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ENTROPY AND INVARIANT MEASURES FOR SKEW PRODUCT MAPS

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Abstract

We study entropy and invariant measures for smooth diffeomorphisms. The main result of the dissertation establishes a theorem on skew product maps with diffeomorphisms on fibers. We show if, for an ergodic invariant measure μ, all Lyapunov exponents along the fibers are non-zero, then any value, between 0 and the metric entropy of μ, is the metric entropy of an ergodic invariant measure for the map. This result generalizes a famous result of A. Katok [17] in which μ is required to be a hyperbolic measure.

To construct the measures of intermediate entropies we find an invariant set on which the induced map is also a skew product which acts like horseshoe maps on fibers. From this set we can construct ergodic measures with the maximal entropy arbitrarily close to the entropy of μ. Since a horseshoe map is conjugate to a full shift, all intermediate entropies can be obtained by changing the weights of different symbols in a continuous way.

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Chapter 1

Introduction

Entropy, which measures the asymptotic rate of orbit divergence, has been one of the centerpieces in dynamics. It characterizes complexity of a system. Positive topological entropy is widely accepted as an indication of chaos. By contrast, unique ergodicity, the property that the system has a unique invariant measure, in which case the time average of any continuous function converges everywhere to the space average with respect to that measure, reflects that such a system is considerably deterministic. It is natural to expect a uniquely ergodic system to have zero topological entropy. However, this is not true in the most general situation. There are many examples , including both symbolic systems and homeomorphisms on manifolds, which have positive topological entropies while being uniquely ergodic (see e.g. [12][5]).

It is not known for smooth system whether unique ergodicity is compatible with positive topological entropy, i.e. diffeomorphisms on compact smooth manifolds. This question was suggested by M. Herman who constructed a C^∞ example that is minimal with positive topological entropy [15] (see Section 3.3). Though being minimal, the example is not uniquely ergodic. In fact it has ergodic measures of arbitrary intermediate entropies. At about the same time A. Katok established a well-known result for smooth systems preserving hyperbolic measures [17]. Later in [18] he showed that these systems have measures of intermediate entropies. Seeing this, Katok made an optimistic conjecture that all smooth systems have such measures. Verifying this is the final goal of our project. In this thesis we make a step of advance by showing that this property holds for skew product maps with hyperbolic diffeomorphisms on fibers.

The problem requires some understanding in two aspects: entropy theory and geometric structures of smooth systems. So in Chapter 2 we introduce some preliminaries of entropy. And in Chapter 3 we discuss Pesin theory as well as the results of Katok and Herman. In Chapter 4 we formulate the main result and give the proof.

Chapter 2

Entropy

2.1 Topological Entropy

We consider a topological space X and a continuous map f. Denote by $h(f)$ the topological entropy of f. There are various definitions of $h(f)$. All these definitions are equivalent when X is compact and metrisable.

2.1.1 Definition Using Open Covers

We assume X is compact. The following definition is due to Adler, Konheim and McAndrew [2].

Definition 2.1.1. Let $\mathcal{U}$ be an open cover of X. Let $N_c(\mathcal{U})$ denote the smallest cardinality of a finite sub-cover of $\mathcal{U}$. Define the entropy of $\mathcal{U}$ by $H(\mathcal{U}) = \log N_c(\mathcal{U})$.

Proposition 2.1.2. *If $\mathcal{U}$ and $\mathcal{V}$ are open covers of X. Let $\mathcal{U} \bigvee \mathcal{V} = \{U \cap V | U \in \mathcal{U}, V \in \mathcal{V}\}$. Then $H(\mathcal{U} \bigvee \mathcal{V}) \leq H(\mathcal{U}) + H(\mathcal{V})$.*

Theorem 2.1.3. *Assume $\mathcal{U}$ is an open cover of X and $f : X \to X$ is a continuous map. Let*

$$\mathcal{U}_f^n = \bigvee_{k=0}^{n-1} f^{-k}(\mathcal{U}).$$

Then

$$h(f, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} H(\mathcal{U}_f^n).$$

The limit exists and is finite.

Remark. By Proposition 2.1.2, $H(\mathcal{U}_f^n)$ is sub-additive. So the limit exists and is finite. Moreover,

$$h(f,\mathcal{U}) = \inf_{n\geq 1} \frac{1}{n} H(\mathcal{U}_f^n).$$

Definition 2.1.4. The topological entropy of f is given by

$$h(f) = \sup\{h(f,\mathcal{U}) | \mathcal{U} \text{ is an open cover of } X\}.$$

Definition 2.1.5. For $U \subset X$, we define the diameter of U to be

$$\text{diam}(U) = \sup\{d(x,y) | x, y \in U\}.$$

If $\mathcal{U}$ is a collection of subsets of X, then the diameter of $\mathcal{U}$ is

$$\text{diam}(\mathcal{U}) = \sup_{U\in\mathcal{U}} \text{diam}(U).$$

Theorem 2.1.6.

$$h(f) = \lim_{\epsilon\to 0} \sup\{h(f,\mathcal{U}) | \text{diam}(\mathcal{U}) < \epsilon\}.$$

Definition 2.1.7. Assume X and Y are compact spaces and $f : X \to X$ and $g : Y \to Y$ are continuous. We say that g is a factor of f if there is $\phi : X \to Y$ called a semiconjugacy such that ϕ is a continuous map onto Y and $\phi \circ f = g \circ \phi$.

A semiconjugacy is called a conjugacy if it is a homeomorphism. In this case f and g are said to be conjugate and they are factors of each other.

Theorem 2.1.8. *If g is a factor of f then $h(f) \geq h(g)$. If f and g are conjugate then $h(f) = h(g)$.*

Theorem 2.1.9. *If f is a homeomorphism and X is compact, then $h(f^{-1}) = h(f)$.*

2.1.2 Bowen-Dinaburg Definition

This definition was given by Dinaburg [11] and Bowen [7].

Let (X, d) be a metric space, not necessarily compact. Let f be a continuous map on X.

Definition 2.1.10. We define for each $n > 0$ a metric d_f^n on X by

$$d_f^n(x, y) = \max_{0 \leq j \leq n-1} d(f^j(x), f^j(y)).$$

Definition 2.1.11. Let K be a subset of X. A subset E is said to (n, ϵ)-span K with respect to f if for every $x \in K$, there is $y \in E$ such that $d_f^n(x, y) \leq \epsilon$. Let $r_n(f, \epsilon, K)$ denote the smallest cardinality of any (n, ϵ)-spanning set for K with respect to f. Let $r_n'(f, \epsilon, K)$ denote the smallest cardinality of any (n, ϵ)-spanning subset of K for K with respect to f.

Remark. $r_n(f, \epsilon, K)$ is also the minimal number of ϵ-balls in the d_f^n-metric needed to cover K. r_n' is analogous (the balls have centers in K).

Definition 2.1.12. Let K be a subset of X. A subset E is said to be (n, ϵ) separated with respect to f if for distinct points $x, y \in E$ we have $d_f^n(x, y) > \epsilon$. Let $s_n(f, \epsilon, K)$ be the largest cardinality of any (n, ϵ) separated subset of K with respect to f.

Theorem 2.1.13. *If K is compact then*

$$\begin{aligned} h(f, K) &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} r_n(f, \epsilon, K) \\ &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} r_n'(f, \epsilon, K) \\ &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} s_n(f, \epsilon, K). \end{aligned}$$

Definition 2.1.14. The topological entropy of f is

$$h(f) = \sup\{h(f, K) | K \text{ is a compact subset of } X\}.$$

Theorem 2.1.15. *If f is continuous and X is a metric space (not necessarily compact), then for every $m \in \mathbb{N}$, $h(f^m) = mh(f)$.*

Theorem 2.1.16. *If X is a compact metric space and f is continuous, then*

1. *$h(f) = h(f, X)$.*
2. *$h(f)$ does not depend on the metrics chosen on X if they induce the same topology.*

3.

$$h(f) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}r_n(f,\epsilon,X) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}s_n(f,\epsilon,X).$$

4. Definition 2.1.4 and Definition 2.1.14 are equivalent.

Definition 2.1.17. Let $f : X \to X$ be a homeomorphism. We say f is *expansive* with constant β_0 if for every $x, y \in X$, $x \neq y$, there is $n = n(x,y) \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) \geq \beta_0$.

Theorem 2.1.18. *If f be an expansive homeomorphism with constant β_0, then for $\epsilon < \beta_0/2$,*

$$h(f) = \limsup_{n\to\infty}\frac{1}{n}r_n(f,\epsilon,X) = \limsup_{n\to\infty}\frac{1}{n}s_n(f,\epsilon,X).$$

2.1.3 Another Definition by Bowen

Let X be a topological space, not necessarily metrisable or compact. Let f be a continuous map on X. Bowen [8] introduced another definition of topological entropy in this general setting.

Definition 2.1.19. Let Y be a subset of X and $\mathcal{U}$ be a finite open cover of Y. For a set $B \subset X$ we write $B \prec \mathcal{U}$ if B is contained in some element of $\mathcal{U}$. Let $n_{f,\mathcal{U}}(B)$ be the largest nonnegative integer n such that $f^k(B) \prec \mathcal{U}$ for $k = 0, 1, \ldots, n-1$. If $B \not\prec \mathcal{U}$ then $n_{f,\mathcal{U}}(B) = 0$ and if $f^k(B) \prec \mathcal{U}$ for all k then $n_{f,\mathcal{U}}(B) = \infty$. we define

$$\mathrm{diam}_{\mathcal{U}}(B) = \exp(-n_{f,\mathcal{U}}(B)).$$

If $\mathcal{B}$ is collection of subsets of X, then we set

$$\mathrm{diam}_{\mathcal{U}}(\mathcal{B}) = \sup_{B\in\mathcal{B}}\mathrm{diam}_{\mathcal{U}}(B)$$

and for any real number λ,

$$D_{\mathcal{U}}(\mathcal{B},\lambda) = \sum_{B\in\mathcal{B}}(\mathrm{diam}_{\mathcal{U}}(B))^{\lambda}.$$

We define

$$\mu_{\mathcal{U},\lambda}(Y) = \lim_{\epsilon\to 0}\inf\{D_{\mathcal{U}}(\mathcal{B},\lambda) | \mathcal{B} \text{ is a cover of } Y \text{ and } \mathrm{diam}_{\mathcal{U}}(\mathcal{B}) < \epsilon\}.$$

Theorem 2.1.20. *There is a number $h_{\mathcal{U}}(f,Y)$ such that $\mu_{\mathcal{U},\lambda}(Y)=\infty$ for $\lambda < h_{\mathcal{U}}(f,Y)$ and $\mu_{\mathcal{U},\lambda}(Y)=0$ for $\lambda > h_{\mathcal{U}}(f,Y)$*

Definition 2.1.21. For any subset Y of X, define $h_Y(f)$ to be

$$\sup\{h_{\mathcal{U}}(f,Y)|\mathcal{U} \text{ is a finite open cover of } Y\}.$$

The topological entropy of f is

$$h(f)=\sup\{h_{\mathcal{U}}(f,X)|\mathcal{U} \text{ is a finite open cover of } X\}.$$

Remark. We remark that $h_Y(f)$ may be different from $h(f,Y)$ as in Theorem 2.1.13 even if Y is compact. In general if Y is compact then $h_Y(f)\leq h(f,Y)$, and the equality holds when Y is also f-invariant (not necessary).

Moreover, $h_Y(f)=h_Y^X(f)$ depends very much on the ambient space X. [8] provides an example suggested by L. Goodwyn showing that there is a set $Y\subset \mathbb{R}\subset \mathrm{S}^1$ such that for $f(x)=x+1$, $h_Y^{\mathbb{R}}(f)=\infty$ but $h_Y^{\mathrm{S}^1}(f)=0$.

Theorem 2.1.22.

1. *If X is compact, then Definition 2.1.21 is equivalent to Definition 2.1.4.*

2. *If X is metrisable, then Definition 2.1.21 is equivalent to Definition 2.1.14.*

2.2 Decay of Lebesgue Numbers

2.2.1 Dimension of a Set

Assume X is a metric space. Y is a subset of X. There are various definitions of dimensions of Y.

Definition 2.2.1. Let $N_b(\gamma)$ be the minimal number of open γ-balls needed to cover Y. The upper box dimension of Y is

$$D_b^+(Y)=\limsup_{\gamma\to 0} -\frac{\log N(\gamma)}{\log \gamma}$$

and the lower box dimension of Y is

$$D_b^-(Y) = \liminf_{\gamma \to 0} -\frac{\log N(\gamma)}{\log \gamma}.$$

Definition 2.2.2. Define the Hausdorff measure of Y by

$$\mu_\lambda(Y) = \liminf_{\epsilon \to 0} \{\sum_{U \in \mathcal{U}} (r(U))^\lambda | \mathcal{U} \text{ is a cover of } Y \text{ and } \operatorname{diam}(\mathcal{U}) < \epsilon\}.$$

Hausdorff dimension of Y is a number $D_H(Y)$ such that $\mu_\lambda(Y) = \infty$ for $\lambda < D_H(Y)$ and $\mu_\lambda(Y) = 0$ for $\lambda > D_H(Y)$.

