The Pennsylvania State University

The Graduate School

Department of Mathematics

BAND-DOMINATED OPERATORS AND THE STABLE HIGSON CORONA

A Dissertation in

Mathematics

by

Rufus Willett

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2009

The dissertation of Rufus Willett was reviewed and approved* by the following:

John Roe
Professor of Mathematics
Dissertation Adviser
Chair of Committee
Head of the Department of Mathematics

Paul Baum
Evan Pugh Professor of Mathematics

Nathanial Brown
Associate Professor of Mathematics

Nigel Higson
Evan Pugh Professor of Mathematics

Donald Schneider
Distinguished Professor of Astronomy and Astrophysics

*Signatures are on file in the Graduate School.

Abstract

Band-dominated operators form a class of bounded operators on $l^2(\mathbb{Z}^n)$ that have been studied in various forms throughout the twentieth century, and arise naturally in many areas of mathematics. One aspect of work of V.S. Rabinovich, S. Roch and B. Silbermann on this class of operators has been a workable criterion for whether or not such an operator has the Fredholm property. Their work suggests in particular a natural index problem for such operators, which was solved for the case of operators on $l^2(\mathbb{Z})$ by Rabinovich, Roch and J. Roe. This dissertation proves a partial generalization of this result to band-dominated operators with slowly oscillating coefficients on $l^2(\mathbb{Z}^n)$, using the stable Higson corona of H. Emerson and R. Meyer. Part of the interest of this result is that it highlights a difference between the 'slowly oscillating' and general cases that only appears in dimensions higher than one.

This stable Higson corona was introduced by Emerson and Meyer in the last five years in the course of their work on the Baum-Connes and Novikov conjectures. This dissertation gives simpler proofs of some of the properties used in the proof of the index theorem mentioned above. These results are also of independent interest insofar as they reprove certain results on the Baum-Connes and coarse Baum-Connes conjectures.

Table of Contents

List of Figures

Acknowledgments

I would like to thank Mrinal Raghupathi and Guoliang Yu for helpful comments on aspects of the current work after a talk given at Vanderbilt University. I would also like to thank Steffen Roch for interesting discussions and showing me the paper [15].

It is a pleasure to thank all of the members of my committee for their support, comments and encouragement. The mathematical members have also all provided a great deal of inspiration through their teaching, conversation and writing. As importantly, they have created a very enjoyable atmosphere in the Geometric Functional Analysis group at Penn State. Conversations with many of the students in this group, in particular Uuye Otgonbayar, also permeate this thesis; thanks to all.

In addition to the comments above, John Roe encouraged me to come to Penn State in the first place, and was extremely welcoming once I arrived. His support, and patience, have been more than appreciated throughout.

Further back, it is due to the influence of Sue Barnes, Clive Sutton and in particular Glenys Luke that I am in mathematics. I am deeply grateful for their instruction, which constantly informs my own approach.

Finally, I would very sincerely like to thank my family and friends in the UK, and in particular my parents, for continuing to support me and welcome me back, despite the mildly quixotic decision to place myself in a small town halfway around the world :)

.

Chapter 1

Introduction

A classical index theorem

In order to motivate the rest of this piece, we start by discussing a slight variation of the *Gohberg-Krein index theorem.* Chapter 7 of [16] is a good reference for what follows.

Let $l^2(\mathbb{Z})$ be the Hilbert space of square-summable maps from $\mathbb{Z}$ to $\mathbb{C}$, equipped with the fixed orthonormal basis $\{\delta_n : n \in \mathbb{Z}\}$. A bounded linear operator T on $l^2(\mathbb{Z})$ can be uniquely represented as a matrix with entries indexed by $\mathbb{Z} \times \mathbb{Z}$: the $(n, m)^{\text{th}}$ entry is $\langle T\delta_m, \delta_n \rangle$, just as in the case of a linear operator on a finite-dimensional vector space.

There are many interesting classes of operators that can be described in terms of such a matricial representation. For example, figure 1.1 below represents an operator where

- the sequence of matrix coefficients along each diagonal arrow converges to some complex number at infinity;
- there are only finitely many arrows;
- all other matrix coefficients outside the central box are zero.

Such operators form a $*$-algebra, with norm closure a C^*-algebra denoted $\Psi^0(\mathbb{T})$. It is in fact the C^*-algebra of *order zero pseudodifferential operators on the circle* $\mathbb{T}$, and

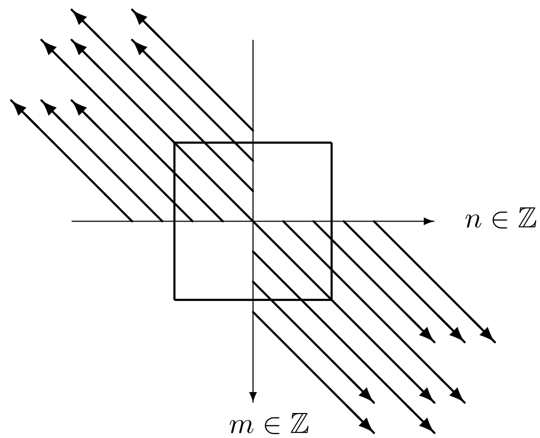

Fig. 1.1. Schematic of an order zero pseudodifferential operator on the circle

is closely related to the *Toeplitz algebra*; we will not, however, use these facts below.

We are going to study *Fredholm* properties of these and similar operators. Recall that an operator T on a Hilbert space $\mathcal{H}$ is said to be *Fredholm* if and only if $Kernel(T) := \{v \in \mathcal{H} : Tv = 0\}$ and $Cokernel(T) := \mathcal{H}/Image(T)$ are both finite dimensional. The fundamental theorem about Fredholm operators is *Atkinson's theorem* as stated below.

Theorem 1.0.1. *An operator T is Fredholm if and only if it is invertible modulo compact operators. Moreover, if this is the case then the* index *of T defined by*

$$Index(T) := dim(Kernel(T)) - dim(Cokernel(T)) \in \mathbb{Z}$$

is invariant under both compact perturbations and small (in operator norm) perturbations of T.

It follows that no finite portion of the matrix representation of an operator T affects its Fredholm properties. If $T \in \mathcal{B}(l^2(\mathbb{Z}))$ is as in the picture above, this suggests that only the 'values at the end of each arrow', i.e. the limits of the corresponding sequences of complex numbers, are relevant to the Fredholm properties of T.

Aiming to develop this idea, define the *limit operator* of T over $-\infty$ to be the operator whose entries on the n^{th} diagonal are all equal to the limit of the 'up-left' arrow on the n^{th} diagonal. Similarly, the limit operator of T over $+\infty$ is the operator with constant diagonal values equal to the limits over the 'down-right' arrows.

Under the Fourier isomorphism $l^2(\mathbb{Z}) \cong L^2(\mathbb{T})$, these limit operators correspond to continuous functions on the circle $\mathbb{T}$ acting on $L^2(\mathbb{T})$ by pointwise multiplication; considered as such, we will denote them by $f^{+\infty}$ and $f^{-\infty}$ respectively. If T is Fredholm, then one can use Atkinson's theorem to show that $f^{+\infty}$ and $f^{-\infty}$ are invertible (i.e. nowhere zero), and moreover that the index of the original T is some homotopy invariant of these functions. There is essentially only one homotopy invariant of a nowhere zero function $f : \mathbb{T} \to \mathbb{C}$: its *winding number*. These comments leads to a slight modification of the *Gohberg-Krein index theorem.*

Theorem 1.0.2. *Let T, $f^{-\infty}$ and $f^{+\infty}$ be as above. Then*

$$Index(T) = -(winding\ number)(f^{-\infty}) - (winding\ number)(f^{+\infty}).$$

One can also view this formula as a very special case of the *Atiyah-Singer index theorem* [4] on elliptic pseudodifferential operators on closed manifolds: it is the case when the manifold is the circle, and the operators are assumed to be order zero.

K-theory

A natural way to prove the index theorem of the previous section is to use K-*theory*. A good introductory reference is [67].

The discussion above can be extended to show the existence of a short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K}(l^2(\mathbb{Z})) \longrightarrow \Psi^0(\mathbb{T}) \xrightarrow{\sigma} C(\mathbb{T}) \oplus C(\mathbb{T}) \longrightarrow 0\ ;$$

this is essentially a theorem of L. A. Coburn. Associated to any such sequence is a six-term exact sequence of K-theory groups

$$\begin{array}{ccccc} K_0(\mathcal{K}(l^2(\mathbb{Z}))) & \longrightarrow & K_0(\Psi^0(\mathbb{T})) & \longrightarrow & K_0(C(\mathbb{T}) \oplus C(\mathbb{T}))\ . \\ {\scriptstyle Ind}\uparrow & & & & \downarrow \\ K_1(C(\mathbb{T}) \oplus C(\mathbb{T})) & \longleftarrow & K_1(\Psi^0(\mathbb{T})) & \longleftarrow & K_1(\mathcal{K}(l^2(\mathbb{Z}))) \end{array}$$

If $T \in \Psi^0(\mathbb{T}^n)$ is a Fredholm operator, then its image under the *symbol map* σ is invertible, so defines a class $[\sigma(T)]$ in $K_1(C(\mathbb{T}) \oplus C(\mathbb{T}))$. The general theory implies that the index of T depends only on the K-theory class $[\sigma(T)]$, and in fact that the map

$$Ind : K_1(C(\mathbb{T}) \oplus C(\mathbb{T})) \to K_0(\mathcal{K}(l^2(\mathbb{Z}))) \cong \mathbb{Z}$$

form the six term exact sequence above sends $[\sigma(T)]$ to $Index(T)$. As one can compute that the group $K_1(C(\mathbb{T}) \oplus C(\mathbb{T}))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, to prove the theorem in the previous section it suffices to check that it is true on a pair of simple generators.

Generalizations and compactifications

The main motivation for this thesis is the work of V.S. Rabinovich, S. Roch and B. Silbermann on Fredholm theory for *band-dominated operators*. Band-dominated operators on $l^2(\mathbb{Z})$ constitute a very wide-ranging generalization of the operators in $\Psi^0(\mathbb{T})$ looked at above: they are norm limits of operators whose matrix coefficients are supported within a pair of diagonal lines like those shown in figure 1.2 below. There are many interesting examples of band-dominated operators in operator theory, and they also play a major rôle in coarse geometry and index theory; see examples 2.1.3 and 2.1.8 below, as well as [56], [63].

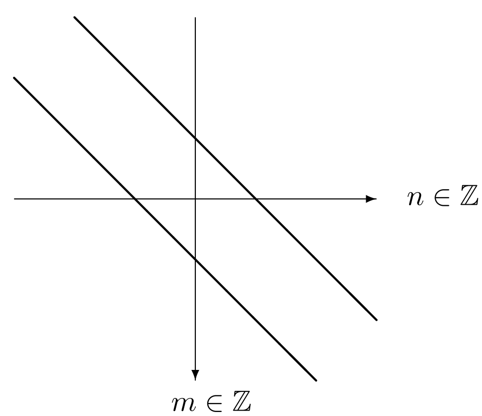

Fig. 1.2. Matricial picture of a class of operators on $l^2(\mathbb{Z})$

There is a notion of 'limit operator' for general band-dominated operators, which was used by Rabinovich, Roch and Silbermann to give a criterion for Fredholmness. As general band-dominated operators can have very complicated behavior along each diagonal, however, one needs much more than the two point set $\{-\infty, +\infty\}$ of 'directions at infinity' to coherently organize the limit operators that arise.

To see how to define such generalized limit operators, note that the two points $\{-\infty, +\infty\}$ define a *compactification* of $\mathbb{Z}$ as in figure 1.3 below.

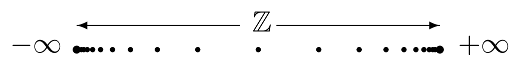

Fig. 1.3. The spherical compactification of $\mathbb{Z}$

It can be thought of as attaching a copy of the zero-dimensional sphere to $\mathbb{Z}$, and is called the *spherical compactification.*

To consider more general 'directions at infinity' one can use larger compactifications of $\mathbb{Z}$; the more complicated the class of diagonals allowed, the more complicated the necessary compactification. If for example *all* band-dominated operators are considered, so all bounded functions are to be allowed as diagonals, one needs the *Stone-Cěch compactification of* $\mathbb{Z}$ to organize all the limit operators. This is the largest possible compactification. If one puts some conditions on the behavior of the diagonals, smaller compactifications can be used; one of the simplest of these is the spherical compactification considered above. We will also consider band-dominated operators on more general discrete groups in what follows. Here geometric conditions on the groups lead to limit operators organized by, for example, the *Gromov boundary* of a *hyperbolic group.*

One of the most important classes of diagonals considered in the work below consists of so-called *slowly oscillating functions*. These are functions whose modulus of continuity decays to zero at infinity, such as $n \mapsto \sin(\sqrt{|n|})$ on $\mathbb{Z}$. The corresponding class of operators is particularly tractable, and has been studied by Rabinovich, Roch and Silbermann, and others. The relevant compactification for the study of their limit operators is the *Higson compactification*, denoted $\partial_h \mathbb{Z}$, which was introduced by N. Higson in the context of index theory [35]. V.M. Deundyak and B.Y. Shteinberg announced an index theorem for this class of operators in [15]; one of our main results is a new proof of this theorem, stated as 5.0.1, 6.1.6 and 6.1.10 below.

The stable Higson corona and Baum-Connes conjecture

A K-theoretic approach to the index theorem of Deundyak and Shteinberg leads one to study the K-group $K^1(\partial_h \Gamma)$. Unfortunately, this is known to be a very complicated object thanks to the work of J. Keesling [47], and A. Dranishnikov and S. Ferry [17].

An attempt to get a grasp on the part of this group that is important for index theory led us to study the *stable Higson corona* of H. Emerson and R. Meyer [24] [22]; this object forms the second major motivation for the work below. As it happens, the stable Higson corona has K-theoretic properties that allow us to prove the index theorem 5.0.1 we are aiming for.

There is much more to say here, however. The work of Keesling, Dranishnikov and Ferry cited above was motivated by connections of the Higson corona to the *Novikov* *conjecture* in manifold theory [25], which predicts homotopy invariance of higher signatures. On the other hand, Emerson and Meyer introduced the stable Higson corona for its connection to the *Baum-Connes conjecture* [5]. This deep conjecture has a variety of applications in topology, geometry and representation theory; in particular it implies the Novikov conjecture.

The properties of the stable Higson corona that we use to prove theorem 5.0.1 are shown by Emerson and Meyer in [22] using some fairly serious machinery. In an attempt to give a simpler proof of these properties, and following a well-trodden route [44] [40] [41] [39], we were led to study the stable Higson corona from a 'homological' point of view. The main results here are theorems 4.3.12 and 4.3.13, which give homotopy invariance properties of the stable Higson corona and some associated constructions.

These are used to reprove some results of Emerson and Meyer on groups of non-positive curvature, in what we hope is a more elementary manner.

Organisation of this piece

Chapter 2 introduces band-dominated operators on $\mathbb{Z}^n$. In particular, we sketch proofs of the results of V.S. Rabinovich, S. Roch and B. Silbermann on Fredholmness, and the index theorem of Rabinovich, Roch and J. Roe which applies in the case $n = 1$. The last section of the chapter details some issues that arise when trying to extend this proof to higher dimensions; these help to motivate the piece as a whole.

Chapter 3 starts by enlarging the class of groups under consideration to that of exact groups. In section 3.1 we extend some comments from section 4 of [65] to give a proof of the basic Fredholmness properties of the corresponding classes of operators. This uses some machinery associated to crossed product C^*-algebras. This section suggests associating an index problem to any equivariant boundary of an exact group, as is made explicit in section 3.2. The rest of the chapter is devoted to discussing some examples, including those associated to operators with slowly oscillating coefficients.

Chapter 4 opens with a discussion of the Higson corona. The connections of this object with the Novikov conjecture are summarized for the reader's convenience in section 4.1. The analogous results of H. Emerson and R. Meyer for the stable Higson corona are then looked at in section 4.2, where we also summarize some connections between the two approaches. Section 4.3 proves certain homological properties of the stable Higson corona; the main results are 4.3.12 and 4.3.13 on homotopy invariance. The remaining sections use these homological properties to study some examples.

Chapter 5 contains most of the proof of theorem the index theorem 5.0.1 of V.M. Deundyak and B.Y. Shteinberg. The techniques involving crossed products from chapter 3 as well as some simple cases of the computations from chapter 4 are important here.

Chapter 6 is split into two distinct sections. Section 6.1 completes the proof of theorem 5.0.1 using *asymptotic morphisms*, and also provides two relatively concrete restatements in theorems 6.1.6 and 6.1.10. Section 6.2 uses known cases of the Baum-Connes conjecture to prove isomorphisms in K-theory that would be useful for extending our index-theoretic results to a larger class of groups.

There are also two appendices. Appendix A sets out some necessary background on coarse categories, which provides the framework for much of chapter 4. Appendix B briefly discusses the class of exact groups and some results we need on crossed product C^*-algebras.

Future directions

- There are a lot more groups to study in detail. It should be interesting to look at connections between the comments on hyperbolic groups in this piece and work of H. Emerson and M. Jury. The papers of Jury were pointed out to me by M. Raghupathi, who suggested that concrete index theorems of some interest to operator theorists may be possible in this direction.

- G. Yu has suggested using the techniques of chapter 4 combined with the fact that an a-T-menable group acts 'nicely' on a Hilbert space (which is $CAT(0)$, but not proper) may allow one to get results on this class of groups. This would not yield any new results on the Baum-Connes conjecture - see the deep work of N. Higson and G. Kasparov [37] - but there *may* (this is very tenuous currently!) be connections with the *stable Borel conjecture* via [33].

- The comments on crossed products in remarks B.3 suggest some interesting questions here. It seems that they may be connected to the important question as to whether the uniform boundedness condition from corollary 2.2.11 is really necessary.

- It is a major hole in the work below that it only applies to the case of band-dominated operators on $\mathbb{Z}^n$ *with slowly oscillating coefficients*. An index theorem that worked for band-dominated operators with arbitrary coefficients would probably not be very computable in general, but seems likely to be of theoretical interest. Note, however, that remark 6.1.9 below suggests that new techniques would be needed in the proof of such a theorem; I once thought a KK-theoretic approach might be feasible here, but unfortunately have no ideas along this line currently.

Chapter 2

Band-dominated operators on $\mathbb{Z}^n$

2.1 Definition and examples

In this section, we approach our central object of study - band-dominated operators on $\mathbb{Z}^n$ - from two different directions: operator theoretic, and coarse geometric.

From the operator theory point of view, band-dominated operators are formed from the following two natural classes of bounded operators on $l^2(\mathbb{Z}^n)$:

- One may consider $l^\infty(\mathbb{Z}^n)$ as a subalgebra of $\mathcal{B}(l^2(\mathbb{Z}^n))$, with $f \in l^\infty(\mathbb{Z}^n)$ acting by pointwise multiplication. In other words, f acts on elements of the standard orthonormal basis of $l^2(\mathbb{Z}^n)$ by

  $$f : \delta_m \mapsto f(m)\delta_m.$$

  Throughout this piece we abuse notation by writing f for both a function and the corresponding multiplication operator.

- If m is an element of $\mathbb{Z}^n$, we denote by U_m the corresponding unitary (left) shift operator, which is defined on basis vectors by

  $$U_m : \delta_l \mapsto \delta_{m+l}.$$

Definition 2.1.1. A *band operator* is a sum

$$\sum_{m\in\mathbb{Z}^n} f_m U_m \tag{2.1}$$

where each f_m is an element of $l^\infty(\mathbb{Z}^n)$ and only finitely many f_m are non-zero. A *band-dominated operator* is an operator norm limit of band operators.

Note moreover that the band operators form a $*$-subalgebra of $\mathcal{B}(l^2(\mathbb{Z}^n))$, whence the band-dominated operators form a C^*-algebra. The C^*-algebra of all band-dominated operators will be denoted $C_u^*(|\mathbb{Z}^n|)$.

The notation '$C_u^*(|\mathbb{Z}^n|)$' comes from the coarse geometric approach to band-dominated operators, and will be explained after lemma 2.1.7 below.

Remarks 2.1.2.

1. A representation as in line (2.1) is unique. Indeed, *any* operator in $\mathcal{B}(l^2(\mathbb{Z}^n))$ can be uniquely (formally) represented in this way, as long as one allows infinitely many non-zero summands. To see this, note that a representation of the form (2.1) uniquely determines, and is uniquely determined by, the matrix coefficients of an operator $T \in \mathcal{B}(l^2(\mathbb{Z}^n))$ via the formula

$$f_m(l) = \langle T\delta_{m+l}, \delta_l\rangle. \tag{2.2}$$

   Note that infinite sums of the form (2.1) need not converge (even in the weak operator topology), however: see part (2) of counterexamples 2.1.4 below.

2. As briefly discussed in the introduction, the name 'band operator' comes from the matrix representation of such an operator on $l^2(\mathbb{Z})$. With respect to this

representation, a band-operator is one with matrix coefficients supported within a pair of diagonal lines like those shown in figure 2.1 below (which is the same as figure 1.2, but repeated for the reader's convenience).

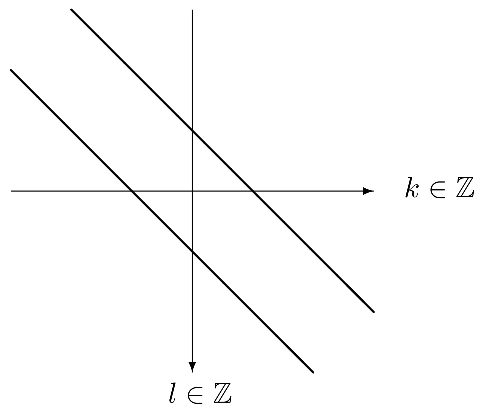

Fig. 2.1. Schematic of a band operator on $l^2(\mathbb{Z})$

In this picture, the m^{th} diagonal (i.e. the one m places above the main diagonal) is the function f_m that appears in a representation as in line (2.1). For this reason, we often call the elements f_m in a representation as in line (2.1) above the *diagonals* of T.

The class of band-dominated operators on $\mathbb{Z}^n$ includes many classical examples from operator theory, operator algebras, and geometric analysis; before continuing, we will discuss some of these below.

Examples 2.1.3. 1. Amongst the simplest band operators are finite sums

$$\sum_{m \in \mathbb{Z}^n} \lambda_m U_m \tag{2.3}$$

where each λ_m is a scalar. Such operators are often said to be *constant down the diagonals.* They can be thought of as elements of the group algebra $\mathbb{C}[\mathbb{Z}^n]$ acting by convolution, or (under Fourier transform) as operators of pointwise multiplication by trigonometric polynomials on $L^2(\mathbb{T}^n)$. Thus the norm closure of the $*$-algebra of such operators can be thought of as either $C^*_r(\mathbb{Z}^n)$, the reduced group C^*-algebra of $\mathbb{Z}^n$, or $C(\mathbb{T}^n)$, the continuous functions on the n-torus. It follows from the theory of Cesàro summation of Fourier series that this C^*-algebra consists precisely of the band-dominated algebras that are constant down diagonals, i.e. band-dominated operators of the form in line (2.3) above where the sum is not necessarily finite.

2. Let $P \in \mathcal{B}(l^2(\mathbb{Z}))$ be the orthogonal projection onto the closed span of $\{\delta_n : n \geq 0\}$. Let f be an element of $C(\mathbb{T})$, thought of as a band-dominated operator which is constant down diagonals as in the previous example. The operator PfP is then band-dominated and called the *Toeplitz operator with symbol* f. The C^*-algebra generated by all such operators is the *Toeplitz algebra.* For a good introduction to these objects and this algebra, see chapter 7 of [16].

3. From another point of view, the image of P is the *Hardy space* $H^2(S^1)$, which consists of those functions in $L^2(S^1) \cong l^2(\mathbb{Z})$ that have a (unique) extension to a holomorphic function on the open unit disc in $\mathbb{C}$.

More generally (see [14] and appendix B in [34]), let $P_n \in \mathcal{B}(l^2(\mathbb{Z}^n))$ be the orthogonal projection onto the closed span of

$$\{\delta_{m=(m_1,...,m_n)} \in l^2(\mathbb{Z}^n) : m_i \geq 0 \text{ for all } i = 1, ..., n\}.$$

Let S^{2n-1} be the unit sphere in $\mathbb{C}^n$. There is a unitary isomorphism between $l^2(\mathbb{Z}^n)$ and $L^2(S^{2n-1})$ defined by sending $\delta_{m=(m_1,...,m_n)}$ to the monomial

$$z = (z_1, ..., z_n) \mapsto \frac{z^m := z_1^{m_1} \cdots z_n^{m_n}}{Volume(S^{2n-1})}. \tag{2.4}$$

The image of P_n can be identified with $H^2(S^{2n-1})$, the elements of $L^2(S^{2n-1})$ that have a (unique) extension to a function holomorphic on the open unit ball in $\mathbb{C}^n$.

Let $f \in C(S^{2n-1})$ act on $L^2(\mathbb{Z}^n)$ by pointwise multiplication. The operator $P_n f P_n$ is called a *(higher dimensional) Toeplitz operator*. As f can be written as a norm limit of linear combinations of the monomials in line (2.4) above as m ranges over $\mathbb{Z}^n$, $P_n f P_n$ is band-dominated. These Toeplitz operators are (special cases of) the operators whose Fredholm properties were studied by Venugopalkrishna in [72].

4. Another case of particular interest is given by *pseudodifferential operators on the n-torus,* $\mathbb{T}^n$. These correspond under Fourier transform to band-dominated operators on $\mathbb{Z}^n$ whose diagonals (the f_m in line (2.1) above) have ‘especially good behavior at infinity’. See section 3.4 below for details, and section 5 of [4] for an overview of pseudodifferential operators on more general closed manifolds .

5. Many other very interesting classes of band-dominated operators are looked at in chapters 3, 4 and 5 of [56], and we refer the reader here for further examples. Some of these (*operators of Wiener type*, *pseudodifference operators*,...) fit into the framework used in this piece.

Counterexamples 2.1.4. 1. It is perhaps worth pointing out that not every operator in $\mathcal{B}(l^2(\mathbb{Z}^n))$ is band-dominated! For example, consider the operator

$$T = \frac{1}{2} + \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{1}{2\pi i m} U_m.$$

This defines a bounded operator on $(l^2(\mathbb{Z}))$: it corresponds under Fourier transform to pointwise multiplication by the descent of the floor function $x \mapsto \lfloor x \rfloor$ from $\mathbb{R}$ to $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. T is constant down diagonals, so part (1) of examples 2.1.3 implies that if it were band-dominated, then the (descent of the) floor function would be continuous on $\mathbb{T}$. As this is not the case, T is not band-dominated.

2. We have seen in part (1) of remarks 2.1.2 that any $T \in \mathcal{B}(l^2(\mathbb{Z}^n))$ can be uniquely formally represented by a sum as in line (2.1). Such a sum need not converge to T, however, even in the weak operator topology, and even if the original T is band-dominated.

   Indeed, if (f_n) is a sequence in $C(\mathbb{T})$ (considered as operators on $l^2(\mathbb{Z})$ via part (1) of examples 2.1.3) that converges to some $f \in C(\mathbb{T})$ in the weak operator topology, then the f_n (considered as functions) converge pointwise to f. Hence if $f \in C(\mathbb{T})$ is such that the partial sums of the expression in line (2.1) above converge in the weak operator topology to f, then f is the pointwise limit of the partial sums in its Fourier series. There are, however, examples of functions $f \in C(\mathbb{T})$ for which this is known to be false; to see existence of such, one can for example use the Baire category theorem.

We will also study band-dominated operators with a somewhat wider range of 'matrix coefficients'.

In order to define these, fix a separable (possibly finite-dimensional) Hilbert space $\mathcal{H}$. Let $\mathcal{K}(\mathcal{H})$ denote the C^*-algebra of compact operators on $\mathcal{H}$, which is isomorphic to either $M_k(\mathbb{C})$ for some $k \in \mathbb{N}$, or $\mathcal{K}$. Let $l^\infty(\mathbb{Z}^n, \mathcal{K}(\mathcal{H}))$ denote the C^*-algebra of bounded functions on $\mathbb{Z}^n$ taking values in $\mathcal{K}(\mathcal{H})$, which acts by pointwise multiplication on the Hilbert space $l^2(\mathbb{Z}^n, \mathcal{H})$. For each $m \in \mathbb{Z}^n$, let U_m be the (left) shift unitary which acts on a function $\xi : \mathbb{Z}^n \to \mathcal{H}$ via the formula

$$(U_m \xi)(l) = \xi(-m + l).$$

Definition 2.1.5. A *band operator with values in* $\mathcal{K}(\mathcal{H})$ is a sum

$$\sum_{m \in \mathbb{Z}^n} F_m U_m \in \mathcal{B}(l^2(\mathbb{Z}^n, \mathcal{H})) \tag{2.5}$$

where each F_m is an element of $l^\infty(\mathbb{Z}^n, \mathcal{K}(\mathcal{H}))$ and only finitely many F_m are non-zero.

A *band-dominated operator with values in* $\mathcal{K}(\mathcal{H})$ is an operator norm limit of band operators with values in $\mathcal{K}(\mathcal{H})$.

If $\mathcal{H}$ has finite dimension k, so $\mathcal{K}(\mathcal{H}) \cong M_k(\mathbb{C})$, then the algebra of band-dominated operators with values in $\mathcal{K}(\mathcal{H})$ is isomorphic to

$$M_k(\mathbb{C}) \otimes C_u^*(|\mathbb{Z}^n|) \cong M_k(C_u^*(|\mathbb{Z}^n|))$$

and will be denoted as such.

If $\mathcal{H}$ has infinite dimension, so $\mathcal{K}(\mathcal{H}) \cong \mathcal{K}$, then the algebra of band-dominated operators with values in $\mathcal{K}(\mathcal{H})$ is denoted $C^*(|\mathbb{Z}^n|)$.

It is perhaps easiest to think of operators in $M_k(C_u^*(|\mathbb{Z}^n|))$ (respectively, $C^*(|\mathbb{Z}^n|)$) as looking just like band-dominated operators as described in part (2) of remarks 2.1.2, but where the matrix coefficients are matrices (respectively, compact operators) rather than scalars. To make this precise, for $m \in \mathbb{Z}^n$, let P_m denote the function

$$\mathbb{Z}^n \to \mathcal{B}(\mathcal{H})$$
$$l \mapsto \begin{cases} 1 & l = m \\ 0 & l \neq m \end{cases}.$$

Having made the identification

$$l^2(\mathbb{Z}^n, \mathcal{H}) \cong \bigoplus_{m \in \mathbb{Z}^n} \mathcal{H},$$

one can define the $(m, l)^{\text{th}}$ matrix coefficient of an operator $T \in \mathcal{B}(l^2(\mathbb{Z}^n, \mathcal{H}))$ to be

$$P_m T P_l \in \mathcal{B}(\mathcal{H}). \tag{2.6}$$

Note that while $M_k(C^*_u(|\mathbb{Z}^n|))$ and $C^*_u(|\mathbb{Z}^n|)$ have very similar properties (for example, they are *Morita equivalent*), $C^*(|\mathbb{Z}^n|)$ is quite different; cf. remark 2.2.8 and section 2.4.

Before moving on to the second of our approaches to band-dominated operators, we should point out here that many classes of operators that are looked at in [56] unfortunately do not fit into our framework. Perhaps most notable amongst these are *pseudodifferential operators on* $\mathbb{R}^n$. Indeed, Rabinovich, Roch and Silbermann have developed a theory that is capable of treating band-dominated operators with matrix coefficients that are much more general than the scalar, matrix and compact operator cases that we consider. Although these more general examples are very interesting, we do not consider them as they do not have the sort of index theoretic properties that form our main focus in this piece.

We also restrict our attention to band-dominated operators on Hilbert spaces throughout, while the authors of [56] consider operators on a much larger class of Banach spaces. This avoids certain technicalities, and allows a somewhat cleaner treatment of many results at the cost of neglecting certain examples. Note that theorem 2.1 from [58] shows that index theoretic properties of a band operator do not change under some variations of the Banach space on which it acts.

The second approach to band-dominated operators proceeds via a geometric notion: the *propagation* of an operator T, which measures 'how far' T can move basis elements. In order to make sense of this, one must choose a metric on $\mathbb{Z}^n$ – for the sake of concreteness we use the restriction of the Euclidean metric on $\mathbb{R}^n$. Later we will use abstract coarse structures for essentially the same purpose – see appendix A.

Definition 2.1.6. The *propagation* of an operator $T \in \mathcal{B}(l^2(\mathbb{Z}^n))$ is

$$prop(T) := \max\{d(m,l) : \langle T\delta_m, \delta_l\rangle \neq 0\}$$

if this exists, and infinity otherwise.

More generally, if T is an operator in $\mathcal{B}(l^2(\mathbb{Z}^n), \mathcal{H})$, then the *propagation of* T is

$$prop(T) := \max\{d(m,l) : P_l T P_m \neq 0\}$$

(cf. line 2.6 above) if this exists, and infinity otherwise.

The following lemma points out the relation between the operator theoretic and coarse geometric approaches to band-dominated operators.

Lemma 2.1.7. *Let $T \in \mathcal{B}(l^2(\mathbb{Z}^n))$ be represented as in line (2.1) above (respectively, line (2.5)). Then*

$$prop(T) = \max\{\|m\| : f_m \neq 0\} \text{ (resp. } \max\{\|m\| : F_m \neq 0\}).$$

Proof. From the definition of propagation and the formula in line (2.2) above, one has that

$$\begin{aligned} prop(T) &= \max\{d(m,l) : \langle T\delta_m, \delta_l\rangle \neq 0\} \\ &= \max\{\|m\| : \langle T\delta_{m+l}, \delta_l\rangle \neq 0 \ \text{ for some } l\} \\ &= \max\{\|m\| : f_m(l) \neq 0 \text{ for some } l\} \\ &= \max\{\|m\| : f_m \neq 0\}. \end{aligned}$$

The proof for operators with more general values as in definition 2.1.5 is essentially the same. □

Hence band operators are exactly those of finite propagation. Indeed, $C_u^*(|\mathbb{Z}^n|)$ is usually defined in coarse geometry as the norm closure of the finite propagation operators. Note that with this definition the group structure has vanished: the collection of band operators depends only on the metric, and in fact, only on the coarse structure defined by the metric; see A.2, example (1). The '$|\cdot|$' in the notation '$C_u^*(|\mathbb{Z}^n|)$' refers to the fact that this algebra depends only the underlying coarse space $|\mathbb{Z}^n|$ rather than the group $\mathbb{Z}^n$; see A.2, example (2).

In coarse geometry, $C_u^*(|\mathbb{Z}^n|)$ is perhaps most commonly known as the *uniform Roe algebra of* $\mathbb{Z}^n$, but has also been called the *rough algebra of* $\mathbb{Z}^n$ (cf. 'rough geometry' as briefly discussed in [63]) and the *uniform translation algebra of* $\mathbb{Z}^n$ ([64], section 4.4). $C^*(|\mathbb{Z}^n|)$ similarly depends only on the coarse structure of $\mathbb{Z}^n$ and is usually called the *Roe algebra of* $\mathbb{Z}^n$ in coarse geometry.

To conclude this section, we give some coarse-geometric / index-theoretic motivations for studying the algebras $C_u^*(|\mathbb{Z}^n|)$ and $C^*(|\mathbb{Z}^n|)$. These are provided mainly as background rather than because they play a central rôle in what follows (cf. sections 4.1 and 4.2, however).

Example 2.1.8. Consider the operator

$$D := -i\frac{d}{dx}$$

on $C^\infty(\mathbb{T})$ ($\mathbb{T}$ has as usual been identified with $\mathbb{R}/\mathbb{Z}$). D is called the *Dirac operator* on $\mathbb{T}$ and is an example of an *elliptic partial differential operator*. It is thus invertible modulo the compact operators (see for example section 6 of [4]), and so has an index in $K_*(\mathcal{K}) = K_0(\mathcal{K}) \oplus K_1(\mathcal{K})$. Odd-dimensionality of $\mathbb{T}$ implies that the index of D lies in $K_1(\mathcal{K}) = 0$ and is thus necessarily trivial (note that we are using *topological* K-theory, which allows us to (formally) treat the even and odd dimensional cases in the same way).

There are, however, ways to get non-trivial 'index data' out of D. For example, D lifts to an operator $\tilde{D}$ on the universal cover $\mathbb{R}$ of $\mathbb{T}$. One can represent both $C^*(|\mathbb{Z}|)$ and $C_u^*(|\mathbb{Z}|) \otimes \mathcal{K}$ on $L^2(\mathbb{R})$, where these algebras can be thought of as consisting of *locally compact* operators on $\mathbb{R}$. Moreover, while $\tilde{D}$ is no longer Fredholm, it *is* invertible modulo both of these algebras of 'locally' compact operators; just as D has an index in $K_*(\mathcal{K})$, then, $\tilde{D}$ has an index in both

$$K_*(C^*(|\mathbb{Z}|)) \text{ and } K_*(C_u^*(|\mathbb{Z}|)).$$

These *higher indices*, or *coarse indices*, turn out to be non-trivial. In fact, in both of these cases $Index(\tilde{D})$ lies in the K_1 component and one can compute that

$$K_1(C^*(|\mathbb{Z}|)) \cong K_1(C_u^*(|\mathbb{Z}|)) \cong \mathbb{Z},$$

with $Index(\tilde{D})$ serving as a generator for both.

