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Abstract

Auctions are among the oldest economic institutions in place—they have been used since antiquity to sell a wide variety of goods, and their basic form has remained unchanged. In this dissertation, I explore the efficiency of common auctions when values are interdependent—the value to a particular bidder may depend on information available only to others—and asymmetric. In this setting, it is well known that sealed-bid auctions do not achieve efficient allocations in general since they do not allow the information held by different bidders to be shared. Open auctions, however, do allow such sharing of information and form the subject of this thesis.

In the first essay, I present a model of the English auction. Typically, in an English auction, say of the kind used to sell art, the auctioneer sets a relatively low initial price. This price is then increased until only one bidder is willing to buy the object, and the exact manner in which this is done varies. In my model a bidder who drops out at some price can "reenter" at a higher price. This feature is realistic, but previous analyses ruled out the possibility of reentry for analytical convenience. The main result is that the English auction with reentry has an efficient equilibrium under weak conditions, a feature not shared by the standard model without reentry. The required conditions are the pairwise single-crossing property—known to be necessary for efficiency—and a new signal intensity condition. These conditions are much weaker than the conditions under which the standard English auction is efficient. Thus the modification is not only a more realistic model of the real-world auction but has superior theoretical properties.

The second essay, written jointly with my colleague, Oleksii Birulin, examines the question of when the standard English auction without reentry has an efficient equilibrium. Maskin (1992) shows that the pairwise single-crossing condition is necessary and sufficient when there are only two bidders. It is known, however, that this condition is not sufficient once there are three or more bidders. We identify a condition that is both necessary and sufficient for efficiency with a general number of bidders. This new condition, called generalized single crossing, is a multilateral version of pairwise single-crossing.

In the third essay, I examine some extensions to situations in which multiple identical objects are to be sold. I present an efficient multi-unit auction that consists of a number of sequential English auctions with reentry and show that this allocates efficiently under quite general circumstances. In each of the individual

auctions all bidders compete simultaneously in the open ascending price format. The distinctive feature of the mechanism is that winners are determined first, and then additional auctions are conducted to determine prices. Total number of auctions depends only on the number of goods to be allocated and not on the number of bidders.

Contents

List of Figures **vii**

Acknowledgments **viii**

1 Introduction **1**
- 1.1 English auctions with reentry . . . 2
- 1.2 Efficiency of the standard English auction . . . 4
- 1.3 Multi-unit open ascending price efficient auction . . . 6
- 1.4 Additional comments . . . 8

2 English auctions with reentry **9**
- 2.1 Preliminaries . . . 9
  - 2.1.1 An illustration . . . 9
  - 2.1.2 The setup . . . 11
  - 2.1.3 Main results . . . 11
- 2.2 English auction with reentry: a first look . . . 14
  - 2.2.1 Rules . . . 14
  - 2.2.2 Information and strategies . . . 15
  - 2.2.3 Equilibrium Strategies . . . 16
  - 2.2.4 Information processing . . . 17
    - Equilibrium play . . . 17
    - Off-equilibrium information processing . . . 18
    - Summary . . . 19
  - 2.2.5 Proof of the Main theorem: simplified case . . . 20
    - An efficient direct mechanism . . . 20
    - Proof . . . 21
- 2.3 English auction with reentry: general case . . . 22
  - 2.3.1 Complete rules for English auction with reentry . . . 23
  - 2.3.2 Equilibrium strategies . . . 24
  - 2.3.3 Proposed equilibrium information processing . . . 25
  - 2.3.4 Results . . . 27
  - 2.3.5 Is (A2) necessary? . . . 29
- 2.4 Conclusion . . . 30

3 Efficiency of the standard English auction **32**
3.1 Preliminaries . . . 32
3.1.1 The English auction . . . 32
3.1.2 Pairwise single-crossing . . . 33
3.2 Generalized single-crossing . . . 34
3.2.1 Results . . . 35
3.2.2 Examples . . . 35
3.3 Sufficiency . . . 37
3.4 Necessity . . . 39
3.4.1 Three-bidder English auction . . . 39
3.4.2 N-bidder English auction . . . 40
3.5 Conclusion . . . 43

4 Multi-unit open ascending price efficient auction **45**
4.1 Preliminaries . . . 45
4.1.1 The environment . . . 45
4.1.2 Overture . . . 47
4.2 An open ascending price efficient auction . . . 48
4.2.1 Structure . . . 48
4.2.2 An illustration . . . 49
4.3 Results . . . 50
4.4 Concluding remarks . . . 52
4.4.1 Variations . . . 52
4.4.2 Searching for the ultimate mechanism . . . 53

Appendix A English auctions with reentry **55**
A.1 Implications of the signal intensity condition (A2) . . . 55
A.2 Properties of $\mathbf{x}(p)$ and $\mathbf{w}(\cdot, p)$ . . . 56
A.3 Special procedures . . . 58
A.3.1 Case 1b. Early entry . . . 58
A.3.2 Case 3. Looking for a partition . . . 60

Appendix B Efficiency of the standard English auction **63**
B.1 Equivalence Lemma . . . 63
B.2 Sufficiency . . . 67
B.3 Necessity . . . 68
B.3.1 Necessity for three bidders . . . 68
B.3.2 Necessity: general case . . . 71
B.3.3 Extensions . . . 76
B.3.4 Necessity for the minimal subset: resumed . . . 78

Bibliography **80**

List of Figures

2.1 Signal intensity vs. single-crossing 12
2.2 Player 1's decision 17

A.1 v_1 is decreasing 61

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Chapter 1

Introduction

Auctions are among the oldest economic institutions in place—they have been used since antiquity to sell a wide variety of goods. Their basic form has remained unchanged—an extraordinary feat stemming from simplicity in practice and remarkable economic properties. The most important of which are capacities to generate high revenues to the seller and to allocate objects efficiently.

Efficiency—an issue of whether at the end of an auction the object is sold to the buyer who values it the most—was almost exclusively a theorist's concern. Nowadays the emphasis is changing. Large-scale privatization of state-owned assets in Russia, China, United Stated, United Kingdom, and many other countries is probably the most quintessential area feeding an interest in the search of mechanisms that obtain efficiency.

A mechanism that lies in the heart of economics—"invisible hand" or competitive market—requires that no participant has a significant effect on the price, so it needs a sufficiently large number of agents to function. Quite often, and especially in privatization, this is not the case. For instance, in the European 3G Telecom auctions for the rights to use certain range of frequencies for telecommunications (see Klemperer (2002)) or in the similar U.S. spectrum auctions there were only a few potential buyers per license. The choice to auction seems only natural.

As known from theory, when no buyer's private information affects how others value the assets—the private values setting, many auction forms are efficient: second-price sealed bid or open ascending price (English) auctions when one object is for sale, Vickrey or Ausubel auctions when many similar objects are for sale, many other forms under specific circumstances (Vickrey (1961), Milgrom & Weber (1982), Ausubel (1997)). When sizeable in value assets are being offered for sale, the interdependent values setting—a buyer's private information can be essential in how others are evaluating the assets—seems to be a more plausible description of the environment. For instance, Porter (1995) reports that in U.S. offshore oil and gas lease auctions, "firms are permitted to gather seismic information prior to the sale. . . . On-site drilling is not permitted, but firms owning adjacent tracts can conduct off-site drilling, which may be informative." Clearly, other firms might be interested in the results of such private tests.

In the interdependent values setting, achieving efficiency is a highly non-trivial problem. All simple auction forms with a remarkable exception of the English auction, that are efficient in the private values setting, cease to be efficient if the values are interdependent. The English auction is efficient under quite restrictive conditions. Known mechanisms that are efficient under most general conditions are complex and almost prohibitively unsuitable for actual implementation.

Why does the seller have to use the efficient mechanism in the first place? If resale is possible, by Coase theorem efficient allocation will be eventually achieved no matter what is the initial mechanism. Coase theorem does not apply under uncertainty. And, with the absence of well-established market for the auctioned products, resale, if happens, most probably will be a result of bilateral or multilateral bargaining under incomplete information. As shown by Myerson & Satterthwaite (1983), an efficient outcome cannot be obtained in general under these conditions. So, if efficiency is the goal, it is important that the mechanism of choice itself is efficient.

This thesis is devoted to the search of a simple and efficient mechanism as broadly applicable as possible. In the first essay, I modify the standard model of an English auction by allowing reentry. As a result, I obtain a model that is more realistic and that is efficient in selling a single object under much weaker conditions than the standard model is. The second essay, written jointly with my colleague, Oleksii Birulin, examines the question of when the standard single-object English auction without reentry has an efficient equilibrium. In the third essay, I extend analysis to the case of multiple identical objects being offered for sale. I present an efficient mechanism that consists of a number of sequential English auctions. The distinctive feature of it is that the winners and the prices they have to pay are determined separately.

1.1 English auctions with reentry

The English, or open ascending price auction, is the most common auction format, first reference to its use dates back to Babylon, 500BC. In an English auction—say of the kind used to sell art—the auctioneer sets a relatively low initial price. This price is then increased, and the exact manner in which this is done varies. In some auctions the bidders themselves call out prices and the auction stops when some bid is not raised any further. In other auctions, the auctioneer raises prices in small increments until only one bidder is willing to buy the object.

The English auction is especially attractive as a mechanism because of its strategic simplicity. The current price is commonly known and a bidder need only to decide whether she wants to buy the object at that price or not. Once the price reaches a level that would make it unprofitable for the bidder to purchase the object she drops out of the bidding. The practical simplicity of the mechanism, when combined with interest stemming from its widespread use—both historical and current—has made the English auction a frequent object of study.

Game theoretic models of the English auction abstract away from many details of the real-world institution. For instance, it is assumed that the price is raised in a continuous fashion by an idealized automated auctioneer—a price clock. The bidders need only indicate whether they are active, that is, willing to buy the object at the current price, or not. More importantly, the strategic problem facing bidders is simplified by requiring that once a bidder drops out of the auction while others are still active, her decision is irrevocable—there are no circumstances under which she can win the object. In what follows, I refer to this as a *standard* English auction. This model was introduced in Milgrom & Weber (1982).

In this essay I consider a more realistic model of the real-world English auction. Specifically, bidders are given the freedom to indicate their willingness to buy at any price and at any time. In particular, a bidder may drop out or exit at a low price and then "reenter" the auction at a higher price.

This model is used to study the problem of allocating a single object among a number of asymmetric, privately informed bidders with *interdependent* values—the *ex post* value of the object to a particular bidder depends on both his own signal and the signals of other bidders. The main question concerns circumstances under which the allocation will be *efficient*, that is, the object will end up in the hands of the bidder with the highest *ex post* value. Note that with private values—when the value depends only on a bidder's own signal—the standard English auction is efficient, that is, it has an equilibrium which always results in an efficient allocation.

It is well known that the possibility of achieving efficiency at all via any mechanism hinges on the so-called single-crossing condition, requiring that a bidder's signal has a greater influence on his value than on some other bidder's value. It is also well known that the standard English auction is efficient when the number of buyers is two and the single-crossing condition is satisfied. When the number of bidders is more than two the standard English auction may not be efficient. An example in Section 2.1.1 below shows that this may happen under quite natural circumstances.

In this essay I argue that the failure of the English auction to achieve efficiency results not from some defect in the institution itself but rather from the way it has been traditionally modeled. In particular, the inefficiency stems from the rule that once a player exits, this decision is permanent. She is not allowed to reenter at a higher price. In the context of private values this restriction is of no consequence. With interdependent values, however, the price carries valuable information about other players' signals, which may be revealed after a player has exited, causing him to regret his exit decision. Thus the restriction of no reentry is substantive.

The main result of this essay is that once the rules of the standard model are amended to allow reentry, the English auction has an efficient equilibrium as long as the *single-crossing* condition and a new condition, called the *signal intensity* condition are satisfied.

Both conditions pertain to situations in which a group of players has the same value and this is the highest among all players. Both specify circumstances in which as a result of an increase in his own signal, a particular member of this group becomes the sole player with the highest value. The single-crossing condition requires that this must happen if the *signals* of the rest of the group are kept fixed whereas the signal intensity condition requires that this happen if their *values* are kept fixed. In this sense, the conditions are dual to one another.

The single-crossing condition is known to be necessary for efficiency and so cannot be avoided. In any case, it appears to be quite weak. I will argue that the signal intensity condition is also relatively weak. It is weaker than the conditions under which the standard English auction has an efficient equilibrium. Moreover, when there are only two players it is implied by the single-crossing condition itself.

A second result of this essay is that whenever the standard English auction has an efficient equilibrium it can be duplicated as an equilibrium in the English auction with reentry. In particular, if the efficient equilibrium of the standard English auction is "regular" as in Maskin (1992) or Krishna (2001), the equilibrium in the English auction with reentry identified in the first result is isomorphic—all actions and information processing are identical.

Both as a model of the real-world institution and on theoretical grounds, the English auction with reentry dominates the standard English auction.

1.2 Efficiency of the standard English auction

The standard model of the English auction, commonly referred to as Milgrom-Weber model, was introduced and extensively analyzed in the classical paper by Milgrom & Weber (1982). The assumption of no reentry, made there, allowed to obtain analytically attractive results and to focus attention on fundamental issues. Irrevocable exits rule out any kind of strategic interactions among bidders and so simplify significantly equilibrium analysis. Moreover, in the original setup of the model, this assumption is not particularly restrictive—in any equilibrium without reentry no bidder ever regrets the decision to exit and so would not want to come back even if it were possible.

If reentry is allowed, bidders can enter and exit strategically. For instance, by exiting earlier a bidder might deceive others into believing that her signal is low; by entering earlier than others expect a bidder might indicate that she is determined to win this object in an attempt to discourage others from continuing. More importantly, with ability to exit and enter freely, the space of messages a bidder can send to others is significantly increased, which potentially makes it easier for the bidders to collude. The English auction with reentry has an efficient equilibrium in spite of all these possibilities. In the actual auction bidders might choose to play differently. Because the standard model is more restrictive, in situations when an option of reentry is not being exercised in equilibrium—and so the equilibrium play in the English auction with or without reentry is the same—the standard model might be a better choice. Therefore, it is quite important to describe conditions under which option of reentry is not exercised and the standard model is efficient.

It is well known (see Maskin (1992)) that the pairwise single-crossing condition is necessary for any mechanism to be efficient. For the standard English auction with two bidders single-crossing is also a sufficient condition for efficiency, with three or more bidders it is not (see Section 2.1.1). Krishna (2001) recently proposed a pair of conditions sufficient for the existence of an efficient equilibrium in the N-bidder case, however, these conditions put relatively strong restrictions on the value functions.

The main objective of this essay is to bring forth the exact requirement for efficiency—the condition that is both necessary and sufficient for existence of an efficient equilibrium in the N-bidder English auction without reentry. We present a fairly intuitive condition—*generalized single-crossing*, and show that it is both necessary and sufficient.

The intuition behind our generalized single-crossing condition extends the intuition which is embodied in the pairwise single-crossing to the case of N bidders. The pairwise single-crossing imposes the following: if starting from a signal profile, where the values of two bidders are equal and maximal, we slightly increase the signal of one of the bidders, his value becomes the highest. This implies that the private information held by the bidders affects their valuations more than the valuations of their competitors. Our condition requires the following: if starting from the signal profile, at which values of some group of bidders are equal and maximal among all the bidders, we slightly increase the signals of some subset of the group, no bidder outside of the subset can have the highest value. The generalized single-crossing condition both implies the pairwise single-crossing and reduces to it in the case of two bidders.

To better understand the role of our condition it is instructive to compare it to the sufficient conditions proposed in Krishna (2001). Krishna introduces two conditions that are sufficient for the existence of an efficient equilibrium: cyclical crossing and average crossing. The English auction proceeds in stages, the stage is characterized by the set of the bidders who are still active. Once one of them drops out, the next stage begins. Cyclical crossing implies that the bidders drop out in the order of their values. Average crossing implies that the value of the bidder who exits, is less than the *average* of the values of the bidders who remain active. Generalized single-crossing implies that the value of the exiting bidder is less than the *maximal* of the values of the remaining bidders.

Two main results of the essay are *Necessity*: if the generalized single-crossing is violated at some interior signal profile, then no efficient equilibrium in the N-

bidder English auction without reentry exists; and *Sufficiency*: if value functions satisfy the generalized single-crossing both in the interior and on the boundary of the signals' domain, then there exists an efficient ex-post equilibrium in the N-bidder English auction without reentry.

1.3 Multi-unit open ascending price efficient auction

In the first two essays the case of single object being offered for sale is analyzed. In practice, many similar objects are often being auctioned simultaneously.[1] Direct simultaneous or sequential application of single-object efficient auctions for sale of multiple objects does not produce an efficient outcome except under very special circumstances. Up to date, only a few efficient constructions are known, all with a relatively limited practical application.

In this essay, I consider a situation when K identical objects are offered for sale. Each bidder's marginal valuation for an additional object is non-increasing in the number of objects in possession, so the objects are substitutes for the bidders. Each bidder receives a one-dimensional private signal that might be of value to the others.[2] To *allocate efficiently* in this setting is to assign objects to the bidders with K highest *ex post* marginal values.

I present a mechanism that consists of a number of sequential single-object English auctions. The distinctive feature of the mechanism is that the winners and the prices they pay are determined separately. The main result is that the proposed auction with the English auction with reentry as the basic mechanism has an ex post equilibrium that is efficient under multi-unit generalizations of single-crossing and signal intensity conditions. Similarly to the single-object case, single-crossing condition is necessary for efficiency—without it the direct efficient mechanism or generalized Vickrey-Clark-Groves (VCG) mechanism does not exist, signal intensity is relatively weak. The mechanism is flexible enough, any single-object efficient mechanism under corresponding conditions—English auction with or without reentry, Vickrey auction in the case of two bidders—can serve as a building block.

The main difficulty in constructing efficient mechanism in the interdependent values setting is that it must guarantee that the winners pay Vickrey prices. Perry & Reny (1999) contains a revenue equivalence theorem establishing the uniqueness of the marginal prices under ex-post incentive compatibility. Therefore the prices

[1] For example, six 20 MHz licenses and six 10 MHz licenses will be offered for sale at FCC (Federal Communications Commission) Auction 31, scheduled to begin on January 14, 2003.

[2] As Jehiel & Moldovanu (2001) show, if the private information received by the bidders prior to the start of the auction is multi-dimensional, no efficient mechanism of any kind exists unless this information can be summarized by the one-dimensional signal.

paid by the winners have to be the same as in VCG mechanism. The existence of the direct efficient auction under single-crossing condition was indirectly shown in Crémer & McLean (1985), Ausubel (1999) contains complete description of the mechanism. Major disadvantage of the direct mechanism is the requirement that the auctioneer knows everything that bidders know about themselves. Dasgupta & Maskin (2000) offer a detail-free "contingent bid" mechanism: a bidder submits the price she is willing to pay for the good conditional on the realized values of all the other bidders, the auctioneer then calculates a fixed point, declares winners and prices. Both the direct mechanism and the "contingent bid" mechanism allocate efficiently provided the multi-unit single-crossing is satisfied. Dasgupta & Maskin (2000) mechanism is a theoretically "perfect" mechanism, it works under most general conditions and not necessarily for identical objects, but it is not practical.

The Vickrey price a winner is obliged to pay for her lth object is obtained by the following counterfactual exercise. True signal of the winner is being lowered until it reaches a level at which her lth marginal valuation becomes equal to the $(K-l+1)$th highest of the marginal valuations of the others. If the bidder were to have this particular signal, the auctioneer would have been indifferent between allocating an object to the bidder (lth for her) or to somebody else. The price the bidder is obliged to pay for this object is the lth marginal value calculated at this "virtual" signal.

In the private values setting, in the process of finding Vickrey prices, the ranking of the marginal values of the others is preserved when the signal of one of the bidders changes. This allows for the open ascending counterpart of Vickrey auction, proposed in Ausubel (1997), to work. Ausubel auction, which is a multi-unit extension of the English auction, relies on clinching rule. A bidder clinches (wins) an object at the price when some other bidder reduces demand and the sum of total demands of the others becomes lower than the number of objects available. Immediately thereafter, the total number of objects available and the demand of the winner are reduced by one. If no other bidder clinches at this price, the auction continues until all objects are allocated.

In the interdependent values setting, since the ranking of the marginal values of the others can change when a signal of a particular bidder increases, Ausubel auction cannot provide Vickrey prices and so cannot be efficient. Perry & Reny (2001) recognize that Ausubel auction is efficient in the important special case of only two bidders in the auction. They propose an open ascending price auction, in which at a given price bidders submit directed demands against every other bidder. The bidder clinches an object when the sum of demands of the others directed at her becomes less than the number of remaining objects. When calculating demand against bidder j, bidder i uses information she inferred about private signals of every other bidder except j from the prices other bidders reduced their demands at. Bidders' values are supposed to be non-degenerate for all K objects, so at the start the demand of every bidder against any other bidder is K. As a result, at the moment a bidder clinches an object, the signals of the others are truthfully revealed in the proposed equilibrium. Since any pair of bidders literally engages in the two-bidder efficient auction with the signals of the others being fixed, winners clinch their objects at Vickrey prices, and so Perry & Reny (2001) obtain efficient auction.

Perry & Reny (2002) presents another construction of the efficient mechanism in the interdependent values setting, which is a generalization of the Vickrey auction and is a sealed bid relative of Perry & Reny (2001).[3] Recognizing that in the interdependent values setting Vickrey auction is efficient with only two bidders present, they propose a two stage auction. In the first stage players make public announcements in a way the others infer their true signals. For example, they may announce their signals directly. In the second stage, all possible pairs of bidders are formed, each pair plays a Vickrey auction. Objects are allocated one at a time, a bidder wins an object if her bid for this object—lth highest of her bids if she has already won $l-1$ objects—in each of the $N-1$ auctions she participates is higher than opponent's bid for this object. After all objects are assigned, prices are determined. A winner of L objects pays the sum of Kth, $(K-1)$th, ..., $(K-L+1)$th highest among the bids submitted against her in all $N-1$ auctions she participated combined.

Perry & Reny (2002), Perry & Reny (2001), and Eső (1999) share a common feature—bidders compete in pairs. While this allows to achieve efficiency, the number of bids or demands of a particular bidder depends on N. The construction presented here has a significant advantage over these mechanisms, all bidders compete simultaneously in each of the single-object auctions. Therefore, at any given moment any bidder has at most one demand for a single object, and so complexity of the bidding does not exceed complexity of the English auction. Note also that the auction presented here unlike the other mechanisms in the case $K = 1$ reduces to a single-object English auction.

1.4 Additional comments

For all basic facts and a brilliant coverage of classic results in auction theory the reader is referred to Krishna (2002). Maskin (2001) provides a survey of the major facts and problems related to efficiency analysis in auctions

Experimental studies suggest that bidders "figure out" equilibrium behavior in the English auction relatively quickly (see Kagel, Harstad & Levin (1987) and Kirchkamp & Moldovanu (2002)) compared to the sealed-bid formats. This is an additional argument toward focusing on the open ascending price constructions in the search for simple and efficient mechanisms.

[3] Eső (1999) contains a mechanism that is very similar to the one Perry & Reny (2002), in it each of the pairs plays English auction instead of the sealed-bid Vickrey auction.

Chapter 2

English auctions with reentry

2.1 Preliminaries

2.1.1 An illustration

Consider a hypothetical "real-life" situation. There is a painting for sale, and there are three bidders: two experienced dealers and a novice art collector. The dealers are experts and have an accurate assessment of the worth of the object to them. The novice collector is unsure about the value and indeed cares about how the dealers value the painting.

Suppose that the painting is sold by means of an open ascending price auction and consider the following scenario. All three participants are active—they keep raising the price or indicating their interest in some other manner—until the price reaches a level such that the novice collector stops bidding. This may be because the price reached the maximum amount he was willing to spend for a nice, but of unknown origin, piece of art, and there was no indication from the behavior of the others that the painting is genuine. But now suppose that the two dealers continue to outbid each other for some time. This may cause the collector to reevaluate his decision—the aggressive bidding of the dealers may convince him that the painting is genuine. Thus the collector may wish to reenter the auction.

The following example is an abstract model of preferences of the collector (player 1) and the dealers (players 2 and 3) intended to capture the scenario outlined above. This example is a simplified version of the example from Perry & Reny (2001).

Example 2.1. *There is a single good for sale and there are three interested players who receive signals s_1, s_2 and s_3 from $[0,1]$ and have the following value functions*

$$\begin{aligned} V_1(s_1,s_2,s_3) &= s_1 + s_2 s_3, \\ V_2(s_1,s_2,s_3) &= s_2, \\ V_3(s_1,s_2,s_3) &= s_3. \end{aligned}$$

Consider what happens in the standard model of the English auction. Obviously, it is dominant for each of players 2 and 3 (the dealers) to stay in until price reaches their signals. Suppose $s_2 = s_3 = \frac{1}{2}$ then $V_1 = s_1 + \frac{1}{4}$, and as long as $s_1 < \frac{1}{4}$ both dealers have the highest value. If $s_2 = s_3 = \frac{4}{5}$ then player 1 (the collector) has the highest value if $s_1 > \frac{4}{5} - \frac{16}{25} = \frac{4}{25}$. Suppose the signal of the collector lies between $\frac{1}{4}$ and $\frac{4}{25}$, say $s_1 = \frac{2}{9}$. If the object is to be allocated efficiently when both dealers' signals are $\frac{1}{2}$ and also when they are $\frac{4}{5}$, player 1 has to drop-out before $p = \frac{1}{2}$, but at the same time stay in at least until $p = \frac{4}{5}$. So, full efficiency is infeasible.[1]

We now argue that the standard English auction is unable to achieve efficiency because it forbids the collector from reentering the auction once he drops out. We showed above that while it is imperative for the collector to exit before the price reaches $\frac{1}{2}$ there are situations in which he would like to be active at $p = \frac{4}{5}$. Giving players the option to exit and enter at will allows player 1 to behave optimally in both situations.

In particular, suppose that player 1 adopts the following strategy in the modified English auction with reentry. Whenever the other players are both active, calculate $w_1(s_1, p) = V_1(s_1, p, p) = s_1 + p^2$—the minimal inferred value at price p. Stay active as long as $w_1(s_1, p) > p$, and be inactive if $w_1 < p$.

For $s_1 = \frac{2}{9}$ the collector has to solve $s_1 + p^2 = p$ and this results in two solutions: $p' = \frac{1}{3}$ and $p'' = \frac{2}{3}$. The proposed strategy implies that when both dealers are active, player 1 should exit at $p' = \frac{1}{3}$ and then reenter at $p'' = \frac{2}{3}$; when some dealer exits first then play as in the "regular" equilibrium of the two-bidder English auction (as in Maskin (1992)).

As we can see, if the current price is below p' or above p'' then the minimal inferred value of the good to the collector exceeds the price. If one of the dealers drops out at such a price, then the auction continues as a two-bidder English auction, and the collector may win the good while paying less than the value, so the expected profit is positive. Thus it is optimal for him to be active when $w_1 > p$.

If the first dealer to drop out, say player 2, does so at a price p_2 in between p' and p'', there is no opportunity for the collector to win the good and pay less. Indeed, $w_1(s_1, p_2) = V_1(s_1, p_2, p_2) = V_3(p_2)$, so $V_1(s) < V_3(s) = s_3$ by the single-crossing condition since $s_3 > p_2$. Therefore, if the collector is active at p_2 and would win the good later, he would have to pay s_3 which is more than the value of the good to him. So, it is optimal for the collector to stay inactive whenever $w_1 < p$.

[1] If the collector chooses to stay until some other player exits first when all players are still actively bidding, then an efficient allocation can be achieved almost always, that is there exists an almost efficient equilibrium which fails to allocate efficiently only at some signal profiles with $s_2 = s_3$. This fact, however, is specific to the particular forms of value functions of players 2 and 3. In the full example in Perry & Reny (2001) with $V_2 = s_2 + \frac{1}{2}s_1$ there are no efficient (or almost efficient) equilibria.

It is easy to see that the resulting allocation is always efficient.

2.1.2 The setup

There is a single good for sale. There are N potential buyers, each of whom receives a signal $s_i \in [0,1]$. Given the signals $\mathbf{s} = (s_1, s_2, \ldots, s_N)$ the value of the object to the player i is $V_i(\mathbf{s})$.[2] Since the value depends on all the signals, this is a situation of *interdependent* values. The valuation functions V_i are assumed to be commonly known to the players.

Let $\mathcal{N}$ denote the set of buyers. For any subset $\mathcal{I}$ of buyers denote $\mathbf{s}_{\mathcal{I}} = (s_i)_{i\in\mathcal{I}}$— the set of signals of players from $\mathcal{I}$. We also write $\mathbf{s} = \mathbf{s}_{\mathcal{N}}$, $\mathbf{s}_{-\mathcal{I}} = \mathbf{s}_{\mathcal{N}\setminus\mathcal{I}}$, and $\mathbf{s} = (\mathbf{s}_{\mathcal{I}}, \mathbf{s}_{-\mathcal{I}})$. Signals are distributed according to the joint probability density function $f(s_1, \ldots, s_N)$ which is assumed to be positive on all of $[0,1]^N$.

For any player i his value function V_i has the following properties: V_i is twice-differentiable, $V_i(0,\ldots,0) = 0$ and

$$\forall i \ \frac{\partial V_i}{\partial s_i} > 0, \qquad \forall j \neq i \ \frac{\partial V_i}{\partial s_j} \geq 0. \tag{2.1}$$

Definition 2.1. Given signals $\mathbf{s}$ the *winners' circle* $\mathcal{I}(\mathbf{s})$ is the set of players with maximal values. Formally,

$$i \in \mathcal{I}(\mathbf{s}) \Longleftrightarrow V_i(\mathbf{s}) = \max_{j\in\mathcal{N}} V_j(\mathbf{s}). \tag{2.2}$$

If the allocation is to be efficient, then the object must go to a member of the winners' circle.

It is assumed that if $s_i = 0$ and there exists a j such that $s_j > 0$ then $i \notin \mathcal{I}(\mathbf{s})$. In words, a player with the lowest possible own signal cannot have the highest value.

It is also assumed that for any $\mathbf{s}$ and a subset of players $\mathcal{J}$ of the winners' circle $\mathcal{I}(\mathbf{s})$ the set of functions $\mathbf{V}_{\mathcal{J}}(\mathbf{s})$ is regular at $\mathbf{s}_{\mathcal{J}}$, that is $\det DV_{\mathcal{J}} \neq 0$, where $DV_{\mathcal{J}}$ is the matrix of partial derivatives (Jacobian), $DV_{\mathcal{J}} = \left(\frac{\partial V_i(\mathbf{s})}{\partial s_j}\right)_{i,j\in\mathcal{J}}$.

2.1.3 Main results

The main result relies on the following two assumptions.

A1 (*single-crossing*) For all $\mathbf{s}$ and any pair of players $\{i,j\} \subset \mathcal{I}(\mathbf{s})$,

$$\frac{\partial V_i(\mathbf{s})}{\partial s_i} > \frac{\partial V_j(\mathbf{s})}{\partial s_i}. \tag{2.3}$$

[2] We denote vectors and sets by bold and caligraphic letters correspondingly. The dimensionality of the vector is omitted when it is clear, $\mathbf{a} \gg \mathbf{b}$ ($\mathbf{a} \geqslant \mathbf{b}$) denotes that $a_i > b_i$ ($a_i \geq b_i$) in every component.