Proposition 2.2.3. *For every $Y \subset X$, we have $D_H(Y) \leq D_b^-(Y) \leq D_b^+(Y)$*

Remark. These definitions of the dimensions do not depend on the metric chosen on the topological space.

2.2.2 Lebesgue Numbers of Open Covers

Assume X be a compact metric space. Let $\mathcal{U}$ be an open cover of X. Throughout this section we assume $X \notin \mathcal{U}$.

Definition 2.2.4. Let

$$\delta(\mathcal{U}, x) = \sup_{U \in \mathcal{U}} (\inf_{y \in X \setminus U} d(x, y)).$$

Then $\delta(\mathcal{U}, x)$ is a continuous function on X taking positive values. since X is compact, it takes its minimum value on X.

$$\delta(\mathcal{U}) = \min_{x \in X} \delta(\mathcal{U}, x) > 0$$

is called Lebesgue number of $\mathcal{U}$.

Proposition 2.2.5. *$\delta(\mathcal{U})$ is the largest number such that for every $x \in X$, the ball centered at x with radius $\delta(\mathcal{U})$ is covered by some element of $\mathcal{U}$.*

Definition 2.2.6. For two open covers $\mathcal{U}$ and $\mathcal{V}$, we say $\mathcal{U}$ is finer than $\mathcal{V}$ if every element of $\mathcal{U}$ is contained in an element of $\mathcal{V}$.

Lemma 2.2.7. *If $\mathcal{U}$ is finer than $\mathcal{V}$ then $\delta(\mathcal{U}) \leq \delta(\mathcal{V})$.*

Lemma 2.2.8. *For any two open covers $\mathcal{U}$ and $\mathcal{V}$,*

$$\delta(\mathcal{U} \bigvee \mathcal{V}) = \min\{\delta(\mathcal{U}), \delta(\mathcal{V})\}.$$

Corollary 2.2.9. *Let $\delta_n = \delta_n(f, \mathcal{U}) = \delta(\mathcal{U}_f^n)$. Then*

$$\delta_n(f, \mathcal{U}) = \min_{0 \leq k \leq n-1} \delta(f^{-k}(\mathcal{U})).$$

2.2.3 Lebesgue Numbers, Entropy and Dimensions

We would like to measure the cells in $\mathcal{U}_f^n$. A more detailed discussion will appear in [31].

Definition 2.2.10. Let $\mathcal{U}$ be an open cover of X. We set

$$h_L(f, \mathcal{U}) = \liminf_{n \to \infty} -\frac{1}{n} \log \delta_n(f, \mathcal{U}),$$

$$h_L(f) = \lim_{\epsilon \to 0} \inf_{\mathrm{diam}(\mathcal{U}) < \epsilon} h_L(f, \mathcal{U})$$

and

$$h_L^+(f, \mathcal{U}) = \limsup_{n \to \infty} -\frac{1}{n} \log \delta_n(f, \mathcal{U}) = \limsup_{n \to \infty} -\frac{1}{n} \log \delta(f^{-n}(\mathcal{U})),$$

$$h_L^+(f) = \sup h_L^+(f, \mathcal{U}).$$

Proposition 2.2.11. *For a given open cover $\mathcal{U}$ and $\epsilon > 0$, if $\mathrm{diam}(\mathcal{U}) < \epsilon$, then for each n, $N_b(\delta_n(f, \mathcal{U})) \geq s_n(f, \epsilon, X)$.*

Proof. Let E be an (n, ϵ)-separated set of cardinality $s_n(f, \epsilon, X)$. If distinct points $x, y \in E$ are covered by the same δ_n-ball, then the ball is covered by some element

$$V = \bigcap_{k=0}^{n-1} f^{-k}(U_{i_k}) \subset \mathcal{U}_f^n,$$

where $U_{i_k} \in \mathcal{U}$ is some element of $\mathcal{U}$ for each k. This implies that $x, y \in V$ and $f^k(x), f^k(y) \in U_{i_k}$. Since $\mathrm{diam}(\mathcal{U}) < \epsilon$, $d(f^k(x), f^k(y)) < \epsilon$ for $0 \leq k \leq n-1$. So $d_f^n(x, y) < \epsilon$, which contradicts the fact that x and y are (n, ϵ)-separated.

So each δ_n-ball can cover at most one point in E. $N_b(\delta_n) \geq s_n(f, \epsilon, X)$. □

Theorem 2.2.12.

$$h_L(f) \geq \frac{1}{D_b^+(X)} \cdot h(f).$$

Proof. If $h(f) = 0$ then it is trivial.

If $h(f) > 0$ then for all small $\epsilon > 0$, as $n \to \infty$, $s_n(f, \epsilon, X) \to \infty$. But for every open cover $\mathcal{U}$ such that $\text{diam}(\mathcal{U}) < \epsilon$, we have $N_b(\delta_n) \geq s_n(f, \epsilon, X)$. Hence $\delta_n \to 0$.

Fix a small $\theta > 0$. For every $N_0 > 0$, by definition of the upper box dimension, there is $\gamma_0 > 0$ and $N_1 > N_0$ such that $\delta_n < \gamma_0$ for all $n > N_1$, and

$$-\frac{\log N_b(\delta_n)}{\log \delta_n} < D_b^+(X) + \theta.$$

Hence

$$-\log \delta_n \cdot (D_b^+(X) + \theta) > \log N_b(\delta_n) \geq s_n(f, \epsilon, X),$$

$$h_L(f, \mathcal{U}) = \liminf_{n\to\infty} -\frac{1}{n} \log \delta_n(f, \mathcal{U}) \geq \frac{1}{D_b^+(X) + \theta} \cdot \lim_{n\to\infty} \frac{1}{n} s_n(f, \epsilon).$$

This is true for every $\mathcal{U}$ with diameter less than ϵ and every $\theta > 0$. Let $\epsilon \to 0$ then the result follows. $\square$

Theorem 2.2.13.

$$h_L^+(f, \mathcal{U}) \geq \frac{1}{D_H(X)} \cdot h_{\mathcal{U}}(f),$$

where $h_{\mathcal{U}}(f)$ is as in Definition 2.1.19.

Proof. Take $K > h_L^+(f, \mathcal{U}) \geq 0$. Then there is n_0 such that for every $n > n_0$,

$$-\frac{1}{n-1} \log \delta_n(f, \mathcal{U}) < K.$$

For $B \in X$, if

$$\text{diam}(B) \leq \exp -K(n-1) < \delta_n(f, \mathcal{U}), \tag{2.1}$$

then $n_{f,\mathcal{U}}(B) \geq n$ since B is contained in some element of $\mathcal{U}_f^n$. (2.1) is satisfied for

$$n \leq -\log(\text{diam}(B))/K + 1.$$

So

$$n_{f,\mathcal{U}}(B) > -\log(\text{diam}(B))/K \text{ and } \text{diam}_{\mathcal{U}}(B) < (\text{diam}(B))^{1/K}. \tag{2.2}$$

Now fix $\lambda > D_H(X)$. By the definition of Hausdorff dimension, $\mu_\lambda(X) = 0$. For every $\epsilon > 0$ small enough (much smaller than $\delta_{n_0}(f,\mathcal{U})$) and every small $\gamma > 0$, there is a cover $\mathcal{B}$ such that $\mathrm{diam}(\mathcal{B}) < \epsilon$ and

$$\sum_{B\in\mathcal{B}} (\mathrm{diam}(B))^\lambda < \gamma.$$

By (2.2) we have

$$D_{\mathcal{U}}(\mathcal{B}, \lambda K) = \sum_{B\in\mathcal{B}} (\mathrm{diam}_{\mathcal{U}}(B))^{\lambda K} < \sum_{B\in\mathcal{B}} (\mathrm{diam}(B))^\lambda < \gamma,$$

while $\mathrm{diam}_{\mathcal{U}}(\mathcal{B}) < \epsilon^{1/K}$. This implies that $\mu_{\mathcal{U},\lambda K}(X) = 0$. Hence $\lambda K > h_{\mathcal{U}}(f)$, whenever $\lambda > D_H(X)$ and $K > h_L^+(\mathcal{U})$. So $D_H(X) \cdot h_L^+(\mathcal{U}) \geq h_{\mathcal{U}}(f)$. □

Corollary 2.2.14.

$$h_L^+(f) \geq \frac{1}{D_H(X)} \cdot h(f).$$

Remark. It is clear that $h_L^+(f,\mathcal{U}) \geq h_L(f,\mathcal{U})$ and $h_L^+(f) \geq h_L(f)$.

Theorem 2.2.15. *For every $n > 0$, $h_L^+(f^n) = nh_L^+(f)$.*

Proof. On one hand, for every finite open cover $\mathcal{U}$ and $m > 0$, by Corollary 2.2.9

$$\delta_m(f^n,\mathcal{U}) = \min_{0\leq j\leq m-1} \delta(f^{-jn}(\mathcal{U})) \geq \min_{0\leq j\leq mn-1} \delta(f^{-j}(\mathcal{U})) = \delta_{mn}(f,\mathcal{U}).$$

So $h_L^+(f^n) \leq nh_L^+(f)$.

On the other hand,

$$\delta_m(f^n,\mathcal{U}_f^n) = \min_{0\leq j\leq m-1} \delta(f^{-jn}(\mathcal{U}_f^n)) = \min_{0\leq j\leq mn-1} \delta(f^{-j}(\mathcal{U})) = \delta_{mn}(f,\mathcal{U}).$$

So $h_L^+(f^n) \geq nh_L^+(f)$. □

Theorem 2.2.16. *If f is Lipschitz with constant $L(f)$, then for every finite open cover $\mathcal{U}$, $h_L^+(f,\mathcal{U}) \leq \max\{\log L(f), 0\}$.*

Proof. Let $L = \max\{L(f), 1\}$. For every $x \in X$ and every $y \in B(x, \delta(\mathcal{U}) \cdot L^{-(n-1)})$,

$$d(f^j(x), f^j(y)) \leq L^j \cdot d(x,y) \leq \delta(\mathcal{U})$$

for $j = 0, 1, \ldots, n-1$. This implies that

$$f^j(B(x, \delta(\mathcal{U}) \cdot L^{-(n-1)})) \subset B(f^j(x), \delta(\mathcal{U})) \subset U_j$$

for some $U_j \in \mathcal{U}$, $j = 0, 1, \ldots, n-1$. So $\delta(\mathcal{U}_f^n, x) \geq \delta(\mathcal{U}) \cdot L^{-(n-1)}$ for every $x \in X$, hence $\delta_n \geq \delta(\mathcal{U}) \cdot L^{-(n-1)}$.

$$h_L^+(f, \mathcal{U}) = \limsup_{n\to\infty} -\frac{1}{n} \log \delta_n \leq \limsup_{n\to\infty} -\frac{1}{n} \log(\delta(\mathcal{U}) \cdot L^{-(n-1)}) = \log L.$$

□

Corollary 2.2.17. (see also [10][24]) *If f is Lipschitz with constant $L(f) > 1$, then*

$$D_H(X) \geq \frac{h(f)}{\log L(f)}.$$

Remark. We note long before Bowen's definition of topological entropy was introduced, Kushnirenko proved a result in a similar form involving the box dimension (see, e.g.[19, Theorem 3.2.9]):

$$h(f) \leq D_b^+(X) \cdot \max\{\log L(f), 0\}.$$

Theorem 2.2.18. *$h_L^+(f)$ is invariant under bi-Lipschitz conjugacy.*

Proof. Let ϕ be a bi-Lipschitz conjugacy between f on X and g on Y. For a finite open cover $\mathcal{U}$ of X and every $x \in X$, there is $U \in \mathcal{U}$ such that

$$U \supset B(x, \delta(\mathcal{U})) \supset \phi^{-1}(B(H(x), \delta(\mathcal{U}) \cdot L(\phi^{-1})^{-1})).$$

Then

$$B(\phi(x), \delta(\mathcal{U}) \cdot L(\phi^{-1})^{-1}) \subset \phi(U) \in \phi(\mathcal{U}).$$

As ϕ is a homeomorphism, this implies

$$\delta(\phi(\mathcal{U})) \geq \delta(\mathcal{U}) \cdot L(\phi^{-1})^{-1}. \tag{2.3}$$

Moreover, ϕ is a conjugacy,

$$g^{-n}(\phi(\mathcal{U})) = \phi(f^{-n}(\mathcal{U}))$$

Replace $\mathcal{U}$ by $g^{-n}(\phi(\mathcal{U}))$ in (2.3), then

$$\delta(g^{-n}(\phi(\mathcal{U}))) \geq \delta(\phi(f^{-n}(\mathcal{U}))) \cdot L(H^{-1})^{-1}.$$

and hence

$$\delta_n(g, \phi(\mathcal{U})) \geq \delta_n(f, \mathcal{U}) \cdot L(\phi^{-1})^{-1}.$$

Taking the upper limit we have $h_L^+(f) \geq h_L^+(g)$. Proof of the other direction is the same. □

Remark. A similar argument shows that $h_L(f)$ is also invariant under bi-Lipschitz conjugacy. In general, these numbers depend on the metric chosen and are not topological invariants. A way to fight with this is taking the infimum over all metrics inducing the same topology. All the inequalities relating entropy and dimensions remain valid.

