A COMPUTABLE THEORY OF DYNAMIC GAMES AND ITS APPLICATIONS

A Thesis in
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by
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Abstract

Recently dynamic competitive service sector problems such as revenue management, dynamic pricing, supply chain management, combined energy generation and distribution, and transportation network flow prediction and control have been the focus of much research activity. In this thesis we describe a general computable theory of dynamic games that relaxes most of the restrictive assumptions of classical nonzero-sum differential game theory, thought to be mandatory in the study of service and utility network problems. We have shown that differential variational inequalities (DVIs), which are infinite dimensional variational inequalities that include state dynamics, controls as well as control constraints, have the same notions of minimum principle, adjoint equations and transversality conditions familiar from the theory of optimal control when relatively mild regularity conditions are imposed. We also show that DVIs may conveniently be used to compute Cournot-Nash-Bertrand equilibria of broad classes of dynamic games where the game-theoretic agents have a forward-looking or anticipatory perspective. We have also shown that when state-dependent time shifts - such as those encountered in some applied problems namely modeling vehicular traffic and supply chain flows - are present, the resulting problems remain surprisingly tractable. A simple fixed-point algorithm combined with descent in Hilbert space for which sub-problem solutions are expressed as pure functions of time allowed us to compute solutions efficiently. Our numerical examples suggest the fixed-point-descent-in-Hilbert-space algorithm may be practical for intermediate size problems without special structure. We have also showed that the fixed-point-descent-in-Hilbert-space algorithm is convergent when suitable small step sizes are employed. We have applied this framework to study a wide class of applied problems. In particular we have studied (a) dynamic revenue management competition, (b) city logistics and supply chain competition, (c) electric power generation-distribution game, and (d) the transportation network flow prediction and congestion option games. In each case we present some structural properties of the games as well as numerical examples that further characterize respective Nash equilibria.
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Chapter 1

Introduction and Motivation

1.1 Dynamic Games and Differential Variational Inequalities

The latter half of the twentieth century saw impressive achievements in the modeling, analysis and computation of competitive static equilibria, as is underscored by the joint award of a Noble Prize to John Nash, John Harsanyi and Reinhard Selten in 1993 for their fundamental work on mathematical games and the relationship of games to equilibrium and optimization. Mathematicians, game theorists, operations researchers, economists, and engineers have employed noncooperative mathematical games and the notion of equilibrium to model virtually every kind of competition. Non-cooperative game theoretic models have been successfully employed to study economic competition in the marketplace, highway and transit traffic in the presence of congestion, regional and global wars, and both intra- and inter-species biological competition. One of the key developments that has made such diverse applications practical is our ability to compute static game-theoretic equilibria as the limit of a sequence of well-defined subproblems solvable by variants of traditional nonlinear programming algorithms. The computation of economic and game theoretic static equilibria has been of great interest in the academic and professional communities ever since the path-breaking paper by Lemke and Howson [1] and the seminal work by Scarf and Hansen.
in the mid-1960’s and early 1970’s. The initial impetus for research on computing equilibria came from the need to empirically analyze general equilibrium theory and to apply this theory to study problems of taxation, unemployment, etc. In recent years, the growth of experimental economics and the use of sophisticated strategic planning models by industry has revitalized the need for efficient methods to analyze and numerically solve models of economic and game theoretic equilibria. The initial methods which were used to compute economic equilibria were all based on the ingenious constructive proof by Lemke and Howson of the existence of an equilibrium point for a bimatrix game, which later have come to be known as fixed point (or homotopy based) methods. The other traditional approach for solving equilibrium models is nonlinear optimization, however, this approach requires very restrictive assumptions on the model in order to work. In summary, the optimization and fixed point approaches each have their own advantages and disadvantages when applied to solve equilibrium models; separately, they either lack the generality or the computational efficiency which is necessary for solving large scale equilibrium problems. In recent years, finite-dimensional variational inequality and nonlinear complementarity problems have emerged as very promising candidates for filling the gap created by the optimization and fixed point approaches. It is with this motivation that we undertake our current research.

In many applications, a static perspective is not appropriate. In particular intermediate disequilibrium states of mathematical games can be intrinsically important. When this is the case, disequilibrium adjustment mechanisms must be articulated, thereby, forcing explicit consideration of time. The archetypal example of a disequilibrium adjustment mechanism is the tatonnement story told to explain how equilibrium prices assuring market clearing arise in microeconomic theory. In still other applications the game-theoretic system of interest displays a so-called moving equilibrium, a notion we discuss in more detail below, wherein equilibrium is enforced at each instant of time although state and control variables will generally fluctuate with time. These fluctuations with respect to time are exactly those needed to maintain the balance of behavioral and economic circumstances
defining the equilibrium of interest. An example of a moving equilibrium is the dynamic user equilibrium problem studied in the field of dynamic traffic assignment.

As a consequence of the importance of disequilibrium adjustment mechanisms and moving equilibria, the modeling of non-cooperative competition in actual socio-technical systems is inherently dynamic. The main body of technical literature relevant to game-theoretic disequilibria and moving equilibria is that pertaining to so-called differential games, a field of inquiry widely held to originate with Isaacs [3]. Although a rather substantial body of literature known as dynamic game theory has evolved from the work of Isaacs [3], that literature continues to be strongly influenced by the emphasis of Isaacs on the relationship of such games to dynamic programming and to the Hamilton-Jacobi-Bellman partial differential equation\(^1\). A consequence of this classical point of view is that full use of the mathematical apparatus of variational inequalities, discovered originally in the context of certain free boundary value problems in mathematical physics, has not occurred in the study of dynamic games. By contrast, in the last fifteen years, variational inequalities have become the formalism of choice for applied game theorists and computational economists solving various static equilibrium models of competition. The “hole” in the dynamic game theory literature owing to this failure to fully exploit the variational inequality perspective is significant, for variational inequalities substantially simplify the study of existence and uniqueness. A variational inequality perspective for infinite dimensional dynamic games also leads directly to function space equivalents of the standard finite dimensional algorithmic philosophies of feasible direction and projection familiar from nonlinear programming.

It is our thesis in this research that many dynamic games may be expressed as infinite dimensional variational inequalities with clearly distinguished state and control variables. The state variables obey state dynamics expressed as ordinary differential equations and the control variables obey either pure control constraints or mixed state-control constraints. Each game agent expresses his/her strategy through subsets of the control variables that

\(^1\) See for example Basar and Olsder [4].
may or may not be shared with other agents, depending on whether collusion occurs. We refer to variational inequalities with such structure as *differential variational inequalities* (DVI), just as some dynamic games are referred to as differential games. We advocate this terminology, despite the fact that Aubin and Cellina [5] use the name *differential variational inequalities* for variational inequalities whose principal functions depend on derivatives of decision variables, since our usage seems very natural and we have discovered no literature derived from Aubin that adopts the Aubin usage. Control theoretic perspective to study dynamic games (without explicit reference to differential variational inequalities) has been advocated in two relatively recent comprehensive books - Sethi and Thompson [6] and Dockner et al. [7]. Both these books study some of the classical problems in economics and management science as potential applications of dynamic games.

In the first half of this report we give a guided tour of how the notion of a DVI may be used to formulate, analyze, and compute solutions of dynamic Cournot-Nash games. In particular, we provide a formal statement of one version of the DVI problem and its relationship to dynamic non-cooperative games. For the sake of brevity, in our exposition, we present a number of results without proof when these are available in our other relevant manuscripts.

Another very interesting class of problems arise in a dynamic game theory where there is a non-zero time difference between the times when an agent takes an action and the time when the impact of that action impacts the state of the game. The problem becomes even more complex when such time differences depend on the states of the game. In our perspective, time-shifted variational inequalities with explicit state dynamics and explicit controls can conveniently be used in the modeling of such systems where the game-theoretic agents have a forward-looking or anticipatory perspective and the emergent behavior is some variety of Cournot-Nash-Bertrand equilibrium, be it static or moving in nature. See Kachani and Perakis [8], Perakis [9], Friesz et al. [10], [11], Friesz and Mookherjee [12] and Dial [13],[14] for a discussion of dynamic traffic models that are or can be placed in the form
of a time-shifted variational inequality with explicit state dynamics and explicit controls. For models of the evolution of supply chains taking the form of a variational inequality with state dynamics, explicit controls and time shifts see Friesz et al. [15]. In this thesis we take the point of view that infinite dimensional variational inequalities with state dynamics among their constraints and having explicit control variables are direct generalizations of optimal control problems.

1.2 Applications Studied

In the second half of the report, we briefly describe the wide range of applications that we have studied as a part of this thesis research namely

1. revenue management and dynamic pricing competition (we have studied two problems: (a) joint pricing, allocation and overbooking decision making in a competitive environment under demand uncertainty and (b) evolutionary game theoretic study of dynamic pricing under competition in a deterministic setting);

2. dynamic competition in city logistics involving the shippers, the carriers and the receivers (a class of problem also known as dynamic urban freight systems);

3. electric power dynamic network games (a network game involving the power generators and the independent system operators in a day ahead market by explicitly considering generators’ ramping costs and ramping constraints);

4. multi-period competition in a supply chain involving the sourcing-production-pricing decisions that a small number of oligopolistic manufacturers make under demand uncertainty and impacts of supply side disruption on the supply chain;

5. dynamic competitive production-distribution model for the manufacturing firms in a network oligopoly; and
6. a variant of the dynamic user equilibrium (DUE) model for transportation network flow prediction. We have also studied the congestion call option games which provides a market based solution to congestion. In our view, the so called ‘congestion call option’ is a combination of the option to travels on a specific path and the associated call option.

This broad spectrum of models can be distinguished from one another according to the following attributes:

1. gaming behaviors of the agents;
2. decision variables (controls) in the hands of the agents;
3. explicit articulation of the constraints;
4. dynamics (an ordinary differential equation or a stochastic differential equation with white noise or jump diffusion process) that describe state of the game;
5. time lags and dual time scales;
6. uncertainties in the decision environment and/or uncertainties arising from incomplete information structure of the game; and
7. algorithmic approaches based on special structure of the problem.

However, amidst this great deal of heterogeneity, in this thesis we have tried to analyze these problems by using our unified computable framework utilizing DVIs. Properties of respective DVIs become handy in establishing some of the properties of these games (e.g. existence and uniqueness etc.).
1.3 Outline of the thesis

The remainder of this thesis is divided in two parts: Part I talks about the foundation blocks behind the computable dynamic game theory where as Part II describes some of the applications where we have applied this framework to study applied problems arising in service sectors as well as infrastructure/utility sectors. Part I consists of Chapter 2; Part II consists of four chapters. In Chapter 3 we introduce three applications of computable dynamic game theory in revenue management problems - joint pricing-allocation-overbooking problem under demand uncertainty, joint dynamic pricing and allocation problem under demand uncertainty and dynamic pricing under evolutionary demand learning. Chapter 4 presents applications in city logistics and supply chain problems. City logistics, often called as urban freight, problems involve the shippers (transporters), the freight carriers and the receivers who are directly, or often time indirectly in competition with each other. City logistics problems are further extended to supply chain settings by explicitly considering the supply network of the shippers, who are manufacturing a family of goods using an array of raw materials (input factors). Chapter 5 presents the electric power network games played over a power distribution network and illustrates how the computable theory using DVIs may be used in efficiently computing Nash equilibria of the game and portraying some complicated strategies of the agents (the generators). In Chapter 6 we show that the computation of dynamic user equilibrium may be achieved without simplifying modeling assumptions using the framework. We also show that securitizing congestion in a road network may be a viable strategy that competes with dynamic tolls, and has greater potential to reduce social cost of congestion. Chapter 7 summarizes the work and outlines some potential future research directions.
Part I

Theory
Chapter 2

A Computable Theory of Dynamic Games

2.1 Variational Inequalities Defined

The finite dimensional variational inequality (VI) may be succinctly stated as follows:

**Definition 2.1** (Finite dimensional variational inequality problem.) Let $X \subseteq \mathbb{R}^n$ be a nonempty subset of $\mathbb{R}^n$ and let $F : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a mapping from $X$ into itself. $VI(X,F)$ is to find a vector $x^* \in X$ such that the following conditions hold

$$F(x^*)^T(y - x^*) \geq 0 \quad \forall y \in X \subseteq \mathbb{R}^n$$

The problem $VI(X,F)$ is closely related to the nonlinear complementarity problem (NCP). Suffice it to say that under appropriate regularity conditions VIs and NCPs are equivalent to one another. Under more stringent conditions, the solution of a VIP is the solution of an NCP. Closely related to the finite dimensional VI is the quasi-variational inequality (QVI) which replaces $X$ with a point-to-set mapping $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$; i.e.
Definition 2.2 (Finite dimensional quasi-variational inequality problem.) Let $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a nonempty point-to-set mapping and let $F : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a mapping from $X$ into itself. QVI$(X, F)$ is to find a vector $x^* \in X(x^*)$ such that the following conditions hold

$$F(x^*)^T (y - x^*) \geq 0 \quad \forall y \in X(x^*)$$

It is of course also possible to articulate the variational inequality problem for an appropriately defined function space. For our purposes, we limit our attention to a non-specific, abstract Hilbert space $V$ so that the following definition obtains:

Definition 2.3 (Infinite dimensional variational inequality problem). Let $X \subseteq V$ be a nonempty subset of the Hilbert space $V$ and let $F : X \subseteq V \rightarrow V$ be a mapping from $V$ into itself. VIP$(X, F)$ is to find a vector $x^* \in V$ such that the following conditions hold

$$F(x^*)^T (y - x^*) \geq 0 \quad \forall y \in X \subseteq V$$

See Harker and Pang [16] for excellent discussions of existence and uniqueness of solutions as well as algorithms for solving the finite dimensional VIP. See Friedman [17] for a detailed presentation of infinite dimensional variational inequalities in function spaces. A parallel literature exists for complementarity problems but a review of it is omitted here for the sake of brevity.

For a survey of recent works related to specific finite dimensional VI applications and algorithms see Ferris and Pang [18]. Algorithms and applications for infinite dimensional VIs are discussed by Kinderlehrer and Stampacchia [19], Friedman [17], Baiocchi and Capelo [20], and Le and Schmitt [21]. Interestingly, some of the most recent applications of infinite dimensional variational inequalities, as well as research on numerical methods for their solution, has occurred in the context of option pricing and various exotic securities used in financial markets; see in particular Wilmott et al [22].
2.2 Differential Variational Inequalities Defined

Before we proceed, let us briefly review few key definitions of function spaces used extensively in rest of the thesis.

Definition 2.4 (Space of square-integrable functions) Space of square-integrable functions, \( L^2[t_0, t_1] \), defined on the segment of the real line \([t_0, t_1] \subseteq \mathbb{R}_+\), is a Banach space which consists of all real-valued functions on \([t_0, t_1] \), and the norm defined by

\[
\|z\|_2 = \left( \int_{t_0}^{t_1} |z(t)|^2 \, dt \right)^{1/2} < \infty
\]

The space \( L^2[t_0, t_1] \) can also be obtained in a direct way by the use of the Lebesgue integral and Lebesgue measurable functions \( z \) on \([t_0, t_1] \) such that the Lebesgue integral of \( |z|^2 \) over \([t_0, t_1] \) exists and is finite. See Kreyszig [23] for an excellent reference.

Definition 2.5 (Sobolev space) Vector space \( \mathcal{H}^1[t_0, t_f] \) is a Sobolev space if

\[
\mathcal{H}^1[t_0, t_f] = \left\{ v \left| v \in \left( L^2[t_0, t_f] \right)^m, \frac{\partial v}{\partial x_i} \in L^2[t_0, t_f] \text{ for all } i = 1, \ldots, m \right\} \right.
\]

We follow Friesz et al [24] in referring to the following differential variational inequality problem as \( DVI(F, f, U, x^0, \Gamma) \):

find \( u^* \in U \) such that

\[
\langle F(x(u^*, t), u^*, t), u - u^* \rangle \geq 0 \text{ for all } u \in U \tag{2.1}
\]

where

\[
u \in U \subseteq (L^2[t_0, t_f])^m \tag{2.2}
\]

\[
x(u, t) = \arg \left\{ \frac{dx}{dt} = f(x, u, t), \; x(0) = x^0, \; \Gamma \left[ x(L), L \right] = 0 \right\} \in (\mathcal{H}^1[t_0, t_f])^n \tag{2.3}
\]
and

$$x^0 \in \mathbb{R}^n$$  \hspace{1cm} (2.4)

$$F : (\mathcal{H}^1 [t_0, t_f])^n \times (L^2 [t_0, t_f])^m \times \mathbb{R}^n_+ \longrightarrow (L^2 [t_0, t_f])^m$$ \hspace{1cm} (2.5)

$$f : (\mathcal{H}^1 [t_0, t_f])^n \times (L^2 [t_0, t_f])^m \times \mathbb{R}^n_+ \longrightarrow (L^2 [t_0, t_f])^n$$ \hspace{1cm} (2.6)

$$\Gamma : (\mathcal{H}^1 [t_0, t_f])^n \times \mathbb{R}^n_+ \longrightarrow (\mathcal{H}^1 [t_0, t_f])^r$$ \hspace{1cm} (2.7)

Note that $(L^2 [t_0, t_f])^m$ is the $m$-fold product of the space of square-integrable functions $L^2 [t_0, t_f]$, while $(\mathcal{H}^1 [t_0, t_f])^n$ is the $n$-fold product of the Sobolev space $\mathcal{H}^1 [t_0, t_f]$. Friesz and Mookherjee [25] have generalized (2.1) that involves the non-differentiability of the principal operator $F(.,.,.)$ as well as differential-difference equations arising from state-dependent time shifts of control variables, which is also briefly outlined here in this report in section 6.11. Such time-shifted DVIs are known to arise as a result of the forward-looking or anticipatory perspective of individual, extremizing game-theoretic agents. Note that the inner product in (2.1) is defined by

$$\langle F(x(u^*), u^*, t), u - u^* \rangle = \int_{t_0}^{t_f} [F(x(u^*), u^*, t)]^T (u - u^*) \, dt \geq 0$$

### 2.3 Qualitative Properties of a DVI

Before we state the regularity conditions for the DVIs, we need to define the directional derivative or a differential in the sense of Gateaux:

**Definition 2.6** (G-differentiability, [26]) Let $V$ be a normed vector space and $J$ be a functional on $V$, we say that $J$ has a directional derivative of a differential in the sense of Gateaux at $v \in V$ in the direction $\varphi \in V$, if

$$\frac{J(v + \theta \varphi) - J(v)}{\theta}$$
has a limit when $\theta \to 0$ (in $\mathbb{R}$). This limit is denoted by $\delta J(v, \varphi)$. Further, if for all $\varphi \in V : \delta J(v, \varphi)$ exists, then $J$ is said to be differentiable in the sense of Gateaux ($G$-differentiable) at $v \in V$.

To analyze (2.1) we will rely on the following notion of regularity:

**Definition 2.7** (regularity of $DVI(F, f, U, x^0, \Gamma)$) We call $DVI(F, f, U, x^0, \Gamma)$ regular if:

1. $x(u, t) : (L^2[t_0, t_1])^m \times \mathbb{R}^1_+ \to (H^1[t_0, t_1])^n$ exists and is continuous and $G$-differentiable with respect to $u$;
2. $\Gamma(x, t)$ is continuously differentiable with respect to $x$;
3. $F(x, u, t)$ is continuous with respect to $x$ and $u$;
4. $f(x, u, t)$ is convex and continuously differentiable with respect to $x$ and $u$;
5. $U$ is convex and compact; and
6. $x^0 \in \mathbb{R}^n$ is known and fixed.

The motivation for this definition of regularity is to parallel as closely as possible those assumptions needed to analyze traditional optimal control problems from the point of view of infinite dimensional mathematical programming. Chapter 10 of Minoux [26] provides a useful introduction to the relationship of infinite dimensional mathematical programming and optimal control theory.

We next note that (2.1) may be restated as the following optimal control problem

$$\min \gamma^T \Gamma [x(t_1), t_1] + \int_{t_0}^{t_1} [F(x^*, u^*, t)]^T u dt$$

(2.8)
subject to

\[
\frac{dx}{dt} = f(x, u, t) \quad (2.9)
\]
\[
u \in U \quad (2.10)
\]
\[
x(t_0) = x^0 \quad (2.11)
\]

where \(x^* = x(u^*, t)\) is the optimal state vector and \(\gamma \in \mathbb{R}^r\) is the vector of dual variables for the terminal constraints \(\Gamma [x(t_1), t_1] = 0\). We point out that this optimal control problem is a mathematical abstraction and of no use for computation, since its criterion depends on knowledge of the variational inequality solution \(u^*\). In what follows we will need the Hamiltonian for \(DVI(F, f, U, x^0, \Gamma)\), namely

\[
H(x, u, \lambda, t) = [F(x^*, u^*, t)]^T u + \lambda^T f(x, u, t) \quad (2.12)
\]

where \(\lambda(t)\) is the adjoint vector that solves the adjoint equations and transversality conditions for given state variables and controls. Note that for a given state vector and a given instant in time (2.12) is convex in \(u\) when \(DVI(F, f, U, x^0, \Gamma)\) is regular. The necessary conditions are stated in the following theorem:

**Theorem 2.1** [necessary conditions for \(DVI(F, f, U, x^0, \Gamma)\)] When regularity in the sense of Definition 2.7 holds, solutions \(u^* \in U\) of \(DVI(F, f, U, x^0, \Gamma)\) must obey:

1. the finite dimensional variational inequality principle:

\[
\left[ F(x^*, u^*, t) + \nabla_u (\lambda^*)^T f(x^*, u^*, t) \right]^T (u - u^*) \geq 0 \quad \forall u \in U; \quad (2.13)
\]

2. the state dynamics

\[
\frac{dx}{dt} = f(x^*, u^*, t) \quad (2.14)
\]
\[
x^*(t_0) = x^0; \quad \text{and} \quad (2.15)
\]
3. the adjoint dynamics

\[
(-1) \frac{d\lambda^*}{dt} = \nabla_x (\lambda^*)^T f(x^*, u^*, t) \quad (2.16)
\]

\[
\lambda^* (t_1) = \gamma^T \frac{\partial \Gamma [x^* (t_1), t_1]}{\partial x^* (t_1)} \quad (2.17)
\]

**Proof.** The Pontryagin minimum principle is a necessary condition for optimal control problem (2.8) through (2.11) so that

\[
u^* = \arg \left\{ \min_{u \in U} H(x^*, u, \lambda^*, t) \right\} \quad (2.18)
\]

for each \(t \in [t_0, t_f]\), which in turn, by virtue of regularity, is equivalent to

\[
[\nabla_u H(x^*, u^*, \lambda^*, t)]^T (u - u^*) \geq 0 \quad u, u^* \in U
\]

Note that

\[
\nabla_u H(x, u, \lambda, t) = F(x^*, u^*, t) + \nabla_x \lambda^T f(x, u, t)
\]

where for given \(u\)

\[
\lambda(u, t) = \arg \left\{ (-1) \frac{d\lambda}{dt} = \nabla_x H(x, u, \lambda, t), \quad \lambda(t_f) = \gamma^T \frac{\partial \Gamma [x(t_f), t_f]}{\partial x(t_f)} \right\}
\]

\[
= \arg \left\{ (-1) \frac{d\lambda}{dt} = \nabla_x [F(x^*, u^*, t)]^T u + \nabla_x \lambda^T f(x, u, t), \quad \lambda(t_f) = \gamma^T \frac{\partial \Gamma [x(t_f), t_f]}{\partial x(t_f)} \right\}
\]

\[
= \arg \left\{ (-1) \frac{d\lambda}{dt} = \nabla_x \lambda^T f(x, u, t), \quad \lambda(t_f) = \gamma^T \frac{\partial \Gamma [x(t_f), t_f]}{\partial x(t_f)} \right\} \quad (2.19)
\]

since \(x(u, t)\) is completely determined by knowledge of the controls \(u\). The theorem follows immediately. \(\blacksquare\)

Note that item 1 of this theorem refers to a *finite* dimensional variational inequality because, as explained in Friesz et al [24], the Pontryagin minimum principle from which it is derived minimizes the associated Hamiltonian for fixed time and fixed state and adjoint variables.

We further note that the following existence result holds:

**Theorem 2.2 (existence of a solution to DVI(\(F, f, U, x^0, \Gamma\)))** When regularity in the sense of Definition 2.7 holds, DVI(\(F, f, U, x^0, \Gamma\)) has a solution.
Proof. By the assumption of regularity $x(u,t)$ is well defined and continuous. So $F(x(u,t), u, t)$ is continuous in $u$. Also by regularity we know $U$ is convex and compact. Consequently, by Theorem 2 of Browder [27], $DVI(F, f, U, x_0^0, \Gamma)$ has a solution. ■

2.3.1 Formulation of a Differential Game as a DVI

In this section we show that a fairly general differential game may be stated as a DVI. Let $\mathcal{F}$ be the set of Cournot-Nash (CN) agents involved in non-cooperative competition. Let us assume $u^f \in (L^2[t_0, t_1])^{\lvert N_f \rvert}$ be the vector of controls exercised by agent $f \in \mathcal{F}$ where the cardinality of $N$ is $\lvert N_f \rvert$. Also the vector of non-own decision variables relative to the agent $f$ is

$$u^{-f} \equiv (u^g : g \neq f) \in (L^2[t_0, t_1])^{q_f}$$

where $q_f = \lvert N \setminus N_f \rvert$. Each agent $f \in \mathcal{F}$ maximizes its present value of utility which we express abstractly as

$$J^f (u^f; u^{-f}) = \Gamma_f (x^f(t_1), t_1) + \int_{t_0}^{t_1} \Psi_f (x, u^f, u^{-f}, t) \, dt \quad (2.20)$$

where the utility function of agent $f$ is $\Psi_f (x, u^f, u^{-f}, t)$, the salvage value for agent $f$ is $\Gamma_f [x^f(t_1), t_1]$, the fixed initial time is $t_0$ and the fixed terminal time is $t_1$. Let $x^f \in (H^1 [t_0, t_1])^{\lvert S_f \rvert}$ denote the vector of state variables associated with agent $f$ and $\lvert S_f \rvert$ is the cardinality of the set $S_f$. The states of the game evolve according to the differential equation

$$\frac{dx^f_i}{dt} = \theta^f_i (x; u^f, u^{-f}, t) \quad \forall i \in S_f$$

$$x^f_i(t_0) = x^f_i(t_0) \in \mathbb{R}^1_+ \quad \forall i \in S_f$$

where

$$\theta^f : (H^1 [t_0, t_1])^{\lvert S_f \rvert} \times (L^2 [t_0, t_1])^{\lvert N_f \rvert} \times (L^2 [t_0, t_1])^{q_f} \times \mathbb{R}^1_+ \rightarrow (L^2 [t_0, t_1])^{\lvert S_f \rvert}$$

Each player $f$ chooses its variables $u^f$ from its set of feasible controls $U^f \subseteq (L^2 [t_0, t_1])^{\lvert N_f \rvert}$. Note that player $f \in \mathcal{F}$ faces the following optimal control problem: with the $u^{-f}$ as
exogenous inputs, compute $u^f$ in order to solve

$$
\max J^f (u^f; u^{-f}) = \Gamma_f (x^f (t_1), t_1) + \int_{t_0}^{t_1} \Psi_f (x^f, u^f; u^{-f}; t) \, dt
$$

subject to

$$
\frac{dx^f_i}{dt} = \theta^f_i (x; u^f, u^{-f}, t) \quad \forall i \in S_f
$$
$$
x^f_i (t_0) = x^{t_0}_{i; f} \quad \forall i \in S_f
$$
$$
u^f \in U^f
$$

Each player is a Cournot-Nash agent that knows and employs the current instantaneous values of the decision variables of other players to make its own non-cooperative decisions. Therefore (2.21), (2.22), (2.23) and (2.24) define a set of coupled optimal control problems, one for each firm $f \in \mathcal{F}$ which gives rise to a dynamic game.

It is assumed in the balance of this section that the dynamic CN game (2.21), (2.22), (2.23) and (2.24) is regular in the sense of the following definition:

**Definition 2.8** The differential game described above is regular if for all agents $f \in \mathcal{F}$

1. $x^f (u^f; u^{-f}) : (L^2 [t_0, t_1])^{\mathcal{N}_f} \times (L^2 [t_0, t_1])^{q_f} \rightarrow (\mathcal{H}^1 [t_0, t_1])^{\mathcal{S}_f}$ exists and is continuous and $G$-differentiable with respect to $u^f$ and $u^{-f}$;

2. $\Psi_f (x, u^f; u^{-f}; t) : (\mathcal{H}^1 [t_0, t_1])^{\mathcal{S}_f} \times (L^2 [t_0, t_1])^{\mathcal{N}_f} \times (L^2 [t_0, t_1])^{q_f} \times \mathbb{R}^1_+ \rightarrow (L^2 [t_0, t_1])^{\mathcal{N}_f}$ is continuously differentiable with respect to $u^f$;

3. $\theta^f (x; u^f, u^{-f}, t) : (\mathcal{H}^1 [t_0, t_1])^{\mathcal{S}_f} \times (L^2 [t_0, t_1])^{\mathcal{N}_f} \times (L^2 [t_0, t_1])^{q_f} \times \mathbb{R}^1_+ \rightarrow (L^2 [t_0, t_1])^{\mathcal{S}_f}$ is continuously differentiable with respect to $x$ and $u^f$;

4. $\Gamma (x^f, t) : (\mathcal{H}^1 [t_0, t_1])^{\mathcal{S}_f} \times \mathbb{R}^1_+ \rightarrow \mathcal{H}^1 [t_0, t_1]$ is continuously differentiable with respect to $x^f$;

5. $U_f \subseteq (L^2 [t_0, t_1])^{\mathcal{N}_f}$ is convex and compact; and
6. $x_f^{t_0} \in \mathcal{H}^{S_f}$

The Hamiltonian associated with the optimal control problem (2.21) - (2.24) is

$$H_f \equiv \Psi_f(x, u_f; u_{-f}; t) + \sum_{i=1}^{S_f} \lambda_i f \cdot \theta_i (x, u_f, u_{-f}, t)$$

(2.25)

while $\lambda_i f$ is the adjoint variable for the dynamics and $\lambda^f \in (\mathcal{H}^1 [t_0, t_1] )^{S_f}$ which is a concatenation of

$$\lambda = (\lambda^f : f \in \mathcal{F})$$

The maximum principle [28] tells that an optimal solution to (2.21) - (2.24) is a triplet

$$\{ u_f^* (t), x_f^* (t) ; \lambda^* f (t) \}$$

that, given $H_f (u_f; x_f; \lambda; u_{-f}; t)$, must satisfy at each time $t \in [t_0, t_1]$

$$u_f^* (t) = \arg \{ \max_{u_f (t) \in U_f} H_f (u_f; x_f; \lambda_f; u_{-f}; t) \}$$

(2.26)

By virtue of regularity (in particular $U_f$ is convex and compact), necessary condition for (2.26) can be expressed in the following variational form (see Minoux Theorem 10.6 [26])

$$\left[ \nabla_{u_f} H_f \left( u_f^*; x_f^*; \lambda^* f; u_{-f}; t \right) \right]^T \left( u_f - u_f^* \right) \leq 0 \quad \forall u_f \in U_f$$

(2.27)

The regularity conditions, especially the requirement that $\Psi_f(x, u_f; u_{-f}; t)$ and $\theta_f(x, u_f, u_{-f}, t)$ be continuously differentiable with respect to $u_f$, make the Hamiltonian $H_f(\cdot)$ differentiable with respect to $u_i f$ for all $i \in N_f$. Therefore condition (2.27) can be restated as

$$\left[ \Phi_f \left( x_f^*; u_f^*; u_{-f}; t \right) \right]^T \left( u_f - u_f^* \right) \leq 0 \quad \forall u_f \in U_f$$

(2.28)

where

$$\Phi_f \left( x_f^*; u_f^*; u_{-f}; t \right) \equiv \frac{\partial \Psi_f (x_f^*; u_f^*; u_{-f}; t)}{\partial u_f}$$

$$+ \sum_{j=1}^{N_f} \lambda_j f \cdot \frac{\partial \theta_j f (x_f^*; u_f^*; u_{-f})}{\partial u_f}$$

$$\Phi_f = \left( \Phi_i^f : i \in N_f \right)$$
Further, adjoint dynamics and state dynamics govern

\[
\frac{\partial H_f}{\partial x_i} (u^*_f; x_f^*; \lambda^*_f; u^{-f}_f; t) = (-1) \frac{d\lambda_i^*_f}{dt} \quad \forall i \in S_f
\]

(2.29)

\[
\frac{\partial H_f}{\partial \lambda_i} (u^*_f; x_f^*; \lambda^*_f; u^{-f}_f; t) = \frac{dx_i^*}{dt} \quad \forall i \in S_f
\]

(2.30)

boundary conditions \(x_i^f(t_0) = x_{i,f}^{t_0}\) \(\forall i \in S_f\)  

(2.31)

Also note that the adjoint dynamics follow

\[
\frac{d\lambda_i^*}{dt} = -\frac{\partial \Psi_f (x^*_f, u_f^*, u^*_D, u^{-f}_D; t)}{\partial x_i^f}
\]

\[
-\sum_{i=1}^{\mid S_f \mid} \lambda_i^* \left( \frac{\partial \theta_i (x^*_f, u_f^*, u^*_D, u^{-f}_D; t)}{\partial x_i^f} \right)
\]

Naturally, the state dynamics obey the initial condition (3.156) while the adjoint dynamics obey a terminal condition given by the transversality conditions:

\[
\lambda_i^f(t_1) = \frac{\partial \Gamma_f [x^f(t_1), t_1]}{\partial x_i^f(t_1)} \quad \forall i \in S_f
\]

Now consider the following DVI which we will show has solutions that are Cournot-Nash equilibria for the agents’ game described above in which individual agent maximizes utility in light of current information about its competitors:

\[
\text{find } u^* \in U \text{ such that }
\]

\[
\sum_{f \in F} \int_{t_0}^{t_1} \left[ \nabla_u H_f^* \right]^T \left( u^f - u^*_f \right) dt \leq 0
\]

(2.32)

for all \(u \in U \equiv \prod_{f \in F} U_f\)

where

\[
H_f^* = H_f \left( u^*_f; x^*_f; \lambda^*_f; u^{-f}_f; t \right)
\]

\[
u = \left( u^f : f \in F \right)
\]
This DVI is a convenient way of expressing the Cournot-Nash game that is our present interest. The variational inequality formulation also provides guidance in devising a computational strategy, as shown by Friesz and Mookherjee [25], Friesz et al. [24] and Pang and Stewart [29]. We briefly describe some of the computational strategies in section 2.8. Note that this variational inequality has clearly defined state and control variables although it is not an optimal control problem and has essentially the similar structure of that of (2.1).

The issue of immediate concern is to formally demonstrate that solutions of this DVI formulation are Cournot-Nash equilibria. In fact we state the following result:

**Theorem 2.3** (DVI formulation of non-cooperative differential games) Any solution of the DVI (2.149) is a differential game-theoretic equilibrium when the regularity conditions 2.8 hold.

**Proof.** We begin by noting that (2.149) is equivalent to the following optimal control problem

\[
\begin{align*}
\max F \left( u^f; u^{-f}; t \right) = & \sum_{f \in F} \sum_{i=1}^{\left| N_f \right|} \int_{t_0}^{t_1} \frac{\partial H^*_{i}}{\partial u_{i}^f} dt \\
\text{s.t.} & \quad u \in U
\end{align*}
\]  

(2.33)

where it is essential to recognize that \( F \left( u^f; u^{-f}; t \right) \) is a linear functional that assumes knowledge of the solution to our game; as such \( F \left( u^f; u^{-f}; t \right) \) is a mathematical construct for use in analysis and has no meaning as a computational device. We note that a necessary condition for the problem (5.35) and (5.36) is

\[
\langle \nabla_u F \left( u^f; u^{-f}; t \right), u^f - u^* \rangle = \sum_{i=1}^{\left| N_f \right|} \int_{t_0}^{t_1} \frac{\partial F \left( \pi^f; \pi^{-f}; t \right)}{\partial u_{i}^f} \left( u_{i}^f - u_{i}^* \right) dt \leq 0
\]  

(2.35)

for all \( u \in U \) since from the regularity condition \( U \) is convex. Furthermore,

\[
\nabla_u F \left( u^f; u^{-f}; t \right) = \left( \frac{\partial H^*_{i}}{\partial u_{i}^f} : i \in N_f \right)
\]  

(2.36)

From (6.2), (3.161) and (2.145), the desired result (2.149) is immediate. ■
Existence of at least one non-cooperative equilibrium

We now state the following existence result:

**Theorem 2.4** (*existence of a solution to (2.149)*) When regularity condition holds, (2.149) has a solution.

As in Theorem 3.9, the formal proof of this existence result is in Browder [27], and depends on the conversion of the DVI to a fixed point problem involving the minimum norm projection and application of Browder’s existence theorem [27].

### 2.4 DVI with State Dependent Time Shifts

#### 2.4.1 Motivation

In the context of dynamic systems comprised of game-theoretic agents, these agents have control of their own (but not necessarily anyone else’s) strategic variables. These strategic variables are self-organizing if observable, having persistent behavioral patterns and hierarchies emerge with the passage of time. In the literature, time-shifted (differential) variational inequalities with explicit state dynamics and explicit controls are known to arise in the modeling of such systems if the game-theoretic agents have a forward-looking or anticipatory perspective and the emergent behavior is some variety of Cournot-Nash-Bertrand equilibrium, be it static or moving in nature. It is also possible to have systems whose time shifts are not only pure function of time, also the state of the system, in which case the system is said to have state dependent time shifts. These mathematical entities often become handy in studying some important practical applications in engineering and economics. For example, Kachani and Perakis [8], Perakis [9], Friesz et al [10], Friesz and Mookherjee [12] and Dial [13] discuss variants of dynamic traffic models that are or can be placed in the form of a time-shifted variational inequality with explicit state dynamics and explicit controls.
For production systems of a related nature see Perakis and Kachani [30]. For models of the evolution of supply chains taking the form of a variational inequality with state dynamics, explicit controls and time shifts see Friesz et al [15]. For models of the Internet expressed as stochastic extensions of the deterministic dynamic variational inequalities studied in this thesis.

2.4.2 Differential Variational Inequalities with State Dependent Time Shifts

We have already seen the following constrained dynamics (2.3) in the statement of DVIs

\[ x(u,t) = \arg\left\{ \frac{dx}{dt} = f(x,u,t), \ x(0) = x^0, \ \Gamma [x(L), L] = 0 \right\} \in (H^1[t_0,t_f])^n \]

Let us consider a modified dynamics which replaces (2.3) by

\[ x(u, uD, t) = \arg\left\{ \frac{dx}{dt} = f(x,u,uD,t), \ x(t_0) = x^0, \ \Gamma [x(t_f), t_f] = 0 \right\} \in (H^1[t_0,t_f])^n \]

(2.37)

where, once again

\[
\begin{align*}
    u & \in U \subseteq (L^2[t_0,\tau])^m \\
    uD & \equiv u(t + D_i(x)) : (H^1[t_0,t_f])^n \times \mathbb{R}_+^m \longrightarrow (L^2[t_0,t_f])^m \\
    f & : (H^1[t_0,t_f])^n \times (L^2[t_0,\tau])^m \times (L^2[t_0,t_f])^m \times \mathbb{R}_+^m \longrightarrow (L^2[t_0,t_f])^n \\
    \Gamma & : (H^1[t_0,t_f])^n \times \mathbb{R}_+^m \longrightarrow (H^1[t_0,t_f])^r
\end{align*}
\]

we assume \( \tau \) is large enough such that

\[ \tau = \sup \{ t + D_i(x(t)) : i \in [1,m], t \in [t_0,t_f] \} \]

Note that \( uD \) is a shorthand for the shifted control vector

\[
uD = \left( \begin{array}{c}
    u_1(t + D_1(x)) \\
    u_2(t + D_2(x)) \\
    \vdots \\
    u_m(t + D_m(x))
\end{array} \right) \in (L^2[t_0,t_f])^m
\]
and

\[ D_i : (\mathcal{H}^1 [t_0, t_f])^n \rightarrow \mathcal{H}^1 [t_0, t_f] \]

for each \( i \in [1, m] \). Additionally we invoke the following regularity condition for the two-point boundary value problem (2.37):

**Definition 2.9 Regular Dynamics.** We shall say the state dynamics operator \( x(u, u_D) \) given by (2.37) is \((x^0, U)\)-regular if the terminal state constraint \( \Gamma [x(t_f), t_f] = 0 \) is reachable from the given initial state \( x^0 \) for all \( u \in U \).

The notation \( x(u, u_D) \) is a direct generalization of that used by Minoux [26] to describe how the Pontryagin minimum principle of optimal control theory may be derived using notions from infinite dimensional mathematical programming; it denotes an operator which determines the state vector for any pair of shifted and un-shifted control vectors. In order to use the operator notation \( x(u, u_D) \), we will invoke \((x^0, U)\)-regularity to ensure that the parametric boundary value problem (2.37) is well posed. Such a regularity condition should not be interpreted as an \textit{a priori} stipulation that the variational inequality to be introduced below has a solution; rather it is a stipulation that the constrained dynamics of (2.37) have a solution for all controls that are considered pertinent to the problem of interest.

### 2.4.3 A Related Optimal Control Problem and Necessary Conditions

Before studying differential variational inequalities with state-dependent time shifts, we need to derive necessary conditions for a related optimal control problem. Let us we consider the following optimal control problem:

\[
\min \Gamma [x(t_f), t_f] + \int_{t_0}^{t_f} G(x, u, u_D, t) dt 
\]

subject to

\[
\frac{dx}{dt} = f(x, u, u_D, t); \quad x(t_0) = x^0 
\]

\[
u \in U
\]

(2.38) \hspace{1cm} (2.39) \hspace{1cm} (2.40)
This is a non-standard optimal control problem, and we will need its necessary conditions. We invoke the following regularity conditions for the above optimal control problem, under which we derive the necessary conditions.

**Definition 2.10 (Regularity conditions of time shifted optimal control problems)** We call the optimal control problem (2.38) - (2.40) regular if:

1. \( u \in U \subseteq (L^2 [t_0, t_f])^m \);
2. \( u_D \in (L^2 [t_0, t_f])^m \);
3. the operator \( x(u, u_D) : (L^2 [t_0, t_f])^m \times (L^2 [t_0, t_f])^m \rightarrow (H^{-1} [t_0, t_f])^n \) is \((x^0, U)\)-regular, continuous and G-differentiable with respect to \( u \) and \( u_D \);
4. \( D_i(x_i) : (H^1 [t_0, t_f])^n \rightarrow H^1 [t_0, t_f] \) is continuously differentiable with respect to \( x_i \) for each \( i \in [1, m] \);
5. \( \Gamma[x, t] : (H^1 [t_0, t_f])^n \times \mathbb{R}^1_+ \rightarrow H^1 [t_0, t_f] \) is continuously differentiable with respect to \( x \);
6. \( G(x, u, u_D, t) : (H^1 [t_0, t_f])^n \times (L^2 [t_0, t_f])^m \times (L^2 [t_0, t_f])^m \times \mathbb{R}^1_+ \rightarrow L^2 [t_0, t_f] \) is continuously differentiable with respect to \( x, u \) and \( u_D \);
7. \( f(x, u, u_D, t) : (H^1 [t_0, t_f])^n \times (L^2 [t_0, t_f])^m \times (L^2 [t_0, t_f])^m \times \mathbb{R}^1_+ \rightarrow (L^2 [t_0, t_f])^n \) is continuously differentiable with respect to \( x, u \) and \( u_D \);
8. \( U \subseteq (L^2 [t_0, t_f])^m \) is convex and compact; and
9. \( x^0 \in \mathbb{R}^n \)

We now state and prove the following necessary conditions for time shifted optimal control problem (2.38) - (2.40):
Theorem 2.5 (Necessary Conditions for Optimal Control with State-Dependent Time Shifts)

When the regulatory conditions of 2.10 hold, then any solution \( u^* \in U \) of the optimal control problem (2.38) through (2.40) obeys the following necessary conditions:

1. The finite dimensional variational inequality principle:

\[
\sum_{i=1}^{m} \frac{\partial H_1^*}{\partial u_i} (u_i - u_i^*) \geq 0 \quad \forall t \in [t_0, D_i(x_i(t_0))], \quad u \in U
\]

\[
\sum_{i=1}^{m} \left\{ \frac{\partial H_1^*}{\partial u_i} + \left[ \frac{\partial H_1^*}{\partial (u_D)} \right] \frac{1}{1 + \sum_{j=1}^{m} \frac{\partial D_j(x^*)}{\partial x_j}} s_i(t) \right\} (u_i - u_i^*) \geq 0
\]

\[
\forall t \in [D_i(x^*(t_0)), t_f + D_i(x^*(t_f))], \quad u \in U
\]

where

\[
s_i(t) = \arg \left[ s = t - D_i(x(s)) \right] \quad \forall t \in [D_i(x^*(t_0)), t_f + D_i(x^*(t_f))], \quad i \in [1, m]
\]

and

\[
H_1^* = H_1(x^*, u^*, u_D^*, \lambda^*, t) = G(x^*, u^*, u_D^*, t) + (\lambda^*)^T f(x^*, u^*, u_D^*, t) \quad \forall t \in [t_0, t_f]
\]

2. The state dynamics

\[
\frac{dx^*}{dt} = f(x^*, u^*, u_D^*, t) ; \quad x^*(t_0) = x^0 ; \quad \text{and}
\]

3. The adjoint dynamics

\[
(-1) \frac{d\lambda^*}{dt} = \nabla_x (\lambda^*)^T f(x^*, u^*, u_D^*, t) ; \quad \lambda^*(t_f) = \frac{\partial \Gamma[x^*(t_f), t_f]}{\partial x}.
\]

Proof. The below proof extends the fixed time shift analysis of Budelis and Bryson [?] to state-dependent time shifts. Note that

\[
x(u, u_D) = x(t_0) + \int_{t_0}^{t} f[x(u, u_D), u, u_D, t] \, dt
\]

It is immediate that

\[
x(u + \theta \rho, u_D + \theta \rho_D) = x(t_0) + \int_{t_0}^{t} f[x(u + \theta \rho, u_D + \theta \rho_D), u + \theta \rho, u_D + \theta \rho_D, t] \, dt
\]
Consequently,

$$\delta x (u, \rho; u_D, \rho_D) = \int_{t_0}^t \left\{ \frac{\partial f [x (u, u_D), u, u_D, t]}{\partial x} \delta x (u, \rho; u_D, \rho_D) + \frac{\partial f [x (u), u, u_D, t]}{\partial u} \delta u (\rho) + \frac{\partial f [x (u), u, u_D, t]}{\partial u_D} \delta u_D (\rho_D) \right\} dt$$

where the G-derivatives of $u$ and $u_D$ obey

$$\delta u (\rho) = \lim_{\theta \to 0} \frac{(u + \theta \rho) - u}{\theta} = \rho$$

$$\delta u_D (\rho_D) = \lim_{\theta \to 0} \frac{(u_D + \theta \rho_D) - u_D}{\theta} = \rho_D$$

Employing the shorthand $y = \delta x (u, \rho; u_D, \rho_D)$, we have the integral equation

$$y = \int_{t_0}^t \left[ \frac{\partial f}{\partial x} y + \frac{\partial f}{\partial u} \rho + \frac{\partial f}{\partial u_D} \rho_D \right] dt$$

(2.41)

It is of course immediate from this integral equation that $y$ obeys

$$\frac{dy}{dt} = \frac{\partial f}{\partial x} y + \frac{\partial f}{\partial u} \rho + \frac{\partial f}{\partial u_D} \rho_D; \quad y(t_0) = 0$$

(2.42)

which is recognized as an initial value problem, verifying that the G-derivative of $x$ is well defined. The G-derivative of $J$ obeys

$$\delta J (u, \rho; u_D, \rho_D) = \left[ \frac{\partial \Gamma [x (t), t]}{\partial x} \delta x (u, \rho; u_D, \rho_D) \right]_{t_0}^{t_f} + \int_{t_0}^{t_f} \left[ \frac{\partial G}{\partial x} \delta x (u, \rho; u_D, \rho_D) + \frac{\partial G}{\partial u} \delta u (\rho) + \frac{\partial G}{\partial u_D} \delta u_D (\rho_D) \right] dt$$

$$= \frac{\partial \Gamma [x (t_f), t_f]}{\partial x} y (t_f) + \int_{t_0}^{t_f} \left[ \frac{\partial G}{\partial x} y + \frac{\partial G}{\partial u} \rho + \frac{\partial G}{\partial u_D} \rho_D \right] dt$$

We introduce adjoint variables $\lambda$ defined by the final value problem

$$-\frac{d\lambda}{dt} = \left( \frac{\partial f}{\partial x} \right)^T \lambda + \left( \frac{\partial G}{\partial x} \right)^T; \quad \lambda (t_f) = \frac{\partial \Gamma [x (t_f), t_f]}{\partial x}$$

(2.43)

so that

$$\delta J (u, \rho; u_D, \rho_D) = \int_{t_0}^{t_f} \left[ -\left( \frac{d\lambda}{dt} \right)^T y - \lambda^T \frac{\partial f}{\partial x} y + \frac{\partial G}{\partial u} \rho + \frac{\partial G}{\partial u_D} \rho_D \right] dt$$

(2.44)
Note that
\[ [\lambda^T y]_{t_0}^{t_f} = [\lambda(t_f)]^T y(t_f) - [\lambda(t_0)]^T y(t_0) \]
\[ = \frac{\partial \Gamma [x(t_f), t_f]}{\partial x} y(t_f) \]
due to (2.43) and the fact that \( y(t_0) = 0 \), so an integration by parts yields
\[ \int_{t_0}^{t_f} \left( \frac{d\lambda}{dt} \right)^T y dt = \int_{t_0}^{t_f} \lambda^T \frac{dy}{dt} dt - \left[ \lambda^T y \right]_{t_0}^{t_f} \]
\[ = \int_{t_0}^{t_f} \lambda^T \left( \frac{\partial f}{\partial x} \cdot y + \frac{\partial f}{\partial u} \cdot \rho + \frac{\partial f}{\partial u_D} \rho_D \right) dt - \frac{\partial \Gamma [x(t_f), t_f]}{\partial x} y(t_f) \]
It follows that
\[ \delta J(u, \rho; u_D, \rho_D) =\]
\[ \frac{\partial \Gamma [x(t_f), t_f]}{\partial x} y(t_f) + \int_{t_0}^{t_f} \left\{ \lambda^T \left[ \frac{\partial f}{\partial x} \cdot y + \frac{\partial f}{\partial u} \cdot \rho + \frac{\partial f}{\partial u_D} \rho_D \right] \right\} dt - \frac{\partial \Gamma [x(t_f), t_f]}{\partial x} y(t_f) \]
\[ = \int_{t_0}^{t_f} \lambda^T \frac{\partial f}{\partial u} \rho dt + \int_{t_0}^{t_f} \lambda^T \frac{\partial G}{\partial u_D} \rho_D dt \]
Defining
\[ H_1(x, u, u_D, \lambda, t) = G(x, u, u_D, t) + \lambda^T f(x, u, u_D, t) \]
we have
\[ \delta J(u, \rho; u_D, \rho_D) = \int_{t_0}^{t_f} \left[ \frac{\partial H_1}{\partial u} \rho + \frac{\partial H_1}{\partial u_D} \rho_D \right] dt \]
as an expression for the G-derivative of the criterion with respect to both \( u \) and \( u_D \). Moreover, terms of the form
\[ \int_{t_0}^{t_f} \frac{\partial H_1}{\partial (u_D)_i} (\rho_D)_i dt = \int_{t_0}^{t_f} \frac{\partial H_1}{\partial (u_D)_i} \delta u_i (t + D_i(x_i)) \]
may be re-expressed by making the change of variables
\[ \Delta_i = t + D_i(x(t)) \iff t = \Delta_i - D_i(x(t)) \]
Because the \( D_i(x) \) are differentiable with respect to \( x_i \), the implicit function theorem gives
\[ \frac{dt}{d\Delta_i} = -\frac{\partial [t - \Delta_i + D_i(x)] / \partial \Delta_i}{\partial [t - \Delta_i + D_i(x)] / \partial t} = \frac{1}{1 + \sum_{j=1}^{m} \frac{\partial D_i(x)}{\partial x_j}} x_j \]
or,
\[
dt = \frac{1}{1 + \sum_{j=1}^{m} \frac{\partial D_i(x)}{\partial x_j} x_j} d\Delta_i
\]  
(2.46)

Note that
\[
t = t_0 \implies \Delta_i = t_0 + D_i(x(t_0));
\]
Putting \(t = t_f \implies \Delta_i = t_f + D_i(x(t_f))\)

Since we may take \(\delta(u_D)_i = 0\) for any time \(t < D_i(x_i(t_0))\), a change of variables based on (2.46) leads to
\[
\int_{t_0}^{t_f} \frac{\partial H_1}{\partial (u_D)_i} (\rho_D)_i dt = \int_{D_i(x_i(t_0))}^{t_f + D_i(x_i(t_f))} \frac{\partial H_1}{\partial (u_D)_i} \delta(u_D)_i dt
\]
\[
= \int_{D_i(x_i(t_0))}^{t_f + D_i(x_i(t_f))} \left[ \frac{\partial H_1}{\partial (u_D)_i} \frac{1}{1 + \sum_{j=1}^{m} \frac{\partial D_i(x)}{\partial x_j} x_j} \right] \delta(u)_i dt
\]
\[
= \int_{D_i(x_i(t_0))}^{t_f + D_i(x_i(t_f))} \left[ \frac{\partial H_1}{\partial (u_D)_i} \frac{1}{1 + \sum_{j=1}^{m} \frac{\partial D_i(x)}{\partial x_j} x_j} \right] \rho_i dt
\]  
(2.47)

where \(s_i(t)\) obeys \(s_i(t) = \arg[s = t - D_i(x_i(s))]\) for any given instant of time \(t\) at which the term
\[
\frac{\partial H_1}{\partial (u_D)_i} \frac{1}{1 + \sum_{j=1}^{m} \frac{\partial D_i(x)}{\partial x_j} x_j}
\]
must be evaluated. Note that the change of variables in (2.47) has re-expressed the G-derivative of \(u_D\) as a derivative of \(u\). We next note that
\[
\int_{t_0}^{t_f} \frac{\partial H_1}{\partial u_i} \rho_i dt = \int_{t_0}^{D_i(x_i(t_0))} \frac{\partial H_1}{\partial u_i} \rho_i dt + \int_{D_i(x_i(t_0))}^{t_f + D_i(x_i(t_f))} \frac{\partial H_1}{\partial u_i} \rho_i dt
\]  
(2.48)

This last result means that for the change of variables introduced above the G-derivative is expressible in terms of \(\rho\); that is
\[
\delta J(u, \rho; u_D, \rho_D) = \delta J(u, \rho; u, \rho) \equiv \delta J(u, \rho)
\]

Using (2.47) and (2.48) we obtain
\[
\delta J(u, \rho) = \int_{t_0}^{D_i(x_i(t_0))} \frac{\partial H_1}{\partial u_i} \rho_i dt + \int_{D_i(x_i(t_0))}^{t_f + D_i(x_i(t_f))} \left\{ \frac{\partial H_1}{\partial (u_D)_i} \frac{1}{1 + \sum_{j=1}^{m} \frac{\partial D_i(x)}{\partial x_j} x_j} \right\}_{t=s_i} \rho_i dt
\]

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Optimality requires \( u^* \in U \) to obey
\[
\delta J (u^*, \rho) \geq 0 \quad \forall \rho \geq 0
\]
which directly yields the desired necessary conditions when it is observed that each direction may be stated as \( \rho = (u - u^*) \) for some \( u \in U \).

The following result, stemming directly from the above theorem, is also important from computational standpoint:

**Corollary 2.1 (Gradient of the Criterion in the Presence of Time Shifts)** For regularity in the sense of Definition 2, the gradient of the criterion (2.38) is

\[
\nabla J (u) = \begin{cases} 
\frac{\partial H_1}{\partial u_i} & \text{if } t \in [t_0, D_i (x_i (t_0))] \\
\frac{\partial H_1}{\partial u_i} + \left[ \frac{\partial H_1}{\partial (u D_i)} \frac{1}{1 + \sum_{j=1}^{m} \frac{\partial H_i (x_j)}{\partial x_j}} s_i(t) \right] & \text{if } t \in [D_i (x^* (t_0)), t_f + D_i (x^* (t_f))] 
\end{cases}
\]

**Proof.** By the Riesz representation theorem we know
\[
\delta J (u^*, \rho) = \langle \nabla J (u^*), (u - u^*) \rangle \quad \forall u \in U
\]
The result is then immediate.

**2.4.4 Statement of a DVI with State Dependent Time Shifts**

With the above background we are now ready to study the following problem:

find \( u^* \in U \) such that
\[
\langle F (x (u^*, u_D, t), u^*, u_D^*, t), u - u^* \rangle \geq 0 \text{ for all } u \in U
\]
where
\[
x (u, u_D) = \arg \left\{ \frac{dx}{dt} = f (x, u, u_D, t), x (t_0) = x^0, u \in U, x (t_f), t_f = 0 \right\} \in (H^1 [t_0, t_f])^n
\]
We refer to (2.51) as a differential variational inequality with explicit controls and time shifts, abbreviated as $DVI(F, f, \Gamma, D, U, x^0)$.

**Necessary Conditions**

To develop necessary conditions for solutions of (2.51) we will rely on the following notion of regularity:

**Definition 2.11** [Regularity of $DVI(F, f, \Gamma, D, U, x^0)$] We call $DVI(F, f, \Gamma, D, U, x^0)$ regular if:

1. $u \in U \subseteq (L^2 [t_0, t_f])^m$;

2. $u_D \in (L^2 [t_0, t_f])^m$;

3. the operator $x(u, u_D) : (L^2 [t_0, t_f])^m \times (L^2 [t_0, t_f])^m \rightarrow (H^1_{\infty} [t_0, t_f])^n$ is $(x^0, U)$-regular, continuous and $G$-differentiable with respect to $u$ and $u_D$;

4. $D_i(x) : (H^1 [t_0, t_f])^n \rightarrow H^1 [t_0, t_f]$ is continuously differentiable with respect to $x_i$, for each $i \in [1, m]$;

5. $\Gamma(x, t) : (H^1 [t_0, t_f])^n \times \mathbb{R}^1_+ \rightarrow (H^1 [t_0, t_f])^r$ is continuously differentiable with respect to $x$;

6. $F(x, u, u_D, t) : (H^1 [t_0, t_f])^n \times (L^2 [t_0, t_f])^m \times (L^2 [t_0, t_f])^m \times \mathbb{R}^1_+ \rightarrow (L^2 [t_0, t_f])^m$ is continuous with respect to $x$ and $u$;

7. $f(x, u, u_D, t) : (H^1 [t_0, t_f])^n \times (L^2 [t_0, t_f])^m \times (L^2 [t_0, t_f])^m \times \mathbb{R}^1_+ \rightarrow (L^2 [t_0, t_f])^n$ is continuously differentiable with respect to $x$, $u$ and $u_D$;

8. $U \subseteq (L^2 [t_0, t_f])^m$ is convex and compact; and

9. $x^0 \in \mathbb{R}^n$. 

30
We next note that (2.51) may be restated as the following optimal control problem

\[
\min_{\gamma} \gamma^T \Gamma [x(t_f), t_f] + \int_{t_0}^{t_f} [F(x^*, u^*, u^*_D, t)]^T u dt \quad (2.53)
\]

subject to

\[
\frac{dx}{dt} = f(x, u, u_D, t); \quad x(t_0) = x^0 \quad (2.54)
\]

\[
u \in U \quad (2.55)
\]

where \( x^* = x(u^*, u^*_D) \) is the optimal state vector and \( \gamma \in \mathbb{R}^r \) is the vector of dual variables for the terminal constraints \( \Gamma [x(t_f), t_f] = 0 \). We point out that this optimal control problem is a mathematical abstraction and of no use for computation, since its criterion depends on knowledge of the variational inequality solution \( u^* \). In what follows we will need the Hamiltonian for (2.53) through (2.55), namely

\[
H_2(x, u, u_D, \lambda, t) = [F(x^*, u^*, u^*_D, t)]^T u + \lambda^T f(x, u, u_D, t) \quad (2.56)
\]

where \( \lambda(t) \) is the adjoint vector that solves the adjoint equations and transversality conditions for given state variables and controls. It is now a relatively easy matter to derive the necessary conditions stated in the following theorem:

**Theorem 2.6 [Necessary Conditions for DVI(\(F, f, \Gamma, D, U, x^0\))]** When regularity in the sense of Definition 3 holds, solutions \( u^* \in U \) of DVI(\(F, f, \Gamma, D, U, x^0\)) must obey:

1. the finite dimensional variational inequality principle:

\[
0 \leq \left[ F_i(x^*, u^*, u^*_D, t) + \sum_{j=1}^{m} \lambda_j \frac{\partial f_i(x^*, u^*, u^*_D, t)}{\partial u_i} \right] (u_i - u^*_i) \quad \forall i \in [1, m], t \in [t_0, D_i(x(t_0))], u \in U
\]

2. the adjoint equation and transversality conditions:

\[
0 \leq \left\{ F_i(x^*, u^*, u^*_D, t) + \sum_{j=1}^{m} \lambda_j \frac{\partial f_j(x^*, u^*, u^*_D, t)}{\partial u_i} + \lambda_j \frac{\partial f_j(x^*, u^*, u^*_D, t)}{\partial u_D_i} \right. \\
\left. + \left[ \lambda_j \frac{\partial f_j(x^*, u^*, u^*_D, t)}{\partial (u_D)_i} \frac{1}{1 + \sum_{j=1}^{m} \frac{\partial D_i(x^*)}{\partial x_j} f_j(x^*, u^*, u^*_D, t)} \right]_{s(t)} \right\} (u_i - u^*_i)
\]
∀i ∈ (1, m), t ∈ [Di (x* (t0)), t_f + Di (x* (t_f))], u ∈ U

2. the state dynamics
\[
\frac{dx^*}{dt} = f (x^*, u^*, u^*_D, t); \quad x^* (t_0) = x^0; \quad \text{and}
\]

3. the adjoint dynamics
\[
(-1) \frac{d\lambda^*}{dt} = \nabla_x (\lambda^*)^T f (x^*, u^*, u^*_D, t); \quad \lambda^* (t_f) = \nu^T \partial \Gamma [x^* (t_f), t_f]
\]

where \( \nu \in \mathbb{R}^r \) is the vector of dual variables for the terminal constraints \( \Gamma [x (t_f), t_f] = 0 \).

**Proof.** Note that the \( DVI (F, f, \Gamma, D, U, x^0) \) is equivalent to the following time-shifted optimal control problem
\[
\min \nu^T \Gamma [x (t_f), t_f] + \int_{t_0}^{t_f} [F (x^*, u^*, u^*_D, t)]^T u dt
\]
subject to
\[
\frac{dx}{dt} = f (x, u, u_D, t); \quad x (t_0) = x^0 \quad u \in U
\]
with Hamiltonian \( H_2 (x, u, u_D, \lambda, t) = [F (x^*, u^*, u^*_D, t)]^T u + \lambda^T f (x, u, u_D, t) \). By virtue of regularity we may apply Theorem 2.5; the necessary conditions follow immediately. □

### 2.5 Stochastic Optimal Control Problem

So far we have considered types of differential games (and associated differential variational inequalities) in which the fundamentals (utility functions, system dynamics, initial state, etc.) do not contain any uncertainty. For some applications this is quite satisfactory, but
for others it presents a severe limitation. We now take a look at stochastic optimal control problems and differential games in which some of the fundamentals involve random variables or stochastic processes. Uncertainty can be incorporated in our basic framework in many different ways, so we shall not give a complete treatment of this matter but concentrate on one form of stochastic models which seem to be most useful in applications. These models make use of so called Wiener process to capture situations characterized by continuous stochastic noise. Wiener processes are also known as Brownian motion or white noise processes and play an important role in many different fields ranging from physics to economics. A Wiener process can be described as

**Definition 2.12 (Wiener process)** A standardized \( k \)-dimensional Wiener process \( z \) with time domain \([t_0, t_1]\)\(^1\) is a continuous-time stochastic process with values in \( \mathbb{R}^k \), that is, \( z : [t_0, t_1] \times \Xi \mapsto \mathbb{R}^k \). Its defining properties are:

1. \( z(t_0, \xi) = z_0 \) for all \( \xi \) in a set of probability 1 where \( z_0 \in \mathbb{R}^k \) is an arbitrary initial value;

2. for any finite sequence of real numbers \((t_1, t_2, \ldots, t_l)\) with \( t_0 \leq t_1 < t_2 < \ldots < t_l \leq t_1 \), it holds that the random variables \( z(t_1, \cdot) \) and \( z(t_{i+1}, \cdot) - z(t_i, \cdot) \), \( i \in [1, l-1] \) are stochastically independent;

3. for all pairs \((s, t)\) of real numbers such that \( t_0 \leq s < t \leq t_1 \), the random variable \( z(t, \cdot) - z(s, \cdot) \) has a normal distribution with mean vector \( 0 \in \mathbb{R}^k \) and covariance matrix \((t-s)I\), where \( I \in \mathbb{R}^{k \times k} \) denotes the \( k \times k \) unit matrix.

In the stochastic control models and differential games to be considered subsequently, it is assumed that the evolution of the state variable \( x \) can be described by a

---

\(^1\)Our notation \([0, T]\) implies \([0, T] = [0, T] \) if \( T \) is a finite number and \([0, T] = [0, \infty) \) if \( T = \infty \).
stochastic differential equation of the form
\[
dx(t) = \mu(u(t), x(t)) \, dt + \sigma(u(t), x(t)) \, dz(t) \tag{2.57}\\
x(t_0) = x_0
\]

We emphasize that any solution to the stochastic differential equation (2.57) is a stochastic process depending on the realizations of \( \omega \in \Xi \). We do not make this dependence explicit but simply write \( x(t) \) instead of correct, but more cumbersome, notation \( x(t, \omega) \). The reader should keep in mind that the value of \( x \) at time \( t \) can not be known in advance without knowing the realization of \( \omega \). A similar remark applies to the control paths. Controls usually depend on the most recent information and are random variables too. However, we simply write \( u(t) \) instead of \( u(t, \omega) \). The precise meaning of the state that \( x(\cdot) \) satisfies the integral equation
\[
x(t) = x_0 + \int_0^t \mu(x(s), u(s), s) \, ds + \int_0^t \sigma(x(s), u(s), s) \, dw(s)
\]
for all \( \xi \) in a set of probability 1. The first integral on the right-hand side is the usual Riemann integral while the second integral has to be interpreted as the limit
\[
\lim_{\delta \to 0} \sum_{l=1}^{L-1} \sigma(x(t_l), u(t_l), t_l) [w(t_{l+1}) - w(t_l)]
\]
where \( 0 = t_1 < t_2 < \cdots < t_L = t \) and
\[
\delta = \max \{|t_{l+1} - t_l| | 1 \leq l \leq L - 1 \}
\]
The feasible control values at time \( t \) have to be chosen from the set \( U(x(t), t) \). For simplicity, let us consider the following one dimensional \( (k = 1) \) stochastic optimal control problem with Wiener process based state dynamics :
\[
J(x(t), t, \infty) = \max_{u(\cdot)} E_t \int_t^\infty e^{-\rho s} F(u(s), x(s), s) \, ds \tag{2.58}
\]
subject to
\[
dx(t) = \mu(u(t), x(t)) \, dt + \sigma(u(t), x(t)) \, dz(t) \tag{2.59}\\
x(t_0) = x_0 \tag{2.60}
\]
where

\[ u = u(t) = u(t, \omega) \]

is the control variable, and

\[ x = x(t) = x(t, \omega) \]

is the state variable, \( \rho \geq 0 \) is the nominal discount rate compounded continuously, \( F(\cdot) \) denotes utility function. \( \mu(\cdot) \) is the drift component and \( \sigma(\cdot) \) is the diffusion component in the state dynamics (2.59), \( x_0 \in \mathbb{R}_+^1 \) is the known initial condition. \( E_t \) denotes expectation conditioned on \( u(t) \) and \( x(t) \). The problem described above is the stochastic analogue of the deterministic optimal control problem. A standard technique to deal with this problem is Bellman’s Principle of Optimality according to which ‘an optimal policy has the property that, whatever the initial state and control are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision’.

### 2.5.1 Maximum Principle in One Dimension

Let us at this moment consider a special case of the problem (5.1) - (2.60) where the problem becomes

\[ J(x(t), t, tf) = \max_{u(\cdot)} E_t \int_t^{tf} F(u(s), x(s), s) \, ds \]  

(2.61)

subject to

\[ dx(t) = \mu(u(t), x(t)) \, dt + \sigma(u(t), x(t)) \, dz(t) \]  

(2.62)

\[ x(t) = x_0 \]  

(2.63)
Using Bellman’s technique of dynamic programming, the problem (2.61) - (2.63) can be analyzed as follows:

\[
J(x(t), t, t_f) = \max_{u(\cdot)} E_t \int_t^{t_f} F(u(s), x(s), s) \, ds
\]

\[
= \max_{u(\cdot)} E_t \int_t^{t+\Delta t} F(u(s), x(s), s) \, ds + \max_{u(\cdot)} E_{t+\Delta t} \int_{t+\Delta t}^{t_f} F(u(s), x(s), s) \, ds
\]

\[
= \max_{u(\cdot)} E_t \left[ \int_t^{t+\Delta t} F(u(s), x(s), s) \, ds + J(x(t+\Delta t), t+\Delta t, t_f) \right]
\]

(2.64)

Our development of subsequent results in this section depends on the following regularity conditions:

**Definition 2.13** (Regularity of (2.61) (2.61) is said to be regular if

1. \( J(x(t), t, t_f) \) is single valued; and

2. \( J(x(t), t, t_f) \) is continuously differentiable (i.e., \( C^1 \))

Now, taking Taylor series expansion of \( J(x(t+\Delta t), t+\Delta t, t_f) \) about the point \((x,t)\). That expansion takes the form

\[
J(x(t+\Delta t), t+\Delta t, t_f) \approx \left[ \frac{1}{2} J_{x} (\Delta x)^2 + \frac{1}{2} J_{tt} (\Delta t)^2 + J_{xt} (\Delta x)(\Delta t) + o(\Delta t) \right]
\]

(2.65)

From (2.64) and (2.65) we can write

\[
J(x(t), t, t_f) = \max_{u(\cdot)} E_t [F(u(s), x(s), s) \Delta t + J(x(t), t, t_f) + J_x \Delta x + J_t \Delta t
\]

\[
\frac{1}{2} J_{xx} (\Delta x)^2 + \frac{1}{2} J_{tt} (\Delta t)^2 + J_{xt} (\Delta x)(\Delta t) + o(\Delta t) \]

(2.66)

Note that we have used the regularity condition 2 of definition 2.13, here assuming that \( J \) has continuous partial derivatives of all orders less than 3 in some open set containing the line segment connecting two points \((x(t), t)\) and \((x(t+\Delta t), t+\Delta t)\). Let (2.62) be
approximated and write
\[ \Delta x = \mu(u(t), x(t)) \Delta t + \sigma(u(t), x(t)) \Delta z + o(\Delta t) \] (2.67)

Inserting (2.67) into (2.66) and using the following multiplication rule
\[
\begin{align*}
(\Delta t)^2 &= 0 \\
(\Delta z)^2 &= \Delta t \\
(\Delta z) \times (\Delta t) &= 0
\end{align*}
\]
when \( \Delta t \to 0 \), we obtain
\[ 0 = \max_{u(\cdot)} E_t \left[ F(u(t), x(t)) \Delta t + \left( J_x \mu + J_t + \frac{1}{2} J_{xx} \sigma^2 \right) \Delta t + J_x \sigma \Delta z + o(\Delta t) \right] \] (2.68)

For notational convenience let
\[ \Delta J = \left( J_x \mu + J_t + \frac{1}{2} J_{xx} \sigma^2 \right) \Delta t + J_x \sigma \Delta z \]
then (2.68) becomes
\[ 0 = \max_{u(\cdot)} E_t \left[ F(u(t), x(t)) \Delta t + \Delta J + o(\Delta t) \right] \] (2.69)
(2.69) is a partial differential equation with boundary condition
\[ \frac{\partial J}{\partial u}(x(t_f), t_f, t_f) = 0 \]
Pass \( E_t \) through parenthesis of (2.69) and after dividing both sides by \( \Delta t \), let \( \Delta t \to 0 \) to conclude
\[ 0 = \max_{u(\cdot)} E_t \left[ F(u(t), x(t)) + J_x \mu(u(t), x(t)) + J_t + \frac{1}{2} J_{xx} \sigma^2 (u(t), x(t)) \right] \] (2.70)
(2.70) is usually written as
\[ -J_t = \max_{u(\cdot)} E_t \left[ F(u(t), x(t)) + J_x \mu(u(t), x(t)) + \frac{1}{2} J_{xx} \sigma^2 (u(t), x(t)) \right] \] (2.71)
and is known as Hamilton-Jacobi-Bellman (HJB) equation of stochastic control theory. Let us proceed further with our analysis. We define the co-state variable \( \lambda(t) \) as
\[ \lambda(t) = J_x (x(t), t, t_f) \] (2.72)
From (2.72) it follows immediately that

\[ \lambda_x = \frac{\partial \lambda}{\partial x} = J_{xx} (x(t), t, t_f) \]  \quad (2.73)

Using (2.72) and (2.73) we may rewrite (2.71) as

\[ -J_t = \max_{u(t)} H \left( x, u, \lambda, \frac{\partial \lambda}{\partial x} \right) \]  \quad (2.74)

where \( H \) is the stochastic version of the Hamiltonian, and is nothing but the functional notation of the expression inside the brackets of (2.71). Assume next that a function \( u(t) \) exists that solves the maximization problem of (2.74) and denote such a function by

\[ u^0 = u^0 \left( x, \lambda, \frac{\partial \lambda}{\partial x} \right) \]  \quad (2.75)

Note that \( u^0 \) is a function of \( x(t) \) and \( t \) alone along the optimum path, because \( J_x \) is a function of \( x(t) \) and \( t \) alone. In the applied control literature, and more specifically in economic applications, \( u^0 \) is called the policy function. Assuming then that a policy function \( u^0 \) exists, (2.74) can be expressed as

\[ -J_t = \max_u H \left( x, u, \lambda, \frac{\partial \lambda}{\partial x} \right) \]

\[ = H \left( x, u^0 \left( x, \lambda, \frac{\partial \lambda}{\partial x} \right), \lambda, \frac{\partial \lambda}{\partial x} \right) \]

\[ = H^0 \left( x, \lambda, \frac{\partial \lambda}{\partial x} \right) \]  \quad (2.76)

this last equation, (2.76), is again a functional notation of the right-hand side expression of (2.71) under the assumption of the existence of an optimum control, \( u^0 \), i.e.,

\[ H^0 \left( x, \lambda, \frac{\partial \lambda}{\partial x} \right) = F \left( u^0, x \right) + \lambda \cdot \mu \left( u^0, x \right) + \frac{1}{2} \frac{\partial \lambda}{\partial x} \sigma^2 \left( u^0, x \right) \]  \quad (2.77)

Equipped with the above analysis, our final goal in this subsection is to derive a system of stochastic differential equations describing the behavior of the state and costate variables (similar to the deterministic optimal control, where the differential equations were ordinary differential equations), i.e., find expressions for \( dx \) and \( d\lambda \). An expression for \( d\lambda \) is almost
readily available to us from (2.62), (2.75) and (5.9); in particular

\[
\begin{align*}
\, dx & = \mu (u^0, x) \, dt + \sigma (u^0, x) \, dz \\
& = H^0_\lambda (x, \lambda, \frac{\partial \lambda}{\partial x}) \, dt + \sigma (u^0, x) \, dz \\
& = H^0_\lambda \, dt + \sigma \, dz
\end{align*}
\] (2.78)

Note that

\[
\frac{\partial H^0_\lambda}{\partial \lambda} = H^0_\lambda = \mu (u^0, x)
\] (2.79)

used in the derivation of (2.78) and this is obvious from (5.9). Next we derive an expression for \(d\lambda\). Use the definition of \(\lambda (t)\) given in (2.72), (2.73) and Itô’s lemma to get

\[
d\lambda = \left[ J_{xt} + J_{xx} \mu + \frac{1}{2} J_{xxx} \sigma^2 \right] \, dt + J_{xx} \sigma \, dz (t)
\] (2.80)

To compute (2.80), we compute \(J_{xt}\) from (2.76) assuming that equality holds for mixed partial derivatives

\[
-J_{xt} = H^0_x + H^0_\lambda \cdot \frac{\partial \lambda}{\partial x} + H^0_{\lambda x} \cdot \frac{\partial^2 \lambda}{\partial x^2}
\] (2.81)

(2.81) is obtained using the relationship

\[
H^0_\lambda = \mu \text{ (from (2.79))}
\]

and \(\frac{\partial \lambda}{\partial x} = J_{xx} \text{ (from (2.73))}\)

(2.81) can further be simplified as

\[
-J_{xt} = H^0_x + \mu \cdot J_{xx} + H^0_{\lambda x} \cdot \frac{\partial^2 \lambda}{\partial x^2}
\] (2.82)

(2.82) is obtained using

\[
H^0_{\lambda x} = \frac{1}{2} \sigma^2 \text{ (from (2.79))}
\]

and \(\frac{\partial}{\partial x} \left( \frac{\partial \lambda}{\partial x} \right) = \frac{\partial}{\partial x} (J_{xx}) = J_{xxx}\).
Finally substituting (2.82) into (2.80) we reach our desired result

\[
\begin{align*}
\frac{d\lambda}{dt} &= \left[ -H^0_x - \mu \cdot J_{xx} - \frac{1}{2} \sigma^2 \cdot J_{xxx} + J_{xx} \mu + \frac{1}{2} J_{xxx} \sigma^2 \right] dt + J_{xx} \sigma dz(t) \\
&= -H^0_x dt + \sigma J_{xx} dz(t)
\end{align*}
\]

(2.83)

Let us summarize the above analysis in the following proposition:

**Proposition 2.7** *(Pontryagin Stochastic Maximum Principle)* Suppose that \( x(t) \) and \( u^0(t) \)
solve for \( t \in [0, t_f] \)

\[
\max_u E_0 \int_0^{t_f} F(u(t), x(t), t) dt
\]

subject to conditions

\[
dx(t) = \mu(u(t), x(t)) dt + \sigma(u(t), x(t)) dz(t), x(t) \text{ given}
\]

Then, there exists a costate variable \( \lambda(t) \) such that for each \( t, t_f \in [0, t_f] \):

1. \( u^0 \) maximizes \( H(x, u, \lambda, \frac{\partial \lambda}{\partial x}) \) where

\[
H(x, u, \lambda, \frac{\partial \lambda}{\partial x}) = F(u, x) + \lambda \cdot \mu(u, x) + \frac{1}{2} \sigma^2(u, x) \cdot \frac{\partial \lambda}{\partial x}
\]

2. the costate function \( \lambda(t) \) satisfies the stochastic differential equation

\[
\frac{d\lambda}{dt} = -H^0_x dt + \sigma(u^0, x) J_{xx} dz(t)
\]

3. and the transversality condition holds

\[
\lambda(x(t_f), t_f) = \frac{\partial J}{\partial x}(x(t_f), t_f, t_f) \geq 0 \\
\lambda(t_f) \cdot x(t_f) = 0
\]

**2.5.2 Generalized Stochastic Optimal Control Problems**

Next we proceed to make some generalizations, we deal with them sequentially in the following order:
1. the two point boundary value problem structure arising from the Pontryagin’s Stochastic Maximum principle

2. presence of a bequest function which has the same interpretation as does the salvage value at the terminal time have for the deterministic optimal control problem

3. when the instantaneous utility function, \( F(u(s), x(s), s) \), is computed as the net present value (i.e., the integrand of the criterion is multiplied by \( e^{-\rho t} \) where \( \rho \) is the constant discount rate compounded continuously)

4. presence of pure control constraints

5. multi-dimensional stochastic optimal control problem with correlated random shocks.

The first three extensions are straightforward, and in fact can readily be extended from our previous analysis, but the remaining two are little more involved and we address them together.

2.5.3 Two Point Boundary Value Problem Formulation

Mathematically speaking, the optimal path obtained from Pontryagin’s Stochastic Maximum Principle is the solution of two stochastic differential equations subject to certain conditions. More specifically, we rewrite these equations together

\[
\begin{align*}
\frac{dx}{dt} &= H^0_x dt + \sigma dz \\
\frac{d\lambda}{dt} &= -H^0_x dt + \sigma J_{xx} dz \\
x(0) &= x_0 \in \mathbb{R}^1 \\
\lambda(x(t_f), t_f) &= 0
\end{align*}
\]

Evidently this is a two point boundary value problem (TPBVP) having initial time condition on state, \( x \) and terminal time constraint on co-state variable, \( \lambda \). Of course, this TPBVP is different than (more difficult to solve as well) from its deterministic analogue as because
of the presence of stochastic differential equations describing the states and the adjoint variables.

2.5.4 Bequest Function

It is easy to generalize proposition 3.150 by introducing a bequest function $\Psi (x(t),t)$. If this were to be the case, the maximization problem would become

$$\max_u E_0 \left[ \int_0^{t_f} F(u(t),x(t),t) \, dt + \Psi(x(t_f),t_f) \right]$$

subject to the same conditions (2.59), (2.60) as before. Proposition 3.150 holds with a new transversality condition, i.e.,

$$\lambda(x(t_f),t_f) = \frac{\partial \Psi(x(t_f),t_f)}{\partial x(t_f)}$$

2.5.5 When the Criterion is NPV of Utility

Generalizing the optimal control problem (2.61) - (2.63) in another direction by allowing discounting, the optimal control problem becomes (5.1) - (2.60). For doing so, we need not repeat the previous analysis. Let,

$$J(x(t),t,t_f) = \max_u E_t \int_t^{t_f} e^{-\rho s} F(u(s),x(s),s) \, ds$$

subject to (2.62) and $x(t)$ is given. Write

$$\phi(x(t),t,t_f) = e^{\rho t} \cdot J(x(t),t,t_f)$$

and also

$$J_t = \frac{d}{dt} \{ e^{-\rho t} \cdot \phi \}$$

$$= -\rho e^{-\rho t} \cdot \phi$$

thus, the Hamilton - Jacobi - Bellman equation is now transformed into

$$\rho \phi = \max_u E_t \left[ F(u(t),x(t)) + \phi_x \mu(u(t),x(t)) + \frac{1}{2} \phi_{xx} \sigma^2(u(t),x(t)) \right]$$

The remaining analysis can then be patterned after.
2.5.6 Multi-dimensional Problem with Correlated Noise

We now consider the following general multidimensional time dependent optimal control problem where control and state vectors are

\[ u = (u_i : i \in [1, m]) \]
\[ x = (x_i : i \in [1, n]) \]

respectively. The optimal control has the form

\[ J(x(t), t, t_f) = \max_u E_t \left[ \Psi(x(t_f), t_f) + \int_t^{t_f} e^{-\rho s} F(u(s), x(s), s) \, ds \right] \]

subject to

\[ dx_i(t) = \mu_i(u(t), x(t)) \, dt + \sigma_i(u(t), x(t)) \, dz_i(t) \text{ for all } i = 1, \ldots, n \quad (2.84) \]

and

\[ x(t) \equiv \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \text{ given} \]

Here the random shocks are assumed to be uncorrelated. If they are correlated, then the associated state dynamics would be

\[ dx_i(t) = \mu_i(u(t), x(t)) \, dt + \sum_{j=1}^{n_i} \sigma_{ij}(u(t), x(t)) \, dz_{ij}(t) \text{ for all } i = 1, \ldots, n \]

where \( dz_{ij} \) are Wiener processes that satisfy

\[ \text{covariance } (dz_{rj}, dz_{sj}) = \rho_{rj,sj} \, dt \]

where \( \rho_{rj,sj} \) is the correlation coefficient which is independent of \( x(t) \) and \( u(t) \). For the time being, we focus on the uncorrelated state dynamics (2.84), the Hamiltonian - Jacobi - Bellman equation for that problem is

\[ -J_t(x(t), t, t_f) = \max_u E_t \left[ F(u(t), x(t)) + [J_x]^T \cdot \mu(u(t), x(t)) + \frac{1}{2} \text{tr} \left( J_{xx} \cdot \sigma(u(t), x(t)) \cdot \sigma^T(u(t), x(t)) \right) \right] \]

\[ = \max_u H(x(t), u(t), \lambda(t), \nabla_x \lambda(t), t) \]

\[ = H^0(x(t), \lambda(t), \nabla_x \lambda(t), t) \]
Note that $\text{tr}(A)$ stands for the trace of the matrix $A$, i.e., if $A = (a_{ij})_{n \times n}$, then $\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$. $\Gamma^T$ denotes transpose and

$$
\mu = \begin{pmatrix} 
\mu_1 \\
\vdots \\
\mu_n 
\end{pmatrix}, \sigma = \begin{pmatrix} 
\sigma_1 \\
\vdots \\
\sigma_n 
\end{pmatrix}
$$

The multidimensional analogue of the costate stochastic differential equation becomes

$$
d\lambda_i(t) = H_{xi}^0 dt + \sum_{j=1}^{n} J_{xixj} \sigma_j dz_j
$$

provided that $dz_i$ is not correlated with $dz_j$ for $i \neq j$.