Ideas like that above have been developed in much greater generality, and have wide-ranging applications to geometry and topology. In some ways the above is a representative (although very simple) example. In others it is not: for example $K_*(C_u^*(\cdot))$ is usually much bigger than $K_*(C^*(\cdot))$. The higher index theory based on (variants of) $C^*(\cdot)$ perhaps starts in [62] and later gave rise to the coarse Baum-Connes conjecture [41]; see also section 4.1. For higher index theory based on (variants of) $C_u^*(\cdot)$, the original references are [59] and [60], while a much more recent one is [70].

2.2 Limit operators and the Fredholm property

In this section, we give an exposition of some of the work of Rabinovich-Roch-Silbermann and Roe on the Fredholm property for band-dominated operators, following the book [56] of the former authors and the paper [65] of the latter. The earlier paper [55] of Rabinovich, Roch and Silbermann contains most of what is relevant to us from [56], and is sometimes easier to read as it works in less generality; nonetheless, references below will be to [56].

We will not give complete proofs in this section, partly as detailed proofs are contained in the references above, but also as analogues to the results here are reproved by somewhat different methods in the next chapter. We will not take the quickest route (this is probably that in [65]) to our intended goal (corollary 2.2.10 and corollary 2.2.11); we hope that taken below is relatively intuitive.

Although this section focuses on $\mathbb{Z}^n$ all statements and sketches of proofs are geared towards the generalizations to the class of *exact groups* that we will look at in chapter 3; see appendix B for a brief discussion of these objects.

Before studying Fredholm operators in $C_u^*(|\mathbb{Z}^n|)$, it is worth checking that interesting examples exist! The operator below is probably the simplest example of a band-dominated operator with non-trivial index.

Example 2.2.1. Consider the 'half-infinite ray' $R = \{(k, 0, ..., 0) \in \mathbb{Z}^n : k \in \mathbb{N}\}$ in $\mathbb{Z}^n$. Let P be the characteristic function of R, considered as a projection operator on $l^2(\mathbb{Z}^n)$. Let $U = U_{-1,0,...,0}$ be the shift in the 'opposite direction' to R. Then the operator

$$F := PUP + (1 - P)$$

is surjective with kernel the span of δ_0; it is thus Fredholm of index one.

We hope the rest of this piece will make clear that $C_u^*(|\mathbb{Z}^n|)$ has a very rich index theory: the above is essentially a 'one-dimensional' example, but many significantly more complicated 'higher-dimensional' examples exist, as follows from parts (3) and (4) of 2.1.3, and also from the contents of chapter 5 and section 6.1.

The following proposition starts to justify the approach of Rabinovich, Roch and Silbermann to Fredholmness for band-dominated operators.

Proposition 2.2.2. *The C^*-algebra of compact operators on $l^2(\mathbb{Z}^n)$, denoted $\mathcal{K}(l^2(\mathbb{Z}^n))$, is contained in $C_u^*(|\mathbb{Z}^n|)$ as a closed ideal.*

Moreover, a band-dominated operator T on $\mathbb{Z}^n$ is compact if and only if its matrix coefficients $\langle T\delta_m, \delta_l\rangle$ tend to zero as $(m, l) \to \infty$ in $\mathbb{Z}^{2n}$.

Comments on the proof. To see that $\mathcal{K}(l^2(\mathbb{Z}^n))$ is contained in $C_u^*(|\mathbb{Z}^n|)$ (as a closed ideal), let P_N be the characteristic function of the ball of radius N in $\mathbb{Z}^n$. Then the sequence $(P_N)_{N=0}^{\infty}$ converges $*$-strongly to the identity, whence for any $K \in \mathcal{K}(l^2(\mathbb{Z}^n))$, $P_N K P_N$ converges to K *in norm*. The operators $P_N K P_N$ are clearly band operators, whence K is band-dominated.

Similarly, to see that the matrix coefficients of $K \in \mathcal{K}(l^2(\mathbb{Z}^n))$ tend to zero at infinity, note that matrix coefficients $\langle K\delta_m, \delta_l\rangle$ with either $\|m\|$ or $\|l\|$ greater than N are bounded above by

$$\max\{\|(1 - P_N)KP_N\|, \|P_N K(1 - P_N)\|, \|(1 - P_N)K(1 - P_N)\|\},$$

and this quantity tends to zero as N tends to infinity.

The hard part of the proof is the converse to this (cf. theorem 2.2.10 from [56] for a proof in the case of $\mathbb{Z}^n$, and lemma 3.2 from [65] for the case of general exact groups). Perhaps the underlying reason for its non-triviality is that it relies on geometric properties of $\mathbb{Z}^n$; indeed, the analogue for general countable discrete groups is not true. This is due to the existence of non-compact *ghost operators*; see proposition B.7 and the comments afterwards. Note that for non band-dominated operators, having matrix coefficients tending to zero at infinity does not imply compactness. □

Now, Atkinson's theorem (1.0.1 above) states that the Fredholm property for a bounded operator depends only on its residue class modulo compact operators. The above proposition thus suggests that information 'at infinity' (in terms of the matrix coefficients of an operator on $l^2(\mathbb{Z}^n)$) may give necessary *and* sufficient conditions for a band-dominated operator being Fredholm.

Similar considerations led the authors of [56] (and others) to the consideration of *limit operators* in this context. In order to state the definition, for $m \in \mathbb{Z}^n$ let V_m denote the operator of *right* shift by m, which is defined on basis vectors by

$$V_m : \delta_l \mapsto \delta_{l-m}. \tag{2.7}$$

As $\mathbb{Z}^n$ is commutative, $V_m = U_m^*$. This is not true for more general groups, however, whence we have introduced right shifts so that our formulas go over to the more general case without change. The definition of limit operator below is taken from section 1.2 of [56].

Definiton 2.2.3. Say $(m^k)_{k=0}^{\infty}$ is a sequence in $\mathbb{Z}^n$ tending to infinity and T a bounded operator on $l^2(\mathbb{Z}^n)$. If the strong limit of the sequence

$$V_{m^k} T V_{m^k}^*$$

exists, then it is called the *limit operator of T along* (m^k).

If T is such that any sequence tending to infinity has a subsequence along which the limit operator of T exists, then T is called *rich*. If both T and T^* have this property, then T is called $*$-*rich*.

Remark 2.2.4. To see why this is relevant to Fredholmness (in the sense of the comments after proposition 2.2.2), consider a band operator, for simplicity on $l^2(\mathbb{Z})$. Its matrix coefficients are thus supported in a band of the form in figure 2.2 below.

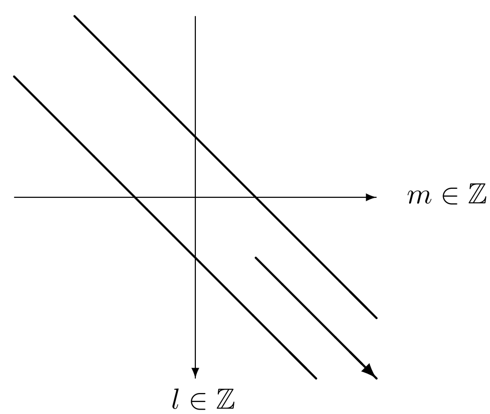

Fig. 2.2. Idea of a limit operator

Say for example that (m^k) is an increasing sequence of positive integers. Conjugating T by the operator V_{m^k} then has the effect of 'moving' this matrix representation in the direction of the arrow shown, with the 'length' of the arrow getting longer as $k \to \infty$. Thus only matrix coefficients 'at infinity' have any effect on the strong limit of the sequence $V_{m^k} T V^*_{m^k}$ (if it exists). For a general band-dominated operator, one can choose a band as above outside of which any matrix coefficients are small; thus again the sequence $V_{m^k} T V^*_{m^k}$ only picks out information at infinity.

Limit operators are often not too difficult to compute in concrete examples, which is part of the appeal of the method of Rabinovich, Roch and Silbermann. The following two examples are taken from [55], section 1.2. While the two are very similar, we hope that comparing them highlights the sort of phenomena that can occur as one moves up in dimension.

Examples 2.2.5.

1. Let P_1 be the orthogonal projection onto the closed span of $\{\delta_m \in l^2(\mathbb{Z}) : m \geq 0\}$. Then the limit operator of P_1 along a sequence $(m^k)_{k=0}^\infty$ exists if and only if the sequence of 'signs' $(m^k/|m^k|)_{k=0}^\infty$ converges to exactly one of ± 1. If this second sequence converges to -1, then the limit operator of P_1 along (m^k) is 0; if it converges to $+1$, the limit operator of P_1 along it is 1.

2. Let P_2 be the orthogonal projection onto the space of $\{\delta_{(m_1,m_2)} \in l^2(\mathbb{Z}^2) : m_2 \geq 0\}$. Let (m^k) be any sequence in $\mathbb{Z}^2$ converging to infinity. The following list covers all limit operators that can occur along (m^k):

   - 1, occurring for example if $(m^k/\|m^k\|)$ converges to a point on the open upper hemisphere of S^1;
   - 0, occurring for example if $(m^k/\|m^k\|)$ converges to a point on the open lower hemisphere of S^1;
   - the characteristic functions of the sets $\{(m_1, m_2) \in \mathbb{Z}^2 : m_2 \geq M\}$ as M ranges over $\mathbb{Z}$, which occur as limit operators along sequences (m^k) such that $(m^k/\|m^k\|)$ converges to ± 1.

The definition of limit operators in 2.2.3 above is perhaps the most intuitive, and also the easiest to use when computing examples. We would like, however, to organize the limit operators of a $*$-rich operator T according to the 'direction' in which they tend to infinity. In this way, the limit operators of T can be organized into a *symbol*, denoted $\sigma(T)$.

The approach we follow is taken from [56], section 2.2.4, and is based on the *Stone-Cěch compactification of* $\mathbb{Z}^n$, denoted $\beta\mathbb{Z}^n$. Recall that $C(\beta\mathbb{Z}^n)$ is isomorphic to $l^\infty(\mathbb{Z}^n)$ (as both have the same universal property), whence points of $\beta\mathbb{Z}^n$ can be identified with multiplicative linear functionals on $l^\infty(\mathbb{Z}^n)$. If one removes $\mathbb{Z}^n$ (identified with point evaluations) from $\beta\mathbb{Z}^n$, the remaining points form the *Stone-Cěch corona of* $\mathbb{Z}^n$, denoted $\partial_\beta\mathbb{Z}^n$.

Points of $\partial_\beta\mathbb{Z}^n$ can be thought of as providing the required notion of 'direction at infinity' in $\mathbb{Z}^n$. Indeed, as $\beta\mathbb{Z}^n$ is the largest compactification of $\mathbb{Z}^n$, it is reasonable to consider $\partial_\beta\mathbb{Z}^n$ as covering all possible 'directions at infinity'. The definition below captures the notion of the limit of an operator T in the direction $\omega \in \partial_\beta\mathbb{Z}^n$ (cf. page 56 in [56]).

Definition-Lemma 2.2.6. Let $f \in l^\infty(\mathbb{Z}^n)$ act on $l^2(\mathbb{Z}^n)$. For each $\omega \in \partial_\beta\mathbb{Z}^n$, define $f^\omega \in l^\infty(\mathbb{Z}^n)$ by

$$f^\omega : m \mapsto \omega(U_m^* f U_m).$$

Let T be any bounded operator on $l^2(\mathbb{Z}^n)$. As in part (1) of remarks 2.1.2, one may formally write T as a sum

$$T = \sum_{m \in \mathbb{Z}^n} f_m U_m.$$

If the sum

$$T^\omega = \sum_{m \in \mathbb{Z}^n} f_m^\omega U_m \tag{2.8}$$

defines a bounded operator on $l^2(\mathbb{Z}^n)$, it is called the *limit operator of T in the direction of ω*.

Finally, if the limit operator of T along a sequence (m^k) exists and ω is any limit point of (m^k) in $\beta\mathbb{Z}^n$, then T^ω exists and is equal to the limit operator of T along (m^k).

Proof of the last statement. Formally write $T = \sum_{m \in \mathbb{Z}^n} f_m U_m$. As the sequence $V_{m^k} T V_{m^k}^*$ converges strongly, it follows that all of the sequences

$$V_{m^k} f_m V_{m^k}^*$$

also converge strongly to some (multiplication) operators, say ϕ_m for each m. Now, by definition of what it means for ω to be a limit point of (m^k) in the Gelfand topology on $\beta\mathbb{Z}^n$ there exists a subnet $(m^i)_{i \in I}$ of (m^k), such that for each $m, l \in \mathbb{Z}^n$

$$f_m^\omega(l) = \omega(U_l^* f_m U_l) = \lim_{i \to \infty} f_m(l + m^i) = \lim_{i \to \infty} (V_{m^i} f_m V_{m^i}^*)(l) = \phi_m(l),$$

where the last equality follows as (m^i) is a subnet of (m^k). This says precisely that the matrix coefficients of $V_{m^k}TV^*_{m^k}$ converge to those of the operator provisionally defined in line (2.8) above. Hence the formula for T^ω *does* define a bounded linear operator, and this operator is the strong limit of $V_{m^k}TV^*_{m^k}$ as required. □

The following theorem lists some basic properties of limit operators. It is essentially a compilation of various results from sections 1.2 through 2.2 from [56].

Theorem 2.2.7.

1. *An operator $T \in \mathcal{B}(l^2(\mathbb{Z}^n))$ is rich (respectively, $*$-rich) if and only if the set $\{V_mTV^*_m : m \in \mathbb{Z}^n\}$ is precompact in the strong (resp. $*$-strong) operator topology.*

2. *If an operator $T \in \mathcal{B}(l^2(\mathbb{Z}^n))$ is rich (resp. $*$-rich), then for any $\omega \in \partial_\beta\mathbb{Z}^n$, the operator T^ω as (provisionally) defined in line (2.8) above exists. Moreover, the map defined on $\beta\mathbb{Z}^n$ by*

   $$m \mapsto V_mTV^*_m \text{ and } \omega \mapsto T^\omega$$

   is continuous for the strong (resp. $*$-strong) operator topology.

3. *Say $S, T \in \mathcal{B}(l^2(\mathbb{Z}^n))$ are rich and ω is any point in $\partial_\beta\mathbb{Z}^n$. Then $S + T$ and ST are also rich and $(S + T)^\omega = S^\omega + T^\omega$, $(ST)^\omega = S^\omega T^\omega$; if T is moreover $*$-rich, then $(T^*)^\omega = (T^\omega)^*$; finally, $\|T^\omega\| \leq \|T\|$.*

4. *Band-dominated operators are $*$-rich.*

5. Define an action of $\mathbb{Z}^n$ on $\partial_\beta \mathbb{Z}^n$ by

$$(m\omega)(f) = \omega(U_m^* f U_m)$$

for all $f \in l^\infty(\mathbb{Z}^n)$ (i.e. this is the usual left action of $\mathbb{Z}^n$ on $\partial_\beta \mathbb{Z}^n$). With respect to this action, for any $m \in \mathbb{Z}^n$ and $\omega \in \partial_\beta \mathbb{Z}^n$, one has that

$$T^{m\omega} = V_m T^\omega V_m^*.$$

Sketch of proof. Throughout we consider only the 'rich' case; the proofs in the '$*$-rich' case are the same, having replaced 'strong operator topology' by '$*$-strong operator topology' where necessary.

For part (1), note that the strong closure of the set $\{V_m T V_m^* : m \in \mathbb{Z}^n\}$ is norm-bounded, whence strong-metrizable. Richness of T is the same as the strong closure of this set being sequentially compact, whence also the same as saying that is is compact.

For part (2), say T is rich, whence $\{V_m T V_m^* : m \in \mathbb{Z}^n\}$ is strongly precompact by part (1). Hence by the universal property of $\beta\mathbb{Z}^n$, there exists a strongly continuous map $\beta\mathbb{Z}^n \to \mathcal{B}(l^2(\mathbb{Z}^n))$ that extends the map

$$\mathbb{Z}^n \to \mathcal{B}(l^2(\mathbb{Z}^n))$$

$$m \mapsto V_m T V_m^*.$$

Temporarily denote the image of ω under this extension by T_ω. Now, by the metrisability of the strong closure of $\{V_m T V_m^* : m \in \mathbb{Z}^n\}$, T_ω is the strong limit of a sequence $V_{m^k} T V_{m^k}^*$ such that ω is a limit point of (m^k); by lemma 2.2.6, then, $T^\omega = T_\omega$.

Part (3) consists of simple applications of the interplay between 'sequence-defined' and 'functional-defined' limit operators as in definition-lemma 2.2.6.

For part (4), note that the strong topology on bounded subsets of $l^\infty(\mathbb{Z}^n)$ corresponds to the topology of pointwise convergence, whence a Cantor diagonal argument shows that multiplication operators are rich. Note moreover that their limit operators are strong limits of multiplication operators, whence also multiplication operators. On the other hand, for a shift operator U_l, $\{V_m U_l V_m^* : m \in \mathbb{Z}^n\} = \{U_l\}$, whence U_l is rich, and all if its limit operators are equal to U_l itself. As these two classes of operators together generate the band-dominated operators, the result follows from part (3).

Part (5) follows from the computation

$$\begin{aligned} f^{m\omega}(l) &= (m\omega)(U_l^* f U_l) = \omega(U_m^* U_l^* f U_l U_m) \\ &= \omega(U_{l+m}^* f U_{l+m}) = f^\omega(l+m) = (V_m f^\omega V_m^*)(l) \end{aligned}$$

and part (3). □

Remark 2.2.8. Essentially the same results are true, with the same proofs, if one considers band-dominated operators with values in a matrix algebra as in definition 2.1.5 above.

If the values are taken to be compact operators, however, then the statement that band-dominated operators are rich in part (4) fails. Indeed, let

$$T = \sum_{m \in \mathbb{Z}^n} F_m U_m$$

be a band-dominated operator where each F_m is a function in $l^\infty(\mathbb{Z}^n, \mathcal{K})$ as in line (2.5) above. Then an elaboration of theorems 2.1.16 and 2.1.18 from [56] (cf. also lemma 9 from [53]) shows that T is rich if and only if for each $m \in \mathbb{Z}^n$, the set

$$\{F_m(l) \in \mathcal{K} : l \in \mathbb{Z}^n\}$$

is relatively compact for the norm topology on $\mathcal{K}$.

The following corollary (of theorem 2.2.7 and proposition 2.2.2) together with Atkinson's theorem 1.0.1 goes back to proposition 2.2.2, and comes a stage closer to connecting limit operators and Fredholmness.

Corollary 2.2.9 (Proposition 2.11 from [65]). *Let T be a band-dominated operator. Then its limit operators all vanish if and only if it is compact.*

Sketch of proof. As T is band-dominated, it can be written $T_0 + T_1$, where T_0 is a band-operator and T_1 has norm at most ϵ. The limit operators of T_0 vanish if and only if its matrix coefficients vanish down diagonals, while the matrix coefficients of T_1 and all of its limit operators are bounded by ϵ. It follows that all limit operators of T vanish if and only if all its matrix coefficients vanish at infinity. By proposition 2.2.2, this occurs if and only if T is compact. □

Now, denote by $C_s(\partial_\beta \mathbb{Z}^n, C_u^*(|\mathbb{Z}^n|))$ the C^*-algebra of functions from $\partial_\beta \mathbb{Z}^n$ to $C_u^*(|\mathbb{Z}^n|)$ which are continuous for the $*$-strong operator topology on the latter. Parts (3) and (4) of theorem 2.2.7 imply that for each $T \in C_u^*(|\mathbb{Z}^n|)$, the 'symbol' $\sigma(T)$ defined by

$$\sigma(T) : \partial_\beta \mathbb{Z}^n \to C_u^*(|\mathbb{Z}^n|)$$

$$\omega \mapsto T^\omega$$

is in $C_s(\partial_\beta \mathbb{Z}^n, C_u^*(|\mathbb{Z}^n|))$ and that $T \mapsto \sigma(T)$ defines a $*$-homomorphism from $C_u^*(|\mathbb{Z}^n|)$ to this algebra. Moreover, corollary 2.2.9 says that the kernel of σ consists precisely of the compact operators. Putting all this together gives the following corollary.

Corollary 2.2.10. *There is an exact sequence*

$$0 \longrightarrow \mathcal{K}(l^2(\mathbb{Z}^n)) \longrightarrow C_u^*(|\mathbb{Z}^n|) \xrightarrow{\sigma} C_s(\partial_\beta \mathbb{Z}^n, C_u^*(|\mathbb{Z}^n|))$$

(σ is not *in general surjective; cf. remark B.3).* □

This in turn leads to a concrete criterion for the Fredholm property. The following corollary is a special case of theorem 2.2.1 from [56], which is perhaps the fundamental result of that book. The proof we give is the same as that of theorem 3.4 from [65].

Corollary 2.2.11. *Let T be a band-dominated operator on $\mathbb{Z}^n$. It is Fredholm if and only if all of its limit operators are invertible, and the norms of their inverses are uniformly bounded.*

Proof. Corollary 2.2.10 and Atkinson's theorem 1.0.1 together imply that T is Fredholm if and only if its symbol $\sigma(T)$ is invertible. Clearly if this is the case, then all T^{ω} are invertible and their inverses are uniformly bounded. Conversely, we need only check that if all T^{ω} are invertible with uniformly bounded inverses, then the map $\omega \mapsto (T^{\omega})^{-1}$ is $*$-strongly continuous. This follows from the identity

$$T^{-1} - S^{-1} = T^{-1}(S - T)S^{-1}$$

and the continuity of multiplication for the $*$-strong topology on bounded subsets. □

Whether the uniform boundedness part of the condition above is really necessary is a major unsolved problem in the theory.

2.3 An index theorem for the case $n = 1$

The last section suggest an index problem. Indeed, we may collate and restate its main results as follows.

Theorem 2.3.1. *A band-dominated operator T on $\mathbb{Z}^n$ is Fredholm if and only if its symbol $\sigma(T) \in C_s(\partial_\beta \mathbb{Z}^n, C_u^*(|\mathbb{Z}^n|)$ is invertible. Moreover, if this is the case then the index of T depends only on $\sigma(T)$.* □

The index problem is then to give a reasonably concrete formula for the index of T in terms of the symbol $\sigma(T)$, or in other words, the limit operators of T combined with information about how they 'fit together'.

There are reasons to be pessimistic about solving this problem. For a start, there is no obvious reason why the symbol $\sigma(T)$ should be any more tractable than T itself. Indeed, section 2.1.5 of [56] shows that *any* band-dominated operator arises as a limit operator of some (other) band-dominated operator. The class of band-dominated operators is also huge; this adds interest to the problem, but also makes it seem unlikely that any general solution will be particularly computable.

Nonetheless, Rabinovich, Roch and Roe have produced an elegant solution for the case $n = 1$ in [54]. In this section we exposit their solution, partly as it motivates the generalizations we pursue in the rest of this piece. To state the main theorem of [54] we need the following definition.

Definition-Lemma 2.3.1. Let $P \in l^{\infty}(\mathbb{Z})$ be the characteristic function of the interval $[0, \infty)$ and $Q = 1 - P$ the characteristic function of $(-\infty, -1] \cap \mathbb{Z}$. Let $F \in C_u^*(|\mathbb{Z}|)$ be Fredholm. Then the operators $Q + PFP$ and $QFQ + P$ are Fredholm.

The *plus-index of* F, denoted $(+Ind)(F)$, is defined to be $Index(Q + PFP)$ and the *minus-index of* F, denoted $(-Ind)(F)$, to be $Index(QFQ + P)$. One moreover has that

$$Index(F) = (-Ind)(F) + (+Ind)(F).$$

Proof. Note first that $C_u^*(|\mathbb{Z}|)$ is generated as a C^*-algebra by $l^{\infty}(\mathbb{Z})$ and the bilateral shift; as P, Q commute with these up to the compact operators, they commute with all elements of $C_u^*(|\mathbb{Z}|)$ up to compact operators. Let $T \sim S$ mean that T and S have the same image in the Calkin algebra. Let G be a paramatrix for F, i.e. G satisfies

$FG \sim GF \sim 1$. Then

$$(Q + PGP)(Q + PFP) = Q + PGPPFP \sim Q + PPGFPP$$

$$= Q + PGFP \sim Q + P1P = Q + P = 1,$$

and similarly $(Q+PFP)(Q+PGP) \sim 1$, whence $Q+PGP$ is a paramatrix for $Q+PFP$ and so the latter is Fredholm. Using the same sort of argument, $QGQ+P$ is a parametrix for $QFQ + P$, so this too is Fredholm.

Finally, note that

$$(QFQ + P)(Q + PFP) = QFQ + PFP \sim QFQ + QFP + PFQ + PFP$$

$$= (P + Q)F(P + Q) = F,$$

whence by the multiplicativity of the index,

$$Index(F) = Index(QFQ + P) + Index(Q + PFP)$$

$$= (-Ind)(F) + (+Ind(F))$$

as required. □

Remark 2.3.2. Another way of viewing the above is to note that the operators P and Q essentially define K-*homology classes* for the algebra $C^*_u(|\mathbb{Z}^n|)/\mathcal{K}$ (see chapter 5 of [43], particularly remark 5.3.9). The word 'essentially' appears in the above as K-homology is usually defined only for separable C^*-algebras, and $C_u^*(|\mathbb{Z}^n|)$ is not separable. Nonetheless, P, Q do still define 'index pairings'

$$Ind_P, Ind_Q : K_1(C_u^*(|\mathbb{Z}^n|)/\mathcal{K}) \to \mathbb{Z}$$

(see [43], section 7.2, all of which is easily adaptable to the case above). Moreover, a Fredholm operator $F \in C_u^*(|\mathbb{Z}|)$ defines a class $[F] \in K_1(C_u^*(|\mathbb{Z}|)/\mathcal{K})$; by definition of the index pairing, $(+Ind)(F)$ and $(-Ind)(F)$ are respectively equal to $Ind_P([F])$ and $Ind_Q([F])$.

We can now state the main theorem of [54].

Theorem 2.3.3. *Say $F \in C_u^*(|\mathbb{Z}|)$ is Fredholm. Let F_+ be* any *limit operator of F along a sequence that converges to $+\infty$, and F_- any limit operator of F along a sequence that converges to $-\infty$. Then*

$$(+Ind)(F) = (+Ind)(F_+) \text{ and } (-Ind)(F) = (-Ind)(F_-).$$

In particular,

$$Index(F) = (-Ind)(F_-) + (+Ind)(F_+).$$

Note that although this theorem is rather elegant, it does not in general give a computable formula. Indeed, its computability depends on being able to find limit operators $F_\pm$ of the original Fredholm F that are simple enough so that the 'secondary index problems' of computing $Index(QF_-Q + P)$ and $Index(Q + PF_+P)$ are accessible. One suspects that this is not always possible, as limit operators need not be any simpler than the original operator itself (cf. for example section 2.1.5 of [56] again). However, there are situations in which it is possible to find such $F_\pm$; in particular, section 3.3 below discusses a class of band-dominated operators on $\mathbb{Z}^n$ whose limit operators *all* have a relatively tractable form. The resulting relatively computable index theorem is stated as theorem 3.3.3.

The proof of theorem 2.3.3 given here is K-theoretic. It is similar to, but not exactly the same as, that in [54]; it is closer to the remark at the top of page 234 in that paper. Roch, Rabinovich and Silbermann also have a rather different proof of theorem 2.3.3 in [57] that uses no K-theory at all. It is not currently entirely clear (to me) what the relationship between the two proofs is.

The following lemma gives a concrete description of the group $K_1(C_u^*(|\mathbb{Z}|))$.

Lemma 2.3.4. *$K_1(C_u^*(|\mathbb{Z}|)) \cong \mathbb{Z}$. Moreover, one can take the class of the bilateral shift $U \in \mathcal{B}(l^2(\mathbb{Z}))$ as a generator.*

Proof. A detailed proof that $K_1(C_u^*(|\mathbb{Z}|)) \cong \mathbb{Z}$ is given in section 2.2 of [54]. The central point is that $C_u^*(|\mathbb{Z}|) \cong l^\infty(\mathbb{Z}) \rtimes_r \mathbb{Z}$ (cf. proposition 3.1.1, where we prove this in much greater generality); one can then use the *Pimsner-Voiculescu exact sequence* (see for example [7], chapter 10) and a computation of $K_*(l^\infty(\mathbb{Z}))$ to complete the proof.

To see that one can take $[U]$ as a generator, let Q be as in definition 2.3.1. As it commutes with $C_u^*(|\mathbb{Z}|)$ up to compact operators, it defines a $\mathbb{Z}$-linear index pairing

$$Ind_Q : K_1(C_u^*(|\mathbb{Z}|)) \to \mathbb{Z}$$

(see [43], section 7.2 again) via the formula $[V] \mapsto Index(QVQ + (1-Q))$ (with amplification to matrix algebras as necessary). Clearly, however, $QUQ + (1-Q)$ has index one. As the index pairing is $\mathbb{Z}$-linear, and $K_1(C_u^*(|\mathbb{Z}|)) \cong \mathbb{Z}$, this is only possible if $[U]$ generates $K_1(C_u^*(|\mathbb{Z}|))$. □

The following corollary is the main ingredient in the proof of theorem 2.3.3.

Corollary 2.3.5. $K_1(C_u^*(|\mathbb{Z}|)/\mathcal{K}) \cong \mathbb{Z}^2$. *Generators can be taken to be the (classes of the) images of* $QUQ + P$ *and* $Q + PUP$ *in the Calkin algebra.*

Proof. Consider the six term exact sequence in K-theory associated to the short exact sequence

$$0 \longrightarrow \mathcal{K}(l^2(\mathbb{Z})) \longrightarrow C_u^*(|\mathbb{Z}|) \longrightarrow C_u^*(|\mathbb{Z}|)/\mathcal{K} \longrightarrow 0 \ ,$$

which looks like

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(C_u^*(|\mathbb{Z}|)) & \longrightarrow & K_0(C_u^*(|\mathbb{Z}|)/\mathcal{K}) \\ {\scriptstyle Index}\uparrow & & & & \downarrow \\ K_1(C_u^*(|\mathbb{Z}|)/\mathcal{K}) & \longleftarrow & \mathbb{Z} & \longleftarrow & 0 \end{array}$$

(one can also compute the two K_0 groups at the top right - they are both uncountable - but this is not relevant to the proof). Note that example 2.2.1 shows that the index map is surjective, however, so there is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_1(C_u^*(|\mathbb{Z}|)/\mathcal{K}) \longrightarrow \mathbb{Z} \longrightarrow 0 \ .$$

The middle group is thus isomorphic to $\mathbb{Z}^2$. Moreover, using lemma 2.3.4 generators can be taken to be the (class of the) image of the bilateral shift U in the Calkin algebra, and (the class of) any operator of index one. Note, however, that

$$(QUQ + P)(Q + PUP) \sim U,$$

whence $[QUQ+P]+[Q+PUP] = [U]$ in $K_1(C^*(|\mathbb{Z}|)/\mathcal{K})$; as moreover $Index(QUQ+P) = 1$, the classes of $QUQ + P$ and $Q + PUP$ generate $K_1(C^*_u(|\mathbb{Z}|)/\mathcal{K})$ as required. □

Proof of theorem 2.3.3. Take any Fredholm $F \in C^*_u(|\mathbb{Z}|)$ and limit operators $F_\pm$ as in the statement of the theorem. Choose $\omega_\pm \in \partial_\beta \mathbb{Z}$ such that $F_\pm = F^{\omega_\pm}$, where notation is as in definition 2.2.6. Note that for any Fredholm G in (a matrix algebra over) $C^*_u(|\mathbb{Z}|)$, theorem 2.2.7 implies that the limit operators $G^{\omega_\pm}$ exist. One may thus define maps $K_1(C^*_u(|\mathbb{Z}|)/\mathcal{K}) \to \mathbb{Z}$ by

$$[\sigma(G)] \mapsto (\pm Ind)(G^{\omega_\pm})$$

(it is not hard to see that these *are* defined at the level of K-theory). Checking on the generators $[QUQ + P]$ and $[Q + PUP]$ shows that these maps agree with the maps

$$[\sigma(G)] \mapsto (\pm Ind)(G),$$

respectively. However, when applied to the class $[\sigma(F)]$, the statement that these two pairs of maps agree is exactly the statement of theorem 2.3.3. □

Remark 2.3.6. Essentially the same index theorem holds (with the same proof) if one considers band-dominated operators with values in a matrix algebra as in definition 2.1.5. Similarly, Rabinovich and Roch [53] have extended the theorem above to the case of rich band-dominated operators with compact values. Recall from remark 2.2.8 that not all compact-valued band-dominated operators are rich; in fact, elaborating slightly on 2.2.8, one can show that the algebra of rich compact-valued band-dominated operators is isomorphic to the stabilization C^*-algebra $C_u^*(|\mathbb{Z}|) \otimes \mathcal{K}$. Using this and the stability of K-theory, it is possible to recover the main results of [53] using the techniques above.

2.4 Problems with extending this to $\mathbb{Z}^n$

There are unfortunately two problems with extending the above proof to $\mathbb{Z}^n$; indeed, I have not been able to do so in full generality. The first is that the analogues of the K-theory groups appearing in lemma 2.3.4 and corollary 2.3.5 are uncountable rank, so a proof by checking a small number of easy examples is no longer possible. The second, and much more serious, could be thought of as the lack of a notion of 'plus and minus infinity' in higher dimensions. We will first discuss the former problem, and a way around it, before moving on to the second problem. Both of these discussions suggest aspects of the eventual approach that we will take.

The first problem is summed up by the following proposition.

Proposition 2.4.1. *For all $n \geq 2$, the K-theory group $K_1(C_u^*(|\mathbb{Z}^n|))$ is uncountable.* □

In fact, this is already true for $K_0(C_u^*(|\mathbb{Z}|))$. We will not give a proof, mainly as the result is not central for the piece and the known proofs are somewhat involved.

One uses traces and the index-theoretic results of [59] and [60] to detect K-theory; another uses the results of [70] and the Baum-Connes conjecture; the most elementary and least elegant uses repeated application of the Pimsner-Voiculescu exact sequence and the isomorphism

$$C_u^*(|\mathbb{Z}^n|) \cong (\cdots(l^\infty(\mathbb{Z}^n)\underbrace{\rtimes_r \mathbb{Z}) \rtimes_r \mathbb{Z}) \rtimes_r \cdots) \rtimes_r \mathbb{Z}}_{n}$$

(cf. the second remark on page 34 of [63]).

There is, however, a way around the problem suggested by the proposition: we may pass to $C^*(|\mathbb{Z}^n|)$, the algebra of band-dominated operators with values in the compact operators as discussed in definition 2.1.5. Remarkably, the K-theory of this algebra is much more tractable than that of $C_u^*(|\mathbb{Z}^n|)$, despite the former being a much 'larger' algebra. The following result is perhaps most easily proved using a Mayer-Vietoris sequence; cf. for example section 6.4 from [43].

Proposition 2.4.2. *One has that*

$$K_i(C^*(|\mathbb{Z}^n|)) \cong \begin{cases} \mathbb{Z} & i = n \ mod \ 2 \\ 0 & i \neq n \ mod \ 2 \end{cases}$$

for all n. □

Let $p \in \mathcal{K}$ be any rank one projection, and P the constant function in $l^\infty(\mathbb{Z}^n, \mathcal{K}) \subseteq C^*(|\mathbb{Z}^n|)$ with value p. Then $C_u^*(|\mathbb{Z}^n|)$ is isomorphic to the 'corner' $PC^*(|\mathbb{Z}^n|)P$. Given a Fredholm operator $F \in C_u^*(|\mathbb{Z}^n|)$, we may thus consider the Fredholm operator

$FP + (1 - P)$ (which has the same index as F) in (the unitization of) $C^*(|\mathbb{Z}^n|)$; this reduces the index problem of interest to one involving only the simple K-theory of $C^*(|\mathbb{Z}^n|)$ (and its quotient by the compact operators).

The second problem we will look at here is more awkward. In a sense, it is conceptual while the first was practical: it asks how to interpret theorem 2.3.3 in a way that is amenable to generalization. To be more explicit, the following list gives the recipe from theorem 2.3.3 for computing the index of some Fredholm F:

1. choose any points $\omega_\pm$ 'at infinity' (i.e. in $\partial_\beta\mathbb{Z}$) over $\pm\infty$;
2. compute the limit operators $F_\pm$ along $\omega_\pm$;
3. compute the 'local-indices' of the operators $F_\pm$ at $\pm\infty$;
4. add these together to get the global index.

Now, note that $\mathbb{Z}$ admits a compactification by a copy of S^0 (the zero dimensional sphere - a two point space) at infinity, and that the universal property of the Stone-Cěch compactification gives a map

$$\partial_\beta\mathbb{Z} \to S^0$$

that can be thought of as sending points of $\partial_\beta\mathbb{Z}$ to plus or minus infinity according to the 'direction in which they tend to infinity'. What point (1) above is asking us to do, then, is choose a splitting of this map. Points (2) and (3) tell us to compute local indices at all (i.e. both!) points of S^0, while point (4) tells us to 'integrate' this local data over S^0 to get the final solution.

$\mathbb{Z}^n$ in fact also admits a natural compactification by a sphere at infinity, although this time it is a copy of the $n-1$ sphere, S^{n-1}. As we will use this compactification later (see particularly section 3.4), we give a precise definition below.

