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A PL-MANIFOLD OF NONNEGATIVE CURVATURE HOMEOMORPHIC TO $S^2 \times S^2$ IS A DIRECT METRIC PRODUCT

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Abstract

Let M^4 be a PL-manifold of nonnegative curvature that is homeomorphic to a product of two spheres, $S^2 \times S^2$. We prove that M is a direct metric product of two spheres endowed with some polyhedral metrics. In other words, M is a direct metric product of the surfaces of two convex polyhedra in $\mathbb{R}^3$.

The background for the question is the following. The classical H.Hopf's hypothesis states: for any Riemannian metric on $S^2 \times S^2$ of nonnegative sectional curvature the curvature cannot be strictly positive at all points. There is no quick answer to this question: it is known that a Riemannian metric on $S^2 \times S^2$ of nonnegative sectional curvature need not be a product metric. However, M.Gromov has pointed out that the condition of nonnegative curvature in the PL-case appears to be stronger than nonnegative sectional curvature of Riemannian manifolds and analogous to some condition on the curvature operator. This dissertation settles the PL-analog of the Hopf's hypothesis as stated above.

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Chapter 1

Introduction

1.1 Background

This work consists of a single structure result for polyhedral 4-manifolds with curvature bounded from below. Given M, a PL-manifold of nonnegative curvature homeomorphic to $S^2 \times S^2$. The claim is that M is a direct metric product. This can be considered a polyhedral version of the Hopf's hypothesis. The classical Heinz Hopf's hypothesis states: for any Riemannian metric on $S^2 \times S^2$ of nonnegative sectional curvature the curvature cannot be strictly positive at all points. Of course, for a PL-manifold there is no distinction between nonnegative and strictly positive curvature. The analogy is: for a Riemannian manifold, being a direct metric product is a particular case of not having strictly positive curvature.

This introduction provides a superficial review of the idea of curvature in Riemannian geometry and mentions several famous rigidity results to illustrate how a condition on curvature can place restraints on the topology of a compact manifold. Note that the Hopf's hypothesis is also of this kind (although not yet a result): if a compact 4-dimensional manifold has positive sectional curvature, then it is not homeomorphic to $S^2 \times S^2$. We then proceed to explain what curvature means for polyhedral spaces, and what polyhedral spaces are in the first place. We fix some terminology in "Preliminaries". An outline of the text is presented afterwards. Since the text consists of the proof of a single result, this is an outline of this proof. Some comments follow about the proof and its possible generalizations, as appropriate to discuss after outlining the structure of the proof.

1.1.1 Curvature and Riemannian geometry

In the study of metric properties of surfaces and manifolds in general one of the central notions is the notion of curvature. There are many different measures of curvature, the easiest being the Gaussian curvature of two-dimensional surfaces. Convex surfaces such as an ellipsoid or an elliptic paraboloid have positive Gaussian curvature, while a hyperbolic paraboloid (a saddle, e.g. $z = x^2 - y^2$) has negative Gaussian curvature. This is most easily seen by comparing the signs of the two eigenvalues of the Hessian of a parameterization of the surface. However, the Gaussian curvature is an intrinsic measure of curvature. It indicates a difference in the intrinsic metric properties of these surfaces, unrelated to a particular embedding in $\mathbb{R}^3$. For example, a geodesic triangle on an ellipsoid has the sum of the angles greater than π, while this sum is less than π on a saddle.

There are several ways to extend the notion of Gaussian curvature to higher dimensions, sectional curvature being the most prominent one. Numerous celebrated structure results connect geometry with topology by demonstrating how curvature bounds lead to topological restrictions, typically of the kind that the manifold in question has to be simply connected or even homeomorphic to a sphere of some dimension.

The following results are supposed to place the Hopf's hypothesis into some minimal context. By the sphere theorem of Berger and Klingenberg, a complete simply connected manifold with positive sectional curvature that is sufficiently constrained (e.g. between $1/4 < K \leq 1$) is homeomorphic to the sphere. By a theorem by Grove and Shiohama, any manifold with sectional curvature ≥ 1 and diameter $> \pi/2$ is homeomorphic to a sphere. Thus, if $S^2 \times S^2$ were to have everywhere positive sectional curvature, it will have to have relatively small curvature at some points and relatively high curvature at some others. [1] is an extensive survey containing these two and many other rigidity results for positive and nonnegative curvature. There is also a construction demonstrating that a Riemannian metric on $S^2 \times S^2$ of nonnegative sectional curvature need not be a product metric (while a product metric cannot have everywhere positive sectional curvature.) For more information on the Hopf's hypothesis one can start with the 1975 article by J.P.Bourgignon [2] containing some related results that were already known more than 30 years ago. Also, [3] contains hundreds of references related to Riemannian geometry. It includes a chapter "From Curvature to Topology", stating the Hopf's hypothesis on the third page (out of a hundred). Quoting from this book published in 2003, Marcel Berger's assessment of the state of the art is: "Thinking for example on Hopf's question on $S^2 \times S^2$ . . . and . . . one might try to classify Riemannian manifolds whose curvature has a given sign, strictly or not . . . for example the classification of positive or nonnegative sectional curvature manifolds is practically completely open." [3][p.573]

1.1.2 Alexandrov geometry and polyhedral spaces

Riemannian geometry is not the only medium for discussing curvature. It is considerably more general to work with curvature bounds of Alexandrov spaces. The notions of Alexandrov spaces of curvature bounded from above or from below can be applied to an arbitrary length space, in particular a geodesic metric space (i.e. with a geodesic curve between any two points). Even more specifically, these notions can be applied to polyhedral geometry — the geometry of spaces made up of flat metric simplices. Thus, a cube and a tetrahedron are both PL-manifolds homeomorphic to S^2 and having nonnegative curvature. The cube has 8 singularities (points where the metric is not flat) with the signed angular defect equal $\pi/2$ at each (the sum of the angles at any vertex of a cube is $3\pi/2$, which is less than 2π by $\pi/2$). This adds up to $4\pi = 2\pi \cdot 2$, 2 being the Euler characteristic of S^2, by an appropriate generalization of the Gauss-Bonnet theorem. For an example of a space of nonpositive curvature, take eight points $O, A_1, A_2, A_3, A_4, A_5, A_6, A_7$ and attach seven equilateral triangles with side 1 as $\triangle OA_1A_2$, $\triangle OA_2A_3$, $\triangle OA_3A_4$, $\triangle OA_4A_5$, $\triangle OA_5A_6$, $\triangle OA_6A_7$, $\triangle OA_7A_1$. This space will be homeomorphic to a disk D^2 and the point O will be a metric singularity with the signed angular defect equal to $-\pi/3$. This space contains *branching geodesics*: the shortest paths from A_1 to both A_4 and A_5 (in fact, to any point inside the triangle $\triangle OA_4A_5$) pass through O. Also, if you consider the geodesic triangle $A_1A_4A_5$, its sum of angles is $2\pi/3 < \pi$, as the angle at A_1 is zero. To get an example of a manifold, take two such disks and identify along their common boundary, the septagon $A_1A_2A_3A_4A_5A_6A_7$. The total signed angular defect will be $7 \cdot (2\pi/3) + 2 \cdot (-\pi/3) = 4\pi$, as before.

