \to \mathbb{R}$ and every $x \in A$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \int \varphi d\mu^+;$$

*3. measure $\mu^+$ is the equilibrium state for the potential $-\log|\mathrm{Jac}(f|_{E^u})|$.*

*4.*

$$h_{\mu^+}(f) = \int \log|\mathrm{Jac}(f|_{E^u})|d\mu^+,$$

*where $h_{\mu^+}(f)$ is the metric entropy of $f$.*

*Remark.* If $f$ is transitive then $\mu^+$ is Bernoulli.

There is also SRB measure $\mu^-$ for $f^{-1}$ that has symmetric properties. If $f$ is volume preserving than $\mu^+ = \mu^-$ coincides with volume.

### A.1.6 Livshits Theorem

The following result will be an indispensable tool in our discussion of smooth conjugacy in hyperbolic dynamics.

**Theorem 32.** *(Livshits) If $f : M \to M$ is a transitive Anosov diffeomorphism and $\varphi_1, \varphi_2 : M \to \mathbb{R}$ are Hölder continuous functions such that*

$$\sum_{i=1}^{p} \varphi_1(f^i(x)) = \sum_{i=1}^{p} \varphi_2(f^i(x)) \text{ whenever } f^p(x) = x$$

*then there is a function $u : M \to \mathbb{R}$, unique up to an additive constant, such that*

$$\varphi_1 - \varphi_2 = u \circ f - u.$$

*Moreover $u$ is Hölder continuous.*

The proof starts with solving for $u$ along a transitive orbit. Once a solution along the orbit is obtained one proves that it is uniformly continuous and hence extends to the whole $M$. The proof of uniform continuity utilizes Höldericity of $\varphi_1 - \varphi_2$ and Anosov Closing Lemma.

In [LMM86] de la Llave, Marco and Moriyón extended the result to smooth setting. They showed that if $\varphi_1$ and $\varphi_2$ are $C^r$, $r \geq 2$, then the transfer function $u$ is $C^{r-\varepsilon}$.

## A.2 Partially Hyperbolic Dynamics

We discuss necessary background from partially hyperbolic dynamics. A nice introduction to the subject is Pesin's book [Pes04]. For a survey on current state of art in the theory see [RHRHU07].

*Definition* 13. A diffeomorphism $f$ of a compact Riemannian manifold is *partially hyperbolic* if there exist numbers $C > 0$ and

$$\lambda < \alpha \leq \beta < \mu \quad \text{with} \quad \lambda < 1 < \mu,$$

and a continuous $Df$-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ satisfying for all $n > 0$

$$\begin{aligned} \|Df^n v\| &\leq C\lambda^n \|v\|, \ v \in E^s, \\ C^{-1}\alpha^n\|v\| \leq \|Df^n v\| &\leq C\beta^n \|v\|, \ v \in E^c, \\ C^{-1}\mu^n\|v\| &\leq \|Df^n v\|, \ v \in E^u. \end{aligned}$$

There is an analogue of Cone Criterium 17 for partially hyperbolic systems. As in Anosov case stable and unstable distributions integrate uniquely to Hölder continuous, absolutely continuous foliations $W^s$ and $W^u$ with smooth leaves. Central distribution is not always integrable and even when it is, the central foliation might be quite pathological.

### A.2.1 Hirsch-Pugh-Shub structural stability

Assume that $f$ is partially hyperbolic and the central distribution $E^c$ is $C^1$. Then it integrates to foliation $W^c_f$.

**Theorem 33** ([HPS77]). *Let $f$ be as above. Then any $g$ sufficiently $C^1$ close to $f$ is partially hyperbolic with central distribution being uniquely integrable to a foliation $W^c_g$ and there is a homeomorphism $h\colon M \to M$ such that $h(W^c_f(x)) = W^c_g(h(x))$ and $h\big(f(W^c_f(x))\big) = g\big(h(W^c_f(x))\big)$.*

Obviously the conjugacy is not unique. It can be composed with any homeomorphism that preserves $W^c_f$ to produce another conjugacy.

## A.2.2 Pathologies of the central foliation

First we will provide an example of $C^\infty$ partially hyperbolic diffeomorphism whose one dimensional central foliation has only $C^1$ leaves. This example was discovered by Rafael de la Llave [Ll92] in the context of smooth-conjugacy problem. It is of great importance to us and we will discuss it from smooth-conjugacy problem point of view later on.

Then we describe two essentially different ways to produce examples of non-absolutely continuous central foliations.