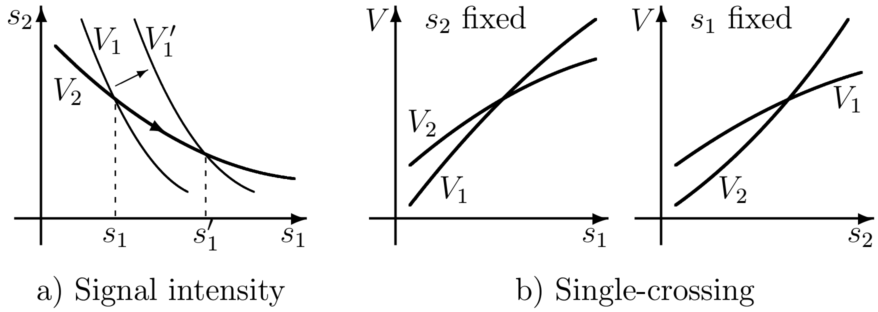

Figure 2.1: Signal intensity vs. single-crossing

The single-crossing condition says that the influence of a player's own signal on his value is greater than the influence of his signal on another player's value. It is required to hold only at signal profiles where both players' values are maximal. The single-crossing condition plays a key role in the analysis of auctions with interdependent values. In particular, if it is not satisfied then there may be no mechanism that allocates the object efficiently (Maskin (1992)).

A2 (*signal intensity*[3]) For all $\mathbf{s}$ and $i \in \mathcal{I}(\mathbf{s})$ there exists an $\varepsilon > 0$ such that for all $\mathbf{s}'$ satisfying (i) $s_i < s_i' < s_i + \varepsilon$; (ii) $\forall j \in \mathcal{I}(\mathbf{s}) \setminus \{i\}$, $V_j(\mathbf{s}') = V_j(\mathbf{s})$ and (iii) $\forall k \notin \mathcal{I}(\mathbf{s})$, $s_k' = s_k$, it is the case that $\mathcal{I}(\mathbf{s}') = \{i\}$.

The signal intensity condition requires that if we increase the signal s_i of some member i of the winners' circle $\mathcal{I}(\mathbf{s})$ and change the signals s_j of other players $j \in \mathcal{I}(\mathbf{s})$ in a way that their values are unchanged, offsetting the effect of the change in s_i, then i's value goes up (the signals of all players $k \notin \mathcal{I}(\mathbf{s})$ are kept fixed). In other words, the combined effect on player i's value, directly from the increase in his own signal and indirectly through the changes in signals of other members of the winners' circle, is positive: the direct effect outweighs the indirect effect.

Figure 2.1 illustrates the difference between the signal intensity and the single-crossing conditions when there are only two bidders in the winners' circle, say 1 and 2. Suppose $\mathbf{s}$ is such that $V_1(\mathbf{s}) = V_2(\mathbf{s}) = V$. If we increase s_1 along the isovalue curve $V_2 = V$ to s_1', the assumption requires that player 1's value V_1' be greater than V. Thus, as shown in Figure 2.1a, the isovalue curves of player 1 must be steeper than the isovalue curves of player 2, that is,

$$-\frac{\partial V_1}{\partial s_1} \Big/ \frac{\partial V_1}{\partial s_2} < -\frac{\partial V_2}{\partial s_1} \Big/ \frac{\partial V_2}{\partial s_2}. \tag{2.4}$$

[3]This condition is so named because it is similar to the notion of factor intensity in the literature on international trade.

As shown in Figure 2.1b, the single-crossing requires that

$$\frac{\partial V_1}{\partial s_1} > \frac{\partial V_2}{\partial s_1} \text{ and } \frac{\partial V_2}{\partial s_2} > \frac{\partial V_1}{\partial s_2}, \tag{2.5}$$

and clearly (2.5) implies, but is not implied by, (2.4). Thus if $\mathcal{I}(\mathbf{s})$ has only *two* players then the signal intensity condition is implied by the single-crossing condition. This reasoning also makes apparent why the single-crossing condition does not imply the signal intensity condition once there are *three or more* players. Now it is possible to move along a player's isovalue surface in a way that the pairwise comparisons of partial derivatives no longer provide enough information.

It is useful to view the signal intensity condition as a *dual* to the single-crossing condition. Single-crossing prescribes what should happen if we fix the *signals* of everyone else in the winners' circle and increase the signal of one particular player—he should become the sole member of the winners' circle. Signal intensity prescribes what should happen if we fix the *values* of everyone else in the winners' circle and increase the signal of one particular player—again, he should become the sole member of the winners' circle.

For a further discussion on the role and implications of the signal intensity assumption see also Section 2.3.5 and Appendix A.1.

Now, I can state the main result of this paper.

Theorem 2.1 (Main). *Under the single-crossing and signal intensity conditions the English auction with reentry has an ex post equilibrium that is efficient.*[4]

The English auction with reentry is, from the perspective of efficiency, superior to the English auction without reentry. First, whenever the latter has an efficient equilibrium so does the former, and this is our second main result. Moreover, as the example in the introduction shows, there are circumstances in which the standard English auction fails to have an efficient equilibrium while the auction with reentry does.

The main result relies on two conditions. The single-crossing condition (A1) is known to be necessary for efficiency. The signal intensity condition (A2) is not necessary but in the dynamic framework of the auction proposed here, guarantees that no player suffers from any regret from exiting or entering. It is well recognized that the question of efficiency is closely linked to the absence of any *ex post* regret. Moreover, the signal intensity assumption is weaker than any other currently known condition under which the standard English auction without reentry possesses an efficient equilibrium.

In fact we can show

[4] A Bayesian-Nash equilibrium is called an *ex post* equilibrium if no player wants change his actions once all the information about the actual realization of the signals becomes commonly known. This notion is closely related to the notion of the robust equilibrium (see Dasgupta & Maskin (2000)), which requires that the strategies remain optimal under any initial distribution of signals. In fact, the presented equilibrium is also robust.

Theorem 2.2. *If the standard English auction without reentry has an efficient equilibrium then so does the English auction with reentry.*

2.2 English auction with reentry: a first look

Allowing the possibility that bidders can exit or enter the auction at will leads to some complications that need to be treated with care.[5] In this section I attempt to convey the main ideas as simply and clearly as possible by, as a first step, neglecting some of these complications. The cost of doing this is not great—if players follow the suggested equilibrium the complications occur only with probability zero. For example, it may be that player 1 who is inactive up to that point, decides to enter at some price p, while player 2 who is active, decides to exit at the same price p. Such simultaneous exits or entries can create complications. The decision of player 1 to enter may have been based on the fact that player 2 was active. If player 1 knew that player 2 was going to exit at p—an event viewed as bad news—he would have stayed out himself. Thus once player 1 enters he may want to exit at the lowest price $p' > p$ which, of course, cannot be defined.

Such complications can also arise as a result of deviations from equilibrium play and as a first pass at the problem, we prohibit such deviations. Later sections are devoted to taking care of all these details.

2.2.1 Rules

The English auction, as modeled in the literature following Milgrom & Weber (1982), is conducted as follows.

1. The auctioneer sets a low initial price, say zero, and constantly raises it. It is convenient to think of an automatic price clock publicly showing the current price.

2. At any price, each player is either active or not. All players are active at a price of zero and a player can exit from the auction at any time. The activity statuses of all players are commonly observed and known. *A player who exits cannot reenter the auction.*

3. The auction ends (the price clock stops) when at most one person is active. The winner is the only remaining person (or is randomly chosen among those who exited last) and pays the price at which the last exit took place.[6]

[5] In the paper the words of each of the following groups mean the same and are used interchangeably: exit, drop out, become inactive; enter, reenter, come back, become active; be in, be active; be out, be inactive.

[6] We shall complete the description of the rules by specifying the outcome when two or more

This paper explores the implications of amending only one of the rules of the English auction. Rule 2 is modified so that the last clause reads:

2M. *A player can exit and reenter the auction at will.*

2.2.2 Information and strategies

It is assumed that at any price p, the history of exits and entries of all players that took place before p is common knowledge. Denote this public information as $H(p)$. Clearly, which players were active or inactive at a particular (time) price before p can be easily reconstructed.

A strategy β_i of player i determines the price level $p_i \equiv \beta_i(s_i, H(p)) > p$ at which player i is going to change his status, enter if he was out, or exit if he was in.

In order to state the suggested equilibrium strategies in the modified English auction, some definitions are needed. These incorporate the inferences that players make about the signals of other players from their exit and entry behavior.

Definition 2.2. Let $\mathcal{A}$ be the set of active players at some price p. The *estimated minimal signals at* p, $\mathbf{x}(p) = (x_1(p), x_2(p), \ldots, x_N(p))$ are defined as follows: (i) $\mathbf{x}(\mathbf{0}) = \mathbf{0}$; (ii) if $j \notin \mathcal{A}$, $x_j(p) = x_j(p_j)$, where $p_j < p$ is the price at which player j last exited; (iii) for all active players, $\mathbf{x}_{\mathcal{A}}(p) = (x_i(p))_{i\in\mathcal{A}}$ is a solution to the system of equations

$$\mathbf{V}_{\mathcal{A}}(\mathbf{x}_{\mathcal{A}}(p), \mathbf{x}_{-\mathcal{A}}(p)) = p. \tag{2.6}$$

Let $w_i(s_i, p) \equiv V_i(s_i, \mathbf{x}_{-i}(p))$ be the *estimated minimal value* of the good to player i when the current price is p.

Suppose that for all i, $\mathbf{x}_{-i}(p)$ represents the minimal signals that i estimates other players to have received. If $\mathbf{x}(p)$ is a consistent set of such estimates, then they have to satisfy (2.6). To see why notice that if for some active player i, $w_i(x_i(p), p) > p$, that is, if the estimated minimal value exceeds the current price, then for all signals $s_i < x_i(p)$ which are close to $x_i(p)$, we have $w_i(s_i, p) > p$. This means that it would be reasonable for player i to be active when his signal were $s_i < x_i(p)$, contradicting the fact that $x_i(p)$ is the minimal signal that i may have received. On the other hand, if $w_i(s_i, p) < p$ then if player i were to play conservatively—and the strategies specified below will call on a player to do so—he would be inactive at p. Thus the estimated minimal signals at p, $\mathbf{x}(p)$ have to satisfy (2.6).

Definition 2.3. The *potential minimal signals at* p, $\mathbf{y}(p) = (y_1(p), \ldots, y_N(p))$ are defined as follows: (i) if $i \in \mathcal{A}$, $y_i(p) = x_i(p)$; (ii) if $j \notin \mathcal{A}$, $y_j(p)$ is the solution to $w_j(y_j(p), p) = p$ or $y_j(p) = 1$ if such a solution does not exist.

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bidders decide to remain active forever, so an auction continues indefinitely. In this case any such bidder receives $-\infty$.

Note that for any active player i, $w_i(x_i(p), p) = w_i(y_i(p), p) = p$. For an inactive player j, $w_j(x_j(p), p) < p$ and thus, $y_j(p) > x_j(p)$. In words, for an inactive player j, $y_j(p)$ is the signal at which j's estimated minimal value equals the price p. Thus $y_j(p)$ represents the hypothetical value of j's signal which would lead him to become active at p, all else being equal.

2.2.3 Equilibrium Strategies

The proposed strategies are based on the estimated minimal values $w_i(s_i, p)$ derived from the history of play up to price p.[7]

Player i should

1. if $w_i(s_i, p) > p$, then be active at p;
2. if $w_i(s_i, p) < p$, then be inactive at p;
3. if $w_i(s_i, p) = p$ and for ε small enough, $w_i(s_i, p-\varepsilon) \leq p-\varepsilon$ and $w_i(s_i, p+\varepsilon) > p+\varepsilon$, then become active at p;
4. if $w_i(s_i, p) = p$ and for ε small enough, $w_i(s_i, p-\varepsilon) \geq p-\varepsilon$ and $w_i(s_i, p+\varepsilon) < p+\varepsilon$, then become inactive at p.

   The strategy can be summarized as: *Exit (Enter) whenever your estimated minimal value crosses the price from above (below).*

Equivalently, the strategies can be defined based on $\mathbf{x}(p)$ and $\mathbf{y}(p)$—Exit, whenever $x_i(p) = y_i(p)$ reaches s_i from below, Enter when $y_i(p)$ reaches s_i (and $x_i(p)$) from above.

Remark 2.1. The reader may wonder whether the strategies proposed above are well defined. The strategies prescribe how a player should change his status based on the estimated minimal values w_i. But these values depend in turn on the estimated minimal signals x_i which depend on the status of a player—active or inactive. The strategies do not explicitly say whether player i is considered to have maintained his status at p while $w_i(s_i, p)$ crosses p, or to have changed his status. As shown below (Lemma A.2 in Appendix A.1), this slight ambiguity causes no difficulties. Assumption (A2) guarantees that if $w_i(s_i, p)$ crosses p when player i is considered active it also crosses p when player i is considered inactive.

Let us revisit example 2.1 for an illustration of the strategies. Figure 2.2 shows the decision rule for player 1 provided players 2 and 3 are still active. The proposed strategy for player 1 with a signal s_1 prescribes that he exit at p' and reenter at p''.

[7] Note that $\mathbf{x}_{-i}(p)$ can be regarded as the set of beliefs player i has at price p. In particular, player i believes at p that $s_j = x_j(p)$ for any inactive player j and for any active player k,

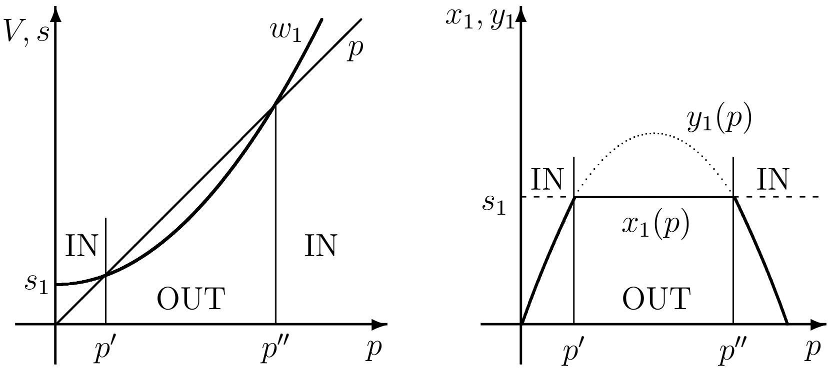

Figure 2.2: Player 1's decision

Remark 2.2. It is possible that for a particular s_i, $w_i(s_i, \cdot)$ would only touch and not cross the 45° line at p. For the moment, let us set aside all such signals s_i in addition to all signal profiles $\mathbf{s}$ that result in simultaneous exits and (or) entries. The probability of such signal profiles is zero. This is because if we fix player i, no matter what are the signals of others, the probability that player i is involved in one of these two situations is zero; a small perturbation in the signal of player 1 necessarily pulls him out of any possible equalities.

2.2.4 Information processing

Equilibrium play

First, I will show that if everyone plays as suggested all $x_i(p)$, $y_i(p)$, $w_i(s_i, p)$ are uniquely defined and satisfy: for any active player i, $w_i(x_i(p), p) = p$; for any inactive player j who did not exit at p, $x_j(p) = s_j < y_j(p)$ and $w_j(s_j, p) < p$.

Suppose, some player k exited or entered at price p^0, and the estimated minimal signals at that price are $x_i^0(p^0)$. Suppose also that these estimates satisfy: (i) for any active player $i \in \mathcal{A}$, $V_i\left(\mathbf{x}^0_{\mathcal{A}}(p^0), \mathbf{x}^0_{-\mathcal{A}}(p^0)\right) = p^0$; (ii) for any inactive player $j \neq k$, $V_j\left(\mathbf{x}^0_{\mathcal{A}}(p^0), \mathbf{x}^0_{-\mathcal{A}}(p^0)\right) < p^0$; (iii) for player k, $V_k\left(\mathbf{x}^0_{\mathcal{A}}(p^0), \mathbf{x}^0_{-\mathcal{A}}(p^0)\right) = p^0$.

Starting from p^0 consider the system (2.6) for the current set of active players $\mathcal{A}$. By full differentiation of (2.6) we get

$$DV_{\mathcal{A}} \cdot \frac{d\mathbf{x}_{\mathcal{A}}}{dp} = \mathbf{1}, \tag{2.7}$$

where $DV_{\mathcal{A}}$ is the matrix of partial derivatives $\left(\frac{\partial V_i(\mathbf{s})}{\partial s_j}\right)_{i,j\in\mathcal{A}}$ and $\mathbf{1}$ is the column of 1s of size $\#\mathcal{A}$.

$s_k \in [x_k(p), 1]$. The joint distribution of $\mathbf{s}_{-i}$ at p is the ex ante distribution of $\mathbf{s}_{-i}$ conditional on $s_k \in [x_k(p), 1]$ for all active $k \neq i$, $s_j = x_j(p)$ for all inactive $j \neq i$, and the true s_i. The proposed strategies depend only on $\mathbf{x}_{-i}(p)$ and not on the distribution of signals.

System (2.7) together with $\frac{dx_j}{dp} = 0$ for any $j \notin \mathcal{A}$ gives a direction of change of the estimated minimal signals $\mathbf{x}(p)$ at the price p

$$\begin{cases} \frac{d\mathbf{x}_{\mathcal{A}}}{dp} = (DV_{\mathcal{A}})^{-1} \cdot \mathbf{1}, \\ \frac{d\mathbf{x}_{-\mathcal{A}}}{dp} = \mathbf{0}. \end{cases} \tag{2.8}$$

For a given initial condition $\mathbf{x}(p^0) = \mathbf{x}^0(p^0)$ system (2.8) has a unique continuous solution $\mathbf{x}(p)$. This solution is considered until some other player changes his status at some $p' > p^0$. After that, the terminal $\mathbf{x}(p')$ will be considered as the initial condition for the new system (2.8), corresponding to the new set $\mathcal{A}$ of active players.

Equilibrium play proceeds as follows. Initially, all players are active at $p = 0$, that is, $\mathcal{A} = \mathcal{N}$. The system (2.8) is solved for $\mathcal{A} = \mathcal{N}$ with the initial condition $\mathbf{x}(\mathbf{0}) = \mathbf{0}$ and all players are active until for some player k, $x_k(p)$ reaches s_k and $w_k(s_k, p)$ crosses p from above. (If $s_k = 0$ player k will exit right before the start of the auction). Player k exits at that point and a new system (2.8) is solved, and remains in effect until some other player exits or player k comes back and so on until the auction ends.

Lemma A.3 in Appendix A.2 guarantees that $\mathbf{x}(p)$ is well-defined: it is positive for all $p > 0$. It is also the case that $w_i(x_i(p), p) = p$ for all (active) players $i \in \mathcal{A}$, and $w_j(x_j(p), p) \leq p$ for all inactive players $j \notin \mathcal{A}$, with the inequality being strict at non-decision points. This follows from the fact, that when player j exits, his $w_j(x_j, p)$ does cross p from above, that is he does not regret exiting. This property is guaranteed by the signal intensity condition and is proved in Lemma A.2 in Appendix A.1. Similarly, there are no regrets about entries.

Off-equilibrium information processing

Suppose player j exits at p_j with estimated minimal signal $x_j(p_j)$. Suppose that at $p > p_j$ his potential estimated signal $y_j(p)$ touches $x_j(p) = x_j(p_j)$ from above. This means that other players expect player j to reenter at p. If player j deviates from the suggested strategy in a manner that is not anticipated, some special attention is needed. There are two possibilities.

Case 1a. Player j does not reenter at p, entering later or, possibly, not at all. Let us call such player j a *dormant* player and define $d_j \equiv x_j(p) = x_j(p_j)$. From the moment $y_j(p)$ reaches d_j from above and player j does not enter as the other players expect him to do, the following procedure applies. Player j is added to $\mathcal{A}$ as if he has entered (is considered by the others as if he is active), his $x_j(p)$ is estimated accordingly, so it is changing with p. This continues until one of the following three situations happens: player j enters; $x_j(p)$ reaches d_j or the auction ends. If player j enters then since $x_j(p)$ is appropriate, that is $V_j(\mathbf{x}(p)) = p$, no additional changes are necessary. If $x_j(p)$ reaches d_j, player j is removed from $\mathcal{A}$ as if he has exited.

Case 1b. Player j enters earlier than expected, at some p when $y_j(p) > x_j(p_j)$. In this case other players adjust player j's estimated minimal signal $x_j(p)$ slowly until a price p' such that $w_j(x_j(p'), p') = p'$. At the same time it is ensured that the auction does not end before the full adjustment takes place. This is achieved by adjusting $x_j(p)$ in a way that guarantees that for some player $i \in \mathcal{A}$, $x_i(p)$ is decreasing with p, so that $x_i(p)$ stays below s_i and player i does not exit. As an exception, full or partial instantaneous adjustment of $x_j(p)$ is possible if the value functions of the other players from $\mathcal{A}$ do not depend on s_j for some range of s_j. For a full description of the procedure see Appendix A.3.1.

The other possible information impasse happens when player j stays too long.

Case 2. Player j does not exit at p_j when $x_j(p_j) = 1$ and $w_j(1,p) < p$ for $p > p_j$. The solution is simple, player j is removed from $\mathcal{A}$ for the purposes of solving (2.8). Nothing happens if he exits in the process. If $w_j(1,p)$ crosses p from below, player j is added back to $\mathcal{A}$.

Summary

During the auction at each price level p for each player i, the other players have an estimate of the signal of player i that is consistent with the current status of player i. If player i is inactive that estimate is presumably the inferred true value of player i's signal. If player i is active, since it is plausible to think that he would have been active with a higher signal as well, the other players have an estimate of minimal possible signal of player i with which it is worth to be active. If all players think like that and behave accordingly, then one can obtain a set of such estimates for all the players at all price levels.

Normally, once a player, say j, exits, the estimate of his presumably true signal s_j is obtained. Given that the others can project future behavior of player j. In particular, they can forecast if and when player j is going to enter again and all other changes of i's status. In fact, the others may *simulate* the behavior of player j after the moment of first exit. It is exactly what they are supposed to do in the presented equilibrium if player j does not enter as projected.

At the same time, players *"forget"* the previously obtained estimate s_j once player j who exited earlier becomes active once again. All estimates of the lowest possible true signals of currently active players are based only on their current status and *not* on the history of previous actions. This is an important feature since the estimate $x_j(p)$ of the lowest possible signal of player j does affect in general the price player j has to pay for the good if he wins. If a higher estimate is taken into account than is necessary to win at a particular realization of signals, player j would have an incentive to shade his actions and exit for the first and all other times earlier, which, in turn, may render the allocation to be inefficient.

The manner in which information is processed is quite intuitive: players update the information they have about the other players only when they are active or become active, that is when players express their willingness to buy the good at the current price.[8]

2.2.5 Proof of the Main theorem: simplified case

An efficient direct mechanism

It is well-known that there exists a direct mechanism which allocates efficiently based on the work of Crémer & McLean (1985). The single-crossing assumption (A1) is sufficient for the existence of an efficient direct mechanism. This mechanism is of interest because equilibrium play in the model of the English auction with reentry will be outcome equivalent to the direct mechanism.

For a given signal profile $\mathbf{s}$, for any player i define a signal s_i^* to be a unique solution to the equation

$$V_i(s_i^*, \mathbf{s}_{-i}) = \max_{j \neq i} V_j(s_i^*, \mathbf{s}_{-i}), \tag{2.9}$$

or set $s_i^* = 1$ if even at $s_i^* = 1$ player i does not have the highest value.

For every player i define p_i^* to be

$$p_i^* = \max_{j \neq i} V_j(s_i^*, \mathbf{s}_{-i}). \tag{2.10}$$

The uniqueness of the solution to (2.9) is guaranteed by the single-crossing condition and by the fact that $V_i(0, \mathbf{s}_{-i}) < \max_{j \neq i} V_j(0, \mathbf{s}_{-i})$ for $\mathbf{s}_{-i} \neq \mathbf{0}$. The solution is interior, $s_i^* < 1$, when $V_i(1, \mathbf{s}_{-i}) \geq \max_{j \neq i} V_j(1, \mathbf{s}_{-i})$; otherwise $s_i^* = 1$.

The prices in (2.10) were identified in Maskin (1992).

Lemma 2.1. *Suppose $i \in \mathcal{I}(\mathbf{s})$ for a signal profile $\mathbf{s}$. Then $p_i^* = \min_j p_j^*$, $V_i(\mathbf{s}) \geq p_i^*$ and the inequality is strict when $\mathcal{I}(\mathbf{s}) = \{i\}$. For any $j \notin \mathcal{I}(\mathbf{s})$, $V_j(\mathbf{s}) < p_j^*$.*

Proof. By the single-crossing, $s_i \geq s_i^*$ and $s_j < s_j^*$, where $j \notin \mathcal{I}(\mathbf{s})$. This implies $V_i(\mathbf{s}) \geq p_i^*$ and $V_j(\mathbf{s}) < V_i(\mathbf{s}) \leq V_i(s_j^*, \mathbf{s}_{-j}) \leq p_j^*$. □

Lemma 2.2. *There exists a direct mechanism that ex post implements an efficient outcome under (A1). Players submit their signals, the winner is player i (or randomly chosen among those) who has the lowest p_i^* and his payment is p_i^*. Others pay* 0.

Proof. It is straightforward from lemma 2.1. □

[8] Note that in special Case 1a, when some player j does not enter when expected, his $x_j(p)$ is lowered temporarily. However, the others remember what it was at the moment of last exit and will keep changing $x_j(p)$ until it reaches that level or player j becomes active. Therefore, no actual updating of information takes place.

Proof

Now we can prove the Main theorem with specified restrictions on initial signal profiles and under a weaker equilibrium concept in which the set of allowed strategies is somewhat limited to prohibit deviations that can result in simultaneous exits and (or) entries. In addition, the strategies that result in an infinite number of entries and exits are also not considered for a moment. An example of such strategy is: exit at $p - \varepsilon$, enter at $p - \varepsilon^2$, exit at $p - \varepsilon^3$, and so on, for some small $\varepsilon > 0$. Therefore, p is a limiting point of switches of status for this particular player, and whether the player is active or inactive right before p cannot be determined. The rules of the auction have to be amended to deal with such cases, which is done in the next section. Meanwhile, it should be noted that in the equilibrium play no matter what the initial $\mathbf{s}$ is, only a finite number of exits and entries by any player is possible, which follows from the boundedness of the first derivative of $w(s_i, p)$ with respect to p, which in turn is guaranteed by the properties of value functions.

Proof of the Main theorem with restrictions. Suppose all bidders except possibly player i follow the strategies proposed in Section 2.2.3. If player i wins at a price p, he has to be active at p and have $w_i(x_i(p), p) = p$. There are only two situations considered previously, Case 1b and Case 2, when he can be active but have $V_j(\mathbf{x}(p)) < p$. If it is Case 2 and he wins, player i obviously pays more than the value of the good to him since $V_j(1, \mathbf{s}_{-j}) < p$. If it is Case 1$b$, by construction of the special procedure, his $x_j(p)$ will be adjusted to have $w_j(x_j(p), p)) = p$ before the auction ends.

If player i wins then the signals of the others have to be truthfully revealed: $x_j(p) = s_j$ for all $j \neq i$. In addition, for any player $j \neq i$, $V_j(x_i(p), \mathbf{s}_{-i}) = w_j(s_j, p) \leq p$ with $w_k(s_k, p) = p$ for at least one player k (who exited at p). This means that $x_i(p) = s_i^*$ and $p = p_i^*$ as defined by (2.9) and (2.10) and player i cannot in any way affect the price he is obliged to pay.

If $s_i > s_i^*$, then by Lemma 2.1 it is player i who has the highest value. He will win the auction and get $V_i(\mathbf{s}) - p_i^* > 0$ if he follows the suggested strategy. If he deviates, his payoff is the same if he still wins the object. Obviously, he is worse off if he does not win. If $s_i < s_i^*$, player i is not supposed to win. If he does, by Lemma 2.1, $V_i(\mathbf{s}) < p_i^*$, so he has to pay more than the value of the object to him.

Obviously, if player i knows what are the true signals of the others, he cannot do better. There is no way he can manipulate the auction, say, to make some players exit without revealing their signals—the procedures of Cases 1a and 1b take care of that. And, if he wins, the signals of the others will be revealed anyway. □

By the revelation principle there exists a direct mechanism that is outcome equivalent to the equilibrium of the English auction with reentry constructed above. This is the mechanism, outlined in the Lemma 2.2.

Remark 2.3. The proposed strategy of player j is the *unique* best response if every other player follows the proposed equilibrium strategies.

This is quite obvious. Suppose player j decides to switch status at p: exit earlier (enter later) or exit later (enter earlier) than p_j, specified by the proposed strategy. One can find an appropriate initial signal profile $\mathbf{s}$ such that all other players exit simultaneously in between p and p_j. As a result, player j either wins the auction but has to pay more than the value of the good to him or does not win when it is efficient and thus profitable for him to do so.

Remark 2.4. If everyone plays according to suggested strategies the set of functions $x_i(p)$ for all active players $i \in \mathcal{A}$ satisfies the same equations as the set of inverse bidding functions $\sigma_i(p)$ for $\mathbf{s}_{-\mathcal{A}} = \mathbf{x}_{-\mathcal{A}}(p)$ as defined in Krishna (2001). Thus parts of the strategies that deal with exits in the proposed equilibrium closely resemble those specified in the regular equilibrium (Krishna (2001) or Maskin (1992)).

In fact, whenever the standard English auction possesses an efficient equilibrium the English auction with reentry has an efficient equilibrium in which the behavior of the players is the same. In particular, in that case no exiting player exercises the option to reenter.

2.3 English auction with reentry: general case

In this section we are considering the most general case, that is, we define equilibrium strategies and information processing for all possible strategy profiles. All kinds of deviations are allowed as well.

In order to accommodate all possible deviations, some of the rules governing play in the English auction with reentry need to be amended. Basically, the amendments are the following. The first amendment (to Rules 1 and 2) is that we now require that in order to change his or his status, a player must *stop* the price clock and a player has the option of doing so at any time. Stopping the clock is a simple and convenient solution to the problems of simultaneous exits and/or entries. For instance, player 1 may want to stay in the auction if some other player 2 is active at p, but wants to be out if 2 exits at p. Without stopping the clock at p player 1 will want to exit the auction as soon as possible if player 2 exits at p. Of course, there is no “as soon as possible” price level. Stopping the clock allows player 1 to exit at the same price as 2.

The second amendment (to Rules 2 and 3) is that the auctioneer can temporarily *suspend* players if their behavior becomes disruptive, in the sense, that they cannot determine their statuses. As an example, suppose that players 1 and 2 exit at some price p, while player 3 enters at the same price. Decisions of players 1 and 2 to exit were based on the assumption that the other would remain active while player 3 would remain inactive. Once they learn that player 3 has entered, there is the possibility that player 1 may want to be active only when player 2 is active but player 2 may want to be active if player 1 is not. With the clock stopped at p, neither player 1 nor 2 can make up their minds, causing a stalemate. In this case, we give the auctioneer the authority to suspend both players 1 and 2 so that the auction may continue.

Finally, Rule 5 below disallows an infinite number of exits and entries by the same player. The auctioneer sets an upper bound to the number of switches of status.

2.3.1 Complete rules for English auction with reentry

1. The auctioneer sets a low initial price, say zero, on a price clock and this is raised.

2. While the clock is ticking, each player is either active, inactive or *suspended.* All players are active at a price of zero. The activity statuses of all players are commonly observed and known. The status of the player can change only when clock is stopped. *A player who is not suspended can stop the clock at any time (say by raising his or his hand).* A suspended player cannot stop the clock.

3. Once the clock stops:

   (a) All suspensions are lifted.

   (b) All players are asked to indicate their intention to be active or inactive once the clock restarts. Players may also indicate that they are undecided. These intentions are communicated to the auctioneer simultaneously and observed by all.

