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ANALYTIC STRUCTURES FOR THE INDEX THEORY OF $\mathbf{SL(3,C)}$

A Thesis in

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Abstract

If G is a connected Lie group, the Kasparov representation ring $KK^G(\mathbb{C}, \mathbb{C})$ contains a singularly important element—the γ-element—which is an idempotent relating the Kasparov representation ring of G with the representation ring of its maximal compact subgroup K. In the proofs of the Baum-Connes conjecture with coefficients for the groups $G = \mathrm{SO}_0(n, 1)$ ([Kas84]) and $G = \mathrm{SU}(n, 1)$ ([JK95]), a key component is an explicit construction of the γ-element as an element of G-equivariant K-homology for the space G/B, where B is the Borel subgroup of G.

In this thesis, we describe some analytical constructions which may be useful for such a construction of γ in the case of the rank-two Lie group $G = \mathrm{SL}(3, \mathbb{C})$. The inspiration is the Bernstein-Gel'fand-Gel'fand complex—a natural differential complex of homogeneous bundles over G/B. The reasons for considering this complex are explained in detail.

For $G = \mathrm{SL}(3, \mathbb{C})$, the space G/B admits two canonical fibrations, which play a recurring role in the analysis to follow. The local geometry of G/B can be modeled on the geometry of the three-dimensional complex Heisenberg group $\mathbb{H}$ in a very strong way. Consequently, we study the algebra of differential operators on $\mathbb{H}$. We define a two-parameter family $H^{(m,n)}(\mathbb{H})$ of Sobolev-like spaces, using the two fibrations of G/B.

We introduce fibrewise Laplacian operators Δ_X and Δ_Y on $\mathbb{H}$. We show that these operators satisfy a kind of directional ellipticity in terms of the spaces $H^{(m,n)}(\mathbb{H})$ for

certain values of (m, n), but also provide a counterexample to this property for another choice of (m, n). This counterexample is a significant obstacle to a pseudodifferential approach to the γ-element for $\mathrm{SL}(3, \mathbb{C})$.

Instead we turn to the harmonic analysis of the compact subgroup $K = \mathrm{SU}(3)$. Here, using the simultaneous spectral theory of the K-invariant fibrewise Laplacians on G/B, we construct a C^*-category $\mathcal{A}$ and ideals $\mathcal{K}_X$ and $\mathcal{K}_Y$ which are related to the canonical fibrations. We explain why these are likely natural homes for the operators which would appear in a construction of the γ-element.

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Acknowledgments

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Chapter 1

Introduction

One of the fundamental examples of a C^*-algebra is the reduced C^*-algebra of a discrete group. It is defined simply: if G is a discrete group then its reduced C^*-algebra C_r^*G is the norm-closed algebra of operators on $L^2(G)$ generated by the regular representation. In the last couple of decades it has been realized that several famous problems in classical topology and geometry could be transformed into questions about the K-theory of reduced C^*-algebras.

The limitation now is that, for a general discrete group G, the reduced C^*-algebra can be very complicated. The holy grail is the Baum-Connes Conjecture [BCH94], which relates the operator K-theory of C_r^*G to a quantity from classical topology. But the conjecture is only known for a relatively small class of discrete groups. For instance, it is not known for the group $\mathrm{SL}(3,\mathbb{Z})$.

In this thesis, we introduce some new tools which are likely to be useful for future work on the Baum-Connes Conjecture. The motivation is work of Kasparov. In [Kas84], Kasparov proved for the semisimple Lie groups $\mathrm{SO}_0(n,1)$ a strong generalization of the Baum-Connes conjecture which is hereditary, in that it passes to any closed subgroup.

With much work, the method was extended by Julg and Kasparov ([JK95]) to the groups $\mathrm{SU}(n,1)$. These groups, like $\mathrm{SO}_0(n,1)$, are rank-one simple Lie groups.

Generalizing the method to higher-rank semisimple Lie groups, however, has been a stumbling block.

In order to motivate the content of this thesis, it is necessary to understand the idea of Kasparov's proof. We will give a survey of that in Chapter 2. For the moment, let us paraphrase the proof in one sentence by saying that one of the key steps is the packaging of some classical homological data—in the case of $\mathrm{SO}_0(2n+1,1)$ it is the de Rham complex for the homogeneous space S^{2n}—into an analytical form—a Fredholm operator, with some additional properties. It is the resulting analytical data which allows us to compute the operator K-theory of the reduced C^*-algebra.

This package of analytical data is called the γ-element for G (it is an element of equivariant K-homology, which we will introduce later). It is the construction of the γ-element for $\mathrm{SL}(3,\mathbb{C})$ which motivates the present work. It is important to note that the construction of γ is not the only part of Kasparov's proof which presents problems for higher-rank Lie groups. But it will be necessary for any future work along Kasparov's lines to have a model for γ similar to those already made in [Kas84] and [JK95]—we will say more about the crucial features of these constructions later in this chapter. Such a model has not been achieved for any higher-rank Lie groups.

The reasons for being optimistic about a possible construction of γ for higher-rank Lie groups are as follows. Firstly, there is the existence of more refined homological tools. Bernstein, Gel'fand and Gel'fand [BGG75] in the 1970s introduced a homological complex tailored to semisimple Lie groups. This will be the centrepiece of the homological side of this thesis. Secondly, there is the introduction of more refined analytical structures. Bernstein, in his 1998 ICM address [Ber98], suggested a method for defining a Sobolev theory tailored to semisimple Lie groups. Bernstein's ideas served as inspiration for the much of what follows, although the approach we take here will be fairly different in character (see Chapters 5 and 6).

In this thesis, we explore the analysis of the differential operators which appear in the Bernstein-Gel'fand-Gel'fand complex for $\mathrm{SL}(3, \mathbb{C})$.

The content of the thesis is as follows. We begin in Chapter 2 with a rapid survey of the Baum-Connes Conjecture, and Kasparov's approach to it. In particular, we define the γ-element. We also describe explicit constructions of the γ-element for the groups $\mathrm{SL}(2, \mathbb{C})$, $\mathrm{SU}(2,1)$ and $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$.

We note from the outset that the last of these examples is particularly relevant to us. In that case, the γ-element is built from the Dolbeault complex of the homogeneous space

$$\mathcal{X} = \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1.$$

But it is crucial to the construction that we use an additional fact: the Dolbeault complex in this case splits as a product of two copies of the Dolbeault complex for $\mathbb{C}\mathrm{P}^1$,

$$\begin{tikzcd}[column sep=tiny] & \Omega^{0,1}\mathbb{C}\mathrm{P}^1 \otimes \Omega^{0,0}\mathbb{C}\mathrm{P}^1 \arrow[dr, "-1\otimes\overline{\partial}"] & \\ \Omega^{0,0}\mathbb{C}\mathrm{P}^1 \otimes \Omega^{0,0}\mathbb{C}\mathrm{P}^1 \arrow[ur, "\overline{\partial}\otimes 1"] \arrow[dr, "1\otimes\overline{\partial}"'] & & \Omega^{0,1}\mathbb{C}\mathrm{P}^1 \otimes \Omega^{0,1}\mathbb{C}\mathrm{P}^1 \\ & \Omega^{0,0}\mathbb{C}\mathrm{P}^1 \otimes \Omega^{0,1}\mathbb{C}\mathrm{P}^1 \arrow[ur, "\overline{\partial}\otimes 1"'] & \end{tikzcd}$$

Stated differently, we have two marked "complex" directions on $\mathcal{X}$ along which to differentiate, given by the fibres of the two coordinate projections,

$$\begin{tikzcd} & \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \arrow[dl, "\tau_X"'] \arrow[dr, "\tau_Y"] & \\ \mathbb{C}\mathrm{P}^1 & & \mathbb{C}\mathrm{P}^1. \end{tikzcd}$$

This split Dolbeault complex is the Bernstein-Gel'fand-Gel'fand complex for the group $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. In Chapter 3 we introduce the Bernstein-Gel'fand-Gel'fand (BGG) complex in generality. The BGG complex is a differential complex which is associated to a complex semisimple Lie group. It is believed that the BGG complex will be useful for γ-element constructions for arbitrary complex semisimple Lie groups.

From Chapter 4 we specialize completely to the group $G = \mathrm{SL}(3,\mathbb{C})$. We will first provide a concrete model of the BGG complex, including a formula in local coordinates. This will begin with a discussion of the geometry of the underlying homogeneous space, $\mathcal{X} = G/B$, where B is the Borel subgroup of lower triangular matrices. This space comes equipped with two fibrations

$$
\begin{tikzcd}
 & \mathcal{X} \arrow[dl, "\tau_X"'] \arrow[dr, "\tau_Y"] & \\
\mathbb{C}\mathrm{P}^2 & & \mathbb{C}\mathrm{P}^2.
\end{tikzcd}
$$

The fibres of these foliations give us two marked directions along which we can differentiate, analogous to those described in the case of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ above.

In the final two chapters we will take two different approaches to studying the BGG complex analytically, both of which are designed to take account of the marked directions just mentioned. The first method, in Chapter 5, is to use a local picture. The geometry of the space $\mathcal{X}$ is locally modelled on the three dimensional complex Heisenberg group $\mathbb{H}$. We will define a bifiltration, *ie* a two-parameter notion of "order", on differential operators on $\mathbb{H}$, and a related family of Sobolev spaces. We will also prove one negative result which shows that this two-parameter order can not be extended to a larger class of operators that one might hope to call "directional pseudodifferential operators."

In Chapter 6 we describe the second approach, which is to use harmonic analysis on the maximal compact subgroup $K = \mathrm{SU}(3)$. In this picture, differentiation along the fibres of the foliations τ_X and τ_Y is related to the action of the Lie subalgebras

$$\mathfrak{s}_X = \begin{pmatrix} \mathfrak{su}(2)_{\mathbb{C}} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathfrak{s}_Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathfrak{su}(2)_{\mathbb{C}}$}} \\ 0 & & \end{pmatrix}$$

of $\mathfrak{su}(3)_{\mathbb{C}}$. We describe the inter-relation of the spectral theory of these two non-commuting subalgebras.

Finally, we define a C^*-subalgebra $\mathcal{A}$ of the bounded operators on $L^2(\mathcal{X})$, as well as two ideals $\mathcal{K}_X$ and $\mathcal{K}_Y$ in $\mathcal{A}$. For future work on the group $\mathrm{SL}(3,\mathbb{C})$, these algebras should play the roles which are fulfilled by the algebras

$$\mathcal{A} = \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}),$$

$$\mathcal{K}_X = \mathcal{K}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$$

and

$$\mathcal{K}_Y = \mathcal{B}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H})$$

(with $\mathcal{H} = L^2(\mathbb{C}\mathrm{P}^1)$) in the construction of the γ-element for $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. We conclude by indicating why this is so.

Chapter 2

The γ-Element

2.1 The Baum-Connes conjecture

Although we will not be attacking the Baum-Connes Conjecture itself in this thesis, it is certainly the motivation for all of the present work. For this reason, we will take the time to provide a quick introduction to the Baum-Connes Conjecture, and the mathematics of Kasparov's approach to the conjecture. This also serves as a convenient narrative in which to introduce many of the basic concepts which will appear in the body of the thesis.

Let us begin by clarifying the ideas of the previous chapter. The place to start—the theory which is underpinning all of this—is Kasparov's analytic development of K-homology, and its generalization, KK-theory. Since the majority of the KK-theory we use will be K-homology, let us begin with that.

In introducing analytic K-homology, it is common to begin with the non-equivariant theory, that is, without the presence of the action of a group G. However, since the presence of the group is fundamental to the Baum-Connes conjecture, we will go for the throat here and make the entire theory equivariant from the start. In the classical topological situation, this would mean working with G-spaces, *ie*, topological spaces equipped with a continuous action of G. Algebraically, this means equipping each algebra of functions with the pull-back action of G. From there it is a short step to define the noncommutative topological analogue of a G-space.

Definition 2.1. A *G-C^*-algebra* is a C^* algebra A with a continuous action of G upon it by $*$-automorphisms. The continuity condition is that for each $a \in A$, the map

$$\begin{aligned} G &\to A \\ g &\mapsto g \cdot a \end{aligned}$$

is continuous.

A C^*-algebra A is $\mathbb{Z}/2\mathbb{Z}$-*graded* (often abbreviated to just *graded*) if it decomposes as a direct sum $A = A^{(0)} \oplus A^{(1)}$ of two closed *-invariant subspaces, such that

$$A^{(i)}.A^{(j)} \subseteq A^{(i+j)} \qquad (i, j \in \mathbb{Z}/2\mathbb{Z}).$$

A Hilbert space $\mathcal{H}$ is $\mathbb{Z}/2\mathbb{Z}$-*graded* if it decomposes as a direct sum $\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$. A representation of A on $\mathcal{H}$ is called *graded* if it respects the gradings of A and $\mathcal{H}$ in the following sense:

$$A^{(i)}.\mathcal{H}^{(j)} \subseteq \mathcal{H}^{(i+j)} \qquad (i, j \in \mathbb{Z}/2\mathbb{Z}).$$

An operator on $\mathcal{H}$ is degree 0 if it preserves the grading subspaces of $\mathcal{H}$, and degree 1 if it interchanges them. When we involve the group G, we will always require that representations of G are representations by degree 0 operators, An automorphism of A is said to be graded if it preserves the subspaces $A^{(0)}$ and $A^{(1)}$. For a graded G-C^*-algebra we require that the automorphisms of the action of G are graded.

Remark 1. For the reader unfamiliar with this material, a grading should be viewed as nothing more than a convenient organizational trick. A graded Hilbert space is really just a pair of Hilbert spaces, $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$. The C^*-algebras we encounter will almost universally be given the trivial grading (*ie*, all elements have degree 0) which means that a graded representation is just a separate representation on each of the two Hilbert spaces. Likewise for representations of the group. The only elements we shall encounter which are not of degree 0 will be certain self-adjoint operators of degree 1. Such an operator decomposes as a pair of mutually adjoint operators interchanging the two Hilbert spaces, which is to say that it could be adequately described as a single operator from $\mathcal{H}^{(0)}$ to $\mathcal{H}^{(1)}$.

Definition 2.2. Let A be a G-C^*-algebra. A *G-equivariant graded Fredholm module over A* is a collection of data $(\mathcal{H}, \pi, \phi, F)$ where:

- $\mathcal{H}$ is a graded Hilbert space,
- π is a representation of G on $\mathcal{H}$,
- ϕ is a *covariant* representation of A on $\mathcal{H}$, that is, ϕ satisfies

$$\pi(g)\phi(a)\pi(g^{-1}) = \phi(g \cdot a) \qquad (a \in A,\ g \in G),$$

- F is a self-adjoint operator on $\mathcal{H}$ of degree 1 which is G-*continuous*, meaning that the map

$$g \mapsto \pi(g) F \pi(g^{-1}) \qquad (g \in G)$$

is continuous in the operator norm,

and such that, for all $a \in A$ and $g \in G$, the operators

(i) $\phi(a)\,(F^2 - 1)$,

(ii) $[\phi(a), F]$, and

(iii) $\phi(a)\,(\pi(g) F \pi(g^{-1}) - F)$

are all compact.

Remark 2. It is common convention to omit mention of any member of the quadruple $(\mathcal{H}, \pi, \phi, F)$ which is deemed “obvious”.

Note that if A is unital, then the factors $\phi(a)$ in (i) and (iii) above may be omitted.

Definition 2.3. The *equivariant K-homology group* $K^0_G(A)$ is the set of G-equivariant graded Fredholm modules over A modulo homotopy. We define an addition operation $\oplus$ on $K^0_G(A)$ by direct sum. It is a theorem that this makes $K^0_G(A)$ into a group (see, for instance, [HR00, §8]).

If X is a locally compact Hausdorff G-space, and $C(X)$ is the associated G-C^*-algebra of functions on X, then we denote

$$K^G_0(X) = K^0_G(C(X)).$$

We have been deliberately vague here about the definition of *homotopy* of Fredholm modules. The most elegant definition is produced using Hilbert modules, which we will introduce in the next section. Instead, let us just note that a common example[1] of a homotopy is a family of graded Fredholm modules $((\mathcal{H}, \pi_t, \phi_t, F_t)$, for $t \in [0,1]$, all on the same Hilbert space, with each of the maps

$$\begin{aligned} t &\mapsto \pi_t(g), \qquad\qquad (g \in G) \\ t &\mapsto \phi_t(a), \qquad\qquad (a \in A) \\ t &\mapsto F_t \end{aligned}$$

being strongly continuous.

Let us illustrate Definition 2.2 with a few examples. Kasparov's original motivation for making this definition was to formalize the properties of elliptic pseudo-differential operators which arose in the proof of the Atiyah-Singer Index Theorem. In that case, $A = C(X)$ for some closed manifold X, with the trivial grading (*ie*, all elements declared to be degree 0). To begin with, let us suppose G is the trivial group, which renders all appearances of G in the definition redundant. If D is a first-order elliptic differential operator between vector bundles E_0 and E_1 over X, then we can form a graded Fredholm module over $C(X)$ as follows. Let $\mathcal{H} = L^2(X; E_0) \oplus L^2(X; E_1)$ be the space of L^2-sections of the bundles (graded according to that decomposition), with

[1] This example almost suffices to characterize the notion of homotopy in Definition 2.2. If one adds a second equivalence relation by introducing the notion of *degenerate* Fredholm modules then one recovers the correct definition of homotopy. See [HR00].

the representation of A by multiplication operators. Put

$$\mathbf{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix},$$

with respect to that decomposition. This is an unbounded, formally self-adjoint operator, which we convert to a bounded operator in a standard way:

$$F = \frac{\mathbf{D}}{\sqrt{1+\mathbf{D}^2}}.$$

We will refer to this procedure as "normalizing" the operator $\mathbf{D}$. That this data defines a Fredholm module is a consequence of the theory of elliptic pseudodifferential operators (see, for instance, [HR00, §10]).

Note that associated to any Fredholm operator F on a Hilbert space $\mathcal{H}$ there is an integer—the Fredholm index,

$$\operatorname{Index}(F) = \dim\ker F - \dim\operatorname{coker} F,$$

which is dependent only on the homotopy class of F. In the present case, since F is self-adjoint the index will be zero, but in the spirit of Remark 1, it has a nontrivial integer invariant:

$$\operatorname{Index}(F) = \dim\ker F_0 - \dim\operatorname{coker} F_0,$$

where F_0 is the component of F mapping $\mathcal{H}^{(0)}$ to $\mathcal{H}^{(1)}$. In this way, we obtain a map

$$\operatorname{Index} : K_0(X) \to \mathbb{Z},$$

for any space X admitting an action of the trivial group (!). If we identify $\mathbb{Z}$ with $K_0(\mathbb{C}) = K_0(C_r^*\{1\})$ (which one would need to do in a natural way[2]), we are starting to see the first hints of the Baum-Connes map.

Of course, this map will not be an isomorphism for arbitrary X. It will be an isomorphism for X being a point, or for X being contractible if we enlarge the class of manifolds considered. In order to make more interesting examples of isomorphisms we need to generalize the above.

If G is a discrete group then often the Baum-Connes map for G can be roughly phrased in the same language. Let us suppose that G is the fundamental group of some closed manifold[3] M, and let X be the universal cover of M. Let D be an elliptic differential operator D between bundles E_0 and E_1 over M. We could, of course, take the ordinary Fredholm index of this operator, but in this scenario an index for D can be defined which has value in $K(C_r^*G)$, rather than $\mathbb{Z}$. Here is a very quick description of that procedure, following [Hig98].

Firstly, pull back the bundles to bundles $\tilde{E}_0$ and $\tilde{E}_1$ over X. The differential operator D lifts to an operator $\tilde{D}$ between these bundles. Next, one expands the bundles $\tilde{E}_0$ and $\tilde{E}_1$ by tensoring with the trivial bundle

$$C_r^*G \times X$$

[2]This glib statement is sweeping an enormous amount under the rug—the missing details basically amount to the Atiyah-Singer Index Theorem itself.

[3]To make this construction work, it would suffice just to have a homomorphism of $\pi_1(M)$ into G.

over X. The operator $\tilde{D}$ acts naturally between these bundles (as $\tilde{D} \otimes 1$). The group G acts "diagonally" on this enlarged bundle by

$$\begin{aligned} G \times (\tilde{E}_0 \otimes C_r^* G) &\rightarrow (\tilde{E}_1 \otimes C_r^* G) \\ g \cdot (v \otimes x) &\mapsto g \cdot v \otimes \lambda(g) x. \end{aligned}$$

If we quotient by this action, we end up with a differential operator on a bundle over M whose fibres are finitely generated projective $C_r^* G$-modules, and a differential operator D_G between them. The kernel and cokernel of this operator will also $C_r^* G$-modules. If we are lucky, they will be finitely generated and projective, and we can put

$$\mathrm{Index}_G(D) = [\ker D_G] - [\operatorname{coker} D_G] \in K(C_r^* G).$$

If not, a perturbation of the kernel and cokernel will be finitely generated projective modules, and we define the index using those instead.

The point is that, by generalizing the standard index construction for elliptic differential operators to the equivariant situation, one can define a more refined index than the standard integer invariant. To generalize further one needs to abstract the analysis from this construction so that it can be applied directly to an equivariant Fredholm module. The result of this abstraction, which was suggested by Baum, Connes and Higson [BCH94], is that for any suitable G-space X there is an analytical index map

$$\mathrm{Index}_G : K_0^G(X) \rightarrow K(C_r^* G). \tag{2.1.1}$$

The essence of the Baum-Connes conjecture is that the collection of such indices completely determines $K(C_r^*G)$.

We should explain what is meant by a "suitable" G-space in the preceding remarks.

Definition 2.4. Let X be a Hausdorff G-space. The action of G on X is *proper* (and X is called a *proper* G-space) if, for every $x, y \in X$, there exist neighbourhoods U_x of x and U_y of y such that the set

$$\{g \in G \quad | \quad g \cdot U_x \cap U_y \neq 0\}$$

is compact.

Definition 2.5. A G-space X is called *G-compact* if the quotient space X/G is compact.

The G-index (2.1.1) can be defined whenever X is a proper G-compact G-space.

For any locally compact group G there is a universal proper G-space, in the following sense: there exists a proper G-space, denoted $\underline{E}G$, such that any proper G-space X admits a continuous G-equivariant map

$$f : X \to \underline{E}G,$$

and any two such maps can be joined by a homotopy of G-equivariant maps. What is more, the space $\underline{E}G$ is unique, up to G-equivariant homotopy.

If the universal proper G-space $\underline{E}G$ is G-compact, then it follows that all G-indices can be realized as images of equivariant Fredholm modules over $\underline{E}G$. In this case, we define the Baum-Connes map (or analytic assembly map) to be the map

$$\mu = \mathrm{Index}_G : K_0^G(\underline{E}G) \to K(C_r^*G).$$

If $\underline{E}G$ is not G-compact, then we need to adjust the left-hand side, by defining

$$RK_0^G(\underline{E}G) = \lim_{\substack{X\subseteq \underline{E}G \\ G\text{-compact}}} K_0^G(X),$$

a direct limit over the directed system of G-compact subsets of $\underline{E}G$. One checks that the G-index is natural with respect to the inclusion of G-invariant subsets, and hence the direct limit of the index maps of all G-compact subspaces of $\underline{E}G$ yields a map

$$\mu : RK_0^G(\underline{E}G) \to K(C_r^*G).$$

Conjecture 2.6 (The Baum-Connes Conjecture). *The Baum-Connes assembly map*

$$\mu : RK_0^G(\underline{E}G) \to K(C_r^*G)$$

is an isomorphism.

For a wealth of examples of the universal spaces $\underline{E}G$, see [BCH94]. We will be interested in one particular case: if G is a connected Lie group, and K is a maximal compact subgroup of G (unique up to conjugacy) then $\underline{E}G = G/K$.

2.2 The Dirac-dual Dirac method

We now turn to Kasparov's approach to the Baum-Connes conjecture. As mentioned earlier, one of the great advantages of Kasparov's approach is that it actually proves a stronger conjecture, in which the equivariant K-homology group $K_0^G(\underline{E}G)$ is allowed to take "coefficients" in an arbitrary G-C^*-algebra.

The key idea to introducing K-homology "with coefficients" is to replace Hilbert spaces, which are modules over $\mathbb{C}$, with Hilbert modules, which are the analogous modules for general C^*-algebras.

Definition 2.7. Let B be a C^*-algebra. Let $\mathcal{E}$ be a right-module over B, *ie* a vector space equipped with an action of B on the right. A *B-valued inner product* on $\mathcal{E}$ is a sesquilinear map

$$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to B$$

(conjugate-linear in the first variable) which satisfies the following analogues of the axioms for a $\mathbb{C}$-valued inner product:

(i) $\langle e, f.b \rangle = \langle e, f \rangle b$ for $b \in B$, $e, f \in \mathcal{E}$,

(ii) $\langle e, f \rangle = \langle f, e \rangle^*$, for $e, f \in \mathcal{E}$,

(iii) $\langle e, e \rangle$ is a positive element of B for all $e \in \mathcal{E}$, and $\langle e, e \rangle = 0$ implies $e = 0$.

A B-valued inner product induces a norm on $\mathcal{E}$ by

$$\|e\| = \|\langle e, e \rangle\|_B^{\frac{1}{2}}.$$

The module $\mathcal{E}$ is called a *Hilbert B-module* if it is complete with respect to this norm.

An operator T on a Hilbert B-module $\mathcal{E}$ is called *adjointable* if there exists an operator T^*, called its *adjoint*, such that

$$\langle e, Tf \rangle = \langle T^* e, f \rangle$$

for all $e, f \in \mathcal{E}$.

The idea of a Hilbert module is important even in the commutative case. If $B = C_0(X)$ for some locally compact topological space X, then the space of continuous sections, vanishing at infinity, of a vector bundle E over X is a module over $C_0(X)$, by pointwise multiplication. A Hilbert module structure on this module is equivalent to a Hermitian structure on the bundle: taking pointwise inner products of two sections yields an inner product valued in $C_0(X)$.

Replacing Hilbert spaces by Hilbert modules in Definition 2.2 leads one to the equivariant KK-theory group $KK^G(A, B)$. Since we will only need the full equivariant KK-theory groups for the background material in this introduction, we will not give the complete definition here. We refer the reader to [Hig90] or [Bla86].

With this in hand, we can now describe the left-hand side of the Baum-Connes conjecture with coefficients. This is the group

$$RKK^G(C(\underline{E}G), A) \quad \stackrel{\text{def}}{=} \lim_{\substack{X \subseteq \underline{E}G \\ G\text{-compact}}} KK^G(C_0(X), A),$$

where A is allowed to be any G-C^*-algebra.

For the right-hand side, we need to form a reduced group C^*-algebra with coefficients in A. This is the reduced crossed-product algebra.

Definition 2.8. Let G be a locally compact topological group, with Haar measure dg, and let A be a G-C^*-algebra. The *convolution algebra of* G *with coefficients in* A is the space $C_c(G, A)$ of continuous compactly-supported A-valued functions on G, equipped with the twisted convolution product

$$f_1 * f_2\,(g) = \int_G f_1(g')\; g' \cdot (f_2(g'^{-1} g))\; dg' \qquad\qquad (f_1, f_2 \in C_c(G, A)).$$

We make a Hilbert A-module $L^2(G; A)$ from $C_c(G, A)$ by completing in the following way. Note that $C_c(G, A)$ is a right A-module, by pointwise right-multiplication. It has a twisted involution $*$ defined by

$$f^*(g) = \delta(g)^{-1}\; g \cdot (f(g^{-1})^*), \qquad\qquad (f \in C_c(G, A))$$

where $\delta : G \to \mathbb{R}^+$ is the modular function of G such that $d(g^{-1}) = \delta(g)^{-1}\, dg$. We define an A-valued inner product by

$$\langle f_1, f_2 \rangle = \int_G f_1^*(g) f_2(g)\; dg \qquad\qquad (f_1, f_2 \in C_c(G, A)).$$

Then $L^2(G; A)$ is the completion of $C_c(G, A)$ with respect to the norm $\|f\| = \|\langle f, f\rangle\|_A^{\frac{1}{2}}$.

Now define a representation of $C_c(A, G)$ on $L^2(A; G)$ by

$$(f.\xi)(g) = \int_G (g^{-1} \cdot (f(g'))) \, \xi(g'^{-1} g) \, dg' \qquad (f \in C_c(G, A), \xi \in L^2(G; A) \ ).$$

This is a representation by adjointable operators, with the adjoint operation corresponding to the involution on $C_c(A, G)$. The *reduced crossed-product algebra* $C_r^*(G; A)$ is the completion of $C_c(G, A)$ in the operator-norm on $L^2(G; A)$.

Defining the Baum-Connes assembly map with coefficients means introducing coefficients into arguments which we have already omitted for brevity. Therefore, the reader is referred [BCH94] for the definition of the map. But assuming an appropriate generalization of the previous G-index maps can be made, we now have:

Conjecture 2.9 (The Baum-Connes Conjecture with Coefficients). *For any G-C^*-algebra, the analytic assembly map*

$$\mu : RKK^G(C_0(\underline{E}G), A) \to K(C_r^*(G; A))$$

is an isomorphism.

The real power of KK-theory is not done justice by describing it as "K-homology with coefficients". The great virtue of the theory is the existence of a product[4]

$$KK^G(A, B) \times KK^G(B, C) \to KK^G(A, C),$$

[4]There are far more general product constructions than that mentioned here. See [Hig90] or [Bla86].

for G-C^*-algebras A, B and C. This product structure lies at the heart of most applications of KK-theory, not the least of which is Kasparov's approach to the Baum-Connes Conjecture.

The algebraic structure which this product endows upon KK-theory is that of a category. More precisely, KK^G is an additive category whose objects are G-C^*-algebras. The KK-theory group $KK^G(A,B)$ is the additive group of morphisms between two specified objects A and B in this category. To provide a different insight into this, let us mention without details that there is a natural construction of an element of $KK^G(A,B)$ from any G-equivariant $*$-homomorphism $\phi : A \to B$. In this way, one can view KK^G as an enlargement of the category of G-C^*-algebras and G-equivariant $*$-homomorphisms (considered modulo homotopy). The additional morphisms in the category KK^G can be explained by the fact that KK-elements are used not to carry C^*-algebra elements from A to B, but to carry K-theory classes from $K(A)$ to $K(B)$. While this job can certainly be done using a *-homomorphism, it can also be achieved with various other constructions.

This is the perspective on KK-theory that we will take for the remainder of this Chapter. It is worth remarking that, as every category should, each object A has an associated identity element, which we denote by 1_A (or if the C^*-algebra A is clear, just 1).

One consequence of this categorical viewpoint is that we have a new notion of equivalence among G-C^*-algebras—one which is weaker than isomorphism. Two G-C^*-algebras A and B are KK^G-equivalent if there exists an invertible morphism in $KK^G(A,B)$. In that case, A and B will have exactly the same equivariant K-theory and K-homology. For example, equivalence in KK^G includes the notion of strong Morita equivalence.

This new idea of equivalence explains the utility of considering the Baum-Connes Conjecture with Coefficients. For if we use well-chosen coefficients, the conjecture can actually become easier to prove. Specifically, if the coefficient algebra A is $A = C_0(X)$ for some proper G-space X then Conjecture 2.9 is known to hold. But now, heuristically, the conjecture should also hold true for coefficients in any G-C^*-algebra which is KK^G-equivalent to $C_0(X)$. In particular, if the algebra $\mathbb{C}$ (with the trivial G-action) is KK^G-equivalent to $C_0(X)$, then the original Baum-Connes Conjecture for G should hold. This idea, when made rigorous, is Kasparov's approach.

Kasparov also provided a candidate for such a KK^G-equivalence when G is a connected Lie group.

Theorem 2.10 (Kasparov). *Let G be a connected Lie group, and K a maximal compact subgroup of G. Then the tangent bundle $X = T(G/K)$ of the symmetric space G/K, is a proper G-space, and there exist elements*

$$\alpha \in KK^G(C_0(X), \mathbb{C}) \text{ and } \beta \in KK^G(\mathbb{C}, C_0(X))$$

such that

$$\alpha\beta = 1 \in KK^G(C_0(X), C_0(X)).$$

The elements α and β are the "Dirac" and "dual Dirac" elements after which Kasparov's method is named.

At this point, it is clear that the element

$$\gamma_G = \beta\alpha \in KK^G(\mathbb{C}, \mathbb{C})$$

is of crucial importance. The element γ_G turns out to be independent of the choice of elements α and β and of the proper G-space X, as long as they satisfy the result of Theorem 2.10. This is the γ-element for the group G, and it is the focus of everything that follows.

We know that if $\gamma_G = 1$, then the Baum-Connes Conjecture holds for G. For instance, $\gamma_G = 1$ for connected amenable Lie groups ([Kas88]). It is also known that $\gamma_G = 1$ for the simple rank-one Lie groups $\mathrm{SO}_0(n, 1)$ and $\mathrm{SU}(n, 1)$, and their products. These latter results were proven in [Kas84], [JK95] by using explicit constructions of the γ-elements as elements in the equivariant K-homology of the homogeneous space G/B, where B is the Borel subgroup on B. We will explain this terminology in the next section.

However, it is also known that $\gamma_G \neq 1$ for any group with property T, and in particular for every higher-rank Lie group. Nevertheless, the γ-element is of fundamental importance in understanding the equivariant KK-theory of Lie groups, as we shall see in the next section, and it will almost certainly play a key role in any approach to the Baum-Connes Conjecture for discrete subgroups of these groups.

2.3 The γ-element of a semisimple Lie group

The goal of the present project, towards which this thesis is a first step, is to provide an explicit model for the γ-element for the group $G = \mathrm{SL}(3,\mathbb{C})$, similar to those already known for the above rank-one Lie groups. What we mean by this is that we wish to provide an explicit $\mathrm{SL}(3,\mathbb{C})$-invariant graded Fredholm module—*ie*, a graded Hilbert space with a representation of $\mathrm{SL}(3,\mathbb{C})$ and a Fredholm operator upon it—whose class in $KK^G(\mathbb{C},\mathbb{C})$ is γ. We desire that this model be of a particular form, which we will describe shortly.

In this section we will describe a method, once again due to Kasparov, for recognizing such a model in the case of a semisimple Lie group G. But before doing so, let us first make a few comments about γ-elements in general.

In fact, let us start with some remarks about the home of the γ-element: the group $KK^G(\mathbb{C},\mathbb{C})$. Because of the product in KK-theory, this KK-group is actually a ring. This ring is of singular importance in equivariant KK-theory. It is often called the *Kasparov representation ring*, for reasons which we will explain shortly, and it earns a special notation: $R(G)$.

To understand the name, we must flesh out the details of its definition. Unwinding Definition 2.2 with the aid of Remark 1, an element of $R(G)$ is given by a pair of unitary representations of G,

$$\pi_0 : G \to \mathcal{H}^{(0)},$$

$$\pi_1 : G \to \mathcal{H}^{(1)},$$

and a G-continuous operator

$$F : \mathcal{H}^{(0)} \to \mathcal{H}^{(1)},$$

which is essentially unitary (*ie*, $F^*F - 1$ and $FF^* - 1$ are compact operators—in particular, F is Fredholm) and which almost intertwines the two representations, in the sense that

$$\pi_1(g)F - F\pi_0(g)$$

is a compact operator for all $g \in G$.

Heuristically, we think of the operator F as instituting a "difference" of the two representations. Consider the case of a compact group, which we now denote by K. In this case, we can replace F by an averaged version,

$$F' = \int_K \pi_1(k)F\pi_0(k)^{-1}\, dk,$$

which is homotopic to F. But now $\pi_1(k)F'\pi_0(k)^{-1} = F'$ for all $k \in K$, so that F is a genuine intertwiner. Being Fredholm, the kernel and cokernel of F' are finite dimensional representations of K. The formal difference

$$(ker F') \ominus (coker F')$$

is a *virtual representation* of K, that is, a direct sum of irreducible representations of K whose multiplicities are permitted to be negative. Virtual representations themselves form a ring under direct sum and tensor product. This is the classical representation ring of K, as known to representation theorists. With some small amount of extra work, the above process shows that the classical representation ring and the Kasparov representation ring are isomorphic for compact groups.

The special role of $R(G)$ in KK^G comes from the fact that, in addition to the KK-product already mentioned, there is also an external product in KK-theory, which is a map

$$KK^G(A,B) \times KK^G(C,D) \to KK^G(A \otimes C, B \otimes D),$$

for G-C^*-algebras A, B, C and D. Since $A \otimes \mathbb{C} = A$ for any G-C^*-algebra A, the external product makes every KK^G-group into a module over the ring $R(G)$. (A point needs to be made about the module action of $R(G)$ upon itself: it does indeed agree with the product already mentioned, which means that we can take products in $R(G)$ without confusion.)

Within this singularly important KK-group $R(G)$, γ_G is a singularly important element. To understand its singular importance, note first that for any subgroup H of G, there is an obvious restriction homomorphism

$$\mathrm{Res}_H^G : KK^G(A,B) \to KK^H(A,B).$$

Theorem 2.11. *Let G be a connected Lie group. The γ-element is an idempotent in $R(G)$. Moreover, if K is a maximal compact subgroup of G, then the restriction map from $R(G)$ to $R(K)$ is split-surjective, with kernel $(1-\gamma)R(G)$. Hence, $R(K) \cong \gamma R(G)$.*

In other words, the γ-element marks out a part of $R(G)$ isomorphic to the ring $R(K)$, which is a classical and well-understood object.

At this point, let us completely restrict our attention to semisimple Lie groups G. Let B denote a Borel (*ie,* minimal parabolic) subgroup of G. In this work we will be almost entirely concerned with the groups $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{R})$, so rather than give general definitions, let us simply note for now that in these groups, B is the subgroup of upper-triangular matrices. General definitions, will be given in Section 3.3.

If X is a proper G-space, then it is also a proper B-space. It follows that the same elements α and β which define the γ-element for G, also serve to define the γ-element for B. In other words,

$$\mathrm{Res}_B^G \gamma_G = \gamma_B.$$

But the group B can be contracted onto its maximal compact subgroup T by a continuous family of automorphisms. This allows any representation of B to be continuously deformed to a representation which factors through T. As a consequence, the restriction map from $R(B)$ to $R(T)$ is an isomorphism. Since $\gamma_T = 1$ by compactness, this shows that $\gamma_B = 1$.

The symmetric space G/B is compact. Because of this, there is a map of C^*-algebras

$$\iota : \mathbb{C} \to C(G/B),$$

including $\mathbb{C}$ as multiples of the unit. With this, one can take any $\xi \in KK^G(C(G/B), \mathbb{C})$, and forget its $C(G/B)$-representation to see it as an element of $KK^G(\mathbb{C}, \mathbb{C})$. This process is denoted by the map $\iota^* : K_0^G(G/B) \to R(G)$. Coupling this with the observation that $\gamma_B = 1$ will suggest that we may look for a model of γ_G as an elliptic differential operator (or some variant thereof) over the space G/B.

To see why this is so, we will need the induction map on equivariant KK-theory. Let H be a subgroup of G. The induction map, in its most elementary form, is a homomorphism

$$\mathrm{Ind}_H^G : R(H) = KK^H(\mathbb{C},\mathbb{C}) \to KK^G(C_0(G/H), C_0(G/H)),$$

which is defined in strong analogy with induction for ordinary group representations (see, for instance, [Bla86, §20.5]). Of course, it can also be generalized enormously. We will need only a small generalization: if X is a G-space which admits a (fixed) G-equivariant map to G/H, then the induction homomorphism can be extended to a map

$$\mathrm{Ind}_H^G : R(H) = KK^H(\mathbb{C},\mathbb{C}) \to KK^G(C_0(X), C_0(X)).$$

The relationship between induction and restriction is as follows.

Lemma 2.12 (Kasparov). [Kas88] *Let G, H and X be as above. The map*

$$\mathrm{Ind}_H^G \mathrm{Res}_H^G : R(G) \to KK^G(C_0(X), C_0(X))$$

is given by

$$\xi \mapsto \xi \otimes 1_{C_0(X)}.$$

The map

$$\operatorname{Res}_H^G Ind_H^G : R(H) \to KK^H(C_0(X), C_0(X))$$

is given by

$$\eta \mapsto \eta \otimes 1_{C_0(X)}.$$

The following theorem encapsulates the technique which we will use to construct the γ-element for a semisimple group G. It is an observation first noted by Kasparov in his work on Lorentz groups [Kas84].

Theorem 2.13. *Suppose that $\theta \in KK^G(C(G/B), \mathbb{C})$ is sent by the map*

$$KK^G(C(G/B), \mathbb{C}) \xrightarrow{\iota^*} R(G) \xrightarrow{\operatorname{Res}_K^G} R(K)$$

to $1 \in R(K)$. Then $\iota^*\theta$ is the γ-element in $R(G)$.

Remark 3. The map $KK^G(C(G/B), \mathbb{C}) \to R(K)$ in the theorem is just a bunch of forgetting. One starts with a G-equivariant Fredholm module over $C(G/B)$, exactly as laid out in Definition 2.2, and then one forgets first the action of $C(G/B)$ and secondly all of the representation of G except for the representation of K.

Proof. In a result such as this, the categorical interpretation of KK^G becomes extremely convenient. Firstly, the homomorphism

$$\iota : \mathbb{C} \to C(G/B)$$

can be interpreted as a KK^G-element in $KK^G(\mathbb{C}, C(G/B))$. From this viewpoint, the map ι^* is given by the product

$$\iota^* : \xi \mapsto \iota\xi.$$

Let us put $\gamma' = \iota^*(\theta) = \iota\theta$. We are to prove that $\gamma' = \gamma$.

We do this by computing the product $\gamma\gamma'$ in two ways. To start, we expand it as an exterior product:

$$\gamma\gamma' = \gamma \otimes \gamma' \in KK^G(\mathbb{C} \otimes \mathbb{C}, \mathbb{C} \otimes \mathbb{C}).$$

Now, if we expand γ' as $\iota\theta$, then the above exterior product can be written as the following composition of morphisms in KK^G:

$$\mathbb{C} \otimes \mathbb{C} \xrightarrow{1\otimes\iota} \mathbb{C} \otimes C(G/B) \xrightarrow{\gamma\otimes 1} \mathbb{C} \otimes C(G/B) \xrightarrow{1\otimes\theta} \mathbb{C} \otimes \mathbb{C}\,.$$

By Lemma 2.12,

$$\gamma \otimes 1_{C(G/B)} = \mathrm{Ind}_B^G \mathrm{Res}_B^G \gamma,$$

and since the restriction of γ to B is 1,

$$\gamma \otimes 1_{C(G/B)} = 1 \otimes 1_{C(G/B)}.$$

Hence $\gamma\gamma' = \gamma'$.

On the other hand, if we expand γ as $\beta\alpha$, then $\gamma\gamma'$ can be written as

$$\mathbb{C} \otimes \mathbb{C} \xrightarrow{\beta\otimes 1} C(X) \otimes \mathbb{C} \xrightarrow{1\otimes\gamma'} C(X) \otimes \mathbb{C} \xrightarrow{\alpha\otimes 1} \mathbb{C} \otimes \mathbb{C}\,,$$

where $X = T(G/K)$ is the proper G-space in Theorem 2.10. Appealing to Lemma 2.12 once more,

$$1 \otimes \gamma' = \operatorname{Ind}_K^G \operatorname{Res}_K^G \gamma'.$$

The restriction of γ' to K is 1 by hypothesis. It follows that $\gamma\gamma' = \gamma$, which completes the proof. $\square$

2.4 Examples of γ-elements

2.4.1 The γ-element for $\mathrm{SL}(2,\mathbb{C})$

The most elementary of all the complex simple Lie groups is $\mathrm{SL}(2,\mathbb{C})$. We are going to be looking in $KK^G(C(G/B),\mathbb{C})$ for an equivariant graded Fredholm module θ, as above. Here is a rough idea for a construction. Recall that the Borel subgroup of $\mathrm{SL}(2,C)$ is the subgroup of upper triangular matrices. Let us put $\mathcal{X} = G/B$, which in this case is the complex projective line, $\mathbb{C}\mathrm{P}^1$. Being a complex manifold, it has a Dolbeault complex:

$$\Omega^{0,0}\mathcal{X} \xrightarrow{\overline{\partial}} \Omega^{0,1}\mathcal{X}$$

The Dolbeault operator $\overline{\partial}$ is an elliptic differential operator, and therefore can be normalized as in Section 2.1 to give a bounded operator F between the spaces of L^2-forms. Moreover, thanks to Hodge theory (see, for instance, [Roe98]), the kernel and cokernel of F are the Dolbeault cohomology groups of $\mathbb{C}\mathrm{P}^1$:

$$H^{0,q}_{\overline{\partial}}(\mathbb{C}\mathrm{P}^1) = \begin{cases} \mathbb{C}, & q = 0 \\ 0, & q = 1 \end{cases}$$

In particular, the kernel will be the space of constant functions, which is K-invariant. In other words, forgetting all but the action of K on this data, the virtual representation of K instituted by F will be the trivial representation. If the details of this rough argument can be worked out then we have constructed the γ-element.

The details of this argument can indeed be worked out, although these details are not completely trivial. The main problem is in defining the representation of the group G. There is a natural representation of G on forms over $\mathcal{X}$ by pull-back. However, this representation is not unitary, because the action of G on $\mathbb{C}\mathrm{P}^1$ admits no invariant probability measure.

The situation is saved by the fact that the action is conformal. Since the concept of conformality is critical to later constructions as well, let us discuss it in generality.