### A.2.2.1 Low smoothness of central foliation

Let $A$ and $B$ be hyperbolic automorphisms of $\mathbb{T}^2$, $Av = \lambda v$, $Bu = \mu u$, with $1 < \mu < \lambda < \mu^2$. Consider the following diffeomorphisms of $\mathbb{T}^4$.

$$L(x, y) = (Ax, By), \tag{A.1}$$

$$\tilde{L}(x, y) = (Ax + \varphi(y)v, By), \quad \varphi(y) = 2\varepsilon\cos(2\pi y_1). \tag{A.2}$$

We look at $L$ and $\tilde{L}$ as at partially hyperbolic diffeomorphisms with two dimensional stable distributions, one dimensional weak unstable distributions $E^{wu}$, $\tilde{E}^{wu}$ and one dimensional strong unstable distributions $E^{su}$, $\tilde{E}^{su}$. We will see that individual leaves of $\tilde{W}^{wu}$ are not $C^2$.

According to Theorem 20 there is a unique conjugacy between $L$ and $\tilde{L}$ in the neighborhood of identity. We will compute it explicitly. Let us look for $h$ of the form

$$h(x, y) = (x + \psi(y)v, y). \tag{A.3}$$

The conjugacy equation $h \circ \tilde{L} = L \circ h$ transforms into a cohomological equation on $\psi$

$$\lambda\psi(y) - \psi(By) = \varphi(y).$$

This equation can be conveniently analyzed using Fourier series. If

$$\varphi(y) = \sum_{k\in\mathbb{Z}^2} \hat{\varphi}_k e^{2\pi i k\cdot y} \quad \text{and} \quad \psi(y) = \sum_{k\in\mathbb{Z}^2} \hat{\psi}_k e^{2\pi i k\cdot y}$$

in the sense of distributions then the conjugacy equation becomes

$$\lambda\hat{\psi}_k - \hat{\psi}_{B^*k} = \hat{\varphi}_k,$$

where $B^* = (B^{-1})^t$.

Let $k_0 = (1,0)^t$. Then $\hat{\varphi}_{k_0} = \hat{\varphi}_{-k_0} = \varepsilon$. Other coefficients are zeroes. Hence $\hat{\psi}_k = 0$ for all $k \notin \{\pm(B^*)^n k_0; n \in \mathbb{Z}\}$. Solving along the dual orbit $\{k_n = (B^*)^n k_0; n \in \mathbb{Z}\}$ gives $\hat{\psi}_{k_n} = 0$, $n > 0$ and $\hat{\psi}_{k_n} = \varepsilon\lambda^{n-1}$, $n \leq 0$. And analogously along the dual orbit of $-k_0$.

The following result will help us to determine smoothness of $\psi$.

**Proposition 34** (e. g. see [Katzn02])**.** *If $\psi\colon \mathbb{T}^d \to \mathbb{R}$ is $C^r$ then $\exists C$ such that*

$$\hat{\psi}(k) \leq \frac{C}{|k|^r}.$$

Obviously $|k_n| \asymp \mu^{|n|}$. Hence if $\psi$ were $C^2$ then

$$\varepsilon\lambda^n \leq \frac{C}{\mu^{2|n|}}, \quad n < 0.$$

This is false by our choice of $\lambda$ and $\mu$. We conclude that $h$ is not $C^2$ but $C^1$ which is again can be seen from the decay of Fourier coefficients.

Now lets go back to $L$ and $\tilde{L}$. Full two dimensional unstable foliations coincide since we shift along $v$. Same is true for strong unstable foliations. By Theorem 33 conjugacy $h$ maps weak unstable foliation $W^{wu}$ into weak unstable foliation $\tilde{W}^{wu}$. Since $W^{wu}$ is a foliation by straight lines we can conclude from the form of $h$ (A.3) that a leaf of $\tilde{W}^{wu}$ is simply the graph of $\psi$ restricted to unstable eigendirection of $B$. Hence this leaf is $C^1$ but not $C^2$.

The lack of smoothness of central leaves is believed to be generic phenomenon. Jiang, Pesin and de la Llave constructed [JPL95] an example on $\mathbb{T}^3$. Their construction is more flexible, it is not intertwined with algebraic structure as the one above.

Consider a hyperbolic automorphism $L$ of $\mathbb{T}^3$ with real spectrum $0 < \lambda_1 < 1 < \lambda_2 < \lambda_3$. Then $\log\lambda_3/\log\lambda_2 \notin \mathbb{Z}$ and let $N = \lfloor \log\lambda_3/\log\lambda_2 \rfloor + 1$. Automorphism $L$ can be viewed as partially hyperbolic system. Then one can find $C^\infty$ diffeomorphism $f$ arbitrarily $C^1$ close to $L$ such that the set $\{x : W_f^{wu}(x)$ is not $C^N\}$ is a residual subset of $\mathbb{T}^3$.