   (c) Players who indicated their intention to be inactive or who were undecided are asked if they wish to change their intentions in light of the information revealed in (b). Undecided players are only allowed to indicate either that they now wish to be active or that they are still undecided.

   (d) If some player changes his intention, then (c) is repeated.

   (e) If no player changes his intention, then these are considered the current statuses. If only one player is undecided, then he must reconsider and must choose to be either active or inactive. Others are not allowed to change their statuses.

   (f) The auctioneer suspends all undecided players if there are at least two active players.

4. The clock restarts as long as there are more than two active players once all players have chosen as in Rule 3. Otherwise, the auction ends and the good is sold at the price showing on the clock. It is awarded to:

(a) The only remaining active person, if there is such a player.

(b) A randomly chosen player among those who exited last, if no active players remain.

5. If the number of exits and entries of a player after his last suspension (if he has been suspended before) or after the start of the auction (if he has never been suspended before) exceeds a commonly known number, pre-announced by the auctioneer, the player is automatically suspended at that price.[9]

To see how these rules work consider an auction with 5 players. Players 1 and 2 are active, player 3 is inactive and players 4 and 5 are suspended at the current price, the price clock is ticking. Later, at a price p, player 1 stops the clock. According to Rule 3 the suspensions of players 4 and 5 are lifted, all players are asked to indicate their intentions as in Rule 3*b*. Suppose players' intentions are as follows: player 2 wants to be active, players 1 and 5—inactive, players 3 and 4 remain undecided. These intentions are observed by all players. Rule 3*c* applies.

If no players are willing to change their intentions then player 2 as the only active player is declared the winner, the auction ends. Suppose instead, player 3, observing that player 2 will be active, decides to be active as well, and he is the only player to change his intentions in the second stage. As a result the intentions of the players are: players 2 and 3 want to be active, players 1 and 5 want to be inactive, player 4 is undecided.

If in the next stage no player changes his intention, by Rule 3*e* player 4 is asked to decide on his status. Once he decides, all players' statuses are fixed and, by Rule 4, the clock restarts. Lastly, suppose in the third stage, player 1 becomes undecided, while player 5 changes his intentions to be active, and no other player wants to change his intentions after this. Then both undecided players 1 and 4 get suspended by Rule 3*f*, the auction restarts with players 2, 3, and 5 being active. Note that Rule 3*c* ensures that after a finite number of stages no player will change his intention.

2.3.2 Equilibrium strategies

The equilibrium strategies proposed in Section 2.2.3 need to be amended only slightly to take account of the fact that some additional decisions have to be made when the price clock stops.

We also have to specify how the auctioneer determines the bound on the number of exits and entries. Actually, any large number will serve its purpose. If in addition, the auctioneer knows the value functions but not the actual signals, the

[9] Note that the mechanism defined by these rules is "detail-free"—the auctioneer is not required to have or acquire any information about the value functions, distribution of the signals or the actual realization of the signals to make the auction work.

auctioneer can compute the number of exits and entries by all the players on the assumption that they follow the proposed strategies. This number depends on the particular realization of the signals $\mathbf{s}$ but it suffices to choose the maximum over all signals.[10]

Player i should behave as follows:

1. if $w_i(s_i, p) > p$, then be active at p or indicate intention to be active if the clock is stopped at p;

2. if $w_i(s_i, p) < p$, then be inactive at p or indicate intention to be inactive if the clock is stopped at p;

3. if $w_i(s_i, p) = p$, then stop the clock at p, if for ε small enough, $w_i(s_i, p-\varepsilon) \leq p-\varepsilon$ and $w_i(s_i, p+\varepsilon) > p+\varepsilon$, or $w_i(s_i, p-\varepsilon) \geq p-\varepsilon$ and $w_i(s_i, p+\varepsilon) < p+\varepsilon$;

4. if $w_i(s_i, p) = p$, and the clock is already stopped,

   (a) then indicate that he is undecided;

   (b) and if forced to make a decision (when i is the only undecided player):

      i. then choose to be inactive, if for ε small enough, $w_i(s_i, p+\varepsilon) \leq p+\varepsilon$;

      ii. then choose to be active, if for ε small enough, $w_i(s_i, p+\varepsilon) > p+\varepsilon$.

2.3.3 Proposed equilibrium information processing

For the general process of calculation of $\mathbf{x}(p)$, $\mathbf{y}(p)$ and $\mathbf{w}(\cdot, p)$ see Section 2.2.4. The only situations which need additional attention are those that deal with the estimation of $\mathbf{x}(p)$ when there are some suspended players and those that involve a particular type of the off-equilibrium behavior, Case 1b, which was considered in Section 2.2.4.

Similar to the analysis of Section 2.2.4, suppose first that all players behave as suggested. If there are no suspended players, $\mathbf{x}(p)$ is calculated as the solution to (2.8) together with an appropriate initial condition, where $\mathcal{A}$ is the set of active players.

Suppose that some players get suspended at p'. Generally, these players are treated as if they are inactive. It is possible that for a suspended player i, if $x_i(p)$ is fixed at $x_i(p')$, for some $p > p'$ during the period of suspension, then $w_i(x_i(p'), p)$ becomes higher than p. This situation is quite similar to the Case 1a of the late reentry, which is considered in Section 2.2.4. Any such player is treated as *dormant* (see Section 2.2.4), he is added to $\mathcal{A}$ and his $x_i(p)$ is adjusted correspondingly until the next stop of the clock or until $x_i(p)$ reaches $d_i = x_i(p')$. The only difference

[10] In fact, the outcome is unaffected no matter what number is chosen by the auctioneer. When this is small Rule 5 will be applied often and there will be many suspensions. We choose the bound in the way specified so that in equilibrium Rule 5 does not come into play.

is that there can be many such dormant players at the same time. While there are no difficulties in dealing with many dormant players simultaneously there is a potential problem with starting this procedure. Namely,

Case 3. Suppose at some p (possibly the price at which the clock was stopped), there is a subset $\mathcal{K}$ of suspended players, such that $w_j(x_j(p), p) = p$ for any $j \in \mathcal{K}$. Additionally suppose that any such player j would have been considered dormant starting at p, given no other player had changed his current extended status: active, inactive, suspended or dormant. If, however, all such players $\mathcal{K}$ are treated as dormant, for some of them $w_i(x_i(p), p)$ may stay lower than p, so $x_i(p)$ should not change. Lemma A.5 in Appendix A.3.2 proves that except for some situations of mere theoretical possibility there exists a partition of suspended players into those who have to be considered dormant and those who do not require any special attention. All dormant players will be added to $\mathcal{A}$ for solution purposes. In those situations where such a partition does not exist a special procedure for determining $\mathbf{x}(p)$, which is described in detail in Appendix A.3.2 will be used. It should be noted that for such situations to arise, some very specific value functions are needed as well as a specific initial signal profile $\mathbf{s}$.

If some player j chooses to deviate, his actions will receive special attention in exactly the same situations and in exactly the same way as considered in Section 2.2.4 as Cases 1a, 1b, 2. The analysis of Cases 1a and 2 remains the same as before.

Case 1b (reconsidered). The previously considered Case 1b deals with a situation where a change of status by a particular player j at some price p is incompatible with the estimate of the signal of player j at p. In the simplified case of Section 2.2 there is only one kind of such situation, when player j enters earlier than expected. In the most general case considered in this section two such situations can arise. Namely, once the clock is stopped at a particular price p, other players may estimate that for some previously inactive (possibly suspended) player j, $w_j(x_j(p), p) < p$. Therefore, if player j chooses to be *undecided* or *active* at this clock stop, the decision is incompatible with the current estimate of $x_j(p)$. If player j chose at some stage to be active his estimated minimal signal will be adjusted in exactly the same way as before. If player j chose to be undecided, in an exceptional case when values of players from $\mathcal{A}$ do not depend on s_j, for some range of s_j that includes $x_j(p)$, an instantaneous, possibly partial, adjustment will take place. There will be no further adjustment if player j is not active once the clock is restarted. For a full description of the adjustment procedure see Appendix A.3.1. Note only that it ensures that player j cannot win the auction before the adjustment is completed.

If some of the proposed procedures of Cases 1b and 2 are applied and there are some suspended players, the procedure of Case 3 can be simultaneously applied as well if needed.

2.3.4 Results

We are ready to formulate and prove our main theorem. The set of rules in Section 2.3.1 describes the proposed English auction with reentry.

Theorem 2.1 (Main). *Under the single-crossing and the signal intensity conditions the English auction with reentry has an ex post equilibrium that is efficient.*

Proof. The arguments presented in Section 2.2.5 in the proof of the simplified version of the result are all valid in the general case. In particular, the main facts that drive the result still stand: If some player i wins the good while the others follow the proposed strategies, then the signals of all the other players must have been revealed truthfully. And, if player i wins the good the price he has to pay does not depend on his own action and is equal to p_i^* from (2.10), the same price as in efficient direct mechanism.

What is needed to be shown is that under the new amended set of rules and new available or not considered previously options for deviations no player j can profitably deviate or create any other kind of problems. Many potential deviations have been already covered in the proof in Section 2.2.5. Deviations that cause simultaneous exits and entries, related to the suspension process and a possibility of creating a converging sequence of stops of the clock are the only additional cases to be considered.

First, we show that the converging sequence of exits and entries is not possible. The fact that the number of exits and entries by all players if finite when everyone follows the proposed strategies follows from the properties of the value functions which imply that the derivatives of $\mathbf{x}(p)$ and, thus, of $\mathbf{w}(\cdot, p)$ are all bounded from above. Using calculus one can show then that $w_j(s_j, p)$ can cross p only finitely many times on a given segment. As a result, some player, say j, has to deviate to cause the converging sequence of switches of status(es). Moreover, by the same reasons as above, he himself has to switch the status in a manner that creates a limiting point of his own switches of status. Rule 5, however, effectively prohibits that, by suspending a player if the number of exits and entries exceeds the bound. In particular, player j will be suspended occasionally, and if the limiting point of switches of statuses exists, so does the limiting point of suspensions of player j. Player j, however, cannot stop the clock while being suspended. Again, it can be shown that, since the others follow the proposed strategies, there can be only a finite number of entries and exits of other players before the supposedly problematic price level p. Since some other player has to stop the clock to end the suspension of player j it is impossible to have a limiting point of suspensions of player j and so, the limiting point of entries and exits of all players.

Second, whenever player j causes a clock stop at which some players, possibly including j, are suspended, the proposed information processing ensures that there is no sudden jump in any of $x_i(p)$ and $w_i(s_i, p)$ for $i \neq j$. Since any suspension of player i can happen only at the clock stop and only when $w_i(s_i, p) = p$, whenever some player i exits or gets suspended, his signal is properly revealed. If several players are suspended at the same time, procedures developed in Section 2.3.3, especially in Case 3, guarantee that $\mathbf{x}(p)$ and $\mathcal{A}$ are at all times such that $\mathcal{A} \subset \mathcal{I}(\mathbf{x}(p))$, thus the system (2.8) can always be resolved.

For the rest of the proof see Section 2.2.5. □

Remark 2.5. The equilibrium constructed here satisfies conditions $B(i) - B(iv)$ and P, amended to account for the continuous nature of the game, that define a *perfect Bayesian equilibrium* in Fudenberg & Tirole (1991) on p. 333. In particular, player k reveals by his own actions no information about player j's signal; beliefs $\mathbf{x}_i(p)$ about player i are the same for all other players; all types of player i have the same beliefs $\mathbf{x}_{-i}(p)$ about the others; and, of course, Bayes' rule is used to determine beliefs whenever possible.

Theorem 2.2. *If the standard English auction without reentry has an efficient equilibrium then so does the English auction with reentry.*

Proof. In the case the standard English auction possesses the "regular" equilibrium β described in Krishna (2001), calculated $\mathbf{x}(p)$ are exactly the same as the inverse bidding strategies $\sigma(p)$ described in Krishna (2001). The same price levels are chosen by the players to stop the clock and become inactive in the presented auction as those prices chosen to exit by the players in the "regular" equilibrium β. The only difference is that in the presented auction the price clock stops at any exit. Additionally, whenever two or more players exit simultaneously, they first become suspended until the next player exits, but they never reenter. Thus the sequence of prices at which bidders become *not* active is exactly the same in both the proposed equilibrium in the English auction with reentry and in the "regular" equilibrium in the standard English auction. Clearly, such an equilibrium is also ex post.

Suppose the standard English auction has an efficient equilibrium of any kind in some setting, not necessarily the one of this paper. To construct an isomorphic Bayesian-Nash equilibrium in the English auction with reentry we only need to specify how players will react to an unexpected entry, what some players may find reasonable to do.

We suggest the following beliefs and the strategy: Whenever (re)entry happens by some player i who had previously exited, other players will believe that the actual signals of all other players are the highest possible, that is they are equal to 1. Any player j is supposed to stay in the auction until $p_j = V_j(s_j, \mathbf{1})$. Clearly, if player i still wins the good, the price he has to pay will be higher than he would have to pay otherwise. Thus, no entries will happen on the equilibrium path and, therefore, such beliefs are consistent. □

It may seem unreasonable that players change their beliefs not only about the reentered player, but about the others as well. In particular, the presented equilibrium may not survive some of the refinements of the equilibrium concept—it does not satisfy $B(iii)$ condition of Fudenberg & Tirole (1991). This set of beliefs is chosen only for the purposes of simplicity, it is possible to construct more complicated equilibrium without involving so harsh changes in the structure of beliefs. In particular, if the payment of the winner in the auction of the standard English auction is never higher than in the direct mechanism, then, once a reentry occurs, the auction may proceed according to the presented strategies and information processing. In such a case there will be some adjustment done first (as in Case 2), sequentially to all players that are in need of adjustment, to receive a set of estimates $\mathbf{x}(p)$.

Remark 2.6. Suppose the standard English auction has an efficient equilibrium. Then, if a player i exits at p, he should not have any immediate ex post regret. If he does then it is not optimal for him to exit. If the information is processed in the manner similar to that in the "regular" equilibrium of the standard English auction, then the signal intensity condition has to hold. This makes the signal intensity condition weaker than any known condition under which the standard English auction has an efficient equilibrium, and weaker than any condition under which a "regular" equilibrium exists. Note also that the signal intensity condition is vacuously satisfied whenever there is a player who has the highest value for all signal vectors.

2.3.5 Is (A2) necessary?

It is well-known that the direct mechanism (see Section 2.2.5) allocates the object efficiently as long as the pairwise single-crossing assumption (A1) alone is satisfied. Since the signal intensity condition (A2) is not needed in the direct mechanism, it is natural to ask what role this assumption plays in ascending price auctions.

The signal intensity condition (A2) allows players to process information in a simple and natural manner—they estimate minimal possible signals $\mathbf{x}(p)$ for other players and use these estimates to make decisions. In particular, once a player, say j, exits, the other players treat his signal as fixed while he is inactive. The signal intensity condition ensures that the imputed value of the object to a newly inactive player j does not rise faster than the imputed values of active players, implying that player j does not suffer from regret when becomes inactive. If the signal intensity condition did not hold player j's imputed value would rise faster than p, which would make player j regret exiting. In addition, since player j would become the only member of the winners' circle $\mathcal{I}(\mathbf{x}(p))$ and the single-crossing assumption (A1) is required to hold only for players in the winners' circle, other players' signals might not be accurately inferred thereafter.

So, what one can do in the absence of the signal intensity assumption? It seems that if players intentionally reduce $x_j(p)$ of any player j, for whom (A2) is violated, at the appropriate time and then use the procedure as in Case $1b$ (see Appendix A.3.1) to return $x_j(p)$ to the previous level, then an efficient equilibrium may possibly be constructed. This way of processing information, however, is unnatural and may need to be repeated a large number of times. Additional complications arise when the number of such players is more than one. Not only one will need to run simultaneous procedures for several players, but also the issues of how to effectively partition the players into those that are considered active and inactive arise.

We conjecture that in the absence of signal intensity assumption (A2) it may still be possible to construct an *ex post* efficient equilibrium, but to do so information has to be processed in a highly unintuitive and complicated manner.

2.4 Conclusion

In an English auction a player faces a simple choice: *To be in or not to be in.* This apparent strategic simplicity is what makes an English auction so attractive both from the perspective of the analyst and from the point of view of actual bidders. This is why it is important to know the strengths and weaknesses of the institution.

From the perspective of an economist an English auction is of special interest since it is known to allocate efficiently and, in some circumstances, simultaneously yield a high revenue to the seller. This paper emphasizes the efficiency aspect. As previous studies have shown, the extensively analyzed variant of an English auction—the standard English auction where exits are irrevocable, may fail to allocate efficiently under quite natural circumstances when the number of buyers is more than two.

This paper shows that the failure to allocate efficiently is not due to an institutional defect. I present a modified model of an English auction in which a buyer who had exited previously can return at will. I have shown that the modified mechanism has a number of remarkable properties. First, it is efficient under the single-crossing and the signal intensity conditions. The former is weak and known to be necessary. The latter ensures that in the dynamic nature of the auction no player regrets his decisions to exit or enter the auction and is also relatively weak—it is implied by any other condition under which the standard English auction is known to possess an efficient equilibrium. In the case of only two active bidders it is guaranteed by the single-crossing condition itself.

Second, the strategic complexity of the English auction with reentry is the same as that of the standard English auction. In fact, the presented equilibrium strategies and the information processing are exactly the same as in the "regular" equilibria of the standard English auction analyzed in Maskin (1992) and Krishna (2001). Moreover, whenever the standard English auction has an efficient equilibrium, regular or not, it can be duplicated—the behavior of the players will be exactly the same—as an efficient equilibrium of the modified auction. Therefore, the modified English auction with reentry augments and subsumes the standard English auction.

Third, since it allows reentry the presented model is obviously a better approximation of the "real-life" auction. The possibility of reentry means that now one can generate and analyze certain kinds of strategic behavior that occur in practice but are not possible to capture in the standard model. For instance, a player may exit "early" or "late" in order to mislead other bidders about his signal.

Finally, the rules of the modified auction presented here are quite flexible. They may be amended so that, for example, reentry may be allowed only at the points when otherwise the good would have been awarded to some player. With this amendment, an efficient equilibrium with the same information processing still exists. Likewise, the rules determining what happens when the price clock stops or even when to stop the clock can also be amended. Thus the mechanism is flexible. This may be a desirable feature for practical purposes.

It remains to be seen how the model presented here can be extended to the case of allocating many goods via an open ascending price auction.

Chapter 3

Efficiency of the standard English auction

This essay is based on the paper "Efficiency of the N-bidder English Auction," written jointly with my colleague Oleksii Birulin.[1]

3.1 Preliminaries

General setup is the same as in Section 2.1.2—N bidders with interdependent values are competing for a single object. Each of the bidders receives a private one-dimensional signal which is essential in calculating valuations of the others. Value functions and the distribution of the signals are commonly known to the bidders.

3.1.1 The English auction

Throughout this essay we analyze properties of the standard English auction (without reentry), which we refer to simply as the English Auction. The equilibrium concepts we use are the same as in Chapter 2—ex post and Bayesian Nash equilibria.

Formal rules of the English auction are presented in Section 2.2.1. At price p all bidders commonly know who was active at every preceding price. This *public history* $H(p)$ can be effectively summarized as a profile of prices at which bidders, inactive at p, have exited, $H(p) = \mathbf{p}_{-\mathcal{M}}$, where $\mathcal{M}$ is the set of active bidders just before p. If no bidder exits at $p \in [p', p'')$ then $H(p') = H(p'')$. Denote with $\bar{H}(p)$ the public history $H(p)$ together with all exits that happen at p. Therefore, if $\bar{H}(p) \neq H(p)$ there exists a bidder who exited at p. All bidders are assumed to be active just before the clock starts at $p = 0$, so $H(0) = \varnothing$.

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In the English auction a strategy for bidder j determines the price at which he would drop out given his signal and given the history of who had dropped out and at what prices. Formally, following Krishna (2001), a bidding strategy for bidder $j \in \mathcal{M}$ is a collection of functions $\beta_j^{\mathcal{M}} : [0,1] \times \mathbb{R}_+^{N-M} \longrightarrow \mathbb{R}_+$, where $\mathcal{M}$ is the set of active bidders just before p, $M = \#\mathcal{M} > 1$. Function $\beta_j^{\mathcal{M}}$ determines the price $\beta_j^{\mathcal{M}}(s_j; H(p))$ at which bidder j will drop out when the set of active bidders is $\mathcal{M}$; j's own signal is s_j; and the bidders in $\mathcal{N} \setminus \mathcal{M}$ have dropped out at prices $H(p) = \mathbf{p}_{-\mathcal{M}} = \{p_j\}_{j \in \mathcal{N} \setminus \mathcal{M}}$. The rules of the English auction require that $\beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}}) > \max\{p_j : j \in \mathcal{N} \setminus \mathcal{M}\}$. If active bidders are able to extract from exit prices $\mathbf{p}_{-\mathcal{M}}$ of inactive bidders their true signals $\mathbf{s}_{-\mathcal{M}}$, the strategies can be equivalently written as $\beta_j^{\mathcal{M}}(s_j; \mathbf{s}_{-\mathcal{M}})$.

3.1.2 Pairwise single-crossing

In what follows we exploit a form of the single-crossing condition that differs from (A1) in Section 2.1.3. The difference is negligible—equalities is (2.3) are permitted. This allows us to define what does it mean that the (pairwise) single-crossing is violated and to cover all possible scenarios.

Definition 3.1. *The pairwise single-crossing condition (SC) is satisfied* if at any $\mathbf{s}$ with $\#\mathcal{I}(\mathbf{s}) \geq 2$, for any pair of bidders $i, j \in \mathcal{I}(\mathbf{s})$,

$$\frac{\partial V_i(s_j, \mathbf{s}_{-j})}{\partial s_j} \leq \frac{\partial V_j(s_j, \mathbf{s}_{-j})}{\partial s_j}. \tag{3.1}$$

Take bidder j that has the highest value. The pairwise single-crossing requires that if the signal of that bidder were slightly higher, bidder j would still have the highest value. In other words, if there is a group of bidders who have equal and maximal values, if the signal of one of the bidders is increased, the effect on the value of that bidder is the highest among all the bidders in the group.

We say that SC is violated at $\mathbf{s}$ with $\#\mathcal{I}(\mathbf{s}) \geq 2$ if there exist bidders $i, j \in \mathcal{I}(\mathbf{s})$ such that $\frac{\partial V_i(s_j, \mathbf{s}_{-j})}{\partial s_j} > \frac{\partial V_j(s_j, \mathbf{s}_{-j})}{\partial s_j}$.

Maskin (1992) establishes that

Claim 3.1 (Maskin, sufficiency). *The pairwise single-crossing is sufficient for the existence of an efficient ex-post equilibrium in the English auction with two bidders.*

Claim 3.2 (Maskin, necessity). *Suppose the pairwise single-crossing is violated at some interior signal profile. Then the English auction does not possess an efficient equilibrium.*[2]

[2]This claim was indicated in Maskin (1992). The rigorous proof can be easily provided.

The following example illustrates that the boundary of the signals' domain should receive a special consideration. Efficient equilibria can exist even when the pairwise single-crossing is violated on the boundary of the signals' domain.

Example 3.1. *Consider the English auction with two bidders with value functions of the form*

$$\begin{aligned} V_1 &= \tfrac{2}{3}s_1 + \tfrac{1}{3}s_2, \\ V_2 &= s_1 + s_2. \end{aligned}$$

There exists an efficient ex post equilibrium.

At the point $s_1 = s_2 = 0$, $V_1 = V_2$, the pairwise single-crossing is violated, while at any other $\mathbf{s}$ it is vacuously satisfied. Strategies $\beta_1(s_1) = s_1$, $\beta_2(s_2) = \infty$—bidder 2 never drops out first, form an ex post equilibrium, which is efficient.

3.2 Generalized single-crossing

For an arbitrary vector $\mathbf{u}$ consider $\mathbf{u} \cdot \nabla V_k(\mathbf{s})$—the derivative of V_k in the direction $\mathbf{u}$, where $\nabla V_k(\mathbf{s}) = \left(\frac{\partial V_k}{\partial s_1}, \frac{\partial V_k}{\partial s_2}, \ldots, \frac{\partial V_k}{\partial s_N}\right)$ is the gradient of $V_k(\mathbf{s})$.

Definition 3.2 (Directional formulation). *The generalized single-crossing condition (GSC) is satisfied* if at any $\mathbf{s}$ with $\#\mathcal{I}(\mathbf{s}) \geq 2$, for any subset of bidders $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$,

$$\mathbf{u} \cdot \nabla V_k(\mathbf{s}) \leq \max_{j \in \mathcal{A}} \{\mathbf{u} \cdot \nabla V_j(\mathbf{s})\} \tag{3.2}$$

for any bidder $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$and any direction $\mathbf{u}$, such that $u_i > 0$for all $i \in \mathcal{A}$ and $u_j = 0$ for all $j \notin \mathcal{A}$.

In words, select any group $\mathcal{A}$ of bidders from $\mathcal{I}(\mathbf{s})$—bidders who have equal and maximal values. Fix any direction along which the signals of bidders from $\mathcal{A}$ are increased. Consider the corresponding increments to the values of bidders from $\mathcal{I}(\mathbf{s})$. *GSC* in the directional formulation requires that for any possible direction, the increments to the values of bidders from $\mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$, are at most as high as the highest increment for the bidders from $\mathcal{A}$.

Note that in the case of $\mathcal{A} = \{j\}$, *GSC* reduces to the pairwise single-crossing.

Definition 3.3 (Equal increments formulation). *GSC is satisfied* if at any $\mathbf{s}$ with $\#\mathcal{I}(\mathbf{s}) \geq 2$, for any subset of bidders $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$,

$$\mathbf{u}^{\mathcal{A}} \cdot \nabla V_k(\mathbf{s}) \leq 1 \tag{3.3}$$

for any bidders $j \in \mathcal{A}$ and $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$, and the unique vector $\mathbf{u}^{\mathcal{A}} = (\mathbf{u}^{\mathcal{A}}_{\mathcal{A}}, \mathbf{0}_{-\mathcal{A}})$, at which

$$\mathbf{u}^{\mathcal{A}} \cdot \nabla V_j(\mathbf{s}) = 1.$$

Existence and uniqueness of the vector $\mathbf{u}^{\mathcal{A}}$ follows from the fact that $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}}$ solves the vector system $DV_{\mathcal{A}}(\mathbf{s}) \cdot \mathbf{u}^{\mathcal{A}}_{\mathcal{A}} = \mathbf{1}_{\mathcal{A}}$, thus $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} = (DV_{\mathcal{A}}(\mathbf{s}))^{-1} \cdot \mathbf{1}_{\mathcal{A}}$. By regularity assumption, $\det DV_{\mathcal{A}}(\mathbf{s}) \neq 0$. We will further refer to $\mathbf{u}^{\mathcal{A}}$ as to the *equal increments vector* corresponding to subset $\mathcal{A}$.

In words, select any group of bidders $\mathcal{A}$ from $\mathcal{I}(\mathbf{s})$. There exists a unique path such that, if signals of bidders from $\mathcal{A}$ are changed along it, their values increase and remain equal to each other. *GSC* in the equal increments formulation requires that along this particular path the value of any other bidder from $\mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$ cannot surpass values of bidders from $\mathcal{A}$.

Lemma B.1 in Appendix B.1 shows that the formulations of *GSC* given in Definitions 3.2 and 3.3 are equivalent. Thus the equal increments formulation of *GSC* is only seemingly less demanding than the directional formulation. In the proofs that follow we will use these formulations interchangeably, whichever is more convenient for the specific argument.

Now we state our main results.

3.2.1 Results

Theorem 3.1 (Sufficiency). *Suppose value functions satisfy* GSC*. Then there exists an* efficient *ex post equilibrium in the N-bidder English auction.*

Definition 3.4. *GSC* condition (in the directional formulation) is *violated* at the signal profile $\mathbf{s}$ for the proper subset $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ and bidder $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$ if there exists a vector $\mathbf{u}$, with $u_i > 0$ for all $i \in \mathcal{A}$, $u_j = 0$ for any $j \notin \mathcal{A}$, such that

$$\mathbf{u} \cdot \nabla V_k(\mathbf{s}) > \max_{i \in \mathcal{A}} \{\mathbf{u} \cdot \nabla V_i(\mathbf{s})\}. \tag{3.4}$$

Similarly a violation of *GSC* condition in the equal increments formulation can be defined. Hereafter whenever we say that *GSC* is violated it means that there exist some $\mathbf{s}$, $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$, and $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$, such that (3.4) holds.

The following result establishes that *GSC* condition is necessary in the sense of

Theorem 3.2 (Necessity). *Suppose* GSC *condition is violated at some interior signal profile. Then no efficient equilibrium in the N-bidder English auction exists.*

3.2.2 Examples

We start with the known example where the English auction fails to allocate efficiently and show that *GSC* is indeed violated there.

Example 3.2 (Perry & Reny (2001)). *Consider the English auction with three bidders with value functions of the form*

$$\begin{aligned} V_1 &= s_1 + s_2 s_3, \\ V_2 &= \tfrac{1}{2}s_1 + s_2, \\ V_3 &= s_3. \end{aligned}$$

There exists no efficient equilibrium.

Perry & Reny (2001) show that the three-bidder English auction has no efficient equilibrium in this example. It is easy to see that *GSC* is violated here. Notice that at the signal profile $\mathbf{s} = (.3, .6, .75)$ all the values are tied. Choose subset $A = \{2, 3\}$ and direction $\mathbf{u} = (0, 1, 1)$. Then, $\mathbf{u} \cdot \nabla V_2(\mathbf{s}) = \mathbf{u} \cdot \nabla V_3(\mathbf{s}) = 1$, while $\mathbf{u} \cdot \nabla V_1(\mathbf{s}) = \frac{\partial V_1(s)}{\partial s_2} + \frac{\partial V_1(s)}{\partial s_3} = 1.35 > 1$.

The next example generalizes the message of Example 3.1 and illustrates that an English auction may possess an efficient equilibrium even when value functions violate *GSC* at the boundary of the signals' domain. In Example 3.3, however, any bidder may have the highest value, hence the existence of efficient equilibrium here is not as trivial as it was in Example 3.1.

Example 3.3. *Consider an English auction with three bidders with value functions of the form*

$$\begin{aligned} V_1 &= s_1 + \tfrac{2}{3}(s_2 + s_3), \\ V_2 &= s_2, \\ V_3 &= s_3. \end{aligned}$$

There exists an efficient ex post equilibrium.

It is clear that *GSC* is violated at $\mathbf{s} = (0, 0, 0)$ for $\mathcal{A} = \{2, 3\}$, bidder 1 and vector $(0, 1, 1)$. There is no other $\mathbf{s}$ at which values of all three bidders are equal. *SC* (or *GSC* for $\#\mathcal{A} = 1$) is clearly satisfied everywhere.

The following strategies form an efficient ex post equilibrium. When all the bidders are active, bidders 2 and 3 drop out when the price reaches their private values, and bidder 1 never drops out first. After one of bidders 2 and 3, say bidder 2, drops out, bidder 3 stays active until the price reaches his private value. Bidder 1 drops out when the price reaches $s_1 + \frac{2}{3}(p + s_2)$, where s_2 is the revealed signal of bidder 2, who had dropped first.

Note that if bidders 2 and 3 follow these strategies and drop out simultaneously, the value of the object to bidder 1 is always higher than the price that she has to pay. Thus the "waiting strategy" is "safe" for bidder 1. Bidders 2 and 3 use their dominant strategies.