2.3 Measure-Theoretic Entropy

In this section we discuss measure-theoretic entropy (or metric entropy).

2.3.1 Kolmogorov-Sinai Definition

The following definition was given by Kolmogorov in 1958 [22] and improved by Sinai in 1959 [28]. It was the first time that entropy was introduced into dynamical systems.

Let (X, Ω, μ) be a probability space. f is a transformation on X preserving μ, i.e. for every $A \in \Omega$, $\mu(f^{-1}(A)) = \mu(A)$.

Definition 2.3.1. Let ξ be a finite partition. Denote by $\xi(x)$ the element of ξ containing the point x. Then the entropy of ξ is

$$H(\xi) = -\sum_{C \in \xi} \mu(C) \log \mu(C) = -\int_X \log \mu(\xi(x)) d\mu.$$

Definition 2.3.2. Let ξ and η be finite measurable partitions, then

$$\xi \bigvee \eta = \{C \cap D | C \in \xi, D \in \eta\}.$$

We set

$$\xi_f^n = \bigvee_{k=0}^{n-1} f^{-k}(\xi).$$

Let

$$h_\mu(f,\xi) = \lim_{n\to\infty} \frac{1}{n} H(\xi_f^n).$$

The metric entropy of f is

$$h_\mu(f) = \sup\{h_\mu(f,\xi) | \xi \text{ is a finite partition of } X\}.$$

Theorem 2.3.3. *If μ is an invariant measure of f, then for $m \in \mathbb{N}$, $h_\mu(f^m) = mh_\mu(f)$. If in addition f is invertible (mod 0) then $h_\mu(f^{-1}) = h_\mu(f)$.*

Definition 2.3.4. Let

$$\mu(A|B) = \frac{\mu(A \cap B)}{\mu(B)}.$$

Define the conditional entropy by

$$H(\xi|\eta) = -\sum_{D\in\eta} \mu(D) \sum_{C\in\xi} \mu(C|D) \log \mu(C|D).$$

Theorem 2.3.5. $h_\mu(f,\xi) = H(\xi|\xi_f^\infty)$

Theorem 2.3.6. (Shannon-McMillan-Breiman) *Let*

$$h_\mu(f,\xi,x) = \lim_{n\to\infty} -\frac{1}{n} \log \mu(\xi_f^n(x)).$$

Then the limit exists for μ-almost every $x \in X$ and

$$h_\mu(f,\xi) = \int h_\mu(f,\xi,x) d\mu.$$

If in addition μ is ergodic then $h_\mu(f,\xi) = h_\mu(f,\xi,x)$ almost everywhere.

Remark. For a proof, see e.g. [23]

2.3.2 Katok's Definition

Analogous to Bowen-Dinaburg definition of topological entropy, A. Katok introduced an equivalent definition of metric entropy [17].

Theorem 2.3.7. *Let μ be an ergodic invariant Borel measure of f on a compact metric space X, then for every $\beta \in (0,1)$,*

$$\begin{aligned} h_\mu(f) =& \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\inf\{r_n(f,\epsilon,K)|\mu(K)>1-\beta\} \\ =& \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\inf\{r_n(f,\epsilon,K)|\mu(K)>1-\beta\} \\ =& \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\inf\{s_n(f,\epsilon,K)|\mu(K)>1-\beta\} \\ =& \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\inf\{s_n(f,\epsilon,K)|\mu(K)>1-\beta\}. \end{aligned}$$

(In the formulas K is a measurable subset and not necessarily compact.)

2.3.3 Local Entropy

It is interesting to see that the metric entropy can be calculated as the integral of a locally defined entropy. This was discovered by Brin and Katok [9].

Theorem 2.3.8. *Let f be a continous map on a compact metric space (X,d). μ is an invariant measure for f. We set*

$$B_n(x,\epsilon) = \{y \in X|d(f^j(x), f^j(y)) < \epsilon, 0 \le j \le n-1\},$$

$$h^+_\mu(x,\epsilon) = \limsup_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x,\epsilon)),$$

and

$$h^-_\mu(x,\epsilon) = \liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x,\epsilon)).$$

If $h_\mu(f)$ is finite, then

$$h_\mu(x) = \lim_{\epsilon\to\infty} h^+_\mu(x,\epsilon) = \lim_{\epsilon\to\infty} h^-_\mu(x,\epsilon)$$

for μ-almost every $x \in X$ and

$$h_\mu(f) = \int h_\mu(x)d\mu.$$

If in addition μ is ergodic, then $h_\mu(f) = h_\mu(x)$ almost everywhere.

2.3.4 Variational Principle

The well known variational principle describes the relation between topological entropy and metric entropy.

Theorem 2.3.9. (Variational Principle) *Let f be a continous map on a compact metric space X, then*

$$\begin{aligned} h(f) =& \sup\{h_\mu(f) | \mu \textit{ is an invariant measure of } f\} \\ =& \sup\{h_\mu(f) | \mu \textit{ is an ergodic measure of } f\}. \end{aligned}$$

2.4 Entropy of a Skew Product

2.4.1 Skew Product

Definition 2.4.1. A skew product map is a map $F = (g, f_x)$ on the product space $X \times Y$ defined by $F(x, y) = (g(x), f_x(y))$. X is called the base and Y is the fiber. Define $\pi : X \times Y \to X$ by $\pi(x, y) = x$. π is called the projection to the base and is a semiconjugacy from F to g.

Proposition 2.4.2. *Let μ be a measure on $X \times Y$. Then μ disintegrates as*

$$\mu = \int \sigma_x d\nu,$$

where ν is a measure on the base X and σ_x, called the conditional measure, is a measure on the fiber $\{x\} \times Y$, for ν-almost every x.

If μ is invariant of F, then for ν-almost every $x \in X$,

$$\sigma_x \circ f_x^{-1} = \sigma_{g(x)}.$$

If μ is F-ergodic then ν is g-ergodic.

2.4.2 Fiber Entropy

For a skew product map, the notion fiber entropy was introduced by Abramov and Rohlin [1].

Let η be a measurable partition of the fiber Y such that

$$\int_X H_x(\eta)d\nu < \infty, \tag{2.4}$$

where $H_x(\eta) = -\sum_{\mathcal{C}\in\eta} \sigma_x(\mathcal{C}) \log \sigma_x(\mathcal{C})$. We put

$$\eta_x^n = \bigvee_{k=1}^{n} f_x^{-1} f_{g(x)}^{-1} \cdots f_{g^{k-1}(x)}^{-1} \eta.$$

Theorem 2.4.3. *For every η satisfying (2.4), let*

$$h^g(f,\eta) = \lim_{n\to\infty} \frac{1}{n} \int_X H_x(\eta_x^n) d\nu.$$

The limit exists and it is finite. Let

$$h^g(f) = \sup\{h^g(f,\eta) | \int_X H_x(\eta)d\nu < \infty\}.$$

$h^g(f)$ is called the fiber entropy. We have

$$h_\mu(F) = h_\nu(g) + h^g(f).$$

Proposition 2.4.4. *If μ is ergodic, then for ν-almost every $x \in X$,*

$$h_x^g(f,\eta) = \lim_{n\to\infty} \frac{1}{n} H(\eta_x^n) = h^g(f,\eta).$$

2.4.3 Analogous Results

For the metric entropy of a skew product, there are analogous results to Theorem 2.3.6 and Theorem 2.3.7.

Theorem 2.4.5. (Belinskaja, [6]) *Let $\eta_x^n(y)$ be the element of η_x^n containing y. For $(x,y) \in X \times Y$, let*

$$h_x^g(f,\eta,y) = \lim_{n\to\infty} -\frac{1}{n} \log \sigma_x(\eta_x^n(y)).$$

Then the limit exists μ-almost everywhere. Moreover,

$$h_x^g(f,\eta) = \int h_x^g(f,\eta,y)d\sigma_x$$

for ν-almost every x and hence

$$h^g(f,\eta) = \int h_x^g(f,\eta,y)d\mu.$$

If μ is ergodic, then for almost every $(x,y) \in X \times Y$,

$$h_x^g(f,\eta,y) = h_x^g(f,\eta) = h^g(f,\eta).$$

Fix d a Riemannian metric on Y. Let $d_n^F(z,z')$ be the metrics defined for z, z' on the fiber $\{x\} \times Y$ by:

$$d_n^F(z,z') = \max_{0 \le i \le n-1} d(F^i z, F^i z').$$

For $x \in X$ and $\beta > 0$, on the fiber $\{x\} \times Y$, let $\mathcal{N}_x^F(n,\epsilon,\beta)$ be the minimal number of ϵ-balls in the d_n^F-metric needed to cover a set of σ_x-measure at least $1-\beta$, and let $\mathcal{S}_x^F(n,\epsilon,\beta)$ be the maximal number of (d_n^F,ϵ)-separated points we can find inside every set of σ_x-measure at least $1-\beta$.

Theorem 2.4.6. *If F is ergodic, then for every $\beta \in (0,1)$ and almost every $x \in X$,*

$$\begin{aligned} h^g(f) &= \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{\log \mathcal{N}_x^F(n,\epsilon,\beta)}{n} = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log \mathcal{N}_x^F(n,\epsilon,\beta)}{n} \\ &= \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{\log \mathcal{S}_x^F(n,\epsilon,\beta)}{n} = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log \mathcal{S}_x^F(n,\epsilon,\beta)}{n}. \end{aligned}$$

Chapter 3

Lyapunov Exponents, Entropy and Hyperbolic Measures

In this chapter we discuss nonuniform hyperbolicity of smooth systems.

3.1 Lyapunov Exponents

3.1.1 Multiplicative Ergodic Theorem

Theorem 3.1.1. (Multiplicative Ergodic Theorem, Oseledec [25]) *Let $f : X \to X$ be a measure preserving transformation of the probability space (X, μ). Let $A : X \to L(\mathbb{R}^k, \mathbb{R}^k)$ be a measurable map from X to the space of all linear operators on $\mathbb{R}^k$. Assume that*

$$\int \log \|A(x)\| d\mu < \infty.$$

Then there is $\Lambda \subset X$ of full measure such that $f(\Lambda) = \Lambda$ and the following properties hold for every $x \in \Lambda$:

1. *There is a measurable function $s : \Lambda \to \mathbb{N}$ with $s \circ f = s$.*

2. *There are measurable functions*

$$\chi_1(x) \le \chi_2(x) \le \cdots \le \chi_k(x).$$

such that for every x they take $s(x)$ different values

$$\chi'_1(x) < \chi'_2(x) < \cdots < \chi'_{s(x)}(x).$$

3. *There is a decomposition of $\mathbb{R}^k$ as*

$$\mathbb{R}^k = \bigoplus_{i=1}^{s(x)} E_i(x)$$

such that for every $v \in E_i(x) \backslash \{0\}$, $i = 1, 2, \ldots, s(x)$,

$$\chi(A, f, x, v) = \lim_{n \to \infty} \frac{1}{n} \log \|A(f^{n-1}(x)) \circ \cdots \circ A(f(x)) \circ A(x)(v)\| = \chi'_i(x). \tag{3.1}$$

4. *For $i = 1, 2, \ldots, s(x)$, we have*

$$\chi_i(f(x)) = \chi_i(x) \text{ and } E_i(f(x)) = A(x)(E_i(x)).$$

5. *For any partition $\{S, S'\}$ of $\{1, \ldots, s(x)\}$, let d_G be Grassmannian metric, then*

$$\lim_{n \to \infty} \frac{1}{n} \log d_G(\bigoplus_{i \in S} E_i(f^n(x)), \bigoplus_{i \in S'} E_i(f^n(x))) = 0. \tag{3.2}$$

All these objects do not depend on the choice of the Riemannian metric.