Definition 2.4.3. The *spherical compactification* of $\mathbb{Z}^n$ is set theoretically equal to the disjoint union $S^{n-1} \sqcup \mathbb{Z}^n$. If S^{n-1} and $\mathbb{Z}^n$ are metrized as subsets of $\mathbb{R}^n$, then a basis for its topology is given by:

- all subsets of $\mathbb{Z}^n$;
- for any $x \in S^{n-1}$ and $R, \epsilon > 0$, the sets

$$W_{x,R,\epsilon} = \{y \in S^{n-1} : d(x,y) < \epsilon\} \cup \{m \in \mathbb{Z}^n : d(x, m/\|m\|) < \epsilon, \|m\| > R\}.$$

The spherical compactification of $\mathbb{Z}^n$ is denoted $\overline{\mathbb{Z}^n}^s$.

There is again a natural quotient map

$$\partial_\beta \mathbb{Z}^n \to S^{n-1},$$

but this time it *does not* split. Hence point (1) above cannot be repeated (at least not exactly) in higher dimensions. In fact, we are not able to carry over the 'four point plan' above to a complete solution in higher dimensions, but the partial solutions we do obtain (see particularly theorem 6.1.6 and remark 6.1.7) can be thought of as following the philosophy above.

In general, we will associate an index problem to each equivariant boundary of an exact group. The proof of the index theorem that we do have for $\mathbb{Z}^n$ will go by way of a reduction to an index problem on the above spherical boundary; it is perhaps not obvious, but this is a special case of the Atiyah-Singer index problem for pseudodifferential operators on compact manifolds.

Chapter 3

Generalizations and special cases

The first aim of this chapter ('generalizations') is to expand the results of section 2.2 on $\mathbb{Z}^n$ to a much larger class of discrete groups. This material is contained in [65]. Our approach is slightly different, however, building on the remarks in section 4 of that paper. Our main tools are C^*-algebra crossed products, combined with some technical results that are descibed in appendix B. This generalization process suggests a large family of index problems, each one associated to an equivariant compactification of a discrete group.

It must be admitted at this point that amongst band-dominated operators on general discrete groups, those on $\mathbb{Z}^n$ are the most interesting in terms of applications to operator theory. Having said this, examples of operator theoretic work on more general groups include some results in the monograph [56] of Rabinovich, Roch and Silbermann on the Heisenberg group, and work of M. Jury on Fuchsian groups [45], [46]. There are also points of contact between the material in this chapter and work of H. Emerson [20] [21], H. Emerson and R. Meyer [23] and others in noncommutative geometry.

The second aim of this chapter ('special cases') is to look at some band-dominated operators associated to specific equivariant compactifications of discrete groups; we hope that this is somewhat motivated by the comments at the end of the last section on the spherical compactification of $\mathbb{Z}^n$. The first of these special cases (section 3.3) covers operators with *slowly oscillating coefficients*, which will be of great importance for much of the rest of this thesis. Although these make sense for general discrete groups, we will focus on the case of $\mathbb{Z}^n$; this leads to index theorems for the corresponding class of operators in chapter 5 and section 6.1. Section 3.4 focuses very specifically on the spherical compactification of $\mathbb{Z}^n$, relating the associated family of band-dominated operators to *pseudodifferential operators* on the n-torus. The remaining two sections consider compactifications associated to geometric conditions on discrete groups. These constructions will provide examples in chapter 4.

Before starting the main part of this chapter, we will generalize the definitions of section 2.1 to the case of a general countable discrete group Γ, and outline the main results of [65]. These are proved in section 3.1 using crossed products.

Just as in the preamble to definition 2.1.1, then, $l^\infty(\Gamma)$ acts on $l^2(\Gamma)$ by pointwise multiplication, and there are unitary (left) shift operators U_g for each $g \in \Gamma$, defined on basis elements by

$$U_g : \delta_h \mapsto \delta_{gh}.$$

We may thus carry over definition 2.1.1 to the current context.

Definition 3.0.1. A *band operator on* Γ is a sum

$$\sum_{g \in \Gamma} f_g U_g \tag{3.1}$$

where each f_g is an element of $l^\infty(\Gamma)$ and only finitely many f_g are non-zero.

A *band-dominated operator* is an operator norm limit of band operators.

The C^*-algebra of all band-dominated operators on Γ will be denoted $C_u^*(|\Gamma|)$.

Many properties of this algebra depend on the group Γ; see for example part (4) of definition-theorem B.5. Nonetheless, Roe [65] was able to prove the analogue of theorem 2.2.10 for the very large class of *exact* groups; see appendix B for a definition and some properties of these groups.

Theorem 3.0.2. *For any countable discrete exact group* Γ*, there is an exact sequence*

$$0 \longrightarrow \mathcal{K}(l^2(\Gamma)) \longrightarrow C_u^*(|\Gamma|) \xrightarrow{\sigma} C_s(\partial_\beta \Gamma, C_u^*(|\Gamma|))$$

(σ *is* not *in general surjective; cf. remark B.3).* □

One can prove this using the approach for $\mathbb{Z}^n$ sketched out in section 2.2. In fact, all the results stated in that section carry over to the class of exact groups; this is why we were far more careful than strictly necessary when distinguishing left and right actions in that section. Note in particular that the analogue of corollary 2.2.11 holds: an operator in $C_u^*(|\Gamma|)$ is Fredholm if and only if all of its limit operators are invertible with uniformly bounded inverses. Nonetheless, we will give a proof below that is independent of these methods, culminating in corollary 3.1.6 below.

3.1 The relevance of crossed product algebras

It is the purpose of this section to prove an analogue of theorem 3.0.2, given as corollary 3.1.4 below. To complete the section, corollary 3.1.6 sketches out the relationship between 3.1.4 and theorem 3.0.2, and indeed shows that theorem 3.0.2 follows from the results below.

In the proof of lemma 2.3.4, we mentioned that $C_u^*(|\mathbb{Z}|)$ has the structure of a crossed product C^*-algebra: it is isomorphic to $l^\infty(\mathbb{Z}) \rtimes_r \mathbb{Z} \cong C(\beta\mathbb{Z}) \rtimes_r \mathbb{Z}$, where $\beta\mathbb{Z}$ is the Stone-Cěch compactification of $\mathbb{Z}$. We will start this section with a general analogue of this structural result.

The next proposition is essentially due to N. Higson and G. Yu; the proof (if not the statement) is exactly the same as that of proposition 5.1.3 from [11].

Proposition 3.1.1. *Let Γ be a countable discrete group. Let α be the natural left shift action of Γ on $l^\infty(\Gamma)$ (which is spatially implemented by the left action of Γ on $l^2(\Gamma)$). Let A be a C^*-subalgebra of $l^\infty(\Gamma) \subseteq \mathcal{B}(l^2(\Gamma))$ that is preserved by α.*

Then the C^*-subalgebra of $\mathcal{B}(l^2(\Gamma))$ generated by elements of the form fU_g, $f \in A$, $g \in \Gamma$, is canonically isomorphic to the reduced crossed product $A \rtimes_r \Gamma$ of A by Γ with respect to the action α.

Proof. To define the reduced crossed product $A \rtimes_r \Gamma$ one may start with any faithful representation of A as in definition B.1; choose the identity representation, say π, on $\mathcal{B}(l^2(\Gamma))$. Let $\tilde{\pi}$ be the resulting representation of $A \rtimes_r \Gamma$ on $l^2(\Gamma) \otimes l^2(\Gamma) \cong l^2(\Gamma \times \Gamma)$;

note that with respect to this isomorphism

$$\tilde{\pi}(fu_g) : \delta_{h,k} \mapsto (\alpha_{(gk)^{-1}} f)(h)\delta_{h,gk} = f(gkh)\delta_{h,gk}.$$

Define now a unitary operator

$$U : l^2(\Gamma \times \Gamma) \to l^2(\Gamma \times \Gamma)$$

by the prescription

$$\delta_{h,k} \mapsto \delta_{h,kh}$$

on basis elements. One computes that for $fu_g \in A \rtimes_{alg} \Gamma$,

$$\begin{aligned}
(U\tilde{\pi}(fu_g)U^*)(\delta_{h,k}) &= U\tilde{\pi}(fu_g)(\delta_{h,kh^{-1}}) \\
&= Uf(gkh^{-1}h)\delta_{h,gkh^{-1}} \\
&= f(gk)\delta_{h,gk} \\
&= (1 \otimes \pi(f)U_g)\delta_{h,k}.
\end{aligned}$$

This says exactly that U conjugates $A \rtimes_{alg} \Gamma$ to the $*$-subalgebra of $\mathcal{B}(l^2(\Gamma \times \Gamma))$ generated by operators of the from $1 \otimes \pi(f)U_g$ for $f \in A$ and $g \in \Gamma$. The C^*-algebra generated by the latter $*$-algebra is canonically isomorphic to the C^*-subalgebra of $\mathcal{B}(l^2(\Gamma))$ generated by the fU_g, whence the result. $\square$

As a simple example, note that if $A = \mathbb{C}$ (acting on $l^2(\Gamma)$ as multiples of the identity), then the proposition says that $\mathbb{C} \rtimes_r \Gamma \cong C_r^*(\Gamma)$. The next corollary is another special case; it is stated more formally as we will refer back to it repeatedly.

Corollary 3.1.2. *For any discrete group* Γ*,* $C_u^*(|\Gamma|)$ *is canonically isomorphic to* $l^\infty(\Gamma) \rtimes_r \Gamma$. □

The following lemma gives a third example that will be important for us, in particular as it starts to relate the above structural result to section 2.2.

Lemma 3.1.3. *Let* A *be as in proposition 3.1.1, and assume that* A *contains* $C_0(\Gamma)$*. Then the isomorphism given by that proposition takes the subalgebra* $C_0(\Gamma) \rtimes_r \Gamma$ *of* $A \rtimes_r \Gamma$ *to* $\mathcal{K}(l^2(\Gamma))$*.*

Proof. Let $p \in C_0(\mathbb{Z}^n)$ denote the point mass at the identity. The image of p under the isomorphism in proposition 3.1.1 is the orthogonal projection onto $\text{span}\{\delta_e\}$. Hence the image of $u_g p u_h^*$ under the isomorphism is the rank-one partial isometry $V_{h,g}$, taking δ_h to δ_g. The operators $u_g p u_h^*$ generate $C_0(\Gamma) \rtimes_r \Gamma$, while the $V_{h,g}$ generate $\mathcal{K}(l^2(\Gamma))$, whence the result. □

Corollary 3.1.4. *If* Γ *is exact, then there exists a short exact sequence*

$$0 \longrightarrow \mathcal{K}(l^2(\Gamma)) \longrightarrow C_u^*(|\Gamma|) \xrightarrow{\sigma_\beta} C(\partial_\beta \Gamma) \rtimes_r \Gamma \longrightarrow 0 \, .$$

Proof. One always has a short exact sequence of commutative C^*-algebras given by

$$0 \longrightarrow C_0(\Gamma) \longrightarrow C(\beta\Gamma) \longrightarrow C(\partial_\beta \Gamma) \longrightarrow 0 \, .$$

Moreover, the maps are equivariant for the natural (left) Γ actions on each algebra. As Γ is an exact group (see definition-theorem B.5), there is a short exact sequence of reduced crossed product algebras

$$0 \longrightarrow C_0(\Gamma) \rtimes_r \Gamma \longrightarrow C(\beta\Gamma) \rtimes_r \Gamma \longrightarrow C(\partial_\beta\Gamma) \rtimes_r \Gamma \longrightarrow 0 \, .$$

The central algebra is isomorphic to $C_u^*(|\Gamma|)$ by corollary 3.1.2, and this isomorphism takes $C_0(\Gamma) \rtimes \Gamma$ to $\mathcal{K}(l^2(\Gamma))$ by lemma 3.1.3; this completes the proof. □

This is not the same exact sequence as in theorem 3.0.2: the right-hand-side algebras are not the same, and indeed the sequence in 3.0.2 is not in general exact at the right-hand-side (cf. remark B.3), while that above is. The relationship between them follows from lemma B.2. Indeed, this lemma gives a canonical inclusion

$$i : C(\partial_\beta\Gamma) \rtimes_r \Gamma \to C_s(\partial_\beta\Gamma, C_u^*(|\Gamma|)).$$

Lemma 3.1.5. *If σ is the symbol map constructed using limit operators, then the following diagram*

$$\begin{array}{ccc} C_u^*(|\Gamma|) & \xrightarrow{\sigma} & C_s(\partial_\beta\Gamma, C_u^*(|\Gamma|)) \\ \| & & \uparrow{\scriptstyle i} \\ C_u^*(|\Gamma|) & \xrightarrow{\sigma_\beta} & C(\partial_\beta\Gamma) \rtimes_r \Gamma \end{array}$$

commutes.

Proof. It suffices to establish commutativity at the level of band-operators. Let $T = \sum f_g U_g$ be a band operator. In what follows,

$$\Big(\sum_{h\in\Gamma} h \mapsto \lambda_h\Big)$$

will denote the operator on $l^2(\Gamma)$ of multiplication by a bounded function $h \mapsto \lambda_h$. The quotient map in the bottom row of the statement followed by i (as in lemma B.2) takes T to the function

$$F : \omega \mapsto \sum_{g\in\Gamma} \Big(h \mapsto \omega(\alpha_{h^{-1}} f_g)\Big)U_g = \sum_{g\in\Gamma} \Big(h \mapsto \omega(U_h^* f_g U_h)\Big)U_g;$$

the latter expression is exactly the limit operator T^ω. □

The following result then proves theorem 3.0.2, and relates it to the material in this section.

Corollary 3.1.6. *If Γ is exact, then there is a short exact sequence*

$$0 \longrightarrow \mathcal{K}(l^2(\Gamma)) \longrightarrow C_u^*(|\Gamma|) \xrightarrow{\sigma} C_s(\partial_\beta\Gamma, C_u^*(|\Gamma|))$$

as in theorem 3.0.2 which moreover fits into a commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}(l^2(\Gamma)) & \longrightarrow & C_u^*(|\Gamma|) & \xrightarrow{\sigma} & C_s(\partial_\beta\Gamma, C_u^*(|\Gamma|)) & & \\ & & \| & & \| & & \uparrow i & & \\ 0 & \longrightarrow & C_0(\Gamma)\rtimes_r\Gamma & \longrightarrow & C(\beta\Gamma)\rtimes_r\Gamma & \longrightarrow & C(\partial_\beta\Gamma)\rtimes_r\Gamma & \longrightarrow & 0 \end{array}.$$

Finally, the image of the map σ *from theorem 3.0.2 sits inside the* Γ*-equivariant maps in* $C_s(\partial_\beta \Gamma, C_u^*(|\Gamma|))$.

Proof. From the previous lemma, there is a commutative diagram

$$\begin{array}{ccccccccc} & & \mathcal{K}(l^2(\Gamma)) & \longrightarrow & C_u^*(|\Gamma|) & \xrightarrow{\sigma} & C_s(\partial_\beta\Gamma, C_u^*(|\Gamma|)) & & \\ & & \| & & \| & & \uparrow i & & \\ 0 & \longrightarrow & \mathcal{K}(l^2(\Gamma)) & \longrightarrow & C_u^*(|\Gamma|) & \xrightarrow{\sigma_\beta} & C(\partial_\beta\Gamma) \rtimes_r \Gamma & \longrightarrow & 0 \end{array}.$$

Exactness of the sequence in theorem 3.0.2 follows from this, and the rest of the corollary is a redrawing of this diagram, combined with lemma B.2. □

This approach to theorem 3.0.2 uses a lot of high-powered machinery, but does have the benefit of giving a description of the image of the symbol map in $C_s(\partial_\beta\Gamma, C_u^*(|\Gamma|))$. This fact is sketched out in section 4 of [65]. It would, however, perhaps be desirable to have a more 'intrinsic' characterization of the image of i; cf. remark B.3.

3.2 Special cases

The results of the previous section suggest a collection of index problems. Specifically, let X be a compactification of Γ such that the (left) Γ action extends to an action on X (in other words, X is an *equivariant* compactification of Γ). Let $\partial X := X \backslash \Gamma$ be the corresponding corona. Exactly the same proofs as given for corollaries 3.1.4 and 3.1.6 above then show the following result.

Proposition 3.2.1. *Let* Γ *be an exact group and* X *an equivariant compactification of* Γ*. Then there exists a commutative diagram of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}(l^2(\Gamma)) & \longrightarrow & C(X) \rtimes_r \Gamma & \xrightarrow{\sigma} & C_s(\partial X, C^*_u(|\Gamma|)) & & . \\ & & \| & & \| & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{K}(l^2(\Gamma)) & \longrightarrow & C(X) \rtimes_r \Gamma & \longrightarrow & C(\partial X) \rtimes_r \Gamma & \longrightarrow & 0 \end{array}$$

Moreover, the image of the 'symbol' map σ *sits inside the* Γ*-equivariant maps in* $C_s(\partial X, C^*_u(|\Gamma|))$.

□

Note that (by proposition 3.1.1) the algebra $C(X) \rtimes_r \Gamma$ can be realized concretely as an algebra of operators on $l^2(\Gamma)$. Thus considered, it consists precisely of the band-dominated operators

$$T = \sum_{g \in \Gamma} f_g U_g$$

where f_g belongs to $C(X)$ for all $g \in \Gamma$; we say that it consists of *band-dominated operators with coefficients in* $C(X)$. Note that the representation above may have infinitely many non-zero terms, and that the sum need not converge (even in the weak operator topology) to T.

Proposition 3.2.1 above suggests an index problem associated to any compactification X of Γ as above: compute the index of a Fredholm operator $F \in C(X) \rtimes_r \Gamma$ in terms of the *symbol* $\sigma(F) \in C_s(\partial X, C^*_u(|\Gamma|))$, which is concretely built out of the limit operators of F. As the algebras $C(X) \rtimes_r \Gamma$ are all subalgebras of $C^*_u(|\Gamma|)$, the index problem for Fredholm operators in this algebra 'contains' all of those associated to other compactifications of Γ; this is why 'special cases' is the title of this section.

Examples 3.2.2. Let Γ^+ be the one point compactification of Γ. Then the associated short exact sequence

$$0 \longrightarrow \mathcal{K}(l^2(\Gamma)) \longrightarrow C(\Gamma^+) \rtimes_r \Gamma \longrightarrow C(\text{point}) \rtimes_r \Gamma \cong C_r^*(\Gamma) \longrightarrow 0$$

splits, whence has no interesting index theory associated.

On the other hand, if Γ is a group having at least one element of infinite order, then example 2.2.1 generalizes naturally to show that $C_u^*(|\Gamma|)$ (or, if you like, the index problem associated to the Stone-Cěch compactification) contains an index one Fredholm operator.

Other compactifications lying 'between' these two extremes may or may not have interesting index theory. Many of those associated to the compactifications introduced in the next four sections *do* seem to contain interesting Fredholm operators, despite the fact that examples of the form 2.2.1 do not in general belong to them.

One has to guess that any given index problem of this form is more likely to be meaningfully solvable when X is some sort of 'reasonable' topological (or geometric) object. Indeed, the proof of our main theorem in chapter 5 proceeds via reduction to a more tractable special case: the index problem associated to the spherical compactification of $\mathbb{Z}^n$ as discussed in definition 2.4.3 above.

The next four sections look at some concretely defined compactifications for certain classes of groups.

3.3 Example: slowly oscillating functions and the Higson corona

This section defines some of our main objects of study. It is for the associated class of operators on $l^2(\mathbb{Z}^n)$ that we prove our main theorem in chapter 5. The relevant compactification makes sense on any countable discrete group Γ, however, and is defined as follows.

Definition 3.3.1. A bounded function Γ is called *slowly oscillating* if for all $h \in \Gamma$

$$|f(gh) - f(g)| \to 0 \text{ as } g \to \infty.$$

The slowly oscillating functions form a C^*-subalgebra of $l^\infty(\Gamma)$ containing $C_0(\Gamma)$, which we denote by $C_h(\Gamma)$. The associated compactification is called the *Higson compactification of* Γ and is denoted $\overline{\Gamma}^h$. The associated corona $\overline{\Gamma}^h \backslash \Gamma$ is denoted $\partial_h \Gamma$ and called the *Higson corona of* Γ.

The terminology 'Higson compactification' (as well as all the h sub- and superscripts) are after N. Higson [35], who first introduced the study of such functions into index theory and K-homology. Indeed, slowly oscillating functions are usually called *Higson functions* in these areas. From this point of view, the above is a special case of definition A.8: in the notation of that definition, the coarse space X is the group Γ equipped with its left-invariant coarse structure. We use the *left* invariant structure here as the *left* action of a group on a Higson corona which is defined with respect to the *right* invariant coarse structure is always trivial; for $\mathbb{Z}^n$ (the main case of interest for us) this distinction is not relevant, but for other groups, the case of non-trivial boundary dynamics seemed the more interesting.

Note that $C_h(\Gamma)$ is invariant under the left Γ action on $l^\infty(\Gamma)$ (and also the right action, but we will not use this). In the case that Γ is exact, proposition 3.2.1 thus gives a short exact sequence

$$0 \longrightarrow \mathcal{K}(l^2(\Gamma)) \longrightarrow C_h(\Gamma) \rtimes_r \Gamma \longrightarrow C(\partial_h \Gamma) \rtimes_r \Gamma \longrightarrow 0\ , \qquad (3.2)$$

and associated index problem. Operators in $C_h(\Gamma) \rtimes_r \Gamma$ (concretely realized as a subalgebra of $C_u^*(|\Gamma|)$) are called *band-dominated operators with slowly oscillating coefficients.* V.S. Rabinovich, S. Roch and B. Silbermann have studied these operators in sections 2.4 and 6.2.2 of [56]. The paper [15] of V. M. Deundyak and B. Y. Shteinberg, which is the inspiration behind chapter 5, studies their index theory.

Part of the motivation for studying operators with slowly oscillating coefficients is that they constitute perhaps the largest class of operators on $l^2(\mathbb{Z})$ for which theorem 2.3.3 of Rabinovich, Roch and Roe gives a *computable* index formula; cf. page 223 of [54]. The following lemma is a first step in this direction.

Lemma 3.3.2. *Say $T \in C_u^*(|\mathbb{Z}|)$ has slowly oscillating coefficients. Then all limit operators of T lie in $C_r^*(\mathbb{Z})$, the (reduced) group C^*-algebra of $\mathbb{Z}$.*

Proof. Consider first the case where T is the operator of multiplication by a single slowly oscillating function f. For any $m, l \in \mathbb{Z}$,

$$|(U_m f U_m^* - f)(l)| = |f(l-m) - f(l)| \to 0 \text{ as } l \to \infty$$

by the slow oscillation condition. In other words, $U_m f U_m^* - f$ is in $C_0(\mathbb{Z})$, whence for any 'direction at infinity' $\omega \in \partial_\beta \mathbb{Z}^n$,

$$f^\omega(m) - f^\omega(e) = \omega(U_m f U_m^* - f) = 0.$$

This says that f^ω is a constant function.

Now, by definition-lemma 2.2.6 and part (3) of theorem 2.2.7, this implies that any limit operator of a band-dominated operator with slowly oscillating coefficients is in the C^*-algebra generated by the unitaries $\{U_m : m \in \mathbb{Z}\}$, which is exactly the reduced group C^*-algebra of $\mathbb{Z}$. □

Let F be a Fredholm band-dominated operator with slowly oscillating coefficients. Theorem 2.3.3 says that with any choice of limit operators $F_\pm$ over $\pm\infty$ respectively, we have that

$$Index(F) = Index(QF\_Q + P) + Index(Q + PF_+P).$$

On the other hand, the above lemma says that $F_\pm$ may be considered (via Fourier transform) to be continuous functions on the circle. Hence the operators PF_+P+Q and $Q + PF\_P$ are simply Toeplitz operators (with continuous symbol) and their indices can be computed via the Gohberg-Krein index theorem: they are equal to the (negative) winding numbers of F_+ and $F\_$ respectively. We may restate this discussion as the following theorem.

Theorem 3.3.3. *Say $F \in C_u^*(|\mathbb{Z}|)$ is Fredholm with slowly oscillating coefficients. Let F_+ be* any *limit operator of F along a sequence that converges to $+\infty$, and $F\_$ any limit* *operator of F along a sequence that converges to $-\infty$. Then*

$$(+Ind)(F) = -(winding\ no.)(F_+) \text{ and } (-Ind)(F) = -(winding\ no.)(F_-).$$

In particular,

$$Index(F) = -(winding\ no.)(F_+) - (winding\ no.)(F_-).$$

This special case of theorem 2.3.3 is significantly more computable: for an example see page 223 of [54].

More generally, one can see that all limit operators of band-dominated operators with slowly oscillating coefficients on $\mathbb{Z}^n$ are in $C^*_r(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$. The proof is basically the same as that of lemma 3.3.2 (it depends only, but crucially, on commutativity of $\mathbb{Z}^n$), so we omit it. In particular, the analogue of the exact sequence in line (3.2) in this case looks like

$$0 \longrightarrow \mathcal{K}(l^2(\mathbb{Z}^n)) \longrightarrow C_h(\mathbb{Z}^n) \rtimes_r \Gamma \xrightarrow{\sigma} C(\partial_h \mathbb{Z}^n \times \mathbb{T}^n) \longrightarrow 0\ ; \tag{3.3}$$

in particular, there is a commutative (if still very large) symbol algebra, and one might hope that a *computable* analogue of theorem 3.3.3 might exist in this case. Something along these lines is contained in the paper [15], and forms the content of our theorems 5.0.1 and 6.1.6; these are perhaps the main results of this piece.

3.4 Example: pseudodifferential operators on the n-torus

For the second concrete class of examples, we restrict attention to $\Gamma = \mathbb{Z}^n$ for some n. Here we look at the band-dominated operators whose coefficients have 'spherical limits at infinity' in the sense of definition 2.4.3. The main result of this section relates the short exact sequence of corollary 3.1.4 to the order zero pseudodifferential operator extension for $\mathbb{Z}^n$; it forms a central part of the proof of theorem 5.0.1.

Throughout this section, $\Psi^0(\mathbb{T}^n)$ denotes the C^*-algebraic closure of the order zero pseudodifferential operators on $\mathbb{T}^n$. To be more precise, in the language of section 5 of [4] we are taking the closure of the algebra $\mathcal{P}^0(\mathbb{T}^n)$ in the operator norm of $\mathcal{B}(L^2(\mathbb{T}^n))$. In fact, however, our approach in this section does not need a precise definition of $\Psi^0(\mathbb{T}^n)$, just some well-known fact about this algebra from [4] and [69].

Under Fourier transform $\Psi^0(\mathbb{T}^n)$ is represented on $l^2(\mathbb{Z}^n)$. Line (5.2) in [4] implies that there is a short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K}(l^2(\mathbb{Z}^n)) \longrightarrow \Psi^0(\mathbb{T}^n) \xrightarrow{\sigma_\Psi} C(S^*\mathbb{T}^n) \longrightarrow 0\,. \tag{3.4}$$

In what follows, σ_Ψ is called the *pseudodifferential symbol map.*

On the other hand, the natural action of $\mathbb{Z}^n$ on itself extends to an action on the spherical compactification $\overline{\mathbb{Z}^n}^s$. In particular, proposition 3.2.1 above implies that there is a short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K}(l^2(\mathbb{Z}^n)) \longrightarrow C(\overline{\mathbb{Z}^n}^s) \rtimes_r \mathbb{Z}^n \longrightarrow C(S^{n-1}) \rtimes_r \mathbb{Z}^n \longrightarrow 0\,. \tag{3.5}$$

However, as the action of $\mathbb{Z}^n$ on the boundary S^{n-1} is trivial,

$$C(S^{n-1}) \rtimes_r \mathbb{Z}^n \cong C(S^{n-1}) \otimes C_r^*(\mathbb{Z}^n) \cong C(S^{n-1} \times \mathbb{T}^n) \cong C(S^*\mathbb{T}^n),$$

the second isomorphism being via the Fourier transform. The short exact sequence in line (3.5) above thus becomes

$$0 \longrightarrow \mathcal{K}(l^2(\mathbb{Z}^n)) \longrightarrow C(\overline{\mathbb{Z}^n}^s) \rtimes_r \mathbb{Z}^n \xrightarrow{\sigma_s} C(S^*\mathbb{T}^n) \longrightarrow 0 \,. \tag{3.6}$$

The map σ_s is called the *spherical symbol map* in what follows.

The main result of this section is the following proposition.

Proposition 3.4.1. *Having concretely realized $\Psi^0(\mathbb{T}^n)$ and $C(\overline{\mathbb{Z}^n}^s) \rtimes_r \mathbb{Z}^n$ on $l^2(\mathbb{Z}^n)$ as above, the two algebras are equal. Moreover, the short exact sequences (3.4) and (3.6) are themselves equal.*

This is no doubt well known to experts in the area (there is a comment to this effect on page 267 of [56], and see also page 303 of [34]), but I am not sure if a proof exists in the literature. It was pointed out to me by J. Roe.

The proof we give is basically contained in the next two lemmas. The only non-trivial aspects are some well-known facts from [69] about closure properties of the class of pseudodifferential operators under taking powers.

Lemma 3.4.2. *$C(\overline{\mathbb{Z}^n}^s) \rtimes_r \mathbb{Z}^n$ is contained in $\Psi^0(\mathbb{T}^n)$.*

Proof. Consider the partial differential operators

$$\frac{\partial}{\partial x_i} \text{ for } i = 1, ..., n \text{ and } \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

on $\mathbb{R}^n$. They are invariant for the action of $\mathbb{Z}^n$, so pass to differential operators (and in particular pseudodifferential operators) of orders one and two on $\mathbb{T}^n$.

Now, Δ is positive and elliptic whence, by results of [69], the operator $(1+\Delta)^{-\frac{1}{2}}$ makes sense and is pseudodifferential of order -1. This implies that the operators

$$\frac{\partial}{\partial x_i}(1+\Delta)^{-\frac{1}{2}}, \ i = 1, ..., n$$

are also pseudodifferential, this time of order zero. Under Fourier transform, they correspond to the multiplication operators

$$e_i : m = (m_1, ..., m_n) \mapsto \frac{m_i}{(1+\|m\|_2^2)^{\frac{1}{2}}}$$

acting on $l^2(\mathbb{Z}^n)$.

Further, note that $\sum_{i=1}^{n} e_i(m)^2 = \|m\|_2^2/(1+\|m\|_2^2)$, from which one can recover $\|m\|_2$; hence from all the $e_i(m)$ one can recover each m_i and hence m itself. This says that the algebra generated by $\{e_i : i = 1, ..., n\}$, separates points in $\mathbb{Z}^n$. Continuing, note that

$$\left| e_i(m) = \frac{m_i}{(1+\|m\|_2^2)^{\frac{1}{2}}} - \frac{m_i}{\|m_i\|_2} \right| \to 0$$

as $m \to \infty$; as the functions $m \mapsto m_i/\|m\|_2$, $i = 1, ..., n$, both extend to $\overline{\mathbb{Z}^n}^s$ and separate points on the boundary S^{n-1}, these properties are true of e_i for $i = 1, ..., n$ as well. Hence the e_i generate a C^*-subalgebra of $C(\overline{\mathbb{Z}^n}^s)$, and by the Stone-Weierstrass theorem this C^*-subalgebra is actually the whole thing.

To recapitulate then, $C(\overline{\mathbb{Z}^n}^s)$ is generated by pseudodifferential operators of order zero, whence is contained in $\Psi^0(\mathbb{T}^n)$. On the other hand, $C^*_r(\mathbb{Z}^n)$ (with respect to its natural representation on $l^2(\mathbb{Z}^n)$) is contained in $\Psi^0(\mathbb{T}^n)$ as the subalgebra $C(\mathbb{T}^n)$. As $C^*_r(\mathbb{Z}^n)$ and $C(\overline{\mathbb{Z}^n}^s)$ together generate $C(\overline{\mathbb{Z}^n}^s) \rtimes \mathbb{Z}^n$ as a C^*-subalgebra of $\mathcal{B}(l^2(\mathbb{Z}^n))$, the lemma follows. □

The next lemma says that this inclusion is consistent with the symbol maps σ_Ψ and σ_s from (3.4) and (3.6) respectively.

Lemma 3.4.3. *The restriction of σ_Ψ from $\Psi^0(\mathbb{T}^n)$ to $C(\overline{\mathbb{Z}^n}^s) \rtimes_r \mathbb{Z}^n$ is equal to σ_s.*

Proof. Following the proof of the above lemma, $C(\overline{\mathbb{Z}^n}^s) \rtimes_r \mathbb{Z}^n$ is generated by the e_i and $C^*_r(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$, so it suffices to check that the two symbol maps agree on these elements. In the case of $C(\mathbb{T}^n)$, this is simple: the image of $f \in C(\mathbb{T}^n)$ under either symbol map is f itself pulled back to all of $S^*\mathbb{T}^n$ via the quotient map to $\mathbb{T}^n$.

Consider again the functions e_i, then. Note first that the symbol of the operators

$$\frac{\partial}{\partial x_i}(1+\Delta)^{-\frac{1}{2}},$$

on $\mathbb{R}^n$ is exactly the map

$$\sigma_i : \mathbb{R}^n \times S^{n-1} \to \mathbb{C}$$

$$(\ x\ \ ,\ \ \xi\ ) \ \mapsto \lim_{t\to\infty} \frac{t\xi_i}{(1+\|t\xi\|_2^2)^{\frac{1}{2}}}.$$

In particular, it is *independent* of the x coordinate in $\mathbb{R}^n$. By the local nature of the pseudodifferential symbol, $\sigma_\Psi(e_i) : (S^*\mathbb{T}^n = \mathbb{T}^n \times S^{n-1}) \to \mathbb{C}$ is given by essentially the same formula

$$(x,\xi) \mapsto \lim_{t\to\infty} \frac{t\xi_i}{(1+\|t\xi\|_2^2)^{\frac{1}{2}}}. \tag{3.7}$$

On the other hand, the image of e_i under the quotient map

$$C(\overline{\mathbb{Z}^n}^s) \to C(\overline{\mathbb{Z}^n}^s)/C_0(\mathbb{Z}^n) \cong C(S^{n-1})$$

is given by the formula

$$\xi \mapsto \lim_{k\to\infty} \frac{(m^k)_i}{(1+\|m^k\|_2^2)^{\frac{1}{2}}},$$

where $(m^k)_{k=0}^{\infty}$ is any sequence in $\mathbb{Z}^n$ converging to ξ. In particular, if ξ is of the form $m/\|m\|_2$, then one may take the sequence $m^k = k \cdot m$. This implies that on the dense set of points of the form of the form $(x, m/\|m\|_2)$ in $S^*\mathbb{T}^n \cong \mathbb{T}^n \times S^{n-1}$, $\sigma_s(e_i)$ is given by

$$(x, m/\|m\|_2) \mapsto \lim_{k\to\infty} \frac{km_i}{(1+\|km\|_2^2)^{\frac{1}{2}}},$$

which is the same formula as appearing in line (3.7) above. Hence $\sigma_s(e_i)$ and $\sigma_\Psi(e_i)$ are continuous functions on $S^*\mathbb{T}^n$ agreeing on a dense subset, so they are the same. □

Proof of proposition 3.4.1. The above two lemmas say that the short exact sequences from (3.4) and (3.6) fit into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}(l^2(\mathbb{Z}^n)) & \longrightarrow & \Psi^0(\mathbb{T}^n) & \xrightarrow{\sigma_\Psi} & C(S^*\mathbb{T}^n) & \longrightarrow & 0\ ; \\ & & \| & & \uparrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K}(l^2(\mathbb{Z}^n)) & \longrightarrow & C(\overline{\mathbb{Z}^n}^s) \rtimes_r \mathbb{Z}^n & \xrightarrow{\sigma_s} & C(S^*\mathbb{T}^n) & \longrightarrow & 0 \end{array}$$

this is only possible (by the five lemma, if you like) if the central vertical arrow is also an equality. □

3.5 Example: the visibility boundary of a $CAT(0)$ space

In this section and the next we look at geometrically defined boundaries for certain classes of groups. These definitions are not used for our results on $\mathbb{Z}^n$ in chapter 5 and section 6.1. Nonetheless, we do get certain K-theoretic results on these boundaries in sections 4.6, 4.7 and 6.2 which mimic part of the results for $\mathbb{Z}^n$, as well as being of interest in their own right.

The first of our examples comes from the action of a group on a $CAT(0)$ space. We assume the reader is familiar with the definition and basic properties of $CAT(0)$ spaces (see e.g. [9]). For ease of reference, we make the following definition.

Definition 3.5.1. A discrete group Γ is called a $CAT(0)$ *group* if it acts properly cocompactly (on the left) by isometries on a proper $CAT(0)$ metric space.

In particular, the following compactification (see section II.8 of [9] for more details, in particular definition 8.5) will be our principal object of study.

Definition 3.5.2. Let X be a proper $CAT(0)$ metric space. Fix a basepoint $x_0 \in X$. It follows from our assumptions that for any $s \geq r \geq 0$, there exists a projection

$$p_{s,r} : \bar{B}(x_0, s) \to \bar{B}(x_0, r)$$

defined by mapping $x \in \bar{B}(x_0, s)$ to the unique point at distance r from x_0 on the (unique) geodesic from x to x_0; see proposition II.2.4 from [9]. Moreover, if $t \geq s \geq r$, then $p_{s,r} \circ p_{t,s} = p_{t,r}$, whence the balls $\bar{B}(x_0, r)$ and maps $p_{s,r}$ form a projective system. Define the *visual compactification of* X, denoted $\overline{X}^v$, to be the inverse limit topological space

$$\overline{X}^v = \varprojlim_r \bar{B}(x_0, r).$$

The *visual boundary of* X, denoted $\partial_v X$, is the associated corona $\overline{X}^v \backslash X$.