3.3 Sufficiency

In this section we show that *GSC* is sufficient for the existence of an efficient equilibrium in the N-bidder English auction. The proof is by construction.[3]

In this equilibrium, for a given public history $H(p) = \mathbf{p}_{-\mathcal{M}}$, active bidders from $\mathcal{M}$ calculate $\sigma(p, H(p))$—a profile of inverse bidding functions that are used to define the strategies. For bidder j, to decide whether to be active at p or not is sufficient to compare her true signal s_j with $\sigma_j(p)$.[4] If $\sigma_j(p) < s_j$ bidder j is suggested to remain active; at the lowest price level p_j such that $\sigma_j(p_j) \geq s_j$ bidder j is suggested to exit. Once bidder j exits at p_j, her true signal $s_j = \sigma_j(p_j)$ is revealed, $\sigma_j(p) = \sigma_j(p_j)$ for any higher price, $p > p_j$.

Now we define the (equilibrium) strategies.

Suppose there exists a profile of functions $\sigma(p, H(p))$, which we call *inferences*, such that:[5]

1. for an inactive bidder $i \in \mathcal{N} \setminus \mathcal{M}$, $\sigma_i(p) = \sigma_i(p_i, H(p_i))$, that is $\sigma_i(p)$ is fixed after bidder i exits at p_i;

2. for any active bidder $j \in \mathcal{M}$, $\sigma_j(p) \in [0, 1]$ solves $V_j(\sigma_j(p), \sigma_{-j}(p)) = p$, if such a solution exists with $\sigma_j(p) \leq 1$, otherwise $\sigma_j(p) = 1$ with $V_j(\sigma_j(p), \sigma_{-j}(p)) < p$.

Thus, for all active bidders, $\sigma_{\mathcal{M}}(p)$ are determined simultaneously, as a solution to

$$\begin{gathered} \mathbf{V}_{\mathcal{M}}(\sigma_{\mathcal{M}}(p), \sigma_{-\mathcal{M}}(p)) \leqslant p\mathbf{1}_{\mathcal{M}}, \quad \sigma_{\mathcal{M}}(p) \leqslant \mathbf{1}_{\mathcal{M}}, \\ \forall j : (V_j(\sigma(p)) - p)(\sigma_j(p) - 1) = 0. \end{gathered} \tag{3.5}$$

For bidder $j \in \mathcal{M}$ *strategy* $\beta_j^{\mathcal{M}} : (s_j, H(p)) \longrightarrow \mathbb{R}_+$ is

$$\beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}}) = \arg\min_p \{\sigma_j(p) \geq s_j\}. \tag{3.6}$$

Strategy β_j can be interpreted as follows. If bidder j is active at p, given the public history $H(p) = \mathbf{p}_{-\mathcal{M}}$ of exits of inactive bidders $\mathcal{N} \setminus \mathcal{M}$, bidder j is supposed to exit the auction at $p_j = \beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}})$, provided no other bidder exits before. If

[3] Milgrom & Weber (1982) propose the ideology of constructing efficient equilibria for the English auction with symmetric bidders; Maskin (1992) extends it to the case of two asymmetric bidders, and Krishna (2001) generalizes it to the case of N asymmetric bidders.

[4] To shorten the notation we are omitting $H(p)$ from the set of arguments, whenever the public history can be implied from the context.

[5] Note that $H(p)$ is a collection of exit prices, $H(p) = \mathbf{p}_{-\mathcal{M}}$, where $\mathcal{M}$ is the set of active bidders. Therefore, $\sigma(p, H(p))$ will be defined for all $p \geq \max_{i \notin \mathcal{M}} p_i$.

Inferences $\sigma(p, H(p))$ are closely related, but are not the same in concept, to estimated minimal signals $\mathbf{x}(p)$ in Chapter 2. In the presented equilibrium below, when it exists, at any p, $\sigma(p, H(p)) = \mathbf{x}(p)$.

the current price p satisfies $p < \beta_j^{\mathcal{M}}(s_j;\mathbf{p}_{-\mathcal{M}})$, bidder j is suggested to maintain an active status; if $p \geq \beta_j^{\mathcal{M}}(s_j;\mathbf{p}_{-\mathcal{M}})$ bidder j is suggested to exit at p.

Once bidder j exits at p_j, other bidders update the public history and, expecting bidder j to follow (3.6), infer $s_j^* = \sigma_j(p_j)$. If $\sigma_j(\cdot)$ is non-decreasing the inferred s_j^* is unique and coincides with true signal s_j. The strategies can then be reformulated as functions of the own and inferred signals of inactive bidders, $\beta_j^{\mathcal{M}}(s_j;\mathbf{s}_{-\mathcal{M}}) = \beta_j^{\mathcal{M}}(s_j;\mathbf{p}_{-\mathcal{M}})$.

To proceed with the sufficiency result we need the following

Lemma 3.1. *Suppose* GSC *is satisfied. Then there exist inferences $\sigma(p, H(p))$, such that each $\sigma_j(\cdot, H(p))$ is continuous and non-decreasing for any $H(p)$, and $\sigma_j(p, \bar{H}(p)) = \sigma_j(p, H(p))$ for all p such that $\bar{H}(p) \neq H(p)$. For any active at $H(p)$ bidder j, $j \in \mathcal{I}(\sigma(p))$ if $\sigma_j(p, H(p)) < 1$.*

Proof. Proof is presented in Appendix B.2. □

The following lemma then proves Theorem 3.1.

Lemma 3.2. *Suppose value functions satisfy* GSC*. Then β defined by (3.6) constitute an* efficient ex-post *equilibrium in the N-bidder English auction.*

Proof. We first show that β are well-defined. For any bidder j, arbitrarily fix exit prices of other bidders, $\mathbf{p}_{-j}$, possibly with $p_i = \infty$ for some bidders. Then one can obtain $\sigma_j(p)$ defined for any $p \geq 0$ as $\sigma_j(p) = \sigma_j(p, H(p))$, where $H(p) = \cup_{p_i<p}\{p_i\}$. Lemma 3.1 shows that $\sigma_j(p)$ is continuos and non-decreasing for any given $\mathbf{p}_{-j}$. Therefore, $p_j = \arg\min_p \{\sigma_j(p) \geq s_j\}$ is unique, so $\beta_j(s_j;\cdot)$ is well defined.

Next, we show that when all the bidders follow strategies (3.6), the good is allocated to the bidder with the highest value. Suppose it is bidder j who wins the good at price p^*. Then, $\sigma_i(p^*) = \sigma_i(p_i) = s_i$ for any $i \neq j$, $\sigma_j(p^*) \leq s_j$, and according to Lemma 3.1

$$V_j(\sigma_j(p^*), \mathbf{s}_{-j}) = \max_{i \neq j} V_i(\sigma_j(p^*), \mathbf{s}_{-j}) = p^*. \tag{3.7}$$

The pairwise single-crossing, $\sigma_j(p^*) \leq s_j$, and equation (3.7) imply that

$$V_j(\mathbf{s}) \geq \max_{i \neq j} V_i(\mathbf{s}) \geq p^*, \tag{3.8}$$

so bidder j is (one of) the bidder with the highest value. Note that price p^*, that bidder j has to pay for the object does not depend on the signal of bidder j.

Finally, we show that β form an ex-post equilibrium. The payoff of bidder j can change as a result of a deviation only if it affects whether she obtains the object. If bidder j wins the object as a result of the deviation, she has to pay $p_j^* = \max_{i \neq j} V_i(\sigma_j(p_j^*), \mathbf{s}_{-j})$. If bidder j were not the winner in the equilibrium, $\sigma_j(p_j^*) \geq s_j$ since $\sigma_j(p)$ is non-decreasing, so $V_j(\mathbf{s}) \leq p_j^*$ and the deviation is not profitable. If as a result of the deviation bidder j is not the winner while she is in the equilibrium, she is possibly forfeiting positive profits according to (3.8). Thus, no profitable deviation exists.

The above argument is valid even if signals $\mathbf{s}$ are commonly known, hence the presented equilibrium is ex-post. □

3.4 Necessity

In this section we establish that *GSC* is necessary for existence of an efficient equilibrium in the N-bidder English auction. We show the following

Theorem 3.2 (Necessity). *Suppose* GSC *is violated at an interior signal profile. Then no efficient equilibrium in the N-bidder English auction exists.*

Actual proof is quite cumbersome and is presented in detail in Appendix B.3. Here we illustrate main ideas behind the proof with the partial analysis of the 3-bidder English auction. Then we provide an outline of the proof in the general case.

3.4.1 Three-bidder English auction

As Claim 3.2 shows, the pairwise single-crossing is necessary for efficiency. Consider a 3-bidder English auction and suppose that the pairwise single-crossing is satisfied with strict inequality for every pair of bidders. Suppose that *GSC* is violated at some interior signal profile $\mathbf{s}'$. We can always relabel the bidders such that *GSC* is violated for $\mathcal{A} = \{2, 3\}$ and bidder 1.

Claim 3.3. *Suppose there are three bidders in the auction, $\mathcal{N} = \{1, 2, 3\}$. Suppose that* GSC *is violated for $\mathcal{A} = \{2, 3\}$ and bidder 1 at the interior signal profile $\mathbf{s}'$, such that $V_1(\mathbf{s}') = V_2(\mathbf{s}') = V_3(\mathbf{s}')$.*

Then, no efficient equilibrium exists with $\beta_2(s_2; \varnothing)$ and $\beta_3(s_3; \varnothing)$ continuous at s_2' and s_3' correspondingly.

Proof. Suppose an efficient equilibrium exists. This requires that any bidder with strictly the highest value must not be the first to drop out.

Consider any history with all three bidders still active, $H(p) = \varnothing$. GSC is violated for $\mathcal{A} = \{2, 3\}$ and bidder 1 at $\mathbf{s}'$, that is there exists direction $\mathbf{u} = (0, u_2, u_3)$ with $u_2 > 0$, $u_3 > 0$, such that for every small enough $\varepsilon > 0$,

$$V_1(\mathbf{s}' + \varepsilon\mathbf{u}) > \max\{V_2(\mathbf{s}' + \varepsilon\mathbf{u}), V_3(\mathbf{s}' + \varepsilon\mathbf{u})\}. \tag{3.9}$$

Thus, efficiency prescribes

$$\beta_1(s_1') > \min\{\beta_2(s_2' + \varepsilon u_2), \beta_3(s_3' + \varepsilon u_3)\}. \tag{3.10}$$

Since $\beta_2(s_2)$ and $\beta_3(s_3)$ are continuous at s_2' and s_3' correspondingly,

$$\beta_1(s_1') \geq \min\{\beta_2(s_2'), \beta_3(s_3')\}. \tag{3.11}$$

First, we show that $\beta_2(s_2') = \beta_3(s_3')$. Suppose not, without loss of generality consider $\beta_2(s_2') < \beta_3(s_3')$. By continuity of $\beta_2(\cdot)$ for $s_2 > s_2'$ sufficiently close to s_2', $\beta_3(s_3') > \beta_2(s_2)$, and by (3.11), $\beta_1(s_1') \geq \beta_2(s_2)$. But by the pairwise single-crossing for $s_2 > s_2'$ close to s_2' the value of bidder 2 is the highest,

$$V_2(s_1', s_2, s_3') > \max\{V_1(s_1', s_2, s_3'), V_3(s_1', s_2, s_3')\}. \tag{3.12}$$

Bidder 2 has the highest value, but drops out (one of the) first, violating efficiency. Thus $\beta_2(s_2') = \beta_3(s_3') \equiv b$, and $\beta_1(s_1') \geq b$.

Next, note that (3.12) implies $\beta_2(s_2) > \min\{\beta_1(s_1'), \beta_3(s_3')\} \geq b$. Therefore,

$$\beta_2(s_2) > b, \quad \beta_3(s_3) > b, \tag{3.13}$$

for $s_2 > s_2'$ and $s_3 > s_3'$ close to s_2' and s_3' correspondingly.

Finally, by (3.9) and continuity of value functions, for a given $\varepsilon > 0$, there exists $\varepsilon_1 > 0$, such that bidder 1 has the highest value at $(s_1' - \varepsilon_1, s_2' + \varepsilon u_2, s_3' + \varepsilon u_3)$. Therefore it has to be that

$$\beta_1(s_1' - \varepsilon_1) > \min\{\beta_2(s_2' + \varepsilon u_2), \beta_3(s_3' + \varepsilon u_3)\}.$$

Then, (3.10) together with (3.13) imply

$$\beta_1(s_1' - \varepsilon_1) > \beta_2(s_2') = \beta_3(s_3').$$

Thus, at $(s_1' - \varepsilon_1, s_2', s_3')$ bidders 2 and 3 drop out simultaneously, and bidder 1 wins, however, by the pairwise single-crossing bidder 1 has the lowest value. □

In Appendix B.3.1 we present the complete analysis of the 3-bidder auction. As an intermediate step there we show that $\beta_2(s_2; \varnothing)$ and $\beta_3(s_3; \varnothing)$ have to be almost everywhere continuous in the neighborhoods of s_2' and s_3' correspondingly.

3.4.2 N-bidder English auction

In the general case of N bidders in the auction our proof incorporates the following three main observations.

1. We proceed from the contrary—we assume that an efficient equilibrium exist, and thus, no matter what is the current public history $H(p)$, a bidder with strictly the highest value must not be the first to drop out. Fix an efficient equilibrium β, $\beta_i(s_i; H(p))$ is the equilibrium strategy of bidder i with s_i.

2. If *GSC* is violated at some interior signal profile, we can find possibly different interior signal profile $\mathbf{s}$, where *GSC* is violated for bidder k and *minimal* subset $\mathcal{A}$, that is the subset that contains the fewest possible number of bidders needed to violate *GSC* (the existence of such a subset is shown in Appendix B.3.2). Moreover, we can find $\mathbf{s}$, minimal subset $\mathcal{A}$, and bidder $k = 1$ (after relabeling), such that these are the only bidders in the winners circle, $\mathcal{A} \cup \{1\} = \mathcal{I}(\mathbf{s})$.

   Our focus will always be on bidders $\mathcal{A} \cup \{1\}$, the signals of the other bidders are fixed at $\mathbf{s}_{-\mathcal{I}(\mathbf{s})}$ throughout the proof.

3. In what follows only bidding functions of bidders from $\mathcal{A} \cup \{1\}$ are compared. We are considering only those histories when all bidders $\mathcal{A} \cup \{1\}$ are active. Suppose bidders $\mathcal{A} \cup \{1\}$ stay forever at any given history, then one can obtain the history of exits of the other bidders. It is predetermined, since their signals are fixed, and is finite with the length of at most $N - \#(\mathcal{A} \cup \{1\})$ exit prices. Therefore, the number of possible histories as vectors of exit prices at which all bidders $\mathcal{A} \cup \{1\}$ are active (with arbitrary signals) is finite.

   For any of these histories we will be comparing bids of bidders $\mathcal{A} \cup \{1\}$ at some signals' realizations. When we state $\beta_i(s_i) > \beta_j(s_j)$ we mean that it must be true *conditional* on the fact that one of the bidders from $\mathcal{A} \cup \{1\}$ drops out first among all active bidders at $H(p)$.

The following presents an outline of the proof. Suppose that the minimal subset $\mathcal{A}$ contains n bidders, and *GSC* is violated at some interior signal profile $\mathbf{s}$ for $\mathcal{A}$ and bidder 1, with $\mathcal{A} \cup \{1\} = \mathcal{I}(\mathbf{s})$.

1. Consider trajectory $\mathbf{s}(t)$ that for each t solves

$$\begin{aligned} V_j(\mathbf{s}(t)) &= V(\mathbf{s}) + t, \text{ for all } j \in \mathcal{A} \cup \{1\}, \\ s_i(t) &= s_i, \text{ for all } i \notin \mathcal{A} \cup \{1\}. \end{aligned}$$

   Such a trajectory exists and is unique, since it can be found as a solution to the differential equation

$$\frac{d\mathbf{s}_{\mathcal{I}(\mathbf{s})}}{dt} = (DV_{\mathcal{I}(\mathbf{s})}(\mathbf{s}))^{-1} \cdot \mathbf{1}_{\mathcal{I}(\mathbf{s})}.$$

   By continuity of value functions and their first derivatives, $\mathcal{A} \cup \{1\} = \mathcal{I}(\mathbf{s}(t))$, and *GSC* is violated at $\mathbf{s}(t)$ for $\mathcal{A}$ and bidder 1 for all t in some neighborhood U_t^0 of $t = 0$. Because we do not impose any *ad hoc* restrictions on the bidding strategies, for example, they are allowed to be discontinuous everywhere, we exploit the entire trajectory $\mathbf{s}(t)$ not just the realization $\mathbf{s} = \mathbf{s}(0)$.

2. Fix any public history $H(p)$ with bidders $\mathcal{A} \cup \{1\}$ still active.

   Consider $\mathbf{s}' = \mathbf{s}(t)$ for an arbitrary $t \in U_t^0$. It can be shown that for any $j \in \mathcal{A}$, there exist $\lim_{s_j \downarrow s'_j} \beta_j(s_j) = b_j(s'_j)$, and these limits are equal for the bidders from $\mathcal{A}$, for any $j \in \mathcal{A}$, $b_j(s'_j) = b(\mathbf{s}') = b(t)$.[6] In addition, for any $j \in \mathcal{A}$ and $s_j > s'_j$ sufficiently close to s'_j, $\beta_j(s_j) \geq b(t)$; and for bidder 1, $\beta_1(s'_1) > b(t)$.

3. Corollary B.2 in Appendix B.1 shows that either (i) for any $j \in \mathcal{A}$, $s_j(t') > s_j(t)$ while $s_1(t') < s_1(t)$ for $t' > t$; or (ii) for any $j \in \mathcal{A}$, $s_j(t') < s_j(t)$ while $s_1(t') > s_1(t)$ for $t' > t$. This, together with the results of Step 3, implies that $b(t)$ is (weakly) monotonic in t. In Case (i) it is non-decreasing, in Case (ii) it is non-increasing.

4. Then, we show that if for some bidder $j \in \mathcal{A}$, $\beta_j(s_j(t)) \neq b(t)$, t has to be a discontinuity point for $b(t)$. Since $b(t)$ is monotonic, it has no more than a countable number of discontinuity points, hence for almost all $t \in U_t^0$, $\beta_j(s_j(t)) = b(t)$ for every $j \in \mathcal{A}$. That is, when the signals of bidders from $\mathcal{A}$ belong to trajectory $s(t)$, bidders from $\mathcal{A}$ almost always exit simultaneously. Remember, this is conditional on the fact that a bidder from $\mathcal{A} \cup \{1\}$ drops out first.

5. Since the number of different histories at which all bidders $\mathcal{A} \cup \{1\}$ are active is finite, there exist at least one of them for which results in Steps 2, 3, and 4 are binding, in particular $b(t)$ is well-defined. There will be at most one such history, $H(p)$, for signals $\mathbf{s}'$ in the neigbourhood of $\mathbf{s}(0)$. Fix it, therefore a bidder from $\mathcal{A} \cup \{1\}$ necessarily drops out first among all active bidders at $H(p)$.

6. Consider two continuity points for $b(t)$, t and t', such that $b(t') \geq b(t)$. In Case (i) above, $t' > t$; in Case (ii), $t' < t$. Then, $s_1(t') < s_1(t)$, $\beta_1(s_1(t')) > b(t') \geq b(t) = \beta_j(s_j(t))$ for all $j \in \mathcal{A}$.

7. If the pairwise single-crossing is strictly satisfied for bidder 1, that is all inequalities concerning $\frac{\partial}{\partial s_1}$ in (3.1) are strict, then at $(s_1(t'), \mathbf{s}_{-1}(t))$ bidder 1 does not have the highest value, and all the bidders with the highest value belong to $\mathcal{A}$. At this realization, bidders from $\mathcal{A}$ drop out simultaneously at $b(t)$, first among all remaining active bidders, and bidder 1 for sure stays longer. Thus, if $b(t) < \infty$, efficiency is not achieved—a contradiction. If $b(t) = \infty$, the auction continues indefinitely—all remaining bidders including $\mathcal{A} \cup \{1\}$ remain active forever. This, however, cannot be in an equilibrium,

[6] The limits are considered in the extended space $\mathbb{R} \cup \{\infty\}$, that is $b(t) = \infty$ is allowed. In any comparisons that follow $\infty > \infty$ is considered to be valid statement, in the sence that, $a > b = \infty$ implies $a = \infty$.

since then all these bidders obtain payoffs of $-\infty$, while by dropping out they are guaranteed 0.

8. If the pairwise single-crossing is only weakly satisfied for bidder 1, more delicate procedures are necessary. In short, we have to disturb the signals of bidders from $\mathcal{A}$ around $\mathbf{s}_{\mathcal{A}}(t)$ to ensure that bidder 1 does not have the highest value at $(s_1(t), \hat{\mathbf{s}}_{-1}(t))$, where $\hat{\mathbf{s}}_{-1}(t)$ is a disturbed profile. Then the arguments similar to those in Step 7 are involved.

To make an exposition as clear as possible the complete proof for the simplest case with only three bidders in the auction is presented separately in Appendix B.3.1. The proof there closely follows the above outline without any additional steps involved. The general case of N bidders is much more technical (Steps 3, 4, and 5 are much more involved) and is presented in Appendix B.3.2.

3.5 Conclusion

The fact that the English auction does allocate efficiently under some circumstances was known before. In this essay we bring forth the exact requirement on value functions—generalized single-crossing condition—both necessary and sufficient for the existence of an efficient equilibrium in the N-bidder English auction. By doing so we generalize both Maskin (1992) results for the two-bidder English auction and the pairwise single-crossing condition.

Given that the pairwise single-crossing arises in many economic applications in incomplete information settings, the natural question is, how broadly can generalized single-crossing be applied? As a direct implication of the results presented here, GSC is necessary for any efficient mechanism that is based on the English auction without reentry. While other mechanisms can be used to achieve efficiency if *GSC* is violated, the standard English auction is potentially the best choice if *GSC* is satisfied. Indeed, if this is the case, the equilibrium behavior in the English auction with or without reentry is the same, but in the latter model bidders can behave strategically, and the space of messages they can send to each other (exiting and/or reentering at certain prices) is enormous. Given that, whenever *GSC* is satisfied, it might be preferable to use an English auction without reentry to provide bidders with as fever opportunities as possible to agree on a collusive outcome.

The pairwise single-crossing guarantees that no bidder is willing to misrepresent her information in the Bayesian-Nash equilibrium of the direct efficient mechanism. We believe that *GSC* is important for any equilibrium concept that involves subsets of bidders or coalitions in the same manner as the pairwise single-crossing is for Bayesian-Nash equilibria, and so it is very promising in this respect.

For example, suppose that *GSC* is violated for subset $\mathcal{A}$ of bidders and bidder i. Then there exist $\mathbf{s}$ and $\mathbf{s}' = (\mathbf{s}'_{\mathcal{A}}, \mathbf{s}_{-\mathcal{A}})$, such that at $\mathbf{s}$ a bidder from $\mathcal{A}$ has the highest value, while at $\mathbf{s}'$ bidder i has, and the signal s'_j of any bidder $j \in \mathcal{A}$ is higher than s_j. Then if efficiency is used as an allocation criterion, if bidders from $\mathcal{A}$ with $\mathbf{s}'_{\mathcal{A}}$ can make a collective decision, they would rather not disclose their information fully. Moreover, if the value of the object to $\mathcal{A}$—as the value of the coalition—is defined as the maximum of the values of its members, generalized single-crossing condition is analogous to the single-crossing condition for $\mathcal{A}$ and bidder 1. This analogy straightforwardly extends to paiwise comparisons of any two coalitions. *GSC* guarantees that no coalition with a "higher type" has incentives to misrepresent its "type" to secure the object.

In the European 3G Telecom Auctions (see Klemperer (2002)) in some of the countries firms were allowed to merge prior to the auction and to submit bids as a coalition. Then the coalition is endowed with a multi-dimensional signal and its value function is an upper envelope of the value functions of the members. We believe that generalized single-crossing condition might be useful in the analysis of the situation like this. First, if *GSC* is not satisfied, it provides incentives for coalitions to form. Second, if *GSC* is satisfied, it might guarantee that the resulting auction is constrained efficient. Indeed, if the assumptions of Eső & Maskin (2000) are satisfied, constrained efficient mechanism exists if the value functions for coalitions as bidders satisfy the pairwise single-crossing with respect to summary statistics (see Eső & Maskin (2000) for relevant definitions). But the latter is guaranteed when the value functions for individual firms satisfy *GSC* with respect to the initial one-dimensional signals.

Chapter 4

Multi-unit open ascending price efficient auction

4.1 Preliminaries

4.1.1 The environment

In the previous chapters we have seen that the English auction is an extraordinary mechanism. It is simple in practice, and it allocates a single object efficiently under quite general circumstances. In this chapter we consider a situation when a number of identical objects is offered for sale. We propose a mechanism that consists of a sequence of one-object English auctions, so it is a relatively simple construction. We show that under multi-unit variants of the single-crossing and signal intensity conditions this mechanism allocates the objects efficiently.

The environment considered is the multi-unit extension of the environment of Section 2.1. There are K homogeneous goods for sale. There are N potential buyers, each of whom receives a signal $s_i \in [0,1]$.[1] Given the signals $\mathbf{s} = (s_1, s_2, \ldots, s_N)$, the marginal value of the kth object to the player i is $V_i^k(\mathbf{s})$. The valuation functions V_i^k are assumed to be commonly known to the players. As in previous chapters, how signals are distributed is not important for the results.

For any player i value functions V_i^k have the following properties: for any k, V_i^k is twice-differentiable, $V_i^k(0, \ldots, 0) = 0$ and

$$\forall i \ \frac{\partial V_i^k}{\partial s_i} > 0, \qquad \forall j \neq i \ \frac{\partial V_i^k}{\partial s_j} \geq 0. \tag{4.1}$$

The marginal value functions are assumed to be non-increasing for each additional object (no complementarities): $V_i^k \geq V_i^{k+1}$ for all $k < K$.

[1] Note that each bidder receives only one signal. Jehiel & Moldovanu (2001) show that if the private signals were multidimensional it would be impossible to achieve efficiency in general by means of any mechanism.

Denote $\mathbf{V^k}$ to be the profile of value functions $V_i^{k_i}$ picked for every bidder $i \in \mathcal{N}$ according to vector $\mathbf{k} = (k_1, \dots, k_N)$. If every bidder $i \in \mathcal{N}$ has been assigned $(k_i - 1)$ objects, then profile $\mathbf{V^k}$ is the set of marginal valuations of all bidders for an additional object. We call $\mathbf{k}$ *admissible* if $k_i \geq 1$ for all i and $0 \leq \sum_{i=1}^N (k_i - 1) < K$, that is the total number of "allocated" objects is less than K. Given $\mathbf{k}$, one can define winners circle $\mathcal{I}^{\mathbf{k}}(\mathbf{s})$ as the set of bidders with maximal values according to $\mathbf{V^k}(\mathbf{s})$.

The main result relies on the following two multi-unit versions of single-crossing (A1) and signal intensity (A2) assumptions. See Section 2.1.3 for the detailed description and comparison of these.

A1K (*single-crossing*) For any admissible $\mathbf{k}$, for all $\mathbf{s}$ and any pair of players $\{i, j\} \subset \mathcal{I}^{\mathbf{k}}(\mathbf{s})$,

$$\frac{\partial V_i^{k_i}(\mathbf{s})}{\partial s_i} > \frac{\partial V_j^{k_j}(\mathbf{s})}{\partial s_i}. \tag{4.2}$$

A2K (*signal intensity*) For any admissible $\mathbf{k}$, for all $\mathbf{s}$ and $i \in \mathcal{I}^{\mathbf{k}}(\mathbf{s})$ there exists an $\varepsilon > 0$ such that for all $\mathbf{s}'$ satisfying (i) $s_i < s_i' < s_i + \varepsilon$; (ii) $\forall j \in \mathcal{I}(\mathbf{s}) \setminus \{i\}$, $V_j^{k_j}(\mathbf{s}') = V_j^{k_j}(\mathbf{s})$ and (iii) $\forall k \notin \mathcal{I}^{\mathbf{k}}(\mathbf{s})$, $s_k' = s_k$, it is the case that $\mathcal{I}^{\mathbf{k}}(\mathbf{s}') = \{i\}$.

Note that these conditions are required to be satisfied only for admissible $\mathbf{k}$—in situations when less than the total number of objects have been "won". The single-crossing condition in this "weak" form is the same condition that is required in Dasgupta & Maskin (2000),[2] and that guarantees existence of the direct efficient mechanism.[3]

As in Section 2.1.2, we also assume that for any admissible $\mathbf{k}$, a player with the lowest possible own signal cannot have the highest value (unless $\mathbf{s} = \mathbf{0}$), and that set of functions $\mathbf{V}_{\mathcal{J}}^{\mathbf{k}_{\mathcal{J}}}(\mathbf{s})$ is regular at $\mathbf{s}_{\mathcal{J}}$ for any $\mathcal{J} \subset \mathcal{I}^{\mathbf{k}}(\mathbf{s})$. These assumptions are not critical for the results, regularity guarantees that equilibrium strategies are well defined, while "no free riders" assumption together with $V_i^{k_i}(\mathbf{0}) = 0$ are only to ensure complete and analytically attractive consideration of the most general case. When they are not satisfied, a similar analysis can be provided, it will give a rise to existence of "waiting" efficient equilibria, where one of the players' strategy is to "wait" (or stay forever) until some other bidder exits the auction. It should be noted, though, that in this case (A1K) and (A2K) will have to be strengthened slightly to ensure proper information exchange in order to obtain efficiency.

[2] In Perry & Reny (2002) and Perry & Reny (2001) a stronger form of single-crossing is required. It has to be satisfied for any pair of bidders with equal values regardless whether they are the members of a winners circle or not.

[3] For a detailed analysis of the direct efficient mechanism see Chapter 17 in Krishna (2002).

4.1.2 Overture

The difficulty in allocating multiple objects efficiently via simple mechanisms arises from the fact that the prices winners have to pay are not trivially determined. If the mechanism has an ex post equilibrium that is efficient, it has to ensure that every winner pays the Vickrey price—the same price she is obliged to pay in the generalized Vickrey-Clark-Groves mechanism.[4]

To illustrate the associated problems consider a simple situation when there are only two objects for sale. Suppose that it is the case that bidders i and k have the first and second highest marginal values at $\mathbf{s}$—they have to be the winners if efficiency is obtained. The Vickrey price, p_i^1, that bidder i has to pay for her object, is equal to her marginal value for the first object, V_i^1, calculated at $(s_i^1, \mathbf{s}_{-i})$, where s_i^1 is the signal with which $V_i^1(s_i^1, \mathbf{s}_{-i})$ would be equal to the *second* highest of all marginal valuations of *the other* bidders. Thus, bidder i with signal $s_i > s_i^1$ would necessarily win at least one object, while bidder i with signal $s_i < s_i^1$ would not be awarded a single object (follows from (A1K)). Any efficient mechanism has to determine both s_i^1 and p_i^1, and this is the core of the problem. With interdependent values, it is not warranted that bidder k has the highest value at $(s_i^1, \mathbf{s}_{-i})$. Consider a case when some other bidder $j \neq k$ has the highest value which is strictly higher than p_i^1. Then, for any s_i' slightly above s_i^1, bidders i and j are the winners. Thus, whether it is bidder i with signal s_i or with signal s_i', any efficient mechanism has to allocate one object to her at the same price p_i^1, but, it also has to allocate the other object to a different player depending on s_i. To differentiate between these two types of bidder i in order to determine the other winner while allocating one object to bidder i is a practically hopeless task for a dynamic mechanism such as an open ascending price auction. The mechanism cannot "promise" one object to bidder i—if s_i' is slightly less than s_i^1, some other bidder, not necessarily k or j, has to win the remaining object. This is exactly the reason why the auction proposed in Ausubel (1997) fails to be efficient when values are interdependent. When the number of objects for sale increases these problems become even worse.