### 2.5.7 Pure Control Constraints and Correlated Noise

We now study a more general problem in stochastic control theory which is formulated as

$$
J(x(t), t, t_f) = \max_u \mathbb{E}_t \left[ \Psi(x(t_f), t_f) + \int_t^{t_f} F(u(s), x(s), s) ds \right]
$$

subject to the conditions

$$
dx_i(t) = \mu_i(u(t), x(t)) dt + \sum_{j=1}^{n} \sigma_{ij}(u(t), x(t)) dz_{ij}(t) \text{ for all } i = 1, \ldots, n \tag{2.85}
$$

$x(t)$ is given

$$
g_i(x(t), u(t), t) \geq 0 \text{ for all } i = 1, \ldots, L
$$

In other words, the control stochastic process $u(t)$ satisfies a set of $L$ inequality constraints.

We also consider correlated shocks, i.e., $dz_{ij}$ are Wiener processes that satisfy

$$
\text{covariance } (dz_{rj}, dz_{sj}) = \rho_{rj,sj} dt
$$

$\rho_{rj,sj}$ is the correlation coefficient which is independent of $x(t)$ and $u(t)$. We proceed with the analysis of this problem and at the end we summarize the results in a proposition, as we did earlier. We write the recursive equation by Bellman’s Principle of Optimality:

$$
J(x(t), t, t_f) = \max_u \mathbb{E}_t \left[ \int_t^{t+\Delta t} F(u(s), x(s), s) ds + J(x(t+\Delta t), t+\Delta t, t_f) \right]
$$
Now, if the approximation
\[ E_t \int_t^{t+\Delta t} F(u(s), x(s), s) \, ds = F(u(s), x(s), s) \Delta t + o(\Delta t) \]
is valid, then we may write
\[ J(x(t), t, t_f) = \max_u [F(u(s), x(s), s) \Delta t + E_t J(x(t+\Delta t), t+\Delta t, t_f) + o(\Delta t)] \]

Put,
\[ \Delta J(t) = J(x(t+\Delta t), t+\Delta t, t_f) - J(x(t), t, t_f) \]

Thus we obtain
\[ 0 = \max_u [F(u(t), x(t), t) \Delta t + E_t \Delta J(t) + o(\Delta t)] \quad (2.86) \]

Using Taylor’s theorem to expand \( \Delta J(t) \) around \((x(t), t)\) (assuming \( J(x(t), t, t_f) \) is regular in the sense Definition 2.13)\)
\[ \Delta J(t) = J_t \Delta t + J_x^T \Delta x + \frac{1}{2} (\Delta x)^T J_{xx} (\Delta x) + o(\Delta t) \quad (2.87) \]

By taking conditional expectation of \( \Delta J(t) \) (conditioned on \( x(t) \) and \( u(t) \), denoted by \( E_t \) in (2.87) we have
\[ E_t \Delta J(t) = J_t \Delta t + E_t \left[ J_x^T \Delta x \right] + E_t \left[ \frac{1}{2} (\Delta x)^T J_{xx} (\Delta x) \right] + o(\Delta t) \quad (2.88) \]

Therefore we will have to compute \( E_t \left[ J_x^T \Delta x \right] \) and \( E_t \left[ \frac{1}{2} (\Delta x)^T J_{xx} (\Delta x) \right] \). We do so next. From state dynamics (2.85) we write
\[ \Delta x_i = \mu_i (u(t), x(t)) \Delta t + \sum_{j_i=1}^{n_i} \sigma_{ij_i} (u(t), x(t)) \Delta z_{ij_i} + o(\Delta t) \quad (2.89) \]
conditional expectation of (2.89) yields
\[ E_t \Delta x_i = E_t \left[ \mu_i (u(t), x(t)) \Delta t + \sum_{j_i=1}^{n_i} \sigma_{ij_i} (u(t), x(t)) \Delta z_{ij_i} + o(\Delta t) \right] \]
\[ = \mu_i (u(t), x(t)) \Delta t + o(\Delta t) \quad (2.90) \]
because
\[ E_t \Delta z_{ij_i} = 0 \text{ for all } i = 1, \ldots, n \text{ and } j_i = 1, \ldots, n_i \]
We can compute

\[
E_t \left[ J_x^T \Delta x \right] = E_t \left\{ \sum_{i=1}^{n} [J_{x_i}]^T \Delta x_i \right\} \\
= \sum_{i=1}^{n} E_t \left[ [J_{x_i}]^T \Delta x_i \right] = \sum_{i=1}^{n} [J_{x_i}]^T E_t (\Delta x_i) \\
= \sum_{i=1}^{n} [J_{x_i}]^T \mu_i (u(t), x(t)) \Delta t + o(\Delta t) \tag{2.91}
\]

the last expression is obtained by using (2.90). Furthermore,

\[
E_t \left[ \frac{1}{2} (\Delta x)^T J_{xx} (\Delta x) \right] = E_t \left\{ \sum_{r=1}^{n} \sum_{s=1}^{n} (\Delta x_r) J_{x_r x_s} (\Delta x_s) \right\} \\
= E_t \left\{ \sum_{r=1}^{n} \sum_{s=1}^{n} \left( \sum_{j_r=1}^{n} \sum_{j_s=1}^{n} J_{x_r x_s} \left( \mu_r \Delta t + \sum_{j_s=1}^{n} \sigma_{r j_s} \Delta z_{r j_s} \right) \right) \right\} \\
= E_t \left\{ \sum_{r=1}^{n} \sum_{s=1}^{n} \left( \sum_{j_r=1}^{n} \sum_{j_s=1}^{n} J_{x_r x_s} \left( \mu_r \Delta t + \sum_{j_s=1}^{n} \sigma_{r j_s} \Delta z_{r j_s} \right) \right) \right\}
\]

We have the following multiplication table (see also Chang [31])

<table>
<thead>
<tr>
<th>Multiplication</th>
<th>\Delta z_{r1}</th>
<th>\Delta z_{r2}</th>
<th>\cdots</th>
<th>\Delta z_{rn}</th>
<th>\Delta t</th>
</tr>
</thead>
<tbody>
<tr>
<td>\Delta z_{s1}</td>
<td>\rho_{s1r1} \Delta t</td>
<td>\rho_{s1r2} \Delta t</td>
<td>\cdots</td>
<td>\rho_{s1rn} \Delta t</td>
<td>0</td>
</tr>
<tr>
<td>\Delta z_{s2}</td>
<td>\rho_{s2r1} \Delta t</td>
<td>\rho_{s2r2} \Delta t</td>
<td>\cdots</td>
<td>\rho_{s2rn} \Delta t</td>
<td>0</td>
</tr>
<tr>
<td>\cdots</td>
<td>\cdots</td>
<td>\cdots</td>
<td></td>
<td>\cdots</td>
<td>\cdots</td>
</tr>
<tr>
<td>\Delta z_{sn}</td>
<td>\rho_{sjr1} \Delta t</td>
<td>\rho_{sjr2} \Delta t</td>
<td>\cdots</td>
<td>\rho_{snrn} \Delta t</td>
<td>0</td>
</tr>
<tr>
<td>\Delta t</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

because of the fact

\[
E_t \{ \Delta z_{r j_r} \Delta z_{s j_s} \} = \rho_{r j_r s j_s} \Delta t + o(\Delta t) \\
E_t \{ \Delta t \Delta z_{r j_r} \} = \Delta t E_t \{ \Delta z_{r j_r} \} = 0 \\
E_t \{ \Delta t \Delta z_{s j_s} \} = \Delta t E_t \{ \Delta z_{s j_s} \} = 0 \\
E_t \{ \Delta t \Delta t \} \rightarrow 0
\]

Now (2.91) becomes

\[
E_t \left[ \frac{1}{2} (\Delta x)^T J_{xx} (\Delta x) \right] = \sum_{r=1}^{n} \sum_{s=1}^{n} \sum_{j_r=1}^{n} \sum_{j_s=1}^{n} J_{x_r x_s} \sigma_{r j_r} \sigma_{s j_s} \rho_{r j_r s j_s} \Delta t + o(\Delta t)
\]

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Collecting the results above, (2.88) becomes

\[ E_t \Delta J(t) = \left\{ J_t \Delta t + \sum_{i=1}^{n} J_{x_i} \mu_i \Delta t + \frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \sum_{j_r=1}^{n_r} \sum_{j_s=1}^{n_s} J_{x,r,s} \sigma_{r,j_r} \sigma_{s,j_s} \rho_{r,j_r,s,j_s} \Delta t + o(\Delta t) \right\} \]

and also

\[-J_t(x(t), t, t_f) \Delta t = \max_u \{ F(u(t), x(t), t) \Delta t + \theta(x(t), u(t), t, t_f) \Delta t + o(\Delta t) \} \]

dividing both sides by \( \Delta t \) and taking the limit \( \Delta t \to 0 \) we get

\[-J_t(x(t), t, t_f) = \max_u \{ F(u(t), x(t), t) + \theta(x(t), u(t), t, t_f) \} \]

where \( \theta \) is defined by

\[ \theta(x(t), u(t), t, t_f) = \sum_{i=1}^{n} J_{x_i} \mu_i + \frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \sum_{j_r=1}^{n_r} \sum_{j_s=1}^{n_s} J_{x,r,s} \sigma_{r,j_r} \sigma_{s,j_s} \rho_{r,j_r,s,j_s} \]

But notice that \( \theta \) is a function of \((x, u, t, t_f)\) since each \( \mu_i \) and \( \sigma_{ij} \) are functions of \((x, u, t, t_f)\).

Therefore we can replace the maximization problem by the simpler one, i.e.,

\[-J_t(x(t), t, t_f) \Delta t = \max_{u \in U(x(t), t)} \{ F(u(t), x(t), t) + \theta(x(t), u(t), t, t_f) \} \]

where the constraint set is defined by

\[ U(x(t), t) = \{ u : g_i(x(t), u, t) \geq 0 \text{ for all } i = 1, \ldots, L \} \]

We summarize the results in the following proposition.

**Proposition 2.8 (The stochastic maximum principle with constraints)** Suppose that \( J(x(t), t, t_f) \) is twice continuously differentiable in \((x, t)\) and that the optimal \( x(t) \) and \( u(t) \) are such that for all \( r \)

\[ E_r \int_{r}^{r+\Delta r} F(x(s), u(s), s) ds = F(x(s), u(s), s) \Delta r + o(\Delta r) \]

Then, at each time \( t \), the optimal control \( u(t) \) solves

\[ \max_{u \in U(x(t), t)} \{ F(u(t), x(t), t) + \theta(x(t), u(t), t, t_f) \} \]

(2.92)
where

\[ U(x(t), t) = \{ u : g_i(x(t), u, t) \geq 0 \text{ for all } i = 1, \ldots, L \} \]

and \( J(x(t), t, t_f) \) must solve the partial differential equation

\[ -J_t(x, t, t_f) = \max_{u \in U(x(t), t)} [F(u(t), x(t), t) + \theta(x(t), u(t), t, t_f)] \quad (2.93) \]

with \( \theta(x(t), u(t), t, t_f) \) defined as

\[ \theta(x(t), u(t), t, t_f) = \sum_{i=1}^{n} J_{x_i} \mu_i + \frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \sum_{i_{rj}=1}^{n} \sum_{j_{rs}=1}^{n} J_{x_r} \sigma_{rj} \sigma_{s_{rj}} \rho_{rj} \sigma_{s_{rj}} \]

and the boundary condition being

\[ J(x(t_f), t_f, t_f) = \Psi(x(t_f), t_f) \]

We can simplify this analysis further, using the costate variables and associated dynamics. Let us define the co-state variable \( \lambda(t) \) as

\[ \lambda^i(t) = J_{x_i}(x(t), t, t_f) \text{ for all } i = 1, \ldots, n \quad (2.94) \]

From (2.94) it follows immediately that

\[ \lambda^i_{x_j} = \frac{\partial \lambda^i}{\partial x_j} = J_{x_i x_j}(x(t), t, t_f) \text{ for all } j = 1, \ldots, n_i \quad (2.95) \]

Vector \( \nabla \lambda \) is the concatenation of

\[ \nabla \lambda = \left( \frac{\partial \lambda^i}{\partial x_j} : i \in [1, n], j \in [1, n_i] \right) \]

Using (2.94) and (2.95) we may rewrite (2.93) as

\[ -J_t = \max_{u \in U(x(t), t)} H(x, u, \lambda, \nabla \lambda) \quad (2.96) \]

where \( H \) is the stochastic version of the Hamiltonian, and is nothing but the functional notation of the expression inside the brackets of (2.93). Assume next that a function \( u(t) \) exists that solves the maximization problem of (2.92) and denote such a function by

\[ u^0 = u^0(x, \lambda, \nabla \lambda) \quad (2.97) \]
Note that $u^0$ is a function of $x(t)$ and $t$ alone along the optimum path, because $J_x$ is a function of $x(t)$ and $t$ alone. Following the same argument as before, we can call $u^0$ the policy function. Assuming then that a policy function $u^0$ exists, (2.96) can be expressed as

$$- J_t = \max_{u \in U(x(t), t)} H(x, u, \lambda, \nabla \lambda) = H(x, u^0(x, \lambda, \nabla \lambda), \lambda, \nabla \lambda) = H^0(x, \lambda, \nabla \lambda)$$

(2.98)

this last equation is again a functional notation of the right-hand side expression of (2.93) under the assumption of the existence of an optimum control, $u^0$, i.e.,

$$H^0(x, \lambda, \nabla \lambda) = F(u^0, x) + \sum_{i=1}^{n} \lambda^i \cdot \mu_i + \frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \sum_{j_r=1}^{n_r} \sum_{j_s=1}^{n_s} \frac{\partial \lambda^r}{\partial x_s} \sigma_{rj_r} \sigma_{sj_s} \rho_{rj_r, sj_s}$$

(2.99)

Equipped with the above analysis, our final goal in this subsection is to derive a system of stochastic differential equations describing the behavior of the state and costate variables (similar to the one we have done before), i.e., find expressions for $dx_i$ and $d\lambda_i$. An expression for $dx_i$ is almost readily available to us from (2.62), (2.75) and (5.9); in particular

$$dx_i = \mu_i(u^0, x) dt + \sum_{j_i=1}^{n_i} \sigma_{ij_i}(u^0, x) dz_{ij_i}(t) \text{ for all } i = 1, \ldots, n$$

$$= H^0_{\lambda^i}(x, \lambda, \nabla \lambda) dt + \sum_{j_i=1}^{n_i} \sigma_{ij_i}(u^0, x) dz_{ij_i}(t)$$

$$= H^0_{\lambda^i} dt + \sum_{j_i=1}^{n_i} \sigma_{ij_i}(u^0, x) dz_{ij_i}(t) \text{ for all } i = 1, \ldots, n$$

(2.100)

Note that

$$\frac{\partial H^0}{\partial \lambda^i} = H^0_{\lambda^i} = \mu_i(u^0, x)$$

(2.101)

used in the derivation of (2.78) and this is obvious from (2.99). Next we derive an expression for $d\lambda_i$. Before we do so, we will need generalized Itô’s lemma which is described below :

**Lemma 2.1 (Generalized Itô’s lemma)** Let $\pi(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^k$ denote a continuous
nonrandom function such that its partial derivatives \( \pi_t, \pi_{x_i}, \pi_{x_ix_j} \) are continuous where

\[
\begin{align*}
\pi_t &= \frac{\partial}{\partial t} \pi(t, x) \\
\pi_{x_i} &= \frac{\partial}{\partial x_i} \pi(t, x) \quad \text{for } i = 1, \ldots, d \\
\pi_{x_ix_j} &= \frac{\partial^2}{\partial x_i \partial x_j} \pi(t, x) \quad \text{for } i, j \leq d
\end{align*}
\]

Suppose that

\[ x(t) = x(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}^d \]

is a process with stochastic differential

\[ dx(t) = f(t)dt + \sigma(t)dz(t) \]

where \( f(t) = f(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}^d \) is measurable in \((t, \omega)\) i.e., measurable in both arguments, and

\[ \sigma(t) = \sigma(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}^d \times \mathbb{R}^m \]

here \( \sigma \) is a \((d \times m)\) matrix-valued function, nonanticipating in \([0, T]\) and finally \( z(t) = z(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}^m \) is a \(m\)-dimensional Wiener process. Let \( y(t) = \pi(t, x(t)) \). Then the process \( y(t) \) also has a differential on \([0, T]\) given by

\[
\begin{align*}
dy(t) &= \left[ \pi_t(t, x(t)) + \pi_x(t, x(t)) f(t) + \frac{1}{2} \sum_i \sum_j \pi_{x_ix_j}(t, x(t)) \left[ \sigma(t) \sigma^T(t) \right]_{ij} \right] dt \\
&\quad + \pi_x(t, x(t)) \sigma(t)dz(t)
\end{align*}
\]

Note that the double summation can also be written as

\[
\sum_i \sum_j \pi_{x_ix_j}(t, x(t)) \left[ \sigma \sigma^T \right]_{ij} = tr(\pi_{xx} \sigma \sigma^T) = tr(\sigma \sigma^T \pi_{xx})
\]

where

\[ \pi_{xx} = \pi_{x_ix_j} \]

is a \((d \times d)\) matrix whose elements. Thus, an alternative expression for \( dy(t) \) is

\[
\begin{align*}
\begin{align*}
\begin{align*}
&= \left[ \pi_t(t, x(t)) + \pi_x(t, x(t)) f(t) + \frac{1}{2} tr(\sigma \sigma^T \pi_{xx}) \right] dt + \pi_x(t, x(t)) \sigma(t)dz(t)
\end{align*}
\end{align*}
\]
Proof. See page 85-89 of Malliaris and Brock [32]. □

Therefore using the definition of $\lambda_i(t)$ given in (2.94), (2.95) and generalized Itô’s lemma to get

$$d\lambda_i = \left[ J_{x,t} + J_{x,x_i} \mu_i + \frac{1}{2} \text{tr} (\sigma \sigma^T J_{xxx})_{ii} \right] dt + \sum_{j=1}^{n_i} J_{x,x_i} \sigma_{ij} \left( u^0, x \right) d\zeta_{ij}(t) \tag{2.102}$$

To compute (2.102), we compute $J_{x,t}$ from (2.98) assuming that equality holds for mixed partial derivatives

$$-J_{x,t} = H^0_{x_i} + \sum_{j=1}^{n} H^0_{\lambda_j} \cdot \frac{\partial \lambda_j}{\partial x_i} + \sum_{j=1}^{n} H^0_{\lambda_j} \cdot \frac{\partial^2 \lambda_j}{\partial x_i^2} \tag{2.103}$$

(2.103) is obtained using definitional relationships namely $H^0_{\lambda_j} = \mu_j$ and $\frac{\partial \lambda_j}{\partial x_i} = J_{x_j x_i}$, which further can be simplified to the following

$$-J_{x,t} = H^0_{x_i} + \sum_{j=1}^{n} \mu_j \cdot J_{x_j x_i} + \sum_{j=1}^{n} H^0_{\lambda_j} \cdot \frac{\partial^2 \lambda_j}{\partial x_i^2} + \sum_{j=1}^{n} \left[ \sigma^T \sigma \right]_{ji} \cdot J_{x_j x_i} \tag{2.104}$$

as because

$$H^0_{\lambda_j} = \frac{1}{2} \left[ \sigma^T \sigma \right]_{ji}$$

and

$$\frac{\partial}{\partial x_i} \left( \frac{\partial \lambda_j}{\partial x_i} \right) = \frac{\partial}{\partial x_i} (J_{x_j x_i}) = J_{x_j x_i}$$
Finally, substituting (2.104) into (2.80) we reach our desired result

\[
d\lambda_i = \left[ -H_0^0 - \sum_{j=1}^{n} \mu_j \cdot J_{x_j x_i} - \frac{1}{2} \sum_{j=1}^{n} [\sigma^T \sigma]_{ji} \cdot J_{x_j x_i x_i} + J_{x_i x_i} \mu_i + \frac{1}{2} \text{tr} \left( \sigma \sigma^T J_{xxx} \right)_{ii} \right] dt \\
+ \sum_{j_i=1}^{n_i} J_{x_j x_i} \sigma_{ij_i} (u^0, x) dz_{ij_i} (t)
\]

(2.105)

\[
= \left[ -H_0^0 - \sum_{j \neq i} \mu_j \cdot J_{x_j x_i} - \frac{1}{2} \sum_{j \neq i} [\sigma^T \sigma]_{ji} \cdot J_{x_j x_i x_i} \right] dt + \\
\sum_{j_i=1}^{n_i} J_{x_j x_i} \sigma_{ij_i} (u^0, x) dz_{ij_i} (t)
\]

(2.106)

Thus we obtain the stochastic differential equations for each of the costate variables. It is possible to obtain closed form solutions of the above stochastic differential equations under special cases (e.g., when the Hamiltonian is in linear or is quadratic in states, \(x_i\) thus making sure \(J_{xxx} = 0\)).

### 2.5.8 Sufficiency Conditions for a Stochastic Optimal Control Problem

We now state sufficient optimality conditions for stochastic optimal control problems of the form defined above. These conditions are based on the HJB equation. Consider an expected value of an integral as the objective functional. Again, we write \(E_{u(\cdot)}\) for the expectation operator because the distribution of the state \(x(\cdot)\) depends on the control path \(u(\cdot)\), through the differential equation

\[
dx(t) = f (x(t), u(t), t) dt + \sigma (x(t), u(t), t) dw(t)
\]

(2.107)

\[
x(0) = x_0
\]

(2.108)

. More specifically, consider the problem of maximizing the objective functional

\[
J(u(\cdot)) = E_{u(\cdot)} \left\{ e^{-rT} S(x(T)) + \int_0^T e^{-rt} F (x(t), u(t), t) dt \right\}
\]

(2.109)

subject to the state dynamics and initial conditions as stated above with the constraints \(u(t) \in U (x(t), t)\). Here, the problem horizon may be finite or infinite such that

\[
S(x) = 0 \text{ for all } x \in X \text{ if } T = \infty
\]
We say that the control path \( u(\cdot) \) is nonanticipating if its value at time \( t \) does not depend on any uncertainty revealed after time \( t \). In particular, \( u(t, \xi) \) must not depend on realizations of the random variables \( w(t + \tau, \xi) \) or \( x(t + \tau, \xi) \) for any strictly positive number, \( \tau \). We can now define feasible and optimal paths, respectively, for a stochastic optimal control problem.

**Definition 2.14** A control path \( u : [0, T) \times \Xi \to \mathbb{R}^m \) is feasible for the stochastic optimal control problem stated above if it is nonanticipating, if there exists a unique solution \( x(\cdot) \) to the stochastic differential equation (2.107), if the constraints \( x(t) \in X \) and \( u(t) \in U(x(t), t) \) are satisfied with probability 1 for all \( t \), and if the integral in (2.109) is well defined. If \( T \) is finite the the control path \( u(\cdot) \) is optimal if it is feasible and if

\[
J(u(\cdot)) \geq J(\tilde{u}(\cdot))
\]

holds for all feasible control paths \( \tilde{u}(\cdot) \). If \( T \) is infinite then a control path \( u(\cdot) \) is a catching up optimal path if it is feasible and if

\[
\lim_{t \to \infty} \left[ J^t(u(\cdot)) - J^t(\tilde{u}(\cdot)) \right] \geq 0
\]

holds for all feasible control paths \( \tilde{u}(\cdot) \). Here, the \( t \)–truncations \( J^t \) are defined analogously to definition provided below, taking into account that we now maximize expected values of integrals.

**Definition 2.15** Consider an optimal control problem in which the objective functional is given by

\[
J(u(\cdot)) = E_{u(\cdot)} \left\{ \int_0^\infty e^{-rt} F(x(t), u(t), t) \, dt \right\}
\]

The \( T \)–truncation of the objective functional, \( J^T(u(\cdot)) \) is defined by

\[
J^T(u(\cdot)) = E_{u(\cdot)} \left\{ \int_0^T e^{-rt} F(x(t), u(t), t) \, dt \right\}
\]

It is clear that in the infinite horizon case one could also consider other optimality criteria. It is not difficult to adapt the following optimality conditions to other criteria.
As in the deterministic model, the only thing that has to be changed is the transversality condition.

**Theorem 2.9** Let $V : X \times [0, T) \mapsto \mathbb{R}$ be a function with continuous partial derivatives $V_t, V_x,$ and $V_{xx}$ and assume that $V$ satisfies the HJB equation

$$rV(x, t) - V_t(x, t) = \max \left\{ F(x, u, t) + V_x(x, t)f(x, u, t) + \frac{1}{2}tr \left[ V_{xx}(x, t)\sigma(x, u, t)\sigma^T(x, u, t) \right] | u \in U(x, t) \right\}$$ (2.110)

for all $(x, t) \in X \times [0, T)$. Let $\Phi(x, t)$ denote the set of controls $u \in U(x, t)$ maximizing the right-hand side of (2.71) and let $u(\cdot)$ be a feasible control path, with corresponding state trajectory $x(\cdot)$ such that $u(t) \in \Phi(x(t), t)$ holds with probability 1 for almost all $t \in [0, T)$.

(i) If $T < \infty$ and if the boundary condition $V(x, T) = S(x)$ holds for all $x \in X$ then $u(\cdot)$ is an optimal control path

(ii) If $T = \infty$ and if either $J$ is bounded and $r > 0$, or $V$ is bounded below and

$$\lim_{t \to \infty} e^{-rt}E_u V(x(t), t) \leq 0$$

holds, then $u(\cdot)$ is a catching up optimal control path

**Proof.** See Dockner et al. [7]. ■

### 2.6 Stochastic Differential Variational Inequality (SDVI)

#### 2.6.1 State Dynamics involving Brownian Motion

Let us consider the following state dynamics which is a stochastic differential equation of Itô type;

$$x(u, t) = \arg \left\{ dx = f(x, u, u_D, t)dt + g(x, u, t)dz; x(t_0) = x^0 \right\} \in \left( \mathcal{H}^1 \left[ t_0, t_f \right] \right)^n$$ (2.111)
where, once again

\[ u \in U \subseteq (L^2 [t_0, t_1])^m \]

\[ f : (H^1 [t_0, t_1])^n \times (L^2 [t_0, t_1])^m \times \mathbb{R}_+ \rightarrow (L^2 [t_0, t_1])^n \]

\[ g : (H^1 [t_0, t_f])^n \times (L^2 [t_0, t_f])^m \times \mathbb{R}_+ \rightarrow (L^2 [t_0, t_f])^n \]

\[ x^0 \in \mathbb{R}^n \]

\[ z : [t_0, t_1] \times \Xi \mapsto \mathbb{R}^n \]

where \( z \) is a standardized \( n \)-dimensional Wiener process. For any sample path realizations of \( z \) (which we denote as \( z_\omega \)), the dynamics can be restated as

\[ x(u, t, \omega) = \arg \{ dx = f(x, u, u_D, t) dt + g(x, u, t)dz; x(0) = x^0 \} \in (H^1 [t_0, t_f])^n \]

thus each time a different trajectory of \( x(\cdot, \omega) \) is realized for every sample path realizations of \( z \).

### 2.6.2 Statement of a SDVI

We refer to the following stochastic differential variational inequality problem corresponding to sample path realization \( \omega \) of Wiener process \( z \) as SDVI \((F, f, g, U, x^0)\)

\[ \text{find } u^*(\omega) \in U \text{ such that} \]

\[ \langle F(x(u^*(\omega), t), u^*(\omega), t), u - u^*(\omega) \rangle \geq 0 \text{ a.e. for all } u \in U \quad (2.112) \]

where

\[ u \in U \subseteq (L^2 [t_0, t_f])^m \quad (2.113) \]

\[ x(u, t) = \arg \{ dx = f(x, u, t) dt + g(x, u, t)dz, \ x(0) = x^0 \} \in (H^1 [t_0, t_f])^n \quad (2.114) \]

and
\[ x^0 \in \mathbb{R}^n \] (2.115)

\[ F : (\mathcal{H}^1 [t_0, t_f])^n \times (L^2 [t_0, t_f])^m \times \mathbb{R}_+^1 \rightarrow (L^2 [t_0, t_f])^m \] (2.116)

\[ f : (\mathcal{H}^1 [t_0, t_f])^n \times (L^2 [t_0, t_f])^m \times \mathbb{R}_+^1 \rightarrow (L^2 [t_0, t_f])^n \] (2.117)

\[ g : (\mathcal{H}^1 [t_0, t_f])^n \times (L^2 [t_0, t_f])^m \times \mathbb{R}_+^1 \rightarrow (L^2 [t_0, t_f])^n \] (2.118)

\[ g : (\mathcal{H}^1 [t_0, t_f])^n \times (L^2 [t_0, t_f])^m \times \mathbb{R}_+^1 \rightarrow (L^2 [t_0, t_f])^n \] (2.119)

Of course for every realization of \( z \), a new (2.112) has to be solved to obtain \( u^*(\omega) \).

**Necessary Conditions for SDVI**

To analyze (2.112) we rely on the following notion of regularity:

**Definition 2.16** *(regularity of SDVI(\( F, f, g, U, x^0, \omega \)))* We call SDVI(\( F, f, g, U, x^0, \omega \)) regular if:

1. \( x(u, t, \omega) : (L^2 [t_0, t_1])^m \times \mathbb{R}_+^1 \times \mathbb{R}^n \rightarrow (\mathcal{H}^1 [t_0, t_1])^n \) exists and is continuous and \( G \)-differentiable with respect to \( u \);

2. \( F(x, u, t) \) is continuous with respect to \( x \) and \( u \);

3. \( f(x, u, t) \) is convex and continuously differentiable with respect to \( x \) and \( u \);

4. \( g(x, u, t) \) is convex and continuously differentiable with respect to \( x \) and \( u \);

5. \( U \) is convex and compact; and

6. \( x^0 \in \mathbb{R}^n \) is known and fixed.

The motivation for this definition of regularity is to parallel as closely as possible those assumptions needed to analyze traditional optimal control problems from the point of view
of infinite dimensional mathematical programming. Chapter 10 of Minoux [26] provides a useful introduction to the relationship of infinite dimensional mathematical programming and optimal control theory.

We next note that (2.112) may be restated as the following fictitious stochastic optimal control problem

$$\min_u \int_{t_0}^{t_1} [F(x(u^*(\omega), t, \omega), u^*(\omega), t)]^T \cdot u \, dt$$  \hspace{1cm} (2.120)$$

subject to

$$dx = f(x, u, t) \, dt + g(x, u, t) \, dz$$ \hspace{1cm} (2.121)$$

$$u \in U$$ \hspace{1cm} (2.122)$$

$$x(t_0) = x^0$$ \hspace{1cm} (2.123)$$

where \( x^* = x(u^*, t, \omega) \) is the optimal state vector. We point out that this optimal control problem is a mathematical abstraction and of no use for computation, since its criterion depends on knowledge of the variational inequality solution \( u^* \). In what follows we will need the stochastic Hamiltonian for \( SDVI(F, f, g, U, x^0, \omega) \), namely

$$H(x, u, \lambda, \frac{d\lambda}{dt}, t) = [F(x(u^*(\omega), t, \omega), u^*(\omega), t)]^T u$$  \hspace{1cm} (2.124)$$

$$+ \lambda^T f(x, u, t) + \frac{1}{2} [\sigma(x, u, t)] \cdot [\sigma(x, u, t)]^T \cdot \frac{\partial \lambda}{\partial x}$$

where \( \lambda(t) \) is the \( n \)-dimensional adjoint vector that solves the adjoint equations and transversality conditions for given state variables and controls. Note that for a given state vector and a given instant in time (2.124) is convex in \( u \) when \( SDVI(F, f, g, U, x^0, \omega) \) is regular.

The necessary conditions are stated in the following theorem:

**Theorem 2.10** (necessary conditions for \( SDVI(F, f, g, U, x^0, \omega) \)) When regularity in the sense of Definition 2.16 holds, solutions \( u^* \in U \) of \( SDVI(F, f, g, U, x^0, \omega) \) must obey:
1. the finite dimensional variational inequality principle:

\[
\begin{bmatrix}
\left[ F(x(u^*(\omega),t,\omega),u^*(\omega),t) \right]^T u + \lambda^T \cdot (\nabla_u f(x,u,t)) \\
+ \frac{1}{2} \nabla_u \left[ [g(x,u,t)] \cdot [g(x,u,t)]^T \right] \cdot \frac{\partial \lambda}{\partial x}
\end{bmatrix}^T (u - u^*(\omega)) \geq 0 \quad \forall u \in U;
\]

(2.125)

2. the state dynamics

\[
dx^* = f(x^*,u^*,t) \, dt + g(x^*,u^*,t) \, dz \quad (2.126)
\]

\[
x^*(t_0) = x^0; \quad \text{and} \quad (2.127)
\]

3. the adjoint dynamics

\[
dl = - \left[ \nabla_x (\lambda^*)^T f(x^*,u^*,t) + \frac{1}{2} [g(x,u,t)] \cdot [g(x,u,t)]^T \cdot \frac{\partial \lambda}{\partial x} \right] dt + g(u^*,x^*,t) \cdot \frac{\partial \lambda}{\partial x} \cdot dz \quad (2.128)
\]

\[
\lambda^*(t_1) = 0 \quad (2.129)
\]

\[
\lambda^*(t_1) = 0 \quad (2.130)
\]

**Proof.** The stochastic version of Pontryagin minimum principle is a necessary condition for optimal control problem (2.120) through (2.123) so that

\[
u^* = \arg \left\{ \min_{u \in U} H \left( x^*, u, \lambda^*, \frac{d \lambda^*}{dt}, t, \omega \right) \right\} \quad (2.131)
\]

for each \( t \in [t_0,t_1] \), which in turn, by virtue of regularity, is equivalent to

\[
\left[ \nabla_u H (x^*, u^*, \lambda^*, t, \omega) \right]^T (u - u^*(\omega)) \geq 0 \quad u, u^* \in U
\]

Note that

\[
\nabla_u H (x,u,\lambda,t) = \left[ F(x(u^*(\omega),t,\omega),u^*(\omega),t) \right]^T u + \lambda^T \cdot (\nabla_u f(x,u,t)) \\
+ \frac{1}{2} \nabla_u \left[ [g(x,u,t)] \cdot [g(x,u,t)]^T \right] \cdot \frac{\partial \lambda}{\partial x}
\]

where for given \( u \)

\[
\lambda(u,t) = \arg \left\{ d \lambda = - \nabla_x H (x^*, u^*, \lambda^*, t, \omega) + [g(u^*, x^*, t)]^T \cdot \frac{\partial \lambda}{\partial x} \cdot dz; \quad \lambda(t_1) = 0 \right\}
\]

(2.132)
since $x(u,t)$ is completely determined by knowledge of the controls $u$. The theorem follows immediately. □

Note that item 1 of this theorem refers to a finite dimensional variational inequality because, as explained in Friesz et al [24], the Pontryagin minimum principle from which it is derived minimizes the associated Hamiltonian for fixed time and fixed state and adjoint variables. The adjoint dynamics is also a stochastic differential equation. We further note that the following existence result holds:

**Theorem 2.11** (existence of a solution to $SDVI(F, f, g, U, x^0, \omega)$) When regularity in the sense of Definition 2.16 holds and the stochastic adjoint dynamics (2.128) - (2.130) has at least a solution, $SDVI(F, f, g, U, x^0, \omega)$ has a solution.

**Proof.** By the assumption of regularity $x(u,t,\omega)$ is well defined and continuous. So $F(x(u^* (\omega), t, \omega), u^*(\omega), t)$ is continuous in $u$. Also by regularity we know $U$ is convex and compact. Further, there exists at least one $\lambda$ that satisfies (2.128) - (2.130). Consequently, by Theorem 2 of Browder [27], $SDVI(F, f, g, U, x^0, \omega)$ has a solution. □

### 2.7 Stochastic Dynamic Games

In this section we consider a stochastic differential game in which the state equation contains a white noise term. Let there be $N$ players and denote by $u^i(t)$ the control value chosen by player $i \in \mathcal{N} \equiv \{1, 2, \ldots, N\}$ at time $t$. We denote the vector of controls used by the opponents of player $i$ by

$$u^{-i}(t) = (u^j(t) : j \in \mathcal{N} \setminus i)$$

If the state of the system at time $t \in [0, T]$ is equal to $x(t) \in X$, player $i$’s set of feasible controls is given by $U^i(x(t), u^{-i}(t), t) \subseteq \mathbb{R}^m$. The state equation for the game is

$$dx(t) = f(x(t), u^1(t), u^2(t), \ldots, u^N(t), t) dt + \sigma(x(t), u^1(t), u^2(t), \ldots, u^N(t), t) dw(t)$$
where \( w(t) \) is a \( k \)-dimensional standardized Wiener process and \( f \) and \( \sigma \) are the functions defined on

\[
\Omega = \{ (x, u^1, \ldots, u^N, t) \mid x \in X, u^i \in U^i(x, u^{-i}, t), t \in [0, T] \}
\]

with values in \( \mathbb{R}^n \) and \( \mathbb{R}^{n \times k} \) respectively. Player \( i \)'s goal is to maximize the following taking \( u^{-i}(t) \) exogenously

\[
J^i(u^i(\cdot), u^{-i}(\cdot)) = E_{u(\cdot)} \left\{ \int_0^T e^{-r^i t} F^i(x(t), u^1(t), \ldots, u^N(t), t) \ dt \right. \\
+ e^{-r^i T} S^i(x(T)) \mid x(0) = x_0 \}
\]

where \( F^i \) is a real valued utility function defined on \( \Omega \), \( S^i \) is a real valued scrap value function defined on \( X \), and \( r^i \) is the discount rate of the player \( i \). The deterministic vector \( x_0 \in X \) is the initial state. We denote the stochastic differential game defined in this way by \( \Gamma(x_0, 0) \). The differential game \( \Gamma(x, t) \) is defined by replacing the time domain \( [0, T] \) by \( [t, T] \) and the initial state \( x_0 \) by \( x \). Note that the Wiener process for the game \( \Gamma(x, t) \) may start at the initial value \( w(t) = 0 \) since the particular choice of the initial value of \( w \) is irrelevant for the differential game.

In the present framework of a system which is disturbed by a white noise process, uncertainty is resolved continuously. Hence only an information structure at least as fine as the Markovian one is up to date with most recent information about the uncertainty of the system. Consequently, we confine the analysis of equilibria in the present model to Markovian Nash equilibria. These equilibria are subgame perfect in a sense made precise in definition provided below.

The exact definition of a Markovian Nash equilibrium for the stochastic game defined above is almost identical to the corresponding definition of the deterministic game. Assume that all players except for player \( i \) determine their actions by the Markovian strategies

\[
u^j(t) = \phi^j(x(t), t), j \neq i
\]

Then, player \( i \)'s decision problem can be rewritten as

\[
\text{Maximize } J^i(u^i(\cdot), \phi^{-i}(\cdot)) = E_{u(\cdot)} \left\{ \int_0^T e^{-r^i t} F^i(x(t), u^i(t), \phi^{-i}(t), t) \ dt \right. \\
+ e^{-r^i T} S^i(x(T)) \mid x(0) = x_0 \}
\]
\[ + e^{-rT} S^i(x(T)) | x(0) = x_0 \]

subject to

\[
\begin{align*}
    dx(t) &= f^i \left( x(t), u^i(t), \phi^{-i}(t), t \right) dt + \sigma^i \left( x(t), u^i(t), \phi^{-i}(t), t \right) dw(t) \quad (2.133) \\
    x(0) &= x_0 \\
    u^i(t) &\in U \left( x(t), \phi^{-i}(t), t \right)
\end{align*}
\]

where

\[
\begin{align*}
    F^i \left( x(t), u^i(t), \phi^{-i}(t), t \right) &= F^i \left( x(t), \phi^1(t), \ldots, \phi^{-i-1}(t), u^i(t), \phi^{i+1}(t), \ldots, \phi^N(t), t \right) \\
    f^i \left( x(t), u^i(t), \phi^{-i}(t), t \right) &= f^i \left( x(t), \phi^1(t), \ldots, \phi^{-i-1}(t), u^i(t), \phi^{i+1}(t), \ldots, \phi^N(t), t \right) \\
    \sigma^i \left( x(t), u^i(t), \phi^{-i}(t), t \right) &= \sigma^i \left( x(t), \phi^1(t), \ldots, \phi^{-i-1}(t), u^i(t), \phi^{i+1}(t), \ldots, \phi^N(t), t \right) \\
    U^i \left( x(t), \phi^{-i}(t), t \right) &= U^i \left( x(t), \phi^1(t), \ldots, \phi^{-i-1}(t), \phi^{i+1}(t), \ldots, \phi^N(t), t \right)
\end{align*}
\]

For any given \((N-1)-\text{tuple} \phi^{-i} = (\phi^j : j \in \mathcal{N} \setminus \{i\})\) of functions \(\phi^j : X \times [0, T) \mapsto \mathbb{R}^{m_j}, j \neq i\), the problem (2.133) is an optimal control model of the form described in the previous section. Markovian Nash equilibria for this differential game are defined in the following obvious way.

**Definition 2.17** The \(N\)-tuple \((\phi^1, \phi^2, \ldots, \phi^N)\) of functions \(\phi^i : X \times [0, T) \mapsto \mathbb{R}^{m_i}, i \in \{1, 2, \ldots, N\}\), is a Markovian Nash equilibrium for the stochastic game defined above if, for each \(i \in \{1, 2, \ldots, N\}\), a (catching up) optimal control path \(u^i(\cdot)\) of the problem (2.133) exists and is defined by the Markovian strategy

\[ u^i(t) = \phi^i(x(t), t) \]

The equilibrium is said to be subgame perfect if, for every pair \((x, t) \in X \times [0, T)\), there exists a Markovian Nash equilibrium \((\psi^1, \psi^2, \ldots, \psi^N)\) of the game \(\Gamma(x, t)\) such that

\[ \psi^i(y, s) = \phi^i(y, s) \]

holds for all \((y, s) \in X \times [0, T)\).
In the following theorem we state conditions which ensure that a given \( N \)-tuple of functions is a subgame perfect Markovian Nash equilibrium. These conditions are very similar to the conditions associated with the deterministic differential game.

**Theorem 2.12** Let \((\phi^1, \phi^2, \ldots, \phi^N)\) be a given \( N \)-tuple of functions \( \phi^i : X \times [0, T) \mapsto \mathbb{R}^{m^i}, i \in \{1, 2, \ldots, N\} \) and make the following assumptions:

(i) for every pair \((y, s) \in X \times [0, T)\) there exists a unique solution \(x_{y,s} : [s,T_i] \times \Xi \mapsto X\) of the stochastic initial value problem

\[
\begin{align*}
\frac{dx(t)}{dt} &= f(x(t), \phi^1(t), \phi^2(t), \ldots, \phi^N(t), t) dt \\
&\quad + \sigma(x(t), \phi^1(t), \phi^2(t), \ldots, \phi^N(t), t) dw(t)
\end{align*}
\]

\[x(s) = y\]

(ii) for all \(i \in \{1, 2, \ldots, N\}\) there exists a function \(V^i : X \times [0, T_i] \mapsto \mathbb{R}\), having continuous partial derivatives \(V^i_t, V^i_x,\) and \(V^i_{xx}\), such that the HJB equations

\[
riV^i(x,t) - V^i_t(x,t) = \max \left\{ F^i(x(t), u^i(t), \phi^{-i}(t), t) \\
+ V^i_x(x(t), u^i(t), \phi^{-i}(t), t) f^i(x(t), u^i(t), \phi^{-i}(t), t) \\
+ \frac{1}{2} tr \left[ V^i_{xx}(x,t) \sigma^i(x,u^i,\phi^{-i},t) \sigma^i(x,u^i,\phi^{-i},t)^T \right] \mid u^i \in U(x, \phi^{-i}, t) \right\}
\]

are satisfied for all \(i \in \{1, 2, \ldots, N\}\) and all \((x,t) \in X \times [0, T)\),

(iii) if \(T < \infty\) then

\[V^i(x,T) = S^i(x)\]

for all \(i \in \{1, 2, \ldots, N\}\) and all \(x \in X\)

(iv) if \(T = \infty\) then for all \(i \in \{1, 2, \ldots, N\}\) either \(V^i\) is a bounded function and \(r^i > 0\), or \(V^i\) is bounded below and \(\limsup_{t \to \infty} e^{-r^i t} E_u(x) V^i(x,y,s(t),t) \leq 0\) along all state
trajectories \( x_{g,s}(\cdot) \) mentioned in condition (i) above²

Denoted by \( \Phi^i(x,t) \) the set of all \( u^i \in U^i(x(t),\phi^{-i}(t),t) \) which maximize the right hand side of (2.134). If \( \phi^i(x,t) \in \Phi^i(x,t) \) holds for all \( i \in \{1, 2, \ldots, N\} \) and all \( (x, t) \in X \times [0, T] \) then \( (\phi^1, \phi^2, \ldots, \phi^N) \) is a subgame perfect Markovian Nash equilibrium. (If \( T = \infty \), optimality is understood in the sense of catching up optimality.)

Proof. Apply Theorem 2.9 to the individual stochastic optimal control problems (2.9) derived from the game \( \Gamma(x_0,0) \) as well as to the corresponding individual stochastic optimal control problems derived from the game \( \Gamma(x,t) \) for all \( (x, t) \in X \times [0, T] \).

Owing to the second order term \( V_{xx}^i(x,t) \) occurring in the HJB equations (2.134), it is usually very difficult to find closed-form solutions for the optimal value functions \( V^i \). Two classes of stochastic games, however, are known in which the solution of the HJB equations is not more difficult than in the deterministic case. The first class of games are stochastic games in which the HJB equations of the corresponding deterministic game, obtained by setting \( \sigma^i(x(t),u^i(t),\phi^{-i}(t),t) = 0 \) in (2.134), have solutions \( V^i \) which are linear in the state \( x \). If this is the case, the same functions \( V^i \) are also solutions to the original HJB equations (2.134) because the term

\[
\text{tr} \left[ V_{xx}^i(x,t) \sigma^i(x(t),u^i(t),\phi^{-i}(t),t) \sigma^i(x(t),u^i(t),\phi^{-i}(t),t)^T \right]
\]

vanishes whenever \( V^i \) is linear with respect to \( x \). Linearity of the optimal value function with respect to the state is also a feature of linear state games.

The second class of differential games with white noise in which the solution of the HJB equations is not more difficult than in the deterministic case is linear quadratic games. In the stochastic framework of this section, a game is called linear quadratic if the integrand in the criterion is quadratic in state and control, function \( f^i(\cdot) \) is linear in states

²\( E_{u(\cdot)} \) denotes the expectation with respect to the probability distribution generated by the control paths \( u(\cdot) \) corresponding to the Markovian strategies \( \phi^i, i \in \{1, 2, \ldots, N\} \)
and controls and the function $\sigma^i$ is linear with respect to the state $x$ and controls $u^i$, for all $i = 1, 2, \ldots, N$. In a game with these properties, the usual guess of quadratic value functions leads to the solutions of the HJB equations.

In the balance of this section we describe how a stochastic differential game can be expressed as a SDVI. To this end, we consider the same stochastic differential game problem stated in section ?? where player $i$’s stochastic optimal control problem takes the form (taking $\phi^{-i}(\cdot)$ as exogenous)

$$\text{Max}_{u^i} J^i(u^i(\cdot), \phi^{-i}(\cdot)) = E_{u(\cdot)} \left\{ \int_0^T e^{-r^it} F^i \left( x(t), u^i(t), \phi^{-i}(t), t \right) dt \right.$$

$$\left. + e^{-r^iT} S^i(x(T)) \mid x(0) = x_0 \right\}$$

subject to

$$\frac{dx(t)}{dt} = f \left( x(t), u^i(t), \phi^{-i}(t), t \right) dt + \sigma \left( x(t), u^i(t), \phi^{-i}(t), t \right) dw(t)$$

$$x(0) = x_0$$

$$u^i(t) \in U^i \left( x(t), \phi^{-i}(t), t \right)$$

If the Hamiltonian associated with this problem is defined as

$$H^i \left( x, u^i, \lambda^i, \frac{\partial \lambda^i}{\partial x}, \phi^{-i}, t \right) = e^{-r^it} F^i \left( x(t), u^i(t), \phi^{-i}(t), t \right) + \lambda^i \cdot f \left( x(t), u^i(t), \phi^{-i}(t), t \right)$$

$$+ \frac{1}{2} \sigma \left( x(t), u^i(t), \phi^{-i}(t), t \right) \cdot \sigma \left( x(t), u^i(t), \phi^{-i}(t), t \right) \sum_{k=1}^n \frac{\partial \lambda^i_k}{\partial x_k}$$

where $\lambda^i$ is the vector of adjoint variables

$$\lambda^i \in \left( \mathcal{H}^1 [0, T] \right)^n$$

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and

\[
U^i (x(t), \phi^{-i}(t), t) : \left( \mathcal{H}^i [0, T] \right)^n \times \left( L^2 [0, L] \right)^n \rightarrow \left( L^2 [0, T] \right)^n
\]

\[x^0 \in \mathbb{R}^n\]  \hspace{1cm} (2.139)

\[
F^i (x(t), u^i(t), \phi^{-i}(t), t) : \left( \mathcal{H}^i [0, T] \right)^n \times \left( L^2 [0, T] \right)^m \rightarrow \left( L^2 [0, T] \right)^m \hspace{1cm} (2.140)
\]

\[
f (x(t), u^i(t), \phi^{-i}(t), t) : \left( \mathcal{H}^i [0, T] \right)^n \times \left( L^2 [0, T] \right)^m \rightarrow \left( L^2 [0, T] \right)^m \hspace{1cm} (2.141)
\]

\[
\sigma (x(t), u^i(t), \phi^{-i}(t), t) : \left( \mathcal{H}^i [0, T] \right)^n \times \left( L^2 [0, T] \right)^m \rightarrow \left( L^2 [0, T] \right)^m \hspace{1cm} (2.142)
\]

\[
S^i (x(T)) : \left( \mathcal{H}^i [0, T] \right)^n \times \mathbb{R}^1_+ \rightarrow \left( \mathcal{H}^i [0, T] \right)^n \hspace{1cm} (2.143)
\]

where

\[m = \sum_{j=1}^{N} n^j\]

We would like to point out that any solution to the stochastic optimal control problem for player \(i\) is a stochastic process depending on the realizations of \(\xi \in \Xi\). We do not make this dependence explicit but simply write \(x(t)\) instead of correct, but more cumbersome, notation \(x(t, \xi)\). The reader should keep in mind that the value of \(x\) at time \(t\) can not be known in advance without knowing the realization of \(\xi\). A similar remark applies to the control paths. Controls usually depend on the most recent information and are random variables too. However, we simply write \(u(t)\) instead of \(u(t, \xi)\).

To analyze (2.135), we will rely on the following notion of regularity:

**Definition 2.18** [regularity of individual agents’ stochastic optimal control problems] We call the stochastic optimal control problem is regular if:

1. \(x (u(p, t), t) : \left( L^2 [t_0, t_f] \right)^m \times \mathbb{R}^1_+ \rightarrow \left( \mathcal{H}^i [t_0, t_f] \right)^n \) exists and is continuous and \(G\)-differentiable with respect to \(u\);

2. \(S^i (x(T))\) is continuously differentiable with respect to \(x\);

3. \(F^i (x(t), u^i(t), \phi^{-i}(t), t)\) is continuous with respect to \(x\) and \(u^i\);
4. \(f(x(t), u^i(t), \phi^i(t), t)\) is convex and continuously differentiable with respect to \(x\) and \(u^i\);

5. \(\sigma(x(t), u^i(t), \phi^i(t), t)\) is convex and continuously differentiable with respect to \(x\) and \(u^i\);

6. \(U^i(x(t), \phi^i(t), t)\) is convex and compact; and

7. \(x^0 \in \mathbb{R}^n\) is known and fixed.

The stochastic maximum principle tells us that an optimal solution to (2.135)-(2.137) is a triplet \(\{u^i(t), x^i(t), \lambda^i(t)\}\) that, given \(H^i(x, u^i, \lambda^i, \partial \lambda^i / \partial x, \phi^i, t)\), must satisfy at each time \(t \in [0, L] :\)

\[
u^i = \arg \max_{u^i \in U_i(x(t), \phi^i(t), t)} H^i(x, u^i, \lambda^i, \partial \lambda^i / \partial x, u^i, t)\]

(2.144)

which in turn, by virtue of regularity, is equivalent to

\[
\left[ \nabla_u H^i(x^i, u^i, \lambda^i, \partial \lambda^i / \partial x, u^i, t) \right]^T (u^i - u^i) \leq 0 \quad \text{for all } u^i, u^i \in U^i
\]

(2.145)

From (2.138)

\[
\left[ \nabla_u \left\{ e^{-r_i t} F^i(x(t), u^i(t), u^i(t), t) + \lambda^i \cdot f(x(t), u^i(t), u^i(t), t) \\
+ \frac{1}{2} \sigma(x(t), u^i(t), u^i(t), t) \cdot \sigma(x(t), u^i(t), u^i(t), t)^T \frac{\partial \lambda^i}{\partial x} \right\} \right]^T (u^i - u^i) \leq 0
\]

(2.146)

for all \(u^i, u^i \in U^i\)

Further, adjoint dynamics and state dynamics govern the stochastic differential equations

\[
\frac{\partial H^i(x^i, u^i, \lambda^i, \partial \lambda^i / \partial x, u^i, t)}{\partial x} dt + \sigma(x^i, u^i, u^i, t) \frac{\partial \lambda^i}{\partial x} dz(t) = (-1) d\lambda^i
\]

(2.147)

\[
\frac{\partial H^i(x^i, u^i, \lambda^i, \partial \lambda^i / \partial x, u^i, t)}{\partial \lambda^i} dt + \sigma(x^i, u^i, u^i, t) dz(t) = dx
\]

(2.148)

while the transversality condition gives

\[
\lambda^i(T) = \frac{\partial S^i(x(T))}{\partial x(T)}
\]
SDVI Formulation

We consider the following SDVI which has solutions that are Markovian Nash equilibria for the dynamic game described above in which individual player maximize its own utility in light of current information about their competitors for a sample path realization \( \xi \in \Xi \)

\[
\text{find } u(\xi)^* \in U \text{ such that } \sum_{i \in N} \int_0^T \left( \sum_{k=1}^{n_i} \frac{\partial H_i^*}{\partial u_k} (u_i^* - u_k^* (\xi)) \right) dt \leq 0 \tag{2.149}
\]

for all \( u \in U = \prod_{i \in N} U^i \)

where

\[
H_i^* = H_i^* \left( x^*, u_i^*, \lambda_i^*, \frac{\partial \lambda_i^*}{\partial x_i}, u_i^{i*}, t \right)
\]

\[
u_i^* = \left( u_i^*; k \in \{1, \ldots, n^i\} \right)
\]

\[
u = \left( u^i; i \in N \right)
\]

Clearly, one would get different solutions of (2.149) for different sample path realizations by varying \( \xi \in \Xi \), therefore the SDVI formulation will be the following:

**Theorem 2.13 (SDVI formulation of non-cooperative stochastic differential games)** For a particular sample path realization \( \xi \in \Xi \), any solution of the SDVI (2.149) is a differential game-theoretic equilibrium when the regularity holds

**Proof.** Begin by noting that (2.149) is equivalent to the following optimal control problem

\[
\max F \left( u^i; u_i^{i*}; t \right) = \sum_{i \in N} \sum_{k=1}^{n_i} \int_0^T \frac{\partial H_i^*}{\partial u_k} u_k^i dt \tag{2.150}
\]

s.t. \( u \in U \) \tag{2.151}

where it is essential to recognize that \( F \left( u^i; u_i^{i*}; t \right) \) is a linear functional that assumes knowledge of the solution to the game; as such \( F \left( u^i; \phi_i^{i*}; t \right) \) is a mathematical construct for use
in analysis and has no meaning as a computational device. Also note that a necessary condition for the problem (5.35) and (5.36) is

\[
\langle \nabla_w F (u^i; \phi^{-i}; t), u^i - u^{i*}(\xi) \rangle \equiv \sum_{k=1}^{n^i} \int_0^t \frac{\partial F (u^{i*}; u^{i*}; t)}{\partial u^i_k} (u^i_k - u^{i*}_k) \, dt \leq 0 \tag{2.152}
\]

for all \( u \in U \) since, from the regularity condition \( U \), is convex. Furthermore,

\[
\nabla_w F (u^{f*}; u^{-f*}; t) = \left( \frac{\partial H^{i*}}{\partial u^i_k} : k \in \{1, \ldots, n^i\} \right) \tag{2.153}
\]

From (6.2), (3.161) and (2.145), the desired result (2.149) is immediate. Since distribution of \( u^*(\xi) \) is essentially obtained (through solving SDVI) by solving a family of (2.149) with different sample path realizations \( \xi \in \Xi \), therefore each such solution would be the Markovian Nash equilibria of the associated dynamic game. And hence, the solution of the SDVI represents the probability distribution of the Markovian Nash equilibria associated with a stochastic differential game played by \( N \) players. Hence the proof.

### 2.8 Algorithms for DVIs

There are many algorithmic choices for solving DVIs with or without time shifts. We outline only three such methods:

1. convert the DVI to a nonlinear complementarity problem (NCP) after making a finite element approximation. Direct solution of the resulting NCP is possible using successive linearizations [33] and an algorithm such as Lemke’s algorithm for solving each linear complementarity subproblem. Alternatively, as we have done for Friesz et al. [24], one may employ the heuristic of finding the primal and dual variables that are most nearly complementary slack using nonlinear programming. If complementary slackness is not achieved, one may resort to successive linearization at any time during the execution of the heuristic. We omit details of this approach here, instead for the curious readers refer to Friesz et al. [24];
2. a fixed point algorithm after recasting a DVI to as appropriate fixed point problem. Standard nonlinear programming solvers may be used to solve the minimum norm projection subproblems after time discretization;

3. descent method in Hilbert space which can be employed within the fixed point algorithm to solve the minimum norm projection subproblems. In this approach, we do not necessarily time discretize the problem at the beginning.

2.8.1 Fixed Point Algorithm

With the preceding background, we are now in a position to state the a result that permits the solution of an appropriately defined differential variational inequality to be re-cast as a fixed point problem. For the sake of brevity we focus on the DVI with no time lag, however, the same set of results can be extended to a DVI with state dependent time shifts. That result is:

**Theorem 2.14** *(fixed point problem)* When regularity in the sense of Definition 2.7 holds, any solution of the fixed point problem

\[ u = P_U [u - \alpha F(x(u,t), u,t)], \]

where \( P_U [.] \) is the minimum norm projection onto the feasible space \( U \subseteq (L^2[t_0, t_f])^m \) and \( \alpha \in \mathbb{R}^+_+ \), is also a solution of \( DVI(F, f, U, x^0, \Gamma) \).

Naturally there is a fixed point algorithm associated with Theorem 5.3; that algorithm is summarized by the following iterative scheme:

\[ u^{k+1} = P_U \left[ u^k - \alpha F \left( x \left( u^k, u^D \right), u^k, u^D, t \right) \right] \] (2.154)

The detailed structure of the fixed point algorithm is:

*The Fixed Point Algorithm*
Step 0. Initialization. Identify an initial feasible solution $u^0 \in U$ and set $k = 0$.

Step 1. Optimal control subproblem. Solve

$$
\min_v J^k(v) = \gamma^T \Gamma [x(t_f), t_f] + \int_{t_0}^{t_f} \frac{1}{2} \left[ u^k - \alpha F(x^k, u^k, t) - v \right]^2 dt
$$

subject to

$$
\frac{dx}{dt} = f(x, v, t)
$$

$$
v \in U
$$

$$
x(t_0) = x^0
$$

and call the solution $u^{k+1}$. Note that in this step it is advantageous to “unfold” and explicitly state the constraints embedded in the operator $x(u, t)$ of statement (??) in order to facilitate computation.

Step 2. Stopping test. If

$$
\|u^{k+1} - u^k\| \leq \varepsilon
$$

where $\varepsilon \in \mathbb{R}_+^1$ is a preset tolerance, stop and declare $u^* \approx u^{k+1}$. Otherwise set $k = k + 1$ and go to Step 1.

Convergence of the above algorithm is guaranteed in certain circumstances by the following result:

**Theorem 2.15** When $DVI(F, f, U, x^0, D)$ is regular in the sense of Definition 2.7 while additionally $F(x, u, t)$ is strongly monotonic for $u \in U$, the fixed point algorithm presented above converges.

Descent in Hilbert Space for the Projection Sub-Problems

It is important to realize that the fixed point algorithm of Section 6.11 can be carried out in continuous time provided we employ a continuous time representation of each subproblem.
(5.40) through (2.158) from Step 1 of the fixed point algorithm. This may be done using a continuous time gradient projection method or a discrete time gradient projection method supplemented by spline approximations. For our present circumstances, that algorithm may be stated as:

Gradient Projection in Hilbert Space

Step 0. Initialization. Pick \( v^{k,0}(t) \in U \) and set \( j = 0 \).

Step 1. Finding state variables. Solve the state dynamics

\[
\frac{dx}{dt} = f(x, v^{k,j}, t) \tag{2.159}
\]

\[
x(0) = x^0 \tag{2.160}
\]

and call the solution

\[
x^{k,j}(t) \tag{2.161}
\]

In the event a discrete time method is used to solve the state dynamics (2.159) and (2.160), curve fitting is used to obtain the continuous time state vector (2.161).

Step 2. Finding adjoint variables. Solve the adjoint dynamics

\[
(-1) \frac{d\lambda}{dt} = \nabla_x H^k \big|_{x=x^{k,j}} \tag{2.162}
\]

\[
\lambda(t_f) = \frac{\partial \Gamma \left[x^{k,j}(t_f), t_f \right]}{\partial x(t_f)} \tag{2.163}
\]

where \( H \) is the augmented Hamiltonian for the optimal control subproblem of the fixed point algorithm based on current information:

\[
H^k = \frac{1}{2} \left[ u^k - \alpha F \left( x^k, u^k, t \right) - v \right]^2 + \lambda^T f \left( x, v^{k,j}, t \right)
\]

Call the solution

\[
\lambda^{k,j}(t) \tag{2.164}
\]

In the event a discrete time method is used to solve the adjoint dynamics (2.162) and (2.163), curve fitting is used to obtain the continuous time adjoint vector (2.164).
Step 3. Finding the gradient. Determine
\[ \nabla_v J^{k,j}(t) \equiv \nabla_v H^k \]

Step 4. Stopping test. For a fixed and suitably small fixed step size
\[ \theta_k \in \mathbb{R}^1_+ \]
determine
\[ v^{k,j+1}(t) = P_U \left[ v^{k,j}(t) - \theta_k \nabla_v J^{k,j} \right] \tag{2.165} \]
In the event a discrete time method is used to solve the above projection subproblem, curve fitting is used to obtain the continuous time control vector (2.165).

Step 5. Stopping test. For \( \varepsilon_2 \in \mathbb{R}^1_+ \), a pre-set tolerance, stop if
\[ \left\| v^{k,j+1} - v^{k,j} \right\| < \varepsilon_1 \]
and declare
\[ v^{k*} \approx v^{k,j+1} \]
Otherwise set \( j = j + 1 \) and go to Step 1.

Note that the above algorithm overcomes the two-point boundary value problem difficulty that is typically associated with the simultaneous determination of state and control variables and that is characteristic of optimal control problems. This is a direct result of determining controls, states and adjoints in a sequential manner. This sequentialness, however, is not an approximation; rather it is a result of the way the original DVI is represented in terms of mappings between appropriately defined Hilbert spaces.

The gradient projection algorithm in Hilbert space has known convergence properties. In fact the following result obtains for our problem:

**Theorem 2.16** If \( DVI(F, f, U, x^0, \Gamma) \) is regular in the sense of Definition 2.7 while the conditions
\[ \langle v - v' + \lambda^T \left[ \nabla_v f(x, v, t) - \nabla_v f(x, v', t) \right], v - v' \rangle \geq \xi \| v - v' \| \tag{2.166} \]
and
\[ \|v - v' + \lambda^T \left[ \nabla_v f (x, v, t) - \nabla_v f (x, v', t) \right] \| \leq \delta \|v - v'\| \]  \tag{2.167}
are satisfied for some \( \xi, \delta \in \mathbb{R}^1_+ \) and all \( v, v' \in U \), then the gradient projection algorithm for the fixed point sub-problem converges.

**Proof:** Note that
\[ \nabla_v J^k (v) = v - u^k + \alpha F \left( x^k, u^k, u_D^k, t \right) + \lambda^T \nabla_v f (x, v, v_D, t) \]
From (2.166) we have
\[ \langle v - u^k + \alpha F \left( x^k, u^k, u_D^k, t \right) + \lambda^T \nabla_v f (x, v, v_D, t) - \nabla_v J^k (v') , v - v' \rangle \geq \xi \|v - v'\| \]
or
\[ \langle \nabla_v J^k (v) - \nabla_v J^k (v') , v - v' \rangle \geq \xi \|v - v'\| \]
which is recognized as a coerciveness condition. Also (2.167) can be similarly re-stated as
\[ \left\| \nabla_v J^k (v) - \nabla_v J^k (v') \right\| \leq \delta \|v - v'\| \]
which is recognized as a Lipschitz condition. Of course
\[ v^{k,j+1} - v^{k*} = P_U \left[ v^{k,j} - \theta_k \nabla_v J^k \left( v^{k,j} \right) \right] - P_U \left[ v^{k*} - \theta_k \nabla_v J^k \left( v^{k*} \right) \right] \]
Because of the contractive nature of the projection operator, we have immediately that
\[ \left\| v^{k,j+1} - v^{k*} \right\|^2 \leq \left\| v^{k,j} - v^{k*} - \theta_k \left( \nabla_v J^k \left( v^{k,j} \right) - \nabla_v J^k \left( v^{k*} \right) \right) \right\|^2 \\
= \left\| v^{k,j} - v^{k*} \right\|^2 + (\theta_k)^2 \left\| \nabla_v J^k \left( v^{k,j} \right) - \nabla_v J^k \left( v^{k*} \right) \right\|^2 \\
- 2\theta_k \langle \nabla_v J^k \left( v^{k,j} \right) - \nabla_v J^k \left( v^{k*} \right) , v^{k,j} - v^{k*} \rangle \]

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Because of coerciveness and the Lipschitz assumption, we have
\[
\|v^{k,j+1} - v^{k*}\|^2 \leq \|v^{k,j} - v^{k*}\|^2 + (\theta_k \delta)^2 \|v^{k,j} - v^{k*}\|^2 - 2\theta_k \xi \|v^{k,j} - v^{k*}\|^2
\]
\[
= \left[1 + (\theta_k \delta)^2 - 2\theta_k \xi\right] \|v^{k,j} - v^{k*}\|^2
\]

We may select \(\theta_k\) such that
\[
1 + (\theta_k \delta)^2 - 2\theta_k \xi < 1
\]
which is equivalent to a non-zero step obeying
\[
\theta_k < \frac{2\xi}{\delta^2},
\]
a condition ensuring the algorithm is a strict contraction mapping.

### 2.9 Numerical Example

In this section we provide three related numerical examples of \(DVIC(F, f, \Gamma, D, U, x^0)\) with state-dependent time shifts to illustrate the fixed point-descent-in-Hilbert-space algorithm we have proposed. That algorithm was implemented in Matlab 6.5 on a Pentium 4 processor desktop computer with 1 GB RAM. The three examples differ from one another according to what type of time shift is employed. In particular we consider both fixed and state-dependent time shifts as well as the absence of time shifts. The run times for these examples were found to be less than 1 minute for the computing hardware described above.
2.9.1 Example 1 (State-Dependent Time Shifts)

Consider a version of \(DVIC(F, f, \Gamma, D, U, x^0)\) involving 3 controls and 2 states:

\[
\begin{align*}
\mathbf{u} &\in (L^2[0, 1])^3; \quad x \in (\mathcal{H}[0, 1])^2; \quad t_0 = 0; \quad t_f = 5; \quad x(0) = \begin{pmatrix} 1 \\ 0.7 \end{pmatrix} \\
F(x, u, u_D) &= \begin{pmatrix} F_1(x, u, u_D) \\ F_2(x, u, u_D) \\ F_3(x, u, u_D) \end{pmatrix} = \begin{pmatrix} x_1^2 - u_1(t + D_1(x)) + u_2(t + D_2(x)) \\ x_2 - u_2(t + D_2(x)) - u_3(t) \\ \frac{1}{10}x_2^2 - u_3^2(t + D_3(x)) \end{pmatrix} \\
f(x, u) &= \begin{pmatrix} f_1(x, u, u_D) \\ f_2(x, u, u_D) \end{pmatrix} = \begin{pmatrix} \frac{1}{5}x_1(t) + \frac{1}{2}u_1(t) + \frac{3}{10}u_2(t + D_2(x)) \\ \frac{1}{2}x_2(t) + \frac{1}{2}u_2(t) - \frac{1}{5}u_3(t + D_3(x)) \end{pmatrix} \\
U &= \{u : 1 \geq u_1 \geq 0.2; \quad 1.2 \geq u_2 \geq 0.2; \quad 1.3 \geq u_3 \geq 0.2\}
\end{align*}
\]

Note that the right hand side of the state dynamics has shifted controls; the shifted control vectors obey

\[
\mathbf{u}_D = \begin{bmatrix} u_1(t + D_1(x)) \\ u_2(t + D_2(x)) \\ u_3(t + D_3(x)) \end{bmatrix}; \quad \mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 80 \\ 85 \\ 90 \end{pmatrix}
\]

\[
D_1(x(t)) = \frac{x_1(t)}{k_1}; \quad D_2(x(t)) = \frac{x_2(t)}{k_2}; \quad D_3(x(t)) = \frac{0.7 \cdot x_1(t) + 0.3 \cdot x_2(t)}{k_3}
\]

We choose the fixed point parameter to obey \(\alpha = 0.05\). A fifth power polynomial was used to express the controls, adjoint variables and state variables as continuous functions of time. Also the nominal decision time interval is \([0, 5]\). The stopping tolerances for both fixed point and descent iterations were set at \(\varepsilon = 10^{-2}\). The combined fixed-point-descent-in-Hilbert-space algorithm converged after 17 fixed point iterations; each of the descent subproblems converged in 10 or fewer iterations. We forgo the detailed symbolic statement of this example and, instead, provide numerical results in graphical form. Figure 2.1 shows the plot of controls \(u^*\) against time (left) and the states \(x^*\) (right).
2.9.2 Example 2 (Fixed Time Shifts)

Next we modify Example 1 so that the shifts do not depend on the states; instead they are fixed. In particular we assume

\[ D_1 = \frac{3.5}{k_1}; \quad D_2 = \frac{2}{k_2}; \quad D_3 = \frac{3}{k_3} \]

and keep all other parameters the same. The solution thus obtained is shown in Figure 2.2 (left) as well as the states \(x^*\) (right). Comparing with Figure 2.1, we notice that fixed shifts tend to produce smaller control fluctuations. In this case, the combined fixed-point-descent-in-Hilbert-space algorithm converged in 15 fixed point iterations; each of the descent subproblems converged in 12 or fewer iterations.

2.9.3 Example 3 (No Time Shifts)

Next we modify Example 1 so that there are no time shifts; that is \(D_1 = D_2 = D_3 = 0\). Figure 2.3 shows the plots of the controls, \(u^*\), that are solutions to \(DVIC(F, f, \Gamma, D, U, x^0)\). States, \(x^*\) are plotted over time in the same figure. The algorithm converged in 12 fixed point iterations, and each of the subproblems converged in 9 or fewer iterations.
Figure 2.2: \((Left) \ u^{\ast} \ vs. \ time \ (t) \ when \ controls \ have \ fixed \ time-shifts \ (Example \ 2), \ (right) \ states, \ x^{\ast} \ vs. \ time \ plot\)

Figure 2.3: \((Left) \ u^{\ast} \ vs. \ time \ (t) \ when \ controls \ have \ no \ time \ shifts \ (Example \ 3), \ (right) \ states, \ x^{\ast} \ vs. \ time \ plot\)
Part II

Applications
Chapter 3

Dynamic Competition in Network Revenue Management

3.1 Organization of the Chapter

In this chapter we show how the computable theory of dynamic games may be applied in the field of network revenue management where the service providers are involved in an oligopoly. We begin by assuming that the service providers are simultaneously making their pricing-allocation-overbooking decisions in the face of uncertain demand. Section 3.2 describes such formulation, some of the structural properties of the equilibrium under mild regularity condition, and of course, a numerical example that illustrates applicability of this perspective. We show in Section 3.3 that when overbooking decisions are excluded from the decision space, resulting pricing-allocation game may also be articulated as a differential variational inequality. In section 3.4 we advocate price-based network revenue management in a Cournot-Nash dynamic game theoretic setting. We assume that the service providers of interest are follow a learning process that is similar to evolutionary game-theoretic dynamics and for which price changes are proportional to their signed excursion from a market clearing price. We stress that in this model firms are setting prices for their
services while simultaneously determining the levels of demand they will serve.

### 3.2 Dynamic Competition Involving Pricing-Allocation-Overbooking Decisions Under Demand Uncertainty

#### 3.2.1 Introduction

As a field of inquiry revenue management (RM), which is also referred to as revenue optimization, is relatively new. RM seeks to identify service provider strategies and tactics that extract all willingness to pay from service customers. Growth of the field of revenue management was boosted by the deregulation of domestic and international airlines in the late 1970’s. Airlines, car rental agencies and hotels typically exercise quantity-based RM techniques by controlling the quantity of services sold to distinct demand categories. By contrast retailers frequently use price-based RM techniques. Revenue management is most challenging and most complex when the market of interest corresponds to dynamic oligopolistic competition among service providers and does not necessarily reach a static equilibrium.

The first comprehensive book on RM, by Talluri and van Ryzin [34], provides a detailed exposition of current price-based and quantity-based RM techniques. Today, as Talluri and van Ryzin [34] point out, RM is widely used and the list of its applications is growing steadily. McGill and van Ryzin [35] provide a detailed survey of the RM research advancements from 1970 to the late 90s. These and other reviews of RM research do not report computable models of overbooking in that allow both quantity-based and revenue-based RM in dynamic game theoretic context for which the service providers compete with one another on a network and demand is uncertain. This chapter presents just such a model as well as an algorithm for its numerical solution. The mathematical formalism employed is that of variational inequalities constrained by equations of state in the form of difference
Before we present the actual model which is the focus of this chapter, it is useful to quickly review some of the literature pertaining to special RM features we wish to capture and which have been difficult to integrate within a single model. Following this brief literature review, the model, an algorithm and a detailed numerical example are presented in separate sections.

3.2.2 Network RM and Overbooking

Network RM arises in airline, railway, hotel and cruise-line service environments where customers enter into service relationships with service providers, and those relationships may be viewed as bundles of resources. Accordingly, each service relationship typically employs a subset of available resources. The relationships among service providers, customers and resources naturally take the form of a network whose topological structure is discussed in section 4.3.1. Overbooking is one of the oldest and most important RM tactics employed by service firms. Overbooking results when a service firm accepts more reservations than its physical capacity can serve to hedge against cancellations and no-shows. Most of the past work on overbooking models considers a single type of service, as noted in Talluri and van Ryzin [34]. By contrast Karaesmen and van Ryzin [36] consider an overbooking model with multiple reservation classes as well as substitution options.

3.2.3 Joint Pricing and Allocation

To our knowledge only very few papers address the competitive joint pricing and resource allocation problem. In particular, Weatherford [37] studies the joint pricing and inventory control problem for an airline in a non-network setting for a single-leg flight with multiple fare classes. Kuyumcu and Garcia-Diaz [38] study the problem of airline revenue management for a large number of booking classes using a large scale binary integer programming
formulation. Furthermore, the research we present in this chapter has been motivated by the work of Bertsimas and de Boer [39] who study a restricted class of joint pricing and resource allocation problems on a network. Their paper provides some instructive structural results about the single firm’s revenue optimization problem that suggest the more general perspective we undertake herein. Also influencing our work is the chapter by Perakis and Sood [40] who consider a joint pricing and resource allocation problem for a single perishable product in an oligopoly using a robust optimization approach to account for demand uncertainty. Also significant is the recent work on generalization of the news vendor problem, especially the papers by Chen et al [41] and Bernstein and Federgruen [42], both of which consider infinite-horizon joint pricing and inventory decision models for competing retailers (newsvendors) from a supply chain perspective. Other influential papers that consider newsvendor-type competition include Parlar [43], Lipman and McCardle [44], Mahajan and van Ryzin [45] and Netessine and Rudi [46].

3.2.4 Perspective of this chapter

In this chapter we consider service firms who provide differentiated, non-substitutable services, set prices for their services, may decline some of the booking requests at any given time, face cancellations and no-shows (with full or partial refunds) and have finite supplies of resources. The demand for each service type is uncertain; the average demand depends on both own and non-own service prices of a given firm and its competitors. A given service firm has to decide both how to allocate its resources (quantity-based RM) and how to set prices for its services (price-based RM) as it seeks to maximize expected revenue. Such a decision environment is similar to that faced by new discount airlines, such as Southwest and Jet Blue.

Our work is distinct from the existing body of RM literature because (i) it employs an integrated game-theoretic approach to pricing, allocation and overbooking in a network oligopoly under demand uncertainty, (ii) it provides an existence result for a pure strategy
Nash equilibrium of the resulting game as well as conditions assuring the uniqueness of that equilibrium, \((iii)\) it provides an efficient and intuitively appealing algorithm to compute equilibrium prices, allocations and overbooking limits together with a proof of convergence in the presence of suitable regularity conditions, \((iv)\) it establishes that at equilibrium the oligopolistic service market exhibits a bias toward under-pricing, and \((v)\) it quantifies the ‘price of anarchy’ in a numerical example of oligopolistic service competition.

3.2.5 Notation and Assumptions

Consider an oligopoly of abstract service providers where the companies experience random demands. Each firm provides a set of services (products). Each network product may be viewed as a bundle of the resources sold with certain terms of purchase and restrictions at a given price. These services are non-substitutable and differentiated. All firms have finite resource capacities. The on-hand reservations are subject to cancellations and no-shows. Cancellations and no-shows may be partially or fully refunded depending on the terms of the relevant service contracts. The booking period is \([0, L]\) which is discretized in \(N\) time segments. At the beginning of period \(t \in [0, N]\) firms set service prices and quantities for sale in that period. Table 3.1 lists the notation will be used throughout our exposition:

From the above notation it is evident that

\[
\bigcup_{i=1}^{\|S\|} C_i = C
\]

Notation will be used to denote controls and states are listed in Table 3.2.

The vectors of decision variables for service \(i\) provided by firm \(f\) are

\[
\begin{align*}
    p_t^f &= \left( p_{i,t}^f : i \in S \right) \\
    u_t^f &= \left( u_{i,t}^f : i \in S \right)
\end{align*}
\]
### Table 3.1: Parameters: Dynamic Pricing-Allocation-Overbooking Model

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F}$</td>
<td>set of firms</td>
</tr>
<tr>
<td>$\mathcal{S}$</td>
<td>set of services each firm provides</td>
</tr>
<tr>
<td>$\mathcal{C}$</td>
<td>set of resources that firms use to provide services</td>
</tr>
<tr>
<td>$\mathcal{C}_i$</td>
<td>set of resources that firms use to provide service $i \in \mathcal{S}$</td>
</tr>
<tr>
<td>$\mathcal{S}_j$</td>
<td>set of services that utilize resource $j \in \mathcal{C}$</td>
</tr>
<tr>
<td>$\mathcal{A}$</td>
<td>resource-service incidence matrix</td>
</tr>
<tr>
<td>$L$</td>
<td>end time of the booking period</td>
</tr>
<tr>
<td>$</td>
<td>\mathcal{C}</td>
</tr>
<tr>
<td>$0 \leq t_n \leq t_N = L$</td>
<td>$n$th time period on the booking horizon, where time step, $\Delta = \frac{L}{N}$</td>
</tr>
<tr>
<td>$p_{i,\text{min}}^f$</td>
<td>minimum price that firm $f$ can charge for service $i \in \mathcal{S}$</td>
</tr>
<tr>
<td>$p_{i,\text{max}}^f$</td>
<td>maximum price that firm $f$ can charge for service $i \in \mathcal{S}$</td>
</tr>
<tr>
<td>$K_j^f$</td>
<td>firm $f$’s capacity for resource type $j \in \mathcal{C}$</td>
</tr>
<tr>
<td>$R_j^f$</td>
<td>partial refunds made by firm $f$ against cancelling resource $j \in \mathcal{C}$</td>
</tr>
<tr>
<td>$\eta_j^f$</td>
<td>firm $f$’s denial of service cost for resource $j \in \mathcal{C}$</td>
</tr>
</tbody>
</table>

### Table 3.2: Control and State Variables: Dynamic Pricing-Allocation-Overbooking Model

<table>
<thead>
<tr>
<th>Variables</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{i,t}^f$</td>
<td>price for the service $i \in \mathcal{S}$ charged by firm $f \in \mathcal{F}$ in time period $t$</td>
</tr>
<tr>
<td>$u_{i,t}^f$</td>
<td>firm $f$’s service level of type $i \in \mathcal{S}$ in time period $t$</td>
</tr>
<tr>
<td>$x_{j,t}^f$</td>
<td>firm $f$’s total allocated resource of type $j \in \mathcal{C}$ in time period $t$</td>
</tr>
<tr>
<td>$d_{i,t}^f(p_{i,t})$</td>
<td>mean demand for service $i \in \mathcal{S}$ from firm $f \in \mathcal{F}$ in time period $t$ when the prevailing price is $p_t$</td>
</tr>
<tr>
<td>$D_{i,t}^f(p_{i,t})$</td>
<td>firm $f$’s realized demand for service $i \in \mathcal{S}$ in time period $t$</td>
</tr>
<tr>
<td>$z_{i,t}^f$</td>
<td>random component associated with demand faced by the firm $f$ for service $i$ in time period $t$</td>
</tr>
</tbody>
</table>
which concatenates to

\[ p^f = (p^f_t : t \in [0, N]) \]
\[ u^f = (u^f_t : t \in [0, N]) \]

The pricing decision variables of firm \( f \)’s competitors for period \( t \) are denoted by the vector

\[ p^{-f}_t = (p^g_t : g \in \mathcal{F} \setminus f) \],

The state variables for firm \( f \) are the vectors of cumulative allocations of resources

\[ x^f_t = (x^f_{j,t} : j \in \mathcal{C}) \]

for period \( t \).

### 3.2.6 Network representation through incidence matrix

The network we are interested in has \( |\mathcal{C}| \) resources and the firm provides \( |\mathcal{S}| \) different services. Each network product is a combination of a bundle of the \( |\mathcal{C}| \) resources sold with certain terms of purchase and restrictions at a given price. The resource-service incidence matrix, \( \mathcal{A} = [a_{ij}] \) is a \( |\mathcal{C}| \times |\mathcal{S}| \) matrix where

\[ a_{ij} = 1 \text{ if resource } i \text{ is used by service } j \]
\[ = 0 \text{ else} \]

Thus, the \( j \)th column of \( \mathcal{A} \), denoted by \( \mathcal{A}_j \), is the incidence vector for service \( j \); while the \( i \)th row, denoted by \( \mathcal{A}^i \), has unity in column \( j \) provided service \( j \) utilizes the resource \( i \). Note that there will be multiple identical columns if there are multiple ways of selling a given bundle of resources, although each could have different revenue values and different demand patterns (Talluri and van Ryzin [34]).
3.2.7 Demand for services

Firm $f$’s realized demand at time period $t$ for service $i$ when the prevailing market price is $p_i$ will be denoted as $D_{i,t}^f(p_{i,t})$. Two types of demand models are predominant in the supply chain, news vendor pricing and revenue management literature: namely, the *additive form* where

$$D_{i,t}^f(p_{i,t}) = d_{i,t}^f(p_{i,t}) + z_{i,t}^f \quad (3.1)$$

and the *multiplicative form* where

$$D_{i,t}^f(p_{i,t}) = d_{i,t}^f(p_{i,t}) \cdot z_{i,t}^f \quad (3.2)$$

where $\{z_{i,t}^f\}$ is a sequence of continuous i.i.d. positive random variables with common pdf $g(\cdot)$ and c.d.f. distribution $F(\cdot)$ respectively; and $d_{i,t}^f(p_{i,t})$ is the expected or average demand faced by the firm $f$ for its service $i$ when the price combination is $p_{i,t}$ at time $t$.

