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PRECISE ASYMPTOTICS FOR PERIODIC ORBITS OF THE GEODESIC FLOW IN NONPOSITIVE CURVATURE

A Thesis in

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Abstract

We establish the most precise asymptotic formula known for the number of homotopy classes of periodic orbits for the geodesic flow. We prove it for any compact manifold of nonpositive curvature with geometric rank one. This extends a celebrated result of G.A. Margulis to the nonuniformly hyperbolic case and strengthens previous results by G. Knieper.

While proving this result, we also manage to carry out Margulis' construction of the measure of maximal entropy for nonpositively curved manifolds with geometric rank one.

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1. Introduction

1.1. **Manifolds of geometric rank one.** Let M be a compact Riemannian manifold with all sectional curvatures nonpositive. Define the notion of (geometric) **rank** as follows: For a vector $v \in TM$, let the rank of v be the dimension of the vector space of parallel Jacobi fields along the geodesic tangent to v. Let the rank of M simply be the minimal rank of all tangent vectors.

Obvious consequences of this definition are that

$$1 \leq \text{rank}(M) \leq \dim(M),$$

that the rank of $\mathbb{R}^k$ with the flat metric is k and that

$$\text{rank}(M \times N) = \text{rank}(M) + \text{rank}(N).$$

Every manifold whose sectional curvature is never zero is automatically rank one. Products with Euclidean n-space clearly have rank at least $n+1$. However, it is possible for a manifold to be everywhere locally a product with a Euclidean space and still have geometric rank one.

It turns out that the rank of a manifold of nonpositive curvature is the algebraic rank of its fundamental group [BaEb].

Apart from manifolds of negative curvature, examples are nonpositively curved surfaces containing flat cylinders or an infinitesimal analogue of a flat cylinder.

Flat Euclidean space is obviously not an example. In higher dimensions, examples include Gromov's (3-dimensional) graph manifolds

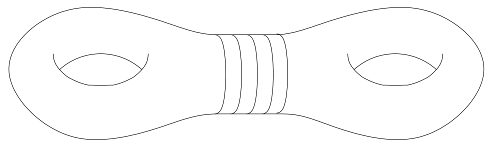

FIGURE 1.1. A surface of geometric rank one with a flat strip and a parallel family of geodesics.

[Gro]. There is an interesting rigidity phenomenon: Every compact 3-manifold of nonpositive curvature whose fundamental group is isomorphic to that of a graph manifold is actually diffeomorphic to that graph manifold [Sch].

We will study properties of spaces with geometric rank one in what follows.

1.2. **Reasons to study these spaces.**

1.2.1. *Rank rigidity.* W. Ballmann [Bal1] and independently K. Burns and R. Spatzier [BuSp] showed that the universal cover of a nonpositively curved manifold can be written uniquely as a product of Euclidean, symmetric and rank one spaces. The first two types are understood, due to P. Eberlein and others. A general introduction to higher rank symmetric spaces is e.g. [Ebe4]; see also [BGS]. For a complete treatment of rank rigidity, see [Bal2].

Thus, in order to understand nonpositively curved manifolds, the most relevant objects to examine are manifolds of dimension at least two with geometric rank one. This becomes even more obvious if one considers the fact that geometric rank one is generic in nonpositive curvature [BBE]. Thus, in a certain sense, "almost all" nonpositively curved manifolds have geometric rank one.

1.2.2. *Limits of hyperbolic systems.* Another reason to study nonpositively curved manifolds is the following. On one hand, strongly hyperbolic systems, particularly geodesic flows on compact manifolds of negative curvature, are well understood since D.V. Anosov; see e.g. [Ano]. Later, P. Eberlein established a condition weaker than negative curvature which still ensures the Anosov property of the geodesic flow [Ebe3]. On the other hand, much less is known about the dynamics of systems lacking strong hyperbolicity. In order to avoid having to create a different (and almost certainly far more complicated) theory than the one for negative curvature from scratch, the most promising way to explore such not strongly hyperbolic systems seems to be to extend existing knowledge of strongly hyperbolic systems to them. Thus it is relevant to study systems in the boundary of the set of hyperbolic systems. Also, since the set of all interesting dynamical systems, particularly the set of all geodesic flows on compact manifolds, is far to vast to analyze at once and only a small part is understood so far, it is important to see what happens at the edge (or frontier) of the set which is understood. Since the open set of geodesic flows on manifolds with negative curvature is "essentially" understood (hyperbolicity is an open property), the edge of our knowledge is marked by the boundary of this set, which is a set of geodesic flows on manifolds of nonpositive curvature. Therefore it is important to study the dynamics of these spaces.

However, the set of nonpositively curved manifolds is larger than just the closure of the set of negatively curved manifolds. This can be seen e.g. as follows: Some nonpositively curved manifolds, such as Gromov's graph manifolds, contain an embedded 2-torus. Thus their fundamental group contains a copy of $\mathbb{Z}^2$. Hence, by Preissmann's theorem, they do not admit *any* metric of negative curvature. Therefore, the investigation in this thesis actually deals with even *more* than the limits of our current knowledge of nearly hyperbolic systems.

1.3. **Statement of the result.** For a periodic geodesic, there may be uncountably many periodic geodesics homotopic to it. However, the number of homotopy equivalence classes of periodic orbits of length at most t is finite for all t. Call this number P_t. Trying to find a concrete and explicit formula for P_t which is accurate for all values of t is completely hopeless, even on very simple manifolds. Nonetheless, in this thesis we manage to derive an asymptotic formula for P_t which tells us how P_t looks like when t is large. One can ask about the asymptotic behavior of this number as $t \to \infty$. We will show (Theorem 10.31, page 80):

$$P_t \sim \frac{1}{ht} e^{ht}$$

where we define the notation $f(t) \sim g(t)$ to mean $\frac{f(t)}{g(t)} \to 1$ as $t \to \infty$. This extends a celebrated result of G.A. Margulis to the case of nonpositive curvature and strengthens previous results by G. Knieper, as we explain in the following section.

2. History

2.1. **Margulis' asymptotics.** The study of the functions P and b, where $b_t(x)$ is the volume of the geodesic ball of radius t centered at x

on manifolds of strictly negative curvature was originated by G.A. Margulis in his dissertation [Mar1]. His influential results were published in [Mar2]. He established that, on a compact manifold of negative curvature,

$$b_t(x) \sim c(x)e^{ht} \tag{2.1}$$

for a continuous function c on M. He also showed that

$$P_t \sim c' \frac{e^{ht}}{t} \tag{2.2}$$

for some constant c'. The exponent h is the topological entropy of the geodesic flow. See [KaHa] for a modern reference on the topic of entropy.

Actually, the observation that the exponential growth rate is the topological entropy does not appear in this publication or in Margulis' thesis. The reason for this omission most likely is that in 1970 entropy was not yet a household concept in dynamics, otherwise Margulis would probably have formulated his results using this quantity. Margulis did point out that if the curvature is constant then the exponential growth rate equals $(n-1)\sqrt{-K}$ and that in this case the function c is constant. In modern notation: $c \equiv \frac{1}{h} = c'$.

2.2. **Beyond strictly negative curvature; Katok's entropy conjecture.** Practically all studies that have since been done are restricted to negative curvature. The reason is that in this case techniques from uniformly hyperbolic dynamics can be applied; see e.g. [PaPo]. So, from the point of view of analysis, this case is much easier to treat.

However, from a geometrical viewpoint, manifolds of nonpositive curvature are a natural object to study. Already in the seventies the investigation of manifolds of nonpositive curvature became the focus of interest. (Also more general classes have been studied since, such as manifolds without *focal points*, i.e. where every parallel Jacobi field with one zero has the property that its length increases monotonically when going away from the zero, or manifolds without *conjugate points*, i.e. such that any Jacobi field with two zeroes is trivial.) A major list of problems that had been posed until that time and which had remained open until then was compiled in 1984 at a MSRI problem session [BuKa], including A. Katok's *entropy conjecture:* The measure of maximal entropy is unique.

One of the first result in this direction is G. Knieper's result [Kni3] that the growth rate of closed geodesics is h, even if the curvature is just nonpositive (instead of strictly negative). The same can be deduced from A. Manning's result [Man1] that the growth rate of volume equals h in nonpositive curvature. This shows in particular that the exponent in Margulis' asymptotics must equal h (and justifies that we have already written Margulis' equations that way). A method for showing that the constant c' in equation 2.2 equals $1/h$ is outlined in C. Toll's dissertation [Tol]. The behavior of the function c in the asymptotic formula 2.1 was investigated by C. Yue in [Yue1] and [Yue2].

It took almost two decades after Knieper's first result (and Manning's result), which in turn was published almost one decade after Margulis' results, until the next step in the analysis of asymptotics of periodic orbits was completed, again by Knieper.

2.3. **Knieper's multiplicative bounds.** In 1996, G. Knieper succeeded in establishing more precise asymptotics for volume and periodic orbits [Kni2]. They are the sharpest results known so far in the case of nonpositive curvature and geometric rank one. They are asymptotics up to a multiplicative constant: there exists a constant C such that for sufficiently large t :

$$\frac{1}{C} \leq \frac{b_t(x)}{e^{ht}} \leq C$$

and

$$\frac{1}{Ct} \leq \frac{P_t}{e^{ht}} \leq C.$$

These results are a groundbreaking success. The most remarkable and useful step in the proof of these asymptotics is the proof of Katok's entropy conjecture. Knieper also demonstrated that the measure of maximal entropy can be obtained via the Patterson-Sullivan construction [Kni1].

Moreover, for the higher geometric rank case he also obtained asymptotic information using rigidity. Namely,

$$\frac{1}{C} \leq \frac{b_t}{t^{(\text{rank}(M)-1)/2}e^{ht}} \leq C.$$

He also has an estimate for closed geodesics for higher geometric rank.

Knieper's results were subsequently sharpened: Using the same method, Knieper proves in his survey [Kni4] that in the rank one case actually

$$\frac{1}{C} \leq \frac{P_t}{e^{ht}/t} \leq C.$$

Still, the quotient of the upper and lower bounds is a constant which cannot be made close to 1.

The question whether in this setup of nonpositive curvature and geometric rank one one can prove more precise multiplicative asymptotics, namely without such multiplicative constants, has remained open so far. In this article we establish this result.

Remark. For non-geodesic dynamical systems no statements providing asymptotics as the ones mentioned in this paper are known. One of the best known results in this setup is that for some prevalent set of such diffeomorphisms the number of periodic orbits of period n is bounded by $\exp(C \cdot n^{1+\delta})$ for some $\delta > 0$ [KaHu].

But even for geodesic flows in the absence of nonpositive curvature it is difficult to count—or even find—closed geodesics. The fact that every compact manifold has even *one* closed geodesic was established only in 1951 by Lyusternik and Fet. In the setup of positively curved manifolds and their kin, one of the strongest known results is H.-B. Rademacher's Theorem from 1990 [Rad] stating that every connected simply connected compact manifold has *infinitely many* closed geodesics for a C^r-generic metric for all $r \in [2, \infty]$.

For the case of Finsler manifolds, there are actually examples of simply connected manifolds that possess only finitely many closed geodesics. On S^2, such examples were constructed by A.B. Katok; see [Zil].

For Riemannian metrics on the two-sphere, existence of many closed geodesics took considerable effort to prove. The famous Lyusternik-Shnirelman Theorem asserts the existence of three closed geodesics. The original proof was somewhat sketchy; detailed proofs were given by W. Ballmann and independently J. Jost [Jos1].

J. Franks [Fra] established that every metric with positive curvature on S^2 has infinitely many (geometrically distinct) geodesics; this is a consequence of his results about area-preserving annulus homeomorphisms.

Existence of infinitely many (geometrically distinct) geodesics on S^2 without requiring positive curvature was shown with variational methods by V. Bangert [Ban].

In this article we derive asymptotics like the ones Margulis obtained for negative curvature. We prove them for nonpositive curvature and geometric rank one using non-uniform hyperbolicity. Hence the same strong statement is true in considerably greater generality. See Theorem 10.31 for the statement of the asymptotic formula.

3. Geometry and dynamics of manifolds of nonpositive curvature and geometric rank one

3.1. Asymptotic geometry.

Definition 3.1. Let M be a compact rank one Riemannian manifold of nonpositive curvature. As is usual, we assume it to be connected and geodesically complete. Let $X = S\tilde{M}$ be the unit sphere bundle of the universal covering of M.

For $v \in X$ let c_v be the geodesic satisfying $c'(0) = v$ (which is hence automatically parameterized by arclength). Here c' of course denotes the covariant derivative $\frac{D}{dt}c$ of c.

Let $g = (g^t)_{t\in\mathbb{R}}$ be the **geodesic flow** on X, i.e., $g^t(v) = c_v'(t)$.

Definition 3.2. We say that $v, w \in X$ are **positively asymptotic** (written $v \sim_+ w$ or simply $v \sim w$) if there exists a constant C such that $\pi d(g^t v, \pi g^t w) < C$ for all $t > 0$.

This is evidently an equivalence relation.

Similarly, we say that $v, w \in X$ are **negatively asymptotic** (written $v \sim_- w$) if there exists a constant C such that $\pi d(g^t v, \pi g^t w) < C$ for all $t < 0$. Of course, v and w are positively asymptotic if and only if $-v$ and $-w$ are negatively asymptotic.

Definition 3.3. We call a vector $v \in X$, as well as the geodesic c_v, **regular** if $\operatorname{rank}(v) = 1$ and **singular** if $\operatorname{rank}(v) > 1$. Clearly the rank is constant along geodesics, i.e.

$$\operatorname{rank}(c_v'(t)) = \operatorname{rank}(c_v'(0))$$

for all $t \in \mathbb{R}$.

Let **Reg** and **Sing** be the sets of regular and singular vectors, respectively.

Remark 3.4. Since rank is semicontinuous in the sense that

$$\operatorname{rank}(\lim_n v_n) \geq \lim_n \operatorname{rank}(v_n),$$

the set **Reg** is open.

Definition 3.5. Since $\tilde{M}$ is of nonpositive curvature, it is diffeomorphic to $\mathbb{R}^n$ by the Hadamard-Cartan theorem, hence to an open Euclidean n-ball. It admits the **compactification**

$$\bar{M} = \tilde{M} \cup \tilde{M}(\infty)$$

where $\tilde{M}(\infty)$, the **boundary at infinity** of $\tilde{M}$, is the set of equivalence classes of positively asymptotic vectors, i.e.,

$$\tilde{M}(\infty) = X/\sim_+.$$

A more accurate but cumbersome way to write $\bar{M}$ would be $\bar{\tilde{M}}$; however, since M is already compact, it is clear that the compactification is that of $\tilde{M}$.

A detailed and thorough (and pleasant to read) description of spaces of nonpositive curvature, even of those not equipped with a manifold structure, can be found in [Bal2].

