RETURN DISTRIBUTIONS OF PRIVATE REAL ESTATE INVESTMENTS

A Thesis in

Business Administration

by

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ABSTRACT

This dissertation investigates the distribution of returns accruing to individual owners of investment real estate property. Previously, most research in investment real estate concentrated upon large institutional owners using finance paradigms, tools and methodology.

Other researchers have questioned the use of finance models, predominantly Modern Portfolio Theory (MPT), for real estate research. None have yet offered an alternative. This dissertation extends the literature by (a) offering a theory of risk based on the assumption of non-normal probability distributions and an economic tradeoff different from the mean-variance rule of MPT; and (b) introducing a data set covering property rarely studied that supports the theory advanced herein.
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"Science would be ruined if (like sports) it were to put competition above everything else, and if it were to clarify the rules of competition by withdrawing entirely into narrowly defined specialties. The rare scholars who are nomads-by-choice are essential to the intellectual welfare of the settled disciplines."


“There is no inevitable connection between the validity of the expected utility maxim and the validity of portfolio analysis based on, say, expected return and variance.”

– Harry M. Markowitz, Portfolio Selection, p. 209
CHAPTER 1

INTRODUCTION

Most of the study of private investor activity has been restricted to investment in financial assets. Microeconomic theory of consumer choice and decision-making under uncertainty is highly developed and finds much empirical support in stock market data. Data providing similar evidence for direct, private investment in real property have not been easy to obtain. As a result there is a paucity of research in this area. Substantial numbers of investors eschew the stock market in favor of real property. Why they do this is unclear. A plausible explanation would be that, over the long term, real property investment offers a higher risk-adjusted return than financial assets. There is little evidence that this is so.

Most research on real estate portfolios has involved those whose capital sources make forming a portfolio a practical reality. Thus, studies abound regarding major institutional investors and the indices containing the returns from holding very large properties. This dissertation is concerned specifically with the activities of individual property investors. Roulac (1995) has suggested that investment real estate markets may be separated along scale or strategic lines. His conclusion was that while size and scale were the obvious differences, “behavioral factors that influence the rationality of their
decision-making processes and the legal differentiation resulting from their status loom larger than do implications of power and economy of scale” (p. 55).

In a piece intended to examine the difference between individual and institutional investing, Markowitz (1991b) confesses that “the ‘investing institution’ which I had most in mind when developing portfolio theory for my dissertation was the open-end investment company or ‘mutual fund’” (p. 2). He recites a host of problems facing the individual that simply don’t apply to the institution, concluding that simulation is perhaps the only practical way to implement portfolio analysis at the individual level.

This dissertation investigates the return distributions of economic agents who choose to invest their discretionary funds in a parcel of income-producing real property rather than in financial instruments. Because the large volume of literature on financial assets comprises the foundation of what is known about personal investments, I will spend considerable time reviewing that literature and foundation. Throughout all of that, it must be remembered that what is investigated here is not Professor Markowitz’s "investing institution". Nor is it an institution that invests in real estate such as a real estate investment trust (REIT) or pension fund, rather it is the individual private investor.

**Historical Background**

With the introduction of Modern Portfolio Theory (MPT) Markowitz (1952) provided academic content and mathematical substance to the timeworn saw: "Don't put all your eggs in one basket." Sharpe (1964) expanded on MPT, adding the Capital Asset Pricing Model (CAPM), a theory of equilibrium pricing for risky assets. When Fama
(1970a) propounded his Efficient Market Hypothesis (EMH) that changes in prices are a random walk, he provided the third leg of the stool that supports what has come to be known as the finance paradigm.¹ Over the second half of the 20th century, a huge body of literature refined and tested these theories. With plentiful data and the largest continuous auction market for financial assets in the world, the U.S. stock market was a natural laboratory for this work. Hence, studies of MPT in the context of financial assets abound over the last four decades.

Over approximately the same period, real estate research was conducted with an eye toward applying MPT to real property markets. Citing the usual list of reasons that real estate is different (sometimes referred to as “the laundry list” and shown in Appendix A), researchers tested EMH and CAPM for real property markets, applying Markowitz technology to real assets with varying levels of success. Not insubstantial in developing a sufficiently large data set were the twin problems of a) obtaining the necessary vector of returns required to implement Markowitz’s technology, and b) forming "the market portfolio" required to test the CAPM. Naturally, the most productive real estate areas for this effort were those in which data, albeit less bountiful than financial asset data, were comparatively plentiful. This invariably left researchers with two domains, each located at the extreme end of the size spectrum: Either they studied the single family housing market where transactions were numerous, frequent, and of public record or they investigated the large, institutional-grade investment property market where corporate

¹ The finance paradigm is a larger and general body of knowledge that is discussed more completely in Chapter 3. For the moment, the reference to MPT is intended to be that portion of the theory flowing from Markowitz (1952).
ownership requires public financial reporting. Largely missing from the scene are what may be described as "mid-range" properties, those too large to be attractive to an owner-occupant and too small to interest institutional investors.

**This Dissertation's Contribution**

This dissertation begins work in the mid-range size of the real estate investment market. It questions the use of Modern Portfolio Theory in this market and suggests alternative motives that induce people to invest in certain forms of real estate. Specifically, I

- Examine the foundation of MPT and the way its underlying assumptions affect the model's predictions. I also discuss the manner in which MPT has been applied to real estate and review the applicable finance, real estate, and statistical literature covering these areas;

- Propose using the family of stable distributions as a way of describing risk in real estate and suggest, for privately held real estate, an alternative to the risk reward tradeoff at work in MPT;

- Introduce the concept of three tiers of real estate based on public vs. private ownership using property size as a proxy for empirical analysis. The market for privately held individual investment is defined as the Tier II market;

- Demonstrate a method of illustrating stable distributions with Fourier transforms;
• Introduce a dataset containing Tier II property, performing tests to determine (a) the nature of return distributions for the data and (b) if property size is related to parameters of the return distributions.

• Suggest implications and opportunities for future research arising from the outcome of the tests and the theory presented.

It should be noted that only one other work of this kind has been done for U.S. real estate. Young and Graff (1995) present similar information on institutional grade commercial property. This dissertation extends their work by applying it to a different property class and size, under different ownership conditions, using different methodology for generating returns and a different methodology for estimating distribution parameters. Young and Graff’s work offered no alternative theory of the type presented here. By enlarging the research to address different market players, offering a theory of the motives of those parties and adding to the empirical evidence of structural differences in the return distributions of real estate, I hope to further distinguish real estate as different from assets that fit neatly into the finance paradigm.

The General Research Question

The literature is sparse on the subject of private real estate investors. Whether from lack of data or lack of interest, the academic community claims little knowledge of this segment of the market. Comparing investments in real estate with investments in

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2 Similar work was done for the Australian market by the same authors.
securities, a list is offered (Appendix A) to demonstrate how real estate is different from financial assets. Most of the items on this list are presented in a light decidedly unfavorable to real estate.\(^3\) Hence, with all the evidence from the stock market and a highly developed theory of personal investment motivation, the primary research question becomes: what motivates private real estate ownership?

Given the apparent legitimacy of the items on the list and the presumption that they do, indeed, not only separate real estate from financial assets but also render real estate less desirable, it would seem that there are more issues associated with the individual ownership of real estate than with ownership of financial assets. Regardless of what assumptions are made or how the questions are framed, we observe a robust market for these assets that shows no sign of going away soon. Yet, compared to publicly owned real property investments or the market for single-family residences, research in this market is fairly limited. This dissertation extends that research in a new direction.

**Challenge to the Dominant Paradigm**

Conventional wisdom in academic finance is that stock market investors care only about return and risk as represented by expected return and variance (often referred to as the "mean-variance rule"). Hence, it is claimed, these investors make decisions solely on the basis of the first two moments of the probability distribution. Since the normal

\(^3\) We accept this characterization for the moment. Discussion in Chapter 7 questions this position. However, it is clear that the need to deal with wasting assets, tenant management, lack of liquidity and high transaction costs, just to name a few, imposes issues on the real estate investor that the stock market investor does not face.
distribution is the only distribution that may be fully described by its first two moments, the finance paradigm depends heavily on returns being normally distributed.

This paper will argue that private real estate investors, in addition to conventional measures of risk and return, care about other parameters, specifically those captured by a broader class of distributions. The general notion of risk as “variance in outcome” may still be appropriate, but limiting the concept of risk to the mathematical construct of statistical variance may be in error.

The Introduction of a New Data Set

Given the ambitious nature of the above, it is only wise to set forth a more achievable goal. As there has been virtually no empirical work in the area of the private real estate investor, this paper provides insight in this area through the analysis of a data set reflecting the buy and sell decisions of participants in this market. Such a data set will allow tests of hypotheses about the primary subject of this paper, a broader form of risk for this class of assets. It will also permit tests of how this risk changes as property size changes.

The Research Hypotheses

I claim that a distinguishing feature of the private real estate investment market is the probability distribution of its returns. This argument may be divided into three parts. The first is a claim that the distribution of investment returns in this market is
meaningfully different from return distributions for financial assets. The second argues that this difference produces unacceptable predictions when modeled under MPT. The third postulates that investors in this market are motivated by a trade-off different from the mean-variance considerations that motivate investors in financial assets.

**Private Real Estate Investment Return Distributions Are “Heavy-Tailed”**

A substantial body of literature exists in the biological and social sciences supporting an assumption that much of the randomness apparent in life is distributed in a shape that approximates a bell curve. Such a curve is known as "normal", or "Gaussian", for Carl Frederick Gauss, the mathematician who developed it.

In financial terms, the shape of a distribution says a great deal about risk. Lately, many have argued that assuming that only one shape - the normal - describes risk is too restrictive. There is a growing literature and substantial anecdotal evidence claiming that many non-normal distributions exist in commodity markets. Those interested in such matters have dubbed their phenomena as “heavy-tailed” to draw attention to the effect of extreme values – outliers – in the distribution. The technical name for these distributions is "stable." The central issue in the dispute over what shape best portrays risk may be illustrated by a series of graphs.
Figure 1–1 reflects a typical normal probability distribution in its usual symmetrical form, with tails rapidly descending to its support on the x-axis.

Figure 1–2 shows a heavy tailed distribution,

Figure 1–2: A “heavy-tailed” distribution
and Figure 1–3 combines the two, with emphasis on the additional variation represented by the difference in the probability mass in the tails of the two distributions. As one can see, the rapid convergence of the tail of the normal distribution ignores a portion of the risk if the actual distribution has a heavy tail. "If asset returns are truly governed by …stable distributions, life is fundamentally riskier than in a Gaussian world." (McCulloch (1996), p. 393).

Figure 1–3: The normal and the heavy-tailed distribution on a single plot

This paper asserts that individual real estate investors face a probability distribution that is heavy tailed and skewed right. If that turns out to be true, the notion of normality and the inexorable regression to the mean that follows from it applies poorly, at best, to real estate investment.
Private Real Estate Investor Portfolios are Not "Efficient Set" Portfolios

A major motivation of this dissertation is the likelihood that non-normal return distributions in the market for private real estate investments lead to the inefficient portfolios. If true, earlier musings about why a market for individual real estate exists becomes even more puzzling. When the laundry list is expanded to exclude the ability to “diversify away” risk by forming portfolios, investors lose another benefit afforded them by the securities market. The general question returns: Why are these assets held?

Having questioned MPT and the mean-variance rule associated with it, I will suggest an alternative. There seems to be a different method by which private real estate investors make decisions. While they are rational risk-averse maximizers like other agents, I will argue that the investor is guided by a different rule related to the shape of the distribution he faces in his market.

Topical Outline of What Follows

After this introduction, Chapter 2 provides a foundation for the remainder of the paper, beginning with a brief description of the development of the finance paradigm, followed by the real estate applications of it. A primer on return distributions is then provided to reacquaint the reader with those technicalities. Finally, I present a short review on the use of the inverse Fourier transform, showing how it can be used to overcome an unwieldy mathematical problem. Chapter 3 covers much of the same ground again, this time through a review of the literature. Here I specifically
acknowledge the contributions of scholars in five different areas. The first part
chronicles the establishment of MPT as the dominant paradigm of finance. The second
part covers real estate applications of the finance paradigm. Third, I look at challenges to
MPT, both from the field of finance and from other disciplines, primarily economics.
Fourth, the theoretical developments in statistics are covered with emphasis on the stable
distribution. Finally, in the last and the shortest portion, I cover the few authors who
have employed stable distributions to explain real estate phenomena.

In order to optimize the value of the data available for this work and make the
task at hand manageable, Chapter 4 introduces a “Tier” concept for thinking about real
estate. This chapter argues that real estate research should be divided into distinct
categories, or Tiers. First, Tier I is made up of the occupied housing market, separating it
from the investment market. The investment market is then divided by size into non-
institutional and institutional components, constituting Tiers II and III, respectively. This
study is exclusively interested in Tier II. An informal survey of institutional investors is
provided suggesting where the boundaries of the Tier II may be, addressing the question:
How small is too small for institutional buyers? Chapter 5 presents a view of private real
estate investment in the Tier II market, based on its particular return distributions. This
chapter begins by discussing how solutions to scientific problems are reached in general,
specifically dealing with those mathematical problems that elude analytical solutions.
The chapter continues with an expansion of the two hypotheses mentioned above, that the
shape of the distribution is heavy tailed and that market participants, rather than forming
efficient set portfolios, ameliorate their risk differently. Chapter 5 concludes by
offering graphical and numerical illustrations to support the theory.

Empirical evidence is presented in Chapter 6, evidence relating to the shape of the
return distributions in Tier II and testing parameters of their distributions under different
conditions. Chapter 7 summarizes, offering implications, discussion and suggestions for
future research.

Following the main text is a series of Appendices. Several of these merely
supplement the text in the usual fashion intended for appendices. Some are more
extensive and take the form of Mathematica notebooks. The intent is to provide those
interested in extending this research with ready-to-use tools, saving researchers who
follow a tedious struggle with baseline programming. The electronic version of this
dissertation provides the Mathematica portions of the appendices in executable form and
the primary data set as an electronic file.
CHAPTER 2

ESSENTIAL FOUNDATION

This chapter presents four "primers" as essential foundation to assist in what follows. The objective is a summary. I will touch lightly on four areas without citing authors or providing exhaustive exposition. Specific references and author citations are found in the next chapter.

Prior to a discussion of the merits of an alternate theory of real estate investing, it is useful to describe the current theory. Therefore, the first section, "Pure Finance," briefly discusses the current status of finance theory in the context of the investment in financial assets. Next, real estate finance is discussed. Specifically, I examine more closely some of the difficulties inherent in applying the finance paradigm to real estate. While the finance paradigm is not without its detractors within the field of finance itself, it is not the aim of this paper to thoroughly discuss or take sides in that argument. Rather, the purpose of the second section is to highlight specific problems, some of which may coincidentally apply generally to the field of finance, that present rough terrain when real estate is the investment in question.

The third section is a primer on statistical distributions. Chapter 1 discussed the notion of a heavy-tailed distribution, one that differs from the normal distribution. When
it comes time to be specific, the most practical class of heavy-tailed distribution is the family of so-called "stable" distributions. These have a critical parameter, \( \alpha \), used to describe their shape. When \( \alpha = 2 \) the stable distribution is also normal, making the normal a special case of stable distribution; thus the normal has a special shape. Much will be made of this special shape later. While some of the discussion of the normal distribution in this section may be elementary or remedial, an attempt is made to draw specific analogues to my area of interest, the stable class of distributions. It is hoped that these analogies will illuminate an otherwise complex mathematical subject. The final primer briefly describes the usefulness of the inverse Fourier transform (IFT), a tool that makes possible some of the intensive numerical analyses and solutions used when working with the stable class of distributions.

**Pure Finance Theory**

As suggested in Chapter 1, the finance paradigm is a three-legged stool consisting of Markowitz’ efficient set portfolio theory, Sharpe’s Capital Asset Pricing Model (CAPM) and Fama’s Efficient Market Hypothesis (EMH). These are major components of Modern Portfolio Theory (MPT), which I will now discuss individually.

**Portfolio Theory**

MPT is based on a set of strong assumptions, one of the most important of which is that returns are normally distributed. This assumption permits a finite variance,
making possible the covariance and correlation coefficients needed to implement MPT. When asset returns are not perfectly correlated, combining them into portfolios is more "efficient" than holding individual assets. When assets are combined, the quadratic nature of asset covariances creates a curve known as the efficient frontier. *Figure 2–1* illustrates how the frontier becomes more convex to the y-axis as the correlation coefficient (derived from the covariance) declines from 1. When correlation is perfect, that is it equals 1, the far right plot is linear because the quadratic term is zero. Return is perfectly linear in risk and there are no gains to be had from diversification. Reading the plots from right to left illustrates how gains from diversification increase as covariance declines. As the plot becomes more convex, more return may be obtained for the same amount of risk or, alternatively, given a fixed return there are opportunities to reduce risk by combining assets.
A major insight flowing from MPT is what is known as "the mean-variance rule." If investors make decisions based solely on mean and variance, the practical consequence is that the first two moments of the distribution are sufficient to form efficient portfolios. Another important ramification is that gains from diversification are available, allowing investors to “diversify away” asset-specific risk.

A second strong assumption of MPT is that investor utility functions are quadratic. If this is so, there is a pleasant symmetry between the derivatives of the utility function and the first two moments of the distribution. If investors seek return and shun

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Figure 2–1: Efficient frontiers reflecting different correlation coefficients ranging, left to right, from -1 to 1 in increments of .2
risk, it would seem that the slope of the first derivative of the utility function would be positive and the slope of the second derivative would be negative. If the first derivative reflects investor preference for mean (return) and the second derivative reflects investor aversion to variance (risk), the mean-variance rule appears to have support in the microeconomics of consumer choice under uncertainty. The symmetry is complete when one assumes both that (a) returns are normally distributed \textit{and} (b) that the utility function is quadratic because (a) the normal distribution can be completely described in terms of its first two moments and (b) there are only two non-zero derivatives for a quadratic utility function.\footnote{\textit{Sharpe, Lintner, Mossin Capital Asset Pricing Model (CAPM)}}

\textbf{Sharpe, Lintner, Mossin Capital Asset Pricing Model (CAPM)}

The CAPM is the second leg of the finance paradigm. As such, it claims that equilibrium and pricing arise from comparing individual asset performance to a market index. Its conclusion is that risk and return are linearly related through the Capital Market Line (CML). The development of the CAPM’s beta as a measure of risk is critically dependent upon the finite variance of the normal distribution.

Not everyone is satisfied with the CAPM. Empirical tests of the CAPM have shown that the SML is flatter than theory predicts, indicating that investors are being compensated for something other than risk at lower levels and are being under-compensated for risk as it rises. Roll (1977) makes an eloquent complaint regarding the
testability of the CAPM. Part of his argument is that many assets, including real estate, are systematically excluded from the market portfolio in empirical tests of the CAPM.

**Fama's Efficient Market Hypothesis (EMH)**

The essence of EMH is that information arrives in the market randomly and is rapidly incorporated into price. With this addition to MPT and CAPM, one finds that price changes are a random walk\(^2\) and returns approximately follow a Normal distribution.\(^3\)

EMH is divided into three forms. “Weak form” presumes that all past information is contained in all past prices. Repeated tests indicate that price change is what is known as a "Markov process," meaning that the best prediction of tomorrow's price is today's price. The conclusion is that there is no money to be made in analyzing and projecting from a historical price series. "Semi-strong" form EMH combines all historical information and presently available public information. Many tests of this second form of EMH indicate that information is so rapidly incorporated into price that it is nearly impossible to make gains, net of transaction costs, by timing one's purchases and sales. Other tests, however, have found some anomalies permitting small advantages, discussed in Chapter 3. Finally, the "strong" form EMH involves all information. After deducting

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1 This area is much more complex and quite controversial. More will be said about this in Chapter 3, but for now I merely state the rationale in the simplest terms to provide foundation.
2 Also, it usually has an upward "drift" due to the intertemporal accumulation of capital.
3 The use of “approximately” here is well chosen. By Fama’s time the assumption of the Normal distribution was being questioned. See Chapter 3.
from all information that portion contained in the semi-strong form, what remains is
private information. There have been few tests of this level of EMH. Due to the
illegality of insider trading, there is an understandable lack of data.

CAPM and EMH are inextricably woven together. The nexus is the assumption that
information arrives randomly; therefore, any two pieces of information are uncorrelated.
Hence, one may assume that price movements caused by that information are also
uncorrelated. From that flows the assumption of the independence of price movements,
permitting a variety of statistical tests that require such an assumption. Roll (1977) also
points out a weakness in this connection, that a test of the EMH is invariably a joint test
of the CAPM and its embedded assumptions about return distributions.

When CAPM and EMH are added to the Markowitz assumptions of portfolio
selection, the finance paradigm makes a neat, tight theoretical package and one that is
best taken as a whole cloth, and applied in places where the assumptions are known to be
approximately true.

Real Estate and Finance Theory

Academic real estate, if a subset of finance, adopts the finance paradigm and
modifies it to accommodate the real asset. Some would say this process is not a smooth
one. Absent a large source of capital, there are practical problems attendant to forming
real estate portfolios. Even if these are overcome empirical difficulties remain. Central
to these is whether asset returns are normally distributed. Since an entire section on
distributions follows this one I will not discuss that area now.
Three other critical elements, common to securities markets but absent for real estate, create empirical problems when applying MPT to real property. The first deals with the allowance of short sales. MPT assumes perfect reversibility, which if unavailable, imposes a “no-short-sale restriction” on the model. This has been handled in the literature but converts the problem from one of linear to non-linear optimization, a complication that is not trivial but neither is it insurmountable. Clearly it is not possible to “short” a parcel of real estate; accordingly, all efficient portfolios must be formed only with “long” transactions in the asset. If risk is measured in part by the number of outcomes possible, certainly a restriction on how the asset may be held limits the number of outcomes, hence impacting the risk equation. Whether this increases or decreases risk is not the subject of this paper, but one sees that risk is affected as MPT’s efficient frontier shrinks to accommodate this restriction.4

Second is what is known as “the integer problem.” MPT assumes perfect divisibility, an assumption easy to maintain in securities where fractional and widely disbursed ownership of the income stream is the goal and consequence of those markets. Accomplishing this is much harder in real estate. The restriction of assets to whole numbers is a thorny mathematical dilemma that defied solution for many years after the publication of Markowitz’s paper. While it is true that considerable amounts of real estate have been “securitized,” a fact that is discussed in detail below, the specific domain of this paper is the non securitized parcel of real property. It will be noted later that

4 As Benninga (1997) states "In general, the efficient frontier with short sales dominates the efficient frontier without short sales. This statement must clearly be so, since the short-sales restriction imposes an extra constraint on the maximization problem" (p. 133).
in institutional reasons account for the absence of securitization in a considerable portion of the real estate market.

Third, MPT assumes perfect liquidity, a feature made possible in a continuous auction market for homogeneous assets such as securities. This feature is seriously lacking in the real estate market, with illiquidity usually found near the top of the laundry list. The important consequence of this omission is that in the stock market, as information arrives continuously and randomly, investors may take immediate action to buy or sell on that information. The inability of real estate investors to do so has implications that will be developed in detail later.

When Roll (1977) critiqued the CAPM, he observed that real estate, among other things, was missing from the market portfolio used in tests of the CAPM. Data limitations doubtless contribute to the reason for the common omission of real estate, but its absence may not be ignored in this context. It is significant that for nearly half a century, constituting perhaps half of the time finance has been an academic subject, assets making up a significant portion of the wealth of humanity have been excluded from the equation connecting risk and return.

Finally, the application of EMH to real estate also raises serious empirical questions. Any item on the laundry list missing from the discussion thus far finds its way into it here. The real estate market has a long, well-earned reputation for being a relatively inefficient market. The evidence, described in more detail in Chapter 3, is mixed. However, most research concludes that real estate information is slowly incorporated into price. Few would contend that the assets are homogeneous. Location-dependent loan
underwriting and site-specific borrowing factors restrict access to capital or make it uneven across regions, market sectors, and time. The real estate market is conducted via private sales in local markets, far from the public and national (now international) markets for securities. Information costs that can be miniscule for stock remain high for real estate. While real estate is affected by national policies influencing macroeconomic factors, a host of micro factors also affect real estate values, including numerous parochial political interests descending to the neighborhood level. For all of these reasons the empirical question of whether EMH may be applied to real estate remains an active area of research.

**Probability Distributions**

The assumption that returns are distributed normally provides a finite variance. This leads to the mathematical concept of covariance so critical to MPT. For Ordinary Least Squares (OLS) regression and the hypothesis testing flowing from it, it is necessary to assume that residuals are independent. Independence of residuals is guaranteed under the assumption of normality. Thus, an assumption of normality is a handy one to make. Accordingly, a review of the basics of probability distributions and normality is warranted.

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Perhaps the most familiar shape in statistics is the bell curve shown in Figure 1-1. The equation that produces this shape, known as a probability density function (pdf), is

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

where

- \( x \) is a random variable, \(-\infty < x < \infty\);
- \( \mu \) is the mean of the distribution;
- \( \sigma \) is the standard deviation of the distribution; and
- \( e \) is the base of the natural logarithm.

This pdf may be used to produce a moment generating function (mgf), a special mathematical expectation of the form

\[ M(t) = e^{\left( \mu t + \frac{\sigma^2 t^2}{2} \right)} \]

Hence, a series of moments may be generated by the sequential differentiation of the mgf. For the normal distribution, interest is often limited to the first four of these, the mean, variance, skewness, and kurtosis.

The normal distribution is actually a special case of a family of distributions known as “stable Paretoian” (SP) distributions. These distributions are unimodal, retain their shape under addition and weighted addition (hence “stable”) and asymptotically approach the Pareto distribution (hence “Paretian”).
Of the infinite number of possible SP distributions, there are only three with known, closed form, probability density functions. The first is the normal distribution, the second is the arctangent or Cauchy distribution, and the third is the Levy distribution. As described below, the Cauchy and Levy distributions are of little interest in a financial setting. With this exception noted, unless the SP distribution is normal, in the limit no moments above the first exist for a broad range of otherwise useful distributions.

This condition does not render stable distributions useless. Stable distributions are fully characterized by four parameters: \( \alpha \in (0,2] \), is known as the characteristic exponent or index of stability; \( \beta \in [-1,1] \), the skewness parameter, is symmetric around the mode when \( \beta = 0 \); \( \gamma > 0 \), is a scale parameter expanding and contracting the distribution around its mode; and \( \delta \in (-\infty,\infty) \), is the location parameter, shifting the distribution left and right along the x-axis.

The most important of these four is \( \alpha \), as it determines the fatness of the tails. As discussed more completely later, the value of \( \alpha \) is very useful for assessing risk in non normal situations. The second most important parameter is \( \beta \); equally vital is its interaction with \( \alpha \), an interaction more fully discussed below and in Chapter 5.

The departure from the normal, the lack of a pdf, and the absence of moments above the first introduce mathematical difficulties. However, while not all probability distributions have a pdf, the characteristic function (ch.f.) always exists. This is true of the SP distribution, which not only has a characteristic function, but several different parameterizations of it. Best suited for the work needed here is the \( S^0 \) variation Nolan
(1998b) provides of the M parameterization given by Zolotarev (1986) (hereafter $S^0(\alpha, \beta, \gamma, \delta)$):

$$E \exp(itX^0) = \begin{cases} \exp \left\{ -\gamma^{\alpha} \left| t \right|^\alpha \left[ 1 + i\beta \left( \tan \frac{\pi \alpha}{2} \right) (\text{sign} \ t) ((\gamma \ | t|)^{1-\alpha} - 1) + i\delta^0 \ t \right] \right\} \text{ where } \alpha \neq 1 \\
\exp \left\{ -\gamma \ | t| \left[ 1 + \beta \frac{2}{\pi} (\text{sign} \ t) (\ln |t| + \ln \gamma) \right] + i\delta^0 \ t \right\} \text{ where } \alpha = 1 \end{cases}$$

$$= \begin{cases} \exp \left\{ -\gamma^{\alpha} \left| t \right|^\alpha \left[ 1 - i\beta \left( \tan \frac{\pi \alpha}{2} \right) (\text{sign} \ t) \right] + i \left[ \delta^0 - \beta \left( \tan \frac{\pi \alpha}{2} \right) \gamma \right] t \right\} \text{ where } \alpha \neq 1 \\
\exp \left\{ -\gamma \ | t| \left[ 1 + \beta \frac{2}{\pi} (\text{sign} \ t) \ln |t| \right] + i \left[ \delta^0 - \beta \frac{2}{\pi} \gamma \ln \gamma \right] t \right\} \text{ where } \alpha = 1 \end{cases}$$

While the above, when compared to the pdf or the mgf of the normal, seems forbidding, it becomes quite manageable upon closer inspection. For instance, it is helpful to know that $\tan \frac{\pi}{2} = 0$. Thus it is easy to see how much the characteristic function simplifies when $\alpha = 2$. As stated in the introduction, this is the special case of the stable distribution where it is normal. The ch.f. then becomes

$$E \exp(itX^0) = \exp \left\{ -\gamma^2 |t|^2 + i\delta^\gamma t \right\},$$

a much more reasonable looking equation that even resembles the normal’s mgf.

Working with SP distributions is further simplified by McCulloch (1978), who states “[i]n financial applications it is universally assumed that $1 < \alpha < 2$” (p. 601 fn 1). Restricting $\alpha$ to this range eliminates the need to consider the Cauchy distribution where $\alpha = 1$ and $\beta = 0$ or the Levy distribution where $\alpha = \frac{1}{2}$ and $\beta = \pm 1$. Thus, my interest is in $\alpha \in (1,2]$ and discussion below will be confined to this range.

It is useful to cast the parameters of the characteristic function of the stable distribution in the light of the moments of the normal distribution using familiar terminology. For instance, $\delta$ is the mean of the stable ch.f. when $\alpha > 1$. When the distribution is symmetric $\beta = 0$ and $\delta$ is also the median (McCulloch (1998a)). For all
parameterizations, $\delta$ is the mode (Nolan (1998c)); hence the best common description of $\delta$ is the “location parameter.”

$\beta$ is the limit value of the ratio of the difference between the tail probabilities (upper – lower) to the sum of those same probabilities. The result of this is that as $\alpha \uparrow 2$, $\beta \to 0$. The practical significance is that, regardless of the value of $\beta$, its meaning diminishes and it becomes harder to estimate as $\alpha$ rises to its maximum permitted value.

The converse of this, critical to the analysis presented in Chapter 5, is that the importance of $\beta$ to the shape of the distribution rises as $\alpha$ falls.

The second moment of the normal distribution, $\sigma^2$, is variance. The variance of the ch.f. of the stable distribution in the special case when it is also the ch.f. of the normal distribution (when $\alpha = 2$) is $2\gamma^2$. The distinction is important when $\alpha < 2$. Then the variance of the distribution is infinite but $\gamma$ remains well defined as a scale parameter that is not, strictly speaking, variance. Rather it is “approximately (though not exactly) the old-fashioned ‘probable error’ or semi-interquartile range, i.e. about half the probability lies between $\delta - \gamma$ and $\delta + \gamma$.” (McCulloch (1998a), p. 361).

All stable distributions maintain their shape under linear transformation, permitting the normalization of the distribution such that $\gamma = 1$ and $\delta = 0$, leaving $\alpha$ and $\beta$ as the parameters of interest. Stable distributions have a second important property in that sums of Stable distributions are themselves Stable. This arises from the Generalized Central Limit Theorem and is useful in the context of finance where observed disturbances are believed to be the cumulative effect of many small disturbances.
Therefore, the ch.f. of the stable distribution is a four-parameter model in which two, $\gamma$ and $\delta$, may be normalized to (1, 0) by a scale and shift, leaving the important $\alpha$ and $\beta$ to describe, respectively, tail thickness and skew. This contrasts with the normal’s first four moments, the first two of which, mean and variance, are often normalized to 0 and 1 and the last two of which, skewness and kurtosis, are typically ignored as they are not needed to describe the distribution.

Many have questioned the validity of assuming the normal distribution for investment returns (see Chapter 3). The natural question then becomes: Why use stable distributions instead? Nolan (1999) recites three reasons for proposing the Stable Paretian (SP) distribution as an alternate:

The first is where there are solid theoretical reasons for expecting a non-Gaussian stable model, e.g. reflection off a rotating mirror yielding a Cauchy distribution, hitting times for a Brownian motion yielding a Levy distribution, the gravitational field of stars yielding the Holtmark distribution; see Feller (1971) for these and other examples. The second reason is the Generalized Central Limit Theorem which states that the only possible non-trivial limit of normalized sums of i.i.d. terms is stable. It is argued that some observed quantities are the sum of many small terms – the price of a stock, the noise in a communication system, etc. and hence a stable model should be used to describe such systems. The third argument for model with stable distributions is empirical: many large data sets exhibit heavy tails and skewness (p. 4).

Other approaches to heavy tails have been mentioned. However, according to McCulloch (1997), “Other leptokurtic distributions, including Student t, mixtures of normals, and the double Weibull, have also been investigated, but these do not have the attractive central limit property or divisibility of stable distributions” (p.74).

Two things deserve special note: One is the effect on the shape of the distribution as $\alpha$ falls; the other is the interaction of $\alpha$ and $\beta$. To better understand the former, note
that a constant, \( \tan \pi \alpha / 2 \), appears in the ch.f. Figure 2–2 plots this term as a function of \( \alpha \). As financial applications restrict the range of \( \alpha \), interest is limited to the lower right portion of the plot as \( \alpha \) ranges downward from 2 to 1. Noticeable downward movement of this function begins after \( \alpha \) passes 1.5 and become more dramatic as it approaches its asymptotic limit at 1. According to Nolan (1999), “As \( \alpha \) decreases, three things occur to the density: the peak gets higher, the region flanking the peak get [sic] lower, and the tails get heavier” (p.9). Notice in the ch.f. that this term is operating on \( \beta \). Thus, a fourth effect of the decline in \( \alpha \) is a more pronounced effect of \( \beta \) on the shape of the distribution.

Figure 2–2: \( \tan (\pi \alpha / 2) \) plotted as a function of \( \alpha \)
Chapter 5 discusses the alpha-beta trade-off in detail, but before proceeding, some intuition is helpful. We view risk as variation in possible outcomes. As such it is considered a "bad" characteristic of any business venture - something to be avoided or requiring compensation. Clearly, however, variation on the positive side of the distribution is more tolerable. Hence, the role of a positive $\beta$ is to take probability mass from the left side and deposit it on the right side. When $\beta > 0$, the right tail is heavier than the left tail. Figure 2–3 modifies Figure 1-3 adding a third distribution that is skewed positive. Note the additional variation on the right that has been taken from the left. It is best to view Figure 2–3 by imagining the normal distribution in the background, behind two stable distributions. The new distribution has a lower peak than the higher, symmetric stable distribution. When the distribution is fat tailed and skewed to the right the investor is compensated by the positive skew for bearing the risk imposed by the greater variation.\footnote{As it requires normality, skewness is not considered in Markowitz portfolio theory. In an attempt to overcome that, Markowitz (1959) and others have noticed a relationship between skewness and semi-variance. Attempts to substitute the latter for the former have met limited success.}
Before undertaking the mathematical complexities of the Stable Distribution in the next section, it is useful to draw a final analogy between the kurtosis in the Normal distribution and $\alpha$ as the measure of tail thickness in the stable distribution. This analogy is admittedly strained but is useful in the big picture. When those using the normal distribution essentially ignore tail behavior and users of the stable distribution make tail behavior the centerpiece of their argument it is natural to ask if there is any connection between these concepts. After all, the normal and the SP are both distributions and the former is a family member of the latter.

