The Pennsylvania State University

The Graduate School

Department of Economics

MONEY CREATION IN A RANDOM-MATCHING MODEL OF MONEY

A Thesis in

Economics

by

Alexei Deviatov

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

December 2002

We approve the thesis of Alexei Deviatov.

Date of Signature

Neil Wallace
Professor of Economics
Thesis Adviser
Chair of Committee

James Jordan
Professor of Economics

Gustavo Ventura
Assistant Professor of Economics

Dmitry Dolgopyat
Assistant Professor of Mathematics

Robert Marshall
Professor of Economics
Head of the Department of Economics

Abstract

I study the effects of lump-sum money creation against the background of the random-matching model of Trejos-Wright (1995) and Shi (1995). That model is interesting for the study of money creation because, alongside with the usual harmful internal margin effects, money creation has beneficial external margin effects. Positive money creation shifts the distribution of money towards the average holdings, thus increasing the frequency of trades in meetings. Molico (1997) demonstrates numerically that beneficial effects are possible in that model. However, Molico assumes a particular bargaining rule, take-it-or-leave-it offers by consumers. That bargaining rule is known to cause too much production in some meetings. Because lump-sum money creation tends to reduce production in meetings with binding producer participation constraints, the beneficial effects he finds may come from offsetting the effects of that bargaining rule. Instead of working with any particular bargaining rule, I consider optima over all implementable outcomes.

In order to keep the optimization problem manageable while enlarging the set of outcomes in that way, I have to make some other compromises. I assume that money is indivisible and that there is a bound on individual holdings - sometimes a low bound but one that always exceeds unity. However, I do permit randomization, which enlarges the set of trades and, thereby, the possible distribution effects. Given randomization, there are two main ways to define the set of implementable outcomes: either ex ante (allowing people to commit to randomization) or ex post (requiring that people go along with each element in the support of the randomization scheme).

Essay 1. "Another Example in which Lump-Sum Money Creation is Beneficial." (Joint with Neil Wallace.) We assume a two-unit upper bound on money holdings and adopt ex post individual rationality as the notion of implementability. The policy is a probabilistic version of the standard helicopter drops followed by proportional reduction in individual holdings. For all discount factors greater than a critical value, we show analytically that the ex ante optimum involves creation of money. This is done by finding the best outcome subject to no money creation and by showing that some creation can improve that outcome. Our results for a two-unit bound on holdings are indicative for what can happen with all higher bounds.

Essay 2. "Optimal Money Creation in a Random-Matching Model with Ex post

Individual Rationality." Although Essay 1 accomplishes the goal of showing that money creation can be helpful, it does not describe the optima. I study the same model (while letting the bound on money holdings be arbitrary) where I do two things. First, I show that, under a mild restriction on the set of implementable outcomes, conditional on the amount of money transferred in a meeting there is no randomization over output, a property I call degeneracy. This degeneracy result facilitates the exploration of the trade-off between harmful and beneficial effects of money creation by way of examples. I compute optimal allocations for examples with a two-unit bound on holdings. These examples are consistent with the conjecture that the optima do not have take-it-or-leave-it offers by consumers in all meetings – the bargaining rule imposed by Molico.

Essay 3. "Money Creation and Optimal Pairwise Core Allocations in a Matching Model." Here I adopt the ex ante pairwise core notion of implementability. In contrast to what happens using the ex post IR notion, now the optimum, even with no money creation, involves binding participation constraints. Therefore, the proof technique of Essay 1 is not applicable. Moreover, it is difficult to get any analytical results. Therefore, I compute numerical examples. In no examples is money creation optimal.

Contents

List of Tables

Acknowledgments

I would like to thank Neil Wallace for his constant guidance and support. I am also grateful to Dmitry Dolgopyat, Sergei Izmalkov, James Jordan, Thomas Sjostrom, and to the participants of the Cornell - Penn State Macro Workshop (April 2000, April 2001) and of the Conference on Monetary Economics at the Federal Reserve Bank of Cleveland (August 2001) for their valuable comments.

Chapter 1

Another Example in which Lump-Sum Money Creation is Beneficial

(Joint with Neil Wallace.)

A standard exercise to perform on monetary models is to subject them to money creation at a rate, where the creation is accomplished through lump-sum transfers, transfers that do not depend on behavior. Representative-agent models with money in utility or production functions or with cash-in-advance constraints generally give results roughly in line with what has come to be called the Friedman rule: the optimum involves not creation, but destruction financed by lump-sum taxes. Models in which money is convincingly essential can give a different answer. We know of two models in which money is convincingly essential and in which lump-sum transfers of money are studied: one is Levine [11] and the generalization of it studied by Kehoe, Levine and Woodford [10]; the other is Molico [15]. Both produce examples in which expansionary policy is beneficial. Here we present another example. We do that because the existing examples are special in ways that may raise doubts about the robustness of the results on beneficial effects.

Levine [11] and Kehoe, Levine and Woodford [10] use a one-good-per-date, pure-exchange model with preference shocks and divisible money. To get money to be essential, they assume that people are anonymous so that only *quid pro quo* spot trades are possible. There are two possible preference realizations at each date and they analyze only equilibria in which at the end of each period all money is held in equal amounts by those who last realized the low preference-for-consumption realization. As the authors make clear, in a model with preference shocks and no risk-sharing arrangements, such degenerate distributions are equilibria only for parameters for which those with high preference-for-consumption realizations want to carry zero wealth from one date to the next. Thus, their analysis leaves open whether beneficial effects of lump-sum money creation could also arise in the more general situation in which precautionary motives for holding money give rise to non-degenerate monetary distributions.

Molico [15] uses a random matching model with divisible money and unbounded individual holdings. As a consequence, he is able to analyze the model only numerically for particular examples. More importantly, he uses a particular bargaining rule: take-it-or-leave-it offers by potential consumers. From the viewpoint of his ex ante welfare criterion, that rule may be a non-optimal way to divide the gains from trade in some meetings. Therefore, part of the role of money creation in his examples may be to counteract a sub-optimal way of dividing the gains from trade in meetings.

We use the same background environment as Molico, but we assume indivisible money and individual holdings bounded at 2 units. That allows us to proceed analytically. Also, because we divide the gains from trade optimally, we are able to isolate the beneficial role of lump-sum money creation. In other words, we can be sure that we are not getting beneficial effects of money creation because we have imposed a sub-optimal trading rule. There is, though, a small price to pay for working with bounded and indivisible money; we must study a probabilistic version of the standard lump-sum money creation policy. We study a model in which holdings are at most 2 units because that is the smallest bound that permits money creation to affect the distribution of holdings in a way that facilitates trade. A plausible conjecture is that the same can happen for all higher bounds.

1.1 Environment

The background environment is a simple random matching model of money due to Shi [16] and Trejos and Wright [17]. Time is discrete and the horizon is infinite. There are $N \geqslant 3$ perishable consumption goods at each date and a $[0, 1]$ continuum of each of N types of agents. A type n person consumes only good n and produces good $n + 1$ (modulo N). Each person maximizes expected discounted utility with discount parameter $\beta \in (0, 1)$. As regards utility in a period, an agent who produces $y \in \mathbb{R}_+$ units of the agent's production good at a date experiences the utility $-y$, while an agent who consumes y units of the agent's consumption good at a date receives the utility $u(y)$. We assume that the function u is strictly concave and increasing, satisfies $u(0) = 0$ and $u'(0) = \infty$, and that there exists $\hat{y} > 0$ such that $u(\hat{y}) = \hat{y}$. At each date, each agent meets one other person at random.

There is only one asset in this economy which can be stored across periods: fiat money. This money is indivisible and no individual can have more than 2 units of money at a time. We assume that an agent's specialization type and individual money holdings are observable. We also assume that agents cannot commit to future actions—that there is sequential individual rationality—and that the agent's history, except as revealed by money holdings, is private.

The pairwise meetings, the inability to commit, the privacy of individual histories, and the perishable nature of the goods imply that any production must be accompanied by a transfer of money. Moreover, the random meetings imply that with positive ex ante probability, there are single-coincidence meetings in which the producer has experienced a long run of being a producer and the consumer has experienced a long run of being a consumer. In such meetings, no matter whether money is bounded or unbounded or indivisible or divisible, the potential consumer will, in general, be unable to offer the producer enough money to induce much production. That opens the way for a potentially beneficial role for redistribution produced through lump-sum creation of money. The redistribution tends to compress the distribution of money holdings and, thereby, lowers the probability of meetings in which producers have a great deal of money and consumers have very little money. In our model with indivisible money and holdings bounded at 2 units, the role of the redistribution is to shift the distribution of money holdings away from the end-points of the support. Of course, as explained below, that potentially beneficial effect of lump-sum money creation may be offset by its undesirable incentive effects.

When first formulated, the randomness of meetings in settings like that described above was adopted because of its simplicity. Here, because the randomness of meetings plays an important role, it ought to be defended on other grounds. The randomness amounts to assuming that people probabilistically encounter consumption opportunities and earnings opportunities. This is a complete-economy version of the kind of uncertainty regarding expenditures and receipts that has long been part of well-known partial equilibrium models of money demand (see, for example, Goldman [7] and Miller and Orr [14]). Moreover, money aside, some such uncertainty has almost always been assumed in inventory theory and in models of precautionary saving. Therefore, it should not be regarded as a strange ingredient of a model of trade. Finally, our result depends on the presence of uncertainty which produces distributions with unbounded support of runs of being a potential consumer and runs of being a potential producer. In single-coincidence meetings between people who have experienced long runs and in the absence of intervention, the potential consumer will not have enough money to induce the potential producer to produce much. While the existence of such runs is important, their source does not seem important; it could be random meetings or something else–for example, preference shocks as in Kehoe, Levine and Woodford [10].

1.2 Policies

We adopt the following timing of events and specification of policies. First there are meetings. After meetings, each person receives one unit of money with probability α. (Those who are at the upper bound and receive a unit must discard it.) Then each unit of money disintegrates with probability δ. Then the next date begins and the sequence is repeated.[1]

This kind of policy is a random version of the standard lump-sum money creation policy. In a model with divisible money, the standard policy is creation of money at a rate with the injections of money handed out lump-sum to people. As is well-known, that policy is equivalent to the following policy: the same injections followed by a reduction in each person's holdings that is proportional to the person's holdings. The proportional reduction is nothing but a normalization (see, for example, Lucas and Woodford [13])[2]. Our policy resembles the second, normalized, policy in two respects. First, the creation part of our policy, the α part, is done on a per person basis, while the disintegration part, the δ part, is proportional to holdings. Second, in a model with divisible money and a nondegenerate distribution of money holdings, the standard policy has two effects: it tends to redistribute real money holdings from those with high nominal holdings to those with low nominal holdings and it has incentive effects by making money less valuable to acquire. Our policy also has these two effects. In particular, as regards incentives, the policy makes producers less willing to acquire money because (a) they may be given money without working for it (the lump-sum transfer part of the policy) and (b) they may lose money for which they have worked (the disintegration part). And, for the same reasons, consumers are more willing to part with money.[3]

Given that the potential beneficial effects of our policy come from redistribution, why not study policies that redistribute directly? The answer is related to the sequential individual rationality that we impose. We interpret that assumption, which in this model is important for the essentiality of monetary exchange, as precluding direct taxes. In particular, it is not feasible to simply take money from people or to force producers to produce. For that reason, we study only non-negative (α, δ) pairs and view any such pair as being accomplished as follows. The creation part is not a problem because it involves giving people something; we view it as accomplished by way of a randomized version of the proverbial helicopter drops of money. The random proportional decline in holdings is accomplished by society's choice of the durability of the monetary object. In a model with divis-

[1] Policies much like ours have been studied, but only for the case in which the upper bound on individual holdings is unity (see, in particular, Li [12]). As we will see, if the bound is unity and if the gains from trade are split optimally, then the policy has no scope for beneficial effects.

[2] However, as Edward J. Green points out, the equivalence could fail in a model which posits costs of changing prices.

[3] In some models, lump-sum transfers of money are equivalent to open-market operations. They are not equivalent here. The equivalence requires Ricardian equivalence and, hence, perfect credit markets. Essentiality of money requires imperfect knowledge of individual histories and, hence, imperfect credit markets (see the discussion in Wallace [18]). Here, and in Levine [11], Kehoe, Levine and Woodford [10], and Molico [15], credit markets are excluded completely by way of the assumptions about privacy of individual histories.

ible money, the proportional reduction could be achieved by using as money an object which physically depreciates at the appropriate rate. Here, because of the indivisibility, we assume that the physical depreciation occurs probabilistically (in a "one-hoss-shay" fashion).[4]

1.3 Implementable allocations and the optimum problem

Given our assumptions, we can restrict attention to what we call *trade meetings*. A trade meeting is a single-coincidence meeting in which the producer does not start with upper-bound money holdings and the consumer starts with positive money holdings. An allocation describes what happens in all such meetings. We restrict attention to allocations that are symmetric across specialization types and are stationary in the following sense: what happens in a trade meeting depends only on the money holdings of the producer and consumer and, in addition, it and the policy, a pair (α, δ), are consistent with a constant and identical distribution of money holdings for each specialization type—a steady state. In a sense to be made precise, we say that such an allocation is implementable if it is also consistent with ex post individual rationality. The optimum problem is to choose an implementable allocation, a policy, and a consistent steady-state initial distribution of money that maximizes ex ante expected utility, utility prior to initial assignments of money. Given the symmetry and the ex ante nature of the criterion, the criterion is a representative-agent criterion.[5]

Although we impose ex post individual rationality, we formulate allocations to permit randomness—to permit different trades in the same kind of meeting. We do this mainly because, with indivisible money, such randomness allows for a much richer set of steady state distributions than would be the case if we required that the same trade be made in all meetings of the same type. In a single-coincidence meeting between a producer with i units of money and a consumer with j units, the set of possible transfers of money is $\mathcal{K}_{ij} = \{0, 1, ... \min(j, 2-i)\}$. For trade meetings in which the producer has i units of money and consumer has j units, we let μ_{ij} on $\mathbb{R}_+ \times \mathcal{K}_{ij}$ denote a measure with the interpretation that if (y, k) is randomly drawn from $\mathbb{R}_+ \times \mathcal{K}_{ij}$ in accordance with measure μ_{ij}, then (y, k) is the suggested trade in that meeting in the sense that it is suggested that y be

[4] Some countries have conducted lotteries in which prizes are awarded to those with currency with serial numbers that match some drawn at random. An alternative way to accomplish the δ part of our policy is through the same kind of lottery except that currency with the matching serial numbers is treated by everyone as being worthless.

[5] In principle, our policies could be analyzed taking as given an arbitrary initial distribution of money holdings. However, then, we would have to study non-stationary policies and allocations and would have no reason to use a representative-agent welfare criterion rather than the Pareto criterion.

produced in exchange for k units of money. We let $\boldsymbol{\mu}$ be the collection of μ_{ij}'s for $(i,j) \in \{0,1\} \times \{1,2\}$.

For each μ_{ij}, it is convenient to define the collection of *k-supports*[6]:

$$\Omega_{ij}^k = (\mathbb{R}_+ \times \{k\}) \cap \operatorname{supp} \mu_{ij},\ k \in \mathcal{K}_{ij}.$$

These represent "k-sections" of the original support of μ_{ij}. The k-supports are disjoint and $\cup_{k\in\mathcal{K}_{ij}} \Omega_{ij}^k = \operatorname{supp} \mu_{ij}$. It is also convenient to let $\lambda_{ij}^k \equiv \mu_{ij}(\Omega_{ij}^k)$, where λ_{ij}^k is the probability that k units of money are transferred in a trade meeting in which the producer starts with i units of money and the consumer with j units. Then we can express the transition matrix for money holdings implied by trades, denoted T, in terms of the λ_{ij}^k as $T = \frac{1}{N}S$, where

$$S = \begin{bmatrix} N - s_{12} - s_{13} & p_1\lambda_{01}^1 + p_2\lambda_{02}^1 & p_2\lambda_{02}^2 \\ p_0\lambda_{01}^1 + p_1\lambda_{11}^1 & N - s_{21} - s_{23} & p_1\lambda_{11}^1 + p_2\lambda_{12}^1 \\ p_0\lambda_{02}^2 & p_1\lambda_{12}^1 + p_0\lambda_{02}^1 & N - s_{31} - s_{32} \end{bmatrix}. \tag{1.1}$$

Here p_i denotes the fraction of each specialization type who start the date with i units of money and the entry in the kth row and lth column, s_{kl}, is N times the probability of a trade that results in transiting from having $k-1$ units of money to having $l-1$ units of money.

According to our sequence of actions, trade is followed first by probabilistic lump-sum creation and then by probabilistic proportional destruction. The transition matrix for the creation part is denoted A and that for the destruction part is denoted D. They are given by

$$A = \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 0 & 1-\alpha & \alpha \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ \delta & 1-\delta & 0 \\ \delta^2 & 2\delta(1-\delta) & (1-\delta)^2 \end{bmatrix}. \tag{1.2}$$

Notice that an individual can be given at most one unit of money, but can lose two units.

We can now express the requirement that $(\boldsymbol{\mu}, \alpha, \delta)$ is consistent with a constant distribution of money holdings. A symmetric distribution of money holdings $\mathbf{p} \equiv (p_0, p_1, p_2)$ is called stationary with respect to $(\boldsymbol{\mu}, \alpha, \delta)$ if it satisfies $\mathbf{p}TAD = \mathbf{p}$.

It is convenient to express the ex post individual rationality restrictions in terms of discounted expected utilities. For $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ that is stationary, the discounted expected utility of an agent who ends up with i units money after the destruction stage, denoted v_i, is constant. We let $\mathbf{v} \equiv (v_0, v_1, v_2)$. Then $\mathbf{v}$ satisfies the following 3-equation system of Bellman equations:

$$\mathbf{v}' = \beta(\mathbf{q}' + TAD\,\mathbf{v}') \tag{1.3}$$

[6]Recall that if μ is a probability measure, the support of μ, denoted $\operatorname{supp} \mu$, is the smallest closed set A such that $\mu(A) = 1$.

where $\mathbf{q}$, the vector of (expected) one period returns from trade, is given by

$$\mathbf{q}' = \begin{bmatrix} -\frac{p_1}{N}\int_{\Omega_{01}^1} y d\mu_{01} - \frac{p_2}{N}[\int_{\Omega_{02}^1} y d\mu_{02} + \int_{\Omega_{02}^2} y d\mu_{02}] \\ \frac{p_0}{N}\int_{\Omega_{01}^1} u(y) d\mu_{01} + \frac{p_1}{N}\int_{\Omega_{11}^1}[u(y) - y] d\mu_{11} - \frac{p_2}{N}\int_{\Omega_{12}^1} y d\mu_{12} \\ \frac{p_0}{N}[\int_{\Omega_{02}^1} u(y) d\mu_{02} + \int_{\Omega_{02}^2} u(y) d\mu_{02}] + \frac{p_1}{N}\int_{\Omega_{12}^1} u(y) d\mu_{12} \end{bmatrix} \tag{1.4}$$

Because T, A, and D are transition matrices and $\beta \in (0, 1)$, the mapping $G(\mathbf{x}) \equiv \beta(\mathbf{q}' + TAD\mathbf{x}')$ is a contraction. Therefore, (1.3) has a unique solution which can be expressed as

$$\mathbf{v}' = \left(\frac{1}{\beta} I - TAD\right)^{-1} \mathbf{q}', \tag{1.5}$$

where I is the 3×3 identity matrix.