In general, one way to define Alexandrov spaces of curvature $\geq \kappa$ or $\leq \kappa$ is by means of comparison triangles, imitating the Toponogov's comparison theorem in Riemannian geometry. A background in Alexandrov spaces is recommended for reading the present work, but not necessary at all. Still for getting some intuition, [4] can serve as a gentle introduction to Alexandrov geometry (and geometry of length spaces in general.) It has a chapter on polyhedral spaces. More references can be found, for example, in [3], already mentioned above.

In fact, for polyhedral spaces, a curvature bound in the sense of Alexandrov spaces translates into a certain condition on the singularities. In the 2-dimensional case, the condition of nonnegative curvature simply means that all angular defects are positive (e.g. a cube or a tetrahedron) and nonpositive curvature means that all angular defects are negative (e.g. a disk made of seven equilateral triangles). This is more complicated in higher dimensions — there are more ways to make up a three-dimensional singularity than simply multiplying a two-dimensional singularities by a line segment. This would be a singularity of codimension 2, while a more sophisticated singularity that cannot be factored would have codimension 3 (by definition). Fortunately, it is known that the condition of nonnegative curvature is known to be equivalent to all conical angles at singularities of codimension 2 being less than 2π. This is in direct analogy with the formulation about angular defects being positive. (This simplification is not true for nonpositive curvature. A reader who is comfortable with the definitions of nonnegative and nonpositive curvature bounds in Alexandrov geometry should be able to come up with a counterexample.)

1.1.3 PL-case of the Hopf's hypothesis

The classical H.Hopf's hypothesis states: for any Riemannian metric on $S^2 \times S^2$ of nonnegative sectional curvature the curvature cannot be strictly positive at all points. It is known that a Riemannian metric on $S^2 \times S^2$ of nonnegative sectional curvature need not be a product metric. This work proves that it is the case for the analogous statement in the polyhedral case, thus settling what can be called a polyhedral version of the Hopf's hypothesis. (Polyhedral spaces are flat almost everywhere, so the notion of strictly positive curvature does not seem to have a polyhedral analog.)

According to [5], M.Gromov has pointed out that the condition of nonnegative curvature in the PL (polyhedral)-case appears to be stronger than nonnegative sectional curvature of Riemannian manifolds and analogous to some condition on the curvature operator. To illustrate, while it is known that a compact simply-connected 4-manifold with a positive curvature operator is homeomorphic to a sphere, positive sectional curvature for a compact 4-manifold only implies (by the Synge's lemma) that the fundamental group is either $\mathbb{Z}^2$ or trivial [2, 6]. This remark leads to the intuition that the premise of the polyhedral Hopf's hypothesis is stronger than the premise in the original conjecture, thus the former is easier, as witnessed by the proof presented here.

1.2 Preliminaries

To fix the terminology: a PL-manifold is a locally finite simplicial complex all whose simplices are metrically flat (convex hulls of finite sets of points in a Euclidean space) that is also a topological manifold. In the compact case, "locally finite" implies "finite", so we are working with some finite simplicial decomposition.

A (metric) singularity in a PL-manifold M^n is a point $x \in M$ that has no flat neighborhood. Metric singularities comprise M_s, the singular locus. $M \backslash M_s$ is a flat Riemannian manifold. More specifically, a singularity of codimension k has a neighborhood that is a direct metric product of an open set in $\mathbb{R}^{n-k}$ with another space, yet no such product for $\mathbb{R}^{n-k+1}$. We will be interested in the case when M is also an Alexandrov space of nonnegative curvature. This condition is known to be equivalent to the following formulation: the link of M_s at each singularity of codimension 2 is a circle of length $< 2\pi$.

It is also useful to recall the definition of a polyhedral Kähler manifold [7]. Let M be an oriented PL-manifold of even dimension $d = 2n$. Suppose that conical angles at codimension 2 faces are either 2π (no singularity) or not a multiple of 2π. Consider the image G of $\pi_1(M^{2n} \backslash M_s^{2n})$ in $SO(2n)$ generated by holonomies of the metric on the complement of the singular locus. A manifold is called polyhedral Kähler (PK) if the group G is contained in a subgroup of $SO(2n)$ conjugate to $U(n)$.

1.3 On the structure of the proof

Given M, a PL-manifold of nonnegative curvature homeomorphic to $S^2 \times S^2$. The claim is that M is a direct metric product. The rest of the text is taken by the proof of this result. The proof is carried out in three stages, so the text is organized in three sections. In section 1 we decompose $M \backslash M_s$ into a local metric product in a consistent way. That is, we establish the existence of two parallel distributions of oriented 2-planes α and β (2-distributions for brevity), foliating $M \backslash M_s$ and orthogonal to each other. In section 2 we classify singularities to decompose a neighborhood of an arbitrary singularity in M_s. Finally, in section 3 we put it all together to decompose M into a direct metric product and argue that the factors are convex polyhedra in $\mathbb{R}^3$.

One natural direction for future research is to attempt to prove other structure results for polyhedral manifolds with a curvature bound, perhaps again following some analogous results or conjectures in Riemannian geometry. One may attempt to reuse or generalize the proof presented in this work for such other results. It may turn out to be difficult, as all three parts are more or less specific to $S^2 \times S^2$ and (except possibly part 3) essentially 4-dimensional.

Section 1 relies on the fact that $b_2(S^2 \times S^2) = 2$, the second Betti number of the manifold in question is 2. It also relies on a specific relationships between the matrix groups $SO(4)$ and $SU(2) \subset SO(4)$ observed under conjugation by different elements of $SO(4)$. One can, of course, continue applying Cheeger's results to obtain information about differential forms in other settings, yet these may not turn out to be so useful without the nice relationship connecting $SO(4)$ with $SU(2)$. Part 2 classifies singularities in a 4-dimensional polyhedral space of nonnegative curvature using also that the manifold in question is a polyhedral Kähler manifold. The case of 4-dimensional polyhedral Kähler manifolds is also special as opposed to the general case. Classifying higher-dimensional singularities, especially without the polyhedral Kähler condition, seems to be a fairly difficult project. Finally, part 3 is using the fact that $S^2 \times S^2$ is compact and simply connected. It is using a number of other specific pieces of information such as that compactness, nonnegative curvature and orientability imply that a polyhedral 2-dimensional surface is homeomorphic to a sphere. However, generalizing most of part 3 to higher dimensions appears feasible, and sometimes straightforward.

It has been pointed out to the author that the proof should benefit from the use of complex geometry. A lot of the discussion of polyhedral Kähler manifolds in [7] is done in terms of complex geometry. Appropriately modifying the linear algebra lemmas in part 1, one may be able to get the same results easier and perhaps get more results, significantly simplifying the presentation in part 3. If this is true and this work can indeed be improved in this direction, this should be a fruitful source of possible generalizations.