Viewed broadly, all deviation from the mean is a form of dispersion. Variance is one measure of that dispersion and kurtosis is another. Kurtosis, the fourth moment of the normal distribution, measures the "flatness" (leptokurtosis) or "peakedness" (platykurtosis) of the distribution. Our interest is in the leptokurtosis because, among
other things, it reduces the accuracy of estimates of the mean. The result, in the case of flatness, is heavy tails. The level of kurtosis found in the Normal distribution (mesokurtosis), is 3. Kurtosis of the heavy-tailed variety is usually expressed in terms of excess kurtosis, that is, the amount of kurtosis in excess of the Normal. Examining more closely the equation used to measure excess kurtosis,

\[
-3 + \frac{\sum (x - \bar{x})^4}{\left(\sum (x - \bar{x})^2\right)^2}
\]

note that this equation is a ratio of the average difference between each observation and the mean taken to the fourth power (the numerator of the ratio) to the square of the squared average deviation or the square of the variance (the denominator of the ratio). Excess kurtosis occurs when the numerator of this ratio rises faster than the denominator does. The cause of this is more outliers or larger outliers or both.

SP distributions do not have kurtosis (or, for that matter, the other common measure of dispersion, variance) in the usual sense. The substitute for variance is a "scale" parameter of the stable distribution's characteristic function. This scale factor, "\(\gamma\)", in the characteristic function is raised to an exponent. That exponent is \(\alpha\). Hence, there is a connection between \(\alpha\) and dispersion; a potent connection determining the shape of the distribution, especially the shape of the tails. To stretch the analogy farther, consider the effect if it were the usual kurtosis. Imagine the kurtosis term [the ratio of the average difference between each observation and the mean taken to the fourth power to the square of the squared average deviation] now raised to a power. The effect is that it
would increase, provided that alpha exceeds 1, which is the range of interest for alpha. This is as far as the kurtosis analogy in the normal case may go, because increases of alpha toward 2 decrease dispersion as the tails thin.

Intuitively, the characteristic function of the stable distribution describes what might be called "the nature" of the stable distribution. That nature includes, and is especially interesting because of, its long and thick tails. Those tails are long because of the "exponented" scaling provided by $\alpha$, magnifying the difference between the mean and outliers in the operation described above.

"Student" (1927) provides a more accurate if less transparent explanation that contains a particularly ingenious memory aid in his footnote reproduced in Figure 2–4:

[I]n normal distributions there is no correlation between the square of the mean and the variance: similarly, in platykurtic* distributions those samples with large variance even tend to have more accurate means. Actually, however, many if not most routine analyses have a leptokurtic error system, possibly because the standard deviation as well as the mean is subject to variation with time…”

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7 Gossett, G.W., Errors of Routine Analysis, *Biometrika*, v. 19 No. 1/2, July, 1927, p. 160
Another way of placing this subject in perspective is to examine the extremes of the range of interest where \( \alpha \) is either 2 or 1. Recall that when \( \alpha = 2 \) the distribution is normal and the tails are quite thin. \( \textit{Figure 2–5} \) plots the normal centered at 0 over the range of \(-5 < x < 5\). Notice that very little of the normal probability mass lies beyond 3. On the other hand, for the Cauchy distribution, \( \alpha = 1 \). According to Nolan (1999), there are, on average, 100 times as many values above 3 in Cauchy samples than there are in the normal case.
There are some further difficulties with stable distributions. Many of these occur when working at the edge of the parameter space. Numerical routines do not work well as $\alpha$ draws very close to 1 or 0. Despite this, Nolan (1998b) reports that $S^{(0),(\alpha,\beta,\gamma,\delta)}$ parameterization estimates are well behaved over the entire parameter space. While all stable distributions are unimodal, it is not known how the mode relates to the parameters. The stable and Paretian densities cross, but locating the point at which they cross is difficult; hence finding when the asymptotic Paretian tail behavior begins is troublesome.

\textit{Figure 2–5:} The normal (0,1) and the Cauchy (1,0) distributions
Whether any of these technicalities is vital depends on the individual setting and the research issues at hand.

Several questions remain. Should a mountain of research based on the assumption of normality be abandoned? Does the juxtaposition of the stable and normal provide a useful comparison between two measures of risk? Are the difficulties of the stable form (no closed form pdf, no moments above the first) insurmountable? I address the last of these in the next section. The remaining questions are covered later.

The Inverse Fourier Tranform

The ch.f. cannot be integrated symbolically. However, subject to certain conditions, any function can be accurately depicted by a sampled version of itself. Fortunately, the pdf and ch.f. of the SP distribution constitute what is known as a Fourier transform pair. Work in the transform space, when combined with the sampling process, can produce a graphical representation of a pdf. This section is intended as a brief introduction to the usefulness of Fourier Transforms. Appendix C elaborates further on the entire process and its combination with sampling theory. Appendix D describes the technicalities of the Fourier and sampling operations that generate the graphics in Chapter 5.

Jean Baptiste Joseph de Fourier (1768-1830) developed a representation of aperiodic signals as linear combinations of complex exponentials. He noticed that it was possible to consider an aperiodic signal as a limiting case of a periodic signal with an
arbitrarily large period. Farlow (1993) provides a practical look at the use of transforms in Figure 2–6.

Figure 2–6: The General philosophy of transforms (source: Farlow (1993) Fig 10.1)

Since the transforms described in Appendix D for the SP work needed in Chapter 5 are complex, the review of transform procedure here is better handled with a simple example similar to Farlow (1993). We begin with a function that is known to have a Fourier transform (FT)

\[ f(x) = e^{-x^2/10}. \]

Figure 2–7 shows a plot of this function.

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8 Granger and Orr (1972) note that "Much of the work that ordinarily uses probability or frequency functions can be carried out in the transform space of the characteristic function" (p.275).
We now define the transformed function, $f(u)$, as the Fourier transform of the original function, $f(x)$,

$$f(u) = \frac{\sqrt{10\pi}}{e^{5u^2/2}}$$

and plot that function in Figure 2–8.

*Figure 2–7: $f(x) = e^{-x^2/10}$*
A useful property of a Fourier Transform is its inverse. The inverse Fourier transform (IFT) of \( f(u) \) returns the original function, \( f(x) \).

The width of a transformed function is proportional to the rate of change of the function before the transform. The extremes (developed further in Appendix C) are represented by an impulse function, DiracDelta\([u-u_1]\), having two important properties. First, it has a value only when \( u = u_1 \), and second, at that value \( \int \text{DiracDelta}(0) = 1 \).
We illustrate the proportionality issue by defining a sharper DF resulting in a wider ch.f.

\[ g(x) = e^{-\frac{x^2}{2}} \]

Plotting them together in *Figure 2–9* reveals that \( g(x) \) is more concentrated at the center than \( f(x) \):

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*Figure 2–9: f(x) and g(x)*

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Having previously transformed \( f(x) \) to \( f(u) \) we now transform \( g(x) \) to \( g(u) \)

\[ g(u) = e^{-\frac{u^2}{2}} \]

and plot both transformed functions in *Figure 2–10*
Using DiracDelta, a table of terms is produced, each of which is individually integratable, making the function transformable. The sampling interval is tested repeatedly and reduced until all significant points of the pdf are plotted and ambiguities (where the function changes direction) are outside the range of interest. The intuition is that we use an iterative process to simulate a shape from specific points of a known function, in this case the IFT of the table of values created by the application of the impulse function to the stable ch.f. The iterations reconcile the shapes depicted in Figure 2–10: $f(u)$ and $g(u)$.
2–9 and Figure 2–10 to a point that the result is the shape we would have were it possible to create a plot directly from a pdf.

Summary

The finance paradigm works well when markets are efficient, homogeneous, easily aggregated and offer liquidity, reversibility and divisibility at minimum transaction costs. Lacking some of these critical features would cast doubts on any paradigm dependent on them. Privately held real estate lacks all of them and it becomes an empirical question as to how damaging the absence of these may be.

An assumption of normality is restrictive. Examining the ramifications of restrictions imposed on models that assume normality is necessary. In Chapter 3 I will cover the advances in statistical theory that answers important parts of these empirical questions.

Next, I discuss the specifics and chronology of academic advances in fields that relate to topics covered here. The literature review provides a brief summary of who-said-what-and-when in finance, economics, real estate, and statistical theory.
CHAPTER 3

REVIEW OF THE LITERATURE

Introduction

As the finance and mathematical statistics disciplines moved forward in the past several decades, academic real estate lagged. A look back at the academic real estate literature reveals that until recently, namely Young and Graff (1995), academics studying real estate spent the majority of their time attempting to apply MPT to the real estate market. A strict application proved difficult due to the “exceptions list” in Appendix A. Hence, much effort went into research based on a variety of “work-arounds,” approximations, and assumptions that often performed poorly as predictors.

The review of the literature relating to the subject of this paper must, therefore, be divided along carefully drawn boundaries. Hence, this chapter will do the following

• Briefly review the finance literature summarizing MPT and a few recent developments.
• Describe the application of MPT to real estate and its varying levels of success.
• Discuss nay-sayers, malcontents and contrarians – those, both inside and outside of the finance discipline proper, who expressed doubts about the validity of the
finance paradigm. I will also inquire into developments in mathematical statistics over the same time period.

- Finally, present a short but growing literature on alternative views of real estate investment and distributions.

Before addressing the specifics, two general ideas that transcend all four of the above areas need mention. First, no literature review of business, economics, and finance would be complete without mention of the seminal work on the nature of risk and uncertainty. Knight (1921) has endured for most of this century as the first and most eloquent work on the difference between risk and uncertainty. This dissertation is primarily an examination of probability distributions in a certain market. Therefore, its focus is on risk, which, according to Knight, is a subset of uncertainty that involves a known, a priori probability distribution. Other than a discussion in the implications section of Chapter 7, no attempt will be made here to reconcile the apparent contradictions that arise when a further subset of risk is made that blurs the line between risk and uncertainty.

Second, much of the focus of this distribution is on assumptions associated with a theory, specifically the assumption of normality in investor return distributions associated with MPT. Accordingly, it is useful to keep in mind the admonition of Friedman (1953):

> the relevant question to ask about the "assumptions" of a theory is not whether they are descriptively "realistic", for they never are, but whether they are sufficiently good approximations for the purpose at hand. And this question can be answered only by seeing whether the theory works, which means whether it yields sufficiently accurate predictions (p.5).

It is in this context that I, and others, have pointed out violations of the assumptions of MPT that produce empirically unsatisfactory results. Traveling through the hierarchy
below, first through finance, then real estate finance, then the cast of doubters, it is necessary look carefully for what the conclusion of these authors means for the subject of this dissertation: Private, individually owned real estate investments. Our question becomes: Does the violation of the assumption of normality produce empirically acceptable results for that specific market?

**The Finance Literature**

Although MPT dominated the literature in the second half of the century, some thought had been given to appropriate investment strategy prior to the 1950s. Graham and Dodd (1934) take credit for providing some science to the art of selecting investments by setting out a list of rules to govern security analysis. Theirs was a one-parameter model concentrated on maximizing the yield on a specific investment. Their view of a portfolio was simplistic: One maximized the return on a portfolio merely by maximizing the return on all of its components. Over- and under-priced securities were the consequence of failing to do one’s homework, the homework necessary to ferret out true value. The Preface of their fourth edition, published in 1962, admits to a price-value dichotomy but warns “the danger lies not so much in the emphasis on future earnings as on a lack of standards used in relating earnings growth to current values” (p. vi.). This 778 page edition appeared 10 years after Markowitz (1952) and contains a mere passing comment about MPT, characterizing it as “[w]hat appears to be the beginning of a new movement…” (p. 523). A fifth edition of *Security Analysis* is available at book stores today. It remains the bible of the value investor, 65 years after its initial publication, yet
the current edition is even more silent on MPT than the original, with no mention of Markowitz at all.

Markowitz (1952) demonstrated that one could improve investment results by combining assets with differing covariances. His main contribution was to double the number of parameters upon which the asset selection depended to two, by showing the way to maximize expected return while minimizing risk. He introduced the “efficient frontier” illustrated in Figure 2-1. Malkiel (1985) described Markowitz as “a scholarly academic ‘computenick’ type” who “discovered…that portfolios of risky (volatile) stocks might be put together in such a way that the portfolio as a whole would actually be less risky than any one of the individual stocks in it.”\(^1\) This is perhaps an understatement given that Professor Markowitz is considered the father of MPT. Certainly he changed the way the world thinks about investing. He was the first to show that risk is a function independent of return and can be “managed.”

A Markowitz student, Sharpe (1964) extended the work to a general equilibrium capital asset pricing model (CAPM), simplifying the daunting matrix algebra necessary to implement MPT.\(^2\) Sharpe introduced a single index linear model to condition the returns of an asset by reference to a “market” of securities. Sharpe’s contribution was the beta, an indication of the relative risk a security contributed to any portfolio. Blume (1970) termed this “the market model,” saying that it “specifies a relationship between the returns for individual assets and a market factor” (p. 155). If (a) return is linear in risk,

\(^1\) p. 194. This simplification does not do Professor Markowitz’s insight justice and, as simplifications often are, is slightly inaccurate.
(b) borrowing and lending takes place at the same rate of interest, and (c) a risk-free asset exists, efficient portfolios may be formed along a line, the Capital Market Line, extending from the risk-free rate through a point on the efficient frontier where the market portfolio may be found.

Markowitz (1952) and Sharpe (1964) having contributed the initial building blocks for the finance paradigm, Fama (1970a) contributed the Efficient Market Hypothesis (EMH), arguing that price change is a result of information randomly reaching the market, thus making price change a “random walk.” Fama separated EMH into the three forms discussed in Chapter 2. The last of these proved relatively untestable but with the publication of Fama, Fisher, Jensen and Roll (1969), a research industry based on “event studies” was launched. These studies repeatedly tested the semi strong form of market efficiency with varying results. Fama (1991) revisits efficient markets, reviewing 20 years of research in this area better than one can hope to do here.

Markowitz (1987) extended his earlier work to encompass the CAPM and address some of the issues raised in the years since his original work. These issues will be discussed in greater depth below.

Jensen and Smith (1984) take the position that there are in fact five, not three, legs to the finance paradigm’s stool, adding Option Pricing Theory and Agency Theory as essential building blocks. These are not insubstantial contributions to finance. However as no option market of the kind that is now in place for securities exists for real assets, and agency theory is predominantly a matter of corporate finance rather than personal

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2 The CAPM is often enough attributed equally to Lintner (1965) and Mossin (1966) as well as to Sharpe
investments, these areas are not covered in depth here. Others contributed MPT refinements. A good example, bearing on the subject of fat tails in the distribution, was that of Elton, Gruber and Padberg (1978), who found that outliers in the portfolio were over-weighted by the Markowitz technique. They created cutoff hurdles for assets to clear before being included in the efficient portfolio. These hurdles were based on a metric they termed “excess return to beta”.

Much of the “normal science,” as Kuhn (1970) refers to it, that followed involved testing a host of anomalies such as securities returns around year-end (“the January effect” Reinganum (1983), Lamoureux and Sanger (1989)), the effect of periods when the market is closed (“the Monday effect” Gerety and Mulherin (1994)), a comparison of returns based on size (“the small firm effect” Keim (1983), Schwert (1983)). These authors found that, contrary to semi strong form EMH, there were some inefficiencies and exploitable regularities in securities markets. Thus, finance broadened its specific knowledge of how markets work under the dominant paradigm of MPT, CAPM, and EMH.

Real Estate Applications of the Finance Paradigm

Wendt and Wong (1965) first suggested that there might be a difference between real estate returns and stock market returns. Their sample came from government
subsidized housing in the apartment sector. They found that there was a difference and that tax effects contributed to that difference. The first attempt to apply portfolio theory to real estate was Friedman (1971), who noticed that the indivisibility problem was a significant complicating factor.

Williams and Findlay (1974) found further fault with the application of finance models to real estate, commenting on the near impossibility of constructing an index and creating their own laundry list of real estate problems that essentially “tuned” the items in Appendix A to the technical issues of MPT (p. 360). Curcio and Gaines (1977) added to the list of complaints, pointing out that revising a portfolio is critical to the maintenance of its efficiency and real estate holdings were difficult and costly to revise. Miles and Rice (1978) joined the chorus shortly thereafter, observing a failure of the model to consider entrepreneurial issues raised by Knight (1921). Findlay, Hamilton, Messner and Yormark (1979) then carefully summarized all of the serious problems associated with adapting finance models to real estate, and few escape their net. Markowitz, Sharpe, Ross, and Friedman are all scrutinized in the context of realities of real estate data and characteristics. While none of the finance theorists fares well, they propose a model based on quadratic integer programming that is in essence an extension of the Markowitz model.

Undeterred, the academic real estate community continued to work around the edges of the finance paradigm. The best way to do this was to study major corporate real estate investors. Miles and Esty (1982) looked at commingled real estate fund performance while Miles and McCue (1982) looked at institutional real estate in general.
Miles and Esty (1982) apparently felt that limiting the field to publicly owned real estate was still not perfect, stating that "trying to borrow stock market technology may be misguided" (p. 67). Shortly thereafter Linneman (1986) used housing data to find real estate to be semi strong form efficient, although he remarked that the finance paradigm was "difficult to implement" for real estate (p. 142). Gau (1984) struggled with time series data for real estate, remarking that "empirical tests of market efficiency require a time series of market prices of real estate assets," acknowledging that collecting such data is virtually impossible (p. 304). Thus, the EMH literature in the private ownership arena is minimal compared to the literature on securitized real estate and miniscule compared to the literature on financial assets.

With the merger of the Frank Russell and National Council of Real Estate Investment Fiduciaries (NCREIF) indices in the early 1980s, data began to grow for institutional grade real estate. Researchers turned their attention from the task of explaining away the exceptions list to inquiring into two sub areas commonly considered to be unique aspects of real estate investing: inflation and taxes. In a variation on the Arbitrage Pricing Theory of Ross (1976), Brueggeman, Chen and Thibodeau (1984) provided a two-index model, the second being inflation. They then further divided that factor, appropriately, into expected and unexpected inflation. Their results, considering the data covered the inflationary 1970s, were not surprising but confirmed that real estate was a hedge against unexpected inflation. Soon after, in a well-timed piece considering the tax reform that was passed by Congress later that year, Webb and Rubens (1986) used an adaptation of Elton, Gruber and Padberg (1978) to show the heavy influence taxes had
on real estate returns, concluding that real estate’s contribution to a portfolio increased during times of increased taxation.

Thereafter, research grew as academics inquired into gains from diversification by property type, region, and in a mixed asset portfolio. Examples of these, respectively, were Grissom, Kuhle and Walther (1987), Hartzell, Shulman and Wurtzebach (1987) and Kuhle (1987). Not everyone went quietly, however. Firstenberg, Ross and Zisler (1988) suggested that portfolio management is no substitute for property-specific management; rather it is intended to supplement the management of on-site dynamics. They also pointed out that portfolio management adds an additional expense tier and asked if it is worth what it costs. Roulac (1988), clarifying the valuation of real estate securities, said that cash flows from real estate securities are not income from portfolio assets, thereby misstating value. He claimed that the difference is site specific financing, management compensation, and other issues peculiar to the real estate form. Zisler (1988) took the interesting view of a parcel of real estate as a portfolio unto itself (leases are like bonds, puts and calls are included, reversion of possession is residual equity claim, etc.) noting a theoretical conflict between portfolio management and property management. Zisler (1988) thought the traditional four-region diversification was "too coarse" and he cautioned that "[t]he careless borrowing of risk measures from other asset classes creates false expectations" (p. 25). In a broad look at the field and its pricing model, Lusht (1988), referring the poor adaptation of real estate to “static equilibrium models of mainstream finance” concluded with “now that we have confirmed what doesn’t work, we should stop [using those models]” (p. 102).
The field did not immediately take Professor Lusht’s recommendation. By the early 1990s, real estate data for institutions had grown to the point that there were several indices, prompting Ross and Zisler (1991) to compare and contrast them. They narrowed the range of real estate risk to 9%-13% from the broader 3%-20% that had been used historically. They found that while volatility in real estate indices was less than the volatility for stock, the cause could be traced to the appraisal-based returns required to generate the real estate index. Their conclusion was that real estate volatility lay between stocks and bonds.

The notion of international real estate investment surfaced at about this time prompting Ziobrowski and Curcio (1991) to study the possibility of forming portfolios across national boundaries. They found that foreign exchange rate fluctuations offset much of the portfolio benefit.

The publication of the first issue of the Journal of Real Estate Portfolio Management appeared in 1995. It included Gold (1995), who asserted that there were limits to MPT for real estate and argued for so-called “fuzzy” efficient frontiers. More articles on international real estate appeared thereafter studying such diverse locations as Japan and Korea (Kim and Suh (1993)), the Netherlands (De Wit (1996)) and Hong Kong (Chiang and Ganesan (1996)); Webb, Chau and Lee (1997); and Brown, Li and Lusht (in press)).

The decade and the century may close by coming full circle. Capozza and Sequin (in Press) are working on a paper tentatively titled “Focus, Transparency and Value” in which they examine REIT diversification. They are unable to detect benefits from
diversification but cannot say that firms holding a diversified set of properties underperform. It may be that Graham and Dodd (1934) will experience a renaissance in the new millennium.

**Doubts about the Finance Paradigm**

Measured in research time, it was not long after Markowitz (1952) that sounds of dissent were heard. The source of doubt was in its mathematics, the very strength of the MPT argument. The opposition came from two areas. In finance the focus was on the assumption of normally distributed returns. The second volley was fired by the economists who were troubled by issues in risk aversion and the specific shape of the utility function. In two parts, this section reviews the literature on this controversy in these separate fields.

**Within the Finance Discipline—Mandelbrot and the Stable Hypothesis**

It could be said that Mandelbrot (1963a) and Mandelbrot (1963b) fired the first shots. Strictly speaking, however, the origin of the misgivings was in the field of mathematical statistics, but Mandelbrot’s contribution appeared in journals representing the fields of finance and economics. In finance proper, Fama represents the most visible of MPT’s early critics. Fama’s Ph.D. dissertation advisor, Mandelbrot, had a keen interest in non normal distributions of all kinds, in and out of the world of business and finance. The Mandelbrot influence precipitated a decade-long inquiry by Fama
questioning the validity of the assumption of normality so critical to the conclusion of MPT. This inquiry began with “Mandelbrot and the Stable Paretian Hypothesis,” (Fama (1963)), based on Fama’s dissertation, in which he develops the problem associated with the infinite variance of SP distributions and all the theoretical difficulties that arise from it. The general conclusion is stated as “a stable Paretian market means, of course, that such a market is inherently more risky for the speculator or investor than a Gaussian market” (p. 427).

Fama (1965a) and Fama (1965b) followed soon thereafter. The first, Fama (1965a), is a theoretical analysis. After acknowledging the formal correctness of the parameter space $0 < \alpha < 2$ in the characteristic function of SP distributions, he says, “This should not, however, be interpreted to mean that diversification is always a meaningful economic concept. We shall see…that diversification only makes economic sense for a narrower range of values of $\alpha$” (p. 410). Fama later concluded that the range is limited to $\alpha \geq 1.5$ in which case increasing the number of securities in the portfolio still moves risk “relatively quickly toward its asymptotic value.”\(^4\) However, it takes more securities than MPT with a normal distribution indicates and the number can grow very large as $\alpha$ falls.

Fama’s summary is worth including in toto:

The main purpose of this paper was to present a formal portfolio model for the case where returns on securities follow stable Paretian distributions with characteristic exponents less than 2. The model shows conditions under which diversification leads to a reduction in the dispersion of the distribution of the return on a portfolio, even though the variance of this distribution is infinite.

\(^4\) p. 414. It should be noted that this decrease is exponential with the fall in $\alpha$. 
Although the model presented is sufficient to accomplish these theoretical goals, there are admittedly difficult problems involved in applying it to practical situations. Most of these difficulties are due to the fact that economic models involving stable Paretian generating processes have developed more rapidly than the statistical theory of stable Paretian distributions. It is our hope that papers like this will arouse the interest of statisticians in exploring more fully the properties of these distributions.5

In the second of his two papers that year, Fama (1965b), performed an empirical test of the theory presented in Fama (1965a) that developed a sort of “neighborhood” argument for portfolio theory when returns are SP distributed. His test used the 30 stocks of the Dow-Jones Industrial Average from late 1957 through the Fall of 1962. He found “[t]he most striking feature…is the presence of some degree of leptokurtosis for every stock.”6 He characterizes his results as “consistent and impressive” (p. 49). Using what were at the time crude (but the best available) estimators, Fama found that the average value of $\alpha$ for the Dow was between 1.7 and 1.9. He soundly rejected the hypothesis of Normality and exhaustively eliminated causes other than distributional. Again, in this paper, an important conclusion is worthy of full recital here:

Finally, from the point of view of the individual investor, the name that the researcher gives to the probability distribution of the return on a security is irrelevant, as is the argument concerning whether variances are finite or infinite. The investor’s sole interest is in the shape of the distribution. That is, the only information he needs concerns the probability of gains and losses greater than given amounts. As long as two different hypotheses provide adequate descriptions of the relative frequencies, the investor is indifferent as to whether the researcher tells him that the distributions of returns are stable Paretian with characteristic exponent $\alpha < 2$ or just long tailed with finite variance (p. 97-98).

5 p. 418. As will be shown, Fama himself undertook some of the work he hoped to inspire statisticians to do. But his concerns were unfounded as the field of mathematical statistics was hard at this task already as will be developed later in this chapter.

6 p. 48. It should be remembered that the non-Gaussian stable distribution with $\alpha < 2$ exhibits leptokurtosis, or “fat tails”.
Fama’s conclusion was that researchers ignore fat tails at their own peril and that the "degree of fatness", as measured by $\alpha$, was important. The message of Fama (1965a) was that MPT technology was “close enough” when $\alpha$ is close to 2 and the message of Fama (1965b) was that, indeed, $\alpha$ was close to 2 for securities. Finance had dodged the bullet, but in the last section of this chapter it will become apparent that real estate was not so lucky.\footnote{Thirty years would pass before someone noticed that real estate return distributions had $\alpha$ values far from 2, far enough to be troublesome.}

Not willing to wait for statisticians to produce estimation methods, Fama drafted a student to assist and the result was Fama and Roll (1968) and Fama and Roll (1971) wherein can be found tables of cumulative distribution functions for $1 < \alpha < 2$ and parameter estimates for the symmetric stable case ($\beta = 0$). This work employed the generalized central limit theorem, expansions due to Bergstrom (1952), and early application of computer speed to produce numerical results where analytical results were impossible. Roll (1970) applied this technology to interest rates with other issues in mind and, as a by-product, found that $\alpha \approx 1.2$ for treasury notes.\footnote{The importance of this finding is unclear, as no one has ever seriously suggested the creation of mean-variance portfolios of treasury notes, except perhaps for immunization purposes. Roll accomplishes a number of other things of greater import and may have merely performed the stable work because the technology was new and he had the data.}

Fama’s last visit to this area appears to be Fama and Miller (1972), a text in which a significant portion of one chapter is devoted to a review of the issues and his prior work. One can only speculate as to why Fama abandoned this line of work, but as described later in this chapter, the mathematics of stable distributions was
underdeveloped until the late 1980s. Fama may have recognized that he had taken this idea as far as possible with current technology and was satisfied to find $\alpha$ as close to 2 as he did. He may have concluded that, notwithstanding the violation of the assumption of normality, predictions of the model for securities met the standard set down by Friedman (1953).

Blume (1970) reviewed Fama’s conclusion that $\alpha$ for securities is between 1.7 and 1.9, commenting about the systematic underestimation of dispersion when this is so, “[t]he smaller the value of $\alpha$, the greater will be the probability of extreme values or geometrically the greater will be the area contained under the tails of the density function” (p. 153). Blume (1970) “tightened” Fama’s estimate range to between 1.7 and 1.8, cautioning against rounding, saying, “The predictive distributions assessed using a value…greater than 1.8 tended to understate the probability of extreme values” (p. 166-68). This “shrinking” of the portion of $\alpha$ parameter space wherein MPT may be safely applied is not trivial. As $\alpha$ is an exponent, a move from 1.5 to 1.8 has a significant effect on the shape of the distribution. Lee and Wu (1985) investigated the stationarity of standard deviation in securities from 1959 through 1979, finding that it was significantly affected by kurtosis. They concluded that the 4th moment of the distribution should not be ignored in option pricing models and that ordinary least squares estimation was not efficient in the presence of fat tails.9

9 The assumption of normality is not strictly required for OLS but is often made because it insures the independence of the error terms from the regressors. McCulloch (1998a) has shown that OLS is inefficient in the presence of heavy tails.
In a very accessible if not scholarly work, Peters (1996) mounts a withering attack on the normality assumption, the random walk argument, and a host of other components of the finance paradigm, claiming that Mandelbrot’s stable hypothesis leads to non linear dynamics and ultimately to chaos theory.\(^{10}\)

One might incorrectly conclude that with the attention given to kurtosis, the 3\(^{rd}\) moment of the distribution, skewness, is ignored in the literature. Not so; a substantial literature exists in this area. The importance of skewness may be diminished by the general agreement that it is a desirable trait of a distribution, less worrisome in the context of risk. Chunhachinda, Dandapani, Hamid and Prakash (1997) took an international look using polynomial goal programming rather than the more usual linear programming, and found that skewness was an aspect of non normality more easily diversified away than kurtosis. Fang and Lai (1997) reinforced the notion that skewness is generally a “good” (its marginal utility is positive) while kurtosis is a “bad” (marginal utility is negative) in the eyes of investors. Golec and Tamarkin (1998) find that bettors’ apparent risk-loving behavior is really an affinity for skewness.

The foregoing representing a non-exhaustive list of doubts expressed about the finance paradigm, one might think little remained to be said. \textit{Au contraire}, the economists were not to be outdone. Theirs was an equally vocal objection, this time coming from a different direction.

\(^{10}\) While well beyond the scope of this dissertation, it is interesting to note two connections of this subject to recent important advances in mathematics. The same \(\alpha\) discussed here as the characteristic exponent of the SP distribution is inverse of the Hurst exponent in chaos theory and, when not an integer, it represents fractional Euclidian space.
In Economics –Risk Aversion and the Utility Function

While he does not consider himself an economist, Mandelbrot (1963a) and Mandelbrot (1963b) published his work in journals of business and economics. The mathematical issues of SP distributions are dealt with in the literature of other disciplines, something that is covered in the next section. In this section I concentrate on objections coming from the field of economics.

Since economics represents the mothership of finance, activities of the surrogate were quickly placed under the microscope. The economists objected to MPT based on a host of problems. Some of these, such as Cootner (1964a) and Cootner (1964b), re-addressed the distributional issues. Most of the new problems raised by economists had to do with utility functions. The seminal work of Pratt (1964) and Arrow (1965) raised the question of risk aversion and showed the importance the exact shape of the utility function assumes in determining just how much and under what conditions investors may be risk averse. Tobin (1965) elaborated MPT’s economic consequences exhaustively, concluding that an investor’s expected utility depended only on two parameters, mean and variance. This offended Borch (1969) and Feldstein (1969), moving them to comment, thus prompting a response from Tobin (1969).

The essence of the Tobin-Borch-Feldstein dispute lies in the combination of (a) the choice of specific form of the utility function and (b) the assumption about how outcomes of various choices made under conditions of uncertainty may be distributed.
Borch (1969) contended that if one makes reasonable assumptions that (a) the marginal utility for money is positive (more is always better) and (b) the second derivative of that same utility function is negative (decreasing marginal returns), “[t]hese two assumptions imply that \( u(x) \) cannot be a polynomial” (p. 2). Borch’s point is that the use of a polynomial creates a contradiction when one attempts to reconcile it with the notion that “a reasonable preference ordering over a set of probability distributions can be represented by a family of indifference curves in the ES-plane.” The only cure for this contradiction is “to take the initial set as the set of all normal distributions” (p. 4).

Feldstein (1969) is more direct, showing Tobin (1965) simply wrong in his indifference curve demonstration and concluding that “if there is more than one asset, an analysis in terms of only mean and variance is not strictly possible unless utility functions are assumed quadratic or probability distributions are severely restricted.” Tobin (1969) responded:

Strictly speaking, the portfolio choices of an expected-utility-maximizing investor can be analyzed in terms of the two parameters, mean and variance, of his subjective probability distributions of the returns from alternative possible portfolios only if one or both of the two following assumptions is met:

(a) the investor’s utility function is quadratic
(b) he regards the \( r_i \) as normally distributed

In the absence of (a), the second assumption is required (p. 13).

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11 Alternate expressions of this same conclusion include “expected value and standard deviation” sometimes abbreviated as “ES”. We also see “mean and standard deviation”. The essential point of the argument is not different under any such alternate expression.

12 p. 2. Here, “ES” refers to “Expected Value-Standard Deviation”.

13 p. 11. “severely restricted” means that they must be normal.
Tobin (1969) responded that the field was in a major transition from a one-parameter to a two-parameter model, a transition he urged his colleagues to see as the big picture. However, he acknowledged that irregularities remained to be worked out.

Rothchild and Stiglitz (1970) refined the definition of risk to four categories, showing that the first three were essentially the same, and the fourth, the mean-variance rule, was different from the other three. Two of the conclusions they reached were that “the ranking by variance and the ranking by expected utility are different” (p. 241) and that “there exist individuals with concave utility functions who are better off with an increase in variance” (p. 242). This presented a clearly defined alternative view of risk to the view provided by MPT. This view was one into which MPT would not easily fit.

Rothchild and Stiglitz (1971), the sequel, opened by dispensing with the mean-variance rule: “The answers of mean-variance are spurious; they hold only if the utility function or the class of distributions is arbitrarily restricted” (p. 67 fn2). They continue by showing the economic content of the use of the other three (which are really one) definitions from their prior paper. They show situations in which variance is greater but at the same time, under their definition, risk is not.

Tsiang (1972), reviewing and concurring with those who came before, sets out properties of a utility function necessary to satisfy the requirements of his predecessors, describing “the ideal utility function” as “somewhere in between the negative exponential function and the constant elasticity function” (p. 357). Using a Taylor expansion he showed that ignoring moments above the second required rapid convergence of the series.
The crucial questions are under what conditions we can expect the expansion of the utility function to converge quickly, and whether these conditions hold in the usual problems to which the E-S analysis is frequently applied…What is necessary for the E-S analysis to be a good approximation is merely that risk should remain small relatively [sic] to the total risk of the individual concerned (p. 356-57).

Levy (1974), working out still more of the irregularities, took exception to some of Tsiang’s conclusions regarding the maximum slopes of indifference curves, using rectangular distributions to show the error. According to Levy (1974), “even if Tsiang’s assumptions for a good approximation are valid, one cannot apply the mean-variance analysis while ignoring all higher distribution’s moments” (p. 434).