We permit each individual to walk away from any realization of $\boldsymbol{\mu}$. In other words, we assume that people in a meeting cannot commit to the outcome of randomization. Therefore, our individual rationality constraints or participation constraints take the following form. If (y_{ij}, k) is in the support of μ_{ij}, then

$$(\mathbf{e}_{i+k} - \mathbf{e}_i) AD\mathbf{v}' - y_{ij} \geqslant 0 \tag{1.6}$$

and

$$(\mathbf{e}_{j-k} - \mathbf{e}_j) AD\mathbf{v}' + u(y_{ij}) \geqslant 0, \tag{1.7}$$

where $\mathbf{e}_l$ is the 3-component coordinate vector with indices running from 0 to 2. The first inequality pertains to the producer and the second to the consumer. We can now summarize the requirements for implementability.[7]

Definition 1.1 $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ *is called implementable if (i)* $\mathbf{p}TAD = \mathbf{p}$ *and (ii) (1.6) and (1.7) hold for all* (y_{ij}, k) *in the support of* μ_{ij}.

Our optimum problem is to maximize ex ante utility. That is, the optimum problem is to choose $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ from among those that are implementable to maximize $\mathbf{p}\mathbf{v}' \equiv W$.[8]

[7] We are claiming that the conditions in Definition 1.1 are necessary and sufficient for weak implementability. For sufficiency, given an allocation that satisfies Definition 1.1, we need to provide a game which has that allocation as an outcome. The game can be a very simple coordination game. The strategy set for each agent in a meeting is {yes, no}. If both say yes to a realization from $\boldsymbol{\mu}$, then they carry it out. If either says no, then there is autarky in that meeting. Obviously, if the participation constraints are satisfied, then saying yes is a subgame perfect Nash equilibrium. Necessity, of course, can only hold in the class of stationary and symmetric allocations we are considering. Then, given the privacy of individual histories and ex post individual rationality, our participation constraints must hold.

[8] Because a maximum may not exist, we should really say that for any $\varepsilon > 0$ we seek an implementable allocation that achieves at least $\sup W - \varepsilon$, where, of course, $\sup W$ is defined over the set of implementable allocations. Our arguments below do not depend on whether a maximum exists.

It is useful in what follows to express the objective W in terms of returns. If we multiply (1.3) by $\mathbf{p}$ and use the fact that $\mathbf{p}TAD = \mathbf{p}$, then we have

$$W = \mathbf{p}\mathbf{v}' = \frac{\beta}{1-\beta}\mathbf{p}\mathbf{q}'.$$

Then, by writing out the product $\mathbf{p}\mathbf{q}'$, we have

$$\begin{aligned} w \equiv \frac{(1-\beta)NW}{\beta} &= p_0 p_1 \int_{\Omega^1_{01}} z(y) d\mu_{01} + p_0 p_2 \int_{\Omega^1_{02}} z(y) d\mu_{02} + \\ & p_0 p_2 \int_{\Omega^2_{02}} z(y) d\mu_{02} + p_1^2 \int_{\Omega^1_{11}} z(y) d\mu_{11} + p_1 p_2 \int_{\Omega^1_{12}} z(y) d\mu_{12}, \end{aligned} \tag{1.8}$$

where $z(y) \equiv u(y) - y$. As one would expect, because there is a producer for each consumer, from an ex ante view utility is a discounted expected value of the function z.

1.4 The result

As noted above, expansionary policy gives rise to two effects. First, it tends to tighten participation constraints for producers and to loosen those for consumers. Second, expansionary policy can change the distribution of money holdings $\mathbf{p}$ to increase the probability of trade meetings. We doubt that anything can be said generally about which effect dominates. We show that the optimum has expansionary policy provided the parameters are such that the participation constraints are not binding at the optimum subject to $\alpha = \delta = 0$. Roughly speaking, we do this in two steps. We describe the optimum for such parameters and for $\alpha = \delta = 0$. Then we show that there are implementable $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ with $\alpha > 0$ that do better. That, of course, implies that for such parameters the optimum is not $\alpha = \delta = 0$. We begin by describing an unconstrained optimum for $\alpha = \delta = 0$.

Lemma 1.1 *If $\alpha = \delta = 0$, then the optimum subject only to condition (i) in Definition 1.1 and condition (ii) for $k = 0$ is a degenerate $\boldsymbol{\mu}$, denoted $\boldsymbol{\mu}^*$, with support $(y^*, 1)$, where $u'(y^*) = 1$. Moreover, the associated optimal $\mathbf{p}$ is $\mathbf{p}^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.*

The proof of Lemma 1.1 and the other proofs are in the appendix. Lemma 1.1 says that if we ignore participation constraints and impose $\alpha = \delta = 0$, then the optimum is a trade of the first-best level of production, that which maximizes $z(y)$, for one unit of money in every trade meeting. Moreover, the best steady-state distribution is uniform, which implies an amount of money per specialization type equal to unity.

The next lemma shows that there is a region of the parameter space, which we describe in terms of the discount factor, for which $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta) = (\mathbf{p}^*, \boldsymbol{\mu}^*, 0, 0)$ is implementable, and, therefore, by Lemma 1.1, is optimal subject to $\alpha = \delta = 0$.

Lemma 1.2 *Let* $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta) = (\mathbf{p}^*, \boldsymbol{\mu}^*, 0, 0)$. *There exists a value of the discount factor,* β^**, given by*

$$\beta^* = \frac{3Ny^*}{3Ny^* + \sqrt{(3y^*)^2 + 4z(y^*)y^*} - 3y^*},$$

such that if $\beta > \beta^*$*, then (1.6) and (1.7) are slack. If* $\beta = \beta^*$*, then (1.6) and (1.7) are slack except for (1.6) for* $i = 1$ *(when the producer has 1 unit of money) which holds at equality.*

Notice that $z(y^*) > 0$ implies that $\beta^* < 1$. Also, β^* is decreasing in $z(y^*)$.

For $\beta \geq \beta^*$, Lemmas 1.1 and 1.2 completely describe the best $(\mathbf{p}, \boldsymbol{\mu})$ subject to $\alpha = \delta = 0$. The final step is to show that for $\beta \geq \beta^*$, there exist $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ with $\alpha > 0$ which are implementable and which imply a higher value of w than the best that can be achieved with $\alpha = 0$. This is done by showing that a relevant derivative of w with respect to α is positive at $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta) = (\mathbf{p}^*, \boldsymbol{\mu}^*, 0, 0)$. For $\beta > \beta^*$, we can compute this derivative while keeping $\boldsymbol{\mu}$ constant at $\boldsymbol{\mu}^*$. Because all the participation constraints are slack at $\beta > \beta^*$, implementability is maintained at $\boldsymbol{\mu} = \boldsymbol{\mu}^*$ as we vary α, and, consequently, $\mathbf{p}$. For $\beta = \beta^*$, in order to maintain implementability as we vary α, we permit output when the producer starts with one unit of money to adjust, but, as in $\boldsymbol{\mu}^*$, one unit of money is transferred in every trade meeting.

Proposition 1.1 *If* $\beta \geq \beta^*$*, then the optimum is not* $\alpha = \delta = 0$.

The proof shows that the distribution $\mathbf{p}$ can be varied from $\mathbf{p}^*$ to one which has more trade meetings, a distribution in which $p_1 > \frac{1}{3}$. The measure of trade meetings is increasing in p_1 because people with one unit of money can be either producers or consumers.

Although our discussion of the resemblance between our policy and the standard policy is meant to convince readers that a positive (α, δ) corresponds to an inflationary policy, we can say a little more about this. A policy that resembles the standard lump-sum creation policy ought to lower the benefits of acquiring money. Those benefits are the differences, $v_1 - v_0$ and $v_2 - v_1$. It is easily shown that they are decreasing in α at $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta) = (\mathbf{p}^*, \boldsymbol{\mu}^*, 0, 0)$. In this sense, our policy is lowering those benefits. One may also wonder what is happening to real balances as we increase α. If $\beta > \beta^*$, then, for sufficiently small α, the price level does not change because in every meeting y^* is exchanged for one unit of money. It turns out that the nominal amount of money, $p_1 + 2p_2$, may either be increasing or decreasing; in particular, $\frac{d(p_1+2p_2)}{d\alpha} = \frac{10-N}{21}$. Thus, real balances may be increasing or decreasing in α. When $\beta = \beta^*$, the price level is increasing in α because the producer with one unit of money produces less as α increases.

Those, of course, are all local statements and do not describe an optimum relative to the optimum constrained by $\alpha = \delta = 0$. It is obvious that for any $\beta \geq \beta^*$, the optimum is such that the participation constraint for producers with one unit of money is binding. That does two things. It tends to make the price level higher. In addition, binding constraints tend to make it better to have less money in the system because less money tends to loosen producer participation constraints. Thus, in the range $\beta \geq \beta^*$, we strongly suspect that real balances at the optimum are lower than real balances at the optimum constrained by $\alpha = \delta = 0$.

1.5 Concluding remarks

An obvious question is what happens if more general individual money holdings are allowed. We have asserted that the main distinction occurs between an upper bound of unity and anything higher. That money creation cannot help when the bound is unity is, in effect, part of the proof of Lemma 1.1. With an upper bound of unity, all possible distributions are implied by varying the constant amount of money per type. With any higher bound, there is scope for affecting the distribution by a money creation scheme. We are confident that we could produce a version of Proposition 1.1 for any finite bound on individual holdings, but we are also confident that the region of the parameter space that is consistent with production of the first-best level of output and trades of one unit of money in all meetings shrinks as the bound gets large (see Camera and Corbae [3] for a closely related result). That, of course, is not to say that the region of the parameter space where expansionary policy helps shrinks as the bound gets large. It says only that our proof technique becomes less applicable.

In this regard, our proof technique seems completely inapplicable if money holdings are unbounded or if money is divisible with or without a bound, because, in these cases, it would seem vacuous to assume that the optimum with a fixed stock of money has no binding constraints. Therefore, for all parameters, we would then be in the general situation of trading off more favorable money distributions against the tightening of producer constraints for meetings with given money holdings. That is, all situations would be like the two-unit bound case when $\beta < \beta^*$. That, in turn, suggests that results for unbounded money holdings or divisible money will be achieved only by way of numerical examples. And, because the optimization is over large spaces in such cases, the numerical analysis will be demanding.

Those remarks are pertinent to a comparison between what we do and what Molico [15] does. As noted above, he works with the same environment, but with divisible and unbounded money holdings. However, rather than dividing the gains from trade optimally in each kind of meeting, he gives all the gains to the consumer so that the producer's participation constraint is always binding. Almost certainly, that way of dividing the gains from trade is not optimal. Therefore, his findings represent some unknown combination of beneficial effects coming from redistribution and other effects which may be offsetting the inefficient division of the gains from trade. Short of carrying out the optimization problem described above, we do not see how to disentangle those effects.[9]

In addition to studying more general individual money holdings, there are variants of our model that could be studied. These include permitting people to hide money, allowing people to commit in a meeting to the outcome of randomization, and permitting cooperative defection by the pair in a meeting. Although the details will differ, we surmise that the possibility of beneficial money creation exists in all these variants[10].

[9] In the case of indivisible money and a unit upper bound, if the parameters are such that the producer's participation constraint is slack at the first-best level of production in the absence of expansionary policy, then whether there exists a beneficial expansionary policy depends on the bargaining rule. Under the bargaining rule used by Molico [15], there does exist an expansionary policy which would reduce output to the first-best level and not affect the probability of trade.

[10] In fact, our results are unaffected by allowing people to hide money because the allocations used in our arguments are such that the trades offered people with i units of money are at least as good as those offered people with $i-1$ units.

Chapter 2

Optimal Money Creation in a Random-Matching Model with Ex Post Individual Rationality

I study money creation in versions of the Trejos-Wright [17] and Shi [16] models with indivisible money and bounded individual holdings. This is a random matching model of money with perishable and divisible produced goods and, in the versions I study, no double coincidences in pairwise meetings. The model is interesting for the study of money creation because it gives rise to a potential trade-off. On the one hand, money creation is harmful for the usual reason that shows up in representative-agent models: money creation makes the acquisition of money less desirable. On the other hand, money creation is beneficial because it can redistribute money toward average holdings so that people on average in meetings are more willing to trade. This accounts for the beneficial effects of money creation demonstrated in Molico [15], and in Deviatov and Wallace [5].

In his Ph.D. dissertation Molico [15] studied money creation with, he claimed, divisible money and unbounded individual holdings. Although that claim is not literally true because he proceeds numerically, he does find examples in which money creation is beneficial. However, he studies only allocations in which the trades are the result of take-it-or-leave-it offers by consumers. Because such trades are known to cause too much production in some meetings, some of the beneficial effects that he finds may come from offsetting the sub-optimal way of dividing the gains from trade.

In Deviatov and Wallace [5] we wanted to determine whether money creation would have a role even if we allowed trades in meetings — the division of the gains from trade in meetings — to be determined optimally from the point of view of ex ante welfare. To do that, we imposed only one restriction — ex post individual rationality in meetings. We allowed randomization and established the following result analytically: for indivisible money and individual holdings limited to the set $\{0, 1, 2\}$ and for all discount factors higher than a critical value, some money creation raises ex ante welfare relative to the best allocation without money creation.[1] Our argument was local and we did not explore the trade-off described above. Indeed, the critical value of the discount factor is such that locally there is no harmful effect of money creation.

Here I do two things. First, I show that conditional on the amount of money transferred in a meeting there is no randomization over output, a property I call degeneracy. This result opens the way to exploration of the trade-off between harmful and beneficial effects of money creation. Second, I undertake such exploration by computing examples of optimal allocations. That is important because it is evident that the optimum always has harmful effects — making it seem from the point of view of someone who ignores distribution effects that there is too much money creation.

One can obtain three different characterizations of the optima which are useful in the computation of examples. First, because the meetings are pairwise, it suffices to consider allocations which have two-point-support conditional measures over output.[2] If B is the bound on money holdings, this leads to a $4M + B + 2$ dimensional optimization problem, where $M \equiv \frac{1}{6}B(B+1)(2B+1)$. Alternatively, if free disposal of goods in meetings is allowed, then it is easily shown that randomization over output is not needed. In that case the dimensionality of the problem is $3M + B + 2$. My degeneracy result reduces it further to $2M + B + 2$ dimensions. The reduction is proportional to the cube of the bound and is, therefore, significant.

One conceivable approach to establishing degeneracy is to replace any non-degenerate distribution over output by its mean. While this would increase the objective, because it is concave, it is not evident how to show that such a non-local alternative also satisfies ex post individual rationality. Therefore, I develop a local argument. First, I devise a way to perturb distributions in terms of a few parameters. The perturbation adjusts the endpoints of the support and creates an atom or adjusts any that exist. Second, in order to carry out the perturbations and to invoke the Kuhn-Tucker theorem's necessary conditions, the allocations under consideration have to be internal. This requirement forces me to consider a subset of allocations, those I call connected. Because this is a proper subset of all ex post individually rational allocations, I also have to argue that it is plausible that the optimum over the larger set is in fact connected.

I use my degeneracy result to compute examples of optimal allocations for the Deviatov-Wallace model. These examples are consistent with their result that

[1] The policy was a probabilistic version of the standard helicopter drops followed by proportional reduction in individual holdings.

[2] The proof is constructive. One simply shows that every implementable allocation can be replaced with another allocation which is implementable, yields the same welfare and uses only two-point-support conditional measures.

money creation is beneficial if agents are patient enough. In addition, the beneficial effects of money creation persist for a while when patience drops below the critical value found in Deviatov and Wallace [5]. Only when agents are impatient enough do the benefits of money creation disappear. Another conclusion is that in the region where money creation is beneficial, the optima do not imply take-it-or-leave-it offers by consumers — the bargaining rule imposed by Molico.

The rest of this chapter is organized as follows: the environment is set out in the next section; the optimum problem is defined in section 2.2; the degeneracy result is contained in section 2.3; examples of the optima are discussed in section 2.4; section 2.5 concludes.

2.1 Environments

Time is discrete and the horizon is infinite. There are $N \geq 3$ perishable consumption goods at each date and a $[0, 1]$ continuum of each of N types of agents. A type n person consumes only good n and produces good $n+1$ (modulo N). Each person maximizes expected discounted utility with discount parameter $\beta \in (0, 1)$. Utility in a period is given by $u(y) - c(x)$, where y denotes consumption and x denotes production of an individual $(x, y \in \mathbb{R}_+)$. The function u is strictly concave, strictly increasing and satisfies $u(0) = 0$, while the function c is convex with $c(0) = 0$ and is strictly increasing. Also, there exists $\hat{y} > 0$ such that $u(\hat{y}) = c(\hat{y})$. In addition, u and c are twice continuously differentiable. At each date, each agent meets one other person at random.

There is only one asset in this economy which can be stored across periods: fiat money. This money is indivisible and no individual can have more than B units of money at any given time, where $2 \leq B < \infty$. Agents cannot commit to future actions, including commitment to outcomes of randomized trades in meetings. Finally, each agent's specialization type and individual money holdings are observable within each meeting, but the agent's history, except as revealed by money holdings, is private.

2.2 Implementable allocations and the optimum problem

The pairwise meetings, the inability to commit, the privacy of individual histories, and the perishable nature of the goods imply that any production must be accompanied by a transfer of money. In every meeting of a potential producer with i units of money and a potential consumer with j units, there is a set, denoted $\mathcal{K}_{ij}$, of feasible money transfers from the consumer to the producer, transfers which are consistent with each person's money holdings being in the set $\{0, 1, ..., B\}$:

$\mathcal{K}_{ij} = \{0, 1, ... \min(j, B - i)\}$. A *trade meeting* is one where $\mathcal{K}_{ij}$ is at least as rich as $\{0, 1\}$ set. For each trade meeting between a producer with i and a consumer with j units of money, trade is represented by a probability measure μ_{ij} on $\mathbb{R}_+ \times \mathcal{K}_{ij}$ with the interpretation that if (y, k) is randomly drawn in accordance with μ_{ij}, then (y, k) is the suggested trade in that meeting.