Remark. If in addition, for almost every x, $A(x)$ is invertible and

$$\int \log \|A^{-1}(x)\| d\mu < \infty,$$

then (3.1) and (3.2) may be replaced by

$$\chi(A, f, x, v) = \lim_{n \to \pm\infty} \frac{1}{n} \log \|A(f^{n-1}(x)) \circ \cdots \circ A(f(x)) \circ A(x)(v)\| = \pm\chi'_i(x)$$

and

$$\lim_{n \to \pm\infty} \frac{1}{n} \log d_G(\bigoplus_{i \in S} E_i(f^n(x)), \bigoplus_{i \in S'} E_i(f^n(x))) = 0.$$

For diffeomorphisms, we can take the measurable function A in the theorem to be the derivative. This provides the following theorem.

Theorem 3.1.2. *Assume f be a diffeomorphism on a k-dimensional compact Riemannian manifold M. Let $df : \mathrm{T}M \to \mathrm{T}M$ be the derivative of f. μ is Borel probability measure invariant of f. Then there is $\Lambda \subset X$ with full measure such that $f(\Lambda) = \Lambda$ and the following properties hold for every $x \in \Lambda$:*

1. *There is a measurable function $s : \Lambda \to \mathbb{N}$ with $s \circ f = s$.*
2. *There are measurable functions*
$$\chi_1(x) \leq \chi_2(x) \leq \cdots \leq \chi_k(x)$$
such that for every x they take $s(x)$ different values
$$\chi'_1(x) < \chi'_2(x) < \cdots < \chi'_{s(x)}(x).$$
3. *There is a decomposition of $\mathbb{R}^k$ as*
$$\mathbb{R}^k = \bigoplus_{i=1}^{s(x)} E_i(x)$$
such that for every $v \in E_i(x) \backslash \{0\}$, $i = 1, 2, \ldots, s(x)$,
$$\chi(f, x, v) = \lim_{n \to \pm\infty} \frac{1}{n} \log \|d_x f^n(v)\| = \pm\chi'_i(x).$$
4. *For $i = 1, 2, \ldots, s(x)$, we have*
$$\chi_i(f(x)) = \chi_i(x) \text{ and } E_i(f(x)) = df(E_i(x)).$$
5. *For any partition $\{S, S'\}$ of $\{1, \ldots, s(x)\}$, let d_G be Grassmannian metric, then*
$$\lim_{n \to \pm\infty} \frac{1}{n} \log d_G(\bigoplus_{i \in S} E_i(f^n(x)), \bigoplus_{i \in S'} E_i(f^n(x))) = 0.$$

Remark. If $F = (g, f_x)$ is a skew product map on $X \times Y$ preserving a measure μ and for almost every x, f_x is a diffeomorphism, then we can take the measurable function A to be the derivative $d_y f_x$ on $\mathrm{T}_y(\{x\} \times Y)$. This provides an analog for skew product maps.

Definition 3.1.3. Every $x \in \Lambda$ is called a Lyapunov regular point. The numbers $\chi_i(x)$, $1 \leq i \leq \dim M$, are called Lyapunov exponents.

Proposition 3.1.4. *If μ is ergodic, then Lyapunov exponents are constant almost everywhere. Denote them by χ_i, $1 \leq i \leq \dim M$.*

Definition 3.1.5. If μ is ergodic and all Lyapunov exponents are non-zero, then μ is said to be hyperbolic.

Definition 3.1.6. A diffeomorphism preserving a hyperbolic measure is called non-uniformly hyperbolic.

3.1.2 Margulis-Ruelle Inequality

As one might expect, Lyapunov exponents are closely related with the metric entropy.

Theorem 3.1.7. (Margulis-Ruelle,[27]) *If f is a diffeomorphism preserving μ, then*

$$h_\mu(f) \leq \int_M \sum_{\chi_i(x)>0} \chi_i(x) d\mu.$$

If in addition μ is ergodic, then

$$h_\mu(f) \leq \sum_{\chi_i>0} \chi_i.$$

Corollary 3.1.8. *If f is a diffeomorphism preserving μ, then*

$$h_\mu(f) \leq -\int_M \sum_{\chi_i(x)<0} \chi_i(x) d\mu.$$

If in addition μ is ergodic, then

$$h_\mu(f) \leq -\sum_{\chi_i<0} \chi_i.$$

Corollary 3.1.9. *Let f be a diffeomorphism on a compact manifold preserving an ergodic measure μ. If $h(f) > 0$, then there are at least one positive and one negative Lyapunov exponents for μ.*

Corollary 3.1.10. *If f is a diffeomorphism on a 2-dimensional compact manifold M, then every ergodic measure of f with positive metric entropy is hyperbolic.*

3.1.3 Pesin Formula

Definition 3.1.11. We say a measure μ_1 is absolutely continuous with respect to μ_2, denoted by $\mu_1 \ll \mu_2$, if for any measurable subset E, $\mu_2(E) = 0$ implies $\mu_1(E) = 0$. We say μ_1 and μ_2 are equivalent if $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$.

Theorem 3.1.12. *Let f be a C^2 diffeomorphism on a compact Riemannian manifold. If μ is an invariant measure which is equivalent to a Riemannian volume, then*

$$h_\mu(f) = \int_M \sum_{\chi_i(x)>0} \chi_i(x) d\mu = -\int_M \sum_{\chi_i(x)<0} \chi_i(x) d\mu.$$

3.2 Hyperbolic Measure

In this section we discuss the properties of a smooth system preserving a hyperbolic measure. We only formulate the results for diffeomorphisms on compact manifolds. If $F = (g, f_x)$ is a skew product map on $X \times Y$ preserving a measure μ and for almost every x, $f_x : Y \to Y$ is a diffeomorphism on the compact manifold Y, the analogous results can be easily formulated.

3.2.1 Regular Neighborhoods

Definition 3.2.1. For $0 < \alpha < 1$, we say f is a $C^{1+\alpha}$ diffeomorphism if its derivative is α-Hölder continuous.

Theorem 3.2.2. *Let M be an l-dimensional compact Riemannian manifold. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism preserving a hyperbolic ergodic measure μ. Denote by $B^k(r)$ the standard Euclidean r-ball in $\mathbb{R}^k$ centered at the origin. There exists a set $\Lambda \subset M$ of full measure such that for every sufficiently small $\epsilon > 0$ and some $\chi > 0$:*

1. *There exists a function $\omega : \Lambda \to (0,1]$ such that*

$$e^{-\epsilon} < \omega(x)/\omega(f(x)) < e^{\epsilon} \quad \text{and} \quad \lim_{|n|\to\infty} \frac{1}{n} \log \omega(f^n(x)) = 0$$

and a collection of embeddings $\Psi_x : B^l(\omega(x)) \to M$ such that for each $x \in \Lambda$, $\Psi_x(0) = x$.

2. *There exist a constant $K > 0$ and a measurable function $C : \Lambda \to \mathbb{R}$ such that for $y_1, y_2 \in B^l(\omega(x))$,*

$$K^{-1} d(\Psi_x(y_1), \Psi_x(y_2)) \leq \|y_1 - y_2\| \leq C(x) d(\Psi_x(y_1), \Psi_x(y_2))$$

with $e^{-\epsilon} < C(f(x))/C(x) < e^{\epsilon}$.

3. *Let $s = \max\{i : \chi_i < 0\}$. The map $f_x := \Psi_{f(x)}^{-1} \circ f \circ \Psi_x : B^s(\omega(x)) \times B^{l-s}(\omega(x)) \to \mathbb{R}^l = \mathbb{R}^s \times \mathbb{R}^{l-s}$ has the form*

$$f_x(u,v) = (A_x u + \eta_{1,x}(u,v), B_x v + \eta_{2,x}(u,v)),$$

where $\eta_{1,x}(0,0) = \eta_{2,x}(0,0) = 0$, $d\eta_{1,x}(0,0) = d\eta_{2,x}(0,0) = 0$ and

$$\|A_x\| < \exp -(\chi - \epsilon), \|B_x^{-1}\| < \exp -(\chi - \epsilon).$$

For $z = (u,v) \in B^l(\omega(x))$, $\eta_x(z) = (\eta_{1,x}(z), \eta_{2,x}(z))$, we have

$$\|d_z \eta_x\| < \epsilon, \|\eta_x\| < \epsilon.$$

Definition 3.2.3. The points $x \in \Lambda$ are called regular points. For each regular point x, the set $\mathcal{R}(x) = \Psi_x(B(\omega(x)))$ is called a regular neighborhood of x. Let $r(x)$ be the radius of the maximal ball contained in the regular neighbor hood $\mathcal{R}(x)$. We call $r(x)$ the size of $\mathcal{R}(x)$.

Theorem 3.2.4. *For each $\beta > 0$ and each sufficiently small $\epsilon(\beta) > 0$, there is a compact set $\Lambda_\beta \subset \Lambda$ such that $\mu(\Lambda_\beta) > 1 - \beta$ and the following conditions hold:*

1. *The functions $x \mapsto \omega(x)$, $x \mapsto C(x)$ and $x \mapsto \Psi_x$ as in Theorem 3.2.2 for $\epsilon = \epsilon(\beta)$, and $x \mapsto r(x)$ are all continuous on Λ_β.*

2. *The decomposition* $T_xM = d\Psi_x\mathbb{R}^k \bigoplus d\Psi_x\mathbb{R}^{l-k}$ *depends continuously on* x *in* Λ_β.

3. *On* Λ_β*, there are bounds:* $\omega_\beta = \min\{\omega(x)\}$*,* $r_\beta = \min\{r(x)\}$*, and* $C_\beta = \max\{C(x)\}$.

Definition 3.2.5. For $\beta > 0$, Λ_β is called a Pesin set.

3.2.2 Admissible Submanifolds and Rectangles

Now we fix a small $\beta > 0$ and a Pesin set Λ_β. We also fix small numbers $\omega < \omega_\beta$, $\gamma > 0$ and $\kappa > 0$.

Definition 3.2.6. An s-dimensional submanifold $W \subset \mathbb{R}^l$ is called an (s,γ)-admissible submanifold if $W = \{(\theta, \phi(\theta)) : \theta \in [-\omega, \omega]\}$, where $\phi : [-\omega, \omega] \to [-\omega, \omega]$ is a C^1 map such that $|\phi(0)| < \kappa$ and $|d\phi| < \gamma$. For every $x \in \Lambda_\beta$, we call $\Psi_x(W) \subset \mathcal{R}(x)$ an (s,γ)-admissible submanifold near x if W is an (s,γ)-admissible submanifold.

Similarly, an $(l-s)$-dimensional submanifold $W \subset \mathbb{R}^l$ is called an (u,γ)-admissible submanifold if $W = \{(\phi(\theta), \theta) : \theta \in [-\omega, \omega]\}$, where $\phi : [-\omega, \omega] \to [-\omega, \omega]$ is a C^1 map such that $|\phi(0)| < \kappa$ and $|d\phi| < \gamma$. For every $x \in \Lambda_\beta$, we call $\Psi_x(W) \subset \mathcal{R}(x)$ a (u,γ)-admissible submanifold near x if W is an (u,γ)-admissible submanifold.

Definition 3.2.7. An (s,γ)-admissible rectangle is defined as a set of points

$$\{(u,v) \in [-\omega,\omega]^2 : v = \lambda\phi_1(u) + (1-\lambda)\phi_2(u), 0 \le \lambda \le 1\}$$

if ϕ_1 and ϕ_2 are admissible (s,γ)-submanifolds. For every $x \in \Lambda_\beta$, we call $\Psi_x(W) \subset \mathcal{R}(x)$ an (s,γ)-admissible rectangle near x if W is an (s,γ)-admissible rectangle. (u,γ)-admissible rectangles are defined similarly.

Theorem 3.2.8. *There is* $\rho > 0$ *such that if* $x, y \in \Lambda_\beta$*,* $d(x,y) < \rho$*, and* V *is an* (s,γ)*-admissible submanifold near* y*, then*

$$\Psi_x(\Psi_y^{-1}(V) \cap \mathcal{R}(x))$$

is part of an (s,γ)-admissible submanifold near x. A similar statement holds for (u,γ)-admissible submanifolds.

Theorem 3.2.9. *For every $x \in \Lambda_\beta$, if $V = \Psi_x(W)$, where $W = \{(\theta, \phi(\theta)) : \theta \in [-\omega,\omega]\}$, is an (s,γ)-admissible submanifold near x, then the connected component of $f^{-1}(V) \cap \mathcal{R}(f^{-1}(x))$ containing $f^{-1}(\Psi_x((0,\phi(0))))$ is an (s,γ)-admissible submanifold near $f^{-1}(x)$.*

If $V = \Psi_x(W)$, where $W = \{(\phi(\theta), \theta) : \theta \in [-\omega,\omega]\}$, is an (u,γ)-admissible submanifold near x, then the connected component of $f(V) \cap \mathcal{R}(f(x))$ containing $f(\Psi_x((0,\phi(0))))$ is an (u,γ)-admissible submanifold near $f(x)$.