Diamond and Stiglitz (1974) added different risk settings to the portfolio choice problem, such as the risk involved in a career change, enriching this area with a discussion of how taxation affects the risk-averse individual. Levy and Markowitz (1979), building on Young and Trent (1969), addressed some of these issues by “fitting the quadratic to three judiciously chosen points” on non quadratic utility functions (p. 309). They then tested this idea with real historical data to find that, at least for those data, the fit is quite a good approximation.14

The economists were not the only ones outside of finance interested in the inner workings of the mean-variance rule. The field of management science took notice of the controversy in the early 1990s, voicing a different concern. Ruefli (1990) and Bromiley (1991) argued about the inverse relationship between mean and variance in which the former claimed that it may only apply over certain periods of time and may not be

14 This should not be surprising if, as Fama (1965a and 1965b) and Blume (1970) demonstrated, the securities were distributed “approximately” normal.
extended to subperiods. Bromiley (1991) objected that, while technically true that annual data disaggregated into subperiods changes the relationships, the point is irrelevant since one tests a theory at a certain level of data and does not claim that it holds for different levels. Ruefli (1991) stood his ground in reply, asserting a misidentification problem. Brockett and Kahane (1992) are the last management scientists on the radar screen in this debate. They reviewed all that has been discussed hereinabove, referring to MPT as “almost folklore” (p. 852) and claimed that the connection between derivatives of the utility function and moments of the probability distribution is an unjustified leap of faith.

Thus, for an individual decision maker, no specification about the derivatives of the utility function (such as risk aversion) can imply skewness preference, aversion to variance, or any other moment preference.

The folklore relating preference properties of the utility function to the moment structure of the desired return distribution is not valid even if the returns are approximately normal (p. 860).

Before leaving the “Doubts…” section a summary is worthwhile. It appears that, under conditions of risk aversion, preference ordering by mean-variance does not lead to the same result as preference ordering by expected utility. It further appears that mean-variance ranking is seriously hampered by non normality. In fact, when tails are fat enough, where \( \alpha < 1.5 \), the mean-variance rule may be disabled.

As the finance and economics fields argued over these issues, at each stage they all carefully skirted a simple, unpleasant fact: for a long time many of these problems did not...

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15 A cursory reading of the SP distribution literature reduces the banter between these two authors to one of more heat than light. Otherwise we must presume that returns in the field of management are not additive.
not have a mathematical solution unless one assumed normality. This was not lost on
mathematicians, who chipped away at this dilemma. It is to their work that I now turn.

**Parallel Advances in Statistical Theory**

Pareto (1897) first noticed that some distributions had heavy tails. He studied
income during a time when economics was dominated by an aging aristocracy and an
emerging merchant class. It appeared to Pareto that higher incomes constituted a greater
share of the right side of the probability mass than a normal distribution predicted,
especially when those incomes were substantial multiples of the mean income. Although
no claim is made that this is the origin of such clichés as “The rich get richer” or “It takes
money to make money,” the intuition is that once a hoard of capital (be it human, non-
human or political) is assembled, the income from it will grow dramatically some of the
time. While the biological scientists had much evidence of regression to the mean, with
Pareto the notion of outliers crept into the thinking of social scientists.

"Student" (1927) noticed that in a host of different repetitive experiments in the
field the tails of the distribution were not normal. But it was Levy (1925) and Levy
(1937) who formalized Pareto’s observations into theorems of probability, providing
needed mathematical foundation. Bergstrom (1952) provided expansions that work
reasonably well in the middle of the parameter space. Gnedenko and Kolmogorov (1968)
provided a characteristic function and proved that the only non trivial limit distribution of
soms of iid random variables must be SP distributed. Feller (1971), considered the source for modern probability theory, treats the work in stable properties and distributions great detail. Hall (1981) discovered an amazing number of errors in the prior work and took the field a long way toward correcting these and standardizing the notation. Zolotarev (1986) provided the derivations that most authors use today and several alternate parameterizations of the characteristic function, each having a different analytic purpose depending upon the application. Granger and Orr (1972) summarized ways one might employ SP distributions in practice and deal with the problem of infinite variance by providing a test for use with large samples. DuMouchel (1971) developed a representation of the SP distribution function by evaluating the inverse Fourier transform of the characteristic function. Chambers, Mallows, and Stuck (1976) provided a method for generating stable random variables. The work of most of the authors in this paragraph makes possible a substantial portion of the demonstrations in Chapter 5 of this dissertation.

For economists, the practical value of many functions in theoretical work depends upon their being unimodal. Considerable ambiguity remains when a multimodal function resists optimization. Hence, Sato and Yamazato (1978) and Yamazato (1978) provided a giant step by proving that all stable densities are unimodal.

Modern finance and economic work with stable distributions began in earnest with McCulloch (1978), who summarized the appeal of the SP process, arguing that nature may shun discontinuities, but the economic environment is artificial and need not

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16 Levy’s contribution is so appreciated that stable distributions are sometimes called “Levy-stable” or “L-
comport with nature. He also pointed out that while distinguishing stable from non-normal leptokurtosis was not easy, that did not constitute a justification for dismissing the stable hypothesis out of hand. Michael (1983) introduced a diagnostic technique for identifying stable properties in a data set that he called a Stabilized Probability Plot.\textsuperscript{17}

Prior to McCulloch (1986), estimating SP distribution parameters was crude and limited to the middle range of the parameter space. McCulloch (1986) provided a non-parametric technique and tables based on the interquartile range that are consistent for values of $\alpha$ as low as .6 and for the entire range of $\beta$. This relaxed the prior restrictions of Fama and Roll (1971) that $\alpha \geq 1$ and $\beta = 0$. McCulloch (1996) followed with perhaps the most comprehensive review and rigorous analysis of the treatment of SP distributions in finance.

The work of Nolan moved the field into the information age. Nolan (1997) introduced a Fortran software package that calculates densities and quantiles, estimates parameters by interquartile range and maximum likelihood methods and generates stable random variates and stabilized probability plots. Nolan (1998b) discusses the various parameterizations, recommending one specifically for numerical work in applications because of its property of being jointly continuous in all four parameters. Fofack and Nolan further clarified the asymptotic properties of SP distributions, showing ways to measure the point at which stable converges to the Pareto under different values of $\alpha$ and $\beta$.

\textsuperscript{17} The use of the root “stable” in “stabilized” is unfortunate as there is no connection with the term “stabilized” in this context and the “stable” distribution.

This section presents a summary of the history and only a fraction of the more recent work in SP distributions. Adler, Feldman and Taqqu (1998) compiled the work of many current scholars presently working at the cutting edge of the field. American University hosted the first annual Heavy Tails Conference in June 1999. The attendance of 200 academics is one indication that work on fat tails has a promising future. Many recent breakthroughs and ongoing work point to a time when the use of stable distributions may be widespread. The normal has not become old fashioned by any means; its useful properties and mathematical tractability guarantee it a long and prosperous future in research. The stable distribution is now available as another tool to be used when models dependent upon an assumption of normality fails to offer "sufficiently good approximations" (Friedman, p. 15).

**Real Estate, Private Real Estate and Stable Paretian Distributions**

This chapter began with the bold claim that “academic real estate has lagged” behind the disciplines from whence it came. It is time to defend that remark.

Nearly 100 years after Pareto first brought it up in an economic context, approximately 70 years after Levy formalized it for mathematicians, and more than 20 years after Fama ended a 10 year period of publishing it, Myer and Webb (1994) said, "The general validity of mean-variance portfolio theory, which is in widespread use today among institutional investors, and many standard statistical tests, depend on the
distribution of returns being normal" (p. 267). Using data from the Russell-NCREIF Property Index, they then proceeded to test for skewness and kurtosis, finding that significant elements of both exist in their data and that tests of normality fail for real estate.

The following year, Young and Graff (1995), using the technology of McCulloch (1986), explicitly tested similar data for stable properties, estimating $\alpha$ as $1.477 \pm 0.038$ (a range of 1.439 to 1.515). They argued that $\alpha$ constitutes site-specific risk and, given its SP distribution, using Fama (1965b) as foundation, they argue that the appropriate degree of risk reduction across multiple factors (locational, economic, etc.) could only be achieved by purchasing most of the institutional-grade properties in the United States – a practical impossibility. This implies that institutional real estate portfolio management must be concerned with the asset-specific risk component of each property included in the portfolio as well as with market/systematic and market sector risk components (p. 254).

Graff, Harrington, and Young (1997) and Graff, Harrington, and Young (1999) extended their work in SP distributions for real estate to Australia, again finding non-normality and a slightly higher (1.588) value for $\alpha$.

Of the small amount of SP work done in real estate, all of it involved major institutional investors and indices containing the returns from holding very large properties. This may or may not bode ill for continued use of MPT for real estate in that class or for real estate in general. The distribution issues associated with real estate in general have been addressed only recently. The private investor has been discussed at most only tentatively and not at all in the context of stable distributions. The contribution of this dissertation begins here.
CHAPTER 4

THE CASE FOR SEGREGATION OF REAL ESTATE RESEARCH INTO TIERS

Introduction

To what degree is the real estate market, as a whole, in fact a whole market? This dissertation argues that it is highly segmented. The simplest example is found in the single-family residential market, the most widespread form of improved real estate. People searching for a home simply do not look at storefronts or industrial buildings. One is usually “in the market” for a certain type of real estate.

Segmented markets are not unique to real estate. The securities market has become highly segmented, offering blue chip investments, high-tech markets, thousands of mutual funds specializing in certain types of assets, and a nearly unlimited derivative market. Indeed, a case can be made that the homogeneity of financial assets is breaking down as niche offerings look for an ever-smaller and unique group of customers.

It will be convenient to segregate real estate first by motive, then by size. My interest is in how real estate investors act. As housing is largely motivated by personal consumption preferences, I choose to eliminate the market for very small, usually owner-occupied properties. I assume that the remaining real estate is motivated by investment
considerations. That remainder will be segregated by size. I shall assume the market for real estate is divided into three major categories, each appealing to an economic class acting exclusively in that market. These categories are described as “Tiers.” The first, Tier I, contains the previously discussed single family, owner-occupied properties. The second is Tier II, my area of interest. It is populated by investment property of more than four dwelling units that is of no interest to institutional investors. Finally, Tier III constitutes that class of, presumably large, properties usually owned by institutions.

It is important to note that Tiers are defined by their ownership and use, not strictly their size. Size is merely a proxy for empirical convenience. The distinction is whether the property is predominantly owner occupied or predominantly securitized. If either is the case, that property falls outside of my interest. Thus Tier II becomes non owner occupied, non securitized investment property.

**Tiers I and III - The Extremes**

The debate about tenure choice in the housing literature is considerable. If anything is resolved, the conclusion is that the acquisition of a home is a consumer purchase with investment aspects. With investment motives secondary, this class of real estate is inappropriate for the kind of analysis contemplated here. Also, even when investment is a more pronounced aspect of the occupied housing question, as in the case where small rentals are purchased adjacent to the home, a meaningful institutional difference exists. Financing for property having four or less units (often referred to as “the one-to-four market”) is both plentiful and supported by a variety of public and quasi-
public agencies. For the purposes of this discussion, Tier I will be defined as owner
occupied residential property. For statistical purposes Tier I eliminates properties
improved with one to four dwelling units from my study.

At the other end of the size spectrum, the Real Estate Investment Trust (REIT)
stands as a visible example of those investors interested only in what is called
“institutional-grade property”\(^1\). Certainly “institutional grade” is a term of art, but it
appears that size is closely associated with what matters to institutions about to purchase
real property. The data in Chapter 6 have comparatively few properties large enough to
be considered institutional grade. Below I discuss the difficulties of carving Tier III out
of the properties remaining after the elimination of Tier I.

It has already been pointed out that there is a data problem in real estate. As
housing is the dominant form of land use in most urban areas, naturally the greatest
quantity of data exists in this sector. Until recently most of this data was drawn from the
single-family market. However, recent improvements in technology have produced
additional data sources that include multifamily housing. For these reasons, my
discussion of tiers will concentrate on the residential investment market. This is not to
suggest that other investment classes (commercial, industrial, land for development) do
not also follow and are influenced by a system of tiers. However, to limit the scope of
this work and make use of the newly plentiful data in a specific sector, multifamily
investment residential real estate will be investigated here.

\(^1\) The term “investment grade” is also heard, usually adding an element of quality or security to the nature
of the property. “Investment grade” and “institutional grade” are often used interchangeably.
**Tier Boundaries**

Using the possibility that institutional factors already define tier boundaries at the extremes, the Tier II market for privately held real estate is defined by default as being all that property not falling into Tier I or Tier III.

Clearly, the nature of establishing tiers requires defining where one tier ends and the other begins. In a three-tiered field there are two boundaries to be set. Institutional financing factors mentioned above make Tier I relatively easy to see. This paper will be satisfied to make four units the upper end of Tier I, consistent with conventions in the field for the “one-to-four market.”

The dividing line between Tier II and Tier III is not as clear. Appendix B describes an informal survey conducted in an attempt to refine the nature of institutional grade. It appears that institutions prefer properties above various limits of dollar value, square feet, or dwelling units. One might simply assume that economies of scale and scope play an important role in this. Anecdotal evidence suggests that transaction costs dominate the decision of how size is determined in acquisition criteria, but little doubt exists that “big” is the general rule. In spite of conflicting standards in this area it appears that for residential investment property the lower bound of Tier III is likely above 100 dwelling units.

If a tier regime exists, I still cannot claim a bright line distinction between Tiers II and III. It is possible for a property owner to purchase an 8-plex and reside in one of the units. It is also possible for a group of local investors to band together to purchase an 80-unit property. Does the former indicate there is wholesale encroachment of Tier I
mentality into Tier II? Does the latter indicate that some sort of institutional thinking
pervades the upper end of Tier II? It is the position of this paper that neither is the case,
rather that these are anomalies in an otherwise settled and consistent market for private
investment. The boundaries may be fuzzy but the tiers exist.

Focusing more closely on what separates Tier II and III discloses an important
factor leading to the question of how and if MPT applies to real estate investing.
Continuing to use REITs as an example, the Tier III market derives its funds from public
sources. REITs and similar institutions function as public trusts, partnerships or
corporations. They are responsible to shareholders and regulated by government
agencies charged with the enforcement of securities laws. They compete for funds with
other public companies involved in the full range of industrial activity typical of an
advanced society. Such funds are national – and to some degree international – in origin
and scope. A large literature on publicly held real estate exists, a substantial portion of
which asks if REITs are stocks or real estate. The consensus is that they are more like
stocks. The question of whether MPT applies to Tier III and whether gains from
diversification in that market are available is unsettled.

It should be noted that Tier III is a recent addition to the economic menu. A
glance at history reveals that all investment property was in private ownership at one
time. This is not insignificant. More will be said in Chapter 7 about how the progress of

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2 The legal nature of the holding entity can be critical, especially when one considers tax consequences. However, this refinement is unimportant to the discussion and all public entities are classed as one.
4 Although Young and Graff (1995, 1997) have made substantial inroads on this issue.
civilization rearranges and distributes property rights leading to the securitization of its real property. Later I will discuss how that relates to my story and what that means to society in general. For now I concentrate on those properties not usually securitized or owner occupied. Since a physical characteristic is required for data purposes, I categorize unsecuritized property on the basis of its size as the middle range of the real estate market and the small end of the real estate investment market.

**Tier II - The Middle of the Field**

Tier II, once the only class of non owner-occupied real estate investment, is now the residual. Its best definition is “what institutions won’t buy.” I speculated earlier in this chapter about why institutions shun certain sizes of property. While this is an interesting question the full answer to which is beyond the scope of this work, I touch on it briefly in Appendix B. Few doubt that institutions are absent from the small property market.

Beyond transaction costs, a more esoteric reason, and one of interest in the context of this discussion, relates to the return distributions in the Tier II market. One may theorize that with the reduction in size, uncertainty increases due to factors inherent in the greater heterogeneity that comes with a portfolio composed of smaller and more numerous parcels. Such an increase in uncertainty may represent to the institution an intolerable reduction in $\alpha$ for the SP distribution. If institutional investors are sensitive to $\alpha$ and aware of its consequences to the gains from diversification sought by forming
portfolios, they may believe that ownership of small properties frustrates their organizational goals.

The Tier II market is a complex opportunity for extending the SP distribution issues, in that practical considerations also impede forming portfolios. Real Estate investments are “lumpy”, meaning that they must be purchased in large bites and are difficult to partition in kind. Simply, while one wishing to invest $50,000 may be able to acquire several different issues of stock, that amount of capital acquires only one apartment building. That building is not only small, in the definition of Tier II, but it is very likely also leveraged. This lumpiness and the problem it causes for diversification may be reflected in the shape of return distributions for this market. We saw in Chapter 2 that the mathematics of SP distributions contribute to the unavailability of gains from diversification, but so does lumpiness. If the shape of the distribution of returns for Tier II property is different, it may be impossible to say to what degree each contributes. This did not burden those who preceded in the study of SP distributions, since they concentrated on Tier III property, a market dominated by institutions with greater access to capital.

Since Markowitz (1952) showed that asset-specific risk could be removed by combining assets, asset selection techniques advocated by such authors as Graham and Dodd (1934) have been pushed to the background. Also since Markowitz, Fama’s EMH has survived many tests of its weak and semi-strong forms to show that the securities markets permit no exploitable market inefficiencies. At first glance, it would appear that
Tier II investors, hard pressed to avoid asset specific risk by forming portfolios, find other benefits in real property investments based on different motives. The next chapter explores that possibility more fully.

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5 I assume that the purchase of stock represents a similar purchase of “equity” in that the leverage is inherent in the capital structure of the firm.
CHAPTER 5

PRIVATE REAL ESTATE INVESTMENT RETURN DISTRIBUTIONS

Introduction

Little doubt exists that investors in securities markets form portfolios composed of financial assets. Following Fama (1965b) and Blume (1970), if $\alpha$ is near 2, these investors enjoy gains from diversification. We observe that institutional real estate owners also form portfolios of real properties. However, according to Young and Graff (1995), there is reason to expect that equivalent gains from diversification are not achieved for owners of Tier III property due to $\alpha \approx 1.5$. It is not known if private investors routinely form real estate portfolios. If they do form a portfolio, my interest is in whether their doing so produces any of the risk/reward efficiencies promised by MPT. On a practical level, due to lumpiness, it appears that private real estate investors have a difficult time following the maxim: “Don’t put all your eggs in one basket.” This dissertation describes an alternative, suggesting that they subscribe to the less well-known adage: “Put all your eggs in one basket and watch that basket closely.”

Yet, the real estate investor is assumed to be a typical, risk-averse, rational maximizer. But I propose that his preferred habitat is a market where returns are
distributed in a way that provides a compensating offset to the unavailable portfolio benefits his brethren in the stock market enjoy. Such a market is characterized by a fat right tail (thus it has a positive skew). Such a distribution offers a different type of risk/reward tradeoff. The thickness of the tail represents more variation (a bad thing) and the fact that it is a right tail places the variation on the positive side of the spectrum (a good thing). The intuition is that an investor does not know how much he will make but he has a higher probability that his return will be positive.

This chapter presents a different view of investment real estate return distributions. The view is that of the individual investor. Because I will rely on numerical rather than analytical solutions, I begin with a discussion of alternate ways available to solve problems. I then return to the stable hypothesis first suggested more than 35 years ago by Fama and Mandelbrot, this time in the context of real estate. I examine more carefully the problems SP distributions cause for MPT’s efficient frontier and for the gains purportedly available from diversification. I then introduce risk aversion and a specific form of utility function, suggesting that a different trade-off is at work for actors in this market. To formalize the intuition provided above, I explain the importance of the fat right tail and show numerically how, holding utility constant, investors are well advised to trade decreases in $\alpha$ for increases in $\beta$.

A Comment about “Solutions”

Classical problem solving using mathematics calls for an analytical solution. Critical to that is the ability to write down mathematical expressions that may be solved
symbolically. Linear problems are among the most easily solved. Mathematicians call the possibility of an analytic solution "tractable." Because of the factors discussed in Chapter 2 and elsewhere in the literature on SP distributions, my interest is in an area where that possibility is absent. There are some, perhaps many, problems that mathematicians have yet to solve, one of which is presented here. Thus, in this dissertation I will employ numeric and graphical demonstrations in lieu of analytical solutions.

Benoit Mandelbrot eloquently set the background for the discussion that follows:

> Everybody’s preference for [linear aggregation] models is of course based on the unhappy but unquestionable fact that mathematics offers few workable non-linear tools to the scientist (Mandelbrot (1963a), p. 424).

Science is a process of discovery, a puzzle-solving activity the ultimate goal of which is a universal solution. Generally, this means a so-called “closed form” analytical solution. Such a solution has at least two desirable qualities. First, one can write down an explicit formula that involves a finite number of operations.¹ Second, the solution has unbounded support, that is \( x \subset (-\infty, \infty) \). If the assumptions necessary to produce a solution of this type are empirically valid, certainly this is a very desirable result.

Given the complexity of the world, it is not surprising that universal answers are rare. Relaxing assumptions that otherwise make a problem mathematically tractable leaves the unpleasant choice between (a) rejecting a theory, (b) having no theory at all or

¹ This excludes infinite series and integrals involving an infinite number of operations. As the next section demonstrates, numerical integration is the key to solving this problem for real estate application.
(c) accepting a theory based on bounded, numerical or graphical solutions. There is a
history of compromise to be recognized here. The field of finance was carved out of
microeconomics by relaxing the assumption of perfect certainty in the theory of perfect
competition. Fortunately for Markowitz and Sharpe, the assumption that the world of
investment returns is distributed normally permitted them to suggest a universal solution
using a linear model that has fared reasonably well in many empirical tests over the past
40 years. But these tests do not perform as well for real estate, and rather than trying to
stretch that theory around real estate data, perhaps it is time to relax an assumption. This
paper asserts that returns in private investment real estate are not distributed normally.
One consequence of this assertion is that nonlinearities are introduced, making the tools
of bounded, numerical, or graphic solutions most appropriate to describe such a theory.

Is a bounded theory less global? I think not, for such a question is merely the
converse of another, equally valid, question: Is an analytic theory too restrictive?
Suppose the boundaries are still well outside the probable range of values that one might
realize? To use an extreme example, suppose a solution may be demonstrated over a
range of returns between ±100% per annum. While the boundaries of this range fall well
short of infinity, even those casually familiar with actual market returns know that
virtually all real estate investments will be captured by an otherwise valid theory bounded
in that way.

In defense of those who came before, the analytical solution had a serious
methodological limitation as recently as 15 years ago. Many numerical solutions require
such a large number of calculations (some approaching an infinite number) that their
implementation was at least tedious and at most presented an impossible time commitment. One can easily appreciate a scientist’s reluctance to devote time to solving puzzles by performing calculations repeatedly for longer than his career. Advancing technology has ameliorated this dilemma. Fast, cheap computing power on the desk of every investigator opens the door to solutions combining the merits of accurate representations of the world and well-grounded numerical theory.

Markowitz (1991b) himself laments the compromise required, describing individual investment analysis as having “characteristics of situations for which simulation methods have proved to be the superior tool in practice.” Markowitz was indeed prescient, for simulation will play a role below in illustrating a serious difficulty individual real estate investors have in implementing MPT.

The Stable Hypothesis

Chapter 2 discussed the general philosophy of transforms. Appendices C and D are more precise about the tools necessary to carry out this analysis. The technical aspects of these tools are not the central issue, as they are merely a mathematical means to overcome the lack of a pdf for SP distributions. The tools permit a theoretical examination of possible non normal return distributions of private real estate investors. It is hoped that this analysis will lead to understanding about how these investors make choices under the kind of uncertainty represented by SP distributions. The analysis shows that investors may be disinclined toward the increased risk represented by the fatter tails (smaller values of $\alpha$) but inclined toward the benefits of a right skew (positive
Finally, it will be shown that, under conditions of fixed utility, there is a trade-off between alpha and beta that is different from the mean-variance tradeoff postulated by MPT.²

The physical sciences, especially mathematics, demand a structured approach to advancing a theory. The procedure is: Definition, Theory, Proof, Remarks. As this chapter offers the theory and a numerical proof, it is useful to begin with some definitions.

One reason we use mathematics to express scientific principles is that our language is imprecise. For instance, the word "portfolio" can mean merely the agglomeration of several different assets under a single ownership. That is not the meaning intended here. We would like to avoid imprecisions by setting forth how certain words will be used in the remainder of this dissertation. The word "portfolio" as used herein is defined as an efficient set portfolio, that is, a Markowitz mean-variance portfolio. The "mean-variance rule" means the strict meaning Markowitz (1991b) intended it to have in the context he intended it, not extended into situations where empirical results provide poor predictions.³ The term "gains from diversification" is intended to have its Markowitz meaning - the benefit arising from combining assets having different covariances such that the risk is reduced for a given level of return. The term "utility" is an abstraction that generally means happiness or an ordinal state of well-being. In the Bernoulli tradition, people are presumed to maximize utility over a bounded range. The "utility function" is

² This α-β tradeoff, in the way approached here, comes with disadvantages, however. Lost are some of the benefits of using all four parameters to describe the distribution.
an unobserved process by which people make choices in order to maximize their utility. “Risk aversion” shall mean the Arrow-Pratt form of absolute risk aversion,

\[ R_A(Y) = -\frac{U''(Y)}{U'(Y)}, \]

an expression that measures the curvature of the utility function. Involving risk aversion makes it necessary to specify an exact form for the utility function.

One more digression is required before I tackle the theory presented here. A substantial portion of what has come before constitute reasons why MPT does not work. There is only so much value in this. Eventually, one must stop criticizing the incumbent paradigm and propose an alternate. However, because the finance paradigm is so entrenched it seems useful to take the available technology described in Chapters 2 and 3 and look at MPT with the horizon expanded to include SP distributions. Recall that the parameter of interest is \( \alpha \), specifically, the behavior of \( \alpha \) as it moves away from the normal case where \( \alpha = 2 \). There are two important effects of a fall in \( \alpha \). One has to do with the shape of the efficient frontier, the other with the number of assets required to achieve gains from diversification by forming portfolios.

Moving \( \alpha \) away from 2 has a dramatic effect on the size and shape of the efficient frontier supposed by MPT. To illustrate, consider the following simulation of efficient frontiers under different values of the first two parameters of the SP distribution. Using a random number generator provided in Fortran by Nolan (1998c) based on McCulloch (1986) and Chambers, Mallows, and Stuck (1976), three sets of three-asset portfolios are

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3 Recall that he said he was thinking about open-ended investment companies.
formed and the efficient frontiers of their various combinations are plotted. Return distributions for the all of the individual assets in each portfolio have, respectively, the ch.f. parameter specification $S^0(2,0,1,0)$, $S^0(1.4,0,1,0)$, and $S^0(1.4, .25,1,0)$. Hence, the first distribution is symmetrical normal, the second is symmetrical heavy tailed and the third is heavy tailed and skewed right. Each portfolio has three assets, each asset reflects a return for 60 consecutive periods. Figure 5–1 shows the familiar parabolic shape of the efficient frontier. 4

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4 One would expect that random numbers would be highly uncorrelated, thereby taking maximum advantage of the curvature produced by the quadratic nature of the construction of the efficient frontier.
Figure 5–2 also shows an efficient frontier for a symmetrical distribution, but this time the distribution is not normal. Rather, it is stable with $\alpha = 1.4$. 

*Figure 5–1*: An efficient frontier of portfolios formed with assets having normally distributed returns.
Both Figure 5–1 and Figure 5–2 show the same parabolic shape. However, the scale for each graph is considerably different. When plotted together on the same scale in Figure 5–3 the efficient frontier for the normal distribution survives in the same size while the efficient frontier for the SP distribution with the heavy tail ($\alpha = 1.4$) shrinks to a mere speck on the graph.

*Figure 5–2:* An efficient frontier of portfolios formed with assets having returns distributed Stable-Paretian
Rescaling the y-axis to obtain a better view of the SP distribution when $\alpha = 1.4$ produces a close-up of the "nose" of the efficient frontiers for the two cases. Figure 5–4 shows that not only is the risk greater for each measure of return when $\alpha$ declines away from 2, but the shape is nearly vertical for the smaller $\alpha$, suggesting that that a broad range of returns is available for a narrow range of risk. Stated differently, in this

Figure 5–3: Efficient frontiers formed by portfolios from two separate sets of assets, each having symmetrical return distributions but one distributed normally and the other distributed Stable-Paretian with $\alpha = 1.4$. 
particular simulation a risk-return trade-off is nearly nonexistent.\(^5\) The effect on gains from diversification is also apparent. Finally, it becomes apparent why \(\alpha < 1\) might produce *losses* from diversification were the curve of the parabola to bend concave to the y-axis.

*Figure 5–4:* The two frontiers of Figure 5-3 with the y-axis rescaled to enlarge the size of the frontier composed of portfolios formed with assets distributed Stable-Paretian

While the mathematical reason for this may be found in Chapter 2, some intuition is helpful at this stage. The essential difference between portfolios formed from assets

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\(^5\) This, of course, is all portrayed in the two-dimensional world of mean-variance, a place where the author
whose returns are distributed normally ($\alpha = 2$) and portfolios formed from assets
having fat tails ($\alpha = 1.4$ in this case) is that the latter is composed of a set of assets in
which outliers have a larger effect on the shape distribution. In such a market, combining
these assets has a different affect on dispersion than combining normally distributed
assets does. For the particular draw depicted in the simulation above, the range of $\sigma^2$
remains fairly constant (Figure 5–4). The return space that matches that small range of
risk also happens to be small in this example, although not as small as that for risk.

Thus far I have concentrated only on the symmetric case where the only variation is
in $\alpha$. The third simulation involves a change in the skewness parameter of the SP
distribution, $\beta$. Figure 5–5 shows that when the same fat-tailed distribution also has a
positive skew, the efficient frontier broadens, curves, and moves to a comparatively more
desirable level, providing both opportunities for more return in the same risk
classification and a curve, albeit small, in the frontier allowing for a risk-reward choice.
Figure 5–5: Efficient frontiers formed by portfolios from two separate sets of assets, one having a normal return distributions and the other distributed Stable-Paretian with $\alpha = 1.4$ and $\beta = .25$ (positive skewness).

Figure 5–6 rescales the y-axis for a close-up that better illustrates the shape of the $S^0(1.4, .25, 1, 0)$ SP distribution.
In summary, Figure 5–7 shows the efficient frontiers for all three simulations on the same plot. It is apparent that moving from the normal to the stable distribution results in a meaningful distortion of any "efficient" frontier generated with MPT technology. The distortion may appear as in the example above or may be reversed showing the normal as the smaller frontier. Chapter 7 discusses the ramifications of this further.

Figure 5–6: The two frontiers of Figure 5–5 with the y-axis rescaled to enlarge the size of the frontier composed of portfolios formed with assets distributed Stable-Paretian
The second effect of $\alpha$ moving away from 2, described by Fama and Miller (1972), is the effect smaller $\alpha$ has on the number of assets needed to reduce or maintain risk. Standardizing a unit of risk at 1, Table 5–1 shows how risk ($\sigma$) changes for a fixed number of assets as $\alpha$ declines toward 1.6 Note that in order, for example, to maintain risk at the 0.100 level, it is necessary to acquire ten times as many assets when $\alpha$ moves from 2 to 1.5. This is clearly impractical for real estate as Young and Graff (1995) pointed out.

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*Figure 5–7: A plot of all three simulations together*
Inverting Fama and Miller's formula, Table 5–2 shows the exponential change in the number of assets required to maintain risk constant for four particular levels of risk as \( \alpha \) declines over a feasible range.

Table 5–1: How a standardized unit of risk is affected by the fall in \( \alpha \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( n=1 )</th>
<th>( n=10 )</th>
<th>( n=100 )</th>
<th>( n=1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.00</td>
<td>1.000</td>
<td>0.010</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>1.75</td>
<td>1.000</td>
<td>0.032</td>
<td>0.006</td>
<td></td>
</tr>
<tr>
<td>1.50</td>
<td>1.000</td>
<td>0.100</td>
<td>0.032</td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td>1.000</td>
<td>0.316</td>
<td>0.178</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>3.162</td>
<td>10.000</td>
<td>31.623</td>
<td></td>
</tr>
</tbody>
</table>

Table 5–2: How the number of assets required, given fixed levels of risk, is affected by the fall in \( \alpha \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \sigma(\alpha, \sigma) = \sigma^{(1-\alpha)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma )</td>
<td>( n=1 )</td>
</tr>
<tr>
<td>2.00</td>
<td>1</td>
</tr>
<tr>
<td>1.80</td>
<td>1</td>
</tr>
<tr>
<td>1.50</td>
<td>1</td>
</tr>
<tr>
<td>1.40</td>
<td>1</td>
</tr>
<tr>
<td>1.20</td>
<td>1</td>
</tr>
</tbody>
</table>

\(^6\) This Table also provides a clue as to why \( \alpha < 1 \) values are not considered in an economic context. Note that diversification becomes counterclockwise when \( \alpha < 1 \).
Mandelbrot to Young and Graff comprises a 35-year story of why not to use MPT for real estate. The field is indebted to Lusht (1988) for trying to end it. The hard part comes next: What is to be done instead? The answer to that question has to do as much with stopping the way we have thought about real estate in the past as it does with thinking about it differently now.

Figures 5–1 through Figure 5–4 and Tables 5–1 and 5–2 all appear to illustrate only the problems with MPT in an SP world. They don't appear to offer much to encourage one to look at the world differently. There is, however, one glimmer of hope in Figure 5–5. That is in the possibility that, whatever frontier that may be formed with SP distributed returns, if it is skewed right it may be northwest of the frontier formed with assets having returns that are normally distributed. I will now take a closer look at the subject of the effect of a positive skew in the SP distribution on the location of the efficient frontier and its implications for investor motives.

The Alpha-Beta Trade-off in an SP Distribution

Following the procedures of the classics works on decision-making under uncertainty, I begin by assuming that investors are rational utility maximizers who view risky alternatives as choosing between lotteries. Each possible return on investment is a “draw” from one of the various available lotteries. Different lotteries are described by the shape of their distribution. In the case at hand, I assert that the four parameters of the

---

7 "now that we have confirmed what doesn’t work, we should stop doing it.” p. 102
SP distribution do a better job of describing the lottery known as real estate investment. I further assume that investors produce utility by the negative exponential function

$$u(x) = 1 - (e^{-1.1x})$$

where \( x \) represents a rate of return presumed to be drawn from a lottery of returns and \( e \) is the base of the natural logarithm. This function has two desirable properties, as illustrated in Figure 5–8. One, the function is zero when \( x \) is zero, an intuitively appealing quality in that investors are probably ambivalent about breaking even.\(^8\) Two, its second derivative is negative, conforming to the conventional belief in diminishing marginal returns.\(^9\)

\[ \partial_{xx} = -0.01(e^{-0.1x}) < 0. \]

---

\(^8\) This is a mere convenience. No generality is lost by the shift of the function in this manner.  
\(^9\) Tsaing (1972) presents two additional properties that are also desirable. These have to do with whether the utility function exhibits constant absolute or relative risk aversion.
Figure 5–8: The negative exponential utility function.

\[ u(x) = 1 - \left( e^{-1x} \right) \]
As paper representation constrains any illustration to two dimensions, it is necessary to fix two of the four parameters of the SP ch.f. in order to plot. Appendix D describes the iterative process leading to fixing $\gamma$ at 1 and $\delta$ at .06, which when done, permits a plot of changes in utility with changes in $\alpha$ and $\beta$ under these bounded conditions. The plots, *Figure 5–9* and *Figure 5–10*, show that utility rises as $\alpha$ and $\beta$ rise.