Let $\mathcal{X}$ be a closed orientable Riemannian manifold of dimension n, with Riemannian metric

$$\langle\,\cdot\,,\,\cdot\,\rangle_x : T_x\mathcal{X} \times T_x\mathcal{X} \to \mathbb{R} \qquad\qquad (x \in \mathcal{X}).$$

Let G be a locally compact group which acts conformally on $\mathcal{X}$, so that for each $g \in G$ there is a function $h_g \in C^\infty(\mathcal{X})$ such that

$$g^*\langle\,\cdot\,,\,\cdot\,\rangle_{gx} = (h_g(x))^2\langle\,\cdot\,,\,\cdot\,\rangle_x.$$

Note that

$$(h_g)^{-1} = g^* h_{g^{-1}}.$$

The Hermitian structure $\langle \cdot , \cdot \rangle$ on $T\mathcal{X}$ induces a Hermitian structure on the dual bundle $T^*\mathcal{X}$, and its tensor powers, in the usual way. We thus obtain a Hermitian structure on the full bundle of differential forms $\bigwedge T^*\mathcal{X}$ (elements of different degree are declared to be orthogonal). The relationship between this inner product structure and the conformal action is as follows: given $\omega, \eta \in \bigwedge^p T_x^*\mathcal{X}$ and $g \in G$,

$$\langle g^*\omega, g^*\eta\rangle_x = h_g(x)^{2p}\langle \omega, \eta\rangle_{gx}. \tag{2.4.1}$$

The Hermitian structure on the bundle gives an inner product on the space of sections by

$$\langle \omega, \eta\rangle = \int_{\mathcal{X}} \langle \omega, \eta\rangle_x \,\mathrm{dVol}(x),$$

for $\omega, \eta \in \Omega\mathcal{X}$. Completing with respect to this inner product yields the space $L^2(\Omega\mathcal{X})$ of L^2-integrable forms on $\mathcal{X}$.

Lemma 2.14. *The pull-back action*

$$g \cdot \omega = (g^{-1})^*\omega$$

of G on $\Omega\mathcal{X}$ becomes a unitary action on $L^2(\Omega\mathcal{X})$ if we modify it by the introduction of the conformal factor, according to the formula

$$\pi(g) : \omega \mapsto (h_{g^{-1}})^{\frac{n}{2}-p} \,.\, (g^{-1})^*\omega \tag{2.4.2}$$

for $\omega \in \Omega^p\mathcal{X}$, and extending linearly.

Proof. Checking this is a computation, as follows. Firstly, the volume form induced by the Riemannian metric (which we have denoted by dVol here) pulls back via the action of $g \in G$ to another volume form, which necessarily differs from the original by a scalar-valued function. That function can be determined from Equation (2.4.1):

$$g^* \,\mathrm{dVol} = h_g(x)^n \,\mathrm{dVol}\,.$$

Using this we have, for $\omega, \eta \in \Omega^p \mathcal{X}$,

$$\begin{aligned}
\langle \pi(g)\omega, \pi(g)\eta \rangle &= \int_{\mathcal{X}} \langle (g^{-1})^* \omega, (g^{-1})^* \eta \rangle_x \; h_{g^{-1}}(x)^{n-2p} \; \mathrm{dVol}(x) \\
&= \int_{\mathcal{X}} \langle \omega, \eta \rangle_{g^{-1}x} \;\; h_{g^{-1}}(x)^{2p} \;\; h_{g^{-1}}(x)^{n-2p} \;\; h_{g^{-1}}(x)^{-n} \; (g^{-1})^* \mathrm{dVol}(x) \\
&= \int_{\mathcal{X}} \langle \omega, \eta \rangle_y \, \mathrm{dVol}(y) \\
&= \langle \omega, \eta \rangle.
\end{aligned}$$

$\square$

If $\mathcal{X}$ is a complex manifold of complex-dimension n (real-dimension $2n$), and G acts conformally by biholomorphic maps, then we work with the sub-bundle of $(0,p)$-forms. Because of the doubled real-dimension, the formula (2.4.2) must be altered to

$$\pi(g) : \omega \mapsto (h_{g^{-1}})^{n-p} . (g^{-1})^* \omega \tag{2.4.3}$$

for $\omega \in \Omega^{0,p} \mathcal{X}$.

Consider the Dolbeault complex of $\mathcal{X}$:

$$\Omega^{0,0}\mathcal{X} \xrightarrow{\overline{\partial}} \Omega^{0,1}\mathcal{X} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{0,n}\mathcal{X}$$

Let us put

$$\mathcal{H} = L^2(\Omega^{0,\bullet}\mathcal{X}) = \bigoplus_{p=0}^{n} L^2(\Omega^{0,p}\mathcal{X}),$$

and grade it by the decomposition into even and odd forms.

Theorem 2.15. *With notation as above, put $D = \overline{\partial}+\overline{\partial}^*$. On $\mathcal{H}$, the normalized operator*

$$F = \frac{D}{\sqrt{1+D^2}}$$

combined with the unitary representation (2.4.3) of G and the pointwise multiplication of $C(\mathcal{X})$ defines a graded G-equivariant Fredholm module over $C(\mathcal{X})$.

The complete proof is given in [Kas84]. Here we will only sketch the proof in order to identify the crucial steps.

Sketch of proof. The key facts to be proven are the commutativity properties of F, namely

(i) that F commutes with the multiplication action of $C(\mathcal{X})$ modulo compacts, and

(ii) that $\pi(g)F\pi(g)^{-1} - F$ is compact for all $g \in G$.

The first item is a standard fact from the theory of pseudodifferential operators. The operator F is an order zero pseudodifferential operator over $\mathcal{X}$, as are multiplication operators. Their commutator is therefore an order -1 operator, which is compact.

The second item uses the first as a component. But first, for $\omega \in \Omega^{0,p}\mathcal{X}$, we compute

$$\begin{aligned}\pi(g)\overline{\partial}\pi(g)^{-1} : \omega \quad &\mapsto \quad (h_{g^{-1}})^{n-(p+1)}\,(g^{-1})^{*}\ \overline{\partial}\left(g^{*}(h_{g^{-1}})^{-(n-p)}\omega\right) \\ &= \quad (h_{g^{-1}})^{-1}\overline{\partial}\omega + \eta\wedge\omega,\end{aligned}$$

where

$$\eta = (h_{g^{-1}})^{n-(p+1)}\ \overline{\partial}\left((h_{g^{-1}})^{-(n-p)}\right) \quad \in \Omega^{0,1}\mathcal{X}.$$

The important point is that

$$\pi(g)\overline{\partial}\pi(g)^{-1} = f\overline{\partial} + E,$$

for $f = 1/h_{g^{-1}}$ and some order zero pseudodifferential operator E. Note that f is independent of p, so that the same equation holds on the total space $\Omega\mathcal{X}$, with some E. Consequently, the operator $D = \overline{\partial} + \overline{\partial}^{*}$ also satisfies a similar identity:

$$\pi(g)D\pi(g)^{-1} = fD + E,$$

for some different order zero operator E.

Now we normalize D, by applying the function

$$\chi(t) = \frac{t}{\sqrt{1+t^{2}}}.$$

Note that

$$\begin{aligned}
\pi(g)F\pi(g)^{-1} &= \chi(\pi(g)D\pi(g)^{-1}) \\
&= \chi(fD+E)
\end{aligned}$$

This last expression can be analyzed using the symbolic calculus. Working modulo operators of order -1, we find

$$\begin{aligned}
\chi(fD+E) &\sim \frac{fD}{\sqrt{f^2D^2}} \\
&\sim \frac{D}{\sqrt{D^2}} \\
&\sim \chi(D),
\end{aligned}$$

which completes the proof.

□

As a consequence, we have:

Theorem 2.16 (Kasparov). *The normalized Dolbeault operator on* $L^2(\Omega^{0,\bullet}\mathbb{C}\mathrm{P}^1)$*, together with the unitarized pull-back representation of Equation (2.4.2), is the* γ*-element for* $\mathrm{SL}(2,\mathbb{C})$*.*

2.4.2 The γ-element for $\mathrm{SU}(2,1)$

The problem which one immediately encounters in trying to generalize the above argument is that most semisimple Lie groups G do not act conformally on their homogeneous spaces G/B. After the family of Lorentz groups $\mathrm{SO}_0(n,1)$, the next cases to consider are the groups $\mathrm{SU}(n,1)$. In this case the action on G/B is still somewhat close to conformal, as we now describe.

Let $G = \mathrm{SU}(2,1)$. Let K be its maximal compact subgroup and B its Borel subgroup. The symmetric space G/K is the 2-dimensional complex hyperbolic space $\mathrm{H}\mathbb{C}^2$. This is a complex manifold upon which G acts biholomorphically. The space G/B is naturally identified with the boundary sphere S^3 of $\mathrm{H}\mathbb{C}^2$ (for instance, after realizing $\mathrm{H}\mathbb{C}^2$ as the unit ball D^4 in $\mathbb{C}^2$). Being the smooth boundary of a Kähler manifold, G/B inherits a contact structure.

A contact structure on a $(2n-1)$-dimensional manifold $\mathcal{X}$ is described by a 1-form τ with the property that $\tau \wedge (d\tau)^n$ is a volume form. We do not intend to give a complete introduction to contact manifolds here. Let us just remark that one of the key consequences of a contact structure is the existence of the codimension-one sub-bundle $\mathcal{Q} = \ker \tau$ of the tangent bundle which is totally non-integrable, in the sense that the vector fields tangent to $\mathcal{Q}$ generate all sections of $T\mathcal{X}$ as a Lie algebra. The contact structure on G/B—and in particular this bundle $\mathcal{Q}$—is preserved by the action of G.

Let us equip $G/B \cong \mathrm{S}^3$ with a K-invariant Riemannian metric. The action of G on S^3 is not conformal. This means that the de Rham complex for S^3 can not be converted into an element of $KK_0^G(G/B)$ in the way of the previous section. However, the action on the sub-bundle $\mathcal{Q}$ is conformal. So an elliptic (or hypoellptic[5]) differential operator on the bundle $\Lambda^\bullet \mathcal{Q}^*$ will allow the construction to go through.

This gives a rough approximation to the construction which Julg and Kasparov used to prove the Baum-Connes Conjecture for the groups $\mathrm{SU}(n,1)$. What is needed is a replacement for the de Rham complex which is tailored to the contact structure. This crucial ingredient was available thanks to earlier work on contact manifolds by Rumin ([Rum94]), who defined a cohomological complex for contact manifolds which has the same cohomology groups as the de Rham complex.

There are, of course, several significant analytical issues which arise in trying to create the equivariant K-homology element from the Rumin complex. Let us mention just one of them here: the Rumin complex is not elliptic. This means that the classical pseudo-differential operator theory which we used in the previous section does not apply. Fortunately, there also exists a pseudodifferential calculus which is tailored to contact manifolds. This is the Heisenberg calculus of Beals and Greiner [BG88]. In the frame-work of the Heisenberg calculus, the complex is "maximally hypoelliptic", which is the appropriate analogue of ellipticity, and this suffices to show that the normalized Rumin differential is Fredholm. The commutativity properties required of an equivariant Fredholm module are also provable.

To summarize, the de Rham complex for G/B, with $G = \mathrm{SU}(n,1)$, can be pared down to guarantee conformality, although this comes at the cost of increased complexity in the pseudodifferential calculus required. It might be hoped that the same situation

[5]Hypoellipticity is a weakening of ellipticity which still guarantees that the operators are Fredholm. The operators in this example will be hypoelliptic. (See [BG88].)

could also be arranged for higher rank Lie groups. Unfortunately this seems difficult at best, and perhaps impossible, for the following reason.

Suppose there existed an operator

$$F : L^2(G/B; E) \longrightarrow L^2(G/B; E)$$

on a bundle E over G/B which, in some hypothetical pseudodifferential calculus, was a pseudodifferential operator of order zero. Suppose, moreover, that the action of G on E were conformal, so that we could define a unitarized action

$$\pi : G \longrightarrow \mathcal{U}\left(L^2(G/B; E)\right)$$

as in Section 2.4.1. This would allow us to prove that

$$\pi(g)F\pi(g)^{-1} - F$$

was of some strictly negative order (and hence was compact), for each $g \in G$. But it is a typical property of pseudodifferential calculi that operators of strictly negative order are not just compact, but Schatten p-class, for some $p < \infty$. The function

$$\begin{aligned} \xi : G &\longrightarrow \mathcal{L}^p \\ g &\mapsto \pi(g)F\pi(g)^{-1} - F \end{aligned}$$

would then be a 1-cocycle for the group G with coefficients in the Banach space $\mathcal{L}^p$ of Schatten p-class operators.

Now, it is conjectured[6] that all higher rank simple Lie groups, have trivial first cohomology with coefficients in any uniformly convex Banach space. If true, this would imply that ξ is in fact a coboundary. In other words, there is some $K \in \mathcal{L}^p$ such that

$$\xi(g) = \pi(g) K \pi(g)^{-1} - K$$

for all $g \in G$. But then

$$\pi(g)(F - K)\pi(g^{-1}) = F - K$$

for all $g \in G$, so that $F - K$ is an intertwiner for the representation π. The representations we are considering—the so-called generalized prinicpal series representations—do not have finite-dimensional direct summands, and this means that $F - K$ would represent the zero element in $R(G)$. But $F - K$ is a compact perturbation of F, and hence is homotopic to F. Thus F cannot represent the γ-element.

This strongly suggests that we will have to relax our expectations of what the γ-element will look like for $\mathrm{SL}(3, \mathbb{C})$. To understand what alternatives exist, it is edifying to consider the example of $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$.

[6] I believe this conjecture should be ascribed to Fisher and Margulis, although I have not actually seen it written as such.

2.4.3 The γ-element for $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$

The action of the group $G = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ is also not conformal. It is, however, a product of two conformal actions in an obvious way. For this group, it is true that $\gamma_G = 1$. The construction of γ_G is a result of the external product in KK-theory. This section is dedicated to an understanding of that product, which will serve as the best inspiration for the results about $\mathrm{SL}(3,\mathbb{C})$ to follow.

Let us put $G_0 = \mathrm{SL}(2,\mathbb{C})$ and let B_0 be the Borel subgroup of upper triangular matrices in G_0. Then the Borel subgroup of $G = G_0 \times G_0$ is $B = B_0 \times B_0$. Let $\mathcal{X}_0 = G_0/B_0 = \mathbb{C}\mathrm{P}^1$ and $\mathcal{X} = G/B = \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$.

We can form the Dolbeault complex for $\mathcal{X}$ as before:

$$\Omega^{0,0} \xrightarrow{\overline{\partial}} \Omega^{0,1}\mathcal{X} \xrightarrow{\overline{\partial}} \Omega^{0,2}\mathcal{X}.$$

Note, however, that the bundle of $(0,1)$-forms splits into two one-dimensional bundles. Specifically, we have

$$\begin{aligned}
\Lambda^{0,0}T^*\mathcal{X} &= \Lambda^{0,0}T^*\mathcal{X}_0 \boxtimes \Lambda^{0,0}T^*\mathcal{X}_0 \\
\Lambda^{0,1}T^*\mathcal{X} &= (\Lambda^{0,1}T^*\mathcal{X}_0 \boxtimes \Lambda^{0,0}T^*\mathcal{X}_0) \oplus (\Lambda^{0,0}T^*\mathcal{X}_0 \boxtimes \Lambda^{0,1}T^*\mathcal{X}_0) \\
\Lambda^{0,2}T^*\mathcal{X} &= \Lambda^{0,1}T^*\mathcal{X}_0 \boxtimes \Lambda^{0,1}T^*\mathcal{X}_0.
\end{aligned}$$

For convenience, let us denote the bundle $\Lambda^{0,i}T^*\mathcal{X}_0 \boxtimes \Lambda^{0,j}T^*\mathcal{X}_0$ over $\mathcal{X}$ by $E_{i,j}$, and the space of L^2-sections of it by $\mathcal{H}_{i,j}$. (We use the Hermitian structure on each $E_{i,j}$ induced from the standard Hermitian structures on the bundles over $\mathcal{X}_0$.)

The Dolbeault complex is now given by a system of differential operators

$$\begin{tikzcd} & C^{\infty}(\mathcal{X}; E_{1,0}) \arrow[dr] & \\ C^{\infty}(\mathcal{X}; E_{0,0}) \arrow[ur] \arrow[dr] & & C^{\infty}(\mathcal{X}; E_{1,1}) \\ & C^{\infty}(\mathcal{X}; E_{0,1}) \arrow[ur] & \end{tikzcd}$$

Each of the bundles is one-dimensional, and hence their Hermitian structures are trivially conformal for the action of G. Therefore, we can define unitarized actions of G on each of the Hilbert spaces $\mathcal{H}_{i,j}$, as in Section 2.4.1. We can also normalize each of the differential operators individually. Let us denote these normalized operators by

$$\begin{tikzcd} & \mathcal{H}_{1,0} \arrow[dr, "c"] & \\ \mathcal{H}_{0,0} \arrow[ur, "a"] \arrow[dr, "b"'] & & \mathcal{H}_{1,1} \\ & \mathcal{H}_{0,1} \arrow[ur, "d"'] & \end{tikzcd} \tag{2.4.4}$$

The question now is whether we can use these operators to construct a Fredholm module. Let us consider the operator a. The differential operator from which a was created is the operator

$$\overline{\partial} \otimes 1 : \Omega^{0,0}\mathcal{X}_0 \otimes \Omega^{0,0}\mathcal{X}_0 \longrightarrow \Omega^{0,1}\mathcal{X}_0 \otimes \Omega^{0,0}\mathcal{X}_0.$$

Hence,

$$a = F \otimes 1,$$

where $F = \overline{\partial}/\sqrt{1+\overline{\partial}^*\overline{\partial}}$ is the normalized Dolbeault operator for $\mathcal{X}_0$. It follows that the operators

$$\begin{array}{ll} a^*a - 1, & \\ aa^* - 1, & \\ [f, a]\,, & (f \in C(\mathcal{X})) \\ \pi(g)a\pi(g)^{-1} - a, & (g \in G) \end{array}$$

are all in $\mathcal{K}(\mathring{\mathcal{H}}) \otimes 1$, where $\mathcal{K}(\mathring{\mathcal{H}})$ denotes the space of compact operators on the graded Hilbert space

$$\mathring{\mathcal{H}} = L^2(\Omega^{0,0}\mathcal{X}_0) \oplus L^2(\Omega^{0,1}\mathcal{X}_0).$$

Analogous statements hold true for the operator d. For b and c, the similar quantities belong to $1 \otimes \mathcal{K}(\mathring{\mathcal{H}})$.

Let us define ideals

$$\begin{aligned} \mathcal{K}_X &= \mathcal{K}(\mathring{\mathcal{H}}) \otimes \mathcal{B}(\mathring{\mathcal{H}}), \\ \mathcal{K}_Y &= \mathcal{B}(\mathring{\mathcal{H}}) \otimes \mathcal{K}(\mathring{\mathcal{H}}) \end{aligned}$$

in the algebra

$$\mathcal{A} = \mathcal{B}(\mathring{\mathcal{H}}) \otimes \mathcal{B}(\mathring{\mathcal{H}}).$$

Note that their intersection is

$$\mathcal{K}_X \cap \mathcal{K}_Y = \mathcal{K}(\mathring{\mathcal{H}}) \otimes \mathcal{K}(\mathring{\mathcal{H}}) = \mathcal{K}(\mathcal{H}),$$

the ideal of compact operators on $\mathcal{H} = \oplus \mathcal{H}_{i,j}$. What we have produced, then, is data similar to that of a graded G-equivariant Fredholm module, but with the ideals $\mathcal{K}_X$ or $\mathcal{K}_Y$ in place of the compact operators.

From this point there are several ways to proceed. The usual way would be to invoke the Kasparov Technical Theorem (see Theorem 3.8.1 and Proposition 9.2.5 of [HR00], for instance) which allows us to adjust the complex (2.4.4) to get a genuine G-equivariant graded Fredholm module. This is the external product in Kasparov's K-homology. But there are also other approaches. For instance, there are alternative bivariant theories to KK-theory, such as Higson's E-theory [CH90] or Dumitraşcu's KE-theory [Dum], in which the external product is defined more simply. There is also the possibility of using the data of the complex (2.4.4) directly in applications by providing an explicit recipe for pairing it with K-theory. We will leave all these possibilities open for the present.

Now, we will begin to study the case of $\mathrm{SL}(3, \mathbb{C})$ in earnest.

Chapter 3

The Bernstein-Gel'fand-Gel'fand Complex

3.1 Introduction

Let G be the group $\mathrm{SL}(3,\mathbb{C})$, and let B be the subgroup of lower triangular matrices. The Bernstein-Gel'fand-Gel'fand (BGG) complex for G is a differential complex over the symmetric space G/B which is naturally associated to the algebraic structure of the group G.

In this chapter we will introduce the BGG complex. Since the construction works in great generality, we will introduce it in the case of an arbitrary complex semisimple Lie group. It is believed that the applicability of the BGG complex to analytic index theory will extend well beyond the case of $\mathrm{SL}(3,\mathbb{C})$.

Definition 3.1. Let G be any Lie group and let $\mathcal{X}$ be a homogeneous space for G. A *homogeneous vector bundle* over $\mathcal{X}$ is a vector bundle $p : E \to \mathcal{X}$ with an action of G by smooth vector bundle maps, such that the following diagram commutes for each $g \in G$:

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p"'] & E \arrow[d, "p"] \\ \mathcal{X} \arrow[r, "g"] & \mathcal{X} \end{tikzcd} .$$

The group G acts on the sections of a homogeneous bundle: if $\sigma \in C^{\infty}(\mathcal{X}; E)$ then we define

$$(g \cdot \sigma)(x) = g(\sigma(g^{-1}x)).$$

Example 3.2. The bundle of p-forms on a homogeneous G-manifold $\mathcal{X}$ is a homogeneous bundle for any $p \in \mathbb{N}$. The action of $g \in G$ on sections is precisely the pull-back by g^{-1}.

Definition 3.3. A differential operator D between two homogeneous vector bundles E and F is G-equivariant if

$$D(g\sigma) = g(D\sigma)$$

for all sections σ of E, and all $g \in G$.

The BGG complex will be a complex of G-equivariant differential operators between homogeneous bundles over G/B. Moreover, each of the homogeneous bundles will decompose G-equivariantly into a direct sum of complex line bundles. From this point of view the BGG-complex is very convenient.

The conception of the BGG complex came from Bernstein, Gel'fand and Gel'fand's work on $\mathfrak{g}$-modules. Their complex was an algebraic homology complex which resolves a finite dimensional representation using Verma modules (which are the universal modules in the category of highest weight modules for $\mathfrak{g}$). It was subsequently observed that the complex also has a geometrical interpretation, which is what we will be using in this thesis.

We begin now by describing this relationship between algebra and geometry.

3.2 Homogeneous vector bundles

The results of this section are quite general. Let G be an arbitrary Lie group, and B any closed subgroup. The quotient space G/B is a smooth manifold (see, *eg*, [War83, Theorem 3.58]). To launch the discussion of homogeneous vector bundles, we describe a particular construction of a homogeneous bundle over G/B. We will see that this construction actually yields all homogeneous vector bundles over G/B.

Notation. For an element x of G, we will denote its image in the quotient G/B by $\underline{x}$. The identity element of G will be denoted by e.

Notation. Let $\lambda : B \to \operatorname{Aut} V$ be a finite-dimensional representation of B. We will use the symbol λ to denote both the representation of B, and the infinitesimal representation of the Lie algebra $\mathfrak{b}$ it induces. This will be the case throughout this work.

Given such a λ, the bundle $G \underset{B}{\times} V$ over G/B will be the quotient of the trivial bundle $G \times V$ over G by the following action of B:

$$b \cdot (x, v) = (xb^{-1}, \rho(b)v) \qquad (b \in B, x \in G, v \in V).$$

In other words, we identify the two vectors (xb, v) and $(x, \rho(b)v)$, which live in different fibres over points in a common B-coset.

Notation. We use the notation $\underline{(x, v)}$ to denote the image of $(x, v) \in G \times V$ in this quotient.

We therefore have a projection map

$$\begin{aligned} \pi : G \underset{B}{\times} V &\rightarrow G/B \\ \underline{(x,v)} &\mapsto \underline{x} \end{aligned}$$

which makes $G \underset{B}{\times} V$ into a vector bundle. The fibres of $G \underset{B}{\times} V$ are isomorphic to V as vector spaces.

The action of G which makes $G \underset{B}{\times} V$ into a homogeneous vector bundle is simply

$$g \cdot \underline{(x,v)} = \underline{(gx,v)} \qquad (g, x \in G, v \in V).$$

Note that the subgroup B maps the fibre over the identity coset $\underline{e} \in G/B$ to itself. With the resulting action of B, this fibre is canonically isomorphic to the original space V, as a representation.

This last observation indicates how to realize any homogeneous vector bundle with the above construction.

Proposition 3.4. *Let E be a homogeneous vector bundle over G/B. Then E is G-equivariantly isomorphic to $G \underset{B}{\times} V$, where $V = E_{\underline{e}}$ is the fibre of E over $\underline{e}$, with its induced B-representation.*

Proof. The isomorphism is given by the map

$$\begin{aligned} G \underset{B}{\times} V &\rightarrow E \\ \underline{(x,v)} &\mapsto x \cdot v. \end{aligned}$$

The map is well-defined since

$$\underline{(xb, b^{-1} \cdot v)} \mapsto x \cdot v.$$

It is an isomorphism on the fibre over $\underline{e}$ by definition, and since it is clearly smooth and G-equivariant, this suffices to prove it is an isomorphism everywhere. $\square$

Example 3.5. The trivial bundle $G/B \times \mathbb{C}$ is a homogeneous bundle with G-action

$$\begin{aligned} g \ : \ G/B \times \mathbb{C} &\ \rightarrow \ G/B \times \mathbb{C} \\ (\underline{x}, v) &\ \mapsto \ (g \cdot \underline{x}, v) \end{aligned}$$

for $\underline{x} \in G/B$, $v \in \mathbb{C}$, and $g \in G$. It is isomorphic to $G \underset{B}{\times} V_0$ where V_0 is the one-dimensional trivial representation of B.

Proposition 3.4 provides an algebraic viewpoint on homogeneous vector bundles. We now want to do the same for G-invariant differential operators between them. Let

$$D : C^{\infty}(G/B; E) \rightarrow C^{\infty}(G/B; F)$$

be a G-invariant differential operator between two homogeneous vector bundles $E = G \underset{B}{\times} V$ and $F = G \underset{B}{\times} W$. It is generally convenient to lift sections of E to B-equivariant sections of the trivial bundle $G \times V$, as in the following proposition.

Proposition 3.6. *Smooth sections of the bundle* $G \underset{B}{\times} V \xrightarrow{\pi} G/B$ *are in one-to-one correspondence with smooth functions* $\tilde{\sigma} : G \to V$ *which satisfy*

$$\tilde{\sigma}(xb^{-1}) = \rho(b)\tilde{\sigma}(x). \tag{3.2.1}$$

Proof. The section corresponding to such a function $\tilde{\sigma}$ is defined by

$$\sigma(x) = \underline{(x, \tilde{\sigma}(x))}.$$

The equivariance condition 3.2.1 ensures that $\sigma(\underline{xb}) = \sigma(\underline{x})$ for all $b \in B$.

Conversely, if $\sigma \in C^\infty(\mathcal{X}; G \underset{B}{\times} V)$, then at each point $\underline{x} \in G/B$, we have

$$\sigma(\underline{x}) = \underline{(y, v)}$$

for some $y \in xB$ and $v \in V$. Rewriting this as

$$\sigma(\underline{x}) = \underline{(x, \rho(x^{-1}y)v)},$$

the second coordinate defines the value of the lifting $\tilde{\sigma}(x)$.

That these two processes are mutually inverse is easily checked. □

Remark 4. We have ignored issues of smoothness throughout this discussion. For more details see, for instance, [War83].

We will use this correspondence a lot in what follows. Frequently we will not distinguish the notation for a section from that for its lift, omitting the decoration $\tilde{}$ used above.

A G-equivariant differential operator is completely determined by its action on functions near $\underline{e} \in G/B$, or more specifically, by its action on jets of sections at $\underline{e}$. We recall the definition of a jet.

Definition 3.7. Let E a smooth vector bundle over a manifold $\mathcal{X}$. A smooth section σ of E is said to vanish to order k at x if, in some (equivalently, in any) trivializing coordinates for E around x, the degree k Taylor expansion of σ at x is zero.

The k-jets of E at x are the equivalence classes of sections, where two sections are equivalent if their difference vanishes to order k.

The ∞-jets of E at x are equivalence classes of sections, where two sections are equivalent if their difference vanishes to order k for all $k \in \mathbb{N}$.

The equivalence class of a section σ in the space of k-jets at x is denoted $J_x^k \sigma$, for $k = 1, 2, \cdots, \infty$. The space of k-jets of E at x is denoted by $J_x^k(E)$.

A differential operator D between two bundles E and F descends to act on jets, although there is a loss of degree in the process. Specifically, if $D : C^\infty(\mathcal{X}; E) \to C^\infty(\mathcal{X}; E)$ is a differential operator of order d, then passing to jets gives a map

$$D : J_x^k(E) \to J_x^{k-d}(F),$$

for all $k \geq d$. Note also that a differential operator of order d is determined completely by its action on the space of d-jets at each point $x \in \mathcal{X}$.

If E is a homogeneous bundle over $\mathcal{X}$, then the action of G on sections also passes to an action of G on jets:

$$g : J^k_x(E) \to J^k_{gx}(E),$$

for all $x \in \mathcal{X}$ and $g \in G$. Since the action of G on $\mathcal{X}$ is transitive, we have the following.

Lemma 3.8. *If D_1 and D_2 are G-equivariant differential operators of order d between homogeneous vector bundles E and F, such that their corresponding maps*

$$D_1, D_2 : J^d_{\underline{e}}(E) \to J^0_{\underline{e}}(E),$$

on d-jets at $\underline{e}$ agree, then $D_1 = D_2$.

Notation. Given a vector space V, the space $C^\infty(G, V)$ of smooth V-valued functions on G admits representations of G by left and right translations representation. These are, respectively,

$$L(g)f(x) = f(g^{-1}x)$$

and

$$R(g)f(x) = f(xg),$$

for $f \in C^\infty(G, V)$ and $g, x \in G$. These induce infinitesimal representations of the Lie algebra $\mathfrak{g}$:

$$L(X)f(x) = -X_R f(x)$$

and

$$R(X)f(x) = X_L f(x),$$

where X_L and X_R are, respectively, the left- and right-invariant vector fields on G generated by $X \in \mathfrak{g}$. Note that, at the point $e \in G$, the fields X_L and X_R agree. We denote the corresponding tangent vector at e by X.

Since B is a connected Lie group, jets of sections of the homogeneous bundle $G \underset{B}{\times} V$ at $\underline{e}$ can be described by an infinitesimal version of Proposition 3.6: they lift to jets of functions $\tilde{\sigma} : G \to V$ which satisfy the infinitesimal equivariance condition

$$R(X)\tilde{\sigma}(e) = -\rho(X)\tilde{\sigma}(e), \tag{3.2.2}$$

for all $X \in \mathfrak{b}$. A fundamental point of this section is a surprisingly elegant description of these jet spaces in terms of highest-weight modules for the Lie algebra $\mathfrak{g}$.

We start with jets of functions $f : G \to V$. Since the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ can be thought of as left-invariant differential operators on G, there is a pairing

$$J_e^{\infty}(G; V) \times (\mathcal{U}(\mathfrak{g}) \otimes V^*) \to \mathbb{C},$$

where V^* is the dual space of V. This pairing is given by

$$\langle J_e^{\infty} f, (A \otimes \phi)\rangle \mapsto \phi(R(A)f(e)), \tag{3.2.3}$$

where $R(A)$ denotes the extension of the infinitesimal right-translation representation to $A \in \mathcal{U}(\mathfrak{g})$. Now, restricting from jets of arbitrary functions to those jets $J^{\infty}_{\underline{e}}\tilde{\sigma}$ which satisfy the infinitesimal B-equivariance condition (3.2.2), the pairing descends to a pairing

$$J^{\infty}_{\underline{e}}(E) \times (\mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*) \to \mathbb{C}.$$

Since the point $\underline{e}$ in $\mathcal{X} = G/B$ is fixed by B under the translation action, the jet space $J^{\infty}_{\underline{e}}(E)$ inherits a representation of B from the left-translation representation of G on $C^{\infty}(\mathcal{X}, V)$. This jet space is not preserved by all of G, but the derivative of the left-translation representation induces an action of $\mathfrak{g}$, as right-invariant differential operators on the lifted jets. Moreover, these two actions are compatible, in the sense that

$$bXb^{-1}J^{\infty}_{\underline{e}}\sigma = (Ad(b)X)J^{\infty}_{\underline{e}}\sigma,$$

for any $b \in B$, $X \in \mathfrak{g}$, and any section σ of E.

The second component $\mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*$ of the pairing also carries compatible actions of B and $\mathfrak{g}$. The action of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*$ is simply by composition on the left. The action of B on $\mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*$ is also, ostensibly, the action of composition on the left, although to make this meaningful, one must define

$$b \cdot (A \otimes \phi) = \mathrm{Ad}(b)A \otimes b \cdot \phi.$$

Lemma 3.9. *The pairing*

$$J^{\infty}_{\underline{e}}(E) \times (\mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*) \to \mathbb{C}$$

defined by (3.2.3) is invariant under the actions of B and $\mathfrak{g}$, in the sense that for any $J^{\infty}_{\underline{e}}\sigma \in J^{\infty}_{\underline{e}}(E)$ and $A \otimes \phi \in \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*$,

$$\langle b \cdot \sigma, b \cdot A \otimes \phi \rangle = \langle \sigma, (A \otimes \phi) \rangle$$

for $b \in B$, and

$$\langle X \cdot \sigma, (A \otimes \phi) \rangle + \langle \sigma, X \cdot (A \otimes \phi) \rangle = 0$$

for $X \in \mathfrak{g}$.

Remark 5. The latter equation is a differential form of invariance.

Proof. This is straightforward, once the notation has been navigated. Let ρ denote the representation of B on V. Consider the first equality. If $A = X_1 \cdots X_n$, where $X_1, \ldots, X_n \in \mathfrak{g}$, then

$$\begin{aligned}
&\langle b \cdot \sigma, b \cdot (A \otimes \phi) \rangle \\
&\quad = (b \cdot \phi)\left((R(\mathrm{Ad}(b)A)\, b \cdot \tilde{\sigma})(e)\right) \\
&\quad = \phi\left(\rho(b^{-1})\frac{d}{dt_1} \cdots \frac{d}{dt_n} L(b)\tilde{\sigma}(b \exp(t_1 X_1) \cdots \exp(t_n X_n) b^{-1})|_{t_1,\ldots,t_n=0}\right) \\
&\quad = \phi\left(\rho(b^{-1})\frac{d}{dt_1} \cdots \frac{d}{dt_n} \tilde{\sigma}(\exp(t_1 X_1) \cdots \exp(t_n X_n) b^{-1})|_{t_1,\ldots,t_n=0}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \phi\left(\frac{d}{dt_1}\cdots\frac{d}{dt_n}\tilde{\sigma}(\exp(t_1X_1)\cdots\exp(t_nX_n))|_{t_1,\ldots,t_n=0}\right) \\
&= \phi\left(R(A)\tilde{\sigma}(e)\right) \\
&= \langle \sigma, (A\otimes\phi)\rangle.
\end{aligned}
$$

For the second equality,

$$
\begin{aligned}
&\langle X\cdot\sigma, (A\otimes\phi)\rangle + \langle\sigma, X\cdot(A\otimes\phi)\rangle \\
&\quad = \phi(R(A)L(X)\tilde{\sigma}(e)) + \phi(R(XA)\tilde{\sigma}(e)) \\
&\quad = \phi(L(X)R(A)\tilde{\sigma}(e)) + \phi(R(XA)\tilde{\sigma}(e)) \\
&\quad = 0,
\end{aligned}
$$

since the left and right invariant vector fields on G agree at e, but the infinitesimal left and right regular representations differ by a sign.

$\square$

Of course, we are only ever interested in differential operators of some given finite order. Let us define the subspace $\mathcal{U}^{(k)}(\mathfrak{g})$ of order-k elements of $\mathcal{U}(\mathfrak{g})$:

$$
\mathcal{U}^{(k)}(\mathfrak{g}) = \mathrm{span}\{X_1X_2\cdots X_j \quad | \quad X_1,\ldots,X_j\in\mathfrak{g} \text{ and } 0\le j\le k\}.
$$

Note that we are including the empty product (where $j=0$), which by convention represents the identity element $1\in\mathcal{U}(\mathfrak{g})$.

Lemma 3.10. *The pairing of Equation (3.2.3) restricts to a non-degenerate pairing*

$$J^k_{\underline{e}}(E) \times (\mathcal{U}^{(k)}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*) \to \mathbb{C}.$$

Proof. That the pairing restricts follows from the fact that if $A \in \mathcal{U}^{(k)}(\mathfrak{g})$, Equation (3.2.3) only depends on the order k part of $J^\infty_{\underline{e}}\tilde{\sigma}$.

Suppose that some k-jet $J^k_{\underline{e}}\sigma$ is annihilated by the pairing. Then its lift $J^k_{\underline{e}}\tilde{\sigma} \in J^\infty_{\underline{e}}(G;V)$ is annihilated by all $A \in \mathcal{U}^{(k)}(\mathfrak{g})$. But it is an immediate consequence of the definition of k-jets that $J^k_{\underline{e}}\sigma$ is nonzero if and only if some differential operator of order less than or equal to k does not annihilate it.

To complete the proof, note that both of the spaces in the above pairing are finite dimensional, and so it suffices to show that they have the same dimension. Choose some basis $\{B_1, \cdots, B_p, X_1, \cdots, X_q\}$ for $\mathfrak{g}$ such that the first p elements are a basis for $\mathfrak{b}$. By the Poincaré-Birkhoff-Witt Theorem (see [Dix96, Theorem 2.1.11]) there is a basis for $\mathcal{U}^{(k)}(\mathfrak{g})$ given by

$$\left\{ X_1^{n_1} \cdots X_q^{n_q} B_1^{n_{q+1}} \cdots B_p^{n_{p+q}} \;\middle|\; \sum_{i=1}^{p+q} n_i \leq k \right\}.$$

It follows that

$$\begin{aligned} \dim(\mathcal{U}^{(k)}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*) &= (\dim V^*) \sum_{j=1}^{k} \dim \mathrm{Sym}^j(\mathfrak{g}/\mathfrak{b}) \\ &= \dim V. \dim \bigoplus_{j=1}^{k} \mathrm{Sym}^j(\mathfrak{g}/\mathfrak{b}), \end{aligned}$$

which is the dimension of the space of Taylor polynomials of degree k on $\mathfrak{g}/\mathfrak{b}$ valued in V, and hence of $J^{\infty}_{\underline{e}}(E)$.

□

Combining this with Lemma 3.9, we see that the two spaces in the pairing of Lemma 3.10 are dual B-modules. (Note, however, that they no longer carry $\mathfrak{g}$-actions.) Therefore, we have the following situation. Any G-equivariant differential operator D between $E = G \underset{B}{\times} V$ and $F = G \underset{B}{\times} W$, is described by the B-equivariant map

$$D_{\underline{e}} : J^k_{\underline{e}}(E) \to F_{\underline{e}} = W.$$

Invoking the above duality, this corresponds to a B-equivariant map

$$W^* \to (\mathcal{U}^{(k)}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*).$$

As a final step, we observe an instance of Frobenius reciprocity.

Proposition 3.11 (Frobenius Reciprocity). *There is a natural identification*

$$\mathrm{Hom}_B(W^*,\ \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*) = \mathrm{Hom}_{(\mathfrak{g},B)}\,(\mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} W^*,\ \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*),$$

Proof. Essentially, the only difference between the two Hom-spaces is that the latter carries with it a copy of $\mathcal{U}(\mathfrak{g})$, in a $\mathcal{U}(\mathfrak{g})$-linear way.

To produce the correspondence explicitly, given $\Psi \in \mathrm{Hom}_B(W^*,\ \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*)$, we can define its corresponding map $\boldsymbol{\Psi} \in \mathrm{Hom}_{(\mathfrak{g},B)}\,(\mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} W^*,\ \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*)$ by

$$\boldsymbol{\Psi}(A \otimes \phi) \stackrel{\text{def}}{=} A \cdot \Psi(\phi).$$

The inverse correspondence takes a map $\boldsymbol{\Psi} \in \mathrm{Hom}_{(\mathfrak{g},B)}(\mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} W^*,\ \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*)$ and restricts it to

$$\Psi(\phi) = \boldsymbol{\Psi}(1 \otimes \phi),$$

for $\phi \in V^*$. $\square$

Summarizing:

Theorem 3.12. *Let V and W be finite-dimensional representations of B. There is a one-to-one correspondence between the set of G-equivariant differential operators between the homogeneous vector bundles $G \underset{B}{\times} V$ and $G \underset{B}{\times} W$, and the set of $\mathfrak{g}$-equivariant maps from $\mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} W^*$ to $\mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V^*$.*

3.3 Structure theory for complex semisimple groups

In the case where G is a complex semisimple Lie group, and B its Borel subgroup, the correspondence of Theorem 3.12 can be elegantly realized as an equivalence of categories. In order to describe this we will need to appeal to the enormous machinery which has been developed to study representations of complex semisimple groups. In this section we will describe some of the key components of this machine, without proof. For complete details, the reader is referred to one of the many texts—for instance, Knapp [Kna86] and Dixmier [Dix96] both cover the requisite material.

Let G be a complex semisimple group. For simplicity let us suppose that G is a group of complex matrices which is closed under conjugate transpose. This certainly includes the few examples we care about. In particular, it includes the groups $\mathrm{SL}(n, \mathbb{C})$, which we will carry as key examples through this section.

The Cartan involution Θ on G is the group automorphism which sends each element to its inverse conjugate transpose. The set of elements fixed by Θ is a compact subgroup of G, denoted by K. The differential of Θ is an involution of the Lie algebra $\mathfrak{g}$, denoted by θ. The invariant subspace of θ is the Lie algebra $\mathfrak{k}$ of K.

Inside the Lie algebra $\mathfrak{g}$, we choose an abelian subalgebra which is maximal amongst all abelian subalgebras preserved by θ. Such a Lie subalgebra is called a *Cartan subalgebra* of $\mathfrak{g}$. We fix a Cartan subalgebra $\mathfrak{h}$ once and for all. This chosen subalgebra provides a reference datum upon which all the structure theory to follow is built. However, the particular choice of $\mathfrak{h}$ does not matter, since the Cartan subalgebra is unique up to conjugation in G.

We can decompose $\mathfrak{h}$ into the $+1$ and -1 eigenspaces of the involution θ. These are real Lie subalgebras of $\mathfrak{h}$. The former, denoted $\mathfrak{m}$, exponentiates to a compact abelian subgroup, *ie* a torus, which we denote by M_0. The latter, denoted $\mathfrak{a}$, exponentiates to a group A which is isomorphic to $\mathbb{R}^k$, for some k.

Example 3.13. For the group $\mathrm{SL}(n, \mathbb{C})$, the subgroup of diagonal matrices is a Cartan subalgebra. Let this be $\mathfrak{h}$. Then $\mathfrak{a}$ and $\mathfrak{m}$ are the Lie subalgebras of diagonal matrices with real and imaginary entries, respectively.

Now let $\rho : G \to \Gamma$ be a finite-dimensional representation of G. We will demand also that ρ be *holomorphic*, meaning that the infinitesimal representation of $\mathfrak{g}$ be complex linear. Since $\mathfrak{h}$ is abelian, ρ decomposes into a direct sum of one-dimensional representations for $\mathfrak{h}$. Each of these is given by a map

$$\lambda \in \mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C}),$$

called a *weight* of the representation ρ. We define the *weight space* of ρ with weight λ by

$$(\Gamma)_\lambda = \{v \in V \mid \rho(H)v = \lambda(H)v \quad \text{for all } H \in \mathfrak{h}\}.$$

Note that these need not be one-dimensional, since weights may occur with multiplicity.

Remark 6. Because the group M_0 is compact, any one-dimensional representation of it must be by scalars of modulus one. Therefore, the image of its Lie algebra $\mathfrak{k}$ under a weight λ must be purely imaginary. Conversely, giving the restriction of a weight to $\mathfrak{k}$ completely determines it, by complex-linearity. The implication of this is that weights can be equivalently described as maps λ for $\lambda \in i\mathfrak{k}^\dagger$, where $\mathfrak{k}^\dagger$ denotes the real dual of $\mathfrak{k}$, that is $\mathrm{Hom}(\mathfrak{k}, \mathbb{R})$. Similarly, weights are also determined by their restriction to maps in $\mathfrak{a}^\dagger = \mathrm{Hom}(\mathfrak{a}, \mathbb{R})$.

We will use whichever interpretation is convenient, the latter one being commonly favoured. The advantage of using $\mathfrak{a}^\dagger$ (or $\mathfrak{k}^\dagger$) is that, being a real vector space, it makes for nice geometrical pictures.

The structure of the Lie algebra $\mathfrak{g}$ can be probed by studying the weight-decomposition of the adjoint representation of $\mathfrak{g}$ on itself:

$$\begin{aligned} \mathrm{ad}(X) : \mathfrak{g} &\to \mathfrak{g} \\ Y &\mapsto [X, Y] \end{aligned}$$

First note that $\mathfrak{h}$ is contained in the 0-weight space $\mathfrak{g}_0$, and in fact $\mathfrak{h} = \mathfrak{g}_0$ by the maximality in the definition of $\mathfrak{h}$. The nonzero weights of the adjoint representation are the *roots* of $\mathfrak{g}$. We will denote the set of roots by Δ.