We focus on the multiplicative demand form in the rest of the chapter, although our results are adaptable to the additive form. The hazard rate function corresponding to c.d.f. $F(\cdot)$ can be expressed as

$$h(\alpha) = \frac{g(\alpha)}{1 - F(\alpha)}$$

Consequently, the generalized failure rate associated with random variable $\alpha$, pdf $g(\cdot)$ and c.d.f. $F(\cdot)$ is

$$\beta(\alpha) = \alpha \cdot h(\alpha) = \frac{\alpha \cdot g(\alpha)}{1 - F(\alpha)},$$

which is known to be increasing in $\alpha$ for most of the well known distributions like uniform, normal, and exponential etc.\(^1\). We will impose a mild restriction on $z_{i,t}^f$ in the spirit of Lariviere and Porteus [48] and Lariviere [47], namely:

**Assumption A1** The random variable $z_{i,t}^f$ is independent of price $p$ and time $t$, identically distributed, and obeys the properties of an increasing generalized failure rate (IGFR), which

\(^1\)See Chen *et al* [41], and Lariviere [47].
are

\[ E \left( z_{i,t}^f \right) = 1 \]
\[ \text{Var} \left( z_{i,t}^f \right) = \left( \tilde{\sigma}_i^f \right)^2 < \infty \]

Ziya et al [49] review random variable regularity assumptions that are predominant in the supply chain and revenue management literature, and conclude that, in practical terms, the IGFR assumption is neither more nor less restrictive than other assumptions. An immediate consequence of Assumption A1 is that the multiplicative demand structure

\[ E \left( D_{i,t}^f (p_{i,t}) \right) = d_{i,t}^f (p_{i,t}) \]

implies that the coefficients of variation of the one-period demands are independent of the price vectors.

We next consider mean demand, \( d_{i,t}^f (p_{i,t}) \) and employ restrictions on mean demand that are very similar to those in Chen et al [41]. In particular we invoke the following:

**Assumption A2** For any firm \( f \in F \) and the service type \( i \in S \), the mean demand \( d_{i,t}^f (p_{i,t}) \) has the following properties for all periods \( t \in [0, N] \):

1. Each price, \( p_{i,t}^f \) is defined on a range \( [p_{i,\text{min}}^f, p_{i,\text{max}}^f] \), where \( p_{i,\text{max}}^f \in \mathbb{R}_{++}^1 \) is the maximum admissible value of \( p_i^f \), and \( p_{i,\text{min}}^f \in \mathbb{R}_{++}^1 \) is the minimum admissible value of \( p_i^f \), while
   
   \[ d_i^f (p_{i,t}) \bigg|_{p_{i,t}^f = p_{i,\text{max}}^f} = 0; \]

2. \( d_{i,t}^f (p_{i,t}) \) depends only on the current (period \( t \in [0, N] \)) prices charged by firm \( f \) and its competitors for service type \( i \);

3. \( d_{i,t}^f (p_{i,t}) \) is continuous, bounded, and differentiable for all \( p_{i,t} \in [p_{i,\text{min}}, p_{i,\text{max}}] \) where
   
   \[ p_{i,t} = \left\{ p_{i,t}^g : g \in \mathcal{F} \right\} \]
4. \( \frac{\partial d_{i,t}(p_{i,t})}{\partial p_{i,t}} < 0 \)

5. The local price elasticity of \( d_{i,t}^f(p_{i,t}) \), defined as
\[
e_i^f \equiv - \left( \frac{\partial d_{i,t}^f}{\partial p_{i,t}} \bigg/ \frac{d_{i,t}^f}{p_{i,t}} \right)
\]
is increasing in \( p_{i,t}^f \); that is
\[
\frac{\partial e_i^f}{\partial p_{i,t}^f} \geq 0;
\]

6. \( \frac{\partial d_{i,t}^f(p_{i,t})}{\partial p_{i,t}^g} > 0 \) for all \( g \neq f \);

7. \( \frac{\partial e_i^f}{\partial p_{i,t}^g} \leq 0 \) for all \( g \neq f \); and

8. \[
\frac{\partial e_i^f}{\partial p_{i,t}^f} + \sum_{g \neq f} \frac{\partial e_i^f}{\partial p_{i,t}^g} \geq 0 \text{ for all } i \in S, f \in F
\]

It is instructive to give qualitative descriptions for each of the separate conditions that comprise Assumption A2. In particular, items 1 and 3 impose bounds that arise from regulations and policy. Item 2 indicate that different services are unsubstitutable and differentiated, however, our results are adaptable to the problem substitutable services. Items 4 and 5 indicate that \( d_{i,t}^f(p_{i,t}) \) is downward sloping in firm \( f \)'s own price and has increasing price elasticity (IPE) relative to \( p_{i,t}^f \). Moreover, linear, exponential and concave forms of the mean demand functions, that are commonly used in the literature, have IPE. Item 6 postulates that when any other firm increases its price for service \( i \), there is a corresponding increment of firm \( f \)'s demand for the service \( i \). Item 7 further requires that an increase in firm \( g \)'s price for service \( i \) not only increases firm \( f \)'s demand, but also decreases firm \( f \)'s price elasticity. Item 8 implies that the local price effect of a price change dominates the cross price effect on the local price elasticity which describes the substitution effect.
It is easy to verify that most of the commonly used demand functions satisfy the restrictions set forth in Assumption A2. In particular, the following demand functions satisfy Assumption A2:

1. **Linear**
   \[
   d_{i,t}^f(p_{i,t}) = \rho_{i,t}^f - \sigma_{i,t}^f \cdot p_{i,t}^f + \sum_{g \in F \setminus f} \gamma_{i,t}^g \cdot p_{i,t}^g
   \]
   where \(\rho_{i,t}^f, \sigma_{i,t}^f, \gamma_{i,t}^f \in \mathbb{R}_{++}^1\) for all \(f \in F, i \in S\) and \(0 \leq t \leq N\).

2. **Logit**
   \[
   d_{i,t}^f(p_{i,t}) = \frac{a_{i,t}^f \exp\left(-b_{i,t}^f \cdot p_{i,t}^f\right)}{\theta_i + \sum_{g \in F} a_{j,t}^g \exp\left(-b_{j,t}^g \cdot p_{j,t}^g\right)}
   \]
   where \(a_{i,t}^f, b_{i,t}^f, \theta_i \in \mathbb{R}_{++}^1\) for all \(f \in F, i \in S\) and \(0 \leq t \leq N\).

3. **Cobb-Douglas**
   \[
   d_{i,t}^f(p_{i,t}) = a_i^f \left(p_{i,t}^f\right)^{-\beta_i^f} \prod_{g \in F \setminus i} \left(p_{i,t}^g\right)^{\beta_{ig}^g}
   \]
   where \(a_i^f > 0, \beta_i^f > 1, \beta_{ig}^g > 0\) for all \(f \in F, i \in S\) and \(0 \leq t \leq N\).

### 3.2.8 Cancellations and Overbooking

In our model we do not distinguish between cancellations and no-shows, since the difference between cancellations and no-shows is simply the timing of these events - a cancellation is a reservation that is withdrawn by a customer strictly prior to the time of service where as a no-show is someone who does not cancel the reservation but does not show up at the time of service. Furthermore, we assume that the refund for cancellations and no-shows is identical. We also assume that the probability of cancellation at a given period is independent of the time the reservation was made. This memoryless nature of cancellations is supported empirically for the airline industry; see Martinez and Sanchez [50].

We additionally assume that customers cancel independently of one another and each customer has the same probability of cancelling. Furthermore, in our model, each
service firm allows its customers to cancel the individual component services that comprise each service relationship between a service provider and a client. That is, cancelling one or more travel legs of a multi-leg itinerary, as well as one or more days of a multi-day hotel room reservation or multi-day car-rental reservation, is allowed. An *a priori* schedule of partial refunds is known for each cancellation scenario. Moreover, the service-resource incidence matrix allows any cancellation to be viewed as a reduction in resources. So, in the narrative that follows, without loss of generality, we refer to cancellations as the reduction of resources and are able to associate a predetermined refund each such reduction.

With the above perspective in mind, we let the refund associated with a cancellation of resource class \( j \) for firm \( f \) be denoted as \( R^f_j \). We also let the random variable \( Z^f_j(x^f_{j,N}) \) be the number of customers holding reservations for service to be provided by firm \( f \) that utilize resource \( j \) and actually show up for some aspect of the original service reservation. This number of customers is the number who survive from the reservation period to the service period. Of course, the number of survivors \( Z^f_j(x^f_{j,N}) \) is a function of the total number of customers holding reservations from firm \( f \) that utilize resource \( j \), namely \( x^f_{j,N} \) which was defined previously. Therefore, the number of cancellations and no-shows impacting resource \( j \) of firm \( f \) is \( x^f_{j,N} - Z^f_j(x^f_{j,N}) \). Furthermore, we will assume, following Karaesmen and van Ryzin [36], that the random variables \( \left\{ Z^f_j(x^f_{j,N}) \right\} \) have the *semi-group* property\(^2\). Lastly, we will assume the \( Z^f_j(x^f_{j,N}) \) for all \( j \in C \) are mutually independent and non-negative.

One model that satisfies these assumptions and is commonly used to study overbooking is the so-called binomial model. Talluri and van Ryzin [34] provide a compelling empirical justification for binomial overbooking models. To create a binomial overbooking model, let \( \alpha^f_j \) denote the probability of survival of resource type \( j \) for firm \( f \) at the end of booking period; that probability is independent of both \( x^f_{j,N} \) and the age of the reservation. Then total show-demand, \( Z^f_j(x^f_{j,N}) \) is binomially distributed with parameters \( \alpha^f_j \) and \( x^f_{j,N} \)

\(^2\)If \( Y_1 \) and \( Y_2 \) are independent and \( Y_1 \) and \( Y_2 \) are stochastically equivalent to (have the same probability distribution as) \( Z^f_1(x^f_{1,N}) \) and \( Z^f_2(x^f_{2,N}) \) respectively, \( Y_1 + Y_2 \) is stochastically equivalent to \( Z^f_1(x^f_{1,N} + x^f_{2,N}) \).
where $x_{j,N}^f$ is a nonnegative number. However, the binomial distribution is not well suited to the continuous analysis and numerical methods emphasized in this chapter. Therefore, we approximate the show-demand distribution as a normal distribution; that is $Z_j^f\left(x_{j,N}^f\right)$ follows a normal distribution with mean $\alpha_j^f \cdot x_{j,N}^f$ and variance $\alpha_j^f \cdot \left(1 - \alpha_j^f\right) \cdot x_{j,N}^f$. Since the $(j, f)$-survival probability $\alpha_j^f$ is independent of $x_{j,N}^f$, the semigroup property is maintained.

When the show demand $Z_j^f\left(x_{j,N}^f\right)$ for resource $j$ exceeds physical capacity $K_j^f$ that the firm possesses, the denial of service cost (or, when relevant, the cost of providing alternative arrangements) is usually high in order to account for direct compensation cost as well as the loss of goodwill. We denote the denial of service cost associated with resource $j$ and firm $f$ by $\eta_j^f$. Therefore, the total expected denial of service cost and cancellation refunds for firm $f$, which is the overbooking cost, may be expressed as

$$OBC_f = E\left[\sum_{j \in C} \eta_j^f \cdot \max\left(Z_j^f\left(x_{j,N}^f\right) - K_j^f, 0\right) + R_j^f \cdot \left(x_{j,N}^f - Z_j^f\left(x_{j,N}^f\right)\right)\right]$$  \hspace{1cm} (3.3)

Since $Z_j^f\left(x_{j,N}^f\right)$ follows a normal distribution with mean $\alpha_j^f \cdot x_{j,N}^f$ and variance $\alpha_j^f \cdot \left(1 - \alpha_j^f\right) \cdot x_{j,N}^f$, the expected denial of service costs may be expressed as

$$\eta_j^f \cdot E\left[\max\left(Z_j^f\left(x_{j,N}^f\right) - K_j^f, 0\right)\right] = \eta_j^f \cdot \sigma_Z \cdot \left[\phi(z) - z \left(1 - \Phi(z)\right)\right]$$

where

$$\sigma_Z = \sqrt{\alpha_j^f \cdot \left(1 - \alpha_j^f\right) \cdot x_{j,N}^f}$$  \hspace{1cm} (3.4)

and $\phi(z)$ and $\Phi(z)$ are the relevant pdf and c.d.f. respectively with

$$z = \frac{K_j^f - \alpha_j^f \cdot x_{j,N}^f}{\sqrt{\alpha_j^f \cdot \left(1 - \alpha_j^f\right) \cdot x_{j,N}^f}}$$

The preceding discussion sets the stage for the following result:

**Lemma 3.1** When each survival probability $\alpha_j^f < 1$ for all $j \in C$ and a given $f \in \mathcal{F}$, the expected overbooking cost $OBC_f$ is monotonically increasing with respect to the total allocation of resources $x_N^f$. 

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Proof. Note that the expected refunds and overbooking costs are separable in resource type. Taking the partial derivative of $OBC_f$ with respect to $x_{j,N}^f$ we obtain

$$\frac{\partial OBC_f}{\partial x_{j,N}^f} = \frac{\partial}{\partial z} \left\{ \eta_j^f \cdot \sigma_Z [\phi (z) - z (1 - \Phi (z))] \right\} \cdot \frac{\partial z}{\partial x_{j,N}^f} + R_j^f \cdot (1 - \alpha_j^f)$$

$$= -\eta_j^f \cdot \sigma_Z (1 - \Phi (z)) \cdot \frac{\partial z}{\partial x_{j,N}^f} + \eta_j^f \frac{\alpha_j^f (1 - \alpha_j^f)}{2} \sqrt{\frac{\beta_j^f}{x_{j,N}^f}} [\phi (z) - z (1 - \Phi (z))]$$

$$+ R_j^f \cdot (1 - \alpha_j^f)$$

After substituting for $\sigma_Z$ using (3.4) and using the fact that $z$ is normally distributed, one obtains after some simplification the following:

$$\frac{\partial OBC_f}{\partial x_{j,N}^f} = \eta_j^f \alpha_j^f \cdot (1 - \Phi (z)) + \frac{\eta_j^f \phi (z)}{2} \sqrt{\frac{\alpha_j^f (1 - \alpha_j^f)}{x_{j,N}^f}} + R_j^f \cdot (1 - \alpha_j^f) \quad (3.5)$$

as required. $\blacksquare$

Figure 3.1 plots $OBC_f$ against cumulative allocation $x_{j,N}^f$ for different survival probabilities $\alpha_j^f$ when there is a single resource bounded from above by 100 resource units.

3.2.9 Revenue Maximization by Providers

The time scale (the booking period) we consider here is short enough hence ignore the time value of money. With the rival firms’ prices $p_t^j$ taken as exogenous to the firm $f \in F$’s discrete time optimal control problem and yet endogenous to the overall equilibrium model, firm $f$ computes its prices $p_t^f$ and allocation of resources $u_t^f$ in order to maximize net revenue generated throughout the booking period

$$\max_{p_t^f, u_t^f} J (p_t^f, u_t^f ; p_t^t) = -OBC_f + E \left[ \sum_{t=0}^{N-1} p_t^f \cdot \min (u_t^f, D_t^f (p_t)) \right] \quad (3.6)$$
Figure 3.1: Expected overbooking penalty and refunds vs. total allocation of a single resource with capacity, $K = 100$ with different survival probabilities

subject to

$$x_{t+1}^f = x_t^f + A \cdot u_t^f \text{ for all } t = 0, \ldots, N - 1$$  \hspace{1cm} (3.7)

$$x_0^f = 0$$  \hspace{1cm} (3.8)

$$p_{t, \min}^f \leq p_t^f \leq p_{t, \max}^f \text{ for all } t = 0, \ldots, N - 1$$  \hspace{1cm} (3.9)

$$u_t^f \geq 0 \text{ for all } t = 0, \ldots, N - 1$$  \hspace{1cm} (3.10)

Readers should note that in this model firms make ‘here-and-now’ type pricing, allocation and overbooking decisions at time 0 prior to observing any demand realizations. The expected overbooking and cancellation cost $OBC_f$ is defined in (3.3). As in a typical RM industry, there is no salvage value of unsold resources at the end of planning horizon which is reflected in this model. The first expression in the right hand side of (3.6) is the total expected denial of service cost and refunds when it is subtracted from the expected revenue generated throughout the booking period, we are left with net revenue generated in that period. Constraints (3.7) are definitional dynamics that describes the rate at which resources are committed. (3.8) is the initial condition in optimal control terminology, in
the present context which says that at the start of the booking period no resources are committed. Service prices are bounded from above and below as in (3.9). The constraint (3.10) is the nonnegativity restriction on the allocation decisions. Realistically, the lower bounds on the allocation variables $u^f_t$ in (3.10) should be strictly positive to ensure that each service provider participates in each period with a strictly positive provision of service. The implication, if this were not true, would be that a firm with nothing to offer in a period could influence the realized demand of its competitors by setting a price. In other words, setting a price would make sense only if there is a nonzero sale in that period. However, our model can be adapted to this situation by modifying expected revenue term in (3.6) to $E \left[ \sum_{t=0}^{N-1} p^f_t \cdot \min \left( u^f_t + u^f_{\text{min}}, D^f_t(p_t) \right) \right]$ and changing the resource capacity to $K^f - A \cdot u^f_{\text{min}} \cdot N$ in (3.3) where $u^f_{\text{min}}$ is the minimum provision of service. We have not imposed any constraint on the direction of price change. Since firms’ optimal control problems are coupled, this gives rise to a Cournot-Nash dynamic games setting. Observe that service provider $f$’s resource allocations $u^f_t$ impacts his own revenue but not that of any of his competitors; where as service price does. Hence a firm only needs information on her competitors’ pricing policies and not information on their allocations - the latter would be unrealistic in practice.

3.2.10 Structural Properties and Variational Inequality Formulation

Structural Properties

We need to study firm $f$’s individual response to other firms’ pricing decisions $p^{-f}$. Expected single period revenue for firm $f$ at time period $t$ can be expressed as

$$R_f(p^f_t, u^f_t, t) \equiv E \left[ p^f_t \cdot \min \left( u^f_t, D^f_t(p_t) \right) \right]$$

(3.11)

**Proposition 3.1** For multiplicative demand (4.77), (3.11) can be expressed as

$$R_f(p^f_t, u^f_t, t) = \sum_{i \in S} \left( p^f_{i,t} \cdot u^f_{i,t} - p^f_{i,t} \cdot d^f_{i,t}(p_{i,t}) \int_0^{u^f_{i,t}} F(\tau) d\tau \right)$$

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Proof. Since $D^f_t ( p_t ) = d^f_t ( p_t ) \cdot z$;

$$E \left[ p^f_t \cdot \min \left( u^f_{i,t}, D^f_t ( p_t ) \right) \right] = \sum_{i \in S} p^f_{i,t} \left[ u^f_{i,t} \cdot \Pr(D^f_t \geq u^f_{i,t}) + d^f_t \int_0^{u^f_{i,t}} z f(z)dz \right]$$

$$= \sum_{i \in S} p^f_{i,t} \left[ u^f_{i,t} \cdot \int_0^{u^f_{i,t}} z f(z)dz + d^f_t \int_0^{u^f_{i,t}} z f(z)dz \right]$$

$$= \sum_{i \in S} p^f_{i,t} \left[ u^f_{i,t} \cdot \left( 1 - \int_0^{u^f_{i,t}} z f(z)dz \right) + d^f_t \int_0^{u^f_{i,t}} z f(z)dz \right]$$

$$= \sum_{i \in S} p^f_{i,t} u^f_{i,t} - \sum_{i \in S} p^f_{i,t} d^f_t \int_0^{u^f_{i,t}} \left( \frac{u^f_{i,t}}{d^f_t} - z \right) f(z)dz$$

using the identity $dF(z) = f(z)dz$

$$E \left[ p^f_t \cdot \min \left( u^f_{i,t}, D^f_t ( p_t ) \right) \right] = \sum_{i \in S} p^f_{i,t} u^f_{i,t} - \sum_{i \in S} p^f_{i,t} d^f_t \int_0^{u^f_{i,t}} \left( \frac{u^f_{i,t}}{d^f_t} - z \right) dF(z)$$

$$= \sum_{i \in S} \left\{ p^f_{i,t} u^f_{i,t} - p^f_{i,t} d^f_t \left[ \frac{u^f_{i,t}}{d^f_t} \cdot F \left( \frac{u^f_{i,t}}{d^f_t} \right) - \int_0^{u^f_{i,t}} z dF(z) \right] \right\}$$

integrating by parts the third term of the right hand side we obtain

$$E \left[ p^f_t \cdot \min \left( u^f_{i,t}, D^f_t ( p_t ) \right) \right] = \sum_{i \in S} \left\{ p^f_{i,t} u^f_{i,t} - p^f_{i,t} d^f_t \left[ \frac{u^f_{i,t}}{d^f_t} \cdot F \left( \frac{u^f_{i,t}}{d^f_t} \right) - \int_0^{u^f_{i,t}} z F(z)dz \right] \right\}$$

$$= \sum_{i \in S} p^f_{i,t} u^f_{i,t} - p^f_{i,t} d^f_t \int_0^{u^f_{i,t}} F(z)dz$$

Hence the proof. □

In the expression for the single period revenue, the component $\sum_{i \in S} p^f_{i,t} u^f_{i,t}$ may be called the riskless component of the revenue for firm $f \in F$. Further, depending on the component $\frac{u^f_{i,t}}{d^f_t ( p_{i,t} )}$, the expected revenue is reduced as the term $\frac{u^f_{i,t}}{d^f_t ( p_{i,t} )}$ increases.
Our analysis is based on the discrete time optimal control theory. Let us now form the discrete time stochastic Hamiltonian as

\[
H_{f,t} = H_f(p_t^f, u_t^f; \lambda_t^f; p_{t-}^f, t) = R_{f,t}(p_t^f, u_t^f, t) + \left( \lambda_{t+1}^f \right)^T \cdot (A \cdot u_t^f)
\]

\[
= \sum_{i \in S} \left( p_{i,t}^f + \left\{ \left( \lambda_{i+1}^f \right)^T \cdot A \right\} \cdot u_{i,t}^f - p_{i,t}^f \cdot d_{i,t}^f \int_0^{u_{i,t}^f} F(z)dz \right) \tag{3.12}
\]

where \( \lambda_t^f \) is the vector of adjoint variables such that

\[
\lambda_t^f = \left( \lambda_{j,t}^f : j \in C \right)
\]

\[
\lambda^f = \left( \lambda_t^f : t \in [1, N] \right)
\]

Adjoint variables \( \lambda_t^f \) may be interpreted as the shadow price of resources. From the maximum principle, at each time period \( t \in [0, N - 1] \) the firm \( f \) seeks to solve the following static optimization problem

\[
\max_{p_t^f, u_t^f} H_f(p_t^f, u_t^f; \lambda_t^f; p_{t-}^f, t) \tag{3.13}
\]

subject to

\[
p_{t}^{f_{\min}} \leq p_{t}^f \leq p_{t}^{f_{\max}} \tag{3.14}
\]

\[
u_{t}^f \geq 0 \tag{3.15}
\]

where the adjoint dynamics arising from the optimal control problem of firm \( f \) is

\[
\lambda_{j,t+1}^f - \lambda_{j,t}^f = - \frac{\partial H_{f,t}}{\partial x_{j,t}^f} = 0 \text{ for all } j \in C
\]

as because Hamiltonian (3.12) is free from the state, hence, \( \lambda_t^f \) remains stationary throughout the trajectory therefore

\[
\lambda_{j,t}^f = \lambda_j^f \text{ for all } t \in [1, N], j \in C \tag{3.16}
\]

and the transversality condition yields

\[
\lambda_j^f = - \frac{\partial OBC_{f,t}}{\partial x_{j,N}^f} \text{ for all } j \in C \tag{3.17}
\]
Figure 3.2: Dependence of shadow price of a resource ($\lambda_j$) on total allocation of that resource ($x_{j,N}$) and survival probabilities ($\alpha_j$); capacity of that resource being 100 units

where $OBC_f$ is defined in (3.3). Figure 3.2 shows the dependence of $\lambda^f$ on total allocated resource ($x_{j,N}^f$) and survival probability of a reservation ($\alpha^f$). If we now assume $c_{i,t}^f \equiv -(\lambda^f)^T A_i$ be the firm $f$’s ‘shadow’ price of providing per unit service type $i \in S$, which is endogenously determined, at each time $t \in [1, N]$. Thus, (3.12) can be rewritten as

$$H_f \left( p_l^f, u_t^f; \lambda^f, p_{-f}^t; t \right) = \sum_{i \in S} \left\{ p_{i,t}^f - c_{i,t}^f \right\} \cdot u_{i,t}^f$$

$$- \sum_{i \in S} p_{i,t}^f \cdot d_{i,t}^f (p_{i,t}) \int_0^{u_{i,t}^f} F(\tau) d\tau$$

Under this condition (5.59)-(3.15) becomes a multi-product newsvendor pricing problem for firm $f$ with endogenously determined prices.

As mentioned before, while a firm’s price $p_l^f$ impacts on the revenues of all other firms, its allocation of resource, $u_t^f$ affects its own revenue only. It thus follows from (3.87) that $u_t^f$ can be calculated in closed form given a firm’s service price $p_l^f$, shadow price of resources $\lambda_l^f$ and competitors’ service prices $p_{-l}^t$. This observation permits us to reduce the noncooperative game in the $(p, u)$-space to one in which retailers compete with a single
instrument, the price $p$. We refer to this game as a reduced revenue management game.

Now we are in a position to explore further shape of the Hamiltonian. In particular, Lemma 3.2 establishes that the Hamiltonian $H_{f,t}$ is strictly concave in $u_{f,t}$. \[ \text{Lemma 3.2} \]

The Hamiltonian $H_f \left( p_i^f, u_i^f; \lambda^f; p_t^{-f}; t \right)$ is strictly concave in $u_i^f$.

**Proof.** To show $H_{f,t}$ is strictly concave in $u_{f,t}$, we need to establish

$$H_{f,t} \left( p_i^f, u_i^f; \lambda^f; p_t^{-f}; t \right) > \mu H_f \left( p_i^f, u_i^f; \lambda^f; p_t^{-f}; t \right) + (1 - \mu) H_f \left( p_i^f, u_i^f; \lambda^f; p_t^{-f}; t \right)$$

where $u^\mu = \mu u_1 + (1 - \mu) u_2$ with $\mu \in [0, 1]$ with is same as

$$\int_0^{d_{i,t}^f(p_i,t)} F(\tau) d\tau < \mu \int_0^{d_{i,t}^f(p_i,t)} F(\tau) d\tau + (1 - \mu) \int_0^{d_{i,t}^f(p_i,t)} F(\tau) d\tau$$

Therefore it will suffice if we can show that $\frac{u_{i,t}^f}{d_{i,t}^f(p_i,t)} F(\tau) d\tau$ is strictly convex in $u_{i,t}^f$ which is true as

$$\frac{\partial}{\partial u_{i,t}^f} \int_0^{u_{i,t}^f} F(\tau) d\tau = \frac{1}{d_{i,t}^f(p_i,t)} F \left( \frac{u_{i,t}^f}{d_{i,t}^f(p_i,t)} \right) > 0$$

which completes the proof. \[ \Box \]

From Lemma 3.2 and the maximum principle, we can obtain $u_{i,t}^f (p_i,t, \lambda^f)$ by equating the first order derivative $\frac{\partial H_{f,t}}{\partial u_{i,t}^f}$ to 0

$$u_{i,t}^f = d_{i,t}^f(p_i,t) \cdot F^{-1} \left( \max \left\{ 1 + A_i^T \cdot \lambda^f, 0 \right\} \right) \text{ for all } i \in S, t \in [0, N - 1] \quad (3.19)$$

where $F^{-1}(\cdot)$ is the inverse of the c.d.f.. Substituting the expression of $u_{i,t}^f$ from (3.19) into the Hamiltonian (3.12) and rearranging terms

$$H_f \left( p_i^f; \lambda^f; p_t^{-f}; t \right) = \sum_{i \in S} p_i^f \cdot d_{i,t}^f(p_i,t) \int_0^{F^{-1}(\mu_{i,t})} \tau f(\tau) d\tau \quad (3.20)$$

where $H_f (p^f; p_{-f}; \lambda^f; t)$ replaces $H_f (p^f; u^f; p_{-f}; \lambda^f; t)$ as $u_i^f$ is derived from $p_i^f, \lambda^f$ and $p_t^{-f}$ where $\mu_{i,t} = \max \left( 1 + A_i^T \cdot \lambda^f, 0 \right)$.

Now note that for $p^f_{i,\min} \leq p^f_{i,t} < (-A_i^T \cdot \lambda^f)$, the value
of the Hamiltonian is always 0. Therefore \( p_{i,t}^f \geq \max \left( p_{i,\text{min}}, -A^T \cdot \lambda^f \right) \) which gives rise to the situation where \( \max \left\{ 1 + \frac{A^T \cdot \lambda^f}{p_{i,t}^f}, 0 \right\} = 1 + \frac{A^T \cdot \lambda^f}{p_{i,t}^f} \); therefore, (3.19) can be expressed as

\[
u_{i,t}^f = d_{i,t}^f (p_{i,t}) \cdot F^{-1} \left( 1 + \frac{A^T \cdot \lambda^f}{p_{i,t}^f} \right)
\]

for all \( i \in S, t \in [0, N - 1] \) and correspondingly \( \mu_{i,t}^f = 1 + \frac{A^T \cdot \lambda^f}{p_{i,t}^f} \).

At this point we can draw some managerial insights from the simple allocation rule (3.19) as follows: let us assume \( z_{i,t}^f \) follows uniform distribution in \([0, 2]\). If \( \lambda_j^f = 0 \) for all \( j \in C_i \) (which implies that none of the resources that service type \( i \) utilizes are constrained), then from (3.19)

\[
u_{i,t}^f = d_{i,t}^f (p_{i,t}) \cdot F^{-1} (1) = 2 \cdot d_{i,t}^f (p_{i,t})
\]

In that case the amount of lost sales in period \( t \) is 0 which is true when firms have an infinite capacity of resources (hence \( \lambda^f \)'s are all 0). On the other hand, if at least one of the resources that are utilized to provide service \( i \) is constrained, it would have a negative shadow price \( (\lambda_j^f < 0 \text{ for some } j \in C_i) \). In that case,

\[
u_{i,t}^f = d_{i,t}^f (p_{i,t}) \cdot F^{-1} \left( p_{i,t}^f + A^T \cdot \lambda_{i+1}^f \right) / p_{i,t}^f = 2 \cdot d_{i,t}^f (p_{i,t})
\]

A fraction of the realized demand may be turned down at time \( t \) with probability \( \Pr (D_{i,t}^f > u_{i,t}^f) \).

Note that the state dynamics (3.139) can now be expressed in terms of \( p^f \) and \( \lambda^f \) as

\[
x_{i+1}^f = x_t^f + A \cdot u_t^f \quad \text{for all } t = 0, \ldots, N - 1
\]

\[
x_{i+1}^f = x_t^f + A \cdot d_{i,t}^f (p_{i,t}) \cdot F^{-1} \left( p_{i,t}^f + A^T \cdot \lambda_{i+1}^f \right) / p_{i,t}^f \quad \text{for all } t = 0, \ldots, N - 1
\]

with the initial condition \( x_0^f = 0 \) we get the following

\[
x_{j,t}^f = \sum_{k=0}^{t-1} \sum_{i \in S} a_{ij} \cdot d_{i,k}^f (p_{i,k}) \cdot F^{-1} \left( p_{i,t}^f + A^T \cdot \lambda_{i+1}^f \right) / p_{i,t}^f \quad \text{for all } j \in C, t = 1, \ldots, N \quad (3.21)
\]
At the end of booking period the total allocation of resource type \( j \) for firm \( f \) is
\[
x_{f,j,N}^j = \sum_{k=0}^{N-1} \sum_{i \in S} \left\{ a_{ij} \cdot d_{i,k}^f(p_{i,k}) \cdot F^{-1} \left( \frac{p_{i,k}^f + A^T \cdot \lambda^f}{p_{i,k}^f} \right) \right\} \text{ for all } j \in \mathcal{C} \tag{3.22}
\]
where the known parameters are \( a_{ij} \)'s, which is nothing but an element of the matrix \( A \), and the unknown variables are \( \lambda^f \) and \( p_{i,k}^f \) (and hence \( d_{i,k}^f(p_{i,k}) \)). When expression (3.22) is used to compute expected overbooking and refund costs (3.3), we can prove that the expected cost increases monotonically with shadow price \( \lambda^f \) as established below.

**Lemma 3.3** When the survival probabilities \( \alpha^f_j < 1 \), the expected refunds and overbooking cost, \( OBC_f \) is (a) monotonically increasing in shadow prices \( \lambda^f \) for given \( p^f \) and \( p^{-f} \); and (b) quasiconcave in \( p_{i,t}^f \) for a given \( \lambda^f \)

**Proof.** Part (a) : In Lemma 3.1 we have already established
\[
\frac{\partial OBC_j}{\partial x_{j,N}^f} > 0
\]
If it can be shown that \( \frac{\partial x_{j,N}^f}{\partial \lambda^f_l} > 0 \) for all \( i \in \mathcal{C} \), this will imply
\[
\frac{\partial OBC_j}{\partial \lambda^f_l} = \frac{\partial OBC_j}{\partial x_{j,N}^f} \cdot \frac{\partial x_{j,N}^f}{\partial \lambda^f_l} > 0 \text{ for all } l \in \mathcal{C}
\]
Differentiating \( x_{j,N}^f \) w.r.t. \( \lambda^f_l \) using (3.22)
\[
\frac{\partial x_{j,N}^f}{\partial \lambda^f_l} = \sum_{k=0}^{N-1} \sum_{i \in S} \left\{ a_{ij} \cdot \frac{d_{i,k}^f(p_{i,k})}{f \left( \frac{p_{i,k}^f + A^T \cdot \lambda^f}{p_{i,k}^f} \right)} \cdot \frac{1}{p_{i,k}^f} \right\} > 0
\]
The last inequality is obtained as the pdf \( f(\cdot) \geq 0 \), \( a_{ij} \) is either 0 or 1. This concludes the first part of the proof.

Part (b) : Differentiating \( x_{j,N}^f \) w.r.t. \( p_{i,t}^f \) using (3.22)
\[
\frac{\partial x_{j,N}^f}{\partial p_{i,t}^f} = a_{ij} \left[ \frac{\partial d_{i,k}^f(p_{i,k})}{\partial p_{i,t}^f} F^{-1} \left( 1 + \frac{A^T \cdot \lambda^f}{p_{i,t}^f} \right) - \frac{d_{i,t}^f}{f \left( F^{-1} \left( 1 + \frac{A^T \cdot \lambda^f}{p_{i,t}^f} \right) \right)} \left( \frac{A^T \cdot \lambda^f}{p_{i,t}^f} \right)^2 \right] \tag{3.23}
\]

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where \( a_{ij} = 1 \) if service \( i \) utilizes resource \( j \) and 0 otherwise. Let \( F^{-1} \left( 1 + \frac{A^T \lambda^f}{p^f_{i,t}} \right) = y \) which implies \( F(y) = 1 + \frac{A^T \lambda^f}{p^f_{i,t}} \), and
\[
\frac{dy}{dp^f_{i,t}} = -\frac{1}{f(y)} \frac{A^T \lambda^f}{(p^f_{i,t})^2} \geq 0
\]
as \( \lambda^f \leq 0 \). Rewriting (3.23), we obtain
\[
\frac{\partial x^f_{j,N}}{\partial p^f_{i,t}} = a_{ij} \frac{d^f_{i,t}}{p^f_{i,t}} \left[ -e^f_{i,t} \cdot y + \frac{1 - F(y)}{f(y)} \right]
\] (3.24)
Using the definition of generalized failure rate \( \beta(y) = \frac{yf(y)}{1-F(y)} \), (3.24) can be further simplified as
\[
\frac{\partial x^f_{j,N}}{\partial p^f_{i,t}} = a_{ij} \frac{d^f_{i,t}}{p^f_{i,t}} y \left[ -e^f_{i,t} + \frac{1}{\beta(y)} \right]
\]
Therefore,
\[
\frac{\partial OBC_j}{\partial p^f_{i,t}} = \sum_{j \in C} \frac{\partial OBC_j}{\partial x^f_{j,N}} \cdot \frac{\partial x^f_{j,N}}{\partial p^f_{i,t}}
\]
\[
= \sum_{j \in C} a_{ij} \frac{\partial OBC_j}{\partial x^f_{j,N}} \cdot \frac{d^f_{i,t}}{p^f_{i,t}} y \left[ -e^f_{i,t} + \frac{1}{\beta(y)} \right]
\]
From Lemma 3.1 we have \( \frac{\partial OBC_j}{\partial x^f_{j,N}} > 0 \) and from item 5 of Assumption A2, \( e^f_{i,t} \) is increasing in \( p^f_{i,t} \). Finally, from Assumption A1 \( \frac{d\beta(y)}{dy} > 0 \) thus
\[
\frac{d}{dp^f_{i,t}} \left( \frac{1}{\beta(y)} \right) = -\frac{1}{\beta^2(y)} \cdot \frac{d\beta(y)}{dy} \cdot \frac{dy}{dp^f_{i,t}} < 0
\] (3.25)
Hence the proof.

We are now going to establish some comparative statics results related to the Hamiltonian (3.20). The following definitions and theorems are needed to establish the results (see Topkis [51] for detailed discussion on supermodularity and strategic complementarities in games).

**Definition 3.1** A function \( f(a_1, a_2, \ldots, a_M) \) has an increasing difference in \((a_i, a_j)\) if
\[
f(a_1, \ldots, a^1_i, \ldots, a_M) - f(a_1, \ldots, a^2_i, \ldots, a_M)
\]
is increasing in \( a_j \) for all \( a^1_i > a^2_i \).
**Definition 3.2** If the function $f$ is twice differentiable, then Definition 5.25 is equivalent to $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$.

**Definition 3.3** A function $f : S \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is supermodular on $S$ if and only if $f$ has increasing differences on $S$.

**Definition 3.4** A function $f : S \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is submodular on $S$ if $-f$ is supermodular.

Increasing differences is interpreted as formalizing the notion of (Edgeworth) complementarity. Having more of one variable increases the marginal returns to having more of the other variable. It turns out that some form of complementarity between endogenous and exogenous variables lies at the heart of any monotone comparative statics conclusion; see Amir [52] for a detailed discussion on applications of supermodularity and complementarity in Economics.

We are now in a position to establish that the Hamiltonian is supermodular in service price, $p$ for a given shadow price.

**Lemma 3.4** When item 7 of Assumption A2 holds, the Hamiltonian $H^i_{f,t}$ is supermodular in service price $p$ for a given $\lambda^f$.

**Proof.** We observe that the Hamiltonian is separable, i.e., $H_{f,t} = \sum_{i \in S} H^i_{f,t}$ where

$$H^i_{f,t} = p^f_{i,t} \cdot d^f_{i,t}(p_{i,t}) \int_0^{F^{-1}(\mu^f_{i,t})} \tau f(\tau) \, d\tau$$

and $H^i_{f,t}$ depends only on $p^f_{i,t}$ for given non-own prices $p^{-f}_{i,t}$ and own shadow prices $\lambda^f$ of resources. Thus

$$\frac{\partial H_{f,t}}{\partial p^f_{i,t}} = \frac{\partial H^i_{f,t}}{\partial p^f_{i,t}}$$
Taking partials of $H_{f,t}$ w.r.t. $p_{i,t}^f$ for some $i \in S$ we obtain

$$
\frac{\partial H_{f,t}^i}{\partial p_{i,t}^f} = \left( p_{i,t}^f \frac{\partial d_{i,t}^f}{\partial p_{i,t}^f} + d_{i,t}^f \right) \cdot \int_0^{F^{-1}(\mu_{i,t}^f)} \tau f(\tau) d\tau - d_{i,t}^f \frac{A_i^T \cdot \lambda_{fi}^f}{p_i^f} \cdot F^{-1}(\mu_{i,t}^f)
$$

$$
= d_{i,t}^f \cdot \int_0^{F^{-1}(\mu_{i,t}^f)} \tau f(\tau) d\tau \left[ (-e_{i,t}^f + 1) - \frac{A_i^T \cdot \lambda_{fi}^f}{p_i^f} \right] \frac{F^{-1}(\mu_{i,t}^f)}{\int_0^{F^{-1}(\mu_{i,t}^f)} \tau f(\tau) d\tau}
$$

$$
= \frac{H_{f,t}^i}{p_{i,t}^f} \left[ (-e_{i,t}^f + 1) - \frac{A_i^T \cdot \lambda_{fi}^f}{p_i^f} \right] \frac{F^{-1}(\mu_{i,t}^f)}{\int_0^{F^{-1}(\mu_{i,t}^f)} \tau f(\tau) d\tau}
$$

(3.26)

where $e_{i,t}^f$ is the local price elasticity. Next, we observe

$$
\frac{\partial^2 H_{f,t}^i}{\partial p_{i,t}^f \partial p_{j,t}^g} = \frac{\partial^2 H_{f,t}^i}{\partial p_{i,t}^f \partial p_{j,t}^j} = 0 \text{ for all } j \neq i \quad (3.27)
$$

$$
\frac{\partial^2 H_{f,t}^i}{\partial p_{i,t}^f \partial p_{j,t}^g} = \frac{\partial^2 H_{f,t}^i}{\partial p_{i,t}^f \partial p_{j,t}^j} = 0 \text{ for all } j \neq i, g \neq f \quad (3.28)
$$

and

$$
\frac{\partial^2 H_{f,t}^i}{\partial p_{i,t}^f \partial p_{j,t}^g} \geq 0 \text{ (from item 7 of assumption A2)}
$$

Hence the proof. 

We also wish to determine the direction of change of an equilibrium point as exogenous parameters (shadow price of the resources) change. Amir’s [52] Theorem 7, stated below, is useful for establishing the result.

**Theorem 3.2** Suppose that the payoff of player $i \in I$ is given by $F_i(a^i, a^{-i}, s)$ where $s \in S$ is a parameter. Assume that

(i) for each $s \in S \subset \mathbb{R}$ the game is supermodular

(ii) $F_i$ has increasing differences in $(a^i, s)$ for each $a^{-i}$

then the extremal equilibria of the game are increasing functions of $s$. 

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Proof. See Theorem 7 of Amir [52].

We can establish a similar result in our case under a mild condition as stated and proved in Lemma 3.5. Recall that \( c^f_i \equiv -\left(\lambda^f \right)^T A_i \) is the ‘shadow’ price of providing per unit service type \( i \in S \) for firm \( f \).

**Lemma 3.5** If
\[
\left. e^f_{i,t} \right|_{p^f_{i,t}=p^f_{i,\min}} \geq \frac{1}{\beta \left( F^{-1} \left( 1 + \frac{4^{f} \lambda^f}{p^f_{i,\min}} \right) \right)} \quad \text{for all } \lambda^f \leq 0,
\]
then the extremal equilibria of the revenue optimization game are increasing functions of service shadow prices, \( c^f \).

**Proof.** We have seen in Lemma 3.4 that for a given \( \lambda^f \) (thus \( c^f \)) the game is supermodular. In addition, if we are able to show that \( H_{f,t} \) has increasing differences in \( (p^f_{i,t}, c^f_i) \) for each \( p^f_{i,t} \), we can use Theorem 3.2 to establish that the extremal equilibria of the revenue optimization game are increasing functions of shadow price of resources, \( \lambda^f \). Differentiating \( H_{f,t} \) by \( c^f_i \) we obtain
\[
\frac{\partial H_{f,t}}{\partial c^f_i} = -d^f_{i,t} \cdot F^{-1} \left( 1 - \frac{c^f_i}{p^f_{i,t}} \right)
\]
(3.29)
Differentiating again (3.29) w.r.t. service prices we observe \( \frac{\partial^2 H_{f,t}}{\partial c^f_i \partial p^f_{i,t}} = 0 \) if \( j \neq i \) and
\[
\frac{\partial^2 H_{f,t}}{\partial c^f_i \partial p^f_{i,t}} = -\frac{\partial d^f_{i,t}}{\partial p^f_{i,t}} \cdot F^{-1} \left( 1 - \frac{c^f_i}{p^f_{i,t}} \right) - \frac{d^f_{i,t}}{f \left( F^{-1} \left( 1 - \frac{c^f_i}{p^f_{i,t}} \right) \right)} \cdot \left( \frac{c^f_i}{p^f_{i,t}} \right)^2
\]
(3.30)
\[
= \frac{d^f_{i,t}}{p^f_{i,t}} \left[ e^f_{i,t} \cdot F^{-1} \left( 1 - \frac{c^f_i}{p^f_{i,t}} \right) - \frac{1}{f \left( F^{-1} \left( 1 - \frac{c^f_i}{p^f_{i,t}} \right) \right)} \cdot \frac{c^f_i}{p^f_{i,t}} \right]
\]
Let \( F^{-1} \left( 1 - \frac{c^f_i}{p^f_{i,t}} \right) = y \) which implies \( \frac{c^f_i}{p^f_{i,t}} = 1 - F(y) \), and
\[
\frac{dy}{dp^f_{i,t}} = + \frac{1}{f(y)} \frac{c^f_i}{(p^f_{i,t})^2} \geq 0
\]
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as \( c_{i}^f = -\left(\lambda^f\right)^T A_i \geq 0 \). Rewriting (3.30) we obtain
\[
\frac{\partial^2 H_{f,t}}{\partial c_{i} \partial p_{i,t}^f} = \frac{d_{i,t}^f}{p_{i,t}^f} \left[ e_{i,t}^f \cdot y - \frac{1 - F(y)}{f(y)} \right] = \frac{d_{i,t}^f}{p_{i,t}^f} y \left[ e_{i,t}^f - \frac{1}{\beta(y)} \right]
\]
Now, we know \( y \geq 0 \), \( e_{i,t}^f \) is increasing in \( p_{i,t}^f \) (item 5 of assumption A2) and \( \frac{1}{\beta(y)} \) is strictly decreasing in \( p_{i,t}^f \) from (3.25). Further, since from the condition \( e_{i,t}^f \left|_{p_{i,t}^f=p_{i,\min}^f} \right. \geq \frac{1}{\beta \left( F^{-1} \left( \frac{1 + A_i^T \lambda^f}{p_{i,\min}^f} \right) \right)} \) we know \( \frac{\partial^2 H_{f,t}}{\partial c_{i} \partial p_{i,t}^f} \bigg|_{p_{i,t}^f=p_{i,\min}^f} \geq 0 \) and more over
\[
\frac{\partial^2 H_{f,t}}{\partial c_{i} \partial p_{i,t}^f} \geq 0 \text{ for all } p_{i,\min}^f \leq p_{i,t}^f \leq p_{i,\max}^f \text{ and } c_{i}^f \geq 0
\]
Therefore, condition (ii) of Theorem 3.2 is satisfied as well, hence the extremal equilibria of the game are increasing functions of \( c^f \).

Now, we are in a position to state and prove the following result regarding the shape of the Hamiltonian.

**Lemma 3.6** If the realized demand satisfies Assumption A1 and items 1-5 of Assumption A2, for any given \( p_t^f \) and \( \lambda^f \), the Hamiltonian \( H_{f,t} \) in (3.20) is quasi-concave in \( p_t^f \) and has a unique maximizer \( p_t^f \left( \mu_t^f \right) \) for all \( t \in [0, N] \)

**Proof.** Since the Hamiltonian is separable and from (3.26)
\[
\frac{\partial H_{f,t}^i}{\partial p_{i,t}^f} = \frac{H_{f,t}^i}{p_{i,t}^f} \left[ -e_{i,t}^f + 1 - \frac{A_i^T \lambda^f}{p_i^f} \cdot \frac{F^{-1} \left( \mu_{i,t}^f \right) \tau f(\tau) d\tau}{F^{-1} \left( \mu_{i,t}^f \right) \tau f(\tau) d\tau} \right]
\]
where \( e_{i,t}^f \) is the local price elasticity. Let us define
\[
\psi \left( p_{i,t}^f, p_t^f - \lambda^f \right) = -\frac{A_i^T \lambda^f}{p_i^f} \cdot \frac{F^{-1} \left( \mu_{i,t}^f \right) \tau f(\tau) d\tau}{F^{-1} \left( \mu_{i,t}^f \right) \tau f(\tau) d\tau}
\]

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Now \( \frac{\partial H_{f,t}}{\partial p_{i,t}^-} = 0 \) is equivalent to \( e_{i,t}^f = 1 + \psi \left( p_{i,t}^f, p_{i,t}^f; \lambda^f \right) \). We know \( e_{i,t}^f \) is increasing in \( p_{i,t}^f \) (from item 5 of assumption A2), therefore if, in addition, we can show that \( \psi \left( p_{i,t}^f, p_{i,t}^f; \lambda^f \right) \) is strictly decreasing in \( p_{i,t}^f \), then \( e_{i,t}^f = 1 + \psi \left( p_{i,t}^f, p_{i,t}^f; \lambda^f \right) \) can have at most one solution.

We will also then be able to show that for given \( p_{i,t}^f \) and \( \lambda^f \), there exists some \( p_{i,t}^{f*} \) such that \( H_{f,t} \) is nondecreasing for \( p_{i,t}^f < p_{i,t}^{f*} \) and nonincreasing for \( p_{i,t}^f > p_{i,t}^{f*} \), hence quasi-concave. Let \( y = A^{-1} \left( \mu_{i,t}^f \right) \), then \( -\frac{A^T \cdot \lambda^f}{p_{i,t}^f} = 1 - F(y) \) and \( \frac{dy}{dp_{i,t}^f} > 0 \) for \( A^T \cdot \lambda^f < 0 \). Taking the derivative of \( \psi \left( p_{i,t}^f, p_{i,t}^f; \lambda^f \right) \) w.r.t. \( p_{i,t}^f \)

\[
\frac{\partial \psi \left( p_{i,t}^f, p_{i,t}^f; \lambda^f \right)}{\partial p_{i,t}^f} = \left[ (1 - \beta(y)) \cdot \int_0^y \tau f(\tau) \, d\tau - y^2 f(y) \right] \frac{1 - F(y)}{[\int_0^y \tau f(\tau) \, d\tau]^2} \cdot \frac{dy}{dp_{i,t}^f} \tag{3.31}
\]

where \( \beta(y) \) is a generalized failure rate. Now consider

\[
\delta(y) = (1 - \beta(y)) \cdot \int_0^y \tau f(\tau) \, d\tau - y^2 f(y)
\]

where it is evident that \( \delta(0) = 0 \). If we can establish that \( \delta'(y) < 0 \) for all \( y > 0 \), then we will be able to establish that \( \delta(y) < 0 \) for all \( y > 0 \), which will also ensure that \( \frac{\partial \psi \left( p_{i,t}^f, p_{i,t}^f; \lambda^f \right)}{\partial p_{i,t}^f} < 0 \) i.e., \( \psi(\cdot) \) is decreasing in \( p_{i,t}^f \). Re-arranging terms we arrive

\[
\delta(y) = (1 - \beta(y)) \cdot \int_0^y \tau f(\tau) \, d\tau - y^2 f(y) \tag{3.32}
\]

where we have used the identity \( \beta(y) [1 - F(y)] = y f(y) \). Taking the derivative of \( \delta(y) \) w.r.t. \( y \)

\[
\delta'(y) = -\beta'(y) \cdot \int_0^y \tau f(\tau) \, d\tau - y^2 f(y) \cdot [1 - F(y)]
\]

here once again we have utilized the identity. Since \( \beta'(y) > 0 \) from IGFR assumption, \( \delta'(y) < 0 \) for all \( y > 0 \) which completes the proof. ■

**Best Response Problem for Firm** \( f \)

From the maximum principle, we have the following best response problem that each firm \( f \) seeks to solve at time \( t \in [0, N - 1] \) with given \( p_{-f}^t \)

\[
\max H_f \left( p_f^t; \lambda^f; p_{-f}^t; t \right)
\]

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subject to
\[
p_{\text{min}}^f \leq p_t^f \leq p_{\text{max}}^f
\]
where \(\lambda^f\) is obtained from (3.17). Using Lemma 3.6, the necessary and sufficient condition for \(p^*_t \left( p_t^-; \lambda^f \right)\) to be a solution of the best response problem can be expressed as the following variational inequality (VI): find \(p^*_t \in \Lambda_f\) such that
\[
\left[ \nabla_{p_t} H_f \left( p_t^*_f; \lambda^f; p_t^-; t \right) \right]^T \cdot \left( p_t^f - p_t^* \right) \leq 0
\]
for all \(p_t^f \in \Lambda_f\) where
\[
K_f = \left\{ p_t^f : p_{\text{min}}^f \leq p_t^f \leq p_{\text{max}}^f \right\}
\]
(3.34)

3.2.11 Market Equilibrium Problem as a Variational Inequality

With the preceding background, we can now formulate the market equilibrium problem using a VI formulation. We combine the variational inequalities (3.33) for each firm \(f \in \mathcal{F}\) and time \(t \in [0, N - 1]\). We define the following feasible space for all service providers.
\[
K = \prod_{f \in \mathcal{F}} K_f
\]
(3.35)
The VI of interest seeks to find a point \(p^* \in K\) such that
\[
\left( \begin{array}{c}
\nabla_{p^1} H_1 \left( p^{1*}; \lambda^{1*}; p^{-1*} \right) \\
\vdots \\
\nabla_{p^{|\mathcal{F}|}} H_{|\mathcal{F}|} \left( p^{|\mathcal{F}|*}; \lambda^{|\mathcal{F}|*}; p^{-|\mathcal{F}|*} \right)
\end{array} \right)^T \cdot \left( \begin{array}{c}
p^{1*} - p^{1*} \\
\vdots \\
p^{|\mathcal{F}|*} - p^{|\mathcal{F}|*}
\end{array} \right) \leq 0
\]
(3.36)
for all \(p \in K\) where
\[
H_f \left( p^f; \lambda^f; p^- \right) = \left\{ H_f \left( p_t^f; \lambda^f; p_t^-; t \right) : t \in [0, N - 1] \right\}
\]
The equilibrium allocation quantities can be computed from
\[
u^*_{i,t} = \frac{d^f_{i,t}}{p^*_t} \cdot F^{-1} \left( 1 + \frac{A^T \cdot \lambda^*}{p^*_t} \right)
\]
for all \(i \in \mathcal{S}, f \in \mathcal{F}\) and \(t \in [0, N - 1]\)
The issue of immediate concern is to formally demonstrate that solutions of (3.36) are Cournot-Nash equilibria of our revenue optimization game. In fact we state and prove the following result which establishes that solving the variational inequality formulation (3.36) is equivalent to solving the market equilibrium problem. This also allows us to also establish existence of a market equilibrium policy.

**Theorem 3.3** The policy arising from the joint variational inequality problem (3.36) and the policy arising from the simultaneous solution of the variational inequalities (3.33) for each firm $f$ are the same.

**Proof.** It is relatively straightforward to show that a policy $p^*$ that solves the variational inequality problem (3.33) for each firm $f \in F$ simultaneously, also solves the joint variational inequality problem (3.36). We will now show the converse, i.e., the solution to joint variational inequality problem (3.36) solves variational inequality problems (3.33) for each firm $f$ simultaneously. That is, if $p^*$ is a solution to joint VI problem (3.36), then for each firm $f \in F$, $p^f$ solves the variational inequality problem (3.33) with competitors’ policies $p^{-f}$ given by $p^{-f^*}$. Own shadow price is computed by solving the equation

$$\lambda^f = -\nabla_{x_N} OBC_f \left(p^{f^*}, p^{-f^*}, \lambda^{f^*} \right)$$

Note that (3.36) is equivalent to the following fictitious mathematical program

$$\max G(p) = \sum_{t=0}^{N-1} \sum_{f \in F} \sum_{i \in S} \frac{\partial H_f \left(p^{f^*}, \lambda^{f^*}; p_t^{-f^*} \right)}{\partial p_{i,t}} p_{i,t}^f \quad (3.37)$$

subject to

$$p_{\text{min}}^f \leq p_t^f \leq p_{\text{max}}^f \quad \text{for all } f \in F, t \in [0, N - 1] \quad (3.38)$$

$$\lambda^f = -\nabla_{x_N} OBC_f \quad \text{for all } f \in F \quad (3.39)$$

where it is essential to recognize that $G(p)$ is a linear functional that assumes knowledge of the solution of (3.36); as such $G(p)$ is a mathematical construct for use in analysis and has no meaning as a computational device. The corresponding necessary and sufficient
conditions for this mathematical program are identical to (3.33) for all \( f \in \mathcal{F} \) as because

\[
\frac{\partial G^*}{\partial p_{i,t}^f} = \frac{\partial H_f \left( p_t^{f^*}; \lambda_t^{f^*}; p_t^{-f^*} \right)}{\partial p_{i,t}^f}
\]

where

\[
G^* = G(p^*) = \sum_{t=0}^{N-1} \sum_{f \in \mathcal{F}} \sum_{i \in \mathcal{S}} \frac{\partial H_f \left( p_t^{f^*}; \lambda_t^{f^*}; p_t^{-f^*} \right)}{\partial p_{i,t}^f} p_{i,t}^{f^*}
\]

hence the proof. \( \blacksquare \)

Existence

We are only interested in pure strategy open-loop Nash equilibrium\(^3\). We note that the following existence result holds:

**Theorem 3.4** There exists at least one equilibrium of the network revenue optimization competition game.

**Proof.** We need to establish that there exists at least one solution of the VI (3.36). Since any solution of (3.36) is a Nash equilibrium of the game (as per Theorem 3.3), then that solution will also be a Nash equilibrium of the game. Note that the strategy space of each firms’ pricing decision for each service is a closed interval, hence \( p \) is a non-empty, compact and convex set of \( \mathbb{R}^{|\mathcal{F}| \times |\mathcal{S}| \times (N-1)} \). Further, \( \left( \nabla_{p_i^f} H_1 \cdots \nabla_{p_i^{|\mathcal{F}|}} H_{|\mathcal{F}|} \right)^T \) is a continuous mapping from \( \mathcal{K} \) into \( \mathbb{R}^{|\mathcal{F}| \times |\mathcal{S}| \times (N-1)} \). Therefore, invoking Theorem 3.1 of Harker and Pang [16] we establish that there exists a solution of (3.36), hence the proof. \( \blacksquare \)

Uniqueness

Before providing conditions for a unique Nash equilibrium, we introduce the following definitions

\(^3\)A pure strategy is a strategy that has no randomly determined choices.
Definition 3.5  The mapping \( \Theta : \mathbb{R}^n \to \mathbb{R}^n \) is said to be strictly monotone over a set \( A \) if

\[
[\Theta(a) - \Theta(b)]^T (a - b) > 0 \text{ for all } a, b \in A
\]

Definition 3.6  If \( \Theta \) is continuously differentiable, \( \Theta : \mathbb{R}^n \to \mathbb{R}^n \) is strictly monotone over set \( A \) if the Jacobian matrix \( \nabla \Theta(a) \) is positive definite for all \( a \in A \).

We have already established existence of at least solution of the VI in Theorem 3.4, in addition, if the principal function of VI, \( \left( \begin{array}{ccc}
\nabla p_1 H_1 & \cdots & \nabla p_{|F|} H_{|F|}
\end{array} \right)^T \), is strictly monotone on \( K \), the VI (3.36) has at most one solution (see Proposition 3.2 of Harker and Pang [16]). This condition is not easily verifiable for a realistic size problem. Fortunately, a lesser restrictive condition exists under which a given solution of the VI (3.36) is isolated in the sense that there exists a neighborhood of the solution with in which no other solution to the solution to the problem exists. The condition for a solution \( p^* \) being locally unique (or isolated) is that the Jacobian matrix \( \left( \begin{array}{ccc}
\nabla p_1 H_1 & \cdots & \nabla p_{|F|} H_{|F|}
\end{array} \right)^T \) evaluated at \( p^* \) is positive definite (Proposition 3.3 of Harker and Pang [16]). Such condition is easily verifiable for a given solution.

3.2.12 Cooperative Equilibrium

We are interested in comparing pricing and allocation strategies of the firms in the event where all firms seek to maximize their aggregate expected revenue throughout the booking period. The single discrete time optimal control problem that the firms collectively seek to solve is the following:

\[
\max_{p, u} J_C (p, u) = \sum_{f \in F} \left\{ -OBC_f + E \left[ \sum_{t=0}^{N-1} p_t f \cdot \min \left( u_t f, D_t^f (p_t) \right) \right] \right\}
\]  

(3.40)
subject to

\[ x_{t+1}^f = x_t^f + A \cdot u_t^f \quad \text{for all } f \in F, \quad t = 0, \ldots, N - 1 \]  
\[ x_0^f = 0 \quad \text{for all } f \in F \]  
\[ p_{t_{\text{min}}}^f \leq p_t^f \leq p_{t_{\text{max}}}^f \quad \text{for all } f \in F, \quad t = 0, \ldots, N - 1 \]  
\[ u_t^f \geq 0 \quad \text{for all } f \in F, \quad t = 0, \ldots, N - 1 \]

From Proposition 3.1, the expected single period revenue is

\[ R_{f,t}(p_t^f, u_t^f, t) = \sum_{i \in S} \left( p_{i,t}^f \cdot u_{i,t}^f - p_{i,t}^f \cdot d_{i,t}^f (p_{i,t}) \int_0^{u_{i,t}^f} F(\tau) \ d\tau \right) \]

In the event of complete collusion, the expected industry-wide single period revenue is

\[ R_t(p_t, u_t) = \sum_{f \in F} R_{f,t}(p_t^f, u_t^f, t) \]

\[ = \sum_{f \in F} \sum_{i \in S} \left( p_{i,t}^f \cdot u_{i,t}^f - p_{i,t}^f \cdot d_{i,t}^f (p_{i,t}) \int_0^{u_{i,t}^f} F(\tau) \ d\tau \right) \]

Therefore the Hamiltonian of the cooperative optimal control problem is

\[ H_t^c(p_t, u_t, \lambda, t) = \sum_{f \in F} \sum_{i \in S} \left[ p_{i,t}^f \cdot u_{i,t}^f + \left( \lambda^f \right)^T A_i \cdot u_{i,t}^f - \sum_{f \in F} \sum_{i \in S} p_{i,t}^f \cdot d_{i,t}^f (p_i) \int_0^{u_{i,t}^f} F(\tau) \ d\tau \right] \]

Let \((\bar{p}, \bar{u})\) be the cooperative service prices and resource allocations respectively for the firms, which are the solutions of the optimal control problem (3.95)-(3.100), where

\[ \bar{p} = \left( \bar{p}_f^i \right)_{f \in F}, \quad \bar{u} = \left( \bar{u}_f^i \right)_{f \in F} \]

Also let the cooperative shadow price of resources be

\[ \bar{\lambda} = \left( \bar{\lambda}_f^i \right)_{f \in F} \]

The analysis of this optimal control problem follows exactly the same path of the non-cooperative case, hence we suppress details. In fact it can be easily verified that \( H_t^1 \) is concave in \( u_{i,t}^f \) and as such the allocation rules remain unchanged, of course, the equilibrium
values of allocation will change with cooperative service price and shadow price of resources. As in the non-cooperative case, we need to solve the following static optimization problem at each time period $t \in [0, N - 1]$ arising from the maximum principle for the optimal control problem (3.95)-(3.100):

$$\max_{p_{\min} \leq p_t \leq p_{\max}} \hat{H}_t^c (p_t, \lambda, t)$$

where $\lambda$ follows the same transversality conditions (3.17) and

$$\hat{H}_t^c (p_t, \lambda, t) = \sum_{f \in \mathcal{F}} \sum_{i \in \mathcal{S}} p_{i,t}^f \cdot d_{i,t}^f (p_{i,t}) \int_0^{F^{-1} \left( \mu_{i,t}^f \right)} \tau f (\tau) d\tau$$

$$= \sum_{f \in \mathcal{F}} H_t^f \left( p_t^f, \lambda^f, p_t^{-f}, t \right)$$  (3.45)

The cooperative service prices can be computed by solving the following VI:

$$\text{find } \tilde{p} \in \mathcal{K} \text{ such that}$$

$$\sum_{t=0}^{N-1} \left[ \nabla_{p_t} \hat{H}_t^c \left( \tilde{p}_t, \tilde{\lambda}_t, t \right) \right]^T \cdot (p_t - \tilde{p}_t) \leq 0 \quad (3.46)$$

for all $p \in \mathcal{K}$

where $\mathcal{K}$ is defined in (3.34) and (3.35).

The next result quantifies the gap between cooperative and non-cooperative services. To differentiate between the non-cooperative and cooperative equilibrium, we will denote non-cooperative equilibrium service price, resource allocation and shadow price of resources by $p^*, u^*$ and $\lambda^*$ respectively.

**Theorem 3.5** *Competition leads to lower equilibrium prices, i.e., $\tilde{p} \geq p^*$*

**Proof.** To establish the claim, we should be able to establish

$$\frac{\partial \hat{H}_t^c (p_t^*, \lambda^*, t)}{\partial p_{i,t}^f} \geq 0 \text{ for all } i \in \mathcal{S}, f \in \mathcal{F}, t \in [0, N - 1]$$  (3.47)
and
\[ \frac{\partial H^f_t(p^f_{i,t}, \tilde{\lambda}^f_{i,t}, \tilde{p}^{-f}_{i,t}, t)}{\partial p^f_{i,t}} \leq 0 \text{ for all } i \in S, f \in F, t \in [0, N - 1] \quad (3.48) \]

To establish (3.47), we will consider 2 cases: (a) \( p^f_{i,t} = p^f_{i,\min} \) and (b) \( p^f_{i,t} > p^f_{i,\min} \).

Case (a): When \( p^f_{i,t} = p^f_{i,\min} \), since cooperative equilibrium has also the same bounds on service prices, \( \hat{p}^f_{i,t} \geq p^f_{i,\min} \).

Case (b): On the other hand, when \( p^f_{i,t} > p^f_{i,\min} \), we need to establish (3.47). From (3.45)
\[ \frac{\partial \tilde{H}^c_t(p^*_t, \lambda^*, t)}{\partial p^f_{i,t}} = \frac{\partial H^f_t(p^f_{i,t}, \lambda^f_{i,t}, \tilde{p}^{-f}_{i,t}, t)}{\partial p^f_{i,t}} + \sum_{g \neq f} \frac{\partial H^g_t(p^g_{i,t}, \lambda^g_{i,t}, p^{-g}_{i,t}, t)}{\partial p^f_{i,t}} \]

since \( p^f_{i,t} > p^f_{i,\min} \) therefore \( \frac{\partial H^f_t(p^f_{i,t}, \lambda^f_{i,t}, \tilde{p}^{-f}_{i,t}, t)}{\partial p^f_{i,t}} \geq 0 \) (if \( p^f_{i,t} \) is a strictly interior point, the partial will be equal to 0 and strictly positive if \( p^f_{i,t} = p^f_{i,\max} \)). Further,
\[ \frac{\partial H^g_t(p^g_{i,t}, \lambda^g_{i,t}, p^{-g}_{i,t}, t)}{\partial p^f_{i,t}} = p^g_{i,t} \cdot \frac{\partial d^g_{i,t}}{\partial p^f_{i,t}} \cdot \int_0^{F^{-1}(m^g_{i,t})} \tau f(\tau) d\tau \geq 0 \text{ (from item 6 of assumption A2)} \quad (3.49) \]

therefore
\[ \frac{\partial \tilde{H}^c_t(p^*_t, \lambda^*, t)}{\partial p^f_{i,t}} \geq 0 \]

To establish (3.48), we once again consider 2 cases: (c) \( \hat{p}^f_{i,t} = p^f_{i,\max} \) and (d) \( \hat{p}^f_{i,t} < p^f_{i,\max} \).

Case (c): When \( \hat{p}^f_{i,t} = p^f_{i,\max} \), automatically the non-cooperative equilibrium cannot be greater than \( \hat{p}^f_{i,t} \).

Case (d): We need to establish (3.48) when \( \hat{p}^f_{i,t} < p^f_{i,\max} \). At this cooperative equilibrium point
\[ \frac{\partial \tilde{H}^c_t(p^*_t, \lambda^*_t)}{\partial p^f_{i,t}} \leq 0 \]

In particular, if \( \hat{p}^f_{i,t} \) is a strictly interior point, the partial will be equal to 0 and strictly
positive if \( \bar{p}^f_{i,t} = p^f_{i,\min} \). Using (3.45)

\[
\frac{\partial H^f_i (\bar{p}^f_{i,t}, \bar{\lambda}^f, \bar{p}^{-f}_{i,t}, t)}{\partial p^f_{i,t}} + \sum_{g \neq f} \frac{\partial H^g_i (\bar{p}^g_{i,t}, \bar{\lambda}^g, \bar{p}^{-g}_{i,t}, t)}{\partial p^f_{i,t}} \leq 0
\]

From (3.49) we know every term inside the summation are non-negative; thus

\[
\frac{\partial H^f_i (\bar{p}^f_{i,t}, \bar{\lambda}^f, \bar{p}^{-f}_{i,t}, t)}{\partial p^f_{i,t}} \leq 0
\]

Therefore, (3.47) says that at the non-cooperative Nash equilibrium point, the joint profit can be further increased if all firms can collude. But no firm will take such strategy unilaterally because it has already made the best response given other firms’ pricing decisions. Further, (3.48) says that if firms adopt cooperative strategies while they are actually involved in non-cooperative equilibrium, they have an incentive to decrease prices to attract more demand. Thus cooperative strategy is clearly not their best response strategy and is not a Nash equilibrium.

### 3.2.13 Fixed Point Algorithm

When the revenue optimization game under consideration is articulated as a VI, we are immediately equipped with a number of powerful and efficient algorithms which either tackles the VI directly or converts the VI to an equivalent nonlinear complementarity problem (NCP) and solves the resulting NCP. See Harker and Pang [16] for an overview of such algorithms. In this chapter we choose to solve the VI using a fixed point formulation. With the preceding background, we are now in a position to state Theorem 5.3 that permits the solution of an appropriately defined variational inequality to be re-cast as a fixed point problem. For the sake of brevity the algorithm development involves the VI (3.36), but the same approach can be extended to (3.46) as well.

**Theorem 3.6** Any solution of the fixed point problem

\[ p = P_K [p - \alpha \cdot F (p, \lambda, t)] , \]
where $P_K[.]$ is the minimum norm projection onto $K$, $\alpha \in \mathbb{R}^1_{++}$, and
\[
F(p, \lambda, t) = \left( -\nabla_{p,f} H_f (p^f; \lambda; p^{-f}) \right)_{f \in \mathcal{F}}
\]
is also a solution of (3.36).

**Proof.** The fixed point problem considered requires that
\[
p = \arg \min_q \left\{ \frac{1}{2} \| p - \alpha \cdot F(p, \lambda, t) - q \| ^2 : q \in K \right\}
\]
where $\alpha \in \mathbb{R}^1_{++}$. That is, we seek the solution of the following mathematical program
\[
\min_q J(q) = \frac{1}{2} [p - \alpha \cdot F(p, \lambda, t) - q]^2
\]
subject to
\[
q \in K
\]
Let us take $q^* \in K$ be a minimum of the above finite dimensional mathematical program and recall that $K$ is convex. Since $J(q)$ is convex and differentiable at $q^* \in K$, a necessary and sufficient condition is
\[
\langle \nabla J(q^*), q - q^* \rangle \geq 0 \text{ for all } q \in K
\]
(3.51)
further
\[
\nabla J(q^*) = (-1) [p - \alpha \cdot F(p, \lambda, t) - q^*]
\]
(3.52)
By virtue of (3.50) $p = q^*$, so (3.52) may be restated as
\[
\nabla J(q^*) = \alpha \cdot F(q^*, \lambda^*, t)
\]
where $\lambda^*$ is obtained by solving the equation (3.17) for given $q^*$. Taken together, (3.51) and (3.52) give
\[
\alpha \langle F(q^*, \lambda^*, t), q - q^* \rangle \geq 0 \text{ for all } q \in K
\]
(3.53)
Because $\alpha$ is positive and constant and $F(p^*, \lambda^*, t) = \left( -\nabla_{p,f} H_f (p^f_*; \lambda_*^*; p^{-f}^*) \right)_{f \in \mathcal{F}}$, (3.53) reduces to
\[
\sum_{f \in \mathcal{F}} \left[ \nabla_{q,f} H_f (q^f_*; \lambda_*^*; q^{-f}_*) \right]^T \cdot (q^f - q^f_*) \leq 0 \text{ for all } q \in K
\]
(3.36) follows immediately, and the theorem is proved. ■

Naturally there is a fixed point algorithm associated with Theorem 5.3; that algorithm is summarized by the following iterative scheme:

\[ p^{k+1} = P_K \left[ p^k - \alpha F \left( p^k, \lambda^k, t \right) \right] \tag{3.54} \]

The detailed structure of the fixed point algorithm is:

**The Fixed Point Algorithm**

**Step 0. Initialization.** Set \( k = 0 \), identify an initial feasible solution \( p^0 \in K \), and find \( \lambda^0 \) by solving (3.17) with given \( p^0 \).

**Step 1. Minimum norm subproblem.** Solve

\[
\min_q J^k(q) = \frac{1}{2} \left[ p^k - \nu \cdot F \left( p^k, \lambda^k, t \right) - q \right]^2 \tag{3.55}
\]

subject to \( p_{\text{min}} \leq q \leq p_{\text{max}} \) \( \tag{3.56} \)

and call the solution \( p^{k+1} \).

**Step 1a. Update shadow prices.** Solve the system of equations for \( \lambda \)

\[
\lambda = - \left[ \nabla_{\mathcal{F}^f} OBC_f \left( p^{k+1}, \lambda, t \right) \right]_{f \in \mathcal{F}} \tag{3.57}
\]

and call the solution \( \lambda^{k+1} \).

**Step 2. Stopping test.** If

\[
\left\| p^{k+1} - p^k \right\| \leq \varepsilon
\]

where \( \varepsilon \in \mathbb{R}_{++}^1 \) is a preset tolerance, stop and declare \( p^* \approx p^{k+1} \). Otherwise set \( k = k + 1 \) and go to Step 1.
We choose the sequence of parameter \( \nu^k \) such that \( \lim_{k \to \infty} \nu^k = 0 \) and \( \lim_{k \to \infty} \sum_{j=0}^{\infty} \nu^j \to \infty \). In particular we choose \( \nu^k = \frac{1}{k} \). Note that the math program (3.55)-(3.56) can be solved in closed form, where

\[
p_{i,t}^{f,k+1} = \max \left( p_{i,min}^f, \min \left( p_{i,t}^{f,k} - \nu^k \cdot F_i^f \left( p^k, \lambda^k, t \right), p_{i,max}^f \right) \right)
\]

for all \( f \in F, i \in S \) and \( t \in [0, N-1] \). The main difficulty in the algorithm above arises in the step 1a where we seek to solve the system of equations (3.57). Using (3.5) and (3.22), expression (3.57) can be rewritten as

\[
\lambda^f_j = -\eta^f_j \alpha^f_j \cdot \left( 1 - \Phi \left( z^k \right) \right) - R^f_j \cdot \left( 1 - \alpha^f_j \right) - \frac{\eta^f_j \phi \left( z^k \right)}{2} \sqrt{\frac{\alpha^f_j \left( 1 - \alpha^f_j \right)}{\sum_{l=0}^{N-1} \sum_{i \in S} \left( a_{ij} \cdot d_{i,l}^f \left( p_{i,l}^k \right) \cdot F^{-1} \left( 1 + \frac{A^T_{i} \cdot \lambda^f, k}{p_{i,l}^k} \right) \right)}}
\]

for all \( j \in C, f \in F \) where \( z = \frac{K^f_j - \alpha^f_j \cdot x_{j,N}^f}{\sqrt{\alpha^f_j \left( 1 - \alpha^f_j \right) x_{j,N}^f}} \) and \( \phi (\cdot) \) and \( \Phi (\cdot) \) are the pdf and c.d.f. of a standard normal variate respectively. Obtaining a closed-form solution of \( \lambda^f \) from (3.59) is very difficult, if not impossible. Therefore, we rely on a heuristic which approximates the solution of (3.59) using first order approximation of \( x_{j,N}^f \) (which appears in the denominator under square root) around current solution of \( \lambda^k \)

\[
\lambda^f_j = -\eta^f_j \alpha^f_j \cdot \left( 1 - \Phi \left( z^k \right) \right) - R^f_j \cdot \left( 1 - \alpha^f_j \right) - \frac{\eta^f_j \phi \left( z^k \right)}{2} \sqrt{\frac{\alpha^f_j \left( 1 - \alpha^f_j \right)}{x_{j,N}^f \left( \lambda^k \right) + \nabla_{\lambda} x_{j,N}^f \left( \lambda^k \right) \cdot \left( \lambda - \lambda^k \right)}}
\]

where

\[
x_{j,N}^f \left( \lambda^k \right) = \sum_{l=0}^{N-1} \sum_{i \in S} \left( a_{ij} \cdot d_{i,l}^f \left( p_{i,l}^k \right) \cdot F^{-1} \left( 1 + \frac{A^T_{i} \cdot \lambda^f, k}{p_{i,l}^k} \right) \right)
\]

and

\[
\nabla_{\lambda} x_{j,N}^f \left( \lambda^k \right) = \sum_{l=0}^{N-1} \sum_{i \in S} \left( \frac{a_{ij} \cdot d_{i,l}^f \left( p_{i,l}^k \right)}{g \left( F^{-1} \left( 1 + \frac{A^T_{i} \cdot \lambda^f, k}{p_{i,l}^k} \right) \right)} \cdot a_{ji} \right)
\]

The resulting fixed point algorithm has the following structure:
The Modified Fixed Point Algorithm

Step 0. Initialization. Set $k = 0, \kappa = 0$, identify an initial feasible solution $p^{0,0} \in K$, and find $\lambda^0$ by solving (3.17) with given $p^{0,0}$

Step 1. Minimum norm subproblem. Solve

$$\min_q J^k(q) = \frac{1}{2} \left[ p^{k,j} - \nu \cdot F \left( p^{k,\kappa}, \lambda^\kappa, t \right) - q \right]^2$$

subject to $\max \left( p_{\min}^f - A^T \lambda^{f,\kappa} \right) \leq q^f \leq p_{\max}^f$ for all $f \in F$

and call the solution $p^{\kappa,k+1}$.

Step 2. Stopping test for the inner problem. If

$$\left\| p^{\kappa,k+1} - p^{\kappa,k} \right\| \leq \varepsilon$$

where $\varepsilon \in \mathbb{R}_{++}$ is a preset tolerance, declare $p^* \approx p^{\kappa,k+1}$ and go to step 3, otherwise update $k = k + 1$ and go to Step 1.

Step 3. Update shadow prices. Update $\lambda^{\kappa+1}$ by solving the following equation

$$\lambda^{f,\kappa+1}_j = -\eta^f_j \alpha^f_j \cdot \left( 1 - \Phi \left( z^f \right) \right) - R^f_j \cdot \left( 1 - \alpha^f_j \right) - \eta^f_j \phi \left( z^k \right) \cdot \alpha^f_j \left( 1 - \alpha^f_j \right)$$

$$- \frac{2 \left( x^f_{j,N} \left( \lambda^\kappa \right) + \nabla_\lambda x^f_{j,N} \left( \lambda^\kappa \right) \cdot (\lambda^{\kappa+1} - \lambda^\kappa) \right)}{\alpha^f_j \left( 1 - \alpha^f_j \right)}$$

Step 4. Stopping test for the outer problem. If

$$\left\| \lambda^{\kappa+1} - \lambda^\kappa \right\| \leq \varepsilon_1$$

where $\varepsilon_1 \in \mathbb{R}_{++}$ is another preset tolerance, stop and declare $\lambda^* \approx \lambda^{\kappa+1}$ and $p^* \approx p^{*,\kappa}$.

Otherwise set $\kappa = \kappa + 1, k = 0$ and go to step 1.
Note that steps 1 and 2 of the above algorithm actually seeks to solve the following VI at some outer iteration $\kappa$ with $\lambda^\kappa$ exogenous: find $p^{*,\kappa} \in \tilde{K}$ such that

$$\sum_{f \in \mathcal{F}} \left[ \nabla_{p^f} H_f \left( p^{f,*,\kappa} ; \lambda^\kappa ; p^{-f,*,\kappa} \right) \right]^T \cdot \left( p^f - p^{f,*,\kappa} \right) \leq 0$$

for all $p \in \tilde{K}$

where

$$\tilde{K} = \prod_{f \in \mathcal{F}} \left\{ p^f : p^f_{\min} \leq p^f \leq p^f_{\max} \right\}$$

We will next show that for a given $\lambda^\kappa$, there exists a unique solution $p^{*,\kappa}$ that solves (3.64). To do so we need to check whether the following system of equations

$$\left( \nabla_{p^f} H_f \left( p^{f,*,\kappa} ; \lambda^\kappa ; p^{-f,*,\kappa} \right) \right)_{f \in \mathcal{F}} = 0$$

has a unique solution. In fact we can state and prove the following result

**Theorem 3.7** If Assumptions A1 and A2 hold, for a given $\lambda^\kappa$, (3.64) has a unique solution

**Proof.** We need to study the negative of the Jacobian matrix. If we can establish that at the point $p^{*,\kappa}$ where (3.65) holds the diagonal terms of the negative Jacobian matrix are strictly positive, off-diagonal terms are nonnegative and the matrix is strictly diagonally dominant, it will follow automatically that the negative Jacobian matrix has all principal minors positive. Hence (3.65) has an unique solution. We know

$$- \frac{\partial^2 H_{f,t}}{\partial p^f_{i,t} \partial p^f_{j,t}} = 0 \text{ for all } i \neq j, f \in \mathcal{F} \text{ and } t \in [0, N - 1]$$

$$- \frac{\partial^2 H_{f,t}}{\partial p^f_{i,t} \partial p^g_{j,t}} = 0 \text{ for all } i \neq j, f \neq g \text{ and } t \in [0, N - 1]$$

$$- \frac{\partial^2 H_{f,t}}{\partial p^f_{i,t} \partial p^f_{i,t}} = \frac{H_{f,t}}{p^f_{i,t}} \cdot \frac{\partial e^f_{j,t}}{\partial p^f_{i,t}} - \frac{1}{p^f_{i,t}} \frac{\partial d^f_{i,t}}{\partial p^f_{i,t}} \left[ \left( -e^f_{i,t} + 1 \right) - \frac{A^T \cdot \lambda^f}{p^f_{i,t}} \cdot \frac{F^{-1}(\mu^f_{i,t})}{\int_0^\tau F^{-1}(\mu^f_{i,t}) \cdot f(t) \, dt} \right]$$

$$\leq 0 \text{ for all } i \in S, f \neq g \text{ and } t \in [0, N - 1]$$

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The last inequality is obtained because at $p^{*\prime}$, \( \frac{\partial H_{f,t}}{\partial p_{t}^{i}} = 0 \) (therefore terms inside the bracket of (3.66) vanish). Further, by Lemma 3.6 we know

\[
\frac{\partial^2 H_{f,t}}{\partial \left( p_{t}^{i} \right)^2} \bigg|_{\frac{\partial H_{f,t}}{\partial p_{t}^{i}} = 0} < 0
\]

Therefore the diagonal terms

\[
- \frac{\partial^2 H_{f,t}}{\partial \left( p_{t}^{i} \right)^2} \bigg|_{\frac{\partial H_{f,t}}{\partial p_{t}^{i}} = 0} > 0
\]

Now, we need to show that

\[
\frac{\partial^2 H_{f,t}}{\partial \left( p_{t}^{i} \right)^2} + \sum_{i,j \in S, f \in F} \frac{\partial^2 H_{f,t}}{\partial p_{t}^{i} \partial p_{t}^{j}} < 0
\]

for all $i \in S$, $f \in F$

i.e., to show

\[
\frac{\partial^2 H_{f,t}}{\partial \left( p_{t}^{i} \right)^2} + \sum_{f,g} \frac{\partial^2 H_{f,t}}{\partial p_{t}^{i} \partial p_{t}^{j}} < 0
\]

for all $i \in S$, $f \in F$

Now,

\[
\frac{\partial^2 H_{f,t}}{\partial \left( p_{t}^{i} \right)^2} = \frac{1}{p_{t}^{i}} \frac{\partial H_{f,t}}{\partial p_{t}^{i}} \cdot \left[ \left( -e_{t}^{f} + 1 \right) - \frac{A_{t}^{T} \cdot \lambda_{t}^{f}}{p_{t}^{i}} \cdot \frac{F^{-1} \left( \mu_{t}^{f} \right)}{\int_{0}^{\tau} f \left( \tau \right) d\tau} \right] + \frac{H_{f,t}^{i}}{p_{t}^{i}} \cdot \left[ \frac{\partial e_{t}^{f}}{\partial p_{t}^{i}} + \frac{A_{t}^{T} \cdot \lambda_{t}^{f}}{ \left( p_{t}^{i} \right)^2 } \cdot \frac{F^{-1} \left( \mu_{t}^{f} \right)}{\int_{0}^{\tau} f \left( \tau \right) d\tau} \right] + \frac{H_{f,t}^{i}}{p_{t}^{i}} \cdot \left[ \frac{A_{t}^{T} \cdot \lambda_{t}^{f}}{ \left( p_{t}^{i} \right)^2 } \cdot \frac{F^{-1} \left( \mu_{t}^{f} \right)}{\int_{0}^{\tau} f \left( \tau \right) d\tau} \right]
\]

\[
= \frac{H_{f,t}^{i}}{p_{t}^{i}} \cdot \left[ \frac{\partial e_{t}^{f}}{\partial p_{t}^{i}} + \frac{A_{t}^{T} \cdot \lambda_{t}^{f}}{ \left( p_{t}^{i} \right)^2 } \cdot \frac{F^{-1} \left( \mu_{t}^{f} \right)}{\int_{0}^{\tau} f \left( \tau \right) d\tau} \right] + \frac{H_{f,t}^{i}}{p_{t}^{i}} \cdot \left[ \frac{A_{t}^{T} \cdot \lambda_{t}^{f}}{ \left( p_{t}^{i} \right)^2 } \cdot \frac{F^{-1} \left( \mu_{t}^{f} \right)}{\int_{0}^{\tau} f \left( \tau \right) d\tau} \right]
\]

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The first term in (3.67) vanishes because
\[
\frac{\partial H_{f,t}}{\partial p_{i,t}^{f,*}} = 0
\]
So,
\[
\frac{\partial^2 H_{f,t}}{\partial (p_{i,t}^f)^2} + \sum_{f,g} \frac{\partial^2 H_{f,t}}{\partial p_{i,t}^f \partial p_{i,t}^g} = -\frac{H_i^{f,t}}{p_i^{f,t}} \left[ \frac{\partial e_f^i}{\partial p_i^{f,t}} + \sum_{g \neq f} \frac{\partial e_f^i}{\partial p_g^{f,t}} \right] + \frac{H_i^{f,t}}{p_i^{f,t}} \left[ \frac{A_T \cdot \lambda_f^i}{(p_i^{f,t})^2} \cdot \frac{F^{-1}(\mu_{i,t}^f)}{\int_0^{F^{-1}(\mu_{i,t}^f)} \tau f(\tau) d\tau} \right] \\
+ \frac{H_i^{f,t}}{p_i^{f,t}} \left[ \left( \frac{A_T \cdot \lambda_f^i}{p_i^{f,t}} \right)^2 \cdot \frac{F^{-1}(\mu_{i,t}^f)}{\int_0^{F^{-1}(\mu_{i,t}^f)} \tau f(\tau) d\tau} \right] \left( F^{-1}(\mu_{i,t}^f) \right)^2 \right]
\]
The terms in the first bracket on the right hand side are non-positive from item 8 of assumption A2. The remaining group of terms has been shown to be negative in the proof of Lemma 3.6 (in particular please refer to (3.31)) after a change of variable \( F(y) = 1 + \left( 1 + \frac{A_T \cdot \lambda_f^i}{p_i^{f,t}} \right) \). Hence
\[
\frac{\partial^2 H_{f,t}}{\partial (p_{i,t}^f)^2} - \sum_{f,g} \frac{\partial^2 H_{f,t}}{\partial p_{i,t}^f \partial p_{i,t}^g} > 0
\]
Therefore, (3.65) has only one solution. From here we conclude that the VI (3.64) also has one solution which can be expressed as
\[
p_i^{f,*} = \max \left( p_i^{f,\min}, \min \left( p_i^{f,k}, p_i^{f,\max} \right) \right)
\]
where \( p_i^{f,k} \) is an element of the unique vector that solves (3.65). Hence the proof. \( \blacksquare \)

So far service price has been allowed to change in either directions between two successive booking periods, in what follows we provide a simple extensions of unidirectional pricing model (either mark-up or mark-down).