*Figure 5–9*: Utility rises as $\alpha$ rises toward 2 (the normal distribution)
Recall that the tail becomes thinner as $\alpha$ rises. I am concerned with the opposite direction - the increased risk associated with a fall in $\alpha$. Hence, one must reverse one's perspective and reconsider Figure 5–9, looking at effect on utility as $\alpha$ falls (when read from right to left) and the tail becomes fatter.

The foregoing offered as a graphic evidence, there is another way to view this problem. We may compute the values of $\alpha$ and $\beta$ under conditions of fixed utility and observe how they change in relation to each other. Table 5–3 shows values of $\alpha$ and $\beta$ when utility is fixed at -.05 ($\gamma = 1$ and $\delta = .06$ as before). Note the inverse relationship.

*Figure 5–10: Utility rises as beta move away from zero in a positive direction*
between $\alpha$ and $\beta$ given a fixed utility. The theory predicts that as alpha falls, beta must rise to keep utility constant. This is the tabular form of the intuition that investors will accept fat tails if they improve their chance of a positive draw.

**Table 5–3: The changing values of $\alpha$ and $\beta$ with utility held constant**

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.66204</td>
<td>0</td>
<td>1</td>
<td>.06</td>
</tr>
<tr>
<td>1.64485</td>
<td>.02</td>
<td>1</td>
<td>.06</td>
</tr>
<tr>
<td>1.62649</td>
<td>.04</td>
<td>1</td>
<td>.06</td>
</tr>
<tr>
<td>1.60691</td>
<td>.06</td>
<td>1</td>
<td>.06</td>
</tr>
<tr>
<td>1.58606</td>
<td>.08</td>
<td>1</td>
<td>.06</td>
</tr>
<tr>
<td>1.56393</td>
<td>.10</td>
<td>1</td>
<td>.06</td>
</tr>
<tr>
<td>1.54051</td>
<td>.12</td>
<td>1</td>
<td>.06</td>
</tr>
<tr>
<td>1.51581</td>
<td>.14</td>
<td>1</td>
<td>.06</td>
</tr>
<tr>
<td>1.48986</td>
<td>.16</td>
<td>1</td>
<td>.06</td>
</tr>
<tr>
<td>1.46270</td>
<td>.18</td>
<td>1</td>
<td>.06</td>
</tr>
<tr>
<td>1.43441</td>
<td>.20</td>
<td>1</td>
<td>.06</td>
</tr>
</tbody>
</table>

A final graphical look at this analysis is the contour plot in *Figure 5–11* where utility is shown constant as $\alpha$ and $\beta$ change. Logically, utility is highest in the northeast quadrant when $\alpha$ is approaching 2 and the skew is at the high end of the range. By
following an almost vertical strip on the plot, one can see how falling $\alpha$ is offset by rising $\beta$, holding utility constant.

---

**Figure 5–11**: Contour plot of constant utility as $\alpha$ and $\beta$ change

---

**Summary**

The discussion above makes several claims: First, the requirement of an analytical solution is unduly restrictive for real estate if returns are SP distributed. Second, investors may view gains from diversification as implausible, either because of practical considerations or the limits of their market’s brand of return distribution. Third, these investors make decisions in an SP world based on a trade-off between tail thickness ($1 < \alpha < 2$) and skewness (positive $\beta$).

Does this add up to a rejection of the mean-variance rule? As the same is defined earlier, it does. Counterintuitive as it may seem, there is no paradox here. The broader
view, perhaps a general mean-variance thought process, is that return is good and risk is bad. This remains. Real estate investors, regardless of their utility functions or the distributions they face, are rational maximizers who seek to maximize return while minimizing risk. It is foolish to argue in general against such an intuitively reasonable maxim.

Markowitz, reflecting on his theory many years later, is quoted as saying, "The only thing I did was bring mathematics up to common sense."\(^{10}\) Note that the common sense came first. The math was applied to a situation (an open-end investment company investing in the stock market) that fit nicely into the model. In one sense, the illustrations above are merely refinements of the prior admonition of others contemplating real estate investing. It appears legitimate to question a particular mathematical approach to selecting investments that fits poorly when the asset is real estate.

If the Markowitz-mean-variance-efficient-set thinking is purged for real estate we start anew, beginning with common sense. After the decision to choose real estate as the investment vehicle, the decision to purchase a particular piece of real estate begins with its most fundamental attribute - its location. I will assume that all of the local economic and environmental issues facing the investor are captured by the location decision. Hence, location becomes the "systematic" portion of risk.\(^{11}\) In a very general sense - not the limited one implied by the definition here - the investor contemplates his risks and rewards in various locations. The general mean-variance thought process (recall this is

---

\(^{10}\) Interview of Harry M. Markowitz, by Ann Perry, August 1, 1999 San Diego Union Tribune
not the "mean-variance rule") certainly enters his or her mind. Will I be exposed to more or less risk in this location or a different one? Will my returns be higher or lower in this location? Are the risks attendant to this location adequately compensated by return? These are all useful and likely questions.

What an investor finds very difficult, if not impossible, to do is take the next step implied by MPT and its mean-variance rule. Because of practical (capital limitation) and theoretical (shape of distributions) considerations he finds forming an efficient set portfolio unlikely. Thus, the investor is unable to "diversify away" remaining risk, the unsystematic portion associated with choosing a specific property, by simply buying many of them in different areas or buying different types in the same area.

I will assume that unsystematic risk as contained in the specific property the investor acquires. The heterogeneous bundle of rights and the combination of characteristics embodied in different buildings offer a wide variety of alternatives, each containing substantial - and different - unsystematic risk.

What is claimed here is that without the ability to form efficient set portfolios or to achieve diversification gains if they try, real estate investors have no choice but to assume non-systematic risk. The inevitable acceptance of non-systematic risk (site-specific risk in real estate terms) that cannot be “diversified away” is subject to a trade-off such as that described in this chapter.

11 Clearly, the hoped for "purging" has not been complete. The notion of systematic and unsystematic risk is embedded in the CAPM, which is rooted in MPT. It is not a matter of a hard habit to break but a matter of these being very useful theories that still have partial applicability to real estate.
Next I will examine some Tier II data. Chapter 6 performs a variety of tests to answer several questions:

- What is the nature of private investor return distributions?
- Are private real estate returns stable?
- How much of the risk is site-specific?
- Is there a difference between Tier II and Tier III other than the obvious size difference?
CHAPTER 6

EMPIRICAL EVIDENCE

The Data

The data are for Tier II property. Such data are becoming more plentiful due to the recent and on-going digitization of this market. Our interest is in the shape of Tier II market real estate return distributions and estimates of the parameters of the stable characteristic function.

It has already been observed that real estate data in sufficient quantity are difficult to obtain. Even when a large enough data set exists, researchers still face one of two problems. Either they have confirmed sales but no intertemporal holding period income or they have intertemporal income on properties that have not sold - problems associated, respectively, with Tier II and Tier III property. Large, publicly held institutions keep meticulous records and deliver regular reports to their constituents, making operating income information comparatively plentiful. On the other hand, these properties typically have long holding periods and infrequent sales. To compute annual returns for Tier III property, appraised values are substituted as hypothetical sales to generate equity reversions for the final cash flow. This leads to the widely discussed "smoothing"
problem. Data for Tier II property are quite different. Operating information is private (if it exists at all) but sales are more frequent. Here, it is common to assume that value is a linear function of income and work with only sale information using the change in price over time as a proxy for return. That is my situation.

Source and Methodology

Source

The data are obtained for the San Diego County multifamily residential market. Included are a total of 6,537 confirmed sales (gross before elimination due to missing variables) occurring during the 183 months between March 1983 and August 1998. There are 4,514 different properties located among 81 different postal zip codes; 94.17% of all sales involved less than 100 units (6,156 of the 6,537 observations). There are 2,013 properties in the data set that sold more than once during the time period. Multiple sales of the same property are shown in Table 6–1.

1 CF. Ross and Zisler (1991). It also raises other issues in a data set when the same properties repeat in the same data set over time. Young and Graff (1995) encountered this problem and so shall I. It may be that this concern is endemic to this type of research and may never be overcome.
Methodology

As the data lack intertemporal income, returns for the Tier II property type will be based on the change in price over time. Returns over a holding period will be calculated as if continuous, given that holding period is measured by days.\(^2\) Thus:

\[
R_t(z) = \frac{1}{n} \left[ \ln \left( \frac{P_t}{P_{t-n}} \right) \right]
\]

\(^2\) The alternative, daily compounding method, \(R_t(z) = \sqrt[n]{\frac{P_t}{P_{t-n}}} - 1\), produces a trivially different result.

### Table 6–1: Sales Repeating in the Data

<table>
<thead>
<tr>
<th>%</th>
<th>Properties</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>46.63%</td>
<td>3048</td>
<td>Properties appearing only once in the data base</td>
</tr>
<tr>
<td>22.58%</td>
<td>1476</td>
<td>1st appearance of a property that sold more than once</td>
</tr>
<tr>
<td>22.58%</td>
<td>1476</td>
<td>2nd appearance of a property that sold twice</td>
</tr>
<tr>
<td>6.75%</td>
<td>441</td>
<td>3rd appearance of a property that sold three times</td>
</tr>
<tr>
<td>1.28%</td>
<td>84</td>
<td>4th appearance of a property that sold four times</td>
</tr>
<tr>
<td>0.18%</td>
<td>12</td>
<td>5th appearance of a property that sold five times</td>
</tr>
<tr>
<td>100.00%</td>
<td>6537</td>
<td>Total properties in the data base</td>
</tr>
<tr>
<td>30.79%</td>
<td>2013</td>
<td>Total properties in the data base with multiple sales</td>
</tr>
</tbody>
</table>
Where:

\[ R_t(z) = \text{the return at the end of the holding period at time } t \text{ at location } z \text{ (zip code);} \]
\[ n = \text{the number of days in the holding period;} \]
\[ P_t = \text{the price at the end of the holding period; and} \]
\[ P_{t-n} = \text{the price at the beginning of the holding period.} \]

My interest is in how asset specific risk varies. Hence, I will employ linear regression to produce a term assumed to model this risk. This term, according to a variation on Eq. 4 of Young and Graff, is the error term in the following regression equation:

\[ R_t(z) = \xi + \eta_t(h(z)) + \epsilon_t(z) \]

Where:

\[ R_t(z) = \text{the return at the end of the holding period at time } t \text{ at location } z \text{ (zip code);} \]
\[ \xi = \text{the intercept;} \]
\[ \eta = \text{coefficient conditioning the mean for location;} \]
\[ h(z) = \text{a dummy variable representing location (zip code); and} \]
\[ \epsilon_t(z) = \text{disturbance term} \]

The error term will be assumed to be the excess return rewarding investors for taking asset-specific risk. Using maximum likelihood estimation provided by Nolan (1997), this asset-specific risk, \( \epsilon_t(z) \), will be fit to the stable distribution, resulting in an estimation of \( \alpha \) for Tier II real estate.

---

3 Young and Graff used property type to condition their mean. Because all of my property is the same type, attributing non-site specific risk to location seems more appropriate.
Descriptive Statistics

Descriptive statistics for selected variables appear in *Table 6–2* with the same
provided in expanded form by Zip Code in Appendix F.
<table>
<thead>
<tr>
<th>Units</th>
<th>Sale_Pr</th>
<th>Age (Yrs)</th>
<th>Lot Size (SF)</th>
<th>Cap Rate</th>
<th>GIM</th>
<th>HP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>29</td>
<td>1,470,823</td>
<td>20</td>
<td>49,660</td>
<td>8.448</td>
<td>7.48</td>
</tr>
<tr>
<td>Standard Error</td>
<td>1</td>
<td>48,348</td>
<td>0</td>
<td>1,836</td>
<td>0.024</td>
<td>0.03</td>
</tr>
<tr>
<td>Median</td>
<td>12</td>
<td>575,000</td>
<td>18</td>
<td>13,210</td>
<td>8.179</td>
<td>7.75</td>
</tr>
<tr>
<td>Mode</td>
<td>8</td>
<td>450,000</td>
<td>20</td>
<td>6,969</td>
<td>7.920</td>
<td>8.33</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>53</td>
<td>3,869,349</td>
<td>15</td>
<td>148,209</td>
<td>1.906</td>
<td>2.01</td>
</tr>
<tr>
<td>Sample Variance</td>
<td>2767</td>
<td>1.50E+13</td>
<td>229</td>
<td>21,965,948,458</td>
<td>3.632</td>
<td>4.05</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>63</td>
<td>367</td>
<td>3</td>
<td>533</td>
<td>14.182</td>
<td>0.25</td>
</tr>
<tr>
<td>Skewness</td>
<td>6</td>
<td>14</td>
<td>1</td>
<td>16</td>
<td>2.218</td>
<td>0.05</td>
</tr>
<tr>
<td>Range</td>
<td>1065</td>
<td>126,915,000</td>
<td>93</td>
<td>6,264,799</td>
<td>32.083</td>
<td>15.21</td>
</tr>
<tr>
<td>Minimum</td>
<td>5</td>
<td>85,000</td>
<td>-</td>
<td>871</td>
<td>0.003</td>
<td>1.69</td>
</tr>
<tr>
<td>Maximum</td>
<td>1070</td>
<td>127,000,000</td>
<td>92</td>
<td>6,265,670</td>
<td>32.086</td>
<td>16.90</td>
</tr>
<tr>
<td>Sum</td>
<td>189930</td>
<td>9,420,619,689</td>
<td>323,586,695</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Count</td>
<td>6537</td>
<td>6,405</td>
<td>5,823</td>
<td>6,516</td>
<td>6,154</td>
<td>4615</td>
</tr>
<tr>
<td>Confidence Level(95.0%)</td>
<td>1</td>
<td>94,778</td>
<td>0</td>
<td>3,599</td>
<td>0.048</td>
<td>0.06</td>
</tr>
</tbody>
</table>
Descriptive statistics for those properties appearing in the data set as repeat sales are shown in Table 6–3.

Table 6–3: Description statistics for continuous returns on property appearing in the data as repeat sales

<table>
<thead>
<tr>
<th>2013 Repeat Sale Continuous Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
</tr>
<tr>
<td>Standard Error</td>
</tr>
<tr>
<td>Median</td>
</tr>
<tr>
<td>Mode</td>
</tr>
<tr>
<td>Standard Deviation</td>
</tr>
<tr>
<td>Sample Variance</td>
</tr>
<tr>
<td>Kurtosis</td>
</tr>
<tr>
<td>Skewness</td>
</tr>
<tr>
<td>Range</td>
</tr>
<tr>
<td>Minimum</td>
</tr>
<tr>
<td>Maximum</td>
</tr>
<tr>
<td>Sum</td>
</tr>
<tr>
<td>Count</td>
</tr>
<tr>
<td>Largest(1)</td>
</tr>
<tr>
<td>Smallest(1)</td>
</tr>
<tr>
<td>Confidence Level(95.0%)</td>
</tr>
</tbody>
</table>

Tests performed

In this section I will describe the propositions tested and the results, each, in two parts. First I will deal with distribution issues, I will then cover size issues.
Distribution issues

The primary tests for distribution issues are those that point to the conclusion that the population from which the sample is drawn is SP distributed. Unfortunately, it is not possible to prove that a data set is stable (Nolan (1998a)). Rather, diagnostics are used to detect non-stability. The two most reliable of these diagnostics are graphical.

First, I prepare and inspect graphical forms of the data. This can be done in several ways. (1) Prepare a histogram of the data and compare it with the shape of a similarly scaled and located normal distribution. (2) Generate a set of points from the random generating methods provided by Nolan (1997) using the parameters estimated from the data, then plot those points to obtain a smoothed plot similar to the histogram. (3) Finally, using the transform techniques described in Appendix D, I create a plot of the simulated pdf for the data, comparing that with a normal distribution. If the data are SP distributed, all should result in the same outcome: Long, heavy tails supports a conclusion of stability.

A second diagnostic involves constructing a stabilized probability plot (Michael (1983)) using STABLE.EXE software developed for that purpose (Nolan (1997)). If the distribution is stable, such a plot, when compared to a diagonal line through the origin, should minimize deviations from the line.

---

4 Although this may sound troubling, Nolan (1998a) reminds us that “testing for normality is still an active field of research” (p.11).
5 The “chicken-and-egg” notion that I am using parameters estimated later to generate these graphics now is not ignored. The processing of seeking stability in a distribution and a sample from it involves interrelated methods, the starting point of which is in the hands of the analyst.
6 This software may be obtained from: http://www.cas.american.edu/~jpnolan/stable.html
There is a third test that is productive only if tails are "very" heavy. Granger and Orr (1972) provide a test for infinite variance. Distributions having $\alpha < 2$ have infinite variance by definition. Samples may or may not have infinite variance depending upon how low $\alpha$ is. The Granger and Orr test is a "converging variance test" where:

$$S_N^2 = \frac{1}{N} \sum_{t=1}^{N} (X_t - \overline{X}_N)^2$$

and

$$\overline{X}_N = \frac{1}{N} \sum_{t=1}^{N} X_t$$

resulting in a plot of $S_N^2$ against $N$. Such a plot will diverge for very heavy tailed samples, another indicator of a stable distribution. However, the plot will converge to a finite value for samples with moderately heavy tails.

If graphical tests are consistent with the hypothesis of stability, the next step is to fit the data to a stable distribution, estimating parameters using STABLE.EXE then computing confidence intervals according to Nolan (1998a).

### Size issues

While the main concern of this paper is whether private real estate investment returns are heavy tailed, there is another area that may be explored as a refinement. This has to do with the value of $\alpha$ as property size changes.

Chapter 4 argues that research in the real estate investment market should be separated into two distinct “tiers” based on property size. The intuition is not difficult as
institutional portfolio managers have constraints and incentives that drive them toward larger properties. It would, however, be intriguing to know if size, risk, heterogeneity and $\alpha$ were somehow related. If small properties exhibit smaller $\alpha$ values, hence more site-specific risk, the normative reasons for institutional property owners to avoid the small property market might support the positive reasons that they actually do.

Two questions may be asked. First, is the Tier II market statistically different from the Tier III market as measured by $\alpha$? Young and Graff (1995) found a Tier III $\alpha$ of $1.477 \pm .038$. If the data studied here are SP distributed, is its $\alpha$ value meaningfully different? The second question is more intricate: *Within Tier II* does $\alpha$ fall with property size? Formally, these two hypotheses are:

**Proposition 1:** $H_0$: Tier III $\alpha$ = Tier II $\alpha$

$H_A$: Tier III $\alpha$ > Tier II $\alpha$

**Proposition 2:** $H_0$: Tier II$_{\text{large}}$ $\alpha$ = Tier II$_{\text{small}}$ $\alpha$

$H_A$: Tier II$_{\text{large}}$ $\alpha$ > Tier II$_{\text{small}}$ $\alpha$
Results

Distribution issues

Graph of the data

As discussed above, three different graphical demonstrations will be made to detect non-stability. Figure 6–1 shows a histogram of the residuals for 2,013 repeat sales. Adopting a variation on the Young and Graff (1995) methodology, the regression that produced these residuals has removed the location effect. Thus, these are assumed to be the returns to site-specific risk encountered by individual real estate owners. In the background is the normal distribution. While not dramatic, there is evidence of long tails, slightly heavier on the right.

Figure 6–1: Histogram of the data with the normal distribution in the background
Figure 6–2 is a plot of a random sample generated with the parameter values developed below. This also shows a heavy tail skewed right.

Finally, Figure 6–3 shows a plot of the "simulated" pdf constructed using the inverse Fourier transform methods of Appendix D. The parameters, $S^0(1.4218, .2579, .000171215, -.0000594559)$, estimated below by the method of Nolan (1997), are used to generate this plot. The normal, as before, shadows the distribution for the sake of comparison. Note the additional variation similar to Figure 1-3 and the shift in variation from left to right similar to Figure 2-3.
Stabilized Probability Plot

As suggested by Nolan (1998a), the Stabilized Probability Plot (Michael, 1983) is constructed and appears as Figure 6–4. Evidence that the data are stable is found when the plot of the points deviates very little from a diagonal line through the origin. Figure 6–4 has such an appearance, further supporting the hypothesis of stability.
The Granger-Orr Test for Infinite Variance

The plot of the running variance in Figure 6–5 shows that the variance quickly converges to a finite value. This, while determinative for the sample, does not necessarily mean the distribution has a finite variance.
Fit of the Data

The hypothesis of stability having graphical support, notwithstanding the failure of the Granger-Orr test to show infinite variance for the sample, the data are fit to a stable distribution using STABLE.EXE provided by Nolan (1997). Table 6–4 shows parameter estimates and confidence intervals for the 90% and 95% levels for $\alpha$ and $\beta$.

Table 6–4: Confidence intervals for estimates of $\alpha$ and $\beta$

<table>
<thead>
<tr>
<th>Conf. Level</th>
<th>n</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>CI$_\alpha$</th>
<th>High</th>
<th>Low</th>
<th>CI$_\beta$</th>
<th>High</th>
<th>Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>90%</td>
<td>2013</td>
<td>1.4218</td>
<td>0.2579</td>
<td>0.0571</td>
<td>1.4789</td>
<td>1.3647</td>
<td>0.0951</td>
<td>0.3530</td>
<td>0.1628</td>
</tr>
<tr>
<td>95%</td>
<td>2013</td>
<td>1.4218</td>
<td>0.2579</td>
<td>0.0681</td>
<td>1.4899</td>
<td>1.3537</td>
<td>0.1133</td>
<td>0.3712</td>
<td>0.1446</td>
</tr>
</tbody>
</table>

Figure 6–5: Granger-Orr test for infinite variance
Confidence intervals are computed from the standard deviation of the parameter based on tables from Nolan (1998a) according to the following formula:

\[ \hat{\theta}_i \pm Z_{\alpha/2} \frac{\hat{\sigma}_{\theta_i}}{\sqrt{n}} \]

where

\[
\hat{\theta}_i \]

is the estimated individual parameter of the parameter vector \( \theta \)

\[ Z_{\alpha/2} \]

is the two-tailed level of confidence desired from the standard z-table and

\[
\hat{\sigma}_{\theta_i}
\]

is the standard deviation of the individual parameter from the table given in Nolan (1998).

As usual, confidence intervals are an artifact of the sample size. As \( n \) grows larger, the Z-table adjustment to the estimated individual parameter becomes smaller.

**Size Issues**

The initial tasks were to (a) determine if the data was heavy tailed, (b) conduct diagnostics to detect evidence of non-stability, and (c) estimate parameters of the distribution from our sample. That done I turn to the questions of size. Recall the first testable hypothesis:

Proposition 1: \( H_0: \text{Tier III } \alpha = \text{Tier II } \alpha \)

\( H_A: \text{Tier III } \alpha > \text{Tier II } \alpha \)

\[ \gamma (.000171215) \text{ and } \delta (-.0000594559) \] which are ignored at this time.
The range of $\alpha$ found by Young and Graff was $1.477 \pm .038$ (range of $1.439 \leftrightarrow 1.515$). Unfortunately, as shown in Figure 6–6 a portion of the range of $\alpha$ for Tier II in this study is also in Young and Graff’s range.

Using Young and Graff as a benchmark may be inappropriate. They used interquartile range methods (McCulloch (1986)) and linear interpolation; where my study uses maximum likelihood methods (Nolan (1998a)) and cubic spline interpolation. Their confidence intervals, if constructed in a fashion similar to that used for this study, would be reduced by their larger sample. However, aggregating observations of the same property over time increased their sample. Their largest single year sample was 1,715 in 1992. They used intertemporal income as observations and assumed sales at appraised values where this study used confirmed sales and assumed value is linear income.

Finally, Young and Graff did not cover residential property while I have only residential property. With these differences noted, the closeness of the point estimates for $\alpha$ and the

![Figure 6–6: Confidence Intervals for $\alpha$](image)
width of the confidence intervals, the null under Proposition 1 still cannot be rejected.

It does appear likely that Tier II $\alpha$ is not above the $\alpha$ value for Tier III.

The second testable hypothesis,

$$\begin{align*}
    \text{Proposition 2: } & H_0: \; \text{Tier II}_{\text{large}} \alpha = \text{Tier II}_{\text{small}} \alpha \\
    & H_A: \; \text{Tier II}_{\text{large}} \alpha > \text{Tier II}_{\text{small}} \alpha
\end{align*}$$

requires the computation of $\alpha$ for smaller properties within the data set for Tier II.

Again, sample size is an issue. Ideally, the data would contain several thousand

observations of buildings in the range of 70 - 100 units and several thousand observations

of buildings in the range of less than 50 units. Having a total of 2013 observations across

all sizes, I am not so fortunate. Table 6–5 reflects the results of estimating $\alpha$ on the

censored data as the number of units drop.
Table 6–5: Values of $\alpha$ for different subsets of the data based on building size

<table>
<thead>
<tr>
<th>Less Than ___ Units</th>
<th>$\alpha$</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>1.3890</td>
<td>1589</td>
</tr>
<tr>
<td>40</td>
<td>1.3993</td>
<td>1679</td>
</tr>
<tr>
<td>50</td>
<td>1.3967</td>
<td>1759</td>
</tr>
<tr>
<td>60</td>
<td>1.4121</td>
<td>1800</td>
</tr>
<tr>
<td>70</td>
<td>1.4103</td>
<td>1830</td>
</tr>
<tr>
<td>80</td>
<td>1.4115</td>
<td>1860</td>
</tr>
<tr>
<td>90</td>
<td>1.4080</td>
<td>1888</td>
</tr>
<tr>
<td>100</td>
<td>1.4136</td>
<td>1907</td>
</tr>
<tr>
<td>200</td>
<td>1.4204</td>
<td>1987</td>
</tr>
<tr>
<td>300</td>
<td>1.4179</td>
<td>2008</td>
</tr>
</tbody>
</table>

Figure 6–7 shows the same information graphically.
While the graph appears to confirm the hypothesis of lower $\alpha$ for smaller properties, restraint is in order. Were the y-axis not so compact the graph would appear to be nearly a horizontal line. Note in Table 6–5 that the number of observations between levels, especially at the larger end, is very small. Clearly, maximum likelihood estimation with sample sizes that small would not be efficient. In fact, the width of the confidence interval may subsume the entire range of $\alpha$ values at all size levels in the data. On the other hand, recall that $\alpha$ is an exponent so small variations, even in the second significant digit to the right of the decimal has an impact.

The decline in $\alpha$ is, of course better than an increase; however, it may be the result of the Fortran algorithm in Nolan's program quitting each time on the same side of the estimate. One approach is to compare the value of $\alpha$ and its confidence interval for the entire data set with the value of the predominant portion of the data set, that below 30

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**Figure 6–7**: The relationship between $\alpha$ and property size
units. Each of these involves a reasonable sample size and the set of small properties
contain the smallest of the buildings in the data set.

Table 6–5 shows that, while all $\alpha$ values for the set containing the smaller sized
properties is below the equivalent values for the full data set, there is considerable
overlap. Table 6–6 shows confidence intervals for subsets of the data for less than 100
unit and less than 30 units. Figure 6–8 illustrates the overlap. Again, due to overlaps in
the confidence intervals, I am reluctant to reject the null hypothesis for Proposition 2.

Table 6–6: Parameter estimates and confidence intervals for subsets of the data

<table>
<thead>
<tr>
<th>Conf. Level</th>
<th>Subset</th>
<th>n</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>CI$_\alpha$</th>
<th>High</th>
<th>Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>90%</td>
<td>All Data</td>
<td>2013</td>
<td>1.4218</td>
<td>0.2579</td>
<td>0.0571</td>
<td>1.4789</td>
<td>1.3647</td>
</tr>
<tr>
<td>90%</td>
<td>&gt;100 units</td>
<td>1907</td>
<td>1.4136</td>
<td>0.2580</td>
<td>0.0586</td>
<td>1.4722</td>
<td>1.3550</td>
</tr>
<tr>
<td>90%</td>
<td>&gt;30 units</td>
<td>1589</td>
<td>1.3890</td>
<td>0.2300</td>
<td>0.0640</td>
<td>1.4530</td>
<td>1.3250</td>
</tr>
<tr>
<td>95%</td>
<td>All Data</td>
<td>2013</td>
<td>1.4218</td>
<td>0.2579</td>
<td>0.0681</td>
<td>1.4899</td>
<td>1.3537</td>
</tr>
<tr>
<td>95%</td>
<td>&gt;100 units</td>
<td>1907</td>
<td>1.4136</td>
<td>0.2580</td>
<td>0.0698</td>
<td>1.4834</td>
<td>1.3438</td>
</tr>
<tr>
<td>95%</td>
<td>&gt;30 units</td>
<td>1589</td>
<td>1.3890</td>
<td>0.2300</td>
<td>0.0762</td>
<td>1.4652</td>
<td>1.3128</td>
</tr>
</tbody>
</table>

Table 6–4 also provides point estimates and confidence intervals for $\beta$. Table 6–6
provides only point estimates for $\beta$ in the small range of the data. The evidence in Table
6–6 does not support the notion of an alpha-beta tradeoff due to the fact that $\beta$ also falls
with the size of the property. However, the same caveat applies regarding wide
confidence intervals due to small sample size. Because little change in $\alpha$ is consistent
with little change in $\beta$, the width of confidence intervals prevents me from drawing
conclusions in this area. Not too much should be made of this until larger sample sizes
and smaller confidence intervals permit us to confirm if, in fact, $\alpha$ is falling as property grows smaller.

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**Figure 6–8:** $\alpha$ Confidence intervals for the smallest units in the data

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**Conclusions**

One conclusion that may be reached is that more study is needed. The results are not exactly “impressive and consistent” like those in Fama (1965b) were. Neither are they trivial. There are grounds for suspecting that Tier II $\alpha$ is lower than Tier III $\alpha$, especially when the property size becomes quite small.

Perhaps the best result is the apparent confirmation that returns in the private investor market are also stable Paretian distributed. Taken with the suspicion that $\alpha$ is smaller, it appears that deviations from the normal are sufficiently large to produce unacceptable empirical results. Thus, it may be productive to continue work to develop models based on other, more empirically useful, assumptions.
In the final chapter I move beyond what these numbers and this sample mean and suggest how research in an entirely different direction may produce real estate insights previously hidden from view by finance models.
CHAPTER 7

CONCLUSIONS, IMPLICATIONS AND SUGGESTIONS FOR FUTURE RESEARCH

Summary

The results in Chapter 6 contain more economic significance than the statistical significance suggests. There is little doubt that returns in the private investor real estate market examined here are non normal. It is very likely that they are stable. The evidence indicates a value of $\alpha$ below 1.5 with 95% confidence. There is ample theoretical evidence that when $\alpha$ drops that low, MPT produces unacceptable empirical results. What remains in doubt is whether the characteristic exponent for small, individually held property is less than that for larger, institutional grade property.

The remainder of this chapter is devoted to three tasks. First, I will present implications of the findings of the research. Second, I will discuss how these findings may impact thinking about real estate differently than that which went before. Finally, I suggest directions for future research.
Implications

Clearly, the most important general finding of this dissertation is the persistence of heavy tails in real estate data. Young and Graff (1995) found them in United States Tier III data, Graff, Harrington, and Young (1997) found them in Australian Tier III data, and it appears here in US Tier II data. Also, "heaviness" in all cases was not marginal as Fama (1965b) found for stocks, rather $\alpha$ is near or below 1.5, a level that implies major difficulties for the finance paradigm.

A more specific finding is the possibility that $\alpha$ is lower and skewed right for smaller properties. Previously, I surmised that institutions shun small projects because of transaction costs or absence of economies of scale. While these may still be valid, $\alpha$ moving downward as property size falls provides another reason, this one based on the shape of its return distributions. However, if the skew is persistent for small properties, institutions, in seeking only large property, also avoid a desirable characteristic. If the second most important finding of this dissertation is the "size-and-skew" factor, the following implications and explanations arise:

* The fact that the evidence here contains a right skew ($\beta = .2579$) supports a notion of a trade-off in Tier II that may not be at work in Tier III. Young and Graff (1995) found both positive and negative $\beta$ values. Tier II skewed right means that investors in that market are being compensated for the higher risk in a way that institutional investors are not.

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1 Their Australian commercial data showed $\alpha = 1.588 \pm .068$ with both positive and negative skewness.
• One would be justified in questioning the actual composition of the right-skewed returns found here. It is believed that these investors manage their own property, thereby adding labor. Hence, the fat right tail may not represent compensation exclusively for capital. Arguably, all management compensation is reflected in Tier III returns. The presence of a fat right tail in Tier II may, in part, account for the compensation to the owner's labor in Tier II that is otherwise implicit in Tier III returns.

• Agency issues may also contribute to the results. The Tier II property owner internalizes agency matters when he does his own management. If agency costs at the institutional level are more than at the individual level due to span of control, incentive and moral hazard issues, this should be reflected in positive returns in Tier II data.

• Roulac (1995) speaks of how individuals capitalize on their size to avoid many of the institutional constraints that affect Tier III investors. For instance, many regulations do not affect small employers. Some small investors are able to avoid employing others altogether. This is another cost factor that explains fat right tails in this data.

• One of the most significant institutional factors affecting real estate is taxation. While this benefit is priced along with all others and may not show up in the form of larger or smaller returns, Tier II owners who add labor, converting their time into equity, have an option to tax defer that income. This option may be seen to affect behavior and influence the decision to acquire real property rather than
other assets as well as the decision to commit more time to ones (currently taxable) employment.

• Refining the size matter further, the drop of $\alpha$ within Tier II as property gets smaller supports a hypothesis of greater heterogeneity among small properties than among large properties.

This dissertation, stated at the outset, has been about the individual investor, the owner of Tier II property. Despite this, it seems that comment is in order about the meaning of these results, if any, to Tier III participants. Institutional portfolio managers spend a meaningful share of their time in due diligence efforts, risk management, and loss control. Among their worries are complaints from investors, questions about the decisions of management and the adequacy of the models upon which management decisions are based. Figure 5-1 through Figure 5-7 present a worrisome picture to the fiduciary who has represented to his investors that the risk they assume will be in accordance with the dictates of MPT. Closer examination with the tools provided here discloses that the "efficient" frontier upon which the client relied may in fact be a fraction of that represented and located in an entirely different place in the mean-variance space than first believed. In short, that which was represented as efficient may actually be far from it.²

There may be good news for those interested in mixed asset portfolios. The combination of normally distributed securities and SP distributed real estate may provide

² I leave it to the legal community to determine the consequences of such an error. However, it would seem that the widespread availability of tools like those in the Appendices make ignorance a poor defense.
more of what is sought by mixing assets. Certainly SP theory described here disturbs the present mathematics of this strategy. Much thought is needed before one can say for sure what effect this has, but the effect can be dramatic. Appendix G provides a Mathematica notebook that permits the user to create output similar to that shown in Figure 5-1 through Figure 5-7. A caution is appropriate, however. The output may look considerably different from the illustrations in Chapter 5 because each replication comes from a different random draw. One may reasonably expect a strong probability that the plots will vary considerably in size, shape, and placement in the mean variance space. Thus, asset managers are presented with a larger set of options for these kinds of portfolios. Whether this will be viewed as welcome news is in doubt considering the consequences to rebalancing and other portfolio maintenance activities resulting from such an expanded set.