Corollary 3.2.10. *For every $x \in \Lambda_\beta$, if V is an (s,γ)-admissible rectangle near x, then the connected component of $f^{-1}(V) \cap \mathcal{R}(f^{-1}(x))$ containing $f^{-1}(x)$ is an (s,γ)-admissible rectangle near $f^{-1}(x)$.*

If V is an (u,γ)-admissible rectangle near x, then the connected component of $f(V) \cap \mathcal{R}(f(x))$ containing $f(x)$ is an (u,γ)-admissible rectangle near $f(x)$.

Theorem 3.2.11. *For every $x \in \Lambda_\beta$, any (s,γ)-admissible submanifold near x intersects any (u,γ)-admissible submanifold near x transversally at exactly one point. Thus any (s,γ)-admissible rectangle near x intersects any (u,γ)-admissible rectangle near x in exactly one connected component.*

3.2.3 Entropy, Periodic Points and Horseshoe

In the well known paper [17] Katok showed:

Theorem 3.2.12. *Let f be a $\mathrm{C}^{1+\alpha}$ diffeomorphism of a compact manifold M, and μ a hyperbolic ergodic measure. Then*

$$\overline{Per}(f) \supset \mathrm{supp}(\mu)$$

and

$$\max(0, \limsup_{n\to\infty} \frac{\ln P_n(f)}{n}) \geq h_\mu(f),$$

where $Per(f)$ is the set of all periodic points of f and $P_n(f)$ the number of periodic points of f with period n (i.e. the number of fixed points of f^n). $\mathrm{supp}(\mu)$ *is the support of μ.*

Corollary 3.2.13. *For any* $\mathrm{C}^{1+\alpha}$ *diffeomorphism* f *of a 2-dimensional compact manifold with positive topological entropy,*

$$\limsup_{n\to\infty} \frac{\ln P_n(f)}{n} \geq h(f).$$

Hence f *is not minimal or uniquely ergodic.*

3.2.4 Measures of Intermediate Entropies

Another remarkable result of Katok is the following [18] [20].

Theorem 3.2.14. *Let* f *be a* $C^{1+\alpha}$ *diffeomorphism on a compact manifold* M*. If* μ *is a hyperbolic ergodic measure and* $h_\mu(f) > 0$*, then for any* $\epsilon > 0$*, there is a hyperbolic horseshoe* $\Lambda \subset \mathrm{supp}(\mu)$ *such that* $h(f|_\Lambda) > h_\mu(f) - \epsilon$*.*

Definition 3.2.15. Let f be a continuous map on a compact metric space X. We say f has measures of intermediate entropies if given any number $a \in [0, h(f))$, there is an ergodic measure μ_a such that $h_{\mu_a}(f) = a$.

If μ is an ergodic measure, then we say f has measures of intermediate entropies for μ if given any number $a \in [0, h_\mu(f))$, there is an ergodic measure μ_a such that $h_{\mu_a}(f) = a$.

Proposition 3.2.16. *If* f *is topologically conjugate to a subshift of finite type, then* f *has measures of intermediate entropies.*

To see this, just note that we can continuous change the weights of different symbols and the entropies obtained range over the whole interval.

Corollary 3.2.17. *Let* f *be a* $C^{1+\alpha}$ *diffeomorphism on a compact manifold* M*. If* μ *is a hyperbolic ergodic measure, then* f *has measures of intermediate entropies for* μ*.*

Corollary 3.2.18. *Let* f *be a* $C^{1+\alpha}$ *diffeomorphism on a 2-dimensional compact manifold* M*, then* f *has measures of intermediate entropies.*

3.3 Herman's Example

3.3.1 Introduction

At the end of the last section we discussed some results of Katok for hyperbolic measures and 2-dimensional diffeomorphisms. We note most of them are not true in full generality. First, there are many examples, including both symbolic systems and homeomorphisms on manifolds, which have positive topological entropies while being uniquely ergodic (see e.g. [12][5]). These maps have no proper compact invariant subset (hence no periodic points).

Even for smooth diffeomorphisms Katok's theorems may fail if a hyperbolic measure is not present. In this section we discuss a C^∞ example constructed by Herman [15] which has prompted fruitful researches.

Consider the C^∞ map $A : \mathbb{T}^1 \to \mathrm{SL}(2, \mathbb{R})$ defined by

$$A(\theta) = A_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix},$$

where $\lambda > 1$ is a fixed number. Let $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1$ be the rotation by $\alpha \in \mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z})$.

Theorem 3.3.1. *There is a dense* G_δ *subset* W *of* $\mathbb{T}^1$*, such that for every* $\alpha \in W$*, the smooth diffeomorphism* $F_\alpha = (R_\alpha, A(\theta))$ *on* $\mathbb{T}^1 \times \mathrm{SL}(2, \mathbb{R})/\Gamma$*, given by* $(\theta, y) \mapsto (\theta + \alpha, A(\theta) \cdot y)$*, is minimal and has positive topological entropy.*

3.3.2 Positivity of Entropy

We first show that Herman's example has positive topological entropy.

Theorem 3.3.2. *For every* $\alpha \in \mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z})$*,* $h(F_\alpha) > 0$*.*

Remark. In fact, $h(F_\alpha) > 0$ for all $\alpha \in \mathbb{T}^1$.

Proof. We need the following theorem.

Denote by

$$D_r^n = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : |z_i| \le r, 1 \le i \le n\}$$

and

$$\mathbb{T}_r^n = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : |z_i| = r, 1 \le i \le n\}$$

the closed ball and torus in $\mathbb{C}^n$ correspondingly. Assume that the map $f : U \to \mathbb{C}^n$ is holomorphic in a neighborhood of D_r^n satisfying $f(0) = 0$, $f(D_r^n) \subset D_r^n$, $f(\mathbb{T}_r^n) \subset \mathbb{T}_r^n$ and $f|_{\mathbb{T}_r^n}$ preserves the Lebesgue measure μ_L. We have:

Theorem 3.3.3. (Herman, [16]) *If A is a holomorphic map in a neighborhood of D_r^n taking values in a Banach algebra $\mathcal{B}$ and generates the cocycle $\mathcal{A}$ over f, then*

$$\lambda^+(f, A) = \liminf_{k\to\infty} \frac{1}{k} \int_{\mathbb{T}_r^n} \log \|\mathcal{A}(z, k)\| d\mu_L \ge \log \sigma(A(0)),$$

where

$$\sigma(B) = \lim_{k\to\infty} \|B^k\|^{1/k} = \inf_{k\ge1} \|B^k\|^{1/k}$$

is the spectral radius of $B \in \mathcal{B}$.

Applying Theorem 3.3.3 (with some change of variables) we obtain

$$\lambda^+(R_\alpha, A_\theta) \ge \log(\frac{\lambda}{2} + \frac{1}{2\lambda}) > 0.$$

(By [3], this is in fact an equality). Let ν be the Lebesgue measure on $\mathbb{T}^1$ and σ the Liouville measure on $\mathrm{SL}(2, \mathbb{R})/\Gamma$. Then the measure $\mu = \nu \times \sigma$ on M is F_α-invariant. By Pesin [26],

$$h_\mu(F_\alpha) \ge \inf_{k\ge1} \frac{1}{k} \int_M \log \|\mathrm{D}F_\alpha^n(x)\| d\mu.$$

A short computation shows that

$$\log \|\mathrm{D}F_\alpha^n(\theta, y)\| \ge 2 \log \|A_{\theta+(n-1)\alpha} \cdots A_{\theta+\alpha} A_\theta\|.$$

So the entropy is at least $2 \log(\lambda/2 + 1/2\lambda)$, hence positive. □

3.3.3 Minimality

Theorem 3.3.4. *There is a dense G_δ subset W of $\mathbb{T}^1$, such that for every $\alpha \in W$, F_α is minimal.*

Proof. We take $\{U_i\}_{i\in\mathbb{N}}$, a countable basis for the topology on M. The map F_α is minimal if and only if for every U_i, there exists $n \in \mathbb{N}$ such that

$$\bigcup_{0\le p\le n} F_\alpha^p(U_i) = M. \tag{3.3}$$

Let $W_i = \{\alpha \in \mathbb{T}^1 : \exists n \in \mathbb{N}$ such that (3.3) holds for F_α and $U_i\}$. The set W_i is open for each i. To see F_α is minimal for a G_δ dense set of α, it is sufficient to show that each W_i is dense.

For each fixed i, we may assume that U_i is of the form $I \times V$ with $I = (a, b) \subset \mathbb{T}^1$ an interval and $V \neq \emptyset$ an open subset in $M_1 = \mathrm{SL}(2, \mathbb{R})/\Gamma$. Take

$$p/q \in \mathbb{Q}/\mathbb{Z} \text{ such that } (p, q) = 1 \text{ and } 1/q < (b - a)/2. \tag{3.4}$$

Denote $A_{\theta+(n-1)\alpha} \cdots A_{\theta+\alpha} A_\theta$ by $A^n_{\alpha,\theta}$. Since $\lambda^+(R_\alpha, A_\theta) > 0$ for all $\alpha \in \mathbb{T}^1$, there exists θ_1 such that $A^q_{p/q,\theta_1}$ is a hyperbolic matrix which is conjugate to a diagonal matrix in $\mathrm{SL}(2, \mathbb{R})$. Every cocycle (R_α, B) on M induces a map on $\mathbb{T}^1 \times \mathbb{RP}^1 = \mathbb{T}^1 \times \mathbb{T}^1$. Denote by ρ the rotation number on the fiber ([14, II and III] or [16, section 5]). Then $\rho(A^q_{p/q,\theta_1}) = \rho_0 \in \mathbb{Z}$. Let $\theta_2 = \sup\{\theta : \rho(A^q_{p/q,\theta}) = \rho_0\}$. By the properties of diffeomorphisms on the circle, $A^q_{p/q,\theta_2}$ as well as $A^q_{p/q,\theta_2+l/q}$, $1 \le l \le q - 1$, must be a parabolic matrix which is conjugate to an element in $\mathrm{SL}(2, \mathbb{R})$ generating the horocycle flow on M_1. By Hedlund [13], the horocycle flow is minimal. Then (3.4) guarantees the existence of $\theta_0 \in I$ such that $(\theta_0, y) \mapsto F^q_{p/q}(\theta_0, y)$ on $\{\theta_0\} \times M_1$ is minimal. Equivalently, there is $N > 0$ such that

$$\bigcup_{0\le k\le N} F^{kq}_{p/q}(I \times V) \supset (a_1, b_1) \times M_1 \supset \{\theta_0\} \times M_1. \tag{3.5}$$

with $(a_1, b_1) \subset (a, b)$. By continuity, for all β in an open neighborhood D of p/q in $\mathbb{T}^1$, relation (3.5) still holds when we replace $F^{kq}_{p/q}$ by F^{kq}_β. Moreover, if $\beta \in D$ is irrational, then

$$\bigcup_{n\in\mathbb{N}} F^n_\beta((a_1, b_1) \times M_1) = \mathbb{T}^1 \times M_1 = M.$$

So, for every rational number p/q satisfying (3.4) (such points form a dense set in $\mathbb{T}^1$), there is an irrational number β which can be arbitrarily close to p/q such that

$\beta \in W_i$. That means W_i is dense. Hence the intersection of all W_i is a dense G_δ set of α such that F_α is minimal. $\square$

3.3.4 Measures of Intermediate Entropies

In the case of Herman's example, there is no proper compact invariant subset since the map is minimal. However, this does not prevent it from having measures of intermediate entropies.

Theorem 3.3.5. *For every $\alpha \in \mathbb{T}^1$, F_α has measures of intermediate entropies.*

Proof. First we note the map on the base has zero entropy. So all entropy comes from the fibers.

By Theorem 3.1.1, There are measurable functions $v_i : \mathbb{T}^1 \to \mathbb{R}^2, i = 1, 2,$ $\|v_i(\theta)\| = 1$ and $l : \mathbb{T}^1 \to \mathbb{R}$ such that

$$\begin{aligned} A_\theta(v_1(\theta)) =& l(\theta)v_1(\theta + \alpha), \\ A_\theta(v_2(\theta)) =& l(\theta)^{-1}v_2(\theta + \alpha). \end{aligned}$$

Then there is a measurable function $Q : \mathbb{T}^1 \to \mathrm{SL}(2, \mathbb{R})$ such that

$$Q(\theta)v_i(\theta) = e_i, e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus

$$H_\theta = Q_{\theta+\alpha}A_\theta Q_\theta^{-1} = \begin{pmatrix} l_\theta & 0 \\ 0 & l_\theta^{-1} \end{pmatrix}$$

is diagonal.