**3.2.14 Simple Extension : Unidirectional Service Pricing**

So far we have considered bidirectional price changes where the service prices charged by firms can be changed arbitrarily from period to period. In what follows we consider the
setting where only markdowns or markups are permitted, but not both. Let us define $r_{i,t}^f$ as
the rate of change of price for the service $i \in S$ charged by firm $f \in \mathcal{F}$ at time $t \in [0, N-1]$ which is the control in the hands of the service provider. Service prices, $p_{i,t}^f$, become the states. To keep the exposition simple we consider the service prices at time 0 exogenous, i.e.,

$$p_0 = \bar{p} \in \mathbb{R}^{[\mathcal{F} \times |S|]}$$

However, this initial condition may also be set endogenously.

### 3.2.15 Price Dynamics

The relationship between the service price and rate of change of price can be articulated through the following definitional relationship:

$$p_{i,t+1}^f = p_{i,t}^f + r_{i,t}^f \quad \text{for all } t \in [0, N-1]$$

(3.68)

$$p_{i,0}^f = \bar{p}_i^f$$

(3.69)

Since prices can only go down or stay flat throughout the booking period, for the company $f$

$$r_{i,t}^f \leq r_i^f \leq 0 \quad \text{for all } t \in [0, N-1]$$

(3.70)

where

$$r_i^f = \left( r_{i}^f : i \in S \right)$$

and $r_{i,t}^f < 0$ is the maximum decrease of price between two consecutive periods. In the event of mark-up pricing, the bounds are changed to

$$0 \leq r_i^f \leq r_{\text{max}}^f$$

where $r_{\text{min}}^f < 0$. We are now in a position to state the coupled optimal control problem faced by each service provider to set its service prices and allocation of resources in the event of markdown pricing.
Service Providers’ Optimal Control Problem Considering Markdown

In the markdown model, each firm must be able to completely and perfectly observe the rate of change of service prices as set by its competitors. We argue that this is no more difficult than observing the service price set by its competitors as we postulate that with the advent of information technology it is becoming increasingly easier for the companies to obtain pricing information about their competitors through support tools like aggregators and comparators etc. With this information it requires just a book-keeping effort for the firms to monitor its competitors’ change of prices and the frequency of such change by looking at the price trajectories. With competitors rate of change of service prices

\[ r^{-f} \equiv \{r^g : g \in \mathcal{F} \setminus f \} \]

taken as exogenous to firm \( f \)'s optimal control problem and yet endogenous to the overall equilibrium problem, firm \( f \) computes its markdown decisions \( r^f \) (hence the service prices) and allocation of resources \( u^f \) in order to maximize revenue generated throughout the booking period

\[
\max_{r^f, u^f} J\left(r^f, u^f; r^{-f}\right) = -OBC_f + E\left[\sum_{t=0}^{N-1} p^f_t \cdot \min\left(u^f_t, D^f_t(p_t)\right)\right]
\]

subject to

\[
x^f_{t+1} = x^f_t + A \cdot u^f_t \quad \text{for all } t \in [0, N-1] \tag{3.72}
\]

\[
x^f_0 = 0 \tag{3.73}
\]

\[
p^f_{t+1} = p^f_t + r^f_t \quad \text{for all } t \in [0, N-1] \tag{3.74}
\]

\[
p^f_0 = \bar{p}^f \tag{3.75}
\]

\[
r^f_{\min} \leq r^f_t \leq 0 \quad \text{for all } t \in [0, N-1] \tag{3.76}
\]

\[
p^f_{\min} \leq p^f_t \quad \text{for all } t \in [0, N-1] \tag{3.77}
\]

\[
u^f \geq 0 \tag{3.78}
\]

Note that this formulation is very similar to that for bidirectional price change in Section 3.2.9 except equations (3.74)-(3.77). The definitive dynamics (3.74) and the initial con-
ditions (3.75) are explained in section 3.3.5. Constraint (3.76) is the bound on the rate of mark-down of service prices. Since service price cannot increase between two consecutive periods, the upper bound on service price is unnecessary in this scenario and is hence omitted in (3.77). Decision variables in the hand of firm $f$ are the rate of markdown ($r^f_i$) and the rate of provision of services ($u^f_i$). The states are the service prices ($p^f_i$) and cumulative allocation of resources ($x^f_i$). Once again, as the firms’ optimal control problems are coupled, this gives rise to a Cournot-Nash dynamic game setting. For the sake of brevity we forego the analysis and VI formulation of the above game. Instead, in what follows we articulate a numerical example motivated from a network airline revenue management problem. We discuss some of the monotonic pricing strategies numerically.

3.2.16 Numerical Example

Service Network and Parameters

As mentioned earlier, the numerical example we present here is motivated by the airline RM problem. We have considered a 2-hub, 6-node and 12-legs airlines network as shown in Figure 3.3, where nodes 2 and 3 are the ‘hub’ nodes. Three firms are competing over this network. To keep the exposition simple we assume that each firm uses the same network; however this can be relaxed with a more general setting where each firm has a different service network. We consider 22 different paths (services) of this network that connect 20 different OD pairs as shown in Table 3.3.

We assume linear average demands for different services

$$d_i^f(p_i, t) = \alpha_i^f(t) - \beta_i^f(t) \cdot p_i^f(t) + \sum_{g \neq f} \gamma_i^{f,g}(t) \cdot p_i^g(t)$$

for all $f \in \mathcal{F}, i \in \mathcal{S}, t \in [t_0, t_1]$ and $\alpha_i^f(t), \beta_i^f(t), \gamma_i^{f,g}(t) \in \mathbb{R}_{++}$. The random variables associated with the stochastic demand are assumed to follow a uniform distribution with a range $[0, 2]$ which is independent of service types and firms. The booking period runs from
Figure 3.3: A six-node, two-hub airline network

<table>
<thead>
<tr>
<th>Service ID</th>
<th>O-D pair</th>
<th>Itinerary</th>
<th>Service ID</th>
<th>O-D pair</th>
<th>Itinerary</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 – 2</td>
<td>1 – 2</td>
<td>12</td>
<td>3 – 5</td>
<td>3 – 5</td>
</tr>
<tr>
<td>2</td>
<td>1 – 3</td>
<td>1 – 3</td>
<td>13</td>
<td>3 – 6</td>
<td>3 – 4 – 6</td>
</tr>
<tr>
<td>3</td>
<td>1 – 4</td>
<td>1 – 2 – 4</td>
<td>14</td>
<td>3 – 6</td>
<td>3 – 2 – 4 – 6</td>
</tr>
<tr>
<td>4</td>
<td>1 – 5</td>
<td>1 – 3 – 5</td>
<td>15</td>
<td>4 – 6</td>
<td>4 – 6</td>
</tr>
<tr>
<td>5</td>
<td>1 – 6</td>
<td>1 – 6</td>
<td>16</td>
<td>5 – 2</td>
<td>5 – 2</td>
</tr>
<tr>
<td>6</td>
<td>2 – 3</td>
<td>2 – 3</td>
<td>17</td>
<td>5 – 3</td>
<td>5 – 3</td>
</tr>
<tr>
<td>7</td>
<td>2 – 4</td>
<td>2 – 4</td>
<td>18</td>
<td>5 – 4</td>
<td>5 – 3 – 4</td>
</tr>
<tr>
<td>8</td>
<td>2 – 5</td>
<td>2 – 3 – 5</td>
<td>19</td>
<td>5 – 6</td>
<td>5 – 3 – 4 – 6</td>
</tr>
<tr>
<td>9</td>
<td>2 – 6</td>
<td>2 – 4 – 6</td>
<td>20</td>
<td>5 – 6</td>
<td>5 – 2 – 4 – 6</td>
</tr>
<tr>
<td>10</td>
<td>3 – 2</td>
<td>3 – 2</td>
<td>21</td>
<td>6 – 5</td>
<td>6 – 5</td>
</tr>
<tr>
<td>11</td>
<td>3 – 4</td>
<td>3 – 4</td>
<td>22</td>
<td>6 – 2</td>
<td>6 – 5 – 2</td>
</tr>
</tbody>
</table>
clock-time 0 to 20. Lower and upper bounds on the service prices are set independent of the identity of firms depending on the number of legs used in a particular itinerary, as shown below

<table>
<thead>
<tr>
<th>No. of legs in the itinerary</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_{i,\text{min}}^f)</td>
<td>25</td>
<td>50</td>
<td>75</td>
</tr>
<tr>
<td>(p_{i,\text{max}}^f)</td>
<td>500</td>
<td>650</td>
<td>725</td>
</tr>
</tbody>
</table>

Each service provider has different, yet mostly comparable capacities of 12 legs as in Table 3.4.

The show demand probabilities are assumed to be independent of resource types, these probabilities are 0.82, 0.85 and 0.81 for firms 1, 2, and 3 respectively. Denial of service costs are assumed to be the same across all firms and resources ($300), so also refunds ($15). We
consider the following different scenarios:

<table>
<thead>
<tr>
<th>Scenarios</th>
<th>Market structure</th>
<th>Pricing rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario a</td>
<td>non-cooperative</td>
<td>bidirectional</td>
</tr>
<tr>
<td>Scenario b</td>
<td>cooperative</td>
<td>bidirectional</td>
</tr>
<tr>
<td>Scenario c</td>
<td>non-cooperative</td>
<td>mark-up</td>
</tr>
<tr>
<td>Scenario d</td>
<td>cooperative</td>
<td>mark-up</td>
</tr>
<tr>
<td>Scenario e</td>
<td>non-cooperative</td>
<td>mark-down</td>
</tr>
<tr>
<td>Scenario f</td>
<td>cooperative</td>
<td>mark-down</td>
</tr>
</tbody>
</table>

For the sake of brevity we forgo the detailed symbolic statement for each scenarios and, instead, provide numerical results in graphical form.
**Equilibrium Service Price Trajectories**

The equilibrium service price trajectories for all 6 scenarios are plotted below.

In figure below we compare competitive pricing with that under collusion by plotting \((\hat{p}_{i,t} - p_{i,t}^{c})\) vs. time for all 22 services and 3 firms where \(\hat{p}\) and \(p^c\) are the cooperative and non-cooperative service prices respectively. We compute the difference for 3 pricing rules: (a) bidirectional, (b) mark-up and (c) mark-down. It is evident from the plot that competition
leads to underpricing of services which is, for certain services, as high as $80 per unit of service.

Equilibrium Allocation Trajectories

The equilibrium offering of different services by the firms over the booking period is shown in figure below, once again we consider 6 different scenarios; hence the plots are grouped.
In figure below we compare competitive allocation with that under collusion by plotting $(\tilde{u}_{i,t} - u^*_{i,t})$ vs. time ($t$) for all 22 services and 3 firms where $\tilde{u}$ and $u^*$ are the cooperative and non–cooperative allocations respectively. We compute the difference for 3 pricing rules accordingly.

![Graphs showing allocations under different scenarios](image-url)
: (a) bidirectional, (b) mark-up and (c) mark-down.

Overbooking Limits

We define the overbooking limit of a resource set by a firm as the excess of total allocation of it at the end of the booking period than firm’s actual capacity of that resource., i.e.,

\[ OL^f_j = \max(x^f_{j,N} - K^f_j, 0) \]

We plot scatter diagrams involving the overbooking limits (y-axis) and actual capacity (x-axis) of all 12 resources of all firms under 6 different scenarios (see figure below) and fit an appropriate polynomial curve that empirically describes the relationship between the overbooking limits and capacities for all 3 firms, in most of the cases we observe complicated nonlinear relationship between these two. This observation illustrates that pricing and allocation decisions strongly affect overbooking decisions, hence simultaneous approach, rather than sequential approach, is required to make these decisions. We also observe that
the overbooking limits are set higher when firms are in competition than cooperation.

![Graphs of different scenarios](Figures)

**Expected Revenue and Price of Anarchy**

We compute the expected net revenue (after subtracting expected overbooking cost and refunds) for the firms under 6 different scenarios and the results are tabulated in Table 3.5.
### 3.2.17 Concluding Remarks

In this chapter we have shown that dynamic oligopolistic service network pricing and resource allocation, cornerstones of RM, may be articulated as variational inequalities. We have provided some of the qualitative properties of the equilibrium. Additionally, we presented the numerical solution of an example service network pricing and resource allocation problem, motivated by the airline industry, using a fixed point algorithm. This example suggests that the variational inequality perspective for dynamic Cournot-Nash games is computationally tractable which is possibly a crucial first step towards a general computable theory of service RM problems, an area which is often overlooked by the researchers and practitioners. Our model may be expanded upon to include more complicated allocation methods such as theft-nesting and bid price controls etc.

### 3.3 Dynamic competition involving Pricing and Allocation Decisions under Demand Uncertainty

In this section we study a joint pricing and resource allocation problem in a network with applications to production planning and airline revenue management. This model is similar
to that of Perakis and Sood [40], though we consider multiple products which leads to the network structure of the problem. We also directly address the demand uncertainty, whereas Perakis and Sood address the uncertainty aspect of the problem by using ideas from robust optimization.

Suppose a set of service providers has a finite supply of resources which it can use to produce multiple products with different resource requirements. The firms do not have the ability of re-stocking their resources during the planning horizon and any unsold resource does not have any salvage value at the end of planning horizon. Further, if realized demand for a service type at any time is more than the rate of provision of service, excess demand is lost. All service providers can set the price for each of their services which are bounded from above and below. The demand for each product is uncertain and depends on own service prices as well as non-own service prices. Each service provider has to decide how to allocate its resources for providing different services and how to price its services to maximize its expected revenue. This research is motivated by the airline pricing problem where the services are combinations of origin, destination and fare class and the resources are seats on flights.

3.3.1 Notation

In this section we pursue continuous time version of the discrete time model of section 3.2. Even though the notations of section 3.2 runs very closely to that of this section, we once again review here the notation in Table 3.6 that we will be using in rest of this section.

from the above notation it is evident that

\[ \bigcup_{i=1}^{\vert S \vert} C_i = C \]

The variables we use are primarily the controls (prices and allocation of resources) and the states (cumulative allocation), as shown in Table 3.7.
Table 3.6: Parameters: Dynamic Pricing-Allocation Model

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F}$</td>
<td>set of firms</td>
</tr>
<tr>
<td>$\mathcal{S}$</td>
<td>set of services each firm provides</td>
</tr>
<tr>
<td>$\mathcal{C}$</td>
<td>set of resources that firms use to provide services</td>
</tr>
<tr>
<td>$\mathcal{C}_i$</td>
<td>set of resources that firms use to provide service $i \in \mathcal{S}$</td>
</tr>
<tr>
<td>$A$</td>
<td>resource-service incidence matrix</td>
</tr>
<tr>
<td>$t_0$</td>
<td>starting time of the booking period (planning horizon)</td>
</tr>
<tr>
<td>$t_1$</td>
<td>end time of the booking period (planning horizon)</td>
</tr>
<tr>
<td>$t \in [t_0, t_1]$</td>
<td>instant of time</td>
</tr>
<tr>
<td>$p_{i,\min}^f$</td>
<td>minimum price that firm $f$ can charge for service $i \in \mathcal{S}$</td>
</tr>
<tr>
<td>$p_{i,\max}^f$</td>
<td>maximum price that firm $f$ can charge for service $i \in \mathcal{S}$</td>
</tr>
<tr>
<td>$u_{i,\min}^f$</td>
<td>minimum rate of provision of service $i \in \mathcal{S}$ offered by firm $f$</td>
</tr>
<tr>
<td>$u_{i,\max}^f$</td>
<td>maximum rate of provision of service $i \in \mathcal{S}$ offered by firm $f$</td>
</tr>
<tr>
<td>$K_{j}^f$</td>
<td>firm $f$'s capacity for resource type $j \in \mathcal{C}$</td>
</tr>
</tbody>
</table>

Table 3.7: Control and State Variables: Dynamic Pricing-Allocation Model

<table>
<thead>
<tr>
<th>Variables</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{i}^f(t)$</td>
<td>price for the service $i \in \mathcal{S}$ charged by firm $f \in \mathcal{F}$ at time $t$</td>
</tr>
<tr>
<td>$u_{i}^f(t)$</td>
<td>firm $f$'s service level of type $i \in \mathcal{S}$ at time $t$</td>
</tr>
<tr>
<td>$x_{j}^f(t)$</td>
<td>firm $f$'s total allocated resource of type $j \in \mathcal{C}$ till time $t$</td>
</tr>
<tr>
<td>$d_{i}^f(p,t)$</td>
<td>mean demand for service $i \in \mathcal{S}$ from firm $f \in \mathcal{F}$ at time $t$ when prevailing price is $p$</td>
</tr>
<tr>
<td>$D_{i}^f(p,t)$</td>
<td>firm $f$'s realized demand of service $i \in \mathcal{S}$ at time $t$</td>
</tr>
<tr>
<td>$z_{i}^f(t)$</td>
<td>random component associated with demand faced by the firm $f$ for service $i$ at time $t$</td>
</tr>
</tbody>
</table>
In the vector notation, decision variables for firm $f$ are

\[ p^f = (p^f_i : i \in S) \]
\[ u^f = (u^f_i : i \in S) \]

Decision variables of firm $f$’s competitors are

\[ p^{-f} = (p^{-g} : g \in \mathcal{F} \setminus f) \]
\[ u^{-f} = (u^{-g} : g \in \mathcal{F} \setminus f) \]

State variables for firm $f$ are the cumulative allocation of resources

\[ x^f = (x^f_j : j \in C) \]
which can be concatenated as

\[ x = (x^f : f \in \mathcal{F}) \]

**Resource-service incidence matrix**

The network we are interested in has $|C|$ resources and the firm provides $|S|$ different services. Each network product is a bundle of the $|C|$ resources sold with certain purchase terms and restrictions at a given price. The resource-service *incidence matrix*, $\mathcal{A} = [a_{ij}]$ is a $|C| \times |S|$ matrix where

\[ a_{ij} = 1 \text{ if resource } i \text{ is used by service } j \]
\[ = 0 \text{ else} \]

Thus the $j$th column of $\mathcal{A}$, denoted $\mathcal{A}_j$ is the *incidence vector* for the service $j$; the $i$th row, denoted $\mathcal{A}^i$, has an entry of one in the column $j$ corresponding to a service $j$ that utilizes the resource $i$. Note that there will be multiple identical columns if there are multiple ways of selling a given bundle of resources, probably with different restrictions. Each would have an identical column in the matrix $\mathcal{A}$, but they could have different revenue values and different demand patterns (Talluri and van Ryzin [34]).
Demand for services

Demand for each service is uncertain. Its expectation only depends on the current market price of services. Firm $f$’s realized demand at time $t$ for service $i$ is $D^f_i(p, t)$ when prevailing market price is $p$. Demands in consecutive periods are independent and nonnegative. Two types of demand models are predominant in the supply chain and news vendor pricing literature:

1. additive form:
   $$D^f_i(p, t) = d^f_i(p, t) + z^f_i$$

2. multiplicative form:
   $$D^f_i(p, t) = d^f_i(p, t) \cdot z^f_i$$

where $z^f_i \geq 0$ is a continuous random variable with known distribution and $d^f_i(p, t)$ is the expected or average demand faced by the firm $f$ for its service $i$ when the price combination is $p$ at time $t$. Nevertheless, to keep the exposition simple, we focus on the multiplicative demand form in the rest of the paper. Probability density function and cumulative probability distribution function of the random variable $z^f_i$ are denoted by $g(z^f_i)$ and $G(z^f_i)$ respectively. In the RM literature it is common to make two major assumptions regarding the nature of the random variable $z^f_i$ and the average demand $d^f_i$. These are:

**Assumption A2.1.** The random variable $z^f_i$ is independent of price, $p$ and time, $t$, and identically distributed with $E(z^f_i) = 1$, $\text{Var}(z^f_i) < \infty$

**Assumption A2.2.** For any firm $f \in F$ and the service type $i \in S$, the mean demand $d^f_i(p, t)$ has the following properties:

1. $d^f_i$ depends only on the prices charges by firm $f$ and its competitors for service type $i$ only

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2. $d^f_i (p,t)$ is continuous, bounded, and differentiable in $p_i$ on the strategy space $[p_i,\min, p_i, \max]$ where

$$p_i = \{ p^g_i : g \in \mathcal{F} \}$$

3. average demand decreases with own service price, i.e., $\frac{\partial d^f_i (p,t)}{\partial p_i} < 0$ and

4. average demand increases with non-own service prices, i.e., $\frac{\partial d^f_i (p,t)}{\partial p_i} > 0$ for all $g \neq f$

Note that

$$E \left( D^f_i (p,t) \right) = d^f_i (p,t)$$

the multiplicative model implies that the coefficients of variation of the one-period demands are constants, independent of the price vectors. This type of multiplicative model was originally proposed by Karlin and Carr [53] and has since been used frequently in the supply chain literature. However, it is also possible to consider a more general demand model in which the standard deviation of demand may fail to be proportional to the mean, i.e., the coefficient of variation may vary as service prices and mean demand volumes vary.

One such demand model is

$$D^f_i (p,t) = d^f_i (p,t) + \sigma^f_i (d^f_i (p,t)) \cdot z^f_i$$

where $\sigma^f_i (\cdot)$ is a general increasing function with $\sigma^f_i (0) = 0$ and $z^f_i \geq 0$ is a continuous random variable (see Bernstein and Federgruen [54] for such nonmultiplicative demand models). It can be easily verified that most of the commonly used demand functions satisfy the requirements in Assumption A2.2. In particular,

1. linear function

$$d^f_i (p,t) = \rho^f_i (t) - \sigma^f_i (t) \cdot p^f_i + \sum_{g \in \mathcal{F} \setminus f} \gamma^f_i (t) \cdot p^g_i$$

where $\rho^f_i (t), \sigma^f_i (t), \gamma^f_i (t) \in \mathbb{R}^{++}_1$ for all $f \in \mathcal{F}$ and $i \in \mathcal{S}$,

2. logit

$$d^f_i (p,t) = \frac{a^f_i (t) \exp \left( -b^f_i (t) \cdot p^f_i \right)}{\sum_{g \in \mathcal{F}} a^f_j (t) \exp \left( -b^f_j (t) \cdot p^f_j \right)}$$
where \( a_i^f(t), b_i^f(t) \in \mathbb{R}^1_{++} \) for all \( f \in \mathcal{F} \) and \( i \in S \).

3. **Cobb-Douglas**

\[
d_i^f(p, t) = a_i^f \left( p_i^f(t) \right)^{-\beta_i^f} \prod_{g \in \mathcal{F} \setminus i} \left( p_i^g(t) \right)^{\beta_i^g}
\]

where \( a_i^f > 0, \beta_i^f > 1, \beta_i^g > 0 \) for all \( f \in \mathcal{F}, i \in S \) and \( t_0 \leq t \leq t_1 \)

In what follows we now describe the revenue optimization problem for the service providers and articulate the Cournot-Nash equilibrium of the dynamic pricing and resource allocation game as a DVI.

### 3.3.2 Service Providers’ Optimal Control Problem

The time scale (booking period) we consider here is short enough that the time value of money does not need to be considered. With competitors’ service prices

\[
p^{-f} \equiv \{ p^g : g \in \mathcal{F} \setminus f \}
\]

taken as exogenous to the firm \( f \in \mathcal{F} \)’s optimal control problem and yet endogenous to the overall equilibrium problem, firm \( f \) computes its prices \( p^f \) and allocation of resources \( u^f \) in order to maximize revenue generated throughout the booking period

\[
\max_{p^f, u^f} J \left( p^f, u^f; p^{-f} \right) = E \left[ \int_{t_0}^{t_1} p^f \cdot \min \left( u^f, D^f(p,t) \right) dt \right]
\]

subject to

\[
\frac{dx^f}{dt} = A \cdot u^f \quad (3.80)
\]

\[
x^f(t_0) = 0 \quad (3.81)
\]

\[
x^f(t_1) \leq K^f \quad (3.82)
\]

\[
p^f_{\min} \leq p^f \leq p^f_{\max} \quad (3.83)
\]

\[
u^f_{\min} \leq u^f \leq u^f_{\max} \quad (3.84)
\]
As in a typical RM industry, there is no salvage value of unsold inventory at the end of planning horizon; all unfulfilled demand is lost. At the same time, firms are restricted from accepting backorders. Our model does not make decisions regarding overbooking, which is a standard practice done in anticipation of cancellations and no-shows. We will later show how this model can be adapted to a model of overbooking. The definitional dynamics (3.80) describe total commitment of resources derived from the rate at which services are provided. The initial condition (3.81) articulates that at the start of the booking period no resources are committed. In the absence of overbooking, the total committed resource at the end of the booking period can not exceed the actual available resource as expressed in constraint (3.82). Constraints (3.83) are simply the upper and lower bound on the prices that the firm is allowed to charge due to some regulation. The lower bounds ($u_{\min} > 0$) on the allocation variables $u^f$ in (3.84) ensure that each service provider participates in each period with a strictly positive provision of service. The implication, if this were not true, would be that a firm with nothing to sell in a period could influence the demand seen by its competitors by setting a price. In other words, setting a price would make sense only if there is a nonzero sale in that period. Please note that we have not imposed any constraint on the direction of price change. Clearly, as firms’ optimal control problems are coupled, this gives rise to a Corunot-Nash dynamic game setting.
3.3.3 Analysis of the Optimal Control Problem and DVI Formulation of the Game

We need to study firm $f$’s best response optimal control problem given its competitors’ service prices $p^{-f}$. The expected instantaneous revenue function for firm $f$ at time $t$ is

$$R_f(p^f, u^f; p^{-f}) = E\left[p^f(t) \cdot \min\left(u^f(t), D^f(p, t)\right)\right]$$

$$= E\left[p^f(t) \cdot \min\left(u^f(t), d^f(p, t) \cdot z^f\right)\right]$$

$$= \sum_{i \in S} \left(p_i^f(t) \cdot w_i^f(t) - p_i^f(t) \cdot d_i^f(p, t) \int_0^{u_i^f(t)} G(\tau) d\tau\right)$$

(3.85)

The component $\sum_{i \in S} p_i^f \cdot w_i^f$ in the instantaneous revenue function (3.85) is called the riskless component of the revenue for firm $f \in F$. Further, depending on the component $\frac{w^f_i}{d_i^f(p, t)}$, the expected revenue is reduced as the term $\frac{w^f_i}{d_i^f(p, t)}$ increases. There is a deterministic optimal control problem that is equivalent to the stochastic optimal control problem of section 3.3.2 because of the following properties: (a) the dynamics are deterministic and (b) a closed-form expression of the objective expectation can be found for multiplicative demand as shown above. Therefore, the equivalent deterministic Hamiltonian associated with firm $f$’s problem is

$$H_f(p^f, u^f; p^{-f}; \lambda^f; t) = \sum_{i \in S} \left(p_i^f(t) \cdot w_i^f(t) - p_i^f(t) \cdot d_i^f(p, t) \int_0^{u_i^f(t)} G(\tau) d\tau\right)$$

$$+ \left(\lambda^f(t)\right)^T \cdot \left(A \cdot u^f(t)\right)$$

where $\lambda^f$ is the vector of adjoint variables which has the interpretation of shadow price of resources in the current context. Let us define

$$\lambda^f = \left(\lambda^f_j : j \in C\right)$$
Therefore, the Hamiltonian can be re-written as

\[
H_f \left( p^f, u^f; p^{-f}; \lambda^f \right) = \sum_{i \in S} \left( p_i^f(t) \cdot u_i^f(t) - p_i^f(t) \cdot d_i^f(p, t) \int_0^{u_i^f(t)} G(\tau) \, d\tau \right) \\
+ \left( \lambda^f(t) \right)^T \left( A \cdot u^f(t) \right) \\
= \sum_{i \in S} \left[ p_i^f(t) + \left( \lambda^f(t) \right)^T A_i \right] \cdot u_i^f(t) \\
- \sum_{i \in S} p_i^f(t) \cdot d_i^f(p, t) \int_0^{u_i^f(t)} G(\tau) \, d\tau
\]  

(3.86)

Now, assume

\[
c_i^f(t) \equiv - \left( \lambda^f(t) \right)^T A_i
\]

to be firm f’s ‘shadow price’ of providing per unit service type \(i \in S\), which is endogenously determined, at each time \(t \in [t_0, t_1]\). Thus, (3.86) can be rewritten as

\[
H_f \left( p^f, u^f; p^{-f}; \lambda^f \right) = \sum_{i \in S} \left\{ p_i^f(t) - c_i^f(t) \right\} \cdot u_i^f(t) \\
- \sum_{i \in S} p_i^f(t) \cdot d_i^f(p, t) \int_0^{u_i^f(t)} G(\tau) \, d\tau
\]  

(3.87)

Under this condition this becomes a multi-product newsvendor pricing problem for firm f with endogenously determined prices.

The adjoint dynamics associated with the optimal control problem of firm f is

\[
\frac{d\lambda_j^f}{dt} = - \frac{\partial H_f}{\partial x_j^f}
\]

\[
= 0
\]

this is because Hamiltonian (3.86) is free from the state, hence, \(\lambda_j^f\) remains stationary throughout the trajectory

\[
\lambda_j^f(t) = \lambda_j^f \quad \forall t \in [t_0, t_1], j \in C
\]
From the terminal time state condition \( x^f(t_1) \leq K^f \) we can form the following complementarity conditions (see Sethi and Thompson [6])

\[
\lambda^f \cdot [x^f(t_1) - K^f] = 0
\]  

(3.88)

and

\[
\lambda^f \leq 0
\]  

(3.89)

\[
x^f(t_1) - K^f \leq 0
\]  

(3.90)

Pontryagin’s maximum principle [28] tells that an optimal solution to (3.138) - (3.143) is a quadruplet

\[
\{ p^f^*(t), u^f^*(t), x^f^*(t), \lambda^f^*(t) \}
\]

that, given \( H_f(p^f, u^f; p^{-f}; \lambda^f) \), must satisfy at each time \( t \in [t_0, t_1] \):

\[
\left( \begin{array}{c}
p^f^*(t) \\ u^f^*(t)
\end{array} \right) = \arg \max_{(p^f(t), u^f(t)) \in \mathcal{K}_f} H_f(p^f, u^f; p^{-f}; \lambda^f)
\]  

(3.91)

where

\[
\mathcal{K}_f = \left\{ (p^f, u^f) : (3.139) - (3.143) \text{ hold} \right\}
\]

\( x^f(u^f, t) \) is the solution of

\[
\frac{dx^f}{dt} = A \cdot u^f, \quad x^f(t_0) = 0, x^f(t_1) \leq K^f
\]  

(3.92)

By virtue of regularity (in particular \( U_f \) is convex and compact), the necessary condition for (3.150) can be expressed in the following variational form (see Minoux Theorem 10.6 [26])

\[
\begin{cases}
\left[ \nabla_{p^f} H_f(p^f^*, u^f^*; p^{-f^*}; \lambda^f^*) \right]^T (p^f - p^f^*) \\
+ \left[ \nabla_{u^f} H_f(p^f^*; u^f*; p^{-f^*}; \lambda^f^*) \right]^T (u^f - u^f^*)
\end{cases} \leq 0 \quad \forall \left( \begin{array}{c} p^f \\ u^f \end{array} \right) \in \mathcal{K}_f
\]  

(3.93)

where

\[
H_f^* = H_f(p^f^*, u^f^*; p^{-f^*}; \lambda^f^*)
\]
Now consider the following DVI which can be shown to have solutions that are Cournot-Nash equilibria for the joint network pricing and resource allocation game described above in which individual service providers maximize expected revenue in light of current information about competitors’ prices:

\[
\begin{align*}
\text{find } & \begin{pmatrix} p^* \\ u^* \end{pmatrix} \in K \text{ such that } \\
\sum_{f \in F} \int_{t_0}^{t_1} \left( \nabla_{p_f} H^*_f \cdot \left( p_f^* - p_f^* \right) + \nabla_{u_f} H^*_f \cdot \left( u_f^* - u_f^* \right) \right) dt \leq 0 \\
\text{for all } & \begin{pmatrix} p \\ u \end{pmatrix} \in K = \prod_{f \in F} K_f
\end{align*}
\]

where

\[
\begin{pmatrix} p \\ u \end{pmatrix} = \begin{pmatrix} [p_f]_{f \in F} \\ [u_f]_{f \in F} \end{pmatrix}
\]

We omit the formal proof of this equivalence relationship between solution of the above DVI and CN equilibrium of the game for the sake of brevity. Curious readers are encouraged to refer to Mookherjee and Friesz [55].

### 3.3.4 Cooperative Equilibrium

We are also interested in comparing pricing and allocation strategies of the firms in the event of perfect collusion where all firms collaborate to maximize their aggregate expected revenue throughout the booking period. The single optimal control problem that the firms collectively seek to solve is the following:

\[
\max_{p,u} J_C (p, u) = \sum_{f \in F} E \left[ \int_{t_0}^{t_1} p_f^* \cdot \min \left( u_f^*, D_f (p, t) \right) dt \right]
\]
subject to

\[ \frac{dx_f}{dt} = A \cdot u^f \quad \forall f \in \mathcal{F} \quad (3.96) \]
\[ x_f(t_0) = 0 \quad \forall f \in \mathcal{F} \quad (3.97) \]
\[ x_f(t_1) \leq K_f \quad \forall f \in \mathcal{F} \quad (3.98) \]
\[ p_{f_{\text{min}}}^f \leq p^f \leq p_{f_{\text{max}}}^f \quad \forall f \in \mathcal{F} \quad (3.99) \]
\[ u_{f_{\text{min}}}^f \leq u^f \leq u_{f_{\text{max}}}^f \quad \forall f \in \mathcal{F} \quad (3.100) \]

Hamiltonian of the above optimal control problem is

\[ H_1(p, u, \lambda) = \sum_{f \in \mathcal{F}} \sum_{i \in \mathcal{S}} \left[ p_i^f \left( \lambda_i^f \right)^T A_i \right] \cdot u_i^f \]
\[ - \sum_{f \in \mathcal{F}} \sum_{i \in \mathcal{S}} p_i^f \cdot d_i^f(p, t) \int_0^{u_i^f} \frac{d_i^f}{d_i^f(p, t)} G(\tau) \, d\tau \]

Let \((\tilde{p}, \tilde{u})\) be the cooperative service prices and resource allocations respectively for the firms where

\[ \tilde{p} = \left( \tilde{p}_f^f \right)_{f \in \mathcal{F}} \]
\[ \tilde{u} = \left( \tilde{u}_f^f \right)_{f \in \mathcal{F}} \]

Cooperative service prices and resource allocations can be computed by solving the following DVI:

\[ \text{find } \left( \begin{array}{c} \tilde{p} \\
\tilde{u} \end{array} \right) \in \mathcal{K} \text{ such that } \]
\[ \int_{t_0}^{t_1} \left( \nabla_p \tilde{H}_1 \cdot (p - \tilde{p}) + \nabla_u \tilde{H}_1 \cdot (u^f - \tilde{u}) \right) \, dt \leq 0 \quad (3.101) \]

for all \( \left( \begin{array}{c} p \\
u \end{array} \right) \in \mathcal{K} \)

where

\[ \tilde{H}_1 = H_1 \left( \tilde{p}, \tilde{u}; \tilde{\lambda} \right) \]

Note that \( \tilde{\lambda} \) is the cooperative shadow price of the resources owned by the firms.
3.3.5 Competitive Model of Mark-down Pricing and Resource Allocation

So far we have considered bidirectional price changes where the service prices charged by firms can be changed arbitrarily from period to period. In what follows we consider the setting where only markdowns are permitted. Let us define \( r_f^i(t) \) as the rate of change of price for the service \( i \in S \) charged by firm \( f \in F \) at time \( t \in [t_0, t_1] \) which is the control in the hands of the service provider. Service prices, \( p_f^i(t) \) become the states. To keep the exposition simple we consider the service prices at time \( t_0 \) exogenous, i.e.,

\[
p(t_0) = \frac{\hat{p}}{\in \mathbb{R}^{|F|\times|S|}}
\]

however, this initial condition may also be set endogenously.

Price Dynamics

The relationship between the service price and rate of change of price can be articulated through the following definitional relationship

\[
\frac{dp_f^i(t)}{dt} = r_f^i(t) \tag{3.102}
\]

\[
p_f^i(t_0) = \hat{p}_f^i \tag{3.103}
\]

Since prices can only go down or stay flat throughout the booking period; for the company \( f \)

\[
r_{min}^f \leq r_f \leq 0 \tag{3.104}
\]

where

\[
r_f = \left( r_f^i : i \in S \right)
\]

and \( r_{min}^f < 0 \) is the maximum decrease of price between two consecutive periods. In the event of mark-up pricing, the bounds are changed to

\[
0 \leq r_f \leq r_{max}^f
\]
where $r_{\text{min}}^f < 0$. We are now in a position to state coupled optimal control problem faced by each service provider to set its service prices and allocation of resources in the event of markdown pricing.

### 3.3.6 Service Providers’ Optimal Control Problem Considering Markdown

In the markdown model, each firm must be able to completely and perfectly observe the rate of change of service prices as set by its competitors. We argue that this is no more difficult than observing service price set by its competitors as we postulate that with the advent of information technology it is becoming increasingly easier for the companies to obtain pricing information about their competitors through support tools like aggregators and comparators etc. With this information it requires just a book-keeping effort for the firms to monitor its competitors’ change of prices and frequency of such change by looking at the price trajectories. With competitors rate of change of service prices

$$r^{-f} \equiv \{ r^g : g \in \mathcal{F}\backslash f \}$$

taken as exogenous to firm $f$’s optimal control problem and yet endogenous to the overall equilibrium problem, firm $f$ computes its markdown decisions $r^f$ (hence the service prices) and allocation of resources $u^f$ in order to maximize revenue generated throughout the booking period

$$\max_{r^f, u^f} J \left( r^f, u^f ; r^{-f} \right) = E \left[ \int_{t_0}^{t_1} p^f \cdot \min \left( u^f, D^f (p, t) \right) dt \right]$$  \hspace{1cm} (3.105)
subject to

\[
\frac{dx^f}{dt} = A \cdot u^f \\
\]

(3.106)

\[x^f(t_0) = 0\]  
(3.107)

\[x^f(t_1) \leq K^f\]  
(3.108)

\[
\frac{dp^f}{dt} = r^f \\
\]

(3.109)

\[p^f(t_0) = \bar{p}^f\]  
(3.110)

\[r^f_{\min} \leq r^f \leq 0\]  
(3.111)

\[p^f_{\min} \leq p^f \leq p^f_{\max}\]  
(3.112)

\[u^f_{\min} \leq u^f \leq u^f_{\max}\]  
(3.113)

Note that this formulation is very similar to that for bidirectional price change in section 3.3.2 except equations (3.109)-(3.112). The definitional dynamics (3.109) and the initial conditions (3.110) are explained in section 3.3.5. Constraint (3.111) is the bound on the rate of mark-down of service prices. Since service price cannot increase between two consecutive periods, upper bound on service price is unnecessary in this scenario and is hence omitted in (3.112). Decision variables in the hand of firm \(f\) are the rate of markdown \((r^f)\) and the rate of provision of services \((u^f)\). The states are the service prices \((p^f)\) and cumulative allocation of resources \((x^f)\). Once again, as firms’ optimal control problems are coupled, this gives rise to a Cournot-Nash dynamic games setting.

### 3.3.7 DVI Formulation of the CN Game

For the sake of brevity we suppress details of analysis of each service provider’s optimal control problem (see Mookherjee and Friesz for a detailed discussion [55]), instead we state the DVI which has solutions that are Cournot-Nash equilibria for the markdown pricing and resource allocation based network RM game described above in which each individual service provider maximizes expected revenue in light of current information about its competitors’
rate of change of prices :

$$\text{find } \left( \begin{array}{c} r^* \\ u^* \end{array} \right) \in \tilde{K} \text{ such that}$$

$$\frac{d}{dt} \left( \begin{array}{c} r^* \\ u^* \end{array} \right) \in \tilde{K}$$

$$\sum_{f \in F} \int_{t_0}^{t_1} \left( \nabla_{r_f} \tilde{H}_f^* \cdot (r_f - r_f^*) + \nabla_{u_f} \tilde{H}_f^* \cdot (u_f - u_f^*) \right) dt \leq 0 \quad (3.114)$$

for all $$\left( \begin{array}{c} r \\ u \end{array} \right) \in \tilde{K} = \prod_{f \in F} \tilde{K}_f$$

where

$$\begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} [r_f]_{f \in F} \\ [u_f]_{f \in F} \end{pmatrix}$$

$$\tilde{H}_f^* = \tilde{H}_f \left( r_f^*, u_f^*, r_f^{-*}; \lambda_f^* \right)$$

$$\tilde{H}_f \left( r^f, u^f; r_f^{-*}; \lambda^f \right) = \sum_{i \in S} \left\{ p_i^f + \left( \lambda^f \right)^T A_i \right\} \cdot u_i^f (t)$$

$$- \sum_{i \in S} p_i^f \cdot d_i^f (p, t) \int_0^{d_i^f (p, t)} G (\tau) d\tau + \sum_{i \in S} r_i^f$$

and

$$\tilde{K}_f = \left\{ \left( r_f, u_f \right) : (3.106) - (3.113) \text{ hold} \right\}$$

### 3.3.8 Cooperative Equilibrium with Markdown Pricing

Similar to the bidirectional pricing case (see section 3.3.4), we are also interested in comparing pricing and allocation strategies of the firms in the event of perfect collusion where all firms collaborate to maximize their aggregate expected revenue throughout the booking period. The single optimal control problem that the firms collectively seek to solve is the following :

$$\max_{r, u} J_C (r, u) = \sum_{f \in F} E \left[ \int_{t_0}^{t_1} p_f \cdot \min \left( u_f, D_f (p, t) \right) dt \right] \quad (3.115)$$
subject to
\[ \frac{dx^f}{dt} = A \cdot u^f \quad \forall f \in \mathcal{F} \quad (3.116) \]
\[ x^f(t_0) = 0 \quad \forall f \in \mathcal{F} \quad (3.117) \]
\[ x^f(t_1) \leq K^f \quad \forall f \in \mathcal{F} \quad (3.118) \]
\[ \frac{dp^f}{dt} = r^f \quad \forall f \in \mathcal{F} \quad (3.119) \]
\[ p^f(t_0) = \tilde{p}^f \quad \forall f \in \mathcal{F} \quad (3.120) \]
\[ r^f_{\text{min}} \leq r^f \leq 0 \quad \forall f \in \mathcal{F} \quad (3.121) \]
\[ p^f_{\text{min}} \leq p^f \leq \text{constant} \quad \forall f \in \mathcal{F} \quad (3.122) \]
\[ u^f_{\text{min}} \leq u^f \leq u^f_{\text{max}} \quad \forall f \in \mathcal{F} \quad (3.123) \]

The Hamiltonian of the above optimal control problem is
\[ H_2(r, u, \lambda) = \sum_{f \in \mathcal{F}} \sum_{i \in \mathcal{S}} \left[ p^f_i + \left( \lambda^f \right)^T A_i \right] \cdot u^f_i \]
\[ - \sum_{f \in \mathcal{F}} \sum_{i \in \mathcal{S}} p^f_i \cdot d^f_i (p,t) \int_0^{d^f_i (p,t)} G(\tau) \, d\tau + \sum_{f \in \mathcal{F}} \sum_{i \in \mathcal{S}} r^f_i \]

Let \((\hat{r}, \hat{u})\) be the cooperative markdown of service prices and resource allocations respectively for the firms where
\[ \hat{r} = \left( \hat{r}^f \right)_{f \in \mathcal{F}} \]
\[ \hat{u} = \left( \hat{u}^f \right)_{f \in \mathcal{F}} \]

Cooperative markdown of service prices and resource allocations can be obtained by solving the following DVI:

\[ \text{find } \begin{pmatrix} \hat{r} \\ \hat{u} \end{pmatrix} \in \tilde{K} \text{ such that } \int_{t_0}^{t_1} \left( \nabla_r H_2 \cdot (r - \hat{r}) + \nabla_u H_2 \cdot (u - \hat{u}) \right) \, dt \leq 0 \quad (3.124) \]

for all \( \begin{pmatrix} r \\ u \end{pmatrix} \in \tilde{K} \)
where

\[ \hat{H}_2 = H_2 (\hat{r}, \hat{u}, \hat{\lambda}) \]

### 3.3.9 Numerical Example

**Problem generation and parameters**

Our numerical example is motivated by the airline RM problem and the network model we use here is the same one we used in section 3.2.16. We have considered a 2-hub 6-node and 12-legs airlines network as shown in Figure 1, where nodes 2 and 3 are the ‘hub’ nodes. Three firms are competing over this network. To keep the exposition simple we assume that each firm uses the same network; however this can be relaxed with a more general setting where each firm has different service network. We consider 18 different O-D pairs of this network and because of simplicity we assume each O-D pair is connected by only one path (itinerary) which are the different services that each firm offers as shown in Table 3.8.

We assume that the average demands for different services follow a linear model. Further-
Table 3.8: Mapping of O-D pair with itinerary : 18 O-D pair

<table>
<thead>
<tr>
<th>Service ID</th>
<th>O-D pair</th>
<th>Itinerary</th>
<th>Service ID</th>
<th>O-D pair</th>
<th>Itinerary</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 − 2</td>
<td>1 − 2</td>
<td>10</td>
<td>3 − 2</td>
<td>3 − 2</td>
</tr>
<tr>
<td>2</td>
<td>1 − 3</td>
<td>1 − 3</td>
<td>11</td>
<td>3 − 4</td>
<td>3 − 4</td>
</tr>
<tr>
<td>3</td>
<td>1 − 4</td>
<td>1 − 2 − 4</td>
<td>12</td>
<td>3 − 5</td>
<td>3 − 5</td>
</tr>
<tr>
<td>4</td>
<td>1 − 5</td>
<td>1 − 3 − 5</td>
<td>13</td>
<td>3 − 6</td>
<td>3 − 4 − 6</td>
</tr>
<tr>
<td>5</td>
<td>1 − 6</td>
<td>1 − 6</td>
<td>14</td>
<td>4 − 6</td>
<td>4 − 6</td>
</tr>
<tr>
<td>6</td>
<td>2 − 3</td>
<td>2 − 3</td>
<td>15</td>
<td>5 − 3</td>
<td>5 − 2</td>
</tr>
<tr>
<td>7</td>
<td>2 − 4</td>
<td>2 − 4</td>
<td>16</td>
<td>5 − 3</td>
<td>5 − 3</td>
</tr>
<tr>
<td>8</td>
<td>2 − 5</td>
<td>2 − 3 − 5</td>
<td>17</td>
<td>5 − 4</td>
<td>5 − 3 − 4</td>
</tr>
<tr>
<td>9</td>
<td>2 − 6</td>
<td>2 − 4 − 6</td>
<td>18</td>
<td>5 − 6</td>
<td>5 − 3 − 4 − 6</td>
</tr>
</tbody>
</table>

more, average demand for a particular type of service provided by a firm depends only on the own and non-own prices for that service, i.e.,

\[ d^f_i(p_i, t) = \alpha_i^f(t) - \beta_i^f(t) \cdot p_i^f(t) + \sum_{g \neq f} \gamma_i^f,g(t) \cdot p_i^g(t) \]

for all \( f \in \mathcal{F}, i \in \mathcal{S}, t \in [t_0, t_1] \) and \( \alpha_i^f(t), \beta_i^f(t), \gamma_i^f,g(t) \in \mathbb{R}_+^{1} \). The random variables associated with the stochastic demand are assumed to follow a uniform distribution with range \([0.3, 1.7]\) which is independent of service types and firms. The booking period runs from clock-time 0 to 10. Lower and upper bounds on the service prices are set independent of the identity of firms depending on number of legs used in a particular itinerary, as shown below.

<table>
<thead>
<tr>
<th>No. of legs in the itinerary</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_{i, \text{min}}^f )</td>
<td>75</td>
<td>150</td>
<td>225</td>
</tr>
<tr>
<td>( p_{i, \text{max}}^f )</td>
<td>500</td>
<td>650</td>
<td>725</td>
</tr>
</tbody>
</table>

Each service provider has different, yet mostly comparable capacities of 12 legs as listed in Table 3.9.
Table 3.9: Leg Capacities : Pricing-Allocation Numerical Example

<table>
<thead>
<tr>
<th>Legs</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm1</td>
<td>300</td>
<td>200</td>
<td>200</td>
<td>400</td>
<td>800</td>
<td>400</td>
</tr>
<tr>
<td>Firm2</td>
<td>320</td>
<td>360</td>
<td>180</td>
<td>320</td>
<td>700</td>
<td>440</td>
</tr>
<tr>
<td>Firm3</td>
<td>280</td>
<td>300</td>
<td>220</td>
<td>380</td>
<td>820</td>
<td>380</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Legs</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm1</td>
<td>400</td>
<td>200</td>
<td>600</td>
<td>200</td>
<td>200</td>
<td>240</td>
</tr>
<tr>
<td>Firm2</td>
<td>500</td>
<td>210</td>
<td>720</td>
<td>300</td>
<td>400</td>
<td>320</td>
</tr>
<tr>
<td>Firm3</td>
<td>440</td>
<td>220</td>
<td>480</td>
<td>280</td>
<td>600</td>
<td>340</td>
</tr>
</tbody>
</table>

**Bidirectional Pricing**

**Competitive Pricing and Resource Allocation** In Figure 3.5, we plot competitive bidirectional equilibrium price trajectories grouped by firms over the planning horizon. The equilibrium rate of provision of services by different firms are plotted vs. time in Figure ???. The average demand for 18 different services (grouped by firms) are plotted vs. time in Figure 3.6.
Figure 3.5: Plot of non-cooperative equilibrium service prices (grouped by firms) vs. time

Plot of non-cooperative equilibrium rate of provision of services (grouped by firms) vs. time
Cooperative Pricing and Resource Allocation  In this section we discuss pricing and allocation strategies for the firms when they are in complete collusion. Figure 3.7 plots collusive bidirectional equilibrium price trajectories grouped by firms over the planning horizon. The equilibrium rate of provision of services by different firms are plotted vs. time in Figure 3.8. The average demand for 18 different services (grouped by firms) are plotted vs. time in Figure 3.9.

Comparison of Competitive vs. Collusive Pricing and Allocations  In Figure 3.10 we compare competitive pricing with that under collusion by plotting $(\bar{p}^i(t) - \bar{p}^*_{i}(t))$ vs. time for all 18 services and 3 firms where $\bar{p}$ and $p^*$ are the cooperative and competitive service prices respectively. It is evident from the plot that competition leads to underpricing of services which is, for certain services, as high as $500 per unit of service (airlines seat in our example). Similarly, we plot in Figure 3.11 the difference between $\bar{u}^f_i(t) - u^*_i(t)$ over time where $\bar{u}$ and $u^*$ are the rates of provision of different services by the firms when they compete with each other and collude respectively. We observe that for most of the service
Figure 3.7: Plot of cooperative equilibrium service prices (grouped by firms) vs. time

Figure 3.8: Plot of rate of provision of services (grouped by firms) vs. time when firms form collusion
types competition leads to over-allocation of resources.

**Comparison of Expected Revenue**  When the firms are in competition, they generate a total revenue of $2,643,900 with the following breakdown of individual’s expected revenue.

<table>
<thead>
<tr>
<th>Firms</th>
<th>Expected Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1</td>
<td>$780,200</td>
</tr>
<tr>
<td>Firm 2</td>
<td>$1,013,900</td>
</tr>
<tr>
<td>Firm 3</td>
<td>$849,800</td>
</tr>
</tbody>
</table>

However, when they collude, their total expected revenue increases to $2,926,400, therefore in this example competition causes an efficiency loss of around 11%.  

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Figure 3.10: Comparison of cooperative service price with competitive service price for 18 different services provided by 3 firms

Figure 3.11: Comparison of cooperative service provision rate with competitive service provision rate for 18 different services provided by 3 firms
Markdown Pricing

**Competitive Pricing and Resource Allocation** When we consider markdown pricing the rate of service provision trajectories show remarkably different qualitative properties compared to bidirectional pricing. In Figure 3.12 we plot competitive markdown price trajectories grouped by firms over the planning horizon. The equilibrium rate of provision of services by different firms are plotted vs. time in Figure 3.13.

**Cooperative Pricing and Resource Allocation** Figure 3.14 describes cooperative markdown price trajectories grouped by firms over the planning horizon. The equilibrium rate of provision of services by different firms are plotted vs. time in Figure 3.15.
Figure 3.13: Plot of non-cooperative equilibrium rate of provision of services (grouped by firms) vs. time under markdown pricing scheme

Figure 3.14: Plot of cooperative equilibrium service prices (grouped by firms) vs. time under markdown pricing
Figure 3.15: Plot of cooperative equilibrium rate of provision of services (grouped by firms) vs. time under markdown pricing scheme

3.3.10 Concluding Remark

We have shown that competitive service pricing and resource allocation, cornerstones of revenue management, may be articulated as differential variational inequalities. Additionally, we presented the numerical solution of an example service network pricing and resource allocation problem, motivated by the airline industry, using a fixed point algorithm. This example suggests that the differential variational inequality perspective for dynamic Cournot-Nash games is computationally tractable.

This work may be expanded upon to include more complicated allocation methods such as theft-nesting. Other possible future work includes modeling scenarios where customers pay for service and are permitted to cancel their order before receiving the service. Such scenarios lead to overbooking policies as seen in the airline industry. In addition to the pricing and resource allocation decisions made in the model presented in this paper, the firms must also determine a level of overbooking, in light of customer no-shows and cancellations, that maximizes the utilization of seats on a flight.
3.4 Dynamic Pricing in a Network Oligopoly with Evolutionary Demand Learning

In this section we advocate price-based network revenue management in a Cournot-Nash dynamic game theoretic setting. This is done by employing a differential variational inequality formulation of a dynamic, non-zero sum evolutionary game that describes oligopolistic competition among service providers who have no variable costs, hold no inventories, accept no backorders, and offer a variety of products. The providers also have fixed upper bounds on output.

The service providers of interest are follow a learning process that is similar to evolutionary game-theoretic dynamics and for which price changes are proportional to their signed excursion from a market clearing price. We stress that in this model firms are setting prices for their services while simultaneously determining the levels of demand they will serve. This is unusual in that, typically, firms in oligopolistic competition are modelled as setting either prices or output flows. The joint adjustment of prices and output is modelled here as determined by comparing the current price to the price that would have cleared the market for the demand that has most recently been served. However, the service providers are unable to make this comparison until the current round of play is completed as knowledge of the total demand served by all competitors is required.

Kachani et al. [56] put forward a revenue management model for service providers to address such joint pricing and demand learning in an oligopoly setting under fixed capacity constraints. The model they consider assumes a service provider’s demand is a linear function of its price and other competitors’ prices; each company learns to set their parameters over time, though the impact of a change in price on demand in one period does not automatically propagate to latter time periods. In our work we allow this impact to propagate to all the time periods down the line.

In most of the revenue management industries, the customers can be classified into
two broad categories (see Dasgupta and Das [57]):

1. Bargain-hunting buyers, who may represent the general public, who are searching for personal or, to a limited extent, business services or products at the most competitive price; these buyers are willing to sacrifice some convenience for the sake of a lower price. Because the services and products are assumed to be homogeneous, if two sellers offer the same price, the tie is broken randomly. In other words, the consumers have no brand preference.

2. Randomly-selecting buyers may represent corporate entities who set a threshold for price and are willing to accept any price below this threshold in the sake of convenience. These buyers are brand conscious and are not very flexible in their schedules.

For the purposes of this paper, we will only be considering bargain-hunting buyers. The randomly selecting buyers are not very sensitive to the price and hence are not considered in this paper.

The non-cooperative dynamic oligopolistic competition amongst the service providing agents may be placed in the form of a dynamic variational inequality (DVI) wherein each individual agent develops for itself a pricing plan that is based on current and past knowledge of non-own price patterns. Each such agent is, therefore, described by an optimal control problem for which the control variables are the price of its various service classes; this optimal control problem is constrained by dynamics that describe how the agent’s share of demand alters with time. The Cournot-Nash assumption that game agents are non-cooperative allows the optimal control problems for individual agents to be combined to obtain a single DVI. This variational inequality may be made stochastic to incorporate random noise in the demand due to change of global and regional scenarios other than price changes, though we do not explore such an extension in this paper. We stress that the reduction of this problem to a DVI is original and a relatively unexplored area. By placing the dynamic gaming behavior that underlies demand fluctuations in the form of a
variational inequality, one has available a substantial arsenal of mathematical and efficient computational tools from functional analysis.

The remainder of the paper is organized as follows: a detailed exposition of our revenue management model of dynamic oligopolistic competition in Section 3.4.1. Section 3.4.5 shows how dynamic revenue management oligopolistic competition may be expressed as a DVI and establishes the existence of a solution to that formulation under plausible regularity conditions. In Section 5.2.10 we outline a set of computational tools to solve this class of DVI and in particular present fixed point algorithm which is used in the next section to solve an example. Section 3.4.7 provides a detailed numerical example to show some interesting behaviors of the agents. Section 4.2.7 summarizes our findings and describes future research.

3.4.1 Description of the Model

A set of service providers are competing in an oligopolistic setting, each with the objective of maximizing their revenue. These service providers have very high fixed costs compared to their relatively low variable or operating costs. Therefore, the providers focus only on maximizing their own revenue.

Each company provides a set of services where each service type is assumed to be homogeneous. For example, the difference between a seat on Southwest and an economy class seat on Jet Blue is indiscernible by customers; the only differences that the customers perceive are the prices charged by the different service providers and restrictions imposed on the ticket (nonrefundable, minimum nights stay etc.). These companies utilize a common set of resources to provide the services. Each firm has a finite capacity of resources which imposes joint constraints for the upper bound of service that the firms can offer.

All service providers can set the price for each of their services. The price that they charge for each service in one time period will affect the demand that they receive.
for that service in the next time period. The price that the service provider charges is compared to the rolling average price of their competitors and the provider’s demand is affected accordingly. The providers must therefore choose prices that create an amount of demand for their services that will maximize their revenue while ensuring that the joint capacity constraints are satisfied.

**Basic Notation**

We denote the set of revenue managing firms as $\mathcal{F}$, each of whom is providing a set of services $\mathcal{S}$. Continuous time is denoted by the scalar $t \in \mathbb{R}_+^1$, while $t_0$ is the finite initial time and $t_1 \in \mathbb{R}_+^1$ the finite terminal time so that $t \in [t_0, t_1] \subset \mathbb{R}_+^1$.

Each firm $f \in \mathcal{F}$ controls price

$$\pi_i^f \in L^2 [t_0, t_1]$$

corresponding to each service type $i \in \mathcal{S}$, where $L^2 [t_0, t_1]$ is the space of square-integrable functions for the real time interval $[t_0, t_1] \in \mathbb{R}_+^1$. The control vector of each firm $f \in \mathcal{F}$ is

$$\pi^f \in (L^2 [t_0, t_1])^{\mid \mathcal{S} \mid}$$

and the concatenation of these vectors is

$$\pi \in (L^2 [t_0, t_1])^{\mid \mathcal{S} \mid \times \mid \mathcal{F} \mid}$$

the complete vector of controls.

We also let

$$N_i^f (\pi, t) : (L^2 [t_0, t_1])^{\mid \mathcal{S} \mid \times \mid \mathcal{F} \mid} \times \mathbb{R}_+^1 \longrightarrow \mathcal{H}^1 [t_0, t_1]$$

denote the demand for service $i \in \mathcal{S}$ of firm $f \in \mathcal{F}$ and define the vector of all such demands for firm $f$ to be

$$N^f \in (\mathcal{H}^1 [t_0, t_1])^{\mid \mathcal{S} \mid}$$
and all such demands for service \(i \in S\) of firm \(f \in F\)

\[ N \in \left( \mathcal{H}^1 [t_0, t_1] \right)^{|S| \times |F|} \]

We will use the notation

\[ \pi^f = (\pi_i^g : i \in S, g \in F - \{f\})^T \in (L^2 [t_0, t_1])^{q_f} \]

for the vector of all non-own prices relative to the firm \(f \in F\), where

\[ q_f = |S| \times (|F| - 1) \]

### 3.4.2 Demand Dynamics

The demand for the service offerings of firm \(f \in F\) evolve according to the following evolutionary game-theoretic dynamics adapted from Feudenberg and Levine [58]:

\[
\frac{dN_i^f(t)}{dt} = \eta_i^f(t) \cdot \left( \bar{\pi}_i(t) - \pi_i^f(t) \right) \quad \forall i \in S, \ f \in F, \ t \in [t_0, t_1] \\
N_i^f(t_0) = K_{i,0} \quad \forall i \in S, \ f \in F
\]

(3.125)  (3.126)

where \(\bar{\pi}_i\) is the moving average price for service \(i \in S\) given by

\[
\bar{\pi}_i = \frac{1}{|F|(t - t_0)} \int_{t_0}^t \sum_{g \in F} \pi_i^g(\tau)d\tau \quad \forall i \in S
\]

while \(K_{i,0} \in \mathbb{R}^{1+}\) and \(\eta_i^f(t) \in \mathbb{R}^{1+}\) are exogenous parameters for each \(i \in S\) and \(f \in F\).

The firms set the parameter \(\eta_i^f\) by analyzing the past demand data and the sensitivity of the demand with respect to price. The demand for a service type \(i\) of a firm \(f\) changes over time in accordance with the excess between the firm’s price and the moving average of all agents’ prices for the particular service. The coefficient \(\eta_i^f\) controls how quickly demand reacts to price changes for each firm \(f\) and service type \(i\). Some providers may specialize in certain services and may be able to adjust more quickly than their competitors.

These dynamics represent a learning mechanism for the firms. As stated here, the dynamics are reminiscent of replicator dynamics (see Hofbauer and Sigmund [59] for detailed
which are used in evolutionary games. The rate of growth of demand, can be viewed as the rate of growth of the firm $f$ with respect to service type $i$. This growth follows the "basic tenet of Darwinism" [59], and may be interpreted as the difference between the fitness (price) of the firm for the service and the rolling average fitness of all the agents for that service.

3.4.3 Constraints

There are positive upper and lower bounds, based on market regulations or knowledge of customer behavior, on service prices charged by firms. Thus we write

$$\pi_{i,\min}^f \leq \pi_i^f \leq \pi_{i,\max}^f \quad \forall i \in S, \quad f \in F$$

where the $\pi_{i,\min}^f \in \mathbb{R}_+$ and $\pi_{i,\max}^f \in \mathbb{R}_{++}$ are known constants. Similarly, there will be a lower bound of zero on the demand for services of each type by each firm as negative demand levels are meaningless; that is

$$N_{i}^f \geq 0 \quad \forall i \in S, \quad f \in F$$

Let $\mathcal{R}$ be the set of resources that the firms can utilize to provide the services, $|\mathcal{R}|$, cardinality of the set $\mathcal{R}$, denotes number of resources at firm’s disposal. We define the incidence matrix (Talluri and Van Ryzin [34]) $\mathcal{A} = [a_{ij}]$ as

$$a_{ij} = \begin{cases} 1 & \text{if resource } i \text{ is used by the service type } j \\ 0 & \text{else} \end{cases}$$

$\mathcal{A}$ is a $|S| \times |\mathcal{R}|$ matrix. Note that the $j$th column of $\mathcal{A}$, denoted as $\mathcal{A}_j$, is the incidence vector for service $j$. Further, the $i$th row, denoted as $\mathcal{A}^i$, has an entry of 1 in column $j$ that uses resource $i$. If there are multiple ways of selling a given bundle of resources, $\mathcal{A}$ will have several identical columns. For example, in the airline case there may be many fare classes for the same itinerary (hence using same set of resources) where as in the hotel revenue management case, there may be many tariff classes for the same duration of stay.
The incidence matrix may also be firm specific as some firms may use different resources to provide the same service. Again, looking at the airlines industry, the flight path between two cities can differ between service providers.

Let $C_j^f$ be firm $f$’s capacity of resource type $j \in \mathcal{R}$. Joint resource-constraints for firm $f$ are

$$A \cdot N_f \leq C_f^f$$

where $C_f^f \in \mathbb{R}_{++}^1$ is the vector of firm $f$’s capacity

$$C_f^f = (C_j^f : j \in \mathcal{R})$$

### 3.4.4 Service Providers’ Extremal Problems

Since revenue management firms have very little variable costs and high fixed costs, each firm’s objective is to maximize revenue which in turn ensures the maximum profit since the variable costs are negligible. We further note that each firm $f \in \mathcal{F}$ faces the following problem: with the $\pi^{-f}$ as exogenous inputs, solve the following optimal control problem:

$$\max_{\pi_f, \pi^{-f}, t} J_f(\pi^f, \pi^{-f}, t) = \int_{t_0}^{t_1} e^{-\rho t} \left( \sum_{i \in \mathcal{S}} \pi_i^f \cdot N_i^f \right) dt - e^{-\rho t_0} \Psi_0^f$$

subject to

$$\frac{dN_i^f}{dt} = \eta_i^f \cdot (\bar{\pi}_i - \pi_i^f) \quad \forall i \in \mathcal{S}$$

$$N_i^f(t_0) = K_{i,0} \quad \forall i \in \mathcal{S}$$

$$\pi_{\text{min}}^f \leq \pi^f \leq \pi_{\text{max}}^f$$

$$N^f \geq 0$$

$$A \cdot N^f \leq C^f$$
where $\Psi^f_0$ is the fixed cost of production for firm $f$, $\rho$ is the nominal discount rate compounded continuously, and

$$
\int_{t_0}^{t_1} e^{-\rho t} \left( \sum_{i \in S} \pi^f_i \cdot N^f_i \right) \, dt
$$

is the net present value (NPV) of revenue. From the familiarity with these dynamics, we may restate them as: for all $f \in \mathcal{F}$

$$
\frac{dN^f_i}{dt} = \eta^f_i \cdot \left( \frac{y_i}{|\mathcal{F}|} \cdot (t - t_0) - \pi^f_i \right) \quad \forall i \in S
$$

$$
\frac{dy_i}{dt} = \sum_{g \in \mathcal{F}} \pi^g_i \quad \forall i \in S
$$

$$
N^f_i(t_0) = K^f_{i,0} \quad \forall i \in S
$$

$$
y_i(t_0) = 0 \quad \forall i \in S
$$

As a consequence we may rewrite the optimal control problem of firm $f \in \mathcal{F}$ as

$$
\max_{\pi^f} J_f(\pi^f, \pi^{-f}, t) = \int_{t_0}^{t_1} e^{-\rho t} \left( \sum_{i \in S} \pi^f_i \cdot N^f_i \right) \, dt - e^{-\rho t_0} \Psi^f_0
$$

subject to

$$
\frac{dN^f_i}{dt} = \eta^f_i \cdot \left( \frac{y_i}{|\mathcal{F}|} \cdot (t - t_0) - \pi^f_i \right) \quad \forall i \in S
$$

$$
\frac{dy_i}{dt} = \sum_{g \in \mathcal{F}} \pi^g_i \quad \forall i \in S
$$

$$
N^f_i(t_0) = K^f_{i,0} \quad \forall i \in S
$$

$$
y_i(t_0) = 0 \quad \forall i \in S
$$

$$
\pi^f_{\min} \leq \pi^f \leq \pi^f_{\max}
$$

$$
N^f \geq 0
$$

$$
A \cdot N^f \leq C^f
$$

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Consequently,

\[ N^f(\pi) = \arg \left\{ \frac{dN^f_i}{dt} = \eta^f_i \cdot \left( \frac{y_i}{|F|(t-t_0)} - \pi^f_i \right) \right\} \]

\[ \frac{dy_i}{dt} = \sum_{g \in F} \pi^g_i; \quad N^f_i(t_0) = K^f_{i,0}, \quad i \in S \]

\[ 0 \leq N^f, \quad A \cdot N^f \leq C^f \]  (3.146)

where we implicitly assume that the dynamics have solutions for all feasible controls. In compact notation this problem can be expressed as: with the \( \pi^-f \) as exogenous inputs, compute \( \pi^f \) in order to solve the following optimal control problem:

\[
\max \quad J_f(\pi^f, \pi^-f, t) \\
\text{subject to} \quad \pi^f \in \Lambda_f \\
\forall f \in F
\]  (3.147)

where \( \Lambda_f = \left\{ \pi^f : (3.139), (3.140), (3.141), (3.142), (3.143), (3.144) \text{ and } (3.145) \text{ hold} \right\} \)

### 3.4.5 DVI Formulation of Network Revenue Management Competition

Each carrier is a Cournot-Nash agent that knows and employs the current instantaneous values of the decision variables of other firms to make its own non-cooperative decisions. Therefore (3.147) defines a set of coupled optimal control problems, one for each firm \( f \in F \).

It is useful to note that (3.147) is an optimal control problem with fixed terminal time and fixed terminal state. Its Hamiltonian is

\[
H_f(\pi^f, N^f; \lambda^f; \sigma^f, \alpha^f, \beta^f; \pi^-f; t) = e^{-\rho t} \left( \sum_{i \in S} \pi^f_i \cdot N^f_i \right) + \Phi_f(\pi^f, N^f; \lambda^f; \sigma^f; \alpha^f; \beta^f; \pi^-f) \\
\]  (3.148)

where

\[
\Phi_f(\pi^f; N^f; \lambda^f; \sigma^f; \alpha^f, \beta^f; \pi^-f) = \left\{ \sum_{i \in S} \lambda^f_i \left[ \eta^f_i \cdot \left( \frac{y_i}{|F|(t-t_0)} - \pi^f_i \right) \right] + \sum_{i \in S} \sigma^f_i \left( \pi^f_i + \sum_{g \in F \setminus f} \pi^g_i \right) \right\} \\
+ \sum_{i \in S} \alpha^f_i (-N^f_i) + \sum_{j \in R} \beta^f_j \left( A^j \cdot N^f \right) \]  (3.149)
while \( \lambda^j_i \in \mathcal{H}^1 [t_0, t_1] \) is the adjoint variable for the dynamics associated with the firm \( f \) with service type \( i \), complete vector of the adjoint variables is
\[
\lambda \in (\mathcal{H}^1 [t_0, t_1])^{|\mathcal{F}| \times |\mathcal{S}|}
\]
adjoint variable associated with the auxiliary state variable \( y_i \) is \( \sigma^f_i \in \mathbb{R}_+^{|\mathcal{S}|} \) for all \( i \in \mathcal{S} \), and \( \alpha^j_i \in \mathbb{R}_+^{|\mathcal{S}|}, \beta^f_j \in \mathbb{R}_+^{|\mathcal{R}|} \) are the dual variables arising from the state space constraints. In compact vector notation \( \sigma \in \mathbb{R}_+^{|\mathcal{S}|}, \alpha^f \in \mathbb{R}_+^{|\mathcal{S}|} \) and \( \beta^f \in \mathbb{R}_+^{|\mathcal{R}|} \). The instantaneous profit for firm \( f \) is \( \sum_{i \in \mathcal{S}} \pi^f_i \cdot N^f_i \). We assume in the balance of this section that the game (3.4.1) is regular in the sense of the following definition:

**Definition 3.7** The dynamic model for revenue management introduced above will be considered regular if all components of the functional \( N(\pi, t) \) are strongly continuous and \( G \)-differentiable.

Using the maximum principle [28] tells us that an optimal solution to (3.4.1) is a sextuplet \( \{ \pi^f* (t), N^f* (t); \lambda^f* (t), \sigma^*, \alpha^f, \beta^f \} \) requiring that the nonlinear program
\[
\max H_f \quad \text{s.t.} \quad \pi^f_{\max} \geq \pi^f \geq \pi^f_{\min}
\]
be solved by every firm \( f \in \mathcal{F} \) for every instant of time \( t \in [t_0, t_1] \) where
\[
\pi^f_{\max} = \left\{ \pi^f_{i, \max} : i \in \mathcal{S} \right\}
\]
\[
\pi^f_{\min} = \left\{ \pi^f_{i, \min} : i \in \mathcal{S} \right\}
\]
Consequently, any optimal solution must satisfy at each time \( t \in [t_0, t_1] \):
\[
\pi^f* = \arg \left\{ \max_{\pi^f_{\min} \leq \pi^f \leq \pi^f_{\max}} H_f \left( \pi^f; N^f, \lambda^f, \sigma, \alpha^f, \beta^f, \pi^{-*f}; t \right) \right\}
\]
which in turn, by virtue of regularity, is equivalent to
\[
[\nabla_{\pi^f} H_f]^T \left( \pi^f - \pi^f* \right) \leq 0 \quad \text{for all} \quad \pi^f_{\min} \leq \pi^f \leq \pi^f_{\max}
\]
where
\[
H_f^* = e^{-\rho t} \left( \sum_{i \in \mathcal{S}} \pi^f_i N^f_i \right) + \Phi^*_f
\]
and
\[ \Phi_f^* = \Phi_f (\pi^f; N^*f; \lambda^f; \sigma^f; \alpha^f; \beta^f; \pi^{-f}) \] (3.153)

From (5.9)
\[ \left[ \nabla_{\pi^f} \left[ e^{-\rho t} \left( \sum_{i \in S} \pi^f_i \cdot N^i \right) \right] + \nabla_{\pi^f} \Phi_f^* \right] \left( \pi^f - \pi^f \right) \leq 0 \] (3.154)

for all \( \pi_{\min}^f \leq \pi^f \leq \pi_{\max}^f \)

Further, adjoint dynamics obey
\[ \frac{d\lambda^f}{dt} = (-1) \frac{\partial H_f^*}{\partial N^f} \] (3.155)
\[ = \pi^f - \alpha^f + \sum_{j \in \mathcal{R}} \beta^f_j \cdot a_{ij} \] (3.156)

Due to absence of terminal time constraints on states, the transversality condition gives
\[ \lambda^f (t_1) = \gamma^T \frac{\partial \Gamma \left[ N^f (t_1), t_1 \right]}{\partial N^f (t_1)} = 0 \]

which gives rise to a two point boundary value problem.

With this preceding background, we are now in a position to create a variational inequality for the non-cooperative competition amongst the firms. We consider the following DVI which has solutions that are Cournot-Nash equilibria for the game described above in which individual firms maximize their revenue in light of current information about their competitors:

find \( \pi^* \in \Omega \) such that
\[ \sum_{f \in \mathcal{F}} \int_{t_0}^{t_1} \left\{ \nabla_{\pi^f} H_f^* \cdot \left( \pi^f - \pi^f \right) \right\} dt \leq 0 \] (3.157)

for all \( \pi \in \Omega = \prod_{f \in \mathcal{F}} \Omega_f \)

where \( H_f^* \) is defined in (3.152)-(3.153) and
\[ \Omega_f = \left\{ \pi^f : \pi_{\max}^f \geq \pi^f \geq \pi_{\min}^f \right\} \]
This DVI is a convenient way of expressing the Cournot-Nash game that is our present interest. The variational inequality formulation also provides guidance in devising a computational strategy, as we show in Section 5.2.10.

Let us now verify that the DVI (3.157) is regular by showing that definition 3.7 adheres to the regularity definition for a general DVI. To this end we make the following observations:

1. the Hamiltonian, $H_f$ as defined in (5.9) is a linear function of the controls $\pi^f$ and states, $N^f$

2. the operator $N(\pi,t)$ has the properties of continuity and G-differentiability from definition (3.7),

3. $\Omega$ is convex and compact,

4. Right hand side of the state dynamics (3.139)-(3.140) are linear in controls $\pi$ and free from states, $N$.

5. there is no terminal time constraint on the states, therefore, $\Gamma (N(t_1),t_1) = 0$. This is motivated from the fact that salvage value of remaining inventory is zero.

Therefore all the conditions (1) - (5) in definition 3.7 are satisfied, so the DVI (3.157) is regular. The next issue of concern is to formally demonstrate that solutions of our DVI formulation are Cournot-Nash (CN) equilibria. To this end we state and prove the following result:

**Theorem 3.8** *(DVI formulation of non-cooperative dynamic revenue management competition)* Any solution of the DVI (3.157) is a differential game-theoretic equilibrium when the regularity in the sense of Definition 3.7 holds.
Proof. We begin by noting that (3.157) is equivalent to the following optimal control problem

\[
\max Z \left( \pi^f; \pi^{-f}; t \right) = \sum \sum \int_{t_0}^{t_1} \frac{\partial H^*_f}{\partial \pi_i} \pi^f_i dt
\]

s.t. \( \pi \in \Omega \) \hspace{1cm} (3.158)

where it is essential to recognize that \( Z (\pi^f; \pi^{-f}; t) \) is a linear functional that assumes knowledge of the solution to our oligopolistic game; as such \( Z (\pi^f; \pi^{-f}; t) \) is a mathematical construct for use in analysis and has no meaning as a computational device. Since \( \Omega \) is convex, using Theorem 10.6 of Minoux [26], we note that a necessary and sufficient condition for the problem (5.35) and (5.36) is

\[
\langle \nabla_{\pi} Z \left( \pi^{f*}; \pi^{-f*}; t \right), \pi - \pi^* \rangle = \sum \sum \int_{t_0}^{t_1} \frac{\partial Z \left( \pi^{f*}; \pi^{-f*}; t \right)}{\partial \pi_i^f} \left( \pi_i^f - \pi_i^{f*} \right) dt \leq 0
\]

for all \( \pi \in \Omega \). Furthermore,

\[
\nabla_{\pi^f} Z \left( \pi^{f*}; \pi^{-f*}; t \right) = \left( \frac{\partial H^*_f}{\partial \pi_i} : i \in S \right)
\]

(3.161)

From (3.160), (3.161) and (3.151), the desired result (3.157) is immediate. \( \blacksquare \)

Existence

We now state and prove the following existence result:

**Theorem 3.9** (existence of a solution to (3.157)) When regularity in the sense of Definition ?? holds, (3.157) has a solution.

Proof. By the assumption of regularity \( N(\pi, t) \) is well defined and continuous. Therefore, the Hamiltonian, \( H_f (\pi^f; N^f; \lambda^f; \sigma, \alpha^f, \beta^f; \pi^{-f}; t) \), is continuous in \( \pi^f \). We also know \( \Omega \) is convex and compact. Consequently, by Theorem 2 of Browder [27], we conclude that (3.157) has a solution. \( \blacksquare \)
3.4.6 Algorithms for DVIs

Numerical Methods

DVIs can be solved by a numerical method. Two of the most commonly used methods are: (1) fixed point algorithm, which is used for this paper, and (2) conversion of the relevant DVIs to a nonlinear complementarity problem (NCP) after making a finite element approximation. Direct solution of the resulting NCP is possible using successive linearizations [33] and an algorithm such as Lemke’s for solving each linear complementarity subproblem. An alternative approach that can be used with method (2) is to employ a heuristic of finding the primal and dual variables that are most nearly complementary slack using nonlinear programming. If complementary slackness is not achieved, one may resort to successive linearization at any time during the execution of the heuristic. Of course, in certain cases it might be possible to reduce the dimensionality exploiting special structures of the problem.

Fixed Point Algorithm

The DVI can be solved by a fixed point algorithm as outlined in this section. This is the method that is used to solve the example problem in section 3.4.7.

**Theorem 3.10** (fixed point problem) When regularity in the sense of Definition (3.7) holds, any solution of the fixed point problem

\[ u = P_U [u - \alpha G (x(u,t), u, t)] , \]

where \( P_U [\cdot] \) is the minimum norm projection onto \( U \subseteq (L^2[t_0,t_1])^m \) and \( \alpha \in \mathbb{R}^{1+} \), is also a solution of DVI\((G, f, U)\).

**Proof.** See Friesz and Mookherjee [25].
Naturally there is a fixed point algorithm associated with Theorem 3.10; that algorithm is summarized by the following iterative scheme:

\[ u^{k+1} = P_U \left[ u^k - \alpha G \left( x \left( u^k, t \right), u^k, t \right) \right] \]  

(3.162)

The detailed structure of the fixed point algorithm is:

*The Fixed Point Algorithm*

**Step 0. Initialization.** Identify an initial feasible solution \( u^0 \in U \) and set \( k = 0 \).

**Step 1. Optimal control subproblem.** Solve

\[
\begin{align*}
\min_v J^k (v) &= \gamma^T \Gamma \left[ x(t_1), t_1 \right] + \int_{t_0}^{t_1} \frac{1}{2} \left[ u^k - \alpha \cdot G \left( x^k, u^k, t \right) - v \right]^2 dt \\
\text{subject to} & \quad \frac{dx}{dt} = f \left( x, v, t \right) \\
& \quad v \in U \\
& \quad x(0) = x^0
\end{align*}
\]

(3.163)-(3.166)

and call the solution \( u^{k+1} \). Note that in this step it is advantageous to “unfold” and explicitly state the constraints embedded in the operator \( x(u, t) \) in order to facilitate computation.

**Step 2. Stopping test.** If

\[ \left\| u^{k+1} - u^k \right\| \leq \varepsilon \]

where \( \varepsilon \in \mathbb{R}_{++}^1 \) is a preset tolerance, stop and declare \( u^* \approx u^{k+1} \). Otherwise set \( k = k + 1 \) and go to Step 1.

Note that each of the subproblems (3.163)-(3.166) of this particular algorithm can be solved using a combined finite element approximation and math programming approach. Convergence of the above algorithm is guaranteed in certain circumstances by the following result:
Theorem 3.11 (Convergence of Fixed Point Algorithm) When $DVI(F, f, U, x^0)$ is regular in the sense of Definition 3.7 while additionally $G(x, u, t)$ is strongly monotonic for $u \in U$, the fixed point algorithm presented above converges.