Chapter 2

Finding parallel 2-distributions

2.1 The idea

In this section we remove all singularities from our consideration and focus on $M \backslash M_s$, a flat Riemannian manifold. The goal of this section is to find two parallel 2-distributions on $M \backslash M_s$. Since $M \backslash M_s$ is a flat Riemannian manifold, one can study differential forms on it, even though differential forms on M itself are not well-defined. Every parallel form (i.e. $\nabla \omega = 0$) on $M \backslash M_s$ is harmonic, L_2, closed and co-closed, as is verified by taking the differential and the codifferential in local (flat) coordinates and integrating in local coordinates (there is a finite flat atlas coming from the PL-structure).

We are only interested in 2-forms on $M \backslash M_s$. We are looking for a 2-form with four distinct eigenvalues to recover two parallel 2-distributions as the two eigenspaces of that form. We are also interested in obtaining a symplectic 2-form on $M \backslash M_s$ to show that M is polyhedral Kähler, as this will help to classify singularities in section 2 . If M is indeed a direct metric product, then there should be plenty of suitable forms found as linear combinations of signed areas of projections onto the first and the second factor, respectively.

The main tool here are J.Cheeger's results for (in particular) PL-manifolds of nonnegative curvature [5]. We are using his results in the following form:

Theorem 1 (J.Cheeger). *Let M^n be a PL-manifold of nonnegative curvature. Let H^i be the space of L_2-harmonic forms on $M \backslash M_s$ that are closed and co-closed.*

Then $dim H^i = b^i(M)$*. Moreover, all forms in* H^i *are parallel.*

What it means for our present discussion, given that $b^2(S^2 \times S^2) = 2$, is that the vector space of parallel forms on $M \backslash M_s$ is 2-dimensional. Pick a basis $\{\omega_1, \omega_2\}$ for this vector space. Thus, we have ω_1 and ω_2 — two parallel 2-forms on $M \backslash M_s$ that are linearly independent (not proportional to each other). We are going to do some linear algebra with these forms in order to obtain two mutually orthogonal 2-distributions (parallel fields of oriented 2-planes). This will prove

Claim 1. *Let* M *be a PL-manifold of nonnegative curvature, homeomorphic to* $S^2 \times S^2$ *and let* M_s *be the singular locus of* M*. Then there are two mutually orthogonal parallel* 2*-distributions (fields of oriented* 2*-planes) on* $M \backslash M_s$*.*

Proof of claim. A parallel 2-form on a flat Riemannian manifold has a well-defined notion of eigenvalues. If we can find a parallel antisymmetric 2-form on $M \backslash M_s$ with four distinct eigenvalues, $\pm\lambda_1 i \neq \pm\lambda_2 i$ (or three for $\lambda_1 = 0$ or $\lambda_2 = 0$), this form immediately gives rise to two fields of planes. One of the (fields of) planes is given by $\{v \in T_x M \mid \exists w : \ \omega(v, w) = \max(a, b) \cdot \|v\| \cdot \|w\|\}$. The other is the orthogonal complement (and also the eigenspace corresponding to the smaller pair of eigenvalues). Since a and b cannot both be 0, one of the planes acquires orientation from the form ω itself. The other can be oriented using the orientation of its orthogonal complement.

Let ω_1 and ω_2 be the two 2-forms on $M \backslash M_s$ not proportional to each other, that came from the Cheeger's results for nonnegative curvature. Assume for a contradiction that all real linear combinations of these two forms have repeating eigenvalues, for otherwise we would immediately obtain two parallel 2-distributions, as desired. (Here "repeating eigenvalues" means that all eigenvalues have multiplicity at least 2. The case $0, 0, \pm\lambda i$ is not counted as repeating eigenvalues for $\lambda \neq 0$). We are going to focus on all holonomies preserving the two given 2-forms and conclude that all these holonomies have to comprise the group $SU(2) \subset SO(4)$ and then necessarily preserve one more 2-form linearly independent of the previous two, a contradiction. (In fact, one of two copies of $SU(2)$ in $SO(4)$: either the canonical one or its conjugate). This is deduced from simple linear algebra done in local flat coordinates.

2.2 The linear algebra

The four following lemmas (1, 2, 3, 4) are technical and straightforward, and are only used to prove Claim 1.

Lemma 1. *If ω is an antisymmetric 2-form with repeating eigenvalues $(\pm\lambda i, \pm\lambda i)$ defined on a $4_{\mathbb{R}}$-dimensional vector space, then its matrix is a scalar multiple of an orthogonal matrix.*

Proof. If the form is nonzero, rescale it to make the eigenvalues equal to $\pm i$ (each with multiplicity 2). The resulting form is given by a matrix A. The form is antisymmetric, so $A^T = -A$. Therefore A can be diagonalized via some unitary matrix, $U^*AU = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \in U(4)$

So A is unitary itself and real-valued, hence orthogonal. □

Lemma 2. *If A is a 4×4 real-valued matrix that is also orthogonal and antisymmetric, then either*

$$A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{bmatrix} \text{ (a matrix of the first kind)}$$

or

$$A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & -c & b \\ -b & c & 0 & -a \\ -c & -b & a & 0 \end{bmatrix} \text{ (a matrix of the second kind)}$$

for some real numbers a, b, c satisfying $a^2 + b^2 + c^2 = 1$.

Proof. The proof is straightforward. Start from an orthogonal matrix of the form

$$\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$$

The columns are normalized:

$$\begin{aligned} a^2 + b^2 + c^2 &= 1 \\ a^2 + d^2 + e^2 &= 1 \\ b^2 + d^2 + f^2 &= 1 \\ c^2 + e^2 + f^2 &= 1 \end{aligned}$$

Consequently, $a^2 = f^2$, $b^2 = e^2$, and $c^2 = d^2$.

The columns are orthogonal to each other:

$$\begin{aligned} bd + ce &= 0 \\ ad &= cf \\ ae + bf &= 0 \\ ab + ef &= 0 \\ ac &= df \\ bc + de &= 0 \end{aligned}$$

Recall that $a^2 = f^2$. The case $a = f = 0$ is easy. Assume this is not the case. If $a = f$, then $c = d$ and $b = -e$, so the matrix is of the first kind. If $a = -f$, then $b = e$ and $c = -d$, and the matrix is of the second kind. The two kinds are easily seen to be mutually exclusive. □

Lemma 3. *Let* $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$, *a particular matrix of the first kind. If B is a matrix of the second kind (as above) and λ, μ are both nonzero real numbers, then $\lambda J + \mu B$ is an antisymmetric matrix with distinct eigenvalues ($\pm\lambda_1 i, \pm\lambda_2 i, \lambda_1 \neq \lambda_2$).*

Proof. $\lambda J + \mu B = \begin{bmatrix} 0 & \lambda + \mu a & \mu b & \mu c \\ -\lambda - \mu a & 0 & -\mu c & \mu b \\ -\mu b & \mu c & 0 & \lambda - \mu a \\ -\mu c & -\mu b & -\lambda + \mu a & 0 \end{bmatrix}$

This matrix is antisymmetric and it is never a multiple of an orthogonal matrix (given that $\lambda, \mu \neq 0$). If $a \neq 0$, compare the norms of different columns. If $a = 0$ and $b \neq 0$, take the dot product of the first column with the fourth column. If $a = 0$ and $c \neq 0$, take the dot product of the first column with the third column. Either way, the matrix has to have distinct eigenvalues by Lemma 1 □