For any measure μ_{ij} it is convenient to consider the collection of conditional measures $\mu_{ij}^k(A) = \mu_{ij}(A \mid k)$, $k \in \mathcal{K}_{ij}$, and their supports Ω_{ij}^k.[3] Then μ_{ij} can be expressed as $\mu_{ij}(A) = \sum_{k \in \mathcal{K}_{ij}} \lambda_{ij}^k \mu_{ij}^k(A)$, where $\lambda_{ij}^k \equiv \mu_{ij}(\Omega_{ij}^k)$, is the probability that k units of money are offered in a meeting. Finally, let p_i be the fraction of agents in each specialization type who start a date with i units of money and let $\mathbf{p} = (p_0, ..., p_B)$. Then, in terms of p_i and λ_{ij}^k, an arbitrary off-diagonal element of the transition matrix T for $\mathbf{p}$ is given by:

$$\pi_{mn} = \begin{cases} \frac{1}{N} \sum_{i=0}^{B-m+n} p_i \lambda_{im}^{m-n} \text{ if } m > n \\ \frac{1}{N} \sum_{j=n-m}^{B} p_j \lambda_{mj}^{n-m} \text{ if } m < n \end{cases} \tag{2.1}$$

where π_{mn} is the probability of a trade that results in transition from having m units of money to having n units. Note that since T is a transition matrix, its diagonal elements are given by $\pi_{mm} = 1 - \sum_{s \neq m} \pi_{ms}$.

In addition to trades I introduce lump-sum money creation. I use the same kind of policy that was studied by Deviatov and Wallace [5]. The policy is a probabilistic version of the proverbial helicopter drops of money. The timing of events in a period is the following. First there are meetings. After meetings, each person receives one unit of money with probability α. (Those who are at the upper bound and receive a unit must discard it.) Then each unit of money disintegrates with probability δ.

This policy has a close resemblance with the standard policy (expansion at a rate). This is shown in Deviatov and Wallace [5], who describe the relationship between the (α, δ)-policy above and the standard policy as follows:

> "In a model with divisible money, the standard policy is creation of money at a rate with the injections of money handed out lump-sum to people. As is well-known, that policy is equivalent to the following policy: the same injections followed by a reduction in each person's holdings that is proportional to the person's holdings. The proportional reduction is nothing but a normalization (see, for example, Lucas and Woodford [13]). Our policy resembles the second, normalized, policy

[3] Recall that if μ is a probability measure, the support of μ, denoted $\operatorname{supp} \mu$, is the smallest closed set A such that $\mu(A) = 1$.

in two respects. First, the creation part of the policy, the α part, is done on a per person basis, while the disintegration part, the δ part, is proportional to holdings. Second, in a model with divisible money and a nondegenerate distribution of money holdings, the standard policy has two effects: it tends to redistribute real money holdings from those with high nominal holdings to those with low nominal holdings and it has incentive effects by making money less valuable to acquire. Our policy also has these two effects. In particular, as regards incentives, the policy makes producers less willing to acquire money because (a) they may be given money without working for it (the lump-sum transfer part of the policy) and (b) they may lose money for which they have worked (the disintegration part). And, for the same reasons, consumers are more willing to part with money."

Similar to trades, creation and destruction parts of the policy yield a pair of transition matrices for money holdings, denoted A and D respectively. According to my description of the policy, A is a two-diagonal matrix where the probability of getting a unit of money, α, is next to and above the main diagonal, and the probability of getting no transfer, $1-\alpha$, is on the main diagonal. Matrix D is lower-triangular where the first i entries in the i-th line comprise the binomial distribution of order i. Thus, the elements of A and D are:

$$a_{ij} = \begin{cases} 1-\alpha, & \text{if } j=i \\ \alpha, & \text{if } j=i+1 \\ 0, & \text{otherwise} \end{cases} \qquad d_{ij} = \begin{cases} \binom{i}{j}\delta^{i-j}(1-\delta)^j, & \text{if } j \le i \\ \\ 0, & \text{otherwise} \end{cases}$$

The stationarity requirement is $\mathbf{p}TAD = \mathbf{p}$.

It is convenient to express individual rationality constraints in terms of discounted expected utilities. For an allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$, where $\boldsymbol{\mu}$ is a stationary collection of measures μ_{ij}, $(i,j) \in \{0,...,B-1\} \times \{1,...,B\}$, discounted expected utility of an agent who ends up with i units of money by the end of the period, denoted v_i, is constant. Then vector $\mathbf{v} \equiv (v_0,...,v_B)$ satisfies the following $B+1$-equation system of Bellman equations:

$$\mathbf{v}' = \beta(\mathbf{q}' + TAD\,\mathbf{v}'), \tag{2.2}$$

where $\mathbf{q}$, the vector of (expected) one period returns from trade, is given by:

$$q_l = \sum_{i=0}^{B-1} \frac{p_i}{N} \sum_{k \in \mathcal{K}_{il}} \lambda_{il}^k \int\limits_{\Omega_{il}^k} u(y)d\mu_{il}^k - \sum_{j=1}^{B} \frac{p_j}{N} \sum_{k \in \mathcal{K}_{lj}} \lambda_{lj}^k \int\limits_{\Omega_{lj}^k} c(y)d\mu_{lj}^k \tag{2.3}$$

and where $l \in \{0,...,B\}$. Note that an individual with no money can only expect to be a producer, an agent with B units can only be a consumer, and anyone else can be either a consumer or a producer.

Because T, A and D are transition matrices and $\beta \in (0,1)$, the mapping $G(\mathbf{x}) \equiv \beta(\mathbf{q}' + TAD\,\mathbf{x}')$ is a contraction. Therefore, (2.2) has a unique solution which can be expressed as

$$\mathbf{v}' = \left(\frac{1}{\beta}I - TAD\right)^{-1}\mathbf{q}' \tag{2.4}$$

where I is the $(B+1) \times (B+1)$ identity matrix.

The ex post notion of individual rationality gives rise to the following definition:

Definition 2.1 *An allocation* $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ *is called implementable if (i)* $\mathbf{p}TAD = \mathbf{p}$, *(ii)* $\mathbf{v}$ *(given by 2.4) is non-decreasing, and (iii)* $p_i p_j > 0$ *and* $(y,k) \in \operatorname{supp}\mu_{ij}$ *imply:*

$$(\mathbf{e}_{i+k} - \mathbf{e}_i)AD\mathbf{v}' - c(y) \geqslant 0 \qquad \textit{and} \qquad (\mathbf{e}_{j-k} - \mathbf{e}_j)AD\mathbf{v}' + u(y) \geqslant 0. \tag{2.5}$$

Here $\mathbf{e}_l$ is the $B+1$-component coordinate vector with indices running from 0 to B. Definition 2.1 says that an allocation is implementable if it is stationary, satisfies free disposal of money and if the ex post gains from trade implied by the allocation are nonnegative.

Finally, our optimum problem is to maximize ex ante utility. That is, the optimum problem, denoted P, is to choose $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ from among those that are implementable to maximize $\mathbf{p}\mathbf{v}' \equiv W$.

It is useful to express the objective W in terms of returns. If I multiply (2.2) by $\mathbf{p}$ and use the fact that $\mathbf{p}TAD = \mathbf{p}$, then I obtain:

$$W = \mathbf{p}\mathbf{v}' = \frac{\beta}{1-\beta}\mathbf{p}\mathbf{q}'$$

Then, by writing out the product $\mathbf{p}\mathbf{q}'$, I get:

$$W = \frac{\beta}{1-\beta}\frac{1}{N}\sum_{i=0}^{B-1}\sum_{j=1}^{B}\sum_{k\in\mathcal{K}_{ij}} p_i p_j \lambda_{ij}^k \int\limits_{\Omega_{ij}^k} z(y)\,d\mu_{ij}^k \tag{2.6}$$

where $z(y) \equiv u(y) - c(y)$. As one would expect, because for every consumer there is a producer, welfare is equal to the net expected discounted utility in all trade meetings.

2.3 Degeneracy of the optima

The objective of this chapter is twofold. First, I want to show that degeneracy of measures μ_{ij}^k holds for solutions to problem P. Second, I want to compute examples of the optima. However, when it comes to degeneracy I cannot work with problem P directly. As I said, I use an argument which is based on perturbations of candidates for the optima. To apply that method, I need the candidates to be internal because otherwise they can not be perturbed and remain implementable. To assure satisfaction of this condition, I consider a subset of implementable allocations — those that satisfy a property I call connectedness. With this additional restriction on the set of allocations, degeneracy holds for all feasible policies. However, to simplify the notation, I proceed with zero money creation in this section. The policy reappears in section 2.4, where I discuss numerical examples.

The formal definition of connectedness is somewhat lengthy and may be difficult to follow at first. Roughly speaking, it requires that an allocation implies a value function consistent with a *willingness* to trade one unit of money in a sufficient number of meetings. Here willingness does not require that actual trades involve transfers of one unit of money, but only that trades of one unit satisfy the participation constraints implied by the allocation. A sufficient number of meetings means that these meetings can be linked into a chain that covers the entire set of money holdings. Here, by describing simple sufficient conditions for connectedness, I suggest that adding the connectedness requirement is likely to be innocuous for problem P.

Given the form of the objective (see (2.6)), one would expect that any solutions to problem P would have trade in many meetings. But, requiring trade in all trade meetings is too restrictive; it may be hard to get trades between poor consumers and rich producers. Fortunately, that is not necessary for connectedness. Instead, the following is sufficient: (i) $(\mathbf{p}, \boldsymbol{\mu})$ implies a concave value function $\mathbf{v}$ and $\mathbf{p}$ has full support; and (ii) trade occurs in all meetings where the consumer is at least as rich as the producer.[4] It is plausible that solutions to problem P satisfy (i) and (ii) and, hence, are connected.[5]

I now turn to the formal definition of connectedness. Let $(\mathbf{p}, \boldsymbol{\mu})$ be an arbitrary allocation. Let $\mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}$ be the set of all pairs (i, j), i being the producer's holdings and j the consumer's holdings, such that agents are *willing to trade* one unit of money. That is:

$$\mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})} = \{(i,j) : \exists\, y \in \mathbb{R}_+ \text{ such that } v_{i+1} - v_i \geq c(y) \text{ and } u(y) \geq v_j - v_{j-1}\} \tag{2.7}$$

Next, I use $\mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}$ to define a correspondence $\Xi_{(\mathbf{p},\boldsymbol{\mu})}$ on the set of money holdings of producers, $\mathcal{I} \equiv \{0, ..., B-1\}$, which gives the post-trade holdings of consumers

[4]The proof that these conditions are sufficient for connectedness is given in lemma B1 in the Appendix.

[5]Another way to get reassurance about the connectedness restriction is by way of a description of the set of allocations that are implementable, but not connected. They tend to be allocations which do not make full use of the set of possible money holdings. For example, for $B = 2$, any non-connected allocation can be achieved with $B = 1$ and two distinct monies (see Aiyagari, Wallace and Wright [1] for examples of such allocations).

implied by $\mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}$. That is:

$$\Xi_{(\mathbf{p},\boldsymbol{\mu})}(i) = \left\{j-1 : (i,j) \in \mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}\right\}$$

Next, let a subset $\mathcal{I}_l$ of $\mathcal{I}$ be called a *block* if the restriction of $\Xi_{(\mathbf{p},\boldsymbol{\mu})}$ to $\mathcal{I}_l \times \mathcal{I}_l$, denoted $\Xi^l_{(\mathbf{p},\boldsymbol{\mu})}$,[6] admits a selection, denoted $\sigma^l_{(\mathbf{p},\boldsymbol{\mu})}$, which is a permutation with a unique orbit.[7] Finally, block $\mathcal{I}_{l_n}$ is said to be *reachable* from $\mathcal{I}_{l_m}$ if it is possible to find a sequence of blocks $\{\mathcal{I}_{l_s}\}_{s=n}^{m-1}$ such that $\mathcal{I}_{l_s} \cap \mathcal{I}_{l_{s+1}} \neq \emptyset$ for all $s = n, ..., m-1$. I can now give the following definition:

Definition 2.2 *An allocation* $(\mathbf{p}, \boldsymbol{\mu})$ *is said to be connected if there exists a collection* $\{\mathcal{I}_{l_s}\}_{s=1}^{m}$ *of blocks such that every block in this collection is reachable from any other and* $\bigcup_{s=1}^{m} \mathcal{I}_{l_s} = \mathcal{I}$.

Now I would like to introduce some additional notation which is used later. If $(\mathbf{p}, \boldsymbol{\mu})$ is implementable and connected, then there are participation constraints implied by both actual trades in meetings and by willingness to trade one unit of money. In particular, implementability implies that if the probability of a transfer of k units of money in a meeting of a producer with i and a consumer with j units, $p_i p_j \lambda^k_{ij}$, is positive, then the participation constraints (2.5) have to hold for every y in the support of conditional measure μ^k_{ij}. Connectedness implies that another group of participation constraints holds for some y in every meeting where agents are willing to trade one unit of money. Therefore, it is convenient to define the following objects:

Definition 2.3 *Given an arbitrary implementable and connected allocation* $(\mathbf{p}, \boldsymbol{\mu})$*, define*

$$\mathbb{Z}^1_{(\mathbf{p},\boldsymbol{\mu})} \equiv \{(i,j,1) : (i,j) \in \mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}\},\ \mathbb{Z}^2_{(\mathbf{p},\boldsymbol{\mu})} \equiv \{(i,j,k) : p_i p_j \lambda^k_{ij} > 0\},$$

and $\mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})} \equiv \mathbb{Z}^1_{(\mathbf{p},\boldsymbol{\mu})} \cup \mathbb{Z}^2_{(\mathbf{p},\boldsymbol{\mu})}$.

Next, observe that if some triplet $(i,j,1)$ is in $\mathbb{Z}^1_{(\mathbf{p},\boldsymbol{\mu})}$ but not in $\mathbb{Z}^2_{(\mathbf{p},\boldsymbol{\mu})}$, then $p_i p_j \lambda^1_{ij} = 0$ and the associated conditional measure, μ^1_{ij}, is empty. It is convenient to replace this empty measure by one with a support whose lower endpoint is positive. Moreover, this replacement is innocuous because $p_i p_j \lambda^1_{ij} = 0$ which

[6] Note that $\Xi^l_{(\mathbf{p},\boldsymbol{\mu})}$ is

$$\Xi^l_{(\mathbf{p},\boldsymbol{\mu})} = \begin{cases} \Xi_{(\mathbf{p},\boldsymbol{\mu})}(i) \cap \mathcal{I}_l \text{ if } i \in \mathcal{I}_l \\ \emptyset \text{ otherwise} \end{cases}$$

[7] Let $\sigma : A \to A$ be a permutation and let $\mathcal{R}$ be an equivalence relation on A such that $a_n \mathcal{R} a_m$ if and only if there exists an integer l such that $a_m = \sigma^l(a_n)$. Then an orbit of σ is an equivalence class of relation $\mathcal{R}$. Note that an arbitrary permutation can have more than one orbit. However, if a permutation has a unique orbit, this orbit necessarily coincides with the set A.

implies that the fictitious support does not affect $\mathbf{v}$ or W. Accordingly, for every $(i,j,1) \in \mathbb{Z}^1_{(\mathbf{p},\boldsymbol{\mu})} \backslash \mathbb{Z}^2_{(\mathbf{p},\boldsymbol{\mu})}$, let μ^1_{ij} be a Dirac measure with support y, where y is any suitable output in the definition of $\mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}$.

Now, let $\underline{y}^k_{ij}$ and $\overline{y}^k_{ij}$ denote the endpoints of the support of measure μ^k_{ij} with the above replacement of empty measures in $\mathbb{Z}^1_{(\mathbf{p},\boldsymbol{\mu})} \backslash \mathbb{Z}^2_{(\mathbf{p},\boldsymbol{\mu})}$. Then, we have the following. If $(\mathbf{p},\boldsymbol{\mu})$ is implementable and connected, then

$$c(\overline{y}^k_{ij}) - (v_{i+k} - v_i) \leq 0 \qquad \textit{and} \qquad (v_j - v_{j-k}) - u(\underline{y}^k_{ij}) \leq 0 \tag{2.8}$$

hold for all $(i,j,k) \in \mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})}$.

I now concentrate on the optimum problem P_0, which is to maximize welfare W subject to $(\mathbf{p},\boldsymbol{\mu})$ being implementable and connected. Most proofs are in the appendix.

First I use connectedness to show that P_0 has solutions. This is done by endowing the space of measures μ^k_{ij} with the weak* topology and by showing that the set of implementable and connected allocations is compact and that the objective W is continuous.

Proposition 2.1 *The optimum problem P_0 has solutions.*

Then, I define two classes of perturbations of non-autarkic probability measures in $\boldsymbol{\mu}$, one class for nondegenerate measures and another for degenerate ones. A measure μ^k_{ij} is called *autarkic* if it has zero support (i.e. $\operatorname{supp}\mu^k_{ij} = \{0\}$). (An allocation is autarkic if all the nonempty measures μ^k_{ij} are autarkic.) Note that autarky is defined as no production rather than no trade. The perturbations adjust measure μ^k_{ij}, but do not affect λ^k_{ij} and, hence, the distribution $\mathbf{p}$. Note that the perturbations do not affect policy parameters α and δ (which I set equal to zero in this section) as well. This is important because it accounts for why degeneracy holds for all feasible policies.

Let μ be a nondegenerate probability measure on $\mathbb{R}_+$ with a bounded support and let $\underline{y}$ and $\overline{y}$ be the endpoints of that support. Let us take six nonnegative numbers: a, b, c, d, x and ε such that $b \geq a + \frac{\overline{y}-\underline{y}}{2}$, $d \geq c + \frac{\overline{y}-\underline{y}}{2}$, $\min(a,c) \leq x \leq \max(b,d)$ and $0 \leq \varepsilon \leq 1$. Also, let us observe that μ can be tautologically written as $\mu = \mu_1 + \mu_2$, where $\mu_1 = \mu_2 = \frac{1}{2}\mu$. Then the perturbation does two things. First, it moves the endpoints $\underline{y}$ and $\overline{y}$ of μ_1 and μ_2 independently to the new positions, a and b for μ_1 and c and d for μ_2, so that the "shapes" of μ_1 and μ_2 (which are those of μ) are preserved. Second, the perturbation creates a mass point x with mass ε within the union of the perturbed supports. That is, the perturbed measure $\widetilde{\mu}$ is obtained from μ via the formula:

$$\widetilde{\mu}(A) = \varepsilon\delta_x(A) + \frac{1}{2}(1-\varepsilon)\left[\mu\left(t_1^{-1}(A)\right) + \mu\left(t_2^{-1}(A)\right)\right] \tag{2.9}$$

where δ_x is a Dirac measure with support x, and t_1 and t_2 are two linear mappings on the real line defined by my requirement that t_1 maps $\underline{y}$ and $\overline{y}$ into a and b and that t_2 maps $\underline{y}$ and $\overline{y}$ into c and d.[8] Note that because I set $b > a$ and $d > c$, the mappings t_1 and t_2 are invertible.