Example 3.14. In $\mathrm{SL}(n, \mathbb{C})$, let E_{ij} be the matrix with all entries zero except for the (i, j)-entry, which is one. The weight spaces of $\mathrm{SL}(n, \mathbb{C})$ are all one-dimensional, spanned by the matrices E_{ij} with $i \neq j$. One can compute that

$$\mathrm{ad}(H)E_{ij} = (H_{ii} - H_{jj})E_{ij},$$

where H_{ii} denotes the (i, i)-entry of $H \in \mathfrak{h}$. We will denote by α_{ij} the root

$$\alpha_{ij}(H) = H_{ii} - H_{jj}.$$

We note now an important fact about root spaces. Let $X \in \mathfrak{g}$ be an element of the root space $\mathfrak{g}_\alpha$. For any representation $\mathfrak{g} \to \mathrm{End}(\Gamma)$, if $v \in V$ is a vector of weight λ, then

$$\begin{aligned} H(Xv) &= [H,X]v + X(Hv) \\ &= \alpha(H)Xv + \lambda(H)Xv \end{aligned}$$

for all $H \in \mathfrak{h}$, so that Xv is a vector of weight $\alpha + \lambda$. In other words,

$$\mathfrak{g}_\alpha \cdot (\Gamma)_\lambda \subseteq (\Gamma)_{\lambda+\alpha}. \tag{3.3.1}$$

We will use this fact frequently.

In particular, applying this to the adjoint representation, we see that the root spaces satisfy $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. In fact, the roots span an integer lattice in $\mathfrak{a}^\dagger$. This is called the *root lattice*, and denoted by Λ_R. The roots do not, however, form a linearly independent generating set for Λ_R. For instance, it is a consequence of semisimplicity that if α is a root of $\mathfrak{g}$, then so is $-\alpha$. But there are usually other linear dependencies between the roots as well.

A linearly independent spanning set of roots $\mathcal{S} \subseteq \Delta$ is called a *system of simple roots* if every root can be written as a linear combination of elements of $\mathcal{S}$ with either all coefficients positive or all negative. It is a theorem that such a set always exists. The choice of a system of simple roots is not unique. The variety of possible choices is described by the Weyl group, which we will describe shortly.

Example 3.15. The set

$$\mathcal{S} = \{\alpha_{i,i+1} \quad | \quad i = 1, \cdots, n-1\}$$

is a system of simple roots for $\mathrm{SL}(n, \mathbb{C})$.

Having fixed some system of positive roots, we can introduce an ordering on the roots. We say that a root (or more generally, a weight) is *positive* if it is a linear combination of simple roots with only positive coefficients. The set of positive roots will be denoted by Δ^+, and the set of positive weights by $(\mathfrak{h}^*)^+$. This allows us to define a partial ordering $\geq$ on Δ by declaring that $\alpha \geq \beta$ if $\alpha - \beta \in (\mathfrak{h}^*)^+$.

Example 3.16. With the above choice of simple roots, the positive roots of $\mathrm{SL}(n, \mathbb{C})$ are those α_{ij} with $i < j$.

From the fact that the spaces $\mathfrak{g}_\alpha$ satisfy $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, it is clear that the direct sum of the positive root spaces is a nilpotent Lie subalgebra of $\mathfrak{g}$. We denote this nilpotent Lie subalgebra by $\mathfrak{n}_+$. Likewise, the negative root spaces span a nilpotent Lie subalgebra $\mathfrak{n}_-$.

The space $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ is a solvable Lie algebra, called the Borel subalgebra. Exponentiating, we obtain the Borel subgroup B of G. It is a maximal connected solvable subgroup of G, and has the property that the symmetric space G/B is compact.

Example 3.17. In $\mathrm{SL}(n, \mathbb{C})$, the nilpotent Lie subalgebra $\mathfrak{n}_+$ is the set of strictly upper triangular matrices, and the Borel subalgebra $\mathfrak{b}$ is the set of traceless upper triangular matrices. The Borel subgroup is the subgroup of upper triangular matrices in $\mathrm{SL}(n, \mathbb{C})$.

The symmetric space G/B is the complete flag variety of $\mathbb{C}^n$, that is,

$$G/B \cong \{(0) = V_0 \leq V_1 \leq \cdots \leq V_n = \mathbb{C}^n \quad | \quad \dim V_i = i\}.$$

This can be seen by observing that $\mathrm{SL}(n, \mathbb{C})$ acts transitively on the space of complete flags, via the usual action of $\mathrm{SL}(n, \mathbb{C})$ on $\mathbb{C}^n$, and that the stabilizer of the flag

$$(0) \leq \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\rangle \leq \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\rangle \leq \cdots \leq \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\rangle$$

is B.

To be even more specific, consider the case of the group $\mathrm{SL}(3, \mathbb{C})$. Using the choice of a simple root system above, we put $\mathcal{S} = \{\alpha_{12}, \alpha_{23}\}$. For notational simplicity, let us put

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We denote the corresponding roots by $\alpha_X = \alpha_{12}$, $\alpha_Y = \alpha_{23}$ and $\alpha_Z = \alpha_{13} = \alpha_X + \alpha_Y$. The six roots of $\mathrm{SL}(3, \mathbb{C})$ are depicted in Figure 3.1. The positive roots are those in the upper-right half plane.

The high degree symmetry in the root system shown in Figure 3.1 is clear, and it is a general phenomenon. As mentioned earlier, if α is a root of G, then so is $-\alpha$. Picking nonzero elements $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$, one can show that their bracket $h_\alpha = [e_\alpha, f_\alpha]$ is nonzero in $\mathfrak{g}_0 = \mathfrak{h}$. Therefore e_α, f_α and h_α span a three-dimensional Lie subalgebra of $\mathfrak{g}$. There is only one three-dimensional semisimple complex Lie algebra,

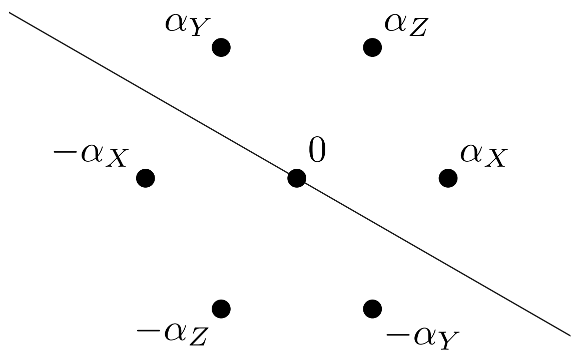

Fig. 3.1. Root system for $\mathrm{SL}(3,\mathbb{C})$.

up to isomorphism, namely $\mathfrak{sl}(2,\mathbb{C})$. Thus, after perhaps rescaling f_α and h_α, we have the relations

$$[h_\alpha, e_\alpha] = 2e_\alpha, \qquad [h_\alpha, f_\alpha] = -2f_\alpha, \qquad [e_\alpha, f_\alpha] = h_\alpha. \tag{3.3.2}$$

This $h_\alpha \in \mathfrak{h}$ is called the *co-root* of α, often denoted $\alpha^\vee$. Note that, by the first relation of (3.3.2), $\alpha(\alpha^\vee) = 2$.

Corresponding to each co-root in $\mathfrak{h}$, there is an annihilating hyperplane in $\mathfrak{a}^\dagger$. These hyperplanes are called *walls*. Reflection in the wall determined by $\alpha^\vee$ is given by the map

$$\begin{aligned} s_\alpha : \mathfrak{a}^* &\to \mathfrak{a}^* \\ \lambda &\mapsto \lambda - \lambda(\alpha^\vee)\alpha. \end{aligned}$$

It is a theorem that these reflections preserve the set of roots Δ. The group of symmetries of $\mathfrak{a}^\dagger$ generated by these reflections is called the *Weyl group* of G, and denoted by W.

Example 3.18. The co-root $h_{ij} = \alpha_{ij}^\vee$ for $\mathrm{SL}(n, \mathbb{C})$ is the diagonal matrix

$$h_{ij} = E_{ii} - E_{jj}.$$

The reflection in the co-root h_{ij} permutes the root system as follows:

$$s_{\alpha_{ij}} : \alpha_{kl} \mapsto \alpha_{\sigma(k)\sigma(l)},$$

where σ is the transposition $(i\ j)$ in S_n. This can be checked by direct computation with the roots. Thus the Weyl group for $\mathrm{SL}(n, \mathbb{C})$ is isomorphic to the symmetric group S_n.

There is also an alternative characterization of the Weyl group. Recall that the Lie subalgebra $\mathfrak{m}$ exponentiates to an abelian subgroup M_0 of the compact subgroup K. Now, let us denote by M the centralizer of M_0 in K, and by M' the normalizer of M_0 in K. The Weyl group is equal to M'/M. For a proof that these two definitions are equivalent, see [Kna86, §IV].

Example 3.19. The subalgebra $\mathfrak{m}$ of $\mathfrak{sl}(n, \mathbb{C})$ is the set of traceless diagonal matrices with purely imaginary entries. Its exponential is the group M_0 of diagonal matrices of determinant one whose diagonal entries are complex of modulus one. It is its own centralizer, so $M = M_0$. The normalizer of M_0 in K is the set of unitary matrices with exactly one nonzero entry in each row and each column. We again see that the Weyl group is the symmetric group S_n.

In this picture, the action of W on the roots is induced from the action of W on the weight spaces, via the adjoint action.

The walls partition $\mathfrak{a}^\dagger$ into regions, called *chambers*. The Weyl group permutes the chambers. Moreover, this action is freely transitive. This allows us to set up a one-to-one correspondence between elements of the Weyl group and Weyl chambers, once we fix a distinguished chamber to correspond to the identity element in W. Such a chamber is furnished by our previous choice of a simple root system: the *fundamental chamber* is defined by the distinguishing property that every λ in its interior satisfies

$$\lambda(\alpha) > 0 \quad \text{for each positive root } \alpha.$$

We will denote the fundamental chamber by $\mathcal{W}(\mathcal{S})$, or just $\mathcal{W}$. Every chamber is the fundamental chamber with respect to some unique choice of a system of simple roots, and thus we see that the Weyl group precisely indexes the possible choices of such a system.

Having fixed a system of real roots, we can endow the Weyl group with the structure of a directed graph. The reflection $s_\alpha \in W$ is called a *simple reflection* if α is a simple root. Then the simple reflections generate the Weyl group.

Definition 3.20. The *length function* on W is the map

$$l : W \to \mathbb{N}$$

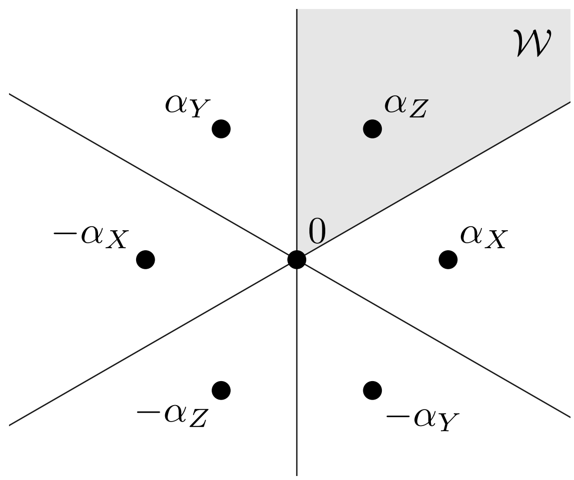

Fig. 3.2. The six Weyl chambers of $\mathrm{SL}(3, \mathbb{C})$. The fundamental chamber $\mathcal{W}$ is shaded.

defined by letting $l(w)$ be the length of the shortest product of simple reflections which equals w.

The set of elements of length k in W is denoted by $W^{(k)}$.

The directed graph will be constructed as follows.

Definition 3.21. For $w_1, w_2 \in W$, we will write $w_1 \to w_2$ if

(i) $l(w_2) = l(w_1) + 1$, and

(ii) $w_2 = s_\alpha w_1$ for some reflection $s_\alpha \in W$ (not necessarily simple).

This yields a directed graph, whose vertices are elements of W and whose directed edges are given by the arrows of Definition 3.21.

The directed graph in turn induces a partial ordering on the Weyl group. We write $w_1 \succeq w_2$ if there is a directed path (possibly trivial) from w_1 to w_2.

Example 3.22. If we pick the simple roots of $\mathrm{SL}(n, \mathbb{C})$ to be $\alpha_{i,i+1}$ $(i = 1, \ldots, n-1)$, then the simple reflections in $W = S_n$ are the transpositions $(i \;\; i{+}1)$.

Figure 3.3 shows the directed graph structure on the roots of the case of $\mathrm{SL}(3, \mathbb{C})$.

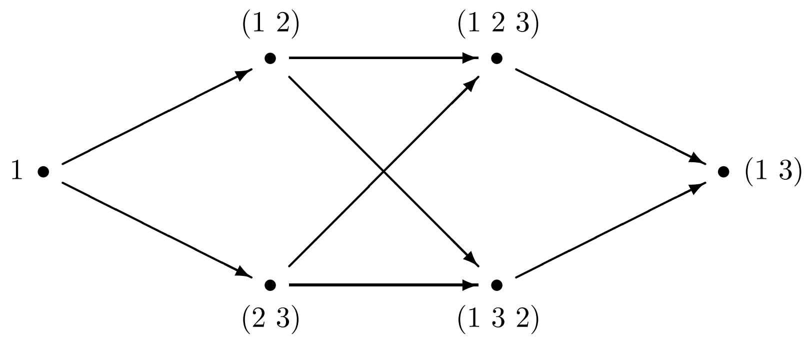

Fig. 3.3. Directed graph structure for the Weyl group of $\mathrm{SL}(3, \mathbb{C})$.

For the remainder of this thesis we will assume that on any complex semisimple group G a choice of a Cartan subalgebra and a system of simple roots has been fixed, once and for all.

3.4 Highest-weight modules

Having dealt with the basic structure theory, let us turn to the representation theory of the semisimple Lie algebra $\mathfrak{g}$.

The name *Verma module* is given to the $\mathfrak{g}$-modules of the form

$$\mathcal{M}_\lambda = \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V_\lambda,$$

where $\lambda \in \mathfrak{h}^*$ is any weight, and V_λ denotes the one-dimensional representation of $\mathfrak{b}$ upon which $\mathfrak{h}$ acts with weight λ, and $\mathfrak{n}_+$ acts trivially. That is a precise, but inelegant, definition of Verma modules. Their real importance is due to their universality as highest-weight modules.

Definition 3.23. A $\mathfrak{g}$-module $\mathcal{M}$ is called a *highest-weight module* with highest weight $\lambda \in \mathfrak{h}^*$ if it is generated by a vector v of weight λ which is annihilated by $\mathfrak{n}_+$.

Proposition 3.24 ([Dix96, Proposition 7.1.8]). *The Verma module $\mathcal{M}_\lambda$ is universal amongst highest-weight modules of highest weight λ,* ie*, for any highest-weight module $\mathcal{M}$ with highest weight λ, there is a surjective $\mathfrak{g}$-module homomorphism $\mathcal{M}_\lambda \to \mathcal{M}$. This map is unique up to a scalar multiple.*

The most obvious examples of highest-weight modules are the finite-dimensional irreducible representations of $\mathfrak{g}$. Why are these highest-weight modules? As remarked earlier, any finite dimensional $\mathfrak{g}$-module Γ decomposes into weight spaces. From (3.3.1), the action of the nilpotent subalgebra $\mathfrak{n}_+$ maps each weight space into another, and Engel's theorem (see [FH91, Theorem 9.9]) tells us that one of these weight spaces must be annihilated by $\mathfrak{n}_+$. Let us denote the weight of this annihilated space by λ. Now $\mathcal{U}(\mathfrak{n}_-)v$ must be all of Γ, for otherwise it would be a proper $\mathfrak{g}$-invariant subspace. Hence Γ is a highest weight module of highest weight λ.

What is more, the finite-dimensional $\mathfrak{g}$-modules are completely classified by their highest-weights, as we will now explain.

Definition 3.25. A weight is *dominant* if it lies in the closure of the fundamental Weyl chamber $\mathcal{W}$.

A weight of $\mathfrak{g}$ is *integral* if its pairing with every coroot of $\mathfrak{g}$ is an integer.

The set of integral weights forms a lattice in $\mathfrak{a}^\dagger$, which we denote by Λ_W. (The root lattice Λ_R of the previous section is a sublattice of this.) The set of dominant integral weights will be denoted by $\Lambda_W^{(\mathrm{Dom})}$.

Proposition 3.26. *There exists a basis for $\mathfrak{a}^\dagger$ comprised of integral weights $\omega_1, \ldots, \omega_n$ with the property that*

$$\overline{\mathcal{W}} = \left\{ \sum_{i=1}^{n} a_i \omega_i \quad | \quad a_i \geq 0 \quad \textit{for all } i \right\}.$$

The weights $\omega_1, \ldots, \omega_n$ are called the *fundamental weights* of $\mathfrak{g}$.

Example 3.27. From our earlier description of the co-roots of $\mathrm{SL}(n, \mathbb{C})$, as

$$h_{ij} = E_{ii} - E_{jj},$$

for $i, j = 1, 2, \ldots$ $(i \neq j)$, it is seen that the integral weights are those weights of the form

$$\lambda : H = \begin{pmatrix} H_{11} & 0 & \cdots 0 & \\ 0 & H_{22} & \cdots 0 & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_{nn} \end{pmatrix} \mapsto \sum_{i=1}^{n} \lambda_i H_{ii},$$

where $\lambda_1, \ldots, \lambda_n$ are integers.

Let e_j denote the integral weight

$$e_j : H \mapsto H_{jj},$$

where H is as above. Note that, since $\mathfrak{sl}(n, \mathbb{C})$ consists of traceless matrices,

$$e_1 + \cdots + e_n = 0.$$

The fundamental weights of $\mathrm{SL}(n, \mathbb{C})$ are the weights

$$\omega_j = e_1 + \cdots + e_j,$$

for $j = 1, \ldots, n-1$.

In particular, for $\mathrm{SL}(3, \mathbb{C})$ the fundamental weights are e_1 and $-e_3 = e_1 + e_2$.

Theorem 3.28. *The set of (equivalence classes) of representations of irreducible finite dimensional $\mathfrak{g}$-modules is in one-to-one correspondence with the set of dominant integral weights for $\mathfrak{g}$. In other words, for each dominant integral weight λ, there is a unique irreducible quotient of the Verma module $\mathcal{M}_\lambda$, which is a finite-dimensional $\mathfrak{g}$-module with highest weight λ.*

For a proof, see Theorem 4.28 and Proposition 5.7 of [Kna86].

Given $\lambda \in \Lambda_W^{(\mathrm{Dom})}$, we will write Γ^λ to denote the unique finite-dimensional irreducible $\mathfrak{g}$-module with highest weight λ.

3.5 The Bernstein-Gel'fand-Gel'fand complex, algebraically

The work of Bernstein-Gel'fand-Gel'fand [BGG75] shows how any finite dimensional $\mathfrak{g}$-module admits a resolution by direct-sums of Verma modules. The authors were originally interested in this resolution from the point of view of algebra. But the correspondence of Theorem 3.12 allows one to reinterpret this algebraic resolution in geometric terms. This geometric viewpoint will be the basis of the ensuing work, but let us begin by motivating the algebraic complex.

For illustration, consider the trivial representation $\Gamma^0 = \mathbb{C}$ of $\mathrm{SL}(3,\mathbb{C})$. Let us change our notation from Section 3.3, putting

$$\begin{aligned} X &= E_{12}, & Y &= E_{23}, & Z &= E_{13}, \\ \bar{X} &= -E_{21}, & \bar{Y} &= -E_{32}, & \bar{Z} &= -E_{31}. \end{aligned}$$

Thus, $\mathfrak{n}_+$ is spanned by X, Y and Z, with $[X,Y] = Z$, and similarly $\mathfrak{n}_-$ is spanned by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$, with $[\bar{X},\bar{Y}] = \bar{Z}$.

Being an $\mathfrak{sl}(3,\mathbb{C})$-module with highest weight 0, the trivial representation admits a realization as a quotient of the Verma module $\mathcal{M}_0$, which we write as

$$0 \longleftarrow \mathbb{C} \longleftarrow \mathcal{M}_0.$$

Let v be a highest-weight vector in $\mathcal{M}_0$.

Now consider the kernel $\mathcal{N}$ of this quotient. Since $\mathcal{M}_0$ is generated by the action of $\mathfrak{n}_-$ on v, it is clear that $\mathcal{N}$ is generated by the vectors $\bar{X}v$, $\bar{Y}v$ and $\bar{Z}v$. And since $\bar{Z}v = \bar{X}\bar{Y}v - \bar{Y}\bar{X}v$, it is generated by just $\bar{X}v$ and $\bar{Y}v$. These two vectors have weights $-\alpha_X$ and $-\alpha_Y$, respectively, and both are annihilated by $\mathfrak{n}_+$ in $\mathcal{N}$. Therefore, we have an exact sequence

$$0 \longleftarrow \mathbb{C} \longleftarrow \mathcal{M}_0 \longleftarrow \begin{array}{c} \mathcal{M}_{-\alpha_X} \\ \oplus \\ \mathcal{M}_{-\alpha_Y} \end{array} .$$

The theorem of Bernstein, Gel'fand and Gel'fand says that this process can be continued, resulting in a resolution of finite length in terms of direct sums of Verma modules.

To approach from a different direction, one can ask a more general question: for which pairs of weights λ and μ does there exist a map of Verma modules $\mathcal{M}_\lambda \to \mathcal{M}_\mu$? The answer comes from the order structure on the Weyl group, which was introduced in Section 3.3.

For notation, we will need to define the weight

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

This weight is an "affine factor" that appears ubiquitously in studying submodules of Verma modules. (A typical appearance is seen in Theorem 3.29 below.) Note that ρ is a *strictly* dominant weight, in that it lies in the interior of the fundamental chamber.

Proposition 3.29. *Let $\lambda \in \mathfrak{a}^\dagger$ be a weight. There is an inclusion of Verma modules $\mathcal{M}_\mu \hookrightarrow \mathcal{M}_\lambda$ if and only if*

(i) *λ is an integral weight,*

(ii) *$\lambda + \rho$ and $\mu + \rho$ lie in the same orbit of the Weyl group, and*

(iii) *there are $w_1, w_2 \in W$ with $w_1 \alpha = \lambda + \rho$, $w_2 \alpha = \mu + \rho$ and $w_1 \succeq w_2$, where α is the unique dominant weight in the orbit of $\lambda + \rho$.*

For a proof, see [Dix96, §7.8].

Therefore, there is a collection of $\mathfrak{g}$-module maps indexed by the directed graph structure on the Weyl group. The amazing result of Bernstein-Gel'fand-Gel'fand is that when these modules are assembled according to the lengths of the corresponding Weyl group elements, this system results in a resolution of a finite-dimensional representation, as follows.

Theorem 3.30 (Bernstein-Gel'fand and Gel'fand [BGG75, Theorem 10.1]). *Let Γ be an irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ. There is an exact sequence of $\mathfrak{g}$-modules*

$$0 \leftarrow \Gamma \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_s \leftarrow 0,$$

where

$$C_k = \bigoplus_{w \in W^{(k)}} \mathcal{M}_{w(\lambda+\rho)-\rho} \ .$$

3.6 The Bernstein-Gel'fand-Gel'fand complex, geometrically

To complete this chapter, we wish to reinterpret the algebraic complex of Theorem 3.30 by means of the correspondence of Theorem 3.12. The results of Section 3.2 can be packaged into the following statement.

Theorem 3.31. *Let G be a complex semisimple Lie group, with Borel subgroup B. There is a contravariant equivalence of categories between the category of G-equivariant differential operators between homogeneous line bundles over G/B and the category of homomorphisms between Verma modules for $\mathfrak{g}$.*

One can now apply this categorical equivalence to the algebraic BGG complex above. One needs to ask what happens to the finite dimensional representation Γ which is being resolved in Theorem 3.30.

Theorem 3.32 (Borel-Weil). *Let K be a complex semisimple group and let λ be a dominant integral weight for G. The irreducible finite-dimensional representation Γ^λ, with highest weight λ, is isomorphic to the space of holomorphic sections of the bundle $G \underset{B}{\times} V_\lambda$, with its natural action of G.*

For a proof, see [Kna86, §7]. We will refer to the map

$$\Gamma^\lambda \hookrightarrow C^\infty(G/B; G \underset{B}{\times} V_\lambda)$$

as the *Borel-Weil inclusion.*

Theorem 3.33 (See [BE89, Theorem 8.3.1]). *Let G be a complex semisimple group, B its Borel subgroup, and let λ be a dominant integral weight for G. There is a resolution of the Borel-Weil inclusion by G-equivariant differential operators,*

$$0 \to \Gamma^\lambda \hookrightarrow C^\infty(G/B; F_0) \to C^\infty(G/B; F_1) \to \ldots \to C^\infty(G/B; F_s),$$

where F_k is the direct sum of homogeneous complex line bundles

$$F_k = \bigoplus_{w \in W^{(k)}} G \underset{B}{\times} V_{\rho - w(\lambda+\rho)}.$$

We will shortly describe the one key example of the BGG complex which will be used in the remainder of this thesis. However, in order to make that example compatible with the ensuing notation, we must digress briefly to make a small change of convention.

3.7 Using the conjugate Borel subgroup

In this chapter we have followed the standard convention for defining the Borel subgroup B, namely that it is the subgroup with Lie algebra

$$\mathfrak{b} = \mathfrak{h} + \mathfrak{n}_+.$$

However, it will be notationally convenient for the remainder of the work to use an alternative choice:

$$\mathfrak{b} = \mathfrak{h} + \mathfrak{n}_-.$$

In the example of the group $\mathrm{SL}(n, \mathbb{C})$, this has the effect of exchanging the upper triangular subgroup for the lower triangular subgroup.

In order to translate between the two conventions, one can simply apply the Cartan involution Θ throughout. We will now list the results of this change of convention.

The primary difference is the class of $\mathfrak{g}$-modules which must be considered in Section 3.4.

Definition 3.34. A *lowest-weight module* is a $\mathfrak{g}$-module which is generated by a vector v which is annihilated by $\mathfrak{n}_-$.

The highest-weight Verma modules must be replaced with lowest weight modules throughout. We will denote these by

$$\overline{\mathcal{M}}_\lambda = \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} V_\lambda.$$

Finite-dimensional $\mathfrak{g}$-modules are lowest-weight modules as well as highest weight modules. A weight λ is called *anti-dominant* if $-\lambda$ is dominant. Then the finite-dimensional $\mathfrak{g}$-modules can be classified by their lowest weights, which are anti-dominant integral weights.

One convenient observation simplifies this issue. Suppose Γ is a finite-dimensional irreducible $\mathfrak{g}$-module, with highest weight λ. Then its dual space Γ^* is also an irreducible $\mathfrak{g}$-module with the contragredient representation, defined by

$$(X\phi)(v) = -\phi(Xv).$$

for $X \in \mathfrak{g}$, $\phi \in \Gamma^*$ and $v \in \Gamma$. If $v_1, \dots, v_n$ is a basis of weight vectors for Γ, then its dual basis $v_1^*, \dots, v_n^*$ is a basis of weight vectors for Γ^*, and the weights of v_j^* will be the negative of that for v_j. Thus, the finite-dimensional $\mathfrak{g}$-module with lowest weight $-\lambda$ is just the dual of the finite-dimensional $\mathfrak{g}$-module with highest weight λ.

The geometric BGG complex becomes as follows: for any dominant integral weight λ, there is a differential resolution

$$0 \to (\Gamma^\lambda)^* \hookrightarrow C^\infty(G/B; F_0) \to C^\infty(G/B; F_1) \to \dots \to C^\infty(G/B; F_s),$$

where F_k is the direct sum of homogeneous complex line bundles

$$F_k = \bigoplus_{w \in W^{(k)}} G \underset{B}{\times} V_{-\rho+w(\lambda+\rho)}.$$

3.8 The BGG complex for $\mathrm{SL}(3, \mathbb{C})$

We conclude this chapter with the most important example for the present work. Let $G = \mathrm{SL}(3, \mathbb{C})$, with B the subgroup of lower triangular matrices, and let $\lambda = 0$. The Weyl group is S_3. The orbit of 0 under the affine Weyl action

$$w : \mu \to -\rho + w(\mu + \rho)$$

is

$$\begin{aligned}
1 \cdot 0 &= 0, & (1\ 2\ 3) \cdot 0 &= -2\alpha_X - \alpha_Y, \\
(1\ 2) \cdot 0 &= -\alpha_X, & (1\ 3\ 2) \cdot 0 &= -\alpha_X - 2\alpha_Y, \\
(2\ 3) \cdot 0 &= -\alpha_Y, & (1\ 3) \cdot 0 &= -2\alpha_X - 2\alpha_Y.
\end{aligned}$$

Therefore, appealing to the directed graph of Figure 3.3, the BGG complex described in the previous section is

$$\begin{tikzcd}
 & & C^\infty(\mathcal{X}; E_{-\alpha_X}) \arrow[r] \arrow[ddr] & C^\infty(\mathcal{X}; E_{-2\alpha_X-\alpha_Y}) \arrow[dr] & \\
\mathbb{C} \arrow[r] & C^\infty(\mathcal{X}; E_0) \arrow[ur] \arrow[dr] & \bigoplus & \bigoplus & C^\infty(\mathcal{X}; E_{-2\alpha_X-2\alpha_Y}) \\
 & & C^\infty(\mathcal{X}; E_{-\alpha_Y}) \arrow[r] \arrow[uur] & C^\infty(\mathcal{X}; E_{-\alpha_X-2\alpha_Y}) \arrow[ur] &
\end{tikzcd}$$

where $\mathcal{X} = G/B$ and E_λ denotes the homogeneous vector bundle $G \underset{B}{\times} V_\lambda$.

Chapter 4

Homogeneous Bundles over $\mathrm{SL}(3, \mathbb{C})$

4.1 The space G/B and its fibrations

Let G be the group $\mathrm{SL}(3, \mathbb{C})$, and B the subgroup of lower triangular matrices. We are interested in the homogeneous space $\mathcal{X} = G/B$. This is a closed complex manifold of three complex dimensions. We will follow the notation for quotient spaces introduced earlier, namely for a point $x \in G$, we will denote its image in G/B by $\underline{x}$. We use e to denote the identity element in G.

With our particular choice of G and B, the quotient space $\mathcal{X}$ has some additional structure. Let us introduce the subgroups

$$P_X = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \in \mathrm{SL}(3, \mathbb{C}) \;\middle|\; \text{each } * \in \mathbb{C} \right\}$$

and

$$P_Y = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \in \mathrm{SL}(3, \mathbb{C}) \;\middle|\; \text{each } * \in \mathbb{C} \right\}$$

of G, both of which contain B. Corresponding to these two subgroups, there are two maps

$$\begin{tikzcd} & G/B \arrow[dl, "\tau_X"'] \arrow[dr, "\tau_Y"] & \\ G/P_X & & G/P_Y \end{tikzcd}$$

to the homogeneous spaces $G/P_X \cong \mathbb{C}\mathrm{P}^2$ and $G/P_Y \cong \mathbb{C}\mathrm{P}^2$. These two maps are fibrations, with fibres isomorphic to $P_X/B \cong P_Y/B \cong \mathbb{C}\mathrm{P}^1$.

Let us denote tangent distributions along the fibres of τ_X and τ_Y by $\mathcal{F}_X$ and $\mathcal{F}_Y$, respectively, *ie*,

$$\begin{aligned}\mathcal{F}_X &= \ker d\tau_X \subset T\mathcal{X},\\ \mathcal{F}_Y &= \ker d\tau_Y \subset T\mathcal{X}.\end{aligned}$$

Note that, since the fibration maps are G-equivariant, the line bundles $\mathcal{F}_X$ and $\mathcal{F}_Y$ are homogeneous subbundles of $T\mathcal{X}$.

In order to get a geometric picture of these fibrations, it is illuminating to look at them in local coordinates. Let N denote the group of upper-triangular matrices with all diagonal entries equal to 1. The map

$$\begin{aligned}\varphi_e : N &\rightarrow \mathcal{X}\\ x &\mapsto \underline{x}\end{aligned} \tag{4.1.1}$$

is an inclusion, diffeomorphic onto a dense open subset of $\mathcal{X}$. This sets up a coordinate patch in $\mathcal{X}$ which is modelled on the three-dimensional complex Heisenberg group N.

One can cover the homogeneous space $\mathcal{X}$ with charts of this form by using the translation action of G. Namely, for any point $\underline{g} \in \mathcal{X}$, we can define the coordinate system

$$\begin{aligned}\varphi_g : N &\rightarrow \mathcal{X} \\ x &\mapsto \underline{gx}.\end{aligned} \tag{4.1.2}$$

Remark 7. In fact, it suffices to use only six such coordinate charts, corresponding to the six elements of the Weyl group. This follows from the PLU-decomposition, which says that any $g \in \mathrm{SL}(3, \mathbb{C})$ can be decomposed (non-uniquely) as

$$g = \tilde{w}nb,$$

with $b \in B$, $n \in N$ and $\tilde{w}$ being a permutation matrix (with signed entries so that its determinant is one). The signs of the entries of $\tilde{w}$ (and even their magnitudes) can be chosen arbitrarily, for if d is a diagonal matrix then

$$g = (\tilde{w}d)(d^{-1}nd)(db)$$

is another decomposition of g of the same form.

The element $\tilde{w}$ belongs to the normalizer $N_K(\mathfrak{m})$ in K of skew-adjoint part of the Cartan subalgebra. By Example 3.19, it corresponds to an element of the Weyl group, that element being given by the permutation represented by the matrix $\tilde{w}$. Therefore, choosing one representative $\tilde{w}$ for each element of the Weyl group, the coordinate patches

$$\phi_{\tilde{w}}(N) = \underline{\tilde{w}N}$$

cover $\mathcal{X}$.

The coordinate patch (4.1.1) sets up an isomorphism between the complex Heisenberg Lie algebra

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a, b, c \in \mathbb{C} \right\}$$

and the tangent space $\mathfrak{g}/\mathfrak{b}$ to $\mathcal{X}$ at $\underline{e}$. The differential of the fibration map τ_X at $\underline{e}$ is the quotient map

$$(d\tau_X)_{\underline{e}} : \mathfrak{g}/\mathfrak{b} \to \mathfrak{g}/\mathfrak{p}_X,$$

where $\mathfrak{p}_X$ is the Lie algebra of P_X. In local coordinates, the kernel of this map is the subspace

$$\left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a \in \mathbb{C} \right\}$$

of $\mathfrak{n}$.

Let us define the elements X_1 and X_2 in $\mathfrak{n}$ by

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$X_2 = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the tangent vectors $d\varphi_e X_1$ and $d\varphi_e X_2$ belong to $\mathcal{F}_X$ at $\underline{e}$. Because τ_X is a G-equivariant map, it follows that in the coordinate chart φ_e, the bundle $\mathcal{F}_X$ is spanned by the left N-invariant vector fields generated by X_1 and X_2. Similarly, $\mathcal{F}_Y$ is tangent to the vector fields generated by

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}.$$

An observation worth noting is that the fields tangent to $\mathcal{F}_X$ and $\mathcal{F}_Y$ generate the entire tangent bundle of $\mathcal{X}$ as a Lie algebra.

4.2 The BGG complex for $\mathrm{SL}(3,\mathbb{C})$, concretely

Consider the Dolbeault complex for $\mathcal{X}$. Because of our choices elsewhere, we will give our Dolbeault complex in terms of holomorphic differentials:

$$\Omega^{0,0}\mathcal{X} \xrightarrow{\partial} \Omega^{1,0}\mathcal{X} \xrightarrow{\partial} \Omega^{2,0}\mathcal{X} \xrightarrow{\partial} \Omega^{3,0}\mathcal{X}.$$

Underlying these spaces are the bundles $\bigwedge^{p,0} T^*\mathcal{X}$, for $p = 0, 1, 2, 3$.

Let us define the smooth G-invariant sub-bundle of $T\mathcal{X}$,

$$\mathcal{Q} = \mathcal{F}_X \oplus \mathcal{F}_Y.$$

We use this to define decompositions of the above bundles.

Firstly, let I^1 be the sub-bundle of $\bigwedge^{1,0} T^*\mathcal{X}$ which annihilates $\mathcal{Q}$, and let $\mathcal{I}^1$ be its space of sections. Thus,

$$\mathcal{I}^1 = \{\omega \in \Omega^{1,0}\mathcal{X} \quad | \quad \omega(V) = 0 \text{ for any section } V \text{ of } \mathcal{Q}\}.$$

If we denote the space of holomorphic sections of the pth exterior power of $\mathcal{Q}^*$ by $\Omega^{p,0}\mathcal{Q}$, for $p \in \mathbb{N}$, then $\Omega^{1,0}\mathcal{Q} = (\Omega^{1,0}T^*\mathcal{X})/\mathcal{I}^1$.

Secondly, let I^2 be the sub-bundle of $\bigwedge^{2,0} T^*\mathcal{X}$ which annihilates $\bigwedge^2 \mathcal{Q}$, and let $\mathcal{I}$ be its space of sections. Therefore,

$$\mathcal{I}^2 = \{\omega \in \Omega^{2,0}\mathcal{X} \quad | \quad \omega(V_1, V_2) = 0 \text{ for any sections } V_1, V_2 \text{ of } \mathcal{Q}\},$$

and $\Omega^{2,0}\mathcal{Q} = (\Omega^{2,0}T^*\mathcal{X})/\mathcal{I}^2$.

The ensuing definitions are complex variants of the operators defined by Rumin in [Rum94].

Definition 4.1. Define a differential operator ∂_R by the composition

$$\partial_R : \Omega^{0,0}\mathcal{X} \xrightarrow{\partial} \Omega^{1,0}\mathcal{X} \longrightarrow \Omega^{1,0}\mathcal{Q}\,.$$

Define another differential operator by

$$\partial_R : \mathcal{I}^2 \longrightarrow \Omega^{2,0}\mathcal{X} \xrightarrow{\partial} \Omega^{3,0}\mathcal{X}\,.$$

These operators will be referred to as *Rumin-Dolbeault differentials.*

We have a diagram of G-equivariant maps

$$
\begin{tikzcd}
 & \mathcal{I}^1 \arrow[d] & \mathcal{I}^2 \arrow[d] \arrow[dr, "\partial_R"] & \\
\Omega^{0,0}\mathcal{X} \arrow[r, "\partial"] \arrow[dr, "\partial_R"'] & \Omega^{1,0}\mathcal{X} \arrow[r, "\partial"] \arrow[d] & \Omega^{2,0}\mathcal{X} \arrow[r, "\partial"] \arrow[d] & \Omega^{3,0}\mathcal{X} \\
 & \Omega^{1,0}\mathcal{Q} & \Omega^{2,0}\mathcal{Q} &
\end{tikzcd}
.
$$

At this point, something surprising happens. The order one differential operator ∂ in the middle of the Dolbeault complex decomposes into an order zero operator and an order two operator. This is explained by the following two lemmas.

Lemma 4.3 is a complex analogue of a result of Rumin ([Rum94]), although the approach we are taking follows [Eas99].

Lemma 4.2. *The composition of maps*

$$
\mathcal{I}^1 \longrightarrow \Omega^{1,0}\mathcal{X} \xrightarrow{\partial} \Omega^{2,0}\mathcal{X} \longrightarrow \Omega^{2,0}\mathcal{Q}
$$

is a $C^\infty(\mathcal{X})$-linear isomorphism.

Proof. Let $\omega \in \mathcal{I}^1$ and $f \in C^\infty(\mathcal{X})$. Then,

$$
\partial(f\omega) = \partial f \wedge \omega + f\partial\omega.
$$

The first term, $\partial f \wedge \omega$, is in $\mathcal{I}^2$, and so is killed in passing to the quotient space $\Omega^{2,0}\mathcal{Q}$. This proves $C^\infty(\mathcal{X})$-linearity.

We need to check that it is an isomorphism. Since the bundles underlying the domain and range of this map are both one-dimensional, we need only check that the underlying bundle map is nonzero at every point. Since the map is G-equivariant, it suffices to check it is nonzero at any point. We therefore work in the coordinate chart φ_e described in the previous section. We will suppress mention of the map φ_e, working directly on N.

Let

$$\begin{aligned} X &= \frac{1}{2}(X_1 - iX_2), \\ Y &= \frac{1}{2}(Y_1 - iY_2), \\ Z &= \frac{1}{2}(Z_1 - iZ_2), \end{aligned}$$

where X_1, X_2, Y_1 and Y_2 are as in the previous section, and

$$\begin{aligned} Z_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ Z_2 &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Let ξ, η and ζ be the holomorphic forms dual to the holomorphic frame X, Y, Z. Then ζ annihilates X and Y, which span the holomorphic part of $\mathcal{Q}$. Hence ζ spans the restriction of $\Omega^{1,0}\mathcal{Q}$ to this chart, as a $C^\infty(N)$-module.

We compute $\partial\zeta$ using Cartan's formula, and the fact that $\overline{\partial}\zeta = 0$. We have

$$i_X \partial\zeta = L_X \zeta - \partial i_X \zeta = L_X \zeta,$$

and hence

$$\partial\zeta(X, Y) = -\zeta([X, Y]) = -1.$$

This shows that

$$\partial\zeta = -\xi \wedge \eta.$$

Since this does not belong to $\mathcal{I}^2$, its image is nonzero in the quotient.

□

Lemma 4.3. *Let $\theta \in \mathcal{Q}^*$. There exists a unique lift $\tilde{\theta} \in \Omega^{1,0}\mathcal{X}$ of θ such that $\partial\tilde{\theta} \in \mathcal{I}^2$.*

Proof. Let $\mathring{\theta}$ be any lift of θ. By the preceding lemma, there is a unique $\omega \in \mathcal{I}^1$ such that

$$\partial\omega = \partial\mathring{\theta} \quad \text{modulo } \mathcal{I}^2.$$

Then the desired lift is

$$\tilde{\theta} = \mathring{\theta} - \omega.$$

Uniqueness follows from the uniqueness of ω. □

Definition 4.4. Define a *Rumin-Dolbeault differential*

$$D_R : \Omega^{1,0}\mathcal{Q} \to J^2$$

by setting

$$D_R\theta = \partial\tilde{\theta},$$

with notation as in the previous lemma.

Note that D_R is a second order differential operator: one must first differentiate $\mathring{\theta}$ to determine $\tilde{\theta}$, and then one applies ∂ to the result.

We also note that D_R is a G-equivariant operator, as follows. Let $g \in G$. Following the notation of the proof of Lemma 4.3, if $\mathring{\theta}$ is a lift of θ, then $g \cdot \mathring{\theta}$ is a lift of $g \cdot \theta$. Therefore,

$$D_R(g \cdot \theta) = \partial(g \cdot \mathring{\theta} - g \cdot \omega) = g \cdot \partial(\mathring{\theta} - \omega) = g \cdot D_R\theta.$$

Let us compute the Rumin-Dolbeault differentials in local "Heisenberg" coordinates. We will work in the coordinate chart given by φ_e of Equation (4.1.1), once again suppressing mention of the map φ_e itself. Let X, Y and Z be the holomorphic vector fields as in the proof of Lemma 4.3, and let ξ, η and ζ be the corresponding holomorphic dual forms.

These form a frame field for the cotangent bundle of this chart. Restricted to this chart, the bundle I^1 is spanned pointwise by ζ, and that I^2 is spanned pointwise by $\xi \wedge \zeta$ and $\eta \wedge \zeta$. We use this frame to furnish splittings of the quotient map, namely, we identify the space $\Omega^{1,0}\mathcal{Q}$ with the space of 1-forms spanned by ξ and η, and identify the space $\Omega^{2,0}\mathcal{Q}$ with the space of 2-forms spanned by $\xi \wedge \eta$.

We computed previously that

$$\partial\zeta = -\xi \wedge \eta.$$

We now show that

$$\partial\xi = \partial\eta = 0.$$

This can again be either using Cartan's formula, as before, or by direct computation in Euclidean coordinates. Let us do the latter.

Put coordinates on the complex Heisenberg group by

$$(x, y, z) \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The group multiplication in these coordinates is given by

$$(x, y, z) \cdot (t_1, t_2, t_3) = (x + t_1, y + t_2, z + t_3 + xt_2).$$

Differentiating this with respect to the complex coordinates t_1, t_2 and t_3 in turn gives the left-invariant vector fields:

$$\begin{aligned} X &= \frac{\partial}{\partial x}, \\ Y &= \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \\ Z &= \frac{\partial}{\partial z}. \end{aligned}$$

The dual forms are

$$\begin{aligned} \xi &= \partial x, \\ \eta &= \partial y, \\ \zeta &= \partial z - x\,\partial y. \end{aligned}$$

Applying ∂ to these 1-forms gives the result.