Remark 3.6. For every $v \in X$ and $p \in M$ there exists some $w_+ \in S_p\tilde{M}$ which is positively asymptotic to v and some $w_- \in S_p\tilde{M}$ which is negatively asymptotic to v. In contrast, the existence of $w_{+-} \in T_p\tilde{M}$ which is simultaneously positively *and* negatively asymptotic to v is rare. Moreover, if $v \sim_+ w$ and $v \sim_- w$ (written $v \sim_{+-} w$) then v, w bound a flat strip, i.e. a totally geodesic embedded copy of $[-a, a] \times \mathbb{R}$ with Euclidean metric, for some number a which is positive if v, w do not lie on the same geodesic. In particular, if $\operatorname{rank}(v) = 1$ (hence c_v is a regular geodesic), then there does not exist such $w \sim_{+-} v$ through *any* base point in the manifold outside c_v. In other words, if $w \sim_{+-} v$ on a rank 1 manifold then $w = g^t v$ for some t. On the other hand, if $\operatorname{rank}(v) > 1$ (hence c_v is a singular geodesic), then v and hence c_v may lie in a flat strip of positive width, and in that case there are vectors $w \sim_{+-} v$ at base points outside c_v.

3.2. **Stable and unstable spaces.**

Definition 3.7. The space TSM, i.e. the tangent bundle of the sphere bundle, admits a natural splitting

$$TSM = E^s \oplus E^u \oplus E^0,$$

i.e. $T_v SM = E^s_v \oplus E^u_v \oplus E^0_v$, where

$$E^0_v := \mathbb{R} \cdot \frac{d}{dt} g^t v|_{t=0},$$

$$E^s_v := \{\xi \in T_v SM : \xi \perp E^0,\ J_\xi \text{ is the stable Jacobi field along } d\pi\xi\},$$

$$E^u_v := \{\xi \in T_v SM : \xi \perp E^0,\ J_\xi \text{ is the unstable Jacobi field along } d\pi\xi\}.$$

(See section 4.2 for a definition of J_ξ.)

Definition 3.8. For $v \in SM$, define $W^s(v)$, the **stable horosphere** based at v, to be the integral manifold of the distribution E^s passing through v. Similarly, define $W^u(v)$, the **unstable horosphere** based at v, via integrating E^u. The flow direction of course integrates to a geodesic, which one might call $W^0(v)$. We write W^i_δ for the δ-neighborhood in W^i $(i = u, s, 0u, 0s, 0)$.

Continuity of these foliations was proved in this form by P. Eberlein [Ebe2] and J.-H. Eschenburg [Esch]:

Theorem 3.9. *Let M be a compact manifold of nonpositive curvature. Then the foliation $\{W^s(v) : v \in SM\}$ of SM by stable horospheres is continuous. The same holds for the foliation $\{W^u(v) : v \in SM\}$ of SM by unstable horospheres.*

During the same years, Eberlein considered similar questions on *Visibility manifolds* [Ebe2].

Note that due to compactness of M, the continuity is automatically uniform. This continuity result was improved by M. Brin [BaPe, Appendix A] to Hölderness on the Pesin sets. For our discussion, uniform continuity is sufficient.

The following result is easier to show in the hyperbolic case (strictly negative curvature) than for nonpositive curvature. In the latter setup, it was demonstrated by Eberlein and is a major theorem ([Ebe1]):

Theorem 3.10. *Let M be a compact rank one manifold of nonpositive curvature. Then stable manifolds are dense. Similarly, unstable manifolds are dense.*

3.3. **Important measures.** The Riemannian structure gives rise to a natural measure λ on SM, called the *Liouville measure*. It is finite since M is compact. It is the prototypical smooth measure, i.e., for any smooth chart $\varphi : U \to \mathbb{R}^{2n-1}$, $U \subset SM$ open, the measure $\varphi_* \lambda$ on a subset of $\mathbb{R}^{2n-1}$ is smoothly equivalent to Lebesgue measure.

The well-known variational principle (see e.g. [KaHa]) asserts that the supremum of the entropies of invariant probability measures on SM is the topological entropy h. The variational principle by itself of course guarantees neither existence nor uniqueness of a measure whose entropy actually equals h. These two facts were established in this setup by G. Knieper [Kni1] by means of the Patterson-Sullivan construction. This construction builds the measure as limit of measures supported on periodic orbits.

This *measure of maximal entropy* is sometimes simply called maximal measure. In the setup of nonpositive curvature, the author thinks that the name *Knieper measure* is appropriate.

For the special case of strictly negative curvature, the measure of maximal entropy was constructed in a different way by G.A. Margulis [Mar3]. His construction builds the measure as the product of limits of measures supported on pieces of stable and unstable leaves. This measure agrees with *Bowen's measure* which is obtained as limit of measures concentrated on periodic orbits. U. Hamenstädt [Ham] gave a geometric description of this measure by projecting distances on horospheres to the boundary at infinity, and this description was immediately generalized to Anosov flows by B. Hasselblatt [Has].

We will show how to carry out a Margulis-type construction to obtain a measure which is well adapted to dynamical properties of the flow. We show that it has maximal entropy and hence agrees with the measure which Knieper has constructed.

Remark. It is part of Katok's entropy conjecture and shown in [Kni1] that $m(\mathbf{Sing}) = 0$ (and in fact that $h(g|_{\mathbf{Sing}}) < h(g)$). In contrast, whether $\lambda(\mathbf{Sing}) = 0$ or not is a major open question; it is equivalent to the famous open problem of ergodicity of the geodesic flow in nonpositive curvature with respect to λ. On the other hand, ergodicity of the geodesic flow in nonpositive curvature with respect to m has been shown to be true by Knieper.

The following was recently established by M. Babillot [Bab]:

Theorem 3.11. *The measure of maximal entropy for the geodesic flow on a compact rank one nonpositively curved manifold is mixing.*

G. Knieper had proved previously ([Kni1]):

Theorem 3.12. *There is a measure of maximal entropy for the geodesic flow on a compact rank one nonpositively curved manifold. Moreover, it is unique.*

4. The measure of maximal entropy: preliminaries

Margulis' results are derived by constructing a special measure which is particularly well adapted to the dynamics of the system under consideration. Knieper showed that a different construction provides this measure in the nonuniform case. Since Margulis' result does not apply, it is not a priori clear that the Knieper measure has all of the relevant properties which the Margulis measure has in the negatively curved case. By generalizing Margulis' construction, we show that those properties are actually present. In order to use it we need to make a few preparatory observations, as well as adapt the notation to a more general setup.

4.1. **Coordinate boxes.**

Definition 4.1. We call an open set $U \subset SM$ of size $\leq \delta$ **regularly coordinated** if for all $v, w \in U$ there are unique x, y such that

$$x \in W^u_\delta(v),\ y \in W^0_\delta(x),\ w \in W^s_\delta(y).$$

In other words, v can be joined to w by means of a unique short three-segment path whose first segment is contained in $W^u(v)$, whose second segment is a piece of a flow line and whose third segment is contained in $W^s(w)$.

Proposition 4.2. *If v is regular then it has a regularly coordinated neighborhood.*

Proof. Some δ-neighborhood V of v is of geometric rank one. Let

$$U = B^s_{\delta/4} g^{[-\delta/4,\delta/4]} B^u_{\delta/4} v.$$

This is contained in V and hence of geometric rank one. It is open since W^0, W^u and W^s are transversal (which is true by rank one, see

Lemma 4.3). Here transversality of the three manifolds means that

$$T_vSM = T_vW^u \oplus T_vW^0 \oplus T_vW^s.$$

By construction, for any $w \in V$, there exists a pair (x, y) such that

$$W^u_\delta(v) \ni x \in W^0_\delta(y),\ y \in W^s_\delta(w).$$

Assume there is another pair (x', y') with this property. From

$$W^u(x) \ni v \in W^u(x')$$

we deduce $x \in W^u(x')$ and from

$$W^0(x) \ni y \in W^s(w),\ w \in W^s(y'),\ y' \in W^0(x')$$

we deduce $x \in W^{0s}(x')$. Hence x and x' bound a flat strip. Since V is of geometric rank one, there is no such strip of nonzero width in U. Hence x and x' lie on the same geodesic. Since $x \in W^u(x')$, these two points are identical.

The same argument with u and s exchanged shows that $y = y'$. Hence the pair (x, y) is unique. □

4.2. **Parallel Jacobi fields.** The following Proposition interprets the definition of regularity:

Proposition 4.3. *The vector $v \in SM$ is regular if and only if $W^u(v)$, $W^s(v)$ and $W^0(v)$ intersect transversely at v.*

Proof. Let $\pi : TM \to M$ be the canonical projection. Let K be the **connection map**, i.e.

$$K\xi := \nabla_{d\pi\xi} Z$$

where ∇ is the Riemannian connection and

$$Z(0) = d\pi\xi, \quad \frac{d}{dt}|_{t=0} Z(t) = \xi.$$

There is a canonical isomorphism

$$I = (d\pi, K)$$

between T_vSM and the set $\mathcal{J}_v$ of Jacobi fields along c_v. It is given by $\xi \mapsto J$ with

$$J(0) = d\pi \cdot \xi,$$

$$J'(0) = K\xi.$$

W^u_v and W^s_v intersect with zero angle at v if and only if there exist

$$\xi \in TW^u(v) \cap TW^s(v) \subset T_vSM.$$

But $\xi \in TW^s(v)$ if and only if J_ξ is the stable Jacobi field along c_v, and $\xi \in TW^u(v)$ if and only if J_ξ is the unstable Jacobi field along c_v. A Jacobi field J is both the stable and the unstable Jacobi field along c_v if and only if J is parallel. The nonexistence of such J perpendicular to c_v is just the definition of rank one. □

We can define a Riemannian metric on SM, the **Sasaki metric**, by setting

$$\langle \xi, \eta \rangle := \langle d\pi\xi, d\pi\eta \rangle + \langle K\xi, K\eta \rangle$$

for $\xi, \eta \in T_v SM$.

4.3. **Existence of holonomies.** This subsection recalls some facts from Margulis, although we will proceed somewhat differently later.

For $L_1, L_2 \subset W^{0u}$ (not necessarily in the same leaf) the holonomy map $\mathbf{H}$ from L_1 to L_2 can be defined by $\{\mathbf{H}(x)\} := W^s_D(x) \cap L_2$ for $x \in L_1$ where D is sufficiently large for the intersection to be nonempty and sufficiently small for it to be just one point. This intersection is well-defined if for all y in L_2 some local intersection $W^s_\gamma \cap W^u_\gamma$ consists of a single point, in particular if W^s and W^{0u} are transversal on L_2.

Lemma 4.4. *Assume that the vector v and the set L are regular, where L is open in a W^{0u}-leaf. Then for $\delta > 0$ sufficiently small and $D \in \mathbb{R}$ sufficiently large there is an open set $U \subset L$ such that $B^{0u}_\delta(v)$ is D-equivalent to U. Moreover, the holonomy between these sets is a homeomorphism.*

Proof. We use regularity and the same transversality argument as in the negatively curved case. Namely: Since $W^s(v)$ is dense, it approximates arbitrarily closely any point of L. By regularity, it is transversal to L, thus it passes through the interior of L. Hence, the holonomy map from a sufficiently small neighborhood of v maps this neighborhood to an open subset of L homeomorphically. □

Even though the preceding statement is what was used by Margulis to prove results in negative curvature, we find it more useful to look at holonomies with less restrictions to size of leaves but with more restrictions on closeness of them. This is done in the following.

4.4. **The Busemann function and conformal densities.**

Definition 4.5. Let $b(.,q,\xi)$ be the **Busemann function** centered at $\xi \in \tilde{M}(\infty)$ and based at $q \in \tilde{M}$. It is given by

$$b(p,q,\xi) := \lim_{p_n \to \xi} (d(q,p_n) - d(p,p_n))$$

and is independent of the sequence $p_n \to \xi$.

Remark 4.6. The function b satisfies

$$b(p,q,\xi) = \lim_{t\to\infty} (d(c_{p,\xi}(t), q) - t)$$

where $c_{p,\xi}$ is the geodesic with $c_{p,\xi}(0) = p$ and $c_{p,\xi}(t) \to \xi$ as $t \to \infty$.

For ξ, p fixed, we have

$$b(p,p_n,\xi) \to -\infty \quad \text{for} \quad p_n \to \xi$$

and

$$b(p,p_n,\xi) \to \infty \quad \text{for} \quad \lim_n p_n \in \tilde{M}(\infty) \setminus \{\xi\}.$$

Clearly, we have

$$b(p,q,\xi) = -b(q,p,\xi).$$

We use the sign convention where $b(p,q,\xi)$ is negative whenever p,q,ξ lie on a geodesic in this particular order.

Definition 4.7. μ_p is a h-dimensional **Busemann density** (also called **conformal density**) if the following are true:

- For all $p \in \tilde{M}$, μ_p is a finite nonzero Borel measure on $\tilde{M}(\infty)$.

- μ_p is equivariant under deck transformations, i.e., for all $\gamma \in \pi_1(M)$ and $S \subset \tilde{M}(\infty)$ we have

$$\mu_{\gamma p}(\gamma S) = \mu_p(S).$$

- When changing the base point of μ_p, the density transforms as follows:

$$\frac{d\mu_p}{d\mu_q}(\xi) = e^{-hb(q,p,\xi)}.$$

Remark 4.8. Knieper has shown in [Kni1] that μ_p is unique up to a multiplicative factor and that it can be obtained by the Patterson-Sullivan construction.

5. Exponentially growing Lebesgue measure of projections: multiplicative bounds

5.1. **Positive density of some projections.**

Definition 5.1. We use the notation $L_t := g^t L$ for a set $L \subset W^{0u}$ (and for $v \in S\tilde{M}$). In particular, $L_\infty := g^\infty L = \lim_{t\to\infty} g^t L$.

Definition 5.2. Let $L \subset W^{0u}$ be a set in the weakly unstable foliation. We call L a **tall** set if it is nonempty, regular and open in W^{0u}.

In particular, any regular vector has a tall neighborhood in its $0u$-leaf. Technically, the empty set is singular, and hence the requirement of nonemptyness in this definition is already implied by regularity.

Definition 5.3. For a set $U \subset \tilde{M}$ we define the **projection** $\text{proj}_p U$ from a point $p \in \bar{M}$ into the boundary at infinity to be the set of endpoints of all geodesic rays emanating from p and passing through the set U.

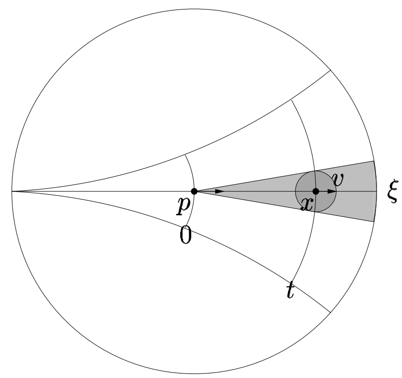

FIGURE 5.1. Projecting into the boundary at infinity.

From Knieper's results we derive:

Theorem 5.4. *There is $\rho = \rho(M)$ such that if $L \subset W^{0u}$ contains a weakly unstable ball B_r^{0u} of radius $r > \rho$ then $\mu_p L_\infty > 0$ for some (hence all) $p \in M$.*

Proof. Knieper showed in [Kni1] that there are constants $\rho, A > 0$ such that balls of radius at least ρ project to sets of $\mu_p > A$, i.e. for any $x \in \tilde{M}$ and $q \in \bar{M}$ it is true that

$$\mu_x \mathrm{proj}_q B_\rho(x) > A.$$

Let $C_0 = C_0(M) \geq 1$ be such that $\pi B^{0u}_{C_0 r}(v) \supset B_r(\pi v)$ for all $v \in X = S\tilde{M}$. C_0 can be chosen to be finite by compactness of M. For $r \leq r_0$, C_0 can also be assumed independent of r.