As we know, the geodesic flow on M_1 is generated by the left action by

$$G_t = \begin{pmatrix} \exp(t/2) & 0 \\ 0 & \exp(-t/2) \end{pmatrix}.$$

So if τ is an invariant measure of the geodesic flow, then τ is also invariant of H_θ. Hence

$$\mu_\tau = \int (\tau \circ Q_\theta) d\nu$$

is an invariant measure of F_α. The geodesic flow is hyperbolic hence has measures of intermediate entropies. So does F_α. $\square$

Chapter 4

Measures of Intermediate Entropies of Skew Products

4.1 Introduction

4.1.1 Formulation of Result

Let $F = (g, f_x)$ be a skew product map on the space $X \times Y$ preserving an ergodic measure $\mu = \int \sigma_x d\nu$. We assume that g is an invertible (mod 0) measure preserving transformation on the probability space (X, ν). Then g is ergodic. For every $x \in X$, f_x is a $C^{1+\alpha}$ diffeomorphism on the compact Riemannian manifold Y, sending σ_x to $\sigma_{g(x)}$.

Theorem 4.1.1. (Sun, [30]) *Assume that $h_\mu(F) > 0$ and $h_\nu(g) = 0$. If for almost every $z = (x, y) \in X \times Y$ and every $v \in T_y(\{x\} \times Y)\backslash\{0\}$, the Lyapunov exponent*

$$\chi(v) = \lim_{n\to\infty} \frac{\log \|df_{g^{n-1}(x)} \cdots df_{g(x)} df_x v\|}{n} \neq 0,$$

then F has measures of intermediate entropies for μ.

Remark. Our proof also works when $h_\nu(g) > 0$. In this case it concludes that F has ergodic measures of arbitrary entropies between $h_\nu(g)$ and $h_\mu(F)$. In addition, we assume g has no periodic point. Otherwise the problem is reduced to Theorem 3.2.14.

4.1.2 Outline of Proof

In the proof we develope Katok's method [20] and combine it with some new ideas about recurrence. To construct measures of intermediate entropies, we find an invariant subset, on which the induced map is also a skew product with horseshoe maps on fibers. By Pesin theory (as discussed in section 3.2), We can first choose some good fibers on which the hyperbolic structure is uniform. Then we can find on each good fiber some points with the same and relatively fast return time such that the return time is integrable and the return map on these points projects to a map on the base. Admissible stable and unstable rectangles of these points and their images can be used to produce horseshoes on fibers. A crucial issue here is that the projected map, which is a return map on the base, is not necessarily the first return in any sense hence may be non-invertible, which obstruct us from constructing invariant measures. To deal with this, we investigated the combinatorial structure of such a return map and established Theorem 4.2.2. This theorem guarantees existence of a subset on which the projected map is invertible and its restriction preserves the measure on the base. So the conditional measures constructed on fibers can be integrated to obtain ergodic measures for the skew product. finally, we managed to control the return time in a subtle way and get good estimates on the entropies of the measures we constructed. Since a horseshoe is conjugate to a full shift, by adjusting weights of different symbols we can obtain arbitrary intermediate entropy.

4.2 Recurrence

In this section we discuss some properties related to nontrivial recurrence of the skew product map. These results serve as key lemmas in the proof of Theorem 4.1.1.

4.2.1 Points with Fast Returns

Theorem 4.2.1 says that, for a subset of positive measure, if the conditional measures are uniformly bounded from below, then on each fiber we can find points that return relatively faster, such that the return time is integrable.

Theorem 4.2.1. (Integrability of Return Time) *Let $P \subset X \times Y$ be a measurable subset. $B = \pi(P) \subset X$ is the projection of P on the base. For $x \in B$, denote $P \cap (\{x\} \times Y)$ by $P(x)$. Assume that $\nu(B) = \nu_0 > 0$ and there is $\sigma_0 > 0$ such that for (almost) every $x \in B$, $\sigma_x(P(x)) > \sigma_0$. Hence $\mu(P) = \mu_0 > \nu_0\sigma_0 > 0$. For (almost) every $z \in P$, denote by $n_l(z)$ the l-th return time of z. Let*

$$P_n^l(x) = \{z \in P(x)|\ n_l(z) \geq n\},\ N_l(x) = \max\{n|\sigma_x(P_n^l(x)) > \sigma_0\}.$$

Then

$$\int_B N_l(x)d\nu < \frac{l}{\sigma_0} < \infty.$$

Remark. $N_l(x)$ is the longest return time for the l-th returns of the points in a subset of conditional measure no less than $\sigma_x(P(x)) - \sigma_0$ in $P(x)$. Or equivalently, $N_l(x)$ is the smallest number such that the set of points in $P(x)$ with l-th return times greater than $N_l(x)$ has conditional measure at most σ_0.

Proof. Since μ is F-invariant and $\mu(P) > 0$, we have

$$0 < \int_P n_1(z)d\mu = \mu(\bigcup_{j=0}^{\infty} F^j(P)) \leq 1.$$

Let F_P be the first return map on P, which preserves μ, then for each k,

$$\int_P n_{k+1}(z)d\mu = \int_P (n_k(F_P(z)) + n_1(z))d\mu = \int_P n_k(z)d\mu + \int_P n_1(z)d\mu.$$

Hence

$$0 < \int_P n_l(z)d\mu = l\int_P n_1(z)d\mu \leq l.$$

Note

$$\int_P n_l(z)d\mu = \sum_{j=1}^{\infty} \mu(P_j),$$

where $P_j = \{z \in P|n_l(z) \geq j\}$ consists of points with l-th return time no less than j. Note $P_j^l(x) = P_j \cap P(x)$.

For every $x \in B$, let $B^j = \{x \in B|N_l(x) \geq j\}$. By definition, for every $j \leq N_l(x)$, $\sigma_x(P_j^l(x)) > \sigma_0$. So $x \in B^j$ iff $\sigma_x(P_j^l(x)) = \sigma_x(P_j \cap P(x)) > \sigma_0$. We have

$$\mu(P_j) = \int_B \sigma_x(P_j \cap P(x))d\nu > \int_{B^j} \sigma_0 d\nu = \nu(B^j) \cdot \sigma_0,$$

hence

$$\int_B N_l(x)d\nu = \sum_{j=1}^{\infty} \nu(B^j) < \sum_{j=1}^{\infty} \frac{1}{\sigma_0}\mu(P_j) \leq \frac{l}{\sigma_0}.$$

□

4.2.2 Kernel of a Random Return Map

We consider a complicated or even randomly selected return map on some set of positive measure, which may be neither injective nor surjective. We show if the return time is integrable, then this return map is one-to-one and measure preserving, on a smaller subset of positive measure. We call this subset the *kernel* of the return map. Moreover, we have an estimate of the integral of the return time, which can be used to estimate the size (measure) of the kernel.

Theorem 4.2.2. *Let g be a measure preserving transformation on a probability space (X, ν). g is invertible and has no periodic point. B is a subset of X with $\nu(B) = \nu_0 > 0$. $N : B \to \mathbb{N}$ is a measurable function such that $\tilde{g}(x) = g^{N(x)}(x) \in B$ for almost every $x \in B$. Assume*

$$\int_B N(x)d\nu = \Sigma_B < \infty. \tag{4.1}$$

Then there is a subset B' of B such that the following holds:

1. *$\nu(B') = \nu_1 > 0$, $\tilde{g}(B') = B'$ and $g_* = \tilde{g}|_{B'}$ is invertible and ν-preserving.*

2. $$\int_{B'} N(x)d\nu \geq \nu(\bigcup_{j=-\infty}^{\infty} g^j(B)).$$

   We call B' the kernel of $\tilde{g}$.

Remark. The assumption that g has no periodic point is not necessary in this theorem. Periodic orbits may be removed as a null set or we can easily find a subset consisting of periodic orbits on which $\tilde{g}$ is invertible.

A result similar to the first part of the theorem has been discussed in [21], where B coincides with X and the measurable function N takes values in $\mathbb{Z}$. The following proof is obtained in [30] independently with a few different tricks.

Proof. With possible loss of a null set we may assume that the first return map g_B on B is defined everywhere on B and invertible. g_B has no periodic point since g has not.

Define a partial order on B: $x_1 \prec x_2$ iff there is $n \geq 0$ such that $g_B^n(x_1) = x_2$, i.e. x_2 is an image of x_1 under iterates of g_B (and g). Since g_B is invertible and has no periodic point, this partial order is well defined.

Let $O^+(x) = \{\tilde{g}^k(x) | k \in \mathbb{N} \cup \{0\}\}$ be the forward $\tilde{g}$-orbit of x. We define an equivalence relation on B: $x_1 \sim x_2$ iff $O^+(x_1) \cap O^+(x_2) \neq \emptyset$, i.e. there are $k_1, k_2 > 0$ such that $\tilde{g}^{k_1}(x_1) = \tilde{g}^{k_2}(x_2)$. Note $x_1 \in O^+(x_2)$ or $x_2 \in O^+(x_1)$ implies $x_1 \sim x_2$, but the converse is not true. Also note within an equivalence class the partial order is a total order, since g_B is invertible and $\tilde{g}^{k_1}(x_1) = \tilde{g}^{k_2}(x_2) = x_3$ implies $g_B^{n_1}(x_1) = g_B^{n_2}(x_2)$ for some n_1 and n_2. We write $x_1 \precsim x_2$ if $x_1 \prec x_2$ and $x_1 \sim x_2$.

Here we adapt some facts we showed in [29].

Lemma 4.2.3. *For every $x \in B$, define*

$$J(x) = \{\tilde{g}(x') : x' \prec x, \text{ and } x \prec \tilde{g}(x')\}.$$

Then for almost every $x \in B$, $J(x)$ is finite.

Proof. Consider

$$I(x) = \{x' \in B : g^k(x') = x \text{ for some } 1 \leq k \leq N(x')\}$$

and

$$I_j = \{x \in B : |I(x)| = j\}.$$

Then we can count the return time and obtain

$$\sum_{j=0}^{\infty} j \cdot \nu(I_j) = \int_B N(x) d\nu < \infty.$$

So for almost every $x \in B$,

$$|J(x)| \leq |I(x) \cap (\{x\} \times Y)| < \infty.$$

□

Lemma 4.2.4. *For almost every $x \in B$, there is $x^* \in B$ such that for every $x' \precsim x$, $x^* \in O^+(x')$, i.e.*

$$H(x) = \bigcap_{x' \precsim x} O^+(x') \neq \emptyset.$$

Moreover, if $x_1 \precsim x_2$ then $H(x_1) \supset H(x_2)$.

Proof. Let

$$J'(x) = \{x'' \in J(x) : x'' \sim x\} \subset J(x).$$

By Lemma 4.2.4, $J'(x)$ is finite for almost every $x \in B$. For every $x' \precsim x$ but $x' \neq x$, there must be $x'' \in J'(x)$ and $x'' \in O^+(x')$. This implies that

$$H(x) = \bigcap_{x' \precsim x} O^+(x') \supset \bigcap_{x' \in J'(x)} O^+(x').$$

It is not hard to see the finite intersection is nonempty.

The last statement is a trivial corollary. □

Lemma 4.2.5. *For almost every $x \in B$, there is a point x' such that $x' \precsim x$ and $x' \neq x$. Hence for almost every $x \in B$, there are infinitely many x' such that $x' \precsim x$.*

Proof. Let

$$B_c = \{x \in B | \text{there is no such } x' \neq x \in B \text{ that } x' \precsim x\}.$$

If $\nu(B_c) > 0$, then by lemma 4.2.4 and Poincaré Recurrence Theorem, there is an element $x_c \in B_c$ such that $J(x_c)$ is finite and

$$O_g^-(x_c) = \{g^{-k}(x_c) : k > 0\} \cap B_c$$

is infinite. For every $x' \in O_g^-(x_c)$,

$$O^+(x') \cap J(x_c) \neq \emptyset.$$

By pigeonhole principle, there is $x'' \in J(x_c)$ such that

$$\{x' \in O_g^-(x_c) : x'' \in O^+(x')\} \subset B_c$$

is an infinite set. But every two distinct elements in this set are equivalent and comparable since their forward orbits contain a common element x'', which contradicts that they are elements of B_c. So we must have $\nu(B_c) = 0$. $\square$

Excluding a null set and its (full) g-orbit (the union is still a null set), we can assume the results in the last two lemmas hold for every $x \in B$.