Now we begin to compute the Rumin-Dolbeault operators. The Dolbeault differential on 0-forms is given by

$$\partial : a \mapsto (Xa)\,\xi + (Ya)\,\eta + (Za)\,\zeta,$$

for $a \in C^\infty(\mathcal{X})$. When we pass to the quotient by $\mathcal{I}^1$ we get

$$\partial_R a = (Xa)\,\xi + (Ya)\,\eta.$$

Next consider D_R. We begin with the form

$$\theta = a\,\xi + b\,\eta, \qquad\qquad (a, b \in C^\infty(\mathcal{X}))$$

lifted from the quotient $\Omega^{1,0}\mathcal{Q}$ by the local splitting mentioned above. Then,

$$\partial\theta = (-Ya + Xb)\,\xi \wedge \eta - (Za)\,\xi \wedge \zeta - (Zb)\,\eta \wedge \zeta.$$

Following the algorithm described in the proof of Lemma 4.3, we take

$$\omega = (-Ya + Xb)\zeta.$$

Hence,

$$
\begin{aligned}
D_R\theta &= \partial(\theta - \omega) \\
&= -(Za)\,\xi\wedge\zeta - (Zb)\,\eta\wedge\zeta + X(-Ya + Xb)\,\xi\wedge\zeta + Y(-Ya+Xb)\,\eta\wedge\zeta \\
&= (-(XY+Z)a + X^2 b)\,\xi\wedge\zeta + (-Y^2 a + (YX - Z)b)\,\eta\wedge\zeta.
\end{aligned}
$$

Finally, if we start with the 2-form

$$\theta = a\,\xi\wedge\zeta + b\,\eta\wedge\zeta \quad \in \mathcal{I}^2,$$

then

$$\partial_R\theta = \partial\theta = (-Ya + Xb)\,\xi\wedge\eta\wedge\zeta.$$

Therefore, the Rumin-Dolbeault complex in these local coordinates is as follows:

$$
\begin{tikzcd}
 & \langle\xi\rangle \arrow[r, "-(XY+Z)"] \arrow[d, dotted, no head] \arrow[ddr, "-Y^2"'] & \langle\xi\wedge\zeta\rangle \arrow[dr, "-Y"] \arrow[d, dotted, no head] & \\
\langle 1\rangle \arrow[ur, "X"] \arrow[dr, "Y"'] & \bigoplus \arrow[d, dotted, no head] & \bigoplus \arrow[d, dotted, no head] & \langle\xi\wedge\eta\wedge\zeta\rangle \\
 & \langle\eta\rangle \arrow[r, "YX-Z"'] \arrow[uur, "X^2"] & \langle\eta\wedge\zeta\rangle \arrow[ur, "X"'] &
\end{tikzcd}
\tag{4.2.1}
$$

Theorem 4.5. *The Rumin-Dolbeault complex for $\mathcal{X}$ is a complex.*

Proof. Since the complex is G-equivariant, it suffices to prove that the compositions of successive operators are zero at a single point in $\mathcal{X}$. We will take advantage of the above coordinate computations. Checking that the collection of maps (4.2.1) is a complex amounts to checking that each of the four diamonds in that diagram anticommute. This is straightforward:

(i) $-(XY+Z)X + X^2Y = X[X,Y] - ZX = 0,$

(ii) $-Y^2X + (YX-Z)Y = Y[X,Y] - ZY = 0,$

(iii) $Y(XY+Z) - XY^2 = [Y,X]Y + YZ = 0,$

(iv) $-YX^2 + X(YX-Z) = [X,Y]X - XZ = 0.$

$\square$

Theorem 4.6. *The Rumin-Dolbeault complex for $\mathcal{X}$ is G-equivariantly isomorphic to the BGG complex of Section 3.8.*

Proof. We have observed that the Rumin-Dolbeault complex is a complex of G-equivariant differential operators between homogeneous line bundles over $\mathcal{X}$. All of the operators are of degree at least one, and in particular are not scalar. Proposition 3.12 and Proposition 3.29 completely constrain the possibilities for the system of operators, up to scalar multiples. The fact that both the Rumin-Dolbeault complex and the BGG-complex are complexes shows that the scalar multiples are consistent.

$\square$

4.3 Compact and nilpotent pictures

Recall from Section 3.2 that the homogeneous vector bundles over G/B are all of the form

$$E = G \underset{B}{\times} V$$

where V is equipped with some representation

$$\mu \ : \ B \longrightarrow End(V)$$

of B. In what follows, we will only need to consider one dimensional bundles, so the representation μ will be a character of B.

Again, we denote by $\underline{(g;v)}$ the element in $E = G \underset{B}{\times} V$ which is the image of $(g,v) \in G \times B$ under the usual quotient map. Recall that sections of E can be described by their lifts to B-equivariant sections of the trivial bundle $G \times V$ over G, as in Proposition 3.6:

$$C^{\infty}(\mathcal{X};E) \cong \{\tilde{\sigma} : G \to V \mid \tilde{\sigma}(gb) = \mu(b^{-1})\tilde{\sigma}(g) \quad \text{for } g \in G, \ b \in B\}.$$

Under this correspondence, the action of G on sections of E is given simply by

$$g_1 \cdot \tilde{\sigma}(g) = \tilde{\sigma}(g_1^{-1} g).$$

This description of E gives a picture which is very convenient for doing global computations. We will also need two other well-known descriptions each of which has its merits.

The first is the so-called "compact picture", which is convenient for global analysis. The Iwasawa decomposition of G allows us to write

$$G = KAN_{-}$$

where $K = SU(3)$ is the maximal compact subgroup of G, A is the group of determinant one diagonal matrices with positive entries, and N_- is the group of lower triangular unipotent matrices. Since $AN_- \subseteq B$, we see that

$$\mathcal{X} = G/B \cong K/M$$

where $M = K \cap B$ is the subgroup of diagonal matrices with entries of modulus one:

$$M = \left\{ \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{pmatrix} \;\middle|\; |w_1| = |w_2| = |w_3| = 1 \text{ and } w_1 w_2 w_3 = 1 \right\}.$$

We get

$$E \cong K \underset{M}{\times} V.$$

It is this picture which most easily allows us to define an inner product on the sections of E. First, fix an inner product on V. In what follows we will have $V = \mathbb{C}$, in which case we use the standard inner product on $\mathbb{C}$. We define the inner products of $\sigma_1,\ \sigma_2 \in C^\infty(\mathcal{X}; E)$ by using the inner product of their lifts to K:

$$\langle \sigma_1,\ \sigma_2 \rangle = \int_K \overline{\tilde{\sigma_1}(k)} \tilde{\sigma_2}(k) dk, \tag{4.3.1}$$

where dk denotes Haar measure on K. The resulting Hilbert space will be denoted by $L^2(\mathcal{X}; E)$.

Remark 8. The following remarks are not essential for what follows, but might provide some context for the definition of inner product (4.3.1).

Firstly, as has been remarked previously, the space $\mathcal{X}$ does not admit a G-invariant metric. But it does of course admit a K-invariant metric, and an accompanying K-invariant volume form. Integration of a function against that volume form is given by

$$\int_{\mathcal{X}} f(x)d\,\mathrm{Vol} = \int_K \tilde{f}(k)dk$$

(up to a normalizing constant), where

$$\tilde{f}(k) = f(\underline{k}).$$

Secondly, the choice of an inner product on V, which is canonically isomorphic to the fibre at $\underline{e}$ of E, extends to a K-invariant Hermitian structure on all of $E = K \underset{M}{\times} V$ by translating by elements of K. The only thing that needs to be checked here is that the inner product on V is respected by the action of the stabilizer of $\underline{e}$ in K, namely M. But since M is compact, $\mu(M) \subseteq S^1$. Once again, it is not possible to make this Hermitian structure G-invariant.

Thus, the above inner product is the usual inner product for a Hermitian bundle over a Riemannian metric, using the natural K-invariant structures.

The other picture, which we will use is commonly called the "noncompact picture" or "nilpotent picture". This is the local picture of G/B based upon the chart φ_e of Section 4.1. Recall that the basic version of such a chart was the inclusion of the nilpotent subgroup

$$N = N_+ = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a,\ b,\ c \in \mathbb{C} \right\}$$

in G. The LU-decomposition for three-by-three matrices gives a decomposition

$$G = NB$$

which holds almost everywhere—that is, except for a set of Haar measure zero, every $g \in G$ can be written uniquely as

$$g = xb$$

for some $x \in N$, $b \in B$. This shows what we have previously claimed—that the chart φ_e includes N into G as an open dense subset.

The fact that we are omitting only a set of measure zero in restricting to this chart implies that we lose nothing of the measure theory of E. In other words, we can compute the inner product of two sections as described in Equation (4.3.1) by working only on this chart. The transferral of the inner product from the compact to the nilpotent picture is a well-known procedure (see, for instance, Knapp [Kna86]). However, the key idea will be useful to us again in the future, so we will conclude this section by recalling the argument.

The key idea is to stop by an intermediate description of the inner product, which is obtained by distributing the formula (4.3.1) along B-cosets. We will need the following properties of the Haar measures for the groups G, B, $N = N_+$ and K, which can be found in [Kna86, §V.6].

The groups G, N and K are unimodular, and we will denote their Haar measures by dg, dn and dk, respectively. Let $d_r b$ and $d_\ell b$ denote the right- and left-invariant measures on B, respectively. The modular function for B is $|\rho|^4$, *ie*,

$$d_\ell b = |\rho(b)|^{-4} d_r b,$$

where ρ is the character of B defined by

$$\begin{aligned} \rho(\exp H) &= \exp(\alpha_X + \alpha_Y)(H), \qquad && (H \in \mathfrak{h}), \\ \rho(n) &= 0, && (n \in N_-) \end{aligned}$$

The products

$$G = KB$$

and

$$G = NB \qquad \text{(almost everywhere)}$$

give rise to formulae

$$dg = dk d_r b$$

and

$$dg = dx d_r b$$

for the Haar measure dg on G.

To describe the formula for the inner product (4.3.1) in the nilpotent picture, we need some notation. Given $g \in G$, let its KAN_{-}-decomposition be

$$g = kan.$$

We then define

$$\beta_K(g) = an$$

to be the "AN_{-}"-component of g.

Proposition 4.7. *The K-invariant inner product of two sections $\sigma_1, \sigma_2 \in C^\infty(\mathcal{X}; E)$ is given by*

$$\langle \sigma_1, \sigma_2 \rangle = \int_N \overline{\tilde{\sigma_1}(x)} \tilde{\sigma_2}(x) |\mu\rho^{-2}(\beta_K(v))|^2 dx,$$

where dx is Haar measure on N. The product $\mu\rho^{-2}$ refers to the product of characters,

$$\mu\rho^{-2}(b) = \mu(b)\rho(b)^{-2}. \qquad (b \in B)$$

Proof. We start with a pair of sections $\sigma_1,\ \sigma_2 \in C^\infty(\mathcal{X}; E)$, which we lift to functions $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ on K. We extend these to functions F_1 and F_2 on G, not by $\mu(B)$-equivariance, but by the formula

$$\begin{aligned} F_i(kb) &= \rho(b)^{-2} \tilde{\sigma}_i(k) \\ &= \mu\rho^{-2}(b) \tilde{\sigma}_i(kb), \end{aligned}$$

for $k \in K$, $b \in B$ and $i = 1, 2$. The inner product will be realized on G by integration of F_1 and F_2 against a weight function of mass one on each B-coset. To this end, choose some $\phi \in C_c(B)$ with $\phi \geq 0$ and

$$\int_B \phi \, d_\ell b = 1.$$

By averaging we may assume that $\phi(mb) = \phi(b)$ for all $m \in M = B \cap K$ and $b \in B$. Now extend ϕ to all of G by defining

$$\phi(kb) = \phi(b) \qquad (k \in K,\ b \in B).$$

Then

$$\int_B \phi(gb) d_\ell b = 1$$

for any $g \in G$.

We get

$$\begin{aligned}
\langle \sigma_1,\ \sigma_2 \rangle &= \int_K \overline{\tilde{\sigma}_1(k)} \sigma_2(k) \left(\int_B \phi(b) d_\ell b \right) dk \\
&= \int_K \int_B \overline{F_1(kb)} F_2(kb) |\rho(b)|^4 \phi(b) \ d_\ell b \ dk \\
&= \int_G \overline{F_1(g)} F_2(g) \phi(\beta_K(g)) \ dg \\
&= \int_N \int_B \overline{F_1(xb)} F_2(xb) |\rho(b)|^4 \phi(b) \ d_\ell b \ dx \\
&= \int_N \overline{F_1(x)} F_2(x) \left(\int_B \phi(b) \ d_\ell b \right) dx \\
&= \int_N \overline{\tilde{\sigma}_1(x)} \tilde{\sigma}_2(x) |\mu\rho^{-2}(\beta_K(x))|^2 \ dx,
\end{aligned}$$

as claimed. $\square$

4.4 The group action

The contents of this section and the next will not actually be needed for the subsequent material in this thesis. However, they will certainly be important for continuing the work which is begun here. The γ-element for $G = \mathrm{SL}(3, \mathbb{C})$ is, of course, a G-equivariant K-homology class. We will eventually need to show that the repackaging of the BGG-complex in this form retains the G-equivariance of the complex.

The disadvantage of working in the compact and nilpotent pictures is that the description of the group action becomes complicated. Nevertheless, we will need to work in these pictures. We include this section and the next to explain the situation.

As before, let V be a one-dimensional representation of B, given by the character μ, and let $E = G \underset{B}{\times} V$. Firstly, working in the compact picture, if σ is a section of E then, for $k \in K$,

$$\begin{aligned} g \cdot \tilde{\sigma}(k) &= \tilde{\sigma}(g^{-1}k) \\ &= \mu(an)^{-1}\tilde{\sigma}(k') \end{aligned} \tag{4.4.1}$$

where

$$g^{-1}k = k'an$$

is the KAN_{-}-decomposition of $g^{-1}k$. The KAN_{-}-decomposition lets us define an action of G on K: for $g \in G$, $k \in K$ we let $g \cdot k$ be the compact part of the KAN_{-}-decomposition of gk. Recalling the notation of the previous section, the "AN_{-}-component" of gk will be denoted by $\beta_K(gk)$, so that

$$gk = (g \cdot k)\, \beta_K(gk).$$

Then Equation (4.4.1) becomes

$$g \cdot \tilde{\sigma}(k) = \mu(\beta_K(g^{-1}k))^{-1} \tilde{\sigma}(g^{-1} \cdot k). \tag{4.4.2}$$

There is a similar formula for the group action the nilpotent picture. Using LU-decomposition we have an (almost everywhere defined) action

$$\begin{aligned} G \times N &\to N \\ (g, x) &\mapsto g \cdot x, \end{aligned}$$

characterized by

$$gx = (g \cdot x)\, \beta_N(gx)$$

for some $g \cdot x \in N$ and $\beta_N(gx) \in B$. Then

$$g \cdot \tilde{\sigma}(x) = \mu(\beta_N(g^{-1}x))^{-1} \tilde{\sigma}(g^{-1} \cdot x). \tag{4.4.3}$$

4.5 Unitary representations

The inner product (4.3.1) on sections of the bundle $E_\mu = G \underset{B}{\times} V_\mu$ is K-invariant, but not G-invariant. This means that the group action on sections is not a unitary representation on $L^2(\mathcal{X}; E_\mu)$. However, since E_μ is a line bundle, any two Hermitian structures on it must only differ by a scalar factor on each fibre, that is, by multiplication by a smooth function. Therefore, for each $g \in G$, there is a function

$$c_g^{(\mu)} \in C^\infty(\mathcal{X}),$$

such that for any sections σ_1 and σ_2 of E_μ,

$$\langle \sigma_1, \sigma_2 \rangle = \langle\, c_g^{(\mu)}\ g \cdot \sigma_1, c_g^{(\mu)}\ g \cdot \sigma_2 \,\rangle.$$

The "conformality factor" $c_g^{(\mu)}$ can be explicitly computed using the trick from the end of Section 4.3. Let σ_1, σ_2 be sections of E, and as before put

$$F_i(kb) = \mu\rho^{-2}(b)\tilde{\sigma}_i(kb) \qquad\qquad (i = 1, 2).$$

Continuing to use the notation of the previous section, we have, for $g \in G$,

$$\begin{aligned} \langle \sigma_1, \sigma_2 \rangle &= \int_G \overline{F_1(g')}\, F_2(g')\, \phi(\beta_K(g'))\; dg' \\ &= \int_G \overline{F_1(g^{-1}g')}\, F_2(g^{-1}g')\, \phi(\beta_K(g_1^{-1}g'))\; dg'. \end{aligned}$$

Writing $g' = kan$ for the KAN_{-}-decomposition of g', we have

$$\begin{aligned} g^{-1}g' &= (g^{-1} \cdot k)\, \beta_K(g^{-1}g') \\ &= (g^{-1} \cdot k)\, \beta_K(g^{-1}k) \qquad \in KB \end{aligned}$$

Hence,

$$
\begin{aligned}
\langle \sigma_1, \sigma_2 \rangle &= \int_K \int_B \overline{F_1(g^{-1} \cdot k)}\, F_2(g^{-1} \cdot k)\, |\rho(\beta_K(g_1^{-1} g))|^{-4}\, \phi(\beta_K(g_1^{-1} g))\, d_r b\, dk \\
&= \int_K \overline{F_1(g^{-1} \cdot k)}\, F_2(g^{-1} \cdot k) \left(\int_B \phi(\beta_K(g^{-1} g))\, d_\ell b \right) dk \\
&= \int_K \overline{\tilde{\sigma}_1(g^{-1} \cdot k)}\, \sigma_2(g_1^{-1} \cdot k)\, |\mu \rho^{-2}(\beta_K(g^{-1} k))|^2\, dk \\
&= \int_K \overline{(g \cdot \tilde{\sigma}_1)(k)}\, (g \cdot \tilde{\sigma}_2)(k)\, |\mu^2 \rho^{-2}(\beta_K(g^{-1} k))|^2\, dk,
\end{aligned}
$$

where the last equality uses (4.4.2). Therefore, if we put

$$c_g^{(\mu)}(\underline{k}) = |\mu \rho^{-1}(\beta_K(g^{-1} k))|^2$$

for $k \in K$, then the map

$$\pi(g) \; : \; \sigma \mapsto c_g^{(\mu)}\, g \cdot \sigma$$

defines a unitary representation of G on $L^2(\mathcal{X}; E)$.

Chapter 5

Differential Operators on the Complex Heisenberg Group

5.1 Introduction

We now commence the analytical study of the differential operators in the BGG resolution associated to $\mathrm{SL}(3,\mathbb{C})$. This will centre on studying the differential operators tangent to the two fibrations $\mathcal{F}_X$ and $\mathcal{F}_Y$ of Section 4.1. In this chapter, we begin by looking at the local structure of G/B, that is, by working in the nilpotent picture of Section 4.3. This means working on the group N_+, which is isomorphic to the complex Heisenberg group.

5.2 The Heisenberg Lie algebra

Let $\mathbb{H}$ be the three-dimensional complex Heisenberg group, realized as the group of unipotent upper triangular complex matrices

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a,\ b,\ c \in \mathbb{C} \right\}. \tag{5.2.1}$$

The Lie algebra of this group is the set of strictly upper triangular matrices

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x,\ y,\ z \in \mathbb{C} \right\}, \tag{5.2.2}$$

with the usual commutator bracket

$$[A, B] = AB - BA.$$

Thus $\mathfrak{h}$ is a six-dimensional real Lie algebra.

Remark 9. Because we will not be dealing with the full group $\mathrm{SL}(3, \mathbb{C})$ in this chapter, there should be no confusion in using $\mathfrak{h}$ as the Heisenberg Lie algebra here, and the Cartan Lie subalgebra of $\mathfrak{g}$ elsewhere.

Let us fix the basis

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}$$

$$Z_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Z_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These basis elements satisfy the commutation relations

$$\begin{aligned} [X_1, Y_1] &= -[X_2, Y_2] &&= Z_1 \\ [X_1, Y_2] &= [X_2, Y_1] &&= Z_2 \end{aligned} \tag{5.2.3}$$

and all other commutators are zero. In particular, the elements Z_1 and Z_2 are central in $\mathfrak{h}$. We will also use the same six symbols to denote the left invariant vector fields on $\mathbb{H}$ which are generated by these Lie algebra elements.

As is usual in complex manifold theory, we now have a confusion of maps which might be considered as “multiplication by i”. Firstly, the real Lie algebra $\mathfrak{h}$ has a complex structure, and hence multiplication by i gives an automorphism of $\mathfrak{h}$ as a vector space, which we denote by J. Explicitly,

$$\begin{aligned} J \ : \ X_1 &\longmapsto X_2 \\ X_2 &\longmapsto -X_1, \quad \textit{etc.} \end{aligned}$$

By left translation, this induces an automorphism of the tangent bundle $T\mathbb{H}$, which we also denote by J. This special notation is crucial, because the map J on $T\mathbb{H}$ should not be confused with the notion of “multiplication by i” on the complex-valued vector fields on $\mathbb{H}$. The former is a rotation of tangent vectors of the manifold $\mathbb{H}$, while the latter is simply a scalar multiplication. In fact, given $f \in C^{\infty}(\mathbb{H})$, one has

$$(JV)f = iVf,$$

for any vector field V on $\mathbb{H}$, if and only if f is a holomorphic function on $\mathbb{H}$ with respect to the complex coordinates (a, b, c).

One formally defines the complex tangent vectors on $\mathbb{H}$ as follows. We can firstly form the complexified Lie algebra

$$\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}.$$

Thus, $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \oplus i\mathfrak{h}$. The operation of $\mathfrak{h}$ by differentiation on smooth functions is extended by letting i act as multiplication by i. In the analogous way, we form the complexified tangent bundle

$$T_{\mathbb{C}}\mathbb{H} = T\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}.$$

The structural automorphism J on $\mathfrak{h}$ extends to a complex linear map on $\mathfrak{h}_{\mathbb{C}}$. Since $J^2 = -1$, $\mathfrak{h}_{\mathbb{C}}$ decomposes into a $+i$ eigenspace and a $-i$ eigenspace for J.

Definition 5.1. The $+i$ eigenspace of J is the space of holomorphic vectors, denoted by $\mathfrak{h}'$. The $-i$ eigenspace is the space of antiholomorphic vectors, denoted by $\mathfrak{h}''$.

These eigenspaces are Lie subalgebras, isomorphic to the real Lie algebra $\mathfrak{h}$.

The complexified Lie algebra $\mathfrak{h}_{\mathbb{C}}$ is a six complex-dimensional space. The basis vectors $X_1,\ X_2,\ Y_1,\ Y_2,\ Z_1,\ Z_2$ of the real Lie algebra become complex-basis vectors for $\mathfrak{h}_{\mathbb{C}}$, satisfying the same commutation relations of Equation (5.2.3).

An alternative choice of basis is given by the three holomorphic elements

$$\begin{aligned} X &= \tfrac{1}{2}(X_1 - iX_2) \\ Y &= \tfrac{1}{2}(Y_1 - iY_2) \\ Z &= \tfrac{1}{2}(Z_1 - iZ_2) \end{aligned} \tag{5.2.4}$$

and the three antiholomorphic elements

$$\begin{aligned} \bar{X} &= \tfrac{1}{2}(X_1 + iX_2) \\ \bar{Y} &= \tfrac{1}{2}(Y_1 + iY_2) \\ \bar{Z} &= \tfrac{1}{2}(Z_1 + iZ_2). \end{aligned} \tag{5.2.5}$$

These basis vectors satisfy the commutation relations

$$[X, Y] = Z \quad \text{and} \quad [\bar{X}, \bar{Y}] = \bar{Z}, \tag{5.2.6}$$

and all other commutators are zero. In particular, holomorphic vectors commute with antiholomorphic vectors.

Again, these six symbols will also denote the left-invariant complex fields on $\mathbb{H}$ generated by them. From earlier comments, or direct computation, one can check that a function on $\mathbb{H}$ is holomorphic with respect to the coordinates (a, b, c) of Equation (5.2.1) if and only if $\overline{V} f = 0$ for every antiholomorphic $\overline{V} \in \mathfrak{h}''$.

5.3 Automorphisms of the Heisenberg group

The Heisenberg group comes equipped with a natural family of endomorphisms, indexed by $\mathbb{R}^2$. Having fixed our realization of the group as lower triangular matrices, these endomorphisms are parameterized as follows:

$$\theta_{s,t} \ : \ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & sa & stc \\ 0 & 1 & tb \\ 0 & 0 & 1 \end{pmatrix} \tag{5.3.1}$$

for $s,\ t \in \mathbb{R}$. The endomorphism $\theta_{s,t}$ is an automorphism if and only if s and t are both nonzero. All of the maps $\theta_{s,t}$ descend to endomorphisms of the Lie algebra:

$$\theta_{s,t} \ : \ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & sx & stz \\ 0 & 0 & ty \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.3.2}$$

Additionally, there is an automorphism of the group defined by

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & -b & c - ab \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}.$$

This exhibits an extra symmetry of the group which is occasionally useful in simplifying proofs. It shows that any fact about vector fields on $\mathbb{H}$ remains true if X and Y are replaced throughout by Y and $-X$, respectively.

5.4 The Algebra of differential operators on $\mathbb{H}$

Differential operators on the Heisenberg group, and more generally on manifolds with a local Heisenberg structure, have been studied in great detail. One of the notable contributions was the pseudodifferential calculus introduced by Beals and Greiner [BG88]. A crucial concept there is the introduction of a non-standard filtration on the algebra of differential operators on $\mathbb{H}$.

In the standard pseudodifferential calculus on Euclidean space, the filtration on the algebra of differential operators is defined to be the weakest filtration such that vector fields have order one. Since a filtered algebra must have the property that

$$\mathrm{Order}(AB) \leq \mathrm{Order}(A) + \mathrm{Order}(B),$$

this suffices to define orders of all differential operators.

In the Heisenberg calculus, one starts with differential operators on the Heisenberg group. Let us use the real Heisenberg group here, for notational simplicity. We can take advantage of the totally non-integrable subbundle $\mathcal{Q}$ of $T\mathbb{H}$ which is spanned pointwise by the left-invariant vector fields generated by

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the definition of the order, we declare only that vector fields tangent to $\mathcal{Q}$ have order one. This suffices to define a filtration on all differential operators since

$$Z = XY - YX.$$

Note that the vector field Z is now an order two operator.

Many of the fundamental results from the classical pseudodifferential calculus remain. In this picture, one ends up with an analogue of an ellipticity, called sub-ellipticity. The prototypical example of a sub-elliptic operator is the *Heisenberg Laplacian*,

$$\Delta_H = -X^2 - Y^2.$$

It is known that sub-elliptic operators, like elliptic operators, are Fredholm as operators on spaces of L^2-sections.

In the context of the local Heisenberg structure we described in Chapter 4, we have even more structure to take advantage of. The subbundle $\mathcal{Q}$ splits canonically into two line bundles, thanks to the fibrations described in Section 4.1. Therefore, we might introduce a bi-filtration on the algebra of differential operators which treats these two directions separately.

The starting point for this is to consider the left-invariant differential operators, acting on the space $C^\infty(\mathbb{H})$ of smooth complex-valued functions on $\mathbb{H}$. These are in one-to-one correspondence with the elements of the universal enveloping algebra $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ of $\mathfrak{h}_{\mathbb{C}}$. The universal enveloping algebra $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ is spanned, as a vector space, by products of the four elements $X, \bar{X}, Y, \bar{Y}$ (including the "empty" product 1). There is a great deal of redundancy in using the set of all such products to span $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$, but this will not concern us here. For ease of discussion we introduce the following terminology.

Definition 5.2. An element of $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ which is a product (perhaps trivial) in the elements X, $\bar{X}$, Y and $\bar{Y}$ is called an *elementary monomial.*

In what follows, we will use the partial ordering on $\mathbb{N} \times \mathbb{N}$ given by

$$(m, n) \leq (m', n') \text{ if and only if } m \leq m' \text{ and } n \leq n'.$$

Definition 5.3. A *bifiltration* of an algebra $\mathcal{A}$ is a collection of linear subspaces $\mathcal{A}^{(m,n)} \subseteq \mathcal{A}$, parameterized by $(m, n) \in \mathbb{N} \times \mathbb{N}$, with the following properties:

(i) $\bigcup_{\mathbb{N}\times\mathbb{N}} \mathcal{A}^{(m,n)} = \mathcal{A}$,

(ii) $\mathcal{A}^{(m_1,n_1)}.\mathcal{A}^{(m_2,n_2)} \subseteq \mathcal{A}^{(m_1+m_2,n_1+n_2)}$ for all (m_1, n_1) and (m_2, n_2) in $\mathbb{N} \times \mathbb{N}$.

Definition 5.4. Define a bifiltration on the algebra $\mathcal{A} = \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ as the weakest bifiltration which satisfies

- $\mathcal{A}^{(0,0)} = \text{span}\{1\}$,
- $\mathcal{A}^{(1,0)} = \text{span}\{1, X, \bar{X}\}$,
- $\mathcal{A}^{(0,1)} = \text{span}\{1, Y, \bar{Y}\}$.

We say that $D \in \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ has order at most (m, n) if $D \in \mathcal{A}^{(m,n)}$.

Note that

$$Z = XY - YX$$

and

$$\bar{Z} = \bar{X}\bar{Y} - \bar{Y}\bar{X},$$

so that Z and $\bar{Z}$ have order $(1, 1)$.

If D is a monomial in $X, \bar{X}, Y, \bar{Y}, Z, \bar{Z}$, then it is clear that

$$\text{Order}(D) \leq (a + c, b + c),$$

where

$$\begin{aligned} a &= \text{total exponent of } X \text{ and } \bar{X} \text{ in } D, \\ b &= \text{total exponent of } Y \text{ and } \bar{Y} \text{ in } D, \\ c &= \text{total exponent of } Z \text{ and } \bar{Z} \text{ in } D. \end{aligned}$$

Moreover, if D can be written as

$$D = \sum_{j=1}^{n} D_j,$$

for any collection of elementary monomials, then at least one of these must have

$$\text{Order}(D_j) \geq (a + c, b + c),$$

as a consequence of the Poincaré-Birkhoff-Witt Theorem. It follows that every such monomial has a well-defined two-parameter order. Specifically,

$$\mathrm{Order}(X^{a_1}\bar{X}^{a_2}Y^{b_1}\bar{Y}^{b_2}Z^{c_1}\bar{Z}^{c_2}) = (a_1 + a_2 + c_1 + c_2, b_1 + b_2 + c_1 + c_2).$$

An element of $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ will be called *homogeneous of order* (m, n) if it is a linear combination of elementary monomials, each of which has order exactly (m, n).

Remark 10. In order to distinguish this two-parameter order from the one-parameter order of the standard Heisenberg calculus, we might have called ours the “bi-order” and referred to “bihomogeneous elements”. However, this terminology seems cumbersome. We will make it clear whenever we use “order” and “homogeneous” in anything other than this two-parameter sense.

The order of a homogeneous element is determinable from its behavior under the morphisms $\theta_{s,t}$ of Section 5.3. Those Lie algebra endomorphisms extend naturally to the universal enveloping algebra $\mathcal{U}(\mathfrak{h})$, and we have the following lemma.

Lemma 5.5. *A left-invariant differential operator A on $\mathbb{H}$ is homogeneous of order (m, n) if and only if*

$$\theta_{s,t}(A) = s^m t^n A \tag{5.4.1}$$

for all $s,\ t \in \mathbb{R}$.

Proof. It is immediate from the definition of $\theta_{s,t}$ that (5.4.1) holds for $A = X,\ \bar{X},\ Y$ or $\bar{Y}$. The fact that $\theta_{s,t}$ is an algebra homomorphism on $\mathcal{U}(\mathfrak{h})$, allows this to be extended to all monomials in $X, \bar{X}, Y, \bar{Y}$, and hence all homogeneous elements.

Conversely, suppose A satisfies (5.4.1) with $m = m_0,\ n = n_0$. One can use the Poincaré-Birkhoff-Witt Theorem to write A as a sum of linearly independent monomials in $X,\ \bar{X},\ Y,\ \bar{Y},\ Z,\ \bar{Z}$. Each of these summands satisfies (5.4.1) for some order (m, n). But since A satisfies (5.4.1) for $(m, n) = (m_0, n_0)$, all of the orders of the summands must equal (m_0, n_0). $\square$

We can easily extend the notion of order to differential operators which are not left-invariant.

Definition 5.6. A differential operator A on $\mathbb{H}$ is said to be of order at most (m, n) if

$$A = \sum_k f_k A_k$$

where each $f_k \in C^\infty(\mathbb{H})$ and each A_k is a homogeneous left-invariant differential operator of order at most (m, n).

Occasionally it will be useful to have a more general notion of order than the bifiltration of Definition 5.4. Recall the notion of an *order ideal*: a subset $\mathcal{I}$ is an order ideal of a partially ordered set $(\mathcal{P}, \leq)$ if

$$x \in \mathcal{I} \text{ and } y \leq x \quad \Rightarrow \quad y \in \mathcal{I}$$

for any $x,\ y \in \mathcal{P}$.

Definition 5.7. Let $\mathcal{I}$ be an order ideal of $(\mathbb{N} \times \mathbb{N}, \leq)$. A differential operator A on $\mathbb{H}$ is said to be of order $\mathcal{I}$ if

$$A = \sum_k f_k A_k$$

where $f_k \in C^\infty(\mathbb{H})$ and each A_k is a homogeneous left-invariant differential operator of order $(m_k, n_k) \in \mathcal{I}$.

In the case where $\mathcal{I} = \langle (m,n) \rangle$ is the order ideal of all (m', n') less than or equal to (m,n), we recover our earlier notion of order.

More generally, we will denote by $\langle (m_1, n_1), \ldots, (m_p, n_p) \rangle$ the order ideal generated by the elements $(m_1, n_1), \ldots, (m_p, n_p) \in \mathbb{N} \times \mathbb{N}$. That is,

$$\begin{aligned} &\Big\langle (m_1, n_1), \ldots, (m_p, n_p) \Big\rangle \\ &\qquad = \Big\{ (m,n) \in \mathbb{N} \times \mathbb{N} \quad | \quad (m,n) \leq (m_j, n_j) \text{ for some } j = 1, \ldots, p \Big\}. \end{aligned}$$

Example 5.8. For $N \in \mathbb{N}$, let

$$[N] = \{ \, (m,n) \in \mathbb{N} \times \mathbb{N} \quad | \quad m + n \leq N \, \}.$$

Then $[N]$ is an order ideal, and the differential operators of order $[N]$ are precisely those which have order at most N in the sense of the ordinary Heisenberg calculus.

5.5 Harmonic analysis of the complex Heisenberg group

At several points in this chapter we will need to appeal to the harmonic analysis of $\mathbb{H}$, the complex Heisenberg group. This is very similar to the harmonic analysis of the real Heisenberg group, and the reader familiar with the latter will recognize the strong resemblance. However, since harmonic analysis on the complex Heisenberg group is not a standard fact, we will provide the details here. This material is a special case of Chapter 6 of [Tay86].

We will start with a description of the unitary representations of $\mathbb{H}$. These are most easily described by the corresponding Lie algebra representations of $\mathfrak{h}$.

Theorem 5.9. *The irreducible unitary representations of* $\mathbb{H}$ *fall into two classes, as follows.*

(i) *One-dimensional representations* $\pi_{(\xi,\eta)}$ *for* $\xi, \eta \in \mathbb{C}$*, defined by*

$$\begin{aligned} \pi_{(\xi,\eta)} : X &\mapsto i\bar{\xi} \quad ; \quad \bar{X} \mapsto i\xi \\ Y &\mapsto i\bar{\eta} \quad ; \quad \bar{Y} \mapsto i\eta \\ Z &\mapsto 0 \quad ; \quad \bar{Z} \mapsto 0. \end{aligned}$$

(ii) *Infinite-dimensional representations* π_w *for* $w \in \mathbb{C}^\times$*, defined on* $L^2(\mathbb{C})$ *by*

$$\begin{aligned} \pi_w : X &\mapsto \frac{\partial}{\partial \zeta} \quad ; \quad \bar{X} \mapsto \frac{\partial}{\partial \zeta} \\ Y &\mapsto iw\zeta \quad ; \quad \bar{Y} \mapsto i\bar{w}\bar{\zeta} \\ Z &\mapsto iw \quad ; \quad \bar{Z} \mapsto i\bar{w} \end{aligned}$$

where ζ *is the complex coordinate function on* $\mathbb{C}$*.*

Proof. Let π be a unitary representation of $\mathbb{H}$. By Schur's Lemma π acts by scalars on the centre $\mathfrak{z}$ of $\mathfrak{h}$. If $\pi(\mathfrak{z}) = \{0\}$ then π factors through the quotient $\mathfrak{h}/\mathfrak{z} \cong \mathbb{C}^2$, and classical harmonic analysis yields case (i).

Otherwise,

$$\pi : \mathfrak{z} \longrightarrow i\mathbb{R},$$

nontrivially. Let $w_1, w_2 \in \mathbb{R}$ be such that

$$\begin{aligned} \pi(Z_1) &= 2iw_1 \\ \pi(Z_2) &= -2iw_2, \end{aligned}$$

and let $w = w_1 + iw_2$. Note that

$$\begin{aligned} \pi(Z) &= \frac{1}{2}\pi(Z_1) - \frac{1}{2}i\pi(Z_2) = iw \\ \pi(\bar{Z}) &= \frac{1}{2}\pi(Z_1) + \frac{1}{2}i\pi(Z_2) = i\bar{w}. \end{aligned}$$

Next, put

$$\begin{aligned} \dot{Y}_1 &= \frac{w_1}{|w|}Y_1 - \frac{w_2}{|w|}Y_2 \\ \dot{Y}_2 &= \frac{w_2}{|w|}Y_1 + \frac{w_1}{|w|}Y_2 \\ \dot{Z}_1 &= \frac{w_1}{|w|}Z_1 - \frac{w_2}{|w|}Z_2 \\ \dot{Z}_2 &= \frac{w_2}{|w|}Z_1 + \frac{w_1}{|w|}Z_2 \end{aligned}$$

With these definitions, the six elements $\dot{X}_1, \dot{X}_2, \dot{Y}_1, \dot{Y}_2, \dot{Z}_1, \dot{Z}_2$ satisfy the same relations as $X_1, X_2, Y_1, Y_2, Z_1, Z_2$. But note that

$$\pi(\dot{Z}_2) = 0.$$

Therefore, π factors through the Lie algebra $\mathfrak{h}/\mathfrak{v}$ where $\mathfrak{v} = \langle \dot{Z}_2 \rangle$. This quotient Lie algebra is isomorphic to the five dimensional real Heisenberg Lie algebra, for which the representation theory is well-known (see [Tay86, Ch.2]). Using that theory, and the fact that

$$\pi(\dot{Z}_1) = 2i|w|,$$

we see that π is isomorphic to the representation on $L^2(\mathbb{R}^2)$ defined by

$$\begin{aligned} \pi : X_1 &\mapsto \frac{\partial}{\partial s} \\ X_2 &\mapsto \frac{\partial}{\partial t} \\ \dot{Y}_1 &\mapsto 2i|w|s \\ \dot{Y}_2 &\mapsto -2i|w|t \\ \dot{Z}_1 &\mapsto 2i|w| \\ \dot{Z}_2 &\mapsto 0 \end{aligned}$$

where s and t are the coordinate functions on $\mathbb{R}^2$. Putting $\zeta = s + it$ gives the representation of case (ii), as we now check:

$$\begin{aligned} \pi : Y_1 &= \frac{w_1}{|w|}\dot{Y}_1 + \frac{w_2}{|w|}\dot{Y}_2 &\mapsto& \quad 2i(w_1 s - w_2 t) \\ Y_2 &= -\frac{w_2}{|w|}\dot{Y}_1 + \frac{w_1}{|w|}\dot{Y}_2 &\mapsto& \quad -2i(w_2 s + w_1 t) \end{aligned}$$

and hence

$$\begin{aligned} \pi(Y) &= i(w_1 s - w_2 t) - (w_2 s + w_1 t) = iw\zeta \\ \pi(\bar{Y}) &= i(w_1 s - w_2 t) + (w_2 s + w_1 t) = i\bar{w}\bar{\zeta}. \end{aligned}$$

□

We refer to the representations π_w of case (ii) in the theorem as the *Schrödinger representations* of $\mathbb{H}$.

Theorem 5.10. *(Plancherel Theorem) Let $u \in L^2(\mathbb{H})$. If π is an irreducible representation of $\mathbb{H}$ then the operator $\pi(u)$ defined by*

$$\pi(u) = \int_{\mathbb{H}} u(n)\pi(n)dn$$

is Hilbert-Schmidt. There exists a measure μ on $\mathbb{C}^\times$ such that for any $u \in L^2(\mathbb{H})$,

$$\|u\|^2 = \int_{\mathbb{C}^\times} \|\pi_w(u)\|_{HS}^2 \, d\mu(w),$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm.

For a proof, see [Tay86].

The Plancherel Theorem allows one to reduce from analysis on $L^2(\mathbb{H})$ to analysis in the representations of $\mathbb{H}$. Suppose A is an element of the universal enveloping algebra $\mathcal{U}(\mathfrak{h})$, which is acting as a differential operator on $L^2(\mathbb{H})$. Or more generally, suppose that A is built from such an element by using functional calculus. The Plancherel formula tells us that, for any u in the domain of A,

$$\begin{aligned}\|Au\|^2 &= \int_{\mathbb{C}^\times} \|\pi_w(Au)\|_{HS}^2 \, d\mu \\ &= \int_{\mathbb{C}^\times} \|\pi_w(A)\pi_w(u)\|_{HS}^2 \, d\mu.\end{aligned}$$

It follows that the operator A on $L^2(\mathbb{H})$ will be bounded if the operators $\pi_w(A)$ on $L^2(\mathbb{C})$ are uniformly bounded in (operator) norm.

We would also like to be able to prove inequalities of the form

$$\|Au\| \leq \|Bu\| \tag{5.5.1}$$

(for all $u \in C^\infty(\mathbb{H})$) by proving the corresponding inequality on each Schrödinger representation:

$$\|\pi_w(A)f\| \leq \|\pi_w(B)f\| \tag{5.5.2}$$

(for all $f \in C^\infty(\mathbb{C})$). This is possible, but in order to make this rigorous, some remarks are in order.

Let $\mathcal{H}$ be a Hilbert space, and $\overline{\mathcal{H}}$ the conjugate Hilbert space. There is an isometric isomorphism of Hilbert spaces between the Hilbert-Schmidt operators on $\mathcal{H}$ and $\mathcal{H} \otimes \bar{\mathcal{H}}$.

Under this isomorphism an elementary tensor $v_1 \otimes \bar{v_2} \in \mathcal{H} \otimes \bar{\mathcal{H}}$ corresponds to the rank-one operator $\langle \bar{v_2}, \bullet \rangle v_1$. If T is an operator on $\mathcal{H}$, then

$$T\left(\langle \bar{v_2}, \bullet \rangle v_1\right) = \langle \bar{v_2}, \bullet \rangle T v_1,$$

which is to say that left multiplication by T on the Hilbert-Schmidt operators corresponds to $T \otimes 1$ on $\mathcal{H} \otimes \overline{\mathcal{H}}$. For this reason, it suffices to consider the action of $\pi_w(A)$ and $\pi_w(B)$ on $L^2(\mathbb{C})$ rather than on the Hilbert-Schmidt operators.

A more subtle problem is the issue of domains. If $u \in L^2(\mathbb{H})$ is of Schwartz class, then the operators $\pi(u) \in \mathcal{L}^2(L^2(\mathbb{C}))$ have Schwartz class integral kernels. Therefore, if the graphs of the operators A and B restricted to the Schwartz class functions on $\mathbb{H}$ are dense in their original graphs, then it will suffice to consider the operators $\pi_w(A)$ and $\pi_w(B)$ with domain the Schwartz class elements of $L^2(\mathbb{C})$. Such is the case when A and B are differential operators, for instance, or for the images of differential operators under the functional calculus. This then allows us to infer (5.5.1) from (5.5.2).

If A and B are homogeneous elements of $\mathcal{U}(\mathfrak{h})$, both of the same order (m, n), then this method of argument becomes even simpler. In that case, the formulas of Theorem 5.9 show that

$$\|\pi_w(A)f\| = |w|^n \|\pi_1(A)f\|$$

and likewise for B. Therefore, (5.5.1) will follow if we can prove (5.5.1) with $w = 1$, *ie*, if

$$\|\pi_1(A)f\| \leq \|\pi_1(B)f\|$$

for all functions f of Schwartz class on $\mathbb{C}$. We will use this observation several times in what follows.

5.6 Sobolev spaces

Now that we have a two-parameter notion of order for differential operators on $\mathbb{H}$, it is very natural to want an accompanying family of Sobolev spaces $H^{(m,n)}(\mathbb{H})$. As usual, the Sobolev spaces will be defined by putting a specific inner product on the space of smooth compactly-supported functions on $\mathbb{H}$. Of course, we will only care about this inner product up to equivalence. Let us recall the definition of equivalence.

Definition 5.11. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V are said to be *equivalent* if there is some constant $C > 1$ such that

$$C^{-1}\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1$$

for all $v \in V$.

Two inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are *equivalent* if the norms they induce on V are equivalent.

In the course of this work, we will introduce a slew of equivalent inner products on $C_c^\infty(\mathbb{H})$ which define the Sobolev spaces $H^{(m,n)}(\mathbb{H})$. The first, and most heavy-handed, uses every elementary monomial in $\mathcal{U}(\mathfrak{h}_\mathbb{C})$ of order at most (m, n).

Let dx denote the Haar measure on $\mathbb{H}$. With respect to the coordinates of (5.2.1), this is just Lebesgue measure on $\mathbb{C}^3$. We use this measure to define the L^2-inner product on $\mathbb{H}$:

$$\langle u, v\rangle_{L^2(\mathbb{H})} \;=\; \int_{\mathbb{H}} \overline{u(x)}v(x)\; dx.$$

This is the fundamental inner product for all the ensuing analysis, and as such we will often refer to it as $\langle\,\cdot\,,\,\cdot\,\rangle$ with no subscript.

Definition 5.12. For each $(m, n) \in \mathbb{N}\times\mathbb{N}$, define the inner product $\langle\cdot,\cdot\rangle_{(m,n)}$ on $C_c^\infty(\mathbb{H})$ by

$$\langle u, v\rangle_{(m,n)} \;=\; \sum_{\mathrm{order}(A)\leq(m,n)} \langle Au, Av\rangle_{L^2(\mathbb{H})} \tag{5.6.1}$$

where A varies over all elementary monomials of order less than or equal to (m, n).