For a measure μ which is degenerate, the perturbation splits its single-point support into two points which, however, are allowed to be the same. Each of these points gets one-half of the mass of measure μ. That is, let g and h be two nonnegative numbers. Then the perturbed measure $\widetilde{\mu}$ is given by

$$\widetilde{\mu}(A) = \frac{1}{2}\delta_g(A) + \frac{1}{2}\delta_h(A). \tag{2.10}$$

Now, given an arbitrary implementable and connected allocation $(\mathbf{p}, \boldsymbol{\mu})$, I define a finite-dimensional optimization problem, denoted $\widetilde{P}_{(\mathbf{p},\boldsymbol{\mu})}$, which is to maximize W by the choice of the parameters $(a_{ij}^k, b_{ij}^k, c_{ij}^k, d_{ij}^k, x_{ij}^k, \varepsilon_{ij}^k, g_{ij}^k, h_{ij}^k)$, one eight-tuple for each nonempty non-autarkic measure μ_{ij}^k in $\boldsymbol{\mu}$, subject to $(\mathbf{p}, \widetilde{\boldsymbol{\mu}})$ being implementable and connected. If $(\mathbf{p}, \boldsymbol{\mu})$ solves P_0, then the null perturbation must solve $\widetilde{P}_{(\mathbf{p},\boldsymbol{\mu})}$. This is the basis for the proof by contradiction showing that every nonempty non-autarkic measure μ_{ij}^k in $\boldsymbol{\mu}$ must be degenerate.

Because this optimization problem is finite-dimensional, it can be analyzed by means of the Kuhn-Tucker theorem. The central hypothesis of that theorem is the constraint qualification: the rank of the Jacobian matrix should be equal to the number of active constraints. The constraint qualification is sufficient to ensure the existence of an open region $\mathcal{U}$ adjacent to the solution point in which all the constraints are relaxed. Existence of such a region allows one to claim that the solution point satisfies the first-order necessary conditions of the Kuhn-Tucker theorem. My approach is to establish existence of $\mathcal{U}$ directly, without appeal to the full rank requirement on the Jacobian matrix.

Lemma 2.1 *Let $(\mathbf{p}, \boldsymbol{\mu})$ be a non-autarkic solution to problem P_0. Let $\widetilde{P}^*_{(\mathbf{p},\boldsymbol{\mu})}$ be the associated perturbation problem $\widetilde{P}_{(\mathbf{p},\boldsymbol{\mu})}$ with the additional restriction that $\varepsilon_{ij}^k \equiv 0$. Let E be the set of all active constraints of problem $\widetilde{P}^*_{(\mathbf{p},\boldsymbol{\mu})}$ at $(\mathbf{p}, \boldsymbol{\mu})$ and assume that E is nonempty. Then there exists a nonempty subset E' of E and multipliers $\xi_s \geq 0$, one for each constraint in E', such that the gradient of the objective W can be written as a linear combination of the gradients of the constraints in E'.*

The proof of the lemma has two major parts. First, connectedness is used to show that any implementable and connected allocation is either autarky or satisfies $\underline{y}_{ij}^k > 0$ for all $(i, j, k) \in \mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})}$. Because I consider non-autarkic solutions to P_0,

[8] That is,

$$t_1(y; a, b) = \frac{a\overline{y} - b\underline{y}}{\overline{y} - \underline{y}} + \frac{b - a}{\overline{y} - \underline{y}}\, y$$

and analogously for t_2.

the latter implies that all the constraints in problem $\widetilde{P}^*_{(\mathbf{p},\boldsymbol{\mu})}$ pertaining to nonnegativity of endpoints are slack. Second, the participation constraints implied by implementability and connectedness are used to show existence of a fixed vector $\mathbf{n}$ in the space of perturbations such that the angle between this vector and the gradient of any of the participation constraints is less than $\frac{\pi}{2}$. These two facts are sufficient to guarantee existence of $\mathcal{U}$. That, in turn, implies that the gradient of W is in the convex hull of the gradients of the active constraints whose edges define E'.

The multipliers ξ_s of Lemma 2.1 can be used to prove the main proposition.

Proposition 2.2 *If* $(\mathbf{p},\boldsymbol{\mu})$ *solves problem* P_0 *and the support of* μ^k_{ij} *is nonempty, then* μ^k_{ij} *is degenerate.*

Proof. Suppose that $(\mathbf{p},\boldsymbol{\mu})$ is a solution to the optimum problem P_0 and that it has at least one nondegenerate measure μ^k_{ij}. Consider the associated perturbation problem $\widetilde{P}_{(\mathbf{p},\boldsymbol{\mu})}$ and let E be the set of active participation constraints of that problem. Let us first assume that E is nonempty.

By Lemma 2.1, $(\mathbf{p},\boldsymbol{\mu})$ satisfies necessary first order conditions for the Kuhn-Tucker theorem for that problem. The constraints are the participation constraints in E' and $\varepsilon^k_{ij} \in [0,1]$ and $x^k_{ij} \in [a^k_{ij}, b^k_{ij}]$. At $\varepsilon^k_{ij} = 0$, the multipliers associated with x^k_{ij} are equal to zero. Therefore, the multiplier associated with the binding constraint, $\varepsilon^k_{ij} = 0$, can be expressed as

$$\sigma = \frac{\partial W}{\partial \varepsilon^k_{ij}} - \sum_{E'} \left[\xi_{s_1} \frac{\partial \left(v_j - v_{j-k} - u(a^k_{ij}) \right)}{\partial \varepsilon^k_{ij}} - \xi_{s_2} \frac{\partial \left(v_{i+k} - v_i - c(b^k_{ij}) \right)}{\partial \varepsilon^k_{ij}} \right]$$

where ξ_{s_1} and ξ_{s_2} are the multipliers from Lemma 2.1.

Note that optimality of $\varepsilon^k_{ij} = 0$ requires that $\sigma \geq 0$ for all $x^k_{ij} \in \left[\underline{y}^k_{ij}, \overline{y}^k_{ij} \right]$, which, because μ^k_{ij} is nondegenerate, is an interval. It follows from (2.9) that:

$$\sigma = \Phi(x^k_{ij}) - \int_{\underline{y}^k_{ij}}^{\overline{y}^k_{ij}} \Phi(y)\ d\mu^k_{ij},$$

where

$$\Phi(y) = z(y) + \frac{\lambda^k_{ij}}{N} [\xi_{s_2}(\mathbf{e}_{i+k} - \mathbf{e}_i) - \xi_{s_1}(\mathbf{e}_j - \mathbf{e}_{j-k})] H \left(p_i\, u(y)\, \mathbf{e}'_j - p_j\, c(y)\, \mathbf{e}'_i \right)$$

and where $\mathbf{e}_i$ denotes i-th coordinate vector and $H = \left(\frac{1}{\beta} I - T \right)^{-1}$. Because the multipliers ξ_{s_1} and ξ_{s_2} are well-defined, $\Phi(y)$ is a continuous function. Moreover, $\Phi(y)$ is non-constant because $u(y)$ and $c(y)$ are linearly independent and because

ξ_{s_1} and ξ_{s_2} can, without loss of generality, be independently scaled. Then, because μ_{ij}^k is a nondegenerate probability measure, straightforward application of the first mean value theorem for the Lebesgue integral yields existence of some $x \in \left[\underline{y}_{ij}^k, \overline{y}_{ij}^k\right]$ such that $\Phi(x) < \int_{\underline{y}_{ij}^k}^{\overline{y}_{ij}^k} \Phi(y)\, d\mu_{ij}^k$. This implies that $\sigma \geq 0$ does not hold for all choices of x_{ij}^k from $\left[\underline{y}_{ij}^k, \overline{y}_{ij}^k\right]$.

If E is empty, then all the participation constraints are slack which implies that the multipliers associated with these constraints are zeros. Then $\Phi(y) = z(y)$ and the above argument applies. ■

Note that the proof of Proposition 2.2 applies to any non-autarkic solution to problem P_0. Recall that autarky is an allocation where all nonempty measures μ_{ij}^k have zero supports and, thus, is degenerate. This means that Proposition 2.2 holds for all possible solutions to problem P_0. However, if Proposition 2.2 is to be of interest, it better be that there is a wide class of environments where these solutions are non-autarkic. It is easy to provide conditions for existence of non-autarkic implementable allocations. These exist if $c'(0) < \frac{1}{\left(\frac{1}{\beta}-1\right)N+1} u'(0)$.[9] Then if, as was argued above, connectedness is an innocuous restriction for problem P, then this condition is also sufficient for existence of non-autarkic solutions to problem P_0.

2.4 Examples

In this section I compute optima for examples with a two-unit bound on individual money holdings. Even for this small bound on holdings, closed-form solutions for the optima are not feasible. I report examples for the case where the utility function is the square root, the cost function is linear, and the number of specialization types is three; that is, $u(x) = \sqrt{x}$, $c(x) = x$, and $N = 3$. The optima are parameterized by the degree of patience of individuals, r, where $r = \frac{1}{\beta} - 1$. I compute examples for all r from 0.01 through 0.2 in increments of 0.01 and also for $r \in \{0.25, 0.3, 0.4, 0.5, 1, 2\}$.

The algorithm is a version of the standard genetic algorithm,[10] which I modified in two ways. First, I made the standard algorithm capable of handling arbitrary inequality and equality constraints. Second, I made the search more efficient by directing mutation into a small region (cone) around the projection of the gradient

[9] This condition is sufficient for existence of non-autarkic implementable allocations in which the support of $\mathbf{p}$ is $\{0, B\}$, trades are limited to transfers of B units of money in meetings of producers with zero and consumers with B units, and in which consumers make take-it-or-leave-it offers to producers.

[10] See Dawid [4] and Houck, Joines and Kay [8] for details.

of the objective onto the subspace orthogonal to the one spanned by the gradients of the active constraints. The algorithm was terminated when the length of that projection became less than the tolerance value. This termination criterion ensures that the first order necessary conditions for the Kuhn-Tucker theorem are (with reasonable precision) satisfied at every terminal point. Because the probability of selection of individuals in the population was set to be an increasing function of the objective, this is sufficient to guarantee that every terminal point is a (local) maximum.

The results for the different r's are consistent with existence of four distinct regions. I define (besides r^* found in Deviatov and Wallace [5]) two other "critical" values for the discount rate (denoted $r^\dagger$ and $r^\ddagger$; with my choice of parameters $r^* = \frac{\sqrt{13}-3}{9} \approx 0.067$, $r^\dagger \approx 0.133$, and $r^\ddagger \approx 0.161$) which separate them. The region $r \in (0, r^*]$ is the region where Deviatov and Wallace showed that money creation is beneficial. If $r \in (r^*, r^\dagger)$, money creation is still beneficial, yet the constrained optimum has binding producer constraints, and the proof technique of Deviatov and Wallace is not applicable. If $r \in (r^\dagger, r^\ddagger)$, money creation is not helpful, however, the optima do not have take-it-or-leave-it offers in all meetings. Finally, for all $r \geq r^\ddagger$ money creation is not helpful and the optima have take-it-or-leave-it offers by consumers in all meetings. Note that I find no examples where money creation is beneficial and the optimum has take-it-or-leave-it offers by consumers in all meetings — the bargaining rule imposed by Molico [15]. In examples where money creation is beneficial, the binding participation constraints are those of producers who have one unit of money and of producers who have no money but trade one unit with consumers who have two units of money.

There are two things that are common to every example. First, one unit of money is transferred with probability 1 in all trade meetings. Second, there are no binding consumer participation constraints. I take advantage of these features to simplify the presentation of examples (see Table 2.1[11]). I omit the probabilities of transfer of money in meetings (the lamdas), suppress superscripts in the notation for outputs, and attach stars (*) to outputs which are part of binding producer constraints. For each value of r in the first row of Table 2.1 I report two optima (I only show the optima for those r where money creation is beneficial). The first column shows the optimum subject to $\alpha = \delta = 0$; the second column shows the unconstrained optimum. Rows 2-3 in Table 2.1 show the optimal policies; rows 4-6 show the distributions of money holdings; rows 7-10 give the outputs; and row 11 gives the welfare gain from adoption of the optimal policies relative to the constrained optima.

In addition, numerical examples suggest the following about optimal allocations. First, in every optimum where money creation is beneficial, the proportion of individuals who hold one unit of money is larger than in the optimum subject

[11] The best quantity of output, the unconstrained maximizer of $z(y)$, is 0.25.

r	0.03		0.06		0.07		0.10		0.13	
α	-	.0787	-	.0555	-	.0480	-	.0258	-	.0045
δ	-	.0523	-	.0374	-	.0325	-	.0177	-	.0031
$\mathbf{p}_0$	1/3	.3342	1/3	.3377	.3337	.3389	.3373	.3426	.3448	.3465
$\mathbf{p}_1$	1/3	.3572	1/3	.3506	.3333	.3484	.3333	.3416	.3332	.3346
$\mathbf{p}_2$	1/3	.3086	1/3	.3117	.3330	.3127	.3294	.3158	.3220	.3189
$\mathbf{y}_{01}$	.25	.2573	.25	.2576	.2497	.2577	.2458	.2580	.2541	.2583
$\mathbf{y}_{12}$	.25	.1945*	.25	.1930*	.2477*	.1924*	.2243*	.1902*	.1946*	.1875*
$\mathbf{y}_{11}$	.25	.1945*	.25	.1930*	.2477*	.1924*	.2243*	.1902*	.1946*	.1875*
$\mathbf{y}_{02}$	.25	.3009*	.25	.2963*	.2530	.2949*	.2833	.2907*	.2894*	.2866*
$\frac{\Delta W}{W}$	2.62%		1.62%		1.27%		0.49%		0.04%	

Table 2.1: Optima

to $\alpha = \delta = 0$. That is, the optimal (α, δ)-policy shifts the distribution towards the mean.

Second, the (α, δ)-policy has incentive effects similar to those of the standard lump-sum policy. The policy tightens producer participation constraints which explains why production is less than the best quantity in meetings where producers have one unit of money. Outputs in meetings where producers have nothing are larger than the best quantity[12] because that helps to relax participation constraints in meetings where producers have one unit. In particular, high y_{02} tends to increase v_2, which, in turn, helps to relax these constraints. The incentive effects of the policy explain why in all examples the set of binding participation constraints is (weakly) larger than the set of binding constraints in examples with $\alpha = \delta = 0$. Note that the policy decreases the total production (GNP). This, however, is welfare improving.

Third, the optimal inflation rate, α, and the welfare gain from adoption of the optimal policy are increasing functions of patience. More patient individuals can tolerate higher inflation without serious incentive effects. This gives rise to distributions which are more concentrated around the mean. These, in turn, increase the number of trades and, hence, welfare. When patience drops below $r^{\dagger}$, the harmful intensive margin effects outweigh the beneficial extensive margin effects of the policy and no money creation is optimal.

Finally, as I already said, all optima have transfers of one unit of money with probability one in all trade meetings. This is consistent (see Definition 2.2) with my conjecture that all optima are connected. I should make it clear that the transfers of one unit are optimal here because of the two-unit bound on holdings. In environments with larger bounds one should not expect that the optima will

[12] This refers to outputs shown in Table 2.1. If r is big enough, production in all meetings is less than the best quantity.

have transfers of one unit, but, as was argued above, it is plausible that they will be connected.

2.5 Concluding remarks

This chapter extends the work of Deviatov and Wallace [5] in two ways. First, I show that the optimum chosen from among allocations which are implementable and connected does not involve randomization over output conditional on the transfer of money in a meeting. Because my proof technique is compatible with the money creation scheme studied in [5], the result can be used to study optimal money creation. Second, I compute numerical examples which provide additional information about the optima. These examples are consistent with what was conjectured in [5]. An important finding is that all optima in examples are connected (in fact, they satisfy sufficient conditions for connectedness in section 2.3). This supports my conjecture that connectedness is an innocuous restriction and that the optima chosen from among implementable allocations satisfy degeneracy.

Chapter 3

Money Creation and Optimal Pairwise Core Allocations in a Matching Model

A minimum test for the usefulness of a monetary model seems to be its ability to study lump-sum money creation. Among such models there seems to be a sharp contrast in results depending on whether there is heterogeneity in asset holdings. Representative agent models tend to yield results which are in line with what has become known as the Friedman rule: the optimal monetary policy is not creation, but destruction financed by lump-sum taxes. Models which make use of heterogeneity do not give a general answer: in some of these models the optimal monetary policy is contractionary, in some other models it is expansionary. Examples of models where it is expansionary include Levine [11] and its generalization by Kehoe, Levine and Woodford [10], Molico [15] and Deviatov and Wallace [5].

Levine [11] and Kehoe, Levine and Woodford [10] use a preference shock model with two realizations for the shock. In that model they produce examples where money creation is beneficial. However, they restrict attention to distributions of money holdings whose supports consist of just two points. To sustain these distributions in equilibrium they make quite special assumptions about parameters.

Deviatov and Wallace [5] and Molico [15] work with the random matching model of Trejos-Wright [17] and Shi [16]. Molico [15] uses a version with divisible money and unbounded holdings. However, he restricts himself to a particular bargaining rule — take-it-or-leave-it offers by consumers. As is well known, this rule can cause too much production in some meetings. Because the policy tightens producer participation constraints, the beneficial effect that he finds may result from offsetting the inefficient division of the gains from trade.

To address that concern, Deviatov and Wallace [5] do not adopt any particular bargaining rule. Instead they work with optima over the entire set of implementable allocations. They define implementable allocations as ones that satisfy ex post individual rationality in meetings. In a model with indivisible money and a two-unit bound on holdings, Deviatov and Wallace [5] show that if individuals are patient enough, then some money creation is better than no money creation. The role of money creation is to shift the distribution of money towards the average holdings to permit more trade to occur. Although Deviatov and Wallace [5] do not give a detailed description of the optima, that is sufficient to demonstrate that the optima have money creation.

Here I study the same environment as in Deviatov and Wallace [5], but I work with a different notion of implementability — the ex ante pairwise core. Because closed-form solutions for the optima cannot be obtained, I proceed numerically. My main finding is that in all examples there are no benefits of money creation. This is in sharp contrast with what is shown in Deviatov and Wallace [5].

The rest of the chapter is organized as follows. In the next section I describe the environment; in section 3.2 I define implementable allocations; in section 3.3 I discuss some general properties of implementable allocations; in section 3.4 I describe the algorithm; in section 3.5 I discuss examples; section 3.6 concludes.

3.1 Environments

The background environment is a simple random matching model of money due to Shi [16] and Trejos and Wright [17]. Time is discrete and the horizon is infinite. There are $N \geq 3$ perishable consumption goods at each date and a $[0, 1]$ continuum of each of N types of agents. A type n person consumes only good n and produces good $n+1$ (modulo N). Each person maximizes expected discounted utility with discount parameter $\beta \in (0, 1)$. Utility in a period is given by $u(y) - c(x)$, where y denotes consumption and x denotes production of an individual ($x, y \in \mathbb{R}_+$). The function u is strictly concave, strictly increasing and satisfies $u(0) = 0$, while the function c is convex with $c(0) = 0$ and is strictly increasing. Also, there exists $\hat{y} > 0$ such that $u(\hat{y}) = c(\hat{y})$. In addition, u and c are twice continuously differentiable. At each date, each agent meets one other person at random.