To summarize, in an efficient mechanism, winners are determined by comparison of marginal values at the actual $\mathbf{s}$, while the prices they have to pay are determined separately, under a counterfactual exercise—bidder i has to pay for the lth object she won the price p_i^l equal to her marginal valuation $V_i^l(s_i^l, \mathbf{s}_{-i})$, where s_i^l is the signal with which she would marginally win l objects—$V_i^l(s_i^l, \mathbf{s}_{-i})$ is equal to the $(K-l+1)$th highest of the marginal valuations of the others.

The proposed auction separates processes of determining winners and prices, and by doing so achieves efficiency.

[4]Perry & Reny (1999) show the uniqueness of the marginal prices under ex-post incentive compatibility.

4.2 An open ascending price efficient auction

4.2.1 Structure

Overview The proposed mechanism consists of a series of sequential single-object English auctions with reentry. These auctions are subdivided into two stages. Stage 1 is composed of K auctions, each determining a winner for one of the objects. During Stage 2 the prices the winners have to pay are determined. The number and particular details of the Stage 2 auctions depend on the results of Stage 1.

Rules Each individual auction is conducted according to the rules of the English auction with reentry described in Section 2.3.1. The identity of the winner and the winning price are publicly announced. So, before the start of a particular auction all public information in every previous auction is commonly known.

Stage 1 This stage spans K auctions. The objects are sold one-by-one. The winner of each auction will at the end obtain an object. Therefore, at given auction, say kth, all bidders know identities of the bidders who have already won $k-1$ objects.

Stage 2 At the end of Stage 1 all K objects are assigned to the bidders. All bidders who have won at least one good are arbitrarily ordered. According to this order, for every winner a number of auctions is run to determine the price for each object won by this bidder. Suppose it is bidder j, who has won L objects. Then for each of his objects, from 1st to Lth, the following procedure applies:

The price for lth object won by bidder j is determined as follows. Every bidder $i \neq j$ is assigned a number $k_i = 1$, k_j is set equal to l. Thus, $\sum_{i=1}^{N}(k_i - 1) = l - 1 < K$. A single-object English auction with reentry is conducted with bidder j being prohibited from exiting. Bidder j necessarily wins this auction at a price p_j. Every runner-up—a bidder who has exited last—has her k_i increased by one. If $\sum_{i=1}^{N}(k_i - 1) < K$, another English auction with reentry is conducted, every runner-up's k_i is increased by one. This is repeated until $\sum_{i=1}^{N}(k_i - 1) \geq K$. The price p_j^l, bidder j has to pay for her lth object is equal to the resulting price in the last auction.

So, how is this supposed to work? In the beginning of every auction in Stage 1, bidders know how many objects were previously allocated and to whom. They obtain the vector $\mathbf{k}$, where for every bidder i, k_i is the number of objects already allocated to i plus 1. In the auction, therefore, bidder i is bidding for her k_ith object, so $\mathbf{V}^{\mathbf{k}}$ is an accurate representation of value functions in accordance with which bidders are going to bid in this auction.

During Stage 2, to determine p_j^l, the procedure described above starts with $\mathbf{k}$ that corresponds to a situation where only bidder j has been allocated $l-1$ objects. Given $\mathbf{k}$, bidders will bid according to $\mathbf{V^k}$. At the end of the first auction, bidder j wins at price p, at which, if the bidders follow suggested strategies as in Section 2.3.2, $p = V_j^l(x_j, \mathbf{s}_{-j}) = \max_{i \neq j} V_i^1(x_j, \mathbf{s}_{-j})$. If the number of runners-up is n, the value $V_j^l(x_j, \mathbf{s}_{-j})$ of bidder j is equal to the nth highest of the marginal values of the remaining bidders. If $n \geq K - l + 1$, the price is found, $p_j^l = p$. Otherwise all runners-up are assigned one object each, a new $\mathbf{k}$ is obtained, and another auction is conducted. In that auction, the final price, p', will not exceed p, and $x_j' \leq x_j$. Thus all marginal values for assigned objects of all runners-up in the previous auction are not lower than p'. Therefore, $V_j^l(x_j', \mathbf{s}_{-j})$ is equal to nth highest of the marginal values of the others, where n is the number of runners-up in the current and the previous auction. All current runners-up are assigned one object each. This is repeated until n exceeds $K - l$, and the Vickrey price p_i^l is the final price of the last auction.

4.2.2 An illustration

Suppose there are 2 objects for sale with three bidders interested. Their value functions are:

$$\begin{aligned} V_1^1 &= 4s_1 + 2s_2, \quad V_1^2 = 3s_1 + s_2; \\ V_2^1 &= 4s_2 + 2s_3, \quad V_2^2 = 3s_2 + s_3; \\ V_3^1 &= 4s_3 + 2s_1, \quad V_3^2 = 3s_3 + s_1. \end{aligned}$$

Suppose the realized signals are $(s_1, s_2, s_3) = (0.8, 0.5, 0.7)$. Thus, $V_3^1 = 4.4$ and $V_1^1 = 4.2$ are the two highest marginal values, so both bidder 1 and bidder 3 has to win one object each to obtain an efficient allocation.

In the proposed mechanism bidders follow equilibrium strategies from Section 2.3.2 according to appropriate $\mathbf{V^k}$. Stage 1 consists of two auctions. In the first auction, each player bids according to V_i^1. It is bidder 2 who drops out first at $p_2 = 3$, revealing $s_2 = 0.5$. Bidder 1 is the next to drop out at $p_1 = 4.2$, bidder 3 is the winner. In the second auction, with bidder 3 bidding according to V_3^2, it is bidder 3, who drops out first at $p_3 = \frac{26}{7}s_3 = 2.6$.[5] Then, bidder 2 drops out at $p_2 = 3.4$, bidder 1 is the winner.

In the second stage suppose the order is $\{3, 1\}$. Then, the first auction (third overall) is conducted according to $\mathbf{k} = (1, 1, 1)$, it coincides with the first auction in Stage 1. Bidder 1 is the runner-up, her k_1 is set to 2 in the second auction (fourth overall). In this auction, bidder 2 drops out first at $p_2 = \frac{26}{5}s_2 = 2.6$. Bidder 1 will exit at $p_1 = 3s_1 + s_2 = 2.9$. This is the price p_3^1 that bidder 3

[5]The estimated minimal signals $\mathbf{x}(p)$ prior to the first exit are: $x_1 = \frac{5}{26}p$, $x_2 = \frac{3}{26}p$, $x_3 = \frac{7}{26}p$. They are obtained by solving (2.6).

has to pay for his object. To verify that it is indeed the Vickrey price, obtain $s_3^1 = 0.325$ that equates V_3^1 to 2.9. At (s_1, s_2, s_3^1) the highest marginal value is V_1^1, while $V_3^1 = V_1^2 = 2.9$—the second highest among marginal values. For comparison, $V_2^1 = 2.65$.

The same is repeated for bidder 1. In the first auction (fifth overall), with $\mathbf{k} = (1, 1, 1)$, bidder 2 drops at 3, and then bidder 3 drops at 4.6 (remember, bidder 1 is always active). The second auction (sixth overall), with $\mathbf{k} = (1, 1, 2)$, coincides with the second auction in Stage 1, so $p_1^1 = 3.4$. It can be easily verified that p_1^1 is the Vickrey price.

4.3 Results

Theorem 4.1. *Under the multi-unit single-crossing and signal intensity conditions the proposed multi-unit open ascending price auction has an ex post equilibrium that is efficient.*

Proof. First, we define the equilibrium strategies. For every bidder i, the following strategy is proposed:

1. At the beginning of each individual auction determine $\mathbf{k}$ and $\mathbf{V^k}$—profile of value functions according to which bidder i and the other bidders are supposed to bid in this auction.

   (a) During Stage 1, k_j is equal to the number of objects assigned to bidder j in preceding auctions increased by 1.

   (b) During Stage 2, k_j is equal to the number assigned by the auctioneer.

2. In each individual auction follow equilibrium strategy for the single-object English auction with reentry proposed in Section 2.3.2 corresponding to $\mathbf{V^k}$.

Second, we show that at the end of Stage 1, the winners are the bidders with K highest marginal values. Indeed, if bidders follow equilibrium strategies from Section 2.3.2, Theorem 2.1 shows that the winner of each individual auction is the bidder with the highest value among $\mathbf{V^k}(\mathbf{s})$. Thus, the winner of the first object is the bidder with the highest marginal value, the winner of the second object is the bidder with the second highest value, and so on.

Third, we show that during Stage 2, the Vickrey prices are determined. Suppose p_l^j, the price bidder j has to pay for lth object she won needs to be determined. Any auction of the described procedure ends at

$$p = V_j^l(x_j, \mathbf{s}_{-j}) = \max_{i \neq j} V_i^{k_i}(x_j, \mathbf{s}_{-j}). \tag{4.3}$$

At p, the bidders in $\mathcal{I}^{\mathbf{k}}(x_j, \mathbf{s}_{-j})$ have equal and maximal values. Since $V_i^k(\cdot) \geq V_i^{k+1}(\cdot)$, for any bidder $i \in \mathcal{I}^{\mathbf{k}}(x_j, \mathbf{s}_{-j})$, $i \neq j$, and any $1 \leq k < k_i$,

$$V_i^k(x_j, \mathbf{s}_{-j}) \geq V_i^{k_i}(x_j, \mathbf{s}_{-j}) = p.$$

Fix some $i \in \mathcal{I}^{\mathbf{k}}(x_j, \mathbf{s}_{-j})$, $i \neq j$. For any bidder $m \notin \mathcal{I}^{\mathbf{k}}(x_j, \mathbf{s}_{-j})$, and any $1 \leq k < k_m$, bidder m with V_m^k must have been a runner-up in one of the previous auctions. In that auction, $k_i' \leq k_i$, and resulting price p' satisfies

$$p' = V_m^k(x_j', \mathbf{s}_{-j}) = V_j^l(x_j', \mathbf{s}_{-j}) \geq V_i^{k_i'}(x_j', \mathbf{s}_{-j}) \geq V_i^{k_i}(x_j', \mathbf{s}_{-j}).$$

Equation (4.3), $V_j^l(x_j', \mathbf{s}_{-j}) \geq V_i^{k_i}(x_j', \mathbf{s}_{-j})$, and (A1K) imply that $x_j' \geq x_j$, and so $p' \geq p$. As a result, again by (A1K), $V_m^k(x_j, \mathbf{s}_{-j}) \geq V_j^l(x_j, \mathbf{s}_{-j})$.

How many bidders have their marginal values higher or equal to V_j^l at $(x_j, \mathbf{s}_{-j})$? For any runner-up $i \in \mathcal{I}^{\mathbf{k}}(x_j, \mathbf{s}_{-j})$, $i \neq j$, after increasing k_i by one, exactly $k_i - 1$ marginal values are higher or equal, for any $m \notin \mathcal{I}^{\mathbf{k}}(x_j, \mathbf{s}_{-j})$, $k_m - 1$ values are. The procedure stops once

$$\sum_{i=1}^{N}(k_i - 1) = \sum_{\substack{i=1 \\ i \neq j}}^{N}(k_i - 1) + l - 1 \geq K.$$

This is exactly the moment when $p_j^l \equiv V_j^l(x_j, \mathbf{s}_{-j})$ is equal to the $(K - l + 1)$th highest marginal value of the other bidders, so p_j^l is the Vickrey price.

At last, we argue that the proposed strategies form an equilibrium. By construction of Stage 2, no bidder can affect her payoff at that moment. Thus, only the number of objects won at Stage 1 affects final payment. Since the resulting prices are the Vickrey prices, bidders receive non-negative payoff for any additional object as long as the marginal valuation of that object is one of the K highest. The payoff from any additional object with marginal valuation not among the K highest is non-positive. Thus, any deviation that results in different number of objects won cannot be profitable.

Presented equilibrium is obviously ex post.

□

An important feature of the proposed mechanism is that the total number of auctions is bounded from above by a number that depends only on K and not on the number of participating bidders as in Perry & Reny (2002), where the total number of two-bidder auctions in the second stage is $N(N-1)/2$. Indeed, Stage 1 involves K auctions, Stage 2 involves at most $K \times K$ auctions—if every winner is a different bidder the maximal value of auxiliary procedures is K per bidder. Some auctions need not be run. For instance, there is no need to repeat first auction in Stage 1. The last auction in Stage 1 immediately determines the Vickrey price that the winner of the last auction has to pay. Therefore for this bidder and her last object no auctions in Stage 2 need to be conducted. Therefore, we have established

Corollary 4.1. *Total number of single-object English auction needed to achieve efficiency does not exceed K^2.*

The proposed mechanism is is quite flexible; it can be built upon any single-object efficient auction with appropriate multilateral extensions of assumptions. For example, if one would like to use English auctions without reentry, generalized single-crossing condition needs to be satisfied for $\mathbf{V^k}$ for any admissible $\mathbf{k}$.

4.4 Concluding remarks

4.4.1 Variations

There is a lot of redundancy in the information exchange in the proposed auction. Indeed, in each individual auction, players bid as if they learn their signals anew. After the winner of the first auction of Stage 1 drops for the first time, the signals of all the bidders are revealed if they follow equilibrium strategies. This suggests that there is a lot of improvement possible in the structure of the proposed auction. I would like to mention two directions.

Variation 1. Proceed as in Stage 1 until the winner of the first auction exits for the first time. Either this happens or the same bidder wins all objects, in which case the minimal value of the signal the winner must receive to win all K objects is inferred. For efficient allocation and for the Vickrey prices it does not matter whether the winner has this inferred signal or her true signal as long as she obtains all the objects.

It is important to note that during this "abridged" Stage 1 the number of admissible $\mathbf{k}$ that might occur and so the number of different auctions that can be conducted is greatly reduced. The winner of all of the auctions in the modified Stage 1 must have the same identity, and so any possible $\mathbf{k}$ can be obtained from $\mathbf{k} = (1, \ldots, 1)$ by changing a single k_i to some other integer. Therefore, a much weaker form of signal intensity assumption, $(A1N)$, suffices for this "abridged" Stage 1 to work—it is required only for these few possible $\mathbf{k}$.

We may presume that at the end of Stage 1, all signals of all bidders are revealed (if the bidders follow proposed strategies). As a result, every single bidder can calculate who are the winners and their prices. Thus, Stage 2 can be conducted in any fashion that elicits this information from the bidders and ensures efficient allocation. In particular, a simpler version of Perry & Reny (2001) auction can be used. There is no need to make inferences, so players can simply reduce their demands in a manner that every winner j clinches her lth object at the Vickrey price p_j^l.

Variation 2. Suppose that instead of playing in person in each of the individual auctions each player is required to submit a program to play. Vector $\mathbf{k}$ or the information about who have won and how many objects are the only inputs. The auctioneer then can sell the objects in the exactly the same manner as proposed in the essay. Prior to the start of any single-object auction, it announces **k**. No other information is given, in particular, programs would not know what has happened in the previous auctions or even how previous auctions looked like. As a result, a program cannot determine whether the current auction is being played in Stage 1 or Stage 2, or if it is a bogus auction, the auctioneer may conduct to make any inferences out of **k** sufficiently noisy. So, in each individual auction a program has much stronger incentives to play than an actual bidder has.

4.4.2 Searching for the ultimate mechanism

What is required from a good mechanism? First, it has to achieve its designed goal. Second, the weaker is the set of conditions under which it works the better. Third, it has to be relatively transparent and simple to play. Last, but not least, it is preferable if players have strong incentives to play.

How does presented construction performs according to these criteria? It is efficient, so it works. Assumptions (A1K) and (A2K) are weak as argued in Chapter 2. The single-crossing condition, exactly in (A1K) specification, is necessary for efficiency.[6] The Perry & Reny (2002) mechanims and Perry & Reny (2001) ascending auction use "strong" form of single-crossing, required to be satisfied for any pair of bidders, not only for the members of the winners circle, and not only for admissible **k**. Thus, the strong form is incomparable with the presented pair of conditions. The signal intensity condition, (A2K), imposes restrictions at exactly the same signal profiles at which weak single-crossing, (A1K), does. Therefore, it might be loosely argued that the conditions presented in this essay are preferable to those in these two papers. Moreover, they can be further weakened if variations of the presented mechanism are employed (Variation 1).

The proposed auction is relatively transparent and simple. First, it employs at most K^2 auctions, a number that does not depend on N. Each individual auction is an English auction, that is, it is simple to play, and it allows for exchange of information. The amount of information that is being carried from one auction to another is minimal. Compared to existing constructions it stands somewhere in the middle. Individual auctions in Perry & Reny (2002), being of two-bidder variety, are much simpler, but the number of those is of the order N^2. An ascending auction in Perry & Reny (2001) allocates all objects simultaneously, that makes it very attractive from strategic point of view—every bidder has strong incentives to bid, but it is also complicated. Every bidder has to submit directed bids against every other bidder, and the process of information updating is static—inferences occur only when somebody adjusts his or her demands. This, in turn, necessitates imposing strong form of single-crossing condition to achieve efficiency. This type of

[6]The single-crossing is also sufficient for efficiency of the Dasgupta & Maskin (2000) and generalized VCG mechanisms.

information processing does not seem to be “natural”—even if no bidder changes her demands, the very increase in the price level has to convey some information.

Note also, that the presented construction reduces to the single-object English auction with reentry when $K = 1$. This is unlike Perry & Reny (2002) and Perry & Reny (2001), that are even with $K = 1$ still have pairwise auctions or pairwise demands.

It seems that, purely from the complexity point of view, mainly because the total number of auctions does not depend on N and each individual auction is relatively simple, the proposed construction is the most appealing among the three. It has, however, similar to Perry & Reny (2002), a major drawback from strategic point of view. After a first few auctions in Stage 1, once two different winners are determined, the signals of everyone are supposedly revealed. Every bidder would know who and at what price has to win the objects. Therefore, the incentives to follow suggested strategies in each of the remaining individual auctions are weak. In particular, every bidder, who does not win a single object, has no incentives to play at all, but her signal and future “correct” behavior is essential in determining winners and prices. As shown in Variation 2, this can be artificially resolved to some extent.

As underscored in Maskin (2001), to find an open auction counterpart to the Vickrey-Clark-Groves mechanism in the case of multiple goods is a very important issue, and great deal of work remains to be done. I believe that the proposed construction is an important step on the way.

Appendix A

English auctions with reentry

A.1 Implications of the signal intensity condition (A2)

First, note that since $\mathbf{V}$ is twice differentiable, the signal intensity condition (A2) is equivalent to the requirement that the directional derivative of $V_i(\mathbf{s})$ with respect to s_i is positive along the path with V_j being fixed for all $j \neq i$, $j \in \mathcal{I}(\mathbf{s})$, and s_k being fixed for all $k \in \mathcal{N} \setminus \mathcal{I}(\mathbf{s})$. Denote $\mathcal{A} = \mathcal{I}(\mathbf{s})$, since $\det DV_{\mathcal{A}} \neq 0$ (value functions are regular) we can write at $\mathbf{x}(p) = \mathbf{s}$

$$\frac{d\mathbf{x}_{\mathcal{A}}}{dp} = B \cdot \frac{d\mathbf{V}_{\mathcal{A}}}{dp}, \tag{A.1}$$

where $B = (b_{ij}) = (DV_{\mathcal{A}})^{-1}$, $\frac{d\mathbf{x}_{\mathcal{A}}}{dp}$ and $\frac{d\mathbf{V}_{\mathcal{A}}}{dp}$ are the columns of $\frac{dx_i}{dp}$ and $\frac{dV_i}{dp}$ for all $i \in \mathcal{A}$, and the signals of players not from $\mathcal{A}$ are fixed, $\frac{dx_k}{dp} = 0$ for $\forall k \notin \mathcal{A}$. If the values of all players from $\mathcal{A} \setminus \{i\}$ are fixed, $\frac{dV_j}{dp} = 0$ for all $j \in \mathcal{A}$, $j \neq i$, we have $\frac{dx_i}{dp} = b_{ii}\frac{dV_i}{dp}$. So, the value of the directional derivative in question is b_{ii}, and the signal intensity condition requires $b_{ii} > 0$.

Lemma A.1. *Given $\mathbf{s} \neq \mathbf{0}$, Assumption (A2) implies that for any proper subset of at least two players $\mathcal{J} \subset \mathcal{I}(\mathbf{s})$ and any player $j \in \mathcal{J}$, $(DV_{\mathcal{J}})^{-1}_{jj} > 0$.*

Proof. The regularity of value functions, $\det DV_{\mathcal{J}} \neq 0$ and $\det DV_{\mathcal{L}} \neq 0$, where $\mathcal{L} = \mathcal{J} \setminus \{j\}$, implies $b_{\mathcal{J},jj} = (DV_{\mathcal{J}})^{-1}_{jj} = \det DV_{\mathcal{L}} / \det DV_{\mathcal{J}} \neq 0$. To show that $b_{\mathcal{J},jj} > 0$ we disturb $\mathbf{s}$ to obtain $\mathbf{s}(\varepsilon)$, at which $\mathcal{J} = \mathcal{I}(\mathbf{s}(\varepsilon))$. Then, by Assumption (A2) at $\mathbf{s}(\varepsilon)$, $b_{\mathcal{J},jj}(\mathbf{s}(\varepsilon)) > 0$. Thus, $\lim_{\varepsilon \to \mathbf{0}} b_{\mathcal{J},jj}(\mathbf{s}(\varepsilon)) = b_{\mathcal{J},jj}(\mathbf{s}) > 0$.

The signals are disturbed as follows. Suppose that players 1 to K are from $\mathcal{K} = \mathcal{I}(\mathbf{s}) \setminus \mathcal{J}$. Note that $s_i > 0$ for all $i \in \mathcal{I}(\mathbf{s})$ since $\mathbf{s} \neq \mathbf{0}$, and no player can have the maximal value with the lowest possible signal. Consider $\mathbf{s}^{(1)} = (\mathbf{s}^{(1)}_{\mathcal{I}(\mathbf{s})}, \mathbf{s}_{-\mathcal{I}(\mathbf{s})})$, where $\mathbf{s}^{(1)}_{\mathcal{I}(\mathbf{s})}$ is constructed as: $s^{(1)}_1 = s_1 - \varepsilon_1$ for some small $\varepsilon_1 > 0$, $\left(s^{(1)}_i\right)_{i \in \mathcal{I}(\mathbf{s}) \setminus \{1\}}$

are such that $V_i(\mathbf{s}^{(1)}) = V_i(\mathbf{s})$ for all $i \in \mathcal{I}(\mathbf{s}) \setminus \{1\}$. If ε_1 is small enough, then by Assumption (A2), V_1 decreases and $\mathcal{I}(\mathbf{s}^{(1)}) = \mathcal{I}(\mathbf{s}) \setminus \{1\}$. Given ε_1, one can in a similar manner construct $\mathbf{s}^{(2)}$, such that $\mathcal{I}(\mathbf{s}^{(2)}) = \mathcal{I}(\mathbf{s}) \setminus \{1, 2\}$ for small enough $\varepsilon_2(\varepsilon_1)$, where the values of players from $\mathcal{I}(\mathbf{s}) \setminus \{1, 2\}$ are fixed, the signals of player 1 and players $\mathcal{N} \setminus \mathcal{I}(\mathbf{s})$ are fixed, and $s_2^{(2)} = s_2^{(1)} - \varepsilon_2$. Proceeding further, one can obtain $\mathbf{s}^{(K)} = \mathbf{s}(\varepsilon)$, for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_K)$, with $\mathcal{I}(\mathbf{s}(\varepsilon)) = \mathcal{J}$. □

The following lemma shows that the signal intensity condition implies that no player would immediately regret his action after he exits or (re)enters the auction.

Consider a signal profile $\mathbf{s}$ and a subset $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ of at least two players, denote $p^0 = V_i(\mathbf{s})$, $i \in \mathcal{I}$. Pick any player $i \in \mathcal{A}$. Let $\mathbf{x}(p)$ be a solution to the system (2.8) for $\mathcal{A}$ with the initial condition $\mathbf{x}(p^0) = \mathbf{s}$. Let $\mathbf{x}^{\mathcal{B}}(p)$ be a solution to the system (2.8) for $\mathcal{B} = \mathcal{A} \setminus \{i\}$ with the same initial condition. The difference between $\mathbf{x}(p)$ and $\mathbf{x}^{\mathcal{B}}(p)$ is that in the first case player i is active, while in the second case he is treated as inactive. If player i is considered inactive his value $V_i(\mathbf{x}(p))$ changes as

$$\frac{dV_i^{\mathcal{B}}}{dp} = \sum_{j \in \mathcal{B}} \frac{\partial V_i}{\partial s_j} \frac{dx_j^{\mathcal{B}}}{dp}. \tag{A.2}$$

Lemma A.2. *With the notation specified above, Assumption (A2) guarantees that player i will not regret his decision to enter or exit. Formally, if $\frac{dx_i}{dp} > 0$ then $\frac{dV_i^{\mathcal{B}}}{dp} < 1$. If $\frac{dy_i}{dp} = \left(1 - \frac{dV_i^{\mathcal{B}}}{dp}\right) / \frac{\partial V_i}{\partial x_i} < 0$, meaning $\frac{dV_i^{\mathcal{B}}}{dp} > 1$, then $\frac{dx_i}{dp} < 0$.*

Proof. Suppose in (2.8), $\frac{dx_i}{dp} > 0$. Substituting $\frac{d\mathbf{V}_{\mathcal{A}}}{dp} = \mathbf{1}$ to (A.1) we obtain

$$0 < \frac{dx_i}{dp} = \sum_{j \in \mathcal{B}} b_{ij} + b_{ii}. \tag{A.3}$$

Now, if we consider system (2.8) for $\mathcal{B}$, we have

$$0 = \frac{dx_i^{\mathcal{B}}}{dp} = \sum_{j \in \mathcal{B}} b_{ij} + b_{ii} \frac{dV_i^{\mathcal{B}}}{dp}. \tag{A.4}$$

Since $b_{ii} > 0$ by Lemma A.1 we must have $\frac{DV_i^{\mathcal{B}}}{dp} < 1$. The other part of the statement follows similarly. □

A.2 Properties of $\mathbf{x}(p)$ and $\mathbf{w}(\cdot, p)$

Lemma A.3. *For all p and every player $i \in \mathcal{I}(\mathbf{x}(p))$, $x_i(p) > 0$ for $p > 0$.*

Proof. To make the proof as straightforward as possible, let us linearly extend all value functions into the domain of negative signals. For each $s_i < 0$ and any player j, let $V_j(s_i, \mathbf{s}_{-i}) = V_j(0, \mathbf{s}_{-i}) + s_i \frac{\partial V_j(0, \mathbf{s}_{-i})}{\partial s_i}$. In case of two or more negative signals the above extension can be done sequentially.

First, we show the result in the neighborhood of $p = 0$ by induction. For $\mathcal{J} = \{i, j\}$ from (2.8) by the single-crossing (where $V_{ij} = \frac{\partial V_i}{\partial s_j}$)

$$\frac{dx_i(\mathbf{0})}{dp} = \frac{V_{jj} - V_{ij}}{V_{ii}V_{jj} - V_{ij}V_{ji}} > 0, \qquad \frac{dx_j(\mathbf{0})}{dp} = \frac{V_{ii} - V_{ji}}{V_{ii}V_{jj} - V_{ij}V_{ji}} > 0. \tag{A.5}$$

Now, suppose for any subset $\mathcal{K} \subset \mathcal{N}$ of K players, $x_i(p) > 0$ for any $i \in \mathcal{K}$, $x_j(p) = 0$ for all $j \notin \mathcal{K}$ for all $p < \varepsilon$ for some $\varepsilon > 0$. We will show that for any arbitrary subset $\mathcal{J} \subset \mathcal{N}$ of $K+1$ players there exists such an open neighborhood of 0, such that $x_i(p) > 0$ for all $i \in \mathcal{J}$ in the solution to (2.8) with $\mathcal{A} = \mathcal{J}$ and $\mathbf{x}(0) = \mathbf{0}$.

Suppose not, that is there exist $\varepsilon > 0$ and a player j, such that in the solution to (2.8) for $\mathcal{A} = \mathcal{J}$, $x_j(p) \le 0$ for $p < \varepsilon$. Then, there exist a (possibly smaller) ε_1 such that $\frac{dx_j}{dp} \le 0$ for $p < \varepsilon_1$. Lemma A.2 implies that $\frac{dV_j^{\mathcal{B}}}{dp} \ge 1$, hence $V_j(\mathbf{x}^{\mathcal{B}}(p)) \ge \max_{i \in \mathcal{B}} V_i(\mathbf{x}^{\mathcal{B}}(p))$ for all $p < \varepsilon_1$, where $\mathcal{B} = \mathcal{J} \setminus \{j\}$. Note that $x_j^{\mathcal{B}}(p) = 0$ and, by the step of induction, $x_i^{\mathcal{B}}(p) > 0$ for all $i \in \mathcal{B}$. Thus, it is either $j \in \mathcal{I}(\mathbf{x}^{\mathcal{B}}(p))$ or $\mathcal{J} \cap \mathcal{I}(\mathbf{x}^{\mathcal{B}}(p)) = \emptyset$. By construction, if $k \notin \mathcal{J}$ then $x_k^{\mathcal{B}}(p) = 0$. So, there exists a player in the winners' circle at $\mathbf{x}^{\mathcal{B}}(p) \ne \mathbf{0}$ with the signal equal to 0, which is a contradiction. Induction is complete.

The above analysis shows that there are no problems with starting the auction. By construction, $\mathcal{A} \subset \mathcal{I}(\mathbf{x}(p))$ for all p. So, if $p > 0$, we must have $x_i(p) > 0$ for all $i \in \mathcal{I}(\mathbf{x}(p))$. Otherwise, let $p^0 > 0$ be the lowest price that $\exists i \in \mathcal{I}(\mathbf{x}(p^0))$ with $x_i(p) = 0$. Such p^0 is well defined since the result holds in the neighborhood of 0, so we have a contradiction to the assumption that a player cannot have the maximal value with the lowest possible signal. □

Lemma A.4. *For any given initial signal profile* $\mathbf{s}$, $\mathbf{x}(p)$ *and* $\mathbf{y}(p)$ *satisfy*

1. *At any price level* p, $V_i(\mathbf{x}(p)) = p$ *for all* $i \in \mathcal{A}$ *and* $V_j(\mathbf{x}(p)) \le p$ *for all* $j \notin \mathcal{A}$, *where* $\mathcal{A}$ *is the set of players used to solve for* $\mathbf{x}(p)$.

2. $\mathbf{x}(p)$, $\mathbf{y}(p)$, *and* $w_i(\cdot, p)$ *are continuous in* p.[1]

Proof. The whole price line (as a time line) can be divided into a sequence of points $p_0 = 0, p_1, \ldots, p_n, \ldots$ such that for $p \in [p_{n-1}, p_n]$ a system like (2.7) is solved for

[1] There is one exception. If player j unexpectedly enters earlier than expected (Case 1*b*), if the signal of player j does not affect values of the other players, then there will be an instantaneous adjustment of $x_j(p)$ at p (see Appendix A.3.1 below). This adjustment concerns player j only, none of the $\mathbf{y}(p)$, $\mathbf{x}_{-j}(p)$, and $w_i(\cdot, \mathbf{x}_{-i})$ for all $i \ne j$ are affected.

some $\mathcal{A}_n$. Since the initial condition for $\mathbf{x}(p)$ at p_{n-1} is the same as the terminal value of $\hat{\mathbf{x}}(p_{n-1})$ of the system solved in the previous segment for $\mathcal{A}_{n-1}$, $\mathbf{x}(p)$ is continuous. Obviously then, $\mathbf{y}(p)$ and $w_i(\cdot, p)$ are continuous as well.