Proof. See Friesz and Mookherjee [12].

The fixed point algorithm we consider here is a heuristic approach as the condition in Theorem 3.11 that the principal function of the DVI, $\frac{\partial H_f}{d\pi_i}$ is strictly monotonic is unlikely to be verifiable for problems of realistic size. Hence, the fixed point algorithm becomes a heuristic – until a more general convergence theory is discovered.

3.4.7 Numerical Example

We consider an abstract network revenue management scenario where 5 service providers are involved in oligopolistic competition. Each of these firms offer a set of 4 services utilizing 5 resources and compete for the market demand of these services. The planning horizon for this problem is $[0, 20]$ with $t_0 = 0$, $t_1 = 10$. The incidence matrix, $A$ is assumed to be the same for all firms and is therefore a $5 \times 4$ matrix. We typically consider

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

It is evident that all the resources, except resource 3, are used to provide 2 or more different services. The sensitivity parameter, $\eta_i^f(t)$ is assumed of the form

$$\eta_i^f(t) = \bar{\eta}_i^f \sin \left( \frac{2t}{t_1} \right) = \bar{\eta}_i^f \sin \left( \frac{t}{10} \right)$$
where, $\bar{\eta}$ is the nominal sensitivity, and $\eta$ takes the values

$$
\eta = 10^{-2} \times
$$

<table>
<thead>
<tr>
<th>Service type, $i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1</td>
<td>10</td>
<td>8</td>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td>Firm 2</td>
<td>11</td>
<td>7</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>Firm 3</td>
<td>20</td>
<td>12</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Firm 4</td>
<td>15</td>
<td>10</td>
<td>15</td>
<td>18</td>
</tr>
<tr>
<td>Firm 5</td>
<td>18</td>
<td>6</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

Each firms’ initial demand (at time $t_0$) for services $K_0$ are shown below

<table>
<thead>
<tr>
<th>Service type, $i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1</td>
<td>10</td>
<td>17.5</td>
<td>22.5</td>
<td>30</td>
</tr>
<tr>
<td>Firm 2</td>
<td>9.5</td>
<td>16.5</td>
<td>20</td>
<td>31</td>
</tr>
<tr>
<td>Firm 3</td>
<td>10.5</td>
<td>17</td>
<td>25</td>
<td>28.5</td>
</tr>
<tr>
<td>Firm 4</td>
<td>11</td>
<td>19</td>
<td>24</td>
<td>31</td>
</tr>
<tr>
<td>Firm 5</td>
<td>10.5</td>
<td>18</td>
<td>23</td>
<td>30.5</td>
</tr>
</tbody>
</table>

As introduced in section 3.4.3, $C^f_j$ denotes the maximum resource of type $j$ available to the firm $f$ with

$$
C = \left( C^f_j : j \in R, f \in F \right)
$$

In this example we consider

<table>
<thead>
<tr>
<th>Resource type, $j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1</td>
<td>300</td>
<td>210</td>
<td>150</td>
<td>60</td>
<td>255</td>
</tr>
<tr>
<td>Firm 2</td>
<td>180</td>
<td>150</td>
<td>120</td>
<td>75</td>
<td>210</td>
</tr>
<tr>
<td>Firm 3</td>
<td>306</td>
<td>225</td>
<td>156</td>
<td>63</td>
<td>240</td>
</tr>
<tr>
<td>Firm 4</td>
<td>270</td>
<td>225</td>
<td>156</td>
<td>66</td>
<td>225</td>
</tr>
<tr>
<td>Firm 5</td>
<td>255</td>
<td>210</td>
<td>165</td>
<td>63</td>
<td>270</td>
</tr>
</tbody>
</table>
It is evident from the data set that resource type 4, which is used in providing service types 2 and 4, is very limited for all firms. We assume the maximum price $\pi_{i,\text{max}}^f$ that firm $f$ can charge for the service type $i$ can not be altered in the short run due to regulations or company policy.

<table>
<thead>
<tr>
<th>Service type, $i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1</td>
<td>85</td>
<td>135</td>
<td>180</td>
<td>205</td>
</tr>
<tr>
<td>Firm 2</td>
<td>75</td>
<td>108</td>
<td>185</td>
<td>210</td>
</tr>
<tr>
<td>Firm 3</td>
<td>86</td>
<td>117</td>
<td>155</td>
<td>201</td>
</tr>
<tr>
<td>Firm 4</td>
<td>90</td>
<td>110</td>
<td>165</td>
<td>205</td>
</tr>
<tr>
<td>Firm 5</td>
<td>92</td>
<td>116</td>
<td>175</td>
<td>218</td>
</tr>
</tbody>
</table>

We also assume that in this pricing-game firms are not authorized to lower the price of any of their services beyond a limit, specified by some regulatory authority. If $\pi_{i,\text{min}}$ is the lower bound of price for service $i \in S$, then $\pi_{\text{min}}$ is the vector of all such prices.

<table>
<thead>
<tr>
<th>Service type, $i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_{i,\text{min}}$</td>
<td>30</td>
<td>40</td>
<td>60</td>
<td>130</td>
</tr>
</tbody>
</table>

Demand has to be nonnegative, i.e.,

$$N^f_t \geq 0$$

The pricing problem that the firms face is to continuously set the prices of their 4 services during time $t \in [0, 10]$. The nominal discount rate is $\rho = 0.05$, compounded continuously. The fixed point stopping tolerance has been set at $\varepsilon = 0.05$ and to assist convergence we choose

$$\alpha = \frac{1}{k}$$

where $k$ is the fixed point major iteration counter.

We forgo the detailed symbolic statement of this example and, instead, provide numerical results in graphical form for the solution which was obtained after 427 fixed point iterations. Figure 3.16 shows the price trajectories for the services set by the firms.
as well as the moving average of price for each service type. Figure 3.17 depicts how the demand for the services of each firm change over time in response to the prices set by the firms. The instantaneous revenues generated over time by the firms are plotted in Figure 3.19. Figure 3.18 plots the excess resource vs. time, grouped by the resource type. The NPV of revenue generated by the firms at the end of the planning horizon are:

<table>
<thead>
<tr>
<th>Firm</th>
<th>NPV of Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1</td>
<td>152,990</td>
</tr>
<tr>
<td>Firm 2</td>
<td>173,860</td>
</tr>
<tr>
<td>Firm 3</td>
<td>271,040</td>
</tr>
<tr>
<td>Firm 4</td>
<td>232,240</td>
</tr>
<tr>
<td>Firm 5</td>
<td>189,320</td>
</tr>
</tbody>
</table>

**Interpretation of Results**

The parameters for this numerical example have been designed in such a way that the model resembles a scenario where service type 1 is a lower priced service and type 4 is higher priced service. Firm 2 is a discounter for service types 1 and 2 with modest capacity where as firm 5 provides more expensive options. Demand for firm 3 is more sensitive to the price compared to others. We can see that demand reacts as we would expect to changes in price. For example, consider service type 2 for firm 1; note that the price is initially set very high, and the demand slowly decreases over time until the price is adjusted downward, at which time we see the demand begin a slow increase. Though this is not very insightful, as we expect this to happen, some other conclusions can be made about the pricing strategies.

A fascinating behavior is observed for all the firms in their price setting mechanisms; even though they initially set different prices for a service, towards the end of the planning horizon the firms start behaving similarly if they are not at their pricing bounds. Some of these pricing strategies are quite complicated as they move away from their bounds. If the
Figure 3.16: Price trajectories for the services charged by different firms and the moving average price, $\tilde{\pi}_i$, for each service type (grouped by service types)
Figure 3.17: Firms’ demand trajectories for different service types (grouped by service types)
Figure 3.18: Excess resource, grouped by the resource type
Figure 3.19: Instantaneous revenue generated by the firms

pricing strategies deviate from the optimal trajectories it may cause the firms to perform suboptimally. Finally, the NPV of cumulative revenue shows that the discounting firm 2 can not capitalize it’s low price structure and lags behind the others in the long run while operating at below the capacity level. Firm 3, which has the highest sensitivity, benefited most in the competition, even though it was not designed as a discounter.

**Pricing Rules**

Note that bi-directional changes of price were allowed in this example. However, we still observe predominant mark-down practices by most of the firms. Prices of the services usually tend go up towards the end of the planning horizon which is due to the finite horizon of this problem; the firms’ decisions have no impact past the final time $t_1$. However, at time beginning of the planning horizon firms set the prices for all the services at the maximum
allowed value, and then start to use mark-downs and promotions at different points of time to boost up demand. Other empirical pricing rules for this game can be devised by experimenting with the input data-set.

**Computational Performance**

As mentioned earlier, the fixed point algorithm converged after 427 major iterations for this numerical example. In Figure 3.20 the relative change from one iteration to the next, expressed as

$$\Delta_k = \| \pi^k - \pi^{k-1} \|$$

is plotted against the iteration counter $k$. It is worth noting that for this particular example even though $\Delta_1 = 669.89$, in the next several iterations $\Delta_k$ decrease very rapidly. The run time for this example is less than 5 minutes using a generic desktop computer with single a Intel Pentium 4 processor and 1 GB RAM. The computer code for the fixed point algorithm is written in MatLab 6.5 and calls a gradient projection subroutine for which the control, state and adjoint variables are determined in the sequential fashion explained earlier.

**3.4.8 Concluding Remarks**

In this section we have provided a foundation game-theoretic model to describe a revenue management problem and predict the dynamic pricing behavior by the firms. The example presented establishes computability of the model and illustrates the importance of faithfully capturing the behavior of the agents in any model of a revenue management system. Even this simple model shows how complicated and rich the behavioral patterns of the firms are and how important it is to steer the firms carefully along the optimal trajectory; even a slight deviation from the optimal trajectory may bring disastrous performance for a firm. The complicated behaviors found for the sample model presented here are a motivation for studying more sophisticated models in the future. The addition of stochasticity, one of
our near term goals, will only serve to make the calculated trajectories more complex and will capture the uncertainty (or noise component) of the demand due to global or seasonal disturbances. Another possible direction would be to consider a network where the firms are spatially separated and there are delays based on the congestion levels of the network. Also, the model could be extended to simultaneously consider pricing and resource allocation. As part of further qualitative analyses, it would be useful to explore the inter-relationships of stability, speed of return to equilibrium, and the general vulnerability of the revenue management systems.

Figure 3.20: (Left) Relative change of controls from one iteration to the next ($\Delta_k$) vs. the iteration counter ($k$), (Right) the zoomed-in view
Chapter 4

Dynamic Competition in Urban Freight Network and Supply Chain

4.1 Introduction and Brief Literature Review

The last 5 years has seen an increased recognition of the importance of urban freight systems and the need to plan for their growth if urban centers are to realize their full potential. In particular it is now widely agreed that dynamic urban freight infrastructure planning models are needed if one is to capture the multiple time scales, as well as the fluctuating demands and costs, characteristic of the urban freight environment. The urban freight transportation system faces major challenges, which are a consequence of demand, supply, economic and environmental factors: (1) its increasing role as the conveyor of goods in an Internet connected world; (2) the increased popularity of Just-In-Time production systems; (3) growing demand for service differentiated systems that enable users to pick the level of service best suited to their needs; (4) the increasing difficulties of adding freight capacity in
congested urban areas, making urban freight service prohibitively expensive; (5) the huge pulse in container traffic due to Malacca-max super containerships that can unload tens of thousands of containers in a short amount of time; (6) increased awareness of the need to mitigate or altogether eliminate the negative externalities produced by freight activity; and (7) recognition of the need for a paradigm shift toward environmentally sustainable logistics and freight technologies that create a good-neighbor image.

Friesz and Holguin-Veras [60] have developed a new family of deterministic dynamic models for planning urban freight capacity enhancements that address some of the challenges described above. Holguin-Veras [61] put forward a simulation based model for the integrative freight market. Furthermore, Holguin-Veras and Thorson [62] model the combined loading and routing problems that the carriers face as a capacitated vehicle routing problem with ‘empty trip-chains’. In Mookherjee et al. [63] we take the service operations management perspective to study competitive urban freight network. Dynamic pricing and inventory management has been an important problem area in operations management which creates an interface between marketing and production/inventory planning – specifically the simultaneous determination of pricing and inventory replenishment strategies in the face of demand uncertainty. See Federgruen and Heching [64] for a combined pricing and inventory management model for a single firm with deterministic demand and Bernstein and Federgruen [65] for a market-based model when demand is random. Furthermore, recent developments in the area of yield and revenue management have demonstrated that major benefits can be derived by complementing a replenishment strategy with the dynamic adjustment of price of the goods and services as a function of its on-hand inventory and the length of its remaining sales season. There exists a significant body of literature in revenue management on dynamic pricing, resource allocation and combined pricing and allocation considering single firm’s problem. Friesz et al. [66] and Perakis and Sood [40] are two pertinent recent papers on joint pricing and resource allocation for service firms who form an oligopoly. Some other relevant papers include Parlar [43], Lipman and McCardle [44], Mahajan and van Ryzin [45] and Netessine and Rudi [46].
In Mookherjee et al. [?] we discuss a model of dynamic pricing of freight services in an urban environment that follows the paradigm set in the field of revenue management for nonlinear pricing in a dynamic, game-theoretic setting. There are many applications of dynamic pricing in a game theoretic setting, however, to our knowledge, not a single study has been done for the urban freight systems from revenue management perspective even though both the suppliers (shippers) and the carriers possess most of the distinct features of airlines, hotel and car rental industries where revenue management concepts are widely practiced. We take a crucial first step in Mookherjee et al. [63] by using the paradigm of revenue management and dynamic pricing while providing a unified decision support environment for the urban freight carriers and the suppliers under competition.

We consider suppliers to be e-retailers who are in oligopolistic competition relative to a common homogeneous product they all produce. Each of the profit maximizing firms has pricing power, and receive price sensitive demand from the receivers. The procurement lead times for the suppliers are assumed fixed so that fixed time shifts are induced in their inventory dynamics. The freight carriers form an oligopoly, have pricing power, employ multiple transportation modes, can consolidate orders from the shippers, and face a dynamic pricing-resource allocation problem in an uncertain environment. The carriers' pricing power helps them to exercise some of the demand management strategies. A lack of transparency of receivers’ demand information complicates the decision environment for the carriers. The receivers set delivery windows and desired delivery volumes, which are deterministic in nature, depending on prevailing prices charged by the receivers. Likewise, a carrier’s demand function depends on its own price as well as its competitors’ prices. The demand for the carriers’ services is generated by the suppliers who wish to ship goods to receivers. Similarly, the carriers must compete with each other to procure freight services.

Strategic interaction among suppliers, carriers and receivers forms an activity-based dynamic freight network. The nodes of this network correspond to the carriers’ bases of operation. The same carrier may have a presence at multiple nodes which are spatially sep-
arated. Suppliers are also located at nodes; they may have production facilities at multiple nodes. Each carrier is connected to a set of suppliers with whom it has business relations; these connections form arcs in the network. Similarly, each carrier is also connected to the set of receivers to which it may deliver. These connections also form arcs in the network.

For the complex decision environment discussed above, we explore two different pricing strategies for the carriers and the suppliers: (a) bidirectional pricing; and (b) monotonic pricing (mark-up or mark-down). The three types of agents referred to above (receivers, carriers and suppliers) have the control and state variables described in the Table 4.1.

The receivers' input factor demands are fixed for the time scale of one abstract “day” (which might be several real days), so the sellers have to compete for that demand which depends on delivered factor prices which in turn depend on transportation prices (tariffs) which are also competitively set. Likewise, each transporter's demand function depends on its own price as well as its competitors' prices. The demand for the transporters is derived from the spatial separation of supply and consumption activities. Similar to the sellers, the transporters must compete with each other to procure this demand for services.

In this formulation we treat receivers as those entities who desire delivery of goods. In particular, receivers dictate the volume of the delivery and the desired time of the delivery of the goods. Demand for the goods and desired time of delivery are taken exogenous to this model as they are considered fixed for the time scale of the model. Our model considers homogeneous goods only; however, this model may be extended to a more general model with nonhomogeneous goods.

The extremal problem for each seller and transporter is formulated as a continuous (discrete) time optimal control problem that depends on the strategies of the other firms. This leads to a set of coupled optimal control problems that describe the game. This set of continuous optimal control problems is then discretized to obtain a set of coupled mathematical programs. Using the Karush-Kuhn-Tucker (KKT) conditions for each mathematical program, the problem can be recast as a nonlinear complementarity problem (NCP).
Table 4.1: Full Blown Dynamic Urban Freight Model Characteristics

<table>
<thead>
<tr>
<th>Agent</th>
<th>Description</th>
<th>Type</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Receiver</td>
<td>Goods desired by receiver</td>
<td>Exogenous</td>
<td>demand depends on delivered price charged by suppliers</td>
</tr>
<tr>
<td>Supplier</td>
<td>Delivered price to receiver</td>
<td>Control</td>
<td></td>
</tr>
<tr>
<td>Supplier</td>
<td>Inventory (backorder)</td>
<td>State</td>
<td></td>
</tr>
<tr>
<td>Supplier</td>
<td>Share of receiver’s demand</td>
<td>Random variable, avg of which is a function of current period retail price charged by a supplier</td>
<td>Induce dynamic pricing strategies for suppliers</td>
</tr>
<tr>
<td>Supplier</td>
<td>Goods deliver to receiver</td>
<td>Control</td>
<td></td>
</tr>
<tr>
<td>Carrier</td>
<td>Price charged to supplier</td>
<td>Control</td>
<td>Carriers are price setters, consolidate orders</td>
</tr>
<tr>
<td>Carrier</td>
<td>Back-logged service</td>
<td>State</td>
<td></td>
</tr>
<tr>
<td>Carrier</td>
<td>Share of supplier’s order</td>
<td>Random variable, average of which is a function of current period price charged by carriers</td>
<td>May also depend on back-logged service</td>
</tr>
<tr>
<td>Carrier</td>
<td>Goods received by receiver</td>
<td>Control</td>
<td>Carriers use different transportation modes</td>
</tr>
</tbody>
</table>
Rest of this chapter is organized as follows: Section 4.2 describes a predictive model of freight flows can be articulated and computed using DVIs. The freight flows are resulting from the activities of the shippers who are in an oligopoly. These shippers simultaneously make production, allocation and shipment decisions for geographically separated markets. In section 4.2.7 we extend the framework by providing pricing power to the suppliers as well as considering transporters as well. In our model we assume that the suppliers and transporters are the Cournot-Nash agents, suppliers (shippers) receive demand from the receivers, transporters have pricing power which helps them to level the demand from peak period to off-peak period, thus helping them to better utilize their capacities and gain more revenue.

4.2 Shipper Dynamic Oligopolistic Network Competition

4.2.1 Motivation and Brief Literature Review

The determination of freight flows between regions is one of the most fundamental problems in transportation systems analysis. In fact models of inter-regional freight flows may serve as the shippers’ submodel in combined shipper-carrier freight models like those originally proposed by Friesz et al. [67], Friesz and Harker [68] and Harker and Friesz [69], [70]. These and other inter-regional freight flow models have assumed an economic environment of either perfect competition or oligopolistic competition. To date all computable inter-regional flow models have been predicated on the notion of a static equilibrium.

In contrast research reported in this section is motivated by two observations:

1. dynamic oligopolistic competition among carriers active on a network is a more realistic economic environment for the study of shipper flows than an environment based on the notion of a static equilibrium; and

2. the most natural mathematical expression of dynamic oligopolistic competition is as
a type of differential variational inequality (DVI) with clearly distinguishable states and controls.

The control variables of the (DVI) we explore in this paper are determined not by the extremizing behavior of a single agent but rather by the competitive market mechanism in which each agent participates. Its state dynamics, when one is considering freight demand to be derived from spatially separated of production and consumption activities connected by a distribution network, as we are in this paper, are differential equations parameterized in terms of controls like production rates, consumption rates and shipping patterns. Although DVI representations have been used by Friesz et al [10] and others to model dynamic traffic assignment for urban road networks, they have never been used before to study shipper flows arising in freight networks.

The research presented in this section advances the understanding and modeling of shippers' flows in several ways:

1. Our shipper dynamic oligopolistic network competition model is the first theoretical framework to integrate the derived demand nature of shippers' flows and the notion of production planning in a dynamic setting.

2. Our inventory dynamics faithfully extend the aspatial state dynamics that have been used extensively in the production planning literature\(^1\) to a network representation.

3. We show that a differential variational inequality representation in continuous time with clearly identified state and control variables is intrinsically well suited for the mathematical articulation of shipper dynamic oligopolistic network competition. This application of the theory of DVIs is wholly original.

4. Using the DVI perspective we are able to establish existence of a shipper dynamic oligopolistic network equilibrium.

\(^1\)See Bensoussan [71] and Bensoussan et al. [72] for a detailed presentation of the classical mathematical theory of production planning.
5. We show that the \textit{DVI} representation of shipper dynamic oligopolistic network competition may be re-cast as either a functional nonlinear complementarity problem or a functional mathematical program, in an appropriate Hilbert space, to form the foundation for a discrete time/finite element numerical approach that may be implemented using off-the-shelf mathematical programming software.

6. We demonstrate the computability of dynamic oligopolistic network equilibria using the functional mathematical programming formulation to quickly solve a test problem involving 4 firms and approximately 1400 variables when a discrete time approximation is employed.

7. Our numerical example shows that the computed trajectories for productions, consumptions and inventories arising from deterministic dynamic oligopolistic network competition may be exceptionally complex even when there are no external phenomena such as seasonality or market shocks.

A detailed exposition of our model of dynamic oligopolistic network competition in Section 4.2.4 shows how dynamic oligopolistic network competition may be expressed as a \textit{DVI} and establishes the existence of a solution to that formulation under plausible regularity conditions. In Section 4.2.5 we present a discrete time approximation for use in conjunction with off-the-shelf mathematical programming software. Section 4.2.6 provides a detailed numerical example. Section 4.2.7 summarizes our findings and describes future research.

The oligopolistic firms of interest, embedded in a network economy, are in Cournot-Nash oligopolistic game theoretic competition according to dynamics that describe the trajectories of inventories/backorders and correspond to flow conservation for each firm at each node of the network of interest. The oligopolistic firms, acting as shippers, compete as price takers in the market for physical distribution services which is perfectly competitive due to its involvement in other markets of the network economy. The time scale we consider
is neither short nor long, but rather of sufficient length to allow output and shipping pattern adjustments but not long enough for firms to re-locate or enter or leave the network economy.

4.2.2 Notation

We employ the notation used in Miller, Friesz and Tobin [73], augmented to handle temporal considerations. We recall that time is denoted by the scalar $t \in \mathbb{R}_+^1$, initial time by $t_0 \in \mathbb{R}_+^1$, final time by $t_1 \in \mathbb{R}_+^1$, with $t_0 < t_1$ so that $t \in [t_0, t_1] \subset \mathbb{R}_+^1$. There are several sets important to articulating a model of oligopolistic competition on a network; these are as follow: $\mathcal{F}$ for firms, $\mathcal{A}$ for directed arcs, $\mathcal{N}$ for nodes and $\mathcal{W}$ for origin-destination (OD) pairs. Subsets of these sets are formed as is meaningful by using the subscript $f$ for a specific firm, $i$ for a specific node, and $w$ for a specific OD pair.

Each firm controls production output rates $q^f$, allocation of output to meet demand $c^f$ and shipping pattern $s^f$. Inventories $I^f$ are state variables determined by the controls. In particular, $c^f$, $q^f$ and $s^f$ constitute concatenations of the following vectors:

$$
c \in (L^2 [t_0, t_1])^{\mathcal{N} \times |\mathcal{F}|}
$$

$$
q \in (L^2 [t_0, t_1])^{\mathcal{N} \times |\mathcal{F}|}
$$

$$
s \in (L^2 [t_0, t_1])^{\mathcal{W} \times |\mathcal{F}|}
$$

$$
I(c, q, s) : (L^2 [t_0, t_1])^{\mathcal{N} \times |\mathcal{F}|} \times (L^2 [t_0, t_1])^{\mathcal{N} \times |\mathcal{F}|} \times (L^2 [t_0, t_1])^{\mathcal{W} \times |\mathcal{F}|}
\rightarrow (H^1 [t_0, t_1])^{\mathcal{N} \times |\mathcal{F}|}
$$

where $L^2 [t_0, t_1]$ is the space of square-integrable functions and $H^1 [t_0, t_1]$ is a Sobolev space for the real interval $[t_0, t_1] \in \mathbb{R}_+^1$.

4.2.3 Model of the Firm

Each firm has the objective of maximizing net profit expressed as revenue less cost and taking the form of an operator acting on allocations of output to meet demands, production rates
and shipment patterns. For each \( f \in \mathcal{F} \), net profit is

\[
\Phi_f(c^f, q^f, s^f; c^{-f}, q^{-f}) = \int_{t_0}^{t_1} e^{-\rho t} \left\{ \sum_{i \in \mathcal{N}} \pi_i \left( \sum_{g \in \mathcal{F}} c^g_i, t \right) c^f_i - \sum_{i \in \mathcal{N}_f} V^f_i (q^f, t) \right. \\
- \sum_{w \in \mathcal{W}_f} r_w(t) s^f_w - \sum_{i \in \mathcal{N}_f} \psi^f_i (I^f_i, t) \left\} \, dt
\]

(4.1)

where \( \rho \in \mathbb{R}_{++} \) is a constant nominal rate of discount, \( r_w \in \mathbb{R}_{++} \) is the freight rate (tariff) charged per unit of flow \( s_w \) for OD pair \( w \in \mathcal{W}_f \), \( \psi^f_i \) is firm \( f \)'s inventory cost at node \( i \), and \( I^f_i \) is the inventory/backorder of firm \( f \) at node \( i \). In (4.1), \( c^f_i \) is the allocation of the output of firm \( f \in \mathcal{F} \) at node \( i \in \mathcal{N} \) to consumption at that node. Our formulation is in terms of flows so we employ the inverse demand functions \( \pi_i(c_i, t) \) where

\[
c_i = \sum_{g \in \mathcal{F}} c^g_i
\]

is the total allocation of output to consumption for node \( i \). Furthermore \( q^f_i \) is the output of firm \( f \in \mathcal{F} \) at node \( i \in \mathcal{N} \). Also \( V^f_i (q,t) \) is the variable cost of production for firm \( f \in \mathcal{F} \) at node \( i \in \mathcal{N} \). Note that \( \theta_f(c^f, q^f, s^f; c^{-f}, q^{-f}) \) is a functional that is completely determined by the controls \( c^f, q^f \) and \( s^f \) when non-own allocations to consumption and non-own production rates

\[
c^{-f} \equiv (c^{f'} : f' \neq f) \\
q^{-f} \equiv (q^{f'} : f' \neq f)
\]

are taken as exogenous data by firm \( f \). The first term of the functional \( \theta_f(c^f, q^f, s^f; c^{-f}, q^{-f}) \) in expression (4.1) is the firm’s revenue; the second term is the firm’s cost of production; the third term is the firm’s shipping costs; and the last term is the firm’s inventory or holding cost.

We also impose the terminal time inventory constraints

\[
I^f_i (t_1) \geq \tilde{K}^f_i \quad \forall f \in \mathcal{F}, i \in \mathcal{N}_f
\]

(4.2)
where the $\tilde{K}^f_i \in \mathbb{R}^{1+}$ are exogenous. All consumption, production and shipping variables are non-negative and bounded from above; that is

$$
C^f \geq c^f \geq 0 \quad (4.3)
$$

$$
Q^f \geq q^f \geq 0 \quad (4.4)
$$

$$
S^f \geq s^f \geq 0 \quad (4.5)
$$

where

$$
C^f \in \mathbb{R}^{\lvert F \rvert}_{++}
$$

$$
Q^f \in \mathbb{R}^{\lvert F \rvert}_{++}
$$

$$
S^f \in \mathbb{R}^{\lvert W_f \rvert}_{++}
$$

Constraints (4.3), (4.4) and (6.24) are recognized as pure control constraints, while (4.2) are terminal conditions for the state space variables. Naturally

$$
\Omega_f = \left\{ (c^f, q^f, s^f) : (4.3), (4.4), (6.24) \right\}
$$

is the set of feasible controls.

Firm $f$ solves an optimal control problem to determine its production $q^f$, allocation of production to meet demand $c^f$, and shipping pattern $s^f$ – thereby also determining inventory $I^f$ via dynamics we articulate momentarily – by maximizing its profit functional $\Phi_f(c^f, q^f, s^f; c^{-f}, q^{-f})$ subject to inventory dynamics expressed as flow balance equations and pertinent production and inventory constraints. The inventory dynamics for firm $f \in F$, expressing simple flow conservation, obey

$$
\frac{dI^f_i}{dt} = q^f_i + \sum_{w \in W^d_i} s^f_{w_i} - \sum_{w \in W^o_i} s^f_{w_i} - c^f_i \quad \forall i \in N_f
$$

(4.6)

$$
I^f_i(t_0) = K^f_i \quad \forall i \in N_f
$$

(4.7)

$$
I^f_i(t_1) \geq \tilde{K}^f_i \quad \forall i \in N_f
$$

(4.8)
where \( K^f_i \in \mathbb{R}^{1+} \) and \( \tilde{K}^f_i \in \mathbb{R}^{1+} \) are exogenous, while \( W_i^d \) is the set of OD pairs with destination node \( i \) and \( W_i^o \) is the set of OD pairs with origin node \( i \). Note that the transportation time for the flow of finished goods is not captured explicitly in the inventory dynamics, however it is accounted for implicitly in the freight rate (tariff) charged per unit of flow. Further, in addition to the terminal time inventory (state) constraints (4.8), the model is general enough to handle inventory constraints over the entire planning horizon \([t_0, t_1]\). For instance, non-negativity of the inventory (state) variables could be imposed to restrict firms from taking backorders.

Consequently,

\[
I(c, q, s) = \arg \left\{ \frac{dI_i^f}{dt} = q_i^f + \sum_{w \in W_i^d} s_i^f - \sum_{w \in W_i^o} s_i^f - c_i^f, \quad I_i^f(t_0) = K_i^f; I_i^f(t_1) = \tilde{K}_i^f \quad \forall f \in \mathcal{F}, i \in \mathcal{N}_f \right\}
\]

where we implicitly assume that the dynamics have solutions for all feasible controls.

With the preceding development, we note that firm \( f \)'s problem is: with the \( c^{-f} \) and \( q^{-f} \) as exogenous inputs, compute \( c^f, q^f \) and \( s^f \) (thereby finding \( I^f \)) in order to solve the following extremal problem:

\[
\max_{c^f, q^f, s^f} \Phi_f(c^f, q^f, s^f; c^{-f}, q^{-f}) \quad \forall f \in \mathcal{F} \quad (4.9)
\]

subject to \((c^f, q^f, s^f) \in \Omega_f\),

where

\[
\Omega_f = \left\{ (c^f, q^f, s^f) : (4.2), (4.3), (4.4), (6.24) \text{ hold} \right\}
\]

also for all \( f \in \mathcal{F} \). That is, each firm is a Cournot-Nash agent that knows and employs the current instantaneous values of the decision variables of other firms to make its own non-cooperative decisions. As such, (4.9) is a Cournot-Nash differential game.
4.2.4 The DVI Formulation of Dynamic Oligopolistic Network Competition

We assume in the balance of this paper that this game is regular in the sense of the following definition:

**Definition 4.1** The dynamic oligopolistic network competition problem introduced above will be considered regular if: (1) each component of the functional $I(c,q,s)$ is strongly continuous and $G$-differentiable; and (2) the inverse demand, production cost and inventory cost functions have continuous partial derivatives.

We note that (4.9) is an optimal control problem with fixed terminal time. Its Hamiltonian is

$$H_f \left( c^f, q^f, s^f, I^f, c^{-f}, q^{-f}; t \right)$$

$$\equiv \Phi_f \left( c^f, q^f, s^f, I^f; c^{-f}, q^{-f}; t \right) + \Psi_f \left( c^f, q^f, s^f, I^f, \alpha^f, \beta^f, \lambda^f \right)$$

where

$$\Phi_f \left( c^f, q^f, s^f, I^f; c^{-f}, q^{-f}; t \right) = e^{-\rho t} \left\{ \sum_{i \in N_f} \pi_i \left( \sum_{g \in \mathcal{I}_f} c^g_i, t \right) c^f_i \right.$$  

$$- \sum_{i \in N_f} V_i^f(q_i, t) - \sum_{w \in \mathcal{W}_f} r_w(t) s^f_w$$

$$- \sum_{i \in N_f} \psi_i^f(I_i^f, t) \right\}$$

(4.10)

and

$$\Psi_f \left( c^f, q^f, s^f, I^f, \alpha^f, \beta^f, \lambda^f \right) = \sum_{i \in N_f} \lambda^f_i \left( q_i^f + \sum_{w \in \mathcal{W}_d} s^f_w - \sum_{w \in \mathcal{W}_o} s^f_w - c^f_i \right)$$

$$+ \sum_{i \in N_f} \alpha^f_i \left( I_i^f - U_i^f \right) - \sum_{i \in N_f} \beta^f_i I_i^f$$

(4.11)

where $\alpha^f_i \in \mathbb{R}_+^1$ and $\beta^f_i \in \mathbb{R}_+^1$ are dual variables for the inventory bound constraints (4.2) while $\alpha^f \in \mathbb{R}^{|N_f|}$ and $\beta^f \in \mathbb{R}^{|N_f|}$; also $\lambda^f_i \in \mathbb{R}_+^1$ is the adjoint variable for the dynamics.
of firm $f$ at node $i$ while $\lambda^f \in (H^1[t_0,t_1])^{\mid N_f \mid}$. Clearly $\Phi_f$ is the instantaneous profit. To interpret $\Psi_f$ we need to understand the relevant dynamic shadow benefits and shadow costs of this model. To that end, recall that, along an optimal trajectory, the adjoint variables obey

$$\chi^f_i = \frac{\partial J_f}{\partial I^f_i}$$

Consequently

$$\Psi_f = \frac{\partial J_f}{\partial I^f_i} dI^f_i + \sum_{i \in N_f} \alpha^f_i \left( I^f_i - a^f_i \right) - \sum_{i \in N_f} \beta^f_i I^f_i$$

which is recognized as the shadow value of dynamic benefits arising from current non-steady state net rates of exportation plus the shadow value of inventory holdings.

Due to regularity the maximum principle takes the form of requiring that the non-linear program

$$\max H_f \quad \text{s.t.} \quad \begin{pmatrix} C^f, Q^f, S^f \end{pmatrix} \geq \begin{pmatrix} c^f, q^f, s^f \end{pmatrix} \geq 0$$

be solved by every firm $f \in F$ for every instant of time $t \in [t_0, t_1]$. Consequently, since the feasible set is convex, using the maximum principle, any optimal solution must satisfy

$$\frac{\partial H^*_f}{\partial c_i} \left( c^f_i - c^f_i \right) \leq 0 \quad \text{(4.12)}$$

$$\frac{\partial H^*_f}{\partial q_i} \left( q^f_i - q^f_i \right) \leq 0 \quad \text{(4.13)}$$

$$\frac{\partial H^*_f}{\partial s_w} \left( s^f_w - s^f_w \right) \leq 0 \quad \text{(4.14)}$$

for every $f \in F$ at every time, $t \in [t_0, t_1]$. Expressions (4.12) through (4.14) provide the primary means for us to verify that the variational inequality formulation presented in the next section is in fact equivalent to Cournot-Nash dynamic oligopolistic network competition.

Familiarity with variational inequalities suggests that the following variational inequality has solutions that are Cournot-Nash equilibria for the game described above in which individual firms maximize net profits in light of current information about their
find \((c^*, q^*, s^*) \in \Omega\) such that

\[
0 \geq \sum_{f \in \mathcal{F}} \int_{t_0}^{t_1} \left[ \sum_{i \in N_f} \frac{\partial \Phi_f^*}{\partial c_i^f} (c_i^f - c_i^{f^*}) + \sum_{i \in N_f} \frac{\partial \Phi_f^*}{\partial q_i^f} (q_i^f - q_i^{f^*}) + \sum_{w \in \mathcal{W}_f} \frac{\partial \Phi_f^*}{\partial s_w^f} (s_w^f - s_w^{f^*}) \right] dt \quad \text{for all} \quad (c, q, s) \in \Omega \quad (4.15)
\]

where

\[
\Phi_f^* = \Phi_f \left( c^*, q^*, s^*, I_f^*; c^{-f}, q^{-f}; t \right) \tag{4.16}
\]

\[
\Omega = \prod_{f \in \mathcal{F}} \Omega_f \tag{4.17}
\]

We note that (4.15) is a differential variational inequality expressing the Cournot-Nash game that is our present interest. This DVI formulation also provides guidance in devising a computational strategy, as we show in Section 4.2.6.

The issue of immediate concern is to formally demonstrate that solutions of (4.15) are Cournot-Nash equilibria. In fact we state and prove the following result:

**Theorem 4.1** (DVI formulation of dynamic oligopolistic network competition) Any solution of the DVI (4.15) is a solution of the dynamic oligopolistic network competition problem when regularity in the sense of Definition 4.1 holds.

**Proof:** We begin by noting that (4.15) is equivalent to the following optimal control problem

\[
\max G(c, q, s) = \sum_{f \in \mathcal{F}} \int_{t_0}^{t_1} \left[ \sum_{i \in N_f} \frac{\partial \Phi_f^*}{\partial c_i^f} c_i^f + \sum_{i \in N_f} \frac{\partial \Phi_f^*}{\partial q_i^f} q_i^f + \sum_{w \in \mathcal{W}_f} \frac{\partial \Phi_f^*}{\partial s_w^f} s_w^f \right] dt
\]

s.t. \((4.2), (4.3), (4.4), (6.24)\) and \((4.6)\),

where it is essential to recognize that \(G(c, q, s)\) is a linear functional that assumes knowledge of the solution to our oligopolistic game; as such \(G(c, q, s)\) is a mathematical construct for
use in analysis and has no meaning as a computational device. The augmented Hamiltonian for this artificial optimal control problem is

$$H_0 = \sum_{f \in \mathcal{F}} \left[ \sum_{i \in \mathcal{N}_f} \frac{\partial \Phi_f^*}{\partial c_i} c^f_i + \sum_{i \in \mathcal{N}_f} \frac{\partial \Psi_f^*}{\partial q_i} q^f_i + \sum_{w \in \mathcal{W}_f} \frac{\partial \Phi_f^*}{\partial s_w} s^f_w \right] + \sum_{f \in \mathcal{F}} \Psi_f$$

The associated maximum principal requires

$$\max H_0 \quad \text{s.t.} \quad (C^f, Q^f, S^f) \geq (c^f, q^f, s^f) \geq 0$$

The corresponding necessary conditions for this mathematical program are identical to (4.12) through (4.14), since

$$\frac{\partial H_0^*}{\partial c^f_i} = \frac{\partial \Phi_f^*}{\partial c^f_i} + \frac{\partial \Psi_f^*}{\partial c^f_i} = \frac{\partial H_f^*}{\partial c^f_i}$$

$$\frac{\partial H_0^*}{\partial q^f_i} = \frac{\partial \Phi_f^*}{\partial q^f_i} + \frac{\partial \Psi_f^*}{\partial q^f_i} = \frac{\partial H_f^*}{\partial q^f_i}$$

$$\frac{\partial H_0^*}{\partial s^f_w} = \frac{\partial \Phi_f^*}{\partial s^f_w} + \frac{\partial \Psi_f^*}{\partial s^f_w} = \frac{\partial H_f^*}{\partial s^f_w}$$

where

$$H_0^* = \sum_{f \in \mathcal{F}} \left[ \sum_{i \in \mathcal{N}_f} \frac{\partial \Phi_f^*}{\partial c^f_i} c^f_i + \sum_{i \in \mathcal{N}_f} \frac{\partial \Phi_f^*}{\partial q^f_i} q^f_i + \sum_{w \in \mathcal{W}_f} \frac{\partial \Phi_f^*}{\partial s^f_w} s^f_w \right] + \sum_{f \in \mathcal{F}} \Psi_f$$

and

$$\Psi_f = \Psi_f \left( c^f, q^f, s^f, I^f, \alpha^f, \beta^f, \lambda^f \right)$$

We now note that the following existence result holds:

**Theorem 4.2 (existence of dynamic oligopolistic network equilibrium)** When the variational inequality of Theorem 4.1 is regular in the sense of Definition 4.1, there exists a solution of the dynamic oligopolistic network competition problem.

**Proof:** Immediate from the existence theorem of a DVI (chapter 2).
4.2.5 Discrete-Time Approximation of Dynamic Oligopolistic Network Competition

The extremal problem (4.9) for all firm \( f \in \mathcal{F} \) may be given the following discrete time approximation:

\[
\max \Phi_f(c^f, q^f, s^f; c^{-f}, q^{-f}) \approx \sum_{t=0}^{N} \tau(t) e^{-\rho t} \cdot \left\{ \sum_{i \in \mathcal{N}_f} \pi_i \left( \sum_{g \in \mathcal{F}} c^g_i(t), t \right) c^f_i(t) \right. \\
- \sum_{i \in \mathcal{N}_f} V^f_i(q^f_i(t), t) - \sum_{w \in \mathcal{W}_f} r_w(t) s^f_w(t) - \sum_{i \in \mathcal{N}_f} \psi^f_i(I^f_i(t), t) \left. \right\}
\]

subject to

\[
I^f_i(t+1) = I^f_i(t) + \Delta \left[ q^f_i(t) + \sum_{w \in \mathcal{W}_i^d} s^f_w(t) - \sum_{w \in \mathcal{W}_i^o} s^f_w(t) - c^f_i(t) \right] \\
\forall t = 0, \ldots, N - 1 \text{ and } i \in \mathcal{N}_f
\]

\[
I^f_i(0) = K^f_i \quad \forall i \in \mathcal{N}_f
\]

\[
I^f_i(N) = \tilde{K}^f_i \quad \forall i \in \mathcal{N}_f
\]

\[
0 \leq c^f \leq C^f
\]

\[
0 \leq q^f \leq Q^f
\]

\[
0 \leq h^f \leq H^f
\]

where \( t \) is now takes on non-negative integer values, \( \Delta \) is the discrete time step that divides the time interval \([t_0, t_1]\) into \( N \) equal segments, and \( \tau(t) \) is the coefficient which arises from a trapezoidal approximation of the present value integral; that is

\[
\tau(t) = 0.5 \text{ for } t = 0 \text{ and } N
\]

\[
= 1 \quad \text{else}
\]

One advantage of time discretization is that we can now completely eliminate state variables (inventories) from the problem by noting that

\[
I^f_i(t+1) = K^f_i + \Delta \cdot \sum_{k=0}^{t} \left[ q^f_i(k) + \sum_{w \in \mathcal{W}_i^d} s^f_w(k) - \sum_{w \in \mathcal{W}_i^o} s^f_w(k) - c^f_i(k) \right] \quad (4.18)
\]

\[
I^f_i(N) = \tilde{K}^f_i \quad (4.19)
\]

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for \( t = 0, \ldots, N - 1 \) and all \( i \in \mathcal{N}_f \). As a consequence one obtains a finite dimensional variational inequality involving only upper and lower bound constraints on the remaining control variables. This finite dimensional variational inequality may be solved by conventional algorithms developed for such problems or a finite dimensional nonlinear complementarity formulation may be created and used to construct a finite dimensional nonlinear mathematical program that is an approximation of (4.15), (4.16), (4.10), (4.9) and (4.17). That finite-dimensional nonlinear mathematical program may be solved using off-the-shelf software as in done in the next section for an example problem.

**A Comment About Path Variables**

It should be noted that one may introduce path flows in the above formulation by re-expressing the state dynamics as

\[
\frac{dI_f}{dt} = q_i^f + \sum_{j \in \mathcal{N}_f} \sum_{p \in P_{ji}} h_p^f - \sum_{j \in \mathcal{N}_f} \sum_{p \in P_{ij}} h_p^f - c_i^f
\]

for every firm \( f \in \mathcal{F} \) and node \( i \in \mathcal{N}_f \), where \( P_{ji} \) is the set of paths connecting nodes \( j, i \in \mathcal{N}_f \) and \( h_p \) is the flow on path \( p \in P_{ji} \). There are corresponding, but quite obvious, changes in the firm’s objective function and the upper and lower bound constraints on its controls. We omit a complete statement of such details for the sake of brevity.

**4.2.6 Numerical Example**

Let us consider a network of 5 arcs, 4 nodes and 4 firms, where a single firm \( f \) is located at each node \( i = 1, 2, 3, 4 \). Consumption of each firm’s output potentially occurs at every node; this consumption may be of local output or of imported output as the network topology permits. Figure 4.1 illustrates the network. The time interval of interest is \([0, 20]\); that is \( t_0 = 0 \) and \( t_1 = 20 \). In this example, firm 1 has an economic presence at all nodes, firm 2
Market 1

Market 2

Market 3

Market 4

Figure 4.1: Network of 5 arcs, 4 nodes and 4 firms

at nodes 2, 3 and 4, firm 3 at nodes 3 and 4 and finally firm 4 at node 4 only. Therefore,

\[ \mathcal{F} = \{1, 2, 3, 4\} \]
\[ \mathcal{N}_1 = \{1, 2, 3, 4\} ; \; \mathcal{N}_2 = \{2, 3, 4\} \]
\[ \mathcal{N}_3 = \{3, 4\} ; \; \mathcal{N}_4 = \{4\} \]

Before time discretization there are 29 controls and 10 state variables associated with this example; these are listed in Table 4.2.

At time \( t_0 = 0 \), every firm has an inventory of 100 units at their respective locations. That is

\[ I_i^f(0) = 100 \quad \text{for} \; f \in \mathcal{F} \; \text{and} \; i \in \mathcal{N}_f \]

In addition, we impose the condition that no backordering is allowed by any firm at any node at the terminal time \( t_1 = 20 \). That is

\[ I_i^f(20) \geq 0 \quad \text{for} \; f \in \mathcal{F} \; \text{and} \; i \in \mathcal{N}_f \]  \hspace{1cm} (4.20)

The inventory dynamics are the flow balance equations:
Table 4.2: Controls and States : Shipper Dynamic Oligopoly Example

<table>
<thead>
<tr>
<th>Controls</th>
<th>States</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1^1$</td>
<td>$I_1^1$</td>
</tr>
<tr>
<td>$c_1^2$</td>
<td>$I_1^2$</td>
</tr>
<tr>
<td>$c_2^2$</td>
<td>$I_3^2$</td>
</tr>
<tr>
<td>$c_3^3$</td>
<td>$I_3^3$</td>
</tr>
<tr>
<td>$c_3^4$</td>
<td>$I_3^4$</td>
</tr>
<tr>
<td>$q_1^1$</td>
<td>$h_1^1$</td>
</tr>
<tr>
<td>$q_2^2$</td>
<td>$h_2^1$</td>
</tr>
<tr>
<td>$q_3^3$</td>
<td>$h_3^1$</td>
</tr>
<tr>
<td>$q_4^4$</td>
<td>$h_4^1$</td>
</tr>
<tr>
<td>$h_1^6$</td>
<td>$h_5^1$</td>
</tr>
<tr>
<td>$h_2^7$</td>
<td>$h_6^1$</td>
</tr>
<tr>
<td>$h_3^8$</td>
<td>$h_7^1$</td>
</tr>
<tr>
<td>$h_4^9$</td>
<td>$h_8^1$</td>
</tr>
<tr>
<td>$h_5^10$</td>
<td>$h_{10}^1$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\frac{dI_1^1}{dt} &= q_1^1 - h_1^1 - h_2^1 - h_3^1 - h_4^1 - h_5^1 - h_6^1 - c_1^1 \\
\frac{dI_2^1}{dt} &= h_1^1 - h_7^1 - h_8^1 - h_9^1 - c_1^2 \\
\frac{dI_3^1}{dt} &= h_2^1 + h_3^1 + h_7^1 - h_{10}^1 - c_1^3 \\
&\vdots\\
\frac{dI_4^1}{dt} &= q_4^4 - c_1^4
\end{align*}
\]

which we only partially enumerate in the interest of saving space. We assume the inverse demands at each node $i$ take the following form:

\[
\pi_i(c_i, t) = \alpha_i - \beta_i (c_i)^m
\]

where $m \in \mathbb{R}_{++}^1$ is a constant. Also $\alpha_i \in \mathbb{R}_{++}^1$ and $\beta_i \in \mathbb{R}_{++}^1$ for all $i$ are constants. The production cost functions are for each firm $f$ have the form

\[
V_i^f = \frac{1}{2} \rho_i^f (q_i^f)^2 + \frac{1}{3} \sigma_i^f (q_i^f)^3
\]

for all $i = 1, \ldots, 4$.

where $\rho_i^f$ and $\sigma_i^f \in \mathbb{R}_{++}^1$ are also constants for all allowed $i$ and $f$. In particular, we consider non-convex production cost to capture the phenomenon of economies of scale. We assume
Table 4.3: Path and Arc Sequence

<table>
<thead>
<tr>
<th>Path</th>
<th>Arc sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>$a_1$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$a_2$</td>
</tr>
<tr>
<td>$p_3$</td>
<td>$a_1, a_3$</td>
</tr>
<tr>
<td>$p_4$</td>
<td>$a_1, a_4$</td>
</tr>
<tr>
<td>$p_5$</td>
<td>$a_1, a_3, a_5$</td>
</tr>
<tr>
<td>$p_6$</td>
<td>$a_2, a_5$</td>
</tr>
<tr>
<td>$p_7$</td>
<td>$a_3$</td>
</tr>
<tr>
<td>$p_8$</td>
<td>$a_4$</td>
</tr>
<tr>
<td>$p_9$</td>
<td>$a_3, a_5$</td>
</tr>
<tr>
<td>$p_{10}$</td>
<td>$a_5$</td>
</tr>
</tbody>
</table>

The holding costs are quadratic and of the form

$$
\psi_i^f = \frac{1}{2} \eta_i^f (I_i^f)^2 \text{ for } f \in \mathcal{F} \text{ and } i \in \mathcal{N}_f
$$

(4.24)

where $\eta_i^f \in \mathbb{R}_{++}$ are constants, again for allowed $i$ and $f$. To construct the path costs, we consider the path listed in Table 4.3

Furthermore, the relevant arc-path incidence matrix is

$$
\Delta = (\delta_{ap}) = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
$$

The associated path costs are

$$
R = \Delta^T r
$$

(4.25)
where
\[ r_{\alpha_i} = A_{\alpha} + B_{\alpha}(f_i)^n \]
is the unit freight rate for arc \( a_j \) and \( A_j, B_j \in \mathbb{R}_{++}^1 \) are constants for \( j = 1, \ldots, 5 \). We impose the following bounds on control variables:
\[ C^f = Q^f = H^f = 75 \]

The individual firms’ instantaneous profit functions are found by substituting (4.22),(4.23),(4.24) and (4.25) in (4.10) where \( \rho \in \mathbb{R}_{++}^1 \) is the fixed discount rate. We solve the corresponding functional mathematical program using a discrete time approximation with \( N = 41 \) equal time steps. Note that the use of the non-convex variable cost function (4.23) will make the resulting mathematical program non-convex. In our calculations we allowed GAMS to exercise its multi-start feature to uncover global optima for which the objective is driven to zero. The solution time for the example presented below is approximately 2 minutes on a Pentium® 4 single-processor computer with 3 major iterations (direction finding) and about 600 minor iterations (line searches) per major iteration. This solution time includes identification of a global minimum. We are certain that we have achieved a global solution because the minimum value of the objective function is very close to zero \((1.3 \times 10^{-8})\); due to how the mathematical program was constructed from the nonlinear complementarity formulation of the relevant DVI, its global minimum is known \emph{a priori} to be precisely zero. The parameters of this model and associated values are tabulated in Table 4.4.

Inventory trajectories are presented in Figure 4.2 which shows substantial backordering by most firms at most nodes. Production rates for the 4 firms are plotted in Figure 4.3 which shows that firms 2 and 3 operate at their full capacity after time \( t = 10 \) until the end of the planning horizon to meet their backorder. Figures 4.4, 4.5 and 4.6 depict the variation of consumption controls (allocations of output to meet demands) over time for firms, nodes and markets. Figure 4.7 presents path flow trajectories. Instantaneous firm-specific profits are presented in Figure 4.8. The net present value for each of the firms
### Table 4.4: Parameters Table: Shipper Dynamic Oligopoly Numerical Example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>0.05</td>
<td>( A_1 )</td>
<td>2</td>
<td>( A_2 )</td>
<td>2</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>2</td>
<td>( A_4 )</td>
<td>2</td>
<td>( A_5 )</td>
<td>2</td>
</tr>
<tr>
<td>( B_1 )</td>
<td>0.9</td>
<td>( B_2 )</td>
<td>0.9</td>
<td>( B_3 )</td>
<td>0.9</td>
</tr>
<tr>
<td>( B_4 )</td>
<td>0.9</td>
<td>( B_5 )</td>
<td>0.9</td>
<td>( \alpha_i ), ( i = 1,3 )</td>
<td>6000</td>
</tr>
<tr>
<td>( \beta_i ), ( i = 1,3 )</td>
<td>3</td>
<td>( \alpha_i ), ( i = 2,4 )</td>
<td>6000</td>
<td>( \beta_i ), ( i = 2,4 )</td>
<td>3</td>
</tr>
<tr>
<td>( \sigma^i ), ( i = 1,\ldots,4 )</td>
<td>1</td>
<td>( \rho^1 )</td>
<td>0.3</td>
<td>( \rho^2, \rho^4 )</td>
<td>0.1</td>
</tr>
<tr>
<td>( \eta^1_2, \eta^1_4, \eta^3_4, \eta^4_4 )</td>
<td>1</td>
<td>( \eta^1_1 )</td>
<td>4</td>
<td>( \eta^1_3, \eta^2_2, \eta^2_3 )</td>
<td>2</td>
</tr>
<tr>
<td>( \eta^2_4, \eta^3_3 )</td>
<td>3</td>
<td>( \rho^3 )</td>
<td>0.2</td>
<td>( t_1 )</td>
<td>20</td>
</tr>
<tr>
<td>( N )</td>
<td>20</td>
<td>( n )</td>
<td>1</td>
<td>( m )</td>
<td>1</td>
</tr>
</tbody>
</table>

![Inventory dynamics](image)

Figure 4.2: Inventory dynamics
4.2.7 Concluding Remarks

We have shown that dynamic oligopolistic network competition is easily and naturally articulated as a differential variational inequality. This DVI formulation immediately suggests an algorithm based on a functional nonlinear program. We have provided an example that suggests this functional mathematical program may well be a practical means of solving non-cooperative dynamic games. We observe complicated temporal behavior from firms even
Figure 4.4: Firm-specific consumption at each node

Figure 4.5: Aggregate firm-specific consumption rates
Figure 4.6: Aggregate nodal consumption rates across all firms

Figure 4.7: All path flows in the network
Figure 4.8: Instantaneous firm-specific profits

though we do not consider complicated, time-varying demand. This is significant for it suggests that firms must be extremely astute and capable of dramatically altering production and distribution schedules if they are to compete in the final goods market successfully.

Future research should focus on laying out a framework to consider multiple time scales and inclusion of traffic congestions in the distribution network which result in time shifted decision variables. Another natural extension would be to stochasticize the inventory dynamics with white noises. The noises arise because of uncertainty in the demand, production and freight flows. The dynamic game associated with the model will be a stochastic differential games and hence one would obtain probability distribution of the Nash equilibria and hence the expectation. The computational strategy should involve parametric differential variational inequalities (PDVI) and stochastic differential variational inequalities (SDVI) - a research area with relatively little literature till date.
Dynamic Urban Freight Network Competition Amongst Sellers and Carriers

In this section we discuss a model of dynamic pricing of freight services in an urban environment that follows the paradigm set in the field of revenue management for nonlinear pricing in a dynamic, game-theoretic setting. There are many applications of dynamic pricing in a game theoretic setting, however, as far as we know, this is the first direct application of results from revenue management to urban freight transport and city logistics. This is not our intention here to review all influential literature on dynamic pricing as a detailed review of the literature is already done in this thesis in Chapter 3. A recent example of dynamic pricing in a game theoretic setting can be seen in Bernstein and Federgruen [42] which addresses dynamic pricing in supply chain competition, or Chen et al [41] who address dynamic pricing in the newsboy vendor problem.

The model we propose involves three classes of spatially separated firms: sellers, transporters and receivers. The sellers are those firms who produce goods that are sold to receivers. The transporters are the firms that are contracted to deliver the goods from the sellers to the receivers. These interactions take place on a network formed by the relationships among the different classes of firms.

We assume that both the sellers and transporters are Cournot-Nash agents in a network economy and they are profit optimizers with pricing power. Each seller of commodities competes with other sellers and each transporter competes with other transporters. However, the sellers and transporters do not compete with each other.

The receivers’ input factor demands are fixed for the time scale of one abstract “day” (which might be several real days), so the sellers have to compete for that demand which depends on delivered factor prices which in turn depend on transportation prices (tarrifs) which are also competitively set. Likewise, each transporter’s demand function depends on its own price as well as its competitors’ prices. The demand for the transporters is derived
from the spatial separation of supply and consumption activities. Similar to the sellers, the transporters must compete with each other to procure this demand for services.

In this formulation we treat receivers as those entities who desire delivery of goods. In particular, receivers dictate the volume of the delivery and the desired time of the delivery of the goods. Demand for the goods and desired time of delivery are taken exogenous to this model as they are considered fixed for the time scale of the model. Our model considers homogeneous goods only; however, this model may be extended to a more general model with nonhomogeneous goods.

The extremal problem for each seller and transporter is formulated as a continuous time optimal control problem that depends on the strategies of the other firms. This leads to a set of coupled optimal control problems that describe the game. This set of continuous optimal control problems is then discretized to obtain a set of coupled mathematical programs. Using the Karush-Kuhn-Tucker (KKT) conditions for each mathematical program, the problem can be recast as a nonlinear complementarity problem (NCP).

4.2.8 Description of the network

We propose a network representation of the sellers, transporters and receivers. As stated previously, the firms are spatially separated. Therefore, consider a network where each facility of the sellers and receivers is represented as a node. Both the sellers and the receivers may have multiple locations and as such may have multiple nodes in the network. If a transporter offers delivery services from a seller node to a receiver node, then an arc exists between those two nodes. If multiple transporters provide service between these two nodes, then an arc will exist for each such transporter, each with a potentially different cost. Therefore, while making a shipment decision, the seller needs to consider only those transporters which connect the seller node to the receiver node of interest.
### Table 4.5: Notation: Parameters: Dynamic Urban Freight Competition

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{S}$</td>
<td>set of sellers</td>
</tr>
<tr>
<td>$\mathcal{C}$</td>
<td>set of transporters</td>
</tr>
<tr>
<td>$\mathcal{R}$</td>
<td>set of receivers</td>
</tr>
<tr>
<td>$\mathcal{N}_s$</td>
<td>set of nodes where seller $s$ is located</td>
</tr>
<tr>
<td>$\mathcal{N}_r$</td>
<td>set of nodes where receiver $r$ is located</td>
</tr>
<tr>
<td>$\mathcal{M}$</td>
<td>set of transportation modes available to each transporter</td>
</tr>
<tr>
<td>$t_0$</td>
<td>start of the planning horizon</td>
</tr>
<tr>
<td>$t_1$</td>
<td>end of the planning horizon</td>
</tr>
<tr>
<td>$t \in [t_0, t_1]$</td>
<td>clock time</td>
</tr>
<tr>
<td>$D_r^i(t)$</td>
<td>amount of goods desired by receiver $r$ at its facility $i \in \mathcal{N}_r$ at time $t$</td>
</tr>
<tr>
<td>$I_{j,0}^s$</td>
<td>starting inventory held by seller $s$ at its location $j \in \mathcal{N}_s$</td>
</tr>
<tr>
<td>$p_{\text{min}}^s$</td>
<td>lower limit of price for firm $s$</td>
</tr>
<tr>
<td>$p_{\text{max}}^s$</td>
<td>upper limit of price for firm $s$</td>
</tr>
<tr>
<td>$q_{j,\text{max}}^s$</td>
<td>upper limit of production at node $j \in \mathcal{N}_s$ of seller $s$</td>
</tr>
<tr>
<td>$\pi_{\text{min}}^c$</td>
<td>lower limit of price for transporter $c$</td>
</tr>
<tr>
<td>$\pi_{\text{max}}^c$</td>
<td>upper limit of price for transporter $c$</td>
</tr>
</tbody>
</table>

#### 4.2.9 Notation

In this section we list the notation we will use in the rest of this section. Table 4.5 lists all parameters and Table 4.6 contains the states and the control variables of the model.
Table 4.6: Notation: States and Control Variables: Dynamic Urban Freight Competition

<table>
<thead>
<tr>
<th>Variables</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_{i,r}^s(t) )</td>
<td>delivered price charged by the seller ( s ) charged to the receiver ( r ) located at node ( i \in \mathcal{N}_r )</td>
</tr>
<tr>
<td>( q_j^s(t) )</td>
<td>production rate of seller ( s ) at location ( j \in \mathcal{N}_s )</td>
</tr>
<tr>
<td>( v_{i,j}^{r,s}(t) )</td>
<td>flow of goods sent by seller ( s ) from its location ( j \in \mathcal{N}_s ) for delivery at receiver ( r ) at its location ( i \in \mathcal{N}_r )</td>
</tr>
<tr>
<td>( I_j^s(t) )</td>
<td>inventory level of seller ( s ) at location ( j \in \mathcal{N}_s ) at time ( t )</td>
</tr>
<tr>
<td>( d_{i,r}^s(p,t) )</td>
<td>demand of goods by receiver ( r ) located at location ( i \in \mathcal{N}_r ) fulfilled by seller ( s )</td>
</tr>
<tr>
<td>( \Psi_j(I_j^s(t)) )</td>
<td>inventory holding cost of seller ( s ) at location ( j \in \mathcal{N}_s ) when inventory level is ( I_j^s(t) )</td>
</tr>
<tr>
<td>( \theta_j^s(q_j^s(t)) )</td>
<td>unit production cost of seller ( s ) located at node ( j \in \mathcal{N}_s ) when production level is ( q_j^s(t) )</td>
</tr>
<tr>
<td>( \pi_{i,j}^{c,r,s}(t) )</td>
<td>price charged by transporter ( c ) for delivering goods from location ( i \in \mathcal{N}_s ) to the location ( j \in \mathcal{N}_r ) at time ( t )</td>
</tr>
<tr>
<td>( \rho_{i,j,m}^{c,r,s}(t) )</td>
<td>flow of goods delivered by transporter ( c ) at time ( t ) to the receiver ( r ) at its location ( i \in \mathcal{N}_r ) using the transportation mode ( m ) shipped by the seller ( s ) from location ( j \in \mathcal{N}_s )</td>
</tr>
<tr>
<td>( x^{c,s}(t) )</td>
<td>total backlogged service of transporter ( c ) for seller ( s ) at time ( t )</td>
</tr>
<tr>
<td>( u_{i,j}^{c,r,s}(\pi(t),x(t)) )</td>
<td>amount of demand of service produced by transporter ( c ) to deliver goods from location ( i \in \mathcal{N}_s ) to the location ( j \in \mathcal{N}_r ) at time ( t )</td>
</tr>
<tr>
<td>( w^{c,s}(x^{c,s}(t)) )</td>
<td>cost of lost goodwill from seller ( s ) for transporter ( c ) due to the level of backlogged shipments at time ( t ).</td>
</tr>
<tr>
<td>( k_m^c(\rho^c(t),t) )</td>
<td>unit transportation cost of transporter ( c ) while using mode ( m ) transferring ( \rho ) units of goods at time ( t )</td>
</tr>
</tbody>
</table>
4.2.10 Seller’s Model

The sellers in this model are in dynamic oligopolistic Cournot-Nash competition. Each seller has the goal of maximizing their own profit over the finite planning horizon \( t \in [t_0, t_1] \). The sellers have the ability to dynamically set the price of the goods they are selling to the receivers. The price that is charged to the receivers can be viewed as the delivered price and includes the price for the goods as well as the price for the transportation. The sellers will then pay the transporters for freight service. Seller \( s \) seeks to solve the following optimal control problem taking \( p^{-s}, \pi^s \), and \( u^s \) as exogenous but endogenous to the overall problem:

\[
\max J^s_1(p^s, q^s, v^s; I^s; p^{-s}, \pi^s, u^s; t) = \int_{t_0}^{t_1} \left\{ d^s(p(t)) \cdot p^s(t) - \theta^s(q^s(t)) - \sum_{j \in N_s} \Psi^s_j(I^s_j(t)) - u^s(\pi(t)) \cdot \pi^s(t) \right\} dt
\]

subject to

\[
\frac{dI^s_j(t)}{dt} = q^s_j(t) - \sum_{r \in R} \sum_{i \in N_r} v^{r,s}_{i,j} \forall j \in N_s
\]

\[
I^s_j(t_0) = I^s_{j,0} \in \mathbb{R} \forall j \in N_s \tag{4.27}
\]

\[
I^s_j(t) \geq 0 \forall j \in N_s, \ t \in (t_0, t_1) \tag{4.28}
\]

\[
\sum_{s \in S} d^{r,s}_{i}(p^r(t)) = D^r_i(t) \forall r \in R, \ i \in N_r \tag{4.29}
\]

\[
d^{r,s}_{i}(p^r(t)) = \sum_{j \in N_s} v^{r,s}_{i,j}(t) \forall r \in R, \ i \in N_r \tag{4.30}
\]

\[
v^s(t) \geq 0 \tag{4.31}
\]

\[
p^s_{\min} \leq p^s(t) \leq p^s_{\max} \tag{4.32}
\]

\[
q^s_{\max} \geq q^s(t) \geq 0 \tag{4.33}
\]
where

\[ p^s = \{ p_{i}^{r,s} : r \in R, \; i \in N_r \} \]
\[ p^r = \{ p_{r}^{r,s} : r \in R, \; i \in N_r \} \]
\[ q^s = \{ q_{j}^{s} : j \in N_s \} \]
\[ v^s = \{ v_{i,j}^{r,s} : j \in N_s, \; r \in R, \; i \in N_r \} \]
\[ I^s = \{ I_{j}^{s} : j \in N_s \} \]
\[ p_{-s} = \{ p_{g}^{s} : g \in S \backslash s \} \]
\[ \pi^s = \{ \pi_{c,i,j}^{c,r,s} : c \in C, \; r \in R, \; i \in N_r, \; j \in N_s \} \]
\[ \pi = \{ \pi^s : s \in S \} \]
\[ u^s = \{ u_{c,i,j}^{c,r,s} : c \in C, \; r \in R, \; i \in N_r, \; j \in N_s \} \]

The controls in this problem are \( p^s \), \( q^s \) and \( v^s \) which are the price, production, and shipments of goods respectively. The states are the inventory levels \( I^s_j \). The first term of the criterion is the revenue generated by the seller represented as the product of the demand and the price. The second and third terms are the production and inventory holding costs respectively. The fourth term is the cost charged by the transporters for delivery services.

The right hand side of the state dynamics (4.27) are flow balance statements that represent the change in inventory as the difference between the amount of goods produced and the amount of goods shipped. The initial inventory for each of the sellers’ nodes is dictated by (4.28). Constraint (4.29) is a pure state constraint to ensure that inventory is non-negative during the planning horizon. Equation (4.30) requires that the sum of all demands for the sellers from a certain receiver be equal to the total demand of that receiver. In equation (4.31), the demand for each seller from a particular receiver must equal the sum of shipments of goods from the sellers different locations to that receiver. Finally, constraints (4.32) - (4.34) are bounds on the controls.

This optimal control problem has joint constraints (4.30) as sellers’ decision variables.
are coupled. Constraints of this type lead to a formulation known as a generalized Nash equilibrium (GNE) and will require special attention when the complementarity conditions for the problem are formed [74].

Receivers’ Demand Model

The demand of goods for each seller $s$ from each receiver $r$ is assumed to be of the linear form

$$d^{r,s}_i (p^{r}_i (t)) = a^{r,s}_1 p^{r}_i (t) + \sum_{g \in S \setminus s} a^{r,s,g}_3 p^{r,g}_i (t) \quad (4.35)$$

where $a^{r,s}_1, a^{r,s}_2, a^{r,s,g}_3 \in \mathbb{R}^+_1$ for all $r \in R$, $s \in S$ and $g \in S \setminus s$. Demand of goods from a particular seller increases as her competitors increase their prices and decreases with her own price. In our work we do not assume any restrictions on the price elasticity and the cross price elasticities.

Inventory Holding Cost and Other Cost Functions

The inventory holding cost for each seller $s \in S$ at a node $j \in N_s$ is assumed to be of the form

$$\Psi^{s}_j (I^s_j) = \frac{1}{2} e^s_j (I^s_j)^2 \quad (4.36)$$

where $e^s_j \in \mathbb{R}^+_1$ for all $s \in S$ and $j \in N_s$.

The unit production cost of each seller $s \in S$ at node $j \in N_s$ is assumed to be of the form

$$\theta^s_j (q^s_j, t) = f^{s}_1 q^s_j (t) + \frac{1}{2} f^{s}_2 (q^s_j (t))^2 + \frac{1}{3} f^{s}_3 (q^s_j (t))^3 \quad (4.37)$$

where $f^{s}_1, f^{s}_2, f^{s}_3 \in \mathbb{R}^+_1$ for all $s \in S$ and $j \in N_s$. 

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4.2.11 Discrete Mathematical Program and NCP Formulation

The optimal control problem (4.26) - (4.34) can be time discretized to obtain a finite dimensional mathematical program for each seller \( s \in S \). This proves useful as we can then investigate the KKT conditions of each mathematical program, but it can also be argued to be a better representation of the problem as receivers have demands at discrete time points. The resulting finite dimensional mathematical program is:

\[
\max J^s_t(p^s_t, q^s_t, v^s_t; I^s_t; p^{-s}_t, \pi^s_t, u^{0,s}_t) = \Delta \sum_{t=0}^{N} \left\{ \sum_{r \in R} (p^r_t)^T \frac{d^r_s(p_t)}{\delta^r_t} q^s_t - \sum_{j \in N_s} \Psi^s_{j,t}(I^s_{j,t}) - \sum_{c \in C} (\pi^c_t)^T u^c_{t,s} \right\}
\]

subject to

\[
I^s_{j,t+1} = I^s_{j,t} + \Delta \left[ q^s_{j,t} - \sum_{r \in R} \sum_{i \in N_r} v^{r,s}_{i,j,n} \right] \quad \forall \ j \in N_s, \ t = 0...N \quad (4.39)
\]

\[
I^s_{j,0} = \overline{I}^s_{j,0} \in \mathbb{R} \quad \forall \ j \in N_s \quad (4.40)
\]

\[
I^s_{j,t} \geq 0 \quad \forall \ j \in N_s, \ t = 0...N \quad (4.41)
\]

\[
\sum_{s \in S} d^{r,s}_{t,t}(p_t) = D^r_{t,t} \quad \forall \ r \in R, \ i \in N_r, \ t = 0...N \quad (4.42)
\]

\[
d^{r,s}_{t,t}(p_t) = \sum_{j \in N_s} v^{r,s}_{i,j,t} \quad \forall \ r \in R, \ i \in N_r, \ t = 0...N \quad (4.43)
\]

\[
v^s_t \geq 0 \quad \forall \ t = 0...N \quad (4.44)
\]

\[
p^s_{\min} \leq p^s_t \leq p^s_{\max} \quad \forall \ t = 0...N \quad (4.45)
\]

\[
q^s_t \geq 0 \quad \forall \ t = 0...N \quad (4.46)
\]

where the subscript \( t \) now denotes a specific time discretization point, \( \Delta \) is the time step size, and \( N = \frac{t_{10} - t_0}{\Delta} + 1 \) is the number of discrete time points.

Note that the states \( I^s_{j,t} \) are merely intermediate variables and can be replaced by the controls \( v^s_t, p^s_t \) and \( q^s_t \) using the following representation.

\[
I^s_{j,t} = I^s_{j,0} + \Delta \sum_{n=0}^{N} \left[ q^s_{j,n} - \sum_{r \in R} \sum_{i \in N_r} v^{r,s}_{i,j,n} \right] \quad (4.47)
\]
Let us use the following vector of controls to simplify the notation.

\[ \psi^s = \begin{pmatrix} v_t^s & p_t^s & q_t^s \end{pmatrix}^T \]

Because all of the constraints (4.39) - (4.46) are linear, Abadie's constraint qualification holds and we may inspect the KKT conditions of the mathematical program (5.59) - (4.46). The KKT identity is:

\[
\nabla \psi^s \begin{pmatrix} -J_1^s(\psi^s; p_t^{-s}, \pi_t^{-c}, u_t^{-c}) + \sum_{i=0}^N \sum_{r \in R_i} \sum_{t \in N_r} \left( \sum_{s \in S} \left( \sum_{i,t} d_{i,t}^r (p_t) - D_{i,t}^r \right) \right) \\
+ \beta_{i,t}^r d_{i,t}^r (p_t) + D_{i,t}^r \\
+ \gamma_{i,t}^r \left( d_{i,t}^r (p_t) - \sum_{j \in N_s} v_{i,j,t}^r \right) + \gamma_{i,t}^s \left( -d_{i,t}^s (p_t) + \sum_{j \in N_s} v_{i,j,t}^s \right) \\
+ \delta_{t}^r \left( -v_t^r \right) + \varepsilon_{t}^r \beta_{i,t}^r \right) \end{pmatrix} = 0
\]

Note that the equations (4.42) and (4.43) are being represented by two inequalities. Also, note that the dual multipliers \( \beta_{i,t}^r \) and \( \beta_{i,t}^s \) do not depend on the specific seller. This is due to the fact that the constraints (4.42) associated with these dual multipliers are joint constraints. That is, the constraint for a specific seller involves the decision variables of that seller as well as the decision variables of competing sellers. This method of using a common dual variable is adapted from the work of Pang [75] and Harker [74].

We can use the following change of variable to further simplify the notation.

\[ p_t^r = p_t^r - p_{min} \geq 0 \]

Manipulating the KKT identity (4.48) results in the following expression

\[
0 = \nabla \psi^s \begin{pmatrix} -J_1^s(\psi^s; p_t^{-s}, \pi_t^{-c}, u_t^{-c}) + \sum_{i=0}^N \sum_{r \in R_i} \sum_{t \in N_r} \left( \sum_{s \in S} \left( \sum_{i,t} d_{i,t}^r (p_t) - D_{i,t}^r \right) \right) \\
+ \beta_{i,t}^r d_{i,t}^r (p_t) + D_{i,t}^r \\
+ \gamma_{i,t}^r \left( d_{i,t}^r (p_t) - \sum_{j \in N_s} v_{i,j,t}^r \right) + \gamma_{i,t}^s \left( -d_{i,t}^s (p_t) + \sum_{j \in N_s} v_{i,j,t}^s \right) \\
+ \delta_{t}^r \left( -v_t^r \right) + \varepsilon_{t}^r \beta_{i,t}^r \right) \end{pmatrix} \cdot \psi^s
\]

\[
= \Theta(\psi^s; \beta_{i,t}^r, \beta_{i,t}^s, \gamma_{i,t}^r, \gamma_{i,t}^s, \varepsilon_{t}^r, \varepsilon_{t}^s, \delta_{t}, \psi^s)
\]

222
where
\[ \vec{\psi}^s = \left( v^s_t, \vec{p}^s_t, q^s_t \right)^T \]

The complementary slackness conditions accompanying the KKT Identity (4.48) give that
\begin{align*}
\xi_{j,t}^s \left( I_{j,t}^s \right) &= 0 \quad \forall \, j \in N_s, \, t = 0...N \\
\beta_{i,t}^{r,+} \left( -\sum_{s \in S} d_{i,t}^{r,s} (p_t) + D_{i,t}^r \right) &= 0 \quad \forall \, r \in R, \, i \in N_r, \, t = 0...N \\
\beta_{i,t}^{r,-} \left( \sum_{s \in S} d_{i,t}^{r,s} (p_t) - D_{i,t}^r \right) &= 0 \quad \forall \, r \in R, \, i \in N_r, \, t = 0...N \\
\gamma_{i,t}^{r,+} \left( -d_{i,t}^{r,s} (p_t) + \sum_{j \in N_s} s_{i,j,t}^{r,s} \right) &= 0 \quad \forall \, r \in R, \, i \in N_r, \, t = 0...N \\
\gamma_{i,t}^{r,-} \left( d_{i,t}^{r,s} (p_t) - \sum_{j \in N_s} s_{i,j,t}^{r,s} \right) &= 0 \quad \forall \, r \in R, \, i \in N_r, \, t = 0...N \\
\delta_{t}^{r,s} \left( v^r_t \right) &= 0 \quad \forall \, r \in R, \, t = 0...N \\
\zeta_{t}^{r,+} \left( p_{\max}^r - \vec{p}_t^r - p_{\min}^r \right) &= 0 \quad \forall \, r \in R, \, t = 0...N \\
\zeta_{t}^{r,-} \left( \vec{p}_t^r \right) &= 0 \quad \forall \, r \in R, \, t = 0...N \\
\eta_{t}^{s,+} \left( q_{\max}^s - q_t^s \right) &= 0 \quad \forall \, t = 0...N \\
\eta_{t}^{s,-} \left( q_t^s \right) &= 0 \quad \forall \, t = 0...N \quad (4.50)
\end{align*}

The manipulated KKT identity (4.49) and complementary slackness conditions
(4.50) can be combined into the following complementary vectors.

\[
\left( \Theta_s \left( \bar{\theta}_s^s, \beta_{i,t}^r, \beta_{i,t}^r, \gamma_{r,s}^r, \gamma_{r,s}^r, \zeta_{s}^s, \eta_t^s \right) \right)
\]

\[
\begin{align*}
0 \leq & \begin{bmatrix}
I_{j,t}^s \\
- \sum_{s \in S} d_{i,t}^{r,s} (p_t) + D_{i,t}^r \\
\sum_{s \in S} d_{i,t}^{r,s} (p_t) - D_{i,t}^r \\
d_{i,t}^r (p_t) + \sum_{s \in S} v_{i,j,t}^{r,s} \\
\sum_{s \in S} v_{i,j,t}^{r,s} \\
p_{\max} - p_{\min}^r \\
p_{\max} - p_{\min}^r \\
q_{\max}^s - q_{t}^s \\
q_{t}^s \\
\end{bmatrix} = G_s (z^s) \perp z^s =
\begin{bmatrix}
\bar{\psi}^s \\
\xi_{s,j,t}^s \\
\beta_{i,t}^r \\
\beta_{i,t}^r \\
\gamma_{r,s}^r \\
\gamma_{r,s}^r \\
\zeta_{s,t}^r \\
\zeta_{s,t}^r \\
\eta_{t}^{s,+} \\
\eta_{t}^{s,+} \\
\end{bmatrix} \geq 0
\end{align*}
\]

\[(4.51)\]

**4.2.12 Extremal Problem for the Transporter**

transporter \( c \) seeks to solve the following optimal control problem

\[
\begin{align*}
\max J_2^c (\pi^c, \rho^c; x^c; \pi^c, \rho^c, v^c; t) &= \int_{t_0}^{t_1} \left\{ u^c (\pi (t)) \cdot \pi^c (t) - \sum_{s \in S} w_s^c (x_s^c (t)) \right. \\
& \quad - \sum_{m \in M} k_m^c (\rho_m^c (t), t) \cdot \rho_m^c (t) \left\} \ dt
\end{align*}
\]

(4.52)
subject to

$$\frac{dx^{c,s}(t)}{dt} = \sum_{r \in R} \sum_{i \in N_r} \sum_{j \in N_s} \left( u^{c,r,s}_{i,j} (\pi^{r,s}_{i,j} (t)) - \sum_{m \in M} \rho^{c,r,s}_{i,j,m} (t) \right)$$

(4.53)

$$x^{c,s} (t_0) = 0 \; \forall \; s \in S$$

(4.54)

$$x^{c,s} (t) \geq 0 \; \forall \; s \in S, \; t \in (t_0, t_1)$$

(4.55)

$$x^{c,s} (t_1) = 0 \; \forall \; s \in S$$

(4.56)

$$\sum_{c \in C} \sum_{m \in M} \rho^{c,r,s}_{i,j,m} = v^{r,s}_{i,j} \; \forall \; r \in R, \; i \in N_r, \; s \in S, \; j \in N_s$$

(4.57)

$$\rho^c \geq 0$$

(4.58)

$$\pi^c_{\min} \leq \pi^c \leq \pi^c_{\max}$$

(4.59)

where

$$\pi^c = \{ \pi^{r,s}_{i,j} : r \in R, \; s \in S, \; i \in N_r, \; j \in N_s \}$$

$$\pi = \{ \pi^c : c \in C \}$$

$$\rho^c_m = \{ \rho^{r,s}_{i,j,m} : r \in R, \; s \in S, \; i \in N_r, \; j \in N_s \}$$

$$\rho^c = \{ \rho^c_m : m \in M \}$$

$$u^c = \{ u^{r,s}_{i,j} : r \in R, \; s \in S, \; i \in N_r, \; j \in N_s \}$$

$$k^c_m = \{ k^{r,s}_{i,j,m} : r \in R, \; s \in S, \; i \in N_r, \; j \in N_s \}$$

$$\pi^{-c} = \{ \pi^g : g \in C \setminus c \}$$

$$\rho^{-c} = \{ \rho^g : g \in C \setminus c \}$$

$$v^s = \{ v^{r,s}_{i,j} : r \in R, \; i \in N_r, \; j \in N_s \}$$

The controls in the above optimal control problem (4.52) - (4.59) are $\pi$ and $\rho$, the price and flow of goods respectively. The states are the backlogged shipments $x^{c,s}$. The first term of the criterion is the transporter’s revenue represented by the product of the demand and price. The second term is the cost of lost goodwill due to backlogged service. Finally, the third term is the cost of transportation.

The states evolve according to the state dynamics (4.53). The right hand side
of the state dynamics are flow balance equations and represent the change in backlogged shipments as the difference between the demanded shipments and the actual shipments made. Equation (4.54) gives that there are no initial backlogged shipments at time $t_0$, while (4.56) gives that there must be no remaining backlogged shipments at the terminal time $t_1$. Constraint (4.55) is a pure state constraint that requires that the level of backlogged shipments be non-negative for all time $t \in (t_0, t_1)$. Equation (4.57) is a joint constraint that requires that the shipments made by all transporters between a particular seller/receiver pair be equal to the shipments made by each seller to the accompanying receiver, $(r_{i,j}^s)$, which is determined in the sellers’ model. This constraint ties the sellers’ and transporters’ models together. As in the sellers’ model, the joint constraints will need special attention.

The last two constraints (4.58) and (4.59) are simply bounds on the controls.

The demand function for services produced by transporter $c \in C$ for seller $s \in S$ to be delivered to receiver $r \in R$ is assumed to be of the linear form

$$u_{i,j}^{c,r,s} \left( \frac{r_{i,j}^s}{s_{i,j}^c} \right) = \omega_{1,i,j}^{c,r,s} - \omega_{2,i,j}^{c,r,s} \pi_{i,j}^c \left( t \right) + \omega_{3,i,j}^{c,g,r,s} \pi_{i,j}^c \left( t \right) + \sum_{g \in C \setminus c} \omega_{4,i,j}^{c,g,r,s} \pi_{i,j}^c \left( t \right)$$

(4.60)

From this demand function, $\omega_{1,i,j}^{c,r,s}$, $\omega_{2,i,j}^{c,r,s}$, $\omega_{3,i,j}^{c,r,s}$, $\omega_{4,i,j}^{c,r,s}$, $\omega_{5,i,j}^{c,r,s} \in \mathbb{R}_+$ for all $c \in C$, $r \in R$, $s \in S$ and $g \in C \setminus c$, we can see that the rate of production of transporter $c$ decreases as its own price increases and increases as its competitors’ prices increase.

The cost function $w^{c,s}(x)$ is associated with lost goodwill due to a transporter’s backlog of service for seller $s \in S$ and is assumed to be of the form

$$w^{c,s}(x) = b_1^{c,s} x^{c,s}(t) + \frac{1}{2} b_2^{c,s} [x^{c,s}(t)]^2$$

(4.61)

where $b_1^{c,s}$, $b_2^{c,s} \in \mathbb{R}_+$ for all $c \in C$ and $s \in S$.