Say that Γ is a $CAT(0)$ group acting on X as in definition 3.5.1. As the topology on $\partial_v X$ does not depend on the choice of basepoint, it follows that the action of Γ on X extends to an action by homeomorphisms on $\overline{X}^v$; see corollary II.8.8 from [9]. Moreover, if $x_0 \in X$ is some choice of basepoint, we may equivariantly embed Γ into X via the orbit map $g \mapsto gx_0$. Taking the union of this orbit with the visual boundary of X gives a compactification $\overline{\Gamma}^X$ of Γ, which is equivariant for the left action of Γ on itself (and also the right action, but we will not use this).

If we assume moreover that Γ is exact, then there is short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow C(\overline{\Gamma}^X) \rtimes_r \Gamma \longrightarrow C(\partial_v X) \rtimes_r \Gamma \longrightarrow 0$$

as in proposition 3.2.1 above.

The following examples are the main motivations.

Examples 3.5.3.

- Let X be a simply connected Riemannian manifold of non-positive sectional curvature. It is a proper $CAT(0)$ space (cf. the appendix to chapter II.1 from [9]). Fix a basepoint $x_0 \in X$. Then the Cartan-Hadamard theorem states that the exponential map $exp : T_{x_0}X \to X$ is a diffeomorphism. Moreover, its restriction to the unit sphere induces a homeomorphism of S^{n-1} with the visual boundary of X.

- In particular, say Γ is the fundamental group of a closed manifold M of non-positive sectional curvature. Then the universal cover X of M is as in the above example, and Γ is thus a $CAT(0)$ group. The previous section can be thought of as the case $M = \mathbb{T}^n$, $X = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$.

- Let G be an almost connected Lie group whose connected component is linear and reductive. Let K be a maximal compact subgroup. Then the symmetric space $X = G/K$ is a manifold of non-positive curvature, so $CAT(0)$ by the above; see chapter II.10 of [9] and references contained there. If Γ is a cocompact lattice in G, then it acts properly cocompactly by isometries on G/K, so is a $CAT(0)$ group. It is moreover automatically exact by results of [30].

- Let X be an affine Bruhat-Tits building, which is a proper $CAT(0)$ space. Then its visual compactification is the same as its *Borel-Serre* compactification; [9], appendix to chapter II.10 and references. This can be used to get results like the above on cocompact lattices in p-adic groups.

3.6 Example: the Gromov boundary of a hyperbolic group

In this section, we discuss the case of *word hyperbolic groups*; see [27]. This class of groups is automatically exact, which follows for example from [66]. We again assume the reader is familiar with these objects (see for example [26], [27] and chapter III.H of [9]). Nonetheless, we give some (slightly non-standard) definitions below to fix conventions.

Definition 3.6.1. Let (X, d) be a metric space and e any point in X (the notation is meant to be reminiscent of the identity element in a discrete group). If x, y are any elements of X, then the *Gromov product of x and y with respect to e*, denoted $\langle x|y\rangle_e$, is defined by

$$\langle x|y\rangle_e = \frac{1}{2}(d(x,e) + d(y,e) - d(x,y));$$

if the basepoint e is understood, we often omit the subscript '$_e$', and write $|x|$ for $d(x, e)$.

Definition 3.6.2. A metric space (X, d) is called *discretely geodesic* if:

- the metric d is integer valued;
- for any $x, y \in X$ there exists a sequence of points $x_0 = x, ..., x_{d(x,y)} = y$ such that $d(x_i, x_i + 1) = 1$ for all $i = 0, ..., d(x, y) - 1$.

A sequence of points as in the second line is called a *geodesic segment* in X.

Definition 3.6.3. Let (X, d) be a proper, discretely geodesic metric space equipped with a basepoint e. If there exists $\delta \geq 0$ such that for all $x, y, z \in X$

$$\langle x|y\rangle \geq \min\{\langle x|z\rangle, \langle y|z\rangle\} - \delta$$

then X is said to be δ-*hyperbolic*. X is said to be *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

Remarks 3.6.4.

1. It is not usual to demand that hyperbolic spaces are discretely geodesic! This simplifies certain technical details in section 4.7 below, however. Moreover, our most important class of examples consists of word hyperbolic groups (see the next comment), and these automatically satisfy the conditions above. One can recover a genuinely geodesic space from a discretely geodesic one by attaching a copy of the interval $[0, 1]$ between any points $x, y \in X$ that satisfy $d(x, y) = 1$.

2. If X and Y are coarsely equivalent discretely geodesic metric spaces, with X hyperbolic then Y is also hyperbolic. In particular, one may say that a finitely generated group Γ is (word) hyperbolic if it is a hyperbolic space with respect to a word metric associated to some (finite) choice of generating set, and this does not depend on the choice involved.

3. Let X be a proper quasi-geodesic metric space in the sense of [26] or [27]. Then it is coarsely equivalent (in fact quasi-isometric) to one of the form in definition 3.6.3 above. In particular, there is no real loss of generality in our definition.

4. In fact, one can define the property of 'being a hyperbolic space' *purely* in terms of a coarse structure on a space X: roughly, one demands that the coarse structure be singly generated (*monogenic* in the language of [64], page 34) whence X is coarsely equivalent to a discretely geodesic space (cf. proposition 2.57 of [64]); one then translates the definitions above into the language of (monogenic) coarse structures. We did not do this because restricting to the language of metric spaces probably makes section 4.7 easier to read, and also as any 'hyperbolic coarse space' is automatically metrizable.

The definition below introduces the *Gromov compactification* of a hyperbolic space. It also outlines some facts that we will need in section 4.7; see chapter 7 of [26] for more details and proofs.

Definition 3.6.5. Let X be a hyperbolic metric space. A *geodesic ray* in X is an isometric map $\gamma : \mathbb{N} \to X$. It is *based at* $x \in X$ if $\gamma(0) = x$. If x, y are any points in X and γ_1, γ_2 are geodesic rays, then the Gromov products with respect to y of x and γ_1, and of γ_1 and γ_2 are defined to be

$$\langle \gamma_1 | x \rangle_y = \lim_{n \to \infty} \langle \gamma_1(n) | x \rangle_y \text{ and}$$

$$\langle \gamma_1 | \gamma_2 \rangle_y = \lim_{n \to \infty} \langle \gamma_1(n) | \gamma_2(n) \rangle_y$$

if these exist, and infinity otherwise. Note that the expressions on the right-hand-side are non-decreasing in n; in particular, the limits exist if they are bounded in n. If the products are taken with respect to the fixed basepoint e of X, then the subscripts will be omitted.

Two geodesic rays γ_1, γ_2 based at e are said to be *equivalent* if their Gromov product (with respect to e) is infinity. The set of equivalence classes $[\gamma]$ of geodesic rays based at e is called the *Gromov boundary of* X and is denoted $\partial_g X$. It can be equipped with a metric d such that

$$c^{-1}e^{-\epsilon\langle\gamma_1|\gamma_2\rangle} \leq d([\gamma_1],[\gamma_2]) \leq ce^{-\epsilon\langle\gamma_1|\gamma_2\rangle} \tag{3.8}$$

for some constants $c, \epsilon > 0$. This metric defines a compact topology on $\partial_g X$.

Finally, the disjoint union of X and $\partial_g X$ can be topologised in such a way that a sequence (x_n) in X converges to $[\gamma] \in \partial_g X$ if and only if

$$\langle x_n|\gamma\rangle \to \infty \text{ as } n \to \infty.$$

The resulting space is a compactification of X called the *Gromov compactification* and denoted $\overline{X}^g$. If X is a word hyperbolic group equipped with a left-invariant word metric, then the natural left action of Γ on itself extends to an action on the Gromov compactification (this is also true of the right action, but we will not use it).

Just as in the case of the Higson corona, we use a *left*-invariant metric (i.e. one that generates the left invariant coarse structure) to define the Gromov boundary of a group; this is as the left action on the boundary defined with respect to a right-invariant word metric is trivial.

Examples 3.6.6.

- Say Γ is the fundamental group of a closed n-dimensional manifold of strictly negative sectional curvature. Then Γ is word hyperbolic, and its Gromov boundary is homeomorphic to S^{n-1}.

- More generally, if Γ acts properly cocompactly by isometries on a simply connected $CAT(-\kappa)$ space X for some $\kappa > 0$, then Γ is word hyperbolic and the Gromov boundary of Γ coincides with the visual boundary of X (see also 1.4 and 1.5 from chapter III.H in [9]).

- From a different, somewhat combinatorial viewpoint, certain *small cancellation* groups are hyperbolic. See for example the appendix to [26].

- In a certain sense, word hyperbolic groups are 'generic' amongst all finitely presented groups; see [51] for a precise statement and proof.

- As far as I am aware, there are no known examples of hyperbolic groups that do not act properly cocompactly on a $CAT(0)$ space; see section 1 of [8]. On the other hand, there are certainly $CAT(0)$ groups that are not hyperbolic (for example, $\mathbb{Z}^n$). Nonetheless, it seemed worthwhile to include hyperbolic groups separately, partly as hyperbolicity is a well-behaved coarse invariant of a group, while the $CAT(0)$ property is not.

Chapter 4

The stable Higson corona

Recall that we are aiming for an index theorem for band-dominated operators with slowly oscillating coefficients on $\mathbb{Z}^n$. An attempt at a K-theoretic proof of this fact together with the short exact sequence in line (3.3) above naturally leads us to study the group $K^1(\partial_h \mathbb{Z}^n \times \mathbb{T}^n)$. Unfortunately, a theorem of Keesling (stated as theorem 4.1.4 below) implies that this group is enormous, making it very difficult to deal with for the sort of proof we would like to give.

Playing historical revisionist, this section is motivated by the first discussion from section 2.4, which pointed out that while $K_*(C^*_u(|\mathbb{Z}^n|))$ is a mess, $K_*(C^*(|\mathbb{Z}^n|))$ is well-behaved. Somehow this suggests (being extremely vague) that 'operators (or functions) with values in the compact operators behave better'.

Continuing with the ill-formed (but often fruitful - cf. Karoubi's conjecture in algebraic K-theory) philosophy in the paragraph above, in this section we will pass from the Higson corona to an analogous object 'with values in the compacts'. This object, a C^*-algebra called the *stable Higson corona*, was introduced by H. Emerson and R. Meyer in a completely different context. Somewhat remarkably, the stable Higson corona really *does* have much better K-theoretic properties than its 'unstable' progenitor. Thus the comparison between the two can in fact be considered as an analogue of the above situation of ‘$C^*(|\mathbb{Z}^n|)$ versus $C_u^*(|\mathbb{Z}^n|)$’. There is a limit to how far this analogy can be pushed, however: see part (3) of remarks 4.2.8 below.

The main portion of this chapter is devoted to studying the K-theory of the stable Higson corona. Most of the K-theoretic results that we prove below follow from those of Emerson and Meyer in [22]. Our methods are different and perhaps simpler, however, in particular using no KK-theory. Special cases of these computations then play a central rôle in the proof of theorem 5.0.1 on $\mathbb{Z}^n$, and we hope that certain of the more general cases will eventually lead to index theorems on a broader class of groups.

The stable Higson corona was in fact introduced by Emerson and Meyer as part of their work on the Novikov and Baum-Connes conjectures; see [24]. The results we get below could be used via ideas of Emerson, Meyer and others to reprove results on Novikov and Baum-Connes: in particular, they imply split injectivity of the Baum-Connes assembly map for a large class of ‘non-positively curved’ groups. Although these results are by now well-known, we hope the proofs below are of some interest; these connections provide another motivation for many of the results in this chapter.

In order to give some background on index theoretic approaches to the Novikov conjecture, and an overview of the K-theory groups of the Higson corona, we first look at the ‘classical’ (i.e. early 90s!) links between these.

4.1 The ‘Higson conjecture’

The so-called ‘Higson conjecture’ (it is not entirely due to N. Higson, and turned out to be false, hence the inverted commas) is related to index theoretic approaches to the Novikov conjecture. Throughout this section, we use the general definition (A.8) of the Higson corona. Similarly, when discussing the Roe algebra, we use definition A.9.

To motivate the Higson conjecture, assume for simplicity that Γ is a group with classifying space $B\Gamma$ realized by a closed manifold (see [41] for a more general treatment). Then if $E\Gamma$ is the universal cover of $B\Gamma$, there is a *coarse assembly map*

$$\mu : K_*(E\Gamma) \to K_*(C^*(|\Gamma|)) \tag{4.1}$$

(see chapter 12 of [43]). The left-hand-side is the K-homology of $E\Gamma$, which is generated by classes of elliptic operators over this space. The assembly map takes the class of an elliptic operator to its 'higher index', as discussed briefly in example 2.1.8.

Now, the image of a class $[D] \in K_*(E\Gamma)$ under the assembly map is a homotopy invariant. In particular, injectivity of μ implies that the class $[D_s] \in K_*(E\Gamma)$ of the lift of the signature operator from $B\Gamma$ to $E\Gamma$ is a homotopy invariant. Thus injectivity of μ can be thought of as a Novikov-type statement, and indeed implies the Novikov conjecture in favourable circumstances; see for example [63].

One approach to proving injectivity of μ proceeds via a study of the Higson corona. Indeed, there is a pairing between the K-theory of the Higson corona $C(\partial_h\Gamma)$ and that of the Roe algebra $C^*(|\Gamma|)$; in the language of section 5.3 of [43], this comes down to the fact that

$$C^*(|\Gamma|) \subseteq \mathcal{D}(C_h(\Gamma)//C_0(\Gamma)),$$

with respect to any fixed ample representation of $C_0(\Gamma)$. See for example proposition 5.18 from [62] for a proof.

Consider now the short exact sequence

$$0 \longrightarrow C_0(E\Gamma) \longrightarrow C_h(E\Gamma) \longrightarrow C(\partial_h E\Gamma = \partial_h \Gamma) \longrightarrow 0\ , \tag{4.2}$$

which gives rise to a K-theory boundary map $\partial : K^*(\partial_h\Gamma) \to K^{*+1}(E\Gamma)$. As moreover $K^*(E\Gamma)$ pairs with the K-homology group $K_*(E\Gamma)$ in the usual way, we get a diagram of pairings

$$\begin{array}{ccccc} K_*(E\Gamma) & & \xrightarrow{\ \mu\ } & & K_*(C^*(|\Gamma|)) \\ \downarrow & & & & \downarrow \\ \otimes & \longrightarrow & \mathbb{Z} & \longleftarrow & \otimes \\ \uparrow & & & & \uparrow \\ K^*(E\Gamma) & & \xleftarrow[\ \partial\ ]{} & & K^{*-1}(\partial_h\Gamma) \end{array} \tag{4.3}$$

(indices of K-groups are taken mod 2).

The following is proved in section 9 of [41] using the construction of the coarse assembly map in terms of Paschke duality. It in turn builds on cohomological ideas from chapter 5 of [62].

Theorem 4.1.1. *If $x \in K_*(E\Gamma)$ and $y \in K^{*-1}(\partial_h\Gamma)$, then*

$$\langle x, \partial(y)\rangle = \langle \mu(x), y\rangle;$$

in other words, diagram 4.3 commutes. □

Corollary 4.1.2. *If ∂ is rationally surjective, then μ is rationally injective.*

Proof. The pairing between $K_*(E\Gamma)$ and $K^*(E\Gamma)$ becomes nondegenerate after tensoring with $\mathbb{Q}$. □

The statement that ∂ is surjective is (a version of what is) called the *Weinberger conjecture*; see 6.33 in [62]. Versions of it have been established for certain classes of groups: see for example [61] and [19]. Also results of section 3 of [17] relate statements of this form to a manifold $E\Gamma$ being hypereuclidean.

The Higson conjecture can then be stated as follows (see 6.35 in [62]).

Conjecture 4.1.3. *The K-theory of $C_h(E\Gamma)$ as in the above situation is $\mathbb{Z}$ (i.e. the inclusion of the constant functions induces an isomorphism on K-theory).*

The six term exact sequence in K-theory associated to the short exact sequence in line (4.2) above shows that the Higson conjecture implies the Weinberger conjecture. That the Higson conjecture might be true is suggested by the fact that the Higson compactification is somewhat similar to the Stone-Cěch compactification, and the latter does have cohomological properties similar to those in the statement of the conjecture.

Unfortunately the Higson conjecture is completely false (see however [18] for an attempt to salvage it by considering only cohomology with finite coefficients). The following theorem is due to Keesling [47] (see also [17]).

Theorem 4.1.4. *With $E\Gamma$ as above, $K_*(C_h(E\Gamma))$ is always uncountable.* □

Amazingly, however, the 'stable version' of the Higson conjecture (i.e. the analogue of 4.1.3 for the 'stable Higson compactification' of Emerson and Meyer stated in 4.2.7 below) *is* known to be true for many groups. Moreover, according to work of Emerson and Meyer it has an even closer connection to the Novikov and Baum-Connes conjectures than the usual Higson conjecture. It is the purpose of the next section to exposit some of their work, after which the rest of this chapter establishes some properties of the stable Higson corona, including certain cases of the 'stable Higson conjecture'.

4.2 The 'stable Higson conjecture' and coarse co-assembly map

Following [24], in this section we will define the stable Higson corona, and sketch out its relationship with the Baum-Connes and Novikov conjectures. This relationship involves the 'stable Higson conjecture'; the inverted commas are there as the somewhat ad-hoc terminology was made up for the current exposition.

It may be helpful to compare the definition below with that of the ('unstable') Higson corona for general coarse spaces given in A.8.

Definition 4.2.1. Let X be a coarse space and $\mathcal{K}$ an abstract copy of the compact operators on a separable, infinite-dimensional Hilbert space.

If $f : X \to \mathcal{K}$ is a bounded function and E is an entourage on X the *variation of f at scale E* is defined to be

$$(\nabla_E f)(x) := \sup_{(x,y)\in E} \|f(x) - f(y)\|.$$

A function $f : X \to \mathcal{K}$ is called a *Higson function* if it is bounded, continuous and if for any entourage E, $(\nabla_E f)(x) \to 0$ as $x \to \infty$ in X.

Equipped with pointwise operations and the supremum norm, the Higson functions form a C^*-algebra, which it is denoted $\bar{\mathfrak{c}}(X)$ and called the *stable Higson compactification of X*. Note that $\bar{\mathfrak{c}}(X)$ contains $C_0(X, \mathcal{K})$ as an ideal; the resulting quotient is denoted

$$\mathfrak{c}(X) := \frac{\bar{\mathfrak{c}}(X)}{C_0(X, \mathcal{K})}$$

and called the *stable Higson corona of* X.

Note that the stable Higson corona is in general much larger than $C(\partial_h X, \mathcal{K})$ (which it contains as a C^*-subalgebra); roughly speaking, the difference is that functions in the latter algebra must have compact image in $\mathcal{K}$.

The next definition gives 'reduced' versions of the above objects that will be very useful for computations. The point (made precise in proposition 4.2.3 below) is that they remove the contribution of the constant functions at the level of K-theory.

Definition 4.2.2. Fix a separable, infinite dimensional Hilbert space $\mathcal{H}$, and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. The *reduced stable Higson compactification*, denoted $\bar{\mathfrak{c}}^r(X)$, is the C^*-algebra of all bounded continuous Higson functions $f : X \to \mathcal{B}(\mathcal{H})$ such that $f(x) - f(y) \in \mathcal{K}(\mathcal{H})$ for all $x, y \in X$.

Note that $\bar{\mathfrak{c}}^r(X)$ contains $C_0(X, \mathcal{K})$ as an ideal; the resulting quotient is denoted $\mathfrak{c}^r(X)$ and called the *reduced stable Higson corona of* X.

The following proposition records some basic properties of these objects; it combines proposition 13, lemma 16 and proposition 33 from [22]. Note that proposition 13 from [22] only proves part (2) for the subcategory of $\mathcal{C}$ with morphisms given by equivalence classes of *Borel* coarse maps; one can, however, extend to all of $\mathcal{C}$ using remark A.6.

Proposition 4.2.3. *1. $X \mapsto \bar{\mathfrak{c}}(X)$ and $X \mapsto \bar{\mathfrak{c}}^r(X)$ define contravariant functors on the category $\mathcal{CC}$ of definition A.10.*

2. $X \mapsto \mathfrak{c}(X)$ *and* $X \mapsto \mathfrak{c}^r(X)$ *define contravariant functors on the category* $\mathcal{C}$ *of definition A.10.*

3. *If* X *is non-compact, the inclusions* $\mathcal{K} \to \bar{\mathfrak{c}}(X)$ *and* $\mathcal{K} \to \mathfrak{c}(X)$ *as constant functions induce injections on* K*-theory. Moreover, there are natural isomorphisms of graded abelian groups*

$$K_*(\bar{\mathfrak{c}}^r(X)) \cong \frac{K_*(\bar{\mathfrak{c}}(X))}{K_*(\mathcal{K})} \text{ and } K_*(\mathfrak{c}^r(X)) \cong \frac{K_*(\mathfrak{c}(X))}{K_*(\mathcal{K})}$$

for any non-compact X. □

Remark 4.2.4. Note that part (3) of the above fails if X is compact. Indeed, in this case $\mathfrak{c}(X) = 0$, while it is not hard to see that $\mathfrak{c}^r(X)$ is isomorphic to the Calkin algebra (on a separable, infinite dimensional Hilbert space). In particular

$$K_i(\mathfrak{c}^r(X)) = \begin{cases} 0 & i = 0 \\ \mathbb{Z} & i = 1 \end{cases} .$$

Now, say Γ is a discrete group and assume for simplicity that the classifying space $B\Gamma$ is realized by a finite CW complex (see [22], particularly section 4, for a more general treatment). The construction of the reduced stable Higson corona gives rise to a short exact sequence of C^*-algebras

$$0 \longrightarrow C_0(E\Gamma, \mathcal{K}) \longrightarrow \bar{\mathfrak{c}}^r(E\Gamma) \longrightarrow \mathfrak{c}^r(E\Gamma) \cong \mathfrak{c}^r(\Gamma) \longrightarrow 0\,, \tag{4.4}$$

whence in turn a six-term exact sequence in K-theory.

Definition 4.2.5. Say Γ has finite classifying $B\Gamma$ realized by a finite CW complex. The *coarse co-assembly map*

$$\mu^* : K_*(\mathfrak{c}^r(\Gamma)) \to K^{*+1}(E\Gamma)$$

is the boundary map associated to the six term exact sequence above (we have used the stability of K-theory to identify $K^{*+1}(E\Gamma) \cong K_{*+1}(C_0(E\Gamma, \mathcal{K}))$.

There is also a natural pairing between the reduced stable Higson corona and Roe algebra of a coarse space X; see the proof of theorem 36 in [22]. With respect to this pairing, one has the analogue of theorem 4.1.1 (which justifies the name 'coarse co-assembly map'). A proof is sketched in theorem 36 of [22].

Theorem 4.2.6. *Let Γ be a discrete group with $B\Gamma$ realized by a finite CW complex Then with respect to the pairing between $K^*(E\Gamma)$ and $K_*(E\Gamma)$ and the pairing between $K_*(C^*(|\Gamma|))$ and $K_{*-1}(\mathfrak{c}^r(\Gamma))$ one has that*

$$\langle \mu(x), y \rangle = \langle x, \mu^*(y) \rangle$$

for all $x \in K_*(E\Gamma)$ and $y \in K_{*-1}(\mathfrak{c}^r(\Gamma))$. □

We state the stable Higson conjecture as follows.

Conjecture 4.2.7. *The K-theory of $\bar{\mathfrak{c}}^r(E\Gamma)$ as in the above situation is* 0.

It implies that the coarse co-assembly is an isomorphism, and thus as in corollary 4.1.2 above that the coarse assembly map in line (4.1) is rationally injective. The following remarks sum up some connections between the pairings discussed in this section and the last.

Remarks 4.2.8. 1. There is a natural inclusion $i : C(\partial_h X) \otimes \mathcal{K} \to \mathfrak{c}^r(X)$. It follows immediately from the formula for the pairing between $K^{*-1}(\partial_h X)$ and $K_*(C^*(X))$ and that for the pairing between $K_{*-1}(\mathfrak{c}^r(X))$ and $K_*(C^*(X))$ that they commute with i in the sense that

$$\langle x, y \rangle = \langle i_*(x), y \rangle$$

for all $x \in K^{*-1}(\partial_h X)$ and $y \in K_*(C^*(X))$. In particular, then, rational nondegeneracy of the pairing between $K^*(E\Gamma)$ and $K_*(E\Gamma)$ implies that

$$\partial : K^{*-1}(\partial_h \Gamma) \to K^*(E\Gamma)$$

is rationally surjective (the 'Weinberger conjecture' holds) if and only if the coarse co-assembly map is rationally surjective. In particular, this suggests that from the point of view of the Novikov conjecture the (often uncountably many) classes in $K^*(\partial_h E\Gamma)$ that are mapped to zero by i_* are relatively uninteresting.

2. If both the coarse assembly and coarse co-assembly maps for a group (with, say, finite classifying space) are isomorphisms, then the pairing between $K_{*-1}(\mathfrak{c}^r(\Gamma))$ and $K_*(C^*(|\Gamma|))$ is rationally nondegenerate.

3. There is also a pairing between $K^{*-1}(\partial_h \Gamma)$ and $K_*(C^*_u(|\Gamma|))$; given the analogies at the start of this chapter, it seems natural to ask if there are circumstances under which (a reduced version of) this pairing is non-degenerate. This fails badly, however, as one can compute directly in the case $\Gamma = \mathbb{Z}$.

Emerson and Meyer also get a result on the relationship between the coarse co-assembly map and the Baum-Connes conjecture that is in a sense rather more precise than the connections discussed above; see [24], corollary 36. The statement below gives an important special case.

Theorem 4.2.9. *Say Γ has finite classifying space $B\Gamma$, and that the coarse co-assembly map as in definition 4.2.5 is an isomorphism (equivalently, that the stable Higson conjecture, $K_*(\bar{\mathfrak{c}}^r(E\Gamma)) = 0$, holds).*

Then Γ has a dual Dirac morphism in the sense of [50]. In particular, the Baum-Connes assembly map for Γ is split injective (with any coefficients), and the (strong) Novikov conjecture holds for Γ. □

4.3 Homological properties of the stable Higson corona

In this section we establish some 'homological properties' of the functors $X \mapsto K_*(\mathfrak{c}^r(X))$ and $X \mapsto K_*(\bar{\mathfrak{c}}^r(X))$ on the categories $\mathcal{C}$ and $\mathcal{CC}$ of definition A.10. The main results are theorems 4.3.12 and 4.3.13, which say that these two functors have certain 'coarse homotopy invariance properties'; cf. [40], section 11 of [39] and chapter 12 of [43] for some similar results in other contexts. In the remaining sections of this chapter, these will be used to compute K-theory groups of some examples; these computations also imply the corresponding cases of the stable Higson conjecture 4.2.7.

Most of the ideas we use come from [44], [43] and [22], and many of the proofs are adapted from these papers and book. Using this approach in this context was suggested to me by J. Roe.

From now on, if $\phi : X \to Y$ is a coarse map, then we denote by

$$\phi^{\sharp} : \mathfrak{c}^r(Y) \to \mathfrak{c}^r(X) \text{ and}$$

$$\phi^{*} : K_*(\mathfrak{c}^r(Y)) \to K_*(\mathfrak{c}^r(X))$$

its image under the functors $\mathfrak{c}^r(\cdot)$ and $K_*(\mathfrak{c}^r(\cdot))$ respectively. We use the same notation for the images of continuous coarse maps under the reduced stable Higson compactification functor and its composition with the K-theory functor, as well as the unreduced versions of all these functors.

We first establish a Mayer-Vietoris type sequence for the functor $X \mapsto K_*(\mathfrak{c}^r(X))$. This requires a notion of excision appropriate to coarse spaces, which is taken from [44], section 1, definition 1.

Definition 4.3.1. Let X be a coarse space. For any subset Y of X and entourage E, let

$$N(Y, E) = \{x \in X : (x, y) \in E \text{ for some } y \in Y\}$$

be the E-neighbourhood of Y in X.

Let $X = A \cup B$ be a decomposition of X into closed subspaces. It is called *coarsely excisive* if for all entourages E on X, there exists an entourage F such that

$$N(A, E) \cap N(B, E) \subseteq N(A \cap B, F).$$

The next lemma is closely related to proposition 1 from section 2 of [44].

Lemma 4.3.2. *Let $X = A \cup B$ be a coarse space equipped with a decomposition into closed subsets. Let $i_A : A \to X$, $i_B : B \to X$, $j_A : A \cap B \to A$, $j_B : A \cap B \to B$ be the associated inclusion maps.*

Then if $X = A \cup B$ is coarsely excisive there is a pullback diagram

$$\begin{array}{ccc} \mathfrak{c}^r(X) & \xrightarrow{i_A^\sharp} & \mathfrak{c}^r(A) \\ \downarrow{\scriptstyle i_B^\sharp} & & \downarrow{\scriptstyle j_A^\sharp} \\ \mathfrak{c}^r(B) & \xrightarrow{j_B^\sharp} & \mathfrak{c}^r(A \cap B) \end{array} \quad . \tag{4.5}$$

Proof. Note first that pullbacks always exist in the category of C^*-algebras and $*$-homomorphisms: the pullback P over the diagram

$$\begin{array}{ccc} P & \longrightarrow & \mathfrak{c}^r(A) \\ \downarrow & & \downarrow{\scriptstyle j_A^\sharp} \\ \mathfrak{c}^r(B) & \xrightarrow{j_B^\sharp} & \mathfrak{c}^r(A \cap B) \end{array}$$

can be concretely constructed as

$$P = \{(a, b) \in \mathfrak{c}^r(A) \oplus \mathfrak{c}^r(B) : j_A^\sharp(a) = j_B^\sharp(b)\},$$

equipped with the obvious projection maps to $\mathfrak{c}^r(A)$ and $\mathfrak{c}^r(B)$.

If Y is a coarse space, denote the image of $f \in \bar{\mathfrak{c}}^r(Y)$ under the quotient map to $\mathfrak{c}^r(Y)$ by $[f]$. To show the existence of the diagram (4.5) above it suffices to prove that the map $\mathfrak{c}^r(X) \to P$ defined by

$$[f] \mapsto ([f|_A], [f|_B]) \tag{4.6}$$

is an isomorphism of C^*-algebras. It is not hard to check that it is a well-defined, injective $*$-homomorphism. Surjectivity is shown below.

Say then that $([g_A], [g_B])$ is an element of P. It follows from the definition of P that there exists $h \in C_0(A \cap B, \mathcal{K})$ such that $g_A - g_B = h$ on $A \cap B$. Surjectivity of the restriction map $C_0(B, \mathcal{K}) \to C_0(A \cap B, \mathcal{K})$ implies that h extends to an element of $C_0(B, \mathcal{K})$, which will also be denoted h. Define a function $f : X \to \mathcal{K}$ by

$$f(x) = \begin{cases} g_A(x) & x \in A \\ g_B(x) + h(x) & x \in B \end{cases}.$$

This is well-defined as $g_A(x) = g_B(x) + h(x)$ on $A \cap B$. If f is in $\bar{\mathfrak{c}}^r(X)$, then $[f] \in \mathfrak{c}^r(X)$ maps to $([g_A], [g_B])$ under the $*$-homomorphism in line (4.6) above. Now, f is continuous and bounded as g_A and $g_B + h$ are, and as they agree on the closed subset of X where both are defined. Moreover, assuming A and B are non-empty (indeed, if one is empty the lemma is trivial), the coarse excisiveness condition forces $A \cap B$ to be non-empty, whence $f(x) - f(y)$ is compact for all $x, y \in X$. It thus suffices to show that f is of vanishing variation.

To this end, let E be an entourage and $\epsilon > 0$. Then there exists an entourage F containing E such that

$$N(A, E) \cap N(B, E) \subseteq N(A \cap B, F).$$

Moreover, by vanishing variation of g_A on A and $g_B + h$ on B, there exist compact subsets $K_A \subseteq A$, $K_B \subseteq B$ such that for all $a \in A \backslash K_A$ (resp. $b \in B \backslash K_B$)

$$|(\nabla_F g_A)(a)| < \epsilon \quad (\text{resp. } |(\nabla_F (g_B + h))(b)| < \epsilon).$$

Adopting the convention that if one of $(\nabla_F f)(x)$, $(\nabla_F g)(x)$ doesn't make sense then we just set it zero, it follows that for all $x \in X \backslash (K_A \cup K_B)$,

$$|(\nabla_E f)(x)| \leq \max\{|(\nabla_F g_A)(x)|, |(\nabla_F (g_B + h))(x)|\} < \epsilon.$$

This shows that f has vanishing variation, completing the proof. □

One also has the corresponding lemma for the compactification functor.

Lemma 4.3.3. *Let $X = A \cup B$ be a proper metric space equipped with a decomposition into closed subsets. Let $i_A : A \to X$, $i_B : B \to X$, $j_A : A \cap B \to A$, $j_B : A \cap B \to B$ be the associated inclusion maps.*

Then if $X = A \cup B$ is coarsely excisive there is a pullback diagram

$$\begin{array}{ccc} \bar{\mathfrak{c}}^r(X) & \xrightarrow{i_A^\sharp} & \bar{\mathfrak{c}}^r(A) \\ \downarrow{\scriptstyle i_B^\sharp} & & \downarrow{\scriptstyle j_A^\sharp} \\ \bar{\mathfrak{c}}^r(B) & \xrightarrow{j_B^\sharp} & \bar{\mathfrak{c}}^r(A \cap B) \end{array} \quad . \tag{4.7}$$

Proof. Essentially the same as (and slightly easier than) that of lemma 4.3.2 □

The second ingredient needed to establish a Mayer-Vietoris sequence is surjectivity of the restriction maps $j_A^\sharp$ and $j_B^\sharp$. Unfortunately I do not know a proof of this in general, but one does have the following lemma for the case of metric spaces.

Lemma 4.3.4. *Let X be a proper metric space and $A \subseteq X$ a closed subspace. Then the restriction $*$-homomorphism*

$$\bar{\mathfrak{c}}^r(X) \to \bar{\mathfrak{c}}^r(A)$$

is surjective.

The proof of lemma 4.3.4 is somewhat technical, and we do not need it to prove our main homotopy invariance results; it is thus postponed until the end of the section. Note that the analogous result for the 'unstable' Higson corona follows in full generality by the same argument as used to prove theorem 1.4 from [19].

The existence of Mayer-Vietoris sequences as set out below follows from the two preceding lemmas.

Proposition 4.3.5. *Let $X = A \cup B$ be a coarsely excisive decomposition and i_A, i_B, j_A, j_B be as in the lemma 4.3.2. Assume that $j_A^\sharp$ and $j_B^\sharp$ are surjective (so in particular this holds if X is metrizable by lemma 4.3.4).*

Then there exists a Mayer-Vietoris sequence

$$\longrightarrow K_i(\mathfrak{c}^r(X)) \xrightarrow{i_A^* \oplus i_B^*} K_i(\mathfrak{c}^r(A)) \oplus K_i(\mathfrak{c}^r(B)) \xrightarrow{j_A^* - j_B^*} K_i(\mathfrak{c}^r(A \cap B)) \longrightarrow .$$

It is moreover natural for morphisms of excisive decompositions in the category $\mathcal{C}$ in the obvious sense.

Proposition 4.3.6. *Let $X = A \cup B$ be a coarsely excisive decomposition and i_A, i_B, j_A, j_B be as in the lemma 4.3.3. Assume that $j_A^\sharp$ and $j_B^\sharp$ are surjective (so in particular this holds if X is metrizable by lemma 4.3.4).*

Then there exists a Mayer-Vietoris sequence

$$\longrightarrow K_i(\bar{\mathfrak{c}}^r(X)) \xrightarrow{i_A^* \oplus i_B^*} K_i(\bar{\mathfrak{c}}^r(A)) \oplus K_i(\bar{\mathfrak{c}}^r(B)) \xrightarrow{j_A^* - j_B^*} K_i(\bar{\mathfrak{c}}^r(A \cap B)) \longrightarrow .$$

It is moreover natural for morphisms of excisive decompositions in the category $\mathcal{CC}$ in the obvious sense.

Emerson and Meyer point out that something like this exists ([22] remark 35), but do not develop it any further.

Proof. This is a special case of the general notion of Mayer-Vietoris sequence associated to a pullback diagram of C^*-algebras (see for example [7], 21.3.2) such that the right-hand and bottom maps are surjective. The naturalness follows from naturalness for this general Mayer-Vietoris sequence with respect to morphisms of pullback diagrams. □

These results are of interest in of themselves, but we will only use a very special case of them as a crutch to theorems 4.3.12 and 4.3.13.

The second ingredient used in the proofs of these results is a vanishing theorem for *coarsely flasque* spaces (cf. [43], lemma 6.4.2). These are defined below.