The corresponding norm will be denoted by $\|\cdot\|_{(m,n)}$.

The completion of $C_c^\infty(\mathbb{H})$ with respect to this norm is denoted $H^{(m,n)}(\mathbb{H})$.

It is clear that whenever $(m', n') \leq (m, n)$ we have

$$\langle u, v\rangle_{(m',n')} \leq \langle u, v\rangle_{(m,n)}$$

for all $u,\ v \in C_c^\infty(\mathbb{H})$. Hence the Sobolev spaces are nested according to the ordering on their indices:

$$H^{(m',n')}(\mathbb{H}) \supseteq H^{(m,n)}(\mathbb{H})$$

if $(m', n') \leq (m, n)$. These inclusions are continuous, and have dense range, since all the Sobolev spaces contain the dense subspace of smooth functions.

The first key property that a Sobolev theory should satisfy is the following lemma.

Lemma 5.13. *If D is a differential operator on $\mathbb{H}$ of order (a,b) which vanishes outside some compact set, then D extends to a bounded operator from $H^{(m+a,n+b)}(\mathbb{H})$ to $H^{(m,n)}(\mathbb{H})$ for any $(m,n) \in \mathbb{N} \times \mathbb{N}$.*

We separate out an important special case of this result.

Lemma 5.14. *Multiplication by $f \in C_c^{\infty}(\mathbb{H})$ is a continuous linear operator on each $H^{(m,n)}(\mathbb{H})$.*

Proof. The key fact is that for any vector field V on $\mathbb{H}$,

$$[V, f] = Vf$$

where f and Vf are interpreted as multiplication operators. Thus, if A is any of the elementary monomials appearing in the definition of $\langle\,\cdot\,,\,\cdot\,\rangle_{(m,n)}$, we have

$$[A, f] = \sum_j f_j A_j$$

for some functions $f_j \in C_c^{\infty}(\mathbb{H})$, and some elementary monomials A_j with

$$\mathrm{order}(A_j) \ < \ \mathrm{order}(A).$$

So for $u \in H^{(m,n)}(\mathbb{H})$,

$$\begin{aligned}
\|Afu\|_{L^2} &\leq \|fAu\|_{L^2} + \sum_j \|f_j A_j u\|_{L^2} \\
&\leq \|f\|_\infty \|Au\|_{L^2} + \sum_j \|f_j\|_\infty \|A_j u\|_{L^2} \\
&\leq C\|u\|_{(m,n)},
\end{aligned}$$

for some constant C. The result follows. $\square$

Proof of Lemma 5.13. Using Lemma 5.14, it suffices to prove the result in the case where D is an elementary monomial. If A is one of the elementary monomials appearing in the definition of $\langle \cdot, \cdot \rangle_{(m,n)}$ then AD has order less than or equal to $(m + a, n + b)$. We see that

$$\|Du\|^2_{(m,n)} = \sum_{\mathrm{order}(A) \leq (m,n)} \|ADu\|^2_{L^2(\mathbb{H})} \leq \|u\|^2_{(m+a,n+b)}.$$

$\square$

More generally, we can make a Sobolev space for any finite order ideal of $\mathbb{N} \times \mathbb{N}$.

Definition 5.15. Given a finite order ideal $\mathcal{I}$ of $\mathbb{N} \times \mathbb{N}$, define

$$\langle u, v \rangle_{\mathcal{I}} = \sum_{\mathrm{order}(A) \in \mathcal{I}} \langle Au, Av \rangle_{L^2(\mathbb{H})}$$

where A varies over elementary monomials with order in $\mathcal{I}$. The induced norm is denoted by $\| \cdot \|_{\mathcal{I}}$, and the completion of $C_c^\infty(\mathbb{H})$ with respect to this norm is denoted $H^{\mathcal{I}}(\mathbb{H})$.

These Sobolev spaces are again nested. Specifically, if $\mathcal{I}$ and $\mathcal{J}$ are finite order ideals, then

$$\mathcal{I} \subseteq \mathcal{J} \quad \Rightarrow \quad H^{\mathcal{I}}(\mathbb{H}) \supseteq H^{\mathcal{J}}(\mathbb{H}).$$

In fact, there is an even stronger relationship. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be finite order ideals in $\mathbb{N}\times\mathbb{N}$. The above fact shows that $H^{\mathcal{I}_1\cup\mathcal{I}_2} \subseteq H^{\mathcal{I}_1} \cap H^{\mathcal{I}_2}$. But also, all of the monomials A in the definition of $\langle\cdot,\cdot\rangle_{\mathcal{I}_1\cup\mathcal{I}_2}$ appear in the definition of either $\langle\cdot,\cdot\rangle_{\mathcal{I}_1}$ or $\langle\cdot,\cdot\rangle_{\mathcal{I}_2}$. Thus

$$H^{\mathcal{I}_1\cup\mathcal{I}_2} = H^{\mathcal{I}_1} \cap H^{\mathcal{I}_2}.$$

This generalizes to any finite collection of order ideals $\mathcal{I}_1, \dots, \mathcal{I}_p$. In particular,

$$H^{\langle(m_1,n_1),\dots,(m_p,n_p)\rangle} = \bigcap_{j=1}^{p} H^{(m_j,n_j)}.$$

Example 5.16. With the notation of Example 5.8, $H^{[N]}(\mathbb{H})$ is the standard (one-parameter) Sobolev space for the Heisenberg calculus, that is, the space of distributions u on $\mathbb{H}$ for which

$$Du \in L^2(\mathbb{H})$$

for every $D \in \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ of order at most N in the sense of the ordinary Heisenberg calculus.

We also have a Sobolev Embedding theorem for these spaces. In order to state this precisely, it is convenient to introduce local Sobolev spaces on $\mathbb{H}$. A distribution u on $\mathbb{H}$ is in $H^{(m,n)}_{\mathrm{loc}}(\mathbb{H})$ if $\varphi u \in H^{(m,n)}_{\mathrm{loc}}(\mathbb{H})$ for any $\varphi \in C_c^\infty(\mathbb{H})$.

Theorem 5.17 (Sobolev Embedding). *If $m, n \geq 3 + p$, then $H_{loc}^{(m,n)}(\mathbb{H})$ embeds continuously into $C^p(\mathbb{H})$. In particular,*

$$\bigcap_{(m,n)\in\mathbb{N}\times\mathbb{N}} H_{loc}^{(m,n)}(\mathbb{H}) = C^{\infty}(\mathbb{H}).$$

Proof. The vector fields $X_1,\ X_2,\ Y_1,\ Y_2,\ Z_1,\ Z_2$, which span $T\mathbb{H}$ at each point, all have order at most $(1,1)$. Thus, any compactly supported differential operator on $\mathbb{H}$ of order d in the ordinary Euclidean sense has order at most (d,d) in the new two-parameter sense. The result then follows from the standard Sobolev Embedding Theorem (see, *eg*, [Tay96]). $\square$

5.7 Alternative descriptions of the Sobolev spaces

Now we will begin to economize in the definition of the Sobolev spaces $H^{(m,n)}(\mathbb{H})$. The first step will be to observe the redundancy of using every single elementary monomial of each order in defining the inner product (5.6.1). Lemma 5.18 will show that it suffices to use a smaller class of each order.

In Section 5.2, we introduced the holomorphic and antiholomorphic Lie algebras, $\mathfrak{h}'$ and $\mathfrak{h}''$. These each generate a subalgebra of the universal enveloping algebra $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$, which we will denote by $\mathcal{U}(\mathfrak{h}')$ and $\mathcal{U}(\mathfrak{h}'')$, respectively. The notation is reasonable since, for instance, the subalgebra $\mathcal{U}(\mathfrak{h}') \subseteq \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ is canonically isomorphic to the universal enveloping algebra of $\mathfrak{h}'$ (see **[Dix96, Section 2.2]**). An element of $\mathcal{U}(\mathfrak{h}') \subseteq \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ will be called *totally holomorphic*, or just *holomorphic*. Elements of $\mathcal{U}(\mathfrak{h}'')$ will be called *(totally) antiholomorphic*.

Since $\mathfrak{h}'$ and $\mathfrak{h}''$ commute in $\mathfrak{h}_{\mathbb{C}}$, it follows that $\mathcal{U}(\mathfrak{h}')$ and $\mathcal{U}(\mathfrak{h}'')$ commute. Therefore, any elementary monomial A in $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ can be separated into a product of holomorphic and antiholomorphic parts. That is,

$$A = A_1 A_2,$$

with $A_1 \in \mathcal{U}(\mathfrak{h}')$ and $A_2 \in \mathcal{U}(\mathfrak{h}'')$.

All of the differential operators we are now dealing with will be considered as unbounded operators on $L^2(\mathbb{H})$. They are defined on the common invariant domain of $C_c^\infty(\mathbb{H})$. The operators $X,\ Y$ and Z have formal adjoints $-\bar{X},\ -\bar{Y}$ and $-\bar{Z}$, respectively. This follows from the formulae (5.2.4) and (5.2.5), and the fact that the left-invariant vector fields X_1, X_2, Y_1, Y_2 are images of the infinitesimal left-regular representation, and so are formally skew-adjoint.

Lemma 5.18. *Let A be any monomial in $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ with order (m, n). Then for some constant $C \geq 0$, we have*

$$\|Au\| \quad \leq \quad C \sum_{\substack{m_1+m_2=m \\ n_1+n_2=n}} \|X_1^{m_1} X_2^{m_2} Y_1^{n_1} Y_2^{n_2} u\| \tag{5.7.1}$$

for all $u \in C_c{}^\infty(\mathbb{H})$.

One of the key steps in the proof is the following observation.

Lemma 5.19. *For any $k \in \mathbb{Z}$, we have*

$$\|Z_1 u\| \leq 2\|(X_1 Y_1 + kZ_1)u\|,$$

$$\|Z_1 u\| \leq 2\|(X_2 Y_2 + kZ_1)u\|,$$

$$\|Z_2 u\| \leq 2\|(X_1 Y_2 + kZ_2)u\|,$$

$$\|Z_2 u\| \leq 2\|(X_2 Y_1 + kZ_2)u\|,$$

for all $u \in C_c{}^{\infty}(\mathbb{H})$.

Proof. We will prove the first of these inequalities. The others are all similar.

The operators X_1 and Y_1, defined on the invariant domain of $C_c{}^{\infty}(\mathbb{H})$, are formally skew-adjoint. Therefore, writing the operator $X_1 Y_1 + kZ_1$ as the sum of its symmetric and antisymmetric parts, we have

$$X_1 Y_1 + kZ_1 = (k + \frac{1}{2})Z_1 + \frac{1}{2}(X_1 Y_1 + Y_1 X_1).$$

Since the symmetric part and the antisymmetric part commute,

$$\begin{aligned} \|X_1 Y_1 u\|^2 &= \|(k + \frac{1}{2})Z_1 u\|^2 + \|\frac{1}{2}(X_1 Y_1 + Y_1 X_1)u\|^2 \\ &\geq \left(\frac{1}{2}\|Z_1 u\|\right)^2. \end{aligned}$$

This proves the lemma. $\square$

Proof of Lemma 5.18. By the Poincaré-Birkhoff-Witt Theorem [Dix96, Theorem 2.1.11], it suffices to consider the case

$$A = X_1^{a_1} X_2^{a_2} Y_1^{b_1} Y_2^{b_2} Z_1^{c_1} Z_2^{c_2}, \tag{5.7.2}$$

where

$$\begin{aligned} m &= a_1 + a_2 + c_1 + c_2, \\ n &= b_1 + b_2 + c_1 + c_2. \end{aligned}$$

The key computations in this proof are the following two applications of Lemma 5.19. For clarity, we will carry them out in the case $b_1 = b_2 = 0$. Firstly, if $c_1 \neq 0$.

$$\begin{aligned} &\|X_1^{a_1} X_2^{a_2} Z_1^{c_1} Z_2^{c_2} u\| \\ &\quad = \|Z_1 X_1^{a_1} X_2^{a_2} Z_1^{c_1-1} Z_2^{c_2} u\| \\ &\quad \leq 2\|(X_1 Y_1 + a_1 Z_1) X_1^{a_1} X_2^{a_2} Z_1^{c_1-1} Z_2^{c_2} u\| \\ &\quad = 2\left\| X_1 \left(Y_1 X_1^{a_1} + [X_1^{a_1},\ Y_1] \right) X_2^{a_2} Z_1^{c_1-1} Z_2^{c_2} u \right\| \\ &\quad = 2\|X_1^{a_1+1} Y_1 X_2^{a_2} Z_1^{c_1-1} Z_2^{c_2} u\| \\ &\quad \leq 2\|X_1^{a_1+1} X_2^{a_2} Y_1 Z_1^{c_1-1} Z_2^{c_2} u\| + 2\|X_1^{a_1+1} a_2 X_2^{a_2-1} Z_1^{c_1-1} Z_2^{c_2+1} u\|. \end{aligned}$$

Secondly, in the case $c_1 = 0$,

$$\begin{aligned}
\|X_1^{a_1} X_2^{a_2} Z_2^{c_2} u\| &= \|Z_2 X_1^{a_1} X_2^{a_2} Z_2^{c_2-1} u\| \\
&\leq 2\|(X_1 Y_2 + a_1 Z_2) X_1^{a_1} X_2^{a_2} Z_2^{c_2-1} u\| \\
&= 2\|X_1^{a_1+1} Y_2 X_2^{a_2} Z_2^{c_2-1} u\| \\
&\leq 2\|X_1^{a_1+1} X_2^{a_2} Y_2 Z_2^{c_2-1} u\| + 2\|X_1^{a_1+1} a_2 X_2^{a_2-1} Z_1 Z_2^{c_2-1} u\|.
\end{aligned}$$

The general situation is obtained by replacing u by $Y_1^{b_1} Y_2^{b_2} u$, whereby the above two inequalities yield

$$\begin{aligned}
&\|X_1^{a_1} X_2^{a_2} Y_1^{b_1} Y_2^{b_2} Z_1^{c_1} Z_2^{c_2} u\| \\
&\leq 2\|X_1^{a_1+1} X_2^{a_2} Y_1^{b_1+1} Y_2^{b_2} Z_1^{c_1-1} Z_2^{c_2} u\| \\
&\quad +2a_2\|X_1^{a_1+1} X_2^{a_2-1} Y_1^{b_1} Y_2^{b_2} Z_1^{c_1-1} Z_2^{c_2+1} u\|
\end{aligned} \tag{5.7.3}$$

for $c_1 \neq 0$, and

$$\begin{aligned}
&\|X_1^{a_1} X_2^{a_2} Y_1^{b_1} Y_2^{b_2} Z_2^{c_2} u\| \\
&\leq 2\|X_1^{a_1+1} X_2^{a_2} Y_1^{b_1} Y_2^{b_2+1} Z_2^{c_2-1} u\| \\
&\quad +2a_2\|X_1^{a_1+1} X_2^{a_2-1} Y_1^{b_1} Y_2^{b_2} Z_1 Z_2^{c_2-1} u\|
\end{aligned} \tag{5.7.4}$$

for $c_1 = 0$.

The two inequalities (5.7.3) and (5.7.4) set up a triple induction in the variables a_1, c_1 and c_2. Firstly, notice that the inequality (5.7.1) is trivial when $c_1 = c_2 = 0$. Next notice that when $a_2 = 0$, the second term on the right-hand side of both (5.7.3) and (5.7.4) are zero. In that case, therefore, these two inequalities allow us to reduce the value of $c_1 + c_2$. By induction, this proves the result whenever $a_2 = 0$.

Finally, fix $a_2 \neq 0$, and suppose (5.7.1) holds for monomials

$$X_1^{a_1'} X_2^{a_2 - 1} Y_1^{b_1'} Y_2^{b_2'} Z_1^{c_1'} Z_2^{c_2'}$$

with arbitrary values of the indices a_1', b_1', b_2', c_1', c_2'. Consider the case of

$$A = X_1^{a_1'} X_2^{a_2} Y_1^{b_1'} Y_2^{b_2'} Z_1^{c_1'} Z_2^{c_2'}.$$

We already saw that (5.7.1) is trivial for $c_1' = c_2' = 0$. If (5.7.1) holds for the monomials

$$A = X_1^{a_1'} X_2^{a_2} Y_1^{b_1'} Y_2^{b_2'} Z_2^{k}$$

with a_1', b_1', b_2' arbitrary and $k \leq c_2' - 1$, then by (5.7.4) it is also true for

$$A = X_1^{a_1'} X_2^{a_2} Y_1^{b_1'} Y_2^{b_2'} Z_2^{c_2'}.$$

Likewise, if it holds for all the monomials

$$A = X_1^{a_1'} X_2^{a_2} Y_1^{b_1'} Y_2^{b_2'} Z_1^{k} Z_2^{c_2'}$$

with a_1', b_1', b_2', c_2' arbitrary and $k \leq c_1' - 1$, then (5.7.3) shows it is also true for

$$A = X_1^{a_1'} X_2^{a_2} Y_1^{b_1'} Y_2^{b_2'} Z_1^{c_1'} Z_2^{c_2'},$$

and this completes the proof. $\square$

Corollary 5.20. *The Sobolev norm* $\|\cdot\|_{(m,n)}$ *is equivalent to the norm defined by*

$$\|u\|^2_{(m,n)} = \sum_{\substack{m_1+m_2\leq m\\ n_1+n_2\leq n}} \|X_1^{m_1}X_2^{m_2}Y_1^{n_1}Y_2^{n_2}u\|^2_{L^2(\mathbb{H})}. \tag{5.7.5}$$

Corollary 5.21. *The Sobolev norm* $\|\cdot\|_{(m,n)}$ *is equivalent to the norm defined by*

$$\|u\|^2_{(m,n)} = \sum_{\substack{a_1+a_2\leq m\\ b_1+b_2\leq n}} \|X^{a_1}\bar{X}^{a_2}Y^{b_1}\bar{Y}^{b_2}u\|^2_{L^2(\mathbb{H})}. \tag{5.7.6}$$

Proof. The change of basis

$$\begin{aligned} X_1 &= X+\bar{X} \\ X_2 &= i(X-\bar{X}) \end{aligned}$$

allows us to write

$$X_1^{m_1}X_2^{m_2}Y_1^{n_1}Y_2^{n_2} = \sum_{\substack{a_1+a_2=m_1+m_2\\ b_1+b_2=n_1+n_2}} C_{a_1,a_2,b_1,b_2}X^{a_1}\bar{X}^{a_2}Y^{b_1}\bar{Y}^{b_2}$$

for some constants C_{a_1,a_2,b_1,b_2}. Therefore,

$$\sum_{\substack{m_1+m_2\leq m\\ n_1+n_2\leq n}} \|X_1^{m_1}X_2^{m_2}Y_1^{n_1}Y_2^{n_2}u\|^2 \leq C \sum_{\substack{a_1+a_2\leq m\\ b_1+b_2\leq n}} \|X^{a_1}\bar{X}^{a_2}Y^{b_1}\bar{Y}^{b_2}u\|^2 \tag{5.7.7}$$

for some constant C. A reverse inequality can be produced similarly. ∎

Since it is extremely cumbersome to maintain separate notations for all of the equivalent norms on $H^{(m,n)}(\mathbb{H})$, from now we will use $\|\cdot\|_{(m,n)}$ to denote any of these equivalent norms. The exact definition will be specified only when it is critical to a computation.

Similarly, we will use the notation

$$\sum_{\mathrm{order}(A)\leq(m,n)}$$

to denote any (finite) sum in which A ranges over at least one elementary monomial of each order less than or equal to (m,n).

5.8 Directional Laplacians

We now have a coherent notion of positive, integral Sobolev spaces to accompany our notion of two-parameter order for differential operators on the Heisenberg group. It is tempting to try to extend this to some kind of pseudodifferential calculus. Unfortunately, this does not seem to be possible. The purpose of this chapter is to describe what can and cannot be achieved.

The prototypical example of a pseudodifferential operator is the inverse of an invertible elliptic differential operator. For instance, if

$$\Delta_{\mathbb{T}^k} = -\frac{\partial^2}{\partial {x_1}^2} - \cdots - \frac{\partial^2}{\partial {x_k}^2}$$

is the Laplacian on the torus $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, then the operator

$$(1 + \Delta_{\mathbb{T}^k})^{-1}$$

is a pseudodifferential operator of order -2, and maps the classical Sobolev space $H^n(\mathbb{T}^k)$ isometrically into $H^{n+2}(\mathbb{T}^k)$.

In our two-parameter situation, we will introduce two "directional Laplacians" corresponding to the vector fields X and Y on $\mathbb{H}$. Geometrically these operators will be assembled from the families of Laplacians along each of the fibres of the fibrations described in Section 4.1. Our definition here, however, will be restricted to the local coordinate space $\mathbb{H}$.

We define

$$\Delta_X = -X_1^2 - X_2^2$$

and

$$\Delta_Y = -Y_1^2 - Y_2^2.$$

At first we can define these as differential operators acting on the domain $C_c^\infty(\mathbb{H})$. From well-known results of representations of Lie groups (see, for instance, Taylor [Tay86]), the operators $X_1,\ X_2,\ Y_1$ and Y_2 are all essentially skew-adjoint as unbounded operators on $L^2(\mathbb{H})$. Thus, the operators $(-X_1^2 - X_2^2)$ and $(-Y_1^2 - Y_2^2)$ are essentially self-adjoint and positive. We will use Δ_X and Δ_Y, respectively, to denote their positive closures. Note also that the complex vector fields $X,\ \bar{X},\ Y$ and $\bar{Y}$ define essentially normal operators on $L^2(\mathbb{H})$, and we have

$$\Delta_X = -\bar{X}X$$
$$\Delta_Y = -\bar{Y}Y.$$

Proposition 5.22. *The operator $(1+\Delta_X)^{-\frac{1}{2}}$ is an isomorphism from $H^{(m,n)}(\mathbb{H})$ to $H^{(m+1,n)}(\mathbb{H})$ for every $(m,n) \in \mathbb{N} \times \mathbb{N}$ with $m \geq n$. Similarly, $(1+\Delta_Y)^{-\frac{1}{2}}$ is an isomorphism from $H^{(m,n)}(\mathbb{H})$ to $H^{(m,n+1)}(\mathbb{H})$ for each $(m,n) \in \mathbb{N} \times \mathbb{N}$ with $n \geq m$.*

Proof. We will prove that, for any $(m,n) \in \mathbb{N} \times \mathbb{N}$, there is $C > 0$ such that

$$\left\| X^{m+1} Y^n (1+\Delta_X)^{-\frac{1}{2}} u \right\|_{L^2(\mathbb{H})} \leq C \left\| X^m Y^n u \right\|_{L^2(\mathbb{H})} \tag{5.8.1}$$

and

$$\left\| X^{m} Y^{n+1} (1+\Delta_Y)^{-\frac{1}{2}} u \right\|_{L^2(\mathbb{H})} \leq C \left\| X^m Y^n u \right\|_{L^2(\mathbb{H})} \tag{5.8.2}$$

for all $u \in C_c{}^\infty(\mathbb{H})$.

Firstly we prove the following commutation relation. Let $F(x)$ be any smooth function on the real line, with growth at infinity bounded by some polynomial. Then $F(\Delta_X)$ is defined on $C_c{}^\infty(\mathbb{H})$, since it is dominated as a positive operator by some power of Δ_X. We claim that

$$[Y, F(\Delta_X)] = \bar{X}ZF'(\Delta_X) \tag{5.8.3}$$

and

$$[X, F(\Delta_Y)] = -\bar{Y}ZF'(\Delta_Y). \tag{5.8.4}$$

as operators on $C_c^\infty(\mathbb{H})$.

Following the observations of Section 5.5, it suffices to check the veracity of (5.8.3) and (5.8.4) on each irreducible representation π_w of $\mathbb{H}$, for $w \in \mathbb{C}^\times$. Let us demonstrate (5.8.4), and then (5.8.3) will follow by symmetry in X and Y. After applying π_w, (5.8.4) becomes

$$\left[\frac{\partial}{\partial\zeta},\ F(|w|^2|\zeta|^2)\right] = -|w|^2\bar{\zeta}\ F'(|w|^2|\zeta|^2).$$

This is just the chain rule.

We will want to make use of a combinatorial formula for commutators of powers. Given two elements V and W of some algebra,

$$V^n W = \sum_{k=0}^{n} \binom{n}{k} ((\operatorname{ad} V)^k W) V^{n-k}, \tag{5.8.5}$$

where $(\operatorname{ad} V)W = [V, W]$. This is readily proven by induction.

Now put $V = Y$ and $W = F(\Delta_X)$ with $F(x) = (1+x)^{\frac{1}{2}}$. The above formula, combined with (5.8.3), gives, for $u \in C_c^\infty(\mathbb{H})$,

$$\begin{aligned}
&\left\|X^{m+1}Y^n F(\Delta_X)u\right\|_{L^2(\mathbb{H})} \\
&\quad\le \sum_{k=0}^{n} \binom{n}{k} \left\|X^{m+1}(\bar{X}Z)^k F^{(k)}(\Delta_X) Y^{n-k} u\right\|_{L^2(\mathbb{H})} \\
&\quad= \sum_{k=0}^{n} \binom{n}{k} \left\|X^{k+1}\bar{X}^k F^{(k)}(\Delta_X) Z^k \bar{X}^{m-k} Y^{n-k} u\right\|_{L^2(\mathbb{H})}.
\end{aligned}$$

Note that this last expression makes sense only as long as $m \geq n$. Using the fact that $\left\|\bar{X}v\right\|_{L^2(\mathbb{H})} = \|Xv\|_{L^2(\mathbb{H})} = \|\Delta_X^{\frac{1}{2}}v\|_{L^2(\mathbb{H})}$ for any $v \in {C_c}^\infty(\mathbb{H})$, we see that

$$\begin{aligned}&\left\|X^{m+1}Y^nF(\Delta_X)u\right\|_{L^2(\mathbb{H})}\\&\qquad\leq \sum_{k=0}^{n}\binom{n}{k}\left\|\Delta_X^{k+\frac{1}{2}}F^{(k)}(\Delta_X)Z^kX^{m-k}Y^{n-k}u\right\|_{L^2(\mathbb{H})}.\end{aligned}$$

Therefore we will be done if we can show that

$$\left\|\Delta_X^{k+\frac{1}{2}}F^{(k)}(\Delta_X)Z^kv\right\|_{L^2(\mathbb{H})} \leq C\left\|X^kY^kv\right\|_{L^2(\mathbb{H})}$$

for some $C > 0$ and all $v \in {C_c}^\infty(\mathbb{H})$.

But $x^{k+\frac{1}{2}}F^{(k)}(x)$ is a bounded function of $x \in \mathbb{R}$, so $\Delta_X^{k+\frac{1}{2}}F^{(k)}(\Delta_X)$ is a bounded operator. Hence it suffices to show that

$$\left\|Z^kv\right\|_{L^2(\mathbb{H})} \leq C\left\|X^kY^kv\right\|_{L^2(\mathbb{H})}$$

for all $v \in {C_c}^\infty(\mathbb{H})$, which is a consequence of Lemma 5.18. □

Proposition 5.22 is not true for any order (m, n). The next proposition gives a strong counterexample using what one might hope was a "pseudodifferential operator of order $(0, -\infty)$".

Recall that, with respect to the coordinates

$$(x, y, z) \mapsto \begin{pmatrix}1 & x & z\\0 & 1 & y\\0 & 0 & 1\end{pmatrix}$$

on $\mathbb{H}$, the left-invariant differential operators X, Y and Z are

$$\begin{aligned} X &= \frac{\partial}{\partial x} \\ Y &= \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \\ Z &= \frac{\partial}{\partial z}. \end{aligned}$$

(See page 92.) The complex submanifold through the point $(x, y, z) \in \mathbb{H}$ which is tangent to Y_1 and Y_2 is

$$\mathrm{F}_Y(x, y, z) = \{\, (x, t, xt + (z - xy)) \quad | \quad t \in \mathbb{C}\,\}.$$

In what follows, if $t = t_1 + it_2$ is a complex variable, then we will use $|dt|^2$ to denote the volume element $dt_1 dt_2$ on $\mathbb{C}$.

Proposition 5.23. *Let $\psi \in C_c^\infty(\mathbb{C})$ such that $\psi \equiv 1$ on the closed unit disk*

$$\overline{\mathbb{D}} = \{\, y \in \mathbb{C} \mid \ |y| \leq 1\}.$$

The operator $S : C_c{}^\infty(\mathbb{H}) \longrightarrow C_c{}^\infty(\mathbb{H})$ defined by

$$Su(x, y, z) = \left(\int_{\mathbb{C}} \ u(x, t, xt + (z - xy)) \ dt \right) \psi(y)$$

does not extend to a bounded operator from $H^{(2,0)}(\mathbb{H})$ to $H^{(2,2)}(\mathbb{H})$.

The same is true even if we restrict the domain of S to smooth functions supported in a fixed compact subset of $\mathbb{H}$ (with nonempty interior).

Remark 11. The value of Su at (x, y, z) depends only on the values of u along the fibre $\mathrm{F}_Y(x, y, z)$. Restricted to each such fibre, S is a smoothing operator—in fact, a rank-one operator.

Proof. Let us consider smooth functions u supported on the unit polydisk,

$$\mathbb{D}^3 = \{\, (x, y, z) \quad | \quad |x|, |y|, |z| < 1\}$$

in $\mathbb{H}$. The proof generalizes to the case of arbitrary polydisks by rescaling, and this implies the result on any subset with nonempty interior.

Note that X is given by an elliptic differential operator along each of the fibres

$$\mathrm{F}_X(x, y, z) = \{(t, y, z) \mid t \in \mathbb{C}\},$$

namely $X = \frac{\partial}{\partial t}$. By standard theory of elliptic boundary value problems, there is a constant $C' > 0$ such that

$$\|u_0(t)\| \leq C' \, \|\frac{\partial}{\partial t} u_0(t)\|,$$

for any smooth u_0 supported on the unit disk $\mathbb{D} \subset \mathbb{C}$. Applying this to a function $u \in C_c^\infty(\mathbb{D}^3)$ restricted to each fibre $\mathrm{F}_X(x, y, z)$, we have that

$$\begin{aligned} \|u\|^2_{(2,0)} &= \|u\|^2 + \|Xu\|^2 + \|X^2 u\|^2 \\ &\leq (1 + C'^2 + C'^4)\|X^2 u\|^2. \end{aligned}$$

Therefore, in order to prove the Proposition, it will suffice to show that there is no constant $C \in \mathbb{R}$ such that

$$\|Z^2 Su\|_{L^2(\mathbb{H})} \leq C\|X^2 u\|_{L^2(\mathbb{H})} \tag{5.8.6}$$

for all $u \in C_c^{\infty}(\mathbb{D}^3)$.

Consider functions of the form

$$u(x, y, z) = a(x)\, b(y)\, c(z)$$

for $a, b, c \in C_c^{\infty}(\mathbb{D})$. Let us fix a once and for all. We choose it such that $a(x) = 1$ for all $|x| \leq \frac{1}{2}$. Then

$$\begin{aligned} \|X^2 u\|^2 &= \|\frac{\partial^2 a}{\partial x^2}(x)\, b(y)\, c(z)\|^2 \\ &= C_1\, \|b(y)c(z)\|^2 \end{aligned}$$

for $C_1 = \|\frac{\partial^2 a}{\partial x^2}\|^2$.

For the left-hand side of (5.8.6),

$$\begin{aligned} Z^2 Su(x, y, z) &= \frac{\partial^2}{\partial z^2} \int_{\mathbb{C}} a(x) b(t) c(xt + z - xy) |dt|^2 \psi(y) \\ &= \left(\int_{\mathbb{C}} b(t) \frac{\partial^2 c}{\partial z^2}(xt + z - xy) |dt|^2 \right) a(x)\psi(y). \end{aligned}$$

Let us suppose temporarily that $b(y)$ is the characteristic function of the disk $\mathbb{D}$. (This is not smooth, so we will have to amend this choice later.) Let us also choose $c(z)$ such that it is smooth, supported on $\mathbb{D}$, $|c| \leq 1$ and

$$c(z) = e^{i\alpha \operatorname{Re}(z)},$$

for all $|z| \leq \frac{1}{2}$, where α is some large positive constant. Note that for $|z| \leq \frac{1}{2}$,

$$\frac{\partial c}{\partial z} = \frac{\partial}{\partial z}\left(e^{\frac{1}{2}i\alpha(z+\bar{z})}\right) = \frac{1}{2}i\alpha e^{i\alpha \operatorname{Re}(z)}.$$

If x, y, z all have modulus less than $\frac{1}{8}$, then

$$|xt + z - xy| \leq \frac{1}{2},$$

for all $t \in \mathbb{D}$. In this case,

$$\frac{\partial^2 c}{\partial z^2}(xt + z - xy) = -\frac{1}{4}\alpha^2 e^{i\alpha \operatorname{Re}(xt)} e^{i\alpha \operatorname{Re}(z-xy)},$$

and hence

$$\begin{aligned}
&\left|\left(\int_{\mathbb{C}} b(t)\frac{\partial^2 c}{\partial z^2}(xt + z - xy)|dt|^2\right) a(x)\psi(y)\right| \\
&\qquad\qquad = \frac{1}{4}\alpha^2 \left|\int_{\mathbb{D}} e^{i\alpha \operatorname{Re}(xt)}|dt|^2\right| \\
&\qquad\qquad = \frac{1}{4}\alpha^2 \left|\int_{\mathbb{D}} e^{i\alpha|x| \operatorname{Re}(t)}|dt|^2\right|, \qquad (5.8.7)
\end{aligned}$$

where the latter follows from the change of variables $t \mapsto \frac{x}{|x|}t$. For any smooth function on $\mathbb{C}$, a corollary of Green's Theorem says that

$$\int_{\mathbb{D}} \frac{\partial f}{\partial t}|dt|^2 = \frac{-1}{2\pi i}\oint_{\partial\mathbb{D}} f\,dt.$$

With this, (5.8.7) becomes

$$\frac{\alpha}{2|x|}\left|\int_{\mathbb{D}} \frac{\partial}{\partial \bar{t}}\left(e^{i\alpha|x|\,\mathrm{Re}(t)}\right)|dt|^2\right| = \frac{\alpha}{4\pi|x|}\left|\oint_{\partial\mathbb{D}} e^{i\alpha|x|\,\mathrm{Re}(t)}dt\right|.$$

Using the change of variables $t \mapsto t^{-1}$, and the fact that on the circle $\partial\mathbb{D}$, $\mathrm{Re}(t) = \mathrm{Re}(t^{-1})$, this equals

$$\frac{\alpha}{4\pi|x|}\left|\oint_{\partial\mathbb{D}} e^{i\alpha|x|\,\mathrm{Re}(t)}t^{-2}dt\right| = \frac{\alpha}{2|x|}\left|J_1(\alpha|x|)\right|,$$

where J_1 is the Bessel function of the first kind.

Returning to the choice of the function $b(y)$, suppose we instead choose $b \in C_0{}^{\infty}(\mathbb{D})$, with $0 \leq b \leq 1$, which approximates the characteristic function of $\mathbb{D}$ in L^1-norm—say,

$$\int_{\mathbb{D}} |1 - b(t)||dt|^2 < \frac{1}{10}.$$

Then we would have

$$\begin{aligned}
\left|\int_{\mathbb{D}} \frac{\partial^2 c}{\partial z^2}(xt+z-xy)|dt|^2\right| - \left|\int_{\mathbb{C}} b(t)\frac{\partial^2 c}{\partial z^2}(xt+z-xy)|dt|^2\right| \\
\leq \left|\int_{\mathbb{D}} (1-b(t))\frac{\partial^2 c}{\partial z^2}(xt+z-xy)|dt|^2\right| \\
\leq \int_{\mathbb{D}} |1-b(t)|\frac{1}{4}\alpha^2|dt|^2 \\
< \frac{\alpha^2}{40}.
\end{aligned}$$

For $0 \leq r \leq 1$, the Bessel function J_1 satisfies $J_1(r)/r > \frac{1}{4}$, so

$$\begin{aligned}
\left|\int_{\mathbb{C}} b(t)\frac{\partial^2 c}{\partial z^2}(xt+z-xy)|dt|^2\right| &> \frac{\alpha}{2|x|}\,|J_1(\alpha|x|)| - \frac{\alpha^2}{40} \\
&> \frac{\alpha^2}{8} - \frac{\alpha^2}{40} \\
&= \frac{\alpha^2}{10}
\end{aligned}$$

when $|x| < \frac{1}{\alpha}$ and $|x|, |y|, |z| < \frac{1}{8}$. We will assume from now on that our choice of large α satisfies $\alpha > 8$.

These computations give us a lower bound on the L^2-norm on Z^2Su. For we have

$$\begin{aligned}
\|ZSu\|^2 &\geq \int_{|x|<\frac{1}{\alpha}} \int_{|y|,|z|<\frac{1}{8}} \left|\int_{\mathbb{C}} b(t)\frac{\partial^2 c}{\partial z^2}(xt+z-xy)|dt|^2 a(x)\psi(y)\right|^2 |dx|^2|dy|^2|dz|^2 \\
&\geq \int_{|x|<\alpha} \frac{\alpha^4}{100}|dx|^2 \\
&= \frac{\pi\alpha^2}{100}.
\end{aligned}$$

On the other hand,

$$\begin{aligned} \|X^2 u\|^2 &= \|\frac{\partial^2 a}{\partial x^2}\|^2 \|b\|^2 \|c\|^2 \\ &\leq \pi^2 \|\frac{\partial^2 a}{\partial x^2}\|^2, \end{aligned}$$

which is independent of α. Since α could be chosen arbitrarily large, this completes the proof. □

Corollary 5.24. *The operator $(1+\Delta_Y)^{-1}$ does not extend to a bounded operator from $H^{(2,0)}(\mathbb{H})$ to $H^{(2,2)}(\mathbb{H})$.*

The same is true even if we restrict the domain of $(1+\Delta_Y)^{-1}$ to compactly supported functions on some bounded subset of $\mathbb{H}$, as in the previous proposition.

Proof. It is convenient to take advantage of the automorphism of $\mathbb{H}$, given in coordinates as

$$\phi : (x, y, z) \mapsto (-y, x, z - xy),$$

which interchanges the role of X and Y. Under this transformation, the operator S of the previous Proposition becomes $S^\phi = \phi^* S(\phi^{-1})^*$, which is given by

$$S^\phi u(x, y, z) = \left(\int_{\mathbb{C}} u(t, y, z) dt \right) \psi(x).$$

Proposition 5.23 says that S^ϕ does not define a bounded operator from $H^{(0,2)}(\mathbb{H})$ to $H^{(2,2)}(\mathbb{H})$. In fact the proof shows more: the operator $Z^2 S^\phi$ does not extend to a bounded operator from $H^{(0,2)}(\mathbb{H})$ to $L^2(\mathbb{H})$, even when restricted to functions supported on a polydisk in $\mathbb{H}$.

We claim that there exists some constant $C \in \mathbb{R}$ such that, for any $u \in C_c^\infty(\mathbb{D}^3)$,

$$\|S^\phi u\|_{L^2(\mathbb{H})} \leq C\|(1+\Delta_X)^{-1}u\|_{L^2(\mathbb{H})}. \tag{5.8.8}$$

This will prove the corollary, since then

$$\begin{aligned} \|Z^2(1+\Delta_X)^{-1}u\| &= \|(1+\Delta_X)^{-1}Z^2u\| \\ &\geq \frac{1}{C}\|S^\phi Z^2 u\| \\ &= \frac{1}{C}\|Z^2 S^\phi u\|, \end{aligned}$$

and this is not bounded as a map to $L^2(\mathbb{H})$, as observed above.

Let $u \in C_c^\infty(\mathbb{D}^3)$, and put

$$v = (1+\Delta_X)^{-1}u.$$

Then the inequality (5.8.8) becomes

$$\|S^\phi(1+\Delta_X)v\|_{L^2(\mathbb{H})} \leq C\|v\|_{L^2(\mathbb{H})}. \tag{5.8.9}$$

Let χ be any smooth, compactly supported function on $\mathbb{H}$, which is equal to one everywhere in $\mathbb{D}^3$. Since $u = (1+\Delta_X)v$ is supported in $\mathbb{D}^3$,

$$S^{\phi}(1+\Delta_X)v = S^{\phi}\,\chi\,(1+\Delta_X)v.$$

The operator $S^{\phi}\,\chi\,(1+\Delta_X)$ decomposes as a family of compactly supported smoothing operators (with uniformly bounded supports) on the fibres $\mathrm{F}_X(x,y,z)$ introduced earlier. As such, it is defines a bounded operator on $L^2(\mathbb{H})$. Letting C be the norm of this operator, the inequality (5.8.9) follows.

$\square$

This counterexample is a serious impediment to hopes for a "two-parameter pseudodifferential calculus" on the Heisenberg group.

Various weakenings of these hopes might also be sufficient for the purposes of index theory of the BGG complex for $\mathrm{SL}(3,\mathbb{C})$. For instance, if we denote the left-most differential in the BGG complex by

$$\partial_1 = \begin{pmatrix} X \\ Y \end{pmatrix} : \quad L^2(\mathcal{X}; E_0) \quad \longrightarrow \begin{array}{c} L^2(\mathcal{X}; E_{\alpha_X}) \\ \oplus \\ L^2(\mathcal{X}; E_{\alpha_Y}) \end{array}$$

with $\mathcal{X} = G/B$, then the associated Laplacian ${\partial_1}^*\partial_1$ is locally modelled by the Heisenberg sub-Laplacian:

$$\begin{aligned} \Delta_H &= -\bar{X}X - \bar{Y}Y \\ &= \Delta_X + \Delta_Y. \end{aligned}$$

This operator dominates both Δ_X and Δ_Y, in the sense of positive unbounded operators, and so it might be hoped that it yields a bounded map

$$(1+\Delta_H)^{-1} : H^{(m,n)}(\mathbb{H}) \longrightarrow H^{(m+2,n)}(\mathbb{H}) \cap H^{(m,n+2)}(\mathbb{H}),$$

for any $(m,n) \in \mathbb{N} \times \mathbb{N}$.

This weaker property also seems likely to fail, based on computational evidence. Providing a counterexample is more difficult. Instead of pursuing this, we will now change our viewpoint to that of global harmonic analysis of differential operators on $\mathcal{X} = G/B$.

Chapter 6

Harmonic Analysis on G/B

6.1 Introduction

As we have seen, one of the key components of each of the constructions of the γ-elements in Sections 2.4.1, 2.4.2 and 2.4.3 is some form of pseudodifferential calculus. We have also remarked at two points so far that creating something as powerful as a pseudodifferential calculus in our present situation seems to be problematic at best. In order to understand what we might replace this component with, let us consider the most elementary example of a pseudodifferential calculus — the pseudodifferential calculus on the circle.

Pseudodifferential calculus was invented to solve problems in partial differential equations, particularly elliptic differential equations. The standard approach to the pseudodifferential calculus, as one would find in an introductory text on pseudodifferential operators, is to come through general manifolds. For the circle, this would mean first introducing a pseudodifferential calculus on the real line. The definitions there are grounded in the Fourier transform on the line, and hence use integral operators for the constructions. Having done this, one uses an atlas of coordinate charts (two charts will suffice, of course) to graft that linear calculus onto the circle.

One of the great advantages of this approach is its enormous generality—it can be used to define a pseudodifferential calculus on any smooth manifold. But it has some notable disadvantages as well. Firstly, because $\mathbb{R}$ is non-compact, there are technicalities in the foundational material—local Sobolev spaces, properly supported operators, and so on—which completely disappear once one has finally transferred the calculus to the circle. Secondly, it ignores the global symmetry in the circle, utilizing only the "local symmetry" inherent in the local Euclidean structure of a manifold.

On the other hand, if we are interested in rotationally invariant differential operators on the circle, such as the de Rham operator or the Laplacian, then one could approach directly from harmonic analysis on the circle itself—that is, from Fourier series. This has the advantage of exploiting the symmetry. It also has the advantage of simplifying the analysis enormously. For instance, considering the Laplacian on the circle from the point of view of classical pseudodifferential theory, all that is immediately apparent is that it is a Fredholm operator, *ie*, that it has finite dimensional kernel and cokernel. But using Fourier analysis, one easily computes that kernel (and identifies it as a representation of $SO(2)$).

Consider a mildly more complicated situation: the Laplace operator Δ on the space $\mathbb{C}\mathrm{P}^1 \cong S^2$. This space admits an isometric action of $\mathrm{SU}(2)$ under which Δ is invariant. Consequently, $L^2(\mathbb{C}\mathrm{P}^1)$ decomposes into finite-dimensional representations of $\mathrm{SU}(2)$, and these representations are eigenspaces of the operator Δ. This is the decomposition into spherical harmonics. In particular, the kernel of Δ is the trivial representation of $\mathrm{SU}(2)$.

Let us follow this train of thought one step further. Consider the space $\mathcal{X} = \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, which is the symmetric space G/B for the group $G = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$.