At $p = 0$, $\mathbf{x}(0) = \mathbf{0}$ and $V_i(\mathbf{x}(0)) = 0$ for any i. This means that the first statement of the lemma is satisfied at $p = 0$ for $\mathcal{A}_0 = \mathcal{N}$. Suppose now, $x(p_n)$ is such that $V_i(x(p_n)) = p_n$ for any $i \in \mathcal{A}_n$ and $V_j(x(p_n)) \leq p_n$ for any $j \notin \mathcal{A}_n$. By construction of the auction, $\mathcal{A}_{n+1}$ is defined in such a way, that in some right neighborhood of p_n, the solution $\mathbf{x}(p)$ to (2.8) with $\mathcal{A}_{n+1}$ satisfies the required properties as well. Again by construction, p_{n+1} is defined as the next price at which someone exits or enters, or the next price at which $\mathcal{A}_{n+1}$ has to be amended to keep $V_i(\mathbf{x}(p)) = p$ for all $i \in \mathcal{A}$ and $V_j(\mathbf{x}(p)) \leq p$ for all $j \notin \mathcal{A}$. In both cases it is either the suggested strategies or the proposed off-equilibrium information processing that guarantees that the required properties are satisfied on the whole $[p_n, p_{n+1}]$. □

A.3 Special procedures

A.3.1 Case 1b. Early entry

Suppose at some p^0 (possibly when the price clock is stopped) some player $j \notin \mathcal{I}(\mathbf{x}(p^0))$ becomes active, intends to be active, or is undecided. Suppose an instantaneous adjustment of $x_j(p^0)$ to $y_j(p^0)$ is made. Typically, set $\mathcal{K} = \mathcal{I}(\mathbf{x}(p^0)) \setminus \mathcal{I}(y_j(p^0), \mathbf{x}_{-j}(p^0))$ of the players, who had the highest value at $\mathbf{x}(p^0)$ but are no longer members of the winners' circle after the adjustment, is not empty. Since (A1)-(A2) are imposed only on the members of the winners' circles, the true signals of players from $\mathcal{K}$ may not be inferred properly thereafter.

The procedure below outlines how $x_j(p)$ can be adjusted gradually. It guarantees that the auction does not end before the full adjustment takes place. This is done by ensuring that $x_i(p)$ decreases for at least some player i during the adjustment.

The whole procedure is based on the following two subprocedures.

Step 1. This is an exceptional case when a full or partial adjustment can be made instantaneously. Suppose for any player $i \in \mathcal{I}(\mathbf{x}(p^0))$, $\frac{\partial V_i(s_j, \mathbf{x}_{-j}(p^0))}{\partial s_j} = 0$ for all $s_j \in [x_j(p^0), t_{ji}]$, that is any increase of j's signal up to t_{ji} has no effect on player i's value. For all $k \notin \mathcal{I}(\mathbf{x}(p^0))$ (including player j) define t_{jk} to be the lowest j's signal at which $V_k(t_{jk}, \mathbf{x}_{-j}(p^0)) = p^0$ or set $t_{jk} = 1$ if $V_k(s_j, \mathbf{x}_{-j}(p^0)) < p^0$ for any s_j. Define $t_j^* \equiv \min_{l \in \mathcal{N}} t_{jl}$. Instantaneously adjust $x_j(p^0)$ to t_j^*. If $t_j^* = t_{jj}$ no additional adjustment is needed, otherwise further steps will follow. If $t_j^* = t_{jk}$ for $k \notin \mathcal{I}(\mathbf{x}(p^0))$, then k becomes a member of the new winners' circle $\mathcal{I}(t_j^*, \mathbf{x}_{-j}(p))$.

Step 2. This is the main subprocedure. It applies whenever there exists $i \in \mathcal{I}(\mathbf{x}(p))$ with $\frac{\partial V_i(s_j, \mathbf{x}_{-j}(p^0))}{\partial s_j} > 0$ for $s_j \in (x_j(p^0),\ x_j(p^0) + \varepsilon)$ for some $\varepsilon > 0$. It is proposed that players find $\mathbf{x}(p)$ as the solution to the following system with an initial condition $\mathbf{x}(p^0)$

$$\begin{cases} \frac{dx_j}{dp} = g(p) \geq 0, \\ \frac{d\mathbf{x}_\mathcal{A}}{dp} = (DV_\mathcal{A})^{-1} \cdot \left(\mathbf{1} - g(p)\frac{\partial \mathbf{V}_\mathcal{A}}{\partial s_j}\right), \\ \frac{d\mathbf{x}_{-\mathcal{A}\setminus\{j\}}}{dp} = \mathbf{0}, \\ \min_{i\in\mathcal{A}} \frac{dx_i}{dp} \leq G < 0, \end{cases} \tag{A.6}$$

where $\mathcal{A} \subseteq \mathcal{I}(\mathbf{x}(p^0))$ is a set of players, who will be considered active, $j \notin \mathcal{A}$. Normally, $\mathcal{A}$ is the set of currently active players. Section A.3.2 below describes what to do in exceptional situations to find such $\mathcal{A}$.

This system is a modified version of (2.8), here $x_j(p)$ has growth rate $g(p)$. Any negative number can serve as G, the above system has a unique solution for $g(p)$ if $\min_{i\in\mathcal{A}} \frac{dx_i}{dp}$ is required to be equal to G. For any $i \in \mathcal{A}$ we have

$$\frac{dV_i}{dp} = 1 = (DV_\mathcal{A})_i \cdot \frac{d\mathbf{x}_\mathcal{A}}{dp} + g(p)\frac{\partial V_i}{\partial s_j},$$

so $g(p)\frac{\partial V_i}{\partial s_j}$ represents the effect of p on V_i through a change in $x_j(p)$. If $\min_{i\in\mathcal{A}} \frac{dx_i}{dp} > G$ then $DV_\mathcal{A} \cdot \frac{d\mathbf{x}_\mathcal{A}}{dp}$ is limited from below for all the players.

If $\exists i \in \mathcal{I}(\mathbf{x}(p^0))$ with $\frac{\partial V_i(\mathbf{x}(p^0))}{\partial s_j} > 0$, then, in particular, $g(p)\frac{\partial V_i}{\partial s_j}$ can be made arbitrarily large, thus at least for some $k \in \mathcal{A}$ (not necessarily i), $\frac{dx_k}{dp}$ has to be lower than G.

If $\frac{\partial V_i(\mathbf{x}(p^0))}{\partial s_j} = 0$ for all $i \in \mathcal{I}(\mathbf{x}(p^0))$, there exist i, such that $\frac{\partial V_i(s_j, \mathbf{x}_{-j}(p^0))}{\partial s_j} > 0$ at $s_j \in (x_j(p), x_j(p) + \varepsilon)$ for some $\varepsilon > 0$. In this case the solution to (A.6) still exists, only $g(p) \to \infty$ when $p \searrow p^0$. We have that $\frac{\partial V_i(\mathbf{z}(p))}{\partial s_j} > 0$ for $p > p^0$ for almost all continuous and piecewise smooth paths $\mathbf{z}(p)$ with $\mathbf{z}(p^0) = \mathbf{x}(p^0)$, $z_j'(p) = g(p) \to \infty$ when $p \searrow p^0$ and $z_i'(p) = \frac{dx_i}{dp}$ as in the system (A.6). So, the requirement $\min_{i\in\mathcal{A}} \frac{dx_i}{dp} \leq G < 0$ can be met by choosing a path $\mathbf{z}(p)$ such that $g(p)\frac{\partial V_i(\mathbf{z}(p))}{\partial s_j} \to g^o > 0$ when $p \searrow p^0$ for at least one player i and large enough g^0.

The resulting $\mathbf{x}(p)$ may not be extendable beyond some $\hat{p}$, with the full adjustment not yet attained at $\hat{p}$. This means that at $\mathbf{x}(\hat{p})$, $\frac{\partial V_i(\mathbf{x}(\hat{p}))}{\partial s_j} = 0$ for all $i \in \mathcal{I}(\mathbf{x}(\hat{p}))$, so another subprocedure will be started at $\hat{p}$, possibly the one of Step 1.

Full procedure can be summarized as follows. Apply an appropriate subprocedure. Repeat if necessary. Stop if player j exits, or if p' is reached such that $w_j(x_j(p'), p') = p'$—the full adjustment takes place. In case some player exits or enters and the clock is not stopped (as in the simplified analysis of Section 2.2), or a player needs to be added or removed from $\mathcal{A}$ (as in Cases 1a, 2 or 3) at some interim p, $g(p)$ is fixed and the corresponding solution is considered until later p, at which the minimal negative growth of any player from previously considered $\mathcal{A}$ equals $G/2$. If some player (not j) exits or enters at that p, different objective can be further set, such as $G/3$. Once this is finished, the procedure continues for a new $\mathcal{A}$.

Note that the above procedure can be easily extended to adjust the minimal estimated signals for more than one player. These adjustments can be done sequentially or simultaneously.

A.3.2 Case 3. Looking for a partition

In this section we establish that the partition of players on active and inactive, in which no player regrets his status, does exist at any given p^0 and appropriate $\mathbf{x}(p^0)$.

Suppose $\mathbf{x}(p^0)$ is such that for all $i \in \mathcal{I}(\mathbf{x}(p^0))$, $w_i(x_i(p^0), p^0) = p^0$. Denote $A = DV_{\mathcal{I}(\mathbf{x}(p^0))}(\mathbf{x}(p^0))$, $B = A^{-1}$. Since players $j \in \mathcal{N} \setminus \mathcal{I}(\mathbf{x}(p^0))$ are of no particular interest—their $x_j(p)$ are fixed and they will stay inactive at least for some time, we omit them from the consideration to save the notation, as if $\mathcal{N} = \mathcal{I}(\mathbf{x}(p^0))$.

Instead of actual value functions consider the linearized at $\mathbf{x}(p^0)$ system of value functions with $DV_{\mathcal{I}(\mathbf{x}(p^0))}(\mathbf{s}) = A$ for all $\mathbf{s}$.

Lemma A.5. *Fix any subset of players $\mathcal{K} \subset \mathcal{I}(\mathbf{x}(p^0))$. For the linearized at $\mathbf{x}(p^0)$ system of value functions there exist a subset $\mathcal{A} \subset \mathcal{I}(\mathbf{x}(p^0))$ of players such that for $p > p^0$ the solution $\mathbf{x}(p)$ to the system (2.8) for $\mathcal{A}$ has the following properties: $\frac{dx_k}{dp} \leq 0, \forall k \in \mathcal{K} \cap \mathcal{A}$; $\frac{dV_k(\mathbf{x}(p))}{dp} \leq 1, \forall k \in \mathcal{K} \setminus \mathcal{A}$.*

Proof. Let $n = \#\mathcal{I}(\mathbf{x}(p^0))$ and $K = \#\mathcal{K}$. Denote $z_i \equiv \frac{dx_i(p)}{dp}$ and $v_i \equiv \frac{dV_i(p)}{dp}$ for all players. We have that $A_i\mathbf{z} = v_i$ and $z_i = B_i\mathbf{v}$, where A_i and B_i are ith rows of matrices A and B correspondingly. Equations $z_i = 0$ and $v_i = 1$ define hyperplanes in the n-dimensional Euclidean space E^n of vectors $\mathbf{v}$.

Lemma states, equivalently, that there exists a point $T \in E^n$, which is an intersection of exactly n hyperplanes $\mathbf{v}_{\mathcal{A}} = 1$ and $\mathbf{z}_{-\mathcal{A}} = 0$, and which lies below hyperplanes $v_k = 1$ for all $k \in \mathcal{K} \setminus \mathcal{A}$, and $z_k = 0$ for $k \in \mathcal{K} \cap \mathcal{A}$. We will prove that such a point exists.

We restrict our attention to subspace $E_{\mathcal{K}}$ of dimensionality K, which is an intersection of E^n and $n - K$ hyperplanes $v_i = 1$ for all $i \in \mathcal{A} \setminus \mathcal{K}$ (these players are considered active and do not regret it for sure). It is enough to find an appropriate point in this subspace. Suppose that all $2K$ hyperplanes are in general position, that is any point which is an intersection of K of them does not belong to any other hyperplane. Consider an arbitrary T, which is an intersection of K hyperplanes, where exactly one of $z_i = 0$ and $v_i = 1$ is fixed for each $i \in \mathcal{K}$. Call any instance of $z_i(T) > 0$ or $v_i(T) > 1$ a *violation*. Among all possible 2^K points pick the one with the least number of violations. If this number is zero, we have found an appropriate partition.

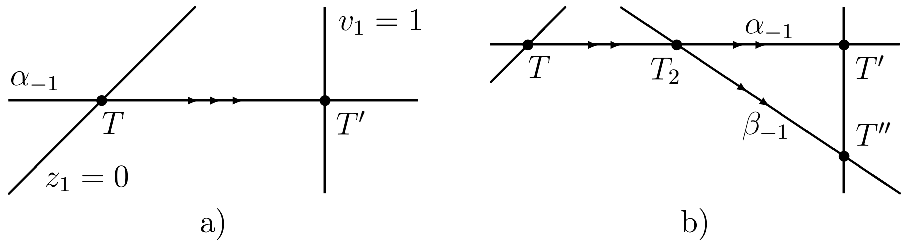

Figure A.1: v_1 is decreasing

Suppose the minimal number of violations is higher than zero and is reached at T. Without loss of generality assume that there is a violation for player 1. Consider the case $z_1(T) = 0$ with $v_1(T) > 1$. The other situation can be considered similarly. Fix the line α_{-1} which is formed by the intersection of the fixed at T hyperplanes of the others (see Figure A.1a). Let T' be the intersection of α_{-1} with $v_1 = 1$. Starting at T, by moving along α_{-1} in the direction of decreasing v_1, T' is necessarily reached . By Lemma A.1, z_1 has to decrease[2], so $z_1(T') < 0$. Thus, if no additional violations were made in the process, we have managed to reduce the number of violations by one, which contradicts the minimality of this number.

By moving from T to T', the other (not fixed) hyperplane of some other player, say 2, may have been crossed and a new violation occurred. Suppose that at T, $v_2(T) = 0$, that is $v_2 = 1$ was fixed at T for player 2. Again, the case of $z_2(T) = 0$ is completely similar. Let β_{-1} be the line in which $z_2 = 0$ is intersected with the fixed hyperplanes for all players from $\mathcal{K} \setminus \{1, 2\}$. Let T_2 be an intersection of β_{-1} and α_{-1}. Start moving from T along α_{-1} by decreasing v_1. If β_{-1} is reached it has to happen at T_2. Starting from T_2 and decreasing v_1 further, by moving along α_{-1} T' is reached, by moving along β_{-1} T'', an intersection of β_{-1} and $v_1 = 1$, is reached (see Figure A.1b). As above, at both T' and T'' there is no violation for player 1 and at one of them there is no violation for player 2 as follows from Lemma A.1. Indeed, if there is a violation for player 2, say at T', meaning $z_2(T') > 0$, by moving along the line connecting T' and T'' toward decreasing z_2, T'' is reached when $z_2 = 0$, but then v_2 has to decrease as well, so $v_2(T'') < 1$.

The above analysis shows that if, when moving along an appropriate line by decreasing v_1, the other (not fixed) hyperplane is reached for some other player j, there is a direction to continue along which no additional violation for player j is created. In fact, if there were a violation for player j it would be corrected. By continuing is this manner until v_1 reaches 1, subsequently changing directions if necessary for any player j whose second hyperplane is reached, a point T^* with less violations than at T is obtained.

[2] To see that, let $\mathcal{J}$ to include player 1 and all players i, for whom $v_i = 1$ is fixed. Lemma A.1 states that once the values of players from $\mathcal{J} \setminus \{1\}$ are fixed and the signals of players not from $\mathcal{J}$ are fixed, the signal and the value of player 1 has to move in the same direction.

If the position of hyperplanes is not general, that is there are points of intersection of more than K hyperplanes in $E_{\mathcal{K}}$, one can disturb them a little, such that (A2) is still satisfied (to keep positive a finite number of coefficients of the finite number of matrices), find the solution and take the limit. □

If at given p, the position of all relevant hyperplanes is general, then one can start (or continue) the information processing at p with the found $\mathcal{A}$, and the required inequalities for players from $\mathcal{K}$ will also be satisfied in the neighborhood of p by continuity of $DV_{\mathcal{A}}$.

The only problem with the above partition that can arise is that the partition may not be extendable beyond current price. This may happen if at the point in $E_{\mathcal{K}}$ space, found in Lemma A.5, both hyperplanes corresponding to the same player intersect for some of the players. Thus, each of such players can be treated either active or inactive. It is possible that no choice of $\mathcal{A}$ at p is dynamically extendable. That is for any $\mathcal{A}$ there will be some player from $\mathcal{K}$, such that either his imputed value increases faster than p if the player is considered inactive or $x_i(p)$ increases if he is considered active. It should be noted that this situation is extremely unlikely, not only several players need to have some specific signals (the probability of this alone is zero), but some special value functions that have these problems at these specific signals are needed as well (some specific conditions on second order derivatives have to be satisfied). Therefore, it is the case of mere theoretical possibility.

If such a case arises the proposed solution is to take all the problematic players and instead of looking at the hyperplanes $z_i = 0$ consider $z_i = \varepsilon_i < 0$ for all such i. That is, starting at p, $x_i(p)$ will be lowered intentionally. The choice of ε_i has to ensure that in the new problem the position of hyperplanes is general. A subset $\mathcal{A}$ found in Lemma A.5 will be used to calculate $\mathbf{x}(p)$ until price reaches $p + \varepsilon$, for some small $\varepsilon > 0$. Starting at $p + \varepsilon$, or at the next stop of the price clock if it happens before, revert to the normal play and by using a procedure of the Case 1b, return an estimate of the minimal possible signal of any affected (who was considered inactive) player i to the level $x_i(p)$.

Appendix B

Efficiency of the standard English auction

B.1 Equivalence Lemma

Lemma B.1. *The formulations of* GSC *given in Definitions 3.2 and 3.3 are equivalent.*

Proof. To shorten the notations we introduce $\mu_k(\mathbf{u}) \equiv \mathbf{u}\cdot\nabla V_k(\mathbf{s})$—the derivative of V_k along the direction $\mathbf{u}$. It is enough to show that the formulations are equivalent at any given $\mathbf{s}$.

($\Longrightarrow$) First we show that whenever *GSC* in the directional formulation is satisfied at $\mathbf{s}$, *GSC* in the equal increments formulation is satisfied at $\mathbf{s}$ as well. It is enough to show that every component of the equal increments vector $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}}$ is non-negative for all subsets $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$.

Step 1. Suppose inequalities in the directional formulation (3.2) are strict for all $\mathcal{A}$. Then we can show that $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \gg 0$ (every component of the vector is strictly positive).

It is done by induction on the number of bidders in $\mathcal{A}$. For $\#\mathcal{A} = 1$, $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} = \left(\frac{\partial V_{\mathcal{A}}}{\partial s_{\mathcal{A}}}\right)^{-1} > 0$.

Suppose for all $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$ with $\#\mathcal{B} \leq n-1$, $\mathbf{u}^{\mathcal{B}}_{\mathcal{B}} \gg 0$. We want to show the same for an arbitrary subset $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ with $\#\mathcal{A} = n$. Suppose there exists $\mathcal{A}$ for which some of the components of $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}}$ are non-positive. Introduce $\mathcal{B} \subset \mathcal{A}$—the subset of bidders for which the corresponding components of $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}}$ are negative, $\mathcal{C} \subset \mathcal{A}$—the subset of bidders for which the corresponding components are zeros and $\mathcal{D} \subset \mathcal{A}$—the subset of bidders for which the corresponding components are positive. By presumption, $\mathcal{B} \cup \mathcal{C} \neq \emptyset$. Obviously, $\mathcal{D}$ is also not empty. Note that $\mathbf{u}^{\mathcal{D}}_{\mathcal{D}} \gg 0$ as $\#\mathcal{D} < n$.

Suppose first that $\mathcal{B}$ is empty, thus, $\mathcal{C} \neq \emptyset$. Then $\mu_j(\mathbf{u}^{\mathcal{D}}) = 1 = \max_{i\in\mathcal{D}}\{\mu_i(\mathbf{u}^{\mathcal{D}})\}$ for $\mathcal{D} = \mathcal{A} \setminus \mathcal{C}$, vector $u^{\mathcal{D}}$, and any bidder $j \in \mathcal{C}$, which contradicts the supposition that the inequalities in (3.2) hold strictly. Thus, $\mathcal{B}$ is not empty (the subset $\mathcal{C}$ may be empty or not). Introduce vector $\mathbf{u} \equiv \mathbf{u}^{\mathcal{A}} - \mathbf{u}'$, where $u'_j = 0$ for $j \notin \mathcal{B}$ and $u'_i = u_i^{\mathcal{A}}$ for $i \in \mathcal{B}$. Clearly all the components of the vector $\mathbf{u}$ corresponding to the subset $\mathcal{D}$ are strictly positive while all the other components are equal to zero. By the construction of set $\mathcal{B}$ all the components of the vector $-\mathbf{u}'$ corresponding to the subset $\mathcal{B}$ are also strictly positive. Now consider bidder $i \in \mathcal{B}$ with the maximal $\mu_i(-\mathbf{u}')$ and bidder $j \in \mathcal{D}$ with the maximal $\mu_j(\mathbf{u})$. *GSC* in the directional formulation for the set $\mathcal{B}$ dictates that $\mu_i(-\mathbf{u}') > \mu_j(-\mathbf{u}')$. *GSC* in the directional formulation for the set $\mathcal{D}$ implies that $\mu_i(\mathbf{u}) < \mu_j(\mathbf{u})$. Since $\mathbf{u}^{\mathcal{A}} = \mathbf{u} - (-\mathbf{u}')$, we have $\mu_i(\mathbf{u}^{\mathcal{A}}) < \mu_j(\mathbf{u}^{\mathcal{A}})$. We have a contradiction since $\mu_i(\mathbf{u}^{\mathcal{A}}) = \mu_j(\mathbf{u}^{\mathcal{A}}) = 1$ by construction of vector $\mathbf{u}^{\mathcal{A}}$ and the fact that $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{D} \subset \mathcal{A}$. Therefore, $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \gg 0$.

Step 2. Suppose that weak inequalities in (3.2) are possible. We will show that $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \geqslant 0$ for all subsets $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$.

Suppose this is not the case, that is, there exists a subset of bidders $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$ such that for some $i \in \mathcal{B}$, $u_i^{\mathcal{B}} < 0$. Then we can slightly perturb the Jacobian of value functions at $\mathbf{s}$, $DV_{\mathcal{I}}(\mathbf{s})$, in the following way: add $\varepsilon > 0$ to every diagonal element,

$$DV'_{\mathcal{I}}(\mathbf{s}) = DV_{\mathcal{I}}(\mathbf{s}) + \varepsilon I_{\#\mathcal{I}}.$$

First note that all inequalities in (3.2) become strict after the perturbation—for any $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ and vector $\mathbf{u}$ from Definitions 3.2a we have $\mu'_j(\mathbf{u}) = \varepsilon \frac{\partial V_j}{\partial s_j} + \mu_j(\mathbf{u}) > \mu_j(\mathbf{u})$ for any bidder $j \in \mathcal{A}$, while $\mu'_i(\mathbf{u}) = \mu_i(\mathbf{u})$ for all $i \notin \mathcal{A}$. At the same time, if ε is small enough, by continuity, $u_i'^{\mathcal{B}}$ must be negative. Clearly, this is a contradiction to the result in Step 1.

($\Longleftarrow$) Now we prove the opposite, the fact that *GSC* in the directional formulation follows from *GSC* in the equal increments formulation. We again use the induction on the number of bidders in $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$.

For $\#\mathcal{A} = 1$ the result is obvious. Fix the subset $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ with $\#\mathcal{A} = n$ and suppose that *GSC* is satisfied at $\mathbf{s}$ in the directional formulation and equal increments formulations for subsets $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$ with $\#\mathcal{B} < n$. Suppose that *GSC* in the directional formulation is violated at $\mathbf{s}$ for $\mathcal{A}$, that is, there exists a vector $\mathbf{u}$ with $u_i > 0$ for all $i \in \mathcal{A}$ and $u_j = 0$ for $j \notin \mathcal{A}$, such that for some $k \in \mathcal{I}(s) \setminus \mathcal{A}$, $\mu_k(\mathbf{u}) > \max_{j\in\mathcal{A}} \mu_j(\mathbf{u})$. Clearly, $\mathbf{u} \neq \mathbf{u}^{\mathcal{A}}$—the equal increments vector for subset $\mathcal{A}$.

Calculate $\mu_i(\mathbf{u})$ for all bidders i from $\mathcal{A}$ and consider $\mathcal{B} \subset \mathcal{A}$ – the subset of bidders who have the highest increments to their values in the direction $\mathbf{u}$. Formally $i \in \mathcal{B} \Leftrightarrow \mu_i(\mathbf{u}) = \max_{j\in\mathcal{A}} \mu_j(\mathbf{u})$. Since $\mathbf{u} \neq \mathbf{u}^{\mathcal{A}}$, $\mathcal{B} \neq \mathcal{A}$. Consider vector $\mathbf{w}_1(t) = \mathbf{u} - t\mathbf{u}^{\mathcal{B}}$, where $\mathbf{u}^{\mathcal{B}}$ is the equal increments vector for $\mathcal{B}$. Since $\#\mathcal{B} < \#\mathcal{A}$, by the induction hypothesis and the argument above, $\mathbf{u}_{\mathcal{B}}^{\mathcal{B}} \geqslant \mathbf{0}$.

At $t = 0$, for any $j \in \mathcal{A} \setminus \mathcal{B}$ and $i \in \mathcal{B}$ we have $\mu_j(\mathbf{w}_1(t)) < \mu_i(\mathbf{w}_1(t)) < \mu_k(\mathbf{w}_1(t))$. Once we start increasing t, that is, decreasing in a special direction the signals of all the bidders from $\mathcal{B}$ only, all $\mu_i(\mathbf{w}_1(t))$, for $i \in \mathcal{B}$, decrease uniformly at rate t, while for any bidder $l \in \mathcal{I}(s) \setminus \mathcal{B}$ (including k) their $\mu_l(\mathbf{w}_1(t))$ decrease at at most the same rate, because *GSC* is satisfied for $\mathcal{B}$. Introduce t_1—the minimal value of $t > 0$ such that: either $\mu_j(\mathbf{w}_1(t)) = \mu_i(\mathbf{w}_1(t))$ for some $j \in \mathcal{A} \setminus \mathcal{B}$ and every $i \in \mathcal{B}$, or $w_{1i}(t) = 0$ for some $i \in \mathcal{B}$. In the latter case, stop. If the former case applies, introduce $\mathcal{C}$—a subset that includes $\mathcal{B}$ and all the bidders $j \in \mathcal{A} \setminus \mathcal{B}$ such that $\mu_j(\mathbf{w}_1(t_1)) = \mu_i(\mathbf{w}_1(t_1))$. Introduce the corresponding $\mathbf{u}^{\mathcal{C}}$, again $\mathbf{u}^{\mathcal{C}}_{\mathcal{C}} \geqslant \mathbf{0}$. Define $\mathbf{w}_2(t) = \mathbf{w}_1(t_1) - t\mathbf{u}^{\mathcal{C}}$. Find the smallest $t_2 > 0$ such that: either $\mu_j(\mathbf{w}_2(t_2)) = \mu_i(\mathbf{w}_2(t_2))$ for some bidder $j \in \mathcal{A} \setminus \mathcal{C}$ and every $i \in \mathcal{C}$, or $w_{2i}(t_2) = 0$ for some $i \in \mathcal{C}$, in which case stop. Again, if the former case applies define $\mathcal{D} \supset \mathcal{C}$. Repeat this procedure until for some bidder $i \in \mathcal{A}$, $w_{mi}(t_m) = 0$. This will take at most $\#\mathcal{A}$ repetitions and may result in all bidders $i \in \mathcal{A}$ having $w_{mi}(t_m) = 0$.

Note that for bidder $k \in \mathcal{I}(s)\backslash\mathcal{A}$, $\mu_k(\mathbf{w}_1(t))$ always decreased at a rate no higher than the rate for bidders from $\mathcal{B}, \mathcal{C}, \ldots$. Thus, $\mu_k(\mathbf{w}_m(t)) > \mathbf{1} = \max_{j\in\mathcal{A}} \mu_j(\mathbf{w}_m(t))$ for all stages m of the procedure and, in particular,

$$\mu_k(\mathbf{w}_m(t_m)) > \max_{j\in\mathcal{A}} \mu_j(\mathbf{w}_m(t_m)). \tag{B.1}$$

If for all $j \in \mathcal{A}$, $w_{mj}(t_m) = 0$, then, by construction, $\mathbf{w}_m(t_m) = \mathbf{0}$, which makes (B.1) impossible. If $\mathbf{w}_m(t_m) \neq \mathbf{0}$, then *GSC* in the directional formulation is violated for the set $\mathcal{A} \setminus \{i\}$, vector $\mathbf{w}_m(t_m)$ and bidder $k \in \mathcal{I}(s) \setminus \mathcal{A}$. Since $\#\{\mathcal{A} \setminus \{i\}\} < n$, this contradicts the induction presumption. □

The following corollary shows that if at some realization $\mathbf{s}$ *GSC* is violated for the subset $\mathcal{A}$ and bidder k in one of the formulations then *GSC* has to be violated in the other formulation, though possibly for some other bidders.

Corollary B.1. GSC *in the directional formulation is violated at* $\mathbf{s}$ *if and only if* GSC *in the equal increments formulation is violated at* $\mathbf{s}$.

If the subset $\mathcal{A}$ contains just one bidder the non-existence of an efficient equilibrium is shown in Maskin (1992). We concentrate on the cases where $\mathcal{A}$ contains more than one bidder. The following Corollary comes handy in the necessity proofs.

Corollary B.2. *At a given* $\mathbf{s}$*, consider an arbitrary* $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ *with* $\#\mathcal{A} = n \geq 2$. *Suppose* GSC *is satisfied at* $\mathbf{s}$ *for any subset* $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$ *with* $\#\mathcal{B} < n$. *For an arbitrary bidder* $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$ *denote* $\mathcal{C} \equiv \mathcal{A} \cup \{k\}$. GSC *is violated at* $\mathbf{s}$ *for* $\mathcal{A}$ *and bidder* k *if and only if (i):* $u^{\mathcal{C}}_k < 0$ *and* $u^{\mathcal{C}}_j > 0$ *for all* $j \in \mathcal{A}$ *or (ii):* $u^{\mathcal{C}}_k > 0$ *and* $u^{\mathcal{C}}_j < 0$ *for all* $j \in \mathcal{A}$.

Proof. By the conditions of the corollary one needs at least $n+1$ bidders to violate *GSC*. Therefore, from the proof of Lemma B.1, $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \geqslant \mathbf{0}$.

If $u^{\mathcal{C}}_k = 0$ then $\mathbf{u}^{\mathcal{C}} = \mathbf{u}^{\mathcal{A}}$ which happens if and only if *GSC* is satisfied (with equality) for bidder k and $\mathcal{A}$. For all other cases it suffices to analyze the case of strict inequalities in (3.3) only. By slightly disturbing the Jacobian as we did in the

Step 2 of the proof of Lemma B.1 we eliminate all equalities. If ε is small enough, whether GSC is violated for bidder k, and the sign of $u_k^{\mathcal{C}}$, remain unchanged.