Let $v \in X$, $r \geq C_0\rho$ be such that $B_r^{0u}(v) \subset L$. Let $x := \pi v$.

For the unique q with $\{q\} = L_{-\infty}$ we get

$$g^{\infty} B_r^{0u}(v) = \mathrm{proj}_q \pi B_r^{0u}(v) \supset \mathrm{proj}_q B_{r/C_0}(x) \supset \mathrm{proj}_q B_\rho(x).$$

Hence

$$\mu_p L_\infty \geq \mu_p \mathrm{proj}_q \pi B_r^{0u}(v) \geq \mu_p \mathrm{proj}_q B_\rho(x) > A > 0.$$

□

Theorem 5.5. *Let $L \subset W^{0u}$ be regular and open in W^{0u} (i.e. tall). Then it satisfies $\mu_p L_\infty > 0$ for some (hence all) $p \in M$.*

Proof. If there exist two vectors $v, w \in L$ for which the iterations $g^t v$, $g^t w$ stay at a bounded distance for all positive t (hence for all t) then the geodesics through v and w bound a flat strip, hence lie on the same geodesic by regularity of L.

Now pick any $v \in L$. Pick $R > 0$ small enough so that $g^{[-R,R]} \bar{B}_R^u v \subset L$. We know that for all $w \in L \cap B^u(v) \setminus \{v\}$:

$$d(\pi g^t v, \pi g^t w) \to \infty \qquad \text{as} \quad t \to \infty.$$

Thus

$$\min_{w \in \partial B_R^u(v)} d(\pi g^t v, \pi g^t w) \to \infty \qquad \text{as} \quad t \to \infty$$

by the compactness of $\partial B_R^u(v)$. In particular, $g^t B_R^u(v)$ contains arbitrarily large u-balls for t sufficiently large. By Theorem 5.4 it follows that $\mu_p L_\infty > 0$. □

5.2. **Projecting distant balls to infinity.** The following theorem uses the results of G. Knieper [Kni1]:

Theorem 5.6 (Projection of distant balls)**.** *Given some arbitrary $p \in \tilde{M}$, for $r > \rho$ there exists some number $a = a(r)$ such that for all $x \in \tilde{M}$:*

$$\frac{1}{a} \leq \frac{\mu_p proj_p B_r(x)}{e^{-ht}} \leq a, \tag{5.1}$$

where $t := d(p, x)$.

Proof. As we pointed out before, Knieper showed that there are constants $\rho = \rho(M), A = A(M) > 0$ such that for any $x \in \tilde{M}$ and $q \in \bar{M}$ and $r > \rho$ it is true that

$$\mu_x \mathrm{proj}_q B_r(x) > A.$$

(Actually, Knieper's formulation assumes that moreover $d(x, p) > r$. Note that the claim of the Theorem is true even without this stipulation since for $d(x, p) < r$ we get $\mu_x \mathrm{proj}_p B_r(x) = \mu_x \tilde{M}(\infty) > A$.)

Recall that μ_x is a finite measure for all x. Thus for all $r > \rho$ there exists $A > 0$ such that for any $q \in \tilde{M}$:

$$A < \mu_x \mathrm{proj}_q B_r(x) \leq \mu_x \tilde{M}(\infty) < \infty.$$

Next we change the base point of μ :

$$\begin{aligned} \mu_p \mathrm{proj}_p B_r(x) &= \int_{\mathrm{proj}_p B_r(x)} 1 d\mu_p(\xi) \\ &= \int_{\mathrm{proj}_p B_r(x)} \frac{d\mu_p}{d\mu_x}(\xi) d\mu_x(\xi) \\ &= \int_{\mathrm{proj}_p B_r(x)} e^{-hb(x,p,\xi)} d\mu_x(\xi). \end{aligned}$$

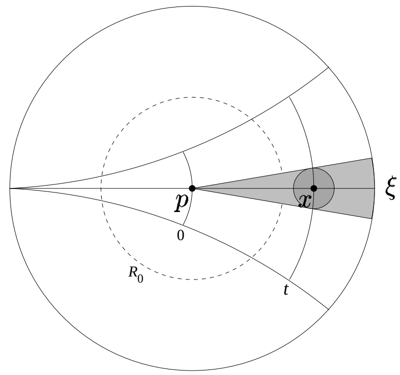

FIGURE 5.2. Projection of a distant ball.

Let $\phi(\xi) := b(x,p,\xi)$. For $\xi_0 := \mathrm{proj}_p x$ we see that $\phi(\xi_0) = d(x,p)$ and $\phi(\xi_0) \geq \phi(\xi)$ for all other $\xi \in \tilde{M}(\infty)$.

We are interested in distant x and p, so assume now that $d(x,p) > R_0$ for some large R_0. Let

$$y := c_{p,\xi} \cap b(x,.,\xi)^{-1}(0),$$

i.e., y is the point on the intersection of the geodesic from p to ξ with the horosphere centered at ξ based at x. In yet different terms, y is given by

$$y = c_{p,\xi}(b(x,p,\xi)).$$

Note that for R_0 sufficiently large, for all $\xi \in \mathrm{proj}_p B_r(x)$ the triangle inequality shows that x and y are at a bounded distance, i.e., we have $d(x,y) < 2r$.

Hence for all $\xi \in \mathrm{proj}_p B_r(x)$ we have

$$0 \leq \phi(\xi_0) - \phi(\xi) \leq C$$

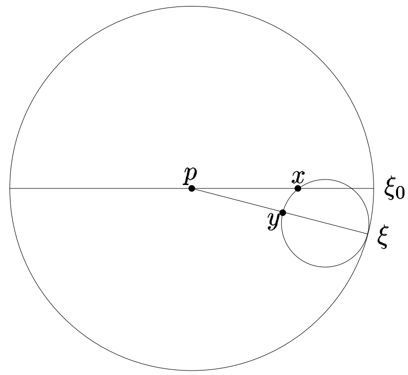

FIGURE 5.3. Construction of y.

independent of p, x (as long as p, x are more than R_0 apart). Therefore

$$1 \geq \frac{e^{-hb(x,p,\xi)}}{e^{-hd(x,p)}} \geq e^{-C}$$

and thus

$$\frac{1}{a} \leq \frac{\mu_p \operatorname{proj}_p B_r(x)}{e^{-ht}} \leq a.$$

So far we have assumed that $d(x,p) > R_0$. However, since the ball of radius R_0 is compact, by increasing a we can ensure that the claim is true also if $d(x,p) \leq R_0$, hence holds for all x, p. □

From this we immediately deduce the following, which is the same formula but where the variables involved have a slightly different meaning:

Corollary 5.7 (Projection of balls in distant leaves)**.** *Let $r > \rho$. Let L be tall and bounded. Let $p \in \tilde{M}$ be arbitrary. Then there exists some number $b = b(p, L)$ such that for all $t \geq 0$ and for all $x \in \pi L_t$ the estimate*

$$\frac{1}{b} \leq \frac{\mu_p proj_p B_r(x)}{e^{-ht}} \leq b$$

holds.

The same statement is true with $B_r(x)$ replaced by $B_r^{0u}(v)$:

Theorem 5.8 (Projection of pieces of distant leaves)**.** *Let $r > \rho$. Let L be tall and bounded. Let $p \in \tilde{M}$ be arbitrary. Then there exists some number $c = c(p, L)$ such that for all $t \geq 0$ and for all $v \in L_t$:*

$$\frac{1}{c} \leq \frac{\mu_p proj_p \pi B_r^{0u}(v)}{e^{-ht}} \leq c. \tag{5.2}$$

5.3. **A multiplicative bound for the growth of Lebesgue measure.** Our efforts are now rewarded by the following Theorem:

Theorem 5.9 (Tall sets grow exponenetially)**.** *If $L \subset W^{0u}$ is bounded and tall then there is a constant $C = C(L)$ such that for all $t \geq 0$:*

$$\frac{1}{C} \leq \frac{\lambda^{0u}(g^t L)}{e^{ht}} \leq C.$$

Proof. Write

$$l_t := \lambda^{0u}(L_t).$$

Fix $p \in \tilde{M}$ and $r > \rho$. From Theorem 5.8 we see that each $0u$-ball of radius r casts a projection from p of μ_p-measure at most ce^{-ht}. Thus it takes at least $\frac{1}{cC_0} e^{ht} \cdot \mu_p \pi L_\infty$ such $0u$-balls of radius r to cover πL_t, with c as in Theorem 5.8, even if their projections to L_∞ do not overlap. Hence we get the lower bound

$$l_t = \lambda^{0u}(g^t L) \geq C_1 e^{ht}$$

with

$$C_1 := \frac{A}{C_0} \cdot \min_{x \in \tilde{M}} \lambda^{0u}(B_r^{0u}(x))$$

where A is as in the proof of Theorem 5.4.

The minimum exists and is positive because the minimum over $\tilde{M}$ is the minimum over M, which is compact. Note that C_1 depends only on L and M, not on t.

Now take any maximal set Z of points in πL_t which are at least distance $2r$ apart from each other and at least at distance r from the boundary of πL_t. (It is easy to see that without the restriction of staying away from the boundary, this set would contain at most C_0' times as many points, where C_0' just depends on the manifold and is independent of L and t.) This set contains at most $ce^{ht} \cdot \mu_p \pi L_\infty$ such points since the balls $(B_r^{0u}(x_i))_{i \in Z}$ are disjoint and project to disjoint subsets of πL_∞, each of which has μ_p-measure at least $\frac{1}{c} e^{-ht}$. Thus it takes at most $ce^{ht} \cdot \mu_p \pi L_\infty$ such $0u$-balls of radius $2r$ to cover πL_t. This gives the upper bound

$$\lambda^{0u}(g^t L) \leq C_2 e^{ht}$$

with

$$C_2 := cC_0 \mu_p(\tilde{M}(\infty)) \cdot \max_{x \in \tilde{M}} \lambda^{0u}(B_{2r}^{0u}(x)).$$

The maximum exists and is finite because the maximum over $\tilde{M}$ is the maximum over M, which is compact. The number C_2 again depends only on L and M, not on t. Hence taking $C := \max(\frac{1}{C_1}, C_2)$ we have proved the claim:

$$\frac{1}{C} \leq \frac{\lambda^{0u}(g^t L)}{e^{ht}} \leq C. \tag{5.3}$$

□

Remark 5.10. The right hand inequality of 5.3 still holds if L is not tall, even if L does not contain any regular vectors. Also, it is true even if L is not open, since any set sits inside a bigger open set.

Remark 5.11. The left hand inequality of 5.3 is not necessarily true if L does not contain a regular vector.

6. Construction of conditionals of the measure of maximal entropy

6.1. **Integration and relative independence of the function.**

Lemma 6.1. *For any compact L which is contained in a W^{0u}-leaf and which is the closure of a tall set and for any $L' \subset W^{0u}$ compact (not necessarily containing regular vectors or the closure of an open set) there exists a constant $C(L, L')$ such that for all $t \geq 0$:*

$$l'_t \leq C(L, L') l_t$$

where $l'_t := \lambda^{0u}(L'_t)$.

Proof. Immediate from Theorem 5.9:

$$l'_t \leq C^{(1)}(L') e^{ht}, \quad l_t \geq \frac{1}{C^{(2)}(L)} e^{ht}$$

where the right inequality is true since L is a tall set. If L' is not a tall set, then the left inequality is still true by Remark 5.10. □

Definition 6.2. Whenever a function f has support in a W^{0u}-leaf, we simply write $\int f$ for $\int_{\text{supp}(f)} f d\lambda^{0u}$.

Corollary 6.3. *For all nonnegative $f_1 \in C(W^{0u})$ with support in a compact set L which is the closure of a tall set and all $L' \subset W^{0u}$ there is a constant $C(L', f_1)$ such that for any bounded measurable function f_2 which is supported on L' and for any $t > 0$:*

$$\int f_2 \circ g^{-t} < C(L', f_1) ||f_2||_\infty \int f_1 \circ g^{-t}.$$

Proof. Choose $\varepsilon \in (0, \max f_1)$ such that $A := \{x : f_1(x) > \varepsilon\}$ still is a tall set (in particular nonempty). Then

$$l_t \leq C(A, L) \cdot \lambda^{0u}(A_t)$$

by Corollary 6.1. Thus

$$\begin{aligned} \int f_2 \circ g^{-t} &\leq ||f_2||_\infty l'_t \\ &\leq C(L, L') ||f_2||_\infty l_t \\ &\leq C(A, L) C(L, L') ||f_2||_\infty \lambda^{0u}(A_t) \\ &\leq \varepsilon^{-1} C(A, L) C(L, L') ||f_2||_\infty \int f_1 \circ g^{-t}. \end{aligned}$$

Choosing $C(L, f_1) := \varepsilon^{-1} C(A, L) C(L, L')$ gives the result. □

In particular, this proof shows that for any tall set L and for any nonzero bounded function f (not necessarily supported on a tall set) the following holds:

$$(6.1) \qquad ||f||_\infty^{-1} \cdot \int f \circ g^{-t} \leq \lambda^{0u}(g^t \text{supp}(f)) \leq C(L, \text{supp}(f)) \cdot l_t.$$

6.2. **Linear functionals on leaves.** Next we can define objects F on $C(W^{0u})$ which are linear functionals in the sense that they satisfy

$$F(a \cdot f_1 + b \cdot f_2) = a \cdot F(f_1) + b \cdot F(f_2)$$

whenever $f_1, f_2 \in C(W^{0u})$ and the sum $f_1 + f_2$ of those functions still is supported on one single leaf:

Definition 6.4. Let $t \geq 0$. Define

$$F_t(f) := \int_{g^t L} f \circ g^{-t} d\lambda^{0u}$$

for a function f with support in $L \subset W^{0u}$. Then F_t is a linear functional, depending on a parameter t.

Remark 6.5. Note that this functional is **positive**, i.e. $F_t(f) \geq 0$ whenever $f \geq 0$.

Remark 6.6. The set C^* of all functionals on $C(W^{0u})$ naturally is equipped with a topology by embedding it into a product of real lines.

Definition 6.7. Given $t \geq 0$, define

$$F'_t := e^{-ht} F_t.$$

Moreover, given numbers $t_i \geq 0$, define

$$\begin{aligned} C_0^* &:= \left\{ \sum_i c_i F'_{t_i} : 0 \leq c_i \leq 1, \sum c_i = 1 \right\} \\ &= \left\{ \sum_i c_i e^{-ht_i} F_{t_i} : 0 \leq c_i \leq 1, \sum c_i = 1 \right\}. \end{aligned}$$

Finally, let

$$C^{\#} := \bar{C}_0^*.$$

($C^{\#}$ is the closure of C_0^*.)

6.3. **Note on Margulis' construction.** Unlike in the hyperbolic case, we have to work without uniform transversality of the stable and unstable foliations and without uniform exponential convergence along stable manifolds. A very detailed description of the construction in the particular case that zero curvature is absent (i.e. on compact manifolds whose curvature is strictly negative) can be found in [KaHa].