Returning to the general and assuming that Tier II $\alpha$ is smaller, this means that site-specific risk is greater in smaller properties. If Young and Graff (1995) urge Tier III investors to pay more attention fundamental analysis of specific acquisitions, Tier II owners are well advised to exercise even more caution in that regard. A lower $\alpha$ also means that whatever practical barriers to diversification Tier II investors face, their gains from diversification are certainly not what MPT predicts. This opens the broader question of what definition of risk is appropriate for Tier II and how it should be measured, a subject discussed more fully in the next section.


**Discussion of Risk in Real Estate**

Perhaps one of the most complex notions of the interconnectedness of modern society is the relationship between risk and reward. It is remarkable that Markowitz and Sharpe were able to simplify it to a linear relationship and that their simplification has survived for so long. Chapter 3 recites the many blows it has taken in the course of justly earning the title "Dominant Paradigm" in finance.

If real estate is a subset of finance, one might be resigned to accept the fit of the paradigm handed down from above. But if real estate is a subset of microeconomics, there is a different view of risk and reward to be considered. That view starts with Knight (1921) and his assertion that bearing risk and undertaking uncertainty are different propositions. Risk is *a priori* quantifiable and may be offset by the payment of insurance premiums to those who make it their business to assume it. Uncertainty, on the other hand, defies quantification and only the profits accruing to the residual claim compensate those who undertake it. In sorting out the mathematics and probabilities, Rothchild and Stiglitz (1970) demonstrated part of the reason that finance clings to the mean-variance rule as its particular measure and definition of risk. When one examines the technicalities, it becomes apparent that, without the restrictive assumptions associated with MPT and the kind of market that accommodates those assumptions, the finance paradigm offers a narrow, rather special case of evaluating risk and reward that may suffice for securities but describes the private real estate market poorly.
An important conclusion of this dissertation is that the nature and dynamics of the private real estate market are so far from the nature and dynamics of the stock market that the MPT model produces empirically unacceptable results. If this conclusion prevails, real estate appears to be a subset of microeconomics. This reintroduces, among other things, the theory of the firm, thus the first implication of this line of thinking is to view the Tier II property as a company. The metaphor that comes to mind is seeing "the building as the firm." The apartment owner is the general manager of a business, an independent entrepreneur bearing uncertainty and collecting the residual, if any.

If one takes that notion to the limit, one finds little science with which to perform analysis. If each undertaking is so unique and so imbued with its own specific and different qualities of uncertainty, nothing even approaching the law of large numbers may ever be applied. It is perhaps this, not the lack of data or lack of interest, which explains why so little research on this market exists.

Perhaps real estate risk and reward takes from both sides of the aisle. To repeat Lusht (1988) we should stop doing what doesn’t work. Beyond that, we should acknowledge that what doesn’t work hasn’t worked for a long time and we are overdue for an alternative.

It is possible that private real estate investors, as compensation for the lack of diversification opportunities, seek the inefficiencies of the real estate market and the opportunity to exploit those inefficiencies. These inefficiencies constitute a form of

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3 This is the purest of theoretical discussion, independent of the politics of academia and other practical realities that may be arrayed against the removal of real estate from finance departments across the land and its placement within economics departments.
uncertainty. Thus, Tier II players may not be risk seekers in the sense that they exhibit counterintuitive behavior because uncertainty is not risk. These market actors may have a very different view of the list shown in Appendix A. They may not view that list as a recitation of obstacles, rather a catalog of opportunities. For instance, illiquidity and long holding periods may translate into the opportunity to make a commitment. Management challenges could become a way to add value by applying entrepreneurial vision to the property (the firm). Lack of diversification might be the chance to focus, to put all one's eggs in one basket and watch that basket very closely.

Of course, without data on owners and some method of looking inside their minds, one is left with conjecture on motives. So a possible line of future research would be to inquire whether items listed in Appendix A are truly viewed as negatives by those who have self-selected into the private real estate market. One might hypothesize that immobility, specialized financing, and durability are all seen as advantages to owners of private real estate.

If the laundry list were to become a list of advantages rather than disadvantages, the complexion of the features distinguishing real estate from financial assets changes. It then becomes easier to take the next step, which is to acknowledge that there are positive nonlinearities at work in real estate. The consequence of such an admission is to exceed the limits of the ability of the MPT risk and reward model to produce acceptable predictions. Real estate no longer would need to suffer the laundry list as a litany of exceptions to the linear, two-dimensional MPT world. This suggests a richer model, one
with more than two parameters. To visualize this effect, recall Figure 5-11, a contour plot of utility for the $S^0(\alpha, \beta, 1, .06)$ distribution. As this is the introductory and exploratory phase of this technology, for expository purposes $\gamma$ was restricted to a low value for Figure 5-11 in order to insure semi straight lines. Recall from Chapter 2 that $\gamma$ is a scale parameter, expanding and contracting the distribution around its mode. As there was never any reason for this other than to provide a quasi-linear demonstration, the restriction of a low $\gamma$ is relaxed in Figure 7–1, increasing it to 3. Figure 7–1 depicts the contour plot of utility for the $S^0(\alpha, \beta, 3, .06)$ distribution. Elsewhere in this dissertation I have cautioned against forcing the stable environment into a two-dimensional illustration that ignores the third and fourth parameters. Figure 7–1 shows the reason. The non-linearities are evident. The ability to vary another parameter has the potential to provide a richer view of how the Tier II market works. Like Fama (1965a) nearly 35 years ago, I hope that advances in statistical theory permitting analysis of situations such as this continue. In cases where stable distributions are common, these tools light the way down an entirely new path.

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4 For excellent primers on the mechanics and mathematics of non-linear dynamic models see Baumol and Benhabib (1989), Cootner (1964b) or Peters (1996)
Tobin (1969) wisely observed that it is difficult to come up with theories that are both general and interesting. To be interesting we need a puzzle that has not been solved. To be general, we need to be able to describe the solution mathematically. To be placed into use it must have a basis in common sense. If there is any "science" in social science it is in its mathematics. If there is any common sense in the world it is in the land. Future real estate research needs a mathematical structure with a strong intuitive appeal.

Thinking about real estate risk and financial risk differently offers several benefits. Studying opposite extremes can be instructive. Those who have avoided the laundry list may be those who have already self-selected out of direct real estate ownership. If those people are investors at all they are investors in financial assets. Their polar opposites are drawn to the laundry list. These are investors who thrive in a market

Figure 7–1: Contour plot of constant utility as $\alpha$ and $\beta$ change
presently without its own paradigm, perhaps seeking out the inefficiencies of a market that accommodates their nature.

Having (a) built a case for using stable distributions as the foundation for a model to describe risk in private real estate investments, (b) said investors in that market are not risk seekers, (c) admitted the distinction between risk and uncertainty, (d) acknowledged elsewhere that a general mean-variance mentality has universal appeal, and (e) recognized the common sense of location, how do we reconcile these?

I propose to divide the subject of acquiring a property into two separate parts: The first is decision about where on the planet to invest, the location decision; the second is the decision to acquire a specific site. I assume that the natural order of things is:

1. The decision to invest in real estate
2. The decision to invest in real estate in a specific location
3. The decision to buy a specific site in the chosen location

Therefore, prior to assuming the risk of a particular site, the investor makes a location decision. That decision is based on a general mean-variance mentality, not a specific mathematical rule but the sense that risk and reward are positively related and vary by location and the clientele associated with different places on earth. The choice between different locations, their microeconomic, social, and political ramifications involve an intuitive mean-variance calculus. Once the investor has decided to invest in a particular neighborhood, his selection of an individual asset is motivated by the parameters of the SP distribution. For the purposes of this dissertation I have simplified this to two parameters, naming it "the alpha-beta rule." However, as seen in Figure 7–1, that simplification is not without problems. Fully expanded, the model has two sets of
parameters: They are (a) the familiar mean-variance rule applied to location and (b) the four parameters of the SP distribution.\(^5\)

Once he has decided to invest in a specific neighborhood, the “uncertainty-seeking” entrepreneur looks for the worst building on a good block. He seeks the opportunity to increase income and value by imposing his will on the property, re-shaping the property rights and adding entrepreneurial skill. It is the urban equivalent of mixing one’s hand with the soil (Locke (1698) p.130). When successful, his investment becomes the large outlier. When it shows up in the researcher's data, the decisions he made result in the heavy right tail.

This is not to say that the real estate market is free of failure. Certainly people lose money. To be right skewed the mean must be to the left of the mode. Hence, large numbers of people lose small amounts of money to balance the few large outliers that add weight to the right tail. One of the implications of the holding period analysis is revealed here. Tier II investors extend their holding period for a variety of different reasons but one of those reasons is not likely so that they can lose money. In an urban setting in the late 20\(^{th}\) century with an expanding population and restrictive growth controls, holding on to a portion of the fixed supply of legally usable land eventually bails out even improvident investors.\(^6\) Result: weight in the right tail.

\(^{5}\) It is acknowledged that at this point the model could be overloaded with parameters. Undoubtedly, should this research continue, one will often hold several of these constant during analysis.

\(^{6}\) If Will Rogers were alive today he might be moved to modify his original recommendation somewhat to "Buy land, they ain't making it anymore and they are making less of it legal every day."
Suggestions for Future Research

If the conclusion is that private real estate investors (a) do not achieve the gains from diversification they seek if they form portfolios, (b) invest in and watch a single basket closely, (c) seek out property in need of maximization, and (d) build equity by adding labor, it would seem that I have merely discovered the obvious. Is it lost effort to trot out all the complexities of the stable characteristic function, the Fourier transforms, the hypothesis testing, and the simulations only to learn that science and logic is a foundation to present insight that is common knowledge? No it is not. It has been observed that economics is only mathematics applied to common sense. Society moves forward when first principles are propounded carefully and a kernel of common knowledge is carefully formalized in a systematic way.

Ample research opportunities exist. With SP distribution tools emerging and more data coming on line every day, researchers may be counted upon to extend this work. A few ideas might include the following:

- There is the necessary extension to different datasets, larger and smaller markets, longer and shorter time periods, other conditions both closely similar to and starkly different from the ones presented here.
- Practitioners in the field commonly assume holding periods are fixed and income change is monotonic. Appendix E contains preliminary evidence on holding periods in Tier II and a tool for varying both holding period and growth rates over time.
• The data used in this dissertation permits an inquiry into investor holding period and market cycles. It would be interesting to know how much success Tier II investors have timing their exits to coincide with market peaks or if the sale decision relates more to personal factors such as tiring of management. The length of their holding period is influenced by factors other than those disclosed by examining the buildings owned. These behavioral factors should be added to the mix for a more complete view.

• Tier II contains a variety of property types, not just the apartments in the data examined here. Other property types should be researched. It would be especially useful to gather data on undeveloped lots to determine if improvements or the lack of improvements affect tail behavior.

• Those interested in Tier III property may wish to improve on Appendix G. This tool is in a very preliminary stage of development. Those with the practical ability to diversify could improve their assessment of risk by using technology suggested in the Appendices to this dissertation. The first improvement on these notebooks would be to accommodate the no-short-sale restriction and the integer problems common to real estate investing.

• Knight (1921) provided for the separation of uncertain prospects into two classifications. With faster computers and larger databases, we may find that the stable frontier represents a semi quantifiable kind of uncertainty.\(^7\) To the extent

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\(^7\) Perhaps this should be "quantifiable kind of semi-uncertainty." I recognize the underdeveloped heuristics in this remark and the paradox involved in quantifying that which by definition is unquantifiable.
we can refine this, we grow closer to being able to describe entrepreneurship in other than behavioral terms.

- There is a growing body of research on the left side of the heavy tail distribution known as Extreme Value Theory (EVT). Primarily interested in the downside of risk, many international financial intermediaries base loss control policies on EVT. But the glass is also half full. The heaviest of the heavy right tails may be found in the field of real estate development, where there is virtually no theory and little data. Combining EVT and real estate in this arena could produce the first theoretical and empirical results for land developers.

- The different taxation strategies for Tier II and Tier III become an empirical issue. While it is logical to presume that tax consequences are captured in price, the fact that the two investor types are covered by significantly different tax rules should be noted. Reasonable questions might include: Do Tier II investors pay a premium for tax benefits?

- A theory of labor contribution to real estate investment requires further elaboration. This would focus on the conversion of human capital to nonhuman capital and its effect on the intertemporal investment outcome. It would also make maximum use of the nonlinear dynamics of this story. Often an apartment building is merely a self-directed retirement plan. If this activity produces greater investor utility through earlier or more comfortable retirement, policymakers may wish to rethink the institutional incentives provided this segment of the market.
• Must this story be limited to real estate? Perhaps not. The evolution of securitization approximately tracks the development of society. To the extent a system of property rights can be more precisely defined, finely divided, and widely distributed, the options and opportunities of a society tend to grow. However, it appears that before uncertainty can be tamed into quantifiable risk, *individuals* must hold all of the property rights to unwieldy ventures.

• Fixing the point at which uncertainty "settles down" to become risk would be an interesting challenge. It is at that point that securitization may begin.

• Perhaps the grandest scale one might imagine for this subject is measuring $\alpha$ for nation states to assess the amount of entrepreneurial opportunity. This idea may be difficult to implement as the data usually only exists after many transactions have been recorded at which time society is usually highly developed.

**Concluding Remarks**

From Lusht (1988) until now real estate has needed to step away from the finance paradigm and forge a path of its own. This dissertation demonstrates the restrictive assumptions of Modern Portfolio Theory produce poor empirical results for real estate. Also suggested here is the shape of a new view of real estate, one with a fat right tail.

The real estate market is heavily affected by its ownership. Due to the separation of ownership and control, the return on financial assets – including Tier III where real
estate is essentially converted into a financial asset – is less influenced by ownership.\(^8\)

The private real estate investment market may be the opposite. Ownership and control are usually concentrated in the same party. There are labor issues, agency cost savings, and entrepreneurship matters influencing returns in this market. In short, a share of stock does not care who its owner is.\(^9\) An apartment building does.

It appears that there is a class of investors for whom the mean-variance rule and modern portfolio theory does not work well. These people compensate by exercising considerable oversight of the daily operations of their investments. Their “hands-on” inclination provides them control. What they might term "sweat equity" is in fact a conversion of human capital into non human capital. They prefer a different kind of risk than stock market investors. It is time to define their form of risk differently than we define risk in securities markets.

---

\(^8\) By this I mean the management of an owner. Certainly returns are affected by the buying and selling activities of shareholders who are in fact owners.

\(^9\) Unless it is part of a controlling block of shares, the exception rather than the rule.
BIBLIOGRAPHY


Pareto, V. (1897). Cours d'Economie Politique. Lausanne, Switzerland.


Journal of Economic Theory, 3(1), 66-84.


Appendix A

THE LAUNDRY LIST

The following represents a nonexhaustive list of why real estate differs from financial assets:

1. Immobility
2. Special legal and institutional factors distinguishing the property rights
3. Heterogeneous in nature
4. Liquidity problems
5. Infrequent sales
6. Private sales
7. No short sales
8. Inefficient market - costly, unique, and perishable information, affecting value that is slowly incorporated into price.
9. High transaction costs
10. High entry costs (lumpiness)
11. Difficult to diversify
12. Not (easily) divisible in kind
13. Long holding Periods
14. Site-specific financing arrangements
15. Special utility value (homes)
16. Tangible, expensive, long-lived physical asset requiring owner care
Appendix B

THE LOWER BOUNDARY OF TIER III REAL ESTATE

In an attempt to refine the boundary between Tier II and Tier III real estate, two Internet surveys were conducted. The first, completed in October 1998, was a survey of property actually owned by institutional investors maintaining a presence on the World Wide Web. Fifteen REITs specializing in residential income property were located. *Table B-1* below shows that the average multifamily complex owned contained 285 units, with the smallest being 211 units. The size of other classes of real estate was similar. Of the total 41 institutional investors, all but one held property that would fit the general description of “large.”
Table B-1: Property Owned by Institutional Investors

<table>
<thead>
<tr>
<th>Type</th>
<th># Projects</th>
<th># Units</th>
<th>Average Size</th>
<th></th>
</tr>
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<tbody>
<tr>
<td>Apartments</td>
<td>54</td>
<td>18,873</td>
<td>340 Units</td>
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</tr>
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<td>233 Units</td>
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<tr>
<td>Apartments</td>
<td>144</td>
<td>42,123</td>
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<td>81</td>
<td>23,960</td>
<td>296 Units</td>
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<tr>
<td>Apartments</td>
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<td>43</td>
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<td>57</td>
<td>13,174</td>
<td>231 Units</td>
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</tr>
<tr>
<td>Apartments</td>
<td>58</td>
<td>12,266</td>
<td>211 Units</td>
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<td>Apartments</td>
<td>8</td>
<td>2,105</td>
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<td>80</td>
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<td>4,816</td>
<td>321 Units</td>
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<tr>
<td>Industrial</td>
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<td>47,675,260</td>
<td>102,970 SF</td>
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<td>545,709</td>
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<tr>
<td>Industrial</td>
<td>32</td>
<td>4,200,000</td>
<td>131,250 SF</td>
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<tr>
<td>Office</td>
<td>118</td>
<td>21,393,404</td>
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<tr>
<td>Office</td>
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<td>1,859,000</td>
<td>143,000 SF</td>
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<tr>
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<td>21</td>
<td>11,500,000</td>
<td>547,619 SF</td>
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<td>9,202,000</td>
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<td>Office</td>
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<td>1,600,000</td>
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<td>3</td>
<td>403,166</td>
<td>134,389 SF</td>
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<td>Office</td>
<td>104</td>
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<td>Office</td>
<td>15</td>
<td>1,702,681</td>
<td>113,512 SF</td>
<td></td>
</tr>
<tr>
<td>Office/Industrial</td>
<td>67</td>
<td>7,300,000</td>
<td>108,955 SF</td>
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</tr>
<tr>
<td>Retail</td>
<td>37</td>
<td>6,848,838</td>
<td>185,104 SF</td>
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</tr>
<tr>
<td>Retail</td>
<td>37</td>
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<td>138,333 SF</td>
<td></td>
</tr>
<tr>
<td>Retail</td>
<td>59</td>
<td>12,000,000</td>
<td>203,390 SF</td>
<td></td>
</tr>
<tr>
<td>Retail</td>
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<td>11,000,000</td>
<td>500,000 SF</td>
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</tr>
<tr>
<td>Retail</td>
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<td>1,153,569</td>
<td>384,523 SF</td>
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<tr>
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<td>849,840 SF</td>
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<tr>
<td>Retail</td>
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<td>92,308 SF</td>
<td></td>
</tr>
</tbody>
</table>
Knowing what assets institutions hold is one indication. Another is what institutions wish to purchase. During the months of June through August 1999 a questionnaire was circulated by e-mail among several hundred members of organizations serving institutional investors. These included the National Association of Real Estate Investment Trusts, The National Council of Real Estate Investment Fiduciaries, the Pension Real Estate Institute and the Real Estate Research Institute. Figure B-1 below is

Your assistance is requested in a research project being conducted by the Institute for Real Estate Research at The Pennsylvania State University. The survey form at the following URL requires only a few seconds to answer three questions. Our goal is to determine the lower bound of “institutional grade” real estate, that is, answer the question: What is the smallest property you will consider for your portfolio? We will share the results of our survey with all who respond and provide return e-mail addresses. All data will be aggregated and no part of the survey or the participant names will be disclosed to any third party.

Thank you for your assistance. Please double click on the URL below to go to the survey form.

http://www.smeal.psu.edu/survey/rjb21/

Figure B–1: Request for participation in survey of institutional property ownership standards

the e-mail sent soliciting participation.

Figure B-2 below shows the questionnaire participants were asked to complete. This was a survey of acquisition criteria. Twenty-six Real Estate Investment Trusts, four Pension Funds and nine “Other” investors responded. The average apartment building size was just under 150 units with only one reporting under 100 units, the average value sought after was slightly above $7 million. Table B-2 shows the results of the survey. Clearly the lower limit for these investors is generally above 100 units. Institutional interest in nonresidential investment properties appears to start at 100,000 square feet.
Please fill out the survey form below. When you are finished, please click on the "Submit" button on the bottom of the page. Thank you very much for your participation in this research project. If you provide your e-mail address we will send you the results of the survey when it is complete.

Please enter your name: __________________________ (last) __________________________ (first) __________________________ (MI)

Title: __________________________

Company: __________________________

E-mail: __________________________

I. Type of Investor: 

C REIT

C Pension Fund

C RE Development Firm

C Public Partnership

Other (please describe): __________________________

II. Type of property your firm typically acquires

(please rank up to three, where 1 = most preferred and 3 = least preferred):

[ ] Apartments, [ ] Retail, [ ] Office [ ] Industrial
III. Minimum property size you will consider:

<table>
<thead>
<tr>
<th>Type of Property</th>
<th>Minimum Physical Size</th>
<th>Measured In</th>
<th>Minimum Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apartment</td>
<td></td>
<td>dwelling units</td>
<td>$</td>
</tr>
<tr>
<td>Retail</td>
<td></td>
<td>square feet</td>
<td>$</td>
</tr>
<tr>
<td>Office</td>
<td></td>
<td>square feet</td>
<td>$</td>
</tr>
<tr>
<td>Industrial</td>
<td></td>
<td>square feet</td>
<td>$</td>
</tr>
<tr>
<td>Other</td>
<td></td>
<td></td>
<td>$</td>
</tr>
</tbody>
</table>

Privacy notice: The information submitted with this form is for the purposes of graduate research in The Smeal College of Business Administration at Penn State, and will not be used for other purposes.

*Figure B–2: Institutional property Internet survey form*
<table>
<thead>
<tr>
<th>Ob#</th>
<th>Inv Type</th>
<th>A pt</th>
<th>Ret</th>
<th>Ofc</th>
<th>Ind</th>
<th>Oth</th>
<th>Property Preference</th>
<th>Size and Value Constraints</th>
<th>Notes</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>Timber, Farmland, REIT, International</td>
<td>AptSz: 10  RetSz: 10  OfcSz: 10  IndSz: 10  OthSz: 10  OthMeas: 10  OthVal: 10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>Timber, Farmland, REIT, International</td>
<td>AptSz: 100  RetSz: 80  OfcSz: 100  IndSz: 100  OthSz: 100  OthMeas: 100  OthVal: 100</td>
<td>Buy multiple properties to form larger investment</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>Timber, Farmland, REIT, International</td>
<td>AptSz: 150  RetSz: 100  OfcSz: 100  IndSz: 100  OthSz: 100  OthMeas: 100  OthVal: 100</td>
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</tr>
<tr>
<td>26</td>
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<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>Timber, Farmland, REIT, International</td>
<td>AptSz: 200  RetSz: 30  OfcSz: 50  IndSz: 100  OthSz: 100  OthMeas: 100  OthVal: 100</td>
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<tr>
<td>5</td>
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</tr>
<tr>
<td>6</td>
<td>2</td>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>Retail Development Sites</td>
<td>AptSz: 75  RetSz: 4</td>
<td>8</td>
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<tr>
<td>7</td>
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<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>Other Type</td>
<td>AptSz: 250  RetSz: 20  OfcSz: 250  IndSz: 200  OthSz: 200  OthMeas: 200  OthVal: 200</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2</td>
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<td>3</td>
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<td>0</td>
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<tr>
<td>9</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>All types of health care facilities</td>
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<td>5</td>
</tr>
<tr>
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<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>Lodging/hotels</td>
<td>AptSz: 150  RetSz: 5</td>
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</tr>
<tr>
<td>12</td>
<td>2</td>
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<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
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<td>3</td>
<td>0</td>
<td>0</td>
<td>Other Type</td>
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</tr>
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<td>1</td>
<td>Parking Garages</td>
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<td>1</td>
<td>1</td>
<td>1</td>
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<td>0</td>
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<td>Ancillary Hospital Facilities (MOB, POB, etc.)</td>
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<td>Health Care</td>
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(cont. on next page)
### Table B-2 (cont.)

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<tr>
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</tr>
<tr>
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<td>3 0 3 2</td>
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<tr>
<td>27</td>
<td>3 1 3</td>
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<tr>
<td>36</td>
<td></td>
</tr>
</tbody>
</table>

*Pension Fund = 1; REIT = 2, Other = 3 |
*Never = 0; Most preferred = 1; Least Preferred = 2 |
*Size in thousands; Value in $Millions |
C:\jb21\Dissertation\Investor\Survey.xls\by Inv Type

### Size and Value Constraints

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<th>Inv Type</th>
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<tr>
<td>26</td>
<td>REIT Maximum (All) 200 10 750 65 250 20 500 20 500 5</td>
</tr>
<tr>
<td>9</td>
<td>Other Minimum (Institutions) 75 2.5 20 1 20 1 20 1 8 2.5</td>
</tr>
<tr>
<td>39</td>
<td>Total Maximum (Institutions) 200 10 750 65 250 20 500 20 500 5</td>
</tr>
</tbody>
</table>
Appendix C*

A BRIEF (AND NECESSARILY INCOMPLETE) EXCURSION INTO FOURIER TRANSFORMS

*The Mathematica Notebooks in this dissertation benefited from the substantial assistance of Marlyn L. Hicks

This is the first of four Mathematica notebooks included in the Appendices of this dissertation. The first two, Appendices C and D, deal with Fourier and Inverse Fourier Transforms, the third, contained in Appendix E, describes a method to model optimal holding period, and the fourth, Appendix G, contains a random number generator that simulates "efficient" frontiers using stable random variates.

All of the notebooks provided were produced with Mathematica version 4.0 and should be executable in Mathematica version 3.0 with few or no modifications. They all may be viewed with MathReader, a free software produced by Wolfram Research and downloadable from http://www.Wolfram.com/MathReader. The electronic version of this dissertation on CD-Rom contains MathReader. Opening the notebook in MathReader is recommended.

Viewing or printing in color considerably enhances the graphics created in these notebooks.

This is the first of two Appendices intended to supplement the main text in the area of Fourier Transforms. Here, Appendix C expands on the general nature of Inverse Fourier Transforms (IFT) briefly discussed in Chapter 2. Appendix D is intended to supplement Chapter 5, covering the more complex application of the IFT to stable distributions and showing how many of the illustrations in this dissertation are produced.

To provide context, I begin by replicating the illustration, also shown in Chapter 2, from Farlow (1993), depicting the reason Fourier Transforms are so helpful

*Printed by Mathematica for Students*
In our case, we wish to illustrate how two parameters of the stable distribution, its shape parameter ($\alpha$) and its skewness parameter ($\beta$) relate to one another. Doing so requires graphing the utility of different risk choices as modeled by different probability distributions. The "obstacle" overcome by the Fourier Transform is the fact that the stable distribution does not have a probability distribution function (pdf). Chapters 10-12 of Farlow constitute a good resource for Fourier review. The ch.f. and the pdf of the stable distribution constitute a Fourier transform pair as described in Farlow.
Below is an example of a Fourier Series, an expansion series in the same general family as a Taylor series. The Fourier Series converts an unmanageable function into a function that can be numerically integrated. The example below is from the Mathematica help file. Here the "unmanagability" is illustrated by the fact that the function is not continuously differentiable.

```
frplt = Plot[t - Round[t], {t, -1.5, 1.5}]
```

The convention for a Mathematica (MM) expression is to spell out most operations, using CapitalLettersButNoSpaces. When preceded by "?", the execution of a MM command returns the on-line help for it. Some of these are provided here for additional clarification.
FourierTrigSeries[expr, t, k] gives the kth order Fourier trigonometric series approximation to the periodic function of t that is equal to expr for \(-\frac{1}{2} \leq t \leq \frac{1}{2}\), and has a period of 1. FourierTrigSeries[expr, t, k, FourierParameters -> \{a, b\}] gives the kth order Fourier trigonometric series approximation to the periodic function of t that is equal to expr for \(-\frac{1}{2} \text{Abs}[b] \leq t \leq \frac{1}{2} \text{Abs}[b]\), and has a period of \(\frac{1}{\text{Abs}[b]}\).

The following is the trigonometric series expansion of order 3 (Note the last character in the command is "3").

\[
\frac{\sin(2\pi t)}{\pi} - \frac{\sin(4\pi t)}{2\pi} + \frac{\sin(6\pi t)}{3\pi}
\]

Note that the plot of this produces the same general shape, changing the amplitude and "rounding out" some of the sharp edges, the next plot shows the two functions together.

\[
\text{frplt3} = \text{Plot[ex1, \{t, -1.5, 1.5\}, Frame->False, Axes->True]}
\]
By increasing the order of the series (note the command below carries the series to the sixth order, thus the last character in the command is therefore "6"), we can more closely approximate the actual function as shown in the next two plots.

```math
ex2 = FourierTrigSeries[t, t, 6]
```

```
\frac{\sin(2\pi t)}{\pi} - \frac{\sin(4\pi t)}{2\pi} + \frac{\sin(6\pi t)}{3\pi} - \frac{\sin(8\pi t)}{4\pi} + \frac{\sin(10\pi t)}{5\pi} - \frac{\sin(12\pi t)}{6\pi}
```

```math
frpltf6 = Plot[ex2, {t, -1.5, 1.5}, Frame -> False, Axes -> True]
```

- Graphics -
A requirement of a Fourier series is that the function must be periodic. Ours is not, in which case we employ a Fourier transform (FT).

To illustrate this with an example much simpler than the one we will encounter for the Stable distribution case, we start with a function, \( f(x) \), that is known to have a Fourier transform. Farlow at page 89 uses a similar example. Note: Mathematica syntax does not allow the use of parentheses in names of functions, hence "fx1" below is the same as \( f(x) \).

\[
fx1 = e^{-x^2/10}
\]

We plot the function, assigning the name "px1" to the plot.
pxl = Plot[fxl, {x, -10, 10}, Frame -> False, Axes -> True]

?FourierOverallConstant

FourierOverallConstant is an obsolete option to FourierTransform and related functions. Given the new option setting FourierParameters -> {a, b}, the FourierOverallConstant option setting now corresponds to the expression Sqrt[Abs[b]] * (2 Pi)^Floor[a/2].
FourierFrequencyConstant is an obsolete option to FourierTransform and related functions, superceded by the constant b in the new option setting FourierParameters → {a, b}.


FourierTransform[expr, t, w] gives a function of w, which is the Fourier transform of expr, a function of t. It is defined by FourierTransform[expr, t, w] = FourierOverallConstant * Integrate[Exp[FourierFrequencyConstant I w t] expr, {t, -Infinity, Infinity}].

Using the variables common to the characteristic function for Stable distributions, the above *Mathematica* description becomes:

FourierTransform[f[x1, x, u] gives a function of
u, which is the Fourier transform of f[x1, a function of x. It is defined by FourierTransform[f[x1, x, u] = FourierOverallConstant*Integrate[Exp[FourierFrequencyConstant I u x] f[x1, {x, -Infinity, Infinity}].

The original function, f[x1], is a function of x, the FT of this function, f(u1) is a function of u (making this whole operation somewhat like a transformation of a variable).

(A note about notation in this monograph: Generally, for our notation, the x is the original function and the u is the transformed function)

The analogy when we apply this methodology to the Stable distribution problem is: The notation for the pdf will be f(x*) and the notation for the characteristic function (ch.f.) will be f(u*), where the symbol " * " has the usual wildcard meaning and normally will be represented by the sequential numbering of functions. (Most texts on partial differential equations use time and frequency domains for their examples. Here, these are, respectively, the pdf and ch.f.)

We now define the transformed function, f(u1), as the FourierTransform of the original function, f(x1). The x is "integrated out", leaving a function of u. The plot of the transformed function, pu1, follows. Note that the range of u (-5,5) is smaller than that of x (-10,10).
The inverse Fourier transform (IFT), as the name implies, returns us to the original function.

\[ \text{fx2} = \text{InverseFourierTransform}[\text{ful}, u, x] \]

\[ e^{-\frac{u^2}{10}} \]

\[ \text{fx2 == fx1} \]

True

The width of a transformed function is proportional to the rate of change of the function before the transform. At the extreme (the reason to be developed later) is the impulse function, DiracDelta, with a FT = 1 (infinite width) and 1 which has a FT = DiracDelta (width = 0).

We illustrate the proportionality issue by defining a sharper pdf resulting in a wider ch.f.

Recall that \( f(xI) = e^{-x^2/10} \) and plots thus:
A sharper pdf would be \( f(x) = e^{-x^2/2} \) with a plot as shown.

\[
\text{fx1a} = \text{E}^{-x^2/2};
\]
\[
\text{px2} = \text{Plot}[\text{fx1a}, \{x, -5, 5\}, \text{Frame} \to \text{False}, \text{Axes} \to \text{True}]
\]

Plotting them together reveals that \( f(x) \) more concentrated at the center (thinnest graph).
Recalling that we have already transformed $f(x_1)$ to produce $f(u_1)$, we now transform $f(x_{1a})$ to $f(u_{1a})$ and plot both transformations.

Plot[{fx1a, fx1}, {x, -10, 10}, PlotRange -> All, Frame -> False, Axes -> True, PlotStyle -> {{Hue[.1], AbsoluteThickness[3]}, {Hue[.8], AbsoluteThickness[5]}}]
We now turn to the impulse function, DiracDelta. It is called an "improper function" because it fails to satisfy a requirement of a function, namely to map each value of \( x \) to a unique value of \( y \). DiracDelta is a function which is 0 for anything other than \( \text{DiracDelta}[0] \) and \( \int \text{DiracDelta}[0] = 1 \). Mathematica’s description from version 3.0 is:

\[ \text{DiracDelta}[x] \text{ is a distribution that is 0 for } x \neq 0 \text{ and satisfies } \int \text{DiracDelta}[x], \{x, -\infty, \infty\} = 1. \text{ DiracDelta}[x_1, x_2, \ldots] \text{ is a distribution that is 0 for } x_1 \neq 0 \| x_2 \neq 0 \| \ldots \text{ and satisfies } \int \text{DiracDelta}[x_1, x_2, \ldots], \{x_1, -\infty, \infty\}, \{x_2, -\infty, \infty\}, \ldots \times = 1. \]

Hence:

\[
\int_{-4}^{4} \text{DiracDelta}[x - 2.9999999999] \, dx
\]

1.
A critical property of the Fourier transform that makes it useful is the existence of its inverse. Together, the FT and the IFT are known as a "Fourier transform pair" (Farlow p. 90).

The IFT is defined below. Following that is the translation of the IFT into the variables we are working with, where \( \omega = w \) and \( t = x \). \( A \) and \( B \) are user defined values with defaults of 1.

\[
f(t) = \frac{B}{2A\pi} \int_{-\infty}^{\infty} F(\omega) e^{-Bi\omega t} d\omega
\]

becomes:

\[
f(x) = \frac{B}{2A\pi} \int_{-\infty}^{\infty} F(w) e^{-Biwx} dw
\]

Note a change of variable here to make this equation consistent with the use of \( w \) as the inverse transform variable below.