In this case, one can define a pair of "directional Laplacians"

$$\Delta_X = \Delta \otimes 1$$

and

$$\Delta_Y = 1 \otimes \Delta$$

on $L^2(\mathcal{X}) \cong L^2(\mathbb{C}\mathrm{P}^1) \otimes L^2(\mathbb{C}\mathrm{P}^1)$. These two differential operators are not elliptic on $\mathcal{X}$, but are elliptic along the fibres

$$\mathbb{C}\mathrm{P}^1 \times \{y\} \qquad\qquad (y \in \mathbb{C}\mathrm{P}^1)$$

and

$$\{x\} \times \mathbb{C}\mathrm{P}^1, \qquad\qquad (x \in \mathbb{C}\mathrm{P}^1)$$

respectively, of the product fibrations of $\mathcal{X}$. They can each be analyzed just as before, this time using harmonic analysis of the compact subgroup $K = \mathrm{SU}(2) \times \mathrm{SU}(2)$. One easily obtains a decomposition of $L^2(\mathcal{X})$ into eigenspaces for each of the two directional Laplacians. The eigenspaces for each of these operators are infinite dimensional, but the two operators commute, so we may consider simultaneous eigenspaces for the operator pair. Doing so yields a decomposition of $L^2(\mathcal{X})$ into finite dimensional pieces. In particular, the mutual kernel of the two operators is the trivial representation of G.

One could phrase the construction of the γ-element for $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ entirely in this language. In particular, the decomposition of $L^2(\mathbb{C}\mathrm{P}^1) \otimes L^2(\mathbb{C}\mathrm{P}^1)$ into mutual eigenspaces for Δ_X and Δ_Y can be seen at the heart of the Kasparov product. In this chapter we provide the foundational material for carrying the idea one step further again, to the case of $\mathrm{SL}(3,\mathbb{C})$.

We will begin by describing the generalities of the harmonic analysis for $\mathrm{SU}(3)$. As in the case of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$, we will introduce a pair of "directional Laplacians", Δ_X and Δ_Y. However, in this case, the two Laplacians will not commute. Nevertheless, we will prove some geometric facts about the spectral decompositions of the two operators which are almost as powerful as those above. We will finish by introducing ideals of operators analogous to the two ideals

$$\mathcal{K}_X = \mathcal{K}(L^2(\mathbb{C}\mathrm{P}^1)) \otimes \mathcal{B}(L^2(\mathbb{C}\mathrm{P}^1))$$

and

$$\mathcal{K}_Y = \mathcal{B}(L^2(\mathbb{C}\mathrm{P}^1)) \otimes \mathcal{K}(L^2(\mathbb{C}\mathrm{P}^1))$$

which appeared in the construction of γ for $\mathrm{SL}(2,C)\times\mathrm{SL}(2,\mathbb{C})$ in Section 2.4.3.

6.2 Decomposition into $\mathrm{SU}(3)$-types

Let $G = \mathrm{SL}(3,\mathbb{C})$, and let

$$E_\lambda = G \underset{B}{\times} V_\lambda$$

be a homogeneous complex line bundle over $\mathcal{X} = G/B$ as described in Section 3.2. Here, V_λ is a one-dimensional representation of B. Specifically, let λ be a character of the Cartan subgroup $H \subseteq G$, and extend it to a representation of $B = HN_-$ by declaring that $\lambda(N_-) = \{1\}$. As previously we will use the same symbol λ to denote the infinitesimal representation of the Lie algebra $\mathfrak{b}$ that λ induces.

Recall that sections of E_λ are identified with functions on G satisfying the B-equivariance condition

$$u(xb) = \lambda(b)^{-1}u(x) \qquad\qquad (x \in G,\, b \in B).$$

As described in Section 4.3, such a function is determined by its values on $K = \mathrm{SU}(3)$. Thus we have the "compact picture": if M is the subgroup of diagonal matrices with diagonal entries of modulus one, then sections of E_λ are identified with functions on K which are M-equivariant, in the sense that

$$u(km) = \lambda(m)^{-1}u(k) \qquad\qquad (k \in K\, m \in M). \tag{6.2.1}$$

Recall that the representation of G on $L^2(\mathcal{X}; E_\lambda)$ is given by

$$(g \cdot u)(x) = u(g^{-1}x). \tag{6.2.2}$$

Therefore, in passing to the compact picture, the action of G is obfuscated, although the action of the subgroup K is still clear.

One big advantage of working in the compact picture, however, is that the space of sections of E_λ depends only on the restriction of λ to $M \subseteq H$. Recall, from Remark 6 on page 62, that holomorphic representations of H are characterized by their restriction to either the compact subgroup M, or the complementary subgroup A. The general (non-holomorphic) characters λ of H are parameterized by writing

$$\lambda = (\mu, \nu)$$

where $\mu \in \Lambda_W \subseteq \mathfrak{h}^*$ is a weight (acting on $\mathfrak{m}$) and $\nu \in \mathfrak{h}^*$ is arbitrary (acting on $\mathfrak{a}$). For more details, see, for instance, [Duf79]. In the compact picture the vector bundle E, as well as the representation of K, is seen to depend only on the discrete parameter μ. For this reason, we will write E_μ for E_λ, when working in the compact picture.

From the compact picture, we can determine the decomposition of the representation space $L^2(\mathcal{X}; E_\mu)$ into K-isotypical pieces. To do so, note that by Equations (6.2.1) and (6.2.2), $L^2(\mathcal{X}; E_\mu)$ is a subspace of the regular representation $L^2(K)$ of K. We can decompose $L^2(K)$ by the Peter-Weyl Theorem. We let $\hat{K}$ denote the set of (isomorphism classes of) irreducible unitary representations of K. For $\pi \in \hat{K}$, let V_π denote its representation space.

Theorem 6.1 (Peter-Weyl). *For any compact group K, the map*

$$\bigoplus_{\pi \in \hat{K}} {V_\pi}^* \otimes V_\pi \longrightarrow L^2(K)$$

defined by

$${\xi_\pi}^* \otimes \xi_\pi \longmapsto \frac{1}{(\dim \Gamma_\pi)^{\frac{1}{2}}} ({\xi_\pi}^*, \pi(\,\bullet\,)\xi_\pi)$$

for $\xi_\pi \in V_\pi$, ${\xi_\pi}^* \in {V_\pi}^*$, *and extending linearly, is an isomorphism of unitary* $K \times K$*-representations. Here the representation on the left-hand side is*

$$k_1 \times k_2 \longmapsto \bigoplus_\pi \pi^*(k_1) \otimes \pi(k_2)$$

(where π^* *is the dual representation to* π*), while on the right-hand side it is given by the left and right regular representations*

$$k_1 \times k_2 \mapsto L(k_1)R(k_2)$$

The functions $({\xi_\pi}^*, \pi(\bullet)\xi_\pi)$ which appear in the theorem are called *matrix coefficients.*

In the case of $K = \mathrm{SU}(3)$, the irreducible representations of K are classified by their highest weights (see Section 3.4). This sets up a one-to-one correspondence between the irreducible representations and the dominant weights of SU(3). Recall that the dominant weights are those in $\Lambda_W \cap \overline{\mathcal{W}}$, where $\overline{\mathcal{W}}$ is the closure of the fundamental Weyl chamber in $\mathfrak{h}^*$. We will write ${\Lambda_W}^{(\mathrm{Dom})}$ for the set of dominant weights.

We will denote the representation with highest weight β by π_β, and its representation space by Γ^β. Then Peter-Weyl becomes

$$L^2(K) \quad \cong \bigoplus_{\beta \in \Lambda_W^{(\mathrm{Dom})}} \Gamma^{\beta *} \otimes \Gamma^\beta .$$

Considering the space $L^2(\mathcal{X}; E_{-\mu})$, the M-equivariance condition (6.2.1) tells us that

$$\begin{aligned} L^2(\mathcal{X}; E_{-\mu}) &= \{ u \in L^2(K) \mid R(T)u = \mu(T)u \text{ for all } T \in \mathfrak{m} \} \\ &\cong \bigoplus_{\beta \in \Lambda_W^{(\mathrm{Dom})}} \Gamma^{\beta *} \otimes (\Gamma^\beta)_\mu \,, \end{aligned} \tag{6.2.3}$$

where $(\Gamma^\beta)_\mu$ is the μ-weight space in the representation Γ^β. This is the decomposition of $L^2(\mathcal{X}; E_{-\mu})$ into K-isotypical pieces. The isomorphism is implemented using matrix coefficients, just as in the Peter-Weyl Theorem.

Remark 12. The μ-weight space of Γ^β is trivial if μ and β are not congruent modulo the root lattice. It will also be trivial if μ lies outside the convex hull of the orbit of β under the Weyl group. We may choose to restrict the direct sum in Equation (6.2.3) accordingly.

6.3 K-equivariant differential operators

The differential operators appearing in the Bernstein-Gel'fand-Gel'fand complex (see Chapter 4) are examples of the extremely small class of G-equivariant differential operators over $\mathcal{X} = G/B$ (*cf.* Theorems 3.29 and 3.31). Now that we are passing to the "compact picture", we will want a good understanding of these operators as members of the much larger class of K-equivariant differential operators.

As preliminary observation, let $V \in \mathfrak{n}'$ be a holomorphic element of the complexification of $\mathfrak{n} = \mathfrak{n}_+$ with weight α. Viewed as a left-invariant differential operator on G, we can let it act on a section

$$u \in C^{\infty}(\mathcal{X}; E_{-\mu})$$

in a manner which we now describe. Firstly, we identify u with a smooth function on G satisfying the B-equivariance property

$$\begin{aligned} Hu &= -\mu(H)u && (H \in \mathfrak{h}) \\ Nu &= 0 && (N \in \mathfrak{n}_-) \end{aligned} \tag{6.3.1}$$

Here H and N are acting on $u \in C^{\infty}(G)$ via the right regular representation, which we will suppress in the notation for clarity. The image of u under V,

$$v = Vu,$$

automatically has an H-equivariance property:

$$\begin{aligned} Hv &= HVu \\ &= V(Hu) + [H, V]u \\ &= \mu(H)Vu + \alpha(H)Vu \\ &= (\mu + \alpha)(H)v. \end{aligned}$$

Since $M \subseteq H$, restricting v to K yields a section of the bundle $E_{-(\mu+\alpha)}$, in the compact picture. In other words, *any* element of $\mathfrak{n}$ of weight α defines a K-invariant differential operator

$$V : C^{\infty}(\mathcal{X}; E_{-\mu}) \longrightarrow C^{\infty}(\mathcal{X}; E_{-(\mu+\alpha)})$$

for each $\mu \in \Lambda_W$.

By generalizing this idea, one might hope to create K-invariant differential operators of higher order by starting with homogeneous elements of $\mathcal{U}(\mathfrak{n}')$. This is certainly possible. If $D \in \mathcal{U}(\mathfrak{n}')$ is homogeneous of weight α, then following through the above construction defines a K-invariant differential operator

$$D : C^{\infty}(\mathcal{X}; E_{-\mu}) \longrightarrow C^{\infty}(\mathcal{X}; E_{-(\mu+\alpha)})$$

for each $\mu \in \Lambda_W$.

However, there is an important nuance here when it comes to composition of such operators. The definition of these K-invariant differential operators involves the restriction of the function Du, which is typically not G-equivariant, to an M-equivariant function on the compact subgroup K. If we were to apply the process a second time, we would need to begin by extending this function on K, in a G-equivariant way, to all of G. Therefore there is no *a priori* guarantee that composition of the K-equivariant differential operators induced from two homogeneous holomorphic elements $D_1, D_2 \in \mathcal{U}(\mathfrak{n}')$ will agree with that K-equivariant differential operator induced from their product $D_1 D_2$.

It turns out, however, that the two operators do agree, at least if we restrict our attention to holomorphic elements of $\mathcal{U}(\mathfrak{n}')$. To understand this, we need a description of the above differential operators in a way which is more intrinsic to the compact picture.

Let

$$\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$$

be the Cartan involution on $\mathfrak{g}$, which for the group $SL(3, \mathbb{C})$ is just the operation of negative conjugate transpose. Extend θ by complex-linearity to $\mathfrak{g}_{\mathbb{C}}$. If $V \in \mathfrak{n}_{\mathbb{C}} = (\mathfrak{n}_{+})_{\mathbb{C}}$ then $\theta V \in (\mathfrak{n}_{-})_{\mathbb{C}}$. Consequently, θV acts trivially on sections of the bundle $E_{-\mu}$. Therefore, if u is a function satisfying the properties (6.3.1), then

$$(V + \theta V)u = Vu.$$

But $V + \theta V \in \mathfrak{k}_{\mathbb{C}}$.

Definition 6.2. Define the "compact realization" map

$$\begin{aligned} \kappa : \mathfrak{n}_{\mathbb{C}} &\rightarrow \mathfrak{k}_{\mathbb{C}} \\ V &\mapsto V + \theta V. \end{aligned}$$

Homogeneous elements of $\mathfrak{k}_{\mathbb{C}}$ have a completely obvious interpretation as left K-invariant differential operators on $C^{\infty}(\mathcal{X}; E_{-\mu})$. Working in the standard picture or the compact picture yields the same operator and hence these operators make sense also for homogeneous elements of $\mathcal{U}(\mathfrak{k}_{\mathbb{C}})$.

There is a natural identification of $\mathfrak{k}_{\mathbb{C}}$ with $\mathfrak{g}$, given by the map

$$K_1 + iK_2 \mapsto K_1 + JK_2,$$

where J is the map of multiplication by i on $\mathfrak{g}$. Moreover, this is an isomorphism of Lie algebras. Using this identification, we have the following extremely convenient observation.

Lemma 6.3. *The map κ sends $\mathfrak{n}'$ to $\mathfrak{n}_+ \subseteq \mathfrak{g}$ and $\mathfrak{n}''$ to $\mathfrak{n}_- \subseteq \mathfrak{g}$.*

Moreover, on holomorphic vectors, the map $\kappa : \mathfrak{n}' \to \mathfrak{n}$ is exactly the inverse of the natural identification of the complex Lie algebra $\mathfrak{n}$ with the holomorphic part of its complexification, as given by

$$V \mapsto \frac{1}{2}(V - iJV),$$

and hence $\kappa|_{\mathfrak{n}'}$ is a Lie algebra homomorphism.

Also, the map $\kappa : \mathfrak{n}'' \to \mathfrak{n}_-$ is given by the inverse of the identification of $\mathfrak{n}$ with the antiholomorphic vectors of $\mathfrak{n}_{\mathbb{C}}$, as given by

$$V \mapsto \frac{1}{2}(V + iJV),$$

followed by Cartan involution $\theta : \mathfrak{n} \mapsto \mathfrak{n}_-$. Thus, $\kappa|_{\mathfrak{n}''}$ is also a Lie algebra homomorphism.

Proof. Note that $\theta J = -J\theta$. So for $V \in \mathfrak{n}_-$,

$$\kappa : \frac{1}{2}(V - iJV) \ \mapsto \ \frac{1}{2}(V + \theta V - iJV + iJ\theta V) \qquad \in \mathfrak{k}_{\mathbb{C}},$$

which identifies with

$$\frac{1}{2}(V + \theta V + V - \theta V) \ = \ V \qquad \in \mathfrak{g}.$$

Similarly,

$$\kappa : \frac{1}{2}(V + iJV) \;\mapsto\; \frac{1}{2}(V + \theta V + iJV - iJ\theta V) \qquad \in \mathfrak{k}_{\mathbb{C}},$$

which identifies with

$$\frac{1}{2}(V + \theta V - V + \theta V) \;=\; \theta V \qquad \in \mathfrak{g}.$$

$\square$

It follows that the "compact realization" map

$$V \mapsto V_K = V + \theta V$$

can be extended to higher order holomorphic operators $D \in \mathcal{U}(\mathfrak{n}')$. Because of the naturality of this map, we will generally suppress mention of κ in the notation. This should not cause confusion, since the K-invariant differential operators induced from D and $\kappa(D)$ are identical.

6.4 K-finite sections

The "compact realization" map also allows us to determine the action of the above differential operators on K-isotypical subspaces of $C^\infty(\mathcal{X}; E_{-\mu})$. Since the operators are K-invariant, Schur's Lemma tells us that they must respect K-types. By the Peter-Weyl decomposition, an isotypical subspace of $L^2(\mathcal{X}; E_{-\mu})$ is isomorphic to

$$\Gamma^{\beta *} \otimes (\Gamma^{\beta})_{-\mu},$$

for some dominant weight β. The action by a homogeneous element $D \in \mathcal{U}(\mathfrak{n}')$ of weight ν on this K-type is therefore given by

$$1 \otimes \pi_\beta(\kappa(D)) : \Gamma^{\beta*} \otimes (\Gamma^\beta)_\mu \longrightarrow \Gamma^{\beta*} \otimes (\Gamma^\beta)_{\mu+\nu}.$$

By Lemma 6.3,

$$\pi_\beta(\kappa D) = \pi_\beta(D),$$

where the right-hand side implicitly uses the well-known correspondence between finite-dimensional unitary representations of K, and finite-dimensional holomorphic representations of G ([Kna86, Proposition 5.7]).

In other words, the essential analytic information in the BGG-complex is carried by the maps

$$\pi_\beta(D) : (\Gamma^\beta)_\mu \longrightarrow (\Gamma^\beta)_{\mu+\nu}$$

between weight-spaces in finite dimensional representations of $\mathfrak{g}$. Thus, the BGG-complex has been reduced to a family of complexes of finite-dimensional spaces. This is analogous to analyzing differential operators on the circle with Fourier series, or differential operators on the 2-sphere with spherical harmonics. Figure 6.1 gives a pictorial indication of the BGG-complex in a typical K-type. (Compare the diagram (4.2.1) on page 94.)

Definition 6.4. Let π be a representation of G on a Hilbert space $\mathcal{H}$. A vector $v \in \mathcal{H}$ is K-*finite* if the orbit of v under $\pi(K)$ spans a finite-dimensional subspace of $\mathcal{H}$.

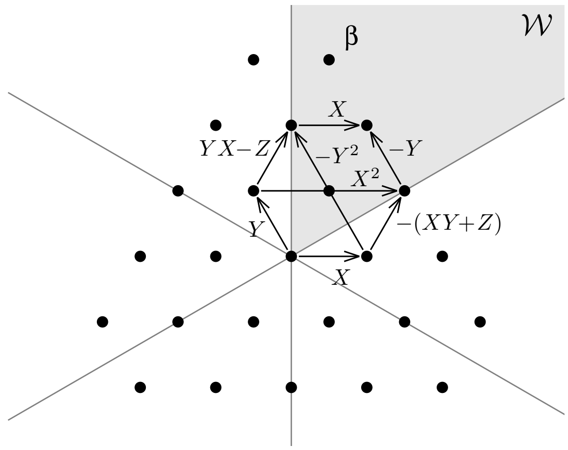

Fig. 6.1. Pictorial description of the BGG-complex in the K-type with highest weight $\beta = 2\alpha_X + 3\alpha_Y$. Dots represent weight spaces, and arrows maps between them.

The space of K-finite vectors in the representation $L^2(\mathcal{X}; E_{-\mu})$ will be denoted by

$$C_f(\mathcal{X}; E_{-\mu}),$$

and its elements will be called *K-finite sections* of $E_{-\mu}$. The K-finite sections correspond to the members of the *algebraic* direct sum

$$\bigoplus_{\beta \in \Lambda_W^{(\mathrm{Dom})}} \Gamma^{\beta *} \otimes (\Gamma^\beta)_\mu,$$

in the Peter-Weyl decomposition.

This space $C_f(\mathcal{X}; E_{-\mu})$, forms a convenient restricted domain for the K-invariant differential operators we have been considering, for the following reason.

Definition 6.5. For any $V \in \mathfrak{k}_{\mathbb{C}} \cong \mathfrak{g}$, we will write $V' = -\theta V$.

Lemma 6.6. *Let V be an element of $\mathfrak{n} \subset \mathfrak{k}_{\mathbb{C}}$ with weight α. For any μ, the K-invariant differential operators*

$$\begin{aligned} V \quad &: \quad C_f(\mathcal{X}; E_{-\mu}) \longrightarrow C_f(\mathcal{X}; E_{-(\mu+\alpha)}) \\ \textit{and} \qquad V' \quad &: \quad C_f(\mathcal{X}; E_{-(\mu+\alpha)}) \longrightarrow C_f(\mathcal{X}; E_{-\mu}), \end{aligned}$$

are formally adjoint as unbounded operators between the corresponding spaces of L^2-sections, and their products $V'V$ and VV' are essentially positive.

Proof. In any finite-dimensional holomorphic representation π of $\mathfrak{k}_{\mathbb{C}} \cong \mathfrak{g}$, the operators $\pi(V)$ and $\pi(V')$ are adjoint, which proves the first claim. Any direct sum of finite-dimensional positive operators has an orthonormal eigenbasis, and hence is essentially positive. $\square$

We could extend the map

$$V \mapsto V'$$

on $\mathfrak{k}_{\mathbb{C}}$ to a conjugate-linear anti-automorphism of $\mathcal{U}(\mathfrak{k}_{\mathbb{C}})$. Then Lemma 6.6 holds for an operator $D \in \mathcal{U}(\mathfrak{n}) \subset \mathcal{U}(\mathfrak{k}_{\mathbb{C}})$ (homogeneous of weight α) in place of V, with the same proof.

6.5 Directional Laplacians on G/B

In what follows, X, Y, Z will denote the elements

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

in $\mathfrak{n}$. Following the observations of the Section 6.3, it will be convenient to identify X, Y and Z with their holomorphic counterparts

$$\frac{1}{2}(X - iJX), \quad \frac{1}{2}(Y - iJY), \quad \frac{1}{2}(Z - iJZ) \quad \in \mathfrak{n}'.$$

Their antiholomorphic counterparts

$$\frac{1}{2}(X + iJX), \quad \frac{1}{2}(Y + iJY), \quad \frac{1}{2}(Z + iJZ) \quad \in \mathfrak{n}''$$

will be denoted $\bar{X}$, $\bar{Y}$ and $\bar{Z}$.

Now let these holomorphic and antiholomorphic vectors act on the sections of some homogeneous vector bundle $E_{-\mu}$. Because we will be working only in the compact picture for the remainder of this chapter, we will follow Lemma 6.3 and identify X, Y and Z with their "compact realizations" $X, Y, Z \in \mathfrak{n} \subseteq \mathfrak{k}_{\mathbb{C}}$. We will likewise identify $\bar{X}, \bar{Y}$ and $\bar{Z}$ with $\theta X = -X'$, $\theta Y = -Y'$, and $\theta Z = -Z' \in \mathfrak{n}_{-} \subseteq \mathfrak{k}_{\mathbb{C}}$.

Let α_X, α_Y and $\alpha_Z = \alpha_X + \alpha_Y \in \mathfrak{h}^*$ be the roots corresponding to the elements X, Y and Z in $\mathfrak{g}$, respectively. Fix a weight $\mu \in \Lambda_W$. As remarked after Lemma 6.6, the differential operators

$$\begin{aligned} X &: C^\infty(\mathcal{X}; E_{-\mu}) \longrightarrow C^\infty(\mathcal{X}; E_{-(\mu+\alpha_X)}) \\ \text{and} \qquad X' = -\bar{X} &: C^\infty(\mathcal{X}; E_{-(\mu+\alpha_X)}) \longrightarrow C^\infty(\mathcal{X}; E_{-\mu}) \end{aligned}$$

are formally adjoint operators between $L^2(\mathcal{X}; E_{-\mu})$ and $L^2(\mathcal{X}; E_{-(\mu+\alpha_X)})$. We define the positive unbounded operator Δ_X to be the self-adjoint extension of the essentially positive operator $X'X$ on $L^2(\mathcal{X}; E_{-\mu})$.

The purpose of the next few sections will be to explain some important features of the spectral theory of Δ_X, and the similarly defined operator

$$\Delta_Y = Y'Y,$$

on the spaces $L^2(\mathcal{X}; E_{-\mu})$.

Remark 13. Although X and X' do not commute as operators on $L^2(K)$, we do have

$$[X, X'] = H_X$$

where

$$H_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{k}_{\mathbb{C}}.$$

This operator H_X acts as a scalar on each $L^2(\mathcal{X}; E_{-\mu})$—specifically,

$$H_X \cdot u = \mu(H_X)\, u, \qquad\qquad (u \in L^2(\mathcal{X}; E_{-\mu})).$$

Therefore, the spectral measures of the operators $X'X$ and XX' are basically the same, differing only by a scalar shift, with that scalar being $\mu(H_X)$.

Thanks to the observations of the previous section, understanding the spectral theory of Δ_X and Δ_Y reduces to an infinite family of finite-dimensional problems, parameterized by the dominant weights of $\mathrm{SL}(3,\mathbb{C})$. In fact since the action of these operators on K-types is given by

$$1 \otimes \pi_\beta(\Delta_X) : \Gamma^{\beta *} \otimes (\Gamma^\beta)_\mu \longrightarrow \Gamma^{\beta *} \otimes (\Gamma^\beta)_\mu$$

and

$$1 \otimes \pi_\beta(\Delta_Y) : \Gamma^{\beta *} \otimes (\Gamma^\beta)_\mu \longrightarrow \Gamma^{\beta *} \otimes (\Gamma^\beta)_\mu,$$

it is clear that we are chiefly interested in the operators $\pi_\beta(\Delta_X)$ and $\pi_\beta(\Delta_Y)$.

For notational simplicity, we will reuse the symbols X, Y, Z, X', Y', Z', Δ_X, and Δ_Y to denote the corresponding finite dimensional operators $\pi_\beta(X)$, $\pi_\beta(Y)$, $\pi_\beta(Z)$, $\pi_\beta(X')$, $\pi_\beta(Y')$, $\pi_\beta(Z')$, $\pi_\beta(\Delta_X)$, and $\pi_\beta(\Delta_Y)$ in $\mathrm{End}(\Gamma^\beta)$, when β is assumed given. When working in particular weight spaces of Γ^β, we will also use the same notation to refer to their restrictions to maps between the appropriate weight spaces.

6.6 The centre of the enveloping algebra of $\mathfrak{su}(3)$.

In the analysis that follows, we will make frequent use of elements in the center of the enveloping algebras of $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$. Central elements in the enveloping algebra play an important role in the representation theory of semisimple Lie groups. Their importance derives from the fact — a consequence of Schur's Lemma — that on any irreducible representation of a Lie group G, with Lie algebra $\mathfrak{g}$, the central elements of $\mathcal{U}(\mathfrak{g})$ act as scalars. In this section we will collect several identities concerning the explicit form of these central elements.

We will begin with the case of $\mathfrak{su}(2)$. We will put $\mathfrak{s} = \mathfrak{su}(2)$ for brevity. Let us briefly recall the basic facts about the finite dimensional irreducible representations of $\mathfrak{s}$. All of this can be found, for instance, in [FH91].

The Cartan subalgebra of $\mathfrak{s}_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ is one-dimensional, spanned by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The weight lattice $\Lambda_W^{\mathfrak{sl}(2,\mathbb{C})}$ is isomorphic to $\mathbb{Z}$, generated by the fundamental weight

$$\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mapsto t.$$

The isomorphism with $\mathbb{Z}$ can be given explicitly by the map

$$\begin{array}{ccc} \Lambda_W^{\mathfrak{sl}(2,\mathbb{C})} & \to & \mathbb{Z} \\ \nu & \mapsto & \nu(H). \end{array}$$

Dominant weights correspond to non-negative integers.

We will follow the usual convention of actually identifying weights for $\mathfrak{sl}(2, \mathbb{C})$ with their corresponding integers under the above map. The irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ with highest weight $b \in \mathbb{N}$ will be denoted by Γ^b. This representation has one-dimensional weight spaces, with weights

$$-b,\ -b+2,\ \ldots,\ b-2,\ b.$$

The centre of $\mathcal{U}(\mathfrak{s}_{\mathbb{C}})$ is a polynomial algebra in one variable, generated by the Casimir element $\Omega_{\mathfrak{s}}$. If

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then the Casimir element of $\mathfrak{s}$ is

$$\Omega_{\mathfrak{s}} = E'E + EE' + \frac{1}{2}H^2.$$

Equivalently,

$$\Omega_{\mathfrak{s}} = 2E'E + \frac{1}{2}H^2 + H.$$

By definition, elements of the Cartan algebra act as scalars on each weight space in a representation of $\mathfrak{s}$. Namely, for a vector v of weight $a \in \mathbb{Z}$,

$$Hv = av.$$

Therefore, when restricted to the a-weight space,

$$\Omega_{\mathfrak{s}} = 2E'E + \frac{1}{2}a^2 + a. \tag{6.6.1}$$

The scalar value of $\Omega_{\mathfrak{s}}$ on an irreducible representation of highest weight b can be determined from this. Since E annihilates any highest weight vector, we get

$$\Omega_{\mathfrak{s}} = \frac{1}{2}b^2 + b, \tag{6.6.2}$$

on the highest weight space, and hence on all of Γ^b. Rewriting Equation (6.6.1), we see that on the weight space $(\Gamma^b)_a$,

$$\begin{aligned} E'E &= \frac{1}{4}((b^2+2b)-(a^2+2a)) \\ &= \frac{1}{4}(b(b+2)-a(a+2)). \end{aligned} \tag{6.6.3}$$

Now we move to the case of $\mathfrak{k} = \mathfrak{su}(3)$. The centre of $\mathcal{U}(\mathfrak{k}_{\mathbb{C}})$ is generated by two elements: the Casimir element Ω and an element of degree three, which we will denote by Ξ. There is a convenient description of both of these elements as follows. For $i, j \in \{1,2,3\}$, let E_{ij} denote the 3×3-matrix all of whose entries are zero except for a 1 in the (i,j)-position. Next, put

$$A_{ij} = \begin{cases} E_{ij} & \text{if } i \neq j, \\ E_{ii} - \frac{1}{3}I & \text{if } i = j. \end{cases}$$

Thus each A_{ij} is in $\mathfrak{su}(3)$. Then the generating central elements are given by

$$\Omega = \sum_{i,j=1}^{3} A_{ij}A_{ji}$$

and

$$\Xi = \sum_{i,j,k=1}^{3} A_{ij}A_{jk}A_{ki}.$$

We want to re-express these in terms of the elements X, Y, Z and so on. Let us put

$$\begin{aligned}
H_X = [X, X'] &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
H_Y = [Y, Y'] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
H_Z = [Z, Z'] &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
A_{11} &= \frac{1}{3}(H_X + H_Z), \\
A_{22} &= \frac{1}{3}(-H_X + H_Y), \\
A_{33} &= \frac{1}{3}(-H_Y - H_Z).
\end{aligned}$$

From this we get

$$\begin{aligned}
\sum_{i=1}^{3} A_{ii}^2 &= \frac{2}{9}(H_X^2 + H_Y^2 + H_Z^2) + \frac{2}{9}(-H_X H_Y + H_Y H_Z + H_X H_Z) \\
&= \frac{4}{9}(H_X^2 + H_Y^2 + H_Z^2) - \frac{1}{9}(H_X + H_Y)^2 - \frac{1}{9}(H_Y - H_Z)^2 - \frac{1}{9}(H_X - H_Z)^2 \\
&= \frac{1}{3}(H_X^2 + H_Y^2 + H_Z^2),
\end{aligned}$$

where the last line follows from $H_Z = H_X + H_Y$. Therefore,

$$\begin{aligned}
\Omega &= X'X + XX' + Y'Y + YY' + Z'Z + ZZ' + \frac{1}{3}(H_X^2 + H_Y^2 + H_Z^2) \\
&= 2(X'X + Y'Y + Z'Z) + \frac{1}{3}(H_X^2 + H_Y^2 + H_Z^2) + H_X + H_Y + H_Z.
\end{aligned} \tag{6.6.4}$$

Consider Γ^β, the irreducible representation of $\mathfrak{su}(3)$ with highest weight β. Applying (6.6.4) to a highest weight vector in Γ^β, we get

$$\Omega = \frac{1}{3}(\beta(H_X)^2 + \beta(H_Y)^2 + \beta(H_Z)^2) + 2\beta(H_Z).$$

Recall that the fundamental weights of $\mathfrak{su}(3)$ are

$$e_1 : \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \mapsto t_1$$

and

$$-e_3 : \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \mapsto -t_3.$$

Thus, β can be written as a linear combination

$$\beta = me_1 + n(-e_3)$$

with $m, n \in \mathbb{N}$. With this notation, the scalar value of the Casimir operator on the irreducible representation Γ^{β} is

$$\begin{aligned}\Omega &= \frac{1}{3}(m^2 + n^2 + (m+n)^2) + 2(m+n) \\ &= \frac{2}{3}(m^2 + mn + n^2 + 3m + 3n). \end{aligned} \tag{6.6.5}$$

We can put this back into Equation (6.6.4) to produce an identity which is valid on any particular weight space of Γ^{β}. If β is as above and $\mu = ae_1 + b(-e_3)$, then on the weight space $(\Gamma^{\beta})_{\mu}$ we have

$$\begin{aligned}&X'X + Y'Y + Z'Z \\ &\qquad = \frac{1}{3}(m^2 + mn + n^2 + 3m + 3n - a^2 - ab - b^2 - 3a - 3b). \end{aligned} \tag{6.6.6}$$

Finally, we carry out similar computations for Ξ. Some rearrangement gives

$$\begin{aligned}\Xi \;=\; & 3(Z'XY + Y'X'Z) - 3(A_{33} - 2)X'X - 3A_{11}Y'Y - 3A_{22}Z'Z \\ & + A_{11}^3 + A_{22}^3 + A_{33}^3 + 6A_{11}^2 - 3A_{22}A_{33} - 3A_{33}^2 + 6A_{11}. \end{aligned}$$

On a particular weight space, with weight $\mu = ae_1 + b(-e_3)$, the elements A_{11}, A_{22} and A_{33} act as scalars—specifically

$$\begin{aligned}
\mu(A_{11}) &= \frac{1}{3}(2a+b), \\
\mu(A_{22}) &= \frac{1}{3}(-a+b), \\
\mu(A_{33}) &= \frac{1}{3}(-a-2b).
\end{aligned}$$

Therefore, on this weight space,

$$\begin{aligned}
\Xi &= 3(Z'XY + Y'X'Z) \\
&\quad + (a+2b+6)X'X - (2a+b)Y'Y + (a-b)Z'Z \\
&\quad + \frac{1}{9}(a-b)(2a^2+5ab+2b^2) + (a+2)(2a+b). \qquad (6.6.7)
\end{aligned}$$

In particular, on the representation Γ^β, with $\beta = me_1 + n(-e_3)$, the scalar value of Ξ is

$$\Xi = \frac{1}{9}(m-n)(2m^2+5mn+2n^2) + (m+2)(2m+n). \qquad (6.6.8)$$

This yields the identity

$$\begin{aligned}
&3(Z'XY + Y'X'Z) + (a+2b+6)X'X - (2a+b)Y'Y + (a-b)Z'Z \\
&\quad = \frac{1}{9}(m-n)(2m^2+5mn+2n^2) + (m+2)(2m+n) \\
&\quad\quad - \frac{1}{9}(a-b)(2a^2+5ab+2b^2) - (a+2)(2a+b) \qquad (6.6.9)
\end{aligned}$$

on the weight space $(\Gamma^\beta)_\mu$.

6.7 Decomposition into $\mathfrak{s}_X$- and $\mathfrak{s}_Y$-types

The operator Δ_X lies not just in the enveloping algebra of $\mathfrak{k}_\mathbb{C}$, but in the enveloping algebra of the smaller Lie algebra.

$$\mathfrak{s}_X = \left\{ \left(\begin{array}{ccc} \multicolumn{2}{c}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| A \in \mathfrak{su}(2)_\mathbb{C} \right\} \cong \mathfrak{sl}(2,\mathbb{C}).$$

Let β be a given dominant weight for $K = \mathrm{SU}(3)$. The $\mathfrak{s}_X$-representation generated by a particular weight space $(\Gamma^\beta)_\mu$ will be referred to as the $\mathfrak{s}_X$-*string* through μ. Pictorially, in the weight-space diagram for Γ^β the $\mathfrak{s}_X$-strings appear as in Figure 6.2.

Remark 14. This is an abuse of terminology. In the general literature, the term $\mathfrak{s}_X$-*string* would be used to refer to the collection of $\mathfrak{sl}(3,\mathbb{C})$-weights of the corresponding $\mathfrak{s}_X$-representations.

To understand the structure of the $\mathfrak{s}_X$-strings, we will need to apply the general theory of finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{C})$ to the subgroup $\mathfrak{s}_X$ of $\mathfrak{sl}(3,\mathbb{C})$. Consider a finite-dimensional irreducible representation Γ^β for $\mathfrak{sl}(3,\mathbb{C})$, where β is a dominant weight for $\mathfrak{sl}(3,\mathbb{C})$. A weight μ for $\mathfrak{sl}(3,\mathbb{C})$ restricts to a weight μ_X for $\mathfrak{s}_X$, namely

$$\mu_X = \mu|_{\mathfrak{h}_X},$$

where

$$\mathfrak{h}_X = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; t \in \mathbb{C} \right\}.$$

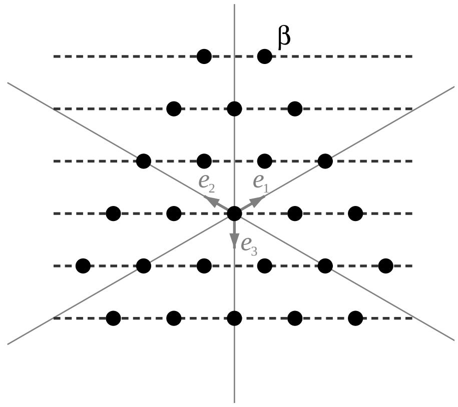

Fig. 6.2. Decomposition of the representation Γ^β into $\mathfrak{s}_X$-strings.

Under the identification of the weight lattice for $\mathfrak{s}_X$ with $\mathbb{Z}$, this weight is

$$\mu_X = \mu(H_X),$$

where

$$H_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The $\mathfrak{s}_X$-strings in Γ^β are typically not irreducible as $\mathfrak{su}(2)$-representations. Our next task will be to describe the decomposition of a particular $\mathfrak{s}_X$-string into irreducibles for $\mathfrak{s}_X$. To do this, we will need to use some geometry of the weight space $\mathfrak{h}^*$ of $\mathfrak{sl}(3,\mathbb{C})$.

Let us begin, though, with some further remarks about finite dimensional representations of $\mathfrak{sl}(2,\mathbb{C})$. As in the previous section, let Γ^b denote the irreducible finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ with highest weight $b \in \mathbb{N}$. As mentioned earlier, the nontrivial weight spaces of Γ^b are all one-dimensional, with weights

$$-b,\ -b+2,\ \dots,\ b-2,\ b.$$

Because of this, it is possible to determine the decomposition of an arbitrary finite dimensional $\mathfrak{sl}(2,\mathbb{C})$-representation V from the dimensions of its weight spaces. Specifically, if we denote the k-weight space of V by V_k, and put

$$N_k = \dim V_k - \dim V_{k+2}$$

for each $k \in \mathbb{N}$, then V contains Γ^k with multiplicity N_k, for each k. That is,

$$V \cong \bigoplus_{k=0}^{\infty} \left(\Gamma^k\right)^{\oplus N_k}.$$

To compute the dimensions of the weight spaces of the $\mathfrak{s}_X$-strings in the irreducible representation Γ^β of $\mathfrak{sl}(3,\mathbb{C})$, we use the Kostant Multiplicity Formula. To state this formula, let G be an arbitrary complex semisimple group, $\mathfrak{g}$ its Lie algebra, Δ^+ its set of positive roots and W its Weyl group. One defines a combinatorial function p on the weights of G by defining $p(\mu)$ to be the number of ways of writing the weight μ as a non-negative integral linear combination of the positive roots,

$$\mu = \sum_{\alpha\in\Delta^+} c_\alpha\,\alpha \qquad\qquad (c_\alpha \in \mathbb{N}).$$

Recall also that W acts on the space of weights of G (see Section 3.3).

Theorem 6.7. (Kostant Multiplicity Formula) *With the above notation,*

$$\dim(\Gamma^{\beta})_{\mu} = \sum_{w \in W} (-1)^{w} p\left(w \cdot (\beta + \rho) - (\mu + \rho)\right)$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^{+}} \alpha$ *is half the sum of the positive roots of* $\mathfrak{g}$*, and* $(-1)^{w}$ *is the determinant of* w *as a transformation of the weight space of* $\mathfrak{g}$*.*

This is a variant of the Weyl character formula. For a proof, see [FH91, §25].

In the case at hand, the group G is $\mathrm{SL}(3, \mathbb{C})$ and the set of positive roots is $\Delta^{+} = \{\alpha_X, \alpha_Y, \alpha_Z\}$. The only dependency in this set is $\alpha_Z = \alpha_X + \alpha_Y$. It follows that if $\mu = a\alpha_X + b\alpha_Y$ $(a, b \in \mathbb{Z})$, then we can write

$$\mu = (a - j)\alpha_X + (b - j)\alpha_Y + j\alpha_Z,$$

for $j = 0, \ldots, min(a, b)$, as long as both a and b are non-negative. Hence,

$$p(\mu) = \begin{cases} 0 & \text{if } a < 0 \text{ or } b < 0, \\ min(a, b) + 1 & \text{otherwise.} \end{cases}$$

It will be convenient to extend the function p on Λ_R to a continuous function on all of the real part of $\mathfrak{h}^{*}$, namely on

$$\mathfrak{h}^{*}_{\mathbb{R}} = \{a\alpha_X + b\alpha_Y \quad | \quad a, b \in \mathbb{R}\}.$$

Therefore, let us define

$$q(a\alpha_X + b\alpha_Y) = \begin{cases} 0 & \text{if } a \leq 0 \text{ or } b \leq 0 \\ min(a,b) & \text{otherwise} \end{cases} \tag{6.7.1}$$

so that $p(\mu) = q(\mu + \rho)$ for $\mu \in \Lambda_R$.

To state the next lemma, we introduce a certain linear function on $\mathfrak{h}^*$. Define ϕ_X on $\mathfrak{h}^*$ by

$$\phi_X(ie_1 + j(-e_3)) = \frac{1}{3}(i + 2j)$$

where e_1 and $-e_3$ are the fundamental weights,

$$e_1 \quad : \quad \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \mapsto t_1$$

$$-e_3 \quad : \quad \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \mapsto -t_3$$

in $\mathfrak{h}^*$, and $i, j \in \mathbb{R}$. The importance of this functional is that

$$\phi_X(\alpha_X) \quad = \quad \phi_X(2e_1 - (-e_3)) = 0$$

and

$$\begin{aligned} \phi_X(\alpha_Y) &= \phi_X(-e_1 + 2(-e_3)) = 1, \\ \phi_X(\alpha_Z) &= \phi_X(\alpha_X + \alpha_Y) = 1. \end{aligned}$$

The first of these identities shows that ϕ_X is constant along each $\mathfrak{s}_X$-string. The other two show that on adjacent $\mathfrak{s}_X$-strings, which differ by a displacement of α_Y or α_Z, the values of ϕ_X differ by one. These values will belong to $\mathbb{Z}$, $\mathbb{Z}+\frac{1}{3}$ or $\mathbb{Z}-\frac{1}{3}$, depending upon the coset of Λ_R in Λ_W which is determined by β. It will be useful to think of ϕ_X as an enumeration of the $\mathfrak{s}_X$-strings.

Lemma 6.8. *Let $\beta = me_1 + n(-e_3)$ be a dominant weight for* $\mathrm{SL}(3,\mathbb{C})$*, and let μ be any weight which occurs non-trivially in Γ^{β}. The $\mathfrak{s}_X$-string through μ in Γ^{β} decomposes into irreducible representations for $\mathfrak{s}_X$ as*

$$\Gamma^{b_0} \oplus \Gamma^{b_0+2} \oplus \cdots \oplus \Gamma^{b_1-2} \oplus \Gamma^{b_1},$$

where

$$\begin{aligned} b_0 &= \left|\phi_X(\mu - (m-n)e_3)\right|, \\ b_1 &= (m+n) - \left|\phi_X(\mu + \frac{1}{2}(m-n)e_3)\right|. \end{aligned}$$

Proof. Fix β as in the statement. Let us define functions

$$\begin{aligned} b_0(t) &= \left|\frac{2}{3}(m-n) + t\,\right|, \\ b_1(t) &= (m+n) - \left|\frac{1}{3}(m-n) - t\,\right|, \end{aligned}$$

so that in the statement of the lemma, $b_0 = b_0(\phi_X(\mu))$ and $b_1 = b_1(\phi_X(\mu))$.

Next put $\mu_0 = -\phi_X(\mu)(e_3)$. Thus μ_0 lies on the line in $\mathfrak{h}^*$ corresponding to the $\mathfrak{s}_X$-string through μ, but has $\mu_0(H_X) = 0$. The weights of this $\mathfrak{s}_X$-string in Γ^β are all of the form

$$\mu + x\alpha_X,$$

with x an integer. Equivalently, they are of the form

$$\mu_0 + x\alpha_X,$$

where either $x \in \mathbb{Z}$ or $x \in \mathbb{Z} + \frac{1}{2}$, according as whether $\mu(H_X)$ is even or odd.

From the preceding remarks about the structure of $\mathfrak{sl}(2, \mathbb{C})$ representations, it suffices to prove that the dimensions of these weight spaces $(\Gamma^\beta)_{\mu_0 + x\alpha_X}$ satisfy:

$$\dim(\Gamma^\beta)_{\mu_0+x\alpha_X} - \dim(\Gamma^\beta)_{\mu_0+(x-1)\alpha_X}$$
$$= \begin{cases} 0, & x \leq -\frac{1}{2}\left(b_1(\phi_X(\mu)) + 1\right), \\ 1, & -\frac{1}{2}\left(b_1(\phi_X(\mu)) + 1\right) < x \leq -\frac{1}{2}b_0(\phi_X(\mu)), \\ 0, & -\frac{1}{2}b_0(\phi_X(\mu)) < x \leq \frac{1}{2}b_0(\phi_X(\mu)), \\ -1, & \frac{1}{2}b_0(\phi_X(\mu)) < x \leq \frac{1}{2}\left(b_1(\phi_X(\mu)) + 1\right), \\ 0, & \frac{1}{2}\left(b_1(\phi_X(\mu)) + 1\right) < x, \end{cases} \tag{6.7.2}$$

with $x \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$, as above. The Kostant Multiplicity Formula tells us that

$$\dim(\Gamma^\beta)_{\mu_0+x\alpha_X} = \sum_{w \in W} (-1)^w q\left(-x\alpha_X - \mu_0 + w \cdot (\beta + \rho)\right),$$

where q is the piecewise linear function of Equation (6.7.1). We can prove the lemma by computing the derivative of this expression, as a function of x.