Remark 6.8. We could have normalized, similar to the way Margulis does, as follows: Let $\mathbf{K} \subset W^{0u}$ be a fixed set which is open in W^{0u}, has compact closure and is a tall set. E.g. a sufficiently large W^{0u}-ball will do. Let $\theta \in C(W^{0u})$ be a fixed continuous and integrable function with $\theta > 1$ on $\mathbf{K}$. Then instead of F_t' use

$$\hat{F}_t(f) := \frac{F_t(f)}{F_t(\theta)}.$$

Our formalism appears slightly simpler, although they are of course similar. We are able to simplify the notation because we already know that the measure we are about to construct has entropy h, hence is the unique maximal measure, a fact not available to Margulis when he pioneered his construction.

6.4. **Uniform bounds for the functionals.**

Proposition 6.9 (Uniform upper bound)**.** *For any $f \in C(W^{0u})$, there exist $C > 0$ such that for all $F \in C^{\#}$ the bound*

$$|F(f)| \leq C$$

holds.

Proof. Define

$$S := \{F'_t : t \in \mathbb{R}, t \geq 0\}.$$

Then the convex hull of S equals C_0^* and $C^{\#}$ is the closure of that.

First of all, fix some tall set L. Then inequality 6.1 shows that

$$\frac{F_t(f)}{||f||_\infty} \leq C(L,K)\lambda^{0u}(L_t)$$

where K is the support of f. By Theorem 5.9 and Remark 5.10, there is a constant C_1 such that

$$\lambda^{0u}(L_t) \leq C_1 e^{ht}.$$

Replacing F_t by F'_t means introducing another factor e^{-ht} and hence the claim

$$|F(f)| \leq C = C(f)$$

is true for all $\phi \in S$.

Second, any $\phi \in C_0^*$ can be written as $\phi = \sum a_i \phi_i$ with $\phi_i \in S$ where $a_i \in [0,1]$ and $\sum a_i = 1$. Hence

$$|\phi(f)| \leq \sum a_i |\phi_i(f)| \leq C(f).$$

So the claim is true for $\phi \in C_0^*$.

Finally, if $\phi \in C^\#$ then $\phi = \lim_i \phi_i$ with $\phi_i \in C_0^*$, thus

$$|\phi(f)| = \lim_i |\phi_i| \leq C(f).$$

Therefore the claim is true for $\phi \in C^\#$ as well. □

Proposition 6.10 (Uniform lower bound)**.** *For any $f \in C(W^{0u})$, $f \geq 0$, such that for some $\varepsilon > 0$ the set $\{x : f(x) > \varepsilon\}$ is a tall set, there exist $C > 0$ such that for all $F \in C^\#$ the bound*

$$1/C \leq |F(f)|$$

holds.

Proof. Let $A := \{x : f(x) > \varepsilon\}$ which we assume to be tall. Then we proceed with a comparison similar to that in the proof of Lemma 6.3:

$$\begin{aligned} \varepsilon C_2 e^{ht} &\leq \varepsilon \lambda^{0u}(L_t) \\ &\leq C(L, A) \int_{A_t} f \circ g^{-t} d\lambda^{0u} \\ &\leq C(L, A) F_t(f). \end{aligned}$$

Hence $F_t'(f) \geq C_2 \varepsilon^{-1} C(L, A)^{-1}$ as claimed. Thus the claim is true for S.

Again, any $\phi \in C_0^*$ can be written as $\phi = \sum a_i \phi_i$ with $\phi_i \in S$ where $a_i \in [0, 1]$ and $\sum a_i = 1$, hence $|\phi(f)| = |\sum a_i \phi_i(f)| \geq C(f)^{-1}$ for some $C(f)$ and therefore the claim is true for $\phi \in C_0^*$ as well.

And as in the previous proof, if $\phi \in C^{\#}$ then $\phi = \lim_i \phi_i$ with $\phi_i \in C_0^*$, thus $|\phi(f)| = \lim_i |\phi_i| \geq C(f)^{-1}$. Therefore the claim is true for $\phi \in C^{\#}$ as well. □

6.5. **The conditional measure as a fixed point.**

Definition 6.11. Let $(G^t)_{t\in\mathbb{R}}$ be the flow on $C^{\#}$ defined by

$$G^t F := F \circ g^t.$$

Let $(\hat{G}^t)_{t\in\mathbb{R}}$ be the flow on $C^{\#}$ defined by

$$\hat{G}^t F := e^{-ht} F \circ g^t.$$

Proposition 6.12. *There is a measure m' which is a fixed point for $\hat{G}^t$.*

Proof. For each t, the map g^t is a smooth diffeomorphism. Hence $\hat{G}^t$ is a continuous map from $C^{\#}$ to itself. $C^{\#}$ is a convex compact subset of a locally convex topological vector space. Thus the Tychonoff fixed point theorem [KaHa] applies and gives a fixed point for $\hat{G}^t$. □

Note that for any $t_1, t_2 \in \mathbb{R} \setminus \{0\}$ the fixed point for $\hat{G}^{t_1}$ is the fixed point for $\hat{G}^{t_2}$. This is clear for $t_1/t_2 \in \mathbb{N}$, hence for $t_1/t_2 \in \mathbb{Q}$ and thus for all t_1, t_2 since $g^t \to \mathrm{Id}$ as $t \to 0$.

Definition 6.13. Later we will denote m' by m^{0u}.

Remark 6.14 (Comment on Margulis' method). Note that m^{0u} (the family of conditional measures on the weakly unstable leaves) is already normalized due to the fact that m^{0u} lies in $C^{\#}$.

The preceding fixed point statement immediately shows the following:

Theorem 6.15 (Uniform expansion on weakly unstable leaves)**.** *The measure m^{0u} satisfies*

$$m^{0u} \circ g^t = e^{ht} \cdot m^{0u}.$$

This is what we actually need for the calculation of the asymptotics of periodic orbits. Before we do so, we verify that these conditionals are indeed those of the maximal measure.

7. Holonomy invariance of the measure on leaves

Definition 7.1. If the leaves $L, L' \subset W^{0u}$ are related by a holonomy map $\mathbf{H}$, i.e. $L' = \mathbf{H}(L)$ and all connecting pieces of W^s between $v \in L$ and $\mathbf{H}(v) \in L'$ are of length at most γ then L and L' are called γ-**holonomic**. If $f_1, f_2 \in C(W^{0u})$ and the support of f_2 is obtained from the support of f_1 via a holonomy $\mathbf{H}$ of length at most γ and $f_2 = f_1 \circ \mathbf{H}$ then f_1 and f_2 are called γ-**equivalent** or also γ-**holonomic**.

Theorem 7.2. *Let M be a manifold of nonpositive curvature with geometric rank one. Then the measure m^{0u} is locally holonomy invariant inside regular neighborhoods.*

In other words, if $L, L' \subset W^{0u}$ are tall and ε-equivalent for ε sufficiently small then

$$m^{0u}(L) = m^{0u}(L').$$

7.1. **Strategy for showing holonomy invariance.** The strategy is as follows: To show that the Jacobian between two holonomic leaves in W^u is 1, we first show in Lemma 7.19 that it is between $1-\varepsilon$ and $1+\varepsilon$ whenever the leaves are δ-close and lie in the same chart. That by itself would not necessarily be sufficient (e.g. this would also be true for Lebesgue measure); we also show that the leaves which initially are δ_0-close (where δ_0 is small but fixed) become δ-close for any $\delta > 0$ as $t \to \infty$. We show this in Lemma 7.20. Then the claim follows by noting that the Jacobian converges to 1 uniformly and that any two holonomic leaves can be broken into pieces of holonomic pairs which lie within one chart. Note that δ_0 is assumed sufficiently small for the arguments to work. However, if we chose it small enough then we can fix it and will then get $\delta \to 0$ as $t \to \infty$.

Remark 7.3. Unlike in the case of strictly negative curvature, we do not have uniform transversality of the stable and unstable manifolds at our disposal. In fact, as we have seen before, the angle between these is zero on the singular set. This restricts our arguments to the regular set. Therefore, when in our setup pieces of holonomic leaves approach each other as $t \to \infty$, it is not a priori clear that their sizes also become very close. Instead, it is necessary to estimate of the size of the "overhanging" part, i.e. the part where the two plaques do not "overlap" in suitable coordinates. This estimate is also achieved in this subsection.

7.2. **Geometric properties of stable and unstable leaves.** Recall that $\pi : S\tilde{M} \to \tilde{M}$ is the canonical projection $\pi v := x$ with $v \in S_x\tilde{M}$.

We start with some elementary considerations:

Lemma 7.4. *$\pi W^u(p) \perp \pi g^{\mathbb{R}} p$ at p. Similarly, $\pi W^s(p) \perp \pi g^{\mathbb{R}} p$ at p.*

Proof. Let $b := b(\pi p, ., g^\infty p)$ be the Busemann function based at πp and centered at $g^\infty p$. Then $\pi W^u(p) = b(\pi p, ., g^\infty p)^{-1}(0)$ and $p = \operatorname{grad} b(\pi p, ., g^\infty p)|_{\pi p}$. The gradient of a function is perpendicular to its level sets. □

Corollary 7.5. *$\pi W^u(p)$ is tangent to $\pi W^s(p)$ at p.*

Proof.

$$T_p \pi W^u(p) = \left(\frac{d}{dt} \pi g^t p|_{t=0} \right)^{\perp} = T_p \pi W^s(p).$$

□

Definition 7.6. Let J_v^s be the **stable Jacobi field** along the geodesic c with $c'(0) = v$, defined by $J_v^s(0)$ and the condition that it is bounded for $t \geq 0$. Similarly, let J_v^u be the **unstable Jacobi field** along the geodesic c with $c'(0) = v$, defined by its initial value at 0 and the condition that it is bounded for $t < 0$.

Unless we specify otherwise, we assume that $|J_v^i(0)| = 1$ (for $i = s, u$), and we reserve the right not to explicitly assign a value to $J_v^i(0)$.

7.3. **Contact structure of the geodesic flow.** The following is well-known (see [Pat]): The geodesic flow has a **contact structure**, i.e. there exists a one-form α on SM, called the **contact form**, such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form. There exists a vector field $V : SM \mapsto TSM$ with $\alpha(V) \equiv 1$, $d\alpha(V, .) \equiv 0$. We can express α as $\alpha_v(\xi) = \langle v, d\pi\xi \rangle$ for $v \in SM$, $\xi \in T_v SM$.

If we disregard the V-direction (i.e. restrict to $\ker(\alpha) = TSM/\mathbb{R}V$), then $d\alpha =: \omega$ is a nondegenerate closed 2-form. We would call it a *symplectic form* if the s- and u-distributions were jointly integrable, but of course they are not.

Note that the whole tangent bundle TM, which is even-dimensional, does admit a **symplectic form** Ω. The restriction of Ω to the odd-dimensional sphere bundle SM acquires a direction of degeneracy, namely the flow direction V. In other words, $\Omega(V, X) = 0$ for all $X \in TSM$.

Analogously to the Sasaki metric, we can write an explicit formula for the form:

$$\omega(\xi, \eta) = \langle d\pi\xi, K\eta \rangle - \langle K\xi, d\pi\eta \rangle.$$

(This is in fact exactly the expression for the **almost complex structure** with respect to the Sasaki metric, but we do not use this structure in the sequel.)

Lemma 7.7. *The spaces $E^u := T_v W^u$ and $E^s := T_v W^s$ are Lagrangian subspaces.*

Proof. We use the formula $\omega(\xi, \eta) = \langle d\pi\xi, K\eta \rangle - \langle K\xi, d\pi\eta \rangle$. Consider $\xi, \eta \in E^u_v$. Find curves x, y representing them, i.e.

$$\xi = \left.\frac{d}{dt}\right|_{t=0} x(t), \; \eta = \left.\frac{d}{dt}\right|_{t=0} y(t), \; x(0) = v = y(0).$$

Then

$$(d\pi, K)(\xi) = \left(\left.\frac{d}{dt}\right|_{t=0} \pi x(t), \nabla_{\frac{d}{dt}|_{t=0}\pi x} x \right)$$

and

$$(d\pi, K)(\eta) = \left(\frac{d}{dt}\bigg|_{t=0} \pi y(t), \nabla_{\frac{d}{dt}|_{t=0}\pi y} y \right).$$

Extend $\frac{d}{dt}\big|_{t=0} \pi x(t)$ to a vector field X in a neighborhood of x such that $X \in T\pi W^u(v)$ on $\pi W^u(v)$ and similarly extend $\frac{d}{dt}\big|_{t=0} \pi y(t)$ to a vector field Y. Note that curves in $W^u(v)$ consist of unit vectors and therefore their derivative is perpendicular to them, i.e. tangential to $\pi W^u(v)$. Thus in particular we have $y \perp X$ at $\pi y(t)$ and $x \perp Y$ at $\pi x(t)$. Thus

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} \langle Y, x \rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \langle Y(x(t)), x(t) \rangle \\ &= \langle \nabla_X Y, x \rangle + \langle Y, \nabla_{\frac{d}{dt}|_{t=0}\pi x} x \rangle \end{aligned}$$

and

$$0 = \langle \nabla_Y X, y \rangle + \langle X, \nabla_{\frac{d}{dt}|_{t=0}\pi y} y \rangle.$$

Hence

$$\begin{aligned} \omega(\xi, \eta) &= \left\langle \frac{d}{dt}\bigg|_{t=0} \pi x(t), \nabla_{\frac{d}{dt}|_{t=0}\pi y} y \right\rangle \\ &\quad - \left\langle \frac{d}{dt}\bigg|_{t=0} \pi y(t), \nabla_{\frac{d}{dt}|_{t=0}\pi x} x \right\rangle \\ &= \langle X, \nabla_{\frac{d}{dt}|_{t=0}\pi y} y \rangle - \langle Y, \nabla_{\frac{d}{dt}|_{t=0}\pi x} x \rangle \\ &= -\langle \nabla_Y X, v \rangle + \langle \nabla_X Y, v \rangle \\ &= \langle \nabla_Y X - \nabla_X Y, v \rangle. \end{aligned}$$

Note that since the Riemannian connection is torsion free,

$$\omega(\xi, \eta) = \langle [X, Y], v \rangle.$$

This term is zero since by Frobenius' theorem, $[X, Y] \in T\pi W^u$ and $v \perp T\pi W^u$. (A similar argument for general normal bundles can be found in [Pat].)

Hence we have shown that E_v^u is an isotropic subspace. The exact same argument with u replaced by s shows that E_v^s is also isotropic. If v is regular, they are disjoint. Their dimension is equal. Hence they are maximal isotropic subspaces, i.e. Lagrangian. □

7.4. **Dynamical proprerties of Jacobi fields.**

Proposition 7.8 (Stable Jacobi fields become short or parallel)**.** *Let v be regular. Then $J_v^u(t)$ (the unstable Jacobi field in the direction v) is unbounded as $t \to \infty$.*

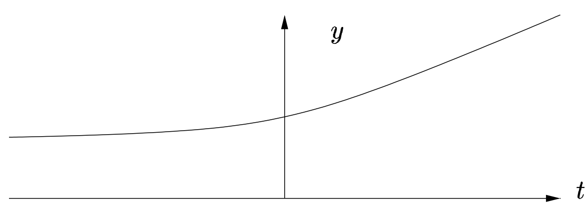

FIGURE 7.1. The length of an unstable Jacobi field is convex.