Lemma 4. *Let J be as above and let C be another matrix of the first kind, not a multiple of J ($a \neq \pm 1$). Let $G \subset O(4)$ be the group of all orthogonal matrices commuting with J and C: $G = \{A | A \in O(4), AJ = JA, AC = CA\}$. Then $G = SU(2) \subset SO(4)$.*

Proof. For notational convenience, certain 2×2 matrices can be abbreviated as complex numbers: $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \leftrightarrow a + bi$. Also let $\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

In this notation, a matrix of the first kind $C = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{bmatrix} =$

$$\begin{bmatrix} -ia & (c - ib)\beta = \beta(c + ib) \\ (-c + ib)\beta = \beta(-c - ib) & -ia \end{bmatrix}.$$

$AJ = JA$ is equivalent to saying that $A = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}$ for some four complex numbers z_1, z_2, z_3, z_4. We are also given that $AC = CA$.

$$AC = \begin{bmatrix} -z_1 ia + z_2(-c + ib)\beta & z_1(c - ib)\beta - z_2 ia \\ -z_3 ia + z_4(-c + ib)\beta & z_3(c - ib)\beta - z_4 ia \end{bmatrix} =$$

$$CA = \begin{bmatrix} -z_1 ia + \beta(c + ib)z_3 & -z_2 ia + \beta(c + ib)z_4 \\ \beta(-c - ib)z_1 - z_3 ia & \beta(-c - ib)z_2 - z_4 ia \end{bmatrix}$$

If z is a complex number, then clearly $\beta z \beta = \overline{z}$ — the complex conjugate of z. Also notice that $\beta^2 = 1$. Then $AC = CA$ is equivalent to four conditions:

$$\overline{z_2(-c + ib)} = (c + ib)z_3$$
$$\overline{z_1(c - ib)} = (c + ib)z_4$$
$$\overline{z_4(-c + ib)} = (-c - ib)z_1$$

$$\overline{z_3(c-ib)} = (-c-ib)z_2$$

Equivalently, $\overline{z_2} = -z_3$ and $\overline{z_1} = z_4$ (we assumed that $a \neq \pm 1$, so $-c \pm ib \neq 0$). The orthogonality of A gives the normalization: $|z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2 = 1$. Lastly, $G = SU(2) \subset SO(4)$ is precisely the set of matrices of the form $\begin{bmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{bmatrix}$ satisfying $|z_1|^2 + |z_2|^2 = 1$.

□

2.3 The conclusion

Any parallel form is preserved by holonomies. Writing down ω_1 and ω_2 in local flat coordinates (and rescaling both of them, if necessary) one obtains two orthogonal antisymmetric 4×4 matrices Ω_1 and Ω_2 that are not proportional to each other (by Lemma 1 — recall that by assumption both forms have repeating eigenvalues). By classification from Lemma 2, there are two kinds of such matrices. Without loss of generality, $\Omega_1 = J$ — a particular matrix of the first kind. Indeed, one can make an appropriate orthogonal change of coordinates, as Ω_1 is a real-valued matrix of a normal operator with imaginary eigenvalues. If Ω_2 is a matrix of the second kind, then $\lambda\Omega_1 + \mu\Omega_2$ has distinct eigenvalues for $\lambda, \mu \neq 0$ (Lemma 3). Otherwise note that all scalar multiples of the matrices of the first kind constitute a 3-dimensional subspace of all real-valued matrices 4×4. Choose Ω_3 that is linearly independent with the previous two matrices and appropriately rescale it in order to make it orthogonal and so a matrix of the first kind, too.

Parallel forms ω_1 and ω_2 are preserved by holonomies and so any matrix in the image of the holonomies in $SO(4)$ commutes with Ω_1 and Ω_2. Any orthogonal matrix commuting with Ω_1 and Ω_2 is in $SU(2)$ and so has to commute with Ω_3 (use Lemma 4 two times). Then any matrix in the image of the holonomies in $SO(4)$ has to commute with Ω_3 as well. We can obtain a third parallel antisymmetric 2-form ω_3 from the form given by the matrix Ω_3 by parallel-translating it to all other points of $M \backslash M_s$. The new form ω_3 is linearly independent with the previous two, leading to the desired contradiction. Indeed, it has already been established that the space of such forms is 2-dimensional as a consequence of $b^2(S^2 \times S^2) = 2$.

This proves Claim 1 about the existence of the desired 2-distributions (two parallel fields of oriented 2-planes on $M\backslash M_s$ orthogonal to each other). □

These two 2-distributions allow us to give a more specific description of all parallel 2-forms on $M\backslash M_s$. Clearly, the signed areas of the projections onto the first and the second of the planes that we have just found constitute two parallel degenerate 2-forms that are not proportional to each other. Hence, they span $H^2(M)$. In appropriate local coordinates these two forms are just $dx_1 \wedge dx_2$ and $dx_3 \wedge dx_4$, respectively. Two 2-distributions can be thought of as the kernels of these two forms. The sum of these two forms, $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ is a symplectic form with repeating eigenvalues $\pm i$. This form yields a pseudocomplex structure on $M\backslash M_s$, so

Lemma 5. *M is a polyhedral Kähler manifold (the definition from [7] has been recalled in the introduction).*

Proof. The matrices representing the holonomies preserving the form $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ commute with J (in the appropriate positively oriented orthogonal basis, where J is the matrix of the 2-form $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$). Commuting with J is equivalent to being in $GL(2, \mathbb{C})$. However, $O(4) \cap GL(2, \mathbb{C}) = U(2)$. (If the basis turns out to be negatively oriented, use $dx_1 \wedge dx_2 - dx_3 \wedge dx_4$ instead.) Thus, the image of the holonomies of M is in a subgroup of $SO(4)$, conjugate to $U(2)$ — precisely what the definiton of a polyhedral Kähler manifold says. (Note that this image is in a subgroup conjugate to $U(2)$, **but not** to $SU(2)$!) □

It has been suggested to the author that the paper can be made shorter and easier to understand by the use of complex geometry, since a polyhedral Kähler manifold of dimension 4 is a complex surface [7]. This is currently used in section 2 for classifying singularities. It has been suggested that this can be used in section 1 to simplify the linear algebra as well as in section 3 by demonstrating that the 2-distributions we have found are actually (singular) holomorphic foliations on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Unfortunately, the author has been unable to implement these suggestions.

Chapter 3

Classifying singularities

The two 2-distributions we have found behave nicely, but are defined only on the nonsingular part $M \backslash M_s$. While we expect $M \backslash M_s$ to be a direct product too, it is easier to factor M as a whole. Fortunately, these two 2-distributions allow us to discern the product structure at the singularities from M_s and, moreover, the 2-distributions turn out to be parallel (respectively, perpendicular) to the fibers of this local product structure.

If you want to skip the details of this local analysis, go straight to the conclusion (Lemma 10).