Let $\tilde{B} = \bigcap_{j=1}^{\infty} \tilde{g}^j(B)$. Then $\tilde{B}$ is a measurable subset of B, consisting of elements that lie in the forward $\tilde{g}$-orbits of infinitely many elements of B. For every $x \in B$, let

$$G(x) = \bigcup_{x' \sim x} H(x').$$

Then for every $\tilde{x} \in G(x)$, there is x' such that $x' \sim x$ and

$$\tilde{x} \in H(x') = \bigcap_{x'' \precsim x'} O^+(x'').$$

By Lemma 4.2.5, the intersection is of infinitely many forward $\tilde{g}$-orbits, which implies $\tilde{x} \in \tilde{B}$. So $G(x) \subset \tilde{B}$ for every $x \in B$.

For $x \in \tilde{B}$, there are infinitely many elements in B such that their forward $\tilde{g}$-orbits pass through x. They also pass through one of the elements in the pre-image $\tilde{g}^{-1}(x)$. But by integrability of return time (4.1), the pre-image $\tilde{g}^{-1}(x)$ consists of finite number of elements. By pigeonhole principle, there must be some element in $\tilde{g}^{-1}(x)$ which lies in infinitely many forward $\tilde{g}$-orbits. Such an element belongs to $\tilde{B}$, hence $\tilde{g}^{-1}(x) \cap \tilde{B}$ is nonempty. The function

$$N'(x) = \min\{N(\tilde{x}) | \tilde{g}(\tilde{x}) = x, \tilde{x} \in \tilde{B}\}$$

is a well-defined measurable function on $\tilde{B}$.

Define $g'(x) = g^{-N'(x)}(x)$. For $x \in \tilde{B}$ note $g'(x) \in \tilde{B}$ and hence $g'(\tilde{B}) \subset \tilde{B}$. So g' is a measurable transformation on $\tilde{B}$. We also note that $\tilde{g}(g'(x)) = x$ for every $x \in \tilde{B}$.

Let $B' = \bigcap_{j=1}^{\infty}(g')^j(\tilde{B})$. Then B' is measurable. On one hand, by definition we have $g'(B') = B'$. For every $x \in B'$, $g'(x) \in B'$ and $\tilde{g}(g'(x)) = x$. This implies $\tilde{g}(B') \supset B'$. On the other hand,

$$\tilde{g}(B') = \tilde{g}(\bigcap_{j=1}^{\infty}(g')^j(\tilde{B})) \subset \bigcap_{j=1}^{\infty}\tilde{g}((g')^j(\tilde{B})) = \bigcap_{j=1}^{\infty}(g')^{j-1}(\tilde{B}) = B'.$$

So $\tilde{g}(B') = B'$.

For every $x \in B$ and every $\tilde{x} \in G(x)$, $\tilde{g}(\tilde{x}) \in G(x) \subset \tilde{B}$. We claim

Lemma 4.2.6. *If $\tilde{x} \in G(x)$, then $g'(\tilde{g}(\tilde{x})) = \tilde{x}$.*

Proof. Assume $g'(\tilde{g}(\tilde{x})) = x_0 \neq \tilde{x}$. Since $\tilde{x} \in G(x)$, there is x' such that $\tilde{x}$ lies on the forward $\tilde{g}$-orbit of every x'' such that $x'' \precsim x'$, while $x_0 \in \tilde{B}$ lies in infinitely many forward $\tilde{g}$-orbits of the elements in the same equivalence class with x'. As g is invertible, there are only finitely many elements between (in the sense of the partial order) x' and x_0. So there must be some (in fact, infinitely many) x_1 such that $x_1 \precsim x'$ and both $\tilde{x}$ and x_0 lie on the forward $\tilde{g}$-orbit of x_1. Assume $\tilde{x} = \tilde{g}^a(x_1)$ and $x_0 = \tilde{g}^b(x_1)$. Then $\tilde{g}^{a+1}(x_1) = \tilde{g}^{b+1}(x_1) = \tilde{g}(\tilde{x})$. As g has no periodic orbit and $\tilde{g}$ as well, we must have $a = b$ and $\tilde{x} = x_0$, which is a contradiction. □

From the lemma we know for every $\tilde{x} \in G(x)$, $(g')^k(\tilde{g}^k(\tilde{x})) = \tilde{x}$, hence $\tilde{x} \in (g')^k(\tilde{B})$ for every positive integer k. This yields

Corollary 4.2.7. *$G(x) \subset B'$ for every $x \in B$.*

Furthermore,

$$\bigcup_{j=-\infty}^{\infty} g^j(B') \supset \bigcup_{j=-\infty}^{\infty} g^j(\bigcup_{x\in B} G(x)) \supset B. \tag{4.2}$$

In particular, B' is nonempty and has positive measure. We shall show $\tilde{g}$ is invertible on B'.

$\tilde{g}|_{B'}$ is surjective since we have showed $\tilde{g}(B') = B'$.

$\tilde{g}|_{B'}$ is injective. If $x \in B'$ then $x \in g'(\tilde{B})$ and there is $\tilde{x} \in \tilde{B}$ such that $x = g'(\tilde{x})$. But $\tilde{x} = \tilde{g}(g'(\tilde{x})) = \tilde{g}(x) \in B'$. This implies that $x = g'(\tilde{g}(x))$ for every $x \in B'$. So if $x_1, x_2 \in B'$ and $\tilde{g}(x_1) = \tilde{g}(x_2) \in B'$, then $x_1 = g'(\tilde{g}(x_1)) = g'(\tilde{g}(x_2)) = x_2$.

$g_* = \tilde{g}|_{B'}$ preserves ν. Let $D_k = \{x \in B' | N(x) = k\}$ for $k = 1, 2, \cdots$. Then $B' = \bigcup_{k=1}^{\infty} D_k$ and $D_i \bigcap D_j = \emptyset$. $g_*(D_i) \bigcap g_*(D_j) = \emptyset$ for $i \neq j$ since g_* is invertible. $g_*|_{D_k} = g^k$ preserves ν. For any measurable subset $E \subset B'$, $E = \bigcup_{1 \leq n < \infty} E_n$, where $E_n = E \bigcap D_n \subset D_n$. We have

$$\nu(g_*(E)) = \sum_{1 \leq n < \infty} \nu(g_*(E_n)) = \sum_{1 \leq n < \infty} \nu(E_n) = \nu(E).$$

This completes the proof of the first part.

For the second part, consider

$$B'' = \bigcup_{k=1}^{\infty} (\bigcup_{j=0}^{k-1} g^j(D_k)) = \bigcup_{j=0}^{\infty} g^j(\bigcup_{k=j+1}^{\infty} D_k) \subset \bigcup_{j=0}^{\infty} g^j(B'). \tag{4.3}$$

Lemma 4.2.8. *B'' is g-invariant, and*

$$B'' = \bigcup_{j=-\infty}^{\infty} g^j(B).$$

Proof. First note for each k, $g^k(D_k) = \tilde{g}(D_k) \subset B' \subset B''$. Then

$$\bigcup_{k=1}^{\infty} g^k(D_k) = \bigcup_{k=1}^{\infty} \tilde{g}(D_k) = \tilde{g}(B') = B' = \bigcup_{k=1}^{\infty} g^0(D_k).$$

So

$$g(B'') = \bigcup_{k=1}^{\infty} (\bigcup_{j=0}^{k-1} g^{j+1}(D_k)) = (\bigcup_{k=1}^{\infty} (\bigcup_{j=1}^{k-1} g^j(D_k))) \cup (\bigcup_{k=1}^{\infty} g^k(D_k)) = B''.$$

From (4.3) and $B' \subset B''$, we have

$$\bigcup_{j=-\infty}^{\infty} g^j(B') \subset \bigcup_{j=-\infty}^{\infty} g^j(B'') = B'' \subset \bigcup_{j=0}^{\infty} g^j(B'),$$

which implies

$$\bigcup_{j=-\infty}^{\infty} g^j(B') = B''.$$

From (4.2),

$$B \subset \bigcup_{j=-\infty}^{\infty} g^j(B') \subset \bigcup_{j=-\infty}^{\infty} g^j(B).$$

So

$$\bigcup_{j=-\infty}^{\infty} g^j(B) = \bigcup_{j=-\infty}^{\infty} g^j(B') = B''.$$

□

Lemma 4.2.8 yields

$$\int_{B'} N(x)d\nu = \sum_{k=1}^{\infty} k \cdot \nu(D_k) \geq \nu(B'') = \nu(\bigcup_{j=-\infty}^{\infty} g^j(B)).$$

□

Corollary 4.2.9. *Let C be a subset of B' of positive measure such that $g_*(C) = C$. If g is ergodic, then*

$$\int_C N(x)d\nu \geq 1.$$

Proof. Let $C_k = \{x \in C | N(x) = k\}$ for $k = 1, 2, \cdots$. Consider

$$C' = \bigcup_{k=1}^{\infty} (\bigcup_{j=0}^{k-1} g^j(C_k)).$$

Similar argument shows $g(C') = C'$. If g is ergodic, then $\nu(C') = 1$. Hence

$$\int_C N(x)d\nu = \sum_{k=1}^{\infty} k \cdot \nu(C_k) \geq \nu(C') = 1.$$

□

4.3 Proof of the Theorem 4.1.1

4.3.1 Regular Tube

Assume on the fiber direction there is no zero Lyapunov exponents. If $h^g(f) > 0$, then there must be at least one positive exponent and one negative exponent. Let us fix small numbers $\epsilon > 0$ and $r > 0$. Guaranteed by discussions in Chapter 3, we have

Proposition 4.3.1. *There is a "Regular Tube" P, which is a measurable subset of $X \times Y$ satisfying the following properties:*

1. $\mu(P) = \mu_0 > 0$.
2. *Let $\pi : P \to X$ be the projection to the base and let $B = \pi(P)$. Then $\nu(B) = \nu_0 > 0$.*
3. *Let $P(x) = P \cap (\{x\} \times Y)$. There is some number $\sigma_0 > 0$ such that, for every $x \in B$, $\sigma_0 < \sigma_x(P(x)) < \sigma_0(1 + r)$.*
4. *For every $x \in B$, there is a rectangle $R(x)$ on the fiber $\{x\} \times Y$ whose diameter is less than $\epsilon/2$. $P(x) \subset R(x) \subset \mathcal{R}(z)$ where $\mathcal{R}(z)$ is the Lyapunov regular neighborhood of some point $z = (x, y) \in X \times Y$ on the fiber $\{x\} \times Y$.*
5. *For every $x \in B$ and $z \in P(x)$, if for some $n > 0$, $F^n(z)$ returns to P, i.e. $g^n(x) \in B$ and $F^n(z) \in P(g^n(x))$, then the connected component of the intersection $F^n(R(x)) \cap R(g^n(x))$ containing $F^n(z)$, denoted by*

$$CC(F^n(R(x)) \cap R(g^n(x)), F^n(z))$$

is an admissible (u, γ)-rectangle in $R(g^n(x))$ and

$$CC(F^{-n}(R(g^n(x))) \cap R(x), z)$$

is an admissible (s, γ)-rectangle in $R(x)$. Moreover, for $j = 0, 1, ..., n$, on the fiber $\{g^j(x)\} \times Y$ we have

$$\mathrm{diam} F^j(CC(F^{-n}(R(g^n(x))) \cap R(x), z)) < \epsilon.$$

6. *Applying Theorem 2.4.6, we may assume that there is some* $m_1 > 0$ *such that for every* $m > m_1$ *and every* $x \in B$*, inside any set of* σ_x*-measure at least* $\sigma_0/2$ *on the fiber* $\{x\} \times Y$*, we can find a* (d^F_m, ϵ)*-separated set with cardinality at least* $\exp m(h_\mu(F) - r)$*.*

Proof. This regular tube can be obtained with the following steps.

1. On almost every fiber, find a regular point $z \in \{x\} \times Y$ and its regular neighborhood $\mathcal{R}(z)$. Take $R(x) \in \mathcal{R}(z)$ with diameter less than $\epsilon/2$.

2. Find m_1 and $B_0 \in X$ with $\nu(B_0) > 0$ such that property (6) holds for $m > m_1$ and $x \in B_0$.

3. Find $P(x) \subset R(x)$ satisfying property (5). There is some $\sigma_0 > 0$ such that $B = \{x \in B_0 | \sigma_x(P(x)) > \sigma_0\} > 0$. For $x \in B$, shrink the size of $P(x)$ properly such that $\sigma_0 < \sigma_x(P(x)) < \sigma_0(1+r)$. $P = \bigcup_{x \in B} P(x)$ is as required.