Proof. Let

$$y(t) := |J_v^s(t)|.$$

Let $c = c(t, s)$ be a variation of geodesics in $\tilde{M}$ associated with the geodesic c_v, i.e. $c(t, 0) = c_v(t)$ and $\frac{\partial}{\partial s} c(t, s) = J_v^s(t)$ and such that

$c(s,.) = (t \mapsto c(s,t))$ is a geodesic for all s close to 0. Then $y(t) = |\frac{\partial}{\partial s} c(t,s)| = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(c(t,s), c(t, s+\varepsilon))$. This expression is a convex function of t since distance along geodesics is convex, i.e. the function $t \mapsto d(c_1(t), c_2(t))$ is convex and in particular $t \mapsto d(c(s,t), c(s, t+\varepsilon))$ is convex.

Hence y is convex (and in particular never zero since it corresponds to the stable Jacobi field). Since y is bounded for $t < 0$, y is either constant or unbounded for $t > 0$. Since v is regular, J_v^s cannot be parallel and perpendicular, hence y cannot be constant. Hence y is unbounded. □

We will write η_t for $dg^t \eta$ and similarly ξ_t.

Proposition 7.9. *Let v be regular. Then*

$$J_v^s(t) \to 0 \quad or \quad \angle(\eta_t, TW^u(g^t v)) \to 0 \qquad as \qquad t \to \infty$$

where $J_v^s = J_\eta$ is the stable Jacobi field along c_v with $d\pi\eta = J_\eta(0), K\eta = J'_\eta(0)$.

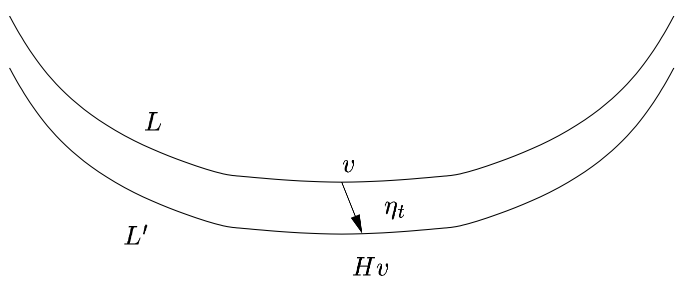

FIGURE 7.2. First possibility: J_v^s is short.

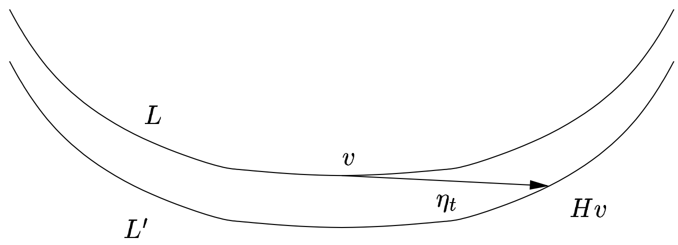

FIGURE 7.3. Second possibility: J^s_v has small angle with the tangent of L.

Proof. The symplectic form ω on SM is invariant. Note that since ω is a 2-form, it satisfies

$$\omega(\xi,\eta) = |\xi|\cdot|\eta|\cdot\angle(\xi,\eta)\cdot\phi(\sigma(\xi,\eta))$$

where $\sigma(\xi,\eta)$ is the 2-plane spanned by (ξ,η) and ϕ is a (continuous) function on the Grassmannian 2-plane-bundle with $|\phi|\leq C$ where C depends only on M. Note that for the minimal choice of C (i.e. $C := \max_{|\xi|=|\eta|=1}\omega(\xi,\eta)$), by Darboux' Theorem for any η there exists ξ such that $\phi(\sigma(\xi,\eta))\geq C/(n-1)$ because we can find symplectic coordinates $P_1,\dots,P_{n-1},Q_1,\dots,Q_{n-1}$ such that $P_1 := \eta$ and then choose $\xi := Q_1$. Since $E^s := TW^s(v)$ and $E^u := TW^u(v)$ are Lagrangian subspaces, ξ has nonzero projection to E^u since it obviously cannot lie entirely in E^s. Hence we may assume ξ to lie in E^u.

Now fix some $\eta\in E^s$. In the sequel, ξ will be an arbitrary element of E^u. Note that by invariance of ω either $\phi(\sigma(\xi,\eta))$ is zero or for all t the term $\phi(\sigma(\xi_t,\eta_t))$ is nonzero. Note that due to the previous Lemma (Lemma 7.8), $|\xi_t|\to\infty$ as $t\to\infty$ for all $\xi\in E^u$. This convergence is uniform on the compact set $S := \{\xi \in E^u : |\xi| = 1\}$. We see that

$$|\eta_t| \cdot \angle(\eta_t, \xi_t) \cdot \phi(\eta_t, \xi_t) = \frac{\omega(\eta_t, \xi_t)}{\xi_t} = \text{const} \cdot |\xi_t|^{-1}.$$

Therefore for $\gamma > 0$ there is $T > 0$ such that for all $t > T$ and for all $\xi \in S$ we have $|\eta_t| \cdot \angle(\xi_t, \eta_t) \cdot \max_\xi \phi(\sigma(\xi_t, \eta_t)) < \gamma$. Hence $|\eta| \cdot \angle(E^u, \eta) < \gamma n / C$. Thus the product $|\eta_t| \cdot \angle(\eta_t, TW^u(g^t v))$ converges to zero. □

Remark 7.10. For $\xi \in W^u$ and $\eta \in W^s$ note that $\angle(\xi, \eta)$ is independent of whether we measure the angle in $\ker \alpha$ or in SM.

Remark 7.11. In the special case of a surface, i.e. $\dim(M) = 2$, it is actually possible to show that of the two possibilities "$J_v^s(t) \to 0$" or "$\angle(\eta_t, TW^u(g^t v)) \to 0$" mentioned in the previous Proposition, the former one always applies, i.e.,

$$J_v^s(t) \to 0 \quad \text{as} \quad t \to \infty.$$

However, we do not need this fact in what follows.

7.5. **Geometric comparisons.**

Proposition 7.12. *Let* $\mathbf{F}, \mathbf{F}^*$ *be foliations of* M *with* C^2*-leaves satisfying* $|c''| < a$ *for any* L *-geodesic and any* L^* *-geodesic* c *parameterized by arc length for any leaf* L *of* $\mathbf{F}$ *and for any leaf* L^* *of* $\mathbf{F}^*$*. Let* c^* *be a geodesic in* $\mathbf{F}^*$ *with* $(c^*)'(0) = v \in SM$*. Then*

$$d(c^*(\varepsilon), B_{2\varepsilon}^{\mathbf{F}}(\pi v)) < 4(a\varepsilon^2 + \varepsilon \angle_{\pi v}(F, F^*).$$

Proof. First note that if N is a C^2-submanifold of M and N satisfies $|c''| < a$ for any N-geodesic c parameterized by arc length (the second derivative is taken in M), then for any $v \in SN$:

$$d(\exp(\varepsilon v), N) < 2a\varepsilon^2.$$

This is true because of the tangency.

Next observe that if M, N are as before and if $\angle(v, N) < \gamma$ then

$$d(\exp(\varepsilon v), N) < 2(a\varepsilon^2 + \varepsilon\gamma).$$

This follows from adding the previous estimate and a linear term, which of course has slope γ.

Now let $\mathbf{F}$ be a foliation of M with C^2-leaves satisfying $|c''| < a$ for any $\mathbf{F}$ -geodesic c parameterized by arc length. Then

$$d(\exp(\varepsilon v), B^{\mathbf{F}}_{2\varepsilon}(\pi v)) < 2(a\varepsilon^2 + \varepsilon\angle(v, \mathbf{F})).$$

This is true because is the previous statement with the leaf through v being the submanifold N.

Now the claim of the Proposition is evident from applying the previous estimate to both foliations and comparing with the corresponding geodesic in M. $\square$

Now we show that we can apply Proposition 7.12 to any piece L with the some uniform constant a:

Proposition 7.13 (Bending of (un)stable leaves is bounded)**.** *Let M be nonpositively curved and compact. Then there exists $a > 0$ such that $|c''| < a$ for any geodesic parameterized by arclength in πL for $L \subset W^s$ or $L \subset W^u$.*

Proof. By compactness, the curvature is bounded below. Note that for the stable Jacobi field J_v^s along $g^t v$ with initial length 1, the initial derivative $|(J_v^s)'(0)|$ is determined up to an error of order $1/R$ by the sectional curvature K along $g^t v$ for $t \in [0, 2R]$. Since K is smooth on M, the sectional curvature K along $g^t v$ for $t \in [0, 2R]$ depends smoothly on v (and of course on t). Hence $|(J_v^s)'(0)|$ is continuous in v up to an error $1/R$. Since R can be chosen arbitrarily large, $|(J_v^s)'(0)|$ is continuous in v. Since SM is compact, this continuity is uniform.

The second derivative $|c''|$ of the horosphere is given (up to a constant c_i depending on the chart) by $|(J_v^s)'(0)|$ where c parameterizes $\pi W_\delta^s(v)$. Since it is continuous, by compactness and since it suffices to consider finitely many charts, $|c''|$ is globally bounded above.

This proves the claim for W^s. Exchanging J_v^s with J_v^u (the unstable Jacobi field) gives the claim for W^u. □

Definition 7.14. For a smooth curve c on $\tilde{M}$ from p to q let

$$\mathrm{Par}_c : T_p\tilde{M} \to T_q\tilde{M}$$

denote the parallel transport along c. We define a distance function $\mathbf{d}$ on $S\tilde{M}$ as follows: For $v, w \in S\tilde{M}$ (not necessarily in the tangent space of the same point) let

$$\mathbf{d}(v, w) := d(\pi v, \pi w) + |w - \mathrm{Par}_{c_{\pi v, \pi w}} v|,$$

where $c_{\pi v, \pi w}$ is the geodesic from πv to πw. Note that $c_{\pi v, \pi w}$ is unique in $\tilde{M}$.

Lemma 7.15. *The following are equivalent:*

(1) $\mathbf{d}(g^t v, g^t w) \to 0 \quad$ *as* $\quad t \to \infty$,

(2) $d(\pi g^t v, \pi g^t w) \to 0 \quad$ *as* $\quad t \to \infty$.

Proof. Since $\mathbf{d}(v,w) \geq d(\pi v, \pi w)$, the implication from (1) to (2) is trivial. For the converse, note that d is convex. This means that for v, w fixed, the function $\phi : t \mapsto d(\pi g^t v, \pi g^t w)$ is convex. Hence the function ϕ is either monotonous or it satisfies $\phi(t) \to \infty$ for $t \to -\infty$ as well as $\phi(t) \to \infty$ for $t \to \infty$. In particular, if $d(\pi g^t v, \pi g^t w) \to 0$ and $|g^t w - \mathrm{Par}_{c_{\pi g^t v, \pi g^t w}} g^t v| \not\to 0$, then v, w are not asymptotic, which contradicts $d(\pi g^t v, \pi g^t w) \to 0$. $\square$

Definition 7.16. Define the distance between two vector spaces $V, V' \subset T_p M$ as

$$\mathbf{d}_H(V, V') := d_H(V \cap S_p M, V' \cap S_p M)$$

where d_H denotes the Hausdorff distance of compact sets.

Lemma 7.17. *Let* $v \in \mathbf{Reg}, w \in \mathbf{Reg} \cap W^s(v)$, $\mathbf{d}(v,w) < \delta/2$. *Let* $L := W^u_\delta(v)$, $L' := W^u_\delta(w)$. *Then*

$$d(g^t v, L'_t) \to 0$$

and

$$d(g^t w, L_t) \to 0$$

as $t \to \infty$.

Proof. The claim is evidently true if $\mathbf{d}(g^t v, g^t w)$ converges to zero.

Now assume that $\mathbf{d}(g^t v, g^t w)$ does not converge to zero. Then Proposition 7.9 shows that if δ is suitably small (but fixed) and if t is sufficiently large then the angle between $\eta \in TW^s(v)$ and $TW^u(v)$ becomes smaller than any $\gamma > 0$. Hence by Proposition 7.12 the distance under consideration becomes smaller than e.g. const $\cdot\, \delta^2$ (in particular smaller than $\delta/2$). Reapplying the argument to $g^t v$ and L'_t shows that the distance becomes arbitrarily small as $t \to \infty$. □

7.6. **Dynamical properties of stable and unstable leaves.**

Definition 7.18. Let $L, L' \subset W^{0u}$ be bounded, tall, δ-holonomic pieces of leaves. We can assume that there exist a tubular neighborhood of L in SM which contains $B^{0u}_\delta L$ and $B^{0u}_\delta L'$. Since we are interested in the case where L and L' are close and holomic, we can assume without loss of generality that for each $v \in L$, the shortest geodesic segment from v to $v' \in L'$ is contained in the tubular neighborhood. Define the **Riemannian projection** $\mathcal{P} : L \to B^{0u}_\delta L'$ to be the vector $\mathcal{P}v := w$ so that $d(w, v)$ is minimal with respect to the Sasaki distance on SM. Let $\operatorname{Jac}\mathcal{P}$ be the **Jacobian** of the Riemannian projection $\mathcal{P}$, given by

$$(\operatorname{Jac}\mathcal{P})(v) := \lim_{\delta_1 \to 0} \frac{\lambda^{0u}(\mathcal{P}(L \cap B_{\delta_1}(v)))}{\lambda^{0u}(L \cap B_{\delta_1}(v))}.$$

Let $\mathcal{P}_t$ be the Riemannian projection from L_t to $B^u_\delta L'_t$. Let Jac $\mathcal{P}_t$ be the Jacobian of $\mathcal{P}_t$.

Similarly, let $\mathcal{P}'_t$ be the Riemannian projection from L'_t to $B^u_\delta L_t$ and let Jac $\mathcal{P}'_t$ be the Jacobian of $\mathcal{P}'_t$.

Now we are going to show that this projection is arbitrarily close to preserving Liouville measure if the leaves sufficiently close:

Lemma 7.19. *For all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $L, L' \subset W^u$ are such that they are tall, bounded, holonomic, δ-close and such that their tangents are δ-close with respect to* $\mathbf{d}$, *then*

$$|\operatorname{Jac} \mathcal{P} - 1| < \varepsilon.$$

Proof. Let N be the perpendicular distribution to the distribution TW^{0u}. This distribution is uniformly continuous since TW^{0u} is.

Let γ be any N-geodesic of length s from L to L' (hence $\gamma'(0) \in L^{\perp}$, $\gamma'(s) \in (L')^{\perp}$).