If $u_k^{\mathcal{C}} > 0$ and $\mathbf{u}_{\mathcal{A}}^{\mathcal{C}} \gg \mathbf{0}$, then consider vector $\mathbf{u}'$ such that $u'_k = 0$ and $\mathbf{u}'_{-k} = \mathbf{u}_{-k}^{\mathcal{C}}$. Note that $\mathbf{u}' \neq \mathbf{u}^{\mathcal{A}}$. Since $\mu_k(\mathbf{u}^{\mathcal{C}}) = \mu_i(\mathbf{u}^{\mathcal{C}}) = 1$ for all $i \in \mathcal{A}$ and GSC is satisfied for $\mathcal{B} = \{k\}$, we have $\mu_k(\mathbf{u}') \leq \min_{i\in\mathcal{A}} \mu_i(\mathbf{u}')$. Suppose GSC is violated for $\mathcal{A}$ and bidder k. Then there exists vector $\mathbf{u}$, $\mathbf{u}_{\mathcal{A}} \gg \mathbf{0}$, $\mathbf{u}_{-\mathcal{A}} = \mathbf{0}$, such that $\mu_k(\mathbf{u}) > \max_{i\in\mathcal{A}} \mu_i(\mathbf{u})$. Consider vector $\mathbf{w}(t) = \mathbf{u} - t\mathbf{u}'$. Obviously, $\mu_k(\mathbf{w}(t)) > \max_{i\in\mathcal{A}} \mu_i(\mathbf{w}(t))$, for $t > 0$. Note that $w_j(t) = 0$ for any t and all $j \notin \mathcal{A}$,.$w_i(0) > 0$ for any $i \in \mathcal{A}$. Since $\mathbf{u}'_{\mathcal{A}} \gg \mathbf{0}$, there exist the smallest $t' > 0$ such that for some $i \in \mathcal{A}$, $w_i(t') = 0$. Then, GSC is violated for bidder k, subset $\mathcal{B} = \mathcal{A} \setminus \{i\}$ and vector $\mathbf{w}(t')$, which is a contradiction since $\#\mathcal{B} < n$. Thus, if $\mathbf{u}_{\mathcal{C}}^{\mathcal{C}} \gg \mathbf{0}$, GSC is satisfied for $\mathcal{A}$ and k.

Similarly, if $u_k^{\mathcal{C}} > 0$ and there exists a bidder $i \in \mathcal{A}$ with $u_i^{\mathcal{C}} > 0$, GSC is satisfied for $\mathcal{A}$ and k.

Next, we show that $\mathbf{u}_{\mathcal{C}}^{\mathcal{C}}$ can have either 1 or n negative components. Suppose this is not true. Denote with $\mathcal{B} \subset \mathcal{C}$ the set of bidders with positive components, with $\mathcal{D} \subset \mathcal{C}$ the set of bidders with negative components; $\mathcal{D} \cup \mathcal{B} = \mathcal{C}$. Consider vector $\mathbf{w}_1(t) = \mathbf{u}^{\mathcal{C}} + t\mathbf{u}^{\mathcal{D}}$. Clearly $\mathbf{u}_{\mathcal{D}}^{\mathcal{D}} \gg \mathbf{0}$, and $\mu_j(\mathbf{u}^{\mathcal{D}}) > \mu_i(\mathbf{u}^{\mathcal{D}})$ for all $j \in \mathcal{D}$ and $i \in \mathcal{B}$ since $\#\mathcal{D} \leq \#C - 2 < a$. Then $\mu_j(\mathbf{w}_1(t)) > \mu_i(\mathbf{w}_1(t))$ for all $t > 0$. There exists the minimal $t_1 > 0$ such that for some $j \in \mathcal{D}$, $\mathbf{w}_{1j}(t) = 0$. Consider the subset $\mathcal{E}$ of bidders $l \in \mathcal{D}$ with $w_{1l}(t) < 0$, and vector $\mathbf{w}_2(t) = \mathbf{w}_1(t_1) + t\mathbf{u}^{\mathcal{E}}$. Increase t until for some bidder $j \in \mathcal{E}$, $w_{2j}(t_2) = 0$. Again, $\mu_j(\mathbf{w}_2(t_2)) > \mu_i(\mathbf{w}_2(t_2))$ for all $j \in \mathcal{E}$ and $i \in \mathcal{B}$. Repeating this procedure we obtain vector $\mathbf{w}_m(t_m)$ such that for all $j \in \mathcal{D}$, $w_{mj}(t_m) = 0$ while for all $i \in \mathcal{B}$, $w_{mi}(t_m) = u_i^{\mathcal{C}} > 0$. There exists a bidder $j \in \mathcal{D}$, with $w_{mj}(0) < 0$. Clearly $\mu_j(\mathbf{w}_m(t_m)) > \mu_i(\mathbf{w}_m(t_m))$ for all $i \in \mathcal{B}$. Therefore GSC is violated for bidder j, subset $\mathcal{B}$, and vector $\mathbf{w}_m(t_m)$, which is a contradiction since $\#\mathcal{B} < n$.

Suppose $\mathbf{u}_{\mathcal{C}}^{\mathcal{C}}$ has 1 negative component—$u_k^{\mathcal{C}} < 0$, so $\mathbf{u}_{\mathcal{A}}^{\mathcal{C}} \gg \mathbf{0}$. Consider vector $\mathbf{u}'$ such that $u'_k = 0$, $\mathbf{u}'_{-k} = \mathbf{u}_{-k}^{\mathcal{C}}$. Since GSC is satisfied for $\mathcal{B} = \{k\}$, $\mu_k(\mathbf{u}') > \max_{i\in\mathcal{A}} \mu_i(\mathbf{u}')$. Therefore, GSC is violated for k, $\mathcal{A}$, and $\mathbf{u}'$. Note that by the argument above GSC is satisfied for any bidder $i \in \mathcal{A}$, $i \neq k$ and subset $\mathcal{C} \setminus \{i\}$.

Suppose $\mathbf{u}_{\mathcal{C}}^{\mathcal{C}}$ has n negative components. There are two cases to consider, $u_k^{\mathcal{C}} < 0$ and $u_k^{\mathcal{C}} > 0$. Suppose first, $u_k^{\mathcal{C}} < 0$, and consider $\mathbf{u}'$ with $u'_k = 0$, $\mathbf{u}'_{-k} = \mathbf{u}_{-k}^{\mathcal{C}}$. There exists at least one other $j \in \mathcal{A}$ with $u_j^{\mathcal{C}} < 0$. Similar to the case: $u_k^{\mathcal{C}} > 0$ and $u_j^{\mathcal{C}} > 0$, suppose GSC is violated for $\mathcal{A}$ and bidder k. There exists vector $\mathbf{u}$, with $\mathbf{u}_{\mathcal{A}} \gg \mathbf{0}$, $\mathbf{u}_{-\mathcal{A}} = \mathbf{0}$, such that $\mu_k(\mathbf{u}) > \max_{i\in\mathcal{A}} \mu_i(\mathbf{u})$. Consider vector $\mathbf{w}(t) = \mathbf{u} + t\mathbf{u}'$. Obviously, $\mu_k(\mathbf{w}(t)) > \max_{i\in\mathcal{A}} \mu_i(\mathbf{w}(t))$ for all $t > 0$. There exists the smallest $t' > 0$ such that for some bidder $l \in \mathcal{A}$, $w_l(t') = 0$. Existence of t' is guaranteed by existence of $j \in \mathcal{A}$ with $u_j^{\mathcal{C}} < 0$. Then, GSC is violated for subset $\mathcal{A} \setminus \{i\}$, bidder k, and vector $\mathbf{w}(t')$, which is a contradiction since $\#\{\mathcal{A} \setminus \{i\}\} < n$.

Now suppose $u_k^{\mathcal{C}} > 0$ while $u_j^{\mathcal{C}} < 0$ for all $j \in \mathcal{A}$. Consider vector $-\mathbf{u}^{\mathcal{C}}$. Then $-u_k^{\mathcal{C}} < 0$ and $-\mathbf{u}_{\mathcal{A}}^{\mathcal{C}} \gg \mathbf{0}$. As above, GSC is violated for $\mathcal{A}$ and bidder k.

It remains to be shown that, once equalities in (3.3) are allowed, if *GSC* is violated for $\mathcal{A}$ and bidder k, then for any $i \in \mathcal{C}$, $u_i^{\mathcal{C}} \neq 0$. We have shown that either $u_k^{\mathcal{C}} > 0$ and $\mathbf{u}_{\mathcal{A}}^{\mathcal{C}} \leqslant \mathbf{0}$, or $u_k^{\mathcal{C}} < 0$ and $\mathbf{u}_{\mathcal{A}}^{\mathcal{C}} \geqslant \mathbf{0}$. Suppose $u_i^{\mathcal{C}} = 0$ for bidders $i \in \mathcal{B} \subset \mathcal{A}$. Clearly, the subset $\mathcal{A} \setminus \mathcal{B} \neq \varnothing$. Then, by the corollary applied to $\mathcal{A} \setminus \mathcal{B}$ and bidder k, *GSC* is violated for $\mathcal{A} \setminus \mathcal{B}$ and k, which is a contradiction since $\#\{\mathcal{A} \setminus \mathcal{B}\} < n$. □

B.2 Sufficiency

Proof of Lemma 3.1. First, we construct inferences $\sigma(p, H(p))$ with $\sigma_j(\cdot, H(p))$ continuous at p for any $H(p)$, and any bidder j active at p has the highest value at $\sigma(p)$ whenever $\sigma_j(p) < 1$. We denote the set of such bidders with $\mathcal{A}(\sigma(p))$.

Suppose at some p^0 with $H(p^0) = \bar{H}(p^0)$ there exists a profile $\sigma^0(p^0)$ that satisfies (3.5), and $\mathcal{A} = \mathcal{A}(\sigma^0(p^0)) \subset \mathcal{I}(\sigma^0(p^0))$. Fix $\sigma_{-\mathcal{A}}(p) = \sigma^0_{-\mathcal{A}}(p^0)$ for $p \geq p^0$. Consider a profile of functions $\sigma(p) = (\sigma_{\mathcal{A}}(p), \sigma_{-\mathcal{A}}(p))$ such that $\sigma_{\mathcal{A}}(p)$ satisfies (3.5) for every $p \in [p^0, p^*]$ for some $p^* > p^0$. Finding a solution $\sigma_{\mathcal{A}}(p)$ to the system

$$\mathbf{V}_{\mathcal{A}}(\sigma_{\mathcal{A}}(p), \sigma^0_{-\mathcal{A}}(p^0)) = p\mathbf{1}_{\mathcal{A}} \tag{B.2}$$

is equivalent to solving the system of differential equations

$$\frac{d\sigma_{\mathcal{A}}}{dp} = (DV_{\mathcal{A}})^{-1}\mathbf{1}_{\mathcal{A}}. \tag{B.3}$$

By Caushy-Peano theorem, there exists a unique continuous solution $\sigma_{\mathcal{A}}(p)$ to the system (B.3) with initial condition $\sigma_{\mathcal{A}}(p^0) = \sigma^0_{\mathcal{A}}(p^0)$, and this solution extends for all $p \leq p^*_{\mathcal{A}}$, where $p^*_{\mathcal{A}}$ is the lowest price at which $\sigma_j(p^*_{\mathcal{A}}) = 1$ for some bidder $j \in \mathcal{A}$.

Suppose *GSC* is satisfied. Then $\sigma_{\mathcal{A}}(p)$ is non-decreasing, and for every $p \in [p^0, p^*]$, $\mathcal{A} \subset \mathcal{I}(\sigma(p))$. As long as $\mathcal{A} \subset \mathcal{I}(\sigma(p))$, $\frac{d\sigma_{\mathcal{A}}}{dp} = \mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \geqslant \mathbf{0}$ (this follows from the proof of Lemma B.1) and $\frac{\partial V_i}{\partial \sigma_{\mathcal{A}}} \frac{d\sigma_{\mathcal{A}}}{dp} \leq 1$ for any $i \notin \mathcal{A}$ by (3.3). Since $\mathcal{A} \subset \mathcal{I}(\sigma(p^0))$, it follows that $\mathcal{A} \subset \mathcal{I}(\sigma(p))$ and $\frac{d\sigma_{\mathcal{A}}}{dp} \geqslant \mathbf{0}$ for all $p \in [p^0, p^*_{\mathcal{A}}]$. Thus we have constructed $\sigma(p, H(p))$ for $p \in [p^0, p^*_{\mathcal{A}}]$.

To extend the construction $\sigma(p, H(p))$ on other prices we need to specify the initial $\sigma^0(p^0)$ and the extension of $\sigma(p, H(p))$ on the prices above $p^*_{\mathcal{A}}$. At $p^0 = 0$ set $\sigma^0(0) = \mathbf{0}$, then $\sigma(p)$ are calculated as above with $\mathcal{A} = \mathcal{N}$, for all $p \in [0, p^*_{\mathcal{N}}]$. At p^0 such that $H(p^0) \neq \bar{H}(p^0)$, define $\sigma^0(p^0) = \sigma(p^0, H(p^0))$.

Obviously, if $\mathcal{A}(\sigma(p, H(p^0))) \subset \mathcal{I}(\sigma^0(p^0))$, then $\mathcal{A}(\sigma^0(p^0)) \subset \mathcal{A}(\sigma(p, H(p^0)))$ and $\mathcal{A}(\sigma^0(p^0)) \subset \mathcal{I}(\sigma^0(p^0))$. Then, we can define $\sigma(p, \bar{H}(p^0))$. Note that proceeding this way allows us to maintain continuity of σ, or more formally, to link $\sigma(\cdot, H(p^0))$ and $\sigma(\cdot, \bar{H}(p^0))$ at the price p^0 where bidders exit the auction.

To extend $\sigma(p, H(p))$ beyond $p^*_{\mathcal{A}}$, we have to solve a new system (B.2) for $\mathcal{A}' = \mathcal{A}(\sigma(p^*_{\mathcal{A}}, H(p^*_{\mathcal{A}}))) \subsetneq \mathcal{A}$ with initial condition $\sigma_{\mathcal{A}'}(p^*_{\mathcal{A}}) = \sigma_{\mathcal{A}'}(p^*_{\mathcal{A}})$. This is repeated until no bidder remains with $\sigma_j(p) < 1$, thereafter $\sigma(p)$ is fixed. □

B.3 Necessity

B.3.1 Necessity for three bidders

Lemma B.2. *An English auction with three bidders does not possess an efficient equilibrium if* GSC *is violated at some interior signal profile.*

Before we proceed with the proof we need to establish a few supporting facts.

Without loss of generality suppose that *GSC* is violated for $\mathcal{A} = \{2, 3\}$ and bidder 1 at $\mathbf{s} \in (0,1)^3$, such that $V_1(\mathbf{s}) = V_2(\mathbf{s}) = V_3(\mathbf{s}) = V(\mathbf{s})$.[1]

Consider trajectory $\mathbf{s}(t)$ that for each t satisfies

$$V_1(\mathbf{s}(t)) = V_2(\mathbf{s}(t)) = V_3(\mathbf{s}(t)) = V(\mathbf{s}) + t,$$

so that $\mathbf{s}(0) = \mathbf{s}$. Such a trajectory exists, is unique, and is defined for small enough t since $\mathbf{V}$ are regular and $\mathbf{s}$ is the internal signal profile. Since *GSC* is violated at $\mathbf{s} = \mathbf{s}(0)$, by continuity of first derivatives of value functions *GSC* is also violated at $\mathbf{s}(t)$ for every small t.

Suppose that in an efficient equilibrium, when all the bidders are active, they follow the strategies $\beta_i(s_i)$, $i = 1, 2, 3$. Then, at any t, such that *GSC* is violated at $\mathbf{s}' \equiv \mathbf{s}(t)$, the following lemma holds.

Lemma B.3. *At every* $\mathbf{s}'$*,*

a) for any bidder $j = 2, 3$*, there exists* $b_j \equiv \lim_{s_j \downarrow s'_j} \beta_j(s_j)$*;*

b) these limits are equal, $b \equiv b_2 = b_3$*, and* $\beta_1(s'_1) > b$*;*

c) for every $s_j > s'_j$ *in some neighborhood of* s'_j*,* $\beta_j(s_j) \geq b_j$ *for* $j = 2, 3$*.*

Proof. Consider $\mathbf{s}'(\tau) \equiv (s'_1, s'_2(\tau), s'_3(\tau))$—a trajectory such that $(s'_2(\tau), s'_3(\tau))$ solves

$$V_2(\mathbf{s}'(\tau)) = V_3(\mathbf{s}'(\tau)) = V(\mathbf{s}') + \tau.$$

Because $\mathcal{A} = \{2, 3\}$ is minimal, $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}}(\mathbf{s}'(0)) \gg 0$ as follows from the proof of Lemma B.1, thus $s'_2(\tau)$ and $s'_3(\tau)$ are increasing. This also implies that

$$V_2(s_2, \mathbf{s}'_{-2}) > V_3(s_2, \mathbf{s}'_{-2}), \quad V_3(s_3, \mathbf{s}'_{-3}) > V_2(s_3, \mathbf{s}'_{-3}), \tag{B.4}$$

for $s_2 > s'_2$ and $s_3 > s'_3$ sufficiently close to s'_2 and s'_3 correspondingly.

We have that (3.3) is violated along this trajectory in some neighborhood of $\tau = 0$, since at $\mathbf{s}'(0)$ the direction of the trajectory is the equal increments vector for $\mathcal{A}$ Therefore, for sufficiently small $\tau > 0$,

$$\beta_1(s'_1) > \min\{\beta_2(s'_2(\tau), \beta_3(s'_3(\tau)\}, \tag{B.5}$$

since it is bidder 1 who has the highest value at $\mathbf{s}'(\tau)$.

[1] The pairwise single-crossing (*GSC* for $\#\mathcal{A} = 1$) has to be satisfied, possibly with weak inequality in (3.1).

Along $\mathbf{s}'(\tau)$ select a sequence of pairs of signals $(s_{2n}, s_{3n}) \downarrow (s'_2, s'_3)$ such that the corresponding sequence of pairs of bids $(\beta_2(s_{2n}), \beta_3(s_{3n}))$ converges as well. Denote

$$b_2 = \lim_{n\to\infty} \beta_2(s_{2n}), \quad b_3 = \lim_{n\to\infty} \beta_3(s_{3n}).$$

Since $[0,1]^3$ is a compact set, such a sequence can always be selected, possibly with $b_j = \infty$ for $j = 2, 3$. Thus, for sufficiently large n,

$$\beta_1(s'_1) > \min\{\beta_2(s_{2n}), \beta_3(s_{3n})\}. \tag{B.6}$$

Moreover, continuity of value functions and (B.6) imply that for such n there exists $\delta(n) > 0$ such that for any $0 < \varepsilon_1 < \delta(n)$ bidder 1 has the highest value, and

$$\beta_1(s'_1 - \varepsilon_1) > \min\{\beta_2(s_{2n}), \beta_3(s_{3n})\}. \tag{B.7}$$

We show first that $b_2 = b_3$. Suppose that the limits are not equal. Without loss of generality consider $b_2 < b_3$. We can find sufficiently small $\varepsilon_1 \geq 0$ such that $V_1(s'_1 - \varepsilon_1, s_2, s'_3) < V_2(\cdot)$. If $\frac{\partial V_1(\mathbf{s}')}{\partial s_2} < \frac{\partial V_2(\mathbf{s}')}{\partial s_2}$, it suffices to choose $\varepsilon_1 = 0$. If $\frac{\partial V_1(\mathbf{s}')}{\partial s_2} = \frac{\partial V_2(\mathbf{s}')}{\partial s_2}$, then $\frac{\partial V_1(\mathbf{s}')}{\partial s_1} > \frac{\partial V_2(\mathbf{s}')}{\partial s_1}$, otherwise the regularity condition is violated for bidders 1 and 2 at $\mathbf{s}'$, so any small enough $\varepsilon_1 > 0$ can be chosen. Define $\hat{\mathbf{s}} = (s'_1 - \varepsilon_1, s_2, s'_3)$. When ε_1 is sufficiently small, $V_1(\hat{\mathbf{s}}) < V_2(\hat{\mathbf{s}})$ and $V_3(\hat{\mathbf{s}}) < V_2(\hat{\mathbf{s}})$, bidder 2 has the highest value at $\hat{\mathbf{s}}$. By continuity of value functions, bidder 2 also has the highest value at $(s'_1 - \varepsilon_1, s_2, s_3)$ if s_3 is chosen close enough to s'_3 (after s_2 and ε_1 are fixed). Therefore,

$$\beta_2(s_2) > \min\{\beta_1(s'_1 - \varepsilon_1), \beta_3(s_3)\} \tag{B.8}$$

at appropriate $(s'_1 - \varepsilon_1, s_2, s_3)$.

Because $(\beta_2(s_{2n}), \beta_3(s_{3n}))$ converges and $b_2 < b_3$, there exists N, such for all $n, k > N$, $\beta_2(s_{2n}) < \beta_3(s_{3k})$. Choose s_2 equal to the element of the sequence, $s_2 = s_{2n}$ with $n > N$. Then, at $\hat{\mathbf{s}} = (s'_1 - \varepsilon_1, s_{2n}, s'_3)$ and at $(s'_1 - \varepsilon_1, s_{2n}, s_{3k})$ (by continuity of value functions) bidder 2 has (strictly) the highest value for s_{3k} sufficiently close to s'_3. At the same time, (B.7) implies that once $k > N$, $\beta_2(s_{2n}) < \min\{\beta_1(s'_1 - \varepsilon_1), \beta_3(s_{3k})\}$, which contradicts (B.8).[2] Thus, $b \equiv b_2 = b_3$.

Now we show that $\beta_j(s_j) \geq b_j$, for $j = 2, 3$ and $s_j > s'_j$ sufficiently close to s'_j. Taking the limits in (B.6) and (B.7) gives $\beta_1(s'_1) \geq b$ and $\underline{\lim}_{s_1 \uparrow s'_1} \beta_1(s_1) \geq b$. Then for s_2 sufficiently close to s'_2, $s_2 > s'_2$, taking the limits in (B.8) with $s_3 = s_3(\varepsilon_1, s_2)$

[2] There is no contradiction per se to efficiency here if $\beta_2(s_{2n}) = \beta_1(s'_1 - \varepsilon_1) = \beta_3(s_{3k}) = \infty$. If this is so, then $\beta_1(s_1) = \infty$ for some range of $s_1 < s'_1$ (since ε_1 can be chosen arbitrarily as long as it is small enough). Then, for all $n > N$, $\beta_2(s_{2n}) = \infty$, and for any s_2 sufficiently close to s'_2, $s_2 > s'_2$ there exists s_{3k} with $\beta_3(s_{3k}) = \infty$, so that $\beta_2(s_2) = \infty$. Repeating the argument for bidder 3, $\beta_3(s_3) = \infty$ for any s_3 sufficiently close to s'_3, $s_3 > s'_3$. This means that with positive probability the auction will continue indefinitely, thus bidding $\beta_i(s_i) = \infty$ can not be an equlibrium strategy because it bears an expected payoff of $-\infty$.

limited to $\{s_{3n}\}_{n=1}^{\infty}$, we obtain $\beta_2(s_2) \geq b$. Repeat the arguments above to obtain c) for bidder 3. Now (B.5) implies that $\beta_1(s_1') > b$.

It remains to show that for any sequence $s_{jm} \downarrow s_j'$, the corresponding sequence of bids $\beta_j(s_{jn})$ converges to b, $j = 2, 3$. Without loss of generality consider bidder 3, and an arbitrary sequence $s_{3m} \downarrow s_3'$ with $\lim_{m\to\infty} \beta_3(s_{3m}) = b'$ (out of any sequence one can choose a converging subsequence if the limit of ∞ is allowed). Then, the arguments above can be repeated with s_{3k} being chosen from the elements of $\{s_{3m}\}_{m=1}^{\infty}$ sequence, to show that $b' = b$. □

For any t define $b(t)$ as $b(s') = b_2(s') = b_3(s')$ for $s' = s(t)$. Corollary B.2 in Appendix B.1 implies for any $t' > t$ that either (i) $s_j(t') > s_j(t)$, for $j = 2, 3$ and $s_1(t') < s_1(t)$, or (ii) $s_j(t') < s_j(t)$, for $j = 2, 3$ and $s_1(t') > s_1(t)$. If (i) holds, Lemma (B.3)c) states $\beta_j(s_j) \geq b(t)$ for any $s_j > s_j(t)$, therefore, $b(t') \geq b(t)$ for any $t' > t$, so $b(t)$ is non-decreasing function. Similarly, if (ii) holds $b(t)$ is non-increasing function.

Lemma B.4. *For almost all t, $\beta_2(s_2(t)) = b(t) = \beta_3(s_3(t))$.*

Proof. Suppose at some t, $\beta_3(s_3(t)) > b(t)$. Taking limits in (B.8) with $s_2 \downarrow s_2' = s_2(t)$ and $\varepsilon_1(s_2) \downarrow 0$ we obtain

$$b_2(t) \geq \min\left[\beta_1(s_1(t)), \beta_3(s_3(t))\right].$$

Since $\beta_1(s_1(t)) > b(t)$ we have $b_2(t) > b(t)$, which is a contradiction.

Suppose $\beta_3(s_3(t)) < b(t)$. Then t is a discontinuity point of $b(t)$ since: (i) $b(t') \leq \beta_3(s_3(t)) < b_3(t)$ for all $t' < t$ if $s_j(t') < s_j(t)$; (ii) $b(t') \leq \beta_3(s_3(t)) < b_3(t)$ for all $t' > t$ if $s_j(t') < s_j(t)$. Similar results hold for $\beta_2(s_2(t))$.

Since $b(t)$ is monotonic, it cannot have more than countable number of discontinuity points. Thus $b(t)$ is continuous almost everywhere. At any t where $b(t)$ is continuous $\beta_2(s_2(t)) = b(t) = \beta_3(s_3(t))$. □

Now we can complete the proof of the necessity claim.

Proof of Lemma B.2. Fix any t such that $\beta_2(s_2(t)) = \beta_3(s_3(t))$. Consider (i) an arbitrary $t' > t$ if $b(t') \geq b(t)$, or (ii) an arbitrary $t' < t$ if $b(t') \geq b(t)$. For any such t' sufficiently close to t, $s_1(t') < s_1(t)$ and $s_j(t') > s_j(t)$ for $j = 2, 3$. By the pairwise single-crossing bidder 1 has the lowest value at realization $(s_1(t'), s_2(t), s_3(t))$ for t' sufficiently close to t. Thus, efficiency prescribes

$$\beta_1(s_1(t')) < \beta_2(s_2(t)) = \beta_3(s_3(t)), \tag{B.9}$$

otherwise bidders 2 and 3 drop out simultaneously while one of them possesses the highest value.

Lemma B.3 suggests that $\beta_1(s_1(t')) > b(t')$. Then, since b is monotonic, using Lemma B.4, we obtain

$$\beta_1(s_1(t')) > b(t') \geq b(t) = \beta_2(s_2(t)),$$

which contradicts (B.9). □

B.3.2 Necessity: general case

In this Section we prove Theorem 3.2. To proceed we will need to establish a number of useful facts along the way. The construction of the proof is from the contrary, so we suppose that an efficient equilibrium exists and then show that this supposition leads to the contradiction.

Claim 3.2 states that *SC* is necessary for efficiency, so the minimal number of bidders needed to obtain a violation of *GSC* has to be at least three.

First, if *GSC* is violated, we can find an interior $\mathbf{s}$, at which *GSC* is violated with the lowest possible number of bidders involved to obtain a violation. Indeed, a number of bidders needed for violation is limited to a finite set from 3 to N, so $\mathbf{s}$ that minimizes this number exists. Since signal profiles at which *GSC* is violated form an open set, one can always find an interior $\mathbf{s}$. Suppose *GSC* is violated at $\mathbf{s}$ for bidders $\mathcal{A}$ and 1 (after relabeling), we call subset $\mathcal{A}$ minimal. If at $\mathbf{s}$, $\mathcal{I}(\mathbf{s}) \neq \mathcal{A} \cup \{1\}$, that is there are some other bidders in the winners circle $\mathcal{I}(\mathbf{s})$, after fixing values of bidders $\mathcal{A} \cup \{1\}$ and slightly decreasing values of all the other bidders from $\mathcal{I}(\mathbf{s})$, say uniformly, we obtain a signal profile at which *GSC* is violated for $\mathcal{A}$ and 1, and who are the only bidders with the highest values. We can do this since $\mathbf{s}$ is internal, so for sufficiently small decrease in values of bidders $\mathcal{I}(\mathbf{s}) \setminus (\mathcal{A} \cup \{1\})$, by regularity of value functions one can recover changes in the signals of all bidders from $\mathcal{I}(\mathbf{s})$ corresponding to the change in values. By continuity, for sufficiently small change in $\mathbf{s}$, *GSC* is still violated (and $\mathcal{A}$ is minimal) and no other bidder joins the winners circle.

Thus, suppose that *GSC* is violated at $\mathbf{s}$ for the minimal subset $\mathcal{A}$ and bidder 1 and, in addition, $\mathcal{I}(\mathbf{s}) = \mathcal{A} \cup \{1\}$. Consider trajectory $\mathbf{s}(t)$ that for each t solves[3]

$$\begin{aligned} V_j(\mathbf{s}(t)) &= V(\mathbf{s}) + t, \text{ for all } j \in \mathcal{A} \cup \{1\}, \\ s_i(t) &= s_i, \text{ for all } i \notin \mathcal{A} \cup \{1\}. \end{aligned}$$

Such a trajectory $\mathbf{s}(t)$ exists and is unique, since it can be found as a solution to the differential equation

$$\frac{d\mathbf{s}}{dt} = (DV_{\mathcal{I}(\mathbf{s})}(\mathbf{s}))^{-1} \cdot \mathbf{1}_{\mathcal{I}(\mathbf{s})}.$$

Since $\mathcal{A} \cup \{1\} = \mathcal{I}(\mathbf{s})$ and *GSC* is violated at $\mathbf{s}$, by the continuity of value functions and their first derivatives, $\mathcal{A} \cup \{1\} = \mathcal{I}(\mathbf{s}(t))$ and *GSC* is violated at $\mathbf{s}(t)$ for $\mathcal{A}$ and bidder 1 for all t in some neighborhood U_t^0 of $t = 0$.

Fix an arbitrary history (a sequence of exits) $H(p)$ with bidders $\mathcal{A} \cup \{1\}$ being active. In what follows we obtain several restrictions on bidding strategies $\beta_i(s_i) = \beta_i(s_i; H(p))$ for $i \in \mathcal{A} \cup \{1\}$. To obtain these results we use only efficiency requirements and no equilibrium arguments. In fact, one can treat these

[3] If it is Case (*ii*) of Corollary B.2, then consider $V_j(\mathbf{s}(t)) = V(\mathbf{s}) - t$. Proceed similarly in the case of any other trajectory considered below.

findings as what should happen on- and off-equilibrium path if efficiency has to be achieved. One can think that these results limit our equilibrium analysis by imposing restrictions on off-equilibrium behavior of the players. This is not so, since in fact no restrictions are imposed, the inequalities we obtain are of the sort: if this history have happened and if no other bidder exits before—this is a condition not a restriction—the strategies of the (some of) remaining bidders have to be related in a certain way. In any potential equilibrium, some histories may never be achieved and in some histories bidders may believe that some other bidders exit first, so in fact they have a lot of options in specifying their β_i. This means, however, that the conditions for inequalities we obtain are not met so these inequalities do not impose any restrictions at all. At the same time, in a particular equilibrium, for a particular realization of signals, there will be a history of exits, the one just prior to the first exit of bidders $\mathcal{A}\cup\{1\}$, for which the inequalities we obtain below must be satisfied.