Definition 4.3.7. Let X be a coarse space. It is *coarsely flasque* if there exists a continuous coarse map $\phi : X \to X$ such that:

- ϕ is close to the identity in the sense of definition A.4;
- for all compact subsets $K \subseteq X$ there exists $N_K \in \mathbb{N}$ such that for all $n \geq N_K$ and $x \in X$, $\phi^n(x) \notin K$;
- for all entourages E there exists an entourage F_E such that for all $(x, y) \in E$ and $n \in \mathbb{N}$, $(\phi^n(x), \phi^n(y)) \in F_E$.

X is called *flasque* (with apologies for the ad-hoc use of this terminology) if it is both coarsely flasque and properly homotopic to the empty set.

The prototypes of flasque spaces are those of the form $X = Y \times [0, \infty)$ where Y is a coarse space, X is equipped with the product coarse structure (see definition A.7) and ϕ is defined by $\phi : (y, t) \mapsto (y, t + 1)$. $Y \times \mathbb{N}$ is an example of a coarsely flasque space that is not flasque (unless Y is empty). The examples we will actually use (in the proofs of theorems 4.3.12 and 4.3.13) are only slightly more complicated than these.

The following now gives a vanishing result for the K-theory of the corona functor; it has as a corollary a similar vanishing theorem for the compactification functor. The proof is essentially the same as that of theorem 30 from [22].

Proposition 4.3.8. *Let X be a coarsely flasque space. Then*

$$K_*(\mathfrak{c}^r(X)) = 0.$$

Proof. There is a separable infinite-dimensional Hilbert space implicit in the definition of $\mathfrak{c}^r(X)$; denote it by $\mathcal{H}$ and decompose it as an infinite direct sum of infinite-dimensional subspaces

$$\mathcal{H} = \oplus_{n\in\mathbb{N}}\mathcal{H}_n.$$

For each n let $V_n : \mathcal{H} \to \mathcal{H}_n$ be a unitary isomorphism and define maps

$$\nu_n : \bar{\mathfrak{c}}^r(X) \to \bar{\mathfrak{c}}^r(X)$$

$$f \mapsto V_n f V_n^*.$$

Let now $\phi : X \to X$ be a map as in the definition of coarsely flasque space and (provisionally) define a map $\nu : \bar{\mathfrak{c}}^r(X) \to \bar{\mathfrak{c}}^r(X)$ via

$$f \mapsto \oplus_{n\in\mathbb{N}}\, \nu_n(f \circ \phi^n).$$

To show that this is well-defined, it suffices to check that:

- νf is continuous. Fix $\epsilon > 0$ and $x \in X$. Let E be an entourage such that

$$U_0 = \{y \in X : (x, y) \in E\}$$

is a compact neighbourhood of x. Let F_E be as given in the definition of ϕ, so in particular $(\phi^n(x), \phi^n(y)) \in F_E$ for all $y \in U_0$ and $n \in N$. As f is of vanishing variation there exists a compact subset K of X such that $(\nabla_{F_E} f)(y) < \epsilon$ for all $y \notin K$. Let N_K be such that $\phi^n(y) \notin K$ for all $y \in X$ and $n > N_K$. Using continuity of f and ϕ there exists a neighbourhood U of x such that for all $n \leq N_K$ and all $y \in U$, $\|f \circ \phi^n(x) - f \circ \phi^n(y)\| < \epsilon$.

Now,

$$(\nu f)(x) - (\nu f)(y) = \oplus_{n\in\mathbb{N}} V_n (f(\phi^n(x)) - f(\phi^n(y))) V_n^*, \tag{4.8}$$

whence for all $y \in U$, the norm of $(\nu f)(x) - (\nu f)(y)$ is bounded by

$$\sup_{n\in\mathbb{N}} \|f(\phi^n(x)) - f(\phi^n(y)\| \leq \epsilon$$

by the assumptions above. This proves continuity of νf.

- $(\nu f)(x) - (\nu f)(y) \in \mathcal{K}$ for all $x, y \in X$. By assumption on ϕ, for any $x, y \in X$, there exists an entourage F on X such that $(\phi^n(x), \phi^n(y)) \in F$ for all n, whence $\|f(\phi^n(x)) - f(\phi^n(y))\|$ converges to zero as $n \to \infty$ (as f is of vanishing variation, and $\phi^n(x), \phi^n(y)$ eventually leave all compact subsets of X). Hence the expression on the right hand side of line (4.8) is a direct sum of compact operators on disjoint spaces with norms converging to zero. It is thus compact.

- νf is of vanishing variation. Let E be an entourage and $\epsilon > 0$. Let F_E be as in the definition of ϕ. We may assume that F_E contains E. Then by vanishing variation of f there exists compact $K \subseteq X$ such that for all $x \in X \backslash K$, $|(\nabla_{F_E} f)(x)| < \epsilon$.

Moreover, by assumption on ϕ, there exists N_K such that for all $n \geq N_K$ and all $x \in X$, $\phi^n(x) \notin K$. As ϕ is assumed close to the identity, there exists an entourage F containing the diagonal such that for all $x \in X$, $(x, \phi(x)) \in F$; define

$$F^n = \underbrace{F \circ F \circ \ldots \circ F}_{n}$$

(cf. definition A.1) and

$$K' = \overline{N\Big(K, \bigcup_{n=1}^{N_K} F^n\Big)},$$

which is compact by compatibility of the coarse structure and topology on X. For all $x \notin K'$ and any $n \in N$:

- if $n < N_K$ then $(x, \phi^n(x)) \in \cup_{n=1}^{N_K} F^n$ and $\phi^n(x) \notin K$;
- if $n \geq N_K$ then $\phi^n(x) \notin K$ by assumption on N_K.

Either way, for any $(x, y) \in E \subseteq F_E$ and any $n \in \mathbb{N}$, the term $V_n(f(\phi^n(x)) - f(\phi^n(y))V_n^*$ in line (4.8) above has norm at most ϵ. The norm of $\nu f(x) - \nu f(y)$ is just the supremum of the norms of the terms in (4.8), however, so it too has norm at most ϵ. Hence νf is of vanishing variation.

Thus ν is indeed well-defined. It also preserves $C_0(X, \mathcal{K})$, so passes to a map $\nu : \mathfrak{c}^r(X) \to \mathfrak{c}^r(X)$.

Now, let f, g be elements of (the unitization of) $\mathfrak{c}^r(X)$ defining some class $[f]-[g] \in K_*(\mathfrak{c}^r(X))$ (any element in this group can be written in this form). Then

$$\nu_*([f]-[g]) = (\nu_0)_*([f]-[g]) \oplus \Big(\Big[\sum_{n\geq 1} \nu_n(f\circ\phi^{n-1})\Big] - \Big[\sum_{n\geq 1} \nu_n(g\circ\phi^{n-1})\Big]\Big), \quad (4.9)$$

where we have used that ϕ induces the identity map on $\mathfrak{c}^r(X)$. Note, moreover, that the map ν_0 defines the identity on K-theory (this is always true of the map induced on K-theory by conjugation by an isometry in the multiplier algebra), while the element $[\sum_{n\geq 1}\nu_n(f\circ\phi^{n-1})]$ is conjugate via another isometry to νf, and similarly with g in place of f. Hence (4.9) above becomes

$$\nu_*([f]-[g]) = [f]-[g]+\nu_*([f]-[g]),$$

so $[f]-[g]=0$ and finally $K_*(\mathfrak{c}^r(X))=0$. □

Proposition 4.3.9. *Let X be a flasque space. Then*

$$K_*(\bar{\mathfrak{c}}^r(X)) = 0.$$

Proof. By the previous proposition, $K_*(\mathfrak{c}^r(X)) = 0$, while $K_*(C_0(X,\mathcal{K})) = 0$ by the fact that X is properly homotopic to the empty set. The result now follows from the K-theory long exact sequence associated to the short exact sequence

$$0 \longrightarrow C_0(X,\mathcal{K}) \longrightarrow \bar{\mathfrak{c}}^r(X) \longrightarrow \mathfrak{c}^r(X) \longrightarrow 0\,.$$ □

Example 4.3.10. We now have enough information to compute K-theory groups in the case of $\mathbb{R}^n$. Indeed, using example 4.2.4, proposition 4.3.5, lemma 4.3.4, proposition 4.3.8 and induction on n one can show that

$$K_i(\mathfrak{c}^r(\mathbb{R}^n)) = \begin{cases} \mathbb{Z} & i = n+1 \bmod 2 \\ 0 & i = n \bmod 2 \end{cases} .$$

The main aim of this section is to combine the results of propositions 4.3.5 and 4.3.8 (resp. 4.3.6 and 4.3.9) to get homotopy invariance results. The notion of homotopy we use is as follows. It is almost the same as that in [39] section 11.

Definition 4.3.11. Let $\phi_0, \phi_1 : X \to Y$ be morphisms in the category $\mathcal{CC}$. They are said to be *elementary continuous coarse homotopy equivalent* if there exists a continuous map $H : X \times \mathbb{R}_+ \to Y$ and a continuous coarse map $X \to \mathbb{R}_+$ denoted $x \mapsto t_x$ such that:

- $H(x, 0) = \phi_0(x)$ for all $x \in X$;
- for all $t \geq t_x$, $H(x, t) = \phi_1(x)$;
- if $Z := \{(x, t) \in X \times \mathbb{R}_+ : t \leq t_x\}$ is equipped with the restriction of the product coarse structure on $X \times \mathbb{R}_+$, then H restricted to Z is coarse.

Such an H is called an *elementary continuous coarse homotopy equivalence* between ϕ_0 and ϕ_1.

Continuous coarse homotopy equivalence is then the equivalence relation on morphisms in $\mathcal{CC}$ generated by elementary coarse equivalence.

To get the definition of *coarse homotopy equivalence*, replace $\mathbb{R}_+$ by $\mathbb{N}$, $\mathcal{CC}$ by $\mathcal{C}$ and remove the word 'continuous'.

Note that coarse homotopy equivalence is a strictly finer equivalence relation than continuous coarse homotopy equivalence: if H is an elementary continuous coarse homotopy equivalence, then its restriction to $X \times \mathbb{N}$ is an elementary coarse homotopy equivalence.

The following proof is taken from [43], proposition 12.4.12.

Theorem 4.3.12. *Let $\phi_0, \phi_1 : X \to Y$ be morphisms in the category $\mathcal{CC}$ that are continuously coarsely homotopic. Then the induced maps*

$$\phi_0^*, \phi_1^* : K_*(\bar{\mathfrak{c}}^r(Y)) \to K_*(\bar{\mathfrak{c}}^r(X))$$

are the same.

Proof. It suffices to show the result when ϕ_0, ϕ_1 are elementary continuous coarse homotopy equivalent. Let $H : X \times \mathbb{R}_+ \to Y$ be an elementary continuous coarse homotopy equivalence, which we extend to all of $X \times \mathbb{R}$ by defining $H(x,t) = H(x,0)$ for $t < 0$. Give $X \times \mathbb{R}$ the product coarse structure (definition A.7) and define

$$Z := \{(x,t) \in X \times \mathbb{R} : x \in X, 0 \leq t \leq t_x\}.$$

Further, define inclusion maps $i_0, i_1 : X \to Z$ by

$$i_0(x) = (x,0) \text{ and } i_1(x) = (x, t_x);$$

i_0 is clearly coarse, while i_1 is coarse by definition of an elementary continuous coarse homotopy equivalence. Note also that H is continuous and coarse when restricted to Z, and that

$$H \circ i_j = \phi_j \text{ for } j = 0, 1. \tag{4.10}$$

Let moreover $\pi : Z \to X$ be the coordinate projection onto the X variable. We first claim that the maps

$$i_0^*, i_1^* : K_*(\bar{\mathfrak{c}}^r(X) \to K_*(\bar{\mathfrak{c}}^r(Z))$$

are in fact equal.

Indeed, extend π as a coordinate projection to all of $X \times \mathbb{R}$. Let $A = \{(x,t) \in X \times \mathbb{R} : t \leq 0\}$, $A' = \{(x,t) \in X \times \mathbb{R} : t \leq t_x\}$ and $B = \{(x,t) \in X \times \mathbb{R} : t \geq 0\}$. Then both of the decompositions $X = A \cup B$ and $X = A' \cup B$ are coarsely excisive. Moreover, it is not too hard to see that the restriction $*$-homomorphisms

$$\bar{\mathfrak{c}}^r(X \times \mathbb{R}) \to \bar{\mathfrak{c}}^r(Y)$$

are surjective for $Y = A, A', B$; for example, in the case of B, any $f \in \bar{\mathfrak{c}}(B)$ extends to a function $\tilde{f} \in \bar{\mathfrak{c}}^r(X \times \mathbb{R})$ via the formula

$$\tilde{f}(x,t) = \begin{cases} f(x,t) & t < 0 \\ f(x,-t) & t \geq 0 \end{cases},$$

and the other cases are similar. Note also that the identity map on $X \times \mathbb{R}$ is a map of such decompositions in the category $\mathcal{CC}$, whence from proposition 4.3.6 there is a commutative diagram of Mayer-Vietoris sequences

$$\begin{array}{ccccccc} \longrightarrow K_i(\bar{\mathfrak{c}}^r(X \times \mathbb{R})) & \longrightarrow & K_i(\bar{\mathfrak{c}}^r(A)) \oplus K_i(\bar{\mathfrak{c}}^r(B)) & \longrightarrow & K_i(\bar{\mathfrak{c}}^r(A \cap B)) \longrightarrow & . \\ \uparrow & & \uparrow & & \uparrow & \\ \longrightarrow K_i(\bar{\mathfrak{c}}^r(X \times \mathbb{R})) & \longrightarrow & K_i(\bar{\mathfrak{c}}^r(A')) \oplus K_i(\bar{\mathfrak{c}}^r(B)) & \longrightarrow & K_i(\bar{\mathfrak{c}}^r(A' \cap B)) \longrightarrow & \end{array}$$

The first vertical arrow is an isomorphism as induced by the identity map. The second is also an isomorphism as both its domain and codomain vanish by flasqueness of A, A' and B, and proposition 4.3.9. Hence the third vertical map is an isomorphism by the five lemma. It is, however, exactly the map $i_0^* : K_*(\bar{\mathfrak{c}}^r(Z)) \to K_*(\bar{\mathfrak{c}}^r(X))$.

Now, π^* is a one-sided inverse to i_0^*; as i_0^* is an isomorphism, π^* is an isomorphic two-sided inverse. As π^* is a one-sided inverse to i_1^* and an isomorphism, i_1^* is an isomorphism, and two-sided inverse to π^*. Hence in particular, i_0^* and i_1^* are both equal to $(\pi^*)^{-1}$, so equal.

To finish the proof, then, looking back at line (4.10) above, we see that

$$i_j^* \circ H^* = \phi_j^* : K_*(\bar{\mathfrak{c}}^r(Y)) \to K_*(\bar{\mathfrak{c}}^r(X))$$

for $j = 0, 1$. As $i_0^* = i_1^*$, however, $\phi_0^* = \phi_1^*$ as claimed. □

Theorem 4.3.13. *Let $\phi_0, \phi_2 : X \to Y$ be morphisms in the category $\mathcal{C}$ that are coarsely homotopic. Then the induced maps*

$$\phi_0^*, \phi_1^* : K_*(\mathfrak{c}^r(Y)) \to K_*(\mathfrak{c}^r(X))$$

are the same.

Proof. Same as that of the previous proposition, but replace $\mathbb{R}_+$ by $\mathbb{N}$, $\mathbb{R}$ by $\mathbb{Z}$, $\bar{\mathfrak{c}}^r(X)$ by $\mathfrak{c}^r(X)$ and remove the word 'continuous'. □

Remark 4.3.14. It will be useful in section 4.7 that one has the same homotopy invariance results for the 'unreduced' functors $X \mapsto K_*(\bar{\mathfrak{c}}(X))$ and $X \mapsto K_*(\mathfrak{c}(X))$. The proofs are essentially the same; we did not state them explicitly above out of fear of too much tedious repetition.

The final result in this section is another vanishing result, closely related to proposition 57 from [22]. It is our main computational tool. It does not seem to be possible to make sense of it without at least a uniform structure on X. We thus restrict to the case of metric spaces for the remainder of this section, including the postponed proof of lemma 4.3.4.

Definition 4.3.15. Let X be a proper metric space. A continuous coarse map $s : X \to X$ is called a *scaling* if for all $x, y \in X$, $d(s(x), s(y)) \leq d(x, y)/2$.

The proof of the following vanishing result is essentially the same as that of theorem 12.4.11 from [43]; it is another Eilenberg swindle.

Proposition 4.3.16. *Say $s : X \to X$ is a scaling. Then the map*

$$s^* : K_*(\bar{\mathfrak{c}}^r(X)) \to K_*(\bar{\mathfrak{c}}^r(X))$$

is zero.

Proof. Choose a decomposition $\mathcal{H} = \oplus_{n\in\mathbb{N}}\mathcal{H}_n$, and corresponding unitaries V_n and conjugating maps ν_n as in proposition 4.3.8. (Provisionally) define a map from $\bar{\mathfrak{c}}^r(X)$ to itself via the formula

$$\nu : f \mapsto \oplus_{n\in\mathbb{N}} \nu_n(f \circ s^n);$$

if this is well-defined, essentially the same Eilenberg swindle as in proposition 4.3.8 completes the proof (note that this map does *not* preserve $C_0(X, \mathcal{K})$, so does not pass to the corona $\mathfrak{c}^r(X)$).

Just as in the earlier proposition, we must check three conditions. The only real observation is that because f is of vanishing variation, it is uniformly continuous, whence the (*global*) modulus of continuity of $f \circ s^n$ gets arbitrarily small as n tends to infinity. The details are below.

- νf is continuous. As s is a contraction and f is uniformly continuous, all the maps $f \circ s^n$ are equicontinuous, whence νf itself is continuous.

- $(\nu f)(x)-(\nu f)(y)$ is compact for all $x, y \in X$. Note that as f is uniformly continuous and for any $x, y \in X$ $d(s^n(x), s^n(y)) \to 0$ as $n \to \infty$, it follows that $(\nu_n(f\circ s^n))(x)-(\nu_n(f \circ s^n))(y)$ tends to zero as $n \to \infty$. Hence $(\nu f)(x) - (\nu f)(y)$ consists of an infinite direct sum of compact operators whose norms tend to zero at infinity. It is thus compact.

- νf is of vanishing variation. Let $R, \epsilon > 0$. As f is uniformly continuous, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $\|f(x) - f(y)\| < \epsilon$. Let N be such that for all $n \geq N$ and all $x, y \in X$ such that $d(x, y) \leq R$, $d(s^n(x), s^n(y)) < \delta$. Further, as f is of vanishing variation, there exists compact $K \subseteq X$ such that if $x, y \notin K$

and $d(x,y) \leq R$, then $\|f(x) - f(y)\| < \epsilon$. Let $K' = \cup_{n<N}(s^n)^{-1}(K)$ (which is compact as s is proper). It follows that for all $x, y \notin K'$ such that $d(x,y) \leq R$, $\|(\nu f)(x) - (\nu f)(y)\| < \epsilon$. Hence νf is of vanishing variation. □

As promised, we finish this section with the proof of lemma 4.3.4.

Proof. Let $f \in \bar{\mathfrak{c}}^r(A)$ and $\epsilon > 0$. As $*$-homomorphisms have closed image, it suffices to prove that there exists $\tilde{f} \in \bar{\mathfrak{c}}^r(X)$ such that $\|\tilde{f}(a) - f(a)\|_{\mathcal{B}(\mathcal{H})} < \epsilon$ for all $a \in A$.

Let first

$$v : A \to \mathbb{R} \cup \{\infty\}$$

be any non-decreasing (with respect to distance from some fixed base-point $a_0 \in A$) function tending to infinity at infinity in A and such that

- $v(a) \leq \sup\{R > 0 : (\nabla_R f)(a) < \epsilon\}$ for all $a \in A$;
- $(\nabla_{v(a)} f)(a) \to 0$ as $a \to \infty$.

Such a v exists by vanishing variation of f. Further, for each $a \in A$, let

$$U_a = B_X(a, \sqrt{v(a)}),$$

the open ball *in* X about a of radius $\sqrt{v(a)}$. Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be any non-decreasing, subadditive function such that $\rho(0) = 0$, $\rho(t) > 0$ for all $t > 0$, $\rho(t) \to \infty$ as $t \to \infty$ and so that for all $x \in X$

$$d(x, A) \leq \rho(d(x, a_0)) \text{ implies } x \in \cup_{a \in A} U_a.$$

One may then define

$$F = \{x \in X : d(x, A) \leq \rho(d(x, a_0))\},$$

a closed subset of X containing A and covered by $\{U_a\}_{a \in A}$. Let $\{\phi_i\}_{i \in I}$ be a locally finite continuous partition of unity on F subordinate to $\{U_a\}$. For each $i \in I$, let $a_i \in A$ be such that ϕ_i is supported in U_{a_i}. Using Tietze's extension theorem, extend each ϕ_i to a $[0, 1]$-valued function defined on all of X and supported in U_{a_i}. Finally, define a function

$$\psi : X \to \mathbb{R}_+$$

$$x \mapsto \max\left\{\frac{\rho(d(x, a_0)) - d(x, A)}{\rho(d(x, a_0))}, 0\right\}$$

(which is supported in F, and equal to one on A) and define

$$\tilde{f} : X \to \mathbb{C}$$

$$x \mapsto \psi(x) \sum_{i \in I} \phi_i(x) f(a_i).$$

The choice of the cover $\{U_a\}$ guarantees that $\|f(a) - \tilde{f}(a)\| < \epsilon$ for all $a \in A$, and moreover $\tilde{f}$ is bounded, continuous, supported in F and satisfies $f(x) - f(y) \in \mathcal{K}$ for all $x, y \in X$. To finish the proof of the lemma it thus suffices to prove that $\tilde{f}$ is of vanishing variation.

To this end, say $x, y \in X$ satisfy $d(x,y) \leq R$ for some $R > 0$. Then

$$\|\tilde{f}(x) - \tilde{f}(y)\|$$

$$\leq |\psi(x)| \Big\| \sum_{i \in I} (\phi_i(x) - \phi_i(y)) f(a_i) \Big\| + |(\psi(x) - \psi(y))| \Big\| \sum_{i \in I} \phi_i(y) f(a_i) \Big\|$$

$$\leq |\psi(x)| \Big\| \sum_{i \in I} (\phi_i(x) - \phi_i(y)) f(a_i) \Big\| + |\psi(x) - \psi(y)| \|f\|_{\overline{\mathfrak{C}}^r(A)}$$

Assume without loss of generality that $\psi(x), \psi(y)$ do not both vanish.

Case (1): Exactly one of $\psi(x), \psi(y)$ is zero.

Without loss of generality, assume that $\psi(x)$ vanishes. Then the above becomes

$$\|\tilde{f}(x) - \tilde{f}(y)\| \leq |\psi(y)| \|f\| = \Big(\frac{\rho(d(y, a_0)) - d(y, A)}{\rho(d(y, a_0))} \Big) \|f\|$$

$$\leq \Big(\frac{\rho(d(x, a_0)) + \rho(R) - d(x, A) + R}{\rho(d(y, a_0))} \Big) \|f\|$$

using subadditivity of ρ. By assumption that $\psi(x) = 0$, this in turn is bounded by

$$\Big(\frac{\rho(R) + R}{\rho(d(y, a_0))} \Big) \|f\|,$$

which gets arbitrarily small as long as y gets far from a_0.

Case (2): Both of $\psi(x), \psi(y)$ are non-zero.

Using a similar argument to that in case (1), $|\psi(x) - \psi(y)|$ can be made arbitrarily small by assuming that x, y are suitably far from a_0. It thus remains to show that the term

$$|\psi(x)| \Big\| \sum_{i \in I} (\phi_i(x) - \phi_i(y)) f(a_i) \Big\| \leq \Big\| \sum_{i \in I} (\phi_i(x) - \phi_i(y)) f(a_i) \Big\|$$

can be made arbitrarily small by assuming x and y are suitably far from a_0. Now, by assumption that $\psi(x) \neq 0 \neq \psi(y)$, it follows that $x, y \in F$, whence there exist $a_x, a_y \in A$ such that $d(a_x, x) \leq \sqrt{v(a_x)}$ and $d(a_y, y) \leq \sqrt{v(a_y)}$. Moreover, as $v(a) \to \infty$ as $a \to \infty$ in A, if $x, y \in X$ are suitably far from a_0, then if $i \in I$ satisfies either $\phi_i(x) \neq 0$ or $\phi_i(y) \neq 0$, then $d(a_i, a_x) \leq v(a_x)$. Finally, then, by assumption that $(\nabla_{v(a)} f)(a) \to 0$ as $a \to \infty$,

$$\Big\| \sum_{i \in I} (\phi_i(x) - \phi_i(y)) f(a_i) \Big\| \leq \Big\| f(a_x) - \sum_{i \in I} \phi_i(x) f(a_i) \Big\| + \Big\| f(a_x) - \sum_{i \in I} \phi_i(y) f(a_i) \Big\|$$

can be made arbitrarily small for a_x suitably far from a_0 (which in turn can be forced by stipulating that x, y are suitably far from a_0). This completes the proof. □

4.4 Geometric boundaries

The aim of the next four sections is to compute $K_*(\mathfrak{c}(X))$ (and also prove the stable Higson conjecture 4.2.7) in certain cases. These results are mainly already contained in, or follow easily from, [22], but the proofs are somewhat different.

We put particular emphasis on the fact that the inclusion of the (stabilised) continuous functions on a *geometric boundary* ∂X of X into $\mathfrak{c}(X)$ induces an isomorphism on K-theory. In some cases, this allows for a description of $K_*(\mathfrak{c}(X))$ in terms of specific geometrically defined generators, although we do not pursue this. In the main cases of interest, the geometric boundaries we use are the *visual* and *Gromov* boundaries introduced in sections 3.5 and 3.6; their existence thus rely on some sort of 'non-positive curvature' assumption on the original space X.

This preliminary section introduces the general setup for the specific results that follow. The next definition is very close to definition 10.6.7 from [62] and also to definition 5.13 in [43].

Definition 4.4.1. Let X be a coarse space and $\overline{X}$ a compactification of X. $\overline{X}$ is called a *geometric compactification* if the canonical inclusion $C_0(X) \to C_h(X)$ extends to an inclusion

$$C(\overline{X}) \to C_h(X).$$

Equivalently, $\overline{X}$ is geometric if any continuous function on it is of vanishing variation when restricted to X. The Higson compactification is itself geometric, and is in a reasonable sense the largest geometric compactification. Further examples are the visual and Gromov compactifications for appropriate X; see lemmas 4.6.2 and 4.7.1.

The following simple lemma extends this inclusion property to the Higson corona and stable Higson corona.

Lemma 4.4.2. *Say $\overline{X}$ is a geometric compactification of X and $\partial X = \overline{X} \backslash X$ the associated corona. Then there are natural inclusions*

$$C(\partial X) \to C(\partial_h X)$$

$$C(\partial X) \otimes \mathcal{K} \to \mathfrak{c}(X)$$

Proof. The first inclusion is immediate from the definition. For the second, note that if $\overline{X}$ is geometric, there are natural inclusions

$$C(\overline{X}, \mathcal{K}) \hookrightarrow C(\overline{X}^h, \mathcal{K}) \hookrightarrow \bar{\mathfrak{c}}(X),$$

where the second inclusion is given by restriction to X (note that it is *never* an equality if X is non-compact). Moreover, these inclusions preserve the ideal $C_0(X, \mathcal{K})$ in each case, and there is thus a natural inclusion of quotients

$$C(\partial X) \otimes \mathcal{K} \cong \frac{C(\overline{X}, \mathcal{K})}{C_0(X, \mathcal{K})} \hookrightarrow \frac{\bar{\mathfrak{c}}(X)}{C_0(X, \mathcal{K})} \cong \mathfrak{c}(X). \qquad \square$$

4.5 Open cones

Our first application is to open cones. From a certain point of view, the cases covered in sections 4.6 and 4.7 below are accessible to our methods because non-positive curvature conditions can be used to show that they have a 'cone-like' structure over a geometric boundary. The class of open cones over compact metric spaces is thus a (simple) prototype for what follows, as well as containing some interesting examples in its own right.

The definition of an open cone we use is as follows.

Definition 4.5.1. Let Y be a compact metric space of diameter at most 2. The *open cone over* Y, denoted $\mathcal{O}Y$, is

$$\mathcal{O}Y := (Y \times \mathbb{R}_+)/(Y \times \{0\})$$

as a topological space. If $s \in \mathbb{R}_+$ and $y \in Y$, a general point in $\mathcal{O}Y$ will be denoted sy. The open cone is equipped with the 'l^1-type' metric

$$d_{\mathcal{O}Y}(sx, ty) := |t - s| + \min\{s, t\} d(x, y). \tag{4.11}$$

Some remarks are perhaps in order here.

Remarks 4.5.2.

1. The expression $d_{\mathcal{O}Y}$ above really is a metric. The condition that the diameter of Y is at most two is important here.

2. As Y is compact, $\mathcal{O}Y$ is a proper metric space.

3. It is perhaps more usual to assume that Y is a subset of the unit sphere in some Hilbert space $\mathcal{H}$, and define the open cone over Y to be

$$\{ty \in \mathcal{H} : t \in \mathbb{R}_+, y \in Y\}$$

equipped with the restriction of the metric from $\mathcal{H}$. If Y does start out as a subset of a sphere in this way, then it is not hard to check that the two definitions of 'open cone' define coarsely equivalent metric spaces.

4. As an example that we will use later, say Y is S^{n-1}, equipped with either its intrinsic metric or the restricted metric from $\mathbb{R}^n$. Then $\mathcal{O}Y$ is coarsely equivalent to $\mathbb{R}^n$ equipped with the Euclidean metric.

5. There is a natural compactification of $\mathcal{O}Y$ defined by attaching a copy of Y at infinity. Indeed, identify $\mathcal{O}Y$ with the space $([0, 1) \times Y)/(Y \times \{0\}) \subset ([0, 1] \times$

$Y)/(Y \times \{0\})$ via some homeomorphism $\rho : [0,1) \to [0,\infty)$ (which must fix zero and be increasing). Then the compactification can be defined to be $[0,1]\times Y/Y\times\{0\}$ (it does not depend on the choice of ρ).

Denote by $\overline{\mathcal{O}Y}$ the compactification of $\mathcal{O}Y$ appearing in the final point above. The following simple lemma says that this is a geometric compactification in the sense of definition 4.4.1.

Lemma 4.5.3. *$\overline{\mathcal{O}Y}$ is a geometric compactification of $\mathcal{O}Y$.*

Proof. Let $R, \epsilon > 0$. Let $\rho : [0,\infty) \to [0,1)$ be some homeomorphism (fixing zero) as in the remark, and note that $\overline{\mathcal{O}Y}$ can be metrized using the expression in line (4.11) above. Let f be a continuous function of $\overline{\mathcal{O}Y}$, so in particular f is uniformly continuous. Let $\delta > 0$ be such that if $d_{\overline{\mathcal{O}Y}}(x_1, x_2) \leq \delta$ then $|f(x) - f(y)| < \epsilon$.

Now, for any $R > 0$ there exists $S > 0$ so that if $t_1, t_2 > S$ then $|\rho(t_1) - \rho(t_2)| < \delta/2$ and $R/S \leq \delta/2$. It follows that if $t_1y_1, t_2y_2 \in \mathcal{O}Y$ satisfy $d_{\mathcal{O}Y}(t_1y_1, t_2y_2) \leq R$ and $t_1, t_2 > S$, then

$$\begin{aligned} d_{\overline{\mathcal{O}Y}}(\rho(t_1)y_1, \rho(t_2)y_2) &= |\rho(t_1) - \rho(t_1)| + \min\{\rho(t_1)\rho(t_2)\}d_Y(y_1, y_2) \\ &\leq |\rho(t_1) - \rho(t_2)| + d_Y(y_1, y_2) \\ &< R/S + \delta/2 < \delta. \end{aligned}$$

Hence $|f(t_1y_1) - f(t_2y_2)| < \epsilon$. ◻

From lemma 4.4.2 then, there is a natural inclusion

$$i^Y : C(Y) \otimes \mathcal{K} \to \mathfrak{c}(\mathcal{O}Y). \tag{4.12}$$

The main result of this section, and the prototype for the main results of the next two sections, is the following.

Proposition 4.5.4. *i^Y as in line (4.12) above induces an isomorphism on K-theory.*

Proof. Note that the map

$$s : \mathcal{O}Y \to \mathcal{O}Y$$

$$ty \mapsto (t/2)y$$

is *both* a scaling and continuously coarsely homotopic to the identity; proposition 4.3.16 and theorem 4.3.12 thus show that the map

$$s^* : K_*(\bar{\mathfrak{c}}^r(\mathcal{O}Y)) \to K_*(\bar{\mathfrak{c}}^r(\mathcal{O}Y))$$

is both zero and an isomorphism. Hence the group appearing in the above line is zero. Note that this statement could be thought of as saying that $\mathcal{O}Y$ satisfies the stable Higson conjecture 4.2.7 (strictly, however, that conjecture involves a group, so this doesn't make literal sense).

Consider now the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(\mathcal{O}Y,\mathcal{K}) & \longrightarrow & C(\overline{\mathcal{O}Y},\mathcal{K}) & \longrightarrow & C(Y,\mathcal{K}) & \longrightarrow & 0\ . \\ & & \Vert & & \downarrow & & \downarrow i^Y & & \\ 0 & \longrightarrow & C_0(\mathcal{O}Y,\mathcal{K}) & \longrightarrow & \bar{\mathfrak{c}}(\mathcal{O}Y) & \longrightarrow & \mathfrak{c}(\mathcal{O}Y) & \longrightarrow & 0 \end{array}$$

$\overline{\mathcal{O}Y}$ is compact and contractible, whence its K-theory is generated by the constant map to any rank one projection in $\mathcal{K}$; by the paragraph at the start of the proof and proposition 4.2.3, the same is true of $\bar{\mathfrak{c}}(\mathcal{O}Y)$. Hence the central vertical map induces an isomorphism on K-theory and so by the long exact sequence in K-theory and the five lemma, i^Y does too. □

4.6 $CAT(0)$ spaces

This section looks at $CAT(0)$ spaces. The *visual compactification* of such a space, as introduced in section 3.5, will be our main tool. The following lemma is well-known, but we couldn't find a proof in the literature so give one one below for completeness.

Lemma 4.6.1. *Let X be as in definition 3.5.2. Then the visual compactification $\overline{X}^v$ of X is contractible.*

Proof. From the definition of the visual compactification, a point in $\overline{X}^v$ is the same thing as a '$[0,\infty)$ indexed set' $(y_r)_{r\in[0,\infty)}$ such that for all $t \geq s \geq r$, $p_{t,r}(y_s) = y_r$. For each $t \in [0,1)$ and any $(y_r) \in \overline{X}^v$, define

$$y_r^t = \begin{cases} y_r & 0 \leq r < \frac{t}{1-t} \\ y_{\frac{t}{1-t}} & \frac{t}{1-t} \leq r < \infty \end{cases} .$$

If $t = 1$, set $(y_r^t) = (y_r)$. The formula

$$H : \overline{X}^v \times [0,1] \to \overline{X}^v$$

$$((y_r), t) \mapsto (y_r^t)$$

defines a contracting homotopy of $\overline{X}^v$. □

Lemma 4.6.2. *Let X be as in definition 3.5.2. Then the visual compactification of X is geometric.*

Proof. See definition 3.5.2 for the notation used below. From the Gelfand-Naimark equivalence of categories and general nonsense,

$$C(\overline{X}^v) = \varinjlim C(\bar{B}(x_0, r)),$$

where the connecting maps in the inductive system are those functorially induced by $p_{s,r}$. It follows that $C(\overline{X}^v)$ has a dense subalgebra consisting of functions f such that there exists $r_0 > 0$ such that for all $x \notin \bar{B}(x_0, r_0)$, $f(x) = f(p_{d(x_0,x),r_0}(x))$. In other words, outside of $\bar{B}(x_0, r_0)$, f is 'constant on radial lines'. It suffices to prove that any such f is of vanishing variation when restricted to X.

To this end, let $R, \epsilon > 0$, and note that f is uniformly continuous on $\bar{B}(x_0, r_0)$ (with r_0 as above), so there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon$. Moreover, it follows from the $CAT(0)$ inequality that there exists $s > 0$ such that for all $x, y \notin \bar{B}(x_0, s)$ with $d(x, y) \leq R$ and all $t \geq d(x, x_0), d(y, x_0), r$ one has that

$d(p_{t,r_0}(x), p_{t,r_0}(y)) < \delta$. Hence

$$|f(x) - f(y)| = |f(p_{t,r_0}(x)) - f(p_{t,r_0}(y))| < \epsilon,$$

whence f is of vanishing variation as required. □

Again, then, there is a canonical inclusion

$$i^v : C(\partial_v X) \otimes \mathcal{K} \to \mathfrak{c}(X).$$

The next proposition is the main result of this section.