From the definition of q, it is easy to check that

$$\frac{\partial}{\partial x} q(x\alpha_X + y\alpha_Y) = \begin{cases} 1 & \text{if } 0 \le x \le y, \\ 0 & \text{otherwise.} \end{cases}$$

Let us change coordinates slightly in this expression. Put

$$\begin{aligned} \delta_X &= \alpha_Y + \frac{1}{2}\alpha_X \\ &= \frac{1}{2}(\alpha_Y + \alpha_Z) \end{aligned}$$

so that $\phi_X(\delta_X) = \phi_X(\alpha_Y) = 1$, but δ_X is orthogonal to α_X. Then

$$\frac{\partial}{\partial x} q(x\alpha_X + y\delta_X) = \begin{cases} 1 & \text{if } -\frac{1}{2}y \le x \le \frac{1}{2}y \\ 0 & \text{otherwise} \end{cases}$$

In other words, as a function of $(x, y) \in \mathbb{R}^2$, $\frac{\partial}{\partial x} q(x\alpha_X + y\delta_X)$ is the characteristic function of the cone

$$\{\, (x, y) \quad | \quad -\frac{1}{2}y \le x \le \frac{1}{2}y \ \ (\text{and } y \ge 0) \,\}.$$

With this description, we see that the function

$$F(x,y) = \frac{\partial}{\partial x} \sum_{w \in W} (-1)^w q\left(-(x\alpha_X + y\delta_X) + w \cdot (\beta + \rho)\right)$$

is an alternating sum of characteristic functions of six cones in $\mathbb{R}^2$. These cones are shown in Figure 6.3.

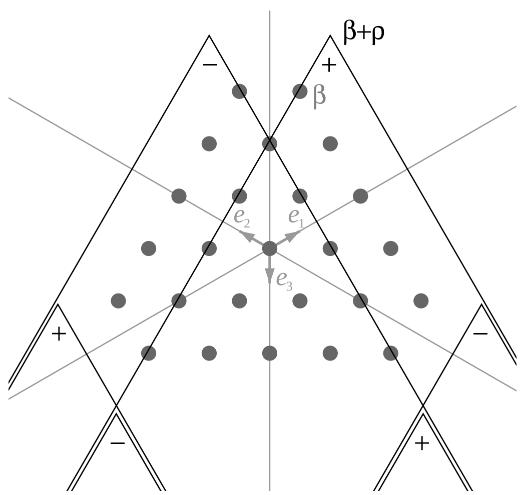

Fig. 6.3. The six cones appearing in the partial derivative of the Kostant multiplicity formula, and their associated signs.

From the diagram, it is clear that the signed sum of the characteristic functions totals zero everywhere except for the two quadrilaterals marked in Figure 6.4, where the total is constant $+1$ and -1 as indicated. The boundaries of the quadrilateral of positive

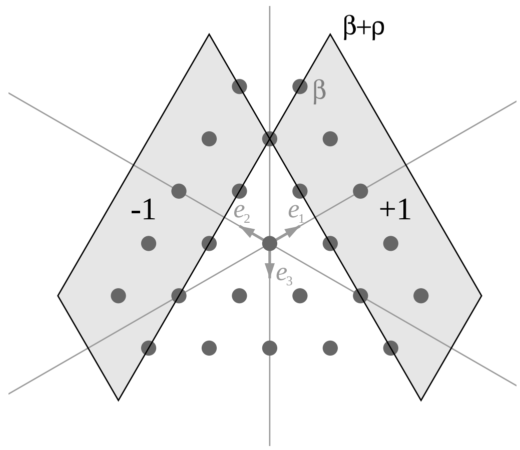

Fig. 6.4. Support of the signed characteristic functions.

sign are the parameterized lines

$$\beta + \rho + t\alpha_Y \qquad (t \in \mathbb{R})$$

$$w_{(2,3)} \cdot (\beta + \rho) + t\alpha_Z \qquad (t \in \mathbb{R})$$

$$\beta + \rho + t\alpha_Z \qquad (t \in \mathbb{R})$$

$$w_{(1,2)} \cdot (\beta + \rho) + t\alpha_Y \qquad (t \in \mathbb{R}).$$

The notation for the Weyl group elements here is that, for $\sigma \in S_3$, w_σ denotes the element which acts on $\mathfrak{h}^*$ by

$$w_\sigma : e_j \mapsto e_{\sigma(j)}$$

for $j = 1, 2, 3$. Specifically, $w_{(2,3)}$ is the reflection in the line through e_1, and $w_{(1,2)}$ is the reflection in the line through e_3.

Putting everything into the coordinate system given by $(x, y) \mapsto x\alpha_X + y\delta_Y$, we get

$$e_1 = (\tfrac{1}{2}, \tfrac{1}{3}), \qquad e_2 = (-\tfrac{1}{2}, \tfrac{1}{3}), \qquad e_3 = (0, -\tfrac{2}{3}),$$
$$\alpha_X = (1, 0), \qquad \alpha_Y = (-\tfrac{1}{2}, 1), \qquad \alpha_Z = (\tfrac{1}{2}, 1),$$

so

$$\begin{aligned}
\beta + \rho &= (m+1)e_1 + (n+1)(-e_3) \\
&= \left(\frac{1}{2}(m+1),\ \frac{1}{3}(m+2n+3)\right), \\
w_{(2,3)} \cdot (\beta + \rho) &= (m+1)e_1 - (n+1)e_2 \\
&= \left(\frac{1}{2}(m+n+2),\ \frac{1}{3}(m-n)\right), \\
w_{(1,2)} \cdot (\beta + \rho) &= (m+1)e_2 + (n+1)(-e_3) \\
&= \left(-\frac{1}{2}(m+1),\ \frac{1}{3}(m+2n+3)\right).
\end{aligned}$$

The four bounding lines of the quadrilateral become

$$2x + y = (m+1) + \frac{1}{3}(m+2n+3) = \frac{4}{3}m + \frac{2}{3}n + 2$$

and

$$2x - y = (m+n+2) - \frac{1}{3}(m-n) = \frac{2}{3}m + \frac{4}{3}n + 2$$

as an upper bound on x, and

$$2x - y = (m+1) - \frac{1}{3}(m+2n+3) = \frac{2}{3}m - \frac{2}{3}n$$

and

$$2x + y = -(m+1) + \frac{1}{3}(m + 2n + 3) = -\frac{2}{3}m + \frac{2}{3}n$$

as a lower bound. In other words, the upper bound on x is

$$\begin{aligned} x &= \frac{1}{2}\min\{\frac{4}{3}m + \frac{2}{3}n + 2 - y,\ \frac{2}{3}m + \frac{4}{3}n + 2 + y\} \\ &= \frac{1}{2}b_1(y) + 1 \end{aligned}$$

and the lower bound is

$$\begin{aligned} x &= \frac{1}{2}\max\{\frac{2}{3}m - \frac{2}{3}n + y,\ \ -\frac{2}{3}m + \frac{2}{3}n - y\} \\ &= \frac{1}{2}b_0(y). \end{aligned}$$

By reflection using $w_{(1,2)}$ we see that the boundaries of the other quadrilateral in Figure 6.4 are

$$\begin{aligned} & & x &= -\frac{1}{2}b_0(y) \\ \text{and} & & x &= -\frac{1}{2}b_1(y) - 1. \end{aligned}$$

Comparing these with the desired values in Equation (6.7.2), we have proven the lemma.

□

There is, of course, an analogous theory for $\mathfrak{s}_Y$-strings, which we define now in the obvious way. Let $\mathfrak{s}_Y$ be the Lie subalgebra of $\mathfrak{su}(3)$

$$\mathfrak{s}_Y = \left\{ \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{A} \end{array} \right) \middle| A \in \mathfrak{su}(2)_{\mathbb{C}} \right\} \cong \mathfrak{sl}(2, \mathbb{C}).$$

In an irreducible $\mathfrak{su}(3)$-representation Γ^{β}, the $\mathfrak{s}_Y$*-string through* μ for a given weight μ of $\mathfrak{su}(3)$ will be the representation of $\mathfrak{s}_Y$ generated by $(\Gamma^{\beta})_{\mu}$.

Define a linear functional ϕ_Y on $\mathfrak{h}^*$ by

$$\phi_Y\left(ie_1 + j(-e_3)\right) = \frac{1}{3}(2i + j),$$

so that

$$\begin{aligned} \phi_Y(\alpha_Y) &= 0 \\ \phi_Y(\alpha_X) &= \phi_Y(\alpha_Z) = 1. \end{aligned}$$

The next lemma is proven in the same way as Lemma 6.8.

Lemma 6.9. *With β and μ as in the preceding lemma, the decomposition of the $\mathfrak{s}_Y$-string through μ into irreducible representations for $\mathfrak{s}_Y$ is*

$$\Gamma^{b_0} \oplus \Gamma^{b_0+2} \oplus \cdots \oplus \Gamma^{b_1-2} \oplus \Gamma^{b_1}$$

where

$$\begin{aligned} b_0 &= |\phi_Y(\mu - (m-n)e_1)|, \\ b_1 &= (m+n) - |\phi_Y(\mu + \frac{1}{2}(m-n)e_1)|. \end{aligned}$$

Definition 6.10. Let Γ^β be an irreducible $\mathfrak{su}(3)$-representation, and μ a weight of $\mathfrak{su}(3)$.

(i) We say a vector $\xi \in (\Gamma^\beta)_\mu$ is *of* $\mathfrak{s}_X$*-type* k (respectively, *of* $\mathfrak{s}_Y$*-type* k) for $k \in \mathbb{N}$, if the representation of $\mathfrak{s}_X$ (respectively, $\mathfrak{s}_Y$) generated by ξ is irreducible with highest weight k.

(ii) The *minimal* $\mathfrak{s}_X$*-type* (respectively, *minimal* $\mathfrak{s}_Y$*-type*) of $(\Gamma^\beta)_\mu$ is the smallest integer k for which there exist vectors of $\mathfrak{s}_X$-type (respectively, $\mathfrak{s}_Y$-type) k in $(\Gamma^\beta)_\mu$.

Remark 15. From time to time, for the ease of stating certain results, we may think of the zero vector as having $\mathfrak{s}_X$-type k for every $k \in \mathbb{N}$. Note, however, that we will certainly ignore the zero vector in the definition of minimal $\mathfrak{s}_X$-type, and also when claiming that a certain space contains no vectors of some given $\mathfrak{s}_X$-type. The same convention shall also be adopted for $\mathfrak{s}_Y$-types.

Lemma 6.8 gives us a formula for the minimal $\mathfrak{s}_X$-type in an entire $\mathfrak{s}_X$-string, but we would also like to have a formula for the minimal $\mathfrak{s}_X$-type in a given weight space $(\Gamma^\beta)_\mu$. If μ and β are as in the two preceding lemmas, then we are looking for the minimal $\mathfrak{s}_X$-type in the $\mu(H_X)$-weight space of the $\mathfrak{s}_X$-representation

$$\Gamma^{b_0} \oplus \Gamma^{b_0+2} \oplus \cdots \oplus \Gamma^{b_1},$$

with b_0 and b_1 as in Lemma 6.8. This is seen to be

$$\max\{\, b_0,\ |\mu(H_X)|\,\}.$$

An extremely convenient formula for this quantity can be given if we introduce some new notation.

Definition 6.11. Let

$$\delta_X = \frac{1}{2}(\alpha_Y + \alpha_Z) = \frac{3}{2}(-e_3)$$

as in the proof of Lemma 6.8, and define an ℓ^∞-type norm $|\cdot|_X$ on $\mathfrak{h}^*$ by

$$|x\alpha_X + y\delta_X|_X = \max\{|2x|, |y|\}$$

for any $x, y \in \mathbb{R}$. Similarly, put

$$\delta_Y = \frac{1}{2}(\alpha_X + \alpha_Z) = \frac{3}{2}e_1$$

and define $|\cdot|_Y$ on $\mathfrak{h}^*$ by

$$|x\alpha_Y + y\delta_Y|_Y = \max\{|2x|,\ |y|\}.$$

Lemma 6.12. *Let $\beta = me_1 + n(-e_3)$ and μ be as in the previous lemma. The minimal $\mathfrak{s}_X$-type of $(\Gamma^\beta)_\mu$ is*

$$|\mu - (m-n)e_3|_X.$$

The minimal $\mathfrak{s}_Y$-type of $(\Gamma^\beta)_\mu$ is

$$|\mu - (m-n)e_1|_Y.$$

Proof. If $\mu = x\alpha_X + y\delta_X$ then

$$\mu - (m-n)e_3 = x\alpha_X + (y + \frac{2}{3}(m-n))\delta_X.$$

On the other hand,

$$|\mu(H_X)| = |2x|,$$

and in the notation of Lemma 6.8,

$$\begin{aligned} b_0 &= |\phi_X(\mu - (m-n)e_3)| \\ &= |y + \frac{2}{3}(m-n)|. \end{aligned}$$

This proves the formula for minimal $\mathfrak{s}_X$-types. The formula for minimal $\mathfrak{s}_Y$-types is proven similarly. ∎

Remark 16. Note that if we fix μ and let $\beta = me_1 + n(-e_3)$ vary, the minimal $\mathfrak{s}_X$-types and $\mathfrak{s}_Y$-types of $(\Gamma^\beta)_\mu$ depend only on the value of $(m-n)$. Moreover, both the minimal $\mathfrak{s}_X$-types and the minimal $\mathfrak{s}_Y$-types tend to infinity as $|m-n|$ goes to infinity. In other words, vectors of a given $\mathfrak{s}_X$-type k occur only in the spaces $(\Gamma^\beta)_\mu$ where β lies within some bounded neighbourhood of the centre line

$$\{t(e_1 - e_3) \quad | \quad t \in \mathbb{R}\}$$

in the fundamental Weyl chamber $\mathcal{W}$.

We also provide an alternative definition of the norms $|\cdot|_X$ and $|\cdot|_Y$. Recall that we have defined functionals ϕ_X and ϕ_Y on $\mathfrak{h}^*$. We define a similar functional ϕ_Z by

$$\phi_Z(ie_1 + j(-e_3)) = i - j,$$

so that

$$\phi_Z(\alpha_Z) = 0$$

and

$$\phi_Z(\alpha_X) = -\phi_Z(\alpha_Y) = 1.$$

Lemma 6.13. *For any* $\mu \in \mathfrak{h}^*$,

$$|\mu|_X = |\phi_Y(\mu)| + |\phi_Z(\mu)|$$

and

$$|\mu|_Y = |\phi_X(\mu)| + |\phi_Z(\mu)|.$$

Proof. The proof amounts to the following relation between the ℓ^1-norm and the ℓ^∞-norm on the plane:

$$\max\{a, b\} = \frac{1}{2}(|a + b| + |a - b|) \qquad (a, b \in \mathbb{R}).$$

For $\mu = x\alpha_X + y\delta_Y$, we get

$$|\mu|_X = \frac{1}{2}(|2x + y| + |2x - y|).$$

On the other hand,

$$\phi_Y(\mu) = \phi_Y(x\alpha_X + \frac{1}{2}y\alpha_Y + \frac{1}{2}y\alpha_Z) = x + \frac{1}{2}y$$

and

$$\phi_Z(\mu) = \phi_Z(x\alpha_X + \frac{1}{2}y\alpha_Y + \frac{1}{2}y\alpha_Z) = x - \frac{1}{2}y,$$

which proves the first of the equalities. The second is proven similarly. □

6.8 Spectral theory of the directional Laplacians

Every weight space in Γ^β has an orthogonal decomposition into $\mathfrak{s}_X$-types, and also an orthogonal decomposition into $\mathfrak{s}_Y$-types. These two decompositions will not be the same, except in trivial cases, and much of what follows is aimed at comparing the two decompositions.

Note that the decomposition of a weight space $(\Gamma^{\beta})_{\mu}$ into $\mathfrak{s}_X$-types is exactly the same as the spectral decomposition of the operator

$$\Delta_X \in \operatorname{End}((\Gamma^{\beta})_{\mu}).$$

This is because the operator Δ_X is in $\mathcal{U}(\mathfrak{s}_X)$, and hence preserves the irreducible components of the $\mathfrak{s}_X$-string through μ. The action of Δ_X on the vectors of $\mathfrak{s}_X$-type k in $(\Gamma^{\beta})_{\mu}$ can be explicitly determined using the results of Section 6.6. Such vectors have $\mathfrak{s}_X$-weight $\mu_X = \mu(H_X)$, and hence Equation (6.6.3) shows that Δ_X acts on them as the scalar

$$\frac{1}{4}\left(k(k+2) - \mu_X(\mu_X+2)\right).$$

Remark 17. Because of this correspondence between vectors of a given $\mathfrak{s}_X$-type and eigenvectors for Δ_X with a particular eigenvalue, and the similar correspondence for $\mathfrak{s}_Y$ and Δ_Y, all the results of this section could be phrased in terms of spectral theory for Δ_X and Δ_Y. Of course, understanding this spectral theory is the aim of this section. However, we will continue to use the terminology of $\mathfrak{s}_X$-types and $\mathfrak{s}_Y$-types, since it is more succinct and makes the proofs of certain results more intuitive.

Considering all the K-types in $L^2(\mathcal{X}; E_{-\mu})$ together, we have now proven that the differential operator Δ_X on $L^2(\mathcal{X}; E_{-\mu})$ has discrete spectrum contained in the set

$$\left\{ \frac{1}{4}\left(k(k+2) - \mu_X(\mu_X+2)\right) \middle| \, k \in \mathbb{N} \right\}.$$

Similarly, the spectrum of Δ_Y on $L^2(\mathcal{X}; E_{-\mu})$ is a subset of

$$\left\{ \frac{1}{4} \left(k(k+2) - \mu_Y(\mu_Y + 2) \right) \middle| \, k \in \mathbb{N} \right\},$$

where

$$\mu_Y = \mu(H_Y) = \mu \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right).$$

In both cases, the multiplicities of all spectral values will be infinite.

A crucial question, though, is how the spectral decompositions of the two operators are related. It is clear that Δ_X and Δ_Y do not commute, so we cannot expect simultaneous diagonalizations of the two operators. Nevertheless, we will eventually prove a result—Proposition 6.18—whose consequences are almost as powerful.

The decomposition of vectors into $\mathfrak{s}_X$-types is clearly respected by the maps

$$X : (\Gamma^\beta)_\mu \longrightarrow (\Gamma^\beta)_{\mu - \alpha_X}$$

and

$$X' : (\Gamma^\beta)_\mu \longrightarrow (\Gamma^\beta)_{\mu + \alpha_X}.$$

This decomposition is not respected by the map

$$Y : (\Gamma^\beta)_\mu \longrightarrow (\Gamma^\beta)_{\mu - \alpha_Y},$$

nor by the corresponding maps for Y', Z and Z'. However, the "lack of respect" is not too bad. Note that

$$\begin{aligned} \alpha_Y(H_X) &= -1, \\ \alpha_Z(H_X) &= 1, \end{aligned}$$

which shows that if the $\mathfrak{s}_X$-types appearing in $(\Gamma^\beta)_\mu$ are all even, then those appearing in $(\Gamma^\beta)_{\mu\pm\alpha_Y}$ and $(\Gamma^\beta)_{\mu\pm\alpha_Z}$ are all odd, and vice versa . With this in mind, we have the following.

Lemma 6.14. *With β and μ as above, let $\xi \in (\Gamma^\beta)_\mu$ be a vector of $\mathfrak{s}_X$-type k. The decompositions of $Y\xi$, $Y'\xi$, $Z\xi$ and $Z'\xi$ into $\mathfrak{s}_X$-types contain only vectors of $\mathfrak{s}_X$-type $(k-1)$ and $(k+1)$.*

Likewise, if $\eta \in (\Gamma^\beta)_\mu$ has $\mathfrak{s}_Y$-type l, then the only $\mathfrak{s}_Y$-types occurring in the decompositions of $X\eta, X'\eta, Z\eta$ and $Z'\eta$ with respect to $\mathfrak{s}_Y$ are $(l-1)$ and $(l+1)$.

Proof. We will prove the result for $Y\xi$ only. The other seven cases are similar.

The key fact that is needed is the following well-known property of representations of $\mathfrak{sl}(2,\mathbb{C})$. (See [FH91, §11].) Let V be a finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$, and let $v \in V$ be a vector of weight $a \in \mathbb{Z}$. Let

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then,

(i) v lies in a direct sum of irreducible subrepresentations with highest weights no more than b if and only if

$$(E')^{\frac{1}{2}(a+b)+1}v = 0.$$

(ii) v lies in a direct sum of irreducible subrepresentations with highest weights no less than b if and only if

$$v = E^{\frac{1}{2}(a+b)}v_0,$$

for some $v_0 \in V$.

With $\mathfrak{s}_X \cong \mathfrak{sl}(2,\mathbb{C})$, we first apply these observations to the vector ξ, which has weight $\mu_X = \mu(H_X)$ with respect to $\mathfrak{s}_X$. Being of $\mathfrak{s}_X$-type k, we have that

$$X^{\frac{1}{2}(\mu_X+k)+1}\xi = 0,$$

and that there is $\xi_0 \in \Gamma^\beta$ such that

$$\xi = (X')^{\frac{1}{2}(\mu_X+k)}\xi_0.$$

From the first of these properties, we see that

$$\begin{aligned}
&X^{\frac{1}{2}(\mu_X+k)+2}(Y\xi) \\
&\quad = Y\left(X^{\frac{1}{2}(\mu_X+k)+2}\right)\xi + \left(\frac{1}{2}(\mu_X+k)+2\right)ZX^{\frac{1}{2}(\mu_X+k)+1}\xi \\
&\quad = 0.
\end{aligned}$$

Since $Y\xi$ is a vector of weight

$$(\mu - \alpha_Y)(H_X) = \mu_X + 1,$$

with respect to $\mathfrak{s}_X$, we have shown that $Y\xi$ contains no $\mathfrak{s}_X$-types greater than $(k+1)$.

From the second property, and the fact that Y commutes with X', we have

$$Y\xi = (X')^{\frac{1}{2}(\mu_X + k)}(Y\xi_0),$$

which shows that $Y\xi$ contains no $\mathfrak{s}_X$-types less than $(k-1)$. $\square$

Let us return, briefly, to consider the K-invariant differential operators which are motivating this discussion. For instance, consider the operators Δ_X and Δ_Y on $C^\infty(\mathcal{X}; E_{-\mu})$. In order to understand the interrelation of their spectral theory, we need to understand the finite-dimensional operators

$$\Delta_X,\ \Delta_Y\ :\ (\Gamma^\beta)_{-\mu} \longrightarrow (\Gamma^\beta)_{-\mu}$$

for each $\beta \in \Lambda_W^{(\mathrm{Dom})}$. It is possible, although painful, to explicitly compute the finite dimensional operators for each Δ_Y with respect to an eigenbasis for Δ_X — that is, in terms of $\mathfrak{s}_X$-types. Fortunately, we will not need that level of detail. Since much of Kasparov's KK-theory works modulo compact operators, we will only need to make calculations on the complement of arbitrarily large finite dimensional subspaces. That is, we will need to consider the operators on Γ^β as the dominant weight β goes to infinity.

Fix a weight $\mu \in \Lambda_W$ and an $\mathfrak{s}_X$-type k. Remark 16 suggests that we parameterize the dominant weights of $\mathfrak{su}(3)$ by

$$\beta = \beta_{s,t} = \begin{cases} se_1 + t(e_1 - e_3) & \text{for} \quad s \geq 0 \\ se_3 + t(e_1 - e_3) & \text{for} \quad s < 0 \end{cases}$$

with $(s,t) \in \mathbb{Z} \times \mathbb{N}$. This organizes the representations Γ^{β} into infinite sequences, one for each fixed value of s, along which the minimal $\mathfrak{s}_X$-types and $\mathfrak{s}_Y$-types of $(\Gamma^{\beta})_{\mu}$ are constant. Moreover, Remark 16 tells us that if we are interested in vectors of a given $\mathfrak{s}_X$-type k in $(\Gamma^{\beta})_{\mu}$, as β varies, then we need only consider a finite number of values of the parameter s, which we can then deal with one at a time. This explains the sequence of dominant weights which is considered in the following technical lemma.

Lemma 6.15. *Fix $k \in \mathbb{N}$ and $\mu \in \Lambda_W$. Fix $s \in \mathbb{Z}$ such that the weight spaces $(\Gamma^{\beta_{s,t}})_{\mu}$ contain vectors of $\mathfrak{s}_X$-type k for all sufficiently large t. For each such t, let ξ_t be a unit vector of $\mathfrak{s}_X$-type k in $(\Gamma^{\beta_{s,t}})_{\mu}$. Then*

$$\lim_{t \to \infty} \frac{1}{t} \|Y \xi_t\| = c, \tag{6.8.1}$$

where $c = c(\mu, k, s)$ is some nonzero constant.

Moreover, if we use Lemma 6.14 to write

$$Y\xi_t = \xi_t' + \xi_t'',$$

where ξ'_t and ξ''_t are vectors of $\mathfrak{s}_X$-type $(k-1)$ and $(k+1)$, respectively, then

$$\begin{aligned} \lim_{t\to\infty} \tfrac{1}{t}\|\xi'_t\| &= c_1 \\ \lim_{t\to\infty} \tfrac{1}{t}\|\xi''_t\| &= c_2 \end{aligned} \tag{6.8.2}$$

for some constants $c_1 = c_1(\mu, k, s)$ and $c_2 = c_2(\mu, k, s)$. The constant c_2 is always nonzero, and c_1 is nonzero unless the minimal $\mathfrak{s}_X$-type of $(\Gamma^{\beta_{s,t}})_{\mu+\alpha_Y}$ is $(k+1)$, in which case $\xi'_t = 0$ for all t.

The above statements remain true if the operator Y is replaced throughout by Y', Z or Z'. It also remains true if $\mathfrak{s}_X$-types are replaced throughout by $\mathfrak{s}_Y$-types, and the operator Y is replaced by X, X', Z or Z'.

Proof. We will prove the result for $\mathfrak{s}_X$-types, working with Y, Y', Z and Z' more or less concurrently. The proof for $\mathfrak{s}_Y$-types is entirely analogous.

We begin with the special cases of $\mu = a(-e_3)$, for $a \in \mathbb{Z}$. For these weight spaces, we can appeal to a certain Weyl group symmetry. Let $w \in W$ be the Weyl group element with representative

$$\tilde{w} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K.$$

Note that w is also the non-trivial Weyl group element for $\mathfrak{s}_X$, and hence it preserves $\mathfrak{s}_X$-types. Furthermore, it preserves the space $(\Gamma^{\beta})_\mu$, which has weight

$$-ae_3(H_X) = 0$$

for $\mathfrak{s}_X$; and it interchanges the spaces $(\Gamma^\beta)_{\mu+\alpha_Y}$ and $(\Gamma^\beta)_{\mu+\alpha_Z}$, which lie in the same $\mathfrak{s}_X$-string and have weights 1 and -1 for $\mathfrak{s}_X$, respectively.

Conjugation by $\tilde{w}$ interchanges Y and Z, up to sign:

$$\begin{aligned} Y^{\tilde{w}} &= -Z \\ Z^{\tilde{w}} &= Y \\ X^{\tilde{w}} &= -X'. \end{aligned}$$

Therefore, we see that for a vector ξ of $\mathfrak{s}_X$-type k in $(\Gamma^\beta)_\mu$,

$$\|Z\xi\| = \|Y^{\tilde{w}}\xi\| = \|\tilde{w}Y\xi\| = \|Y\xi\|.$$

To prove (6.8.1), we consider the action of the Casimir element Ω for $\mathfrak{sl}(3,\mathbb{C})$. The pertinent fact about the Casimir element is the identity (6.6.6), which tells us that on the $a(-e_3)$-weight space in $\Gamma^{\beta_{s,t}}$,

$$\begin{aligned} X'X + Y'Y + Z'Z &= \begin{cases} t^2 + (s+2)t + (\frac{1}{3}s^2 + s - \frac{1}{3}a^2 - a), & s \geq 0 \\ t^2 + (-s+2)t + (\frac{1}{3}s^2 - s - \frac{1}{3}a^2 + a), & s < 0 \end{cases} \\ &= t^2 + O(t). \end{aligned}$$

Note that, by Equation (6.6.3),

$$\langle X'X\xi_t, \xi_t\rangle = \|X\xi_t\|^2 = \frac{1}{4}k(k+2),$$

which is constant, independent of t. Hence,

$$\|Y\xi_t\|^2 + \|Z\xi_t\|^2 = t^2 + O(t). \tag{6.8.3}$$

Since $\|Y\xi_t\| = \|Z\xi_t\|$, this proves (6.8.1), as well as (6.8.1) with Y replaced by Z. The constant c is in both cases equal to $\frac{1}{2}$.

The proof of (6.8.2) for the weight $\mu = a(-e_3)$ is an elaboration of the above argument. We will firstly need to spell out some parity issues regarding the action of $\tilde{w}$ on the weight spaces mentioned earlier. On the space $(\Gamma^\beta)_\mu$, $\tilde{w}$ acts as $+1$ on the vectors of $\mathfrak{s}_X$-type $2j$ with j even, and -1 on the vectors of $\mathfrak{s}_X$-type $2j$ with j odd. This can be confirmed by direct computation in a model for the finite-dimensional irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$.

The similar parity issue for the spaces $(\Gamma^\beta)_{\mu+\alpha_Y}$ and $(\Gamma^\beta)_{\mu+\alpha_Z}$, is as follows. If $\eta \in (\Gamma^\beta)_{\mu+\alpha_Y}$ is a vector of $\mathfrak{s}_X$-type $2j-1$, then $X\eta$ and $\tilde{w}\eta$ are both vectors of the same $\mathfrak{s}_X$-type in $(\Gamma^\beta)_{\mu+\alpha_Z}$, and hence are scalar multiples of one another. Specifically,

$$X\eta = \pm j\tilde{w}\eta,$$

where the sign is $+$ if j is even, and $-$ if j is odd.

Let ξ_t have $\mathfrak{s}_X$-type $k = 2j$. Note that, if we write

$$Y\xi_t = \xi_t' + \xi_t'' \tag{6.8.4}$$

as in the statement of the lemma, then the $\mathfrak{s}_X$-type decomposition of $Z\xi_t$ is

$$\begin{aligned} Z\xi_t &= -Y^{\tilde{w}}\xi_t \\ &= \tilde{w}\left(-Y\tilde{w}^{-1}\xi_t\right) \\ &= (-1)^{j+1}\left(\tilde{w}\xi_t' + \tilde{w}\xi_t''\right). \end{aligned} \tag{6.8.5}$$

The $\mathfrak{s}_X$-type decomposition of $XY\xi_t$ is

$$\begin{aligned} XY\xi_t &= X\xi_t' + X\xi_t'' \\ &= (-1)^j\, j\, \tilde{w}\xi_t' + (-1)^{j+1}(j+1)\tilde{w}\xi_t''. \end{aligned} \tag{6.8.6}$$

Now we consider the order three element Ξ of center of the enveloping algebra of $\mathfrak{sl}(3,\mathbb{C})$, as described in Section 6.6. Equation (6.6.9) gives

$$\begin{aligned} & 3(Z'XY + Y'X'Z) + (2a+6)X'X - aY'Y - aZ'Z \\ &= \begin{cases} (s+3)t^2 + (s^2+5s+6)t + (\frac{2}{9}s^3 + 2s^2 + 4s + \frac{2}{9}a^3 - 2a), & s \geq 0 \\ (s+3)t^2 + (-s^2 - s + 6)t + (\frac{2}{9}s^3 - 2s + \frac{2}{9}a^3 - 2a), & s < 0 \end{cases} \\ &= (s+3)t^2 + O(t). \end{aligned}$$

Once again, using the fact that $X'X$ acts as a fixed scalar on vectors of $\mathfrak{s}_X$-type k in $(\Gamma^{\beta_{s,t}})_\mu$, we get that

$$6\,\mathrm{Re}\langle XY\xi_t, Z\xi_t\rangle - a\|Y\xi_t\|^2 - a\|Z\xi_t\|^2 = (s+3)t^2 + O(t).$$

Using (6.8.4), (6.8.5) and (6.8.6), and the fact that the Weyl group element $\tilde{w}$ acts unitarily, we get

$$-6j\|\xi_t'\|^2 + 6(j+1)\|\xi_t''\|^2 - 2a(\|\xi_t'\|^2 + \|\xi_t''\|^2) = (s+3)t^2 + O(t).$$

Thus,

$$(-6j - 2a)\|\xi_t'\|^2 + (6j - 2a + 6)\|\xi_t''\|^2 = (s+3)t^2 + O(t). \tag{6.8.7}$$

Also, Equation (6.8.3) implies that

$$\|\xi_t'\|^2 + \|\xi_t''\|^2 = \frac{1}{2}t^2 + O(t), \tag{6.8.8}$$

since the left-hand side of this is $\|Y\xi_t\|^2$. We now have a pair of linear equations in $\|\xi_t'\|^2$ and $\|\xi_t''\|^2$. In matrix form,

$$\begin{pmatrix} 1 & 1 \\ -6j - 2a & 6j - 2a + 6 \end{pmatrix} \begin{pmatrix} \|\xi_t'\|^2 \\ \|\xi_t''\|^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t^2 \\ (s+3)t^2 \end{pmatrix} + O(t).$$

The matrix on the left is singular only if $j = -\frac{1}{2}$. But j is an integer, and hence the pair of equations can be solved to give (6.8.2).

It remains to check whether the coefficients c_1 and c_2 of t^2 in the solution are nonzero. Inverting the two-by-two matrix above, we see that the constant c_1 is zero only if

$$\frac{1}{2}(6j - 2a + 6) - (s+3) = 0,$$

that is, if

$$k = 2j = \frac{2}{3}(a+s).$$

If this occurs then, by Lemma 6.8, the minimal $\mathfrak{s}_X$-type for $(\Gamma^{\beta_{s,t}})_{\mu+\alpha_Y}$ is

$$|\frac{2}{3}s + \phi_X(a(-e_3) + \alpha_Y)| = |\frac{2}{3}(a+s) + 1| = k+1,$$

as claimed. On the other hand, the constant c_2 is zero only if

$$\frac{1}{2}(6j + 2a) + (s+3) = 0,$$

so that

$$k = 2j = -\frac{2}{3}(a+s) - 2.$$

But then the minimal $\mathfrak{s}_X$-type for $(\Gamma^{\beta_{s,t}})_\mu$ is

$$|\frac{2}{3}s + \phi_X(a(-e_3))| = \frac{2}{3}|a+s|,$$

and this is strictly greater than k, giving a contradiction.

This proves the result for Y and Z, with $\mu = a(-e_3)$. The result for Y' and Z' follows from a symmetry argument. If we identify $\Gamma^{\beta *}$ with Γ^β using the inner product, then the identity map on Γ^β is a conjugate-linear map between Γ^β and $(\Gamma^\beta)^*$. The contragredient representation π_β^* on $(\Gamma^\beta)^*$ is defined by

$$\pi_\beta{}^*(V) = -\pi_\beta(V)^*, \qquad (V \in \mathfrak{k}_{\mathbb{C}}).$$

The identity map sends the μ-weight space for Γ^β to the $(-\mu)$-weight space for the contragredient representation, and preserves $\mathfrak{s}_X$-types. Since

$$\begin{aligned} \pi_\beta{}^*(Y) &= -\pi_\beta(Y') \\ \text{and} \qquad \pi_\beta{}^*(Z) &= -\pi_\beta(Z'), \end{aligned}$$

this gives the desired result, with $\mu = ae_3 \ \ (a \in \mathbb{Z})$, for the operators

$$\begin{aligned} Y' &: (\Gamma^\beta)_\mu \longrightarrow (\Gamma^\beta)_{\mu-\alpha_Y}, \\ Z' &: (\Gamma^\beta)_\mu \longrightarrow (\Gamma^\beta)_{\mu-\alpha_Z}. \end{aligned}$$

It is also easy to extend to the adjoints of any of the four operators considered so far, namely to

$$\begin{aligned} Y' &: (\Gamma^\beta)_{\mu+\alpha_Y} \longrightarrow (\Gamma^\beta)_\mu, \\ Z' &: (\Gamma^\beta)_{\mu+\alpha_Z} \longrightarrow (\Gamma^\beta)_\mu, \\ Y &: (\Gamma^\beta)_{\mu-\alpha_Y} \longrightarrow (\Gamma^\beta)_\mu, \\ Z &: (\Gamma^\beta)_{\mu-\alpha_Z} \longrightarrow (\Gamma^\beta)_\mu, \end{aligned}$$

with $\mu = a(-e_3) \ \ (a \in \mathbb{Z})$. For suppose $\xi_t \in (\Gamma^\beta)_{\mu-\alpha_Y}$ is a unit vector of $\mathfrak{s}_X$-type k, and let η_t be a unit vector in $(\Gamma^\beta)_\mu$ of $\mathfrak{s}_X$-type $(k-1)$. (If $(\Gamma^\beta)_\mu$ has minimal $\mathfrak{s}_X$-type

$(k+1)$, put $\eta_t = 0$.) Then, with

$$Y\xi_t = \xi'_t + \xi''_t$$

as usual, we have

$$\begin{aligned} \|\xi'_t\| &= |\langle Y\xi_t, \eta_t\rangle| \\ &= |\langle \xi_t, Y'\eta_t\rangle| \\ &= \|\eta''_t\|, \end{aligned}$$

where we are writing

$$Y'\eta_t = \eta'_t + \eta''_t,$$

with η'_t and η''_t being vectors of $\mathfrak{s}_X$-type $(k-2)$ and k, respectively. The asymptotics of $\|\xi'_t\|$ therefore follow from the earlier results, applied to η_t. A similar argument works for $\|\xi''_t\|$, and also for the other three operators, Z, Y' and Z', listed above.

By now we have proven the lemma for each of the four operators Y, Z, Y' and Z' on exactly one weight space in each $\mathfrak{s}_X$-string. Specifically, we have proven it in the unique weight-space in that $\mathfrak{s}_X$-string for which either the domain or target weight space has weight $\mu = a(-e_3)$ for some $a \in \mathbb{Z}$. The domains and targets, therefore, have $\mathfrak{s}_X$-weights either 0 or ± 1, which means that they contain vectors of every $\mathfrak{s}_X$-type that occurs in that $\mathfrak{s}_X$-string.

The final step is to extend away from these central weight spaces. If μ is arbitrary, and $\xi_t \in (\Gamma^\beta)_\mu$ has $\mathfrak{s}_X$-type k, then either

$$\xi_t = X^n \zeta_t \tag{6.8.9}$$

or

$$\xi_t = X'^n \zeta_t, \tag{6.8.10}$$

for some $n \in \mathbb{N}$, and some ζ_t of $\mathfrak{s}_X$-type k which belongs to one of the weight spaces where the lemma is already proven. Let us suppose (6.8.9)—the case of (6.8.10) is essentially the same.

The norm of ζ_t is independent of t—it depends only on n and the $\mathfrak{s}_X$-type k. If

$$Z\zeta_t = \zeta_t' + \zeta_t''$$

where ζ_t' and ζ_t'' are of $\mathfrak{s}_X$-type $(k-1)$ and $(k+1)$, respectively, then the expression

$$Z\xi_t = X^n Z\zeta_t = X^n\zeta_t' + X^n\zeta_t''$$

is a similar decomposition for $Z\xi_t$. The norm $\|X^n\zeta_t'\|$ is a multiple of $\|\zeta_t'\|$, depending on $(k-1)$ and n, but independent of t, and likewise for $\|X^n\zeta_t''\|$ and $\|\zeta_t''\|$. This proves the result.

This method also works to prove the desired asymptotics for $Y'\xi_t$, since Y' also commutes with X^n. To prove the result for $Y\xi_t$, we note once again that the map Y is the adjoint of a map Y' for which we have already proven the result. We can therefore follow an earlier argument to obtain the result for Y. The same trick works for Z', completing the proof.

$\square$

Lemma 6.16. *Fix $\mu \in \Lambda_W$. Let $s \in \mathbb{Z}$ be such that $(\Gamma^{\beta_{s,t}})_\mu$ is nonzero for all $t \gg 0$. For each $t \gg 0$, let ξ_t and η_t be unit vectors in $(\Gamma^{\beta_{s,t}})_\mu$ of $\mathfrak{s}_X$-type k and $\mathfrak{s}_Y$-type l, respectively, for some fixed k and l. Then*

$$\lim_{t\to\infty} \langle \xi_t, \eta_t \rangle = 0.$$

Proof. Actually, we will prove a more specific result: that for some constant $C = C(\mu, k, l, s)$

$$|\langle \xi_t, \eta_t \rangle| \leq C\, t^{-|\phi_Z(\mu - se_3)|+1} \tag{6.8.11}$$

where ϕ_Z is the functional introduced prior to Lemma 6.13.

As in the proof of the previous lemma, we will begin by proving the result for a special case of μ—this time we work with the weights

$$\mu = \beta_{s,\tau}$$

for $\tau \in \mathbb{Z}$. For convenience, let us write $\mu^{(\tau)} = \beta_{s,\tau}$. We will only give the proof for $s \geq 0$. The case of $s < 0$ can be proven similarly, or deduced by using the contragredient representation

$$(\Gamma^{\beta_{s,t}})^* \cong \Gamma^{\beta_{-s,t}}.$$

Therefore, let us consider the weight $\mu^{(\tau)}$ in the representation $\Gamma^{\beta_{s,t}}$. We begin by considering a unit vector $\xi_t^{(\tau)}$ of minimal $\mathfrak{s}_X$-type in $(\Gamma^{\beta_{s,t}})_{\mu^{(\tau)}}$. Note that

$$\phi_X(\mu^{(\tau)}) = \phi_X(se_1 + \tau\alpha_Z) = \frac{1}{3}s + \tau$$

and hence, according to Lemma 6.8, the minimal $\mathfrak{s}_X$-type in the $\mathfrak{s}_X$-string through $\mu^{(\tau)}$ is

$$|\frac{2}{3}s + \phi_X(\mu^{(\tau)})| = s + \tau.$$

The $\mathfrak{s}_X$-weight of $(\Gamma^{\beta_{s,t}})_{\mu^{(\tau)}}$ is

$$\mu^{(\tau)}(H_X) = s + \tau.$$

Hence, $\xi_t^{(\tau)}$ is a highest weight vector of the smallest irreducible $\mathfrak{s}_X$-representation in its $\mathfrak{s}_X$-string.