First we want to show the results similar to Lemma B.3 and Lemma B.4. An enormous amount of new complications arises once the number of bidders in $\mathcal{A}$ increases. To obtain these results we first show the following Lemma.

Lemma B.5. *Consider a subset $\mathcal{A}$ of the set of active bidders $\mathcal{M}$ and an interior signal profile $\mathbf{s}'$ such that $\mathcal{A}\subset\mathcal{I}(\mathbf{s}')$. Suppose that a)* GSC *condition is satisfied (with strict inequalities) for $\mathcal{A}$ and all proper subsets of $\mathcal{A}$ at $\mathbf{s}'$ and b) for some range of signals $s_j > s'_j$ for all $j\in\mathcal{A}$,*

$$\beta_j(s_j) < b_{-\mathcal{A}}(\mathbf{s}') \equiv \min_{k\in\mathcal{M}\setminus\mathcal{A}}\{\beta_k(s'_k)\}\,.$$

Then efficiency is violated: there exists a realization where all the bidders $j\in\mathcal{A}$ drop out simultaneously while having the highest value.

Corollary B.3. *Suppose that for the set $\mathcal{C}\subset\mathcal{M}$ there exist sequences of bids $\beta_k(s_{kn})\longrightarrow b_k(s'_k)$ when $s_{kn}\downarrow s'_k$ for all $k\in\mathcal{C}$.*

Suppose that a) GSC *condition is satisfied (with strict inequalities) for $\mathcal{A}$ and all proper subsets of $\mathcal{A}$ at $\mathbf{s}'$ and b) for some range of signals $s_j > s'_j$ for all $j\in\mathcal{A}$,*

$$\beta_j(s_j) < B_{-\mathcal{A}}(\mathbf{s}') \equiv \min\{\min_{k\in\mathcal{C}\setminus\mathcal{A}}\{b_k(s'_k)\}\,,\min_{k\notin\mathcal{C}\cup\mathcal{A}}\{\beta_k(s'_k)\}\}.$$

Then efficiency is violated: there exists a realization where all the bidders $j\in\mathcal{A}$ drop out simultaneously while having the highest value.

Proof. By induction on the number of bidders in the set $\mathcal{A}$.

Suppose $\mathcal{A}=\{i,j\}$.

i) Take *any* sequences of signals $s_{jn}\downarrow s'_j$ and $s_{in}\downarrow s'_i$ so that the corresponding sequences of bids $\beta_j(s_{jn})\longrightarrow b_j(s'_j)$ and $\beta_i(s_{in})\longrightarrow b_i(s'_i)$. Then it must be that $b_j(s'_j) = b_i(s'_i)$. Suppose not, that is for some pair of sequences $b_j(s'_j) < b_i(s'_i)$.

Then efficiency is violated. Take a signal s_{jn} such that $\beta_j(s_{jn}) < b_i(s'_i)$. By the single-crossing bidder j has the highest value at $(s_{jn}, \mathbf{s}'_{-j})$. We can pick s_{im} close enough to s'_i such that bidder j still has the highest value and $\beta_j(s_{jn}) < \beta_i(s_{im})$. As a result, bidder j drops out first violating the efficiency.

Similarly if $\beta_j(s_j) < b_j(s'_j) = b_i(s'_i)$ for any $s_j > s'_j$, then efficiency is violated. Thus $\beta_j(s_j) \geq b_j(s'_j)$ for any $s_j > s'_j$.

ii) There exists a trajectory $\mathbf{s}_{\mathcal{A}}(\tau) = (s_i(\tau), s_j(\tau))$ for $\tau \geq 0$ such that the values:

$$V_j(\mathbf{s}_{\mathcal{A}}(\tau), \mathbf{s}'_{-\mathcal{A}}) = V_i(\mathbf{s}_{\mathcal{A}}(\tau), \mathbf{s}'_{-\mathcal{A}}) = V_i(\mathbf{s}') + \tau.$$

Since *GSC* is satisfied for $\mathcal{A}$ at $\mathbf{s}'$, $s_i(\tau)$ and $s_j(\tau)$ are increasing. Therefore, $\mathcal{A} = \mathcal{I}(s_i(\tau), s_j(\tau), \mathbf{s}'_{-\mathcal{A}})$ and *GSC* is also satisfied at $(s_i(\tau), s_j(\tau), \mathbf{s}'_{-\mathcal{A}})$ for some range of $\tau \geq 0$.

Notice that for some range of $\tau \geq 0$ the corresponding bids $\beta_i(s_i(\tau)), \beta_j(s_j(\tau))$ are still lower than $b_k(s'_k)$ for all $k \notin \mathcal{A}$.

Consider an arbitrary τ from the mentioned range. Step **i)** suggests that for any τ $b(\tau) \equiv b_i(s_i(\tau)) = b_j(s_j(\tau))$ and for any $\tau' > \tau$, $b(\tau') \geq b(\tau)$. Thus $b(\tau)$ is monotonic.

iii) Suppose $\beta_i(s_i(\tau)) > b_i(s_i(\tau)) = b_j(s_j(\tau))$, then by the argument similar to the one in step **i)** there exists a realization where bidder j has the highest value but drops out first.

Suppose $\beta_i(s_i(\tau)) < b_i(s_i(\tau))$. Then τ is a discontinuity point for $b(\tau)$ since $b_i(s_i(\tau')) \leq \beta_i(s_i(\tau)) < b_i(s_i(\tau))$ for all $\tau' < \tau$.

Since $b(\tau)$ is monotonic it cannot have more than countable number of discontinuity points. Thus for almost every τ, $\beta_j(s_j(\tau)) = b(\tau) = \beta_i(s_i(\tau))$. Thus both bidders i and j have the highest value and both are the first to drop out.

Lemma for $\mathcal{A} = \{i, j\}$ is established.

iv) To prove Corollary notice that by continuity of value functions all the strict inequalities in steps **i)-iii)** are preserved if the signals $\mathbf{s}_{\mathcal{B}n}$ are taken sufficiently close to the realization $\mathbf{s}'_{\mathcal{B}}$.

Suppose $\mathcal{A}$ contains K elements and the result of Lemma B.5 and Corollary B.3 is established for any set of smaller size.

v) Take *any* sequences $s_{jn} \downarrow s'_j$ such that the corresponding sequences of bids $\beta_j(s_{jn}) \longrightarrow b_j(s'_j)$ for all $j \in \mathcal{A}$. Then it must be the case that $b_j(s'_j) = b_i(s'_i)$ for all $i, j \in \mathcal{A}$. If not, define $b_{\min} \equiv \min_{j \in \mathcal{A}} \left\{ b_j(s'_j) \right\}$. There exist a subset $\mathcal{C} \subset \mathcal{A}$, such that for any $i \in \mathcal{C}$ *any* sequence $\beta_i(s_{im})$ converges to $b_{\min}$ when $s_{im} \downarrow s'_i$, while $b_j(s'_j) > b_{\min}$ for any $j \in \mathcal{A} \setminus \mathcal{C}$. The fact that $\mathcal{C}$ exists and contains at least two bidders follows from the argument similar to the one in step **i)**.

Then by Corollary B.3 for set $\mathcal{C}$ there exist a signal profile $\left(\mathbf{s}_{\mathcal{C}}, \mathbf{s}'_{-\mathcal{C}}\right)$, such that $\beta_i(s_i) = \beta < b_{-\mathcal{C}}(\mathbf{s}')$ for any $i \in \mathcal{C}$ and $\mathcal{C} = \mathcal{I}\left(\mathbf{s}_{\mathcal{C}}, \mathbf{s}'_{-\mathcal{C}}\right)$. We can pick elements of any of the applicable converging sequences close enough to $\mathbf{s}'_{-\mathcal{C}}$, such that bidder(s) with the highest value belongs to set $\mathcal{C}$, but all bidders from set

$\mathcal{C}$ drop out simultaneously, violating efficiency. So, the limits of any sequences $b_j(s'_j) = b_i(s'_i)$ for all $i, j \in \mathcal{A}$.

In addition $\beta_j(s_j) \geq b_j(s'_j)$ for any $s_j > s'_j$ and all $j \in \mathcal{A}$ as in step **i)**. Similar to the argument in the above paragraph, $b(\tau) < b_{-\mathcal{A}}(\mathbf{s}')$.

vi) As in step **ii)** there exists a trajectory $\mathbf{s}_{\mathcal{A}}(\tau)|_{\tau \geq 0}$, such that $\mathcal{A} = \mathcal{I}(\mathbf{s}_{\mathcal{A}}(\tau), \mathbf{s}'_{-\mathcal{A}})$ and $b(\tau)$ is monotonic. As in step **iii)** if for some bidder $j \in \mathcal{A}$, $\beta_j(s_j(\tau)) < b(\tau)$ then τ is a discontinuity point of $b(\tau)$. Monotonic function cannot have more than countable number of discontinuity points. Hence $\beta_j(s_j(\tau)) = b(\tau) < b_{-\mathcal{A}}(\mathbf{s}')$ for all $j \in \mathcal{A}$ for almost all τ.

Corollary follows. □

Recall that by assumption *GSC* is violated for the minimal set $\mathcal{A}$ at the profile $\mathbf{s}$; and all the bidders other than $\mathcal{I}(\mathbf{s})$ are either inactive or that none of the $\beta_{\mathcal{M}\setminus\mathcal{I}(\mathbf{s})}(s)$ is the minimal at $\mathbf{s}$. We can use Lemma B.5 to prove the following result. Denote $\mathbf{s}' \equiv \mathbf{s}(t)$.

Lemma B.6. *At every* $\mathbf{s}'$,

a) for any bidder $j \in \mathcal{A}$, for all converging sequences $s_{jn} \downarrow s'_j$ the corresponding sequences of bids $\beta_j(s_{jn})$ converge and have the same limit $b_j(s'_j)$;

b) for any bidders $i, j \in \mathcal{A}$, $b_j(s'_j) = b_i(s'_i) < \beta_1(s'_1)$;

c) for any bidder $j \in \mathcal{A}$, for every $s_j > s'_j$, $\beta_j(s_j) \geq b_j(s'_j)$.

Proof. Define $\mathbf{s}'(\tau) = (s'_1, \mathbf{s}'_{-1}(\tau))$—a trajectory such that $(\mathbf{s}'_j(\tau))$ solves

$$V_j(\mathbf{s}'(\tau)) = V(\mathbf{s}') + \tau : \text{ for all } j \in \mathcal{A}.$$

Note that (3.3) is violated along this trajectory in some neighborhood of $\tau = 0$, since at $\mathbf{s}'(0)$, the direction of the trajectory is the direction used in equal increments formulation of *GSC*. As for the case of three bidders along the direction $\mathbf{u}$ select a converging sequence of signal profiles $\mathbf{s}_{\mathcal{A}n} \downarrow \mathbf{s}'_{\mathcal{A}}$. (With $\mathbf{s}'_{\mathcal{A}}$ we denote the signals of the bidders from the set $\mathcal{A}$). Denote the corresponding sequence of profiles of bids $\beta_{\mathcal{A}}(s_{\mathcal{A}n})$.

Since *GSC* is violated at $\mathbf{s}'(0)$, bidder 1 (with signal s'_1) has the highest value at any element of the sequence starting from some large enough N. Thus, efficiency prescribes

$$\beta(s'_1) > \min_{j \in \mathcal{A}} \{\beta_j(s_{jn})\}.^4 \tag{B.10}$$

From $\beta_{\mathcal{A}}(s_{\mathcal{A}n})$ with slight abuse of notations we can select a converging subsequence $\beta_{\mathcal{A}}(s_{\mathcal{A}n})$. Such a subsequence exists since any of the bids is limited from below by $\mathbf{0}$ and from above by the maximal possible value of the good. Denote the limit of the subsequence with $\mathbf{b}_{\mathcal{A}}(s'_{\mathcal{A}})$.

First consider the subsequence $\beta_{\mathcal{A}}(s_{\mathcal{A}n})$.

[4] Convergence of $\beta_{\mathcal{A}}(s_{\mathcal{A}n})$ suggests that the bids $\beta_k(s'_k)$ of the bidders from $\mathcal{M}\setminus\mathcal{I}(\mathbf{s})$ are higher than the min in (B.10), hence we just suppress these bids.

Suppose that contrary to **a)** there exists a *proper* subset $\mathcal{J} \subset \mathcal{A}$ such that $b_j(s'_j) = b_i(s'_i)$ for $i, j \in \mathcal{J}$ ($i = j$ is allowed) and

$$b_j(s'_j) < B_{\mathcal{M}\setminus\mathcal{J}}(\mathbf{s}') \equiv \min\{\min_{k\in\mathcal{A}\setminus\mathcal{J}}\{b_k(s'_k), \min_{l\in\mathcal{M}\setminus\mathcal{I}(\mathbf{s})} \beta_l(s'_l)\},$$

where $j \in \mathcal{J}$. Inequality (B.10) and convergence of $\beta_{\mathcal{A}}(s_{\mathcal{A}n})$ then imply $\beta_1(s'_1) \geq b$.

i) We first establish that $\beta_1(s'_1) > b$.

Suppose that $\beta_1(s'_1) = b_j(s'_j)$ for $j \in \mathcal{J}$. Then $\beta_j(s_j) \geq b_j(s'_j)$ for any $j \in \mathcal{J}$ and any $s_j > s'_j$. Suppose the contrary, $\beta_j(s_j) < b_j(s'_j) = \beta_1(s'_1)$ for some $j \in \mathcal{J}$. Then by the pairwise single-crossing for all such s_j:

$$V_j(s_j, \mathbf{s}'_{-j}) > \max_{k\neq j}\{V_k(s_j, \mathbf{s}'_{-j})\}.$$

and by the continuity of value functions there exist signals s_{kn}, for all $k \neq j, 1$ close enough to s'_k so that: at $(s_j, \mathbf{s}_{kn})$ bidder j still has the highest value, but the lowest bid.

Thus $\beta_j(s_j) \geq b_j(s'_j)$ for any $s_j > s'_j$ and in particular $\beta_j(s_{jn}) \geq \beta_1(s'_1)$ for all $j \in \mathcal{J}$. Convergence of $\beta_{\mathcal{A}}(s_{\mathcal{A}n})$ guarantees that for high enough n, $\min\{\beta_{\mathcal{A}}(s_{\mathcal{A}n})\}$ is reached at some $j \in \mathcal{J}$. But then (B.10) implies that $\beta_1(s'_1) > b$.

Now we know that there exists a converging subsequence $\beta_j(s_{jn})$ so that

$$\min\{\beta_1(s'_1), B_{\mathcal{M}\setminus\mathcal{J}}(\mathbf{s}')\} > b_j(s'_j) : \text{for all } j \in \mathcal{J} \subset \mathcal{A}.$$

The left hand side of the above inequality is in fact $B_{-\mathcal{J}}(\mathbf{s}')$ from the statement of Corollary B.3.

ii) By the argument similar to the one in step **v)** in the proof of Lemma B.5 we can show that efficiency dictates that for any bidder $j \in \mathcal{J}$, *any* sequence $\mathbf{s}_{\mathcal{A}n} \downarrow \mathbf{s}'_{\mathcal{A}}$, the corresponding sequence of bids $\beta_j(s_{jn})$ converges to $b_j(s'_j)$ and $b_j(s'_j) = b_i(s'_i)$ for all $i, j \in \mathcal{J}$.

Thus, there exist ranges of signals $s_j > s'_j$ for each $j \in \mathcal{J}$, such that

$$\beta_j(s_j) < B_{-\mathcal{J}}(\mathbf{s}').$$

The conditions of Corollary B.3 are satisfied. Hence there exist a realization such that all the bidders from the set $\mathcal{J}$ drop out simultaneously while their values are the highest.

The proof above was given for any proper $\mathcal{J} \subset \mathcal{A}$. Thus $b_j(s'_j) = b_i(s'_i)$ for the converging subsequence $\beta_{\mathcal{A}}(s_{\mathcal{A}n})$ for all $i, j \in \mathcal{A}$. By the argument similar to **i)** above we can show that $\beta_1(s'_1) > b_j(s'_j)$ for $j \in \mathcal{A}$.

iii) Now take an arbitrary sequence $\mathbf{s}_{\mathcal{A}n} \downarrow \mathbf{s}'_{\mathcal{A}}$ so that the corresponding sequence $\beta'_{\mathcal{A}}(s_{\mathcal{A}n}) \longrightarrow \mathbf{b}'_{\mathcal{A}}(s'_{\mathcal{A}})$. Suppose that for bidders j from some subset $\mathcal{K} \subset \mathcal{A}$, $b'_j(s'_j) \neq b_j(s'_j)$. Then by the argument similar to the one in step **v)** in the proof of Lemma B.5 we can find a contradiction to efficiency.

iv) Now suppose that the sequence of bids does not converge. Then from this sequence we can select two subsequences converging to different limits. Repeating the argument in step **iii)** we can find a contradiction to efficiency.

Thus the results **a)** and **b)** are established for all sequences $\mathbf{s}_{\mathcal{A}n} \downarrow \mathbf{s}'_{\mathcal{A}}$.

c) The fact that $\beta_j(s_j) \geq b_j(s'_j)$: for all $s_j > s'_j$ for all $j \in \mathcal{A}$ now follows from the pairwise single-crossing. □

B.3.3 Extensions

Lemma B.7 is an extension of Lemma B.3, although the proof for N bidders relies on the fact that GSC is satisfied with strict inequalities for all proper subsets of $\mathcal{A}$. No such assumption is in fact needed, but the complete proof for the case of N bidders appears very cumbersome. Here we just sketch how the argument can be adjusted to cover the cases of equalities. The key point in the proofs of Lemmas B.3 and B.7 is the following: for every proper subset $\mathcal{B}$ of the set $\mathcal{A}$ we have to show that there exists a realization, where all the bidders from $\mathcal{B}$ drop out, while some of them (and only bidders from $\mathcal{B}$) have the highest value. Clearly the presence of equalities in GSC makes this task harder. In the case of three bidders (where $\mathcal{A} = \{2, 3\}$) the only proper non-empty subsets of $\mathcal{A}$ are individual bidders. For a given bidder, say bidder 2, we have considered the case $\frac{\partial V_1}{\partial s_2} = \frac{\partial V_2}{\partial s_2}$. The other possibility would be $\frac{\partial V_3}{\partial s_2} = \frac{\partial V_2}{\partial s_2}$. With three bidders in the auction this case can be easily ruled out, because then GSC is violated for bidder 1 and $\mathcal{A}' = \{2\}$, which contradicts minimality of $\mathcal{A}$. With more than three bidders the situations of this sort—"sidewise equalities"—have to be dealt with. Below we illustrate how this can be done. Claims proven below enable us to extend the proof of necessity of GSC to the case where equalities in GSC are not precluded.

Claim B.1. *Suppose* GSC *in equal increment formulation is violated for the set* $\mathcal{A}$ *and bidder* 1 *at* $\mathbf{s}'$. *Suppose for the subset* $\mathcal{B} \subset \mathcal{A}$ *and bidder* 1 GSC *is satisfied with weak inequality. Then there exists a realization* $\mathbf{s}$ *close to* $\mathbf{s}'$, *with* $s_1 \leq s'_1$, $\mathbf{s}_{\mathcal{A}} \geqslant \mathbf{s}'_{\mathcal{A}}$, *such that* $1 \notin \mathcal{I}(\mathbf{s})$.

Proof. Consider vector $\mathbf{u}$, such that GSC in the equal increments formulation is violated for set $\mathcal{A}$ and bidder 1 at $\mathbf{s}'$. Note that $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \gg \mathbf{0}$ for all $j \in \mathcal{A}$, otherwise the subset $\mathcal{A}$ is not minimal. Consider vector $\mathbf{u}^{\mathcal{B}}$ from the equal increments formulation of GSC for subset $\mathcal{B}$. By supposition, for all $j \in \mathcal{B}$

$$\mathbf{u}^{\mathcal{B}} \cdot \nabla V_1(\mathbf{s}') = \mathbf{u}^{\mathcal{B}} \cdot \nabla V_j(\mathbf{s}') = 1.$$

We claim that $\frac{\partial V_1}{\partial s_1}(\mathbf{s}') > \min_{j \in \mathcal{B}} \left\{ \frac{\partial V_j}{\partial s_1}(\mathbf{s}') \right\}$. Suppose instead $\frac{\partial V_1}{\partial s_1}(\mathbf{s}') = \frac{\partial V_j}{\partial s_1}(\mathbf{s}')$ for all $j \in \mathcal{B}$. Then, for the subset $\mathcal{C} = \{1\} \cup \mathcal{B}$, we have $DV_{\mathcal{C}}\mathbf{e}^1_{\mathcal{C}} = \mathbf{1}_{\mathcal{C}}$, where $\mathbf{e}^1 = (1, 0, ..., 0)$ is the unit vector for bidder 1, $\mathbf{e}^1_{\mathcal{C}}$ is the projection of $\mathbf{e}^1$ onto the subspace of signals of bidders from $\mathcal{C}$, and $\mathbf{1}$ has all components equal to 1.

In addition $DV_{\mathcal{C}}\mathbf{u}^{\mathcal{B}}_{\mathcal{C}} = \mathbf{1}_{\mathcal{C}}$. Clearly $\mathbf{e}^1_{\mathcal{C}} \neq \mathbf{u}^{\mathcal{B}}_{\mathcal{C}}$, and therefore $\det DV_{\mathcal{C}} = 0$, which contradicts the assumption of regularity.

By continuity of the first derivatives of value functions the same inequality, $\frac{\partial V_1}{\partial s_1}(\mathbf{s}'(\tau)) > \min_{j\in\mathcal{B}}\left\{\frac{\partial V_j}{\partial s_1}(\mathbf{s}'(\tau))\right\}$, holds along $\mathbf{s}'(\tau) = (s'_1, \mathbf{s}'_{-1}(\tau))$—a trajectory such that $s'_i(\tau) = s'_i$ for all $i \notin \{1\} \cup \mathcal{B}$, and $(\mathbf{s}'_j(\tau))_{j\in\{1\}\cup\mathcal{B}}$ solves

$$V_j(\mathbf{s}'(\tau)) = V(\mathbf{s}') + \tau.$$

Consider $\mathbf{s}'(\tau)$ for small enough τ. It can not be the case that $\mathcal{I}(\mathbf{s}'(\tau)) = \{1\}$, otherwise *GSC* is violated for $\mathcal{B}$. If $1 \notin \mathcal{I}(\mathbf{s}'(\tau))$, the statement is shown. If $\{1\} \subset \mathcal{I}(\mathbf{s}'(\tau))$, we can slightly reduce the signal of bidder 1 and there will exist a bidder from $\mathcal{B}$ with higher value. □

Claim B.2. *Suppose that* GSC *is violated for bidder* 1 *and set* $\mathcal{A}$ *at* $\mathbf{s}'$*,* $\mathcal{I}(\mathbf{s}') = \{1\}\cup\mathcal{A}$*. Then for any proper subset* $\mathcal{B} \subset \mathcal{A}$*, there exists a signal profile* $\mathbf{s} = (\mathbf{s}_{\mathcal{A}}, \mathbf{s}'_{-\mathcal{A}})$ *with* $\mathbf{s}_{\mathcal{A}} \geqslant \mathbf{s}'_{\mathcal{A}}$ *close to* $\mathbf{s}'_{\mathcal{A}}$*, such that* $\mathcal{I}(\mathbf{s}) \subset \{1\} \cup \mathcal{B}$*.*

Proof. It is enough to show that for any $\mathcal{B}$ there exists vector $\mathbf{u}$, with $u_i \geq 0$ for all $i \in \mathcal{A}$, $u_k = 0$ for all $k \notin \mathcal{A}$, such that $\mathbf{u}\cdot\nabla V_j(\mathbf{s}') = 1$ for $j \in \mathcal{B}$ and $\mathbf{u}\cdot\nabla V_i(\mathbf{s}') < 1$ for $i \in \mathcal{A} \setminus \mathcal{B}$.

The proof is by induction on the size of $\mathcal{B}$. First we show that the statement is correct if $\#\mathcal{B} = \#\mathcal{A}-1$ and then prove the claim for any $\#\mathcal{B}$ with the presumption that it is true for any larger subset.

Consider vector $\mathbf{v}$ from the equal increment formulation of *GSC* for subset $\mathcal{B}$. Since $\mathcal{A}$ is minimal, it must be the case that $\mathbf{v}\cdot\nabla V_k(\mathbf{s}') \leq 1 = \mathbf{v}\cdot\nabla V_j(\mathbf{s}')$, for any $k \in \mathcal{A} \setminus \mathcal{B}$, $j \in \mathcal{B}$.

If $\#\mathcal{B} = \#\mathcal{A} - 1$, then for $\{k\} = \mathcal{A} \setminus \mathcal{B}$, $\mathbf{v}\cdot\nabla V_k(\mathbf{s}') < 1$, otherwise *GSC* is violated for bidder 1 and $\mathcal{B}$, contradicting minimality of $\mathcal{A}$.

Consider the case $\#\mathcal{B} < \#\mathcal{A} - 1$ and assume that the claim is correct for all larger subsets. Define $\mathcal{C} \subset \mathcal{A}\setminus\mathcal{B}$ to be a subset of bidders k, such that $\mathbf{v}\cdot\nabla V_k(\mathbf{s}') < 1$. If $\mathcal{C} = \emptyset$, then similarly to the above *GSC* is violated for $\mathcal{B}$. If $\mathcal{C} = \mathcal{A} \setminus \mathcal{B}$, then it is enough to choose $\mathbf{u} = \mathbf{v}$. If not, define $\mathcal{D} = \mathcal{C} \cup \mathcal{B}$. By induction there exists vector $\mathbf{u}^{\mathcal{D}}$, such that $\mathbf{u}^{\mathcal{D}}\cdot\nabla V_j(\mathbf{s}') = 1$ for $j \in \mathcal{D}$ and $\mathbf{u}^{\mathcal{D}}\cdot\nabla V_i(\mathbf{s}') < 1$ for $i \in \mathcal{A}\setminus\mathcal{D}$. Then any linear combination $\mathbf{u} = \lambda\mathbf{v} + (1-\lambda)\mathbf{u}^{\mathcal{D}}$, with $0 < \lambda < 1$ suffices to be chosen. □

Corollary B.4. *Suppose that* GSC *is violated for bidder* 1 *and set* $\mathcal{A}$ *at* $\mathbf{s}'$*,* $\mathcal{I}(\mathbf{s}') = \{1\} \cup \mathcal{A}$*. Then for any proper subset* $\mathcal{B} \subset \mathcal{A}$*, there exists a signal profile* $\mathbf{s}$ *close to* $\mathbf{s}'$ *such that* $\mathcal{I}(\mathbf{s}) \subset \mathcal{B}$*, with* $\mathbf{s}_{-\mathcal{I}(\mathbf{s}')} = \mathbf{s}'_{-\mathcal{I}(\mathbf{s}')}$*,* $s_1 \leq s'_1$*,* $\mathbf{s}_{\mathcal{A}} \geqslant \mathbf{s}'_{\mathcal{A}}$*.*

Proof. It is enough to find a vector $\mathbf{v}$ such that $\mathbf{v}\cdot\nabla V_k(\mathbf{s}') < \max_{j\in\mathcal{B}}\mathbf{v}\cdot\nabla V_j(\mathbf{s}')$ for all $k \in \mathcal{I}(\mathbf{s}') \setminus \mathcal{B}$.

Take vector $\mathbf{u}$ from Claim B.2, we know that $\mathbf{u}\cdot\nabla V_i(\mathbf{s}') < 1 = \max_{j\in\mathcal{B}}\mathbf{v}\cdot\nabla V_j(\mathbf{s}')$ for all $i \in \mathcal{A} \setminus \mathcal{B}$. If $\mathbf{u}\cdot\nabla V_1(\mathbf{s}') < 1$, set $\mathbf{v} = \mathbf{u}$. If $\mathbf{u}\cdot\nabla V_1(\mathbf{s}') = 1$, consider $\mathbf{v} = \mathbf{u}-\varepsilon\mathbf{e}^1$. If ε is small enough, for all $i \in \mathcal{A}\setminus\mathcal{B}$, $\mathbf{v}\cdot\nabla V_i(\mathbf{s}') < \max_{j\in\mathcal{B}}\mathbf{v}\cdot\nabla V_j(\mathbf{s}')$ is still preserved, and by Claim B.1, $\mathbf{v}\cdot\nabla V_1(\mathbf{s}') < \max_{j\in\mathcal{B}}\mathbf{v}\cdot\nabla V_j(\mathbf{s}')$ as well. □

B.3.4 Necessity for the minimal subset: resumed

We still have to find a realization where the *actual bids* of all the bidders from $\mathcal{A}$ coincide.

Denote $b_j(t) \equiv b_j(s'_j(t))$. From Corollary B.2 it follows that for any $t' > t$ the corresponding $s_j(t') > s_j(t)$, for all $j \in \mathcal{A}$.[5] For any $s_j > s_j(t)$, $\beta_j(s_j) \geq b_j(t)$. Therefore, $b_j(t') \geq b_j(t)$. Thus $b_j(t)$ is monotonic function.

Lemma B.7. *For almost all t, $\beta_j(s_j(t)) = b_j(t) = b_i(t) = \beta_i(s_i(t))$ for all $i, j \in \mathcal{A}$.*

Proof. Fix any t. Suppose $\beta_j(s_j(t)) > b_j(t)$ for the bidders from the set $\mathcal{J}$. Applying Lemma B.5 to the subset $\mathcal{A} \setminus \mathcal{J}$ we can show that there exists a realization where all the bidders from the subset $\mathcal{A} \setminus \mathcal{J}$ have the highest value but drop out simultaneously.

Suppose $\beta_j(s_j(t)) < b_j(t)$. Then t is a discontinuity point for $b_j(t)$ since $b_j(t') \leq \beta_j(s_j(t)) < b_j(t)$ for all $t' < t$. The first inequality follows from result c) in Lemma B.6.

Since $b_j(t)$ for all $j \in \mathcal{A}$ is monotonic it cannot have more than countable number of discontinuity points. Thus $b_j(t)$ are continuous almost everywhere. At any t where all $b_j(t)$ are continuous $\beta_j(s_j(t)) = b_j(t) = b_i(t) = \beta_i(s_i(t))$ for all $i, j \in \mathcal{A}$.

□

Lemma B.8. *Suppose* GSC *condition is strictly violated at the interior signal profile for the minimal subset $\mathcal{A}$. Then the N-bidder English auction does not possess an efficient equilibrium.*

Proof. Fix any t so that $\beta_j(s_j(t)) = \beta_i(s_i(t))$ for all $i, j \in \mathcal{A}$. Consider $t' > t$. We know that $s_1(t') < s_1(t)$ and $s_j(t') > s_j(t)$ for all $j \in \mathcal{A}$. By the single-crossing bidder 1 has the lowest value at the realization $(s_1(t'), \mathbf{s}_{\mathcal{A}}(t))$ for t' sufficiently close to t. Thus efficiency prescribes $\beta_1(s_1(t')) < \beta_j(s_j(t))$ (for all $j \in \mathcal{A}$.), otherwise all the bidders from the subset $\mathcal{A}$ drop out simultaneously while one of them possesses the highest value.

But result b) in Lemma B.6 suggests that $\beta_1(s_1(t')) > b_j(s_j(t')$. The monotonicity of b_j then ensures that $\beta_1(s_1(t')) > b_j(s_j(t')) \geq b_j(s_j(t)) = \beta_j(s_j(t))$ for all $j \in \mathcal{A}$. The equality follows from Lemma B.7.

Thus we have reached a contradiction. □

[5] Remember that for both Case (i) and Case (ii) the parametrization of the trajectory is chosen such that for any $j \in \mathcal{A}$, $s_j(t)$ is increasing (see Footnote 3).

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