The next function returns a table which is a function of \([u,w]\). Note that values are 0 except where \([w-u] = 0\). The restriction of the range of \( u \) to \([-3,3]\) and the sampling interval of 1 (resulting in integers only) results in seven terms.

The improvement is that this function is integrable term by term, hence transformable. Note that \( u \) is evaluated at points where \([w-u] = 0\), while \( a \) is unevaluated.

This procedure is valid only if \( a \) is independent of \( u \) and \( w \), \( \partial_{u,w} a = 0 \).

Note that the result of the IFT is as predicted by

\[
f(x) = \frac{B}{2A\pi} \int_{-\infty}^{\infty} F(w) e^{-Biwx} dw \text{ where } F[w] = \text{DiracDelta}[w-u].
\]

\[
f[u2a] := \text{Table}[a \text{ DiracDelta}[w-u], \{u,-3,3,1\}]
\]
where \( \text{DiracDelta}[w-u] = \text{DiracDelta}[0] \)

When we specify a value for \( w \), as below where \( w = 2 \), we get zero for all values except when \( \text{DiracDelta}[w-u]=\text{DiracDelta}[0] \)

\[
\text{fx2a} = (1 / (2 \pi)) \int_{-\infty}^{\infty} \text{fu2a} e^{-iwx} \, dw
\]

\[
\left\{-3 a e^{3ix} / 2\pi, -a e^{ix} / \pi, -a e^{ix} / 2\pi, 0, a e^{-ix} / \pi, a e^{-ix} / 2\pi, 3 a e^{-3ix} / 2\pi \right\}
\]

One may integrate a single term to check...

\[
\text{fx2b} = (1 / (2 \pi)) \text{Integrate}\left[-3 a \text{DiracDelta}[3 + w] e^{-iwx}, \{w, -\infty, \infty\}\right]
\]

\[
-3 a e^{3ix} / 2\pi
\]

...or perform \textit{Mathematica}'s IFT on it

\[
\text{fx2c} = \text{InverseFourierTransform}[\text{fu2a}, w, x]
\]

\[
\left\{-3 a e^{3ix} / \sqrt{2\pi}, -a e^{ix} \sqrt{2\pi}, -a e^{ix} / \sqrt{2\pi}, 0, a e^{-ix} \sqrt{2\pi}, a e^{-ix} / \sqrt{2\pi}, 3 a e^{-3ix} / \sqrt{2\pi} \right\}
\]

We can convert \( f(x2a) \) to seven complex trigonometric functions.

\(? \text{ExpToTrig} \)

\text{ExpToTrig}[\text{expr}] \text{ converts exponentials in expr to trigonometric functions.}

\text{ExpToTrig}[\text{fx2a}]

\[
\left\{-3 a \cos[3x] / 2\pi, -a \sin[3x] / \pi, -a \cos[2x] / \pi, -a \sin[2x] / \pi, -a \cos[x] / 2\pi, -a \sin[x] / \pi, -3 a \cos[3x] / 2\pi, -3 a \sin[3x] / \pi \right\}
\]

When we specify a value for \( w \), as below where \( w = 2 \), we get zero for all values except where \( \text{DiracDelta}[w-u]=\text{DiracDelta}[0] \)
fu2a /. w -> 2

\{0, 0, 0, 0, 0, 2 \text{DiracDelta}[0], 0\}

Setting \(a = 1\), carrying out the IFT on \(f(u2a)\) table and summing the terms produces a pdf (plotted next) but not a very useful one. Recall that in this case the sampling interval is 1.

\[
fx3 = \sum_{n=1}^{7} fx2c[[n]] /. a \rightarrow 1
\]

\[
e^{-2 i x} \sqrt{\frac{2}{\pi}} - e^{2 i x} \sqrt{\frac{2}{\pi}} + \frac{e^{-i x}}{\sqrt{2 \pi}} - \frac{e^{i x}}{\sqrt{2 \pi}} + \frac{3 e^{-3 i x}}{\sqrt{2 \pi}} - \frac{3 e^{3 i x}}{\sqrt{2 \pi}}
\]
Returning to $f(uL)$ we apply the sampling procedure as above, then the IFT to the samples. The range for the sampling procedure is determined from the plot of $f(uL)$ to cover all significant points (in this case $u = +/- 2$ or $f = +/- 1/\pi$). The sampling interval below is .1, determining the unambiguous range of $x$ after IFT. The range of $x$ is $1$/sampling interval. The ambiguity, in sampling theory, is known as aliasing. In the above example we have deliberately undersampled to show the effect. The effect is that the function changes direction within the range of $x$, resulting in the multi-modal shape. Our goal is to eliminate the direction change within the range of interest, or to move the ambiguity outside the range. We do this by reducing the size of the sampling interval.

[Note for later: When using this methodology on stable distributions, the variable "t" in Nolan's ChF becomes $u$ which is defined as $u = 2\pi f$ (as shown below), where $f$ is the frequency (an engineering convention). Alternatively, $2\pi$ could have been the constant "B" in the IFT (See Shannon for derivation and proof). The 2 arises from the fact that samples in the DF domain are complex, therefore each sample counts as two, the real value and the imaginary value.]

Recalling the original function and its FT:
Defining the variable "u" as we will do in the Stable distribution case, we multiply the DiracDelta function times the FT of the original function

\[ u = 2\pi f; \]
\[ fu2 = \text{DiracDelta}[w - u] \times \text{ful} \]

The problem with our first attempt to simulate a pdf (resulting in the multi-modal plot), sampled from the table created where

\[
\begin{align*}
\text{fu2a} &= \text{Table}\{\text{a DiracDelta}[w-u]u, \{u,-3,3,1\}\},
\end{align*}
\]

was that we undersampled. The pdf is ambiguous at the points \( x = +/-1/\text{SI} \), where SI is the sampling interval - in this case 1. The unambiguous range must include all significant values of the pdf. As the transforms are not known to exist for the type of functions we are working in and as we know of no analytical method of determining the sampling interval that produces all significant values prior to the transform, the method we use is to plot the transform to determine if the pdf is very near 0 well within the ambiguous range. If it is not, we shorten the sampling interval.

The following reduces the sampling interval from 1 to .1. Note the last character in the command line is ".1".

\[
\begin{align*}
\text{fu3} &= \text{Table}\{\text{fu2}, \{f, -1/\pi, 1/\pi, .1\}\}
\end{align*}
\]

With a revised sample, \( f(u3) \), we again perform the IFT on it.
We then sum them.

\[ fx5 = \sum_{n=1}^{\text{Length}[fx4]} \text{fx4}[[n]] \]

Calibration is done by noting that the value of the original function, \( f(x_1) = e^{-x^2/10} \), should be 1 at \( x = 0 \). We need a divisor, \( mx1 \) below, that will produce that result in sum of the IFT. The result is a plot which, while closer to the shape we are interested in, still falls short.

\[ mx1 = \text{Abs}[fx5 /. x -> 0] \]

While it is possible to calculate the constant "A" for the IFT for a specific situation, it is easier to supply the calibration constant after the transform. Here we relied on our knowledge of the function value at \( x = 0 \). When we work with the Stable ch.f. we will apply a more general rule, relying on the knowledge that \( \int_{-\infty}^{\infty} \text{PDF} \, dX \) should = 1 and adjust the function accordingly.
And we see by plotting both functions that in the x range +/- 5 the result obtained above does not match our original function, plotted at the beginning as px1. Points of ambiguity appear at x = -5 and x = 5. We will eliminate this problem by shortening the sampling interval such that we move the points of ambiguity outside our range of interest.
We again shorten the sampling interval, this time from .1 to .05. The x range is now +/- 10. Running the same steps as before, producing a different divisor (mx2), results in a plot that shows congruence.

```
fu4 = Table[fu2, {f, -1/π, 1/π, .05}];

fu2;

fx6 = InverseFourierTransform[fu4, w, x];

Length[fx6]

fx7 = Sum[fx6[[n]], {n, 1}];

mx2 = fx7 /. x -> 0

3.18308
```
Note that, while difficult to see in the output below (the actual result we are seeking), the two graphs - the plot of our original function \( px1 \) and the plot of our transposed function \( px3 \) are superimposed on each other, indicating that our Fourier transform successfully replicated the amplitude of the original function.

\[
px4 = \text{Plot}[\text{Abs}[fx7/mx2], \{x, -10, 10\}]
\]
Appendix D

STABLE DISTRIBUTIONS FOR INVESTMENT REAL ESTATE RETURNS IN TIER II

<< Calculus`
<< Statistics`
<< Graphics`
<< Graphics`FilledPlot`
<< Graphics`Arrow`
Off[General::spell1];
Off[General::spell];
I. Mathematics of Expected Value with Stable, non-Normal Distributions

This monograph is intended to demonstrate that one may expect values of $\alpha$ between 1.2 and 1.8 and $\beta$ between 0 and .2 for distributions of investment returns on privately held real estate. It also extends the illustration contained in Appendix C to the case involving stable distributions. Finally, it attempts to provide more insight into the special form of discrete Fourier transforms used in this dissertation. This notebook was used to produce many of the illustrations in the text.

In general, the probability distribution function (pdf) and the characteristic function (ch.f.) of a random variable form a Fourier transform pair defined by the equations:

\[
F(u) = \int_{-\infty}^{\infty} f(x) e^{iux} \, dx,
\]

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{-iux} \, du
\]

where $x$ and $u$ are respectively the probability distribution variable and the angular variable of the characteristic function.

For many distributions, including almost all of the stable distributions of interest here, the above integrations cannot be performed analytically and a closed analytical pdf is not defined. Therefore, stable distributions are usually defined by stating their ch.f.

It is sometimes desirable, as in our case, to study the effects on the pdf of varying the parameters of the cf. It is possible, subject the restrictions of sampling theory, to construct a sampled version of the ch.f. by evaluating the ch.f. at equally spaced values of the angle variable $u$. The usual procedure is to assign values to all parameters before sampling, which results in a numeric series. This series is then transformed into a numeric series in $x$ using discrete fourier transform techniques. This can be accomplished to a high degree of accuracy provided the samples are properly spaced and cover all the significant range of the ch.f.

One of the drawbacks of using the numeric approach is that all parameters must be evaluated prior to the application of the transform procedure. We have elected to use a variation of the usual discrete transform which allows evaluation after transformation.
procedure is outlined below.

We will use the the \(S(\alpha,\beta,\gamma,\delta,0)\) parameterization as suggested by Nolan (1998 page 189 Eq. #3) substituting \(u\) for \(t\), \(\gamma\) for \(\sigma\), and \(\delta\) for \(\mu\).

\[
fu := \exp\left[-\gamma^\alpha \text{Abs}[u]^\alpha (1 + i \beta \text{Sign}[u] \tan[\pi \alpha / 2] (\gamma \text{Abs}[u]^{1 - \alpha} - 1)) + \delta u I \right]
\]

Although \(fu\) is not directly integrable, we can apply the transform at the selected sampling points by use of an impulse function, \(\text{DiracDelta}[u-u1]\), which has value only when \(u= u1\). The integration is performed over a band of +/- \(\epsilon\) about each point where \(u= u1\), then the limit is calculated as \(\epsilon \to 0\). The result is a list of terms which are the Fourier transforms of each sample of \(u\). \(x\) is now the independent variable. The pdf is the sum of these terms, \(ffx1\) below. Mathematica performs a change of variable \((u1 \to 2\pi f, \text{to correct the scaling})\), evaluates the terms at selected points, \(-5 \leq f \leq 5\) in steps of .01, and sums the terms in the \(ffx1\) operation below.

\[
\text{idf} = \text{Limit}\left[\frac{1}{2\pi} \int_{u1-\epsilon}^{u1+\epsilon} \text{DiracDelta}[u-u1] (fu / . u \to u1) e^{-iu1x} \, du, \epsilon \to 0\right]
\]

\[
= \frac{e^{-i \cdot u1 \cdot x \cdot i \cdot u1 \cdot \gamma^\alpha \text{Abs}[u1]^\alpha (1 - \beta \cdot (\gamma \text{Abs}[u1]^{1 - \alpha}) \text{Sign}[u1] \tan[\pi \alpha / 2])}}{2\pi}
\]

Note, below, that a table of values, \(ffx1\), is produced with a specific range, \([-5,5]\), and step value (.01).

\[
ffx1 = \text{Apply}[\text{Plus}, \text{Table}[\{\text{idf} / . u1 \to 2\pi f\}, \{f, -5, 5, .01\}]]
\]

To correct amplitude scaling, we make use of the fact that \(\int_{-\infty}^{\infty} x \, dx = 1\).

We can perform the integration numerically after assigning values to the parameters. The scaling factor is independent of the assigned parameters so long as the integration is performed over a range which covers all significant values of the pdf.
The value of the constant term "m1" above is ONLY useful for that certain range and step value (in ffx1) specified above. This constant, a divisor of the function that creates the "simulated" pdf, must be reset if range or step value changes.

After application of the scaling factor, we are able to plot the pdf for a wide range of parameters. The "simulated" probability distribution function, above, is now operational within the bounds of the range and step value leading to the m1 constant above. Below, a stable distribution is plotted using input values for $\alpha$, $\beta$, $\gamma$ and $\delta$.

We can represent the approximation to the normal using $\alpha = 2$ and $\beta = 0$
Below is a symmetric, stable distribution
We can change the parameters to show a right "fat tail" for the stable distribution with a positive skew, below.

Plotting them together, we can illustrate the additional variation exposed by heavy tails that is "cut off" by the normal tails.
We can combine a symmetric and skewed stable plot illustrate how variation is moved from the left to the right as $\beta$ increases.
Below, we remove the normal from the background to reveal only the two stable plots.
The prior illustration may be clearer with the only the normal plot in the background, behind the filled plot of the two stable distributions.
Two factors are critical to the successful application of these procedures: the sampling range and interval in the ch.f. domain. As in any sampled data process, aliasing occurs if the sampling interval exceeds that required for the necessary range of the transformed variable. The sampling range in the ch.f. domain must include all significant values for the parameters of interest.

Because of the characteristics of the transformation (rapid convergence in the ch.f. domain implies a wide range in the pdf domain and visa versa), the number of samples required tends to be constant - about 1000 samples are adequate for most sceneros, but more samples may be required for a wide range of parameters. If a larger range of x is required, the sampling interval may be increased in proportion, i.e. \{f,-5,5,.01\} or \{f,-0.5,0.5,.001\} or \{f,-50,50,.1\}, provided the f range as determined below is adequate.

The f range can be readily tested by assigning the minimum values of interest to \(\sigma\) and \(\gamma\) (the rate of convergence is relatively insensitive to \(\beta\) and \(\delta\), but values must be provided), then evaluating the function \(|f|\) over a range of f sufficient to include all significant values of \(|f|\). To illustrate:
The plot above shows that the range $f +/-$ 5 is adequate for $\gamma = .2$, but marginal for $\gamma = .1$.

The unaliased range of $x$ is equal to $1/df$, where $df$ is the sampling interval of $f$, which, in our example, is $1/0.01$ or 100 or $+/-$ 50. Actually, $df * dx = 1/n$ where $n$ is the number of samples, but the number of samples of $x$ is the same as the number of samples of $f$ so the total range of $x = n* dx = 1/df$).

The effect of aliasing is shown when we plot $x$ over the range of 0 to 100 for $df = .01$. Note that the plot 50-100 is a mirror image of 0-50. This type of plot is the easiest test test for aliasing.

A. Plotting Expected Values against $\alpha$ for different $\beta$ values
Using the tools at hand, we now wish to investigate expected values under differing stable distributions. To make this manageable, \( \gamma \) and \( \delta \) are fixed at, respectively, 3 and .06 leaving only \( \alpha \) and \( \beta \) varying. The plot below shows expected values for \{\alpha, 1.2, 1.8\} and \{\beta, -1, 1\}.

The steps are:

1. Define the pdf with fixed \( \gamma \) and \( \delta \) (pdf1);

2. From the definition of expected value for a continuous random variable, numerically integrate the product of \( x \) \( f(x) \) where \( f(x) \) is the pdf (pdf1);

3. Create a table (tvs1) using a range of values for \( \alpha \) and \( \beta \), here \{\alpha, 1.2, 1.8\} in steps of .1 and \{\beta, -1, 1\} in steps of .2;

4. Working from the list created by the table, produce an object (tvf1) that can be plotted.

\[
\text{pdf}[\alpha, \beta, \gamma, \delta] = \text{Abs}\left[\frac{xfx1}{ml}\right];
\]

\[
\text{pdf1} = \text{pdf}[\alpha, \beta, 3, .06];
\]

\[
\text{pvs1} := \text{NIntegrate}[\text{Evaluate}[x \ \text{pdf1}], \{x, -30, 30\}];
\]

\[
\text{tvs1} = \text{Table}[\text{pvs1}, \{\alpha, 1.2, 1.8, .1\}, \{\beta, -1, 1, .2\}];
\]

\[
\text{tvf1} = \text{ListInterpolation}[	ext{tvs1},
\{\{1.2, 1.8\}, \{-1, 1\}\}, \text{InterpolationOrder} \to \{6, 9\}];
\]

[Mathematica's explanation of the functions used above are shown below. Double click on far right cell bracket if not visible.]

\textbf{ListInterpolation}

\textit{ListInterpolation[array]} constructs an InterpolatingFunction object which represents an approximate function that interpolates the array of values given. \textit{ListInterpolation[array, [[xmin, xmax], [ymin, ymax], ...]]} specifies the domain of the grid from which the values in array are assumed to come.

\textbf{InterpolationOrder}

\textit{InterpolationOrder} is an option to \textit{Interpolation} and \textit{ListInterpolation}. \textit{InterpolationOrder -> n} specifies interpolating polynomials of order \( n \). \textit{InterpolationOrder -> [n1,n2,...]} specifies interpolating polynomials of order \( n1, n2, \ldots \) for dimensions 1,2,..., respectively.
Below the upper and lower "corners" of the plane are turned in when the range of $\alpha$ is $1.2 < \alpha < 1.8$. Changing that range to $1.3 < \alpha < 1.7$ overcomes that problem. To view the plot delete "DisplayFunction\rightarrow Identity" from the statement.

```
Plot3D[tvf1[\alpha, \beta], \{\alpha, 1.2, 1.8\}, \{\beta, -1, 1\}, BoxRatios \rightarrow \{1, 1, 1\},
AxesLabel \rightarrow \{"alpha", "beta", "Expected value"\},
DisplayFunction \rightarrow Identity]
```

Continuing to work with the Mathematica object created above from the table (tvs1), we can plot expected values over a range of $\alpha$ between 1.2 and 1.8 for different values of $\beta$ over its feasible range, 0 to .2 in this case. Here, $\gamma$ remains at a value of 3.

```
pual =
Plot[\{tjf1[\alpha, 0], tvf1[\alpha, .04], tvf1[\alpha, .08], tvf1[\alpha, .1], tvf1[\alpha, .14],
tvf1[\alpha, .18], tvf1[\alpha, .2], .06\}, \{\alpha, 1.2, 1.8\}, GridLines \rightarrow Automatic,
TextStyle \rightarrow \{FontSize \rightarrow 12, FontFamily \rightarrow Times\}, PlotStyle \rightarrow
\{\{Hue[1], AbsoluteThickness[3]\}, \{Hue[2], AbsoluteThickness[3]\},
\{Hue[3], AbsoluteThickness[3]\}, \{Hue[4], AbsoluteThickness[3]\},
\{Hue[5], AbsoluteThickness[3]\}, \{Hue[6], AbsoluteThickness[3]\},
\{Hue[7], AbsoluteThickness[3]\}, \{Hue[9], AbsoluteThickness[3]\}\},
PlotLegend \rightarrow \{"\beta=0", .04, .08, .1, .14, 1.8, .2, "ev=.06"\},
LegendPosition \rightarrow \{1.2, -.5\}, LegendSize \rightarrow \{.6, .8\},
PlotLabel \rightarrow \"[\alpha, \beta, 3, .06]\", Frame \rightarrow Automatic,
FrameLabel \rightarrow \{\alpha, expected value\}]
```

The importance of the above is that we can see expected value falling for all values of $\beta$ with the rise in $\alpha$. Or, we can ask if, to some degree, at a constant expected value, an increase in $\alpha$ is offset by an increase in $\beta$. We also see that expected value $= \delta$ when the distribution is symmetrical ($\beta = 0$).

Below, in a contour plot where ev is constant throughout each "band", we see expected value move upward in the direction of the northwest quadrant with rise in $\beta$ as $\alpha$ declines.
pucev1 = ShowLegend[ContourPlot[Evaluate[tvf1[α, β]], {α, 1.2, 1.8}, {β, 0, .2}], 
  Contours -> {.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5}, 
  ColorFunction -> Hue, DisplayFunction -> Identity, 
  TextStyle -> {FontSize -> 12, FontFamily -> Times}, 
  Epilog -> {{{Text["ev=.5", {1.52315, 0.0348515}]}, {Text["ev=2", 
    {1.43655, 0.130151}]}, {Text["ev=4.5", {1.27, 0.189055}]}}}, 
  FrameLabel -> {α, β}, {Hue[(1 - #) & , 10, "4.5", "<.5"}, 
  LegendPosition -> {1.2, -5}, LegendSize -> {.6, 1.2}]

- Graphics -

Below, the entire operation above is repeated, creating pdf2. The only change is that γ = 1 rather than γ = 3. We get the same general results displayed by straighter curves on a compressed scale for the y axis.

pdf2 = pdf[α, β, 1, .06];
pvs2 := NIntegrate[Evaluate[x pdf2 ], {x, -30, 30}];
tvs2 = Table[pvs2, {α, 1.2, 1.8, .1}, {β, -1, 1, .2}];
tvf2 = ListInterpolation[tvs2,
  {{1.2, 1.8}, {-1, 1}}, InterpolationOrder -> {6, 9}];
Plot3D[tvf2[α, β], {α, 1.2, 1.8}, {β, -1, 1}, BoxRatios -> {1, 1, 1},
  AxesLabel -> {"alpha", "beta"}, "Expected value"}, 
  DisplayFunction -> Identity]

- SurfaceGraphics -
Pua2 = 
Plot[{tvf2[α, 0], tvf2[α, .04], tvf2[α, .08], tvf2[α, .1], tvf2[α, .14], 
tvf2[α, .18], tvf2[α, .2], .06}, {α, 1.2, 1.8}, GridLines -> Automatic, 
TextStyle -> {FontSize -> 12, FontFamily -> Times}, PlotStyle -> 
{{Hue[.1], AbsoluteThickness[3]}, {Hue[.2], AbsoluteThickness[3]}, 
{Hue[.3], AbsoluteThickness[3]}, {Hue[.4], AbsoluteThickness[3]}, 
{Hue[.5], AbsoluteThickness[3]}, {Hue[.6], AbsoluteThickness[3]}, 
{Hue[.7], AbsoluteThickness[3]}, {Hue[.9], AbsoluteThickness[3]}}, 
PlotLegend -> {"β=0", .04, .08, .10, .14, .18, .20, "ev=.06"}, 
LegendPosition -> {1.2, -.5}, LegendSize -> {.4, .8}, 
PlotLabel -> "[α,β,1,.06]", Frame -> Automatic, 
FrameLabel -> {α, expected value}]

B. Plotting Expected Values against β for different α values

Next, returning to a value of 3 for γ, we examine how plots of different values of α change in expected value with changes in β. The results are consistent with the plots above. For all values of α, expected values rise with increases of β. For a fixed expected value, a fall in β may be offset by a decrease in α.
puβ1 = Plot[{{tvf1[1.2, β], tvf1[1.4, β], tvf1[1.6, β], tvf1[1.8, β], .06},
{β, 0, .2}, GridLines -> Automatic,
TextStyle -> {FontSize -> 12, FontFamily -> Times},
PlotStyle -> {{Hue[.1], AbsoluteThickness[3]},
{Hue[.3], AbsoluteThickness[3]}, {Hue[.7], AbsoluteThickness[3]},
{Hue[.9], AbsoluteThickness[3]}},
PlotLegend -> {"α=1.2", 1.4, 1.6, 1.8, "ev=.06"},
LegendPosition -> {1.2, -5}, LegendSize -> {.4, .8},
PlotLabel -> "[α,β,3,.06]", Frame -> Automatic,
FrameLabel -> {beta, expected value}]

If γ is changed to 1, as before, the y axis scale is again compressed.
\( p_{\beta 2} = \text{Plot}[[\text{tvf2}[1.2, \beta], \text{tvf2}[1.4, \beta], \text{tvf2}[1.6, \beta], \text{tvf2}[1.8, \beta], .06], \\
\{\beta, 0, .2\}, \text{GridLines} \rightarrow \text{Automatic}, \\
\text{TextStyle} \rightarrow \{\text{FontSize} \rightarrow 12, \text{FontFamily} \rightarrow \text{Times}\}, \\
\text{PlotStyle} \rightarrow \{\{\text{Hue}[.1], \text{AbsoluteThickness}[3]\}, \\
\{\text{Hue}[.3], \text{AbsoluteThickness}[3]\}, \{\text{Hue}[.7], \text{AbsoluteThickness}[3]\}, \\
\{\text{Hue}[.9], \text{AbsoluteThickness}[3]\}, \{\text{Hue}[.1], \text{AbsoluteThickness}[3]\}\}, \\
\text{PlotLegend} \rightarrow \{"\alpha=1.2", 1.4, 1.6, 1.8, "ev=.06"\}, \\
\text{LegendPosition} \rightarrow \{1.2, -5\}, \text{LegendSize} \rightarrow \{.4, .8\}, \\
\text{PlotLabel} \rightarrow \"[\alpha, \beta, 1.06]\", \text{Frame} \rightarrow \text{Automatic}, \\
\text{FrameLabel} \rightarrow \{\beta, \text{expected value}\}\)
II. The Economics of These Mathematics

All of the above having dealt with mathematical technicalities, our interest is in how economic man (a rational maximizing investor with a utility function having a second derivative < 0) acts in the face of differing distributions. That is, how does one choose between lotteries when the \( \alpha \) and \( \beta \) of those lotteries change?

Recall that \( \alpha = 2 \) and \( \beta = 0 \) results in the normal distribution. Under that special case of the stable distribution, variance and covariance exist, gains from diversification are possible and practical. As \( \alpha \) falls, there is more probability mass in the tails, they become fatter and gains from diversification become less achievable. As \( \alpha \) moves closer to 1, practical considerations make diversification in real estate more difficult. At the \( \alpha = 1.5 \) level there is a suggestion that gains from diversification in real estate are a practical impossibility. When \( \alpha < 1 \) diversification actually introduces more risk and becomes counterproductive (a result so hard to explain that we generally impose the restriction \( 1 < \alpha < 2 \) in financial applications).

On the other hand, \( \beta > 0 \) means the distribution is skewed to the positive side, meaning a greater likelihood of a positive draw in the lottery. The intuition is that under fat right tail conditions I don’t know if I will make a little or a lot but I know I am less likely to lose than if the distribution were symmetric or skewed left (\( \beta < 0 \)).

We begin by assuming an investor's utility function is \( 1 - e^{-1x} \), having a shape and second derivative as shown below.

\[
\text{Plot[Evaluate[1 - E^{-1x}], \{x, -10, 10\},}
\text{Frame -> False, Axes -> True, AxesLabel -> \{x, utility\}]}
\]

Verifying that \( \partial_{xx} < 0 \):

\[
\text{D[1 - E^{-1x}, \{x, 2\}] // TraditionalForm}
\]

\[-0.01e^{-0.1x}\]
Carrying out the procedure as before for expected values, replacing x with the utility function, beginning (again as before) with a fixed value of $\gamma = 3$ and $\delta = .06$ we create the necessary table.

Below, again the most dramatic effect of the change in fu is that the lower corner is turned up when the range of $\alpha$ is $\{1.2, 1.8\}$. This is cured when the range of the plot is reduced (as before) to $\{\alpha, 1.3, 1.7\}$. I note that the table is still over the old range of $\alpha$.

```mathematica
dfs = pdf[\alpha, \beta, 3, .06];
pvs = NIntegrate[Evaluate[dfs (1 - E^{-x})], \{x, -30, 30\}];
tvs = Table[pvs, \{\alpha, 1.2, 1.8, .1\}, \{\beta, -1, 1, .2\}];
tv = ListInterpolation[tvs, 
   \{\{1.2, 1.8\}, \{-1, 1\}\}, InterpolationOrder \rightarrow \{6, 9\}];

Plot3D[tv[\alpha, \beta], \{\alpha, 1.3, 1.7\}, \{\beta, -1, 1\},
   AxesLabel \rightarrow \{"alpha\", "beta\", "utility\"\},
   BoxRatios \rightarrow \{1, 1, 1\}, DisplayFunction \rightarrow Identity]
```

A. Plotting Utility against $\alpha$ for different $\beta$ values

Things are not a simple as before. Only when the distribution is symmetrical does utility rise monotonically with increases in alpha. In the lower ranges of alpha as $\beta$ rises, the function is not well behaved.
This reality is reflected in the contour plot below.
However, when \( \gamma = 1 \) as below, utility rises monotonically with increases in \( \alpha \) for all displayed values of \( \beta \). May we conclude that monotonicity results from small values of \( \gamma \)?
pua3 =
Plot[{tvf3[α, 0], tvf3[α, .04], tvf3[α, .08], tvf3[α, .1], tvf3[α, .14],
  tvf3[α, .18], tvf3[α, .2]}, {α, 1.2, 1.8}, GridLines -> Automatic,
  TextStyle -> {FontSize -> 12, FontFamily -> Times},
  ImageSize -> {72 6, 72 4}, PlotStyle -> {{Hue[.1], AbsoluteThickness[1]},
  {Hue[.2], AbsoluteThickness[1]}, {Hue[.3], AbsoluteThickness[1]},
  {Hue[.4], AbsoluteThickness[1]}, {Hue[.5], AbsoluteThickness[1]},
  {Hue[.6], AbsoluteThickness[1]}, {Hue[.7], AbsoluteThickness[1]}},
PlotLegend -> {"β=0", .04, .08, .1, .14, .18, .2},
LegendPosition -> {1.2, -.5}, LegendSize -> {.4, .8},
PlotLabel -> "[α,β,1,.06]", Frame -> Automatic,
FrameLabel -> {α, utility}]

- Graphics -

Note the semi-linear nature of the contour plot
B. Plotting Utility against $\beta$ for different $\alpha$ values

Here complications appear as $\beta$ grows larger. When $\gamma = 3$, for the low values of $\alpha$, there is a crossover point, below at $\beta = .08$. In the next graph, where $\gamma = 1$, this problem disappears and the utility of all displayed values of $\alpha$ are rising in $\beta$.

```mathematica
puB = Plot[{{tvf[1.2, $\beta$], tvf[1.4, $\beta$], tvf[1.6, $\beta$], tvf[1.8, $\beta$]}},
{\beta, 0, .2}, GridLines -> Automatic, TextStyle -> {FontSize -> 12, FontFamily -> Times}, PlotStyle ->
{{Hue[.1], AbsoluteThickness[2]}, {Hue[.3], AbsoluteThickness[2]},
{Hue[.7], AbsoluteThickness[2]}, {Hue[.8], AbsoluteThickness[2]}},
PlotLegend -> {"$\alpha=1.2$", 1.4, 1.6, 1.8}, LegendPosition -> {1.2, -.5},
LegendSize -> {.4, .8}, PlotLabel -> "[$\alpha, $\beta, 3, .06]", Frame -> Automatic, FrameLabel -> {beta, utility}]
```
Graphics

pucl = ShowLegend[ContourPlot[tf3[α, β], {α, 1.2, 1.8}, {β, 0, .2},
   Contours -> {-0.07, -0.06, -0.05, -0.04, -0.03, -0.02, -0.01},
   ColorFunction -> Hue, DisplayFunction -> Identity,
   TextStyle -> {FontSize -> 12, FontFamily -> Times} Epilog ->
   {{Text["\(u = -0.07\)"], {1.29, .01]},
     Text["\(u = -0.04\)"], {1.45, .1]},
     Text["\(u = -0.1\)"], {1.7, .19}}},
   FrameLabel -> \([\alpha, \beta, 1, 0.06]\) \}, {Hue[\#] & , 8, "\textless\textappprox;0.07", "\textgreater\textappprox;0.01",
   LegendPosition -> \{1.2, \textminus0.5\}, LegendSize -> \{0.6, 1.2\}]

Graphics
puβ3 = Plot[{tvf3[1.2, β], tvf3[1.4, β], tvf3[1.6, β], tvf3[1.8, β]},
{β, 0, .2}, GridLines -> Automatic,
TextStyle -> {FontSize -> 12, FontFamily -> Times},
ImageSize -> {72 6, 72 4}, PlotStyle ->
{{Hue[.1], AbsoluteThickness[1]}, {Hue[.3], AbsoluteThickness[1]},
{Hue[.7], AbsoluteThickness[1]}, {Hue[.8], AbsoluteThickness[1]}},
PlotLegend -> {"α=1.2", 1.4, 1.6, 1.8,}, LegendPosition -> {1.2, -.5},
LegendSize -> {.4, .8}, PlotLabel -> "[α,β,1,.06]",
Frame -> Automatic, FrameLabel -> {beta, utility}]

- Graphics -

C. Finding the α-β tradeoff under constant utility

Holding utility constant at -.05 we create a table of values for α and β over the ranges suggested above. Error Messages are harmless and may be ignored. Tinkering with the range of α and β will produce a variety of results in the tables below. The goal is to change these ranges until the table at the end produces values of α in the permissible range.

\[
\text{u05 = Table[FindRoot[tvf[α, β] == - .05, \{α, \{1.4, 1.8\}\}], \{β, 0, .2, .02\}];}
\]

\[
\text{u05a = α /. u05;}
\]
u05b = Table[u05a[[n + 1]], 0 + .02 n], {n, 0, 10}] // TableForm

<table>
<thead>
<tr>
<th>n</th>
<th>0.02 n</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.26008</td>
</tr>
<tr>
<td>0.02</td>
<td>2.10431</td>
</tr>
<tr>
<td>0.04</td>
<td>2.07508</td>
</tr>
<tr>
<td>0.06</td>
<td>2.05969</td>
</tr>
<tr>
<td>0.08</td>
<td>2.05036</td>
</tr>
<tr>
<td>0.1</td>
<td>1.79367</td>
</tr>
<tr>
<td>0.12</td>
<td>2.0431</td>
</tr>
<tr>
<td>0.14</td>
<td>2.04544</td>
</tr>
<tr>
<td>0.16</td>
<td>2.05352</td>
</tr>
<tr>
<td>0.18</td>
<td>2.06838</td>
</tr>
<tr>
<td>0.2</td>
<td>2.08902</td>
</tr>
</tbody>
</table>

u05 = Table[FindRoot[tvf[α, β] == -.05, {α, {1.4, 1.8}}], {β, 0, .2, .02}];

u05a = α /. u05;

u05b = Table[u05a[[n + 1]], 0 + .02 n], {n, 0, 10}] // TableForm

<table>
<thead>
<tr>
<th>n</th>
<th>0.02 n</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.26008</td>
</tr>
<tr>
<td>0.02</td>
<td>2.10431</td>
</tr>
<tr>
<td>0.04</td>
<td>2.07508</td>
</tr>
<tr>
<td>0.06</td>
<td>2.05969</td>
</tr>
<tr>
<td>0.08</td>
<td>2.05036</td>
</tr>
<tr>
<td>0.1</td>
<td>1.79367</td>
</tr>
<tr>
<td>0.12</td>
<td>2.0431</td>
</tr>
<tr>
<td>0.14</td>
<td>2.04544</td>
</tr>
<tr>
<td>0.16</td>
<td>2.05352</td>
</tr>
<tr>
<td>0.18</td>
<td>2.06838</td>
</tr>
<tr>
<td>0.2</td>
<td>2.08902</td>
</tr>
</tbody>
</table>

We note that falling $\alpha$ is offset by rising $\beta$ until the higher levels of $\alpha$ where the direction reverses. One solution is to restrict the alpha range to {$\alpha$, 1.4, 1.6}.

u05c = Table[FindRoot[tvf[α, β] == -.05, {α, {1.4, 1.6}}], {β, 0, .2, .02}];

u05d = Table[u05c[[n + 1]], 0 + .02 n], {n, 0, 6}] // TableForm

<table>
<thead>
<tr>
<th>α</th>
<th>0.02</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.26008</td>
<td>0</td>
</tr>
<tr>
<td>2.10431</td>
<td>0.02</td>
</tr>
<tr>
<td>2.07508</td>
<td>0.04</td>
</tr>
<tr>
<td>2.05969</td>
<td>0.06</td>
</tr>
<tr>
<td>2.51833</td>
<td>0.08</td>
</tr>
<tr>
<td>1.79171</td>
<td>0.1</td>
</tr>
<tr>
<td>1.88677</td>
<td>0.12</td>
</tr>
</tbody>
</table>
The value of $\alpha$ above still being outside the permissible range, using a lower value for $\gamma (tfv3)$ is another solution. Below, with $\gamma = 1$, under conditions of a fixed utility, $\alpha$ falls with a rise of $\beta$ over the entire range, however the range is rather narrow and at the low end of the parameter space. So far the only way to handle this is a series of iterations testing the limits.

```
u05e = Table[FindRoot[tfv3[\alpha, \beta] == -.05, \{\alpha, \{1.4, 1.8\}\}], \{\beta, 0, .2, .02\}];
u05f = Table[u05e[[n + 1]], 0 + .02 n], \{n, 0, 10\}] // TableForm
```

\begin{verbatim}
\begin{tabular}{ll}
\alpha & 1.36655 0 \\
\alpha & 1.34871 0.02 \\
\alpha & 1.33031 0.04 \\
\alpha & 1.31136 0.06 \\
\alpha & 1.29188 0.08 \\
\alpha & 1.27185 0.1 \\
\alpha & 1.25131 0.12 \\
\alpha & 1.23026 0.14 \\
\alpha & 1.20875 0.16 \\
\alpha & 1.18679 0.18 \\
\alpha & 1.16443 0.2 \\
\end{tabular}
\end{verbatim}
Appendix E

TIER II PROPERTY OWNERS' HOLDING PERIOD

The primary question of this dissertation is the nature and shape of Tier II return distributions. Many secondary questions arose in the course of the investigation. Some of these have to do with Tier II investor holding period. For instance:

- How long do Tier II investors hold their properties?
- What influences the length of time investors hold properties?
- Why do some properties sell repeatedly or more often than others?
- What is the relationship, if any, of holding period to market cycles?