□

4.3.2 Control of Return Time

We start with a regular tube P. Applying Theorem 4.2.1, we can find a measurable section $q : B \to P$, $\pi \circ q = Id$ such that

$$\int_B N_1(x) d\nu \leq \frac{1}{\sigma_0},$$

where $N_1(x)$ is the first return time of $q(x)$.

Denote by χ_P the characteristic function of the measurable set P. Consider the sets

$$\mathcal{A}_n = \{z \in X \times Y | \text{ For every } k \geq n, \sum_{i=1}^{k} \chi_P(F^i(z)) < k\mu_0(1 + \frac{r}{3}) \\ \text{and } \sum_{i=1}^{k(1+r)} \chi_P(F^i(z)) > k\mu_0(1 + \frac{2r}{3})\}.$$

(numbers like $k(1 + r)$ are rounded to the nearest integer, if needed). Since μ is ergodic, Birkhoff Theorem tells us for almost every z,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_P(F^i(z)) = \mu(P) = \mu_0,$$

which implies

$$\lim_{n\to\infty} \mu(\mathcal{A}_n) = 1.$$

Let $\mathcal{B}_n = \{x|\sigma_x(P(x)\cap\mathcal{A}_n) > \sigma_0(1-r)\}$. Then as $n\to\infty$,

$$\nu(B\backslash\mathcal{B}_n) \to 0 \text{ and } \int_{B\backslash\mathcal{B}_n} N_1(x)d\nu \to 0.$$

There is m_0 and a measurable subset $B_1 \subset B$ with the following properties

1. For $x \in B_1$, let $P^{(m)}(x) = P(x)\cap\mathcal{A}_m$. For $m > m_0$, $\sigma_x(P^{(m)}(x)) > \sigma_0(1-r)$.

2. $\nu(B_1) > \nu_0(1-r)$. Let $B_2 = B\backslash B_1$. Then

$$\int_{B_2} N_1(x)d\nu < r.$$

Now we fix $m > \max\{m_1, m_0\}$. For convenience, denote by $\mathcal{K} = [m\mu_0(1+\frac{r}{2})]$ the integer part of $m\mu_0(1+\frac{r}{2})$. For large m,

$$m\mu_0(1+\frac{r}{3}) < \mathcal{K} < m\mu_0(1+\frac{2r}{3}).$$

For every $x \in B_1$, $P^{(m)}(x) > \sigma_0(1-r) > \sigma_0/2$, by property (6) of the regular tube, there is a (d_m^F, ϵ)-separated set $E(x) \subset P^{(m)}(x)$ with cardinality

$$|E(x)| > \exp m(h_\mu(F) - r).$$

For $z \in E(x) \subset \mathcal{A}_m$, the $\mathcal{K}$-th return time of z to P is an integer number between $m+1$ and $m(1+r)$. So there is $V(x) \subset E(x)$ with cardinality

$$|V(x)| = [\frac{1}{mr}\exp m(h_\mu(F) - r)]$$

and the $\mathcal{K}$-th return times for points in $V(x)$ are the same, denoted by $N(x)$. The set $\bigcup_{x\in B_1} V(x)$ can be chosen to be the union of $[\frac{1}{mr}\exp m(h_\mu(F)-r)]$ measurable sections over B_1.

$N(x)$ is a measurable function on B_1. We extend $N(x)$ to a measurable function on B: for $x \in B_2$, let $N(x) = N_1(x)$. Consider the map $\tilde{g}(x) = g^{N(x)}(x)$. $\tilde{g}(x)$ is well defined on B and $\tilde{g}(B) \subset B$. Moreover,

$$\int_B N(x)d\nu \leq \int_{B_1} N(x)d\nu + \int_{B_2} N_1(x)d\nu < m(1+r)\cdot \nu_0 + r < \infty.$$

Applying Theorem 4.2.2 we can find the kernel $B_3 \subset B$ of positive measure such that the $\tilde{g}$ restricted on B_3 is invertible and preserves μ. We can assume that $g_* = \tilde{g}|_{B_3}$ is ergodic (with respect to the measure induced by ν) by taking an ergodic component of positive ν-measure.

Let $B_4 = B_3 \cap B_1$ and $B_5 = B_3 \cap B_2$. Let $\mathcal{G}(x)$ be the first return map on B_4 with respect to g_*. Then $\mathcal{G}$ is invertible and preserves ν. Define measurable functions ρ and l on B_4 such that $\mathcal{G}(x) = g^{\rho(x)}(x) = g_*^{l(x)}(x)$. For $x \in B_4$,

$$\rho(x) = \sum_{j=0}^{l(x)-1} N(g_*^j(x))$$

and

$$\int_{B_4} \rho(x)d\nu = \int_{B_3} N(x)d\nu.$$

Note $B_3 = B_4 \cup B_5 \subset B_4 \cup B_2$. From Corollary 4.2.9,

$$1 \leq \int_{B_3} N(x)d\nu \leq \int_{B_4} N(x)d\nu + \int_{B_2} N(x)d\nu \leq m(1+r)\cdot \nu(B_4) + r.$$

So

$$\nu(B_4) \geq \frac{1-r}{m(1+r)}$$

and the average return time

$$\begin{aligned}\frac{1}{\nu(B_4)}\int_{B_4}\rho(x)d\nu \leq& \frac{m(1+r)\cdot\nu(B_4)+r}{\nu(B_4)}\\ =& m(1+r)+\frac{r}{\nu(B_4)}\\ \leq& \frac{m(1+r)}{1-r}.\end{aligned}$$

4.3.3 Construction of Horseshoes

We are going to construct a skew product map with base $\mathcal{G}$ on B_4 and horseshoes on fibers. we mostly follow the argument of Katok (for details, see [20] or [4, Section 15.6]). The novelty here is that in our case, for $x \in B_4$ and $z \in V(x)$, $\mathcal{F}(z) = F^{\rho(x)}(z)$ does not necessarily return to $R(g^{\rho(x)}(x)) = R(\mathcal{G}(x))$. To live with this we consider the orbits of g_* and use the admissible rectangles around the points $q(g_*^k(x))$, $k = 1, 2, \dots, l(x) - 1$, to carry over the horseshoe structure until it finally returns to $R(\mathcal{G}(x))$.

For every $x \in B_3$ and $z \in \{x\} \times Y$, let $F_*(z) = F^{N(x)}(z) \in \{g_*(x)\} \times Y$. Note F_* is invertible on $B_3 \times Y$ since F and g_* are invertible. If in addition $z \in P(x)$, then $F_*(z)^{\pm 1} \in P(g_*^{\pm 1}(x))$.

For $x \in B_4$, we note $g_*^k(x) \in B_5$ for $k = 1, 2, \dots, l(x) - 1$ and $g_*^{l(x)}(x) \in B_4$. If $z \in V(x)$, then the connected component

$$CC(R(x) \cap F_*^{-1}(R(g_*(x))), z)$$

is an admissible (s, γ)-rectangle in $R(x)$. As $F_*(z) = F^{N(x)}(z)$ is the $\mathcal{K}$-th return of z to P and $N(x) > m$, and points in $V(x)$ are (d_m^F, ϵ)-separated, from property (5) of the set P, we can conclude that this connected component contains no other points in $V(x)$. So there are $[\frac{1}{mr}\exp m(h_\mu(F) - r)]$ such connected components, each of which contains exactly one point in $V(x)$.

Let

$$S_0(z) = CC(R(x) \cap F_*^{-1}(R(g_*(x))), z)$$

and for $k = 1, 2, \ldots, l(x) - 1$, define by induction

$$S_k(z) = CC(F_*(S_{k-1}(z)) \cap F_*^{-1}(R(g_*^{k+1}(x))), q(g_*^k(x))) \subset F_*(S_{k-1}(z)).$$

Then for each k, $S_k(z)$ is part of an admissible (s, γ)-rectangle in $R(g_*^k(x))$ such that $F_*(S_k(z))$ is an admissible (u, γ)-rectangle in $R(g_*^{k+1}(x))$. Moreover, $F_*^{-(l(x)-1)}(S_{l(x)-1}(z)) \subset S_0(z)$. So we can select for each $z \in V(x)$ a point $u(z) \in F_*^{-(l(x)-1)}(S_{l(x)-1}(z))$. Then

$$\mathcal{F}(u(z)) = F_*^{l(x)}(u(z)) \in F_*(S_{l(x)-1}(z)) \subset R(g_*^{l(x)}(x)) = R(\mathcal{G}(x))$$

and

$$CC(\mathcal{F}(R(x)) \cap R(\mathcal{G}(x)), \mathcal{F}(u(z))) = F_*(S_{l(x)-1}(z)) \subset \mathcal{F}(S_0(z))$$

is an admissible (u, γ)-rectangle in $R(\mathcal{G}(x))$. Note that $\mathcal{F}$ is invertible and $S_0(z)$ are disjoint for different $z \in V(x)$. So there are $[\frac{1}{mr} \exp m(h_\mu(F) - r)]$ such rectangles. Likewise, the pre-images

$$\mathcal{F}^{-1}(CC(\mathcal{F}(R(x)) \cap R(\mathcal{G}(x)), \mathcal{F}(u(z)))) = CC(R(x) \cap \mathcal{F}^{-1}(R(\mathcal{G}(x))), u(z))$$

are $[\frac{1}{mr} \exp m(h_\mu(F) - r)]$ disjoint admissible (s, γ)-rectangles in $R(x)$.

Let $U(x) = \{u(z) | z \in V(x)\}$. For $x \in B_4$, consider the set

$$\Lambda(x) = \bigcap_{n \in \mathbb{Z}} \mathcal{F}^{-n}\Big(\bigcup_{z' \in U(\mathcal{G}^n(x))} CC(R(\mathcal{G}^n(x)) \cap \mathcal{F}^{-1}(R(\mathcal{G}^{n+1}(x))), z')\Big) \subset R(x).$$

Then $\Lambda(\mathcal{G}(x)) = \mathcal{F}(\Lambda(x))$. $\Lambda(x)$ is the intersection of infinitely many layers, each of which consists of $[\frac{1}{mr} \exp m(h_\mu(F) - r)]$ disjoint connected components. If $z'' \in \Lambda(x)$, then for each $n \in \mathbb{Z}$, $\mathcal{F}^n(z'')$ belongs to exactly one of the connected components in

$$\bigcup_{z' \in U(\mathcal{G}^n(x))} CC(R(\mathcal{G}^n(x)) \cap \mathcal{F}^{-1}(R(\mathcal{G}^{n+1}(x))), z').$$

Meanwhile, for any sequence $\{z_n \in U(\mathcal{G}^n(x))\}_{n\in\mathbb{Z}}$, The intersection

$$\bigcap_{n\in\mathbb{Z}} \mathcal{F}^{-n}(CC(R(\mathcal{G}^n(x)) \cap \mathcal{F}^{-1}(R(\mathcal{G}^{n+1}(x))), z_n))$$

contains exactly one point in $\Lambda(x)$. Therefore, $\Lambda = \bigcup_{x\in B_4} \Lambda(x)$ is invariant of $\mathcal{F}$ and $\mathcal{F}|_\Lambda = (\mathcal{G}, \mathcal{H})$, with the base $\mathcal{G}$ on B_4 and $\mathcal{H}$ on the fibers conjugate to the full shift on $[\frac{1}{mr} \exp m(h_\mu(F) - r)]$ symbols.

4.3.4 Estimate of Entropy

As $\mathcal{G}$ is ergodic, Λ carries many ergodic invariant measures for $\mathcal{F}|_\Lambda$ of the form

$$\frac{1}{\nu(B_4)} \int_{B_4} \tau_x d\nu,$$

where τ_x for each $x \in B_4$ is supported on $\Lambda(x)$ and $\tau_x \circ \mathcal{F}^{-1} = \tau_{\mathcal{G}(x)}$. Entropies of these measures vary from 0 to the topological entropy of the full shift which equals

$$\log[\frac{1}{mr} \exp m(h_\mu(F) - r)].$$

Ergodic measures of arbitrary intermediate entropies can be obtained by properly assigning weights to different symbols for the shift. These measures induce ergodic invariant measures of F. The average return time is

$$\frac{1}{\nu(B_4)} \int_{B_4} \int_{\Lambda(x)} \rho(x) d\tau_x d\nu = \frac{1}{\nu(B_4)} \int_{B_4} \rho(x) d\nu \leq \frac{m(1+r)}{(1-r)}.$$

So the measures we constructed has the maximal entropy no less than

$$\log[\frac{1}{mr} \exp m(h_\mu(F) - r)] \cdot \frac{1-r}{m(1+r)},$$

which is arbitrarily close to $h_\mu(F)$ as $r \to 0$ and $m \to \infty$. This completes the proof of Theorem 4.1.1.

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