Proposition 4.6.3. *The inclusion i^v defined above induces an isomorphism on K-theory. Moreover the stable Higson conjecture 4.2.7 holds for a group that acts freely properly cocompactly by isometries on a $CAT(0)$ space.*

Proof. Fix any basepoint $x_0 \in X$. Then any $x \in X$ is connected to x_0 via a unique geodesic, say $\gamma_x : [0, d(x_0, x)] \to X$. Assume that each γ_x is oriented so that $\gamma_x(0) = x_0$, $\gamma_x(d(x_0, x)) = x$, and define a map

$$s : X \to X$$

$$x \mapsto \gamma_x(d(x_0, x)/2),$$

which is a continuous coarse map. It follows immediately from the CAT(0) inequality that s is a scaling. Moreover, s is continuously coarsely homotopic to the identity via the map

$$H : X \times \mathbb{R}_+ \to X$$

$$(x,t) \mapsto \begin{cases} \gamma_x(d(x_0,x) - t) & 0 \le t < d(x_0,x)/2 \\ \gamma_x(d(x_0,x)/2) & t \ge d(x_0,x)/2 \end{cases} .$$

Just as in proposition 4.5.4, it follows that $K_*(\bar{\mathfrak{c}}^r(X)) = 0$. Note in particular that if Γ acts freely, properly, cocompactly and by isometries on X then X can itself be taken as an $E\Gamma$. Thus the above implies the stable Higson conjecture for Γ.

To complete the proof, then, invoke lemma 4.6.1 and proceed exactly as in the second half of the proof of proposition 4.5.4. □

4.7 Gromov hyperbolic spaces

Our next result is the analogous isomorphism for spaces that are hyperbolic in the sense of Gromov. There is a significant amount of overlap between the main result of this section and that of the last, but neither follows from the other.

In order to set up the main result, we first have the following lemma, which is contained in proposition 2.3 from [61].

Lemma 4.7.1. *The Gromov compactification of a hyperbolic space is geometric.* □

As before, then, if X is a hyperbolic space, then there is a a natural inclusion

$$i^g : C(\partial_g\Gamma) \otimes \mathcal{K} \to \mathfrak{c}(\Gamma),$$

and our main result in this section is the following.

Proposition 4.7.2. *The inclusion i^g above induces an isomorphism on K-theory.*

Our proof is almost exactly the same as that Higson and Roe give of the coarse Baum-Connes conjecture in section 8 of [41]. However, our machinery is somewhat different to theirs, and the reference above is not that detailed, so we provide a complete proof below.

The proof proceeds by reducing from a hyperbolic space to the case of an open cone over its Gromov boundary, and then applying proposition 4.5.4.

I was not able to produce a proof that proceeds directly along the lines of those in sections 4.5 and 4.6. In fact, up to some technicalities, a δ-hyperbolic space X admits a sort of 'coarse scaling': a map $s : X \to X$ satisfying $d(s(x), s(y)) \leq d(x, y)/2 + 16\delta$. Passing to the *Rips complex* of X (see chapter 4 of [26] for the definition of this object and proofs of some of its properties), s can be assumed continuously coarsely homotopic to the identity. However, I cannot see how to adapt the proofs of the previous sections when the constant 16δ is greater than zero.

For the remainder of this section fix a δ-hyperbolic space X equipped with a basepoint e and Gromov boundary $\partial_g X$ with metric $d_{\partial_g X}$ as in definition 3.6.5, which we assume by rescaling to have diameter at most 2. The proof of proposition 4.7.2 proceeds via *exponential* and *logarithm* maps as defined below.

Definition 4.7.3. Let $\mathcal{O}\partial_g X$ be the open cone over the Gromov boundary of X, and

$$\mathcal{O}_{\mathbb{N}}\partial_g X := \{t[\gamma] \in \mathcal{O}\partial_g X : t \in \mathbb{N}\}$$

the subset of 'integral points'. For each $[\gamma] \in \partial_g X$ choose a representative geodesic ray γ based at e. Define then the *exponential map* by

$$exp : \mathcal{O}_{\mathbb{N}} \partial_g X \to X$$

$$n[\gamma] \mapsto \gamma(n).$$

Let also $I \subseteq X$ be the image of the exponential map. For each $x \in I$, let γ_x be some choice of geodesic ray based at e (as used in the definition of exp) that passes through x. Define the *logarithm map* by

$$log : I \to \mathcal{O}_{\mathbb{N}} \partial_g X$$

$$x \mapsto \gamma_x(|x|).$$

Our aim is to show that exp induces an isomorphism

$$\text{`}exp^*\text{'} : K_*(\mathfrak{c}^r(X)) \xrightarrow{\cong} K_*(\mathfrak{c}^r(\mathcal{O}_{\mathbb{N}} \partial_g X)) \; ; \tag{4.13}$$

from here it is not too hard to complete the proof of proposition 4.7.2, which is is done in lemma 4.7.10. We show the existence of this isomorphism in two stages:

- first, show that the map 'exp^*' (which we have not yet defined!) induces an isomorphism

$$\text{`}exp^*\text{'} : K_*(\mathfrak{c}^r(I)) \longrightarrow K_*(\mathfrak{c}^r(\mathcal{O}_{\mathbb{N}} \partial_g X)) \; ; \tag{4.14}$$

- second, show that the inclusion $i : I \hookrightarrow X$ induces an isomorphism

$$i^* : K_*(\mathfrak{c}^r(X)) \xrightarrow{\cong} K_*(\mathfrak{c}^r(I)) \ . \tag{4.15}$$

Note, however, that *exp* is not in general a coarse map: its expansiveness can get arbitrarily bad. Indeed, the map 'exp^*' in the above is not that functorially induced by *exp*, but rather built out of the construction in lemma 4.7.5 below. We need the following simple geometric lemma.

Lemma 4.7.4. *Fix $N \in \mathbb{N}$, and say that γ_1, γ_2 satisfy*

$$d_{\partial_g X}([\gamma_1], [\gamma_2]) < c^{-1} e^{-\epsilon N}$$

for constants c, ϵ as in line (3.8) above. Then for all $n \leq N$ one has that

$$d_X(\gamma_1(n), \gamma_2(n)) \leq 4\delta.$$

Intuitively, the lemma says that if $[\gamma_1], [\gamma_2]$ are close in $\partial_g X$ then γ_1, γ_2 must stay close for a significant portion of their lengths.

Proof. Using the inequalities in line (3.8) above, one has

$$c^{-1} e^{-\epsilon \langle \gamma_1 | \gamma_2 \rangle} \leq d_{\partial_g X}([\gamma_1], [\gamma_2]) < c^{-1} e^{-\epsilon N},$$

whence $\langle \gamma_1 | \gamma_2 \rangle > N$. Fix $M \in \mathbb{N}$ large enough so that $M \geq N$ and $\langle \gamma_1(M) | \gamma_2(M) \rangle > N$.

Let now $n \leq N$. Using definition 3.6.3

$$\langle \gamma_1(m) | \gamma_2(M) \rangle \geq \min\{\langle \gamma_1(M) | \gamma_2(M) \rangle, \langle \gamma_1(m) | \gamma_1(M) \rangle\} - \delta$$

$$\geq \{N, m\} - \delta = m - \delta.$$

Moreover,

$$\langle \gamma_1(m) | \gamma_2(m) \rangle \geq \min\{\langle \gamma_1(m) | \gamma_2(M) \rangle, \langle \gamma_1(m) | \gamma_2(M) \rangle\} - \delta$$

$$\geq \{m - \delta, m\} - \delta = m - 2\delta.$$

Substituting in the definition of the Gromov product, this says that

$$\frac{1}{2}(m + m - d_X(\gamma_1(m), \gamma_2(m)) \geq m - 2\delta,$$

whence $d_X(\gamma_1(m), \gamma_2(m)) \leq 4\delta$ as required. □

Lemma 4.7.5.
- *there exists a contractive non-decreasing map* $r : \mathbb{N} \to \mathbb{N}$ *such that* $r(0) = 0$, $r(n) \to \infty$ *as* $n \to \infty$ *and so that if* $\rho : \mathcal{O}_{\mathbb{N}}\partial_g X \to \mathcal{O}_{\mathbb{N}}\partial_g X$ *is defined by* $\rho(m[\gamma]) = r(m)[\gamma]$ *then* $exp \circ \rho$ *is a coarse map.*
- *log is a coarse map.*

Proof. Look first at *exp.*

For each $n \in \mathbb{N}$, let $\delta(n)$ be a Lebesgue number for the cover

$$\{B_{\mathcal{O}_{\mathbb{N}}\partial_g X}(n[\gamma], c^{-1}e^{-\epsilon n}) : [\gamma] \in \partial_g X\}$$

of $n\partial_g X \subseteq \mathcal{O}_{\mathbb{N}}\partial_g X$. Note that $\delta : \mathbb{N} \to \mathbb{R}$ is then a strictly positive, non-increasing function. Define moreover a sequence $(n_k)_{k=0}^{\infty}$ by

$$n_k = \begin{cases} 0 & k = 0 \\ \min\{n \in \mathbb{N} : \frac{k^2}{n} \leq \min\{\delta(2k), 1\}\} & k > 0 \end{cases},$$

and define $r : \mathbb{N} \to \mathbb{N}$ by setting $r(n) = k$ if and only if $n \in [n_k, n_{k+1})$. Note that $r(0) = 0$ and $r(n) \to \infty$ as $n \to \infty$.

Now, let ρ be defined as in the statement of the lemma with r as above. Note that $exp \circ \rho$ is proper, as exp and ρ both are. Say that $N \in \mathbb{N}$ and that $m_1[\gamma_1], m_2[\gamma_2]$ are points in $\mathcal{O}_{\mathbb{N}}\partial_g X$ such that $d_{\mathcal{O}_{\mathbb{N}}\partial_g X}(m_1[\gamma_1], m_2[\gamma_2]) \leq N$. Say without loss of generality that $m_1 \geq m_2$, and assume that both are greater than n_N. Then

$$\begin{aligned} d_{\mathcal{O}_{\mathbb{N}}\partial_g X}(\rho(m_1[\gamma_1]), \rho(m_2[\gamma_2])) &\leq |r(m_1) - r(m_2)| + r(m_2) d_{\partial_g X}([\gamma_1], [\gamma_2]) \\ &\leq |r(m_1) - r(m_2)| + \frac{r(m_2)}{m_2} d_{\partial_g X}(m_1[\gamma_1], m_2[\gamma_2]) \\ &\leq |r(m_1) - r(m_2)| + \frac{r(m_2)^2}{m_2} \\ &\leq |r(m_1) - r(m_2)| + \delta(r(m_2) + r(m_2)). \end{aligned}$$

Hence one has that

$$
\begin{aligned}
d_X(exp \circ \rho(m_1[\gamma_1]), exp \circ \rho(m_2[\gamma_2])) \\
&\leq |r(m_1) - r(m_2)| + d(exp(r(m_1)[\gamma_1]), exp(r(m_2)[\gamma_2])) \\
&\leq |m_1 - m_2| + 4\delta \\
&\leq d_{\mathcal{O}_{\mathbb{N}}\partial_g X}(m_1[\gamma_1], m_2[\gamma_2]) + 4\delta.
\end{aligned}
$$

Finally, then, for *any* $x, y \in \mathcal{O}_{\mathbb{N}}\partial_g X$ such that $d(x, y) \leq N$, one has that

$$
\begin{aligned}
d(exp(x), exp(y)) &\leq d(x, y) + 4\delta + \text{diameter}(exp(B(0, n_N)) \\
&= d(x, y) + 4\delta + 2n_N.
\end{aligned}
$$

This completes the proof that $exp \circ \rho$ is coarse.

log is much simpler! Indeed, it is clearly proper. Say that $x, y \in I$ and that $d_X(x, y) \leq N$. Then

$$
d_{\mathcal{O}_{\mathbb{N}}\partial_g X}(log(x), log(y)) = ||x| - |y|| + \min\{|x|, |y|\} d_{\partial_g X}([\gamma_x], [\gamma_y])
$$

using the definition of the open cone metric from line (4.11). Using line (3.8), this in turn is bounded by

$$
N + \min\{|x|, |y|\} c e^{-\epsilon \langle \gamma_x | \gamma_y \rangle}. \tag{4.16}
$$

On the other hand, note that

$$\langle \gamma_x | \gamma_y \rangle \geq \langle x | y \rangle = \frac{1}{2}(|x| + |y| - d(x,y)) \geq \min\{|x|, |y|\} - N.$$

This implies that line (4.16) above is bounded by

$$N + c \sup_{t \in \mathbb{R}_+} \left(te^{-\epsilon t} \right) e^{\epsilon N},$$

which completes the proof. $\square$

The map appearing in line (4.13) above can now be defined to be that functorially induced by $exp \circ \rho : \mathcal{O}_{\mathbb{N}} \partial_g X \to X$. The following lemma essentially completes the proof of the isomorphism in line (4.14) above.

Lemma 4.7.6.
- *$log \circ exp \circ \rho$ is coarsely homotopic to the identity.*
- *$exp \circ \rho \circ log$ is coarsely homotopic to the inclusion $I \hookrightarrow X$.*

Proof. Let first $n[\gamma]$ be any point of $\mathcal{O}_{\mathbb{N}} \partial_g X$. Then

$$(log \circ exp)(m[\gamma]) = log(\gamma(m)) = \gamma'(m)$$

where γ' is some geodesic ray based at e and passing through $x = \gamma(m)$. Hence in particular

$$\langle \gamma | \gamma' \rangle \geq \langle \gamma(m) | \gamma'(m) \rangle = m$$

whence $d_{\partial_g X}([\gamma],[\gamma^{\prime}]) \leq ce^{-\epsilon m}$ and moreover

$$d_{\mathcal{O}_{\mathbb{N}}\partial_g X}(m[\gamma], log(exp(m[\gamma])) = d_{\mathcal{O}_{\mathbb{N}}\partial_g X}(m[\gamma], m[\gamma^{\prime}]) \leq cme^{-\epsilon m}$$

which is bounded independently of m. This says that $log \circ exp$ is close to the identity. As moreover ρ is coarsely homotopic to the identity via the map

$$H : \mathcal{O}_{\mathbb{N}}\partial_g X \times \mathbb{N} \to \mathcal{O}_{\mathbb{N}}\partial_g X$$

$$(n[\gamma], m) \mapsto \begin{cases} (r(n)+m)[\gamma] & 0 \leq m \leq n - r(n) \\ n[\gamma] & m > n - r(n) \end{cases},$$

this completes the proof of the first part.

For the second part consider the map defined by

$$H : I \times \mathbb{N} \to I$$

$$(x, n) \mapsto \begin{cases} exp((r(|x|)+m)[\gamma_x]) & 0 \leq m \leq |x| - r(|x|) \\ exp(log(x)) & m > |x| - r(|x|) \end{cases}.$$

As $exp \circ log$ simply *is* the inclusion $I \hookrightarrow X$, it suffices to show that that H is a coarse homotopy, and for this to show that H restricted to the set

$$Z := \{(x, n) \in I \times \mathbb{N} : 0 \leq n \leq |x| - r(|x|)\}$$

is a coarse map. Clearly this restriction of H is proper, while control over its expansiveness follows from convexity of the metric in a hyperbolic space (proposition 21 from chapter 2 of [26]). □

If the image of the map *exp* were dense (or coarsely dense), we would have now established the isomorphism in line (4.13). This need not be true in general, however: consider a copy of $\mathbb{N}$ with the interval $[0, n] \cap \mathbb{N}$ attached at the point n for all n. We thus need to prove the isomorphism in line (4.15) above.

The following two lemmas achieve this.

Lemma 4.7.7. *Let S be a subset of a discretely geodesic metric space Y. S is called* coarsely convex *if there exists $C \geq 0$ such that for any points $s, t \in S$, any geodesic segment $[s, t]$ between them is contained in the C-neighbourhood of S.*

Say now that Y is a hyperbolic space and that $S \subseteq Y$ a coarsely convex subset with respect to the constant C. Then there exists a retraction $\pi : Y \to S$ such that

$$d(\pi(x), \pi(y)) \leq d(x, y) + 2C + 8\delta.$$

for all $x, y \in Y$.

The map π is in fact unique up to closeness, but we will not need this. One can also do somewhat better than the estimate above: see lemma 7.3.D from [27], but the slightly simpler lemma above is enough for our purposes.

Proof. Define π by sending a point $x \in Y$ to *any* $s \in S$ realising $d(x, S)$; clearly this is a retraction. Assume for contradiction that there exist $x, y \in Y$ such that

$$d(\pi(x), \pi(y)) > d(x, y) + 2c + 8\delta.$$

Let γ be a geodesic segment between $\pi(x)$ and $\pi(y)$, so in particular γ lies within the C-neighbourhood of S. As $\pi(x)$ minimises the distance from x to a point in S and using lemma 17 and proposition 21 from chapter 2 of [26], one has that

$$\begin{aligned} d(x, \pi(x)) \leq d(x, \gamma) + C \leq \langle \pi(x) | \pi(y) \rangle_x + 4\delta + C \\ = \frac{1}{2}(d(x, \pi(x)) + d(x, \pi(y)) - d(\pi(x), \pi(y))) + C + 4\delta. \end{aligned}$$

Rearranging,

$$\begin{aligned} d(x, \pi(x)) \leq d(x, \pi(y)) - d(\pi(x), \pi(y)) + 2C + 8\delta \\ < d(x, \pi(y)) - d(x, y)) \leq d(y, \pi(y)); \end{aligned}$$

by symmetry however, one also has $d(y, \pi(y)) < d(x, \pi(x))$. Contradiction. □

Lemma 4.7.8. *The inclusion $i : I \hookrightarrow X$ is a coarse homotopy equivalence.*

Proof. Note first that I is coarsely convex with respect to the constant 4δ. Indeed, given $x, y \in I$, there are geodesic rays γ_x, γ_y based at e passing through them. Any geodesic between x and y must pass within 4δ of Image(γ_x)∪Image(γ_y) using 4δ-thinness of geodesic triangles (proposition 21 from chapter 2 of [26]). Let then $\pi : X \to I$ be any retraction as given in the previous lemma. As $\pi \circ i$ is the identity on I, it suffices to prove that π is coarsely homotopic to the identity.

To this end choose a geodesic segment, say γ^x, from x to $\pi(x)$ for each $x \in X$ and define

$$H : X \times \mathbb{N} \to X$$

$$(x, n) \mapsto \begin{cases} \gamma^x(n) & 0 \leq n \leq d(x, \pi(x)) \\ \pi(x) & n > d(x, \pi(x)) \end{cases}.$$

It suffices now to prove that π is proper, and that H restricted to the set

$$Z := \{(x, n) \in X \times \mathbb{N} : 0 \leq n \leq d(x, \pi(x))\}$$

is coarse; indeed, given these facts, H is a coarse homotopy in the sense of definition 4.3.11.

Assume first for contradiction that π is not proper. Then there must be a sequence of points (x_n) tending to infinity in X and such that $\pi(x_n)$ remains bounded for all n. Passing to a subsequence of (x_n), assume that it converges to some point $[\gamma] \in \partial_g X$, where γ is a geodesic ray based at e, and image(γ) is contained in I. Set now $y_n = \gamma(\langle x_n | \gamma \rangle)$. As (x_n) tends to infinity and $\pi(x_n)$ remains bounded there exists a constant C such that $|x_n| - C \leq d(x, \pi(x_n))$ for all n. Moreover, as y_n is in I and using the triangle inequality

$$|x_n| - C \leq d(x, \pi(x_n)) \leq d(x_n, y_n) \leq |x_n| - |y_n|,$$

whence $|y_n| \leq C$ for all n. However, $|y_n| = \langle x|\gamma\rangle \to \infty$ as $n \to \infty$ by definition of what it means for x_n to converge to $[\gamma]$ (see definition 3.6.5). Contradiction.

To complete the proof, we will show that H restricted to Z is coarse. Proposition 25 from chapter 2 of [26] and lemma 4.7.7 show that

$$d(H(x,n),H(y,m)) \leq d(x,y) + 8\delta + 4\delta + 2\delta$$

for all $(x,n),(y,n) \in Z$. Properness of H follows from properness of π. $\square$

Corollary 4.7.9. $\exp \circ \rho$ *induces an isomorphism*

$$(exp \circ \rho)^* : K_*(\mathfrak{c}^r(X)) \xrightarrow{\cong} K_*(\mathfrak{c}^r(\mathcal{O}_{\mathbb{N}}\partial_g X)) .$$

Proof. Indeed, using lemmas 4.7.6 and 4.7.8 and theorem 4.3.13 (more precisely the related version in remark 4.3.14) the maps

$$i^* : K_*(\mathfrak{c}^r(X)) \xrightarrow{\cong} K_*(\mathfrak{c}^r(I))$$

$$(exp \circ \rho)^* : K_*(\mathfrak{c}^r(I)) \xrightarrow{\cong} K_*(\mathfrak{c}^r(\mathcal{O}_{\mathbb{N}}\partial_g X))$$

are both isomorphisms. The map in the statement of the lemma is their composition. $\square$

From now on in this section, we identify $K_*(\mathfrak{c}(\mathcal{O}\partial_g X))$ and $K_*(\mathfrak{c}(\mathcal{O}_{\mathbb{N}}\partial_g X))$ via the canonical coarse equivalence $\mathcal{O}_{\mathbb{N}}\partial_g X \hookrightarrow \mathcal{O}\partial_g X$. To complete the proof of proposition 4.7.2, let

$$i^{\partial_g X} : C(\partial_g X) \otimes \mathcal{K} \to \mathfrak{c}^r(\mathcal{O}_{\mathbb{N}}\partial_g X)$$

be as in line (4.12) (having applied the identification above). Proposition 4.5.4 and corollary 4.7.9 imply that there are isomorphisms in K-theory

$$K_*(C(\partial_g X) \otimes \mathcal{K}) \xrightarrow{i_*^{\partial_g X}} K_*(\mathfrak{c}(\mathcal{O}_{\mathbb{N}} \partial_g X)) \xleftarrow{exp^*} K_*(\mathfrak{c}(X)) \,.$$

The next lemma thus completes the proof of proposition 4.7.2.

Lemma 4.7.10. *As maps on K-theory, $i_*^g = ((\exp \circ \rho)^*)^{-1} \circ i_*^{\partial_g X}$.*

Proof. It is sufficient to prove that $(\exp \circ \rho)^* \circ i_*^g = i_*^{\partial_g X}$ as maps $K_*(C(\partial_g X) \otimes \mathcal{K}) \to K_*(\mathfrak{c}(\mathcal{O}\partial_g X))$. Let $f \in C(\partial_g X) \otimes \mathcal{K}$ represent a K-theory class $[f]$ (the K-groups of this algebra are generated by such elements).

Extend f to some $\tilde{f} \in C(\overline{X}^g) \otimes \mathcal{K}$. Then $\exp^* \circ i_*^g[f]$ is represented by the image of the function

$$f_1 : \mathcal{O}\partial_g X \to \mathcal{K}$$

$$t[\gamma] \mapsto \tilde{f}(\gamma(r(t))) \quad \text{(which tends to } f([\gamma]) \text{ as } t \to \infty)$$

under the quotient by the functions $X \mapsto \mathcal{K}$ that vanish at infinity (this follows from the proof of proposition 4.2.3). On the other hand, $i_*^{\partial_g X}[f]$ is represented by the image of the map

$$f_2 : \mathcal{O}\partial_g X \to \mathcal{K}$$

$$t[\gamma] \mapsto f([\gamma])$$

under the quotient by the functions that vanish at infinity (this map is not well-defined at zero, but that doesn't matter up to taking the quotient).

It suffices to show that $f_1 - f_2$ vanishes at infinity on $\mathcal{O}_{\mathbb{N}}\partial_g X$. Say otherwise for contradiction, and let $n_k[\gamma_k]$ be a sequence of points tending to infinity in $\mathcal{O}\partial_g X$ on which $f_1 - f_2$ is bounded below. Passing to a subsequence if necessary, assume that $[\gamma_k]$ converges to some $[\gamma]$ in $\partial_g X$. The it is not hard to check that both $f_1(n_k[\gamma_k])$ and $f_2(n_k[\gamma_k]))$ converge to $f([\gamma])$. This contradiction completes the proof. □

Remark 4.7.11. Note that in the previous two sections, we established a version of the stable Higson conjecture 4.2.7 directly, and *use* it to show that the inclusion of some geometric boundary into the stable Higson corona induces an isomorphism on K-theory. We have not proved the stable Higson conjecture for (torsion free) hyperbolic groups above, but it does follow from our results.

Indeed, assume that Γ is a torsion free word hyperbolic group. Then by théorème 12 from chapter 4 of [26] the *Rips complex* $P_d(\Gamma)$ provides a model for $E\Gamma$ for d suitably large. Consider then the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(P_d(\Gamma),\mathcal{K}) & \longrightarrow & C(\overline{P_d(\Gamma)}^g,\mathcal{K}) & \longrightarrow & C(\partial_g\Gamma,\mathcal{K}) & \longrightarrow & 0\,. \\
 & & \Vert & & \downarrow & & \downarrow{\scriptstyle i^g} & & \\
0 & \longrightarrow & C_0(P_d(\Gamma),\mathcal{K}) & \longrightarrow & \bar{\mathfrak{c}}(P_d(\Gamma)) & \longrightarrow & \mathfrak{c}(\Gamma) & \longrightarrow & 0
\end{array}$$

The first and third vertical maps induce isomorphisms on K-theory, whence the second does by the five lemma. However, the Gromov compactification of $P_d(\Gamma)$ is contractible by theorem 1.2 in [6]. This thus proves the stable Higson conjecture, whence also that the coarse co-assembly map is an isomorphism.

Chapter 5

An index theorem for $\mathbb{Z}^n$, after Deundyak and Shteinberg

In this section we go most of the way to proving our main result (theorem 5.0.1) on indices of band-dominated operators on $\mathbb{Z}^n$ with slowly oscillating coefficients. The result is a special case of that in the announcement [15] of V.M. Deundyak and B.Ya. Shteinberg. The proof we give is very different, however, at least insofar as I understand the hints at a proof given in [15].

We do not complete the proof in this chapter, however: this is done in section 6.1 using *asymptotic morphisms.* The approach in 6.1 also produces a more computable 'index formula'; see theorems 6.1.6 and 6.1.10. Indeed, it would have been faster to use asymptotic morphisms throughout, but we thought it might be preferable to use this section to give at least a statement in line with [15].

The proof of theorem 5.0.1 given below proceeds via a reduction to the Atiyah-Singer index theorem for the n-torus; proposition 3.4.1 thus plays a central rôle. Apart from Atiyah-Singer, the other main ingredient is the computation of the K-theory of the stable Higson corona given in proposition 4.5.4 for open cones.

In order to state the theorem, define

$$X := \{x \in \mathbb{R}^n : \|x\| \geq 1\}, \tag{5.1}$$

and let $\overline{X}^h$ denote its Higson compactification. Include $\mathbb{Z}^n$ in X via any map including the non-zero integer points in $\mathbb{R}^n$ in the usual way. Any two such maps are close coarse equivalences, whence this prescription induces a canonical isomorphism of Higson coronas by theorem A.10. We thus have inclusions

$$i_\infty : \partial_h \mathbb{Z}^n \to \overline{X}^h \text{ and } i_1 : S^{n-1} \to \overline{X}^h$$

at infinity and one respectively. Note moreover that X is homeomorphic to $S^{n-1} \times [1, \infty)$, which is properly homotopy equivalent to the empty set. The K-theory long exact sequence for the pair $(\overline{X}^h, \partial_h \mathbb{Z}^n)$ thus implies that i_∞ induces an isomorphism on K-theory, and so one may define a map

$$\beta := i_1^* \circ (i_\infty^*)^{-1} : K^*(\partial_h \mathbb{Z}^n) \to K^*(S^{n-1}). \tag{5.2}$$

Note moreover that β induces a map

$$(\beta \times 1) : K^*(\partial_h \mathbb{Z}^n \times \mathbb{T}^n) \to K^*(S^{n-1} \times \mathbb{T}^n).$$

Here then is the main result of this chapter.

Theorem 5.0.1. *Let $k \geq 1$ and $F \in M_k(C_h(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n)$ be a Fredholm band-dominated operator on $\mathbb{Z}^n$ with slowly oscillating coefficients and values in $M_k(\mathbb{C})$. Let $[\sigma(F)] \in K^1(\partial_h \mathbb{Z}^n \times \mathbb{T}^n)$ be the K-theory class of its (invertible) symbol (cf. line 3.3 above, and preceding comments).*

Let $Ind_T : K^1(S^{n-1} \times \mathbb{T}^n) \to \mathbb{Z}$ *be the Atiyah-Singer topological index map. Then*

$$Index(F) = Ind_T \circ (\beta \times 1)[\sigma(F)].$$

As stated earlier, the results of section 6.1 allow a more computable looking presentation; see theorems 6.1.6 and 6.1.10. The fact that we consider matrix-valued operators is important: corollary 6.1.8 below says that there exist operators in $M_k(C_h(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n)$ with non-zero index if and only if $k \geq n$.

The proof of theorem 5.0.1 basically follows from consideration of a commutative diagram that we introduce in line (5.3) below. Note first that $\overline{\mathbb{Z}^n}^s$ is a geometric compactification of $\mathbb{Z}^n$ in the sense of definition 4.4.1, i.e. there is an injective $*$-homomorphism

$$C(\overline{\mathbb{Z}^n}^s) \to C_h(\mathbb{Z}^n).$$

This map induces all the vertical arrows between the bottom and middle rows in the diagram in line (5.3) below. Fix moreover a rank-one projection $p \in \mathcal{K}$ and define inclusions by

$$C(X) \to C(X) \otimes \mathcal{K}$$

$$f \mapsto f \cdot p;$$

combining these with the second inclusion from lemma 4.4.2 above gives rise to the maps between the middle and top rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(\mathbb{Z}^n, \mathcal{K}) & \longrightarrow & \bar{\mathfrak{c}}(\mathbb{Z}^n) & \longrightarrow & \mathfrak{c}(\mathbb{Z}^n) & \longrightarrow & 0\,. \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle j^h} & & \\ 0 & \longrightarrow & C_0(\mathbb{Z}^n) & \longrightarrow & C_h(\mathbb{Z}^n) & \longrightarrow & C(\partial_h \mathbb{Z}^n) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle j^s} & & \\ 0 & \longrightarrow & C_0(\mathbb{Z}^n) & \longrightarrow & C(\overline{\mathbb{Z}^n}^s) & \longrightarrow & C(S^{n-1}) & \longrightarrow & 0 \end{array} \tag{5.3}$$

Now, consider the stable Higson compactification of X as in line (5.1) above. Just as in that case, there are maps 'at infinity and one'. These are now quotient maps, however, as they occur at the level of C^*-algebras rather than spaces; we denote them

$$q^\infty : \bar{\mathfrak{c}}(X) \to \mathfrak{c}(\mathbb{Z}^n) \text{ and } q^1 : \bar{\mathfrak{c}}(X) \to C(S^{n-1}) \otimes \mathcal{K}.$$

Just as before, the fact that $C_0(X, \mathcal{K})$ is K-theoretically trivial implies that q^∞ is an isomorphism; we may thus form

$$\beta_S := (q^1_*) \circ (q^\infty_*)^{-1} : K_*(\mathfrak{c}(\mathbb{Z}^n)) \to K_*(C(S^{n-1}) \otimes \mathcal{K}) \tag{5.4}$$

(the 'S' is for 'stable').

The first lemma below relates β and β_S, while the second applies the ideas of chapter 4 to show that β_S is an isomorphism.

Lemma 5.0.2. *Let β, β_S and j^h be as in lines (5.2), (5.4) and (5.3) above respectively. They are related by the formula*

$$\beta = \beta_S \circ j^h_* : K_*(C_h(\mathbb{Z}^n)) \to K_*(C(S^{n-1})),$$

where we have identified $C(\partial_h \mathbb{Z}^n)$ and $C_h(\partial_h \mathbb{Z}^n) \otimes \mathcal{K}$ on the level of K-theory in order to make sense of this.

Lemma 5.0.3. *$j^h \circ j^s : C(S^{n-1}) \otimes \mathcal{K} \to \mathfrak{c}(\mathbb{Z}^n)$ induces an isomorphism on K-theory, with inverse β_S.*

Moreover, there are maps between $K_*(C(S^{n-1} \times \mathbb{T}^n) \otimes \mathcal{K})$ and $K_*(\mathfrak{c}(\mathbb{Z}^n) \otimes C(\mathbb{T}^n))$ functorially induced by $j^h \circ j^s$ and β_S which also induce isomorphisms on K-theory.

The proof of these two lemmas is postponed to section 6.1. The basic idea in that section is to realize both β and β_S more explicitly, in fact as the maps induced on K-theory by certain asymptotic morphisms.

Assuming these two lemmas, we may now complete the proof of theorem 5.0.1. The maps in diagram (5.3) above are all $\mathbb{Z}^n$-equivariant. We may thus take crossed products by $\mathbb{Z}^n$ everywhere, getting a new commutative diagram. Moreover, as $\mathbb{Z}^n$ is an exact group (see definition-theorem B.5), this operation preserves exactness of the rows so the new diagram looks like

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (C_0(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n) \otimes \mathcal{K} & \longrightarrow & \bar{\mathfrak{c}}(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n & \longrightarrow & \mathfrak{c}(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow {\scriptstyle j^h \rtimes_r 1} & & \\
0 & \longrightarrow & C_0(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n & \longrightarrow & C_h(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n & \longrightarrow & C(\partial_h \mathbb{Z}^n) \rtimes_r \mathbb{Z}^n & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow {\scriptstyle j^s \rtimes_r 1} & & \\
0 & \longrightarrow & C_0(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n & \longrightarrow & C(\overline{\mathbb{Z}^n}^s) \rtimes_r \mathbb{Z}^n & \longrightarrow & C(S^{n-1}) \rtimes_r \mathbb{Z}^n & \longrightarrow & 0
\end{array} \quad (5.5)$$

Note, however, that one can make the following identifications in the above:

- proposition 3.4.1 identifies the bottom line with the Atiyah-Singer order zero pseudodifferential operator extension for $\mathbb{T}^n$;
- all the $\mathbb{Z}^n$ actions in the right-hand column are trivial (the proof is essentially the same as that of lemma 3.3.2), whence '$\cdot \rtimes_r \mathbb{Z}^n$' has the same effect as '$\cdot \otimes C(\mathbb{T}^n)$';
- by lemma 3.1.3, $C_0(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n \cong \mathcal{K}$.

Diagram (5.5) above thus becomes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}\otimes\mathcal{K} & \longrightarrow & \bar{\mathfrak{c}}(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n & \longrightarrow & \mathfrak{c}(\mathbb{Z}^n)\otimes C(\mathbb{T}^n) & \longrightarrow & 0\,. \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle j^h\otimes 1} & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & C_h(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n & \longrightarrow & C(\partial_h \mathbb{Z}^n \times \mathbb{T}^n) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle j^s\otimes 1} & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \Psi^0(\mathbb{T}^n) & \longrightarrow & C(S^*\mathbb{T}^n) & \longrightarrow & 0
\end{array} \tag{5.6}$$

This in turn gives rise to a commutative diagram of six-term exact sequences in K-theory. We will be interested only in the 'index map' portion of these exact sequences.

To this end, note that the map

$$\mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$$

$$K \mapsto K \otimes p$$

between the middle and top rows on the left-hand-side of (5.6) above induces an isomorphism on K-theory: indeed, it induces the canonical stabilization isomorphism. Hence the portion of the commutative diagram of six term exact sequences containing the index maps looks like

$$\begin{array}{ccc} K_1(\mathfrak{c}(\mathbb{Z}^n) \otimes C(\mathbb{T}^n)) & \xrightarrow{Ind_S} & \mathbb{Z}\,. \\ {\scriptstyle (j^h \times 1)_*}\uparrow & & \| \\ K^1(\partial_h \mathbb{Z}^n \times \mathbb{T}^n) & \xrightarrow{Ind} & \mathbb{Z} \\ {\scriptstyle (j^s \times 1)_*}\uparrow & & \| \\ K^1(S^*\mathbb{T}^n) & \xrightarrow{Ind_T} & \mathbb{Z} \end{array} \tag{5.7}$$

Here the horizontal maps are all the connecting maps ('index maps') in the K-theory long exact sequence; the bottom one has been identified with the Atiyah-Singer topological index map using proposition 3.4.1.

We are now finally ready to prove theorem 5.0.1.

Proof of theorem 5.0.1. Let F be as in theorem 5.0.1. It suffices to compute $Ind[\sigma(F)]$, which is equal to

$$Ind_S \circ (j^h \times 1)_*[\sigma(F)] = Ind_S \circ (j^h \times 1)_* \circ (j^s \times 1)_* \circ (\beta_S \times 1) \circ (j^h \times 1)_*[\sigma(F)]$$

using naturality of the index map in K-theory and lemma 5.0.3. Using naturality of the index map again, this is in turn equal to

$$Ind_T \circ (\beta_S \times 1) \circ (j^h \times 1)_*[\sigma(F)].$$

Finally, applying lemma 5.0.2, this is equal to

$$Ind_T \circ (\beta \times 1)[\sigma(F)],$$

which proves theorem 5.0.1. □

Chapter 6

Two applications of bivariant K-theory

In this chapter we will apply some ideas from bivariant K-theory, mainly to compute certain maps between K-theory groups.

The purpose of section 6.1 is to complete the proof of theorem 5.0.1 using *asymptotic morphisms*. This method also allows for a much more computable-looking statement of theorem 5.0.1: see theorems 6.1.6 and 6.1.10.

Section 6.2 looks at more general groups. Roughly speaking, the proof of theorem 5.0.1 proceeds via a reduction from the (complicated) Higson corona of $\mathbb{Z}^n$ to its simpler spherical boundary; it is the purpose of section 6.2 to extend this 'reduction' process to a much larger class of geometric boundaries associated to discrete groups.