3.1 Singularities of codimension 2

Since M is a polyhedral Kähler manifold as established above (Lemma 5), we can use the following result from [7]:

Lemma 6 (D.Panov)**.** *Let M^4 be a $4_{\mathbb{R}}$-dimensional polyhedral Kähler manifold. Then there are no singularities of codimension 3 (all singularities have to have codimension 2 or 4).*

Proof. The proof uses the fact that the singular locus of M is a holomorphic subspace of M in the sense of Kähler structure, and also some Morse theory. See [7, proposition 3.3]. □

There can only be finitely many singularities of codimension 4 (they all have to be vertices of the simplicial decomposition of M). The locus of singularities of codimension 2 with the induced intrinsic metric is a flat 2-dimensional manifold (since near every singularity of codimension 2 M can be decomposed into a product of a flat space with a 2-cone) that is also a subset of the two-skeleton Σ^2. It remains to add that codimension 4 singularities cannot be isolated from the rest of the singular locus.

Lemma 7. *A singular point $x \in M_s$ of any codimension cannot be isolated (from the rest of the singular locus). In other words, if a point in a 4-dimensional PL-manifold has a flat pinched neighborhood, this point is not a singularity (has a flat neighborhood).*

Proof. This proof is here for the sake of completeness. The statement is very simple, although not as trivial as it appears.

The link at x is homeomorphic to S^3. It is also a space of curvature $\kappa \geq 1$ because M itself has nonnegative curvature. Moreover, the link at x is a space of curvature $\kappa \leq 1 + \epsilon$ for any $\epsilon > 0$. This is true, since a sufficiently small triangle in the link witnessing $\kappa \not\leq 1 + \epsilon$ would also witness that a pinched neighborhood of x (that is supposed to be flat) is not a space of nonpositive curvature — a contradiction.

Because of the properties of Alexandrov spaces of curvature bounded from above, $1 \leq \kappa \leq 1 + \epsilon$ for any $\epsilon > 0$ does imply that $\kappa \equiv 1$ for the link. So the link at x is a space of constant curvature 1, thus the standard 3-sphere. The cone over this sphere is $\mathbb{R}^4$ and so x is not a singularity, as claimed. □

Note that this argument fails in the two-dimensional case (S^1 is not simply connected). Indeed, singularities of a 2-dimensional PL-manifold are always of the highest possible codimension and yet always isolated from one another.

At this point, we can conclude that the singular locus of M consists of several triangles that are also faces in the simplicial decomposition of M, where the vertices of these triangles may be singularities of codimension 4, but at all other points the singular locus is a flat 2-dimensional manifold. It turns out that the singularities have to be aligned in accord with the two parallel 2-distributions that we have just found.

Lemma 8. *Let $x \in M_s$ be a singularity of codimension 2. By definition, M can be factored near this singularity as $C \times \mathbb{R}^2$ (this factoring is unique — just take all* *geodesics passing through the origin). Then the fibers of this factoring are parallel (respectively, perpendicular) to the two 2-distributions found above.*

Proof. Consider an Euler vector field near x stretching the metric away from the singular locus. This vector field is parallel to the conical fibers and is directed away from the vertex of such a conical fiber. Let ω be a 2-form such that its kernel is one of our 2-distributions. ω is a parallel form, so in particular it is preserved by the holonomy, resulting from going around x any number of times. Take any nonsingular point near x and two tangent vectors, parallel to one and (respectively) the other of the fibers of the unique local product structure near x, say u and v.

After going around x, u is unchanged, but v is turned by some angle and becomes $\hat{v}$. Yet $\omega(u, v) = \omega(u, \hat{v})$, so $\omega(u, v - \hat{v}) = 0$. By choosing v appropriately we can make $v - \hat{v}$ to have any direction in its plane. Thus, $\omega(u, v) = 0$ if u and v are parallel to different fibers ($(C, *)$ and $(*, \mathbb{R}^2)$). For vector in the kernel, its projections onto the two fibers also belong to the kernel. Given that the kernel of ω is 2-dimensional and that it is preserved by holonomies, it is easy to see that the kernel is indeed parallel to one of the fibers. □

3.2 Singularities of codimension 4

Lemma 9. *Let $x \in M_s$ be a singularity of codimension 4. Then this singularity can be factored as a product of two conical singularities, aligned in accord with the two 2-distributions found above.*

Proof.

A direct proof using straightforward geometric arguments can be carried out in three steps.

1. Prove that the singular locus near x consists of x itself and two connected components of codimension 2 (connected by x) that are pinched cones over the two singular circles in the link

2. Construct a continuous map from a conical neighborhood of x in M into a product of two cones (via two projections)

3. Prove that this map is an isometry

There is a much shorter alternative proof presented at the end of this section (subsection 3.2.4). It uses more machniery, such as the universal cover.

3.2.1 Proof: singular locus has two components

Proof of 1. Recall that a codimension 4 singularity cannot be isolated (Lemma 7). As we know from the work of Dmitri Panov, in a 4-dimensional polyhedral Kähler manifold we cannot have any singularities of codimension 3 (Lemma 6). The locus of singularities of codimension 2 is a flat 2-dimensional manifold (as follows from the unique factoring of a codimension 2 singularity), and also this locus is a subset of the 2-skeleton of M. All that implies that the singular locus near x is x itself and several (finitely many) connected singular components of codimension 2, all looking like pinched cones with x in the center.

Assume there is only one such connected component. Recall that it has to be parallel to one of our 2-distributions (Lemma 8). All holonomies near x can be generated by going around singularities of codimension 2. In particular, the tangent vectors from $T(M \backslash M_s)$ that are parallel to the mentioned 2-distribution are preserved by all holonomies near x. So we can find a parallel vector field near x that is also parallel to the same 2-distribution and hence to the singular locus. Integrating this field we obtain infinite geodesics, so we can split the singularity by the Splitting Theorem for Alexandrov spaces of nonnegative curvature, contradicting the assumption that x has codimension 4. This proves that there are at least two connected components of the singular locus of codimension 2 and moreover that there is at least one component parallel to one 2-distribution and at least one parallel to the other.

Now consider the link of M at x viewed as all rays emanating from x in a cone over the link itself. The singular rays (by definition) are those without a neighborhood in the link isometric to a piece of the standard 3-sphere. Clearly, the singular rays comprise several circles. These circles are closed geodesics. This suffices to be checked locally at any point as all points of a circle look the same, and so it suffices to notice that the locus of singularities in M of codimension 2 consists of singular triangles (except, perhaps, their vertices) that are simplicial faces of M and thus totally geodesic. Moreover, the link is an Alexandrov space of curvature

$\kappa \geq 1$. By the properties of closed geodesics in such spaces, the distance from every ray (point in the link) to every singular circle is at most $\pi/2$. If the distance between a ray r and a singular circle is exactly $\pi/2$, then this is the distance from r to every ray compsiring this singular circle.

The distance between two singular circles parallel to different 2-distributions is precisely $\pi/2$. This is because every shortest path in a neighborhood of x in M from a point on one ray emanating from x (belonging to one of the singular circles in the link) to the closest point on another ray emanating from x (belonging to the other singular circle in the link) has to go to x implying the orthogonality of the rays in question. Otherwise this path would have to be orthogonal to one of the 2-distributions and parallel to the other, so anyway start from going along the original ray directly to x.