<<need to make $l_{1,m}^{c,r,s}$ and $l_{2,m}^{c,r,s}$ parameters that change with time>>

The transportation cost function $l_{m}^{c,r,s} \left( \sum_{s \in S} \sum_{r \in R} \rho_{i,j,m}^{c,r,s}, t \right)$ represents the unit transportation cost for transporter $c \in C$ shipping goods from seller $s \in S$ at node $j \in N_s$
4.2.13 Discrete Mathematical Program and NCP Formulation

The optimal control problem (4.52) - (4.59) can be time discretized, as in the seller model, to obtain the following finite dimension mathematical program.

\[
\begin{align*}
\max \ J_c^2 \left( \psi^c_t, \rho^{c,s}_t; \pi^c_t, x^{c,s}_t; v^c_t \right) &= \Delta \sum_{t=0}^{N} \left\{ u^c_t(\pi_t) \cdot \pi^c_t - \sum_{s \in S} w^{c,s}_t(x^{c,s}_t) \right. \\
&\quad \left. - \sum_{m \in M} k^{c,t}_m (\rho^{c,t}_m) \cdot \rho^c_{m,t} \right\} 
\end{align*}
\]

subject to

\[
\begin{align*}
x^{c,s}_{t+1} &= x^{c,s}_t + \Delta \sum_{r \in R} \sum_{i \in N_r} \sum_{j \in N_s} \left( u^{c,r,s}_{i,j,t} (\pi_t) - \sum_{m \in M} \rho^{c,r,s}_{i,j,m,t} \right) \quad \forall \ s \in S, \ t = 0, \ldots, N - 1 \\
x^{c,s}_0 &= 0 \quad \forall \ s \in S \\
x^{c,s}_t &\geq 0 \quad \forall \ s \in S, \ t = 1, \ldots, N - 1 \\
x^{c,s}_N &= 0 \quad \forall \ s \in S \\
\sum_{c \in C} u^{c,r,s}_{i,j,t} (\pi_t) &= v^{r,s}_t \quad \forall \ r \in R, \ i \in N_r, \ s \in S, \ j \in N_s, \ t = 0, \ldots, N \\
\rho^{c,t}_m &\geq 0 \quad \forall \ m \in M, \ t = 0, \ldots, N \\
\pi^{c,s}_t &\leq \pi^{c,s}_t \leq \pi^{c,s}_t \quad \forall \ s \in S, \ t = 0, \ldots, N 
\end{align*}
\]

where \( t \) now denotes a specific time discretization point, \( \Delta \) is the time step size, and \( N = \frac{t_f - t_0}{\Delta} + 1 \) is the number of discrete time points. These time discretization parameters do not necessarily need to be kept the same for both the seller and transporter models.

Note that the states \( x^{c,s}_t \) can be represented in terms of the controls \( \rho_t \) and \( \pi_t \) using the following representation.

\[
x^{c,s}_t = x^{c,s}_0 + \Delta \sum_{n=0}^{t-1} \left[ \sum_{r \in R} \sum_{j \in N_r} \sum_{i \in N_s} \left( u^{c,r,s}_{i,j,t} (\pi_t) - \sum_{m \in M} \rho^{c,r,s}_{i,j,m,t} \right) \right] 
\]
Using the following vector to simplify notation

$$\psi^c = \left( \rho^{c,r,s}_{i,j,m,t} \quad \pi^c_{r,s,t} \right)^T$$

we can inspect the KKT conditions. The constraints (4.63) - (4.69) for the transporters’ problem are linear as in the sellers’ problem, so again Abadie’s constraint qualification holds. The KKT identity is

$$\nabla \psi^c \left\{ -J^c (\psi^c; \rho^c) + \vartheta^{s,+} (x^c_N) + \vartheta^{s,-} (-x^c_N) + \sum_{t=1}^{N-1} \left( \sum_{s \in S} (\theta^s_t (-x^c_t)) \right) \right. \\
\left. + \sum_{t=0}^{N} \left[ \lambda^{r,s,+}_{i,j,t} \left( \sum_{c \in C} u^{c,r,s}_{i,j,t} (\pi_t) - v^r_{i,j,t} \right) + \lambda^{r,s,-}_{i,j,t} \left( - \sum_{c \in C} u^{c,r,s}_{i,j,t} (\pi_t) + v^r_{i,j,t} \right) \right. \\
+ \nu^c_{i,j,t} \left( -\pi^c_{r,s,t} \right) + \nu^{c,r,s,-}_{i,j,t} (\pi^c_{r,s,t} - \pi^{c,r,s}_{max}) + \mu^{c,r,s}_{m,t} (\rho^{c,r,s}_{m,t}) \right] \right\} = 0 \quad (4.71)$$

where the following change of variable is used to simplify the notation.

$$\bar{\pi}^c_{r,s,t} = \pi^c_{r,s,t} - \pi^{c,r,s}_{\text{min}} \geq 0$$

Manipulating the KKT identity in (4.71), we can obtain the following statement.

$$0 = \nabla \psi^c \left\{ -J^c (\psi^c; \rho^c) + \vartheta^{c,r,s,+} (x^c_N) + \vartheta^{c,r,s,-} (-x^c_N) + \sum_{t=1}^{N-1} \left( \sum_{s \in S} (\theta^s_t (-x^c_t)) \right) \right. \\
+ \sum_{t=0}^{N} \left[ \lambda^{r,s,+}_{i,j,t} \left( \sum_{c \in C} u^{c,r,s}_{i,j,t} (\pi_t) - v^r_{i,j,t} \right) + \lambda^{r,s,-}_{i,j,t} \left( - \sum_{c \in C} u^{c,r,s}_{i,j,t} (\pi_t) + v^r_{i,j,t} \right) \right. \\
+ \nu^c_{i,j,t} \left( -\pi^c_{r,s,t} \right) + \nu^{c,r,s,-}_{i,j,t} (\pi^c_{r,s,t} - \pi^{c,r,s}_{max}) \left. \right] \psi^c \right\}$$

$$= \Theta^c \left( \bar{\psi}^c; \phi^c_{i,j,t}, \vartheta^{c,s,+,+}, \vartheta^{c,s,-}, \lambda^{s,+,+}_{i,j,t}, \rho^{c,r,s}_{i,j,m,t}, \nu^{c,r,s,-}_{i,j,t} \right) \cdot \bar{\psi}^c \quad (4.72)$$

where

$$\bar{\psi}^c = \left( \rho^{c,r,s}_{i,j,m,t} \quad \pi^c_{r,s,t} \right)^T$$

The complementary slackness conditions associated with the KKT identity (4.71)
are

\[ \varphi_t^s (x_{t,c,s}^c) = 0 \quad \forall s \in S, \ t = 1 \ldots N - 1 \]
\[ \varphi^s, + (-x_{N,c,s}^c) = 0 \quad \forall s \in S \]
\[ \varphi^s, - (x_{N,c,s}^c) = 0 \quad \forall s \in S \]
\[ \lambda_{i,j,t}^{s,+} \left( - \sum_{c \in C} u_{i,j,t}^{c,r,s} (\pi_t) + v_{i,j,t}^{r,s} \right) = 0 \quad \forall s \in S, \ i \in N_r, \ s \in S, \ j \in N_s, \ t = 1 \ldots N \]
\[ \lambda_{i,j,t}^{s,-} \left( \sum_{c \in C} u_{i,j,t}^{c,r,s} (\pi_t) - v_{i,j,t}^{r,s} \right) = 0 \quad \forall s \in S, \ i \in N_r, \ s \in S, \ j \in N_s, \ t = 1 \ldots N \]
\[ \nu_{i,j,t}^{s,+} (\bar{\pi}_{i,j,t}^{c,s}) = 0 \quad \forall s \in S, \ t = 1 \ldots N \]
\[ \nu_{i,j,t}^{s,-} (-\bar{\pi}_{i,j,t}^{c,s} + \pi_{\min}^{c,s} + \pi_{\max}^{c,s}) = 0 \quad \forall s \in S, \ t = 1 \ldots N \]
\[ \mu_{m,t} (\rho_{m,t}^c) = 0 \quad \forall t = 1 \ldots N \]

(4.73)

Similar to the sellers’ model, the dual multipliers \( \lambda_{i,j,t}^{s,+} \) and \( \lambda_{i,j,t}^{s,-} \) are common among the transporters due to the joint constraints (4.57).

Combining the complementary slackness conditions (4.73) with the manipulation of the KKT identity (4.72), we can create the following two vectors that are complementary to one another.

\[
\begin{pmatrix}
\Theta_c \left( \bar{\psi}^c; \varphi_{t,c,s}^c, \varphi_{t,c,s,-}, \varphi_{t,c,s,+}, \lambda_{i,j,t}^{s,+}, \lambda_{i,j,m,t}^{r,c,s} \right) \\
\end{pmatrix}
\]

0 \leq

\[
\begin{pmatrix}
0 \\
- \sum_{c \in C} u_{i,j,t}^{c,r,s} (\pi_t) + v_{i,j,t}^{r,s} \\
- \sum_{c \in C} u_{i,j,t}^{c,r,s} (\pi_t) - v_{i,j,t}^{r,s} \\
-x_{c,s}^N \\
-x_{c,s}^N \\
\bar{\pi}_{i,j,t}^{c,r,s} \\
\bar{\pi}_{i,j,t}^{c,r,s} \\
\pi_{\min}^{c,r,s} + \pi_{\max}^{c,r,s} \\
(\rho_{m,t}^{c,r,s})
\end{pmatrix}
\]

= \( G_c (z^c) \perp z^c \)

\[
\begin{pmatrix}
\hat{\psi}^c \\
\varphi_{t,c,s}^c \\
\varphi_{t,c,s,-} \\
\varphi_{t,c,s,+} \\
\lambda_{i,j,t}^{r,s,+} \\
\lambda_{i,j,t}^{r,s,-} \\
\nu_{t,c,r,s,+} \\
\nu_{t,c,r,s,-} \\
\mu_{m,t}^{c,r,s}
\end{pmatrix}
\]

\( \geq 0 \)

(4.74)
4.2.14 Complete NCP Formulation

The complete non-linear complementarity problem (NCP) describing the Cournot-Cournot game is created by concatenating the complementarity conditions from (4.51) and (4.74) that were obtained through the analysis of the seller and transporter models.

\[
G(z) = \begin{pmatrix}
G_s(z^s) \\
G_c(z^c)
\end{pmatrix} \perp z \begin{pmatrix}
z^s \\
z^c
\end{pmatrix}
\]  

(4.75)

Such a complementarity problem can be solved using a commercial solver such as PATH via a modeling language such as GAMS. Because both the seller and transporter models are linear in the constraints, we may use the sequential linearization option in PATH to solve this complementarity problem and be guaranteed convergence for strictly monotic principal functions [75].

4.2.15 Illustrative Numerical Example

In this section, we will describe an example problem containing 3 sellers, 3 transporters, and 3 receivers. Each seller and receiver is located at only one node independent of all others. The 3 transporters can each deliver from any seller to any receiver. This leads to a fully connected network representation. For this example, only one mode of transportation is considered. The ranges of parameters used in the model are listed in Table 4.7.

This example was formulated as a nonlinear complementarity problem in GAMS and solved using the PATH solver with the above parameters. PATH was run with the options to sequentially linearize the problem and use a Lemke’s type algorithm to solve the linearized problem at each iteration. The time for solution was less than 15 seconds on a Pentium 4 desktop with 1GB of RAM. Some of the solution outputs are displayed below.

In figure below on the left, we can see that the sellers start with a positive inventory and then quickly sell it off to keep inventory as close to zero as possible, thereby keeping inventory holding costs low. Figure below on the right shows that the sellers start off with
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Range</th>
<th>Parameter</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_{1}^{r,s})</td>
<td>47 – 57</td>
<td>(b_{1}^{c,s})</td>
<td>0.05 – 0.15</td>
</tr>
<tr>
<td>(a_{2}^{r,s})</td>
<td>0.45 – 0.525</td>
<td>(b_{2}^{c,s})</td>
<td>0.02 – 0.12</td>
</tr>
<tr>
<td>(a_{3}^{r,s,g})</td>
<td>0.025 – 0.075</td>
<td>(\omega_{1}^{c,r,s})</td>
<td>9 – 10</td>
</tr>
<tr>
<td>(e_{j}^{s})</td>
<td>0.45 – 0.55</td>
<td>(\omega_{2}^{c,r,s})</td>
<td>0.45 – 0.525</td>
</tr>
<tr>
<td>(f_{1,j}^{s})</td>
<td>0.25 – 0.35</td>
<td>(\omega_{3}^{c,g,r,s})</td>
<td>0.1 – 0.15</td>
</tr>
<tr>
<td>(f_{2,j}^{s})</td>
<td>0.05 – 0.15</td>
<td>(I_{j,0}^{s})</td>
<td>35 – 75</td>
</tr>
<tr>
<td>(f_{3,j}^{s})</td>
<td>0</td>
<td>(D_{r}^{t})</td>
<td>50 – 70</td>
</tr>
<tr>
<td>(l_{1,m}^{c,r,s})</td>
<td>15 – 15.5</td>
<td>(\Delta)</td>
<td>0.5</td>
</tr>
<tr>
<td>(l_{2,m}^{c,r,s})</td>
<td>0.3 – 0.4</td>
<td>(N)</td>
<td>21</td>
</tr>
<tr>
<td>(p_{\min})</td>
<td>0</td>
<td>(\pi_{\min})</td>
<td>0</td>
</tr>
<tr>
<td>(p_{\max})</td>
<td>100</td>
<td>(\pi_{\max})</td>
<td>75</td>
</tr>
<tr>
<td>(q_{\max})</td>
<td>100</td>
<td>(q_{\min})</td>
<td>0</td>
</tr>
</tbody>
</table>
little production since they want to sell of their existing inventory. The sellers then begin to
ramp up production and take on a just-in-time production scheme that keeps their inventory
level near zero as seen in figure on the left. The sellers do not produce anything in the final
time period because of the finite planning horizon. There is no advantage to producing
anything because there is no following period to sell it in.

![Supplier Inventory](image1)
![Supplier Production](image2)

To conserve space, we only give the graphs for one seller’s prices and realized demand
below. The other two sellers display qualitatively similar results for both price and realized
demand. The figure below on the left displays the delivered price charged by seller 1 to each
of the three receivers, while figure below on the right displays the seller 1’s realized demands
from the three receivers. We can see that the solution behaves as we would expect; as the
price charged to a specific receiver decreases between two time periods, the realized demand
increases for that same time interval.
Price of goods charged by supplier 1

Figures below show the cumulative backlog of shipments as demanded by the three sellers in each time period for transporters 1, 2 and 3 respectively. We can see that the transporters start and end with no backlogged service as the model dictates. However, they do allow some service backlogs to find the best tradeoff between the loss-of-goodwill and transportation costs.
Carrier 3 Backlog

Further, figures below show the pricing strategies followed by transporters 1, 2 and 3 respectively for each of the receiver, seller pairs. One of the interesting details of the solution is that transporters 1 and 3 both have times when they set their prices to zero (the lower bound in this example.) transporter 2 always follows a strictly positive pricing policy for all of the services it offers. However, transporters 1 and 3 charge higher prices than transporter 2 for some of their services.
4.2.16 Concluding Remark

We have presented a dynamic Cournot-Cournot game theoretic model of dynamic pricing in an urban freight setting involving two classes of Cournot-Nash agents. The model was formulated as a set of coupled continuous time optimal control problems that were discretized and then recast as a nonlinear complementarity problem. We used an off-the-shelf commercial solver (PATH) to quickly and efficiently solve a numerical example.

Many extensions can be made to this model. One example is to include nonhomogeneous goods with different time values. Another obvious extension is to add stochasticity in the model. The receiver demand is a natural candidate for introducing uncertainty to the model.

4.3 Competition in a Supply Chain Network Involving Freight Shippers

In this section we further illustrate the equilibrium behavior of the freight shippers considering their respective supply chains. As before, we assume that these shippers (manufacturers) are involved in an oligopoly, produce a family of products utilizing a common
product platform, set price for their products. In this section we take into account the underlying supply chain network for each of these shippers who are manufacturers - a network arising from the activities of the suppliers who provide input factors to the shippers. From this point onwards we will interchangably use manufacturers and shippers conveying the same meaning. The suppliers under consideration are finite in number and each of them offer multiple input factors using a general production facility which in turn imposes capacity of the input factors that each one can offer at any point in the time.

We present a finite horizon stochastic dynamic game involving competing shippers and their suppliers. The dynamic supply chain game we consider involves a small number of make-to-order shippers who form an oligopoly and produce a family of products using a common product platform. They set the prices of their final differentiated products, accept backorders and face price elastic random demands for their finished goods from the receivers. Furthermore, these firms procure their input factors (components) from a small number of suppliers and hold inventory of the input factors. The capability of any given supplier to provide any given input factor has a finite upper bound. Consequently, explicit production lead times are intrinsic to the strategies of the manufacturers.

Moreover, the competing manufacturing firms face a finite planning horizon arising from relatively short selling periods (for style goods) and also from fast product obsolescence (for electronics component manufacturers). Therefore, each of the competing manufacturers we model needs to make combined sourcing, production and pricing decisions at the beginning of the planning horizon that they follow until the end of the planning horizon. The resulting non-cooperative dynamic game is a generalized Nash game whose equilibrium may be conveniently articulated as a restricted finite dimensional variational inequality after time discretization. We establish existence of at least one pure strategy Nash equilibrium and outline some of the qualitative properties of that equilibrium. We also outline an efficient algorithm to compute equilibrium of the game.
4.3.1 Notation

We use the following usual notations in the balance of this chapter, as far the superscript and subscripts, \( f \) denotes a firm, \( i \) a product, \( j \) input factor (raw material), and \( g \) denotes a supplier. Table 4.8 describes the parameters where as Table 4.9 lists control and state variables of the model.

It is evident that
\[
\bigcup_{i=1}^{|S|} C_i = C
\]

It is evident from above that
\[
\bigcup_{i=1}^{|S|} C_i = C
\]

The variables we use are primarily the controls (prices, production volume and procurement of raw materials) and the states (inventory of finished goods and inventory of input factors).

In the vector notation, decision variables for firm \( f \) are
\[
\begin{align*}
p^f_t & = \left( p^f_{i,t} : i \in S \right) \\
y^f_t & = \left( y^f_{i,t} : i \in S \right) \\
u^f_t & = \left( u^f_{j,g,t} : j \in G, g \in G \right)
\end{align*}
\]

which concatenates to
\[
\begin{align*}
p^f & = \left( p^f_{t} : t \in [0, N] \right) \\
y^f & = \left( y^f_{t} : t \in [0, N] \right) \\
u^f & = \left( u^f_{t} : t \in [0, N] \right)
\end{align*}
\]

Decision variables of firm \( f \)'s competitors are denoted as
\[
\begin{align*}
p^{-f} & = \left( p^g_{t} : g \in F \backslash f \right) \\
u^{-f} & = \left( u^j_{t} : j \in F \backslash f \right)
\end{align*}
\]

State variables for firm \( f \) are
\[
I^f_i = \left( I^f_{i,t} : i \in S \right)
\]
Table 4.8: Notations : Parameters : Supply Chain Competition

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>set of manufacturers</td>
</tr>
<tr>
<td>$S$</td>
<td>set of differentiated products each firm provides</td>
</tr>
<tr>
<td>$G$</td>
<td>set of suppliers from whom manufacturers procure raw materials</td>
</tr>
<tr>
<td>$C$</td>
<td>set of resources that firms use to provide services</td>
</tr>
<tr>
<td>$C_i$</td>
<td>set of resources that firms use to provide service $i \in S$</td>
</tr>
<tr>
<td>$A$</td>
<td>activity matrix</td>
</tr>
<tr>
<td>$L$</td>
<td>end of planning horizon</td>
</tr>
<tr>
<td>$0 \leq t_n \leq t_N = L$</td>
<td>$n$th time period on the planning horizon, where time step, $\Delta = \frac{L}{N-1}$</td>
</tr>
<tr>
<td>$p_{f,\text{min}}^i$</td>
<td>minimum price that firm $f$ can charge for product $i \in S$</td>
</tr>
<tr>
<td>$p_{f,\text{max}}^i$</td>
<td>maximum price that firm $f$ can charge for product $i \in S$</td>
</tr>
<tr>
<td>$K_j^f$</td>
<td>supplier $g$’s production capacity for input factor $j \in C$</td>
</tr>
<tr>
<td>$h_{f,i,t}^+$</td>
<td>holding cost of finished good $i \in S$ for firm $f$ for period $[t, t + 1]$</td>
</tr>
<tr>
<td>$h_{f,i,t}^-$</td>
<td>backorder cost of finished good $i \in S$ for firm $f$ for period $[t, t + 1]$</td>
</tr>
<tr>
<td>$K_{f,g,t}^j(y)$</td>
<td>unit procurement cost of input factor $j$ for firm $f \in F$ procuring from supplier $g \in G$ at time $t$ (including fixed ordering and transportation cost) when total order volume from that supplier is $y$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>penalty associated with any unfulfilled order at the end of planning horizon</td>
</tr>
<tr>
<td>$\psi_{f,t}^j(r)$</td>
<td>single period holding cost of the raw material $j$ (input factors) for firm $f$</td>
</tr>
</tbody>
</table>
Table 4.9: Notations : Supply Chain Competition : Control and State Variables

<table>
<thead>
<tr>
<th>Variables</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{i,t}^f$</td>
<td>price for product $i \in S$ charged by firm $f \in F$ during period $t$</td>
</tr>
<tr>
<td>$y_{i,t}^f$</td>
<td>firm $f$’s production of product $i \in S$ during period $t$</td>
</tr>
<tr>
<td>$u_{j,g,t}^f$</td>
<td>firm $f$’s procurement of resource $j$ from supplier $g$ at time $t$</td>
</tr>
<tr>
<td>$I_{i,t}^f$</td>
<td>firm $f$’s finished goods inventory of product $i$ at the beginning of period $t$</td>
</tr>
<tr>
<td>$d_{i,t}^f(p_t)$</td>
<td>mean demand for product $i \in S$ from firm $f \in F$ during the period $t$ when prevailing price is $p_t$</td>
</tr>
<tr>
<td>$D_{i,t}^f(p_t)$</td>
<td>firm $f$’s realized demand for service $i \in S$ at time period $t$</td>
</tr>
<tr>
<td>$z_i^f$</td>
<td>random component associated with demand faced by the firm $f$ for service $i$ at time period $t$ which are i.i.d.</td>
</tr>
<tr>
<td>$r_{j,t}^f$</td>
<td>firm $f$’s inventory of input factor $j \in C$ at time $t$</td>
</tr>
</tbody>
</table>

Network representation through activity matrix

The network we are interested in has $|C|$ input factors and the firm manufacture $|S|$ differentiated products. The activity matrix, $A = [a_{ij}]$ is a $|C| \times |S|$ matrix where

$$a_{ij} = \begin{cases} 1 & \text{if input factor } i \text{ is used to manufacture product } j \\ 0 & \text{else} \end{cases}$$

Thus the $j$th column of $A$, denoted $A_j$ is the incidence vector for the product $j$; the $i$th row, denoted $A^i$, has an entry of one in the column $j$ corresponding to a input factor $j$ that utilizes the resource $i$.

**Demand for products**

Firm $f$’s realized demand at time period $t$, $D_{i,t}^f(p_{i,t})$ for service $i$ when prevailing market price is $p_t$. Two types of demand models are predominant in the supply chain, news vendor
pricing and revenue management literature: the additive form where

\[ D_{i,t}^f (p_{i,t}) = d_{i,t}^f (p_{i,t}) + z_{i,t}^f \]  \hspace{1cm} (4.76)

and the multiplicative form where

\[ D_{i,t}^f (p_{i,t}) = d_{i,t}^f (p_{i,t}) \cdot z_{i,t}^f \]  \hspace{1cm} (4.77)

where \( \{ z_{i,t}^f \} \) is a sequence of continuous i.i.d. positive random variables with common pdf \( g (\cdot) \) and cdf distribution \( F (\cdot) \) respectively; and \( d_{i,t}^f (p_{i,t}) \) is the expected or average demand faced by the firm \( f \) for its service \( i \) when the price combination is \( p_{i,t} \) at time \( t \).

In the supply chain and revenue management literature it is common to make two major assumptions regarding the nature of the random variable \( z_{i,t}^f \) and the average demand \( d_{i}^f \). These are:

**Assumption A2.1.** The random variable \( z_{i,t}^f \) is independent of price, \( p \) and time, \( t \), and identically distributed with \( E ( z_{i,t}^f ) = 1 \), \( \text{Var}( z_{i,t}^f ) < \infty \)

**Assumption A2.2.** For any firm \( f \in F \) and the service type \( i \in S \), the mean demand \( d_{i}^f (p,t) \) has the following properties:

1. \( d_{i}^f \) depends only on the prices charges by firm \( f \) and its competitors for service type \( i \) only

2. \( d_{i}^f (p,t) \) is continuous, bounded, and differentiable in \( p_i \) on the strategy space \([p_{i,\min}, p_{i,\max}]\)
   where
   \[
p_i = \{ p_i^g : g \in \mathcal{F} \}
   \]

3. average demand decreases with own service price, i.e., \( \frac{\partial d_{i}^f (p,t)}{\partial p_i} < 0 \) and

4. average demand increases with non-own service prices, i.e., \( \frac{\partial d_{i}^f (p,t)}{\partial p_g} > 0 \) for all \( g \neq f \)

Note that

\[ E \left( D_{i}^f (p,t) \right) = d_{i}^f (p,t) \]

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the multiplicative model implies that the coefficients of variation of the one-period demands are constants, independent of the price vectors.

The multiplicative model implies that the coefficients of variation of the one-period demands are constants, independent of the price vectors. This type of multiplicative model was originally proposed by Karlin and Carr [53] and has since been used frequently in the supply chain literature. However, it is also possible to consider a more general demand model in which the standard deviation of demand may fail to be proportional to the mean, i.e., the coefficient of variation may vary as service prices and mean demand volumes vary. One such demand model is

\[ D_{i,t}^f (p,t) = d_{i,t}^f (p,t) + \sigma_{i,t}^f \left( d_{i,t}^f (p,t) \right) \cdot z_{i,t}^f \]  

where \( \sigma_{i,t}^f (.) \) is a general increasing function with \( \sigma_{i,t}^f (0) = 0 \) and \( z_{i,t}^f \geq 0 \) is a continuous random variable (see Bernstein and Federgruen [54] for such nonmultiplicative demand models). For the demands models (4.76) and (4.78) we must have

\[ E \left( z_{i,t}^f \right) = 0, \ Var \left( z_{i,t}^f \right) < \infty \]

In the balance of the chapter we assume \( D_{i,t}^f (p_t) \) follows nonmultiplicative form (4.78)

\[ D_{i,t}^f (p_t) = d_{i,t}^f (p_t) + \sigma_{i,t}^f \left( d_{i,t}^f (p_t) \right) \cdot z_{i,t}^f \]

where \( d_{i,t}^f (p,t) \) is a linear function and \( z_{i,t}^f \) follows normal distribution with mean 0 and finite variance \( \sigma_{i,t}^2 \).

### 4.3.2 Models for the average demand

Followings are the commonly used average demand functions used in the literature:

1. **Linear function**

\[ d_{i,t}^f (p,t) = \rho_{i,t}^f - \sigma_{i,t}^f \cdot p_{i,t}^f + \sum_{g \in F \setminus f} \gamma_{i,t}^g \cdot p_{i,t}^g \]

where \( \rho_{i,t}^f (t), \sigma_{i,t}^f (t), \gamma_{i,t}^f (t) \in \mathbb{R}^{1+} \) for all \( f \in F \) and \( i \in S \).
2. Logit
\[ d_{i,t}^f (p, t) = \frac{a_i^f \exp \left( -b_i^f (t) \cdot p_{i,t}^f \right)}{\sum_{g \in F} \alpha_{i,t}^g \exp \left( -b_i^g \cdot p_{i,t}^g \right)} \]
where \( a_i^f (t), b_i^f (t) \in \mathbb{R}^+ \) for all \( f \in F \) and \( i \in S \),

3. Cobb-Douglas
\[ d_{i,t}^f (p, t) = a_i^f \left( p_{i,t}^f \right)^{-\beta_i^f} \prod_{g \in F \setminus i} \left( p_{i,t}^g \right)^{\beta_i^g} \]
where \( a_i^f > 0, \beta_i^f > 1, \beta_i^g > 0 \) for all \( f \in F, i \in S \) and \( t_0 \leq t \leq t_1 \),

4. Attraction Model (which assumes a fixed market size and firms compete over market share, common in marketing literature)
\[ d_{i,t}^f (p, t) = M_t \frac{a_i^f \left( p_{i,t}^f \right)}{\sum_{g \in F} \sum_{j \in S} \alpha_{j,t}^g \left( p_{j,t}^g \right) + a_0} \]
where \( M_t \) is the fixed market size at time \( t \) and \( a_i^f \) is the attraction parameter and \( a_0 \) is the attraction parameters corresponding to no purchase option. Typically, \( a_i^f \left( p_{i,t}^f \right) \)

\[ a_i^f \left( p_{i,t}^f \right) = \exp \left\{ -\alpha_i^f \cdot p_{i,t}^f \right\} \] (MNL model)

or
\[ a_i^f \left( p_{i,t}^f \right) = \kappa_i^f \cdot \left( p_{i,t}^f \right)^{-\alpha_i^f} \] (multiplicative competitive interaction)

where \( \alpha_i^f, \kappa_i^f > 0 \).

4.3.3 Shipper \( f \)'s extremal problem

Taking competitors’ finished goods price \( p^{-f} \) and procurement volumes \( u^{-f} \) as exogenous, firm \( f \) seeks to solve the following discrete time stochastic optimal control problem

\[ J_f \left( p^f, y^f, u^f; p^{-f}; u^{-f} \right) = \max_{p^f, y^f, u^f} E \left[ -\delta \left( I_N \right) - \sum_{t=0}^{N-1} \left\{ p_{i,t}^f \cdot D_i^f (p_t) - K_{f,t} (u_{f,t}) \cdot u_{f,t} - \psi_i^f \left( r_t^f \right) \right\} - h_f^+ \left[ I_{f,t} - D_i^f (p_t) \right]^+ - h_f^- \left[ I_{f,t} - D_i^f (p_t) \right]^- \right] \] (4.79)
subject to

\[ I_{t+1}^f = I_t^f + y_t^f - D_t^f (p_t) \text{ for all } t = 0, \ldots, N - 1 \] (4.80)

\[ I_0^f = \gamma_0^f \in \mathbb{R}^{|S|}_+ \] (4.81)

\[ r_{t+1}^f = r_t^f + \sum_{g \in G} u_{g,t}^f - A \cdot y_t^f \text{ for all } t = 0, \ldots, N - 1 \] (4.82)

\[ r_0^f = \nu_0 \in \mathbb{R}^{|C|}_+ \] (4.83)

\[ r_t^f \geq 0 \text{ for all } t = 1, \ldots, N \] (4.84)

\[ u_{g,t}^f + \sum_{j \neq f} u_{j,t}^f \leq K_t^g \text{ for all } t = 0, \ldots, N - 1 \] (4.85)

\[ u_t^f \geq 0 \text{ for all } t = 0, \ldots, N - 1 \] (4.86)

\[ p_{\min}^f \leq p_t^f \leq p_{\max}^f \text{ for all } t = 0, \ldots, N - 1 \] (4.87)

\[ y_t^f \geq 0 \text{ for all } t = 0, \ldots, N - 1 \] (4.88)

(4.80) is the flow balance equation of finished goods inventory, where as (4.81) is the initial condition. Since these firms may accept backorders, the finished goods inventory can be negative as well for one or more time periods. However, to eliminate the always favorable strategy of accepting orders without ever fulfilling them from the model, we append a very high penalty cost for any unfulfilling orders at the end of the planning horizon, \( E \left[ -\delta \left( I_N^f \right) \right] \), in the objective function. The input factor balance equations are expressed in (4.82), whereas (4.83) indicates the initial states of the input factor bins. Since input factors are associated with the physical production system, (4.84) imposes a behavioral restriction which prevents input factor inventory to ever become negative. Equation (4.85) is the joint capacity constraints arising from the procurement decision of the firm \( f \) and its competitors. Procurement decisions have to be nonnegative, as captured in (4.86), so also production decisions (4.88). The lower and upper bounds on price are imposed via (4.87). Non-zero lower bounds on the production variables \( u_t^f \) in (4.86) will ensure that each shipper participates in each period with a strictly positive provision of service. The implication, if this were not true, would be that a firm with nothing to sell in a period could influence the demand seen by its competitors by setting a price. In other words, setting a
price would make sense only if there is a nonzero sale in that period. We have not imposed any constraint on the direction of price change. Clearly, as firms’ optimal control problems are coupled, this gives rise to a Cournot-Nash dynamic game setting.

4.3.4 Structural Properties of the Extremal Problem

We observe some rich structural properties of shippers’ extremal problems which facilitates computations and further analysis. First we observe that under the nonmultiplicative demand model with the Gaussian random noise, finished goods inventories at any point also follow normal distributions as illustrates in Lemma 4.1.

Lemma 4.1 Finished goods inventory for product $i$ owned by firm $f$ at time $t$, $I_{i,t}^f$ follows normal distribution with mean $\gamma_0 + \sum_{k=0}^{t-1} (y_{i,k}^f - D_{i,k}^f(p_k))$ and variance $\left( \sum_{k=0}^{t-1} \left[ \sigma_{i,k}^f \left( d_{i,k}^f(p_k) \right) \right]^2 \right) \cdot \hat{\sigma}_{i,f}^2$ for all $t = 1, \ldots, N$.

Proof. Using (4.80) and (4.81)

$$I_{i,t}^f = \gamma_0 + \sum_{k=0}^{t-1} \left[ y_{i,k}^f - D_{i,k}^f(p_k) \right] \tag{4.89}$$

Since $D_{i,t}^f(p_t) = d_{i,t}^f(p_t) + \sigma_{i,t}^f \left( d_{i,t}^f(p_t) \right) \cdot z_t^f$, therefore (4.89) can further be simplified as

$$I_{i,t}^f = \gamma_0 + \sum_{k=0}^{t-1} \left[ y_{i,k}^f - d_{i,k}^f(p_k) + \sigma_{i,k}^f \left( d_{i,k}^f(p_k) \right) \cdot z_t^f \right]$$

since sum of two or more normal distributions is also a normal distribution, the result is immediate. 

Lemma 4.2 expresses the one period backordering cost in terms of expected inventory holding cost which in turn facilitates exposition.
Lemma 4.2 The expected backordering cost for period $t$ can be restated as

$$h_f^- \cdot E \left( \left[ I_{f,t} - D^t_f (p_t) \right]^- \right) = h_f^- \left[ d_{i,t}^f (p_t) - \gamma_0^f - \sum_{k=0}^{t-1} (y_{i,k}^f - d_{i,k}^f (p_k)) \right] + h_f^- \cdot E \left( \left[ I_{f,t} - D^t_f (p_t) \right]^+ \right)$$

Proof. We begin with the following identity

$$h_f^- \cdot E \left( [I_{f,t} - D^t_f (p_t)]^- \right) = h_f^- \cdot E \left( [I_{f,t} - D^t_f (p_t)]^+ \right) + h_f^- \cdot E \left( \sum_{k=0}^{t-1} (y_{i,k}^f - d_{i,k}^f (p_k)) \right)$$

From Lemma 4.1 we observe that inventory at time $t$, $I_{f,t}$ follows normal distribution with mean $\gamma_0^f + \sum_{k=0}^{t-1} (y_{i,k}^f - d_{i,k}^f (p_k))$ and variance $\left( \sum_{k=0}^{t-1} \left[ \sigma_{i,k}^f \cdot (d_{i,k}^f (p_k)) \right]^2 \right)$. and for the nonmultiplicative demand

$$E \left( D^t_f (p_t) \right) = d_{i,t}^f (p_t)$$

therefore, re-writing (4.90)

$$h_f^- \cdot E \left( [I_{f,t} - D^t_f (p_t)]^- \right) = h_f^- \cdot E \left( [I_{f,t} - D^t_f (p_t)]^+ \right) + h_f^- \cdot E \left( \sum_{k=0}^{t-1} (y_{i,k}^f - d_{i,k}^f (p_k)) \right)$$

hence the proof. 

Based on the above two lemmas, Proposition (4.3) derives an expression for the single period expected inventory holding cost.

Proposition 4.3 For the nonmultiplicative demand model (4.78) the expected inventory holding cost at time $t$ is

$$h^+ \cdot E \left( [I_{f,t} - D^t_f (p_t)]^+ \right) = h^+ \left[ \tilde{\mu}_t \cdot \left( 1 - \Phi \left( -\frac{\tilde{\mu}_t}{\tilde{\sigma}_t} \right) \right) + \tilde{\sigma}_t \cdot \phi \left( -\frac{\tilde{\mu}_t}{\tilde{\sigma}_t} \right) \right]$$

where $\Phi \left( \cdot \right)$ and $\phi \left( \cdot \right)$ are the cdf and pdf of a standard normal variate with $\tilde{\mu}_t \equiv \gamma_0^f + \sum_{k=0}^{t} (y_{i,k}^f - d_{i,k}^f (p_k))$, and $\tilde{\sigma}_t^2 \equiv \sum_{k=0}^{t} \left[ \sigma_{i,k}^f \cdot (d_{i,k}^f (p_k)) \right]^2$.
Proof. Since \( I_{f,t} \) follows normal distribution with mean \( \gamma_{0}^{f} + \sum_{k=0}^{t-1} \left( y_{i,k}^{f} - d_{i,k}^{f} (p_{k}) \right) \) and variance \( \left( \sum_{k=0}^{t-1} \left[ \sigma_{i,k}^{f} \left( d_{i,k}^{f} (p_{k}) \right) \right]^{2} \) \), therefore \( I_{f,t} - D_{t}^{f} (p_{t}) \) also follows normal distribution with mean

\[
\gamma_{0}^{f} + \sum_{k=0}^{t} \left( y_{i,k}^{f} - d_{i,k}^{f} (p_{k}) \right)
\]

and variance

\[
\sum_{k=0}^{t} \left[ \sigma_{i,k}^{f} \left( d_{i,k}^{f} (p_{k}) \right) \right]^{2}
\]

so, the expected inventory holding cost is

\[
h^{+} \cdot E \left[ I_{f,t} - D_{t}^{f} (p_{t}) \right]^{+} = h^{+} \int_{0}^{\infty} \tau f(\tau) d\tau
\]

where \( \tau \) follows normal distribution with mean, \( \bar{\mu}_{t} \equiv \gamma_{0}^{f} + \sum_{k=0}^{t} \left( y_{i,k}^{f} - d_{i,k}^{f} (p_{k}) \right) \) and standard deviation, \( \bar{\sigma}_{t} \equiv \sqrt{\sum_{k=0}^{t} \left[ \sigma_{i,k}^{f} \left( d_{i,k}^{f} (p_{k}) \right) \right]^{2}} \). After substituting

\[
\kappa = \frac{\tau - \bar{\mu}_{t}}{\bar{\sigma}_{t}}
\]

where \( \kappa \) is a standard normal variate we obtain

\[
h^{+} \cdot \left[ \int_{0}^{\infty} \tau f(\tau) d\tau \right] = h^{+} \cdot \left[ \int_{-\bar{\mu}_{t}}^{\infty} \left( \bar{\mu}_{t} + \bar{\sigma}_{t} \cdot \kappa \right) f(\kappa) d\kappa \right]
\]

\[
= h^{+} \cdot \bar{\sigma}_{t} \left[ \int_{-\bar{\mu}_{t}}^{\infty} \left( \frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}} + \kappa \right) \phi(\kappa) d\kappa \right]
\]

\[
= h^{+} \cdot \bar{\sigma}_{t} \cdot \Phi^{1} \left( -\frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}} \right)
\]

where \( \Phi^{1} \left( -\frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}} \right) \) is the standard normal loss function which can be expressed as (see Zipkin [76])

\[
\Phi^{1} \left( -\frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}} \right) = \frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}} \int_{-\frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}}}^{\infty} f(\kappa) d\kappa + \phi \left( -\frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}} \right)
\]

\[
= \frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}} \int_{-\frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}}}^{\infty} f(\kappa) d\kappa + \phi \left( -\frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}} \right)
\]

\[
= \frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}} \left( 1 - \Phi \left( -\frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}} \right) \right) + \phi \left( -\frac{\bar{\mu}_{t}}{\bar{\sigma}_{t}} \right)
\]
therefore, the expected inventory holding cost is

\[ h^+ \cdot \left[ \int_0^\infty \tau f(\tau) d\tau \right] = h^+ \left[ \bar{\mu}_t \cdot \left( 1 - \Phi \left( \frac{\bar{\mu}_t}{\bar{\sigma}_t} \right) \right) + \bar{\sigma}_t \cdot \phi \left( -\frac{\bar{\mu}_t}{\bar{\sigma}_t} \right) \right] \]

hence the proof. ■

Further, we can obtain an expression for the expected bequest function as described in Proposition 4.4.

**Proposition 4.4** For the nonmultiplicative demand model (4.78),

\[ E \left[ \delta \left( I_N^f \right)^- \right] = \delta \cdot \left[ \Phi \left( \frac{\bar{\mu}_{N-1}}{\bar{\sigma}_{N-1}} \right) - \bar{\sigma}_{N-1} \cdot \phi \left( -\frac{\bar{\mu}_{N-1}}{\bar{\sigma}_{N-1}} \right) \right] \]

where \( \Phi(\cdot) \) and \( \phi(\cdot) \) are the cdf and pdf of a standard normal variate (with mean 0 and variance 1), with \( \bar{\mu}_{N-1} \equiv \gamma_0^f + \sum_{k=0}^{N-1} \left( y_{i,k}^f - d_{i,k}^f (p_k) \right), \) and \( \bar{\sigma}_{N-1}^2 \equiv \sum_{k=0}^{N-1} \left[ \sigma_{i,k}^f \left( d_{i,k}^f (p_k) \right) \right]^2. \)

**Proof.** Since \( I_N^f \) follows normal distribution with mean \( \gamma_0^f + \sum_{k=0}^{N-1} \left( y_{i,k}^f - d_{i,k}^f (p_k) \right) \) and variance \( \sum_{k=0}^{N-1} \left[ \sigma_{i,k}^f \left( d_{i,k}^f (p_k) \right) \right]^2 \), therefore

\[ E \left[ \delta \left( I_N^f \right)^- \right] = \delta \cdot \int_{-\infty}^{0} \tau f(\tau) d\tau \]

where \( \tau \) follows normal distribution with mean, \( \bar{\mu}_N \equiv \gamma_0^f + \sum_{k=0}^{N-1} \left( y_{i,k}^f - d_{i,k}^f (p_k) \right) \) and standard deviation, \( \bar{\sigma}_N \equiv \sqrt{\sum_{k=0}^{N-1} \left[ \sigma_{i,k}^f \left( d_{i,k}^f (p_k) \right) \right]^2}. \) After substituting

\[ \kappa = \frac{\tau - \bar{\mu}_{N-1}}{\bar{\sigma}_{N-1}} \]

where \( \kappa \) is a standard normal variate we obtain

\[ \delta \cdot \int_{-\infty}^{0} \tau f(\tau) d\tau = \delta \cdot \left[ \int_{-\infty}^{\frac{\bar{\mu}_{N-1} - \bar{\sigma}_{N-1} \cdot \kappa}{\bar{\sigma}_{N-1}}} \left( \bar{\mu}_{N-1} + \bar{\sigma}_{N-1} \cdot \kappa \right) f(\kappa) d\kappa \right] \]

\[ = \delta \cdot \bar{\mu}_{N-1} - \delta \cdot \left[ \int_{-\frac{\bar{\mu}_{N-1} - \bar{\sigma}_{N-1} \cdot \kappa}{\bar{\sigma}_{N-1}}}^{\infty} \left( \bar{\mu}_{N-1} + \bar{\sigma}_{N-1} \cdot \kappa \right) f(\kappa) d\kappa \right] \]

(4.91)
Observe that an expression for the second term in the right hand side of (4.91) can be obtained by using Proposition 4.3, therefore
\[
\delta \int_{-\infty}^{0} \tau f(\tau) d\tau = \delta \cdot \mu_{N-1} - \delta \cdot \left( \mu_{N-1} \cdot \left( 1 - \Phi \left( -\frac{\mu_{N-1}}{\sigma_{N-1}} \right) \right) + \sigma_{N-1} \cdot \phi \left( -\frac{\mu_{N-1}}{\sigma_{N-1}} \right) \right) 
\]
\[
= \delta \cdot \Phi \left( -\frac{\mu_{N-1}}{\sigma_{N-1}} \right) - \delta \cdot \sigma_{N-1} \cdot \phi \left( -\frac{\mu_{N-1}}{\sigma_{N-1}} \right) 
\]
\[
= \delta \cdot \left( \Phi \left( -\frac{\mu_{N-1}}{\sigma_{N-1}} \right) - \sigma_{N-1} \cdot \phi \left( -\frac{\mu_{N-1}}{\sigma_{N-1}} \right) \right) 
\]
which completes the proof.

In the light of the Lemmas 4.1 and 4.2 and the Propositions 4.3 and 4.4, we can rewrite the criterion of each manufacturer \( f \)'s discrete time optimal control problem as
\[
J_f \left( p^f, y^f, u^j; p^{-f}; u^{-f} \right) = \max_{p^f, y^f, u^f} \left\{ \begin{array}{l}
-\delta \cdot \left( \Phi \left( -\frac{\mu_{N-1}}{\sigma_{N-1}} \right) - \sigma_{N-1} \cdot \phi \left( -\frac{\mu_{N-1}}{\sigma_{N-1}} \right) \right) \\
+ \sum_{t=0}^{N-1} \left\{ p^f_t \cdot d^f_t (p_t) - K_{f,t} (u_{f,t}) \cdot u_{f,t} - \psi_{f,t} (r^f_t) \right\} \right. \\
\left. - \left( h_f^+ + h_f^- \right) \cdot \left\{ \mu_t \cdot \left( 1 - \Phi \left( -\frac{\mu_t}{\sigma_t} \right) \right) + \sigma_t \cdot \phi \left( -\frac{\mu_t}{\sigma_t} \right) \right\} \right\} 
\]
(4.92)

The feasible space of controls for firm \( f \) can be expressed as
\[
K_f \left( u^{-f} \right) = \left\{ \left( p^f, u^f, y^f \right) : \begin{array}{l}
r_{t+1} = r_t^f + \sum_{g} u_{g,t}^f - A \cdot y_t^f \text{ for all } t = 0, \ldots, N - 1 \\
r_0^f = \nu_0 \in \mathbb{R}_+ \\
r_t^f \geq 0 \text{ for all } t = 1, \ldots, N \\
\left( \sum_{j \neq f} u_{g,t}^j \right) \leq K_{f,t} - \left( \sum_{j \neq f} u_{g,t}^j \right) \text{ for all } t = 0, \ldots, N - 1 \\
u_t^f \geq 0 \text{ for all } t = 0, \ldots, N - 1 \\
p_{\text{min}}^f \leq p_t^f \leq p_{\text{max}}^f \text{ for all } t = 0, \ldots, N - 1 \\
y_t^f \geq 0 \text{ for all } t = 0, \ldots, N - 1 \end{array} \right\} 
\]
(4.93)

Note that the feasible space for the firm \( f \) is coupled with some of the decision variables \( (u_{g,t}^j) \) of its competitors through (4.85), therefore the multi-period game that we consider here is in fact a generalized Nash equilibrium (GNE) game\(^2\) which is a noncooperative game with nondisjoint strategy sets.

\(^2\)See Harker [74] for further details.
Best Response Problem for Firm $f$

Applying Pontryagin’s maximum principle to the extremal problem (4.92) - (4.93), we have the following best response problem that each firm $f$ seeks to solve at time $t \in [0, N - 1]$ with given $p_t^- f$

$$\max H_f \left( p_t^f; \lambda_f; p_t^- f; t \right)$$

subject to

$$p_{t \min}^f \leq p_t^f \leq p_{t \max}^f$$

$$r_0^f = \nu_0 \in \mathbb{R}_{+}^{\left| G \right|}$$

$$r_t^f \geq 0 \text{ for all } t = 1, \ldots, N$$

$$y_t^f \geq 0 \text{ for all } t = 0, \ldots, N - 1$$

$$u_t^f \geq 0 \text{ for all } t = 0, \ldots, N - 1$$

$$u_{g,t}^f \leq K_t^g - \left( \sum_{j \neq f} u_j^t \right) \text{ for all } t = 0, \ldots, N - 1$$

where Hamiltonian is formed as

$$H_f \left( p_t^f; u_t^f; g_t^f; \lambda_f; p_t^- f, u_t^- f; t \right) =$$

$$\begin{array}{c}
-\delta \cdot \left[ \Phi \left( \frac{\tilde{\mu}_{N-1}}{\sigma_{N-1}} \right) - \tilde{\sigma}_{N-1} \cdot \phi \left( \frac{\tilde{\mu}_{N-1}}{\sigma_{N-1}} \right) \right] \\
+ \sum_{t=0}^{N-1} \left\{ p_t^f \cdot d_t^f (p_t) - K_{f,t} (u_{f,t}) \cdot u_{f,t} - \psi_{f,t} (r_t^f) \right. \\
\quad - \left. h_t^f \cdot \gamma_t \cdot \left[ d_t^f (p_t) - \sum_{k=0}^{t-1} \left( y_{t,k}^f - d_{t,k}^f (p_k) \right) \right] \right\} \\
\quad \left( h_t^f + h_t^- \right) \cdot \left\{ \tilde{\mu}_t \cdot \left( 1 - \Phi \left( \frac{\tilde{\mu}_t}{\sigma_t} \right) \right) + \tilde{\sigma}_t \cdot \phi \left( \frac{\tilde{\mu}_t}{\sigma_t} \right) \right\} \\
+ \lambda_t^f \cdot \left( \sum_{g \in G} u_{g,t}^f - A \cdot y_t^f \right)
\end{array}$$

where $\lambda_t^f$ is the adjoint variable (shadow price) of input factors. The necessary and sufficient condition for

$$\begin{bmatrix}
\left[ p_t^f \right]^* (p_t^- f, u_t^- f, \lambda_f) \\
\left[ u_t^f \right]^* (p_t^- f, u_t^- f, \lambda_f) \\
\left[ y_t^f \right]^* (p_t^- f, u_t^- f, \lambda_f)
\end{bmatrix}$$

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to be a solution of the best response problem can be expressed as the following variational inequality (VI):

\[
\text{find } \begin{bmatrix} p^f_* \\ u^f_* \\ y^f_* \end{bmatrix} \in \mathcal{K}_f \left(u^{-f} \right) \text{ such that }
\begin{bmatrix}
\nabla p^f_t H_f \\
\nabla u^f_t H_f \\
\nabla y^f_t H_f
\end{bmatrix}^T \begin{bmatrix}
p^f_t - p^f_* \\
u^f_t - u^f_* \\
y^f_t - y^f_*
\end{bmatrix} \leq 0 \text{ for all } \begin{bmatrix} p^f \\ u^f \\ y^f \end{bmatrix} \in \mathcal{K}_f \left(u^{-f} \right)
\]

(4.94)

where the feasible space for firm \( f \) is defined as

\[
\mathcal{K}_f (u^{-f}) = \left\{ (p^f, u^f, y^f) : \begin{align*}
& r^f_0 = \nu_0 \in \mathbb{R}^{C_f} \\
& r^f_t \geq 0 \text{ for all } t = 1, \ldots, N \\
& u^f_{g,t} \leq K^g_t - \left( \sum_{j \neq f} u^j_{g,t} \right) \text{ for all } t = 0, \ldots, N - 1 \\
& u^f_t \geq 0 \text{ for all } t = 0, \ldots, N - 1 \\
& p_{\min}^f \leq p^f_t \leq p_{\max}^f \text{ for all } t = 0, \ldots, N - 1 \\
& y^f_t \geq 0 \text{ for all } t = 0, \ldots, N - 1
\end{align*} \right\}
\]

### 4.3.5 Market Equilibrium Problem as a Variational Inequality

With the preceding background, we can now formulate the market equilibrium problem as a VI. To do so we combine the variational inequalities (4.94) for each firm \( f \in \mathcal{F} \) and time \( t \in [0, N - 1] \). However, since the strategy set of the firms are not disjoint, hence we take the restricted multiplier approach, as put forward by Harker [74] to create disjoint strategy set for each firms.

As the constraint

\[
u^f_{g,t} \leq K^g_t - \left( \sum_{j \neq f} u^j_{g,t} \right) \text{ for all } t = 0, \ldots, N - 1
\]

is a joint constraint which is problematic, under a suitable constraint qualification such as Abadie’s (as the feasible set is convex), we obtain the following KKT conditions which are

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necessary for \( s^f \) being the optimal for firm \( f \)'s stationary problem (5.17) - (5.18).

\[
0 = -\nabla u^f H_f \left( p^f_t; u^f_t; y^f_t; \lambda^f; p_t^-f, u_t^-f; t \right) - \beta_{g,t} + \gamma^f_{g,t} \tag{4.95}
\]

\[
0 \leq \beta \perp \left( u^f + \sum_{g \in \mathcal{F}\setminus i} u^g - K \right) \leq 0 \tag{4.96}
\]

\[
0 \leq \gamma \perp \left( -u^f \right) \leq 0 \tag{4.97}
\]

where \( L_f \left( p^f_t; u^f_t; y^f_t; \lambda^f; p_t^-f, u_t^-f; \beta, \gamma^f; t \right) \) is agent \( f \)'s 'semi-Lagrangian function' defined as

\[
L_f \left( p^f_t; u^f_t; y^f_t; \lambda^f; p_t^-f, u_t^-f; \beta; t \right) = H_f \left( p^f_t; u^f_t; y^f_t; \lambda^f; p_t^-f, u_t^-f; t \right) \tag{4.98}
\]

\[
+ \beta^T \cdot \left( u^f + \sum_{g \in \mathcal{F}\setminus i} u^g - K \right) + \gamma^f \cdot \left( -u^f \right)
\]

Note that, as discussed in Harker [74], we use the restricted multiplier formulation which is applicable to all generalized Nash games where the players have the same joint constraint. In this case we use dual multiplier \( \beta_{g,t} \) associated with the joint sourcing capacity constraint. This dual multiplier \( \beta_i \) is dependent only on the supplier \( g \) and time \( t \), and is common to all firms,

\[
\beta = \left( \beta_{g,t} \right)_{g \in \mathcal{G}, t \in [0,N]}
\]

Therefore, concatenating \(|\mathcal{F}|\) number of KKT systems, we obtain a characterization of a generalized Cournot-Nash equilibrium as a partitioned VI. In order to formulate the latter VI, let us define the function \( \Theta \left( s^f, \tau^f; q^f; s^-f; \lambda^f, \tau^f, \varsigma^f-, \varsigma^f+; t \right) \) as

\[
\Theta \left( p_t; u_t; y_t; \lambda; \beta; t \right) = \left( \nabla u^f H_f \left( p^f_t; u^f_t; y^f_t; \lambda^f; p_t^-f, u_t^-f; t \right) + \beta \right)_{f=1}^{\left| \mathcal{F} \right|} \tag{4.99}
\]

\[
\begin{aligned}
\mathbf{z} &= \left( \begin{array}{c} \left( u^f \right)_{f=1}^{\left| \mathcal{F} \right|} \\
\beta \end{array} \right)
\end{aligned}
\]
Now, the reduced feasible space is

\[
\mathcal{K}_f = \left\{ (p^f, u^f, y^f) : \begin{align*}
    r_0^f &= \nu_0 \in \mathbb{R}_+^{|C|} \\
    r_t^f &\geq 0 \text{ for all } t = 1, \ldots, N \\
    u_{g,t}^f &\leq K_t^g - \left( \sum_{j \neq f} u_{g,t}^j \right) \text{ for all } t = 0, \ldots, N - 1 \\
    u_t^f &\geq 0 \text{ for all } t = 0, \ldots, N - 1 \\
    p_{\text{min}}^f &\leq p_t^f \leq p_{\text{max}}^f \text{ for all } t = 0, \ldots, N - 1 \\
    y_t^f &\geq 0 \text{ for all } t = 0, \ldots, N - 1
\end{align*} \right\}
\]

We define the following feasible space for all service providers.

\[
\mathcal{K} = \prod_{f \in \mathcal{F}} \mathcal{K}_f
\]

The VI of interest seeks to find a point \((p^*, u^*, y^*)^T \in \mathcal{K}\) such that

\[
\begin{pmatrix}
    \nabla_p H_f \left( p^{1*}; u^{1*}; y^{1*}; \lambda^1; p_t^{-1}, u_t^{-1}; \beta, t \right) \\
    \vdots \\
    \nabla_{p|\mathcal{F}|} H_f \left( p|\mathcal{F}|^{*}; u|\mathcal{F}|^{*}; y|\mathcal{F}|^{*}; \lambda|\mathcal{F}|; p_t^{-|\mathcal{F}|}, u_t^{-|\mathcal{F}|}; \beta, t \right)
\end{pmatrix}^T
\begin{pmatrix}
    p^{1*} - p^1 \\
    \vdots \\
    p|\mathcal{F}| - p|\mathcal{F}|^{*}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
    \nabla_{y} H_f \left( p^{1*}; u^{1*}; y^{1*}; \lambda^1; p_t^{-1}, u_t^{-1}; \beta, t \right) \\
    \vdots \\
    \nabla_{y|\mathcal{F}|} H_f \left( p|\mathcal{F}|^{*}; u|\mathcal{F}|^{*}; y|\mathcal{F}|^{*}; \lambda|\mathcal{F}|; p_t^{-|\mathcal{F}|}, u_t^{-|\mathcal{F}|}; \beta, t \right)
\end{pmatrix}^T
\begin{pmatrix}
    y^{1*} - y^1 \\
    \vdots \\
    y|\mathcal{F}| - y|\mathcal{F}|^{*}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
    \nabla_{u} H_f \left( p_t^{1*}; u_t^{1*}; y_t^{1*}; \lambda^1; p_t^{-1}, u_t^{-1}, t \right) + \beta \\
    \vdots \\
    \nabla_{u|\mathcal{F}|} H_f \left( p_t|\mathcal{F}|^{*}; u_t|\mathcal{F}|^{*}; y_t|\mathcal{F}|^{*}; \lambda|\mathcal{F}|; p_t^{-|\mathcal{F}|}, u_t^{-|\mathcal{F}|}, t \right) + \beta
\end{pmatrix}^T
\begin{pmatrix}
    u^{1*} - u^1 \\
    \vdots \\
    u|\mathcal{F}| - u|\mathcal{F}|^{*}
\end{pmatrix} \geq 0
\]

for all \((p, u, y)^T \in \mathcal{K}\) where

\[
H_f \left( p^f; u^f; y^f; \lambda^f; p^{-f}, u^{-f} \right) = \left\{ H_f \left( p_t^f; u_t^f; y_t^f; \lambda^f; p_t^{-f}, u_t^{-f}, t \right) : t \in [0, N - 1] \right\}
\]

and

\[
\mathcal{K} = \prod_{f \in \mathcal{F}} \mathcal{K}_f
\]
The issue of immediate concern is to formally demonstrate that solutions of (4.101) are Cournot-Nash equilibria of our revenue optimization game. In fact we state and prove the following result which establishes that solving the variational inequality formulation (4.101) is equivalent to solving the market equilibrium problem. This also allows us to also establish existence of a market equilibrium policy.

**Theorem 4.5** The policy arising from the joint variational inequality problem (4.101) and the policy arising from the simultaneous solution of the variational inequalities (4.94) for each firm $f$ are the same.

**Proof.** We begin with invoking Theorem 3 of Harker [74] which establishes that for games where each agents have identical joint constraints, solution of the variational inequality (4.99) is a quasi-variational inequality solution (4.94).

Now, it is relatively straightforward to show that a policy $(p^*, u^*, y^*)^T$ that solves the variational inequality problem (4.94) for each firm $f \in F$ simultaneously, also solves the joint variational inequality problem (4.101). We will now show the converse, i.e., the solution to joint variational inequality problem (4.101) solves variational inequality problems (4.94) for each firm $f$ simultaneously. That is, if $(p^*, u^*, y^*)^T$ is a solution to joint VI problem (4.101), then for each firm $f \in F$, $(p^*, u^*, y^*)^T$ solves the variational inequality problem (4.94) with competitors’ policies $[p^{-f}, u^{-f}, y^{-f}]^T$ given by $[p^{-f^*}, u^{-f^*}, y^{-f^*}]^T$. Note that
(4.101) is equivalent to the following fictitious mathematical program

$$\max G(p, u, y, \beta) = \sum_{t=0}^{N-1} \sum_{f \in \mathcal{F}} \sum_{i \in S} \frac{\partial H_f(p^f; u^f; y^f; \lambda^f; p_t^f; u_t^f; \beta, t)}{\partial p_{i,t}^f} p_{i,t}^f \quad (4.102)$$

$$+ \sum_{g \in G} \left( \frac{\partial H_f(p^f; u^f; y^f; \lambda^f; p_t^f; u_t^f; \beta, t)}{\partial u_{g,i,t}^f} + \beta_{g,t}^* \right) u_{g,i,t}^f \quad (4.103)$$

$$+ \frac{\partial H_f(p^f; u^f; y^f; \lambda^f; p_t^f; u_t^f; \beta, t)}{\partial y_{i,t}^f} y_{i,t}^f + \left( u^f + \sum_{g \in \mathcal{F} \setminus \mathcal{F}} u^g - K \right) \quad (4.104)$$

subject to

$$p_{\min}^f \leq p_t^f \leq p_{\max}^f \text{ for all } f \in \mathcal{F}, t \in [0, N-1] \quad (4.105)$$

$$r_t^f \geq 0 \text{ for all } t = 1, \ldots, N \quad (4.106)$$

$$u_t^f \geq 0 \text{ for all } t = 0, \ldots, N-1 \quad (4.107)$$

$$y_t^f \geq 0 \text{ for all } t = 0, \ldots, N-1 \quad (4.108)$$

where it is essential to recognize that $G(p, u, y, \beta)$ is a linear functional that assumes knowledge of the solution of (4.101); as such $G(p, u, y, \beta)$ is a mathematical construct for use in analysis and has no meaning as a computational device. The corresponding necessary and sufficient conditions for this mathematical program are identical to (3.33) for all $f \in \mathcal{F}$ as because

$$\frac{\partial G^*}{\partial p_{i,t}^f} = \frac{\partial H_f(p^f; u^f; y^f; \lambda^f; p_t^f; u_t^f; \beta, t)}{\partial p_{i,t}^f}$$

$$\frac{\partial G^*}{\partial u_{i,t}^f} = \frac{\partial H_f(p^f; u^f; y^f; \lambda^f; p_t^f; u_t^f; \beta, t)}{\partial u_{i,t}^f} + \beta_{g,t}^*$$

$$\frac{\partial G^*}{\partial y_{i,t}^f} = \frac{\partial H_f(p^f; u^f; y^f; \lambda^f; p_t^f; u_t^f; \beta, t)}{\partial y_{i,t}^f}$$

$$\frac{\partial G^*}{\partial \beta} = u^* - K$$

hence the proof. ■

As far as existence results are concerned, we are only interested in pure strategy open-loop Nash equilibrium\(^3\). We note that the following existence result holds:

\(^3\)A pure strategy is a strategy that has no randomly determined choices.
**Theorem 4.6**  There exists at least one equilibrium of this supply chain competition game.

**Proof.** We need to establish that there exists at least one solution of the VI (4.101). Since any solution of (4.101) is a Nash equilibrium of the game (as per Theorem 4.5), then that solution will also be a Nash equilibrium of the game. Note that ‘do-nothing’ is a feasible strategy for all the firms where they do not produce or procure anything \( u_t = y_t = 0 \) for all \( t \in [0, N] \), but set prices for their products to clear off initial inventories, hence the strategy set is a non-empty, compact and convex set. Further, the principal function of the VI is a continuous mapping from \( \mathcal{K} \) into \( \mathbb{R}^{\left|\mathcal{F}\right| \times |\mathcal{S}| \times (N-1)} \). Therefore, invoking Theorem 3.1 of Harker and Pang [16] we establish that there exists a solution of (4.5), hence the proof.

\[ \blacksquare \]

### 4.3.6 Numerical Example

**Description of supply chain network**

Let us consider a supply chain network involving two manufacturers (shippers), each of them producing two differentiated products from three input factors. There are two suppliers, each of whom can offer all 3 input factors. Input factor 1 is used in either products (the common product), input factor 2 is used in product 1 only where as the input factor 3 is used in product 2. Bill of material in this simplified case is shown below

<table>
<thead>
<tr>
<th></th>
<th>Input Factor 1</th>
<th>Input Factor 2</th>
<th>Input Factor 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product 1</td>
<td>√</td>
<td>√</td>
<td>×</td>
</tr>
<tr>
<td>Product 2</td>
<td>√</td>
<td>×</td>
<td>√</td>
</tr>
</tbody>
</table>

Cost structures are such that the input factor 2 is more expensive than Tinput factor 3, there by product 1 is more expensive (unit production cost) than product 2. The bipartite graph describing the resulting supply chain network is shown in Figure 4.9.
Both the firms are making pricing-production-procurement decisions for 10 future time periods. The demand model we consider in this example is nonmultiplicative, the average demand has a linear form, whereas the standard deviation of the product demand \( \sigma_i^f (d_i^f (p,t)) \) follows power law with exponent being 0.5. We suppress the numerical values of every parameter of the problem, rather provide the key information of the input variables. Firms 1 and 2 have some and comparable finished goods inventories and input factor inventories at the beginning. The total ordering cost of every input factor has a fixed component and a variable component which decreases linearly with order quantity (which approximates the quantity discount). One period finished goods inventory holding costs are approximately 20\% of the base price in the previous period, similarly, single period back-ordering cost is 50\% of the base price in the previous period. The raw material handling (and storage) costs are quadratic functions of input factor bin quantity. In general, supplier 2 charges higher than the supplier 1, but has higher capacity. Supplier 1 has lower capacity but offers input factors at a cheaper price.
Equilibrium Behavior

Figure 4.10 shows the price charged by the firm 1 for its two product offerings. As product 2 is more costly to make, firm 1 initially sets a higher price for this product, but towards the end of the selling season takes a deeper discount to clear inventory. As Figure 4.11 describes, firm 2 also shows the similar pricing pattern as it sets higher initial price for product 2, but because its demand function has higher own price elasticity as well as slighter higher initial inventory, firm 2 takes the deeper discount earlier than the firm 1, which in turn forces the firm 1 to take the discount. Firm 1 produces very little product 2 as it has higher initial inventory and it was little more expensive for firm 1 to produce product 2 compared to firm 1. Figure 4.12 shows the production quantity strategies for firm 1, figure 4.13 shows the
Figure 4.12: Production quantities as adopted by firm 1

Figure 4.13: Production decisions for firm 2

production strategies for firm 2. Figure 4.14 depicts the sourcing strategies for the firms from two suppliers. Since Firm 1 produces small amount of product 2, it orders small input factor 3 (only used in product 2) in small quantity. Supplier 2 must operate on full capacity for most of the time (joint capacity constraint for this supplier will be binding for those time periods) and also observe that both manufactures prefer supplier 2 as it is cheaper. The input factor inventories 4.15 shows the input factor inventory levels over the planning horizon for firm 1 (top) and 2 (bottom). We observe that both the manufacturers try to keep input factors inventory as small as possible; however, in general, firm 2 maintains more inventory than firm 1.
Impacts of Supply Side Disruptions

In this section we study the impacts of supply side disruptions on the equilibrium behaviors of the firms by disruption events. We focus on the impacts of supply disruptions for the manufacturers described above by considering the following two scenarios: (a) the prediction of a supply chain disruption allows the firms to revise their strategies before it happens; and (b) there is no interruption prediction but rather unanticipated supply chain disruption. We quantify the impacts of such disruptions when one or more firms adopt some or all of the following risk mitigating strategies:

- dynamic pricing (dramatically influencing customers' product selection when the cheapest product is unavailable and some innovative pricing strategies adopted by the firms to cater some of these demands by more expensive products).

- a flexible supply base (enabling a firm to source input factors from geographically...
Figure 4.15: Input factor inventory levels for firm 1 (top) and firm 2 (bottom).
Figure 4.16: Impacts of disruption on pricing decisions of firm 1

separated suppliers, thus providing greater flexibility);

- make-and-buy (shifting production between an in-house production facility and a third party vendor).

In the simulation study we have excluded the scenarios of multiple concurrent or consecutive disruptions. In general, we consider a global disruption impacting both suppliers, hence no supply was possible during disruption. When there is no advance information available, disruption occurs at time $t = 2$ with stated duration of 3 periods. At that point, both the manufacturers simultaneously revisit their strategies (at $t = 2$) with disruption information. We assume that the manufacturers have enough time and computational resources to recalculate their strategies at $t = 2$ and start applying the new strategies from $t = 3$ onwards. We assume that the resolution of the disruption was relatively rapid, which happens at $t = 4$. Hence manufacturers once again revisit their strategies at time $t = 4$ for rest of the planning horizon. Figure 4.16 illustrates the impacts of disruption on the pricing strategies of firm 1.
We observe that after the disruption is resolved, the equilibrium pricing decisions for the product 2 catches up with the original pricing plan at the end of the life, however, not quite so for the product 1. Hence empirically we observe that the disruption induces a whole new Nash equilibrium of the game. Similar impacts are observed for firm 2 as Figure 4.17 demonstrates. However, the equilibrium price for either products of firm 2 are distinct before and after disruptions throughout the remaining lifecycle. Similarly, Figures 4.18 and 4.19 illustrates the impacts of disruptions on the production quantity decisions for firm 1 and 2 respectively. The impacts of disruptions on the procurement decisions of firm 2 from supplier 1 is illustrated in Figure 4.20. Through this simple simulation, we empirically observe that the impacts of disruption are more profound in the absence of advanced information and the global disruptions impact all manufacturers. Local disruptions, which affects only one supplier, impact more heavily the manufacturers who rely heavily on that supplier. Under most scenarios, after disruption is resolved, firms tend to charge higher. We also find that the firms offer deep discounts for expensive products when an inexpensive substitutable product becomes unavailable during disruption. Further, firms tend to procure and produce more after disruption is resolved to offset some of the lost business.
Figure 4.18: Impacts of disruption on production quantity decisions for firm 1

Figure 4.19: Impacts of disruption on production quantity decisions for firm 2
Figure 4.20: Impacts of disruptions on procurement decisions made by firm 1 procuring from supplier 2
Chapter 5

Electric Power Network Games and Impacts of Network Disruptions

5.1 Organization of the Chapter

In this chapter we discuss the application of computable dynamic game theory in formulating and solving the electric power network games played over a power distribution network. The typical planning horizon we consider in our formulation is 24 hours as the regional demand of electricity varies across hours of a day. Over different days there is notable periodicity. For example, the demand for power at a specific hour of a day is very similar to the demands at the same hour on different days. In section 5.2 we discuss one such model, where we explicitly consider the ramping constraints as well as generator’s capacity constraints in the formulation. Section 5.3 describes the impacts of the disruption of power grids on the market equilibrium. We simulate the equilibrium behavior of the agents (the generators) on a realistic network involving multiple generators separated geographically in western
5.2 Dynamic Electric Power Network Games with Ramping, Capacity and Sales Constraints

5.2.1 Introduction and Brief Literature Review

One of the major areas of application of complementarity and variational inequalities-based models of economic equilibria is electric power markets; more so since this economically crucial industry underwent a transition from tight regulation to intense competition subject to loose regulatory constraints. It is not our intention, nor does the space permit, to list or discuss all the previous models. However, Daxhalet and Smeers [77], Day et al [78], and Hobbs and Helman [79] summarize the relatively recent literatures on the complementarity and variational inequalities-based models of electric power equilibrium problems. Unlike other engineered systems, technology and cost information is widely available for power industries which facilitates modeling. At the same time, the unique characteristics of electricity transmission, such as Kirchhoff’s power and voltage laws, present intriguing challenges to modelers and systems engineers. Kirchhoff’s laws arise from the inherent characteristic of flow of power in an electric network, namely the energy balance (known as the first law) and voltage law (also known as the second law).

An important feature omitted in most complementarity-based models is the possibility that an electricity producer can recognize joint constraints wherein the possible solution space for one player is affected by the decisions of the other players. For instance, regulators might impose an upper bound upon the market share of the few largest producers in some markets, or upon the proportion of transmission capacity that is sold to such producers, as is the case for transmission capacity into the Netherlands. Perhaps the most important example would be a recognition by a generator that its sales and generation are limited
by available transmission capacity less that capacity which is already taken up by sales and generation by other producers. When considering elaborate network topologies, very complex sets of constraints on sales and generations may arise.

Another important feature missing from all existing electric power equilibrium models is the consideration of ramping costs. There are some instances of ramping cost-based models for the monopolistic firm, namely by Wang and Shahidehpour [80], Shresta et al. [81], and Tanaka [82]. Wang and Shahidehpour [80] consider a decomposed model for optimal generation scheduling of a cost minimizing monopolistic power generator explicitly considering ramping costs. However, the market perspective is missing in that model. On the other hand, Shresta et al [81] consider a dynamic model for strategic use of ramping rates beyond elastic limits in a power producer’s self-dispatch in a power market with exogenous price and demand. Tanaka [82], on the other hand, focuses on derivation of a pricing policy that achieves the optimal rate of a demand change by explicitly considering the ramping cost. All three of these papers do not consider the network topology or Kirchhoff’s laws.

Meanwhile, nearly all of the power market equilibrium models fail to take a dynamic perspective. However these models do not consider shortrun constraints and costs associated with changing generators output. Large system generators can take up to ten or more hours to ramp up to full output, yet power demands can change drastically from one hour to the next. In order to match those changes, quick-starting but costly combustion turbines are turned on and off. The results is that the highest price spikes often occurs not during the periods of high demand, but during the early morning or late afternoon when the demand is changing most quickly. Figure 5.1 and 5.2 show a typical daily load profile for the California Independent System Operator (ISO), and the price spikes that result from ramping constraints and costs respectively. These dynamic market phenomena can not be captured by the typical static formulation of equilibrium models that omit consideration of time (see, e.g., Oren and Ross [83]).

We believe here that electric power systems display a so-called moving-equilibrium,
Figure 5.1: Load profile in CalISO market in a typical summer day

Figure 5.2: Corresponding spot market price in CalISO
wherein an equilibrium is enforced at each instant of time although state and control variables will generally fluctuate with time. These fluctuations with respect to time are exactly those needed to maintain the balance of behavioral and economic circumstances defining the equilibrium of interest.

In the current paper we have tried to tie all these missing components together. In doing so we consider a dynamic model of generalized Cournot-Nash (CN) equilibria for oligopolistic competition on an electric power network. The generalized CN equilibrium problem is an extension of the CN equilibrium problem, in which each player’s strategy set is dependent on the rival players’ strategies. The electric power network we consider here consists of spatially distributed markets, generating firms and an ISO. The ISO is the principal agent for electricity transmission who receive fees from the generators. The ISO sets the transmission fees in order to efficiently clear the market for transmission capacity. We do not consider additional players in the markets, such as an allocator of inputs to production and an arbitrager. We also do not consider refinements such as price function conjectures. Nonetheless, our basic dynamic model of generalized CN equilibrium has some unique features that were overlooked in earlier papers; namely dynamics, ramping constraints and costs.

In the next section we put forward an optimal control based formulation of generators’ profit maximization model explicitly considering the ramping constraints and costs as well as the joint constraints arising from the market capacity constraints. We also show how such models give rise to a joint Nash game in a dynamic sense, and the equivalent dynamic quasi variational inequality formulation, solution of which is a solution of the Nash game. We also describe a variant of fixed point algorithm designed to solve this problem and illustrate the efficacy of such algorithm through a numerical example.
5.2.2 The Generating Firms’ Problems

The oligopolistic firms of interest are power generating firms embedded in a network economy. These firms are in CN oligopolistic game theoretic competition where each firm is attempting to maximize its own profits while adhering to both physical and regulatory constraints. The firms’ profits are its revenues less costs. Instantaneous revenues are equal to regional sales in the market level times the corresponding nodal (market) prices, and the costs include generation costs, ramping costs and transmission fees, the latter paid to the ISO. Due to the inherent properties of an electrical power network, it is assumed that a firm located at some node of the network can supply energy to any other node (market) of the network.

The physical constraints arise from the physical characteristics of the actual power network as well as the physical attributes of the generating facilities. The flows of power in the network must obey Kirchhoff’s power and voltage laws. The generating facilities have both lower and upper bounds on level of power generation as well as lower and upper bounds on the rate at which the output of the generating units can be adjusted. Regulatory constraints may arise in the form of an upper limit on the total power provided to a particular market by all of the firms. The generating firms compete as price takers in the electric power markets which is perfectly competitive due to its involvement in other markets of the network economy.

Representation of Kirchhoff’s Laws

Rather than explicitly considering Kirchhoff’s laws, in this paper we will use the simpler representation of power transfer distribution factors (PTDFs) as seen in Helman [84]. These PTDFs can be computed offline and indicate what fraction of power will flow over each edge of the network when injecting 1 MW at node \( i \) and extracting 1 MW from node \( j \). Because the system is linear, the PTDFs from \( j \) to \( i \) are simply the negative of the PTDFs from \( i \) to
j. By considering an arbitrary hub node, we can compute the PTDFs for a 1 MW injection at the hub and a 1 MW extraction at node i for all nodes in the network. The PTDF for an injection at node i and an extraction at node j can be represented as the negative of the PTDF from the hub to node i plus the PTDF from the hub to node j. In addition to using the PTDF representation, flow balance is enforced at each node; the sum of power flows into a node must equal the sum of power flows out of that node. Using this representation greatly simplifies the constraint set for each firm’s problem.

5.2.3 Notation

We primarily employ the notation used in Miller, Friesz and Tobin [73], augmented to handle temporal considerations. Time is denoted by the scalar \( t \in \mathbb{R}_+ \), initial time by \( t_0 \in \mathbb{R}_+ \), final time by \( t_1 \in \mathbb{R}_++ \), with \( t_0 < t_1 \) so that \( t \in [t_0, t_1] \subset \mathbb{R}_+ \). Other notation involved with our model is summarized in Table 5.1 and 5.2.

5.2.4 The Generating Firms’ Problem Formulation

Time Scale

The regional demand of electricity varies across hours of a day. Over different days there is notable periodicity. For example, the demand for power at a specific hour of a day is very similar to the demands at the same hour on different days. In particular, we assume the period \( T \) to be 24 hours which is the planning horizon for our model. Because of the relatively short time-scale, we do not consider the time value of money in our models.
Table 5.1: Notation : Parameters : Electric Power Network Competition

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N}$</td>
<td>set of nodes, excluding the hub, denoted $H$</td>
</tr>
<tr>
<td>$\mathcal{A}$</td>
<td>set of bi-directional arcs</td>
</tr>
<tr>
<td>$H$</td>
<td>hub node</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>set of firms</td>
</tr>
<tr>
<td>$\mathcal{N}_f$</td>
<td>set of nodes where generating firm $f \in \mathcal{F}$ has power generators</td>
</tr>
<tr>
<td>$\mathcal{G}(i,f)$</td>
<td>set of power generating facilities owned by firm $f \in \mathcal{F}$ located at node $i \in \mathcal{N}_f$</td>
</tr>
<tr>
<td>$T_a$</td>
<td>transmission capacity on arc $a \in \mathcal{A}$</td>
</tr>
<tr>
<td>$PDF_{ia}$</td>
<td>power transfer distribution factor for node $i$ on arc $a$, describing the megawatt (MW) increase in flow on arc $a$ resulting from 1 MW of power withdrawal at $i$ and 1 MW of injection at $H$ due to Kirchhoff’s law</td>
</tr>
<tr>
<td>$CAP^f_j$</td>
<td>generation capacity of plant $j \in \mathcal{G}(i,f)$ located at node $i \in \mathcal{N}_f$ owned by firm $f \in \mathcal{F}$</td>
</tr>
<tr>
<td>$\sigma_i$</td>
<td>regional sales cap at market $i \in \mathcal{N}$</td>
</tr>
<tr>
<td>$R^{f+}_{j}$</td>
<td>upper bound of ramping rate of the generator unit $j \in \mathcal{G}(i,f)$ located at node $i \in \mathcal{N}_f$ owned by firm $f \in \mathcal{F}$</td>
</tr>
<tr>
<td>$R^{f-}_{j}$</td>
<td>lower bound of ramping rate of the generator unit $j \in \mathcal{G}(i,f)$ located at node $i \in \mathcal{N}_f$ owned by firm $f \in \mathcal{F}$</td>
</tr>
</tbody>
</table>
Table 5.2: Notation: States and Control Variables: Electric Power Competition

<table>
<thead>
<tr>
<th>Variables</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_i^f$</td>
<td>rate of sales of power (MWatt-hour) at node $i \in \mathcal{N}$ by firm $f \in \mathcal{F}$</td>
</tr>
<tr>
<td>$q_j^f$</td>
<td>generation rate at of the generator unit $j \in \mathcal{G}(i,f)$ located at node $i \in \mathcal{N}_f$ owned by firm $f \in \mathcal{F}$</td>
</tr>
<tr>
<td>$y_i$</td>
<td>amount of transmission in megawatts from hub $H$ to node $i$</td>
</tr>
<tr>
<td>$w_i$</td>
<td>wheeling fee or price from hub $H$ to node $i$ ($$/megawatt)</td>
</tr>
<tr>
<td>$r_j^f$</td>
<td>ramping rate of the generator unit $j \in \mathcal{G}(i,f)$ located at node $i \in \mathcal{N}_f$ owned by firm $f \in \mathcal{F}$</td>
</tr>
<tr>
<td>$\alpha_a$</td>
<td>dual variables of transmission constraint in ISO’s problem associated with arc $a$</td>
</tr>
<tr>
<td>$\beta_i$</td>
<td>dual variables associated with joint sales capacity constraint, 1 for each node</td>
</tr>
</tbody>
</table>

Revenue and Cost Components

We consider the inverse demand function for market $i \in \mathcal{N}$ to be

$$\pi_i \left( \sum_{g \in \mathcal{F}} s_{i,g}^q, t \right)$$

In particular, we assume that the inverse demand is separable in the sense that the price at market $i$ only depends on the consumptions at that market; i.e.,

$$\pi_i \left( \sum_{g \in \mathcal{F}} s_{i,g}^q, t \right) = P_{0,i}(t) - \frac{Q_{0,i}(t)}{P_{0,i}(t)} \cdot \left[ \sum_{g \in \mathcal{F}} c_{i,g}^q \right]$$

where the coefficients $P_{0,i}(t), Q_{0,i}(t)$ vary with time of the day reflecting the pattern of energy using activities. This results in price rises during high load periods. The revenue that firm $f$ is generating at time $t$ is therefore given by

$$\sum_{i \in \mathcal{N}} \pi_i \left( \sum_{g \in \mathcal{F}} s_{i,g}^q, t \right) \cdot s_i^f$$
where the summation is over all markets as it is assumed that any generating firm can
service any market.

The costs that the generating firm bears are:

1. The *generation cost* (i.e., the cost per unit power generation) for some generation unit
   \( j \in \mathcal{G} (i, f) \) is denoted by
   \[ V^f_j \left( q^f_j, t \right) \]
   and typically has a fixed component and a variable component. Generation cost is
   usually quadratic of the form
   \[ V^f_j \left( q^f_j, t \right) = \mu^f_j + \tilde{\mu}^f_j \cdot q^f_j + \frac{1}{2} \hat{\mu}^f_j \cdot \left( q^f_j \right)^2 \]
   where \( \mu^f_j, \tilde{\mu}^f_j, \hat{\mu}^f_j \in \mathbb{R}^1_{++} \) for all \( f \in \mathcal{F}, j \in \mathcal{G} (i, f) \).

2. The *ramping cost* is obtained from the fatigue effect of the rotors (which in turn affects
   the life span of the rotor). However, the ramping cost is negligible if the magnitude
   of power change is less than some elastic range; that is, there is a range in which the
   generation rate can be adjusted that causes minimal wear on the rotors and is thus
   considered cost free. The slope of the cost curve also depends on the ramp-up time.
   Therefore, in general we may use the function
   \[ \Phi^f_j \left( r^f_j, t \right) = \frac{1}{2} \gamma^f_j \left[ \max \left( 0, \left| r^f_j \right| - \xi^f_j \right) \right]^2 \]
   to represent the ramping cost associated with some generation unit \( j \in \mathcal{G} (i, f) \) when
   the ramping rate is \( r^f_j, \xi^f_j \in \mathbb{R}^1_{++} \) is the elastic threshold of the unit and \( \gamma^f_j \) is the cost
   coefficient which depends on the ramp-up time. In this case, we are using a symmetric
   cost for ramping up and ramping down, though this is not necessary in general. In
   general, with asymmetric ramp-up and ramp-down costs we may have
   \[ \Phi^f_j \left( r^f_j, t \right) = \frac{1}{2} \gamma^f_- j \left[ \max \left( 0, r^f_j - \xi^f_- j \right) \right]^2 + \frac{1}{2} \gamma^f_+ j \left[ \max \left( 0, -r^f_j + \xi^f_+ j \right) \right]^2 \]
   where \( \gamma^f_- j, \gamma^f_+ j \) are the cost coefficients during ramp-up and ramp-down respectively
   and \( \xi^f_- j, \xi^f_+ j \) are the elastic thresholds during ramp-up and ramp-down respectively.
3. The wheeling fee \( w_i(t) \) is paid to the ISO for transmitting 1 MWatt-hour of power from the hub to market \( i \) at time \( t \). The wheeling fee is endogenously determined to enforce the market clearing conditions by the ISOs.