There is only one asset in this economy which can be stored across periods: fiat money. This money is indivisible and no individual can have more than B units of money at any given time, where $2 \leq B < \infty$. Agents cannot commit to future actions except commitment to outcomes of randomized trades in meetings. Finally, each agent's specialization type and individual money holdings are observable within each meeting, but the agent's history, except as revealed by money holdings, is private.

3.2 Implementable allocations and the optimum problem

The pairwise meetings, the inability to commit, the privacy of individual histories, and the perishable nature of the goods imply that any production must be accompanied by a positive probability of receiving money. In every meeting of a potential producer with i units of money and a potential consumer with j units, there is a set, denoted $\mathcal{K}_{ij}$, of feasible money transfers from the consumer to the producer, transfers which are consistent with each person's money holdings being in the set $\{0, 1, ..., B\}$: $\mathcal{K}_{ij} = \{0, 1, ... \min(j, B - i)\}$. A *trade meeting* is one where $\mathcal{K}_{ij}^{+} \equiv \mathcal{K}_{ij} \backslash \{0\}$ is nonempty. For each trade meeting between a producer with i and a consumer with j units of money, trade is represented by a probability measure μ_{ij} on $\mathbb{R}_{+} \times \mathcal{K}_{ij}$ with the interpretation that if (y, k) is randomly drawn in accordance with μ_{ij}, then (y, k) is the suggested trade in that meeting. Let $\boldsymbol{\mu}$ be the collection of measures μ_{ij} corresponding to trade meetings.

For any measure μ_{ij} it is convenient to consider the collection of conditional measures $\mu_{ij}^{k}(A) = \mu_{ij}(A \,|\, k)$, $k \in \mathcal{K}_{ij}$, and their supports Ω_{ij}^{k}.[1] Then μ_{ij} can be expressed as $\mu_{ij}(A) = \sum\limits_{k \in \mathcal{K}_{ij}} \lambda_{ij}^{k} \mu_{ij}^{k}(A)$, where $\lambda_{ij}^{k} \equiv \mu_{ij}(\Omega_{ij}^{k})$, is the probability that k units of money are offered in a meeting. Finally, let p_i be the fraction of agents in each specialization type who start a date with i units of money and let $\mathbf{p} = (p_0, ..., p_B)$. Then, in terms of p_i and λ_{ij}^{k}, an arbitrary off-diagonal element of the transition matrix T for $\mathbf{p}$ is given by:

$$\pi_{mn} = \begin{cases} \frac{1}{N} \sum\limits_{i=0}^{B-m+n} p_i \lambda_{im}^{m-n} \text{ if } m > n \\ \frac{1}{N} \sum\limits_{j=n-m}^{B} p_j \lambda_{mj}^{n-m} \text{ if } m < n \end{cases} \tag{3.1}$$

where π_{mn} is the probability of a trade that results in transition from having m units of money to having n units. Note that since T is a transition matrix, its diagonal elements are given by $\pi_{mm} = 1 - \sum\limits_{s \neq m} \pi_{ms}$.

In addition to trades there is lump-sum money creation. I use the same kind of policy that was studied by Deviatov and Wallace [5]. The policy is a probabilistic version of the proverbial helicopter drops of money. The timing of events in a period is the following. First there are meetings and trades. Next, each person receives one unit of money with probability α. (Those who are at the upper bound and receive a unit must discard it.) Then each unit of money disintegrates with probability δ.

[1] Recall that if μ is a probability measure, the support of μ, denoted $\operatorname{supp} \mu$, is the smallest closed set A such that $\mu(A) = 1$.

This policy has a close resemblance with the standard policy (expansion at a rate) which is followed by proportional reduction (normalization, see e.g. Lucas and Woodford [13]) in individual holdings. The standard policy shifts the distribution of money holdings towards the mean and makes money less desirable to acquire because poor producers are less willing to produce for money (because they get a transfer without production) and rich consumers are more willing to part with money (because they lose some of its value). The (α, δ)-policy of Deviatov and Wallace [5] has these effects as well.

Similar to trades, creation and destruction parts of the policy yield a pair of transition matrices for money holdings, denoted A and D respectively. According to my description of the policy, A is a two-diagonal matrix where the probability of getting a unit of money, α, is next to and above the main diagonal, and the probability of getting no transfer, $1-\alpha$, is on the main diagonal. Matrix D is lower-triangular where the first i entries in the i-th line comprise the binomial distribution of order i. Thus, the elements of A and D are:

$$a_{ij} = \begin{cases} 1-\alpha, & \text{if } j = i \\ \alpha, & \text{if } j = i+1 \\ 0, & \text{otherwise} \end{cases} \qquad d_{ij} = \begin{cases} \binom{i}{j}\delta^{i-j}(1-\delta)^j, & \text{if } j \leq i \\ \\ 0, & \text{otherwise} \end{cases}$$

The stationarity requirement is $\mathbf{p}TAD = \mathbf{p}$.

It is convenient to express individual rationality constraints in terms of discounted expected utilities. For an allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$, that is stationary, discounted expected utility of an agent who ends up with i units of money at the end of the period, denoted v_i, is constant. Then vector $\mathbf{v} \equiv (v_0, ..., v_B)$ satisfies the following $B+1$-equation system of Bellman equations:

$$\mathbf{v}' = \beta(\mathbf{q}' + TAD\,\mathbf{v}'), \tag{3.2}$$

where $\mathbf{q}$, the vector of (expected) one period returns from trade, is given by:

$$q_l = \sum_{i=0}^{B-1} \frac{p_i}{N} \sum_{k \in \mathcal{K}_{il}} \lambda_{il}^k \int\limits_{\Omega_{il}^k} u(y) d\mu_{il}^k - \sum_{j=1}^{B} \frac{p_j}{N} \sum_{k \in \mathcal{K}_{lj}} \lambda_{lj}^k \int\limits_{\Omega_{lj}^k} c(y) d\mu_{lj}^k \tag{3.3}$$

and where $l \in \{0, ..., B\}$. Note that an individual with no money can only expect to be a producer, an agent with B units can only be a consumer, and anyone else can be either a consumer or a producer.

Because T, A, and D are transition matrices and $\beta \in (0,1)$, the mapping $G(\mathbf{x}) \equiv \beta(\mathbf{q}' + TAD\,\mathbf{x}')$ is a contraction. Therefore, (3.2) has a unique solution which can be expressed as

$$\mathbf{v}' = \left(\frac{1}{\beta}I - TAD\right)^{-1} \mathbf{q}' \tag{3.4}$$

where I is the $(B+1)\times(B+1)$ identity matrix.

Let

$$\Pi_{ij}^{p} \equiv \sum_{k\in\mathcal{K}_{ij}} \lambda_{ij}^{k}\left((\mathbf{e}_{i+k}-\mathbf{e}_{i})\, AD\mathbf{v}' - \int_{\Omega_{ij}^{k}} c(y)d\mu_{ij}^{k} \right) \tag{3.5}$$

be the expected gain from trade for the producer with i units of money in a meeting with a consumer with j units and let

$$\Pi_{ij}^{c} \equiv \sum_{k\in\mathcal{K}_{ij}} \lambda_{ij}^{k}\left((\mathbf{e}_{j-k}-\mathbf{e}_{j})AD\mathbf{v}' + \int_{\Omega_{ij}^{k}} u(y)d\mu_{ij}^{k} \right) \tag{3.6}$$

be the expected gain from trade for the consumer in that meeting (where $\mathbf{e}_l$ is the $B+1$-component coordinate vector with indices running from 0 to B).

The ex ante pairwise core notion of implementability gives rise to the following definition:

Definition 3.1 *An allocation* $(\mathbf{p},\boldsymbol{\mu},\alpha,\delta)$ *is called implementable if (i)* $\mathbf{p}TAD=\mathbf{p}$, *(ii)* $\mathbf{v}$ *(given by 3.4) is non-decreasing, (iii) the participation constraints*

$$\Pi_{ij}^{p}\geq 0 \qquad \text{and} \qquad \Pi_{ij}^{c}\geq 0 \tag{3.7}$$

hold, and (iv) there exists a vector $\boldsymbol{\theta}\in[0,1]^{B^2}$ *such that for all pairs* (i,j) *corresponding to trade meetings, measure* μ_{ij} *solves*

$$\max_{\mu_{ij}} \left(\Pi_{ij}^{p}\right)^{1-\theta_{ij}} \left(\Pi_{ij}^{c}\right)^{\theta_{ij}}. \tag{3.8}$$

where the value function, $\mathbf{v}$*, is taken as given.*

Definition 3.1 says that an allocation is implementable if (i) it is stationary, (ii) satisfies free disposal of money, (iii) the ex ante gains from trade implied by the allocation are nonnegative, and (iv) there is no incentive for defections by pairs in meetings.

Finally, our optimum problem is to maximize ex ante utility. That is, the optimum problem, denoted P, is to choose $(\mathbf{p},\boldsymbol{\mu},\alpha,\delta)$ from among those that are implementable to maximize $\mathbf{p}\mathbf{v}'\equiv W$.

It is useful to express the objective W in terms of returns. If I multiply (3.2) by $\mathbf{p}$ and use the fact that $\mathbf{p}TAD=\mathbf{p}$, then I obtain:

$$W=\mathbf{p}\mathbf{v}'=\frac{\beta}{1-\beta}\mathbf{p}\mathbf{q}'$$

Then, by writing out the product $\mathbf{pq}'$, I get:

$$W = \frac{\beta}{1-\beta}\frac{1}{N}\sum_{i=0}^{B-1}\sum_{j=1}^{B}\sum_{k\in\mathcal{K}_{ij}} p_i p_j \lambda_{ij}^k \int_{\Omega_{ij}^k} z(y) d\mu_{ij}^k \tag{3.9}$$

where $z(y) \equiv u(y) - c(y)$. As one would expect, because for every consumer there is a producer, welfare is equal to the net expected discounted utility in all trade meetings.

3.3 General results

In their paper on lotteries, Berentsen, Molico and Wright [2] give a complete characterization of the ex ante pairwise core for the case of one-unit bound on holdings. Here I use their results to show that every implementable allocation has no randomization over output; each conditional measure μ_{ij}^k is degenerate and does not depend on k. The proof is the same as that of Proposition 3 in Berentsen, Molico and Wright [2], so I do not reproduce it here. Degeneracy follows immediately from concavity of $u(x)$ and $-c(x)$. Independence on k follows from concavity of the Nash product (3.8) in Definition 3.1.

These results imply that the expressions for the gains from trade (3.5) and (3.6) and for welfare (3.9) can be simplified. Welfare is

$$W = \frac{\beta}{1-\beta}\frac{1}{N}\sum_{i=0}^{B-1}\sum_{j=1}^{B} p_i p_j z(y_{ij}) \tag{3.10}$$

where y_{ij} denotes output in a meeting of producer with i and consumer with j units of money.

Degeneracy of conditional measures implies that the optimum problem P is finite dimensional. This allows me to characterize the pairwise core in terms of the necessary first order conditions for maximization of the Nash product. Because of concavity of the latter these necessary conditions are also sufficient. If an allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ has $y_{ij} > 0$ in all trade meetings,[2] then the first order conditions can be conveniently written as

$$\left[(e_{j-k} - e_j) + \frac{u'(y_{ij})}{c'(y_{ij})}(e_{i+k} - e_i)\right] AD\mathbf{v}' \begin{array}{l} \geq 0 \text{ if } \lambda_{ij}^k = \overline{\lambda}_{ij}^k \\ = 0 \text{ if } 0 < \lambda_{ij}^k < \overline{\lambda}_{ij}^k \\ \leq 0 \text{ if } \lambda_{ij}^k = 0 \end{array} \tag{3.11}$$

[2] A sufficient condition for this is that $AD\mathbf{v}'$, where $\mathbf{v}$ is the value function implied by an implementable allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$, is strictly increasing and that $u'(0) = \infty$ and $c'(0) = 0$.

for all pairs (i, j) corresponding to trade meetings and transfers of positive amounts of money k, where $\overline{\lambda}_{ij}^{k} \equiv 1 - \sum_{s \in \mathcal{K}_{ij}^{+} \setminus \{k\}} \lambda_{ij}^{s}$.

The first order conditions (3.11) yield a set of constraints which an implementable allocation must satisfy in addition to the participation constraints in Definition 3.1. If the value function $AD\mathbf{v}'$ implied by an implementable allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta)$ is strictly concave, then (3.11) has implications for the level of output in some meetings. In particular, if $\lambda_{ij}^{k} > 0$ and $k \geq j - i$ for some $k \in \mathcal{K}_{ij}^{+}$, then $y_{ij} \leq y^{*}$, the unconstrained maximizer of $z(y)$. (Notice that if $j \geq i - 1$, that is if the producer's holdings are one unit less than the consumer's or larger, then trade implies $\lambda_{ij}^{k} > 0$ for some $k \geq j - i$.) In the examples below, $B = 2$, so the only meetings in which output can exceed y^{*} are those between a producer with zero and a consumer with two units of money.

3.4 The algorithm

With the ex ante pairwise core notion of implementability the optima always have some binding participation constraints. If individuals are patient enough, the optima also have randomization over how much money is transferred in meetings. This implies that some of the constraints in (3.11) are also binding. Because these constraints are complicated functions of an allocation, closed-form solutions for the optima are out of reach even for the case of a two-unit bound on holdings. That is why I compute a set of examples.

The optimization problem P falls within the class of problems generally referred to as "nonlinear programming problems", for which many standard routines are available. However, as one can see, the constraints in (3.11) are discontinuous.[3] Another difficulty is that the mapping $F(\mathbf{p}) \equiv \mathbf{p}TAD - \mathbf{p}$ is ill-behaved at $\alpha = \delta = 0$.[4] This precludes application of routines which require continuous differentiability of objective and constraints, such as sequential quadratic programming. I overcome this difficulty by designing a hybrid algorithm which combines genetic and conventional smooth optimization techniques.

There are three main steps in this algorithm.

- **Step 1.** Create an initial population of allocations.

[3]Each constraint in (3.11) is equivalent to

$$\left[(\mathbf{e}_{j-k} - \mathbf{e}_{j}) + \frac{u'(y_{ij})}{c'(y_{ij})}(\mathbf{e}_{i+k} - \mathbf{e}_{i})\right] AD\mathbf{v}' + (\mathrm{sign}(\lambda_{ij}^{k}) - \mathrm{sign}(\overline{\lambda}_{ij}^{k} - \lambda_{ij}^{k}))\vartheta_{ij}^{k} = 0$$

and $\vartheta_{ij}^{k} \leq 0$, where sign$(x)$ is the sign function, and ϑ_{ij}^{k} is a slack variable.

[4]See Deviatov and Wallace [5], who study the properties of that mapping for $B = 2$.

- **Step 2.** Amend the population by replacing the worst allocations by better ones.

- **Step 3.** Check if the termination criterion is satisfied for the best allocation in the population. If yes, terminate. If no, return to step 2.

In step 1 I create a matrix where each row is an allocation. Allocations in the initial population are picked randomly among those which satisfy ex ante individual rationality. The size of the population is a parameter of the algorithm.

To amend the population in step 2 I use several genetic operators. These operators are called selection, crossover and mutation. I use standard selection and crossover operators, a subset of those described in Houck, Joines and Kay [8]. However, I modify the standard mutation operator. The standard operator alters a single allocation (called "the parent") to produce another allocation (called "the child"). The operator I use is a composition of two independent operators.

The first one is applied only if the parent has at least one of the transfer probabilities λ_{ij}^k at its upper or lower bound or if it has $\alpha = \delta = 0$. The operator pushes a random subset of these variables into the interior. If a better allocation is produced, it replaces the parent in the population. This simple mutation deals with discontinuity of the constraints in (3.11) and with ill behavior of the mapping $F(\mathbf{p})$ at zero.

The second operator alters only those of the transfer probabilities and policy pairs which are already in the interior. There, because all constraints are twice continuously differentiable, application of smooth methods is possible. This leaves a range of possibilities for what this second operator can be. In particular, one can run a few iterations of a sequential quadratic routine or of the BFGS algorithm[5] (as long as these iterations remain in the interior). The operator I adopt makes use of the gradients in the following way.

First, I compute (reduced) gradients of the objective and of all active constraints. Then I compute an orthogonal projection of the gradient of the objective onto the subspace orthogonal to the one spanned by the gradients of the active constraints. After that I randomly pick a search direction in the neighborhood (small cone) of that projection. Going in that search direction is likely to improve the objective and does not violate (at least by much) the active constraints. The child is obtained from the parent by moving along the search direction. However, this procedure often leads to violation of some constraints even if the parent satisfies all the constraints. In this case the objective implied by the child is reduced by some value which is proportional to the amount by which the constraints are violated. If the penalty parameter is large, even a small violation is costly, and the child dies out of the population quickly. If the parent itself violates constraints

[5] See Judd [9] for further details.

by large amounts, then the search direction is chosen to move the child closer to the feasible region regardless of what happens to the objective. Because the initial population is chosen randomly, this is important in the beginning of search. In other words, the second operator first pushes allocations towards satisfaction of the pairwise core conditions; then it drives the population to the optimum.

The termination criterion in step 3 is based on the first order conditions for the Kuhn-Tucker theorem. If the length of the projection of the gradient of the objective onto the subspace orthogonal to that spanned by the gradients of the active constraints is less than the tolerance value, the necessary conditions for the theorem are (approximately) satisfied. Because the probability of selection of parents in the population is an increasing function of the objective, this is sufficient to guarantee that every terminal point is a (local) maximum.

3.5 Examples

I use the above algorithm to compute optima for examples with a two-unit bound on individual money holdings. I compute two sets of examples. In all the examples, the utility function, $u(x)$, is x^κ; the cost function, $c(x)$, is x; and the number of specialization types, N, is 3. The examples are computed for various κ and various degrees of patience, r, where $r = \frac{1}{\beta} - 1$.

Several things are common to every example. First, there are no binding consumer participation constraints and there are no examples where money creation is beneficial. The latter may not be merely coincidental.

Consider an implementable allocation $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta) \equiv x$ with $(\alpha, \delta) > 0$. Next, consider another allocation $(\mathbf{p}', \boldsymbol{\mu}', \alpha', \delta') \equiv x'$ with the same outputs and with $\alpha' < \alpha$ and $\delta' < \delta$ such that TAD and hence $\mathbf{p}$ are unchanged. (One can show that it is sufficient to adjust λ^1_{11} alone and that there exists a unique direction in the (α, δ) plane such that x' remains in the core.) By (3.10) it follows that x and x' yield the same welfare. In addition, the replacement of x by x' tends to relax producer participation constraints and to tighten consumer participation constraints. But since the optima tend not to have binding consumer participation constraints, the replacement tends to slacken the relevant constraints. Then, if the replacement makes all of the producer participation constraints be slack, continuity implies that it is possible to find an implementable allocation which is better than x. A formal argument along these lines is difficult because it is difficult to show that the optima do not have binding consumer participation constraints. (This, however, is not surprising because, as demonstrated in Berentsen, Molico and Wright [2], money has no value if the gain from trade for consumers is zero.)