The data for the dissertation's primary question includes a vector of repeat sale returns. Because the organization of the data in this fashion requires a time period over which to calculate returns, it also provides information for the secondary questions above. The reason for performing additional analysis is to explore as fully as possible the activities of these investors simply because so little is known about them. The holding period questions are not the result of nor are they expected to support the theory presented in the body of the dissertation. Indeed, little theory beyond street clichés is advanced in this Appendix. However, if those clichés, however vernacular they may be, describe the motives of Tier II investors, the analysis that follows sheds more light in the area where few have gone before.
The analysis of real estate investments is fraught with difficulties. These are not improved by the practice in the field of fixing, ignoring or assuming away variables in the return equation. One often hears "Timing is everything", but like the weather, nobody seems to do anything about it. Practitioners typically project a real estate investment over a fixed time horizon of, say, five years. If alternatives are required, the analyst may project for longer periods of time such as 10 or 15 years. The fact that these are fixed and divisible by five is an artificial and unrealistic constraint. The enigmatic, timeworn advice "Buy low; Sell high" says nothing about the period between these two acts. Nor should it, as the enigma lies in the timing decision. The decision about that time period is what separates the successful investor from his less successful brethren.

Two other perversions of the analysis process, often found in the field, are (a) fixing rates of growth in income over the holding period and (b) assuming that going-in and going-out capitalization rates are the same. It can be shown that these two assumptions reduce discounted cash flow (DCF) analysis to simple capitalization rate theory, essentially neutralizing the benefit of varying cash flows that DCF offers.

Finally, the relationship between the growth in income (therefore value) and the discount rate is a complex one. In the limit, if these are the same they cancel each other out. Life is made interesting by the fact that they are never the same. Their difference and especially their difference at the beginning and end of holding period provide much more realistic snapshots of what a real estate investment might produce.

We have the technology to model holding periods more carefully. At the end of this Appendix there is a Mathematica notebook that produces a broad range of outcomes given
varied rates of income growth, discount rates, and capitalization rates over different holding periods. Much more thought needs to be given to these issues before a theory of Tier II optimal holding period emerges. For now, the remainder of this Appendix will concentrate on learning as much as we can about how long Tier II investors actually do hold their property.

**Methodology for Holding Period Issues**

Two research questions will be approached, one having to do with the length of the holding period, the other having to do with frequency of sales. In the former, linear regression (OLS) will be performed on properties appearing in the data set more than once (those having a nonzero holding period). The dependent variable will be holding period (number of days a property is owned) and independent variables will include physical (age, lot size, number of units), location (zip code), and financial (capitalization rate, GIM, leverage) variables.

For frequency-of-sale analysis, discrete dependent variable methods, multinomial and binomial logit, will be employed. Independent variables will be those that do not change (or change minimally) over time, namely physical and location variables.

Theory and some preliminary analysis indicate that variance in this data is not constant. Therefore, t-statistics for significance will be calculated in the standard fashion and using the White (1980) heteroscedastic consistent method.
Our first questions are: Do any or all characteristics relate to how long investors hold property? Do they affect the number of times a property is sold? We will begin with a simple OLS regression testing.

Proposition 1: H₀: Characteristics are not a factor in holding period
Hₐ: Some or all characteristics affect investor holding period

To test for the effect of location and characteristics on the number of times a property sells, the following multinomial logit model was used:

\[
P(y - 1 = 0 \mid x) = \frac{1}{1 + e^{(b(1)'x)} + \ldots + e^{(b(m)'x)}}
\]

\[
P(y - 1 = j \mid x) = e^{(b(j)'x)} P(y = 0 \mid x)
\]

Where

\( Y \) = the number of time the property repeated in the data base (1 = the first repeat sale, hence the first time it repeated in the data base but the second sale of the property during the time period covered by the data);
\( x \) = the independent variable of interest;
\( m \) = the maximum number of times any property repeated (3) in the data set;
\( j \) = the individual repetition in the data set; and
\( b \) = the parameter vector.

This tests the following:

Proposition 2:

\( H₀: \) The number of times a property sold in the time period covered by the data set was not influenced by characteristics or location

\( Hₐ: \) The number of times a property sold in the time period covered by the data set was influenced by characteristics or location
A second approach performed a binomial logit test using the model:

\[ P(y = 1 | x) = F(b(1)x(1) + \ldots + b(m)x(m)) \]

Where:

\[ F(u) = \frac{1}{1 + e^{-u}}; \]

- \( y \) = the binary choice 1 (the property repeated only once) or 2 (the property repeated more than once);
- \( x \) = a vector of regressors, the independent variables of interest;
- \( m \) = the maximum number independent variables of interest (11) effecting \( y \), including the intercept (units, lot size and eight zip codes); and
- \( b \) = the parameter vector.

This tests the following:

**Proposition 3:**

- \( H_0 \): The fact that a property sold repeatedly (more than twice) in the time period covered by the data set was not influenced by characteristics or location
- \( H_A \): The fact that a property sold repeatedly (more than twice) in the time period covered by the data set was influenced by characteristics or location

**Results**

The average holding period was just under four and one half years (1609 days). I might speculate that this represents average exhaustion time for the small investor.

Alternatively, given the industry's long history of projecting investment performance over a five-year time horizon, it may be that investors are conditioned to dispose of property within
that time frame. Of the properties repeating in the data, 73.31% were for properties that repeated only once in the data (sold twice = repeated once). The most number of times a property was sold, for the purposes of these tests, was four.¹

The only consistent property characteristics having data in all fields for this test were the number of units in the project, the lot size and the location. These are the only three things associated with a multifamily property that may be expected to remain constant over long periods of time.² Hence, the test for significance of characteristics on holding period and frequency of sale were limited to these three. I will summarize the findings in the text below. Detailed results may be found at the end of this appendix.

Linear regression of holding period in days on number of units, lot size, and location returned only number of units and one zip code as significant. A test for joint significance of all zip codes included failed to reject the null hypothesis that they were all zero. Hence, for Proposition 1, we may conclude that larger buildings may be held longer, a notion that also rings true for Tier III property. Stated differently, one may infer from the regression that smaller buildings are turned over more often. Finally, it appears that location does matter where holding period is concerned.

In order to perform the multinomial logit test, only those zip codes containing at least one property that had sold four times could be used.³ This reduced the number of zip codes to eight. We find that Units and LotSz are insignificant. Although some of the zip code

¹ A few properties in the data set sold five times but these could not be used in the model as the Newton iterative method would not converge with the larger matrix required if these properties were included.
² Certainly location does not change but in the rare case the other two may (units may be demolished or added; lot size may be increased by assemblage or decreased by partial dispositions). Such rare events were ignored.
³ Lacking this the matrix has perfect collinearity and will not invert.
locations are individually significant, the test of joint significance for all zip codes failed to reject. Hence we may infer that location does not matter for the question of the number of times a property sells.

A second approach performed a binomial logit test. The results of this test may also be found in Appendix E. In this case we also conclude that the number of units in the building and the lot size have no significant impact on whether a property sells repeatedly.4 We cannot, however, say the same about location. Although only three of the eight zip codes were individually significant, the test for joint significance rejected the null that all zip code variables were jointly zero. It may be that frequency of sale (resold more than once), the more general question, is influenced by location even if the number of times a property is sold (high number of resales vs. low number of resales) is not. There is a strong feeling among the public and in the field that location matters in real estate. Thus it is comforting that two of the three tests involving location confirmed the intuition.

Finally, the linear relationship between holding period and capitalization rate was tested. A previous study by this author for a smaller data set (542 observations within three zip codes) over a longer period of time (21 years between 1970 and 1990) showed that holding period increased during times of falling capitalization rates. The procedure was to compare "going-in" or acquisition capitalization rates with "going-out" or disposition capitalization rates. A dummy variable was assigned to those properties for which the going-out rate was higher than the going-in rate. It was found that there is a negative

4 There is a slight distinction from the prior multi-nominal logit test. The number of times a property sells is a discrete variable ranging from $0 \rightarrow \infty$. Whether a property in a repeat sale data set appears more than once is a binary variable.
relationship between the movement of capitalization rates and holding period. The intuition behind this is reasonable. Capitalization rates falling over time means that value is rising faster than income. If one assumes that (a) the owner has to work at improving his property to increase its income but (b) value increases not driven by income increases are exogenous, times when values rise faster than rents (falling capitalization rates) are times when the investor enjoys an unearned increment in the return. Thus, at least while the ride is going up and he expects it to continue, he is well advised to hold the property longer.

The data set used in this study produced mixed results. Regression was performed with holding period as the independent variable on a dummy variable (1 = capitalization rate rising; 0 = capitalization rate falling) for 1858 observations over the 183 months between March 1983 and August 1998.\(^5\) The independent variable is significant and positive, indicating that investors hold property longer when capitalization rates are rising (income is rising faster than value). This is the opposite of the prior result and puzzling. The data used here covers a large portion of the time studied earlier and a larger geographic area (all zip codes rather than just three) of the same region (San Diego County).

The more recent data used in this dissertation contains a period of one of the worst real estate recessions in recent memory (1992-1996). Those years were not included in the prior study. One might readily agree that the period of time covered by the current data (1983-98) contained a disproportionately higher number of "bad years" for real estate investing than the period of time covered by the prior study (1970-90). As a second test of the question of

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5 The data was censored to eliminate those less than .05 and more than .15 as unlikely to be determined by income. Some properties at the end of the building’s useful life sell for land value, others are condominiums
how investors adjust holding period in the face of changing capitalization rates over some of the same years but a larger geographical area, a second regression was run on the first half of the data set, comprising 882 observations from March 1983 through December 1991. Again the dummy variable was significant but the sign was negative, consistent with the prior study and intuition. Another interpretation of the effect of rising capitalization rates on holding period is that value could be falling while income remains constant. This was the case in the early 1990s in San Diego County. Under this interpretation one may conclude that investors extend holding period either (a) when values are rising faster than rent to enjoy unearned returns associated with the good fortunes of timing; or (b) when values are falling despite constant rent to merely hold on to an income producing asset until the market "re-discovers" it and values once again increase. This not so remarkable finding reduces to the conclusion that investors extend holding periods for different reasons.

**Interpretation**

It is not surprising that characteristics do not affect holding period. Nor is it a shock to learn that buildings in some locations sell more frequently than others. The capitalization rate/holding period relationship is a useful insight for those who wish to take a different approach to measuring expectations. For those making projections for investors this

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sold in bulk. Prices observed for such sales are not determined by income. As it is the income relationship we are interested in, these sales are screened out.

6 We are not alone in considering the effect of the recent recession in real estate. Young and Graff (1995) completely eliminated 1991 from one of their tests for the same reason.
appendix argues for improving the performance estimates by abandoning static or monotonic assumptions in favor of more realistic dynamic or cyclical assumptions.

The holding period analysis, a digression here, should be extended to see if the negative draws in the SP distribution are related to short tenure and or financing, two intuitively reasonable explanations for why people lose money in real estate investments.

This appendix is supplemented with a Mathematica notebook ("AppndxE.nb") that provides theoretical insight to holding period issues.
Detailed Results

Results of Linear Regression (OLS) test of the effect of location and characteristics on length of holding period

Model variables:

<table>
<thead>
<tr>
<th></th>
<th>y = HP</th>
<th>x(1) = Units</th>
<th>x(2) = LotSz</th>
<th>x(3) = 91977</th>
<th>x(4) = 92020</th>
<th>x(5) = 92026</th>
<th>x(6) = 92054</th>
<th>x(7) = 92103</th>
<th>x(8) = 92104</th>
<th>x(9) = 92105</th>
<th>x(10) = 92115</th>
<th>x(11) = 1</th>
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<tr>
<td>x(1) = Units</td>
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<td>x(2) = LotSz</td>
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<td>x(3) = 91977</td>
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<td>x(4) = 92020</td>
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<td>x(8) = 92104</td>
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<td>x(10) = 92115</td>
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</tbody>
</table>

Available observations: t = 1 -> 2013
minus 12 missing values in between = 2001 Chosen

OLS estimation results for Y = HP

<table>
<thead>
<tr>
<th>Variables</th>
<th>OLS estimate</th>
<th>t-value</th>
<th>H.C. t-value(*)</th>
<th>s.e.</th>
<th>H.C. s.e.(*)</th>
<th>[p-value]</th>
<th>H.C. <a href="*">p-value</a></th>
</tr>
</thead>
<tbody>
<tr>
<td>x(1) = Units</td>
<td>3.689542</td>
<td>3.255</td>
<td>2.963*</td>
<td>1.1334E+00</td>
<td>1.2453E+00</td>
<td>.00115</td>
<td>.00305</td>
</tr>
<tr>
<td>x(2) = LotSz</td>
<td>-.00096</td>
<td>-.213</td>
<td>-.216</td>
<td>4.5095E-04</td>
<td>4.4512E-04</td>
<td>.00115</td>
<td>.00305</td>
</tr>
<tr>
<td>x(3) = 91977</td>
<td>140.198011</td>
<td>.848</td>
<td>.946</td>
<td>1.6533E+02</td>
<td>1.4814E+02</td>
<td>.00115</td>
<td>.00305</td>
</tr>
<tr>
<td>x(4) = 92020</td>
<td>-1.866354</td>
<td>-.018</td>
<td>-.018</td>
<td>1.0127E+02</td>
<td>1.0180E+02</td>
<td>.00115</td>
<td>.00305</td>
</tr>
<tr>
<td>x(5) = 92026</td>
<td>197.418916</td>
<td>.872</td>
<td>.725</td>
<td>2.2652E+02</td>
<td>2.7218E+02</td>
<td>.00115</td>
<td>.00305</td>
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<tr>
<td>x(6) = 92054</td>
<td>-134.572138</td>
<td>-1.201</td>
<td>-1.365</td>
<td>1.1208E+02</td>
<td>9.8573E+01</td>
<td>.00115</td>
<td>.00305</td>
</tr>
<tr>
<td>x(7) = 92103</td>
<td>-106.426157</td>
<td>-.778</td>
<td>-.755</td>
<td>1.3672E+02</td>
<td>1.4096E+02</td>
<td>.00115</td>
<td>.00305</td>
</tr>
<tr>
<td>x(8) = 92104</td>
<td>-.431907</td>
<td>-.005</td>
<td>-.005</td>
<td>8.2218E+01</td>
<td>8.5041E+01</td>
<td>.00115</td>
<td>.00305</td>
</tr>
<tr>
<td>x(9) = 92105</td>
<td>31.652390</td>
<td>.429</td>
<td>.463</td>
<td>7.3865E+01</td>
<td>6.8431E+01</td>
<td>.00115</td>
<td>.00305</td>
</tr>
<tr>
<td>x(10) = 92115</td>
<td>-159.892821</td>
<td>-1.842</td>
<td>-2.063*</td>
<td>8.6822E+01</td>
<td>7.7487E+01</td>
<td>.00115</td>
<td>.00305</td>
</tr>
<tr>
<td>x(11) = 1</td>
<td>1526.775604</td>
<td>39.386</td>
<td>37.283</td>
<td>3.8764E+01</td>
<td>4.0951E+01</td>
<td>.00115</td>
<td>.00305</td>
</tr>
</tbody>
</table>

(*) Based on White's heteroskedasticity consistent variance matrix. The p-values are two-sided. The standard p-value = P(|t(1990)|>|t-value|), where t(1990) is t distributed with 1990 degrees of freedom.
The H.C. p-values are based on the normal approximation:
H.C. p-value = P(|u|>|H.C. t-value|), where u is N(0,1) distributed.

Standard error of the residuals: 10.495787E+002
Residual sum of squares (RSS): 21.922145E+008
Total sum of squares (TSS): 22.479142E+008
Overall F test: F(10,1990): 5.06
p-value = 0.00000
Significance levels: 10% 5%
Critical values: 1.6 1.83
Conclusions: reject reject

R-square: 0.024778
Adjusted R-square: 0.019878
Effective sample size (n): 2001

Jarque-Bera/Salmon-Kiefer test: 256.981295
Null hypothesis: The errors are normally distributed
Null distribution: Chi-square(2))
p-value = 0.00000
Significance levels: 10% 5%
Critical values: 4.61 5.99
Conclusions: reject reject

Breusch-Pagan test: 31.220560
Null hypothesis: The errors are homoskedastic
Null distribution: Chi-square(10)
p-value = 0.00054
Significance levels: 10% 5%
Critical values: 15.99 18.31
Conclusions: reject reject

x(1) = Units 3.689542 3.255 2.963
x(2) = LotSz -.000096 -.213 -.216
x(3) = 91977 140.198011 .848 .946(*)
x(4) = 92020 -1.866354 -.018 -.018(*)
x(5) = 92026 197.418916 .872 .725(*)
x(6) = 92054 -134.572138 -1.201 -1.365(*)
x(7) = 92103 -106.426157 -.778 -.755(*)
x(8) = 92104 -.431907 -.005 -.005(*)
x(9) = 92105 31.652390 .429 .463(*)
x(10) = 92115 -159.892821 -1.842 -2.063(*)
x(11) = 1 1526.775604 39.386 37.283

Test of the null hypothesis that the parameters indicated by (*) are jointly zero:
F-test statistic: 0.95 Null distribution: F(8,1990)
p-value = 0.47619
Significance levels: 10% 5%
Critical values: 1.67 1.94
Conclusions: accept accept
Wald test: 7.57
Asymptotic null distribution: Chi-square(8)
p-value = 0.47619
Significance levels: 10% 5%
Critical values: 13.36 15.51
Conclusions: accept accept

Test result on the basis of the heteroskedasticity consistent variance matrix:
Wald test: 9.14
Asymptotic null distribution: Chi-square(8)
p-value = 0.33078
Significance levels: 10% 5%
Critical values: 13.36 15.51
Conclusions: accept accept
Results of Multinomial logit test of effect of location and characteristics on the number of times a property sells

Multinomial logit model:
Model variables:

<table>
<thead>
<tr>
<th>Variable</th>
<th>ML estimate of b(.) (t-value)</th>
<th>[p-value]</th>
</tr>
</thead>
<tbody>
<tr>
<td>x(1) = Units</td>
<td>b(1,1) = 0.0018351 (0.56)</td>
<td>[p-value = 0.57325]</td>
</tr>
<tr>
<td>x(2) = LotSz</td>
<td>b(1,2) = -0.0000015 (-1.00)</td>
<td>[p-value = 0.31541]</td>
</tr>
<tr>
<td>x(3) = 92020</td>
<td>b(1,3) = 0.3999260 (1.77)</td>
<td>[p-value = 0.07612]</td>
</tr>
<tr>
<td>x(4) = 92026</td>
<td>b(1,4) = 0.0295640 (0.05)</td>
<td>[p-value = 0.95841]</td>
</tr>
<tr>
<td>x(5) = 92054</td>
<td>b(1,5) = 0.5776539 (2.39)*</td>
<td>[p-value = 0.01685]</td>
</tr>
<tr>
<td>x(6) = 92103</td>
<td>b(1,6) = 0.1253653 (0.38)</td>
<td>[p-value = 0.70192]</td>
</tr>
<tr>
<td>x(7) = 92104</td>
<td>b(1,7) = 0.0638880 (0.33)</td>
<td>[p-value = 0.74222]</td>
</tr>
<tr>
<td>x(8) = 92105</td>
<td>b(1,8) = 0.3418127 (2.05)*</td>
<td>[p-value = 0.03996]</td>
</tr>
<tr>
<td>x(9) = 92115</td>
<td>b(1,9) = 0.2665826 (1.35)</td>
<td>[p-value = 0.17581]</td>
</tr>
<tr>
<td>x(10) = 91977</td>
<td>b(1,10) = 0.3360526 (0.89)</td>
<td>[p-value = 0.37282]</td>
</tr>
<tr>
<td>x(11) = 1</td>
<td>b(1,11) = -1.3377389 (-14.34)</td>
<td>[p-value = 0.00000]</td>
</tr>
<tr>
<td>x(1) = Units</td>
<td>b(2,1) = 0.0096627 (0.90)</td>
<td>[p-value = 0.37034]</td>
</tr>
<tr>
<td>x(2) = LotSz</td>
<td>b(2,2) = -0.0000119 (-1.66)</td>
<td>[p-value = 0.09607]</td>
</tr>
<tr>
<td>x(3) = 92020</td>
<td>b(2,3) = 0.6542811 (1.41)</td>
<td>[p-value = 0.15973]</td>
</tr>
<tr>
<td>x(4) = 92026</td>
<td>b(2,4) = 0.9019449 (0.83)</td>
<td>[p-value = 0.40674]</td>
</tr>
<tr>
<td>x(5) = 92054</td>
<td>b(2,5) = 0.6438238 (1.29)</td>
<td>[p-value = 0.19650]</td>
</tr>
<tr>
<td>x(6) = 92103</td>
<td>b(2,6) = 0.9395381 (1.97)*</td>
<td>[p-value = 0.04897]</td>
</tr>
<tr>
<td>x(7) = 92104</td>
<td>b(2,7) = 0.0256487 (0.06)</td>
<td>[p-value = 0.94988]</td>
</tr>
<tr>
<td>x(8) = 92105</td>
<td>b(2,8) = 0.3682711 (1.08)</td>
<td>[p-value = 0.27912]</td>
</tr>
<tr>
<td>x(9) = 92115</td>
<td>b(2,9) = 0.2581856 (0.63)</td>
<td>[p-value = 0.52670]</td>
</tr>
<tr>
<td>x(10) = 91977</td>
<td>b(2,10) = 0.3360526 (0.89)</td>
<td>[p-value = 0.37282]</td>
</tr>
<tr>
<td>x(11) = 1</td>
<td>b(2,11) = -1.3377389 (-14.34)</td>
<td>[p-value = 0.00000]</td>
</tr>
</tbody>
</table>

Available observations: t = 1 -> 2013 minus 12 missing values in between = 2001 Chosen.
x(10)=91977  b(2,10) = 1.0754464  (1.68)
[p-value = 0.09355]

x(11)=1   b(2,11) = -2.9550608 (-14.04)
[p-value = 0.00000]

x(1)=Units    b(3,1) = 0.0269658  (1.21)
[p-value = 0.22694]

x(2)=LotSz     b(3,2) = -0.0000155 (-1.02)
[p-value = 0.30632]

x(3)=92020      b(3,3) = 3.1081530  (2.46)*
[p-value = 0.01370]

x(4)=92026      b(3,4) = 4.5332218  (2.82)*
[p-value = 0.00480]

x(5)=92054      b(3,5) = 3.3550368  (2.67)*
[p-value = 0.00751]

x(6)=92103      b(3,6) = 2.7683421  (1.91)
[p-value = 0.05608]

x(7)=92104      b(3,7) = 1.6994085  (1.17)
[p-value = 0.32612]

x(8)=92105      b(3,8) = 2.123634  (1.76)
[p-value = 0.05608]

x(9)=92115      b(3,9) = 1.8194135  (2.27)
[p-value = 0.01370]

x(10)=91977     b(3,10) = 3.4648033  (2.38)*
[p-value = 0.01733]

x(11)=1        b(3,11) = -6.8796406 (-6.28)
[p-value = 0.00000]

[The two-sided p-values are based on the normal approximation]
*Significant at the 95% level

Log likelihood: -1.42615693500E+003
Sample size (n): 2001

Test for joint significance of location variables

x(1)=Units     b(1,1) = 0.0018351  (0.56)

x(2)=LotSz     b(1,2) = -0.0000015 (-1.00)

x(3)=92020      b(1,3) = 0.3999260  (1.77)*

x(4)=92026      b(1,4) = 0.0295640  (0.05)*

x(5)=92054      b(1,5) = 0.5776539  (2.39)*

x(6)=92103      b(1,6) = 0.1253653  (0.38)*

x(7)=92104      b(1,7) = 0.0638880  (0.33)*

x(8)=92105      b(1,8) = 0.3418127  (2.05)*

x(9)=92115      b(1,9) = 0.2665826  (1.35)*

x(10)=91977     b(1,10) = 0.3360526  (0.89)*

x(11)=1        b(1,11) = -1.3377389 (-14.34)

[The two-sided p-values are based on the normal approximation]
*Significant at the 95% level

Log likelihood: -1.42615693500E+003
Sample size (n): 2001

Test for joint significance of location variables
<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter</th>
<th>Coefficient</th>
<th>Standard Error</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{10}$=91977</td>
<td>$b(2,10)$</td>
<td>1.0754464</td>
<td>0.0269658</td>
<td>(1.68) (*)</td>
</tr>
<tr>
<td>$x_{11}$=1</td>
<td>$b(2,11)$</td>
<td>-2.9550608</td>
<td>0.0000155</td>
<td>(-14.04)</td>
</tr>
<tr>
<td>$x_{1}$=Units</td>
<td>$b(3,1)$</td>
<td>0.0269658</td>
<td>(-1.02)</td>
<td></td>
</tr>
<tr>
<td>$x_{2}$=LotSz</td>
<td>$b(3,2)$</td>
<td>-0.0000155</td>
<td>(-1.21)</td>
<td></td>
</tr>
<tr>
<td>$x_{3}$=92020</td>
<td>$b(3,3)$</td>
<td>3.1081530</td>
<td>(2.46) (*)</td>
<td></td>
</tr>
<tr>
<td>$x_{4}$=92026</td>
<td>$b(3,4)$</td>
<td>4.5332218</td>
<td>(2.82) (*)</td>
<td></td>
</tr>
<tr>
<td>$x_{5}$=92054</td>
<td>$b(3,5)$</td>
<td>3.550368</td>
<td>(2.67) (*)</td>
<td></td>
</tr>
<tr>
<td>$x_{6}$=92103</td>
<td>$b(3,6)$</td>
<td>2.7683421</td>
<td>(1.91) (*)</td>
<td></td>
</tr>
<tr>
<td>$x_{7}$=92104</td>
<td>$b(3,7)$</td>
<td>1.6994085</td>
<td>(1.17) (*)</td>
<td></td>
</tr>
<tr>
<td>$x_{8}$=92105</td>
<td>$b(3,8)$</td>
<td>2.2123634</td>
<td>(1.76) (*)</td>
<td></td>
</tr>
<tr>
<td>$x_{9}$=92115</td>
<td>$b(3,9)$</td>
<td>1.8194135</td>
<td>(1.27) (*)</td>
<td></td>
</tr>
<tr>
<td>$x_{10}$=91977</td>
<td>$b(3,10)$</td>
<td>3.4648033</td>
<td>(2.38) (*)</td>
<td></td>
</tr>
<tr>
<td>$x_{11}$=1</td>
<td>$b(3,11)$</td>
<td>-6.8796406</td>
<td>(-6.28)</td>
<td></td>
</tr>
</tbody>
</table>

Test of the null hypothesis that the parameters indicated by (*) are jointly zero:

Wald test: 29.74
Asymptotic null distribution: Chi-square(24)
p-value = 0.19343
Significance levels: 10% 5%
Critical values: 33.2 36.42
Conclusions: accept accept
Results of Binomial logit test of effect of location and characteristics on the fact that a property is resold more than once in 15 years

Logit model:
Model variables:

<table>
<thead>
<tr>
<th>Variable</th>
<th>ML estimate of $b(.)$</th>
<th>(t-value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(1)=\text{Units}$</td>
<td>$b(1)= -0.0021930$</td>
<td>(-0.67)</td>
</tr>
<tr>
<td>$x(2)=\text{LotSz}$</td>
<td>$b(2)= 0.0000020$</td>
<td>(1.31)</td>
</tr>
<tr>
<td>$x(3)=91977$</td>
<td>$b(3)= -0.5182681$</td>
<td>(-1.54)</td>
</tr>
<tr>
<td>$x(4)=92020$</td>
<td>$b(4)= -0.4716261$</td>
<td>(-2.26)*</td>
</tr>
<tr>
<td>$x(5)=92026$</td>
<td>$b(5)= -0.2974455$</td>
<td>(-0.61)</td>
</tr>
<tr>
<td>$x(6)=92054$</td>
<td>$b(6)= -0.6385785$</td>
<td>(-2.84)*</td>
</tr>
<tr>
<td>$x(7)=92103$</td>
<td>$b(7)= -0.3792964$</td>
<td>(-1.34)</td>
</tr>
<tr>
<td>$x(8)=92104$</td>
<td>$b(8)= -0.0827083$</td>
<td>(-0.46)</td>
</tr>
<tr>
<td>$x(9)=92105$</td>
<td>$b(9)= -0.3735603$</td>
<td>(-2.41)*</td>
</tr>
<tr>
<td>$x(10)=92115$</td>
<td>$b(10)= -0.2881367$</td>
<td>(-1.57)</td>
</tr>
<tr>
<td>$x(11)=1$</td>
<td>$b(11)= 1.1616644$</td>
<td>(13.24)</td>
</tr>
</tbody>
</table>

[The two-sided p-values are based on the normal approximation]

*Significant at the 95% level

Log likelihood: $-1.14944956193E+003$
Sample size (n): 2001
Test for joint significance of location variables

\[
\begin{align*}
\text{x(1)=Units} & \quad b(1)= -0.0021930 (-0.67) \\
\text{x(2)=LotSz} & \quad b(2)= 0.0000020 (1.31) \\
\text{x(3)=91977} & \quad b(3)= -0.5182681 (-1.54) (*) \\
\text{x(4)=92020} & \quad b(4)= -0.4716261 (-2.26) (*) \\
\text{x(5)=92026} & \quad b(5)= -0.2974455 (-0.61) (*) \\
\text{x(6)=92054} & \quad b(6)= -0.6385785 (-2.84) (*) \\
\text{x(7)=92103} & \quad b(7)= -0.3792964 (-1.34) (*) \\
\text{x(8)=92104} & \quad b(8)= -0.0827083 (-0.46) (*) \\
\text{x(9)=92105} & \quad b(9)= -0.3735603 (-2.41) (*) \\
\text{x(10)=92115} & \quad b(10)= -0.2881367 (-1.57) (*) \\
\text{x(11)=1} & \quad b(11)= 1.1616644 (13.24)
\end{align*}
\]

Test of the null hypothesis that the parameters indicated by (*) are jointly zero:

Wald test: 17.26
Asymptotic null distribution: Chi-square(8)
p-value = 0.02754
Significance levels: 10% 5%
Critical values: 13.36 15.51
Conclusions: reject reject
Regression results for test of significance of change in capitalization rate on holding period

Regression below was performed on a data set covering San Diego County over the period March, 1983 - August, 1998.

Regression Statistics
- Multiple R: 0.338810805
- R Square: 0.114792761
- Adjusted R Square: 0.114315818
- Standard Error: 992.294305
- Observations: 1858

ANOVA

<table>
<thead>
<tr>
<th></th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>Significance F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>1</td>
<td>236989178.6</td>
<td>236989178.6</td>
<td>240.6841649</td>
<td>3.92257E-51</td>
</tr>
<tr>
<td>Residual</td>
<td>1856</td>
<td>1827506665</td>
<td>984647.9877</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>1857</td>
<td>2064495844</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Coefficients

<table>
<thead>
<tr>
<th></th>
<th>Coefficients</th>
<th>Standard Error</th>
<th>t Stat</th>
<th>P-value</th>
<th>Lower 95%</th>
<th>Upper 95%</th>
<th>Lower 95.0%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>1233.367196</td>
<td>32.27933615</td>
<td>38.20918714</td>
<td>3.6523E-236</td>
<td>1170.059559</td>
<td>1296.674833</td>
<td>1170.059559</td>
</tr>
<tr>
<td>CRDum</td>
<td>714.3907451</td>
<td>46.04813661</td>
<td>15.513999</td>
<td>3.92257E-51</td>
<td>624.0791379</td>
<td>804.7023523</td>
<td>624.0791379</td>
</tr>
</tbody>
</table>
Regression below was performed on a data set covering San Diego County over the period March, 1983 - December, 1991.