**Constraints**

1. Each firm must balance sales and generation for all time \( t \in [t_0, t_1] \) as we do not consider storage of electricity

\[
\sum_{i \in \mathcal{N}} s_i^f(t) = \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} q_j^f(t)
\]

2. The sales of power at every market must be nonnegative:

\[
s_i^f(t) \geq 0 \text{ for all } t \in [t_0, t_1]
\]

3. The output level of each generating unit is bounded from above and below as

\[
0 \leq q_j^f(t) \leq \text{CAP}_{j}^f
\]

for all \( i \in \mathcal{N}_f, j \in \mathcal{G}(i,f) \) and \( t \in [t_0, t_1] \) where \( \text{CAP}_{j}^f \in \mathbb{R}^{1,1} \). The upper bound is imposed from the physical constraints of the generator.

4. Total sales by all the firms at a particular market may be bounded from above by some regulatory authority. This can be represented by the constraint of the form

\[
\sum_{f \in \mathcal{F}} s_i^f(t) \leq \sigma_i \text{ for all } i \in \mathcal{N}, t \in [t_0, t_1]
\]

where \( \sigma_i \in \mathbb{R}_{1,1} \) is the regional sales cap. Note that this constraint makes the equilibrium problem a generalized Nash equilibrium problem as with this constraint each firms’ strategy set (here for sales variable) is dependent on the rival players’ strategy (competitors sales variable).

5. The ramping rate for every generation unit is bounded from above and below, imposed by the very characteristic of the plant,

\[
R_{j}^{f-} \leq r_{j}^f(t) \leq R_{j}^{f+}
\]
for all $i \in \mathcal{N}_f$, $j \in \mathcal{G}(i,f)$ and $t \in [t_0, t_1]$

**Firms’ Extremal Problem**

With the transmission (wheeling) fee $w_i$ and the rival firms’ sales

$$s^{-f} \equiv \{s^g : g \in \mathcal{F} \setminus f\}$$

taken as exogenous to the firm $f \in \mathcal{F}$’s optimal control problem and yet endogenous to the overall equilibrium model, firm $f$ computes its nodal sales $s^f$ and generations $q^f$ in order to:

$$\max J(s^f, q^f; s^{-f}; t) = \int_{t_0}^{t_1} \left\{ \sum_{i \in \mathcal{N}} \pi_i \left( \sum_{g \in \mathcal{F}} s^g_i, t \right) \cdot s^f_i \right\} \ dt$$

subject to

$$\frac{dq^f_j}{dt} = r^f_j \text{ for all } i \in \mathcal{N}_f, j \in \mathcal{G}(i,f)$$

$$q^f_j(t_0) = q^f_{j,0} \in \mathbb{R}^+ \text{ for all } i \in \mathcal{N}_f, j \in \mathcal{G}(i,f)$$

$$R^f_j^- \leq r^f_j \leq R^f_j^+ \text{ for all } i \in \mathcal{N}_f, j \in \mathcal{G}(i,f)$$

$$0 \leq q^f_j \leq \text{CAP}^f_j \text{ for all } i \in \mathcal{N}_f, j \in \mathcal{G}(i,f)$$

$$s^f_i \geq 0 \text{ for all } i \in \mathcal{N}$$

$$s^f_i + \sum_{g \in \mathcal{F} \setminus i} s^g_i \leq \sigma_i \text{ for all } i \in \mathcal{N}$$

$$\sum_{i \in \mathcal{N}} s^f_i = \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} q^f_j$$

Let us review the formulation from an optimal control theory perspective. Constraints of type (5.8) are called the mixed constraints as they involve both states ($q^f_j$) and controls ($s^f_i$). (5.4) and (5.6) are the pure control constraints, and (5.6) are the state-space constraints. The right hand side of the dynamics, (5.2), are free from states and linear in controls (ramping rates).
5.2.5 The Differential Quasi Variational Inequality (DQVI) Formulation

Let us begin by noting that (5.1), (5.2), (5.3), (5.4), (5.5), (5.6), (5.7) and (5.8) form a standard form optimal control problem for firm $f$, where the controls for the firm are

$$s^f = \left\{ s^f_i : i \in \mathcal{N} \right\}$$

$$r^f = \left\{ r^f_j : i \in \mathcal{N}_f, j \in \mathcal{G}(i,f) \right\}$$

and the states are

$$q^f = \left\{ q^f_j : i \in \mathcal{N}_f, j \in \mathcal{G}(i,f) \right\}$$

Note that we have state space constraints in the form

$$0 \leq q^f \leq \text{CAP}^f$$

where

$$\text{CAP}^f = \left\{ \text{CAP}^f_j : i \in \mathcal{N}_f, j \in \mathcal{G}(i,f) \right\}$$

are known constants. The Hamiltonian associated with this optimal control problem is

$$H_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, t \right) = \sum_{i \in \mathcal{N}} \pi_i \left( \sum_{g \in \mathcal{F}} s^g_i, t \right) \cdot s^f_i - \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} \left[ V^f_j \left( q^f_j, t \right) + \Phi^f_j \left( r^f_j, t \right) \right]$$

$$- \sum_{i \in \mathcal{N}} w_i \cdot \left( s^f_i - \sum_{j \in \mathcal{G}(i,f)} q^f_j \right) + \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} \lambda^f_j \cdot r^f_j$$

(5.9)

where the feasible set of controls for firm $f$ is expressed as

$$\Omega_f = \left\{ \left( s^f, r^f \right) : (5.4),(5.5),(5.6), (5.7) \text{ and } (5.8) \text{ hold} \right\}$$

Note that we have not yet explicitly considered the state space constraints and the mixed constraints in the Hamiltonian. This is accomplished by forming the Lagrangian as (see
Sethi and Thompson [6]

\[ L_f \left( s^f, r^f; q^f; s^{-f}, \lambda^f, \tau^f-, \tau^f+, \varsigma^f-, \varsigma^f+; t \right) = H_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, t \right) \]

\[ + \tau^f- \left( \sum_{i \in \mathcal{N}} s^f_i - \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i, f)} q^f_j \right) \]

\[ + \tau^f+ \left( - \sum_{i \in \mathcal{N}} s^f_i + \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i, f)} q^f_j \right) \]

\[ + \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i, f)} \left\{ \varsigma^f_j - q^f_j + \varsigma^f_j \cdot \left( \text{CAP}^f_j - q^f_j \right) \right\} \]

where \( \tau^f- \in \mathbb{R}_+^1, \tau^f+ \in \mathbb{R}_+^1, \varsigma^f_j \in \mathbb{R}_+^1 \) and \( \varsigma^f_j \in \mathbb{R}_+^1 \) are dual variables. \( \tau^f- \) and \( \tau^f+ \) satisfy the complementarity slackness conditions

\[ \tau^f- \geq 0, \quad \tau^f- \left( \sum_{i \in \mathcal{N}} s^f_i - \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i, f)} q^f_j \right) = 0 \] (5.11)

\[ \tau^f+ \geq 0, \quad \tau^f+ \left( - \sum_{i \in \mathcal{N}} s^f_i + \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i, f)} q^f_j \right) = 0 \] (5.12)

Also, \( \varsigma^f_j \) and \( \varsigma^f_j \) satisfy the conditions

\[ \varsigma^f_j \geq 0, \quad \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i, f)} \varsigma^f_j \cdot q^f_j = 0, \quad \varsigma^f_j \leq 0 \] (5.13)

\[ \varsigma^f_j \geq 0, \quad \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i, f)} \varsigma^f_j \cdot \left( \text{CAP}^f_j - q^f_j \right) = 0, \quad \varsigma^f_j \leq 0 \] (5.14)

where

\[ \varsigma^f_+ = \left\{ \varsigma^f_j : i \in \mathcal{N}_f, \; j \in \mathcal{G}(i, f) \right\} \]

\[ \varsigma^f_+ = \left\{ \varsigma^f_j : i \in \mathcal{N}_f, \; j \in \mathcal{G}(i, f) \right\} \]

Note that these conditions on the multipliers \( \varsigma^f_+ \) and \( \varsigma^f_+ \) arise from the state space constraints where as (5.11) - (5.12) arise from the mixed contraints.

The adjoint variables \( \lambda^f \) follow the dynamics

\[ \frac{d\lambda^f_i}{dt} = - \frac{\partial L^*_f}{\partial q^*_j} \text{ for all } i \in \mathcal{N}_f, \; j \in \mathcal{G}(i, f) \]

\[ = \frac{\partial \Phi^f_j \left( r^*_j, t \right)}{\partial q^*_j} + \tau^f- - \tau^f+ - \varsigma^f_+ + \varsigma^f_- \text{ for all } i \in \mathcal{N}_f, \; j \in \mathcal{G}(i, f) \] (5.15)
and the transversality condition yields

\[ \lambda^f_j(t_1) = 0 \] (5.16)

Therefore, for given set of controls \( r^f \), the state variables \( q^f \) and adjoint variables \( \lambda^f \) can be determined in a sequential fashion; note that this is not an approximation. This observation will facilitate our computational efforts.

Also note that constraints (5.7) are joint constraints involving \( s^f \) and \( s^{-f} \). We handle these constraints in the following way as outlined by Pang [75], though we are applying this to a dynamic game.

From the maximum principle (see Bryson and Ho [28]) we know that the necessary condition (which we will later establish to be a sufficient condition as well) for the quadruplet \( \{ s^f_s(t), r^f_s(t); q^f_s(t); \lambda^f_s(t) \} \) (as well as the dual multipliers \( \tau^f_-, \tau^f_+, \zeta^f_-, \zeta^f_+ \)) being an optimal solution to the optimal control problem (5.1) - (5.7) is that the nonlinear program

\[
\max L_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, t \right)
\]

subject to \( \{ s^f(t), r^f(t) \} \in \tilde{\Omega}_f \)

be solved for each instant \( t \in [t_0, t_1] \) where \( \tilde{\Omega}_f \) is a set formed by pure control constraints

\[
\tilde{\Omega}_f = \left\{ \left( s^f, r^f \right) : \text{(5.4), (5.6) and (5.7) hold} \right\}
\]

This is a stationary problem. Further, we write this problem as

\[
\max L_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, t \right)
\] (5.17)

subject to

\[
s^f + \sum_{g \in \mathcal{F}_i} s^g \leq \sigma \] (5.18)

\[
R^f_- \leq r^f \leq R^f_+ \] (5.19)

\[
s^f \geq 0 \] (5.20)
Note that, even though we only have control constraints in (5.18), (5.19) and (5.20), constraint (5.18) is a joint constraint which is problematic. Under a suitable constraint qualification such as Abadie’s (as the feasible set is convex), we obtain the following KKT conditions which are necessary for $s^f$ being the optimal for firm $f$’s stationary problem (5.17) - (5.18).

\[ 0 = -\nabla_{s^f} \tilde{L}_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, t \right) - \gamma^f \quad (5.21) \]

\[ 0 \leq \beta \perp \left( s^f + \sum_{g \in \mathcal{F} \setminus i} s^g - \sigma \right) \leq 0 \quad (5.22) \]

\[ 0 \leq \gamma^f \perp (-s^f) \leq 0 \quad (5.23) \]

where $\tilde{L}_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \zeta^f, \zeta^f + \beta; t \right)$ is agent $f$’s ‘semi-Lagrangian function’ defined as

\[ \tilde{L}_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \zeta^f, \zeta^f + \beta; t \right) = L_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \zeta^f, \zeta^f + \beta; t \right) + \beta^T \left( s^f + \sum_{g \in \mathcal{F} \setminus i} s^g - \sigma \right) \quad (5.24) \]

Note that, as discussed in Harker [74], we use the restricted multiplier formulation which is applicable to all generalized Nash games where the players have the same joint constraint. In this case we use dual multiplier $\beta_i$ associated with the regional sales cap constraint. This dual multiplier $\beta_i$ is dependent only on the region (market) $i$ and is common to all firms.

Therefore, concatenating $|\mathcal{F}|$ number of KKT systems, we obtain a characterization of a generalized Cournot-Nash equilibrium as a partitioned DVI. In order to formulate the latter DVI, let us define the function $\Theta \left( s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \zeta^f, \zeta^f + \beta; t \right)$ as

\[ \Theta \left( s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \zeta^f, \zeta^f + \beta; t \right) = \begin{pmatrix} \nabla_{s^f} L_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \zeta^f, \zeta^f + \beta; t \right) + \beta \left( s^f + \sum_{g \in \mathcal{F} \setminus i} s^g - \sigma \right) \\ s^f \end{pmatrix}_{f=1}^{|\mathcal{F}|} \]

\[ z = \begin{pmatrix} s^f_{f=1}^{|\mathcal{F}|} \\ \beta \end{pmatrix} \]
Further, considering the complementarity conditions (5.11), (5.12), (5.13) and (5.14), we can define \( \tilde{\Theta} \left( s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \varsigma^f; \varsigma^f +; t \right) \) as

\[
\tilde{\Theta} \left( s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \varsigma^f; \varsigma^f +; t \right) = \begin{pmatrix}
\nabla_{s^f} L_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \varsigma^f; \varsigma^f +; t \right) + \beta)^{\left| f \right|}
\left( s^f + \sum_{g \in F \setminus f} s^g - \sigma \right)
\left( - \sum_{i \in N_i} s_i^f + \sum_{i \in N_f} \sum_{j \in G(i,f)} q_j^f \right)^{\left| f \right|}
\left( \sum_{i \in N_i} s_i^f \right)^{\left| f \right|}
\left( -q^f \right)^{\left| f \right|}
\left( -CAP^f + q^f \right)^{\left| f \right|}
\end{pmatrix}^{\left| f \right|}
\]

(5.25)

The solution of the following partitioned DVI described below is also the generalized Cournot-Nash equilibrium of the game described above taking the wheeling fee \( w \) as exogenous:

\[
\begin{aligned}
\text{find } (\tilde{z}^*, r^*) \in \Lambda \\
\int_{t_0}^{t_1} \left\{ \tilde{\Theta} \left( \tilde{z}^*, r^f; t \right) \right\}^T (\tilde{z} - \tilde{z}^*) dt + \sum_{f \in F} \int_{t_0}^{t_1} \left[ \nabla_{r^f} L_f \left( \tilde{z}^*, r^f; t \right) \right]^T \left( r^f - r^f \right) dt \leq 0
\end{aligned}
\]

(5.27)

where

\[
\Lambda_f = \left\{ r^f : \text{constraints (5.4) hold} \right\}
\]

\[
\kappa = \left\{ \tilde{z} : \tilde{z} \geq 0 \right\}
\]
\[ \Lambda = \prod_{f \in \mathcal{F}} \Lambda_f \]

### 5.2.6 The ISO and Transmission Fees

It was mentioned earlier that the transmission (wheeling) fees \( w \) are set by the ISO in order to efficiently clear the market for transmission capacity. Specifically, taking \( w \) as exogenous to its problem, at every instant \( t \in [t_0, t_1] \) the ISO seeks to solve the following linear program to determine the transmission flows \( y \) in order to

\[
\max_{y(t)} J_2(t) = \sum_{i \in \mathcal{N}} y_i(t) \cdot w_i(t)
\]

subject to

\[
\sum_{i \in \mathcal{N}} PDF_{ia} \cdot y_i(t) \leq T_a \text{ for all } a \in \mathcal{A} \quad (\alpha_a)
\]

where we write dual variables in the parentheses next to the corresponding constraints. Recall that \( \mathcal{A} \) is the arc set of the electric power network, \( T_a \) is the transmission capacity on arc \( a \in \mathcal{A} \) and \( PDF_{ia} \) are the power transmission distribution factors that describe how much MW flow occurs through arc \( a \) as a result of a unit MW injection at an arbitrary hub node and a unit withdrawal at node \( i \). In a linearized DC power flow model, which is the basis of the above model, the PTDF factors are assumed to be constant and are unaffected by the load of the transmission line. Therefore the principle of superposition applies. The decision variables \( y_i(t) \) denote transfers of power in MWatt by the ISO from a hub node to the node (market) \( i \in \mathcal{N} \) at time \( t \in [t_0, t_1] \). In this particular formulation we ignore transmission loss, however, our model is general enough and does not prohibit us to consider non-linear losses. In the case of losses, either the ISO or the firms involved in the transaction should account for the losses and a book-keeping effort is required.
5.2.7 The Market Clearing Conditions

To clear the market, the transmission flows $y_i$ must balance the net sales at each node (market), therefore

$$y_i(t) = \sum_{f \in \mathcal{F}} \left( s_i^f(t) - \sum_{j \in \mathcal{G}(i,f)} q_j^f(t) \right)$$

for all $i \in \mathcal{N}$

therefore, re-writing (5.54) - (5.55) we have

$$\max J_2(t) = \sum_{i \in \mathcal{N}} \sum_{f \in \mathcal{F}} \left( s_i^f(t) - \sum_{j \in \mathcal{G}(i,f)} q_j^f(t) \right) \cdot w_i(t)$$

subject to

$$\sum_{i \in \mathcal{N}} \sum_{f \in \mathcal{F}} \left( s_i^f(t) - \sum_{j \in \mathcal{G}(i,f)} q_j^f(t) \right) \leq T_a \text{ for all } a \in \mathcal{A}$$

Therefore, the optimality condition of the linear program (5.57) - (5.58) can be written down as

$$w_i = \sum_{a \in \mathcal{A}} \text{PDF}_{i+k} \cdot \alpha_a(t) \text{ for all } i \in \mathcal{N}$$

$$0 \leq \alpha_a(t) \downarrow T_a - \sum_{i \in \mathcal{N}} \sum_{f \in \mathcal{F}} \left( s_i^f(t) - \sum_{j \in \mathcal{G}(i,f)} q_j^f(t) \right) \geq 0 \text{ for all } a \in \mathcal{A}$$

To articulate the complete DQVI formulation let us define the following vector

$$v(s, q; t) = \left( T_a - \sum_{i \in \mathcal{N}} \text{PDF}_{i+k} \cdot \sum_{f \in \mathcal{F}} \left( s_i^f(t) - \sum_{j \in \mathcal{G}(i,f)} q_j^f(t) \right) \right)_{a=1}^{\mathcal{A}}$$

The Complete Restricted DVI Formulation

Putting together the generating firms' optimality conditions, the ISO's problem and the market clearing condition, we obtain the complete formulation of the market equilibrium problem as the following DQVI:

$$\text{find } (z^*, r^*, \alpha^*) \in \tilde{\mathcal{A}} \text{ such that}$$

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\[
\int_{t_0}^{t_1} \left[ \tilde{\Theta} \left( \tilde{z}^*, r^{f*}; t \right) \right]^T (\tilde{z} - \tilde{z}^*) \, dt + \sum_{j \in \mathcal{F}} \int_{t_0}^{t_1} \left[ \nabla_{r^f} L_f \left( \tilde{z}^*, r^{f*}; t \right) \right]^T (r^f - r^{f*}) \, dt
\]

\[
- v (s^*, q^*; t) \cdot (\alpha - \alpha^*) \} \, dt \leq 0
\]

for all \((z, r, \alpha) \in \tilde{\Lambda}\)

where

\[
\tilde{H}_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, \alpha, t \right) = \sum_{i \in N} \pi_i \left( \sum_{g \in \mathcal{F}} q^f_g \right) - \sum_{i \in N_f} \sum_{j \in \mathcal{G}(i,f)} \left[ V^f_j \left( q^f_j, t \right) + \Phi^f_j \left( r^f_j, t \right) \right]
\]

\[
- \sum_{i \in N} \left( \sum_{a \in \mathcal{A}} PDF^a_{ik} \cdot \alpha_a \left( t \right) \right) \cdot \left( q^f_i - \sum_{j \in \mathcal{G}(i,f)} q^f_j \right) + \sum_{i \in N_f} \sum_{j \in \mathcal{G}(i,f)} \lambda^f_j \cdot r^f_j
\]

and \(L_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \tau^{f-}, \tau^{f+}, \zeta^{f-}, \zeta^{f+}; t \right)\) and \(\tilde{\Theta} \left( s^{f*}, r^{f*}; q^{f*}; s^{-f*}; \lambda^{f*}, t \right)\) are defined in the same way as in (5.10) and (5.25) respectively, only replacing \(H_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, t \right)\) by \(\tilde{H}_f \left( s^f, r^f; q^f; s^{-f}; \lambda^f, \alpha, t \right)\). Further, \(\tilde{\Lambda}\) is defined as

\[
\tilde{\Lambda}_f = \left\{ \left( \tilde{z}, r^f, \alpha \right) : \begin{array}{c}
\tilde{z} \geq 0 \\
R^{-} \leq r^f \leq R^{+} \\
\zeta^{f-} \leq 0 \\
\zeta^{f+} \leq 0
\end{array} \right\}
\]

\[
\tilde{\Lambda} = \prod_{f \in \mathcal{F}} \tilde{\Lambda}_f
\]

Note that the last two constraints can be rewritten as

\[
\frac{d \zeta^{f-}}{dt} = u^{f-} \quad (5.33)
\]

\[
\frac{d \zeta^{f+}}{dt} = u^{f+} \quad (5.34)
\]

where \(u^{f-}\) and \(u^{f+}\) are dummy control variables. We treat these dynamics explicitly in our numerical algorithm (implicit fixed point algorithm) as discussed in Section 5.2.10.
5.2.8 Equivalence between the GNEP and the DVI (5.32)

Before we establish the equivalence between the GNEP and the restricted DVI formulation provided in (5.32), we need to establish that the necessary conditions for the optimal control problem (5.1) - (5.8) are indeed also sufficient conditions. The lemma stated below establishes the sufficiency condition.

**Lemma 5.1** (Sufficiency conditions for the optimal control problem) For a given $s^{-f}$ let $(s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \tau^f, \varsigma^f, \varsigma^f)$ satisfy the necessary conditions (5.11), (5.12), (5.13), (5.14), (5.15), (5.16), (5.17), (5.19) and (5.20), then $(s^f, r^f; q^f)$ is optimal.

**Proof.** We begin with defining an augmented adjoint variable

$$\tilde{\lambda}_j^f(t) = \lambda_j^f(t) + \varsigma_j^- \frac{\partial q_j^f}{\partial q_j^f} + \varsigma_j^+ \frac{\partial (CAP_j^f - q_j^f)}{\partial q_j^f}$$

for all $i \in N_f$, $j \in F(i,f)$ and $t \in [t_0, t_1]$. We also observe that for a given $s^{-f}$, the instantaneous revenue is concave in the generating firm’s own sales $s^f$, unit generation cost is convex in $q^f$, unit ramping cost is also convex in ramping rate $r^f$ and transhipment cost is linear in $s^f$ and $q^f$. Also, the right hand side of the dynamics (5.2) of firm $f$ is linear in the firm’s own ramping rate $r^f$ and $\tilde{\lambda}_j^f$ is free from states. Using the fact that the negative of a convex function is a concave function, we observe that the modified Hamiltonian $\tilde{H}_f(s^f, r^f; q^f; s^{-f}; \tilde{\lambda}^f, t)$

$$\tilde{H}_f(s^f, r^f; q^f; s^{-f}; \tilde{\lambda}^f, \alpha, t) = \sum_{i \in N} \pi_i \left( \sum_{g \in F} s_{gi}^f, t \right) \cdot s_i^f - \sum_{i \in N_f} \sum_{j \in F(i,f)} \left[ V_j^f(q_j^f, t) + \Phi_j^f(r_j^f, t) \right]$$

$$- \sum_{i \in N} \left( \sum_{a \in A} PDF_{ik} \cdot \alpha_a(t) \right) \cdot \left( s_i^f - \sum_{j \in F(i,f)} q_j^f \right) + \sum_{i \in N_f} \sum_{j \in F(i,f)} \tilde{\lambda}_j^f \cdot r_j^f$$

is concave in $(s^f, r^f; q^f)$ at each $t \in [t_0, t_1]$. Also observe that the state-space constraints (5.5) are linear constraints as are the mixed constraints (5.8). Further, there is no terminal
We now state and establish the equivalence between the GNEP and the DVI (5.32).

**Theorem 5.1 (Equivalence between the GNEP and DVI)** Under a suitable constraint qualification (e.g., for every \( f \) and \( t \in [t_0, t_1] \), there exists \( \tilde{s}^f, \tilde{r}^f \) and \( \tilde{q}^f \) such that \( \tilde{q}^f > 0, \tilde{q}^f < \text{CAP}^f, \tilde{s}^f + \sum_{g \in F_i} s^g > \sigma, \tilde{s}^f > 0, \sum_{i \in N} s^i > \sum_{i \in N_f} \sum_{j \in G(i,f)} q^i_j \) and \( \sum_{i \in N} s^i < \sum_{i \in N_f} \sum_{j \in G(i,f)} q^i_j \), the tuple \((s^*, q^*)\) is a GNE if and only if there exists \( \beta^*, \tau^-, \tau^+, \varsigma^-, \varsigma^+, \alpha^* \) such that \((\beta^*, \tau^-, \tau^+, \varsigma^-, \varsigma^+, \alpha^*, s^*, q^*)\) is a solution of the DVI (5.32).

**Proof.** We begin by noting that (5.32) is equivalent to the following optimal control problem

\[
\max J_3\left(z, r^f, r^{-f}, \alpha, t\right) = \int_{t_0}^{t_1} \left\{ \tilde{\Theta} \left( \tilde{z}^*, r^f \alpha^*; t \right) \tilde{z} + \nabla_{r^f} L_f \left( \tilde{z}^*, r^f \alpha^*; t \right) r^f - v(s^*, q^*; t) \alpha + \sum_{f \in F} \left[ \mu^{-} \right]^T u^{-} + \sum_{f \in F} \left[ \mu^{+} \right]^T u^{+} \right\} dt \tag{5.35}
\]

s.t. \( (z, r, \alpha) \in \tilde{\Lambda} \tag{5.36} \)

where it is essential to recognize that \( J_3\left(z, r^f, r^{-f}, \alpha, t\right) \) is a ‘linear’ functional that assumes knowledge of the solution to our oligopolistic game; as such \( J_3\left(z, r^f, r^{-f}, \alpha, t\right) \) is a mathematical construct for use in analysis and has no meaning as a computational device. The augmented Hamiltonian for this artificial optimal control problem is

\[
H_0 = \tilde{\Theta} \left( \tilde{z}^*, r^f \alpha^*; t \right) \tilde{z} + \nabla_{r^f} L_f \left( \tilde{z}^*, r^f \alpha^*; t \right) r^f - v(s^*, q^*; t) \alpha + \sum_{f \in F} \left[ \mu^{-} \right]^T u^{-} + \sum_{f \in F} \left[ \mu^{+} \right]^T u^{+}
\]

where \( \mu^{-} \) and \( \mu^{+} \) are the auxiliary adjoint variables associated with the dynamics (5.33) and (5.34) respectively. The associated maximum principal requires

\[
\max H_0
\]

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subject to

\[ R^- \leq r \leq R^+ \]
\[ \dot{z} \geq 0 \]
\[ \alpha \geq 0 \]

for all \( t \in [t_0, t_1] \). The corresponding necessary and sufficient (as the Hamiltonian is linear in the controls) conditions for this optimal control problem are identical to (5.37) through (5.39), since

\[ \nabla_\tilde{z} H^*_0 = \tilde{\Theta} \left( \tilde{z}^*, r^f*, \alpha^*; t \right) \] (5.37)
\[ \nabla_{r^f} H^*_0 = \nabla_{r^f} L_f \left( \tilde{z}^*, r^f*; t \right) \] (5.38)
\[ \nabla_\alpha H^*_0 = -v(s^*, q^*; t) \] (5.39)

where

\[ H^*_0 = \left[ \tilde{\Theta} \left( \tilde{z}^*, r^f*, \alpha^*; t \right) \right]^T \cdot \tilde{z}^* + \sum_{f \in \mathcal{F}} \left[ \nabla_{r^f} L_f \left( \tilde{z}^*, r^f*; t \right) \right]^T \cdot r^f* \]
\[ -v(s^*, q^*; t) \cdot \alpha^* + \sum_{f \in \mathcal{F}} \left[ \mu^{f^-} \right]^T u^{f-} + \sum_{f \in \mathcal{F}} \left[ \mu^{f+} \right]^T u^{f+} \]

which are identical to the necessary conditions for the firm \( f \)'s optimal control problem. Further, from Lemma 5.1 we know these are the sufficient conditions for firm \( f \)'s optimal control problem, hence the desired result (5.32) is immediate. ■

### 5.2.9 Existence

We also establish the existence of a solution of the GNEP. Since we have already established in Theorem 5.1 the equivalence between the GNEP and the DVI (5.32), it is sufficient to show that the DVI (5.32) has a solution. We state and prove the following result.

**Theorem 5.2** (Existence of a solution of the GNEP) Under a suitable constraint qualification (e.g., for every \( f \) and \( t \in [t_0, t_1] \), there exists \( \tilde{s}^f, \tilde{r}^f \) and \( \tilde{q}^f \) such that \( \tilde{q}^f > 0 \))
0, $q^f < \text{CAP}_f$, $\bar{s}^f + \sum_{g \in F_{i}} s^g < \sigma$, $\bar{s}^f > 0$, $\sum_{i \in N} s^f_i > \sum_{i \in N_f} \sum_{j \in G(i,f)} q^f_j$ and $\sum_{i \in N} s^f_i < \sum_{i \in N_f} \sum_{j \in G(i,f)} q^f_j$, the GNEP has a solution.

**Proof.** The formal proof of this existence result is in Browder [27], and depends on the conversion of the DVI to a fixed point problem involving the minimum norm projection and application of Browder’s existence theorem. We know $q(r, t)$ is well defined and continuous. So the principal functions of the DVI, i.e., $\tilde{\Theta}(\cdot)$, $\nabla_r L_f(\cdot)$ and $\upsilon(\cdot)$ are continuous in the controls $s$ and $r$. Also since the feasible control set for each player is convex and compact, their cartesian products is also convex and compact. Consequently, by Theorem 2 of Browder [27], DVI (5.32) has a solution. Further, under a suitable constraint qualification, we can invoke the Theorem 5.1 which ascertains that the GNEP also has a solution. Hence the proof.

5.2.10 Algorithms for the DVI

We employ a version of an implicit fixed point algorithm to compute the equilibria of the model. There is a fixed point form of $DVI(F, f, U, x^0, D)$. In particular we state and prove the following result:

**Theorem 5.3** (fixed point formulation of the DVI (5.32)) The DVI (5.32) is equivalent to the following fixed point problem:

$$\begin{pmatrix}
\tilde{z} \\
r \\
\alpha
\end{pmatrix} = P_{\tilde{\Lambda}} \begin{pmatrix}
\begin{pmatrix}
\tilde{z} \\
r \\
\alpha
\end{pmatrix} - \alpha_{fp} \begin{pmatrix}
\tilde{\Theta}(\tilde{z}, r^f, \alpha; t) \\
\nabla_r L_f(\tilde{z}, r^f; t) \\
\upsilon(s; t)
\end{pmatrix}
\end{pmatrix}$$

where $P_{\tilde{\Lambda}}[\cdot]$ is the minimum norm projection onto $\tilde{\Lambda}$ and $\alpha_{fp} \in \mathbb{R}^1_{++}$.

**Proof.** We observe that the right hand side of the dynamics (5.2) for each generator’s optimal control problem is linear in controls ($r^f$) and is thus convex. Further, the principal
functions of the DVI, namely $\tilde{\Theta}(\tilde{z}, r^f, \alpha; t)$, $\nabla_{r^f} L_f (\tilde{z}, r^f; t)$ and $v(s, q; t)$ are continuous with respect to both the controls $(s^f, r^f)$ and states $(q^f)$; hence, all the regularity conditions of Definition 2 of Mookherjee et al [85] hold. Using Theorem 3 of Mookherjee et al [85] we obtain the desired result.

Naturally there is an associated fixed point algorithm based on the iterative scheme

\[
\begin{pmatrix}
  z^{k+1}
  \\
  r^{k+1}
  \\
  \alpha^{k+1}
\end{pmatrix} = P_{\hat{\Lambda}} \begin{bmatrix}
  z^k
  \\
  r^k
  \\
  \alpha^k
\end{bmatrix} - \alpha_{fp} \begin{pmatrix}
  \tilde{\Theta}(\tilde{z}^k, r^{f,k}, \alpha^k; t) \\
  \nabla_{r^f} L_f (\tilde{z}^k, r^{f,k}; t) \\
  v(s^{k}; t)
\end{pmatrix}
\]

where $k$ in the superscript denotes the counter for the fixed point iteration. The detailed structure of the fixed point algorithm is as follows:

**Fixed Point Algorithm**

**Step 0.** Initialization: identify an initial feasible solution $\begin{pmatrix}
z^0 \\
r^0 \\
\alpha^0
\end{pmatrix}$ $\in \tilde{\Lambda}$ and set $k = 0$.

**Step 1.** Solve optimal control subproblem:

\[
\min_{(\tilde{z}, r, \alpha)} J^k(\tilde{z}, r, \alpha) = \frac{1}{2} \int_{t_0}^{t_1} \left[ \left( \tilde{z}^k - \alpha_{fp} \tilde{\Theta}(\tilde{z}^k, r^{f,k}, \alpha^k; t) - \tilde{z} \right)^2 + (r^k - \alpha_{fp} \nabla_{r^f} L_f (\tilde{z}^k, r^{f,k}; t) - r)^2 + (\alpha^k - \alpha_{fp} v(s^k; t) - \alpha)^2 \right] dt
\]

subject to

\[
\begin{align*}
\frac{d\varsigma^-}{dt} & = u^-, \quad \frac{d\varsigma^+}{dt} = u^+ \\
\varsigma^- & \leq 0, \quad u^+ \leq 0 \\
R^- & \leq r \leq R^+ \\
\tilde{z} & \geq 0 \\
\alpha & \geq 0 \\
\varsigma^-(t_0) & = \varsigma^{0-}, \varsigma^+(t_0) = \varsigma^{0+}
\end{align*}
\]
Call the solution \( \begin{pmatrix} z^{k+1} \\ r^{k+1} \\ \alpha^{k+1} \end{pmatrix} \)

**Step 2.** Stopping test: if

\[
\left\| \begin{pmatrix} z^{k+1} \\ r^{k+1} \\ \alpha^{k+1} \end{pmatrix} - \begin{pmatrix} z^k \\ r^k \\ \alpha^k \end{pmatrix} \right\| \leq \varepsilon
\]

where \( \varepsilon \in \mathbb{R}_{++} \) is a preset tolerance, stop and declare \( \begin{pmatrix} \tilde{z}^* \\ r^* \\ \alpha^* \end{pmatrix} \approx \begin{pmatrix} z^{k+1} \\ r^{k+1} \\ \alpha^{k+1} \end{pmatrix} \). Otherwise set \( k = k + 1 \) and go to Step 1.

The convergence of this algorithm is guaranteed if \( \tilde{\Theta} (\tilde{z}, r^f, \alpha; t) \), \( [\nabla_r L_f (\tilde{z}, r^f; t)] \) and \( v(s; t) \) are strongly monotonic for all \( \begin{pmatrix} \tilde{z} \\ r \\ \alpha \end{pmatrix} \in \tilde{\Lambda} \), see Friesz and Mookherjee (2005) for a detailed discussion.

### 5.2.11 Numerical Example

**Description of the Network and Choice of Parameters**

Let us consider a 3 arc 3 node electric power network where a regional market is located at each nodes and there are 3 firms involved in the oligopolistic competition. The power network is illustrated in Figure 5.3.

We assume that each firm has 2 generation units at each of the nodes with different capacities and ramping rates. Therefore, each firm has a total of 6 generation units which are geographically separated. We consider inverse demand parameters (namely price and
Figure 5.3: 3 node 3 arc power network with 3 firms each having 6 generating units

quantity intercept) to be time varying with their nominal components being

<table>
<thead>
<tr>
<th>Market, i</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{0,i}$</td>
<td>40</td>
<td>35</td>
<td>32</td>
</tr>
<tr>
<td>$Q_{0,i}$</td>
<td>5000</td>
<td>4000</td>
<td>6200</td>
</tr>
</tbody>
</table>

and $P_{0,i}(t)$ and $Q_{0,j}(t)$ are estimated from the load profiles of 5.1 for all $t \in [t_0, t_1]$. PDF values associated with the network are as follows

<table>
<thead>
<tr>
<th>Arc</th>
<th>(1, 2)</th>
<th>(1, 3)</th>
<th>(2, 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node 1</td>
<td>0.33</td>
<td>0.67</td>
<td>0.33</td>
</tr>
<tr>
<td>Node 2</td>
<td>-0.33</td>
<td>0.33</td>
<td>0.67</td>
</tr>
<tr>
<td>Node 3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Generation capacities, $CAP^f_j$ (in MWatt) of different generation units are shown below

<table>
<thead>
<tr>
<th>Firm 1</th>
<th>Unit 1</th>
<th>Unit 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node 1</td>
<td>1000</td>
<td>500</td>
</tr>
<tr>
<td>Node 2</td>
<td>750</td>
<td>500</td>
</tr>
<tr>
<td>Node 3</td>
<td>800</td>
<td>400</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Firm 2</th>
<th>Unit 1</th>
<th>Unit 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node 1</td>
<td>750</td>
<td>500</td>
</tr>
<tr>
<td>Node 2</td>
<td>500</td>
<td>600</td>
</tr>
<tr>
<td>Node 3</td>
<td>400</td>
<td>500</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Firm 3</th>
<th>Unit 1</th>
<th>Unit 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node 1</td>
<td>600</td>
<td>500</td>
</tr>
<tr>
<td>Node 2</td>
<td>1000</td>
<td>400</td>
</tr>
<tr>
<td>Node 3</td>
<td>1200</td>
<td>400</td>
</tr>
</tbody>
</table>
It is evident from the above tables that each firm has a combination of high capacity and low capacity generators. The ramping rates, $r_j^f$, are bounded from above and below. We assume these bounds are symmetric

$$R_j^{f+} = -R_j^{f-}$$

for all $f \in \mathcal{F}$, $i \in N_f$, and $j \in \mathcal{G}(i, f)$. The upper bounds on ramping rates for the generation units are shown below

<table>
<thead>
<tr>
<th>Firm</th>
<th>Unit 1</th>
<th>Unit 2</th>
<th>Firm 2</th>
<th>Unit 1</th>
<th>Unit 2</th>
<th>Firm 3</th>
<th>Unit 1</th>
<th>Unit 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node 1</td>
<td>58</td>
<td>336</td>
<td>Node 1</td>
<td>84</td>
<td>330</td>
<td>Node 1</td>
<td>290</td>
<td>340</td>
</tr>
<tr>
<td>Node 2</td>
<td>84</td>
<td>340</td>
<td>Node 2</td>
<td>336</td>
<td>290</td>
<td>Node 2</td>
<td>58</td>
<td>380</td>
</tr>
<tr>
<td>Node 3</td>
<td>70</td>
<td>400</td>
<td>Node 3</td>
<td>400</td>
<td>340</td>
<td>Node 3</td>
<td>35</td>
<td>365</td>
</tr>
</tbody>
</table>

If we compare ramping rate bounds and generation capacities of the units, it will be evident that units having higher capacity typically have slower ramping capability and vice versa. Also, ramping costs ($/\text{MWatt-hour}$) associated with the faster ramping machines are higher compared to their slower counterparts. We tabulate unit ramping cost $\gamma_j^f$ parameters below

<table>
<thead>
<tr>
<th>Firm</th>
<th>Unit 1</th>
<th>Unit 2</th>
<th>Firm 2</th>
<th>Unit 1</th>
<th>Unit 2</th>
<th>Firm 3</th>
<th>Unit 1</th>
<th>Unit 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node 1</td>
<td>2.14</td>
<td>6.82</td>
<td>Node 1</td>
<td>4.50</td>
<td>6.72</td>
<td>Node 1</td>
<td>6.00</td>
<td>6.90</td>
</tr>
<tr>
<td>Node 2</td>
<td>4.50</td>
<td>6.75</td>
<td>Node 2</td>
<td>6.75</td>
<td>5.93</td>
<td>Node 2</td>
<td>2.20</td>
<td>8.60</td>
</tr>
<tr>
<td>Node 3</td>
<td>5.50</td>
<td>8.70</td>
<td>Node 3</td>
<td>8.74</td>
<td>6.80</td>
<td>Node 3</td>
<td>1.54</td>
<td>8.65</td>
</tr>
</tbody>
</table>

Elastic limits for the generators, $\xi_j^f$, are usually not very dependent on the capacities of the generators, which is also evident from below where we list values of $\xi_j^f$ for the generators (in MWatt)

<table>
<thead>
<tr>
<th>Firm</th>
<th>Unit 1</th>
<th>Unit 2</th>
<th>Firm 2</th>
<th>Unit 1</th>
<th>Unit 2</th>
<th>Firm 3</th>
<th>Unit 1</th>
<th>Unit 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node 1</td>
<td>65</td>
<td>57</td>
<td>Node 1</td>
<td>60</td>
<td>55</td>
<td>Node 1</td>
<td>55</td>
<td>53</td>
</tr>
<tr>
<td>Node 2</td>
<td>60</td>
<td>54</td>
<td>Node 2</td>
<td>61</td>
<td>59</td>
<td>Node 2</td>
<td>65</td>
<td>51</td>
</tr>
<tr>
<td>Node 3</td>
<td>62</td>
<td>52</td>
<td>Node 3</td>
<td>51</td>
<td>55</td>
<td>Node 3</td>
<td>67</td>
<td>52</td>
</tr>
</tbody>
</table>
The regional sales capacities in each of the 3 markets are assumed to be

<table>
<thead>
<tr>
<th>Market, $i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market CAP, $\sigma_i$ (MWatt)</td>
<td>3000</td>
<td>3200</td>
<td>2900</td>
</tr>
</tbody>
</table>

Coefficients associated with the linear component of generation costs of the units, $\mu_j^f$ ($$/\text{MWatt})$, are assumed to be

<table>
<thead>
<tr>
<th>Firm 1</th>
<th>Unit 1</th>
<th>Unit 2</th>
<th>Firm 2</th>
<th>Unit 1</th>
<th>Unit 2</th>
<th>Firm 3</th>
<th>Unit 1</th>
<th>Unit 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node 1</td>
<td>15</td>
<td>15</td>
<td>Node 1</td>
<td>15.2</td>
<td>14.7</td>
<td>Node 1</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>Node 2</td>
<td>14.5</td>
<td>15</td>
<td>Node 2</td>
<td>15.1</td>
<td>14.9</td>
<td>Node 2</td>
<td>14.8</td>
<td>14.8</td>
</tr>
<tr>
<td>Node 3</td>
<td>14.7</td>
<td>15.2</td>
<td>Node 3</td>
<td>15</td>
<td>15.1</td>
<td>Node 3</td>
<td>15.3</td>
<td>15</td>
</tr>
</tbody>
</table>

We typically assume all coefficients associated with the quadratic component of generation costs of units belonging to a firm to be the same with

\[
\hat{\mu}_j^1 = 0.08 \quad \text{for all } i \in \mathcal{N}_1, j \in \mathcal{G} (i, 1) \\
\hat{\mu}_j^2 = 0.07 \quad \text{for all } i \in \mathcal{N}_2, j \in \mathcal{G} (i, 2) \\
\hat{\mu}_j^3 = 0.075 \quad \text{for all } i \in \mathcal{N}_3, j \in \mathcal{G} (i, 3)
\]

Transmission capacities of the arcs are assumed to be the following

<table>
<thead>
<tr>
<th>Arc, $a$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transmission Capacity, $T_a$ (MWatt)</td>
<td>130</td>
<td>150</td>
<td>160</td>
</tr>
</tbody>
</table>

Our planning horizon in this example is 24 hours with $t_0 = 0$ and $t_1 = 24$. The initial generation rates at $t_0 = 0$ are

\[
q_{j,0}^1 = 150 \quad \text{for all } i \in \mathcal{N}_1, j \in \mathcal{G} (i, 1) \\
q_{j,0}^2 = 175 \quad \text{for all } i \in \mathcal{N}_2, j \in \mathcal{G} (i, 2) \\
q_{j,0}^3 = 160 \quad \text{for all } i \in \mathcal{N}_3, j \in \mathcal{G} (i, 3)
\]
which implies that at the beginning all the generators for a firm were operating at the same level. This choice is intentional, as we want to study the impact of ramping rates on the generators.

**Sales, Ramping Rates, Generation Rates and Market Prices**

**Performance of the algorithm** We forgo the detailed symbolic statement of this example and, instead, provide numerical results in graphical form for the solution which was obtained after 399 fixed point iterations. We choose the convergence parameter

\[ \alpha_{fp} = \frac{1}{k} \]

where \( k \) is the iteration counter, and pre-set tolerance

\[ \epsilon = 0.5 \]

which are the parameters for the fixed point algorithm (Mookherjee et al [25]). In Figure 5.4 the relative change from one iteration to the next, expressed as

\[ \Delta_k = \left\| \begin{pmatrix} \tilde{z}^{k+1} \\ r^{k+1} \\ \alpha^{k+1} \end{pmatrix} - \begin{pmatrix} \tilde{z}^k \\ r^k \\ \alpha^k \end{pmatrix} \right\| \]

is plotted against the iteration counter \( k \). It is worth noting that for this particular example even though \( \Delta_1 = 5816.9 \), in the next several iterations \( \Delta_k \) decreases very rapidly. The run time for this example is less than 10 minutes using a generic desktop computer with single a Intel Pentium 4 processor and 1 GB RAM. The computer code for the fixed point algorithm is written in MatLab 6.5 and calls a gradient projection subroutine for which the control, state and adjoint variables are determined in the sequential fashion explained in Section 5.2.10.
Figure 5.4: Performance of fixed point iteration: (left) plot of $\Delta_k$ vs. $k$ (semi-log scale); (right) zoomed in view of the same, red line: pre-set tolerance, $\epsilon$

Ramping rate trajectories

In Figures 5.5, 5.6 and 5.7 we plot ramping rate trajectories of different power units owned by firm 1, 2 and 3 respectively. As a general trend, we observe that the gas turbines are ramped up and down at the full capacity (ramping capacity constraints are binding at those points) to meet up the market demand in a short notice.

Power generation rates

Figures 5.8, 5.9 and 5.10 plot corresponding generation rate trajectories of different power units owned by firm 1, 2 and 3 respectively. As expected, our simulated result demonstrates that in equilibrium coal units are operated at least 50% of their respective capacities for throughout the day. However, the expensive gas turbines (marked as unit 2 in the figures) are only used only during peak load periods.
Figure 5.5: The equilibrium ramping strategies for firm 1 (grouped by the nodes)
Figure 5.6: The equilibrium ramping strategies for firm 2 (grouped by the nodes)
Figure 5.7: The equilibrium ramping strategies for firm 3 (grouped by the nodes)
Figure 5.8: Generation rates of firm 1 over time (grouped by the location and generator type)
Figure 5.9: Generation rates of firm 2 over time (grouped by the location and generator type)
Figure 5.10: Generation rates of firm 3 over time (grouped by the location and generator type)
Figure 5.11: Total regional sales over time at 3 different markets (regions)

Regional sales

We plot the total regional sales by all 3 firms at every regions are plotted over time in Figure 5.11. We observe that the joint sales capacities constraints were binding in market 2 and 3 right after 8 pm.

Market price

Market price of electricity (expressed in $/MWatt) at 3 different regions (markets) are plotted over time in Figure 5.12.
Figure 5.12: Spot market price (at equilibrium) of power at 3 different markets (regions) at different times of a day
5.2.12 Concluding Remarks

We present a dynamic Cournot-Nash model of competition among the power generators on a transmission network that includes two new features in each generator’s profit maximization optimal control problem: (i) a set of joint constraints arising from the joint regional sales cap; (ii) explicit consideration of generators’ ramping costs. These extensions have important economic and operational applications. The joint sales cap constraints induce additional analytical and computational challenges. To overcome those, we provide a restricted formulation involving partitioned differential variational inequalities. We have shown the equivalence relationship between solutions of this DVI formulation with the Cournot-Nash equilibrium of the model, and shown that under very mild regularity conditions there exists at least one equilibrium. We also discuss a version of implicit fixed point algorithm which may be employed to compute our model.

5.3 Impacts of Infrastructure Disruptions on Electric Power Network Equilibrium

5.3.1 Introduction and Motivation

Once again we consider an electricity power market consisting of electric power generating firms and an independent service operator (ISO) on an underlying power system network. The competing generating firms wish to allocate power generated at potentially multiple locations to different markets in order to maximize their profits. The ISO’s job is to efficiently clear the markets for power by setting wheeling fees for the transmission of power along the arcs of the network. To keep the exposition simple and facilitate the use of off-the-shelf complementarity problem solvers (PATH), here we formulate the extremal problem of every generator as well as the ISO as a discrete time mathematical program which relies on the actions of their competitors. The collection of these coupled discrete time mathematical
programs describe the dynamic Cournot-Nash game that we are interested in. This game is represented by a nonlinear complementarity problem (NCP) through the analysis of the necessary conditions for each problem. Representing the problem as an NCP allows us to make direct use of powerful commercial solvers such as PATH for efficient computation, in turn making it useful to the research of big events in electric power systems.

We think of big (or extreme) events as those events with a very large deviation from an expected event. These events typically have a negative effect on system performance and may arise due to belligerent agents or system failures. Designing and planning for extreme events is not a trivial matter. A clear tradeoff exists between the costs of designing and planning for an extreme event and the costs of managing the event if it occurs. We have witnessed such extreme events in recent years. Some examples include the 2003 electricity blackout in the northeast United States and southeast Canada and hurricane Katrina in 2005 which greatly affected multiple states in the southern United States. Another older extreme event of interest is the 1977 blackout in New York City which was the result of a lightning strike. This brings us to another point being that a localized extreme event may have a global impact in a network setting. It is our belief that the model proposed in this paper may be used to understand how local disruptions in an electric power system effect the flows of power throughout the network over time as well as the economic impacts of such disruptions on the market equilibrium. We are also interested to study the rate of return of the system to the equilibrium once the disruption is resolved and normalcy is restored. We take a simulation approach, as described in section 5.3.3, to study such effects.

The model put forth in this paper is deterministic, however a simulation type experimentation method is proposed for generating perturbations to the parameters to test the effects of extreme events. Although not stressed in this paper, the proposed simulation method makes use of a Gumbel distribution as described in Montroll and Badger [86]. The model may also be used directly to test different scenarios including failures of transmission lines and power generating facilities. Because the model is dynamic in nature, it is possible
to see how the system reacts to a failure as well as how it returns to equilibrium as a failed component is brought back into service. This is also of great interest as it allows the modeler to directly model any scenario they intuit.

Of particular interest when modeling electric power markets is the availability of data on market demands, electric power technology and generation costs for electric power systems that need not be approximated or derived as in other applications; in many cases independent system operators (ISO) publish daily load profiles, market prices and other information on a daily basis through their website. This paper makes use of such information for a numerical example that is modeled around an actual power system in northwest Europe formed by Belgium, France, Germany and the Netherlands. This network includes 15 nodes and 12 generating firms. Of the 12 generating firms, eight are distinct firms, while four represent conglomerations of remaining firms in each country. In order to allow a linearized DC load flow representation of the network, eight of the nodes experience no demand or generation. The data available for this network includes the PTDF values for each arc, transmission line capacities and generation costs. To the best of our knowledge, this is the first dynamic electric power equilibrium model to use this data set.

5.3.2 Electric Power Model

The generators’ extremal problem we consider here runs very closely with the one described in section 5.2. Unlike section 5.2, we do not consider the market level joint sales cap constraints arising from regulations imposed exogenously. As mentioned previously, each firm’s extremal problem is represented as a discrete time mathematical program with linear constraints. Each firm’s extremal problem is dependent upon the actions of their competitors and results in a set of coupled discrete time mathematical programs that define the game. The necessary conditions for this set of problems are analyzed and used to formulate a nonlinear complementarity problem (NCP.) The NCP can be solved directly using a commercial solver or a sequential linearization scheme can be employed where a Lemke’s type
algorithm is used to solve each resulting linear complementarity subproblem.

5.3.3 A Simulation Approach to Big Events

We envision a simulation type approach to modeling big events using the electric power model described in this paper. The model described herein may be envisioned as a day ahead model that is run by generating firms for planning and bidding purposes. The generating firms may solve this model based on their perception of the network and demands. The resulting solution will dictate their commitments to the ISO to provide energy and for what price the next day. However, as the plan is implemented the following day, the ISO is able to immediately realize variations in the system caused by various externalities or big events and will adjust the wheeling fees charged to the generating firms to transmit their power across the network. Therefore, the generating firms will experience a different profit than was predicted by the day ahead model.

A simulation approach could thus be used to repeatedly run the model and compare the day ahead predicted profits with the realized profits. Each simulation run will vary the parameters of the system according to some distribution. In particular, to model big events a Gumbel distribution may be used to vary the parameters of the model as is described by Montroll and Badger [86].

5.3.4 Generating Firm’s Problem

We are now in a position to discuss the problem faced by each generating firm. Each firm’s extremal problem is formulated as a discrete time mathematical program with an objective of maximizing the profit of the firm. The strategy of each firm is dependent upon the actions of its competitors, thus we adopt the notation for variables controlled by another firm, specifically the allocation to consumption, as \( c^{-f} \) where

\[
c^{-f} : c^g \text{ for all } g \in \mathcal{F} \text{ such that } g \neq f
\]
With the preceding information, we may introduce the single generating firm’s extremal problem as

$$
\max J_1(c^f, q^f; r^f, c^- f, w) = \sum_{t=0}^{N} \sum_{i \in \mathcal{M}} \left\{ \pi_{i,t} \left( \sum_{g \in \mathcal{F}} c^g_{i,t} \right) \cdot c^f_{i,t} - V^f_{i,t} (q^f_{i,t}) \right\} - w_{i,t} \cdot \left( c^f_{i,t} - q^f_{i,t} \right)
$$

subject to

$$
\sum_{i \in \mathcal{M}} q^f_{i,t} = \sum_{i \in \mathcal{M}} c^f_{i,t} \text{ for all } t = 0 \ldots N \tag{5.48}
$$

$$
\pi_{i,t} = a_{i,t} - b_{i,t} \sum_{g \in \mathcal{F}} c^g_{i,t} \text{ for all } i \in \mathcal{M}, t = 0 \ldots N \tag{5.49}
$$

$$
V^f_{i,t} = \max \left( m^1_{1,i} q^f_{i,t} + b^1_{1,i}, m^2_{2,i} q^f_{i,t} + b^2_{2,i} \right) \text{ for all } i \in \mathcal{M}, t = 0 \ldots N \tag{5.50}
$$

$$
r^f_{i,t} = q^f_{i,t-1} - q^f_{i,t} \text{ for all } i \in \mathcal{M}, t = 1 \ldots N \tag{5.51}
$$

$$
0 \leq q^f_{i,t} \leq q^f_{i,\text{max},t} \text{ for all } i \in \mathcal{M}, t = 0 \ldots N \tag{5.52}
$$

$$
r^f_{i,\text{min}} \leq r^f_{i,t} \leq r^f_{i,\text{max}} \text{ for all } i \in \mathcal{M}, t = 1 \ldots N \tag{5.53}
$$

The objective (5.47) is simply a profit maximization that is the difference between the revenue generated at all nodes minus the costs of generation and the wheeling fee. The wheeling fee is the cost (or profit) of transmitting power outside of the generating node and is dictated by the independent service operator’s (ISO) model. Equation (5.48) is a flow balance equation which states that all power produced must be allocated to consumption in each period as we do not consider the storage of electricity. Equation (5.49) is the inverse demand, or market price, for electricity which is dependent upon the allocation to consumption of all firms at that market with $a_i, b_i \in \mathbb{R}^{1+}_+$ for all $i \in \mathcal{M}$. Equation (5.50) is a two piece linear generation cost function with $m^1_{1,i}, m^2_{2,i}, b^1_{1,i}, b^2_{2,i} \in \mathbb{R}^{1+}$ for all $f \in \mathcal{F}$ and $i \in \mathcal{M}$. Equation (5.51) are discrete dynamics which define the ramping rates for the generators, while (5.53) represents the bounds on the ramping rate of a generator between two time periods with $-r^f_{i,\text{min}} \in \mathbb{R}^{1+}, r^f_{i,\text{max}} \in \mathbb{R}^{1+}$ for all $f \in \mathcal{F}$ and $i \in \mathcal{M}$. Finally, constraint (5.52) gives the bounds on generation capacity with $q^f_{i,\text{max},t} \in \mathbb{R}^{1+}$ for all $f \in \mathcal{F}$,
\( i \in \mathcal{M} \) and \( t = 0...N \). Note that the parameter for the generation capacity may change with time. This allows us to test the effects of a reduction in generation capacity for some number of time periods.

Because we will be analyzing the necessary conditions for each firm’s discrete time mathematical program in order to formulate the game as an NCP, we need to have differentiability of all constraints. Therefore, equation (5.50) requires some special attention. Instead of using the form expressed above as

\[
V_{i,t}^f = \max \left( m_{1,i}^f q_{i,t}^f + b_{1,i}^f, m_{2,i}^f q_{i,t}^f + b_{2,i}^f \right)
\]

we can instead regard \( V_{i,t}^f \) as a variable and construct the following inequalities.

\[
\begin{align*}
V_{i,t}^f & \geq m_{1,i}^f q_{i,t}^f + b_{1,i}^f \\
V_{i,t}^f & \geq m_{2,i}^f q_{i,t}^f + b_{2,i}^f
\end{align*}
\]

Because the problem is to maximize profit and these two equations will always have a non-negative value with a negative coefficient in the objective, the cost will always lie on the lower envelope formed by the original equation as increasing above that would further reduce the the value of the objective (5.47).

The resulting discrete time mathematical program after these changes is:

\[
\max J_1 \left( c^f, q^f, V^f, r^f; c^{-f}, w \right) = \sum_{t=0}^{N} \sum_{i \in \mathcal{M}} \left\{ \pi_{i,t} \cdot c_{i,t}^f - V_{i,t}^f - w_{i,t} \cdot \left( c_{i,t}^f - q_{i,t}^f \right) \right\}
\]
subject to

\[
\sum_{i \in \mathcal{M}} q_{i,t}^f = \sum_{i \in \mathcal{M}} c_{i,t}^j \quad \text{for all } t = 0 \ldots N \\
\pi_{i,t} = a_{i,t} - b_{i,t} \sum_{y \in \mathcal{F}} c_{i,t}^y \quad \text{for all } i \in \mathcal{M}, \quad t = 0 \ldots N \\
V_{i,t}^f \geq m_{1,i}^f q_{i,t}^f + b_{1,i}^f \quad \text{for all } i \in \mathcal{M}, \quad t = 0 \ldots N \\
V_{i,t}^f \geq m_{2,i}^f q_{i,t}^f + b_{2,i}^f \quad \text{for all } i \in \mathcal{M}, \quad t = 0 \ldots N \\
r_{i,t}^f = q_{i,t}^f - q_{i,t+1}^f \quad \text{for all } i \in \mathcal{M}, \quad t = 1 \ldots N \\
0 \leq q_{i,t}^f \leq q_{i,t}^{f,\max} \quad \text{for all } i \in \mathcal{M}, \quad t = 0 \ldots N \\
r_{i,t}^{f,\min} \leq r_{i,t}^f \leq r_{i,t}^{f,\max} \quad \text{for all } i \in \mathcal{M}, \quad t = 1 \ldots N
\]

5.3.5 ISO’s problem

It was mentioned earlier that the wheeling fees, \( w \) are set by the ISO in order to efficiently clear the market for transmission capacity. As we have seen, these wheeling fees are taken exogenous to the generating firms’ extremal problems and impacts their revenues. These wheeling fees are determined by the ISO for each time period according to the solution of the following linear program. The ISO wishes to determine the transmission flows \( y \) in order to

\[
\max J_2 (t) = \sum_{i \in \mathcal{N}} y_{i,t} \cdot w_{i,t} \quad (5.54)
\]

subject to

\[
\sum_{i \in \mathcal{N}} PDF_{i,a} \cdot y_{i,t} \leq T_{a,t} \quad \text{for all } a \in \mathcal{A} \quad (5.55)
\]

where \( \mathcal{A} \) is the arc set of the electric power network, \( T_{a,t} \) is the transmission capacity on arc \( a \in \mathcal{A} \) at time \( t \) and \( PDF_{i,a} \) is the power transmission distribution factor (PTDF) that describes how much MW flow occurs through arc \( a \) as a result of a unit MW injection at an arbitrary hub node and a unit withdrawal at node \( i \). It is important to note that the parameters \( T_{a,t} \) representing the transmission capacities of the arcs may change with time; this makes it possible to create scenarios where a transmission line is disabled for some
number of time periods in order to observe the effects on the equilibrium. The decision variables $y_{i,t}$ denote transfers of power in MWatt by the ISO from a hub node to the node $i \in \mathcal{N}$ at time period $t$.

To clear the market, the transmission flows $y_{i,t}$ must balance the net sales at each node. Therefore the net transmission into a market is given by

$$y_{i,t} = \sum_{f \in \mathcal{F}} \left( c_{i,t}^f - q_{i,t}^f \right) \text{ for all } i \in \mathcal{M} \quad (5.56)$$

Re-writing (5.54) - (5.55) we obtain the following set of mathematical programs, one for each time period of interest.

$$\max J_{2,t} = \sum_{i \in \mathcal{N}} \sum_{f \in \mathcal{F}} \left( c_{i,t}^f - q_{i,t}^f \right) \cdot w_{i,t} \quad (5.57)$$

subject to

$$\sum_{i \in \mathcal{N}} \sum_{f \in \mathcal{F}} \left( c_{i,t}^f - q_{i,t}^f \right) \leq T_a \text{ for all } a \in \mathcal{A} \quad (5.58)$$

In this particular formulation we ignore transmission loss, however, our model is general enough to consider non-linear losses. In the case of losses, either the ISO or the firms involved in the transaction should account for the losses and a book-keeping effort is required.

5.3.6 Formulation of NCP

We will form the nonlinear complementarity problem by analyzing the necessary conditions for both the generating firms’ problems as well as the ISO’s problems. Because we have only linear constraints, Abadie’s constraint qualification holds and we can inspect the Karush Kuhn Tucker (KKT) conditions. We will first inspect the necessary conditions for the generating firms and then the ISO. These conditions are then combined into the complete NCP representation.
Complementarity Conditions for Generating Firms

The mathematical program (in standard form) that we wish to examine is

\[
\min J_1 \left( c^f, q^f, V^f; c^{-f}, w \right) = -\sum_{t=0}^N \sum_{i \in \mathcal{M}} \left\{ \left( a_i - b_i \sum_{g \in \mathcal{G}} c^g_{i,t} \right) \cdot c^f_{i,t} - V^f_{i,t} \right. \\
-w_{i,t} \cdot \left( c^f_{i,t} - q^f_{i,t} \right) \right\} 
\]

subject to

\[
\sum_{i \in \mathcal{M}} q^f_{i,t} - \sum_{i \in \mathcal{M}} c^f_{i,t} \leq 0 \text{ for all } t = 0 \ldots N \quad (\zeta^+_{t}) \quad (5.60)
\]
\[
-\sum_{i \in \mathcal{M}} q^f_{i,t} + \sum_{i \in \mathcal{M}} c^f_{i,t} \leq 0 \text{ for all } t = 0 \ldots N \quad (\zeta^-_{t}) \quad (5.61)
\]
\[
-V^f_{i,t} + m^f_{1,i} q^f_{i,t} + b^f_{1,i} \leq 0 \text{ for all } i \in \mathcal{M}, t = 0 \ldots N \quad (\gamma^f_{i,t}) \quad (5.62)
\]
\[
-V^f_{i,t} + m^f_{2,i} q^f_{i,t} + b^f_{2,i} \leq 0 \text{ for all } i \in \mathcal{M}, t = 0 \ldots N \quad (\eta^f_{i,t}) \quad (5.63)
\]
\[
-c^f_{i,t} \leq 0 \text{ for all } i \in \mathcal{M}, t = 0 \ldots N \quad (\phi^f_{i,t}) \quad (5.64)
\]
\[
-V^f_{i,t} \leq 0 \text{ for all } i \in \mathcal{M}, t = 0 \ldots N \quad (\delta^f_{i,t}) \quad (5.65)
\]
\[
-q^f_{i,t} \leq 0 \text{ for all } i \in \mathcal{M}, t = 0 \ldots N \quad (\rho^f_{i,t}) \quad (5.66)
\]
\[
q^f_{i,t} - q^f_{i,\text{max}} \leq 0 \text{ for all } i \in \mathcal{M}, t = 0 \ldots N \quad (\sigma^f_{i,t}) \quad (5.67)
\]
\[
-q^f_{i,t} \leq q^f_{i,t-1} + r^f_{i,t-1} \text{ for all } i \in \mathcal{M}, t = 0 \ldots N \quad (\mu^f_{i,t}) \quad (5.68)
\]
\[
q^f_{i,t} - q^f_{i,t-1} - r^f_{i,\text{max}} \leq 0 \text{ for all } i \in \mathcal{M}, t = 0 \ldots N \quad (\theta^f_{i,t}) \quad (5.69)
\]

where the variables in parenthesis are the dual variables associated with each inequality.

Note that equation (5.48) is now being represented by two inequalities. Also note that the ramping rates \( r^f_{i,t} \) have been replaced by their representation in terms of generation \( q^f_{i,t} \).

As stated previously, Abadie’s constraint qualification holds, and thus, we may inspect the KKT conditions of the above mathematical program (5.59) - (5.69). The KKT identities
can be found to be

\[ 0 = 2b_{i,t}c^f_{i,t} - a_{i,t} + b_{i,t} \sum_{g \in F, g \neq f} c^g_{i,t} - w_{i,t} - \zeta^+ f - \zeta^- f - \phi^f_{i,t} \]  
\[ 0 = w_{i,t} + \zeta^+ f - \zeta^- f + m^f_{1,i}\gamma^f_{i,t} + m^f_{2,i}\eta^f_{i,t} - \rho^f_{i,t} + \sigma^f_{i,t} - \mu^f_{i,t} + \theta^f_{i,t} \]  
\[ 0 = 1 - \gamma^f_{i,t} - \eta^f_{i,t} - \delta^f_{i,t} \] (5.70)

(5.71)

(5.72)

with the following accompanying complementarity conditions:

\[ 0 \leq \left[ - \sum_{i \in M} q^f_{i,t} + \sum_{i \in M} c^f_{i,t} \right] \perp \zeta^+_t \geq 0 \] (5.73)

\[ 0 \leq \left[ \sum_{i \in M} q^f_{i,t} - \sum_{i \in M} c^f_{i,t} \right] \perp \zeta^-_t \geq 0 \] (5.74)

\[ 0 \leq \left[ V^f_{i,t} - m^f_{1,i} \left( \sum_{i \in M} c^f_{i,t} \right) - b^f_{1,i} \right] \perp \gamma^f_{i,t} \geq 0 \] (5.75)

\[ 0 \leq \left[ V^f_{i,t} - m^f_{2,i} \left( \sum_{i \in M} c^f_{i,t} \right) - b^f_{2,i} \right] \perp \eta^f_{i,t} \geq 0 \] (5.76)

\[ 0 \leq c^f_{i,t} \perp \phi^f_{i,t} \geq 0 \] (5.77)

\[ 0 \leq q^f_{i,t} \perp \rho^f_{i,t} \geq 0 \] (5.78)

\[ 0 \leq \left[ -q^f_{i,t} + q^f_{i,t,max} \right] \perp \sigma^f_{i,t} \geq 0 \] (5.79)

\[ 0 \leq \left[ q^f_{i,t} - q^f_{i,t-1} - r^f_{i,min} \right] \perp \mu^f_{i,t} \geq 0 \] (5.80)

\[ 0 \leq \left[ -q^f_{i,t} + q^f_{i,t-1} + r^f_{i,max} \right] \perp \theta^f_{i,t} \geq 0 \] (5.81)

We can create another set of complementarity conditions making use of the KKT identities (5.70) - (5.72).

\[ 0 \leq c^f_{i,t} \perp \left[ 2b_{i,t}c^f_{i,t} - a_{i,t} + b_{i,t} \sum_{g \in F, g \neq f} c^g_{i,t} - w_{i,t} - \zeta^+ f - \zeta^- f \right] \phi^f_{i,t} \geq 0 \] (5.82)

\[ 0 \leq q^f_{i,t} \perp \left[ w_{i,t} + \zeta^+ f - \zeta^- f + m^f_{1,i}\gamma^f_{i,t} + m^f_{2,i}\eta^f_{i,t} + \rho^f_{i,t} + \sigma^f_{i,t} - \mu^f_{i,t} + \theta^f_{i,t} \right] \rho^f_{i,t} \geq 0 \] (5.83)

\[ 0 \leq V^f_{i,t} \perp \left[ 1 - \gamma^f_{i,t} - \eta^f_{i,t} \right] \delta^f_{i,t} \geq 0 \] (5.84)
We may now state the NCP for the generating firms’ problems using the complementarity conditions from (5.73) - (5.84) as

\[
\begin{bmatrix}
2b_{i,t}c_{i,t}^f - a_{i,t} + b_{i,t} \sum_{g \in \mathcal{F}, g \neq f} c_{g,i,t}^g - w_{i,t} - \zeta_t^f + \zeta_t^{-f} \\
w_{i,t} + \zeta_t^f - \zeta_t^{-f} + \gamma_{i,t} f_{i,t} + \delta_{i,t} f_{i,t} + \sigma_{i,t}^f - \mu_{i,t}^f + \theta_{i,t}^f \\
1 - \gamma_t^f - \eta_t^{f} \\
- \sum_{i \in \mathcal{M}} q_i^f + \sum_{i \in \mathcal{M}} c_{i,t}^f \\
\sum_{i \in \mathcal{M}} q_i^f - \sum_{i \in \mathcal{M}} c_{i,t}^f \\
V_{i,t}^f - m_{1,i}^f \left(q_i^f - b_{1,i}^f \right) \\
V_{i,t}^f - m_{2,i}^f \left(q_i^f - b_{2,i}^f \right) \\
- q_{i,t}^f + q_{i,max}^f \\
q_{i,t}^f - q_{i,t-1}^f - r_{i,min}^f \\
- q_{i,t}^f + q_{i,t-1}^f + r_{i,max}^f \\
\end{bmatrix} = F_1(z) \perp z = \begin{bmatrix}
\nu_{i,t}^f \\
\phi_{i,t}^f \\
\psi_{i,t}^f \\
\delta_{i,t}^f \\
\mu_{i,t}^f \\
\sigma_{i,t}^f \\
\theta_{i,t}^f \\
\end{bmatrix}
\tag{5.85}
\]

5.3.7 Complementarity Conditions for the ISO

Returning to the ISO’s problem of interest, we have the mathematical program below.

\[
\max J_{ISO,t} = \sum_{i \in \mathcal{N}} \sum_{f \in \mathcal{F}} y_{i,t}^f \cdot w_{i,t}
\]

subject to

\[
\sum_{i \in \mathcal{N}} PDF_{i,a} \cdot \sum_{f \in \mathcal{F}} y_{i,t}^f \leq T_{a,t} \text{ for all } a \in \mathcal{A} \quad (\alpha_{a,t})
\]

We may again inspect the KKT conditions for this mathematical program as the constraints are linear and Abadie’s constraint qualification holds. The KKT identity for this problem is

\[
-w_{i,t} + \sum_{a \in \mathcal{A}} PDF_{i,a} \cdot \alpha_{a,t} = 0
\]
Making use of (5.56), we obtain to the following conditions

\[ w_{i,t} = \sum_{a \in A} PDF_{i,a} \cdot \alpha_{a,t} \quad \text{for all } i \in \mathcal{M} \]  

\[ 0 \leq \alpha_{a,t} \perp T_{a,t} - \sum_{i \in \mathcal{N}} PDF_{i,a} \cdot \sum_{f \in \mathcal{F}} (c_{i,t}^f - q_{j,t}^f) \geq 0 \quad \text{for all } a \in A \]  

where (5.86) gives the wheeling fee and (5.87) gives the complementary conditions for the ISO. The nonlinear complementarity problem for the ISO is thus

\[ \left[ T_{a,t} - \sum_{i \in \mathcal{N}} PDF_{i,a} \cdot \sum_{f \in \mathcal{F}} (c_{i,t}^f - q_{j,t}^f) \right] = F_2 (\alpha) \perp \alpha \geq 0 \]  

**Complete NCP Formulation**

The overall NCP formulation is completed by concatenating the complementarity conditions for the generating firms (5.85) with those from the ISO (5.88).

\[ \begin{bmatrix} F_1 (z) \\ F_2 (\alpha) \end{bmatrix} = G (y) \perp y = \begin{bmatrix} z \\ \alpha \end{bmatrix} \geq 0 \]

This NCP formulation can be represented and efficiently solved in a commercial software package such as GAMS utilizing the PATH solver. An algorithm may also be devised to solve this NCP by sequentially linearizing while solving each resulting LCP using a Lemke’s type algorithm; note that this type of algorithm is built into PATH and is easily implemented.

**5.3.8 Numerical Example**

For our numerical examples, we want to test two different scenarios. The first examples involve the removal of an arc from the network and the second involves the removal of a generator. In both cases, we compare the social welfare between the full capacity network and the reduced capacity network. Because we are using PTDFs, we can simulate the removal of an arc by reducing the capacity of that arc to some very small number. By doing this, we do not need to recalculate the PTDFs for all arcs as would be the case if
we completely removed the arc from the network. The model was solved with all arcs at full capacity, and then an arc was removed and the model was rerun to see what the effects were. The social welfare is calculated as

$$SW = \sum_{t=0}^{\mathcal{N}} \sum_{i \in \mathcal{M}} CS_{i,t} - \sum_{f \in \mathcal{F}} V_{i,t}^{f*}$$

where $CS$ is the consumer’s surplus, represented by

$$CS_{i,t} = \int_0^{\Sigma_{g \in \mathcal{F}} c_{i,t}^{g*} a_{i,t} - b_{i,t}x} dx$$

and $V_{i,t}^{f*}$ and $c_{i,t}^{g*}$ are equilibrium values. Note that the amount paid by the consumers and the wheeling fees are not included in the calculations as they are transfers of money and cancel out.

We are also interested in how efficiently this model can be solved. As such, we have created a numerical example based on the northwest European electricity market formed by Belgium, France, Germany and the Netherlands. This network is comprised of 15 nodes, 28 flowgates (transmission lines) and 12 generating firms. As mentioned previously, eight of the generating firms are distinct and supply the largest fraction of power, while the other four represent conglomerations of the remaining generating firms in each country. For the purposes of the numerical example, the set of firms $\mathcal{F} = \{1, \ldots, 12\}$, the set of nodes is $\mathcal{N} = \{1, \ldots, 15\}$ and the set of nodes at which there are markets is $\mathcal{M} = \{4, 5, 6, 8, 9, 14, 15\}$.

We consider a time horizon of one day with 24 discrete time periods. Synthetic data was created for the inverse demand parameters to represent the change in loads throughout a day. The representative data was obtained from CalISO. The bounds for the ramping rates of the generators are also synthetic. The data used for the two piece linear generation cost function, generation capacitites, PTDF values and transmission line capacities was obtained from the Energy research Centre of the Netherlands (ECN) and is given in tables 1 - 4. The network is illustrated in Figure 5.13 with the nodes and flowgates (arcs) enumerated. Table ??-5.16 describe essential characteristic of the network as well as the cost structure.
Figure 5.13: Network Illustration

<table>
<thead>
<tr>
<th>Firm</th>
<th>Node</th>
<th>Segment 1</th>
<th>Segment 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>LB1 MW</td>
<td>UB1 MW</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>2832</td>
<td>6.4</td>
</tr>
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<td>6576</td>
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<td>72617</td>
<td>4.0</td>
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<td>0</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>33823</td>
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</tr>
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<td>8</td>
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<td>22541</td>
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</tr>
<tr>
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<td>15</td>
<td>0</td>
<td>0</td>
</tr>
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<td>7</td>
<td>15</td>
<td>6550</td>
<td>6.2</td>
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<td>7910</td>
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</tr>
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<td>8</td>
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<tr>
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<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
</tr>
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<td>11.2</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
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</table>

Figure 5.14: Data for two piece linear generation cost function
### Figure 5.15: PTDF data continued

<table>
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<tr>
<th>Line</th>
<th>Capacity</th>
<th>Line</th>
<th>Capacity</th>
<th>Line</th>
<th>Capacity</th>
<th>Line</th>
<th>Capacity</th>
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<td>22</td>
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</tr>
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<td>9</td>
<td>20000</td>
<td>10</td>
<td>20000</td>
<td>17</td>
<td>641</td>
</tr>
<tr>
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<td>20000</td>
<td>11</td>
<td>20000</td>
<td>18</td>
<td>1842</td>
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<td>20000</td>
<td>19</td>
<td>641</td>
<td>26</td>
<td>1326</td>
</tr>
<tr>
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<td>1842</td>
<td>20</td>
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<td>1842</td>
</tr>
<tr>
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<td>2971</td>
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<td>267</td>
<td>21</td>
<td>1842</td>
<td>28</td>
<td>1842</td>
</tr>
</tbody>
</table>

PTDF data computed with node 15 as the hub

The first set of results are shown in Table 5.17 below. The data in this table represents the social welfare for the full capacity network, as well as the difference between the social welfare of the network at full capacity and that of the network with each arc removed. The change in SW is computed as

\[
\text{Change in SW} = \text{Full Capacity SW} - \text{SW with arc removed}
\]

We would expect that the social welfare would decrease as transmission capacity is removed from the network. However, there are cases where the removal of an arc results in an increase in social welfare.
<table>
<thead>
<tr>
<th>Arc Removed</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change in SW</td>
<td>1.506E+07</td>
<td>0.000E+00</td>
<td>0.000E+00</td>
<td>3.362E+07</td>
<td>1.984E-04</td>
<td>-4.341E+06</td>
<td>5.882E+04</td>
</tr>
<tr>
<td>Arc Removed</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>Change in SW</td>
<td>2.585E+06</td>
<td>2.453E+06</td>
<td>5.773E+05</td>
<td>5.064E+05</td>
<td>-4.728E+05</td>
<td>0.000E+00</td>
<td>0.000E+00</td>
</tr>
<tr>
<td>Arc Removed</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>21</td>
</tr>
<tr>
<td>Change in SW</td>
<td>0.000E+00</td>
<td>-3.999E+05</td>
<td>0.000E+00</td>
<td>0.000E+00</td>
<td>-1.484E+07</td>
<td>-3.510E+06</td>
<td>6.554E+06</td>
</tr>
<tr>
<td>Arc Removed</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
<td>26</td>
<td>27</td>
<td>28</td>
</tr>
<tr>
<td>Change in SW</td>
<td>-1.538E+06</td>
<td>-2.053E+07</td>
<td>1.743E+07</td>
<td>2.180E+07</td>
<td>1.280E+06</td>
<td>-2.049E+07</td>
<td>1.984E-04</td>
</tr>
</tbody>
</table>

Figure 5.17: Changes in net social welfare when different arcs of the network are removed due to disruption effects

Figure 5.18: Allocation to Consumption of Firms at node 4

increase to social welfare; this type of observation is typically termed the Braess paradox. In this case, we see that the when arcs 6, 12, 16, 19, 20, 22, 23 and 27 are removed, the social welfare actually increases, counter to what we expect.

To give an example of the equilibrium solution, we give some figures below. Figure 5.18 shows the allocation to consumption patterns by each firm at node 4 at equilibrium under full capacity.

In Figure 5.18, we can see that firm 6 has the highest allocation to consumption throughout the day. Also note that there are two predominant "humps" in the allocation to
consumption. These arise from the data used for the inverse demand function and represents the increase in load experienced at the beginning of the day when people arrive to work and at the end of the day when people arrive at their homes. Correspondingly, we see in Figure 3 that firm 6 has the highest production at node 4.

As mentioned previously, we also test the effects on the equilibrium of removing a generator for a number of consecutive time periods. Specifically, we removed the generator owned by firm 2 at node 4 during time periods 5, 6 and 7. We first solved the problem to find the equilibrium when all generators and transmission lines are working. The problem was then solved again with the said generator crippled. The equilibriums were then compared to see what the effects of such a disruption were. The model was formulated in GAMS and solved with the PATH solver using the default settings. The solution time for this problem is under 20 seconds for one run on a 2.8GHz Intel Pentium 4 processor machine with 512MB of RAM.

Figure 5.20 below shows the equilibrium allocation to consumption by firm 2 for both the non-disrupted (before) and disrupted (after) cases. As we can see, the allocation
to consumption by firm 2 is drastically affected by this disruption. However, as when the generating facility becomes operational again, the allocation to consumption pattern returns to the equilibrium that was established with no disruption. Meanwhile, we see in Figure ?? that firm 1 visibly increases its allocation to consumption at nodes 4 and 15 during the time periods during and following the disruption to the generation of firm 2; changes in behavior can be seen at the other nodes too, though it is much more subtle. Examining the changes observed by other firms at other nodes provides similar behavior.

Figure 5.20: Allocation to Consumption of Firm 2 Before and After
We can also look at the change in generation that is experienced during this disruption. Figure 5.21 shows the equilibrium generation pattern for firm 2. We can see that the generator at node 4 drops to zero for time periods 5, 6 and 7 and then begins to slowly ramp back up according to the ramping rate constraints. The generation level finally reaches that of the non-disrupted equilibrium around time period 14. We also see that firm 2 turns on generator 5, though it is very limited in output, during the disruption. Figure 5.22 shows the equilibrium generation of firm 1 in both the disrupted and non-disrupted cases. It is clear that firm 1 experiences changes to its generation at both nodes 5 and 9 due to the disruption occurring in firm 1’s generation at node 4. Similar results can be seen for the other generating firms as well.

5.3.9 Conclusions and Future Work

We have put forth a dynamic game theoretic model of oligopolistic competition in an electric power network setting. This model is formulated to lend itself to efficient computation in order to facilitate the research of big events in such an electric power system. An outline
Figure 5.21: Generation of Firm 2 Before and After

Figure 5.22: Generation of Firm 1 Before and After
of a possible simulation route was provided which may be realistically carried out for the
testing of extreme events. We have also shown how a local disruption in the system can
have global implications for the system as is witnessed in the numerical example.
Chapter 6

Dynamic User Equilibrium and Securitization of Congestion

6.1 Brief Literature Review of Dynamic Traffic Assignment

Dynamic Traffic Assignment (DTA) is the positive (descriptive) modeling of time varying flows of automobiles on road networks consistent with established traffic flow theory and travel demand theory. In this paper Dynamic User Equilibrium (DUE) is one type of DTA wherein the unit travel cost, including early and late arrival penalties, of travel for the same purpose is identical for all utilized route and departure time pairs. In the context of planning, DUE is usually modelled for the within-day time scale based on demands established on a day-to-day time scale.

In the last several years, much effort has been expended to develop a theoretically sound formulation of dynamic network user equilibrium (DUE) which is also a canonical form acceptable to scholars and practitioners alike. DUE models tend to be comprised of four essential submodels:
1. a model of path delay;

2. flow dynamics;

3. flow propagation constraints; and

4. a route/departure-time choice model.

Peeta and Ziliaskopoulos [87], in a comprehensive review of DTA and DUE research, note that there are several published models comprised of the four submodels named above. We are interested in this paper in showing how such DUE models may be numerically solved using the theory of infinite dimensional mathematical programming and the notion of a fixed point in Hilbert space. To that end we focus on two infinite dimensional variational inequality formulations of the DUE problem reported in Friesz et al. [11] and Friesz et al [10] that have much in common with other published models. In fact the Friesz et al. [11] and Friesz et al [10] formulations are more computationally demanding than most if not all other DUE models because of the complicated path delay operators, equations of motion and time lags they embody. As such the algorithmic results reported herein should work as well or better when adapted to other DUE models, including those for which path delay is determined by a nonlinear response surface or by simulation for a so-called rolling horizon.

That is, the model presented herein has previously been studied by Friesz et al [11] and Friesz et al [10] from the point of view of various alternative mathematical representations. By necessity we must recap the model being used to illustrate our computational approach in Section 6.2. This is done as expeditiously as possible, but is nonetheless somewhat detailed since we must give a foundation for certain of the undesirable properties of DUE models that make their solution challeging. In contrast, our algorithmic presentation is new, for we have not previously reported on algorithms for solving the Friesz et al DUE model. Moreover, the algorithmic approach we take – namely, a fixed point algorithm whose subproblems are solved by descent in Hilbert space – departs significantly from other algorithms proposed for DUE.
The computation of dynamic user equilibrium has tended to be seen in the DUE literature as so fundamentally difficult that expedient simplifying modeling assumptions are justified. We take the opposite approach: fundamental properties of DUE cannot be ignored simply for the sake of computability. Two DUE features have been particularly over-simplified in computational studies: (1) the intrinsically nested nature of path delays, wherein delays are functions of functions which in turn are functions of functions, with the depth of nesting being determined by the number of arcs in a path; and (2) the time shifted natures of arc inflows/outflows needed to capture the fact that finite, state-dependent arc transit times arise when one contructs a rigorous model of flow propagation. We show via an analysis of the fixed point algorithm as well as a numerical example presented herein that both nested delay operators and state-dependent time shifts may be attacked head-on using relative simple extensions of known mathematical results.