The second thing common to all the examples is that in a meeting of a producer with no money and a consumer with two units, one unit of money changes hands with probability one. I take advantage of these common features to simplify presentation of examples in the tables below. I omit the policy variables α and

r	0.01	0.07	0.11	0.12	0.15	0.19	0.20	0.30	0.40	0.50
p_0	.2181	.2704	.2938	.3023	.3286	.3570	.3645	.4242	.4635	.4830
p_1	.5884	.4974	.4566	.4466	.4226	.3998	.3951	.3573	.3303	.3135
p_2	.1935	.2322	.2496	.2511	.2488	.2432	.2404	.2185	.2062	.2035
λ_{01}	$.2023^{\dagger}$	$.4143^{\dagger}$	$.5774^{\dagger}$	$.6291^{\dagger}$	$.7820^{\dagger}$	$.9884^{\dagger}$	1	1	1	1
λ_{12}	$.3357^{\dagger}$	$.6763^{\dagger}$	$.9478^{\dagger}$	1	1	1	1	1	1	1
λ_{11}	$.1218^{\dagger}$	$.2537^{\dagger}$	$.3516^{\dagger}$	$.3804^{\dagger}$	$.4577^{\dagger}$	$.5432^{\dagger}$	$.5613^{\dagger}$	$.7259^{\dagger}$	$.8761^{\dagger}$	1
y_{01}	$.25^*$	$.25^*$	$.25^*$	$.25^*$	$.25^*$	$.25^*$	$.2408^*$	$.1585^*$	$.1136^*$	$.0844^*$
y_{12}	$.25^*$	$.25^*$	$.25^*$	$.2401^*$	$.1870^*$	$.1390^*$	$.1301^*$	$.0730^*$	$.0452^*$	$.0293^*$
y_{11}	$.0908^*$	$.0938^*$	$.0927^*$	$.0913^*$	$.0855^*$	$.0755^*$	$.0730^*$	$.0529^*$	$.0395^*$	$.0293^*$
y_{02}	.6876	.5306	$.4331^*$	$.3974^*$	$.3196^*$	$.2529^*$	$.2408^*$	$.1585^*$	$.1136^*$	$.0844^*$

Table 3.1: Optima with $u(x) = \sqrt{x}$

δ and the probabilities of transfer of money in meetings of producers with nothing and consumers with two units (λ_{02}^1 and λ_{02}^2). I also suppress superscripts in the notation for the other transfer probabilities (λ_{01}^1, λ_{12}^1 and λ_{11}^1). I attach stars (*) to outputs which correspond to binding producer participation constraints and daggers (†) to the transfer probabilities which correspond to binding first order constraints in (3.11).

The first set of examples shows how optima change with patience. Here I fix $\kappa = \frac{1}{2}$ and vary r. This choice implies that the best quantity of output, y^*, is 0.25. I compute examples for all r from 0.01 through 0.25 in increments of 0.01 and for $r \in \{0.3, 0.35, 0.4, 0.5\}$. I report a subset of these examples in Table 3.1. The examples are consistent with the existence of four different regions with respect to the degree of patience r. If r is low enough, then the optima have randomization over the transfers of money in all three trade meetings where transfers of only one unit are feasible. If r belongs to the second region, the optima have randomization over the transfers of money only in meetings where the consumers have one unit. In meetings of producers with one and consumers with two units, money changes hands with probability one. In the next region the optima have randomization over the transfers of money in meetings where both producers and consumers have one unit. Finally, if r is big enough, one unit of money changes hands with probability one in all trade meetings. The examples are consistent with the transfer probabilities λ_{12}, λ_{01}, and λ_{11} being decreasing functions of patience.

In addition, the examples are consistent with the optima having at most one nonbinding producer participation constraint, the one in meetings of producers with nothing and consumers with two units of money. In a meeting of a producer with one unit and a consumer with two, lowering the probability of handing over money raises v_2. That is helpful because it loosens producer constraint in $(i, j) =$

κ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
p_0	.1357	.1845	.2148	.2357	.2510	.2623	.2727	.3243	.4796
p_1	.7593	.6703	.6114	.5682	.5343	.5067	.4812	.4268	.3572
p_2	.1050	.1452	.1738	.1961	.2147	.2310	.2461	.2489	.1632
λ_{01}	$.0300^\dagger$	$.0803^\dagger$	$.1457^\dagger$	$.2232^\dagger$	$.3108^\dagger$	$.4058^\dagger$	$.5180^\dagger$	$.8995^\dagger$	1
λ_{12}	$.1622^\dagger$	$.2618^\dagger$	$.3514^\dagger$	$.4331^\dagger$	$.5103^\dagger$	$.5821^\dagger$	$.6640^\dagger$	1	1
λ_{11}	$.0247^\dagger$	$.0596^\dagger$	$.0998^\dagger$	$.1432^\dagger$	$.1888^\dagger$	$.2359^\dagger$	$.2898^\dagger$	$.4430^\dagger$	$.6134^\dagger$
y_{01}	$.0744^*$	$.1337^*$	$.1791^*$	$.2172^*$	$.2500^*$	$.2789^*$	$.3046^*$	$.3277^*$	$.1624^*$
y_{12}	$.0744^*$	$.1337^*$	$.1791^*$	$.2172^*$	$.2500^*$	$.2789^*$	$.3046^*$	$.3051^*$	$.1411^*$
y_{11}	$.0118^*$	$.0305^*$	$.0509^*$	$.0717^*$	$.0925^*$	$.1130^*$	$.1330^*$	$.1352^*$	$.0866^*$
y_{02}	.4792	.5326	.5549	.5726	.5859	.5969	$.5881^*$	$.3645^*$	$.1624^*$

Table 3.2: Optima with $u(x) = x^\kappa$

$(1,1)$ meeting, which, in turn, allows a decrease in λ_{11} and, thus, an increase in p_1 (and, thereby, in the frequency of trade). Because λ_{11} is low, the participation constraint in $(i,j) = (1,1)$ meeting is binding and the output is low.

A smaller probability of giving up money in $(i,j) = (0,1)$ meeting lowers v_0 which helps to relax the producer constraint in $(i,j) = (0,2)$ meeting. This allows a higher y_{02} which, again, pushes up v_2. This accounts for why y_{02} is so high. The same kind of effect on v_2 could be achieved with a positive λ^0_{02}, but that would reduce λ^1_{02} and, hence, the inflow into p_1.

The second set of examples shows how optima change with risk aversion. Here I fix $r = 0.04$ and vary κ. I compute examples for all κ from 0.1 through 0.9 in the increments of 0.1. These examples are reported in Table 3.2.[6] A general finding here is that the optima change with κ in a similar way as they change with patience. In particular, the transfer probabilities λ_{12}, λ_{01}, and λ_{11} are decreasing functions of risk aversion.

In Table 3.3[7] I present some comparison of the two notions of implementability: the ex ante pairwise core and ex post IR notions. Note that, even though every allocation which satisfies ex post IR satisfies ex ante IR, there is no subset result for the allocations with these two notions of implementability.[8] Nevertheless, the ex ante notion is in some sense weaker because it allows for randomization over the

[6] The best quantity of output varies with κ. In the Table 3.2 the best quantity is equal to y_{01} for all κ except $\kappa = 0.9$ for which this quantity equals 0.3487.

[7] For each value of r the first column shows the optimum with ex post IR and the second column shows the optimum with ex ante pairwise core notions of implementability. The last row shows welfare improvement relative to optima with ex post IR notion of implementability.

[8] Examples of ex post individually rational allocations which fail to satisfy the first order conditions for ex ante pairwise core include the best allocations under no policy described in Deviatov and Wallace [5] for $r \le r^*$.

r	0.02		0.06		0.10		0.15		0.25	
α	.0866	0	.0555	0	.0258	0	0	0	0	0
δ	.0573	0	.0374	0	.0177	0	0	0	0	0
p_0	.3330	.2310	.3377	.2649	.3424	.2857	.3525	.3286	.4094	.3979
p_1	.3595	.5678	.3506	.5084	.3416	.4680	.3330	.4226	.3279	.3744
p_2	.3075	.2012	.3117	.2267	.3158	.2463	.3145	.2488	.2627	.2277
λ_{01}	1	$.2385^\dagger$	1	$.3802^\dagger$	1	$.5244^\dagger$	1	$.7820^\dagger$	1	1
λ_{12}	1	$.3948^\dagger$	1	$.6217^\dagger$	1	$.8572^\dagger$	1	1	1	1
λ_{11}	1	$.1441^\dagger$	1	$.2323^\dagger$	1	$.3211^\dagger$	1	$.4577^\dagger$	1	$.6262^\dagger$
y_{01}	.2572	$.25^*$	.2576	$.25^*$	.2580	$.25^*$	.2667	$.25^*$	$.1853^*$	$.1927^*$
y_{12}	$.1950^*$	$.25^*$	$.1930^*$	$.25^*$	$.1902^*$	$.25^*$	$.1715^*$	$.1870^*$	$.0919^*$	$.0959^*$
y_{11}	$.1950^*$	$.0913^*$	$.1930^*$	$.0934^*$	$.1902^*$	$.0937^*$	$.1715^*$	$.0855^*$	$.0919^*$	$.0619^*$
y_{02}	$.3025^*$	.6473	$.2963^*$	.5459	$.2907^*$	$.4763^*$	$.2783^*$	$.3196^*$	$.1853^*$	$.1927^*$
$\frac{\Delta W}{W}$	19.5%		14.2%		11.2%		7.62%		3.22%	

Table 3.3: Optima with ex ante pairwise core versus optima with ex post IR notions of implementability

amount of money transferred in meetings. This, to some extent, mimics divisibility of money. With the ex post IR notion individuals agree to every realization in the support of randomized trades which makes randomization costly. Deviatov [6], who computes examples of optima with the ex post IR notion, finds that in all the examples one unit of money changes hands with probability one in all trade meetings.

Given that pattern of trade, the only way to enlarge the set of feasible distributions is by means of the policy which accounts for the beneficial effects found in Deviatov and Wallace [5]. However, money creation never allows to achieve distributions which are concentrated around the average holdings to such an extent as those that are feasible with the ex ante pairwise core notion. That is why optima with ex ante pairwise core notion yield a considerably higher welfare (the difference is shown in the last row of Table 3.3).

3.6 Concluding remarks

The results in this chapter stand in sharp contrast to these in Deviatov and Wallace [5] and in Deviatov [6]. There money creation can be beneficial, whereas here there are no examples where money creation is optimal. The disparity is due entirely to the distinction between committing or not committing to randomization. (The imposition of ex post pairwise core restrictions which allow for the trading pair to defect to any deterministic trade does not change the results in [5] and [6].)

This disparity is interesting because as the bound on individual money holdings gets large, randomization plays a smaller and smaller role and, in the limit, no role. Then, the two notions of implementability coincide. The uniformity of the numerical finding of no beneficial money creation using the ex ante notion leads me to surmise that it is the general result that will survive in the limit. Of course, surmising such a result and proving it are very different.

Bibliography

[1] Aiyagari R., Wallace N., Wright R. "Coexistence of Money and Interest-Bearing Securities." *Journal of Monetary Economics,* 37 (1996), 397-420.

[2] Berentsen A., Molico M., Wright R. "Indivisibilities, lotteries and monetary exchange." Working paper, University of Pennsylvania, 2000.

[3] Camera G, Corbae D. "Money and price dispersion." *International Economic Review,* 40 (1999), 985–1008.

[4] Dawid H. Adaptive learning by genetic algorithms. 2nd edition. Springer Verlag, Berlin, 1999.

[5] Deviatov A., Wallace N. "Another example in which lump-sum money creation is beneficial." *Advances in Macroeconomics,* 1 (2001), Article 1.

[6] Deviatov A. "Optimal Money Creation in a Random-Matching Model with Ex Post Individual Rationality." Working paper, Pennsylvania State University, 2002.

[7] Goldman S. "Flexibility and the demand for money." *Journal of Economic Theory*, 9 (1974), 203-222.

[8] Houck C., Joines J., Kay M. "A genetic algorithm for function optimization: a MATLAB implementation." Working paper, North Carolina State University, 1996.

[9] Judd K. Numerical methods in Economics. MIT Press. Cambridge and London, 1998.

[10] Kehoe T., Levine D., Woodford M. "The optimum quantity of money revisited." In Economic Analysis of Markets and Games: Essays in Honor of Frank Hahn. Dasgupta P. et al -eds. MIT Press. Cambridge and London, 1992, 501-526.

[11] Levine D. "Asset trading mechanisms and expansionary policy." *Journal of Economic Theory,* 54 (1991), 148-164.

[12] Li V. "The optimal taxation of fiat money in search equilibrium." *International Economic Review,* 36 (1995), 927-942.

[13] Lucas R E Jr., Woodford M. "Real effects of monetary shocks in an economy with sequential purchases." University of Chicago Working Paper, 1994.

[14] Miller M, Orr D. "A model of the demand for money for money by firms." *Quarterly Journal of Economics*, 80 (1966), 413-35.

[15] Molico M. "The distribution of money and prices in search equilibrium." Working paper, University of Pennsylvania, 1997.

[16] Shi S. "Money and prices: a model of search and bargaining." *Journal of Economic Theory,* 67 (1995), 467-496.

[17] Trejos A., Wright R. "Search, bargaining, money and prices." *Journal of Political Economy,* 103 (1995), 118-141.

[18] Wallace, N. "Introduction to modeling money and studying monetary policy." *Journal of Economic Theory,* 81 (1998), 223-231.

Appendix A

Appendix to Chapter 1

Lemma 1.1 *If* $\alpha = \delta = 0$*, then the optimum subject only to condition (i) in Definition 1.1 and condition (ii) for* $k = 0$ *is a degenerate* $\boldsymbol{\mu}$*, denoted* $\boldsymbol{\mu}^*$*, with support* $(y^*, 1)$*, where* $u'(y^*) = 1$*. Moreover, the associated optimal* $\mathbf{p}$ *is* $\mathbf{p}^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$*.*

Proof.

The steady-state condition, which becomes $\mathbf{p}T = \mathbf{p}$ under $\alpha = \delta = 0$, does not involve the outputs. Therefore, for given λ_{ij}^k, a necessary condition for maximizing W is $y = y^*$ for all y in the support of $\boldsymbol{\mu}$. Then W which satisfies that necessary condition can be written as

$$w = z(y^*)[p_0 p_1 \lambda_{01}^1 + p_0 p_2 \left(\lambda_{02}^1 + \lambda_{02}^2\right) + p_1^2 \lambda_{11}^1 + p_1 p_2 \lambda_{12}^1]. \qquad \text{(A.1)}$$

Whenever $k = j - i$, the constraint $\mathbf{p}T = \mathbf{p}$ does not depend on λ_{ij}^k because money holdings are being exchanged. And since the λ_{ij}^{j-i} appear in (A.1) with non-negative coefficients, we set them at their maxima; namely, $\lambda_{01}^1 = \lambda_{12}^1 = 1$ and $\lambda_{02}^2 = 1 - \lambda_{02}^1$. It follows that w which satisfies necessary conditions for a maximum can be written

$$w = z(y^*)[p_0 p_1 + p_0 p_2 + p_1^2 \lambda_{11}^1 + p_1 p_2].$$

Thus, the problem is to maximize $F(\mathbf{p}, \lambda_{11}^1, \lambda_{02}^1) = p_0 p_1 + p_0 p_2 + p_1^2 \lambda_{11}^1 + p_1 p_2$ subject to $\mathbf{p}T = \mathbf{p}$.

Now $\mathbf{p}$ either has full support or not. If not, then either $p_1 = 0$ or $\lambda_{11}^1 = 0$. (If $p_1 > 0$ and $\lambda_{11}^1 > 0$, then $\mathbf{p}$ has full support because there is an inflow into holdings of both 0 and 2 as a result of trade between producers and consumers with one unit.) If $p_1 = 0$ or $\lambda_{11}^1 = 0$ and $\mathbf{p}$ does not have full support, then the objective F has the form $p_i p_j$ for $i \neq j$, the maximum of which is $\frac{1}{4}$.

We now find the maximum over full support $\mathbf{p}$'s. Consider the Lagrangian

$$L = p_0 p_1 + p_0 p_2 + p_1^2 \lambda_{11}^1 + p_1 p_2 - \psi(\lambda_{11}^1 p_1^2 - \lambda_{02}^1 p_0 p_2) - \nu \left(\sum p_i - 1\right), \qquad \text{(A.2)}$$

where ψ and ν are non-negative multipliers and where we have inserted the explicit form of the constraint, $\mathbf{p}T = \mathbf{p}$. This constraint reduces to the single equation, $\lambda_{11}^1 p_1^2 = \lambda_{02}^1 p_0 p_2$, which says that the outflow from holdings of 1 unit is equal to inflow. Because $\mathbf{p}$ has full support, the first order conditions with respect to the p_i hold at equality. They are

$$p_0 : p_1 + p_2 + \lambda_{02}^1 \psi p_2 - \nu = 0, \tag{A.3}$$

$$p_1 : p_0 + p_2 + 2\lambda_{11}^1 (1 - \psi) p_1 - \nu = 0, \tag{A.4}$$

$$p_2 : p_0 + p_1 + \lambda_{02}^1 \psi p_0 - \nu = 0. \tag{A.5}$$

Again because $\mathbf{p}$ has full support, either $\lambda_{11}^1 = \lambda_{02}^1 = 0$ or $\lambda_{11}^1 > 0$ and $\lambda_{02}^1 > 0$. In the first case, it follows from (A.3)-(A.5) that the maximum of $F(\mathbf{p}, \lambda_{11}^1, \lambda_{02}^1)$ is attained at $\mathbf{p}^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Inserting this and $\lambda_{11}^1 = 0$ into F implies that the value of F is $\frac{1}{3}$.

For the second case ($\lambda_{11}^1 > 0$ and $\lambda_{02}^1 > 0$), we substitute from $\lambda_{11}^1 p_1^2 = \lambda_{02}^1 p_0 p_2$ directly into the objective. Then, because the remaining constraint does not involve λ_{02}^1 and because the resulting objective is increasing in λ_{02}^1, we conclude that the optimum in this case has $\lambda_{02}^1 = 1$. Then the sum of (A.3) and (A.5) minus twice (A.4) gives

$$2p_1 - (p_0 + p_2 + 4\lambda_{11}^1 p_1) + \psi(p_0 + p_2 + 4\lambda_{11}^1 p_1) = 0,$$

which can be written as

$$1 - \psi = \frac{2p_1}{p_0 + p_2 + 4\lambda_{11}^1 p_1} > 0. \tag{A.6}$$

From (A.2), we have

$$\frac{\partial L}{\partial \lambda_{11}^1} = p_1^2 (1 - \psi) > 0.$$

where the inequality follows from (A.6). Therefore, $\lambda_{11}^1 = 1$. Also, if we subtract (A.3) from (A.5), we get $p_2 = p_0$. This and $p_1^2 = p_2 p_0$, the explicit form of $\mathbf{p}T = \mathbf{p}$ with $\lambda_{11}^1 = \lambda_{02}^1 = 1$, imply $\mathbf{p} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Therefore, the maximum of F in this case is attained at $\mathbf{p}^*$ and is equal to $\frac{4}{9}$.