The distance between any two rays is strictly less than π (no geodesic passes through x as the singularity is of codimension 4 — again using the Splitting Theorem for Alexandrov spaces of nonnegative curvature). So shortest paths in M near x correspond to shortest paths between the corresponding rays in the link in the obvious way (as in any cone of diameter $< \pi$). Notice also that singular points (from M_s) near x in M belong to singular rays (in the sense of having no neighborhood in the link isometric to a piece of a standard sphere), while nonsingular points belong to nonsingular rays. M is a PL-manifold of nonnegative curvature, so any shortest path in M between two nonsingular points consists entirely of nonsingular points. Consequently, any shortest path in the link between two nonsingular rays may only contain nonsingular rays. Similarly, any shortest path between two singular rays has only singular or only nonsingular intermediate rays.

Observe that some singular and some nonsingular rays are parallel to one of the 2-distributions. If two rays r_1 and r_2 are parallel to the same 2-distribution, *so are all rays in any shortest path connecting them in the link*. Indeed, just consider the angular sector consisting of all rays in this path — the angle between the rays is less than π so the plane of the sector is precisely the plane of the 2-distribution in question. To put it more abstractly, all rays parallel to either of the 2-distributions form a convex subspace of the link. One corollary is that the singular circles are totally geodesic subspaces of the link. Another one is that there are only two singular circles in the link (otherwise take two so that their rays are parallel to the same 2-distribution and connect any two rays from them by a path in the link). Similarly, no nonsingular ray can be parallel to either 2-distribution. So the distance in the link from any nonsingular ray r to either of the two singular circles (in fact, to any ray in these circles) is strictly less than $\pi/2$. Indeed, if it were $\pi/2$ (to at least two rays in a singular circle — recall the remark about closed geodesics), the ray r would be orthogonal to one and so parallel to the other 2-distribution, hence it would be a singular ray itself.

All this implies that the singular locus near x consists of x itself and two components of codimension 2 that are pinched cones over the two singular circles in the link. □

3.2.2 Proof: constructing a map

Proof of 2. Taking any point near x, we can project it onto both components (by finding the closest point). Since this is the same as finding the closest ray in a singular circle to a given ray, we can conclude that having x as one of the two projections is equivalent to being a singular point. Moreover, both projections are unique (and so the operations of taking both projections are well-defined). Indeed, asume that there were a nonsingular point p near x with two shortest paths from p to one of the singular components, say $[pu]$ and $[pu']$. The angle between these two paths at p cannot be π: we are looking for shortest paths from a point in a cone to some set of rays in this cone; any such shortest path should start from locally decreasing the radial distance (coordinate along the rays of the cone) so two shortest paths cannot run in the opposite directions. But if the angle is less than π, we use the same argument as before: $[pu]$ and $[pu']$ are orthogonal to the same 2-distribution, hence upu' defines a plane (leaf) that is parallel to the other 2-distribution yet does not pass through x ($p \in M$ is nonsingular, so the ray $[xp)$ is not parallel to either 2-distribution. Indeed, a leaf containing x and a point p on the ray $[xp)$ would have to contain the whole ray — a contradiction). It allows us to move p along the bisector of upu', decreasing the distance from p to the singular component in question and yet keeping the nonuniqueness of a shortest path. Eventually p runs into the singular locus, but upu' does not pass through x, so p will run into a codimension 2 singularity orthogonal to upu' (from the component to which we are measuring distances). Yet when p is close to this singular component, the uniqueness of a shortest path is clear — a contradiction.

Therefore, we get a well-defined continuous mapping from a conical neighborhood of x in M into a product of two cones (via two projections). □

3.2.3 Proof: the map is an isometry

Proof of 3. It sends x to the origin in the product and the rest of the singular locus into the two cones (factors) in the product. Clearly, at any nonsingular point near x in M this mapping is a local isometry (use 2-distributions). Now consider a codimension 2 singular point p near x. What happens with this mapping near p — is it a local isometry too? Take a nonsingular point u near p so that $[up]$ is orthogonal to the singular component containing p. The claim is that the length of $[up]$ is preserved under the projection (p is projected into x while u is projected into some other point; we are only interested in one projection as the other projection for u and p is the same: p). It is easy to see that $[up]$ is projected into a straight segment (i.e. a radial segment in a 2-cone). Choose any $v \in [up]$ sufficiently close to p and cover $[uv]$ with finitely many appropriate open neighborhoods — the projection of $[uv]$ is (locally) a geodesic and hence a shortest path, since we know that projection is a local isometry outside of the singular locus and the projection of $[uv]$ is a radial segment. By continuity, the length of $[up]$ is preserved, too.

Now take two points u and u' near p (still a codimension 2 singular point) such that $[up]$ and $[u'p]$ are both orthogonal to the singular component containing p and project both onto the same cone as before (the other singular component). Draw segments along which we projected: $[uw]$ and $[u'w']$. We already verified that the lengths stay the same: $|up| = |wx|$ and $|u'p| = |w'x|$. Clearly, $[uw]$ and $[u'w']$ are parallel to the singular component along which we are projecting. If we start moving u along $[uw]$ and u' along $[u'w']$, and also p towards x, $|up|$ and $|u'p|$ stay the same and so the distance between u and u' locally stays the same, too! Thus, by continuity (and compactness) the distance between u and u' is the same as between w and w'. This shows that our map is a local isometry at codimension 2 singularities as well. □

Therefore, the map as defined on a pinched conical neighborhood of x in M (that is simply connected and a topological manifold) into a pinched direct metric product of two appropriate cones is a local isometry and thus an isometry. □

3.2.4 Alternative proof

Proof.

This proof was inspired by the proof of [7, proposition 3.9].

Consider E, the enveloping map for M [7, definition 3.4]. By definition, E is a local isometry from the universal cover of $M \backslash M_s$ to $\mathbb{C}^2$, where $\mathbb{C}$ is the set of complex numbers. There is some freedom in the choice of E. So use this freedom to make sure that E makes the (images of the) 2-distributions α and β parallel to the coordinate planes $z_1 = 0$ and $z_2 = 0$ (in $\mathbb{C}^2$), respectively. After this is ensured at a single point in $M \backslash M_s$, it will be automatically true throughout the whole $M \backslash M_s$.

Focus on a single singularity in M_s that has codimension 4. Without loss of generality, M is a single polyhedral Kähler-cone, i.e. the tangent cone at the singularity in question. Denote by O the vertex of the cone, now the only codimension 4 singularity in M_s. M_s is a connected set, hence so is the image, $E(M_s)$ (called the *branching set* of E). $M_s \backslash \{O\}$ may have many connected components, yet each of these components will be sent by E into a connected subset of a horizontal or a vertical complex line in $\mathbb{C}^2$. Here "horizontal" means "parallel to $z_2 = 0$", and "vertical" means "parallel to $z_1 = 0$". From that it is clear that $E(M_s)$ is entirely contained in two complex lines, one horizontal and one vertical, intersecting at $E(O)$. Denote these lines by L_1 and L_2, respectively.