Let c be a geodesic in SM with $c'(0) = \gamma'(0)$. Then c crosses L' at some parameter value σ with $|\sigma - s| < \sigma^2$ and $|\angle(c'(\sigma), L') - \pi/2| < \sigma^2$ by uniform continuity of the two distributions. Hence $d(c(\sigma), \mathcal{P}c(0)) < \sigma^2$ whenever the distance δ between L and L' was small enough. Thus for r sufficiently small, $\lambda^{0u}\mathcal{P}B_r/\lambda^{0u}B_r$ is arbitrarily close to 1. $\square$

From this we deduce that if we start with holonomic leaves at a fixed distance and wait sufficiently large, $\mathcal{P}_t$ will again be arbitrarily close to preserving Liouville measure:

Corollary 7.20. *Let $L, L' \subset W^u$ be such that they are tall with compact closure, holonomic, δ_0-close and such that their tangents are δ_0-close w.r.t.* $\mathbf{d}$. *For all $\varepsilon > 0$ there exists $T > 0$ such that whenever $t > T$ then*

$$|\operatorname{Jac} \mathcal{P}_t - 1| < \varepsilon.$$

Proof. Simply observe that by Lemma 7.17, for any $v \in \bar{L}$, after flowing for sufficiently large time t, v_t is δ-close to L'_t for δ arbitrarily small. Observe that by compactness of $\bar{L}$, this can be guaranteed uniformly for all v in L. Hence for all $\delta > 0$ there exists $T > 0$ such that $t > T$ implies that for all $v \in L$ the estimate $d(v_t, L'_t) < \delta$ holds. In particular, L_t and L'_t are δ-holonomic. Increasing T further we can assure that also the tangents of L_t and L'_t are δ-close. Hence L_t and L'_t satisfy the assumptions of Lemma 7.19, which shows that the Riemannian Jacobian after flowing time t is smaller than ε. □

We have already seen that the leaves L, L' get arbitrarily close everywhere except possibly on the δ-neighborhood of their boundaries. Next we will show that this neighborhood does not contribute a substantial part.

Lemma 7.21 (No boundary effects). *Let L, L' be bounded, tall and δ_0-holonomic. Define*

$$O(t) := L_t \setminus \mathcal{P}'_t(L'_t), \quad O'(t) := L'_t \setminus \mathcal{P}_t(L_t).$$

Then

$$\frac{\lambda^{0u} O(t)}{\lambda^{0u} L_t} \to 0, \quad \frac{\lambda^{0u} O(t)}{\lambda^{0u} L_t} \to 0$$

as $t \to \infty$.

Proof. First note that it suffices to show the claim for *strictly* stable holonomic leaves $L, L' \subset W^u$. It suffices to consider the case that $L = B^u_\gamma(v)$ for some arbitrarily small γ. It is obvious that for each $\varepsilon_2, \varepsilon_3 > 0$ the variable γ can be chosen such that the stable measure of the annulus

$$A := \bar{B}^u_{\gamma+\varepsilon_2}(v) \setminus B^u_{\gamma-\varepsilon_2}(v)$$

satisfies $m^u(A)/m^u(B^u_\gamma) < \varepsilon_3$; this is because the ball $B^u_{\gamma+\varepsilon_2}(v)$ of finite measure can be cut into arbitrarily many nested annuli, so some of the annuli must have small measure. Note that the convexity of the distance along geodesics implies that if $w \neq w'$ are in A then $d(w_t, w'_t) \to \infty$, and in particular this distance $d(w_t, w'_t)$ will become larger than δ_0 (hence larger than the distance between L and L'). Since the set

$$K := \{(w, w') : w, w' \in B^{0u}_{2\gamma}(v), d(v, w) = \gamma + \varepsilon_2/2,\ d(v, w') = \gamma - \varepsilon_2/2\}$$

is compact and disjoint from the diagonal, we deduce that $d(w_t, w'_t) \to \infty$ uniformly for $(w, w') \in K$. Thus we can choose T such that for all $t > T$ and all $(w, w') \in K$ the lower bound $d(w_t, w'_t) > \delta_0$ holds.

On the other hand, since distances along weakly stable fibers are nonincreasing, the distance $d(v_t, v'_t)$ is bounded by δ_0 for all positive time t. We have shown before that for t sufficiently large, the graphs are arbitrarily close. Thus the "overhang" $O(t)$ is contained in a δ_0-neighborhood of ∂L_t. Hence for t large enough, $O(t)$ is contained in A_t, since A_t expands without bounds. Therefore the uniform expansion property implies that

$$\lim_{t\to\infty} \frac{m^u(O(t))}{m^u(B^u_\gamma(v))} \leq \lim_{t\to\infty} \frac{m^u(A_t)}{m^u(g^t B^u_\gamma(v))} = \frac{m^u(A)}{m^u(B^u_\gamma(v))} < \varepsilon_3.$$

Since ε_3 is arbitrary, this shows that

$$\lim_{t\to\infty} m^u(O(t))/m^u(B^u_\gamma(v)) = 0$$

as claimed. Analogously we see for $L' = B^u_\gamma(v)$ that

$$\lim_{t\to\infty} m^u(O'(t))/m^u(B^u_\gamma(v')) = 0.$$

□

Now we have all the components to formulate the immediate precursor to holonomy invariance:

Theorem 7.22 (Asymptotic equality of Liouville measure for holonomic pieces)**.** *Let L, L' be bounded, tall and holonomic. Then*

$$\lim_{t\to\infty} \frac{l_t}{l'_t} = 1.$$

Proof. Corollary 7.20 shows that this is true for large t up to an error term at the boundary. Lemma 7.21 shows that this error term is asymptotically zero. □

Lemma 7.23. *Let L, L' be tall, holonomic and bounded. Then*

$$\frac{m^{0u}(L)}{m^{0u}(L')} = \lim_{t\to\infty} \frac{\lambda^{0u}(L_t)}{\lambda^{0u}(L'_t)}.$$

Proof. The characteristic function on L, L' can be approximated by continuous functions. Thus we can write

$$\frac{F_t\chi_L}{F_t\chi_{L'}} = \frac{\int \chi_L d\lambda^{0u}}{\int \chi_{L'} d\lambda^{0u}} = \frac{l_t}{l'_t} \to 1$$

as $t \to \infty$. Hence clearly also

$$\frac{F'_t \chi_L}{F'_t \chi_{L'}} = \frac{e^{-ht} F_t \chi_L}{e^{-ht} F_t \chi_{L'}} \to 1$$

as $t \to \infty$. Thus the term

$$\frac{\sum_i c_i F'_{t_i} \chi_L}{\sum_i c_i F'_{t_i} \chi'_L}$$

is arbitrarily close to 1 if $t_i > T_0$ for all i and T_0 is sufficiently large. Now let F^j be a sequence converging to m^{0u} with each F^j of the form $\sum_i c_i F'_{t_i}$. Then we can assume that $t_i > T_0$ for all i and for sufficiently large j, where T_0 is sufficiently large. Hence $\frac{F^j \chi_L}{F^j \chi_{L'}}$ is arbitrarily close to 1 for j large. □

Now we can finish the proof of Theorem 7.2 (holonomy invariance of m^{0u}):

End of the proof of holonomy invariance. The previous statement (Theorem 7.22) shows that the quotient of Liouville measures, $\lambda^{0u}(L)/\lambda^{0u}(L')$, converges to 1. Lemma 7.23 shows that this quotient converges to

$$m^{0u}(L)/m^{0u}(L').$$

Hence $m^{0u}(L)/m^{0u}(L') = 1$. □

8. Assembling the measure of maximal entropy from measures on leaves

Now we are able to show the following:

Theorem 8.1. *There exists a measure m^{0u} on tall sets with compact closure in W^{0u}, which extends to a measure m^{0u} on* **Reg** *satisfying:*

- $m^{0u}(g^t(U)) = e^{ht} m^{0u}(U)$,
- $0 < m^{0u}(U) < \infty$ *for any* $U \subset W^{0u}$ *tall (and in particular nonempty),*
- γ*-equivalent tall sets* U_1, U_2 *in* W^{0u} *have the same* m^{0u}*-measure.*

Proof. Define

$$T(p) := \{\text{tall sets with compact closure in } W^{0u}(p)\},$$

$$T := \bigcup_{p \in M} T(p).$$

Let

$$C_U(T) := \{f \in C(W^{0u}) : \text{ supp}(f) \subset \bar{U}\}$$

for $U \in T$.

We have seen that m' (hereafter called m^{0u}) is a positive linear functional on $C_U(W^{0u})$. A suitable version of the Hahn-Banach Theorem shows that m' extends to $C(\bar{U})$. This extension is still positive, thus an appropriate version of the Riesz Representation Theorem [Fol] shows that there is a measure m_U on $\bar{U}$ such that $m'(f) = \int f dm_U$ for all $f \in C_U(W^{0u})$. $\square$

Remark 8.2. Nonpositivity of the curvature suffices to see that there exist $t_0 = t_0(M) > 0$, $r_0 = r_0(M) > 0$ such that for all $U \in W^u$ with diameter at most r_0 and for all $0 \leq t < t' \leq t_0$ the iterates $g^t U$ and $g^{t'} U$ are disjoint. Hence we can define:

Definition 8.3.

$$m^u(U) := m^{0u}(L_U)$$

where

$$L_U := g^{[0,t_0]}U.$$

Remark 8.4. This gives a measure on W^u. This definition is valid whether U is regular or not, although we will be interested in m^u (and m^s, see below) on the regular set.

Proposition 8.5 (Uniform expansion on W^u).

$$m^u \circ g^t = e^{ht} \cdot m^u.$$

Proof. Immediate from the analogous property of m^{0u} :

$$\begin{aligned}(m^u \circ g^t)(U) &= m^{0u}(g^t L_U) \\ &= e^{ht} m^{0u}(L_U) \\ &= e^{ht} m^u(U).\end{aligned}$$

□

Remark 8.6. If we apply time reversal (i.e. considering the flow $\bar{g}$ with $\bar{g}^t := g^{-t}$), then W^u of $\bar{g}$ is W^s of g and analog for W^{0u} etc. Thus the preceding construction gives another measure m^s (and a measure m^{0s}) with the same properties as time is reversed. In particular:

Proposition 8.7 (Uniform contraction on W^s).

$$m^s \circ g^t = e^{-ht} \cdot m^s.$$

Lemma 8.8. *For all $\varepsilon > 0$ there is $\gamma > 0$ such that for γ-equivalent sets A_1, A_2 we have*

$$\left|\frac{m^s(A_1)}{m^s(A_2)} - 1\right| < \varepsilon.$$

Thus for all $\gamma > 0$ there is C such that for γ-equivalent sets A_1, A_2 we have

$$1/C < m^s(A_1)/m^s(A_2) < C.$$

The same statement is true with m^s replaced by m^u.

Proof. As in [KaHa], we consider $g^{[0,t_0]}A_1$ and $g^{[0,t_0]}A_2$ and use holonomy invariance for those, after cutting off the non-overlapping part. □

We are now able to define a product measure on the regular set as follows. The regular set is open. As we have shown in Theorem 4.3, a local product structure exists in the neighborhood of any regular point. Thus if v is regular, there is a regular neighborhood U of v of the form $U = U^s \times U^{0u}$.

Definition 8.9. Let $O \subset U$ be regular and open. Define a function α_O by

$$\alpha_O(q) := m^s((\{q\} \times U^s(p)) \cap O).$$

This function can easily be verified to be semicontinuous and thus integrable on U. Therefore we can define:

Definition 8.10.

$$m_q(L) := m^{0u}(L \times \{q\}),$$

$$m(O) := \int \alpha_O dm_q.$$

This gives a well-defined measure on the regular set, since holonomy invariance shows that m_q does in fact not depend on q.

In the sequel we consider m to be normalized to 1.

Lemma 8.11. *The measure m has maximal entropy, i.e.*

$$h(m) = h_{\text{top}}.$$

Proof. This follows immediately from the property

$$m^{0u}(g^t(B)) = e^{ht} \cdot m^{0u}(B).$$

□

Theorem 8.12. *The measure m which we have constructed is the Knieper measure.*

Proof. Since the measure of maximal entropy is unique [Kni1] and both our measure and the Knieper measure have maximal entropy, these measures are the same. □

Remark 8.13. The uniqueness of the maximal measure shows that m has a unique extension to the entire SM given by $m(U) := 0$ for all $U \subset \mathbf{Sing}$. We denote this extension again by m.

9. A direct proof of uniform expansion/contraction of the conditionals of the Knieper measure

The preceding discussion was to a large part motivated by the task of showing that the measures m^{0u} and m^u expand uniformly with t (Theorem 6.15 and Proposition 8.5) and hence that the measures m^s and

m^{0s} contract uniformly with t (Proposition 8.7). Since the (maximal) measure we constructed is equal to the one obtained by the Knieper construction, it is reasonable to try to find a proof of these properties that uses only the arguments in Knieper's construction. In this section we present such a proof.

The measures dealt with here will have the same names as the ones defined before (they are the same objects, after all). However, we will only use definitions of them which are either given by Knieper or by us here in this section.

Definition 9.1 (Knieper's definition of the measure of maximal entropy)**.** Let μ_p be the Busemann density. Let

$$\Pi : S\tilde{M} \to M(\infty) \times M(\infty), \quad \Pi(v) = (v_\infty, v_{-\infty})$$

be the projection of a vector to both endpoints of the corresponding geodesic. Then the Knieper measure of a set $A \subset SM$ (we can without loss of generality assume A to be regular) is given by

$$m(A) := \int_{\xi,\eta \in M(\infty), \xi \neq \eta} \mathrm{len}(A \cap \Pi^{-1}(\xi,\eta)) e^{-h(b(p,q,\xi)+b(p,q,\eta))} d\mu_p(\xi) d\mu_p(\eta),$$

where $q \in \Pi^{-1}(\xi,\eta)$ and p arbitrary.

9.1. **Heuristic discussion of the conditionals.** Given a vector $v \in S\tilde{M}$ with base point p, we want to put a conditional measure m^u on the stable horosphere $b(p,.,\xi)^{-1}(0)$ given by v. This horosphere is centered at $\xi := v_\infty$. What we need for this conditional is a multiplier with respect to some measure on this horosphere.

Note that the set of points q on the horosphere is parameterized by the set $\tilde{M}(\infty) \setminus \xi$, hence our desired term should depend on $\eta := \mathrm{proj}_\xi(q)$, i.e. be a multiple of $d\mu_x(\eta)$ for some x. The only canonical choice for x is p. Clearly the whole horosphere has infinite m^u-measure, but μ_x is finite for any x. Thus the multiplier of $d\mu_p$ has to have a singularity, and clearly this has to happen for $\eta \to \xi$ since that is the direction where all but a compact piece of the horosphere is mapped. The term $e^{-hb(p,q,\eta)}$ has just the right singularity since $\eta \to \xi$ means $q \to \xi$ and then by the basic properties of the Busemann function our term $e^{-hb(p,q,\eta)}$ converges to infinity. Thus we should investigate

$$m_p(q) := e^{-hb(p,q,\eta)} d\mu_p(\eta).$$

In the next subsection, we prove that this is the right term for dm^s.

9.2. **Formally precise discussion of the coditionals.**

Definition 9.2. For $v, w \in S\tilde{M}$, let

$$dm^u_v(w) := e^{-hb(\pi v, \pi w, w_\infty)} \cdot d\mu_{\pi v}(w_\infty),$$

$$dm^s_v(w) := e^{-hb(\pi v, \pi w, w_{-\infty})} \cdot d\mu_{\pi v}(w_{-\infty}).$$

Proposition 9.3. *dm^s_v, dm^s_v and dt are the conditionals of the Knieper measure.*

Proof. Observe that

$$\begin{aligned} dt\, dm_v^u(w)\, dm_v^s(w) &= dt\, e^{-h(b(\pi v,\pi w,w_\infty)+b(\pi v,\pi w,w_{-\infty}))} \\ &\quad \cdot d\mu_{\pi v}(w_\infty) d\mu_{\pi v}(w_{-\infty}) \\ &= dt\, e^{-h(b(p,q,\xi)+b(p,q,\eta))} d\mu_p(\xi) d\mu_p(\eta) =: E \end{aligned}$$

with $p := \pi v,\ q := \pi w,\ \xi := w_\infty,\ \eta := w_{-\infty}$. The formula already agrees with Knieper's formula in Definition 9.1, but the meaning of the parameters does not yet: In Definition 9.1, p and to some extend q are arbitrary in $\tilde{M}$, while in the latter formula they are fixed. Thus we need to show that if we change them within the range allowed in Definition 9.1, the value of E does not change.