Direct comparison of the three values of maximized objective completes the proof. ■

Lemma 1.2 *Let* $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta) = (\mathbf{p}^*, \boldsymbol{\mu}^*, 0, 0)$. *There exists a value of the discount factor,* β^*, *given by*

$$\beta^* = \frac{3Ny^*}{3Ny^* + \sqrt{(3y^*)^2 + 4z(y^*)y^*} - 3y^*},$$

such that if $\beta > \beta^*$, then (1.6) and (1.7) are slack. If $\beta = \beta^*$, then (1.6) and (1.7) are slack except for (1.6) for $i = 1$ (when the producer has 1 unit of money) which holds at equality.

Proof.

The proof proceeds by explicit computation of the v_i at $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta) = (\mathbf{p}^*, \mu^*, 0, 0)$. In particular, we have

$$v_1 - v_0 = \frac{2\beta[3N(1-\beta)u(y^*) + 4\beta u(y^*) + 2\beta y^*]}{3[N(1-\beta)+2\beta][3N(1-\beta)+2\beta]} \equiv h_1(\beta)$$

and

$$v_2 - v_1 = \frac{2\beta[3N(1-\beta)y^* + 4\beta y^* + 2\beta u(y^*)]}{3[N(1-\beta)+2\beta][3N(1-\beta)+2\beta]} \equiv h_2(\beta).$$

It follows that $h_1(\beta)$ and $h_2(\beta)$ are defined and continuous on $[0, 1]$, are strictly increasing, satisfy $h_1(\beta) > h_2(\beta)$, $h_1(0) = h_2(0) = 0$,

$$h_1(1) = \frac{2u(y^*) + y^*}{3} \in (y^*, u(y^*))$$

and

$$h_2(1) = \frac{2y^* + u(y^*)}{3} \in (y^*, u(y^*)),$$

where all the inequalities follow from $y^* < u(y^*)$. It follows that consumer participation constraints are slack at all $\beta \in (0, 1)$. It also follows that there exists a unique $\beta \in (0, 1)$ such that $h_2(\beta) = y^*$. Denote this β^*. Then, aside from the explicit claim about the expression for β^*, all the remaining claims follow from the assertions about $h_1(\beta)$ and $h_2(\beta)$. The explicit expression for β^* is obtained by solving the equation, $h_2(\beta^*) = y^*$.■

Proposition 1.1 *If $\beta \geq \beta^*$, then the optimum is not $\alpha = \delta = 0$.*

Proof.

We compute a derivative of W with respect to α and evaluate it at $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta) = (\mathbf{p}^*, \boldsymbol{\mu}^*, 0, 0)$. The only requirement is that implementability is maintained as we vary α. In computing the derivative, we keep all money transfers in meetings as they are under $\boldsymbol{\mu}^*$. That is, one unit of money is transferred in every trade meeting in. It follows that the trade matrix T has the form

$$T^* = \begin{bmatrix} 1 - \frac{p_1+p_2}{N} & \frac{p_1+p_2}{N} & 0 \\ \frac{p_0+p_1}{N} & 1 - \frac{p_0+2p_1+p_2}{N} & \frac{p_1+p_2}{N} \\ 0 & \frac{p_0+p_1}{N} & 1 - \frac{p_0+p_1}{N} \end{bmatrix}.$$

The mapping from (α, δ) to $\mathbf{p}$ that satisfies $\mathbf{p}T^*AD = \mathbf{p}$ is not well-behaved at $\alpha = \delta = 0$. At $\alpha = \delta = 0$, there is a one dimensional set of $\mathbf{p}$'s that are distributions and that satisfy $\mathbf{p}T^*AD = \mathbf{p}$. They can be thought as being generated by the set of alternative amounts of money per type, the interval $[0,2]$. For $(\alpha,\delta) > 0$ and in a neighborhood of $\alpha = \delta = 0$, we can show that there is a unique $\mathbf{p}$ that satisfies $\mathbf{p}T^*AD = \mathbf{p}$ so that the mapping from (α,δ) to $\mathbf{p}$ is a function in that neighborhood, and, moreover, is a differentiable function. After doing that, we will find the unique direction in the (α,δ) plane along which that unique solution approaches $\mathbf{p}^*$ as $(\alpha,\delta) \to 0$. Finally, we will compute the derivative of $\mathbf{p}$ along that direction and evaluate it at $\mathbf{p}^*$. That, in turn, will allow us to show that welfare is increasing along that direction.

The conditions $\sum p_i = 1$ and $\mathbf{p}T^*AD = \mathbf{p}$ can be written as the following system of three equations in three unknowns:

$$p_1 + p_2 + p_0 = 1, \tag{A.7}$$

$$-\xi_1 p_1 + \xi_2 p_2 - \xi_3 p_0 = 0, \tag{A.8}$$

$$\frac{1}{N}(1-2\alpha)(1-\delta)^2[p_1^2 - p_0 p_2] + \alpha(1-\delta)^2 p_1 - \delta(2-\delta)p_2 = 0, \tag{A.9}$$

where

$$\begin{aligned}
\xi_1 &\equiv (1-\delta)[(\alpha-\delta)(1-\alpha) - \alpha^2\delta] = \alpha - \delta + o(\alpha,\delta) \\
\xi_2 &\equiv \delta[\alpha\delta + 2(1-\alpha)(1-\delta)] = 2\delta + o(\alpha,\delta) \\
\xi_3 &\equiv \alpha(1-2\alpha)(1-\delta)^2 = \alpha + o(\alpha,\delta)
\end{aligned}$$

and where $o(\alpha,\delta)$ denotes terms of order higher than (α,δ).

Because (A.7) and (A.8) are linear, they can be solved uniquely for p_1 and p_2 (in terms of p_0) if

$$\det\begin{pmatrix} 1 & 1 \\ -\xi_1 & \xi_2 \end{pmatrix} = \xi_2 + \xi_1 \neq 0.$$

From the expressions for the ξ_i, it follows that $\xi_2 + \xi_1 = \delta + \alpha + o(\alpha,\delta) > 0$. Therefore, we have,

$$p_1 = \frac{\xi_2 - (\xi_3 + \xi_2)p_0}{\xi_2 + \xi_1} \text{ and } p_2 = \frac{\xi_1 + (\xi_3 - \xi_1)p_0}{\xi_2 + \xi_1}.$$

If we substitute these into (A.9), the result is a quadratic equation in p_0 or, more simply, x, which we write as $f(x) = ax^2 + bx + c = 0$, where

$$a = -\frac{1}{N}(1-2\alpha)(1-\delta)^2[(\xi_1+\xi_2)(\xi_3-\xi_1) - (\xi_2+\xi_3)^2]$$

$$b = -\frac{1}{N}(1-2\alpha)(1-\delta)^2[\xi_1(\xi_1+\xi_2) + 2\xi_2(\xi_2+\xi_3)] - \alpha(1-\delta)^2(\xi_1+\xi_2)(\xi_2+\xi_3) - \delta(2-\delta)(\xi_1+\xi_2)(\xi_3-\xi_1)$$

$$c = \frac{1}{N}(1-2\alpha)(1-\delta)^2 (\xi_2)^2 + \alpha(1-\delta)^2\xi_2(\xi_1+\xi_2) - \delta(2-\delta)\xi_1(\xi_1+\xi_2)$$

We can rewrite these coefficients as:

$$\begin{aligned} a &= \frac{1}{N}(\alpha^2 + 3\alpha\delta + 3\delta^2) + o(\alpha^2, \alpha\delta, \delta^2) \\ b &= -\frac{1}{N}(\alpha^2 + 4\alpha\delta + 7\delta^2) + o(\alpha^2, \alpha\delta, \delta^2) \\ c &= \frac{1}{N}4\delta^2 + o(\alpha^2, \alpha\delta, \delta^2) \end{aligned}$$

Then $f(0) = \frac{1}{N}4\delta^2 + o(\alpha^2, \alpha\delta, \delta^2) > 0$ and $f(1) = -\frac{1}{N}\alpha\delta + o(\alpha^2, \alpha\delta, \delta^2) < 0$. Therefore, there exists a unique solution for p_0 consistent with $\mathbf{p}$ being a distribution. That is, for $(\alpha, \delta) > 0$ and in a neighborhood of 0, there exists a unique solution for $\mathbf{p}$. Moreover that solution is differentiable because the coefficients of f are differentiable functions of the parameters.

Now that we have established properties of the mapping from (α, δ) to $\mathbf{p}$ in the neighborhood of $(\alpha, \delta) = 0$, we can proceed by differentiating $\mathbf{p}T^*AD = \mathbf{p}$ and evaluating the result at $\alpha = \delta = 0$ and $\mathbf{p}^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. This gives the following system of equations:

$$\frac{1}{N}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{pmatrix} dp_0 \\ dp_1 \\ dp_2 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ -1 & 2 \end{bmatrix}\begin{pmatrix} d\alpha \\ d\delta \end{pmatrix} \tag{A.10}$$

Because the first and third components of the left-hand side vector are identical, this system has solutions if and only if $d\delta = \frac{2}{3}d\alpha$. In other words, the direction $\delta = \frac{2}{3}\alpha$ is the unique direction such that $\mathbf{p} \to \mathbf{p}^*$ when $(\alpha, \delta) \to 0$. (Existence of this path can be confirmed from the quadratic equation $f(x) = 0$. In particular, if we set $\delta = \frac{2}{3}\alpha$ and let $\alpha \to 0$, then $f(x) \to 39x^2 - 61x + 16 = 0$, whose roots are $\frac{16}{13}$ and $\frac{1}{3}$.) Using $\sum dp_i = 0$, it follows from (A.10) that

$$dp_1 = \frac{N}{9}d\alpha \tag{A.11}$$

along the direction $\delta = \frac{2}{3}\alpha$. As we now show, this is enough to conclude that it is possible to raise welfare with some $(\alpha, \delta) > 0$. The argument is slightly different for $\beta > \beta^*$ and $\beta = \beta^*$.

$\beta > \beta^*$. Here, the participation constraints are slack at $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta) = (\mathbf{p}^*, \boldsymbol{\mu}^*, 0, 0)$ (see Lemma 1.2). Therefore, we can vary (α, δ) from 0 with $\delta = \frac{2}{3}\alpha$ while fixing $\boldsymbol{\mu} = \boldsymbol{\mu}^*$ without violating those constraints. It follows that the derivative of w with $\boldsymbol{\mu} = \boldsymbol{\mu}^*$along the $\delta = \frac{2}{3}\alpha$ path and evaluated at $(\mathbf{p}, \boldsymbol{\mu}, \alpha, \delta) = (\mathbf{p}^*, \boldsymbol{\mu}^*, 0, 0)$, is given by

$$\frac{dw}{d\alpha} = z(y^*)\left[\frac{dp_0}{d\alpha}(p_1+p_2) + p_0(\frac{dp_1}{d\alpha}+\frac{dp_2}{d\alpha}) + \frac{dp_1}{d\alpha}(p_1+p_2) + p_1(\frac{dp_1}{d\alpha}+\frac{dp_2}{d\alpha})\right]$$

At $\mathbf{p} = \mathbf{p}^*$, this becomes

$$\begin{aligned}\frac{dw}{d\alpha} &= \frac{z(y^*)}{3}\left[2\frac{dp_0}{d\alpha}+\frac{dp_1}{d\alpha}+\frac{dp_2}{d\alpha}+2\frac{dp_1}{d\alpha}+\frac{dp_1}{d\alpha}+\frac{dp_2}{d\alpha}\right] \\ &= \frac{2z(y^*)}{3}\frac{dp_1}{d\alpha}\end{aligned}$$

where the last equality uses $\sum \frac{dp_i}{d\alpha} = 0$. This and (A.11) give the result.

$\beta = \beta^*$. Here to maintain implementability as we vary (α, δ), we adjust the supports of the μ_{11} and μ_{12} components of $\boldsymbol{\mu}$, while keeping all other components at their $\boldsymbol{\mu}^*$ values. We let the support of μ_{11} and μ_{12} be degenerate at $(y_1, 1)$, where y_1 is determined by the binding producer participation constraint

$$(\mathbf{e}_2 - \mathbf{e}_1)AD\left(\frac{1}{\beta^*}I - T^*AD\right)^{-1}\mathbf{q}' - y_1 = 0 \tag{A.12}$$

with

$$\mathbf{q}' = \frac{1}{N}\begin{bmatrix} -(p_1+p_2)\,y^* \\ p_0u(y^*) + p_1\left[u(y_1) - y_1\right] - p_2y_1 \\ p_0u(y^*) + p_1u(y_1)\end{bmatrix}.$$

With $(p_0, p_1, p_2) = \mathbf{p}(\alpha, \frac{2}{3}\alpha)$ given by the unique differentiable solution established above, we can write (A.12) as $g(\alpha, y_1) = 0$, where $g(0, y^*) = 0$ and where

$$\partial g(0, y^*)/\partial y_1 = \frac{2\beta^*\left[\beta^* + (1-\beta^*)N\right]}{4\beta^*\left[\beta^* + 2(1-\beta^*)N\right] + 3(1-\beta^*)^2N^2} - 1 \in (-1, -\frac{1}{2}).$$

It follows from the implicit function theorem that for α in a neighborhood of 0, the y_1 that satisfies (A.12) is a differentiable function of α.

Since the $\boldsymbol{\mu}$ we are now using continues to have degenerate supports, the objective function (1.8) can be written as:

$$w = p_0(p_1+p_2)z(y^*) + p_1(p_1+p_2)z(y_1)$$

Then the derivative of welfare with respect to α evaluated at $\alpha = \delta = 0$ and $\mathbf{p} = \mathbf{p}^*$ is

$$\frac{dw}{d\alpha} = \frac{2z(y^*)}{3}\frac{dp_1}{d\alpha} + \frac{4}{9}z'(y^*)\frac{dy_1}{d\alpha}.$$

This differs from the corresponding expression for $\beta > \beta^*$ by the presence of an additional term. However, because the derivative $\frac{dy_1}{d\alpha}$ exists and because $z'(y^*) = 0$, this additional term is zero. Therefore, the result again follows from (A.11).■

Appendix B

Appendix to Chapter 2

Lemma B1. *If* $(\mathbf{p}, \boldsymbol{\mu})$ *is implementable and is such that (i)* $\mathbf{p}$ *has full support and the associated value function* $\mathbf{v}$ *is concave and (ii)* $\lambda_{ij}^k > 0$ *for some* $k \geq 1$ *in all meetings in which* $j \geq i$*, where* j *is money holdings of the consumer and* i *those of the producer, then* $(\mathbf{p}, \boldsymbol{\mu})$ *is connected.*

Proof. First, I show that concavity of the value function implies that trade in a meeting implies *willingness* to trade one unit in that meeting. Because $(\mathbf{p}, \boldsymbol{\mu})$ is implementable and $\mathbf{p}$ has full support, $\lambda_{ij}^k > 0$ implies that there exists $y \geq 0$ such that

$$kc\left(\frac{y}{k}\right) \leq c(y) \leq v_{i+k} - v_i \leq k(v_{i+1} - v_i)$$

and

$$ku\left(\frac{y}{k}\right) \geq u(y) \geq v_j - v_{j-k} \geq k(v_j - v_{j-1})$$

where in each display the second inequality follows from implementability. The outer inequalities imply willingness to trade $\frac{y}{k}$ for one unit of money.

Therefore, by hypothesis (ii) of the lemma, $\mathbb{G}_{(\mathbf{p},\boldsymbol{\mu})}$ contains all pairs of money holdings (i, j) with $j \geq i$. Now for each $i \in \{1, ..., B-1\}$ consider the set $\mathcal{I}_i \equiv \{i-1, i\}$. This is a block because $j = i+1$ and $j = i$ satisfy $j \geq i$ and because the associated permutation, $\sigma_{(\mathbf{p},\boldsymbol{\mu})}^i = \begin{pmatrix} i-1 & i \\ i & i-1 \end{pmatrix}$, has a unique orbit. Finally, these blocks are mutually reachable and jointly cover the set $\{0, ..., B-1\}$ of money holdings of producers. ■

Proposition 2.1 *The optimum problem* P_0 *has solutions.*

Proof. I need to show that the set of implementable and connected allocations, Γ, is nonempty and compact and that the objective W is continuous.

To see that Γ is nonempty, observe that autarky is always in Γ. The fact that $\operatorname{supp} \mu_{ij}^k = \{0\}$ for all nonempty measures μ_{ij}^k in $\boldsymbol{\mu}$ implies that the associated value function, $\mathbf{v}$, is zero and money has no value. Then, because $y = 0$ satisfies participation constraints for all i, j, k, autarky is implementable and connected.

To demonstrate compact valuedness of Γ, it suffices to show that Γ is closed valued and that all of the supports of measures μ_{ij}^k are bounded[1]. Consider a converging net of implementable and connected allocations, $(\mathbf{p}, \boldsymbol{\mu})_r$, and let $(\mathbf{p}, \boldsymbol{\mu})$ be its limit. The choice of the topology implies that $\mathbf{p}_r \to \mathbf{p}$ and $\left(\lambda_{ij}^k\right)_r \to \lambda_{ij}^k$ for all i, j, k. This and continuity of the function $g(\mathbf{p}, \boldsymbol{\lambda}) \equiv \mathbf{p}T - \mathbf{p}$ imply that $\mathbf{p}T = \mathbf{p}$ and the limiting distribution $\mathbf{p}$ is stationary.

To show that the limit $(\mathbf{p}, \boldsymbol{\mu})$ is implementable and connected, let us first consider all converging nets $(\mathbf{p}, \boldsymbol{\mu})_r$ such that starting from some r, $\mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})_r}$ is a constant set, denoted $\mathbb{Z}$. Then, because $\operatorname{supp} \mu_{ij}^k \subseteq \lim_r \left(\operatorname{supp}\left(\mu_{ij}^k\right)_r\right)$ and because $\mathbf{v}_r \to \mathbf{v}$, all participation constraints in (2.8) hold in the limit, and $(\mathbf{p}, \boldsymbol{\mu})$ is implementable and connected. To see that the constancy of $\mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})_r}$ is without loss of generality, consider an arbitrary converging net $(\mathbf{p}, \boldsymbol{\mu})_r$. Because for every r, $\mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})_r}$ is a subset of $\{0, ..., B-1\} \times \{1, ..., B\}^2$, which is finite, there exists some set $\mathbb{Z}$ and a subnet $(\mathbf{p}, \boldsymbol{\mu})_{r_s}$ with the property that $\mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})_{r_s}} = \mathbb{Z}$. Then, because a net converges if and only if every subnet converges to the same limit, $(\mathbf{p}, \boldsymbol{\mu})$, the constancy of $\mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})_r}$ is without loss of generality.