An immediate consequence of this is that

$$X\xi_t^{(\tau)} = 0.$$

Also, one can compute that

$$\begin{aligned} XX'\xi_t^{(\tau)} &= X'X\xi_t^{(\tau)} + H_X\xi_t^{(\tau)} \\ &= (s+\tau)\xi_t^{(\tau)}. \end{aligned}$$

Next, consider the vector $Z\xi_t^{(\tau)}$. This lives in the weight space

$$(\Gamma^{\beta_{s,t}})_{\mu^{(\tau)}+\alpha_Z} = (\Gamma^{\beta_{s,t}})_{\mu^{(\tau+1)}}.$$

Since the minimal $\mathfrak{s}_X$-type of this space is $(s+\tau+1)$, Lemma 6.14 shows that $Z\xi_t^{(\tau)}$ is actually a scalar multiple of $\xi_t^{(\tau+1)}$. We should like to compute this scalar multiple, or rather its modulus (which is all that is well-defined because of the possible choices of the unit vectors $\xi_t^{(\tau)}$). Since $X\xi_t^{(\tau)} = 0$,

$$Z\xi_t^{(\tau)} = X(Y\xi_t^{(\tau)}).$$

Now, $Y\xi_t^{(\tau)}$ is also a vector of $\mathfrak{s}_X$-type $(s+\tau+1)$ since this is the minimal $\mathfrak{s}_X$-type in the entire $\mathfrak{s}_X$-string through $\mu^{(\tau+1)}$. Therefore,

$$\begin{aligned}\|Z\xi_t^{(\tau)}\|^2 &= \langle X'X(Y\xi_t^{(\tau)}), Y\xi_t^{(\tau)}\rangle \\ &= (s+\tau+1)\|Y\xi_t^{(\tau)}\|^2.\end{aligned}$$

Now we use the Casimir operator for $\mathfrak{su}(3)$. By Equation 6.6.4,

$$\Omega = 2(X'X + Y'Y + Z'Z) + \mathbf{H},$$

where

$$\mathbf{H} = \frac{1}{3}(H_X^2 + H_Y^2 + H_Z^2) + H_X + H_Y + H_Z \in \mathcal{U}(\mathfrak{h}).$$

Let us denote the scalar value of $\mathbf{H}$ on the weight space $(\Gamma^{\beta_{s,t}})_{\mu^{(\tau)}}$ by $\mu^{(\tau)}(\mathbf{H})$. If we identify Ω with its scalar value on $\Gamma^{\beta_{s,t}}$, we get

$$\begin{aligned}\Omega &= \langle \Omega \xi_t^{(\tau)}, \xi_t^{(\tau)} \rangle \\ &= 2\|Y\xi_t\|^2 + 2\|Z\xi_t\|^2 + \langle \mathbf{H}\xi_t^{(\tau)}, \xi_t^{(\tau)} \rangle \\ &= 2(s+\tau+2)\|Y\xi\|^2 + \mu^{(\tau)}(\mathbf{H}).\end{aligned}$$

Therefore,

$$\|Y\xi_t\|^2 = \frac{1}{2(s+\tau+2)}\left(\Omega - \mu^{(\tau)}(\mathbf{H})\right),$$

and hence

$$\|Z\xi_t\|^2 = \frac{(s+\tau+1)}{2(s+\tau+2)}\left(\Omega - \mu^{(\tau)}(\mathbf{H})\right).$$

Next, a similar calculation must be carried out for vectors $\eta_t^{(\tau)}$ of minimal $\mathfrak{s}_Y$-type in $(\Gamma^{\beta_{s,t}})_{\mu^{(\tau)}}$. Once again $\eta_t^{(\tau)}$ is a highest weight vector for the minimal $\mathfrak{s}_Y$-type in its $\mathfrak{s}_Y$-string with that $\mathfrak{s}_Y$-type being τ. It follows that $Z\eta_t^{(\tau)}$ is a scalar multiple of $\eta_t^{(\tau+1)}$. That scalar multiple can be similarly computed, with the result that

$$\|Z\eta_t^{(\tau)}\|^2 = \frac{(\tau+1)}{2(\tau+2)}\left(\Omega - \mu^{(\tau)}(\mathbf{H})\right).$$

Now, since $X\xi_t^{(\tau)} = 0$ and $Y\eta_t^{(\tau)} = 0$, we can compute

$$\begin{aligned}
\langle Z\xi_t^{(\tau)}, Z\eta_t^{(\tau)}\rangle &= \langle (X'X + Y'Y + Z'Z)\xi_t^{(\tau)}, \eta_t^{(\tau)}\rangle \\
&= \frac{1}{2}\left\langle (\Omega - \mathbf{H})\xi_t^{(\tau)}, \eta_t^{(\tau)} \right\rangle \\
&= \frac{1}{2}\left(\Omega - \mu^{(\tau)}(\mathbf{H})\right)\langle \xi_t^{(\tau)}, \eta_t^{(\tau)}\rangle
\end{aligned}$$

and hence

$$\left|\left\langle \xi_t^{(\tau)}, \eta_t^{(\tau)} \right\rangle\right| = \sqrt{\frac{(s+\tau+1)(\tau+1)}{(s+\tau+2)(\tau+2)}}\; \left|\left\langle \xi_t^{(\tau+1)}, \eta_t^{(\tau+1)} \right\rangle\right|.$$

To complete this case of the result, notice that $\xi_t^{(t)}$ and $\eta_t^{(t)}$ are both highest weight vectors with norm one in the $\mathrm{SL}(3,\mathbb{C})$-representation $\Gamma^{\beta_{s,t}}$. Therefore,

$$|\langle \xi_t^{(t)}, \eta_t^{(t)}\rangle| = 1.$$

Hence,

$$\begin{aligned}
|\langle \xi_t^{(0)}, \eta_t^{(0)}\rangle| &= \prod_{\tau=1}^{t} \sqrt{\frac{(s+\tau)\tau}{(s+\tau+1)(\tau+1)}} \\
&= \sqrt{\frac{(s+1)}{(s+t+1)(t+1)}}\,,
\end{aligned}$$

which decays like t^{-1} as t tends to infinity. This proves (6.8.11) for $\mu = \mu^{(0)}$, and k and l minimal.

The remainder of this proof consists of generalizing from this basic case by using a series of inductive arguments. Firstly, suppose μ is any weight, such as $\mu^{(0)}$ above, for which (6.8.11) is known when the $\mathfrak{s}_X$-type k of $\xi_t \in (\Gamma^{\beta_{s,t}})_\mu$ and the $\mathfrak{s}_Y$-type l of $\eta_t \in (\Gamma^{\beta_{s,t}})_\mu$ are both minimal. We will generalize to arbitrary $\mathfrak{s}_X$-types and $\mathfrak{s}_Y$-types in $(\Gamma^{\beta_{s,t}})_\mu$.

Fix k and l, with k greater than the minimal $\mathfrak{s}_X$-type in $(\Gamma^{\beta_{s,t}})_\mu$. Suppose we know (6.8.11) for unit vectors η_t of $\mathfrak{s}_Y$-type l and unit vectors ξ_t of $\mathfrak{s}_X$-type k', for any $k' < k$. Choose unit vectors η_t of $\mathfrak{s}_Y$-type l and ξ_t of $\mathfrak{s}_X$-type $(k-2)$ in $(\Gamma^{\beta_{s,t}})_\mu$. Note that

$$\Delta_Y \eta_t = a\eta_t$$

for some constant $a \in \mathbb{R}$ (independent of t). By applying Lemma 6.15 twice, we also have

$$\Delta_Y \xi_t = Y'Y\xi_t = p(t)\xi_t^- + q(t)\xi_t + r(t)\xi_t^+,$$

where ξ_t^- and ξ_t^+ are unit vectors in $(\Gamma^{\beta_{s,t}})_\mu$ of $\mathfrak{s}_X$-type $(k-4)$ and k, respectively, and p, q and r are scalar functions of t with

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t^2} p(t) &= p_0, \\
\lim_{t\to\infty} \frac{1}{t^2} q(t) &= q_0, \\
\lim_{t\to\infty} \frac{1}{t^2} r(t) &= r_0,
\end{aligned}$$

for some nonzero constants $p_0, q_0, r_0 \in \mathbb{R}$. An exception needs to be made here if $(k-2)$ is the minimal $\mathfrak{s}_X$-type in $(\Gamma^{\beta_{s,t}})_\mu$, in which case we have $\xi_t^- = 0$ for all t, and $p(t)$

becomes redundant. Now, we have

$$
\begin{aligned}
a\langle \xi_t, \eta_t \rangle &= \langle \Delta_Y \xi_t, \eta_t \rangle \\
&= p(t)\langle \xi_t^-, \eta_t \rangle + q(t)\langle \xi_t, \eta_t \rangle + r(t)\langle \xi_t^+, \eta_t \rangle,
\end{aligned}
$$

and hence

$$
\langle \xi_t^+, \eta_t \rangle = \frac{-p(t)}{r(t)}\langle \xi_t^-, \eta_t \rangle + \frac{a - q(t)}{r(t)}\langle \xi_t, \eta_t \rangle.
$$

Since $p(t)/r(t)$ and $(a-q(t))/r(t)$ are bounded as $t \to \infty$, this proves (6.8.11) for vectors of $\mathfrak{s}_X$-type k and $\mathfrak{s}_Y$-type l. A similar argument allows an induction in the $\mathfrak{s}_Y$-type l of η_t.

Now it only remains to prove (6.8.11) for the minimal $\mathfrak{s}_X$-type and $\mathfrak{s}_Y$-type in each weight space of $(\Gamma^{\beta_{s,t}})_\mu$. This is done by a very similar inductive argument to that just used. However, a little bit of careful bookkeeping needs to be done in order to translate it from a heuristic idea to a precise proof.

Let k be the minimal $\mathfrak{s}_X$-type and l the minimal $\mathfrak{s}_Y$-type in $(\Gamma^{\beta_{s,t}})_\mu$. Suppose (6.8.11) is known for the weight $\mu+\alpha_Y$ in place of μ. Note that there are two possibilities for the minimal $\mathfrak{s}_X$-type k' of $(\Gamma^{\beta_{s,t}})_{\mu+\alpha_Y}$: either $k' = k-1$ or $k' = k+1$. We will have to deal with each case separately. But first let us understand when each of the two cases occurs.

By Lemma 6.12,

$$
k = |\mu - se_3|_X
$$

and

$$k' = |\mu + \alpha_Y - se_3|_X.$$

Using Lemma 6.13,

$$k - k' = |\phi_Z(\mu - se_3)| - |\phi_Z(\mu + \alpha_Y - se_3)|,$$

since $\phi_Y(\alpha_Y) = 0$. In other words, the difference between the minimal $\mathfrak{s}_X$-types of the weights μ and $\mu + \alpha$ is exactly the same as the difference between the exponents appearing in Equation (6.8.11).

We will need to make an assumption about the minimal $\mathfrak{s}_Y$-types in the argument to follow.

Assumption 6.17. The minimal $\mathfrak{s}_Y$-type of $(\Gamma^{\beta_{s,t}})_\mu$ is not smaller than the minimal $\mathfrak{s}_Y$-type of $(\Gamma^{\beta_{s,t}})_{\mu+\alpha_Y}$.

Now we begin the induction.

<u>Case I</u> : $k = k' + 1$.

Let $\eta_t \in (\Gamma^{\beta_{s,t}})_\mu$ be unit vectors of $\mathfrak{s}_Y$-type l. Because of Assumption 6.17, we can write

$$Y'\eta_t = a\eta_t' \tag{6.8.12}$$

for some unit vector $\eta_t' \in (\Gamma^{\beta_{s,t}})_{\mu+\alpha_Y}$ of the same $\mathfrak{s}_Y$-type, and some fixed constant a.

Let ξ_t' be a unit vector of minimal $\mathfrak{s}_X$-type k' in $(\Gamma^{\beta_{s,t}})_{\mu+\alpha_Y}$ for each t. Using the fact that $k > k'$, Lemma 6.15 shows that

$$Y\xi_t' = p(t)\xi_t$$

where the ξ_t are unit vectors of type $k = k' + 1$ in $(\Gamma^{\beta_{s,t}})_\mu$, and $p(t) \in \mathbb{R}$ with

$$\lim_{t\to\infty} \frac{1}{t} p(t) = p_0$$

for some $p_0 \neq 0$. In particular, $|p(t)^{-1}| \leq C_1 t^{-1}$ for some $C_1 \in \mathbb{R}$ and all $t \gg 0$. We get

$$\begin{aligned} \langle \xi_t, \eta_t \rangle &= \frac{1}{p(t)} \langle Y\xi_t', \eta_t \rangle \\ &= \frac{a}{p(t)} \langle \xi_t', \eta_t' \rangle. \end{aligned}$$

Now we can apply the inductive hypothesis to get

$$\begin{aligned} |\langle \xi_t, \eta_t \rangle| &\leq aC_1 t^{-1} C t^{-|\phi_Z(\mu+\alpha_Y - se_3)|-1} \\ &\leq C' t^{-|\phi_Z(\mu - se_3)|-1} \end{aligned}$$

for some $C' \in \mathbb{R}$, by using the remarks preceding Assumption 6.17.

<u>Case II</u> : $k = k' - 1$.

Again, let $\eta_t \in (\Gamma^{\beta_{s,t}})_\mu$ be a unit vector of $\mathfrak{s}_Y$-type l, for each $t \gg 0$. The implication of Assumption 6.17 is that $(\Gamma^{\beta_{s,t}})_\mu$ also contains unit vectors η_t' of $\mathfrak{s}_Y$-type l.

They satisfy

$$Y\eta_t' = a\eta_t$$

for some nonzero constant $a \in \mathbb{R}$, at least after adjusting each η_t' by a complex multiple of modulus one.

Now we let $\xi_t \in (\Gamma^{\beta_{s,t}})_\mu$ be unit vectors of $\mathfrak{s}_X$-type k. Lemma 6.15 shows that

$$Y'\xi_t = p(t)\xi_t',$$

where $\xi_t' \in (\Gamma^{\beta_{s,t}})_{\mu+\alpha_Y}$ are unit vectors of $\mathfrak{s}_X$-type $k' = k+1$, and $p(t) \in \mathbb{R}$ with

$$\lim_{t\to\infty} \frac{1}{t} p(t) = p_0$$

for some $p_0 \in \mathbb{R}$. Hence,

$$|p(t)| \leq C_1 t$$

for some constant $C_1 \in \mathbb{R}$ and all t. Then

$$\begin{aligned} \langle \xi_t, \eta_t \rangle &= \frac{1}{a} \langle \xi_t, Y\eta_t' \rangle \\ &= \frac{p(t)}{a} \langle \xi_t', \eta_t' \rangle. \end{aligned}$$

By the inductive hypothesis, we see that

$$\begin{aligned} |\langle \xi_t, \eta_t \rangle| &\leq C_1 t\, C t^{-|\phi_Z(\mu+\alpha_Y - se_3)|-1} \\ &\leq C' t^{-|\phi_Z(\mu - se_3)|-1}, \end{aligned}$$

for some new constant C', by again using the remarks preceding Assumption 6.17.

That concludes the inductive argument for the operator Y, which allows us to transfer the result (6.8.11) from $(\Gamma^{\beta_{s,t}})_{\mu+\alpha_Y}$ to $(\Gamma^{\beta_{s,t}})_\mu$ as long as the minimal $\mathfrak{s}_Y$-type does not decrease in the transition (Assumption 6.17). By the same methods, we can prove an inductive step for Y' which allows us to transfer (6.8.11) from $(\Gamma^{\beta_{s,t}})_{\mu-\alpha_Y}$ to $(\Gamma^{\beta_{s,t}})_\mu$, as long as the minimal $\mathfrak{s}_Y$-type does not decrease in the transition. Likewise we can induce the result (6.8.11) from $(\Gamma^{\beta_{s,t}})_{\mu\pm\alpha_X}$ to $(\Gamma^{\beta_{s,t}})_\mu$, as long as the minimal $\mathfrak{s}_X$-type does not decrease when we move to $(\Gamma^{\beta_{s,t}})_\mu$. What remains to be done, therefore, is some combinatorics in the weight lattice to check that every weight space $(\Gamma^{\beta_{s,t}})_\mu$ can be reached by steps of these kinds, starting from the weight $\mu^{(0)} = se_1$, for which we have proven (6.8.11).

We have already remarked that the minimal $\mathfrak{s}_X$-type of $(\Gamma^{\beta_{s,t}})_{\mu^{(0)}}$ is the minimal $\mathfrak{s}_X$-type of the entire $\mathfrak{s}_X$-string passing through $\mu^{(0)}$. Thus, inducing the result from the weight $\mu^{(0)}$ to the weights $\mu^{(0)} + m\alpha_X$ with $m \in \mathbb{Z}$ presents no problem. The minimal $\mathfrak{s}_Y$-type of $(\Gamma^{\beta_{s,t}})_{\mu^{(0)}+m\alpha_X}$ is

$$|\mu^{(0)} + m\alpha_X - se_1|_Y = |m\alpha_X|_Y = |m|,$$

by Lemma 6.13. Moving from these weight spaces with a displacement of $n\alpha_Y$ ($n \in \mathbb{Z}$), we find that the minimal $\mathfrak{s}_Y$-type of the new weight space is

$$\begin{aligned} |m\alpha_X + n\alpha_Y|_Y &= |(m-n)\alpha_X - n\alpha_Z|_Y \\ &= |m-n| + |n| \\ &\geq |m|. \end{aligned}$$

That is, the minimal $\mathfrak{s}_Y$-type of $(\Gamma^{\beta_{s,t}})_{\mu^{(0)}+m\alpha_X}$ is the minimal $\mathfrak{s}_Y$-type for its entire $\mathfrak{s}_Y$-string. Therefore we can prove (6.8.11) for any weight

$$\mu = \mu^{(0)} + m\alpha_X + n\alpha_Y \qquad (m, n \in \mathbb{Z}),$$

completing the proof. □

Proposition 6.18. *Fix $\mu \in \Lambda_W$ and $k, l \in \mathbb{N}$. For any $\epsilon > 0$, there are only finitely many dominant weights β for* SU(3) *for which there exist unit vectors $\xi, \eta \in (\Gamma^{\beta})_{-\mu}$ of $\mathfrak{s}_X$-type k and $\mathfrak{s}_Y$-type l, respectively, such that*

$$|\langle \xi, \eta \rangle| \geq \epsilon.$$

Proof. In the remarks preceding Lemma 6.15, we observed that we only need consider $\beta_{s,t}$ for finitely many values of s, and then we can appeal to Lemma 6.16 □

Since $\mathfrak{s}_X$-types correspond to eigenvalues of Δ_X, and $\mathfrak{s}_Y$-types correspond to eigenvalues of Δ_Y, Proposition 6.18 says that the eigenspaces for Δ_X and Δ_Y on $L^2(\mathcal{X}; E_{-\mu})$ are almost orthogonal on the complement of a finite dimensional subspace.

6.9 Properly supported operators

We now define some operator algebraic structures related to the directional Laplacians Δ_X and Δ_Y. We begin by setting notation for the spectral projections of Δ_X and Δ_Y. As in the previous section, we will phrase everything in the language of $\mathfrak{s}_X$–types and $\mathfrak{s}_Y$-types. We extend the definitions of $\mathfrak{s}_X$-type and $\mathfrak{s}_Y$-type to the spaces $L^2(\mathcal{X}; E_{-\mu})$ in the following natural way.

Definition 6.19. Let $\mu \in \Lambda_W$. Embed $L^2(\mathcal{X}; E_{-\mu})$ into $L^2(K)$ as a subspace of M-equivariant functions, as in Equation (6.2.1), and let $\mathfrak{s}_X$ and $\mathfrak{s}_Y$ act on $L^2(K)$ by the restriction of the right-regular representation of $\mathfrak{k}_{\mathbb{C}}$. A section $u \in L^2(\mathcal{X}; E_{-\mu})$ is said to be of $\mathfrak{s}_X$-type k (respectively, $\mathfrak{s}_Y$-type k) if the representation of $\mathfrak{s}_X$ (respectively, $\mathfrak{s}_Y$) it generates in $L^2(K)$ is irreducible with highest weight k.

Remark 18. A section $u \in L^2(\mathcal{X}; E_{-\mu})$ has $\mathfrak{s}_X$-type k if and only if u has a Peter-Weyl decomposition

$$u = \bigoplus_{\beta} \sum_{j} ({\xi_{\beta,j}}^* \otimes \xi_{\beta,j}) \in \bigoplus_{\beta \in \Lambda_W^{(\mathrm{Dom})}} \Gamma^{\beta *} \otimes (\Gamma^{\beta})_{\mu}$$

where each $\xi_{\beta,j} \in (\Gamma^{\beta})_{\mu}$ is a vector of $\mathfrak{s}_X$-type k in the sense of Definition 6.10.

The space of vectors of any given $\mathfrak{s}_X$-type k in $L^2(\mathcal{X}; E_{-\mu})$ is infinite dimensional, if nonempty. Vectors of different $\mathfrak{s}_X$-types are orthogonal.

Definition 6.20. Fix $\mu \in \Lambda_W$. For $k \in \mathbb{N}$, we define

$$P_k^{(\mu)} \in \mathcal{B}(L^2(\mathcal{X}; E_\mu))$$

to be the orthogonal projection onto the space of sections of $\mathfrak{s}_X$-type k in E_μ. Similarly, we define

$$Q_k^{(\mu)} \in \mathcal{B}(L^2(\mathcal{X}; E_\mu))$$

to be the orthogonal projection onto the space of sections of $\mathfrak{s}_Y$-type k in E_μ.

More generally, if $A \subseteq \mathbb{N}$ is any set of positive integers, we let

$$P_A^{(\mu)} = \sum_{k \in A} P_k^{(\mu)}$$

and

$$Q_A^{(\mu)} = \sum_{k \in A} Q_k^{(\mu)}.$$

The sum makes sense in the strong operator topology, since all of the projections being summed are orthogonal. We will also write

$$\begin{aligned} P_{[k]}^{(\mu)} &= P_{\{0,\ldots,k\}}^{(\mu)}, \\ Q_{[k]}^{(\mu)} &= Q_{\{0,\ldots,k\}}^{(\mu)} \end{aligned}$$

for $k \in \mathbb{N}$.

If the weight μ is understood as given, we will just write P_A and Q_A for $P_A^{(\mu)}$ and $Q_A^{(\mu)}$, and so on.

These are, of course, the spectral projections for Δ_X and Δ_Y on $L^2(\mathcal{X}; E_\mu)$.

Lemma 6.21. *Fix $\mu \in \Lambda_W$. For any finite sets $A, B \subseteq \mathbb{N}$, $P_A Q_B$ and $Q_B P_A$ are compact operators.*

Proof. Since

$$P_A = \sum_{k \in A} P_k$$

and similarly for Q_B, it suffices to prove that $P_k Q_l$ and $Q_l P_k$ are compact for $k, l \in \mathbb{N}$.

Let $\epsilon > 0$. Let V_0 be the finite dimensional subspace

$$V_0 \;=\; \bigoplus_{\beta \in \Sigma} \Gamma^{\beta *} \otimes (\Gamma^\beta)_{-\mu} \quad \subseteq \quad L^2(\mathcal{X}; E_\mu),$$

where Σ is the finite set of dominant weights for $\mathfrak{k}$ which satisfies the conditions of Proposition 6.18. Let S be the orthogonal projection onto V_0. Note that both P_k and Q_l commute with S, since they both respect the Peter-Weyl decomposition of $L^2(\mathcal{X}; E_\mu)$.

Now,

$$\begin{aligned} \|P_k Q_l (1-S)\| &= \sup_{\|u\|=\|v\|=1} |\langle Q_l(1-S)u, P_k(1-S)v\rangle| \\ &< \epsilon, \end{aligned}$$

by the definition of V_0. Therefore, writing

$$P_k Q_l = P_k Q_l S + P_k Q_l (1 - S)$$

exhibits $P_k Q_l$ as the sum of a finite rank operator and an operator of norm less than ϵ. Since ϵ was arbitrary, $P_k Q_l$ is compact. Taking adjoints proves that $Q_l P_k$ is compact. □

Corollary 6.22. *If $F_1, F_2 \in C_0([0, \infty))$, then $F_1(\Delta_X) F_2(\Delta_Y)$ and $F_2(\Delta_Y) F_1(\Delta_X)$ are compact operators.*

Proof. Lemma 6.21 implies the result if F_1 and F_2 are characteristic functions of bounded sets in $\mathbb{R}$. Since these span a dense subspace in $C_0([0, \infty))$, the result follows by an approximation argument. □

The following definition is inspired by Roe's theory of C^*-algebras for coarse spaces (see [Roe03]).

Definition 6.23. Let μ_1, μ_2 be weights for $\mathfrak{k} = \mathfrak{su}(3)$. A bounded linear operator $T : L^2(\mathcal{X}; E_{\mu_1}) \longrightarrow L^2(\mathcal{X}; E_{\mu_2})$ is called *properly supported for X* if, for every $k \in \mathbb{N}$, there exists $k' \in \mathbb{N}$ such that

$$\left(1 - P^{(\mu_2)}_{[k']}\right) T P^{(\mu_1)}_{[k]} = 0$$

and

$$P^{(\mu_2)}_{[k]} T \left(1 - P^{(\mu_1)}_{[k']}\right) = 0.$$

It is *properly supported for* Y if, for every $k \in \mathbb{N}$ there exists $k' \in \mathbb{N}$ such that

$$\left(1 - Q^{(\mu_2)}_{[k']}\right) T Q^{(\mu_1)}_{[k]} = 0$$

and

$$Q^{(\mu_2)}_{[k]} T \left(1 - Q^{(\mu_1)}_{[k']}\right) = 0.$$

For the next definition, we put $\mathcal{H}_1 = L^2(\mathcal{X}; E_{\mu_1})$ and $\mathcal{H}_2 = L^2(\mathcal{X}; E_{\mu_2})$.

Definition 6.24. (i) The norm-closure in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ of the set of operators that are properly supported for X (respectively, Y) will be denoted by $\mathcal{A}_X(\mathcal{H}_1, \mathcal{H}_2)$ (respectively, $\mathcal{A}_Y(\mathcal{H}_1, \mathcal{H}_2)$).

(ii) We define

$$\mathcal{A}(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{A}_X(\mathcal{H}_1, \mathcal{H}_2) \cap \mathcal{A}_Y(\mathcal{H}_1, \mathcal{H}_2).$$

(iii) Denote by $\mathcal{K}_X(\mathcal{H}_1, \mathcal{H}_2)$ the norm-closure in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ of the set

$$\bigcup_{k=0}^{\infty} P^{(\mu_2)}_{[k]} \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) P^{(\mu_1)}_{[k]},$$

and by $\mathcal{K}_Y(\mathcal{H}_1, \mathcal{H}_2)$ the norm-closure of

$$\bigcup_{k=0}^{\infty} Q^{(\mu_2)}_{[k]} \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) Q^{(\mu_1)}_{[k]}.$$

(iv) When $\mu_1 = \mu_2$ we will denote the above operator spaces by $\mathcal{A}_X(\mathcal{H})$, $\mathcal{A}_Y(\mathcal{H})$, $\mathcal{A}(\mathcal{H})$, $\mathcal{K}_X(\mathcal{H})$ and $\mathcal{K}_Y(\mathcal{H})$, where $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$.

We will also sometimes write $\mathcal{A}_X(\mu_1, \mu_2)$ for $\mathcal{A}\left(L^2(\mathcal{X}; E_{\mu_1}), L^2(\mathcal{X}; E_{\mu_2})\right)$, and so on.

Remark 19. (For the reader familiar with Roe algebras). The C^*-algebra which $\mathcal{A}_X(\mathcal{H})$ most closely resembles is the Roe algebra of the space $\mathbb{N}$ (or actually, of the space of $\mathfrak{s}_X$-types of $\mathcal{H} = L^2(\mathcal{X}; E_\mu)$) endowed with the indiscrete coarse structure. A subset C of $\mathbb{N} \times \mathbb{N}$ is controlled for the indiscrete coarse structure if it is proper, that is if for each $k \in \mathbb{N}$, the sets

$$\begin{aligned} &\{\, k' \quad | \quad (k, k') \in C \,\} \\ \text{and} \qquad &\{\, k' \quad | \quad (k', k) \in C \,\} \end{aligned}$$

are finite. Note, however, that we do not impose the condition of local compactness on elements of $\mathcal{A}_X(\mathcal{H})$. The set $\mathcal{K}_X(\mathcal{H})$ is analogous to the ideal in $C^*(\mathbb{N})$ of norm limits of operators supported close to $0 \in \mathbb{N}$, as defined in [HRY93].

For organizational reasons, it is useful to think of the set of operators

$$\bigcup_{\mu_1, \mu_2 \in \Lambda_W} \mathcal{B}\left(L^2(\mathcal{X}; E_{\mu_1}), L^2(\mathcal{X}; E_{\mu_2})\right)$$

as a C^*-category, over the set of objects Λ_W. We will denote this C^*-category simply by $\mathcal{B}$. In the same vein, we will put

$$\mathcal{A}_X = \bigcup_{\mu_1, \mu_2 \in \Lambda_W} \mathcal{A}_X\left(L^2(\mathcal{X}; E_{\mu_1}), L^2(\mathcal{X}; E_{\mu_2})\right)$$

and similarly for $\mathcal{A}_Y, \mathcal{A}, \mathcal{K}_X$ and $\mathcal{K}_Y$. For more on C^*-categories, see [Mit02].

Lemma 6.25. *An operator $T \in \mathcal{B}$ belongs to $\mathcal{K}_X$ if and only if $P_{[k]}TP_{[k]} \to T$ as $k \to \infty$. Similarly, $T \in \mathcal{K}_Y$ if and only if $Q_{[k]}TQ_{[k]} \to T$ as $k \to \infty$.*

Proof. Suppose $T \in \mathcal{K}_X$. For any $\epsilon > 0$, there exists $k \in \mathbb{N}$ and $S \in P_{[k]}\mathcal{B}P_{[k]}$ such that $\|S - T\| < \epsilon$. Then, for all $l > k$,

$$\begin{aligned} \|T - P_{[l]}TP_{[l]}\| &\leq \|T - S\| + \|P_{[l]}(S - T)P_{[l]}\| \\ &\leq 2\epsilon. \end{aligned}$$

Therefore $P_{[k]}TP_{[k]} \to T$. The converse is obvious.

The same kind of argument proves the statement for $\mathcal{K}_Y$. □

Theorem 6.26. *$\mathcal{A}_X, \mathcal{A}_Y, \mathcal{A}, \mathcal{K}_X$ and $\mathcal{K}_Y$ are C^*-subcategories of $\mathcal{B}$. The intersection of $\mathcal{K}_X$ and $\mathcal{K}_Y$ is the C^*-category $\mathcal{K}$ of compact operators in $\mathcal{B}$.*

Furthermore, $\mathcal{K}_X$ is an ideal in $\mathcal{A}_X$, $\mathcal{K}_Y$ is an ideal in $\mathcal{A}_Y$, and both $\mathcal{K}_X$ and $\mathcal{K}_Y$ are ideals in $\mathcal{A}$.

Proof. Since $\mathcal{K}_X$ is the norm-closure of a nested union of C^*-subcategories of $\mathcal{B}$, it is clearly a C^*-category. Similarly for $\mathcal{K}_Y$.

Lemma 6.21 shows that, for any S and T in $\mathcal{B}$, and any $k, l \in \mathbb{N}$,

$$(P_{[k]}SP_{[k]})(Q_{[l]}TQ_{[l]}) \in \mathcal{K}.$$

Such operators are norm-dense in $\mathcal{K}_X\mathcal{K}_Y$, and hence

$$\mathcal{K}_X \cap \mathcal{K}_Y = \mathcal{K}_X\mathcal{K}_Y \subseteq \mathcal{K}.$$

For the reverse inclusion, note that the projections $P_{[k]}$ converge to the identity in the strong operator topology as $k \to \infty$, and similarly for $Q_{[l]}$ as $l \to \infty$. It follows that, for any rank-one operator $T \in \mathcal{B}$,

$$P_{[k]}TP_{[k]} \to T$$

and

$$Q_{[l]}TQ_{[l]} \to T$$

in the norm topology. Therefore all rank-one operators belong to the C^*-category $\mathcal{K}_X \cap \mathcal{K}_Y$, and hence so do all compact operators.

Next we will show that $\mathcal{A}_X$ is the C^*-category of multipliers of $\mathcal{K}_X$ in $\mathcal{B}$, *ie*,

$$\mathcal{A}_X = \{\, T \in \mathcal{B} \quad | \quad T\mathcal{K}_X,\ \mathcal{K}_XT \subseteq \mathcal{K}_X \,\}.$$

This will imply both that $\mathcal{A}_X$ is a C^*-category, and that $\mathcal{K}_X$ is an ideal in $\mathcal{A}_X$.

Suppose T is properly supported for X. Given $k \in \mathbb{N}$, let k' be as in Definition 6.23. We may take $k' \geq k$. Then, for any $K \in P_{[k]}\mathcal{B}P_{[k]}$,

$$TK = P_{[k']}TKP_{[k']}$$

and

$$KT = P_{[k']} KT P_{[k']}.$$

Thus left and right multiplication by T preserve $\bigcup_{k=0}^{\infty} P_{[k]} \mathcal{B} P_{[k]}$, and hence its norm closure $\mathcal{K}_X$ also. By continuity, every $T \in \mathcal{A}_X$ is a multiplier of $\mathcal{K}_X$.

For the reverse inclusion, suppose that $T \in \mathcal{B}$ multiplies $\mathcal{K}_X$. We will inductively define multipliers T_n of $\mathcal{K}_X$, close to T in norm, which are increasingly close to being properly supported for X in the following sense: there is a sequence of positive integers, $\{j(k)\}_{k \in \mathbb{N}}$ such that,

$$k < n \qquad \Rightarrow \qquad \begin{cases} P_{[j(k)]}{}^{\perp} T_n P_{[k]} & = & 0 \\ P_{[k]} T_n P_{[j(k)]}{}^{\perp} & = & 0. \end{cases} \tag{6.9.1}$$

Fix $\epsilon > 0$. Put $T_0 = T$. Now suppose multipliers $T_0, \ldots, T_n$ of $\mathcal{K}_X$ and integers $j(0), \ldots, j(n-1)$ have been defined, satisfying property (6.9.1). Since T_n multiples $\mathcal{K}_X$, $T_n P_{[n]}$ and $P_{[n]} T_n$ are in $\mathcal{K}_X$. By Lemma 6.25, this means that there exists an integer $j(n) \in \mathbb{N}$ such that

$$\begin{aligned} \|P_{[j(n)]}(T_n P_{[n]}) P_{[j(n)]} - (T_n P_{[n]})\| &< 2^{-n-1}\epsilon \\ \|P_{[j(n)]}(P_{[n]} T_n) P_{[j(n)]} - (P_{[n]} T_n)\| &< 2^{-n-1}\epsilon. \end{aligned}$$

We may assume $j(n) \geq n$, in which case the above gives

$$\begin{aligned} \|P_{[j(n)]}{}^{\perp} T_n P_{[n]}\| &< 2^{-n-2}\epsilon \\ \|P_{[n]} T_n P_{[j(n)]}{}^{\perp}\| &< 2^{-n-2}\epsilon. \end{aligned} \tag{6.9.2}$$

We define

$$T_{n+1} = T_n - P_{[j(n)]}{}^{\perp} T_n P_{[n]} - P_{[n]} T_n P_{[j(n)]}{}^{\perp}.$$

We see immediately that

$$\|T_{n+1} - T_n\| \leq 2^{-n-1}\epsilon \tag{6.9.3}$$

and hence each T_n defined this way is within distance ϵ of $T = T_0$.

Next we show that T_{n+1} multiplies $\mathcal{K}_X$. By the definition of T_{n+1}, it suffices to show that $P_{[j(n)]}{}^{\perp} T_n P_{[n]}$ and $P_{[n]} T_n P_{[j(n)]}{}^{\perp}$ multiply $\mathcal{K}_X$. Let $l \in \mathbb{N}$ and $K \in P_{[l]} \mathcal{B} P_{[l]}$. Considering $P_{[j(n)]}{}^{\perp} T_n P_{[n]}$ first, we have

$$K P_{[j(n)]}{}^{\perp} T_n P_{[n]} \in P_{[l]} \mathcal{B} P_{[n]} \subseteq \mathcal{K}_X.$$

Also, since T_n multiplies $\mathcal{K}_X$, for any $\delta > 0$ there exists $m \in \mathbb{N}$ (we may assume $m \geq l$) such that

$$\|P_{[m]} T_n K P_{[m]} - T_n K\| < \delta$$

and hence

$$\|P_{[m]}{}^{\perp} T_n K P_{[m]}\| < \delta.$$

Therefore,

$$\begin{aligned}&\left\| P_{[m]} \left(P_{[j(n)]}{}^{\perp} T_n P_{[n]} K \right) P_{[m]} - \left(P_{[j(n)]}{}^{\perp} T_n P_{[n]} K \right) \right\| \\ = \;& \left\| P_{[j(n)]}{}^{\perp} \left(P_{[m]}{}^{\perp} T_n P_{[n]} K P_{[m]} \right) \right\| < \delta.\end{aligned}$$

Since δ was arbitrary, this shows

$$P_{[j(n)]}{}^{\perp} T_n P_{[n]} K \in \mathcal{K}_X.$$

Hence, $P_{[j(n)]}{}^{\perp} T_n P_{[n]}$ multiplies $\mathcal{K}_X$. A similar argument shows that the operator $P_{[n]} T_n P_{[j(n)]}{}^{\perp}$ also multiplies $\mathcal{K}_X$.

Finally, we show that T_{n+1} satisfies property (6.9.1). Since the projections $P_{[k]}$ all commute, (6.9.1) is satisfied (with T_{n+1} in place of T_n) for all $k < n$, by using the same property for T_n. For $k = n$,

$$P_{[j(n)]}{}^{\perp} T_{n+1} P_{[n]} = -P_{[j(n)]}{}^{\perp} P_{[n]} T_n P_{[j(n)]}{}^{\perp} P_{[n]}$$

and

$$P_{[n]} T_{n+1} P_{[j(n)]}{}^{\perp} = -P_{[n]} P_{[j(n)]}{}^{\perp} T_n P_{[n]} P_{[j(n)]}{}^{\perp},$$

and both of these are zero, since we chose $j(n) \geq n$. This proves the claim about the operators T_n.

Now put

$$T_{\infty} = \lim_{n \to \infty} T_n.$$

The limit exists by Equation (6.9.3), and moreover

$$\|T - T_\infty\| < \epsilon.$$

For each $k \in \mathbb{N}$,

$$P_{[j(k)]}{}^{\perp} T_\infty P_{[k]} = \lim_{n\to\infty} P_{[j(k)]}{}^{\perp} T_n P_{[k]} = 0$$

and similarly

$$P_{[k]} T_\infty P_{[j(k)]}{}^{\perp} = \lim_{n\to\infty} P_{[k]} T_n P_{[j(k)]}{}^{\perp} = 0,$$

so T is properly supported for X. Since ϵ can be chosen arbitrarily small, $T \in \mathcal{A}_X$.

Finally, consider $\mathcal{A}$. Being the intersection of $\mathcal{A}_X$ and $\mathcal{A}_Y$, it is clearly a C^*-category. If we can show that $\mathcal{K}_X$ is a subset of $\mathcal{A}$, then we will know it is an ideal, since it is an ideal of $\mathcal{A}_X$. But

$$\mathcal{K}_X \mathcal{K}_Y = \mathcal{K} \subseteq \mathcal{K}_Y,$$

which is to say that $\mathcal{K}_X$ multiplies $\mathcal{K}_Y$. Therefore, $\mathcal{K}_X \subseteq \mathcal{A}_Y$, and hence $\mathcal{K}_X \subseteq \mathcal{A}$. Proving that $\mathcal{K}_Y \subseteq \mathcal{A}$ is similar.

$\square$

To understand why Definition 6.24 should be interesting, let us return briefly to the case of the group $G = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. For this group, we take $\mathcal{X} = G/B = \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$. Let Δ denote the Laplacian on $\mathbb{C}\mathrm{P}^1$, and define the operators

$$\Delta_X = \Delta \otimes 1$$

and

$$\Delta_Y = 1 \otimes \Delta$$

on $L^2(\mathcal{X}) = L^2(\mathbb{C}\mathrm{P}^1) \otimes L^2(\mathbb{C}\mathrm{P}^1)$.

The maximal compact subgroup of G is $K = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Define Lie subalgebras $\mathfrak{s}_X$ and $\mathfrak{s}_Y$ of $\mathfrak{k}_\mathbb{C}$ by

$$\mathfrak{s}_X = \mathfrak{su}(2)_\mathbb{C} \oplus \{0\}$$

and

$$\mathfrak{s}_Y = \{0\} \oplus \mathfrak{su}(2)_\mathbb{C},$$

both of which are isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. We can decompose the space $\mathcal{H} = L^2(\mathcal{X})$ according to its $\mathfrak{s}_X$ and $\mathfrak{s}_Y$-types, and again these will correspond to the eigenspace decompositions for the directional Laplacians Δ_X and Δ_Y. In this case, though, the two decompositions can be made simultaneously, since Δ_X and Δ_Y commute.

With this set-up, we can define the projections $P_{[k]}$ and $Q_{[k]}$, and the algebras $\mathcal{A}_X(\mathcal{H})$, $\mathcal{A}_Y(\mathcal{H})$, $\mathcal{A}(\mathcal{H})$, $\mathcal{K}_X(\mathcal{H})$ and $\mathcal{K}_Y(\mathcal{H})$ in exact analogy with Definitions 6.20, 6.23 and 6.24 above.

For any k, the algebra $P_{[k]}\mathcal{B}(\mathcal{H})P_{[k]}$ is the algebra of bounded operators on the Hilbert space $P_{[k]}\mathcal{H}$. Let $\mathring{P}_{[k]}$ denote the projection onto functions of $\mathfrak{s}_X$-types $0,\ldots,k$ in $\mathring{\mathcal{H}} = L^2(\mathbb{C}\mathrm{P}^1)$. Then

$$P_{[k]}\mathcal{H} = (\mathring{P}_{[k]}\mathring{\mathcal{H}}) \otimes \mathring{\mathcal{H}}.$$

Since $\mathring{P}_{[k]}\mathring{\mathcal{H}}$ is a finite-dimensional space, it follows that

$$P_{[k]}\mathcal{B}(\mathcal{H})P_{[k]} = \mathcal{B}(P_{[k]}\mathcal{H}) = \mathcal{B}(\mathring{P}_{[k]}\mathring{\mathcal{H}}) \otimes \mathcal{B}(\mathring{\mathcal{H}}).$$

Hence

$$\mathcal{K}_X = \overline{\bigcup_{k\in\mathbb{N}} P_{[k]}\mathcal{B}(\mathcal{H})P_{[k]}} = \mathcal{K}(\mathring{\mathcal{H}}) \otimes \mathcal{B}(\mathring{\mathcal{H}}).$$

Similarly,

$$\mathcal{K}_Y = \overline{\bigcup_{k\in\mathbb{N}} Q_{[k]}\mathcal{B}(\mathcal{H})Q_{[k]}} = \mathcal{B}(\mathring{\mathcal{H}}) \otimes \mathcal{K}(\mathring{\mathcal{H}}).$$

These algebras are the ideals of $\mathcal{B}(\mathring{\mathcal{H}}) \otimes \mathcal{B}(\mathring{\mathcal{H}})$ which one encounters when forming the Kasparov product in the standard construction of the γ-element for $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$, as in Section 2.4.3. The algebra $\mathcal{A}_X$ is the multiplier of $\mathcal{K}(\mathring{\mathcal{H}})\otimes\mathcal{B}(\mathring{\mathcal{H}})$, which is somewhat larger than the algebra $\mathcal{B}(\mathring{\mathcal{H}}) \otimes \mathcal{B}(\mathring{\mathcal{H}})$. Similarly for $\mathcal{A}_Y$.

Returning now to the group $G = \mathrm{SL}(3,\mathbb{C})$, we will conclude with a simple application which indicates the prospective role of the C^*-categories above.

Proposition 6.27. *Let μ be an integral weight for $SL(3,\mathbb{C})$, and let $\mathcal{H} = L^2(\mathcal{X}, E_{-\mu})$. If f is a continuous function on $\mathcal{X}$, viewed as a multiplication operator on $\mathcal{H}$, then $f \in \mathcal{A}(\mathcal{H})$.*

If $\chi : [0,\infty) \to \mathbb{R}$ is a continuous function which tends to one at infinity, then

$$[\chi(\Delta_X), f] \in \mathcal{K}_X$$

and

$$[\chi(\Delta_Y), f] \in \mathcal{K}_Y.$$

Proof. To begin with, suppose that f is a function corresponding to an elementary tensor in the Peter-Weyl decomposition (6.2.3). Namely, suppose that we have vectors

$$\xi \in (\Gamma^{\beta})_0,$$

and

$$\xi^* \in \Gamma^{\beta *},$$

for some dominant integral weight β, and that f is defined by

$$f(\underline{k}) = (\xi^*, \pi_{\beta}(k)\xi).$$

Consider a section u of $E_{-\mu}$, defined in a similar way:

$$\tilde{u}(k) = (\eta^*, \pi_{\omega}(k)\eta),$$

for some

$$\eta \in (\Gamma^{\omega})_{\mu}$$

and

$$\eta^* \in \Gamma^{\omega *},$$

where π_ω is the finite-dimensional irreducible representation of G with highest weight ω, on the space Γ^ω. Then,

$$\tilde{f}\tilde{u}(k) = (\xi^* \otimes \eta^*, \pi_\beta \otimes \pi_\omega(k)(\xi \otimes \eta)).$$

Suppose further that ξ has $\mathfrak{s}_X$-type k and η has $\mathfrak{s}_X$-type l. Then $\xi \otimes \eta$ lies in an $\mathfrak{s}_X$-subrepresentation of $\Gamma^\beta \otimes \Gamma^\omega$ which is isomorphic to the $\mathfrak{s}_X$-representation

$$\Gamma^k \otimes \Gamma^l.$$

If $l \geq k$, then

$$\Gamma^k \otimes \Gamma^l \cong \Gamma^{k-l} \otimes \Gamma^{k-l+2} \otimes \cdots \otimes \Gamma^{k+l},$$

while if $l \leq k$,

$$\Gamma^k \otimes \Gamma^l \cong \Gamma^{l-k} \otimes \Gamma^{l-k+2} \otimes \cdots \otimes \Gamma^{l+k}$$

(see [FH91]). Either way, the decomposition of $\xi \otimes \eta$ into $\mathfrak{s}_X$-types will contain only vectors with $\mathfrak{s}_X$-types in $\{l-k, \ldots, l+k\}$.

It follows that, for any l,

$$P^\perp_{[l+k]} f P_{[l]} = 0$$

and

$$P_{[l]} f P^\perp_{[l+k]} = 0.$$

Hence f is properly supported for X. By taking finite sums of such functions f, it follows that multiplication by any K-finite function f is properly supported for X. Since the

K-finite functions are dense in $C(\mathcal{X})$, all multiplication operators $f \in C(\mathcal{X})$ belong to $\mathcal{A}_X$. A similar argument shows that every $f \in C(\mathcal{X})$ also belongs to $\mathcal{A}_Y$, and hence all multiplication operators belong to $\mathcal{A}$.

For the second result, we show that

$$\psi(\Delta_X) \in \mathcal{K}_X,$$

for any $\psi \in C_0(\mathbb{R})$. This will prove the result because

$$[\chi(\Delta_X), f] = -[1 - \chi(\Delta_X), f],$$

where $1 - \chi \in C_0(\mathbb{R})$, and since f multiplies $\mathcal{K}_X$ we will be done.

Recall that Δ_X and Δ_Y have discrete spectra (see p. 196). If ϕ restricts to the characteristic function of a point on the spectrum of Δ_X, then $\phi(\Delta_X) = P_k \in \mathcal{K}_X$, for some $k \in \mathbb{N}$. Since these functions span a dense subspace of $C_0(\mathbb{R})$, the result follows.

A similar argument works for $\mathcal{K}_Y$.

$\square$

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Vita

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