Lemma 9.4. *The term E does not change if p is replaced by an arbitrary point in $\tilde{M}$ and q by any point in $\tilde{M}$ on the geodesic $c_{\eta\xi}$ from η to ξ.*

As we mentioned before, saying that $\Pi^{-1}(\xi,\eta)$ is a geodesic already is a simplification, but a fully justified one since we need to deal only with the regular set.

Proof. q can be allowed to be anywhere on $c_{\eta\xi}$: Replacing $q =: c_{\eta\xi}(s)$ by $q' =: c_{\eta\xi}(s')$ changes $b(p,q,\xi)$ to $b(p,q',\xi) = b(p,q,\xi) - (s'-s)$ since we move the distance $s'-s$ closer to ξ. It also changes $b(p,q,\eta)$ to $b(p,q',\eta) = b(p,q,\eta) + (s'-s)$ since we move the distance $s'-s$ away from to η. Thus E does not change under such a translation of q.

Now fix q anywhere on $c_{\eta\xi}$ and replace p by some completely arbitrary $p' \in \tilde{M}$. Note that

$$\begin{aligned} d\mu_{p'}(\xi) &= e^{hb(p',p,\xi)} d\mu_p, \\ d\mu_{p'}(\eta) &= e^{hb(p',p,\eta)} d\mu_p, \\ b(p',q,\xi) &= b(p,q,\xi) + b(p',p,\xi), \\ b(p',q,\eta) &= b(p,q,\eta) + b(p',p,\eta). \end{aligned}$$

Thus

$$e^{-h(b(p',q,\xi)+b(p',q,\eta))} d\mu_{p'}(\xi) d\mu_{p'}(\eta) = e^{-h(b(p,q,\xi)+b(p,q,\eta))} d\mu_p(\xi) d\mu_p(\eta).$$

Hence E does not change if p is changed to any arbitrary point. □

This concludes the proof of Proposition 9.3. □

9.3. **Uniform expansion/contraction of the conditionals.** Recall that w_t denotes $g^t w$.

Theorem 9.5 (Uniform expansion/contraction of the conditionals). *For all $t \in \mathbb{R}$ and all $v, w \in S\tilde{M}$ we have*

$$dm_v^u(w_t) = e^{ht} \cdot dm_v^u(w),$$

$$dm_v^s(w_t) = e^{-ht} \cdot dm_v^s(w).$$

Proof.

$$\begin{aligned} dm_v^s(w_t) &= e^{-hb(\pi v, \pi w_t, w_{-\infty})} d\mu_{\pi v}(w_{-\infty}) \\ &= e^{-h(b(\pi v, \pi w, w_{-\infty}) + b(\pi w, \pi w_t, w_{-\infty}))} d\mu_{\pi v}(w_{-\infty}) \\ &= e^{-hb(\pi v, \pi w, w_{-\infty}) - ht} d\mu_{\pi v}(w_{-\infty}) \\ &= e^{-ht} \cdot e^{-hb(\pi v, \pi w, w_{-\infty})} d\mu_{\pi v}(w_{-\infty}) \\ &= e^{-ht} \cdot dm_v^s(w). \end{aligned}$$

Similarly

$$\begin{aligned} dm_v^u(w_t) &= e^{-hb(\pi v, \pi w_t, w_{+\infty})} d\mu_{\pi v}(w_{+\infty}) \\ &= e^{-h(b(\pi v, \pi w, w_{+\infty}) + b(\pi w, \pi w_t, w_{+\infty}))} d\mu_{\pi v}(w_{+\infty}) \\ &= e^{-hb(\pi v, \pi w, w_{+\infty}) + ht} d\mu_{\pi v}(w_{+\infty}) \\ &= e^{ht} \cdot dm_v^u(w). \end{aligned}$$

This shows the desired uniform expansion of m^u and the uniform contraction of m^s. □

From this we also immediately see the uniform expansion of m^{0u} and the uniform contraction of m^{0s} since dt is evidently invariant under g^t.

9.4. **A direct proof of holonomy invariance of the conditionals of the Knieper measure.** Another important property of the conditional measures on the leaves is holonomy invariance. The proof which we have presented before is quite involved, so it is natural to ask whether there is another one that uses Knieper's formula for the measure of maximal entropy and the arguments about conditionals used in this section. In this subsection we prove that this is indeed the case, i.e., we give a direct (and considerably shorter) proof of holonomy invariance.

Remark 9.6. The proof presented above has the advantage that it is applicable in a somewhat more general setting than the geodesic flow on rank one manifolds of nonpositive curvature. While we formulated it in the setup of nonpositively curved manifolds of geometric rank one, when suitably reformulated, the arguments in the proof are appropriate for non-geodesic flows satisfying suitable cone conditions. Hence it has greater generalization power than the proof presented in the following, which, while considerably shorter, substantially uses features of nonpositively curved manifolds with geometric rank one.

We formulate holonomy invariance on infinitesimal unstable pieces here, but of course this is equivalent to holonomy invariance that deals with pieces of leaves of small but positive size. We consider positively asymptotic vectors w, w' and calculate the infinetesimal $0u$-measure on corresponding leaves. We let v, v' be some (arbitrary) base points used as parameters for the pieces of leaves, so that w lies in the same $0u$-leaf of v and similarly w' in that of v'. The factor dt is evidently invariant, so we do not have to mention it any further.

Theorem 9.7 (Holonomy invariance of the conditionals of the measure of maximal entropy)**.**

$$dm_v^u(w) = dm_{v'}^u(w')$$

whenever $v' \in W^s(v)$, $w' \in W^s(w)$, $w \in W^{0u}(v)$ *and* $w' \in W^{0u}(v')$.

Proof. Note that the equation $w' \in W^s(w)$ is equivalent to the two equations

$$w'_\infty = w_\infty,$$

$$b(\pi w, \pi w', w_\infty) = 0.$$

The latter equation is equivalent to $b(p, \pi w, w_\infty) = b(p, \pi w', w_\infty)$ for all $p \in \tilde{M}$. Clearly

$$\begin{aligned} dm^u_{v'}(w') &= e^{-hb(\pi v', \pi w', w_\infty)} d\mu_{\pi v'}(w'_\infty) \\ &= e^{-hb(\pi v', \pi w', w_\infty)} d\mu_{\pi v'}(w_\infty). \end{aligned}$$

Now

$$\begin{aligned} b(\pi v', \pi w', w_\infty) &= b(\pi v', \pi v, w_\infty) + b(\pi v, \pi w', w_\infty) \\ &= b(\pi v', \pi v, w_\infty) + b(\pi v, \pi w, w_\infty) \end{aligned}$$

and $d\mu_{\pi v'}(w_\infty) = e^{-hb(\pi v, \pi v', w_\infty)} d\mu_{\pi v}(w_\infty)$. Thus

$$\begin{aligned} dm^u_{v'}(w') &= e^{-h(b(\pi v', \pi w', w_\infty) + b(\pi v, \pi v', w_\infty))} d\mu_{\pi v}(w_\infty) \\ &= e^{-h(b(\pi v', \pi v, w_\infty) + b(\pi v, \pi w, w_\infty) + b(\pi v, \pi v', w_\infty))} d\mu_{\pi v}(w_\infty) \\ &= e^{-hb(\pi v, \pi w, w_\infty)} d\mu_{\pi v}(w_\infty) \\ &= dm^u_v(w). \end{aligned}$$

□

Corollary 9.8.

$$dm^s_v(w) = dm^s_{v'}(w')$$

whenever $v' \in W^u(v)$, $w' \in W^u(w)$, $w \in W^{0s}(v)$ *and* $w' \in W^{0s}(v')$.

Proof. This is the same proof as before with u and s exchanged and with w_∞ and $w_{-\infty}$ exchanged. □

Note that m^{0u} (and m^{0s}) are holonomy invariant but not m^u (or m^s) due to expansion (resp. contraction) in the flow direction.

10. Counting closed geodesics

In this final section we are going to count the periodic geodesics on M. The method used here is a generalization of the method which, for the special case of negative curvature, was outlined in [Tol] and provided with more detail in [KaHa].

Definition 10.1. Let f and g be expressions depending on t and ε. Write

$$f \sim g$$

if $f(t)/g(t) \to 1$ as $t \to \infty$. In other words, $f/g - 1 = o(1)$ (here 1 is understood to be a (constant) function of t). Write

$$f \bowtie g$$

if $f/g - 1 = O(\varepsilon)$ as $\varepsilon \to 0$. We write

$$f \cong g$$

if there exists f' with $f \sim f' \bowtie g$, i.e. if $f/g = (1+O(\varepsilon))(1+o(t \mapsto 1))$.

10.1. **The flow cube.** Let $z \in \mathbf{Reg}$. Choose sufficiently small ε and δ such that $\delta < \varepsilon$, $B_{4\varepsilon}(z) \subset \mathbf{Reg}$ and such that further requirements on the smallness of these which we will mention later are satisfied.

Definition 10.2. Let the **flow cube** be

$$A := \bar{B}^s_\delta(g^{[0,\varepsilon]}(\bar{B}^u_\delta(z))) \subset \mathbf{Reg}.$$

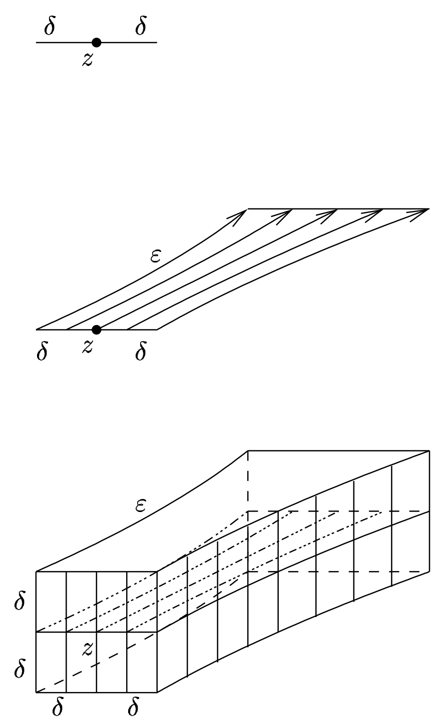

FIGURE 10.1. The flow cube A: an unstable neighborhood of z (top) is iterated (center) and a stable neighborhood of that is formed (bottom).

The cube A will be fixed in the following arguments. In particular, ε and δ are considered fixed (although subject to restrictions on their size). Afterwards, we will consider what happens when $\varepsilon \to 0$.

The local product structure in $\mathbf{Reg}$ now reads:

Lemma 10.3. *For all* $p, q \in A$ *there exists a unique* $\beta \in (-\varepsilon, \varepsilon)$ *such that*

$$B^s_\varepsilon(p) \cap B^u_\varepsilon(g^\beta q)$$

is one point.

Definition 10.4. We denote this point by $[p, q]$.

Definition 10.5. Let the **depth** τ be defined on A by

$$x \in g^{\tau(x)} B^u_\delta(z)$$

if $x \in F$ and

$$\tau(x) := \tau(B^s_\delta(x) \cap F)$$

otherwise.

Clearly $\tau \in (0, \varepsilon)$.

Lemma 10.6. *For all* $x \in A$, $y \in B^u_\delta(x)$:

$$|\tau(y) - \tau(x)| < \varepsilon^2.$$

Proof. The foliation W^u is uniformly continuous. □

Lemma 10.7. *Choosing* δ *smaller than* γ *in Lemma 8.8 we get a constant* K *with*

$$\left| \frac{m^s(B^s x)}{m^s(B^s y)} - 1 \right| < K\varepsilon$$

for arbitrary x, y *in* A.

10.2. **Size of stable fibers.**

Lemma 10.8 (Size of stable fibers)**.** *There is a function $\sigma = \sigma(t)$ such that*

$$m^s g^t B^s(x) \bowtie \sigma(t)$$

for all $x \in A$.

For all x, y in A we have

$$m^s g^t B^s(x) \bowtie m^s g^t B^s(y).$$

Proof. Lemma 8.8 shows that $m^s B^s(x) \bowtie m^s B^s(y)$. From this we deduce

$$\begin{aligned} m^s g^t B^s(x) &= e^{ht} m^s B^s(x) \\ &\bowtie e^{ht} m^s B^s(y) \\ &= m^s g^t B^s(y). \end{aligned}$$

Thus we can pick some arbitrary $z \in A$ and define $\sigma(t) := m^s g^t B^s(z)$. The previous estimate shows that this definition does not depend on z (up to $\bowtie$-equivalence). □

We think of $\sigma(t)$ as the size of a stable fiber from the box A after flowing it for time t. The previous Lemma states that it makes no difference which fiber we choose.

Although we could in principle estimate $\sigma(t)$, this is not necessary because the following arguments are such that $\sigma(t)$ eventually cancels. Thus it is not strictly necessary to know its value for any particular

t. Nonetheless, the following Remark shows that it is asymptotically equal to const$\cdot e^{-ht}$:

Remark 10.9. Let $F := B^{0u}_{\varepsilon}(z) \cap A$. The local product structure and the flow invariance of m give

$$\begin{aligned} m(A) &= m(g^t A) \\ &\cong e^{ht} m^{0u}(F)\sigma(t). \end{aligned}$$

Proof. This follows immediately from the uniform expansion of m^{0u} on the weakly unstable leaves, in particular when applied to $g^t F$. □

10.3. **Expansion at the boundary.**

Definition 10.10. For the cube A as above, we call

$$\bar{B}^s_\delta(g^{[0,\varepsilon]}(\partial B^u_\delta(z)))$$

the **side** of the cube,

$$(\partial B^s_\delta)(g^{[0,\varepsilon]}(\bar{B}^u_\delta(z)))$$

the **top/bottom**,

$$\bar{B}^s_\delta(g^0(\bar{B}^u_\delta(z)))$$

the **back** and

$$\bar{B}^s_\delta(g^\varepsilon(\bar{B}^u_\delta(z)))$$

the **front** of the cube.

For $x \in A$ define

$$s(x) := \sup \{r : B^u_r x \subset A\}$$

to be the **distance from the side** of the flow cube.

Lemma 10.11 (expansion of sides)**.** *There exists a monotonous positive function $S : \mathbb{R} \to \mathbb{R}$ satisfying $S(t) \to 0$ as $t \to \infty$ and such that if $s(x) > S(t)$ and if $x \in A$ and $g^t x \in A$ then*

$$B^u_\delta(x) \cap A \subset g^t B^u_{S(t)}(x). \tag{10.1}$$

That means that if $g^t A$ intersects A at some point not too close to the side boundary then the intersection goes all the way from one side to the other.

Proof. By nonpositivity of the curvature, W^u_δ noncontracts, i.e., pieces of it are nondecreasing in length. This follows from Proposition 7.8, which shows that this is true even infinitesimally.