#### A.2.2.2 Non-absolutely continuous central foliations: preserving central exponents

The example below is the first example of non-absolutely continuous foliation and is due to Anatole Katok.

Let $A$ be a hyperbolic automorphism of $\mathbb{T}^2$ with a fixed point $P$. Consider a family $\{f_t; t \in S^1 = [0,1]/\sim\}$ of area preserving $C^1$ small perturbations of $A$ such that

1. $f_{1-t} = f_t$;
2. $f_t$ depends smoothly on $t$;
3. $f_t(P) = P$, $t \in S^1$;
4. The larger eigenvalue $\lambda_t$ of $Df_t(P)$ is strictly decreasing on $[0, 1/2]$.

All diffeomorphisms in the family are ergodic with respect to volume. Given two diffeomorphisms $f_t$ and $f_s$ from the family there is unique conjugacy $h_{ts}$, $h_{ts} \circ f_t = f_s \circ h_{ts}$.

Ergodicity implies that $h_{ts}$ is either singular or preserves area.

Assume that $h_{ts}$ preserves area. Consider an unstable foliation box around $P$. Then $h$ preserves conditional density on almost every unstable plaque. Conditional density depends continuously on the plaque. Together with continuity of $h$ this implies that $h$ preserves conditional density on every plaque. In particular $h$ preserves continuous density on unstable plaque passing through $P$. Hence $h$ is $C^1$ when restricted to this plaque. Differentiate the conjugacy equation at $P$ along unstable direction we get $\lambda(t) = \lambda(s)$.

If $s \neq t$, $s \neq 1 - t$ i. e. $\lambda(s) \neq \lambda(t)$. Then $h_{ts}$ is singular with respect to the area.

For every $t$ consider the set $E_t$ of generic points of $f_t$. A point is *generic* if Birkhoff averages of any continuous function are defined and are equal to the integral of the function with respect to the area. If $x \in E_t$ then $x$ is generic for $f_s$ with respect to $(h_{ts})_*$(area). Hence $x \notin E_s$ unless $s = t$ or $s = 1 - t$.

Define $f\colon \mathbb{T}^3 \to \mathbb{T}^3$ by the formula $f(x,t) = (f_t(x),t)$. Map $f$ is $C^1$ close to $(x,t) \mapsto (Ax,t)$ and hence is partially hyperbolic with central foliation $W^c$ by compact leaves. The set of generic points $E = \cup E_t$ has full volume and intersect every leaf of $W^c$ at most at two points (so called Fubini's nightmare). It follows that $W^c$ is not absolutely continuous.

#### A.2.2.3 Non-absolutely continuous central foliations: perturbing central exponents

First example of this type is due to Michael Shub and Amie Wilkinson [SW00]. We describe it on $\mathbb{T}^3$ again although everything works in much greater generality.

Let $f$ be a volume preserving stably ergodic partially hyperbolic diffeomorphism $C^1$-close to a partially hyperbolic automorphism of $\mathbb{T}^3$ with non-negative central exponent. The central leaves are diffeomorphic to $S^1$ or $\mathbb{R}$.

The central Lyapunov exponent is given by the formula

$$\lambda_c(f) = \int \log D^c(f) \mathrm{d}vol.$$

It is shown in [BB03] that $f$ can be perturbed locally to $\tilde{f} = f \circ h$ with bigger central exponent. Map $h$ depends on $f$ and is identity outside a small ball $B$. Inside of $B$ $h$ is $C^1$ close to identity and preserves center-unstable foliation.

Now Mañé's argument works as follows. Let $E$ be the set of Lyapunov regular points for $\tilde{f}$. Assume that central leaves are circles. If some leaf intersects $E$ by a set of positive Lebesgue measure then the leaf must be exponentially expanded by $\tilde{f}$ but there is a uniform bound on the lengths of the leaves due to compactness. Hence every leaf intersect $E$ by a set of Lebesgue measure zero and the foliation is non-absolutely continuous.

In contrast to the previous example it is not immediately clear how the conditional measures look like. It was shown in [RW01] that they are again atomic with a uniform bound on number of atoms.

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- Smooth conjugacy of Anosov diffeomorphisms on higher dimensional tori, *Journal of Modern Dynamics,* 2, no. 4, 645-700 (2008).

- On diffeomorphisms Hölder conjugate to Anosov ones, *to appear Ergodic Theory and Dynamical Systems.*

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