A specific algorithmic contribution of this paper is that we show how computation in continuous time facilitates the numerical analysis of DUE models that employ time shifts to model flow propagation in a theoretically rigorous fashion. Heretofore, the treatment of time shifts in the DUE literature has been *ad hoc*, at best. By contrast, we describe an implicit fixed point computational scheme that, for each main iteration, approximates any decision variable with shifted argument as a pure, continuous function of time.

As our presentation of the DUE model and algorithm we emphasize is crafted to make such an implicit fixed point approximation possible, it is worthwhile to explain the basic notions involved before we introduce the complicated notation characteristic of DUE models. To do this, let us suppose one is faced at iteration \( \ell \) with the need to evaluate a control

\[
u [t + \Delta (x (t))]\]

for which \( t \) denotes continuous time, \( x \) denotes a state variable, and the time shift \( \Delta \) is state-dependent, as explicitly noted by the form of (6.1). Further suppose that in iteration
\[ \Delta (x(t)) \approx \sum_{j=0}^{r} \alpha_{j}^{\ell-1}(t)^{j} \]  
\[ u(t) \approx \sum_{k=0}^{s} \beta_{k}^{\ell-1}(t)^{k} \]  

where \( s, r, \alpha_{j}^{\ell-1}, \) and \( \beta_{k}^{\ell-1} \) are arbitrary names of parameters and the polynomial forms are meant merely to be illustrative. For such a circumstance we form the following approximation of the shifted control

\[ u^{\ell}(t+\Delta) \approx \sum_{k=0}^{s} \beta_{k}^{\ell-1} \left(t + \sum_{j=0}^{r} \alpha_{j}^{\ell-1}(t)^{j}\right)^{k} \equiv \bar{u}_{\ell}(t) \]

a pure function of time meant for use in iteration \( \ell \). It is important to note that the continuous time forms (6.2) and (6.3) may be either the direct result of a continuous time analysis or they may be polynomials fit to the output of a discrete time approximation. In either case it is possible to form the approximation \( \bar{u}_{\ell}(t) \).

### 6.2 The Differential Variational Inequality (DVI) Formulation

In this section we re-cast the Friesz et al (1993) model in the form of a differential variational inequality in preparation for the proposal and testing of a fixed point algorithm in continuous time to be presented subsequently. The primary components of the model are those mentioned previously: (1) a model of path delay; (2) flow dynamics; (3) flow propagation constraints; and (4) a route/departure-time choice model.

An arbitrary path \( p \in \mathcal{P} \) of the network of interest is

\[ p \equiv \{a_{1}, a_{2}, \ldots, a_{i-1}, a_{i}, a_{i+1}, \ldots, a_{m(p)}\} \]

where \( \mathcal{P} \) is the set of all paths and \( m(p) \) is the number arcs of \( p \in \mathcal{P} \). The set of all network arcs is denoted by \( \mathcal{A} \). We also let \( t_{e} \) denote the time at which flow exits an arc, while \( t_{d} \) is
the time of departure from the origin of the same flow. The exit time function \( \tau_{a_i}^p(t) \) therefore obeys

\[
t_e = \tau_{a_i}^p(t_d)
\]

The relevant arc dynamics are

\[
\begin{align*}
\frac{dx_{a_i}^p(t)}{dt} &= g_{a_{i-1}}^p(t) - g_{a_i}^p(t) \quad \forall p \in \mathcal{P}, i \in [1, m(p)] \quad (6.4) \\
x_{a_i}^p(t_0) &= x_{a_i,0}^p \quad \forall p \in \mathcal{P}, i \in [1, m(p)] \quad (6.5)
\end{align*}
\]

where \( x_{a_i}^p \) is the traffic volume of arc \( a_i \), \( g_{a_i}^p \) is flow exiting arc \( a_i \) and \( g_{a_{i-1}}^p \) is flow entering arc \( a_i \) of path \( p \in \mathcal{P} \). Also, \( g_{a_0}^p \) is the flow exiting the origin of path \( p \); by convention we call this the flow of path \( p \) and use the symbolic name

\[ h_p = g_{a_0}^p \]

Furthermore

\[
\delta_{a,p} = \begin{cases} 
1 & \text{if } a_i \in p \\
0 & \text{if } a_i \notin p
\end{cases}
\]

so that

\[
x_a(t) = \sum_{p \in \mathcal{P}} \delta_{ap} x_{a_i}^p(t) \quad \forall a \in \mathcal{A}
\]

(6.6)

is the total arc volume.

Arc unit delay is

\[ D_a(x_a) \quad \forall a \in \mathcal{A} \]

That is, arc delay depends on the number of vehicles in front of an auto as that auto enters an arc. Of course total path traversal time is

\[
D_p(t) = \sum_{i=1}^{m(p)} \left[ \tau_{a_i}^p(t) - \tau_{a_{i-1}}^p(t) \right] = \tau_{a_{m(p)}}^p(t) - t \quad \forall p \in \mathcal{P}
\]
It is expedient to introduce the following recursive relationships that must hold in light of the above development:

\[ \tau_{a_1}^p = t + D_{a_1} [x_{a_1} (t)] \quad \forall p \in \mathcal{P} \]

\[ \tau_{a_i}^p = \tau_{a_{i-1}}^p (t) + D_{a_i} [x_{a_i} \left( \tau_{a_{i-1}}^p (t) \right)] \quad \forall p \in \mathcal{P}, i \in [2, m(p)] \]

from which we have the nested path delay operators first proposed by Friesz et al [11]:

\[ D_p (t, x) \equiv \sum_{i=1}^{m(p)} \delta_{a_i p} \Phi_{a_i} (t, x) \quad \forall p \in \mathcal{P} \]

where

\[ x = \left( x_{a_i}^p : p \in \mathcal{P}, i \in [1, m(p)] \right) \]

and

\[ \Phi_{a_1} (t, x) = D_{a_1} [x_{a_1} (t)] \]
\[ \Phi_{a_2} (t, x) = D_{a_2} [x_{a_2} (t + \Phi_{a_1})] \]
\[ \Phi_{a_3} (t, x) = D_{a_3} [x_{a_3} (t + \Phi_{a_1} + \Phi_{a_2})] \]
\[ \vdots \]
\[ \Phi_{a_i} (t, x) = D_{a_i} [x_{a_i} (t + \Phi_{a_{i-1}} + \ldots + \Phi_{a_1})] = D_{a_i} [x_{a_i} \left( t + \sum_{j=1}^{i-1} \Phi_{a_j} \right)] \]

To ensure realistic behavior, we employ asymmetric early/late arrival penalties

\[ F [t + D_p (t, x) - t_A] \]

where \( t_A \) is the desired arrival time and

\[ t + D_p (t, x) > t_A \implies F [t + D_p (t, x) - t_A] = \chi^L (t, x) > 0 \]
\[ t + D_p (t, x) < t_A \implies F [t + D_p (t, x) - t_A] = \chi^E (t, x) > 0 \]
\[ t + D_p (t, x) = t_A \implies F [t + D_p (t, x) - t_A] = 0 \]
\[ \chi^L (t, x) > \chi^E (t, x) \]
We combine the actual path delays and arrival penalties to obtain the effective delay operators
\[ \Psi_p(t, x) = D_p(t, x) + F[t + D_p(t, x) - t_A] \quad \forall p \in \mathcal{P} \quad (6.7) \]

Elementary manipulations based on the chain rule and the assumption that arc delay operators are differentiable lead to
\[ g_{a_1} (t + D_{a_1} [x_{a_1}(t)]) (1 + D'_{a_1} [x_{a_1}(t)] \dot{x}_{a_1}) = h_p(t) \quad \forall p \in \mathcal{P} \quad (6.8) \]
\[ g_{a_i} (t + D_{a_i} [x_{a_i}(t)]) (1 + D'_{a_i} [x_{a_i}(t)] \dot{x}_{a_i}) = g_{a_{i-1}} (t) \quad \forall p \in \mathcal{P}, i \in [2, m(p)] \quad (6.9) \]
These are proper flow progression constraints derived in a fashion that makes them completely consistent with the chosen dynamics and point queue model of arc delay. These constraints involve a state dependent time lag \( D_{a_1} [x_{a_1}(t)] \) but make no explicit reference to the exit time functions. These flow propagation constraints describe the expansion and contraction of vehicle platoons; they were first presented by Friesz et al [88]. Astarita [89], [90] independently proposed flow propagation constraints that may be readily placed in the above form.

The flow conservation constraints are
\[ \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t) \, dt = Q_{ij} \quad \forall (i, j) \in \mathcal{W} \quad (6.10) \]
where \( Q_{ij} \) is the fixed travel demand for OD pair \((i, j)\), \([t_0, t_f] \subset \mathbb{R}_+^1\) is the departure time window, \( \mathcal{P}_{ij} \) is the set of paths connecting that OD pair, and \( \mathcal{W} \) is the set of all OD pairs. Of course, we have
\[ \mathcal{P} = \bigcup_{(i, j) \in \mathcal{W}} \mathcal{P}_{ij} \]
We also enforce the following non-negativity restrictions:
\[ x \geq 0 \quad g \geq 0 \quad h \geq 0 \quad (6.11) \]
for all times within the analysis period, where \( x \) has been defined previously and

\[
g = (g^p_i : p \in P, i \in [1, m(p)])
\]

\[
h = (h_p : p \in P)
\]

As a consequence

\[
\Omega = \{ (g, h) : (6.4), (6.5), (6.8), (6.9), (6.10) \text{ and } (6.11) \text{ hold} \}
\]

is the relevant feasible region.

The infinite dimensional variational inequality formulation for dynamic network user equilibrium itself is: find \((g^*, h^*) \in \Omega\) such that

\[
\langle \Psi(t, x(h^*)), (h - h^*) \rangle = \sum_{p \in P} \int_{t_0}^{t_f} \Psi_p(t, x(h^*)) \left[ h_p(t) - h^*_p(t) \right] dt \geq 0
\]

for all \((g, h) \in \Omega\), all of whose solutions Friesz et al [10] show are dynamic user equilibria\(^1\). In particular the solutions of (6.12) obey

\[
\Psi_p(t, x^*) > \mu_{ij} \implies h^*_p(t) = 0
\]

\[
h^*_p(t) > 0 \implies \Psi_p(t, x^*) = \mu_{ij}
\]

for \( p \in P_{ij} \). We call a flow pattern satisfying (6.13) and (6.14) a dynamic user equilibrium. The behavior described by (6.13) and (6.14) is readily recognized to be a type of Cournot-Nash non-cooperative equilibrium. It is important to note that these conditions do not describe a stationary state, but rather a time varying flow pattern that is a Cournot-Nash equilibrium (or user equilibrium) at each instant of time.

The variational inequality (6.12) has been called a variational inequality control problem by Bernstein et al [91]. This name can be confusing, since it suggests to some

\(^1\)Although we have purposely suppressed the functional analysis subtleties of the formulation, it should be noted that (6.12) involves an inner product in a Hilbert space, namely \((L^2[0, T])^{\left| P \right|}\).
that the model is one of system optimization with centralized control and network-wide congestion minimization, when in fact flow patterns calculated from it are user optimized. Consequently, we now prefer to call (6.12) a differential variational inequality (DVI) formulation of DUE as it is constrained by state dynamics in the form of ordinary differential equations that depend on users’ controls and takes the form of an infinite dimensional variational inequality.

6.3 The Variational Inequality (VI) Formulation

In this section we explain how the differential variational inequality may be re-stated as an infinite dimensional variational inequality without explicit dynamics. This sets the stage for obtaining a fixed point representation.

If, in the DVI formulation developed above, one uses the arc dynamics (6.4) to eliminate state variables and recognizes that arc exit flows are completely determined by path flows (departure rates) and state variables according to the flow propagation constraints (6.8) and (6.9), the problem statement may be made more concise. In particular the arc dynamics and flow propagation constraints may be embedded in the effective path delay operators which are then given the symbolic names

$$\Psi_p(t, h) \quad \forall p \in \mathcal{P}$$

Then, as shown in Friesz et al. [11], any solution of the following variational inequality is also a solution of the DUE problem: find $h^* \in \Lambda$ such that

$$\langle \Psi(t, h^*), (h - h^*) \rangle = \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*) \left[ h_p(t) - h_p^*(t) \right] \, dt \geq 0 \quad (6.15)$$

for all $h \in \Lambda$, where

$$\Lambda = \left\{ h \geq 0 : \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t) \, dt = Q_{ij} \quad \forall (i, j) \in \mathcal{W} \right\} \quad (6.16)$$
The advantage of this formulation is that it subsumes almost all DUE models regardless of the arc dynamics, flow propagation and arc delay functions employed; it is the formulation originally put forward by Friesz et al [11]. However, it would be a mistake to think that somehow (6.15) is an “easier” formulation since the delay operators are generally not knowable in closed form; in fact the delay operators may be non-analytic and may need to be derived from simulation models using response surface methodology or a rolling horizon philosophy.

6.4 Comparison of the DVI and VI Formulations

There are several important points to be made regarding the two formulations (6.12) and (6.15):

1. The differential variational inequality DUE model (6.12) is very similar to others reported on the literature since, as already noted in Section ??, it has four main sub-models: (a) a model of path delay, (b) arc dynamics, (c) flow propagation constraints and (d) a routing principle.

2. Although (6.12) has great intuitive appeal, demonstrating that any solution of it is a dynamic user equilibrium under mild regularity conditions is non-trivial. In particular, using the theory of optimal control, one must derive necessary conditions for differential variational inequalities with state-dependent time shifts as presented in Friesz et al [10].

3. A demonstration of the correctness of the alternative formulation (6.15) is based on functional analysis and requires the measure theoretic arguments presented in Friesz et al [11].

4. The arc dynamics (6.4) are intrinsically approximate. They are only exact in the limit of zero arc length. Shorter and more numerous arcs should yield a more accurate
solution, although no numerical testing of that conjecture has been conducted for any DUE model based on arc dynamics.

6.5 FIFO

Enforcing the first-in-first-out (FIFO) queue discipline is thought by many to be essential to the mathematical statement of the DUE problem. FIFO-ness is assured by the conditions

\[ 1 + D'_a \left[ x_a(t) \right] \dot{x}_a(t) \geq 0 \quad \forall a \in \mathcal{A} \tag{6.17} \]

with equality allowed only when there is no arc exit flow, as explained in Zhu and Marcotte [92]. The flow propagation constraints (6.8) and (6.9) together with nonnegativity (6.11) assure that (6.17) must hold.

6.6 Existence

Note that if the integrals in (6.16) are interpreted as Lebesgue and the flow conservation constraints themselves are interpreted as valid almost everywhere, the finiteness of each \( Q_{ij} \) is not enough to assure bounded path flows. This observation is the fundamental hurdle to proving existence without the \textit{a priori} invocation of bounds on path flows.

Existence results for DUE are most general if based on the formulation (6.15). Theorem 2 of Browder [27] for the existence of solutions of variational inequalities in topological vector spaces can be applied if the operators \( \Psi_p(t, h) \) for all \( p \in \mathcal{P} \) can be shown to be continuous and the feasible region \( \Lambda \) can be shown to be compact. Zhu and Marcotte [92] show, for the arc dynamics and arc delay submodel we have presented in Section 6.2, that under rather mild regularity conditions the path delay operators are continuous. However, to assure boundedness they make the assumption that departure rates are \textit{a priori} bounded from above. Under this last rather stringent restriction they show that a dynamic user equilibrium exists. Missing from the DUE literature is a proof of existence for regularity
conditions imposed entirely on the path delay operators without the invocation of a priori boundedness of path flows.

At this point we observe that the DUE problem, when articulated as a DVI is actually a special case of the DVI with state dependent time shifts, which is developed and articulated in Chapter 2 of this thesis. In particular we present a fixed point algorithm for DVI s with state-dependent time shifts. We need to specialize the fixed point algorithm for application to the DUE problem.

6.7 Applying the Fixed Point Algorithm to the DVI Formulation

In order to apply the formalism developed in Chapter 2, we make the following observations/assumptions:

1. the controls are \( g \in (L^2 [t_0, t_f])^{n_1} \) and \( h \in (L^2 [t_0, t_f])^{[P]} \) where
\[
n_1 = \sum_{p=1}^{[P]} m(p);
\]

2. the state variables are the traffic volumes
\[
 x^p_{a_i} \quad \forall p \in P, i \in [1, m(p)];
\]

3. the arc delays
\[
 D_{a_i}(x_{a_i}) \quad \forall p \in P, i \in [1, m(p)];
\]

appear as explicit time shifts in the flow propagation constraints (6.8) and (6.9);

4. the operator
\[
x(g, g_D, h, h_D)
\]
has the properties of continuity and G-differentiability, where $g_D$ and $h_D$ have the obvious definitions

$$g_D \equiv g(t + D(x))$$
$$h_D \equiv h(t + D(x))$$

and of course

$$x \equiv (x_{a_i}^p : p \in \mathcal{P}, i \in [1, m(p)])$$
$$D \equiv (D_{a_i} : p \in \mathcal{P}, i \in [1, m(p)])$$

5. $D(x)$ is continuously differentiable with respect to $x$;

6. $\Psi_p(t, h^*)$ is strongly monotone with respect to $h$;

7. a coerciveness condition like (2.166) holds; and

8. a Lipschitz condition like (2.167) holds.

We stress that conditions (1) through (5) above are reasonable and assure that the fixed point problem whose solution is a dynamic user equilibrium is well defined. By contrast conditions (6), (7) and (8) assure that the fixed point algorithm converges. Conditions (6), (7) and (8) are unlikely to be verifiable for problems of realistic size. In that event, the fixed point algorithm becomes a heuristic – until a more general convergence theory is discovered.

### 6.8 Numerical Example

In what follows, we consider the 5 arc, 4 node network shown in Figure 6.1. The forward star array and arc delay functions for this network are summarized in Table 6.1.
Figure 6.1: 5-arc 4-node traffic network

Table 6.1: Forward Star Array and Arc Delay Functions

<table>
<thead>
<tr>
<th>Arc name</th>
<th>From node</th>
<th>To node</th>
<th>Arc Delay, $D_a \left( x_a(t) \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{2} + \left( \frac{x_{a_1}}{70} \right)$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>1</td>
<td>3</td>
<td>$1 + \left( \frac{x_{a_2}}{150} \right)$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>2</td>
<td>3</td>
<td>$\frac{1}{2} + \left( \frac{x_{a_3}}{100} \right)$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>2</td>
<td>4</td>
<td>$1 + \left( \frac{x_{a_4}}{150} \right)$</td>
</tr>
<tr>
<td>$a_5$</td>
<td>3</td>
<td>4</td>
<td>$\frac{1}{2} + \left( \frac{x_{a_5}}{100} \right)$</td>
</tr>
</tbody>
</table>
Table 6.2: Path Flows, Arc Exit Flows and Path Specific Arc Traffic Volumes

<table>
<thead>
<tr>
<th>Paths</th>
<th>Path Flows</th>
<th>Arc Exit Flows</th>
<th>Traffic Volume of Arcs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>$h_{p_1}$</td>
<td>$g_{a_1}, g_{a_4}$</td>
<td>$x_{a_1}^{p_1}, x_{a_4}^{p_1}$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$h_{p_2}$</td>
<td>$g_{a_2}, g_{a_5}$</td>
<td>$x_{a_2}^{p_2}, x_{a_5}^{p_2}$</td>
</tr>
<tr>
<td>$p_3$</td>
<td>$h_{p_3}$</td>
<td>$g_{a_3}, g_{a_5}$</td>
<td>$x_{a_3}^{p_3}, x_{a_5}^{p_3}$</td>
</tr>
</tbody>
</table>

There is a travel demand of $Q = 75$ units from node 1 (origin) to node 4 (destination). There are 3 paths connecting the single OD pair formed by nodes 1 and 4, namely:

$$
\mathcal{P}_{14} = \{p_1, p_2, p_3\}
$$

$$
p_1 = \{a_1, a_4\}
$$

$$
p_2 = \{a_2, a_5\}
$$

$$
p_3 = \{a_1, a_3, a_5\}
$$

The controls (path flows and arc exit flows) and states (path-specific arc traffic volumes) associated with the network are listed in Table 6.2.

The planning horizon is $t_0 = 0, t_f = 10$, hence the time interval considered is $[0, 10]$, while the desired arrival time is $t_A = 5$. We employ the symmetric early/late arrival penalty

$$
F \left[ t + D_p(x, t) - t_A \right] = [t + D_p(x, t) - t_A]^2
$$

Further, without any loss of generality, we take

$$
x_{a_i}^p(0) = 0 \quad \forall p \in \mathcal{P}, i \in [1, m(p)]
$$

The fixed point stopping tolerance has been set at

$$
\varepsilon = 0.01
$$

and to assist convergence we choose

$$
\alpha = \frac{1}{k}
$$

where $k$ is the fixed point iteration counter.
Note that Friesz et al. [12] show that arc delay functions that are linear in volume induce arc delays that are nonlinear in flow; in particular delay as a function of flow is a hyperbola. Moreover, there is no aspect of our analysis or computational approach that prevents the use of arc delay functions that are nonlinear in volume. Indeed, the only difficulty in the use of the arc delay functions that are nonlinear in volume is that no measurements exist for the determination of their parameters.

We forgo the detailed symbolic statement of this example and, instead, provide numerical results in graphical form for the solution after 77 iterations of the fixed point algorithm. Figures 6.2, 6.3, and 6.4 depict path flows and arc exit flows for paths $p_1$, $p_2$, and $p_3$ defined above. Cumulative traffic volume on the network’s 5 different arcs are plotted against time in Figure 6.5 where

\[
x_{a_1}(t) = x_{a_1}^{p_1}(t) + x_{a_1}^{p_3}(t) \\
x_{a_2}(t) = x_{a_2}^{p_2}(t) \\
x_{a_3}(t) = x_{a_3}^{p_3}(t) \\
x_{a_4}(t) = x_{a_4}^{p_1}(t) \\
x_{a_5}(t) = x_{a_5}^{p_2}(t) + x_{a_5}^{p_3}(t)
\]

for all time $t \in [0, 10]$. Figure 6.5 presents arc volumes derived from activity on all paths and corresponding to the departure rates.

When, for path $p_1$, we compare the effective path delay operator (6.7) with path flow (which is the departure rate) by plotting both for the same time scale, Figure 6.6 is obtained. This figure shows that departure rate peaks when the associated effective path delay achieves a local minimum, thereby demonstrating that an user equilibrium has been found. Similar comparisons are made for paths $p_2$ and $p_3$ in Figures 6.7 and 6.8 respectively.

In Figure 6.9 the relative change from one iteration to the next, expressed as

\[
\Delta_k = \left\| g^k - g^{k-1} \right\| \leq \varepsilon,
\]
Figure 6.2: Path and arc exit flows for path $p_1$

is plotted against the iteration counter $k$. It is worth noting that for this particular example even though $\Delta_1 = 2.13 \times 10^4$, the next several iterations very rapidly decrease $\Delta_k$. The run time for this example is less than 3 minutes using a generic desktop computer with dual Intel Xeon processors and 2 GB RAM. The computer code for the fixed point algorithm is written in MatLab 6.5 and calls a gradient projection subroutine for which control, state and adjoint variables are determined in the sequential fashion explained in Section ??.

### 6.9 Extensions

We observe that the above formalism of dynamic user equilibrium may be extended to account for stochastic phenomena, including both imperfect and incomplete information. We conclude this paper by applying the formalism to create two entirely new formulations of dynamic user equilibrium when: (1) there are dual time scales (day-to-day and within-day);
Figure 6.3: Path and arc exit flows for path $p_2$

Figure 6.4: Path and arc exit flows for path $p_3$
Figure 6.5: Cumulative traffic volume on each arc

Figure 6.6: Comparison of path flows and associated unit travel costs for path $p_1$
Figure 6.7: Comparison of path flows and associated unit travel costs for path $p_2$

Figure 6.8: Comparison of path flows and associated unit travel costs for path $p_3$
Figure 6.9: Relative change of controls from one iteration to the next ($\Delta_k$) vs. the iteration counter ($k$)

and (2) demand information is uncertain.

### 6.9.1 Dual Time Scales (day-to-day and within-day)

Let $\tau \in \Upsilon \equiv \{1, 2, ..., L\}$ be one typical day within the planning horizon, and take the length of each day to be $\Delta$, while the clock time within each day $\tau$ is presented by $t \in [(\tau - 1) \Delta, \tau \Delta]$ for all $\tau \in \{1, 2, ..., L\}$. The planning horizon consists of $L$ consecutive days. We assume the travel demand for each day changes based on the moving average of congestion experienced over previous days. We postulate that the travelling demand $Q_{ij}^\tau$ for day $\tau$ between a given O-D pair $(i, j) \in \mathcal{W}$ determined by the following system of difference equations:

\[
Q_{ij}^{\tau+1} = \left[ Q_{ij}^\tau - \eta_{ij} \left\{ \sum_{p \in \mathcal{P}_{ij}} \sum_{j=0}^{\tau-1} \int_{(j-1)\Delta}^{j\Delta} \Psi_p \left[ h^*, g^* \right] dt \right\} + \frac{\sum_{p \in \mathcal{P}_{ij}} \sum_{j=0}^{\tau-1} \int_{(j-1)\Delta}^{j\Delta} \Psi_p \left[ h^*, g^* \right] dt}{|\mathcal{P}_{ij}| \cdot \tau \cdot \Delta} - \chi_{ij} \right]^{+} \quad \forall \tau \in \{1, 2, ..., L\}
\]

\[
Q_{ij}^1 = \tilde{Q}_{ij}
\]
where \( \hat{Q}_{ij} \in \mathbb{R}_+ \) is the fixed traveling demand for the O-D pair \((i, j) \in W\) for the first day. The operator \([x]^+\) is equivalent to \(\max\{0, x\}\).

### 6.9.2 Uncertain Travel Demand Information

Once again let us assume \( \tau \in \Upsilon \equiv \{1, 2, ..., L\} \) be one typical day within the planning horizon, and take the length of each day to be \( \Delta \), while the clock time within each day \( \tau \) is presented by \( t \in [(\tau - 1) \Delta, \tau \Delta] \) for all \( \tau \in \{1, 2, ..., L\} \). where the planning horizon consists of \( L \) consecutive days. Here we assume that the travel demand for each day is a random variable in the following multiplicative form

\[
\hat{Q}_{ij}^\tau = Q_{ij}^\tau \cdot z_{ij}
\]

where \( \hat{Q}_{ij}^\tau \) is the realized travel demand on day \( \tau \) between the OD pair \((i, j) \) where as \( z_{ij} \) is the random variable. To keep exposition simple we assume that distribution of \( z_{ij} \) is known exactly, however, it can further be generalized to have only partial information (e.g., first and second moments) about \( z_{ij} \). The average travel volume, \( Q_{ij}^\tau \) may be computed from (6.18).

### 6.9.3 Numerical Example

Consider the same numerical example as illustrated in Section 6.8. As opposed to a single day model, we now consider the planning horizon to be 4 day-long (i.e., \( L = 4 \)), the length of each day being \( \Delta = 24 \) hours, as usual. The desired arrival time for commuters is \( T_A = 13 \) (1:00 PM of every day). Figures below depict departure rates and arc exit flows for paths \( p_1, p_2 \) and \( p_3 \) respectively. The congestion threshold is assumed to be identical for all commuters irrespective of their route of choice, which is 10 units in this particular
example. Travel demand on day 1 is assumed to be 75 units.

Path and arc exit flows for path 1

Path and arc exit flows for path 2
Path and arc exit flows for path 3
Cumulative traffic volumes on the 5 different arcs are plotted against time in figure below where

\[
\begin{align*}
    x_{a_1}(t) &= x_{a_1}^{p_1}(t) + x_{a_1}^{p_3}(t) \\
    x_{a_2}(t) &= x_{a_2}^{p_2}(t) \\
    x_{a_3}(t) &= x_{a_3}^{p_3}(t) \\
    x_{a_4}(t) &= x_{a_4}^{p_2}(t) \\
    x_{a_5}(t) &= x_{a_5}^{p_2}(t) + x_{a_5}^{p_3}(t)
\end{align*}
\]
for all time $t \in [0, L\Delta]$.

Cumulative arc volume vs. time

Note that the effective path delay operator provides the unit travel cost along a path $p$ at time $t$. Figure below analyzes the effective delay and flow for path $p_2$ by plotting both for the same time scale which shows that path flow is maximal when the associated unit travel cost (effective path delay) is at its well defined minimum.

Comparison of path flows and associated unit travel costs for path $p_2$

Net travel demand and demand reduction are plotted below against the same time scale (day) which clearly demonstrates that more commuters switch to alternative mode (e.g., telecommuting) as their rolling average experience of congestion increases with passage of
6.10 The Philosophy Behind the Congestion Call Option

6.10.1 Multiple Time Scales

Let \( \tau \in \mathcal{Y} \equiv [0,1,\ldots,L] \) be one typical day within the planning horizon, and take the length of each day to be \( \Delta \), while the clock time within each day \( \tau \) is represented by \( t \in [(\tau - 1) \Delta, \tau \Delta] \) for all \( \tau \in [1,\ldots,L] \). The planning horizon consists of \( L \) consecutive days.

6.10.2 A Sub-model of Demand Reduction Through Telecommuting

We assume all travelers are headed to work and each traveler using the transport network has a qualitatively identical real option; he/she could have stayed home and worked via the Internet. Moreover, productivity of the worker varies depending on whether he/she is telecommuting or traveling to work. Travelers decide whether or not to telecommute based on their moving average of congestion experienced over previous days and also on

Net travel demand and demand reduction
restrictions imposed on the maximum number of people who may telecommute on a given day. We postulate that the volume of network travelers $R_{ij}^\tau$, who decide to telecommute between a given OD pair $(i, j) \in W$ at the beginning of day $\tau$ is determined by the following system of constrained difference equations:

$$
R_{ij}^{\tau+1} = R_{ij}^\tau + \eta_{ij}^\tau \left[ \sum_{p \in P_{ij}} \sum_{j=0}^{\tau-1} \int \Psi_p[t,x(h^*,g^*)] dt \right] - \chi_{ij} 
$$

(6.19)

$$
R_{ij}^0 = 0, \quad \sum_{i \in W_o} R_{ij}^\tau \leq \gamma_j \cdot \left( \sum_{i \in W_o} Q_{ij} \right) \quad \text{and} \quad R_{ij}^\tau \geq 0 \quad \forall \tau \in [0, L] 
$$

(6.20)

where $Q = (Q_{ij} : (i, j) \in W)$ is a known trip table, $P_{ij}$ is the set of paths connecting OD pair $(i, j)$ and $W_o$ is the set of origin nodes considered. The first term in the parenthesis of (6.19) is the moving average of effective travel delay at the beginning of day $\tau + 1$ when the traveler decides whether or not to telecommute; $\chi_{ij}$ is the threshold of congestion set by the $(i, j)$-travelers, and $\gamma_j$ is a policy determined parameter. Note that more people will switch to telecommuting as the moving average of congestion exceeds their threshold and vice versa. The constraints (6.20) articulate the requirement laid down by the the central authority (or by the employer in some cases, especially that of the Federal workforce in Washington) limiting the total number of people who can telecommute. Therefore, the total number of people who travel to work at node $j$ from node $i$ on day $\tau$ is

$$
Q_{ij}^\tau = Q_{ij} - R_{ij}^\tau 
$$

(6.21)

6.10.3 Option Trading

We postulate that there exists an information system that identifies routes and provides perfect path delay information. Network users (commuters) decides to make a physical trip to work are left with two choices:
he/she may purchase a congestion call option to travel on a route or

may travel on the same route by paying at the end of travel when he/she reaches
destination at the prevailing price.

The congestion call option is sold for a given route when departure from the origin
occurs. Price of the option is set by the central authorities like municipal planning orga-
nizations (MPOs). We postulate that the option pricing decision by the central authority is
influenced by the current valuation of the option as obtained from the Black-Scholes type
partial differential equation described below. We also assume that the option trading starts
at the beginning of the morning commute to work and continues till a pre-speciﬁed time
at which every option purchaser has returned home. At the time of expiry, the traveler,
who purchased the option, makes the decision whether or not to exercise the option (based
on the actual payoff). If the traveler decides to exercise the option, he/she gets a refund
which is equal to the difference between the current asset price and strike price. Natu-
rally uncertainty exists concerning the call option prices, which are described by geometric
Brownian motion, and depends on the history of congestion up to that point. The call
option prices impart, according to Itô’s lemma, randomness to the departure rates and to
the arc exit ﬂows even if the route and departure time choice mechanism is deterministic.
Fluctuations of call option prices arise from a myriad of causes. Perhaps the most obvious
factor inﬂuencing option price is the level of congestion as expressed by effective path delay.
Indeed, the ﬂuctuation of congestion option price volatility is assumed to vary with the level
of congestion. We further assume that the time of day drift and delay-dependent volatility
are available in identical form to all travelers for use in decision-making. Such a congestion
call option is a contract between the seller (writer of the call) and traveler (buyer of the
call) that allows the buyer to have access to a travel route and a departure time as well as
information on the travel time to his/her destination. Moreover, the call options considered
for this example are of the so-called European type, meaning that they may be exercised
only at a pre-speciﬁed moment of expiry. Clearly, for the notion of a congestion option
to be useful, appropriate technology facilitating the writing and exercising of congestion options must be in place. Our purpose is not to design or describe the implementation of such technology but rather to develop a mathematical model that can explore the impacts of congestion call options and – perhaps at some future date – to form the basis of a decision support system for managing a market for congestion puts as well as congestion calls. Owing to the well known put-call parity relationship, the results of our mathematical analysis of the call option that is the focus of this paper may be immediately adapted to the study and pricing of congestion puts. In order to evaluate the potential of congestion calls to influence the net social costs of congestion, we need to value such options. We follow the now standard analysis to value an option whose dynamics constitute a Wiener process\textsuperscript{2}, but make important modifications to that analysis in order to capture the essential features of a congested transportation network. In fact the partial differential equation we obtain for congestion call value is not the much studied Black-Scholes equation but rather a different partial differential equation.

Since transport network users could have telecommuted, when an individual decides to make a physical trip to work, that trip must have a value equal to or higher than the value of time expended during the commuting experience. To value time we use the commuters’ wage rate for each origin-destination pair, although any other value of time could be used. In reality only a fraction of the total population of travelers would belong to the class of potential telecommuters we have described. However, we have purposely striven to keep our exposition simple so that those unfamiliar with option pricing may fully grasp the notion of a congestion option. In particular, modeling several distinct transportation network user classes for each origin-destination pair would gain us little in understanding how to model congestion call options and present notational complexity that would hamper understanding. For these reasons we consider only a single user class for each origin-destination pair, namely telecommuters with an identical value of time.

\textsuperscript{2}There are many published expositions of option valuation analysis based on Itô’s lemma. One such reference is Wilmott et al. [22].
6.10.4 Stochastic Differential Equations Describing Asset Prices

The aforementioned access to path \( p \) is thought of as an “asset” with real or virtual market price \( S_p(t) \) at time \( t \) evolving according to the geometric Brownian motion

\[
\frac{dS_p(t, h^*)}{S_p(t)} = \mu_p(t) \, dt + \sigma_p^\tau \left[ D_p(t, x(h^*, t)) , t \right] dB_p \quad \forall p \in \mathcal{P}, t \in [0, L\Delta] \tag{6.22}
\]

\[
S_p(0) = K_p \in \mathbb{R}_{++}^1 \quad \forall p \in \mathcal{P} \tag{6.23}
\]

where \( \mu_p(t) \) is the deterministic drift that depends on the time of day, and \( \sigma_p^\tau (\cdot) \) is the associated volatility during day \( \tau \), and \( dB_p \) is a Brownian (white noise) disturbance. The volatility for path \( p \) will in general depend on time of the day path delays for that path. In (6.22) \( dB_p \) denotes the white noise component of Brownian motion for all \( p \in \mathcal{P} \). Note also that the initial asset prices \( K_p \in \mathbb{R}_{++}^1 \) for all \( p \in \mathcal{P} \) are known constants. Furthermore, \( S \) is the vector of all asset prices; that is

\[
S \equiv (S_p : p \in \mathcal{P})'
\]

It is widely known that the stochastic differential equation (6.22) with the initial condition (6.23) has an analytic solution

\[
S_p(t, h^*) = K_p \cdot \exp \left( \int_0^t \left( \mu_p^\tau (\cdot) - \frac{(\sigma_p^\tau \left[ \Psi^\tau_p(t, x(h^*, t)) , t \right])^2}{2} \right) \, dt + \sigma_p^\tau \left[ \Psi^\tau_p(t, x(h^*, t)) , t \right] B_p \right)
\]

We imagine a situation where network users have perfect and complete information about everything except call option prices. These users are described by a dynamic user equilibrium that devolves from deterministic arc dynamics and deterministic arc and path delays as well as the from the values of relevant congestion call options. Uncertainty exists concerning the call option prices, which are described by Brownian motion. Naturally these call option prices impart, according to Itô’s lemma, randomness to the departure rates and the arc exit flows. Fluctuations of call option prices arise from a myriad of causes. For a given origin-destination pair \((i, j) \in \mathcal{W}\), perhaps the most obvious factor influencing volatility for a specific path \( p \in \mathcal{P}_{ij} \) is own traffic expressed as \( h_p \); also important are non-own traffic \( h_q \) for paths \( q \in \mathcal{P}_{ij} \) that have arcs in common with path \( p \in \mathcal{P}_{ij} \). In Section
6.12 we present a simple model (??) for congestion option price volatility that is based on the above observations.

**Estimation of Drift and Volatility**

The functions $\mu^\tau_p(\cdot)$ and $\sigma^\tau_p(\cdot)$ may be based on theoretical arguments or inferred statistically. For example, since the time scales likely to be considered will be short, the drift may be thought of as changing very gradually and in proportion to the moving average of path flow (departure rate) for the path of interest. Furthermore, the volatility will climb sharply with rising congestion (effective path delay) until a jam density is reached at which time volatility will drop sharply toward zero.

As mentioned earlier, a congestion call option is sold for a given route when departure from the origin occurs. A congestion call option is exercised at the end of the day (at a pre-specified time) when all the network users return from work. To evaluate the potential of congestion calls to influence the net social costs of congestion, one may follow the now standard analysis, based on the Itô calculus, of option valuation but with important modifications explained below.

**6.10.5 The Value Function for the Congestion Call Option**

We are now in a position to analyze the path specific value function of the congestion call option using Ito’s Lemma and the standard arbitrage-does-not-occur condition, provided we properly define the notion of arbitrage associated with telecommuting. We derive a partial differential equation with proper initial and boundary conditions, solution of which provides the value function of the option for a particular path.
Itô’s Lemma

It is useful to remember that since the underlying stochastic process for asset prices is a Wiener process, we know

$$(dB_p)^2 \rightarrow dt$$ as $$dt \rightarrow 0$$ so that $$dB_p = O\left(\sqrt{dt}\right)$$ (6.25)

for all $$p \in \mathcal{P}$$. As a consequence, our dynamics (6.22) lead to

$$(dS_p)^2 = (S_p \mu_p dt + S_p \sigma_p dB_p)^2 = (\sigma_p S_p)^2 (dB_p)^2 + 2 \mu_p \sigma_p S_p^2 \cdot dt \cdot dB_p + (\mu_p S_p)^2 (dt)^2$$

$$\simeq \sigma_p^2 S_p^2 (dB_p)^2 \rightarrow \sigma_p^2 S_p^2 dt$$ (6.26)

when we retain only first order terms. On the basis of this result, Itô’s lemma tells us that for all $$p \in \mathcal{P}$$

$$dF(S_p, t) = \sigma_p S_p \frac{\partial F(S_p, t)}{\partial S_p} dB_p + \mu_p S_p \frac{\partial F(S_p, t)}{\partial S_p} dt + \frac{1}{2} (\sigma_p S_p)^2 \frac{\partial^2 F(S_p, t)}{\partial (S_p)^2} dt + \frac{\partial F(S_p, t)}{\partial t} dt$$ (6.27)

for any function $$F(S_p, t)$$ that has the first and second derivatives appearing on the right hand side of (6.27).

The Value Function for a Path

With the preceding background we are ready to begin construction of the congestion call value function. In particular we express the net value of the congestion portfolio to travelers using path $$p$$ instead of other routes as

$$\pi_p = V_p - \Delta_p \cdot S_p \quad \text{for } (i, j) \in \mathcal{W}, p \in \mathcal{P}_{ij}$$ (6.28)

where $$V_p$$ is the valuation of the call option represented by path $$p$$ and $$\Delta_p$$ is a scaling factor chosen to eliminate stochasticity from the partial differential equation describing value of the option (the negative of which has an interpretation of the number of underlying assets
held in the portfolio). Note that the value function depends on the asset price and the time of departure; that is

\[ V_p = F_p \left[ S_p(t), t \right] \quad \text{for} \ (i,j) \in \mathcal{W}, p \in \mathcal{P}_{ij} \quad (6.29) \]

The value measure \( V_p \) captures far more than the traditional unit cost of flow notions emphasized in the transportation network modeling literature; in particular, it implicitly reflects the risk associated with volatile congestion levels. Clearly, taking \( \Delta_p \) to be fixed during the time step \( dt \), one may write

\[ d\pi_p = dV_p - \Delta_p \cdot dS_p \]

We apply Itô’s Lemma to obtain

\[ dV_p = \sigma_p[t, \Psi_p] S_p \frac{\partial V_p}{\partial S_p} dB_p + \mu_p(t) S_p \frac{\partial V_p}{\partial S_p} dt + \frac{1}{2} (\sigma_p[t, h] S_p)^2 \frac{\partial^2 V_p}{\partial (S_p)^2} dt + \frac{\partial V_p}{\partial t} dt \]

As a consequence

\[
\begin{align*}
    d\pi_p & = \sigma_p[t, \Psi_p] \frac{\partial V_p}{\partial S_p} dB_p + \mu_p(t) S_p \frac{\partial V_p}{\partial S_p} dt + \frac{1}{2} (\sigma_p[t, \Psi_p] S_p)^2 \frac{\partial^2 V_p}{\partial (S_p)^2} dt + \frac{\partial V_p}{\partial t} dt - \Delta_p \cdot dS_p \\
    & = \sigma_p[t, \Psi_p] \left( \frac{\partial V_p}{\partial S_p} - \Delta_p \right) dB_p + \mu_p(t) S_p \left( \frac{\partial V_p}{\partial S_p} - \Delta_p \right) dt + \frac{1}{2} (\sigma_p[t, \Psi_p] S_p)^2 \frac{\partial^2 V_p}{\partial (S_p)^2} dt + \frac{\partial V_p}{\partial t} dt
\end{align*}
\]

We now pick the scale to obey

\[ \Delta_p = \frac{\partial V_p}{\partial S_p} \quad \forall p \in \mathcal{P} \quad (6.30) \]

so that

\[ d\pi_p = \frac{1}{2} (\sigma_p[t, h])^2 \frac{\partial^2 V_p}{\partial (S_p)^2} dt + \frac{\partial V_p}{\partial t} dt \quad \forall p \in \mathcal{P} \quad (6.31) \]

**The Arbitrage-Does-Not-Occur Condition**

We postulate that people have distinct in-office and at-home marginal productivities with respect to time. We assume no productive work occurs during the commute, so the value of the option to travel must equal the net productivity loss associated with commuting; that is

\[ d\pi_p = \left( MP_{ij}^{office} - MP_{ij}^{home} \right) dt \quad \forall p \in \mathcal{P}_{ij} \quad (6.32) \]
for each $p \in \mathcal{P}_{ij}$ and $M_{ij}^{\text{office}}$ and $M_{ij}^{\text{home}}$ are the office and at-home marginal productivities, respectively. Combining (6.31) and (6.32) leads to the following system of partial differential equations that describe evolution of the congestion call for every origin-destination pair $(i,j) \in \mathcal{W}$:

$$
\frac{1}{2} (\sigma_p[t,h] S_p)^2 \frac{\partial^2 V_p}{\partial (S_p)^2} + \frac{\partial V_p}{\partial t} - \left( M_{ij}^{\text{office}} - M_{ij}^{\text{home}} \right) = 0 \quad \forall p \in \mathcal{P}_{ij} \quad (6.33)
$$

**Initial and Boundary Conditions** To keep the exposition simple, we assume that the expiry time of the option is fixed at midnight $(\tau \Delta)$ for all potential commuters. Therefore the relevant boundary conditions for the European congestion call option are

$$
V_p^\tau [S_p(\tau \Delta), \tau \Delta] = \nu \max [S_p(\tau \Delta) - \alpha_p, 0] \quad \forall (i,j) \in \mathcal{W}, p \in \mathcal{P}_{ij} \quad (6.34)
$$

$$
V_p[0,t] = 0 \quad \forall p \in \mathcal{P} \quad (6.35)
$$

and

$$
V_p^\tau [S_p(\tau \Delta), \tau \Delta] \rightarrow S_p^\tau(\tau \Delta) \quad \text{as} \quad S_p^\tau(\tau \Delta) \rightarrow \infty \quad \forall p \in \mathcal{P} \quad (6.36)
$$

where $\alpha_p$ is the strike price of the option associated with path $p$ and $\nu$ is the known value of time. Note that (6.34) ensures that at the time of expiry the option would not be exercised if the net value of the option $S_p(T) - \alpha_p$ is negative. Similarly, if asset price becomes 0 at any time $t$, it will stay at 0 for rest of the time as per the geometric brownian motion (6.22) - (6.23). Under such circumstances, the option becomes meaningless, and hence value of option becomes 0 as per (6.35). Furthermore, (6.36) tells us that if the underlying asset price becomes very high, the exercise price becomes very small compared to the asset price and, hence, the value of the option is, in the sense of a limit, the asset price.

**6.10.6 Dynamic Equilibrium and the Stochastic Differential Variational Inequality (SDVI) Problem**

We assume that the network travelers are risk-neutral in selecting a route and departure time. A risk neutral user purchases an option only if the expected financial liability arising from the purchase of the option is negative (which implies a positive net cash flow.)
Otherwise, the user simply takes the same route without purchasing the option. We use a
dynamic Cournot-Nash equilibrium to describe the decisions of all agents, both those pur-
chasing option and those not (still take the same route and pay the prevailing asset price at
the time of arrival). However, because of the Brownian motion we consider in describing the
price fluctuations of travel alternatives arising from the unobservable aspects of individual
travelers’ utility functions over time, the formulation becomes a stochastic differential (that
is, dynamic) variational inequality (SDVI). Let $\kappa$ be the premium charged by the central
authority to decide purchase price of an option which then becomes $\kappa \cdot V^T_p (S_p (t, h^*), t)$.
This is the price one has to pay at time $t$ to purchase the option. When $\kappa > 1$, a premium
is charged where as $\kappa < 1$ corresponds to a scenario where the central authority provides
subsidy to encourage more travelers to purchase an option. Therefore, $\kappa$ is a decision vari-
able in the hand of the central authority, however, we consider this as exogenous for our
user equilibrium problem.

**Net Trip Value For The Option Purchasers**

Expected net trip value for the option purchasers corresponding to a departure time $t$ can
be expressed in terms of their expected return at the time of expiry and price they paid to
purchase the option

$$EV^T_1 (t, h^*) = E_t \left[ (S^*_p (\tau \Delta) - \alpha_p)^+ | S^*_p (t) \right] - \kappa \cdot V^T_p (S^*_p (t), t)$$  \hspace{1cm} (6.37)

where the first term in the right hand side is the expected payoff at the time of expiry conditioned over the current asset price and the second term is the price to be paid at
time $t$ to purchase the option (and depart from the origin). We use the usual notation
$(y)^+ \equiv \max (y, 0)$. Note that since asset price follows geometric Brownian motion, asset
price distribution follows a *log-normal* distribution, and thus the conditional expectation
may be obtained as a closed form expression. However, as this is done in this paper, one
can also obtain the conditional expectation numerically using finite element approximation.
Net Trip Value For Non-option Purchasers

If a traveler does not purchase an option, he/she would pay a price equals to the prevailing market price at the time he/she reaches destination. Therefore, the net expected trip cost for such travelers is the expected asset price at the time of their arrival conditioned over the current asset price. Thus, the expected net trip value in this case is

\[
EV_2^\tau (t, h^*) = -E_t \left[ S_p^*(t + D_p(t, h^*)) \right] S_p^*(t) \tag{6.38}
\]

where \(D_p(t, h^*)\) is the path delays along path \(p\) corresponding to a departure time \(t\).

6.10.7 Properties of Modified Dynamic User Equilibrium

Travelers departing at time \(t\) will purchase the option when \(EV_1^\tau (t, h^*) > EV_2^\tau (t, h^*)\) and vice versa. Therefore, the definition of a dynamic user equilibrium is modified to consider the net expected trip value. In particular, the modified DUE problem must obey:

1. non-trivial path flow requires maximal net value:

\[
h_p > 0 \implies -v \cdot F [t + D_p(x^*, t) - T_A] + \max \left( EV_1^\tau (t, h^*), EV_2^\tau (t, h^*) \right) = \xi_{ij}
\]

\[
\forall (i,j) \in W, p \in P_{ij}, \tau \in [1, L] \text{ and } t \in [(\tau - 1) \Delta, \tau \Delta]
\]

2. net trip value has a well-defined maximum:

\[
-F [t + D_p(x^*, t) - T_A] + \max \left( EV_1^\tau (t, h^*), EV_2^\tau (t, h^*) \right) \leq \xi_{ij}
\]

\[
\forall (i,j) \in W, p \in P_{ij}, \tau \in [1, L] \text{ and } t \in [(\tau - 1) \Delta, \tau \Delta]
\]

3. flows, volumes, and maximal trip value are non-negative:

\[
h \geq 0
\]

\[
x \geq 0
\]

\[
\xi \geq 0
\]
Articulation of an infinite dimensional variational inequality consistent with the above formulation of stochastic DUE requires that we define

\[
\Lambda = \left\{ (h, g): (6.4), (6.5), (6.8), (6.9), (6.19), (6.20), (6.11), (6.24), (6.22), (6.23), (6.33), (6.34), (6.35) \text{ and } (6.36) \text{ hold} \right\}
\]

Consequently we may characterize dynamic user equilibrium in the presence of congestion options as follows: for a sample path \( \omega \), which is a realization of the geometric Brownian disturbances on the right-hand sides of (6.22), we solve the following stochastic differential variational inequality (SDVI):

\[
\text{find } \{h^*(\omega), g^*(\omega)\} \in \Lambda \text{ such that } \sum_{(i,j) \in W} \sum_{p \in P_{ij}} \sum_{j=0}^{L-1} \int_{j \Delta}^{(j+1) \Delta} \left\{ \frac{vF[t + D_p(x^*, t) - T_A]}{\text{EV}_{\mathcal{I}}^T(t, h^*)} - \max \left[ E_t \left[ \left( S_p^* (\tau \Delta) - \alpha_p \right)^+ | S_p^* (t, \omega) \right] \right] - \kappa V_p^T \left( S_p^* (t, \omega), t \right), \right. \\
\left. -E_t \left[ S_p^* (t + D_p(t, h^*)) | S_p^* (t, \omega) \right] \right\} \cdot (h_p - h_p^*(\omega)) dt \geq 0
\]

for all \((h, g) \in \Lambda\)

where \( v \) is the value of time, \( S(t, \omega), h(\omega) \) and \( g(\omega) \) denote respectively the realizations of asset prices, path flows and arc departure rates for sample path \( \omega \in \Omega \), where as \( \Omega \) is the set of all possible sample paths. Clearly, (6.39) is a SDVI where the uncertainties are associated with the asset price \( S_p^* (t, \omega) \) and value function of the congestion option \( V_p \left[ S_p^* (t, \omega), t \right] \).

Furthermore, we may obtain the distributions of \( \{h^*(\omega), g^*(\omega)\} \) where for a given outcome (realization) \( \tilde{\omega} \in \Omega \), \( \{h^*(\tilde{\omega}), g^*(\tilde{\omega})\} \) is a solution of the SDVI (6.39).
6.10.8 Fokker-Planck Equation and Expected Net Congestion Costs

The probability density function of asset price $S_p(t)$, conditioned over $S_p(0) = K_p$, is

$$
\psi(z,t,K_p) = \lim_{\Delta z \to 0} \frac{P\{z < S_p(t) < z + \Delta z | S_p(0) = K_p\}}{\Delta z}
$$

(6.40)

for every path $p \in \mathcal{P}$. The entities (6.40) are actually conditional transition probabilities. In fact, the Fokker-Planck equation is a forward partial differential equation whose solutions are the conditional transition probabilities (6.40). The Fokker-Planck equation\(^3\) associated with the geometric Brownian motion (6.22) and initial condition (6.23) is

$$
\frac{1}{2} \sigma^2_p(t,h^*) S^2_p(h^*,t) \frac{\partial^2}{\partial S^2_p} \psi(a,t,K_p) = \mu_p(t,h^*) S_p(h^*,t) \frac{\partial}{\partial S_p} \psi(a,t,K_p) + \frac{\partial}{\partial t} \psi(a,t,K_p)
$$

for all $p \in \mathcal{P}$

(6.41)

the initial and boundary conditions are:

$$
\psi(a,0,K_p) = \delta(a - K_p)
$$

(6.42)

and

$$
\begin{align*}
\psi(\infty,t,K_p) &= 0 \text{ for all } t \in [0,L], p \in \mathcal{P} \\
\psi(0,t,K_p) &= 0 \text{ for all } t \in [0,L], p \in \mathcal{P}
\end{align*}
$$

(6.43)

(6.44)

where $\delta(\cdot)$ is the dirac delta function.

Closed Form Solution of (6.41)

For a given sample path $\omega$, the path flows $h^*(\omega)$ can be expressed as a pure function of time as explained in Friesz and Mookherjee\(^{[12]}\). Hence the volatility and drift coefficients $\sigma(h^*(\omega),t)$ and $\mu(h^*(\omega),t)$ may also be expressed as pure functions of time. With the above observations in mind, it is possible to show that the conditional probability density function of asset price $S_p(t)$, conditioned over the initial condition follows a log-normal

\(^3\)See Montrol and Badger\(^{[?]}\) for an explanation and derivation of the Fokker-Planck equation.
distribution, therefore \( \ln (S_p(t)) \) follows normal distribution with mean

\[
m_p(h^* (\omega), t) \equiv \ln (K_p) + \int_0^t \left[ \mu_p(h^* (\omega), \kappa) - \frac{1}{2} \sigma_p^2(h^* (\omega), \kappa) \right] d\kappa
\]

and variance

\[
\eta_p^2(h^* (\omega), t) \equiv \int_0^t \sigma_p^2(h^* (\omega), \kappa) d\kappa
\]

This partial differential equation enjoys the closed form solution\(^4\)

\[
\psi(a, t, K_p) = \frac{1}{a \sqrt{2\pi \cdot \eta_p^2(h^* (\omega), t)}} \exp \left( -\frac{1}{2} \left[ \frac{\log (a) - m_p(h^* (\omega), t)}{\eta_p(h^* (\omega), t)} \right]^2 \right) \text{ for } a > 0
\]

\[
= 0 \text{ otherwise}
\]

Making substitutions for \( \psi(a, t, K_p) \) in the previously derived expressions for \( m_p(h^* (\omega), t) \) and \( \eta_p^2(h^* (\omega), t) \), we obtain for \( a > 0 \)

\[
\psi(a, t, K_p) = \frac{1}{a \sqrt{2\pi \cdot \int_0^t \eta_p^2(h^* (\omega), \kappa) d\kappa}} \\
\cdot \exp \left( -\frac{\left[ S_p(t, \omega) - \ln (K_p) - \int_0^t (\mu_p(h^* (\omega), \kappa) - \frac{1}{2} \sigma_p^2(h^* (\omega), \kappa)) d\kappa \right]^2}{2 \int_0^t \sigma_p^2(h^* (\omega), \kappa) d\kappa} \right)
\]

Expressions for Conditional Expectations

As mentioned before, asset price dynamics follow a geometric Brownian motion, the asset price distribution conditioned over its current value follows a lognormal distribution which can be obtained by solving the Fokker Planck partial differential equation, as discussed in section 6.10.8. We need to obtain the expressions of \( E_t \left[ (S_p^* (\tau \Delta) - \alpha_p)^+ | S_p^* (t, \omega) \right] \) and \( E_t \left[ S_p^* (t + D_p (t, h^*)) | S_p^* (t, \omega) \right] \).

\( ^4 \text{See Wilmott et al [22] for the derivation of the closed form solution using the Fourier transform.} \)
bution with mean

\[ m_p \equiv \ln \left( S_p^* (t, \omega) \right) + \int_t^{\tau \Delta} \left[ \mu_p (h^* (\omega), \kappa) - \frac{1}{2}\sigma_p^2 (h^* (\omega), \kappa) \right] d\kappa \quad (6.46) \]

and variance

\[ \eta_p^2 \equiv \int_t^{\tau \Delta} \sigma_p^2 (h^* (\omega), \kappa) d\kappa \quad (6.47) \]

We use an indicator function \( I(\cdot) \) such that for any event \( A \), \( I(A) = 1 \) if the event occurs and \( I(A) = 0 \) otherwise. For any such event \( A \) we have

\[ E(I(A)) = P(A) \]

We can hence write

\[
E_t \left[ (S_p^* (\tau \Delta) - \alpha_p)^+ \right| S_p^* (t, \omega)] = E_t \left[ (S_p^* (\tau \Delta) - \alpha_p) \cdot I \left( S_p^* (\tau \Delta) \geq \alpha_p \right) \right| S_p^* (t, \omega)] \\
= E_t \left[ S_p^* (\tau \Delta) \cdot I \left( S_p^* (\tau \Delta) \geq \alpha_p \right) \right| S_p^* (t, \omega)] \\
- \alpha_p P \left( (S_p^* (\tau \Delta) \geq \alpha_p) \right| S_p^* (t, \omega) \right) \quad (6.48)
\]

Now, to derive an expression \( P \left( (S_p^* (\tau \Delta) \geq \alpha_p) \right| S_p^* (t, \omega) \right) \), let us define \( z \) as the standard normal random variable with CDF \( \Phi(\cdot) \). Therefore \( S_p^* (\tau \Delta) \) can be expressed using \( m_p \) and \( \eta_p \) as defined respectively in (6.46) and (6.47) as

\[ S_p^* (\tau \Delta) = \exp (m_p + \eta_p \cdot z) \quad (6.49) \]

It follows automatically

\[
P \left( (S_p^* (\tau \Delta) \geq \alpha_p) \right| S_p^* (t, \omega) \right) = P \left\{ m_p + \eta_p \cdot z \geq \ln (\alpha_p) \right\} \\
= P \left\{ z \geq \frac{\ln (\alpha_p) - m_p}{\eta_p} \right\} \\
= 1 - \Phi \left( \frac{\ln (\alpha_p) - m_p}{\eta_p} \right) \\
= \Phi \left( \frac{m_p - \ln (\alpha_p)}{\eta_p} \right) \quad (6.50)
\]
Next, we need to get an expression of $E_t \left[ S^*_{p} (\tau \Delta) \cdot I \left( S^*_{p} (\tau \Delta) \geq \alpha_p \right) \right]$. Using the expression for $S^*_{p} (\tau \Delta)$ in (6.49), we have

$$E_t \left[ S^*_{p} (\tau \Delta) \cdot I \left( S^*_{p} (\tau \Delta) \geq \alpha_p \right) \right] = \frac{1}{\sqrt{2\pi}} \int_{m_p-\ln \alpha_p}^{m_p+\eta_p} e^{m_p+\eta_p \cdot \kappa} \cdot e^{-\frac{\kappa^2}{2}} d\kappa$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\kappa \geq -\frac{m_p-\ln \alpha_p}{\eta_p}} e^{-\frac{1}{2} (\kappa-\eta_p)^2} \cdot e^{m_p+\eta_p \cdot \kappa} d\kappa$$

$$= e^{m_p+\eta_p \cdot \frac{\eta_p}{2}} \cdot P \left\{ z + \eta_p \geq - \left( \frac{m_p - \ln (\alpha_p)}{\eta_p} \right) \right\}$$

$$= e^{m_p+\eta_p \cdot \frac{\eta_p}{2}} \cdot \Phi \left( \eta_p + \frac{m_p - \ln (\alpha_p)}{\eta_p} \right)$$

where $m_p$ and $\eta_p$ are defined in (6.46) and (6.47) respectively.

**Expression For $E_t \left[ S^*_{p} (t + D_p (t, h^*)) \right]$:** The expression for $E_t \left[ S^*_{p} (t + D_p (t, h^*)) \right]$ can also be obtained in closed form after observing that $S^*_{p} (t + D_p (t, h^*))$ conditioned over $S^*_{p} (t, \omega)$ also follows a lognormal distribution; therefore ln $(S^*_{p} (t + D_p (t, h^*)))$ follows normal distribution with mean

$$\tilde{m}_p \equiv \ln (S^*_{p} (t, \omega)) + \int_{t}^{t+D_p (t, h^*)} \left[ \mu_p (h^* (\omega), \kappa) - \frac{1}{2} \sigma_p^2 (h^* (\omega), \kappa) \right] d\kappa$$

and variance

$$\tilde{\eta}_p^2 \equiv \int_{t}^{t+D_p (t, h^*)} \sigma_p^2 (h^* (\omega), \kappa) d\kappa$$

Therefore,

$$E_t \left[ S^*_{p} (t + D_p (t, h^*)) \right] = \tilde{m}_p$$

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6.11 A Fixed Point Algorithm

To solve (6.39) for a specific sample path realization \( \omega \) we employ a fixed point algorithm, which is stated below.

*Fixed Point Algorithm For Solving (6.39)*

**Step 0. Realization of Random Noise.** Generate random noise components \( dB^\omega_p (t) \sim N (0, dt) \) corresponding to the asset-price-sample-path \( \omega \) for all \( t \in [0, L\Delta] \).

**Step 1. Initialization.** Identify an initial feasible solution \((h^0, g^0) \in \Lambda \) and set \( k = 0 \).

**Step 2. Sample Path Realization.** Generate an asset-price-sample-path realization using (6.22) and (6.23) with \( h^k (\omega) \) as the path flow vector, and thereby update the asset prices, \( S^k_p (t, \omega) \), for \( 0 \leq t \leq L\Delta \) and all \( p \in \mathcal{P} \).

**Step 3. Solve the PDE.** Solve the PDE (6.33)-(6.36) with the volatility \( \sigma_p (t, h^k (\omega)) \) to obtain the current path-specific congestion call value operators \( V_p [S^k_p (t, \omega), t] \), for all \((\tau - 1) \Delta \leq t \leq \tau \Delta, \tau \in [1, L] \) and all \( p \in \mathcal{P} \).

**Step 3a. Compute Demand Reduction.** Using \( h^k (\omega) \) as the path flow vector solve constrained difference equations (6.19) - (6.20) to compute \( R^\tau_{ij} \) and \( Q^\tau_{ij} \).

**Step 4. Form and Solve Fixed Point Problem.** Re-state the DUE problem as a fixed point problem based on the call value operators and effective travel demand as found in Steps 3 and 3a respectively, solve iteratively\(^5\). Call the solution \((h^{k+1} (\omega), g^{k+1} (\omega))\).

**Step 5. Stopping test.** If

\[
\left\| \begin{pmatrix} h^{k+1} (\omega) \\ g^{k+1} (\omega) \end{pmatrix} - \begin{pmatrix} h^k (\omega) \\ g^k (\omega) \end{pmatrix} \right\| \leq \varepsilon
\]

\(^5\)See Friesz and Mookherjee [12] for the details of converting a differential variational inequality into a fixed point problem in Hilbert space.
where \( \varepsilon \in \mathbb{R}^{1}_{++} \) is a preset tolerance, stop and declare \( h^* (\omega) \approx h^{k+1} (\omega) , g^* (\omega) \approx g^{k+1} (\omega) \). Otherwise set \( k = k + 1 \) and go to Step 2.

Note that by generating different random noise components, \( dB^\omega_p (t) \) and thereby different sample paths \( \omega \) in step 0, we will get different solutions \( h^* (\omega) , g^* (\omega) \). It is also important to realize that the fixed point algorithm described above may be carried out in continuous time provided we employ a continuous time representation of the solution of each subproblem from Step 4. This may be done using a continuous time gradient projection method, as in Friesz and Mookherjee [12], to deal with time-shifted controls.

We further comment that it is unlikely that the path value operators will always be strictly monotonic. In that event, as explained in [12], the fixed point algorithm used for Step 4 may fail to converge and, consequently, the algorithm we have proposed for solving \( (6.39) \) is a heuristic.

### 6.11.1 Approximating the Expected Value of Net Congestion Costs

To compute the average net social congestion costs, we use the transition probabilities obtained through the solution of \( (6.41) \) through \( (6.44) \), as well as the net values of the congestion calls corresponding to each sample path of interest. Therefore, let us denote

\[
V_p^{\text{net}} (\omega, t) = vF [t + D_p (x^*, t) - T_A] - \Gamma (t) \cdot \left\{ \max \left[ S_p^* (\tau \Delta, \omega) - \alpha_p, 0 \right] - \kappa V_p^\tau \left( S_p^* (t, \omega), t \right) \right\} \\
+ \{1 - \Gamma (t)\} \cdot S_p^* (t + D_p (t, h^*), \omega)
\]

be the congestion cost at time \( t \in [0, L\Delta] \) associated with the sample path \( \omega \in \Omega \), where \( \Gamma (t) \) is the indicator function expressing whether people departed at time \( t \) bought the option or not i.e.,

\[
\Gamma (t) = \begin{cases} 
1 & \text{if } EV^\tau_1 (t, h^*) \geq EV^\tau_2 (t, h^*) \\
0 & \text{otherwise}
\end{cases}
\]
then the expected net congestion cost is

$$EV^{net} = \sum_{(i,j) \in W} \sum_{p \in P} \sum_{j=0}^{L-1} \left( (j+1) \Delta \int_{z=-\infty}^{z=+\infty} V^{net}_p(\omega, t) \psi_p(z, t, K_p) \, dz \right) dt$$

(6.55)

Using an ensemble of a large number of samples one expects that

$$EV^{net} \approx \frac{1}{R} \sum_{(i,j) \in W} \sum_{p \in P} \sum_{\omega \in \Omega} \sum_{j=0}^{L-1} \left( (j+1) \Delta \int_{z=-\infty}^{z=+\infty} V^{net}_p(\omega, t) \psi_p(S_p^*(t, \omega), t, K_p) \, dt \right)$$

(6.56)

is a good approximation, where $R$ is of course the number of sample path realizations considered.

### 6.12 Numerical Example

Numerical solution of the differential variational inequality (6.39) using the fixed point algorithm introduced in Section 6.11 requires that we know how the value function evolves over time for a given sample path of congestion security prices. We postulate that volatility depends on departure rates; consequently, for a given sample path realization, each partial differential equation (6.33) is coupled to the differential variational inequality (6.39).

Using the current value of departure rates, our software code employs the current path specific volatility to solve (6.33) with appropriate boundary and initial conditions. The value function $V_p(S_p, t)$ thus obtained is then returned to the main program to solve the subproblem of the fixed point scheme. Therefore, at each fixed point major iteration, the stochastic differential equations (6.22) and (6.23) along with the partial differential equations (6.33) are solved using the current path flow vector $(h^k)$ and arc exit flow vector $(g^k)$ where $k$ is the index of major (fixed point) iterations. In particular, the partial differential equation (6.33) is solved using Matlab’s pdepe parabolic partial differential equation solver.

Our presentation of numerical results is divided in four sections (i) DUE results for a 5 arc, 4 node network having a single OD pair and 3 feasible paths with no options being
traded and telecommuting is not a feasible option to the commuters; (ii) the same network with no options being traded but some of the commuters can telecommute on a given day; (iii) the same network with telecommuting a choice to the commuters and a market for congestion is created for a subset of routes; and finally (iv) the same network with option trading for one of the three paths with telecommuting a feasible choice. We compare the path flows for both cases and compute the net congestion costs with and without option trading and telecommuting. The example presented below establishes that congestion call option trading can lower the social costs of congestion.

6.12.1 Multi-period DUE model : No telecommuting and options

When telecommuting is not a viable choice to the travelers, OD travel demand stays unchanged hence flow patterns remain identical for all days. In this case each of the within-day subproblems are decoupled and may be solved independently as outlined by Friesz and Mookherjee [12]. In what follows, we consider the 5 arc, 4 node network shown below. The forward star array and arc delay functions $D_a(x_a(t))$ for all 5 arcs of the network are contained in the following table:

<table>
<thead>
<tr>
<th>Arc name</th>
<th>From node</th>
<th>To node</th>
<th>Arc Delay, $D_a(x_a(t))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{2} + \frac{x_{a1}}{70}$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>1</td>
<td>3</td>
<td>$1 + \frac{x_{a2}}{150}$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>2</td>
<td>3</td>
<td>$\frac{1}{7} + \frac{x_{a3}}{100}$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>2</td>
<td>4</td>
<td>$1 + \frac{x_{a4}}{150}$</td>
</tr>
<tr>
<td>$a_5$</td>
<td>3</td>
<td>4</td>
<td>$\frac{1}{7} + \frac{x_{a5}}{100}$</td>
</tr>
</tbody>
</table>
The 5-arc 4-node traffic network with (1, 4) being the OD-pair
There is a travel demand of $Q = 75$ units from node 1 (origin) to node 4 (destination).
There are 3 paths connecting nodes 1 through 4, namely

\[ \mathcal{P}_{14} = \{p_1, p_2, p_3\} \]
\[ p_1 = \{a_1, a_4\} \]
\[ p_2 = \{a_2, a_5\} \]
\[ p_3 = \{a_1, a_3, a_5\} \]

We consider the planning horizon to be 4 days (i.e., $L = 4$) and the length of each day is $\Delta = 24$ hours. The desired arrival time for commuters is $T_A = 13$ (1:00 PM of every day). The controls (path flows and arc exit flows) and states (arc traffic volumes) are enumerated in the following table:

<table>
<thead>
<tr>
<th>Paths</th>
<th>Path Flows</th>
<th>Arc Exit Flows</th>
<th>Traffic Volume of Arcs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>$h_{p_1}$</td>
<td>$g_{a_1}^{p_1}, g_{a_4}^{p_1}$</td>
<td>$x_{a_1}^{p_1}, x_{a_4}^{p_1}$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$h_{p_2}$</td>
<td>$g_{a_2}^{p_2}, g_{a_5}^{p_2}$</td>
<td>$x_{a_2}^{p_2}, x_{a_5}^{p_2}$</td>
</tr>
<tr>
<td>$p_3$</td>
<td>$h_{p_2}$</td>
<td>$g_{a_1}^{p_3}, g_{a_3}^{p_3}, g_{a_5}^{p_3}$</td>
<td>$x_{a_1}^{p_3}, x_{a_3}^{p_3}, x_{a_5}^{p_3}$</td>
</tr>
</tbody>
</table>

We consider the symmetric early/late arrival penalty

\[ F [t + D_p (x, t) - T_A] = [t + D_p (x, t) - T_A]^2 \]
Furthermore, without any loss of generality, we take the initial traffic volumes on every arc to be zero:

\[ x_{ai}^p(0) = 0 \quad \forall p \in \mathcal{P}, i \in [1, m(p)] \]

We forgo the detailed symbolic statement of this example and instead provide numerical results in graphical form for an essentially exact solution achieved after 29 iterations of the algorithm. Figures below depict departure rates and arc exit flows for paths \( p_1, p_2 \) and \( p_3 \) respectively.

Path and arc exit flows for path 1

Path and arc exit flows for path 2

Path and arc exit flows for path 3
Cumulative traffic volumes on the 5 different arcs are plotted against time in figure below where

\[
\begin{align*}
x_{a1}(t) &= x_{a1}^{p1}(t) + x_{a1}^{p3}(t) \\
x_{a2}(t) &= x_{a2}^{p2}(t) \\
x_{a3}(t) &= x_{a3}^{p3}(t) \\
x_{a4}(t) &= x_{a4}^{p4}(t) \\
x_{a5}(t) &= x_{a5}^{p5}(t) + x_{a5}^{p3}(t)
\end{align*}
\]

for all time \( t \in [0, L\Delta] \).

Cumulative arc volume vs. time

Note that the effective path delay operator in (??) gives the unit travel cost along a path \( p \) at time \( t \). Figure below analyzes the effective delay and flow for path \( p_1 \) by plotting both for the same time scale which shows that path flow is maximal when the associated unit travel cost (effective path delay) is at its well defined minimum. Similar comparisons are also
made for paths $p_2$ and $p_3$.

Path flows and unit travel costs for path $p_1$  

Similar comparison for path $p_2$

Comparison of path flows and associated unit travel costs for path $p_3$

6.12.2 Multi-period DUE model: Telecommuting without option

When telecommuting is an alternative to some of the travelers, number of travelers decide to telecommute at the beginning of day $\tau$ is given by the constrained difference equations (6.19) - (6.20). proportionality constant ($\eta_{ij}$) associated with (6.19) as 0.06. The threshold of congestion set by the $(1,4)$-travelers is $\chi_{14} = 20$. The policy determined parameter $\gamma_4$ is set at 0.2 which says that the all employers located at node 4 allow upto 20% of their total work force to telecommute on any given day. We once again forgo the detailed symbolic statement of this example and instead provide numerical results in graphical form for an
essentially exact solution achieved after 32 iterations of the algorithm. Figures below respectively depict departure rates and arc exit flows for paths $p_1$, $p_2$, and $p_3$. 

Path and arc exit flows for path 1

Path and arc exit flows for path 2

Path and arc exit flows for path 3
Next we plot cumulative traffic volumes on the 5 different arcs against time.

Cumulative arc volume vs. time
As before, we analyze the effective delay and flow for paths $p_1$, $p_2$ and $p_3$ by plotting both for the same time scale in figures below respectively; these plots show that path flow is maximal when the associated unit travel cost (effective path delay) is at its well defined minimum.

Path flows and unit travel costs for path $p_1$  

Similar comparison for path $p_2$
Comparison of path flows and associated unit travel costs for path $p_3$

Net travel demand and demand reduction are plotted below against the same time scale (day) which clearly demonstrates that more commuters switch to telecommuting as their rolling average experience of congestion increases with passage of time.

6.12.3 Market for Congestion with Telecommuting

When we assume that there exists a market for congestion, travelers choose to decide to take a particular route need to purchase that right at the prevailing asset price. We consider a market for trading rights to travel along path $p_1$; no such market exists for paths $p_2$ and
We assume the deterministic drift $\mu_{p_1}(t)$ and path specific volatility $\sigma_{p_1}(t,h)$ have the following form:

$$
\begin{align*}
\mu_{p_1}(t) &= \alpha_{p_1} + \beta_{p_1} \cdot t \\
\sigma_{p_1}(t, D_p^r(t, x(h,t))) &= \gamma_{p_1} + \eta_{p_1} \cdot D_p^r(t, x(h,t)) + \kappa_{p_1} \cdot [D_p^r(t, x(h^*, t))]^2
\end{align*}
$$

(6.57) (6.58)

where $\alpha_{p_1}, \gamma_{p_1}, \eta_{p_1} \in \mathbb{R}_{++}^1$ and $\beta_{p_1}, \kappa_{p_1} \in \mathbb{R}^1$. We choose $\alpha_{p_1} = 0.03$, $\beta_{p_1} = -0.001$, $\gamma_{p_1} = 0.2$, $\eta_{p_1} = 0.02$ and $\kappa_{p_1} = -3 \times 10^{-5}$. From our choice of parameters, it is evident that deterministic drift is overshadowed by the path specific volatility. Also, the volatility associated with a path has nonlinear relationship with own path delay. Our model of volatility also expresses the tendency of volatility to drop after a jam density is reached.

We generate sample paths using a traditional Euler scheme with time discretization:

$$
S_{p_1,t} = S_{p_1,t-1} + \Delta_t \cdot \mu_{p_1,t} + B_{p_1,t} \cdot \sigma_{p_1,t} \quad \forall t \in [1, N]
$$

$$
S_{p_1,0} = K_{p_1}
$$

where the entire planning horizon is discretized using time step $\Delta_t = \frac{1}{4}$. The noise, $B_{p_1,t}$ follows a normal distribution with mean 0 and standard deviation $\Delta_t$. We choose $K_{p_1} = 4$ and generate a number of sample paths. We plot variations of asset price vs. time for six such sample paths.

Plot of asset price ($S_{p_1}$) vs. time associated with path $p_1$ for 6 sample path realizations.
The numerical results we provide here in graphical form corresponds to an essentially exact solution achieved after 18 iterations of the fixed point algorithm. Figures below depict departure rates and arc exit flows for paths $p_1$, $p_2$ and $p_3$ respectively.

Path and arc exit flows for path 1

Path and arc exit flows for path 2

Path and arc exit flows for path 3
Cumulative traffic volumes on the 5 different arcs are plotted against time in figure below.

Cumulative arc volume vs. time

Further we analyze the effective delay and flow for paths $p_1$, $p_2$ and $p_3$ by plotting both for the same time scale in figures below respectively; these figures show that path flow is maximal when the associated unit travel cost (effective path delay) is at its well defined minimum.

Path flows and unit travel costs for path $p_1$  

Similar comparison for path $p_2$
Comparison of path flows and associated unit travel costs for path $p_3$

Net travel demand and demand reduction are plotted below against the same time scale (day) for 4 sample paths in the figure below.

Net travel demand and demand reduction for 4 sample paths

### 6.12.4 Congestion option with telecommuting

Now we consider the same network, but with a European call option traded for path $p_1$; no options are traded for paths $p_2$ and $p_3$. Drift and volatility associated with the asset
price dynamics are assumed to be the same as described in Section 6.12.3. We again forgo the detailed symbolic statement of this example and instead provide numerical results in graphical form for an essentially exact solution after 34 iterations of the fixed point algorithm. In each main iteration we solve the partial differential equation

$$\frac{1}{2} \left( \sigma_{p1} \left[ t, h^k \right] \right)^2 \frac{\partial^2 V_{p1}}{\partial (S_{p1})^2} + \frac{\partial V_{p1}}{\partial t} - \left( MP_{ij}^{office} - MP_{ij}^{home} \right) = 0$$

with appropriate initial and boundary conditions where $k$ is the iteration index. Figure below is a response surface corresponding that depicts option value as a function of time and traffic activity.

Plot of value of the option ($V_{p1}(t, h^*_p)$), asset price ($S_{p1}(t, h^*_p)$) and time ($t$) for path $p_1$

We use the standardized value of time $\nu = $ $8$, while the difference in the marginal productivity of each traveler commuting from origin 1 to destination 4 is 5 and the call exercise price is $E_{p1} = 3.5$.

When call option trading occurs as described above, Figures below respectively
depict departure rates and arc exit flows for the paths $p_1$, $p_2$ and $p_3$. 

Path and arc exit flows for path 1

Path and arc exit flows for path 2

Path and arc exit flows for path 3

Cumulative traffic volumes on the 5 different arcs of the network are plotted against time
below in the figure below.

Cumulative arc volume vs. time

We compare net unit congestion cost and path flow in the figures below.

Path flows and unit travel costs for path $p_1$

Similar comparison for path $p_2$
Comparison of path flows and associated unit travel costs for path $p_3$

Further, net travel demand and demand reduction are plotted below against the same time scale (day) for 4 sample paths.

Net travel demand and demand reduction for 4 sample paths

6.12.5 Comparison of Congestion Costs

For a specific sample path, we compute the net congestion costs with the options and repeat the process for 10 sample paths keeping all the parameters unchanged. For computing the transition probabilities $\psi(a, t, K_p)$ corresponding to a sample path $\omega$, we use the closed form
Table 6.3: Comparison of Four Scenarios

<table>
<thead>
<tr>
<th>Cases</th>
<th>Cong. cost</th>
<th>$Q_{14}$</th>
<th>$R_{14}^1$</th>
<th>$\tilde{Q}_{14}^1$</th>
<th>Qualitative analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario (a)</td>
<td>1,196</td>
<td>75</td>
<td>0</td>
<td>0</td>
<td>Identical flow pattern</td>
</tr>
<tr>
<td>Scenario (b)</td>
<td>1,185</td>
<td>75</td>
<td>0</td>
<td>1.45 3.45 4.56 75 73.5 71.5 70.4</td>
<td>Increases cong. locally slightly dec global cong.</td>
</tr>
<tr>
<td>Scenario (c)</td>
<td>1,104</td>
<td>75</td>
<td>0</td>
<td>2.76 5.81 8.94 75 72.2 69.2 66.1</td>
<td>Decreases global cong. cost</td>
</tr>
<tr>
<td>Scenario (d)</td>
<td>1,077</td>
<td>75</td>
<td>0</td>
<td>2.78 5.36 7.92 75 72.2 69.6 67.1</td>
<td>Further decreases exp cong cost, on avg less telecom than scenario (c)</td>
</tr>
</tbody>
</table>

expression (??) for the solution of the partial differential equation (6.41) through (6.44) with initial condition $K_p = 4$ and drift and volatility based on $h_{p1}^*(t, \omega)$ as previously explained.

Table 6.3 compares efficacy of congestion options in reduction of net social congestion by considering four scenarios: (a) no telecommuting, (b) telecommuting without options, (c) a market for congestion with telecommuting; and (d) telecommuting with options.

We see that introduction of telecommuting only decreases the total congestion cost by about 1%, primarily because local congestion on some paths increases in some days. However, introduction of the call option lowers the expected net social cost of congestion (calculated over 10 sample paths) by 9% and also induces more telecommuting. On the other hand, if we just have a market for congestion (scenario c), that too has the potential to reduce social cost of congestion.
6.13 Concluding Remarks

We have presented an overview of one version of the dynamic network user equilibrium problem that is central to the field of dynamic traffic assignment. We have shown that when this problem is represented as an infinite dimensional differential variational inequality with well-defined states, and controls, it may be solved using the notions of a fixed point and infinite dimensional mathematical programming. In particular we solve the subproblems of the fixed point algorithm using descent in Hilbert space. In applying these perspectives to the Friesz et al DUE model we found that control variables (arc entry and exit flows) which involve state-dependent time shifts may be readily dealt with using an implicit fixed point scheme. This is significant for it allows theoretically rigorous flow propagation constraints to be employed within a non-simulation based DUE model. We have also shown how any dynamic traffic assignment model of choice may be modified to consider congestion call option. We promote the concept of securitization of congestion in a within-day time scale and have illustrated that such a system may have better potential to alleviate the social cost of congestion.

In closing we remark that we have taken a crucial first step in dynamic traffic assignment by developing a model that integrates the within-day (short) time scale of the DUE model presented with day-to-day (long) time scale models that describe work and residential choice. Such a dual time-scale model has dynamics comprised of both differential and difference equations; the foundations of dual time-scale models are discussed in Friesz et al [93] and Friesz et al [94] and will be the subject of our future research.
Chapter 7

Conclusions

From a theoretical perspective, we have shown that differential variational inequalities that include state dynamics as well as controls and control constraints have associated with them the same notions of minimum principle, adjoint equations and transversality conditions familiar from the theory of optimal control when relatively mild regularity conditions are imposed. We have also shown that when state-dependent time shifts – such as those encountered in some applied problems namely modeling vehicular traffic and supply chain flows – are present the resulting problems remain surprisingly tractable. We have also encountered stochasticity in a specific form – when the uncertainty arises in the state dynamics, which is a stochastic differential equation of Itô type. We have shown, under mild regularity conditions, that stochastic differential variational inequalities that include stochastic differential equations as state dynamics, as well as controls and control constraints have similar notions of minimum principle, adjoint dynamics (stochastic differential equations) and transversality conditions familiar from the theory of stochastic optimal control. A simple fixed point algorithm combined with descent in Hilbert space for which subproblem solutions are expressed as pure functions of time allowed us to compute solutions efficiently. Our numerical examples suggest the fixed-point-descent-in-Hilbert-space algorithm may be practical for intermediate size problems without special structure. We also showed that
the fixed-point-descent-in-Hilbert-space algorithm is convergent when suitable small step sizes are employed. We have applied this framework to a wide class of applied problems. These problems can broadly be classified as the problems belong to the service sector and infrastructure systems.

In future there is a need to conduct more extensive algorithm tests as well as adapt our findings in this thesis to more specific applied game theory and service engineering problems. In future applications we should look for specific problem structures that may make it possible to combine the fixed-point-descent-in-Hilbert-space algorithm of this thesis with notions of decomposition so that truly large and very large applied models may be solved. There is an an opportunity to investigate the computational implications of variable step sizes.
Bibliography


VITA

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