Hence it is clear that if the claimed inclusion holds for some $t = t_0$ then it holds for all $t \geq t_0$. So the smallest $S = S(t)$ that satisfies the inclusion can already be assumed to be monotonously decreasing. If the smallest such S would not converge to zero, it would require the existence of a flat strip of width

$$\liminf_{t \to \infty} S(t) = \lim_{t \to \infty} S(t) > 0,$$

which would intersect $\bar{A}$. By the regularity of a neighborhood of $\bar{A}$, this cannot happen, since no such strip can have width > 0. $\square$

Remark 10.12. The convergence of S to zero in the previous proof is no longer necessarily exponential, as opposed to the strongly hyperbolic case. However, we do not need this property of exponential convergence.

10.4. **Intersection components and orbit segments.**

Definition 10.13. Let A'_t be the set of $x \in A$ with $s(x) > S(t)$ and $\tau(x) \in (\varepsilon^2, \varepsilon - \varepsilon^2)$.

Thus A'_t is the set A with a small neighborhood of the sides and of the front and back end removed.

Definition 10.14. Let Φ_t be the set of all **full components of intersection** at time t: If I is a connected component of $A'_t \cap g^t(A'_t)\}$ define

$$\Phi_t^I := \left\{ \bigcup_{x \in I} g^{[-\varepsilon,\varepsilon]} x \cap A \cap g^t(A) \right\}$$

and

$$\Phi_t := \{\Phi_t^I : I \text{ is a connected component of } A'_t \cap g^t(A'_t)\}.$$

Define the equivalence classes

$$[x]_0 := g^{[-\varepsilon,\varepsilon]} x \cap A$$

for $x \in A$. We call $[x]_0$ the **orbit segment** through x. Clearly each such segment has length $\bowtie \varepsilon$.

Lemma 10.15. *For every orbit segment of length $\varepsilon - 2\varepsilon^2$ in A_t' that belongs to a periodic orbit of period in $[t - \varepsilon + 2\varepsilon^2, t + \varepsilon - 2\varepsilon^2]$ there exsits a unique $I \in \Phi_t$ through which the orbit segment passes.*

Proof. Existence: If $g^l o = o$ for an orbit segment o of length $\varepsilon - 2\varepsilon^2$ in A_t' that belongs to a periodic orbit of period $l \in [t-\varepsilon+2\varepsilon^2, t+\varepsilon-2\varepsilon^2]$ then o also passes through $g^t A_t'$, hence through some component of $A_t' \cap g^t A_t'$.

Uniqueness: Assume that o passes through $I, J \in \Phi_t$, $I \neq J$, i.e. $o(a) \in I$, $o(b) \in J$, $|b - a| < \varepsilon$. Then o passes through I', J' where I', J' are the connected components corresponding to I, J respectively. Choose a fundamental domain of SM containing A. Since I', J' correspond to different components, we see that if we lift A_t' from SM to $S\tilde{M}$ then the lifts of I', J' in the lift of $g^t A_t'$ lie in different fundamental domains. Hence there is a path in $g^t A_t'$ from the lift of $o(a)$ to that of $o(b)$. On the other hand, the orbit subsegment $o|_{[a,b]}$ lies in one fundamental domain. Therefore the loop in SM obtained by taking a path in $g^t A_t'$ from $o(a)$ to $o(b)$ and closing it with $o|_{[a,b]}$ is homotopically nontrivial. But g^{-t} of this loop would lie in A, which is contractible. Since g^{-t} is a homeomorphism, this is a contradiction. □

In the other direction, we have:

Lemma 10.16. *For every $I \in \Phi_t$ there exists a unique periodic orbit with period in $[t - \varepsilon,\ t + \varepsilon]$ and a unique segment on that orbit passing through I.*

In other words, up to a small error, intersection components correspond to homotopy classes of periodic orbits, and of the orbit segments that belong to this periodic orbit, just one orbit segment goes through any particular full component of intersection.

Proof. Φ_t consists of disjoint connected subsets of A, as seen in the previous proof. For t sufficiently large, each has measure at least $\frac{1}{2}m(A)^2$ by mixing, hence there are only finitely many. Fix any one of them and denote it by K.

Define $X_0 := K$ and inductively

$$X_{j+1} := \{x \in [x']_0 : x' \in X_j' \cap g^t X_j'\}$$

where $X_j' := X_j \cap A_t'$. This is a monotone sequence of compact connected sets which are topologically closed balls. Hence the intersection $X := \bigcap_{j \geq 0} X_j$ is topologically a closed ball.

Similarly, define $Y_0 := K$ and

$$Y_{j+1} := \{x \in [x']_0 : x' \in Y_j' \cap g^{-t} Y_j'\}$$

where $Y_j' := Y_j \cap A_t'$. This is again a monotone sequence of topologically closed balls and hence ence the intersection $Y := \bigcap_{j \geq 0} Y_j$ is topologically also a closed ball.

Since X and Y are closed under the bracket, their intersection is again topologically a ball. Hence the projection to the back boundary $Q := X \cap Y \cap \partial_0 A$ is again a closed ball. Note that our constructions of $s(x)$ and A_t' imply that the Poincare map is continuous on Q and its image is contained in Q. Thus by the Brouwer fixed point theorem it has a fixed point, which shows the existence of a periodic orbit through I.

Now assume there are two fixed points. Hence there are two closed geodesics c_1, c_2 with periods l_1 and l_2 so that $c_1(kl_1)$ and $c_2(kl_2)$ are close for all $k \in \mathbb{N}$. Hence c_1 and c_2 are at bounded distance. By the flat strip theorem, they bound a flat strip of width $a \geq 0$. However, since A has geometric rank one, the only flat strips that can pass through A are of width 0, i.e. single geodesics. Thus c_1 and c_2 are the same periodic orbit. Since we can assume that different segments on it that pass through A are at least 2ε apart, the orbit segment also is unique. $\square$

10.5. **Intersection thickness.**

Definition 10.17. For $I \in \Phi_t$, let

$$I' := \bigcup_{x \in A,\, g^t x \in I} [x]_0.$$

Define the **thickness** of I by

$$\theta(I) := \varepsilon - \sup\{\tau(x) : x \in g^t(I' \cap \partial_0 A)\}$$

for such I which intersect $\partial_0 A = \bar{B}^s_\delta(\bar{B}^u_\delta(z))$ (the back boundary of A) and

$$\theta(I) := \inf\{\tau(x) : x \in g^t(I' \cap \partial_\varepsilon A)\}$$

for such I which intersect $\partial_\varepsilon A = \bar{B}^s_\delta(g^\varepsilon(\bar{B}^u_\delta(z)))$ (the front boundary of A).

Lemma 10.18. *The average thickness of full components of intersection $I \in \Phi_t$ is $\cong \varepsilon/2$.*

Proof. Take a full component of intersection I. Cut it along flow lines in

$$N := \left\lfloor \frac{1}{\varepsilon} \right\rfloor$$

equal pieces, i.e. let

$$A_i := \{v \in A : \tau(v) \in [i\varepsilon/N, (i+1)\varepsilon/N)\}$$

for $i = 0, \dots, N-1$. By mixing, $m(A_i \cap g^t A_0)$ is asymptotically independent of i as $t \to \infty$. Hence the number of components of intersection is asymptotically independent of i. Since every complete intersection component of $(A_i \cap g^t A_0)$ has depth τ with

$$|\tau - i\varepsilon/N| < \varepsilon/N,$$

we see that the average is $\varepsilon/2$ up to an error of order ε^2. The same reasoning applies if A_0 is changed to A_{N-1}. □

10.6. A Bowen-type property of the Knieper measure.

Definition 10.19. Let P_t be the number of homotopy classes of closed geodesics of length at most t. Let $P_t(A)$ be the number of homotopy classes of closed geodesics of length at most t that intersect A. Let P'_t be the number of *regular* closed geodesics of length at most t.

Remark 10.20. When we say "geodesic," we mean with parameterization (although always by arclength and modulo adding a constant to the parameter). Thus any geometric curve is counted as two geodesics (i.e. periodic orbits for the geodesic flow), namely one for each direction. Therefore in our notation, “geodesic” and “periodic orbit for the geodesic flow” are the same.

Clearly

$$P_t(A) \subset P'_t \subset P_t$$

for any t. We will show that for large t, these are asymptotically actually the same.

Lemma 10.21.

$$P'_t \sim P_t.$$

Proof. Singular geodesics have a smaller exponential growth rate because the singular set has entropy smaller than h for any dimension of M [Kni2]. □

In the case that M is a surface, the growth rate of **Sing** is zero, since a parallel perpendicular Jacobi field gives rise to the largest Liapunov exponent being zero.

Definition 10.22. Let μ_t be the arclength measure on all regular periodic orbits of length at most t, normalized to 1. I.e.:

$$\mathbf{P}_t := \{\text{regular closed geodesics of length} \leq t\},$$

$$\mathbf{P}_t(A) := \{\text{geodesics in } \mathbf{P}_t \text{ which pass through } A\},$$

$$\mu_t := \frac{1}{\operatorname{card}(\mathbf{P}_t)} \sum_{c \in \mathbf{P}_t} \frac{1}{\operatorname{len}(c)} \delta_c,$$

$$\mu_t^A := \frac{1}{\operatorname{card}(\mathbf{P}_t(A))} \sum_{c \in \mathbf{P}_t(A)} \frac{1}{\operatorname{len}(c)} \delta_c.$$

Theorem 10.23. *For any weak limit μ of $(\mu_t)_{t>0}$ and for any weak limit μ^A of $(\mu_t^A)_{t>0}$ we have*

$$m \sim \mu$$

and

$$m \sim \mu^A.$$

In other words, for any weak limit of $(\mu_{t_k}^A)_{t_k \in \mathbb{R}}$ and any measurable S, the following holds:

$$\lim_{k \to \infty} \mu_{t_k}^A(S) = m(S).$$

Similarly with μ^A replaced by μ.

Proof. It is known [Kni1] that m can be obtained as a weak limit of the measures μ_{t_k} of the Borel probability measures supported on $\mathbf{P}_{t_k}$. The singular closed geodesics can be neglected because the singular set has entropy smaller than h. Hence any weak limit of μ_t equals m.

Since

$$P_t(A) \geq Ce^{-ht}/t$$

[Kni1, Remark after Theorem 5.8], any weak limit of the measures $\mu_{t_k}^A$ concentrated on $\mathbf{P}_{t_k}^A$ has entropy h. Since the maximal measure is unique, any such weak limit equals m. $\square$

Corollary 10.24.

$$P_t(A) \sim P_t.$$

Remark 10.25. This means that we can approximate the Knieper measure m of a measurable set S by its μ_{t_k}-measure for k sufficiently large.

Moreover, when counting orbits, a suitable small local product cube A will suffice to count periodic orbits in such a way that the fraction of those not counted will converge to zero as the period of these orbits becomes large. We use this fact in the proof of Proposition 10.29.

10.7. **Counting intersections and periodic orbits.**

Proposition 10.26. *The number $N(A,t)$ of full components of intersection of A with g^tA satisfies*

$$N(A,t) \cong 2e^{ht}m(A).$$

Proof. First note that

$$m(I) \bowtie \frac{\theta(I)}{\varepsilon} m^{0u}(F)\sigma(t)$$

for a component of intersection I since m^s of the pieces of stable fibers in I is equal to $\sigma(t)$ up to an error term in ε and since the m^{0u}-measure of I is the same as that of F ("I intersects A all the way from side to side") except that the depth of the intersection is not ε but $\theta(I)$. Since the average of $\theta(I)$ is asymptotically $\varepsilon/2$, we get

$$m(I) \cong \frac{1}{2}\sigma(t)m^{0u}(F).$$

Since the measure of $A \cap g^tA$ is the sum of the measures of the intersection components and there are $N(A,t)$ of those, we get

$$m(A \cap g^tA) \cong \frac{1}{2}N(A,t)\sigma(t)m^{0u}(F).$$

Note that by the mixing property of g :

$$m(A \cap g^t A) \cong e^{ht} m(A) \sigma(t) m^{0u}(F).$$

The claim follows from combining the two previous equations. □

Definition 10.27. Let $\mathbf{P}_{t,\varepsilon}$ be the set of *regular* geodesics of length l with $l \in (t-\varepsilon, t+\varepsilon]$.

Let $P_{t,\varepsilon}$ be the number of elements in $\mathbf{P}_{t,\varepsilon}$.

Remark 10.28. $P_{t,\varepsilon}$ is finite because there is only one regular geodesic in each homotopy class.

Theorem 10.29. *The number of regular closed geodesics with prescribed length is given by the asymptotic formula*

$$P_{t,\varepsilon} \cong \frac{\varepsilon N(A,t)}{t m(A)}.$$

Proof. By Lemma 10.23, for a typical closed geodesic c of length t with t sufficiently large,

$$\frac{\int_{\dot{c} \cap A} d\mathrm{len}}{\mathrm{len}(c)} \cong m(A).$$

Here "typical" means that the number of closed geodesics of length at most t that have this property is asymptotically the same as the number of all closed geodesics of length at most t.

Hence such a geodesic (which consists of t/ε segments of length ε) will have asymptotically $m(A)t/\varepsilon$ segments of length ε intersecting A. Thus

$$P_{t,\varepsilon} \cong \frac{\varepsilon N(A,t)}{t m(A)}$$

as claimed. □

Proposition 10.30.

$$P_{t,\varepsilon}(A) \cong \frac{2\varepsilon e^{ht}}{t}.$$

Proof. Proposition 10.26 and Theorem 10.29 combined yield the claim. □

10.8. **The main result.** The desired asymptotic formula is now derived in the following.

Theorem 10.31 (Precise asymptotics for periodic orbits)**.** *Let M be a Riemannian manifold of nonpositive curvature whose rank is one. Then the number P_t of homotopy classes of closed geodesics of length at most t is asymptotically given by the formula*

$$P_t \sim \frac{e^{ht}}{ht}$$

where $\sim$ means that the quotient converges to 1 as $t \to \infty$.

Proof. We use the standard limiting process

$$\int_a^b f(x)dx \bowtie \sum_{i=\lfloor a/2\varepsilon \rfloor}^{\lfloor b/2\varepsilon \rfloor} 2\varepsilon f((2i+1)\varepsilon)$$

for suitable functions f (e.g. if f is continuous and piecewise monotonous, as is the case here). Choose some fixed number $K > 0$. Since we can ignore all closed geodesics of length at most K for the asymptotics, we see that for $t > K$:

$$\begin{aligned}
P'_t &\cong P_t(A) \\
&\cong \sum_{i=\lfloor K/2\varepsilon \rfloor}^{\lfloor t/2\varepsilon \rfloor} P_{(2i+1)\varepsilon,\varepsilon} \\
&\cong \sum_{i=\lfloor K/2\varepsilon \rfloor}^{\lfloor t/2\varepsilon \rfloor} 2\varepsilon \frac{e^{h(2i+1)\varepsilon}}{(2i+1)\varepsilon} \\
&\cong \int_K^t \frac{e^{hx}}{x} dx \\
&= \left. \frac{e^{hx}}{hx} \right|_K^t + \int_K^t \frac{e^{hx}}{hx^2} dx \\
&\cong \frac{e^{ht}}{ht} - \frac{e^{hK}}{hK} \\
&\cong \frac{e^{ht}}{ht}.
\end{aligned}$$

Hence

$$P_t \sim \frac{e^{ht}}{ht}$$

since $P'_t \sim P_t$ and since in the last formula there is no dependence on ε. This concludes the proof. □

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