6.1 Asymptotic morphisms

To call this section an application of 'bivariant K-theory' is perhaps disingenuous: we will not use E-theory, the full-blown bivariant theory built out of asymptotic morphisms, but rather a very concrete instance of the action of asymptotic morphisms on K-theory. In order to keep the exposition reasonably self-contained, and to establish which versions of certain constructions we are using, the basic definitions and proofs of (most of) the properties we need are given below.

Definition 6.1.1. Let A, B be C^*-algebras. Denote by $C_b([1,\infty), B)$ the continuous bounded functions from $[1,\infty)$ to B, which forms a C^*-algebra when equipped with pointwise operations and the supremum norm. Similarly, let $C_0([1,\infty), B)$ denote the C^*-algebra of continuous functions from $[1,\infty)$ to B that vanish at infinity.

An *asymptotic morphism from A to B*, denoted $\alpha : A \rightsquigarrow B$, is a $*$-homomorphism

$$\alpha : A \to \frac{C_b([1,\infty), B)}{C_0([1,\infty), B)}$$

Given $\alpha : A \rightsquigarrow B$ as above, let

$$s : \frac{C_b([1,\infty), B)}{C_0([1,\infty), B)} \to C_b([1,\infty), B) \tag{6.1}$$

be *any* set theoretic section of the quotient map. For each $t \in [1,\infty)$ let $e_t : C_b([1,\infty), B) \to B$ be the point evaluation at t and define

$$\alpha_t := e_t \circ s \circ \alpha : A \to B \tag{6.2}$$

The maps α_t depend on s, but not in a way that will be important for what follows. Note that for each $a \in A$ the map $t \mapsto \alpha_t(a)$ is continuous and for each $a_1, a_2 \in A$ and $\lambda, \mu \in \mathbb{C}$

- $\alpha_t(a_1^*) - \alpha_t(a_1)^*$
- $\alpha_t(a_1 a_2) - \alpha_t(a_2)\alpha_t(a_2)$
- $\alpha_t(\lambda a_1 + \mu a_2) - \lambda\alpha_t(a_1) - \mu\alpha_t(a_2)$

all tend to zero as $t \to \infty$. In some treatments of asymptotic morphisms, e.g. chapter 25 from [7], an asymptotic morphism is *defined* to be a family $\{\alpha_t\}_{t\in[1,\infty)}$ satisfying the properties above, while what we have defined would be called an *asymptotic equivalence class of asymptotic morphisms*; we follow [29] for our basic definitions.

Now, the properties above say that the maps $\alpha_t : A \to B$ get 'close' to being a $*$-homomorphism as $t \to \infty$. In particular, if $p \in A$ is a projection, then $\alpha_t(p)$ gets close to having the properties of a projection as $t \to \infty$ in the sense that

- $\alpha_t(p)^2 - \alpha_t(p)$
- $\alpha_t(p)^* - \alpha_t(p)$

both tend to zero as $t \to \infty$. Using the functional calculus, one can thus build a path of projections out of $\alpha_t(p)$ for all t suitably large; by the homotopy invariance of K-theory, this path of projections defines an element of $K_0(B)$. Suitably elaborating on this idea, one sees that asymptotic morphisms induce maps on K_0 groups.

We will mainly use the maps induced on K_1 groups by asymptotic morphisms, however, so give a detailed definition below which basically follows 25.1.6 from [7]; see chapter 1 of [29] or chapter 25 of [7] for a formal definition of the action of asymptotic morphisms on K_0 groups.

Definition-Lemma 6.1.2. Let $\alpha : A \rightsquigarrow B$ be an asymptotic morphism between C^*-algebras, which we may assume unital by extending it to the unitizations. Let s be any set-theoretic section as in line (6.1) above, which we (temporarily) assume unital. Let $\{\alpha_t : A \to B\}_{t\in[1,\infty)}$ be the family of maps defined in line (6.2) above.

Now, say $x \in M_k(A)$ is an invertible element representing a class in $K_1(A)$. We may apply α_t to x entry-wise get elements $\alpha_t(x) \in M_k(B)$ (note that we have abused notation slightly, also writing 'α_t' for this amplification of the original α_t to matrix algebras). Then for all t suitably large

$$\|\alpha_t(x)\alpha_t(x^{-1}) - 1\| < 1 \text{ and } \|\alpha_t(x^{-1})\alpha_t(x) - 1\| < 1,$$

whence both $\alpha_t(x)\alpha_t(x^{-1})$ and $\alpha_t(x^{-1})\alpha_t(x)$ are invertible in $M_k(B)$. This implies (elementary algebra) that $\alpha_t(x)$ itself is invertible for all t suitably large.

Provisionally define a map α_* on K_1 groups by sending $[x] \in K_1(A)$ to any of the classes $[\alpha_t(x)] \in K_1(B)$ for t suitably large.

Then α_* is well-defined and depends only on α.

Proof that this does define a map between K_1 groups. Fix first an invertible $x \in M_k(A)$. Note that all the elements $\alpha_t(x)$ (t suitably large) are homotopic through invertibles, whence $\alpha_t(x)$ defines the same element of $K_1(B)$ for all t suitably large. Moreover, if α'_t is defined via another splitting s', it follows that

$$\alpha_t(x) - \alpha'_t(x) \to 0 \text{ as } t \to \infty$$

whence the classes defined by $\alpha_t(x)$ and $\alpha'_t(x)$ in $K_1(B)$ are the same for all t suitably large.

To complete the proof, it suffices to show that $\alpha_*[x]$ does not depend on the chosen representative of the class $[x] \in K_1(A)$; this may be done by checking the properties in proposition 8.1.5 from [67] (technically, this proposition refers to unitaries while we are working with general invertibles, but this makes no difference). The only non-trivial condition to check is that if x, y are homotopic invertibles in $M_k(A)$ for some k, then $\alpha_*[x] = \alpha_*[y]$.

Now, we have already shown that for fixed x the class $[\alpha_t(x)] \in K_1(B)$ does not depend on the section s chosen in the definition of α_t. Hence by the Bartle-Graves selection theorem (cf. [7], 25.1.5) we may choose a continuous section s that moreover gets uniformly close to a $*$-homomorphism on compact sets as $t \to \infty$, and use this to define α_t. With such a choice of s, the maps α_t map homotopies of invertibles to homotopies of invertibles for all t suitably large. □

Remark 6.1.3. The fact that we may use any section s to define the maps α_t makes asymptotic morphisms enormously computationally convenient. Indeed, given a class $[x] \in K_1(A)$ we may choose s *based on* x. One can often do so in a way that makes the computation of the class $[\alpha_t(x)]$ (for t suitably large) relatively simple. We will repeatedly use this observation below.

Our main application of the theory above will come from the following asymptotic morphisms. Indeed, comparing these to the maps β and β_S from lines (5.2) and (5.4) in chapter 5 above essentially completes the proof of theorem 5.0.1.

In the proofs given below, we will focus on the K_1 case, the K_0 case being basically the same.

Definition 6.1.4. For each $t \in [1, \infty)$, evaluating at the sphere S_t of radius t in $\mathbb{R}^n$ defines a $*$-homomorphism

$$C_h(\mathbb{R}^n) \to C_b([1, \infty), C(S^{n-1})).$$

Passing to the quotient by $C_0(\mathbb{R}^n)$ on the left hand side, and making the canonical identifications

$$C(\partial_h \mathbb{Z}^n) \cong C(\partial_h \mathbb{R}^n) \cong \frac{C_h(\mathbb{R}^n)}{C_0(\mathbb{R}^n)},$$

defines a $*$-homomorphism

$$\alpha : C(\partial_h \mathbb{Z}^n) = \frac{C_h(\mathbb{R}^n)}{C_0(\mathbb{R}^n)} \to \frac{C_b([1, \infty), C(S^{n-1}))}{C_0([1, \infty), C(S^{n-1}))},$$

or in other words, an asymptotic morphism $\alpha : C(\partial_h \mathbb{Z}^n) \rightsquigarrow C(S^{n-1})$.

Similarly, starting the same construction with the stable Higson compactification of $\mathbb{R}^n$ defines an asymptotic morphism

$$\alpha^S : \mathfrak{c}(\mathbb{Z}^n) \rightsquigarrow C(S^{n-1} \otimes \mathcal{K}),$$

the 'stable version' of α.

The following lemma is the main point of these maps: they are a more computable way of defining the map β of Deundyak and Shteinberg, and its analogue β_S.

Lemma 6.1.5. *Let β and β_S be as in lines (5.2) and (5.4) respectively. Then*

$$\beta = \alpha_* \text{ and } \beta_S = \alpha^S_*.$$

Proof. We focus on the K_1-case for β_S and α^S; the proofs for K_0 and the 'unstable' versions are essentially the same. Recall that $\beta_S = q^1_* \circ (q^\infty_*)^{-1}$, so it suffices to prove that

$$\alpha^S_* \circ q^\infty_* = q^1_* : K_1(\bar{\mathfrak{c}}(X)) \to K_1(C(S^{n-1} \otimes \mathcal{K}),$$

where X is as in line (5.1) above.

Let then $[x]$ be a class in $K_1(\bar{\mathfrak{c}}(X))$. Its image under the left-hand-side above is that of $\widetilde{q^\infty(x)}$ restricted to the sphere of radius t for any t suitably large, where $\widetilde{q^\infty(x)}$ is *any* lift of $q^\infty(x)$ to $\bar{\mathfrak{c}}(X)$ (the fact that any lift will work is a consequence of remark 6.1.3). We may, however, take x itself for this lift, so $\alpha^S_* \circ q^\infty_*[x]$ can be represented by x restricted to the sphere of radius t for any t suitably large. Any of these choices is homotopic to $q^1(x)$, however. □

Having thus realized β and β_S as the maps induced by α and α^S on K-theory, we can now prove lemmas 5.0.2 and 5.0.3 above, thus completing the proof of theorem 5.0.1.

Proof of lemma 5.0.2. Lemma 6.1.5 shows that 5.0.2 is equivalent to the equality

$$\alpha_* = \alpha^S_* \circ j^h_* : K_1(C_h(\mathbb{Z}^n)) \to K_1(C(S^{n-1})).$$

The map $\alpha^S_* \circ j^h_*$ takes a class $[f]$ to the class of $\tilde{f}p + (1-p)$ restricted to the sphere of radius t for t suitably large, where $\tilde{f}$ is some lift of f to $\overline{\mathbb{R}^n}^h$ and $p \in \mathcal{K}$ is any rank one projection. However, the canonical stabilization isomorphism $K_1(C(S^{n-1})) \cong K_1(C(S^{n-1}) \otimes \mathcal{K})$ can be realized by sending a class $[x] \in K_1(C(S^{n-1}))$ to the class of $[xp + (1-p)]$ in $K_1(C(S^{n-1}) \otimes \mathcal{K})$. Hence $\tilde{f}p + (1-p)$ also represents the image of $[f]$ under α_*. □

Proof of lemma 5.0.3. The first part of this lemma is equivalent to the statement that $j^h \circ j^s : C(S^{n-1}) \otimes \mathcal{K} \to \mathfrak{c}(\mathbb{Z}^n)$ induces an isomorphism on K-theory, and that α^S_* is its inverse.

Consider $\mathbb{R}^n$ as the open cone over S^{n-1} as in example 4.5.2, part (4). Then, up to the stabilization isomorphism in K-theory and the canonical identification of $\mathfrak{c}(\mathbb{Z}^n)$ and $\mathfrak{c}(\mathbb{R}^n)$, the map $j^h \circ j^s$ is the same as the map $i^{S^{n-1}}$ of line (4.12). Hence by proposition 4.5.4 it is an isomorphism.

Consider moreover the map induced on K_1 by $\alpha^S \circ j^h \circ j^s$. Up to the stabilization isomorphism again, this takes a class $[f]$ to the restriction of any lift $\tilde{f}$ of f to the sphere of radius t for t suitably large. However, by remark 6.1.3, and as f is just a function on S^{n-1}, we may assume that $\tilde{f}$ simply *is* f on all spheres of radius t for $t \geq 1$. Hence $\alpha^S \circ j^h \circ j^s$ induces the identity on K-theory.

This says that α^S_* is a one-sided inverse to $j^h_* \circ j^s_*$; as the latter map is an isomorphism, however, α^S_* must be a two-sided inverse as required.

To complete the proof, it suffices to show that the $*$-homomorphism

$$(j^h \circ j^s) \otimes 1 : C(S^{n-1}) \otimes \mathcal{K} \otimes C(\mathbb{T}^n) \to \mathfrak{c}(\mathbb{Z}^n) \otimes C(\mathbb{T}^n)$$

and asymptotic morphism

$$\alpha^S \otimes 1 : \mathfrak{c}(\mathbb{Z}^n) \otimes C(\mathbb{T}^n) \to \frac{C_b([1,\infty), C(S^{n-1}) \otimes \mathcal{K} \otimes C(\mathbb{T}^n))}{C_0([1,\infty), C(S^{n-1}) \otimes \mathcal{K} \otimes C(\mathbb{T}^n))}$$

induce mutually inverse isomorphisms at the level of K-theory. A slight adaptation of proposition 4.5.4 shows that $((j^h \circ j^s) \otimes 1)_*$ is an isomorphism, however, and then the same argument as above shows that $(\alpha^S \otimes 1)_*$ is a one-sided, whence also two-sided, inverse. □

The following theorem is a first attempt at a more computable statement of theorem 5.0.1 .

Theorem 6.1.6. *Let $k \geq 1$ and $F \in M_k(C_h(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n)$ be a Fredholm band-dominated operator on $\mathbb{Z}^n$ with slowly oscillating coefficients and values in $M_k(\mathbb{C})$. Then its index may be computed by the following 'recipe':*

- *Using the fact that $\partial_h \mathbb{Z}^n = \partial_h \mathbb{R}^n$, extend all the matrix entries of $F = (f_{ij})$ to functions on $\mathbb{R}^n$ in any convenient manner (as concrete slowly oscillating functions on $\mathbb{Z}^n$ are often given as restrictions of slowly oscillating functions on $\mathbb{R}^n$, this may not be as difficult as it first sounds). Denote by $\tilde{f}_{ij}$ the extension of f_{ij} so obtained.*

- *For some $t_0 \in [1,\infty)$ and all $t \geq t_0$, the matrix $(\tilde{f}_{ij})$ will be invertible when restricted to the sphere of radius t. It thus defines a function, which we denote $\sigma_t(F)$, from $S^{n-1} \times \mathbb{T}^n$ to $GL(k,\mathbb{C})$. Assume $\sigma_t(F)$ is smooth by suitably approximating it.*

- *Let* $ch_k \in \Omega^*(GL(k,\mathbb{C}))$ *be the (concretely defined) Chern form. For all* $t \geq t_0$*, the index of* F *is then equal to*

$$\int_{S^{n-1}} \int_{\mathbb{T}^n} \sigma_t(F)^* ch_k.$$

Proof. Looking back at theorem 5.0.1 and using lemma 6.1.5, it suffices to prove that the recipe above computes the integer

$$Ind_T \circ (\alpha_* \times 1)[\sigma(F)].$$

Now, by the definition of α, and with the extra flexibility provided by remark 6.1.3, the element $(\alpha_* \times 1)[\sigma(F)] \in K^1(S^*\mathbb{T}^n)$ is equal to the class of $\sigma_t(F)$ for any t suitably large. It thus suffices to show that the Atiyah-Singer topological index map Ind_T : $K^1(S^*\mathbb{T}^n) \to \mathbb{Z}$ is computed by the formula

$$[x] \mapsto \int_{S^{n-1}} \int_{\mathbb{T}^n} x^* ch_k.$$

Up to multiplication by the Todd class of $T\mathbb{T}^n \otimes \mathbb{C}$ (the complexified tangent bundle of $\mathbb{T}^n$), however, this is precisely the formula for Ind_T given at the top of page 177 of [3]. Moreover, as $\mathbb{T}^n$ is trivialisable, this Todd class is 1. □

Remark 6.1.7. We may restate the recipe above in a way that more closely resembles the discussion in section 2.4 as follows:

1. choose a suitably good approximation of $\partial_h \mathbb{Z}^n$ by a sphere at infinity;

2. at each point x of this sphere, compute the 'limit operator'

$$\sigma_t(F_x) : \mathbb{T}^n \to GL(k, \mathbb{C})$$

of F;

3. compute the 'local index data'

$$\int_{\mathbb{T}^n} \sigma_t(F_x)^* ch_k$$

at each point $x \in S^{n-1}$;

4. integrate this local data over all of S^{n-1} to get the global index.

Theorem 6.1.6 has the following corollary.

Lemma 6.1.8. *There exist Fredholm operators in $M_k(C_h(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n)$ of non-zero index if and only if $k \geq n$.*

Proof. For the integral

$$\int_{S^{n-1}} \int_{\mathbb{T}^n} \sigma_t(F)^* ch_k$$

to be non-zero, the Chern form ch_k must have a component in dimension $2n - 1$ (i.e. the dimension of $\mathbb{T}^n \times S^{n-1}$). This is the case only if $k \geq n$.

On the other hand, for a proof that there *is* an index one Fredholm operator in $M_n(\Psi_0(\mathbb{T}^n))$ (and hence in $M_k(C_h(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n)$ for all $k \geq n$), see appendix A in [68]. □

Remark 6.1.9. Recall that example 2.2.1 shows that there are index one band-dominated Fredholm operators with *non*-slowly-oscillating coefficients and with *scalar* values on

$l^2(\mathbb{Z}^n)$ for all n. The above lemma is thus of interest as it suggests that any index theorem applying to *all* band-dominated operators on $\mathbb{Z}^n$ would have to be significantly different from theorems 5.0.1 and 6.1.6. Note that this 'dimension problem' does not apply in the case $n = 1$ studied in [54].

In the special case that $k = n$, we may equivalently state theorem 6.1.6 as follows.

Theorem 6.1.10. *Let $F \in M_n(C_h(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n)$ be a Fredholm band-dominated operator on $\mathbb{Z}^n$ with slowly oscillating coefficients and values in $M_n(\mathbb{C})$. Then its index may be computed by the following 'recipe':*

- *Using the fact that $\partial_h \mathbb{Z}^n = \partial_h \mathbb{R}^n$, extend all the matrix entries of $F = (f_{ij})$ to functions on $\mathbb{R}^n$. Denote by $\tilde{f_{ij}}$ the extension of f_{ij} so obtained.*
- *For some $t_0 \in [1, \infty)$ and all $t \geq t_0$, the matrix $(\tilde{f_{ij}})$ will be invertible when restricted to the sphere of radius t. It thus defines a function, which we denote $\sigma_t(F)$, from $S^{n-1} \times \mathbb{T}^n$ to $GL(n, \mathbb{C})$.*
- *For any $t \geq t_0$, consider the map $\sigma_t^0(F) : S^{n-1} \times \mathbb{T}^n \to S^{2n-1}$ defined by projecting the image of $\sigma_t(F)$ to the normalised first column of $GL(n, \mathbb{C})$. Hence $\sigma_t^0(F)$ has a degree as a map between oriented manifolds of the same dimension. One has then that*

$$Index(F) = Degree(\sigma_t^0(F))$$

  for any $t \geq t_0$.

Note that if $F \in M_k(C_h(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n)$ for any $k \geq n$, then $\sigma(F) : \partial_h \mathbb{Z}^n \times \mathbb{T}^n \to M_k(\mathbb{C})$ can be homotoped to a map $\sigma(F) : \partial_h \mathbb{Z}^n \times \mathbb{T}^n \to M_n(\mathbb{C}) \oplus \{1_{k-n}\}$, where 1_{k-n} is the identity in $k-n$ dimensions (see page 239 of [2]). Having performed, this homotopy, $Index(F)$ can be computed by the process above. This idea combined with lemma 6.1.8 provides an index formula for any k, but unfortunately the presence of the (non-trivial) phrase 'can be homotoped' in the above mean that it is not very constructive.

Proof. As with theorem 6.1.6, the proof boils down to showing that the Atiyah-Singer topological index map can be computed by the 'degree prescription' above. This is essentially proved in appendix A of [68]. See also [2] for a very accessible exposition of similar ideas. □

As a concluding comment to this section, we indicate how to apply some of the ideas above more generally. One has the following construction, generalising definition 6.1.4.

Example 6.1.11. Let X be a proper $CAT(0)$ metric space as in definition 3.5.2, and S_t be the 'sphere' of points at distance t from some fixed choice of basepoint x_0. In certain examples (e.g. manifolds of non-positive curvature, like the case of $\mathbb{R}^n$ underlying definition 6.1.4), these spheres are all homeomorphic, but this need not be the case: consider for example the infinite three-regular tree. Nonetheless, the definition of $\partial_v X$ guarantees the existence of inclusion $*$-homomorphisms $i_t : C(S_t) \to C(\partial_v X)$ for all $t \geq 0$.

Now, one might hope that evaluating at S_t and composing with i_t for all $t \in [1, \infty)$ defines a $*$-homomorphism,

$$\bar{\mathfrak{c}}(X) \to C_b([1, \infty), C(\partial_v X) \otimes \mathcal{K})$$

but this is *not* true in general (the three regular tree provides a counterexample again). However, passing to the quotient by $C_0(X, \mathcal{K})$ on the left-hand-side, this prescription *does* define a $*$-homomorphism

$$\alpha : \mathfrak{c}(X) = \frac{\bar{\mathfrak{c}}(X)}{C_0(X, \mathcal{K})} \to \frac{C_b([1, \infty), C(\partial_v X) \otimes \mathcal{K})}{C_0([1, \infty), C(\partial_v X) \otimes \mathcal{K})},$$

or in other words an asymptotic morphism $\alpha : \mathfrak{c}(X) \rightsquigarrow C(\partial_v X) \otimes \mathcal{K}$. This asymptotic morphism defines the inverse to i^v_* on the level of K-theory for essentially the same reasons as in the proof of lemma 5.0.3 above.

6.2 The Baum-Connes conjecture

A central stage in the proof of theorem 5.0.1 is that the natural inclusion

$$(C(S^{n-1}) \otimes \mathcal{K}) \rtimes_r \mathbb{Z}^n \to \mathfrak{c}(\mathbb{Z}^n) \rtimes_r \mathbb{Z}^n$$

induces an isomorphism on K-theory. In proving this, we are helped immensely by the triviality of the $\mathbb{Z}^n$ actions; thus we do not have to do much work in the proof of lemma 5.0.3 to use the isomorphism

$$K^*(S^{n-1}) \to K_*(\mathfrak{c}(\mathbb{Z}^n))$$

to produce a corresponding isomorphism of the K-theory groups of the crossed product algebras. Now, for a more general class of groups Γ we showed in sections 4.4 to 4.7

that there are ('nice') geometric boundaries $\partial\Gamma$ such that the (equivariant) inclusions $C(\partial\Gamma)\otimes\mathcal{K}\to\mathfrak{c}(\Gamma)$ induce isomorphisms $K^*(\partial\Gamma)\cong K_*(\mathfrak{c}(\Gamma))$.

One is thus led to ask whether the corresponding inclusions of crossed product algebras

$$(C(\partial\Gamma)\otimes\mathcal{K})\rtimes_r\Gamma\to\mathfrak{c}(\Gamma)\rtimes_r\Gamma$$

also induce isomorphisms on K-theory. The purpose of this section is to answer this question: we apply known cases of the Baum-Connes conjecture to establish that the inclusions of crossed product algebras above in many cases do induce isomorphisms on K-theory.

The main result of this section follows. The proof we give is not too difficult in of itself, but makes use of very deep results, due to J. Chabert, S. Echterhoff, N. Higson, G. Kasparov, H. Oyono-Oyono and J-L. Tu (amongst others); see [12] and [13] (these two are the only papers that we use directly) and the references contained therein.

Proposition 6.2.1. *Say Γ is a torsion free countable discrete group equipped with a metrizable geometric boundary $\partial\Gamma$ such that the canonical (Γ-equivariant) inclusion $i: C(\partial\Gamma)\otimes\mathcal{K}\to\mathfrak{c}(\Gamma)$ induces an isomorphism on K-theory. Say moreover that either:*

1. *Γ satisfies the Baum-Connes conjecture with arbitrary coefficients, or*

2. *Γ acts amenably on $\partial\Gamma$.*

Then i as above induces an isomorphism on K-theory of crossed product algebras

$$i_*: K_*(C(\partial\Gamma)\rtimes_r\Gamma)\xrightarrow{\cong}K_*(\mathfrak{c}(\Gamma)\rtimes_r\Gamma)\,.$$

In the final line above, we have made the canonical identifications

$$K_*((C(\partial\Gamma)\otimes\mathcal{K})\rtimes_r\Gamma)=K_*((C(\partial\Gamma)\rtimes_r\Gamma)\otimes\mathcal{K})=K_*(C(\partial\Gamma)\rtimes_r\Gamma).$$

Similarly, we will identify $C(\partial\Gamma)$ and $C(\partial\Gamma)\otimes\mathcal{K}$ on the level of K-theory whenever convenient in what follows.

The hypotheses cover several interesting classes of groups: fundamental groups of closed manifolds of negative curvature and torsion free lattices in semisimple Lie groups provide relatively 'classical' examples. Groups acting properly cocompactly by isometries on CAT(0) cube complexes and torsion free word hyperbolic groups also provide interesting (and large) classes of examples. In some of these cases, the K-theory groups of $C(\partial\Gamma)\rtimes_r\Gamma$ are computable using, for example, the techniques of [23] (cf. remark 6.2.4 below).

The hypothesis that Γ is torsion free seems likely to be unnecessary: roughly speaking, if it were omitted we would need to demand that the inclusion $C(\partial\Gamma)\otimes\mathcal{K}\to\mathfrak{c}(\Gamma)$ induces an isomorphism on K-theory that is 'equivariant with respect to finite subgroups of Γ'. I have not checked the details at this stage.

The next lemma constitutes the first step in the proof. It is basically a special case of theorem 1.5 from [13], as the proof makes clear.

Lemma 6.2.2. *Say Γ is a countable torsion free discrete group. Say moreover that A, B are separable Γ-algebras and $\phi : A\to B$ an equivariant $*$-homomorphism that induces* *an isomorphism on K-theory. Then the map*

$$\phi_* : K_*^{top}(\Gamma, A) \to K_*^{top}(\Gamma, B)$$

induced by ϕ is also an isomorphism.

Proof. Remark 1.2 from [13] says, in the language of that paper, that the families of functors

$$\{C_0(Z) \mapsto KK_*^H(C_0(Z), A)\}_{H=\Gamma,\{e\}} \quad \text{and}$$

$$\{C_0(Z) \mapsto KK_*^H(C_0(Z), B)\}_{H=\Gamma,\{e\}}$$

are both *going down functors* for Γ. Moreover ([13] definition 1.4), the map $\phi : A \to B$ induces a *going down transformation* between them. To get the result of the lemma, then, it suffices by [13], theorem 1.5, to show that for any finite dimensional real vector space V, the map

$$\phi_* : KK_*(C_0(V), A) \to KK_*(C_0(V), B)$$

induced by ϕ is an isomorphism. However up to a shift in degree Bott periodicity canonically identifies this map with the map $\phi_* : K_*(A) \to K_*(B)$ induced on K-theory by ϕ. Hence it is an isomorphism by assumption. □

The basic idea of the proof of proposition 6.2.1 (which was explained to me by P. Baum and N. Higson) is to use the above lemma and Baum-Connes with coefficients $C(\partial\Gamma)$ and $\mathfrak{c}(\Gamma)$ to get the result. However, the latter algebra is not separable (or even

σ-unital), so one cannot quite apply the usual version of the conjecture directly. It would be possible to use a version of the Baum-Connes conjecture with non-separable coefficients in what follows; for example it is true that both the domain and codomain of the Baum-Connes assembly map are continuous in the coefficient algebra, and this would suffice for our purposes (cf. for example the proof of lemma 2.10 in [36]). Exactly what we need does not seem to appear explicitly in the literature, however, so the following (admittedly somewhat ad-hoc) lemma seemed a simpler way to get around this issue.

Lemma 6.2.3. *Say $B \subseteq A$ is an inclusion of C^*-algebras that induces an isomorphism on K-theory, and say that B is separable. Then there exists a net $(A_i)_{i\in I}$ of separable C^*-subalgebras of A such that*

1. *each A_i contains B;*
2. *for each i, both inclusions*
$$B \to A_i \to A \tag{6.3}$$
induce isomorphisms on K-theory;
3. *$A = \cup_{i\in I} A_i$.*

Moreover, if Γ is a countable discrete group and X a metrizable locally compact Γ space, and if $B \subseteq A$ is an inclusion of $\Gamma - C(X)$ algebras, then the A_i can be chosen to be $\Gamma - C(X)$ algebras such that the inclusions in line (6.3) above are compatible with this extra structure.

Proof. Take $(A_i)_{i\in I}$ be the collection of *all* separable C^*-subalgebras of A satisfying conditions (1) and (2) from the lemma, and such that the inclusions in line (6.3) above are compatible with whatever $\Gamma - C(X)$ structure is present. Note that the collection (A_i) is non-empty as it contains B. It suffices to show that (A_i) forms a directed system (when partially ordered by inclusion) satisfying condition (3).

To say that (A_i) is directed means that for all elements A_i, A_j of this family there exists some $D \in (A_i)_{i\in I}$ such that $A_i, A_j \subseteq D$. Fix $A_i, A_j \in (A_i)_{i\in I}$ and let D_0 be the smallest C^*-subalgebra of A containing A_i and A_j, and closed under the Γ and $C(X)$ actions; note that D_0 is a separable C^*-algebra. As D_0 contains B, the inclusion $D_0 \to A$ induces a surjective map on K-theory, but it may not be injective. For any $x \in K_*(D_0)$ that maps to zero in $K_*(A)$ there must be a path of elements in (the unitization of) A that 'shows it to be zero'. There will be only countably many such paths (as $K_*(D_0)$ is countable), so the smallest C^*-subalgbera of A containing D_0 and all of these paths (in its unitization), and that is moreover closed under the Γ and $C(X)$ actions, is still separable. Call it D_1. Again, the map on K-theory induced by the inclusion of D_1 into A will be surjective, but may not be injective; repeat the above procedure, getting another separable C^*-subalgebra of A, say D_2.

Repeating this procedure countably many times, define

$$D = \overline{\cup_{n\in\mathbb{N}} D_n},$$

which is a separable $\Gamma - C(X)$ C^*-subalgebra of A that contains A_i and A_j. As it contains B, the map induced on K-theory by the inclusion $D \subseteq A$ must again be surjective. It must also be injective, however. Indeed, say that $x \in K_*(D)$ maps to zero in $K_*(A)$. x

can be represented by elements from (the unitization of) some D_N, and thus vanishes in $K_*(D_{N+1})$ (and so also $K_*(D)$) by construction of the D_n's.

To show property (3) from the lemma one may use exactly the same technique: given any $a \in A$, define D_0 to be the smallest $\Gamma - C(X)$ subalgebra of A containing B and a. Repeating the same inductive procedure gives $D \in (A_i)_{i \in I}$ containing a. □

Proof of proposition 6.2.1. Apply the above lemma to the inclusion $C(\partial\Gamma) \to \mathfrak{c}(\Gamma)$ to get a net $(A_i)_{i \in I}$ of $\Gamma - C(\partial\Gamma)$ algebras with the properties listed there. Say first that Γ satisfies condition (2) in the statement of the proposition. Then by theorem 0.4 from [12], the Baum-Connes assembly map for Γ with coefficients in any A_i

$$\mu_{A_i} : K_*^{\text{top}}(\Gamma, A_i) \to K_*(A_i \rtimes_r \Gamma) \tag{6.4}$$

is an isomorphism, as is that for $C(\partial\Gamma)$

$$\mu_{C(\partial\Gamma)} : K_*^{\text{top}}(\Gamma, C(\partial\Gamma)) \to K_*(C(\partial\Gamma) \rtimes_r \Gamma). \tag{6.5}$$

Of course, these isomorphisms also hold if Γ satisfies condition (1) from the statement.

The inclusions $C(\partial\Gamma) \to A_i$ induce isomorphisms

$$K_*^{\text{top}}(\Gamma, C(\partial\Gamma)) \xrightarrow{\cong} K_*^{\text{top}}(\Gamma, A_i) \tag{6.6}$$

for all i by lemma 6.2.2. Moreover, by naturality of the Baum-Connes assembly map, there exists for each $i \in I$ a commutative diagram

$$\begin{array}{ccc} K_*^{\mathrm{top}}(\Gamma, C(\partial\Gamma)) & \xrightarrow{\cong} & K_*^{\mathrm{top}}(\Gamma, A_i) \\ \downarrow \mu_{C(\partial\Gamma)} & & \downarrow \mu_{A_i} \\ K_*(C(\partial\Gamma) \rtimes_r \Gamma) & \longrightarrow & K_*(A_i \rtimes_r \Gamma) \end{array} ,$$

where the horizontal maps are induced by the inclusions, and the vertical ones are Baum-Connes assembly maps. The top map is an isomorphism by line (6.6) above, while the two vertical ones are isomorphisms by lines (6.4) and (6.5) above. Hence the inclusions $C(\partial\Gamma) \to A_i$ all induce isomorphisms

$$K_*(C(\partial\Gamma) \rtimes_r \Gamma) \to K_*(A_i \rtimes_r \Gamma) \tag{6.7}$$

of crossed product algebras.

Finally, as the reduced crossed product functor is continuous with respect to directed systems with injective connecting maps, one has that $\mathfrak{c}(\Gamma) \rtimes_r \Gamma = \lim_i (A_i \rtimes_r \Gamma)$. Using continuity of the K-functor, then, there is an isomorphism

$$\lim_{i \in I} K_*(A_i \rtimes_r \Gamma) \to K_*(\mathfrak{c}(\Gamma) \rtimes_r \Gamma).$$

Note however that line (6.7) implies in particular that all the connecting maps in the directed system (of graded abelian groups) on the left are isomorphisms. Hence *any* of the inclusions $A_i \rtimes_r \Gamma \to \mathfrak{c}(\Gamma) \rtimes_r \Gamma$ induces an isomorphism on K-theory, whence (by line (6.7) again) so too does the composition of inclusions

$$C(\partial\Gamma) \rtimes_r \Gamma \to A_i \rtimes_r \Gamma \to \mathfrak{c}(\Gamma) \rtimes_r \Gamma,$$

for any i, completing the proof. □

The following remark gives some examples, and an interesting consequence.

Remark and examples 6.2.4. It is natural to ask whether $\mathfrak{c}(\Gamma) \rtimes_r \Gamma$ is coarsely invariant, at least on the level of K-theory. For example, this is true of the closely related Roe algebra $C^*(|\Gamma|) \cong l^\infty(\Gamma, \mathcal{K}) \rtimes_r \Gamma$, as well as of $\mathfrak{c}(\Gamma)$ itself. Coarse invariance does not hold, however, as the following example shows.

Let Γ_g be the fundamental group of the surface of genus g; for $g > 1$, these groups all act properly cocompactly by isometries on the hyperbolic plane, so in particular are all coarsely equivalent. Moreover, the natural circular boundary of the hyperbolic plane provides a geometric boundary $\partial\Gamma_g$ satisfying the conditions of either proposition 4.6.3 or proposition 4.7.2. As each Γ_g satisfies Baum-Connes with coefficients (as e.g. it is a-T-menable), we may apply proposition 6.2.1 to get isomorphisms

$$K_*(C(\partial\Gamma_g) \rtimes_r \Gamma_g) \cong K_*(\mathfrak{c}(\Gamma) \rtimes_r \Gamma_g)$$

for all g. In [23], example 34, however, Emerson and Meyer show (not trivial!) that the left hand side is isomorphic to $\mathbb{Z}/(2g-2)\mathbb{Z} \oplus \mathbb{Z}^{2g+1}$, whence the right-hand-side cannot be coarsely invariant. A similar example is provided by free groups of rank at least two (which are all coarsely equivalent) and their Gromov boundaries (cf. [23], example 33):

here one sees that

$$K_*(\mathfrak{c}(F_n) \rtimes_r F_n) \cong K_*(C(\partial_g F_n) \rtimes_r F_n) \cong \mathbb{Z}/(n-1)\mathbb{Z} \oplus \mathbb{Z}^n.$$

Indeed, the paper [23] contains a wealth of techniques to compute, and examples of computations of, K-theory groups of the form $K_*(C(\partial\Gamma) \rtimes_r \Gamma)$.

Appendix A

Functors on coarse categories

In this appendix we define the notion of a *coarse structure*, and two associated categories. What follows below is mainly taken from the monograph [64].

Coarse structure does for large scale structure what topology does for small scale structure: it abstracts the relevant (and only the relevant) information contained in a metric structure on a set.

The extra generality given by using coarse structures is not really important for us: all of the examples of coarse spaces that we use are metrizable (see the first point in examples A.2 below), and therefore we would not have lost anything by restricting attention to the classical case of metric spaces. Nonetheless, we feel that dealing directly with abstract coarse structures is helpful as:

- coarse structures ignore much of the irrelevant (for our purposes) information contained in a metric structure;
- dealing directly with coarse structures makes it easier to tell when a given property is 'coarsely invariant';
- our main objects of study are discrete groups, and while these have an (more precisely, two) intrinsic coarse structure(s), they do not have an intrinsic metric.

In order not to stray too far from intuition, we will spell out what the definitions in this section mean in terms of a metric.