To demonstrate boundedness of supports, let us consider an arbitrary block $\mathcal{I}_l$ and write down incentive compatibility constraints (2.8), which pertain to selection $\sigma_{(\mathbf{p},\boldsymbol{\mu})}^l$ from $\Xi_{(\mathbf{p},\boldsymbol{\mu})}$:

$$c(\overline{y}_{ij}^1) \leq v_{i+1} - v_i, \qquad \textit{and} \qquad v_j - v_{j-1} \leq u(\underline{y}_{ij}^1)$$

all $i \in \mathcal{I}_l$. Because $\sigma_{(\mathbf{p},\boldsymbol{\mu})}^l$ is a permutation and selection from $\Xi_{(\mathbf{p},\boldsymbol{\mu})}$, for each j, which shows up in the above collection of the participation constraints, it is possible to find a unique i such that $j - 1 = \sigma_{(\mathbf{p},\boldsymbol{\mu})}^l(i)$. Adding up separately producer and consumer constraints and taking the latter into account, one obtains:

$$\begin{aligned} \sum_{i \in \mathcal{I}_l} c(\overline{y}_{i(\sigma^l(i)+1)}^1) &\leq \sum_{i_m \in \mathcal{I}_l} (v_{i+1} - v_i) \\ \sum_{i \in \mathcal{I}_l} \left(v_{\sigma^l(i)+1} - v_{\sigma^l(i)}\right) &\leq \sum_{i \in \mathcal{I}_l} u(\underline{y}_{i(\sigma^l(i)+1)}^1). \end{aligned} \tag{B.1}$$

Because $\sigma_{(\mathbf{p},\boldsymbol{\mu})}^l$ is a permutation, the two sums of gains from trades in (B.1) are equal, which yields:

$$\sum_{i \in \mathcal{I}_l} \left[u\left(\underline{y}_{i(\sigma^l(i)+1)}^1\right) - c\left(\overline{y}_{i(\sigma^l(i)+1)}^1\right)\right] \geq 0.$$

[1]Recall that if topology on the space of probability measures $\mathcal{P}(X)$ is the weak* topology, then $\mathcal{P}(X)$ is compact if and only if X is compact.

Note that by definition, $\underline{y}_{ij}^k \leq \overline{y}_{ij}^k$, which, together with the properties of utility and cost functions, yields:

$$c\left(\overline{y}_{ij}^1\right) - u\left(\underline{y}_{ij}^1\right) \leq \left(|\mathcal{I}_l| - 1\right)\left[u(y^*) - c(y^*)\right],$$

all $i \in \mathcal{I}_l$, where y^* is a unique solution to $u'(y) = c'(y)$ and $|\mathcal{I}_l|$, $|\mathcal{I}_l| \leq B$, is the size of block $\mathcal{I}_l$. Then properties of $u(y)$ and $c(y)$ guarantee that $\overline{y}_{ij}^1$ is finite for all $i \in \mathcal{I}_l$ and, because $\mathcal{I}_l$ is arbitrary, all supports that correspond to transfer of one unit and are a part of some $\sigma_{(\mathbf{p},\boldsymbol{\mu})}^l$ are bounded. Boundedness of all other supports follows immediately from consumer constraints in (2.8), from $v_n - v_m = \sum_{l=1}^{n-m}(v_{n-l+1} - v_{n-l})$ and $\bigcup_l \mathcal{I}_l = \mathcal{I}$, and from free disposal of money.

Finally, recall that $u(y)$ and $c(y)$ are continuous. Because the supports, Ω_{ij}^k, are bounded and because each of the spaces of probability measures μ_{ij} on $\mathbb{R}_+ \times \mathcal{K}_{ij}$ is endowed with the weak* topology, continuity of the objective W is immediate. ■

Lemma 2.1 *Let $(\mathbf{p}, \boldsymbol{\mu})$ be a non-autarkic solution to problem P_0. Let $\widetilde{P}_{(\mathbf{p},\boldsymbol{\mu})}^*$ be the associated perturbation problem $\widetilde{P}_{(\mathbf{p},\boldsymbol{\mu})}$ with the additional restriction that $\varepsilon_{ij}^k \equiv 0$. Let E be the set of all active constraints of problem $\widetilde{P}_{(\mathbf{p},\boldsymbol{\mu})}^*$ at $(\mathbf{p}, \boldsymbol{\mu})$ and assume that E is nonempty. Then there exists a nonempty subset E' of E and multipliers $\xi_s \geq 0$, one for each constraint in E', such that the gradient of the objective W can be written as a linear combination of the gradients of the constraints in E'.*

Proof. By assumption, $(\mathbf{p}, \boldsymbol{\mu})$ is non-autarkic, implementable and connected. I first show that $\underline{y}_{ij}^k > 0$ for all $(i, j, k) \in \mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})}$. Suppose to the contrary, that there exists a triplet $(i, j, k) \in \mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})}$ such that $\underline{y}_{ij}^k = 0$. By (2.8), it follows that in this case $v_j - v_{j-k} = 0$, which implies that $v_j - v_{j-1} = 0$. Because $(\mathbf{p}, \boldsymbol{\mu})$ is connected, there exists a block, $\mathcal{I}_l$, such that $j - 1 \in \mathcal{I}_l$. Then $v_j - v_{j-1} = 0$ implies that $\overline{y}_{i_1 j_1}^1 = 0$, where $i_1 = j - 1$ and $j_1 = \sigma_{(\mathbf{p},\boldsymbol{\mu})}^l(i_1) + 1$. From $\overline{y}_{i_1 j_1}^1 = 0$ it follows that $\underline{y}_{i_1 j_1}^1 = 0$, which implies that $v_{j_1} - v_{j_1 - 1} = 0$. Continuing this process recursively, one obtains

$$v_{\left(\sigma^l\right)^m (j-1)+1} - v_{\left(\sigma^l\right)^m (j-1)} = 0$$

for $m = 1, 2, \ldots$. Because $\sigma_{(\mathbf{p},\boldsymbol{\mu})}^l$ has a unique orbit, which spans the block $\mathcal{I}_l$, the process will cycle at $m = |\mathcal{I}_l| \leq B$, and then $v_{i+1} - v_i = 0$ for all $i \in \mathcal{I}_l$. Note that this implies that no production takes place in return for one unit of money in meetings which pertain to permutation $\sigma_{(\mathbf{p},\boldsymbol{\mu})}^l$.

If two blocks $\mathcal{I}_{l_1}$ and $\mathcal{I}_{l_2}$ overlap and one of the two has $\underline{y}_{ij}^k = 0$, then $v_{i+1} - v_i = 0$ for all $i \in \mathcal{I}_{l_1} \cup \mathcal{I}_{l_2}$. Finally, because connectedness requires that every block is reachable from any other and because these blocks jointly cover $\mathcal{I}$, the value function $\mathbf{v}$ is zero. That, in turn, implies that $(\mathbf{p}, \boldsymbol{\mu})$ is autarkic, a contradiction.

I now construct a vector $\mathbf{n}$ whose inner product with the gradients of the constraints in problem $\widetilde{P}^*_{(\mathbf{p},\boldsymbol{\mu})}$ is positive. The vector $\mathbf{n}$ is obtained by stacking vectors $\mathbf{l}^k_{ij}$, one for each $(i,j,k) \in \mathbb{Z}_{(\mathbf{p},\boldsymbol{\mu})}$. The construction of $\mathbf{l}^k_{ij}$ differs depending on whether μ^k_{ij} is or is not degenerate.

Let us first consider a nondegenerate measure μ^k_{ij}. Without loss of generality, I can assume that $a^k_{ij} \le c^k_{ij} \le b^k_{ij} \le d^k_{ij}$. Then the 4×2 block of the Jacobian matrix which corresponds to the perturbation of μ^k_{ij} can be written as

$$J = \begin{bmatrix} -\left(\mathbf{e}_{i+k} - \mathbf{e}_i\right) H \frac{\partial \mathbf{q}'}{\partial a^k_{ij}}, & \left(\mathbf{e}_j - \mathbf{e}_{j-k}\right) H \frac{\partial \mathbf{q}'}{\partial a^k_{ij}} - u'(a^k_{ij}) \\ -\left(\mathbf{e}_{i+k} - \mathbf{e}_i\right) H \frac{\partial \mathbf{q}'}{\partial b^k_{ij}}, & \left(\mathbf{e}_j - \mathbf{e}_{j-k}\right) H \frac{\partial \mathbf{q}'}{\partial b^k_{ij}} \\ -\left(\mathbf{e}_{i+k} - \mathbf{e}_i\right) H \frac{\partial \mathbf{q}'}{\partial c^k_{ij}}, & \left(\mathbf{e}_j - \mathbf{e}_{j-k}\right) H \frac{\partial \mathbf{q}'}{\partial c^k_{ij}} \\ c'(d^k_{ij}) - \left(\mathbf{e}_{i+k} - \mathbf{e}_i\right) H \frac{\partial \mathbf{q}'}{\partial d^k_{ij}}, & \left(\mathbf{e}_j - \mathbf{e}_{j-k}\right) H \frac{\partial \mathbf{q}'}{\partial d^k_{ij}} \end{bmatrix}. \tag{B.2}$$

Now let us take some vector $\mathbf{l} \equiv (-l_a, l_b, l_c, l_d) \in \mathbb{R}^4$. The scalar products of $\mathbf{l}$ and the columns of J are given by

$$l_d c'(d^k_{ij}) + \left(\mathbf{e}_{i+k} - \mathbf{e}_i\right) H \left(l_a \frac{\partial \mathbf{q}'}{\partial a^k_{ij}} - l_b \frac{\partial \mathbf{q}'}{\partial b^k_{ij}} - l_c \frac{\partial \mathbf{q}'}{\partial c^k_{ij}} - l_d \frac{\partial \mathbf{q}'}{\partial d^k_{ij}} \right)$$

and

$$l_a u'(a^k_{ij}) - \left(\mathbf{e}_j - \mathbf{e}_{j-k}\right) H \left(l_a \frac{\partial \mathbf{q}'}{\partial a^k_{ij}} - l_b \frac{\partial \mathbf{q}'}{\partial b^k_{ij}} - l_c \frac{\partial \mathbf{q}'}{\partial c^k_{ij}} - l_d \frac{\partial \mathbf{q}'}{\partial d^k_{ij}} \right).$$

Note that these products are positive if I can find $l_a > 0$ and $l_d > 0$ such that

$$l_a \frac{\partial \mathbf{q}'}{\partial a^k_{ij}} - l_b \frac{\partial \mathbf{q}'}{\partial b^k_{ij}} - l_c \frac{\partial \mathbf{q}'}{\partial c^k_{ij}} - l_d \frac{\partial \mathbf{q}'}{\partial d^k_{ij}} = \mathbf{0}. \tag{B.3}$$

To show that such a choice of $\mathbf{l}$ is possible, let us first write out the derivatives of the vector $\mathbf{q}$ (evaluated at $a^k_{ij} = c^k_{ij} = \underline{y}^k_{ij}$ and $b^k_{ij} = d^k_{ij} = \overline{y}^k_{ij}$). These are

$$\begin{aligned} \frac{\partial \mathbf{q}'}{\partial a^k_{ij}} &= \frac{\partial \mathbf{q}'}{\partial c^k_{ij}} = \frac{\lambda^k_{ij}}{2N} \left[p_j \underline{\eta}^k_{ij} \mathbf{e}'_i - p_i \underline{\gamma}^k_{ij} \mathbf{e}'_j \right], \\ \frac{\partial \mathbf{q}'}{\partial b^k_{ij}} &= \frac{\partial \mathbf{q}'}{\partial d^k_{ij}} = \frac{\lambda^k_{ij}}{2N} \left[p_j \overline{\eta}^k_{ij} \mathbf{e}'_i - p_i \overline{\gamma}^k_{ij} \mathbf{e}'_j \right] \end{aligned} \tag{B.4}$$

where

$$\underline{\gamma}_{ij}^k = \int_{\underline{y}_{ij}^k}^{\overline{y}_{ij}^k} u'(y)\frac{\overline{y}_{ij}^k - y}{\overline{y}_{ij}^k - \underline{y}_{ij}^k}\ d\mu_{ij}^k, \qquad \underline{\eta}_{ij}^k = \int_{\underline{y}_{ij}^k}^{\overline{y}_{ij}^k} c'(y)\frac{\overline{y}_{ij}^k - y}{\overline{y}_{ij}^k - \underline{y}_{ij}^k}\ d\mu_{ij}^k$$

$$\overline{\gamma}_{ij}^k = \int_{\underline{y}_{ij}^k}^{\overline{y}_{ij}^k} u'(y)\frac{y - \underline{y}_{ij}^k}{\overline{y}_{ij}^k - \underline{y}_{ij}^k}\ d\mu_{ij}^k, \qquad \overline{\eta}_{ij}^k = \int_{\underline{y}_{ij}^k}^{\overline{y}_{ij}^k} c'(y)\frac{y - \underline{y}_{ij}^k}{\overline{y}_{ij}^k - \underline{y}_{ij}^k}\ d\mu_{ij}^k. \tag{B.5}$$

Observe that because μ_{ij}^k is nondegenerate, all four integrals in (B.5) are strictly positive. Then, because the expected cost of production for producer and the expected utility of consumption for consumer show up only in the i-th and j-th entries of $\mathbf{q}$, (B.3) gives rise to the following linear 2-equation system:

$$\lambda_{ij}^k \begin{bmatrix} p_j\underline{\eta}_{ij}^k & p_j\overline{\eta}_{ij}^k \\ p_i\underline{\gamma}_{ij}^k & p_i\overline{\gamma}_{ij}^k \end{bmatrix} \begin{bmatrix} l_c \\ l_b \end{bmatrix} = \lambda_{ij}^k \begin{bmatrix} p_j\underline{\eta}_{ij}^k & -p_j\overline{\eta}_{ij}^k \\ p_i\underline{\gamma}_{ij}^k & -p_i\overline{\gamma}_{ij}^k \end{bmatrix} \begin{bmatrix} l_a \\ l_d \end{bmatrix}$$

Notice that $l_c = l_a$ and $l_b = -l_d$ is a solution, which implies that $l_a > 0$ and $l_d > 0$ is possible.

If measure μ_{ij}^k is degenerate, then the analogue of (B.3) is

$$l_g\frac{\partial \mathbf{q}'}{\partial g_{ij}^k} - l_h\frac{\partial \mathbf{q}'}{\partial h_{ij}^k} = \mathbf{0} \tag{B.7}$$

where the derivatives of $\mathbf{q}$ are

$$\frac{\partial \mathbf{q}'}{\partial g_{ij}^k} = \frac{\lambda_{ij}^k}{2N}\left[p_j\underline{\eta}_{ij}^k\mathbf{e}_i' - p_i\underline{\gamma}_{ij}^k\mathbf{e}_j'\right] \qquad \text{and} \qquad \frac{\partial \mathbf{q}'}{\partial h_{ij}^k} = \frac{\lambda_{ij}^k}{2N}\left[p_j\overline{\eta}_{ij}^k\mathbf{e}_i' - p_i\overline{\gamma}_{ij}^k\mathbf{e}_j'\right]$$

with $\underline{\gamma}_{ij}^k = \overline{\gamma}_{ij}^k = u'(\overline{y}_{ij}^k)$ and $\underline{\eta}_{ij}^k = \overline{\eta}_{ij}^k = c'(\overline{y}_{ij}^k)$. Therefore, (B.7) reduces to

$$\begin{array}{l} \lambda_{ij}^k p_j\overline{\gamma}_{ij}^k(l_g - l_h) = 0 \\ \lambda_{ij}^k p_i\overline{\eta}_{ij}^k(l_g - l_h) = 0 \end{array}.$$

Obviously, $l_g = l_h = 1$ satisfies this equation.

Thus, we have the vector $\mathbf{n}$ whose inner product with the gradients of the constraints in problem $\widetilde{P}^*_{(\mathbf{p},\boldsymbol{\mu})}$ is positive. Because the objective and constraints are continuously differentiable, existence of $\mathbf{n}$ is equivalent to existence of an open region $\mathcal{U}$ in the space of perturbations where all the constraints in (2.8) are relaxed. Because $(\mathbf{p}, \boldsymbol{\mu})$ solves P_0, it follows that the gradient of the objective is in the convex hull of the gradients of the active constraints. Finally, because the number of constraints in (2.8) does not exceed the number of degrees of freedom provided by perturbations, the edges of that convex hull are linearly independent. ■

ALEXEI DEVIATOV

Address: Department of Economics
608 Kern Graduate Building
The Pennsylvania State University
University Park, PA 16802

Telephone: (814) 865-1108
E-mail: Deviatov@psu.edu
Website: www.personal.psu.edu/ayd1

Curriculum Vitae

CITIZENSHIP:

- Russian Federation

EDUCATION:

- Ph.D., Economics, The Pennsylvania State University, University Park, PA, expected December 2002
- M.A., Economics, New Economic School, Moscow, Russia, July 1996, Master Thesis with Distinction
- B.Sc., Mechanics and Applied Mathematics, Moscow State University, Moscow, Russia, June 1992

Ph.D. THESIS:

- Money Creation in a Random-Matching Model of Money.
  Thesis Adviser: Neil Wallace

FIELDS:

- **Primary:** Macroeconomics, Monetary Economics
- **Secondary:** Econometrics, Game Theory

PAPERS:

- Another Example in which Lump-Sum Money Creation is Beneficial (joint with Neil Wallace), A*dvances in Macroeconomics*, Vol. 1: No. 1, (2001), Article 1.
- Optimal Money Creation in a Random-Matching Model with Ex post Individual Rationality.
- Money Creation and Optimal Pairwise Core Allocations in a Matching Model.

GRANTS & FELLOWSHIPS:

- TACIS Action for Cooperation in the Field of Economics Programme 1995 (T95-4001-R). Banking and central banking in emerging capital markets: case of studies of Russia and Kazakhstan. Grant Coordinator.

TEACHING EXPERIENCE:

- Instructor, Intermediate Macroeconomics, Summer 2001
- T.A., two-semester sequence of Ph.D. Macroeconomic Theory
- T.A., Introduction to Statistics and Econometrics

PRESENTATIONS & OTHER PROFESSIONAL ACTIVITIES:

- "Optimal Money Creation in a Random-Matching Model with Ex Post Individual Rationality"
  - Conference on Monetary Economics, Federal Reserve Bank of Cleveland, August 2001
  - Cornell-Penn State Macro Workshop, April 2001
- "Another Example in Which Lump-Sum Money Creation is Beneficial"
  - Cornell-Penn State Macro Workshop, April 2000

REFERENCES:

- Neil Wallace
- James Jordan
- B. Ravikumar