Regression Statistics
Multiple R 0.03560021
R Square 0.001267375
Adjusted R Square 0.000132452
Standard Error 551.437618
Observations 882

ANOVA

<table>
<thead>
<tr>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>Significance F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>1</td>
<td>339571.5831</td>
<td>339571.5831</td>
<td>1.116705257</td>
</tr>
<tr>
<td>Residual</td>
<td>880</td>
<td>267593432.9</td>
<td>304083.4465</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>881</td>
<td>267933004.5</td>
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Coefficients

<table>
<thead>
<tr>
<th></th>
<th>Coefficients</th>
<th>Standard Error</th>
<th>t Stat</th>
<th>P-value</th>
<th>Lower 95%</th>
<th>Upper 95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>1004.64759</td>
<td>21.39992985</td>
<td>46.94630297</td>
<td>7.3646E-242</td>
<td>962.6467552</td>
<td>1046.648426</td>
</tr>
<tr>
<td>CRDum</td>
<td>-45.4870399</td>
<td>43.04457216</td>
<td>-1.056742758</td>
<td>0.290919013</td>
<td>-129.9689981</td>
<td>38.99491831</td>
</tr>
</tbody>
</table>
Regression below was performed on a data set covering only three zip codes (92103, 92104 and 92116) in San Diego County over the period 1970 - 1990 inclusive.

Regression Statistics

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiple R</td>
<td>0.164565943</td>
</tr>
<tr>
<td>R Square</td>
<td>0.02708195</td>
</tr>
<tr>
<td>Adjusted R Square</td>
<td>0.025280249</td>
</tr>
<tr>
<td>Standard Error</td>
<td>1336.77076</td>
</tr>
<tr>
<td>Observations</td>
<td>542</td>
</tr>
</tbody>
</table>

ANOVA

<table>
<thead>
<tr>
<th></th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>Significance F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>1</td>
<td>26860327.21</td>
<td>26860327.21</td>
<td>15.0313305</td>
<td>0.000118762</td>
</tr>
<tr>
<td>Residual</td>
<td>540</td>
<td>964956275.3</td>
<td>1786956.065</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>541</td>
<td>991816602.6</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Coefficients

<table>
<thead>
<tr>
<th></th>
<th>Coefficients</th>
<th>Standard Error</th>
<th>t Stat</th>
<th>P-value</th>
<th>Lower 95%</th>
<th>Upper 95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>2045.479769</td>
<td>101.6327989</td>
<td>20.12617768</td>
<td>1.23128E-67</td>
<td>1845.835593</td>
<td>2245.123944</td>
</tr>
<tr>
<td>CRDUM</td>
<td>-477.5502295</td>
<td>123.174369</td>
<td>-3.877025987</td>
<td>0.000118762</td>
<td>-719.5099671</td>
<td>-235.5904918</td>
</tr>
</tbody>
</table>
This notebook addresses one of the little noticed variables in real estate projection and analysis - holding period. While equity growth rates, capitalization rates, interest rates and changes in operating inputs (rents, expenses, vacancy, etc.) are routinely varied, holding period is often assumed to be fixed. Modeling an acquisition based on a 5 or 10 year holding period is common. It would seem that the analysis would be improved by "tuning" holding period to market cycles. The technique used here will consider the direction of capitalization rate movement (up or down) during holding period as the index of the market cycle.

Variable definitions are:

- \( er \) = equity reversion (full value as we assume no debt) calculated as \( \frac{cf_1}{cr_i} \)
- \( npv \) = net present value
- \( pd \) = initial investment, made at \( t=0 \). We initially assume that property is debt free
- \( cf_0 \) = initial cash flow to be received at the end of year 1
- \( cf_n \) = subsequent year cf, compounded annually based on growth rate, \( g \)
- \( cr_i \) = "going in" capitalization rate
- \( cr_o \) = "going out" capitalization rate
- \( r \) = discount rate (when \( pv = 0 \) this is the internal rate of return)
- \( g \) = growth rate for cash flow, as described above
- \( k \) = length holding period (year of sale, presumed to be at the end of the year)

Compounding of cash flows is offset one year. The idea is that an investor on January 1 makes a decision to purchase based on his expectation of the income to be received at the end of the year. Thus, \( cf_0 \) is the initial cash flow to be received at the end of year one. The first year of compounding is 12 months later at the end of year two, hence only \( n-1 \) years of compounding cash flows are in a holding period. This matter has repercussions when considering the equity reversion at the time of sale. Value still compounds for the full \( n \) periods. The rationale is that at the time of sale, the next investor anticipates the end-of-year income (to be received 12 months after his acquisition) in making his acquisition decision. So the first investor collects the first cash flow without any compounding and \( n-1 \) years of compounding cash flows but prices the property on the basis of \( n \) compounded cash flows.

The examples below assume a going-in cap rate of 8%, a ten year holding period, collection of a first year \( (cf_0) \) cash flow and 9 years of cash flows compounding at the rate of 4%, followed by a sale based on the next year's (yet-to-be-collected) cash flow at a going-out cap rate of 10%. The discount rate is 10%. The examples are then modified to show what happens when the direction of cash flows reverses, when growth rates are cyclical and when holding period is extended.
For each illustration below, several outputs are provided: The equation for present value is illustrated without specifying values for r or k, then inserting a value for r, then inserting a value for both r and k. For illustration in two dimensions, all but two must be fixed. For illustration in three dimensions, all but three must be fixed. For simplicity initial cash flows are all shown at $500

\[
cfn = cf0 \cdot (1 + g)^{n-1};
\]
\[
er = cf0 \cdot (1 + g)^k / cro;
\]
\[
pd = cf0 / cri;
\]
\[
npv := \frac{er}{(1 + r)^k} + \left( \sum_{k=1}^{n} \text{Evaluate} \left[ \frac{cfn}{(1 + r)^n} \right] \right) - pd
\]

\[nvp1 = npv /. \{g -> .04, cf0 -> 500, cri -> .08, cro -> .1\}\]
\[nvp1a = npv /. \{g -> .04, cf0 -> 500, cri -> .08, cro -> .1, r -> .1\}\]
\[nvp1b = npv /. \{g -> .04, cf0 -> 500, cri -> .08, cro -> .1, r -> .1, k -> 10\}\]
\[nvp1c = npv /. \{g -> .04, cf0 -> 500, cri -> .08, cro -> .1, r -> .1, k -> 10, n -> 1\}\]

\[-6250. + 5000. \cdot 1.04^k (1 + r)^{-k} + \frac{500 \cdot (-1 + 1.04^k \cdot \left( \frac{1}{1.04} \right)^k)}{0.04 - r} \]

\[-6250. + 5000. \cdot 0.945455^k - 8333.33 \cdot (-1 + 0.945455^k)\]

\[181.006\]
\[181.006\]

Below, the limit of the sum of discounted cash flows is merely the sum of the uncompounded cash flows when the growth in income is equal to the discount rate because the factor \((1 + r)^{n-1}\) appears in both the numerator and the denominator, thus canceling and leaving only \(\sum cf0 \cdot k\)

The message here, not surprisingly, is that \(r\) and \(g\) must be different to make this process meaningful. Of course, they always are because they are quite different concepts. "\(g\)" is the result of local macroeconomic factors, the microeconomy of the neighborhood and [central to our story] the entrepreneurial effort of the owner to deal with site specific risk. "\(r\)" is the cost of capital influenced by many national factors such as expected inflation, interest rates, the return on other investments such as financial assets, etc.

\(?\) Limit

\(\text{Limit[expr, x->x0]}\) finds the limiting value of \(\text{expr}\) when \(x\) approaches \(x0\).
Checking the cash flows individually, here is the 9th cash flow in our example. For Excel users it may be comforting to set this example up in columns and rows, use Excel functions and arrive at the same answers as the next two below:

\[ \text{cf0} \cdot k \]

Checking the example's reversion and its discounted present value:

\[ \text{cf0} \cdot (1 + g)^k / \text{cro} \cdot .1, \text{cf0} \rightarrow 500, g \rightarrow .04, n \rightarrow 10, k \rightarrow 10 \]
\[ \% / (1 + r)^k \cdot .1, r \rightarrow .1, k \rightarrow 10 \]

7401.22
2853.49

Returning to the general, we plot present values against various levels of \( r \) and \( k \):
As expected, pv rises with longer holding periods and lower discount rates. In the combined graph below, where pv = 0, the intersection with the pv function shows what happens to zero NPV outcomes as holding period and discount rates change.
Next, the plot of the present value function over time at different discount rates shows the expected monotonic increase of all plots with the higher plots at the lower discount rates. The reason this is expected is that the asset is constantly increasing in value at a rate greater than the discount rate is reducing it. Note that
early years have an NPV less than zero. Recall that for npv1 the rate of growth, \( g \), is given at 4\%. The steepness of the curve reflects the difference between the rate of growth and the discount rate.

\[
\text{Plot}[[\text{npv1} \rightarrow 0.025, \text{npv1} \rightarrow 0.05, \text{npv1} \rightarrow 0.075, \text{npv1} \rightarrow 0.1], \\
\{k, 1, 10\}, \text{Frame} \rightarrow \text{True}, \text{PlotStyle} \rightarrow \\
\{\{\text{Hue}[0.1], \text{AbsoluteThickness}[3]\}, \{\text{Hue}[0.3], \text{AbsoluteThickness}[3]\}, \\
\{\text{Hue}[0.5], \text{AbsoluteThickness}[3]\}, \{\text{Hue}[0.7], \text{AbsoluteThickness}[3]\}\}, \\
\text{PlotLegend} \rightarrow \{"r = 0.025", 0.05, 0.075, 0.10\}, \text{LegendPosition} \rightarrow \{1.2, -0.5\}, \\
\text{LegendSize} \rightarrow \{0.6, 0.8\}, \text{FrameLabel} \rightarrow \{"Years", "Present Value"\}]
\]

Up until now we have assumed a given rate of cash flow growth and plotted net present value against changing discount rates and length of holding period. We now turn to the alternative, taking a discount rate as given and plotting npv against rate of cash flow growth and length of holding period. The results are similar.

\[
\text{npv2} = \text{npv} /. \{r \rightarrow 0.05, \text{cf0} \rightarrow 500, \text{cri} \rightarrow 0.08, \text{cro} \rightarrow 0.1\} // \text{Simplify}
\]

\[
-6250 + 5000 \cdot 0.952381^k (1 + g)^k + \frac{500 \cdot (-1 + 0.952381^k (1 + g)^k)}{-0.05 + g}
\]
pkr3 = Plot3D[npv2, {k, 1, 10}, {g, 0.02, 0.10},
    BoxRatios -> {1, 1, 1.5}, ViewPoint -> {-1.83, -2.2, 1.58}, AxesLabel ->
    {"holding period", "cf growth rate", "npv"}]

- SurfaceGraphics -
What follows are some rough sensitivity tests to find out what happens to npv when variables change. To check range issues we change the example inputs to reflect more realistic values. Typically, discount rates are 15 - 20 percent per annum. When we use the lower bound of this we have a negative net present value. These assets will not be sold unless expectations are that cap rates will fall over the holding period and/or growth rates are assumed to be higher.
\[
npv3 = npv / \{ g \to .04, cf0 \to 500, cri \to .08, cro \to .1 \}
\]
\[
npv3a = npv / \{ g \to .04, cf0 \to 500, cri \to .08, cro \to .1, r \to .15 \}
\]
\[
npv3b = npv / \{ g \to .04, cf0 \to 500, cri \to .08, cro \to .1, r \to .15, k \to 10 \}
\]

\[-6250 + 5000 \cdot 1.04^k (1 + r)^{-k} + \frac{500 \left( -1 + 1.04^k \left( \frac{1}{1+r} \right)^k \right)}{0.04 - r} \]

\[-6250 + 5000 \cdot 0.904348^k - 4545.45 \left( -1 + 0.904348^k \right) \]

\[-1538.23\]
A reversal of the direction of cap rate movement is one solution. When going in cap rate is 10% and going out cap rate is 8% we return to a positive npv:
\[\text{npv4} = \text{npv} /. \{g \rightarrow .04, \text{cf}0 \rightarrow 500, \text{cri} \rightarrow .1, \text{cro} \rightarrow .08\}\]
\[\text{npv4a} = \text{npv} /. \{g \rightarrow .04, \text{cf}0 \rightarrow 500, \text{cri} \rightarrow .1, \text{cro} \rightarrow .08, r \rightarrow .15\}\]
\[\text{npv4b} = \text{npv} /. \{g \rightarrow .04, \text{cf}0 \rightarrow 500, \text{cri} \rightarrow .1, \text{cro} \rightarrow .08, r \rightarrow .15, k \rightarrow 10\}\]

\[-5000 + 6250 \cdot 1.04^k (1 + r)^{-k} + \frac{500 \left(-1 + 1.04^k \left(\frac{1}{1 + r}\right)^k\right)}{0.04 - r}\]

\[-5000 + 6250 \cdot 0.904348^k - 4545.45 \left(-1 + 0.904348^k\right)\]

169.137
When cap rates are rising from 8% to 10% over the holding period we can overcome this with an extremely high cash flow growth (an unrealistic rate in the neighborhood of 9% per annum is required to return to a positive npv with a 15% discount rate). Below, with cap rates unchanged over the holding period, an income growth rate of 5.5% is necessary to return to a positive npv, although this is true only below 15% discount rates.
\begin{verbatim}
npv5 = npv /. {g -> .055, cf0 -> 500, cri -> .1, cro -> .1}
npv5a = npv /. {g -> .055, cf0 -> 500, cri -> .1, cro -> .1, r -> .15}
npv5b = npv /. {g -> .055, cf0 -> 500, cri -> .1, cro -> .1, r -> .15, k -> 10}

-5000. + 5000. 1.055^k (1 + r)^k -5000. + 5000. 0.917391^k - 5263.16 (-1 + 0.917391^k)

152.045
\end{verbatim}
What we have yet to explore is how changes in holding period affect maladjusting capitalization rates and low income growth rates. Returning to the 4% income growth and "no change" cap rate conditions and doubling the holding period still produces a negative NPV with a 15% discount rate. Further tests show that \( npv = 0 \) at about a 14% discount rate.
\[ npv6 = npv /\{g \rightarrow .04, cf0 \rightarrow 500, cri \rightarrow .1, cro \rightarrow .1\} \]
\[ npv6a = npv /\{g \rightarrow .04, cf0 \rightarrow 500, cri \rightarrow .1, cro \rightarrow .1, r \rightarrow .14\} \]
\[ npv6b = npv /\{g \rightarrow .04, cf0 \rightarrow 500, cri \rightarrow .1, cro \rightarrow .1, r \rightarrow .14, k \rightarrow 20\} \]

\[
-5000. + 5000. 1.04^k (1 + r)^{-k} + \frac{500 \left(-1 + 1.04^k \left(\frac{1}{1+r}\right)^k\right)}{0.04 - r}
\]

\[
-5000. + 5000. 0.912281^k - 5000. (-1 + 0.912281^k)
\]

\[-9.09495 \times 10^{-13}\]
One of the abuses of discounted cash flow analysis is the use of monotonic growth rates. Rather than assume that growth rates never change, we model it as a periodic function with cycles keyed to a sine wave such as the one below:
cycle1 = 0.04 + 0.04 Sin[2 Pi n / 3]; sinplot1 = Plot[cycle1, {n, 0, 10}]

Note that we can change the length of the cycle by reducing the divisor...

cycle2 = 0.04 + 0.04 Sin[2 Pi n / 2];
sinplot2 = Plot[cycle2, {n, 0, 10}, PlotStyle -> {Hue[0.6]}]

...and we can change the severity of the cycle by changing the coefficient
When plotted together we see three very different types of cycles

Using cycle1 for change in income making holding period, n, the variable in the growth function, we have a cyclical growth pattern with an upward drift. Clearly, the timing of exit is important. Substitution of cycles 2 or 3 (not shown here but possible for those with Mathematica) produces meaningfully different result, especially in later years.
The two dimensional plot, below, of present value at different stages given different discount rates, shows that present value increases and the spread between different outcome increases as holding period increases. The example below uses cycle1 and a change to different frequency and amplitude of the cycle is, of course, a very important one.
Below, the fixed portion of the cash flow growth rate is removed. Note that negative present values dominate in the early years with negative PVs the rule throughout an entire ten year holding period at higher discount rates.
npv8 := npv /. \{g \to .04 \text{Sin}[2 \pi n/3], cf0 \to 500, cri \to .08, cro \to .1\} // Simplify;

npv9 = npv8 /. n \to k;

pkr10 = Plot3D[npv9, \{k, 1, 10\}, \{r, .02, .10\},
BoxRatios \to \{1, 1, 1.5\}, ViewPoint \to \{-2.630, -2.200, 1.980\},
AxesLabel \to \{"holding period", "discount rate", "npv"\}]

- SurfaceGraphics -
The next inquiry is about the effect of a capitalization rate changing over the holding period. With an oscillating growth in income, what can we say about the holding period effect? The intuition is that investors delay their sales until values are up and npv is positive. One way for this to be illustrated is to have the cap rate going in the "right" direction at the time of sale.

We return to the condition of the fixed base of .04 for the g function and reverse capitalization rate direction, going in at 10% and out at 8%. The graph is the same shape but all of it is higher on the npv axis with all PVs positive. We really have done little more here than illustrate the mathematical nature of "buy low, sell high" where "low" and "high" are determined by the relationship between going-in and going-out cap rates.
\[ \text{npv9} := \text{npv} /\{g \to 0.04 + 0.04 \sin[2 \pi n/3], \text{cf0} \to 500, \text{cri} \to 1, \text{cro} \to 0.08\} \]
Simplify;

\[ \text{npv10} = \text{npv9} /\{n \to k\}; \]

\[ \text{pkr11} = \text{Plot3D}[\text{npv10}, \{k, 1, 10\}, \{r, 0.02, 0.10\}, \]
BoxRatios \to \{1, 1, 1.5\}, ViewPoint \to \{-2.630, -2.200, 1.980\},
AxesLabel \to \{"holding period", "discount rate", "npv\}]; \]
We return to the "rising cap rate" condition that the investor so wants to avoid. This time we use more realistic discount rates, plotting \( r \) over a range of 12 - 18% producing substantially all negative npv results.

```
Plot[{npv10 /. r -> .025, npv10 /. r -> .05, npv10 /. r -> .075, npv10 /. r -> .1}, {k, 1, 10}, PlotStyle ->
  {{Hue[.1], AbsoluteThickness[3]}, {Hue[.3], AbsoluteThickness[3]},
   {Hue[.5], AbsoluteThickness[3]}, {Hue[.7], AbsoluteThickness[3]}},
  PlotLegend -> {"r=0.025", 0.05, 0.075, 0.10},
  LegendPosition -> {1.2, -.5}, LegendSize -> {.6, .8},
  AxesLabel -> {"Years", "Present Value"}]
```
npv10 := npv /. 
   {g -> .04 + .04 Sin[2 \pi n / 3], cf0 -> 500, cri -> .08, cro -> .1} // Simplify; 
npv11 = npv10 /. n -> k; 
pkr12 = Plot3D[npv11, {k, 1, 10}, {r, .12, .18}, 
   BoxRatios -> {1, 1, 1.5}, ViewPoint -> {-2.630, -2.200, 1.980}, 
   AxesLabel -> {"holding period", "discount rate", "npv"}]
Where does this leave us? We have eliminated some of the more obvious situations and we see that the critical path to a positive NPV is through the "proper" direction of cap rate over the holding period. The investor extends his holding period until he enjoys a market in which the cap rate upon his exit is lower than the cap rate he used to value the property at the time of purchase.

Some of the complexity of this arises from the three way connection between three rates, $r$, $g$ and $cr$:

1. Increasing cash flow at a high rate of growth, $g$, delivers expectations of more rent raises, encouraging buyers to pay more (accepting the seller's lower cap rate) in anticipation of greater future cash flow.

2. Cap rates and discount rates are related as they both represent proxies for the cost of capital and the "shadow price" of other investments.

3. The direction of going in and going out cap rates has a substantive effect on the final $r$ and therefore the NPV.

For a century of real estate analysis the assumption has been that holding period
is fixed and \textit{a priori} known. Everyone recognizes that such an assumption is faulty, but no one has known what to do about it. A few - a very few - have suggested that going in and going out cap rates are different, but nobody has ever linked cap rates and holding periods in a structured way.

We now have the capability of modeling not only variation in holding period but "tuning" holding period to market cycles. Investors whose business acumen foments accurate projections always succeed because "timing is everything". The rest of us can employ the tools like those above to assess the risk of being wrong about our probability beliefs in the area of capitalization rates and market cycles. By looking closely at holding period we can also assess the time it may take to "ride out our mistakes".

There are numerous implications to this. One of the more obvious and more important is the setting of maturity dates on financing. Another is the effect on representations to investors about expected time horizons. Tools such as these, once in wide circulation, become part of proper due diligence and are soon considered to be included in the minimum industry standard of care one should use prior to making investment recommendations.
### Appendix F

**DATA BY ZIP CODE**

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Variable | N | Mean | Std Dev | Minimum | Maximum
---------|---|------|---------|---------|--------
REP=2     | 1478 | 29  | 45      | 5       | 448    |
| HP       | 1477 | 148 | 1105    | 16      | 5173   |
| SALE_PR  | 1478 | 1375959 | 2717990 | 127000 | 34425000 |

REP=3     | 440  | 26  | 38      | 5       | 385    |
| HP       | 440  | 1527 | 941     | 1       | 4687   |
| SALE_PR  | 440  | 1046174 | 1989654 | 113500 | 31900000 |

REP=4     | 87   | 20  | 20      | 7       | 114    |
| HP       | 87   | 1360 | 867     | 1       | 3637   |
| SALE_PR  | 87   | 692114 | 692176  | 165000 | 4000000 |

REP=5     | 11   | 29  | 30      | 8       | 105    |
| HP       | 11   | 1170 | 626     | 302     | 2436   |
| SALE_PR  | 11   | 935773 | 918917  | 170000 | 2800000 |
Appendix G

SIMULATOR AND RANDOM NUMBER GENERATOR FOR STABLE DISTRIBUTIONS

Chapter 5 illustrates various "efficient" frontiers that may be created when $\alpha < 2$ and $\beta > 0$. Those illustrations employed the random number generator in the Fortran program, Stable.exe provided by Nolan (1997). This notebook replicates that random number generator, permitting the faster creation of a wider variety of possible portfolios. This notebook is in the experimental phase. Enhancements are expected in the future. Monitoring the author's Web site for updates is suggested.

The procedure is as follows:

1. Generate several series of stable quasi-random variables using the method of McCulloch (1998) based on Chambers, Mallows and Stuck (1976) using any value for $\alpha \in [1,2]; \beta \in [-1,1]$.
2. Form a portfolio from three sets, each with $\alpha = 2, \beta = 0$ (asset201, asset 202, asset 203), draw the efficient frontier.
3. Repeat the process, each time with a different $\alpha$. (numbering sets 141, 142,143 for $\alpha = 1.4$ asset1, $\alpha = 1.4$ asset2, $\alpha = 1.4$ asset3, etc.)

The goal is to show the change in the "efficient" frontier as $\alpha$ declines.

Stable Random variates are computed using Chambers, Mallows and Stuck (1976) using the form and notation of McCulloch (1998) p.373, Adler, et al, editors

It is necessary to use Mathematica's Finance Essentials Pack to plot the efficient frontiers.
We require \(u\) and \(v\) to be two iid uniform \((0,1)\) quasi-random variables. We begin with two vectors of 1000 uniform random variables to serve as \(u\) and \(v\):

```mathematica
fxran :=
  
  \[ u = \text{Table}[	ext{Random[]}, \{1000\}]; \]
  
  \[ v = \text{Table}[	ext{Random[]}, \{1000\}]; \]
  
  \[ w = \log[1/u]; \]
  
  \[ \theta = \pi (v - .5); \]
  
  \[ w = \cos[(1 - \alpha) \theta] + \beta \tan[\frac{\pi \alpha}{2}] \sin[(1 - \alpha) \theta]; \]
  
  \[ z = \frac{w \cos[\theta]}{\sin[\alpha \theta] + \beta \tan[\frac{\pi \alpha}{2}] \cos[\alpha \theta]} \left( 1 + \frac{\sigma^2}{\alpha} \right); \]

\[
\begin{align*}
\text{asset201} &= \text{fxran} /\{\alpha \to 2, \beta \to 0\}; \\
\text{TableForm[} &\{\text{Mean[asset201]}, \text{Max[asset201]}, \text{Min[asset201]}, \text{Variance[asset201]}\}, \\
\text{TableHeadings} \to \{\{"Mean", "Max", "Min", "\sigma^2"}\}\}
\end{align*}
\]

\[
\begin{align*}
\text{Mean} &= -0.031602 \\
\text{Max} &= 5.38354 \\
\text{Min} &= -4.69863 \\
\sigma^2 &= 2.05703
\end{align*}
\]

\[
\begin{align*}
\text{asset202} &= \text{fxran} /\{\alpha \to 2, \beta \to 0\}; \\
\text{TableForm[} &\{\text{Mean[asset202]}, \text{Max[asset202]}, \text{Min[asset202]}, \text{Variance[asset202]}\}, \\
\text{TableHeadings} \to \{\{"Mean", "Max", "Min", "\sigma^2"\}\}\}
\end{align*}
\]

\[
\begin{align*}
\text{Mean} &= 0.0629079 \\
\text{Max} &= 3.73526 \\
\text{Min} &= -3.60465 \\
\sigma^2 &= 1.90033
\end{align*}
\]
asset203 = fxran /. \{\alpha \to 2, \beta \to 0\};

TableForm[
{Mean[asset203], Max[asset203], Min[asset203], Variance[asset203]},
TableHeadings \to \{\"Mean\", \"Max\", \"Min\", \"\sigma^2\"\}]

Mean       -0.0139417
Max        4.20358
Min        -4.74127
\sigma^2   2.16015

We now form a portfolio of these three assets, each having returns that are normally distributed

pflo20 = Transpose[\{asset201, asset202, asset203\}];

We now have three assets, each with 1000 return observations, forming a portfolio for which we can compute a return vector (meanvect) and covariance matrix (cov), determine efficient set portfolios and plot.

MeanVector[pflo20] // MatrixForm

\[
\begin{pmatrix}
-0.031602 \\
0.0629079 \\
-0.0139417
\end{pmatrix}
\]

cov20 = CovarianceMatrix[pflo20]

\[
\begin{pmatrix}
2.05497 & 0.0821849 & -0.00822309 \\
0.0821849 & 1.89843 & -0.0612739 \\
-0.00822309 & -0.0612739 & 2.15799
\end{pmatrix}
\]

EfficientFrontier[MeanVector[pflo20], cov20, x];
\( p20 = \text{ParametricPlot[} \)
\( \text{EfficientFrontier[MeanVector[pflo20], cov20, x], \{x, -1, 1\}] \)

ParametricPlot::ppcom : Function EfficientFrontier[MeanVector[pflo20], cov20, x]
\text{cannot be compiled; plotting will proceed with the uncompiled function.}

We now require a stable NON NORMAL draw where \( \alpha = 1.4 \) and \( \beta = 0 \)

\( \text{asset141 = fxran /. \{\alpha \to 1.4, \beta \to 0\};} \)
\( \text{TableForm[\{Mean[asset141], Max[asset141], Min[asset141],} \)
\( \text{Variance[asset141]\}, TableHeadings \to \{\{"Mean", "Max", "Min", "\sigma^2\"\}\};} \)

\( \text{asset142 = fxran /. \{\alpha \to 1.4, \beta \to 0\};} \)
\( \text{TableForm[} \)
\( \text{\{Mean[asset142], Max[asset142], Min[asset142], Variance[asset142]\},} \)
\( \text{TableHeadings \to \{\{"Mean", "Max", "Min", "\sigma^2\"\}\]} \)

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<td>Min</td>
<td>-117.927</td>
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<td>( \sigma^2 )</td>
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asset143 = fxran /. \{\alpha \to 1.4, \beta \to 0\};

TableForm[
  \{Mean[asset143], Max[asset143], Min[asset143], Variance[asset143]\},
  TableHeadings -> {"Mean", "Max", "Min", "\(\sigma^2\)"}]

Mean  0.0391002
Max   115.539
Min   -55.1529
\(\sigma^2\)  32.1088

We now form a portfolio of these three assets, each having returns that are stable distributed with \(\alpha = 1.4, \beta = 0\)

pflo14 = Transpose[\{asset141, asset142, asset143\}];

We now have three assets, each with 1000 return observations, forming a portfolio for which we can compute a return vector (meanvect) and covariance matrix (cov), determine efficient set portfolios and plot.

MeanVector[pflo14] // MatrixForm

\[
\begin{pmatrix}
-0.244499 \\
-0.305473 \\
0.0391002
\end{pmatrix}
\]

cov14 = CovarianceMatrix[pflo14]

\[
\begin{pmatrix}
28.5618 & 0.113326 & 0.292627 \\
0.113326 & 23.9762 & 0.226258 \\
0.292627 & 0.226258 & 32.0767
\end{pmatrix}
\]

EfficientFrontier[MeanVector[pflo14], cov14, x];

pl4 = ParametricPlot[EfficientFrontier[MeanVector[pflo14], cov14, x],
  \{x, -1, 1\}, PlotStyle -> Hue[0.9]]

ParametricPlot::ppcom : Function EfficientFrontier[MeanVector[pflo14], cov14, x]
cannot be compiled; plotting will proceed with the uncompiled function.
We now require a stable NON NORMAL draw where $\alpha = 1.7$ and $\beta = 0$

\begin{verbatim}
asset171 = fxran /. \{\alpha \to 1.7, \beta \to 0\};
TableForm[
  \{Mean[asset171], Max[asset171], Min[asset171], Variance[asset171]\},
  TableHeadings -> {"Mean", "Max", "Min", "\sigma^2"}]
Mean 0.0430247
Max 28.9206
Min -21.4236
\sigma^2 6.63094

asset172 = fxran /. \{\alpha \to 1.7, \beta \to 0\};
TableForm[
  \{Mean[asset172], Max[asset172], Min[asset172], Variance[asset172]\},
  TableHeadings -> {"Mean", "Max", "Min", "\sigma^2"}]
Mean 0.151413
Max 43.3347
Min -12.5065
\sigma^2 7.1868
\end{verbatim}
\texttt{asset173 = \texttt{fxran} /. \{\alpha \rightarrow 1.7, \beta \rightarrow 0\};}

\texttt{TableForm[}
\texttt{\{Mean[asset173], Max[asset173], Min[asset173], Variance[asset173]\},}
\texttt{TableHeadings \rightarrow \{\{"Mean", "Max", "Min", \"\sigma^2\"\}\}}
\texttt{]}

\begin{tabular}{llll}
Mean & 0.0331019 \\
Max & 44.5818 \\
Min & -35.0795 \\
\sigma^2 & 7.2496 \\
\end{tabular}

We now form a portfolio of these three assets, each having returns that are stable distributed with $\alpha = 1.7$ and $\beta = 0$

\texttt{pflo17 = Transpose[\{asset171, asset172, asset173\]];}

We now have three assets, each with 1000 return observations, forming a portfolio for which we can compute a return vector (meanvec) and covariance matrix (cov), determine efficient set portfolios and plot.

\texttt{MeanVector[pflo17] // MatrixForm}

\begin{pmatrix}
0.0430247 \\
0.151413 \\
0.0331019 \\
\end{pmatrix}

\texttt{cov17 = CovarianceMatrix[pflo17]}

\begin{pmatrix}
6.62431 & 0.0345242 & 0.287894 \\
0.0345242 & 7.17961 & -0.0448546 \\
0.287894 & -0.0448546 & 7.24235 \\
\end{pmatrix}

\texttt{EfficientFrontier[MeanVector[pflo17], cov17, x];}

\texttt{p17 = ParametricPlot[EfficientFrontier[MeanVector[pflo17], cov17, x],}
\texttt{\{x, \texttt{-1, 1}\}, \texttt{PlotStyle \rightarrow Hue[.6]}}

\texttt{ParametricPlot::ppcom : Function \texttt{EfficientFrontier[MeanVector[pflo17], cov17, x]} cannot be compiled; plotting will proceed with the uncompiled function.}

- Graphics -

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We now require a stable NON NORMAL draw where $\alpha = 1.5$ and $\beta = 0$

```mathematica
asset151 = fxran /. {\alpha -> 1.5, \beta -> 0};
TableForm[
{Mean[asset151], Max[asset151], Min[asset151], Variance[asset151]},
TableHeadings -> {{"Mean", "Max", "Min", "\sigma^2"}}]
```

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0745317</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max</td>
<td>49.1665</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Min</td>
<td>-25.9035</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>11.1429</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

```mathematica
asset152 = fxran /. {\alpha -> 1.5, \beta -> 0};
TableForm[
{Mean[asset152], Max[asset152], Min[asset152], Variance[asset152]},
TableHeadings -> {{"Mean", "Max", "Min", "\sigma^2"}}]
```

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.104162</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max</td>
<td>58.8586</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Min</td>
<td>-26.8752</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>12.0163</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Graphics -
asset153 = fxran /. \{\alpha \to 1.5, \beta \to 0\};

TableForm[
  \{Mean[asset153], Max[asset153], Min[asset153], Variance[asset153]\},
  TableHeadings \to \{"Mean", "Max", "Min", "\sigma^2"\}]

Mean  0.41151
Max   356.237
Min   -47.0428
\sigma^2  135.267

We now form a portfolio of these three assets, each having returns that are normally distributed

pflo15 = Transpose[{asset151, asset152, asset153}];

We now have three assets, each with 1000 return observations, forming a portfolio for which we can compute a return vector (meanvect) and covariance matrix (cov), determine efficient set portfolios and plot.

MeanVector[pflo15] // MatrixForm

\begin{pmatrix}
  0.0745317 \\
  0.104162 \\
  0.41151
\end{pmatrix}

cov15 = CovarianceMatrix[pflo15]

\begin{pmatrix}
  11.1317 & 0.149934 & -1.16065 \\
  0.149934 & 12.0042 & -0.692109 \\
  -1.16065 & -0.692109 & 135.131
\end{pmatrix}

Clear[x]

EfficientFrontier[MeanVector[pflo15], cov15, x];

p15 = ParametricPlot[EfficientFrontier[MeanVector[pflo15], cov15, x],
  \{x, -1, 1\}, PlotStyle \to Hue[.3]]

ParametricPlot::ppcom : Function EfficientFrontier[MeanVector[pflo15], cov15, x]
cannot be compiled; plotting will proceed with the uncompiled function.
ShowLegend[Show[p20, p17, p15, p14, DisplayFunction -> Identity],
{{GrayLevel[.1], "α=2.0"}, {Hue[.9], "α=1.7"},
 {Hue[.3], "α=1.5"}, {Hue[.6], "α=1.4"}},
LegendPosition -> {1.2, -.5}, LegendSize -> {.6, .8}]

- Graphics -

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Education

The Pennsylvania State University, Ph.D. in Business Administration (Concentration: Real Estate, Minor: Economics)

San Diego State University, MBA, Emphasis Finance, 1994

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Employment

1972- Present - Self-Employed owner/operator of commercial investment real estate firm in San Diego, CA

Publications:


"Zijn onroerend goed aandelen onroerend goed of aandelen?" (Are REITs Stocks or Real Estate? A Review of the Issues) VOGON Journal (The Netherlands), January 1997