This appendix also establishes the categories $\mathcal{C}$ and $\mathcal{CC}$ which provide the natural framework for the results of chapter 4.

Recall the following set operations (they are in fact the operations of the *pair groupoid*, but this need not concern us here). Let X be a set and $E, F \subseteq X \times X$. Then the *diagonal*, *composition of E and F* and *inverse of E* are the sets

$$\Delta = \{(x, x) \in X \times X : x \in X\}$$

$$E \circ F = \{(x, z) \in X \times X : \exists y \in X \text{ such that } (x, y) \in E \text{ and } (y, z) \in F\}$$

$$E^{-1} = \{(x, y) \in X \times X : (y, x) \in E\}.$$

Here then is the definition of a coarse structure (cf. definition 2.3 from [64]).

Definition A.1. Let X be a set. A *coarse structure* on X is a collection $\mathcal{E}$ of subsets $E \subseteq X \times X$ such that:

1. $\Delta \in \mathcal{E}$;
2. all finite subsets of $X \times X$ are in $\mathcal{E}$;
3. if $E \in \mathcal{E}$ and $F \subseteq E$, then $F \in \mathcal{E}$;
4. if $E, F \in \mathcal{E}$, then $E \circ F \in \mathcal{E}$;
5. if $E, F \in \mathcal{E}$, then $E \cup F \in \mathcal{E}$;
6. if $E \in \mathcal{E}$, then $E^{-1} \in \mathcal{E}$.

The sets in a coarse structure will be called *entourages* or *controlled sets*.

Definition 2.3 from [64] does not stipulate that a coarse structure contains all finite sets; in the language of definition 2.11 from that book, condition (2) above is equivalent to our demanding that all our coarse spaces be *connected*.

Note that any collection C of subsets of $X \times X$ generates a coarse structure in the obvious way: simply take the intersection of all coarse structures containing C. The following two cases are our most important examples.

Examples A.2. 1. Say X is a metric space. Then the sets

$$E_R = \{(x, y) \in X \times X : d(x, y) \leq R\}$$

for $R \geq 0$ generate a coarse structure. This is the motivating example for definition A.1. A coarse structure $\mathcal{E}$ is called *metrizable* if there exists a metric on X such that $\mathcal{E}$ is the coarse structure associated to the metric by the above construction.

2. Say Γ is a (discrete) group. Then the sets

$$E_S = \{(g, h) \in \Gamma \times \Gamma : gh^{-1} \in S\}$$

as S ranges over finite subsets of Γ generate a coarse structure. It is called the *right-invariant coarse structure on* Γ, the name coming from the fact that for any finite $S \subseteq \Gamma$ and $g, h \in \Gamma$, $(g, h) \in E_S$ if and only if $(gk, hk) \in E_S$ for all $k \in \Gamma$. Similarly, the *left-invariant coarse structure on* Γ is generated by the sets

$$E_S = \{(g, h) \in \Gamma \times \Gamma : g^{-1}h \in S\}.$$

These structures are not in general the same, but they are coarsely equivalent via the map $g \mapsto g^{-1}$ (see definition A.5 below); we use both of these coarse structures at different points in the main piece. We sometimes write $|\Gamma|$ for a group Γ considered only as a coarse space.

3. More generally, say X is a topological space equipped with a left action of Γ by homeomorphisms. For each compact set K in X, define

$$E_K = \{(gx, x) \in X \times X : x \in K, g \in \Gamma\}.$$

   The *action coarse structure* on X is the coarse structure generated by the E_K. In the case $X = \Gamma$, this is the same as the right-invariant coarse structure. In the case X is a proper metric space on which Γ acts properly, cocompactly and by isometries, this is the same as the metric coarse structure on X.

A coarse structure is metrizable if and only if it is countably generated: see theorem 2.55 from [64]. In particular, note that the right (or left) invariant coarse structure on a (discrete) group Γ is metrizable if and only if the group itself is countable. In particular, if Γ is finitely generated, then one can in fact metrize these coarse structures by some choice of (left or right) word metric; see for example the start of section 2.3 from [73] for the construction of word metrics, as well as metrics on general countable discrete groups. Note however that these metrics are not intrinsic to the group, while the coarse structure is.

Our coarse structures will always be compatible with a topology in the sense of the following definition (cf. definition 2.22 from [64]). Following proposition 2.16 from

[64], if X is equipped with a coarse structure $\mathcal{E}$, then a subset B of X is called *bounded* if $B \times B$ is an entourage. If the coarse structure comes from a metric, 'bounded' has its usual meaning.

Definition A.3. Say X is a topological space equipped with a coarse structure $\mathcal{E}$. The topology and coarse structure are said to be *compatible* if:

1. some neighbourhood of the diagonal is an entourage;
2. all bounded sets are relatively compact.

A locally compact, paracompact Hausdorff topological space equipped with a compatible coarse structure will be called a *coarse space*.

Note that as a consequence of condition (1) above and conditions (2) and (4) from definition A.1, all compact sets in a coarse space are bounded. Thus in a coarse space, a set is bounded if and only if it is relatively compact.

In terms of examples A.2 above, the coarse structure associated to a metric is compatible with the topology associated to the metric if and only if the metric is *proper*, meaning that closed balls are compact. The right (or left) invariant coarse structure on a discrete group is always compatible with its (discrete) topology, as the bounded sets are precisely the finite sets. Similarly, the action coarse structure is always compatible with a locally compact, paracompact and Hausdorff topology on X.

We will now introduce some classes of maps between coarse spaces.

Definition A.4. Say $f : X \to Y$ is a map between coarse spaces. It is called a *coarse map* if the map $(x, x) \mapsto (f(x), f(x))$ takes entourages to entourages, and if the pullback under f of bounded sets in Y are bounded in X.

Two coarse maps $f, g : X \to Y$ are called *close* if the set $\{(f(x), g(x)) \in Y \times Y : x \in X\}$ is an entourage.

If (X, d_X) and (Y, d_Y) are proper metric spaces, then a map $f : X \to Y$ is coarse if and only if both:

- for all $R > 0$ there exists $S > 0$ such that if $d_X(x_1, x_2) \leq R$, then $d_Y(f(x_1), f(x_2)) \leq S$;
- f pulls back bounded (equivalently, relatively compact) sets to bounded (relatively compact) sets.

Note that closeness is an equivalence relation on coarse maps. Here then is the definition of the categories we will be working with.

Definition A.5.

- The category $\mathcal{C}$ (for 'coarse') has for objects the coarse spaces. Its morphisms are closeness classes of coarse maps. An isomorphism in $\mathcal{C}$ is called a *coarse equivalence.*
- The category $\mathcal{CC}$ (for 'continuous coarse') has for objects the coarse spaces. Its morphisms are the continuous coarse maps (*not* up to closeness).

Remark A.6. In some works, for example [62] and [22], the authors consider a category where the morphisms are closeness classes of *Borel* coarse maps; temporarily denote the subcategory of $\mathcal{C}$ thus defined by $\mathcal{C}_b$. It is occasionally useful to note that any functor on $\mathcal{C}_b$ automatically extends to one on $\mathcal{C}$.

Indeed, let X be a coarse space. Then our topological assumptions imply that there exist a discrete subspace $Y \subseteq X$ equipped with a Borel retract $r : X \to Y$ and an entourage E on X such that $N_E(Y) = X$ (this says that Y is *coarsely dense* in X). If then $f : X \to Z$ is any coarse map, the map $f \circ r : X \to Z$ is Borel (as f restricted to Y is continuous), and close to the original f. Moreover, up to closeness it does not depend on the choice of Y and r. If then $\mathcal{F}$ is a functor on $\mathcal{C}_b$, we may *define* the image of f under $\mathcal{F}$ to be $\mathcal{F}(f \circ r)$; the comments above show that this does not depend on the pair (Y, r), and agrees with $\mathcal{F}(f)$ whenever the latter makes sense.

We will also need the following definition of a product coarse structure, which corresponds to the usual (i.e. category theoretical) notion of 'product' in the category $\mathcal{C}$.

Definition A.7. Let X and Y be coarse spaces. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the coordinate projections. The *product coarse structure* on $X \times Y$ is the coarse structure consisting of all $E \in (X \times Y) \times (X \times Y)$ such that both $(\pi_X \times \pi_X)(E) \subseteq X \times X$ and $(\pi_Y \times \pi_Y)(E) \subseteq Y \times Y$ are entourages for the coarse structures on X and Y respectively.

As an example of functors on the coarse category (and as we use them in section 4.1), we give definitions of the Roe algebra and Higson corona that make sense for any coarse space. This generalizes definitions 2.1.5 and 3.3.1 (which are the case where the coarse space is a group equipped with its right or left invariant coarse structure respectively). These definitions have been relegated to this appendix as we do not make much use of them in the main piece.

Definition A.8. Let X be a coarse space. If $f : X \to \mathbb{C}$ is a bounded function and E is an entourage the *variation of f at scale E* is defined to be

$$(\nabla_E f)(x) = \sup_{(x,y)\in E} |f(x) - f(y)|.$$

A function $f : X \to \mathbb{C}$ is called a *Higson function* if it is bounded, continuous and if for any entourage E, $(\nabla_E f)(x) \to 0$ as $x \to \infty$ in X.

The Higson functions form a C^*-subalgebra, denoted $C_h(X)$, of the C^*-algebra of all bounded functions on X. They also contain $C_0(X)$, whence by the Gelfand-Naimark theorem they correspond to the continuous functions on some compactification $\overline{X}^h$ of X. This compactification is called the *Higson compactification of X*, and the associated corona $\partial_h X := \overline{X}^h \backslash X$ the *Higson corona of X*.

Definition A.9. Let X be a coarse space. A representation of $C_0(X)$ on a Hilbert space $\mathcal{H}$ is called *ample* if it is faithful, and if no non-zero $f \in C_0(X)$ acts as a compact operator.

An operator T on $\mathcal{H}$ is called *locally compact* if fT and Tf are compact for all $f \in C_0(X)$. T is said to be of *finite propagation* if there exists an entourage $E \subseteq X \times X$ such that whenever $f, g \in C_0(X)$ satisfy $\operatorname{supp}(f \times g) \cap E = \emptyset$, then $fTg = 0$.

The locally compact, finite propagation operators form a $*$-algebra on X. Its closure is called the *Roe algebra of X* and denoted $C^*(X)$.

The Higson corona defines a functor on the category $\mathcal{C}$. The Roe algebra does not: it does not depend on the representation $\mathcal{H}$, but only up to non-canonical isomorphism, and there are no canonical choices of $*$-homomorphisms associated to morphisms in either

$\mathcal{C}$ or even $\mathcal{CC}$. Nonetheless, the assignment $X \mapsto K_*(C^*(X))$ *does* define a functor on $\mathcal{C}$. We state these results as a theorem.

Theorem A.10. *The following are covariant functors on the coarse category $\mathcal{C}$:*

1. $X \mapsto \partial_h X$, *from $\mathcal{C}$ to the category of compact Hausdorff spaces and continuous maps;*

2. $X \mapsto K_*(C^*(X))$, *from $\mathcal{C}$ to the category of graded abelian groups and graded homomorphisms.*

See [64], proposition 2.41, for what is essentially a proof of part (1) and [43], section 6.3, for a proof of part (2).

Finally we will give a general definition of the *uniform Roe algebra.* This definition does not make sense for all coarse spaces, nor does it define a functor on the categories where it is defined. Coarsely equivalent spaces do have Morita equivalent uniform Roe algebras, however; see section 2 of [10].

Definition A.11. A coarse space X is called *uniformly locally finite* if it is equipped with the discrete topology and if for each entourage E on X there is a uniform bound on the cardinalities of the sets

$$\{y \in X : (x, y) \in E\} \text{ and } \{y \in X : (y, x) \in E\}$$

as x ranges over X. For example, a discrete group equipped with either its right or left invariant coarse structure is uniformly locally finite.

Let X be a uniformly locally finite coarse space. A bounded operator T on $l^2(X)$ is of *finite propagation* if there exists an entourage E such that $\langle T\delta_x, \delta_y \rangle = 0$ whenever $(x, y) \notin E$ (i.e. all matrix coefficients of T are supported in E).

The *uniform Roe algebra of* X, denoted $C_u^*(X)$, is the C^*-subalgebra of $\mathcal{B}(l^2(X))$ generated by the finite propagation operators.

Appendix B

Crossed products and exact groups

In this appendix we start with a precise definition of the reduced crossed product of a C^*-algebra A by a discrete group Γ, which seemed helpful to establish conventions. We next use this definition to prove that if A is commutative then the reduced crossed product admits an embedding into a certain fixed point algebra; this simple fact is used in section 3.1.

The remainder of the appendix defines the class of *exact groups* and sketches some of their properties.

Definition B.1. Let A be a C^*-algebra equipped with an action α of a discrete group Γ by $*$-automorphisms. Let $A \rtimes_{alg} \Gamma$ be the $*$-algebra of formal linear combinations

$$\sum_{g \in \Gamma} a_g u_g,$$

where the a_g are elements of A which are non-zero for only finitely many $g \in \Gamma$ and the u_g are formal symbols. Addition is defined using addition in A, while multiplication and adjoint are defined using the corresponding operations in A and subject to the additional rules

$$u_g u_h = u_{gh}$$

$$u_g a = \alpha_g(a) u_g$$

$$u_g^* = u_{g^{-1}}.$$

Now, given a faithful representation $\pi : A \to \mathcal{B}(\mathcal{H})$, define a representation $\tilde{\pi}$ of A on $\mathcal{H} \otimes l^2(\Gamma)$ by

$$(\tilde{\pi} a)(v \otimes \delta_g) = \pi(\alpha_{g^{-1}} a)(v) \otimes \delta_g.$$

Define moreover operators

$$\tilde{\pi}(u_g) \in \mathcal{B}(\mathcal{H} \otimes l^2(\Gamma))$$

by

$$\tilde{\pi}(u_g) : v \otimes \delta_h \mapsto v \otimes \delta_{gh}$$

One can check that these formulas for $\tilde{\pi}$ define a (faithful) representation of $A \rtimes_{alg} \Gamma$ on $\mathcal{H} \otimes l^2(\Gamma)$. The *reduced crossed product norm* on $A \rtimes_{alg} \Gamma$ is the induced operator norm on this algebra (one can show that it is independent of the original π).

The *reduced crossed product of A by* Γ, denoted $A \rtimes_r \Gamma$, is the completion of $A \rtimes_{alg} \Gamma$ in this norm.

Note that $A \rtimes_r \Gamma$ depends on the action α; nonetheless, we do not include it in the notation to avoid too much clutter, and as it is always clear which action we have in mind.

The following lemma is used in chapter 3 to relate crossed products to limit operators. Recall that if X is a compact topological space then $C_s(X, C_u^*(|\Gamma|))^\Gamma$ denotes the $*$-strongly continuous functions from X to $C_u^*(|\Gamma|)$ that are Γ-equivariant in the sense that

$$F(gx) = V_g F(x) V_g^*$$

for all $x \in X$ and $g \in \Gamma$. Throughout the proof, if g, h are elements of Γ, then $e_{g,h}$ denotes the rank one matrix unit acting on $l^2(\Gamma)$ by

$$e_{g,h} : \delta_k \mapsto \begin{cases} \delta_h & k = g \\ 0 & k \neq g \end{cases}.$$

Lemma B.2. *Let Γ act on a compact Hausdorff space X. Then there is a canonical inclusion $C(X) \rtimes_r \Gamma \to C_s(X, C_u^*(|\Gamma|))^\Gamma$.*

Proof. For each $f \in C(X)$, define a map $F : X \to C_u^*(|\Gamma|)$ by

$$F(x) = \sum_{h \in \Gamma} (\alpha_{h^{-1}} f)(x) e_{h,h}.$$

F is then $*$-strongly continuous, so in $C_s(X, C_u^*(|\Gamma|))$ (note that F need *not* be continuous for the norm topology on $C_u^*(|\Gamma|)$ - this is what the '$_s$' is doing in the statement of the lemma). Abuse notation slightly by writing $U_g \in C_s(X, C_u^*(|\Gamma|))$ for the constant function taking all $x \in X$ to the unitary $U_g \in C_u^*(|\Gamma|)$. The formula

$$\sum_{g \in \Gamma} f_g u_g \mapsto \sum_{g \in \Gamma} F_g U_g \tag{B.1}$$

then defines a map from $C(X)\rtimes_{alg}\Gamma$ to $C_s(X, C_u^*(|\Gamma|))^\Gamma$ as one may readily check. It now suffices to prove that this map extends to an injection of $C(X)\rtimes_r\Gamma$ into $C_s(X, C_u^*(|\Gamma|))^\Gamma$.

To do this, fix a faithful representation of $C(X)$ on some Hilbert space $\mathcal{H}$, which we will use to define $C(X)\rtimes_r\Gamma$ as in definition B.1. Splitting $\mathcal{H}$ into cyclic representations, write it as

$$\mathcal{H} = \oplus_{i\in I} L^2(X, \mu_i)$$

for some collection $(\mu_i)_{i\in I}$ of measures on X (if we were only interested in second countable spaces X, we could assume that $\mathcal{H} = L^2(X, \mu)$ for some μ). Then there are canonical isomorphisms

$$\mathcal{H}\otimes l^2(\Gamma) \cong \oplus_{i\in I} L^2(X)\otimes l^2(\Gamma) \cong \oplus_{i\in I} L^2(X, l^2(\Gamma)).$$

Now, each $L^2(X, l^2(\Gamma))$ is equipped with an obvious representation of $C_s(X, C_u^*(|\Gamma|))$; let π be the direct sum representation of this algebra on $\mathcal{H}\otimes l^2(\Gamma)$. One then checks that π composed with the inclusion in line (B.1) above is the representation $\tilde{\pi}$ of $C(X)\rtimes_{alg}\Gamma$ on $\mathcal{H}\otimes l^2(\Gamma)$ from definition B.1. This completes the proof of the lemma. □

Remark B.3. The cases of the above lemma of most interest to us are the inclusions

$$C(\partial_\beta\mathbb{Z}^n)\rtimes_r\mathbb{Z}^n \to C_s(\partial_\beta\mathbb{Z}^n, C_u^*(|\mathbb{Z}^n|))^{\mathbb{Z}^n}$$

and

$$C_u^*(|\mathbb{Z}^n|) \cong C(\beta\mathbb{Z}^n)\rtimes_r\mathbb{Z}^n \to C_s(\beta\mathbb{Z}^n, C_u^*(|\mathbb{Z}^n|))^{\mathbb{Z}^n}.$$

The second of these is actually an isomorphism, which one can prove using part (1) of theorem 2.2.7. The first is not in general surjective however. Indeed, consider the case $n = 2$, and any map $f : S^1 \to C(\mathbb{T}^2)$ which is bounded, has image consisting of real-valued functions, and is pointwise, but *not* uniformly, continuous. There is a canonical quotient $\partial_\beta \mathbb{Z}^2 \twoheadrightarrow S^1$ arising from the spherical compactification of $\mathbb{Z}^2$, and the composition

$$F : \partial_\beta \mathbb{Z}^n \twoheadrightarrow S^1 \to^f C(\mathbb{T}^2) \cong C^*(\mathbb{Z}^2) \subseteq C^*_u(|\mathbb{Z}^2|)$$

defines an element of $C_s(\partial_\beta \mathbb{Z}^2, C^*_u(|\mathbb{Z}^2|))^{\mathbb{Z}^2}$. *If* F came from the inclusion of $C(\partial_\beta \mathbb{Z}^2) \rtimes_r \mathbb{Z}^2$, then the results of chapter 3 would imply that it was the symbol of a band-dominated operator with slowly oscillating coefficients, and in particular would actually be continuous for the *norm* topology on $C(\mathbb{T}^2) \cong C^*(\mathbb{Z}^2)$. This, however, is not the case.

Motivated in part by this example it would be interesting to give a precise characterization of the image of $C(X) \rtimes_r \Gamma$ inside $C_s(X, C^*_u(|\Gamma|))^\Gamma$ for general X and Γ. Indeed, assume that Γ acts amenably on X (which is the case for most of the particular examples that we are interested in). Then one can show that there are canonical inclusions

$$C(X, C^*_u(|\Gamma|))^\Gamma \subseteq C(X) \rtimes_r \Gamma \subseteq C_s(X, C^*_u(|\Gamma|))^\Gamma,$$

the second of which is provided by the lemma above (the first is more subtle, and may not be true without some sort of amenability assumption). There are non-trivial examples where either, none or both of these inclusions is / are the identity. It would seem to be interesting to give a precise characterization of when these identities occur (section 11.5.3 from [64] discusses the case when X is a point), but I do not know at this stage how to do so.

The remainder of this section will define and give some properties of exact groups. The next definition may seem somewhat ad-hoc at first, but these and similar objects come up surprisingly often; see for example chapter 3 in [73] and chapter 4 and 5 of [11], for example, which rely on these and similar ideas.

Definition B.4. Let Γ be a discrete group. A map $K : \Gamma \times \Gamma \to \mathbb{C}$ is called a *positive type kernel* if for any finite set $\{g_1, ..., g_n\} \subseteq \Gamma$, the matrix $[K(g_i, g_j)]_{i,j=1}^n$ is positive.

We can now state the main definition of this section.

Definition-Theorem B.5. Let Γ be a countable discrete group. Then the following are equivalent:

1. if $C_r^*(\Gamma)$ is the reduced group C^*-algebra, then the functor $\cdot \otimes C_r^*(\Gamma)$ is exact on the category of C^*-algebras, i.e. if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of C^*-algebras, then so too is

$$0 \longrightarrow A \otimes C_r^*(\Gamma) \longrightarrow B \otimes C_r^*(\Gamma) \longrightarrow C \otimes C_r^*(\Gamma) \longrightarrow 0\ ;$$

2. the functor $\cdot \rtimes_r \Gamma$ is exact on the category of $\Gamma - C^*$-algebras, i.e. if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of $\Gamma - C^*$-algebras (this includes the fact that the arrows are Γ-equivariant $*$-homomorphisms), then

$$0 \longrightarrow A \rtimes_r \Gamma \longrightarrow B \rtimes_r \Gamma \longrightarrow C \rtimes_r \Gamma \longrightarrow 0$$

is a short exact sequence of C^*-algebras;

3. for any entourage $E \subseteq \Gamma \times \Gamma$ for the (right-invariant) coarse structure on Γ and any $\epsilon > 0$ there exists an entourage $F \subseteq \Gamma \times \Gamma$ and a positive type kernel $K : \Gamma \times \Gamma \to \mathbb{C}$ supported in F such that for all $(g, h) \in E$

$$|K(g, h) - 1| < \epsilon.$$

4. the algebra $C^*_u(|\Gamma|)$ of band-dominated operators on Γ is *nuclear*, i.e. for any C^*-algebra A there is a unique cross-norm on the algebraic tensor product of A and $C^*_u(|\Gamma|)$.

If Γ satisfies these conditions, it is called *exact*.

Examples B.6. There are no known explicit examples of non-exact groups, but Gromov has argued in [28] that they must exist on probabilistic grounds; a fully detailed proof is yet to be written down, but most experts believe the result to be correct. Many classes of discrete groups are known to be exact, including:

- word hyperbolic groups;
- amenable groups;
- linear groups (over any field)

(see the comments and references in chapter 3 of [73]).

The theorem above is a compilation of results due to C. Anantharaman-Delaroche [1], E. Guentner and J. Kaminker [32] [31], N. Higson and J. Roe [42], E. Kirchberg [48], E. Kirchberg and S. Wasserman [49], N. Ozawa [52] and J.-L. Tu [71]. The first condition is probably the most standard one, while the second is the most important for us, especially insofar as it underlies section 3. The third and fourth conditions are included mainly for interest.

Note in particular that the third condition implies that exactness is a *coarse invariant*: it is preserved by isomorphisms in the full subcategory of $\mathcal{C}$ from definition A.5 whose objects are (countable) discrete groups equipped with their (right-invariant) coarse structures. In fact, this condition does not use the group structure at all, and indeed 'exactness' makes sense for any coarse space. When adapted to general coarse spaces it is usually called *property A*. This property was originally defined in [74] by G. Yu (before it was known to be related to exactness as defined in part one above); see also [73] for an overview of property A.

We will not prove it here, but state the following result for reference. A detailed proof is given in section 11.5.2 of [64]. The first part is proved using Schur multipliers constructed out of positive type kernels as appearing in part (3) of theorem B.5.

Proposition B.7. *Let X be a countable uniformly locally finite coarse space (see definition A.11). An operator $T \in C^*_u(X)$ is called a* ghost *if its matrix coefficients*

$$\langle T\delta_x, \delta_y \rangle, \quad x, y \in X$$

tend to zero as x, y tend to infinity

Let Γ be a countable discrete exact group and T be an operator in $C^*_u(|\Gamma|)$. Then T is a ghost if and only if it is compact.

On the other hand, there exist examples of coarse spaces X (certain so-called expander graphs*) such that $C^*_u(X)$ contains non-compact ghost operators.* □

Gromov's random groups from [28] ('weakly') contain coarsely embedded expander graphs. The results of section 7 of [38] show that if Γ is such a group, then $C^*_u(|\Gamma|)$ contains a projection p that is a non-compact ghost operator. For such groups, theorem 3.0.2 fails.

References

[1] C. Anantharaman-Delaroche. Amenability and exactness for dynamical systems and their C^*-algebras. *Transactions of the American Mathematical Society*, 354:4153–4178, June 2002.

[2] M. Atiyah. Algebraic topology and elliptic operators. *Communications on Pure and Applied Mathematics*, 20:237–249, 1967.

[3] M. Atiyah and R. Bott. The index problem for manifolds with boundary. In *Bombay Colloquium on Differential Analysis*, pages 175–186, 1964.

[4] M. Atiyah and I. Singer. The index of elliptic operators I. *The Annals of Mathematics*, 87(3):484–530, 1968.

[5] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K-theory of group C^*-algebras. *Contemporary Mathematics*, 167:241–291, 1994.

[6] M. Bestvina and G. Mess. The boundary of negatively curved groups. *Journal of the American Mathematical Society*, 4(3):469–481, July 1991.

[7] B. Blackadar. *K-Theory for Operator Algebras*. Cambridge University Press, second edition, 1998.

[8] M. Bridson. Non-positive curvature and complexity for finitely presented groups. In *Proceedings of the International Congress of Mathematicians*, volume II, pages 961–987. EMS Publishing House, Zürich, 2006.

[9] M. Bridson and A. Haefliger. *Metric Spaces of Non-postive Curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften*. Springer, 1999.

[10] J. Brodzki, G. Niblo, and N. Wright. Property A, partial translation structures and uniform embeddings in groups. *Journal of the London Mathematical Society*, 76(2):479–497, 2007.

[11] N. Brown and N. Ozawa. *C^*-Algebras and Finite-Dimensional Approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, 2008.

[12] J. Chabert, S. Echterhoff, and H. Oyono-Oyono. Shapiro's lemma for topological K-theory of groups. *Commentarii Mathematici Helvetici*, 78(1):203–225, February 2003.

[13] J. Chabert, S. Echterhoff, and H. Oyono-Oyono. Going-down functors, the Künneth formula, and the Baum-Connes conjecture. *Geometric and Functional Analysis*, 14(3):491–528, June 2004.

[14] L. Coburn. Toeplitz operators on odd-dimensional spheres. In *Proceedings of a conference on operator algebras*, pages 345–352, 1973.

[15] V. M. Deundyak and B. Y. Shteinberg. Index of convolution operators with slowly varying coefficients on abelian groups. *Functional Analysis and its Applications*, 19(4):321–323, October 1985.

[16] R. G. Douglas. *Banach Algebra Techniques in Operator Theory*. Academic Press, 1972.

[17] A. Dranishnikov and S. Ferry. On the Higson-Roe corona. *Russian Mathematical Surveys*, 52(5):1017–1028, 1997.

[18] A. Dranishnikov, S. Ferry, and S. Weinberger. An étale approach to the Novikov conjecture. *Communications on Pure and Applied Mathematics*, 61(2):139–155, 2007.

[19] A. Dranishnikov, J. Keesling, and V. V. Uspenskii. On the Higson corona of uniformly contractible spaces. *Topology*, 37(4):791–803, 1998.

[20] H. Emerson. The Baum-Connes conjecture, noncommutative Poincaré duality and the boundary of a free group. *International Journal of Mathematics and the Mathematical Sciences*, 38:2425–2445, 2003.

[21] H. Emerson. Noncommutative Poincaré duality for boundary actions of hyperbolic groups. *Journal für die reine und angewandte Mathematik*, 564:1–33, 2003.

[22] H. Emerson and R. Meyer. Dualizing the coarse assembly map. *Journal de l'Institut de Mathématiques de Jussieu*, 5:161–186, 2006.

[23] H. Emerson and R. Meyer. Euler characteristics and Gysin sequences for group actions on boundaries. *Mathematische Annalen*, 334(4):853–904, April 2006.

[24] H. Emerson and R. Meyer. A descent principle for the Dirac dual Dirac method. *Topology*, 46:185–209, 2007.

[25] S. Ferry, A. Ranicki, and J. Rosenberg. A history and survey of the Novikov conjecture. In S. Ferry, A. Ranicki, and J. Rosenberg, editors, *Novikov conjectures, index theory theorems and rigidity*, volume 1, pages 7–66. Oberwolfach, Cambridge University Press, 1993.

[26] E. Ghys and P. de la Harpe, editors. *Sur les groupes hyperboliques d'après Mikhael Gromov*. Number 83 in Progress in mathematics. Birkhäuser, 1990.

[27] M. Gromov. Hyperbolic groups. In S. M. Gertsen, editor, *Essays in group theory*, number 8 in M.S.R.I. Publications, pages 75–263. Springer, 1987.

[28] M. Gromov. Random walks in random groups. *Geometric and Functional Analysis*, 13(1):73–146, February 2003.

[29] E. Guentner, N. Higson, and J. Trout. Equivariant E-theory. *Memoirs of the American Mathematical Society*, 148(703), 2000.

[30] E. Guentner, N. Higson, and S. Weinberger. The Novikov conjecture for linear groups. *Publications Mathematiques de l'Institut Hautes Études Scientifiques*, 101:243–268, 2005.

[31] E. Guentner and J. Kaminker. Addendum to 'exactness and the Novikov conjecture'. *Topology*, 41(2):411–418, 2002.

[32] E. Guentner and J. Kaminker. Exactness and the Novikov conjecture. *Topology*, 41(2):411–418, 2002.

[33] E. Guentner, R. Tessera, and G. Yu. Decomposition complexity and the stable Borel conjecture. *Preprint*, 2009.

[34] W. Helton and R. E. Howe. Traces of commutators of integral operators. *Acta Mathematica*, 135(1):271–305, December 1975.

[35] N. Higson. On the relative K-homology theory of Baum and Douglas. *Preprint*, 1989.

[36] N. Higson and E. Guentner. Group C^*-algebras and K-theory. In *Noncommutative Geometry*, number 1831 in Springer Lecture Notes, pages 137–252. Springer, 2004.

[37] N. Higson and G. Kasparov. E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. *Inventiones Mathematicae*, 144:23–74, 2001.

[38] N. Higson, V. Lafforgue, and G. Skandalis. Counterexamples to the Baum-Connes conjecture. *Geometric and Functional Analysis*, 12:330–354, 2002.

[39] N. Higson, E. Pedersen, and J. Roe. C^*-algebras and controlled topology. *K-Theory*, 11:209–239, 1997.

[40] N. Higson and J. Roe. A homotopy invariance theorem in coarse cohomolgy and K-theory. *Transactions of the American Mathematical Society*, 345:347–365, 1994.

[41] N. Higson and J. Roe. On the coarse Baum-Connes conjecture. *London Mathematical Society Lecture Notes*, 227:227–254, 1995.

[42] N. Higson and J. Roe. Amenable group actions and the Novikov conjecture. *Journal für die reine und angewandte Mathematik*, 519:143–153, 2000.

[43] N. Higson and J. Roe. *Analytic K-homology*. Oxford University Press, 2000.

[44] N. Higson, J. Roe, and G. Yu. A coarse Mayer-Vietoris principle. *Mathematical Proceedings of the Cambridge Philosophcal Society*, 114:85–97, 1993.

[45] M. T. Jury. C^*-algebras generated by groups of composition operators. *Indiana University Mathematics Journal*, 56(6):3171–3192, 2007.

[46] M. T. Jury. The Fredholm index for elements of Toeplitz-compositon C^*-algebras. *Integral equations and operator theory*, 58(3):341–362, July 2007.

[47] J. Keesling. The one dimensional Cěch cohomology of the Higson compactification and its corona. *Topology proceedings*, 19:129–148, 1994.

[48] E. Kirchberg. On subalgebras of the CAR algebra. *Journal of functional analysis*, 129:35–63, 1995.

[49] E. Kirchberg and S. Wasserman. Exact groups and continuous bundles of C^*-algebras. *Mathematische Annalen*, 315(2):169–203, October 1999.

[50] R. Meyer and R. Nest. The Baum-Connes conjecture via localisation of categories. *Topology*, 45(2):209–259, March 2006.

[51] A. Y. Ol'shanskii. Almost every group is hyperbolic. *International Journal of Algebra and Computation*, 2(1):1–17, March 1992.

[52] N. Ozawa. Amenabe actions and exactness for discrete groups. *Comptes Rendus de l'Academie des Sciences - Mathematics*, 330:691–695, 2000.

[53] V. S. Rabinovich and S. Roch. The Fredholm index of locally compact band-dominated operators on $L^p(R)$. *Integral equations and operator theory*, 57(2):263–281, February 2007.

[54] V. S. Rabinovich, S. Roch, and J. Roe. Fredholm indices of band-dominated operators on discrete groups. *Integral equations and operator theory*, 49:221–238, 2004.

[55] V. S. Rabinovich, S. Roch, and B. Silbermann. Fredholm theory and finite section method for band-dominated operators. *Integral equations and operator theory*, 30:425–495, 1998.

[56] V. S. Rabinovich, S. Roch, and B. Silbermann. *Limit Operators and Their Applications in Operator Theory*, volume 150 of *Operator Theory: Advances and Applications*. Birkhäuser, 2004.

[57] V. S. Rabinovich, S. Roch, and B. Silbermann. The finite sections approach to the index formula for band-dominated operators. In *Recent Advances in Operator Theory and Applications*, volume 187 of *Operator Theory: Advances and Applications*, pages 185–193. Birkhäuser, 2009.

[58] S. Roch. Band-dominated operators on l^p-spaces: Fredholm indices and finite sections. *Acta Scientarum Mathematicarum*, 70:783–797, 2004.

[59] J. Roe. An index theorem on open mainfolds, I. *Journal of differential geometry*, 27:87–113, 1988.

[60] J. Roe. An index theorem on open mainfolds, II. *Journal of differential geometry*, 27:115–136, 1988.

[61] J. Roe. Hyperbolic metric spaces and the exotic cohomology Novikov conjecture. *K-Theory*, 4(6):501–512, 1990.

[62] J. Roe. Coarse cohomology and index theory on complete Riemannian manifolds. *Memoirs of the American Mathematical Society*, 104(497), July 1993.

[63] J. Roe. *Index Theory, Coarse Geometry and Topology of Manifolds*, volume 90 of *CBMS Conference Proceedings*. American Mathematical Society, 1996.

[64] J. Roe. *Lectures on Coarse Geometry*, volume 31 of *University Lecture Series*. American Mathematical Society, 2003.

[65] J. Roe. Band-dominated Fredholm operators on discrete groups. *Integral equations and operator theory*, 51:411–416, 2005.

[66] J. Roe. Hyperbolic groups have finite asymptotic dimension. *Proceedings of the American Mathematical Society*, 133(9):2489–2490, 2005.

[67] M. Rørdam, F. Larsen, and N. Laustsen. *An Introduction to K-Theory for C^*-Algebras*. Cambridge University Press, 2000.

[68] R. T. Seeley. Integro-differential operators on vector bundles. *Transactions of the American Mathematical Society*, 117:167–204, May 1965.

[69] R. T. Seeley. Complex powers of an elliptic operator. In *Proceedings of the symposium in pure mathematics*, volume 10. American Mathematical Society, 1971.

[70] J. Spakula. Uniform K-homology theory. *Preprint*, 2009.

[71] J.-L. Tu. Remarks on Yu's property A for discrete metric spaces and groups. *Bulletin de la Société Mathematique Françaiss*, 129:115–139, 2001.

[72] U. Venugopalkrishna. Fredholm operators associated with strongly pseudo-convex domains on C^n. *Journal of functional analysis*, 9:349–373, 1972.

[73] R. Willett. Some notes on property A. In G. Arzhantseva and A. Valette, editors, *Limit of Graphs in group theory and computer science*. EPFL press, 2009.

[74] G. Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Inventiones Mathematicae*, 139(1):201–240, January 2000.

Vita

Rufus Willett was born in Penrith, a small town in northern England, in 1983. While a teenager he became interested in travel, language and, a little later, mathematics. The latter two led to a first class Bachelor's degree in mathematics and philosophy from the University of Oxford. While he was at Oxford, Glenys Luke introduced Rufus to John Roe, who encouraged starting a Ph.D. in mathematics at the Pennsylvania State University, and subsequently became Rufus's adviser. Awards won by Rufus during his academic career include the Gibbs prize from Oxford and the Jack and Eleanor Petit scholarship from Penn State, as wells as awards for teaching and encouraging a sense of community. He has recently accepted a postdoctoral position from Vanderbilt University in Nashville, Tennessee.