The restriction of $E : E^{-1}(\mathbb{C}^2 \backslash (L_1 \cup L_2)) \mapsto \mathbb{C}^2 \backslash (L_1 \cup L_2)$ is a covering map. Lift the direct metric product structure from $\mathbb{C}^2 \backslash (L_1 \cup L_2)$ to $E^{-1}(\mathbb{C}^2 \backslash (L_1 \cup L_2))$. This can be done consistently, as the lift will go through the 2-distributions α and β in $M \backslash M_s$, aligned in accord with the product structure of $\mathbb{C}^2$ by the choice of E.

Consequently, $E^{-1}(\mathbb{C}^2 \backslash (L_1 \cup L_2))$ is a direct metric product itself, and the covering splits into two factors. It actually is the universal covering (other possible coverings of $\mathbb{C}^2 \backslash (L_1 \cup L_2)$ do not work). We can conclude that E is also a covering and, in fact, the universal covering of $E(M\backslash M_s)$. This implies $\mathbb{C}^2\backslash(L_1 \cup L_2) = E(M\backslash M_s)$, because the deck transformation group has to be the same in both cases, and $E(M_s) = L_1 \cup L_2$, not a proper subset of it.

It remains to conclude that the same product structure can be carried to $M\backslash M_s$ (which is, recall, a polyhedral Kähler-cone). The holonomies of $M\backslash M_s$ can be generated by going around singularities of codimension 2. This ensures that the covering map from the universal cover of $M\backslash M_s$ onto $M\backslash M_s$ itself respects the lifted product structure. The transition from $M\backslash M_s$ to M can be done by taking the metric completion. The fact that the factors are aligned along the two 2-distributions follows easily. □

3.3 Summary

This completes the preliminary phase. The useful part of this analysis is summarized in the following lemma that will be used extensively in the final part of the argument.

Lemma 10. *Let M be a PL-manifold of nonnegative curvature, homeomorphic to $S^2 \times S^2$. Then at every point $p \in M$, M can be locally represented in a unique way as a product $C_1 \times C_2$ of two 2-cones with conical angles $2\pi\alpha_1 \leq 2\pi$ and $2\pi\alpha_2 \leq 2\pi$ such that this decomposition is aligned along our two 2-distributions. More precisely, for every nonsingular point $(x, y) \in C_1 \times C_2$ near p ($x \neq 0 \in C_1, y \neq 0 \in C_2$) the two 2-distributions at (x, y) are parallel to the fibers $(C_1, *)$ and $(*, C_2)$, respectively. Lastly, there is a uniform bound $\delta > 0$ such that every $p \in M$ has a neighborhood containing the ball $B_\delta(p)$ that again has a unique factoring with the factors aligned along the 2-distributions.*

Proof. The flat case is obvious. The codimension 2 case is handled by Lemma 8. There is no codimension 3 case (Lemma 6). The existence of factoring in the codimension 4 case is handled by Lemma 9. To prove uniqueness, identify factors as codimension 2 singularities.

The "lastly" part is clear, since M is a finite simplicial complex. Note that if δ is sufficiently small, the factors will still be $C_1 \times C_2$, yet now p need not be the vertex of either cone. □

Chapter 4

Decomposing M into a product

What Lemma 10 says is that M has a local decomposition near every point, singular or not. This decomposition is consistent throughout M: two points that are sufficiently close to each other have *the same* local decomposition. Moreover, the holonomies on the nonsingular part $M \backslash M_s$ respect this local factorization, aligned along the two 2-distributions coming from parallel 2-forms. The rest of the argument is very similar to the de Rham decomposition theorem; in particular, it is crucial that $M = S^2 \times S^2$ is simply connected.

4.1 Integrating: a leaf

Fix one of the two 2-distributions mentioned throughout the text and call it α. We are going to learn to integrate this distribution not just on $M \backslash M_s$, but integrate it in some sense on M. Take any point $x \in M$, possibly a singular point. Construct a "leaf" $L_\alpha(x) \subset M$ — the smallest subset of M containing x and closed under a certain operation. Start from adding x to $L_\alpha(x)$. Use the local decomposition at x given by Lemma 10 and choose the fiber parallel to α. Take the points in M near x that belong to this fiber and add them to $L_\alpha(x)$, too. Continue this operation until every point in $L_\alpha(x)$ is there with a neighborhood of its appropriate fiber, parallel to α.

The resulting set $L_\alpha(x)$ is a 2-dimensional topological manifold "immersed" in M, called the leaf of α passing through x.

Lemma 11. *Any such leaf $L_\alpha(x) \subset M$ is a convex polyhedron, i.e. a compact simply-connected 2-dimensional PL-manifold of nonnegative curvature. (This is "without loss of generality": we actually prove that this is true for any $L_\alpha(x)$ or for any $L_\beta(x)$. Here β is the other 2-distribution that is orthogonal to α.)*

Proof of lemma. It is clear that $L_\alpha(x)$ is a 2-dimensional PL-manifold of nonnegative curvature from the way such leaves were defined. Using the lower bound δ in Lemma 10, we see that $L_\alpha(x)$ has no boundary and is a complete metric space. The leaf is oriented as the 2-distribution α is oriented. To prove compactness we need the following:

Claim 2. *There are no nonsingular leaves (in the sense of the intrinsic PL-metric).*

Proof of claim. Indeed, assume for a contradiction that $L_\alpha(x)$ is a flat leaf in its intrinsic metric (while all its points may be singular in M). Since the leaf has no boundary, it may be a plane, a cylinder, or a flat torus. Every point $y \in L_\alpha(x) \subset M$ has a neighborhood from Lemma 10 that contains the ball $B_\delta(y) \subset M$ (is not too small) and has a unique factoring, where one of the factors is a neighborhood of y in the leaf. Then the other factor will be the same for all $y \in L_\alpha(x)$, and in a canonical way. This is clear when the leaf is isometric to $\mathbb{R}^2$ and hence simply connected. For the cases of a torus and a cylinder it becomes true if we view a torus (or a cylinder) as the image of $\mathbb{R}^2$ "immersed" via a local isometry.

This allows us to define a normal parallel field of directions on the leaf and move the leaf in this direction — that is, any normal direction. Here it is crucial that δ is a uniform constant for all points in M, hence for all points in the leaf. (Recall that we view a nonsingular leaf not just as a set, but as an "immersion" of a plane. So of course, during the movement a plane (the image) may become a torus, or vice versa.) What can be an obstacle for such an operation?

If the leaf is within distance δ from a codimension 4 singularity or from a codimension 2 singularity that is orthogonal (not parallel) to the leaf, the leaf itself must have a singularity (use Lemma 10). Assume that it never happens. If all codimension 2 singularities are parallel to the leaf, without loss of generality replace α with β, the orthogonal complement of α. (So if we cannot prove the statement for any $L_\alpha(x)$, we will instead prove it for any $L_\beta(x)$.) Certainly, M must have some codimension 2 singularities (it must have some singularities by the

Gauss-Bonnet theorem, and then use Lemma 7 and Lemma 6). It is easy to argue that at any given moment all points of a leaf will be singularities of codimension 2, or all points of a leaf will be nonsingular points from $M \backslash M_s$. After moving the leaf around, it will span all of M (contradicting the existence of singularities of codimension 2 orthogonal to the leaf) or stop near such a singularity. Then the local factorization of M near such a singularity will contradict the assumption that the leaf itself is nonsingular. □

So any leaf has singularities. However, any leaf can only have finitely many singularities. We can say that each singularity "carries some angular defect" that is the difference between the conical angle at this singularity and 2π. Any such "angular defect" can only be a number from some fixed list of numbers between 0 and 2π, coming from the finite simplicial decomposition of M. In the case of a compact leaf these angular defects add up to 4π. They cannot add up to more than that in the noncompact case either. TO prove that, we can assume without loss of generality that the leaf is simply connected. (If we conclude for a contradiction that the universal cover is compact, so is the leaf itself.) A simply connected noncompact leaf is topologically a plane. If the angular defects at different singularities add up to more than 2π, then the circumference of a sufficiently large circle around any point in this leaf ("plane") decreases with some fixed rate as the radius increases, thus cannot increase indefinitely. This implies finite diameter and hence compactness, leading to the desired contradiction.

One can try to find a constant D such that any point in $L_\alpha(x)$ is within distance D from some singularity in this leaf in the intrinsic metric. If this is possible, choose a sequence of points $c_n \in L_\alpha(x)$ that are further than n from any singularity in this leaf. M is compact, so choose a converging subsequence $c_{n_k} \to c \in M$. Again using the local decomposition of M one can see that the leaf $L_\alpha(c)$ has no singularities — a contradiction.

So, every point in the leaf $L_\alpha(x)$ is not further than M from some singularity and there are finitely many such singularities — say, q. The leaf has nonnegative curvature, so its area is at most $q\pi D^2 < \infty$. Finite area clearly implies compactness. Compactness, nonnegative curvature and orientability imply that the leaf is homeomorphic to S^2. □

We are going to focus on the set of all such leaves in M.

4.2 The metric space of all leaves

Lemma 12. *For any $x, y \in M$,*

$dist(L_\alpha(x), L_\alpha(y)) = dist(x, L_\alpha(y)) = dist(L_\alpha(x), y)$.

Proof. Suffices to show that for all $x, \hat{x}, y \in M$ such that $L_\alpha(x) = L_\alpha(\hat{x})$, also $dist(x, L_\alpha(y)) = dist(\hat{x}, L_\alpha(y))$. This can be proved locally, for x and $\hat{x}$ close to each other. The leaves are compact, so for a given x we can find $z \in L_\alpha(y)$ closest to x: $dist(x, z) = dist(x, L_\alpha(y))$. Take any geodesic $[xz]$. It arrives to z parallel to one of the fibers of the local product decomposition from Lemma 10, for otherwise it would not be a geodesic. Hence, it goes along this fiber all the way from x to z.

Pick any $\hat{x}$ from the same leaf ($L_\alpha(x) = L_\alpha(\hat{x})$) that is close to x: $dist(x, \hat{x}) < \delta$ where δ is the constant from Lemma 10. Only choose $\hat{x}$ such that $dist(x, \hat{x}) = dist_{leaf}(x, \hat{x})$ — they are equally close in the intrinsic metric of the leaf. Then we can easily move the geodesic $[xz]$ using the local product structure (chosen canonically at all points) to obtain a segment $[\hat{x}\hat{z}]$ of the same length. Therefore, $dist(\hat{x}, L_\alpha(y)) \leq dist(x, L_\alpha(y))$, and this implies what we need. □

Consequently, all leaves form a connected metric space *Leaves* with the metric

$$dist_{Leaves}(L_\alpha(x), L_\alpha(y)) =^{def}= dist_M(x, L_\alpha(y))$$

This metric is strictly intrinsic — for two leaves l_1 and l_2 it is easy to find l_3 in between: $dist_{Leaves}(l_1, l_2)/2 = dist_{Leaves}(l_1, l_3) = dist_{Leaves}(l_2, l_3)$. Since M is a disjoint union of different leaves, this yields a natural mapping $M \mapsto Leaves$ where x is sent to $L_\alpha(x)$. This mapping is continious (because it is 1-Lipshitz) and onto, so *Leaves* is compact. *Leaves* is also simply connected: any loop in *Leaves* can be lifted to a path $\gamma : [0, 1] \mapsto M$ with $\gamma(0)$ and $\gamma(1)$ in the same leaf: $L_\alpha(\gamma(0)) = L_\alpha(\gamma(1))$. Connect $\gamma(0)$ with $\gamma(1)$ by a path in this leaf and contract the resulting loop in M.

4.3 The decomposition

This allows us to sharpen the statement of Lemma 10.

Lemma 13. *There is $\epsilon > 0$ (smaller than δ from Lemma 10) such that for every point $p \in M$, a neighborhood of p in M can be factored (again along the 2-distributions) into a product of the ϵ-neighborhood of p in $L_\alpha(p)$ with the ϵ-neighborhood of $L_\alpha(p)$ in Leaves.*

Proof. Since all leaves are compact, they have finite area and a sufficiently small neighborhood of p will intersect $L_\alpha(p)$ only along the fiber of the decomposition parallel to α (and not along several parallel fibers). Choose such an ϵ and use $\epsilon/3$ to make sure the distances between different leaves measured within this neighborhood are indeed true distances.

It is easy to choose a uniform ϵ for all points, as the maximal ϵ that works for a given p is a 3-Lipschitz function of p, and M is compact. All details follow easily from Lemma 12. □

It remains to argue that this gives us the desired decomposition.

Theorem 2. *M is a direct metric product of any leaf $L_\alpha(x)$ with the space of all leaves, Leaves.*

Proof. Let ϵ be as in Lemma 13. Pick any leaf $l \in Leaves$ and let U be the $\epsilon/2$-neighborhood of l in *Leaves*. Let $Z = f^{-1}(U) \subset M$ be the set of all points in M that are closer to the leaf l than $\epsilon/2$. Here f is the projection $M \mapsto Leaves$ sending x to $L_\alpha(x)$. It is clear from Lemma 12 and Lemma 13 that for every $x \in Z$ there is exactly one $y \in l$ closest to x ($dist(x,l) = dist(x,y)$). Hence, Z is homotopy equivalent to l and as such is simply connected.

Lemma 13 gives a local isometry of Z with $l \times U$ (this isometry is well-defined as l is simply connected), and this is also a global isometry (as both spaces are simply connected — U is an $\epsilon/2$ - neighborhood of a point in a 2-cone).

This implies that all leaves are isometric (l is isometric to all leaves in U, and *Leaves* is connected). Any curve in *Leaves* defines an isometry between the two leaves it connects. This isometry is trivial for a closed curve that is shorter than $\epsilon/2$ and *Leaves* is simply connected. Hence, all leaves are isometric to each other in a canonical way (fix a leaf l_0 and connect it to ever other leaf via any curve). So M is locally isometric to $l_0 \times Leaves$ and both sides are simply connected, hence it is indeed a true isometry. □

We have established that $M \simeq L \times Leaves$, where L is a convex polyhedron in $\mathbb{R}^3$ (Lemma 11). *Leaves* is a PL 2-dimensional manifold of nonnegative curvature (from local product structure — Lemma 13). We also know that *Leaves* is connected, simply connected and compact (see remarks after the space *Leaves* was defined). So it is also a convex polyhedron in $\mathbb{R}^3$, and we are done.

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