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PARTIALLY HYPERBOLIC PHENOMENA IN DYNAMICAL SYSTEMS WITH DESCRETE AND CONTINUOUS TIME

A Thesis in

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Abstract

In chapter 3 we prove that any compact manifold of dimension $n \geq 3$ has a completely hyperbolic volume preserving Bernoulli flow. This is a joint work with H. Hu and Ya. Pesin. In chapter 4 we prove that there is a manifold of dimension at least four which possesses a hyperbolic ergodic map near the identity map. This is a joint work with H. Hu. In chapter 5 we prove that stable accessibility is generic among suspension flows with the roof function in $C^r_+(M)$. This result was proven by M. Brin but is not available in english.

Table of Contents

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Chapter 1

Introduction

A fundamental problem in the ergodic theory is to determine if a particular map is ergodic and, in general, how frequent the ergodic maps are. One classical result of the Kolmogorov, Arnold, Moser theory (KAM) shows that ergodicity is not a generic property in the space of measure preserving diffeomorphic maps, $Diff^r_\mu(M)$. In fact, there are open sets of maps which have sets of positive volume that consist of invariant tori.

On the other hand, ergodicity is "inevitable" for maps that have strong hyperbolic behavior. In 1967 Anosov showed [A1] that volume preserving uniformly hyperbolic diffeomorphisms (that are called Anosov diffeomorphisms) are stably ergodic. The first example of a non-hyperbolic stably ergodic diffeomorphism, though, appeared only in 1994 when M. Grayson, C. Pugh and M. Shub published their article "Stably ergodic diffeomorphisms" where they have shown that the time-1 map of the geodesic flow of a surface S of constant negative curvature is stably ergodic. A. Wilkinson generalized this result for manifolds with variable negative curvature [W].

THEOREM 1.1. *Let F^t be a geodesic flow on a compact manifold M of negative curvature. Then for any $t > 0$ the map F^t is ergodic and partially hyperbolic.*

Another example of stably ergodic maps includes skew products over Anosov diffeomorphisms with a compact connected Lie group G or so called $G-$extensions (that is maps g on a manifold M that admit a fibering with the projection $\pi : M \to N$ such that for any $x \in M$ and

$g \in G$, $\pi(f(x)) = h\pi(x)$ and $fR_g f(x) = R_g f(x)$, where $R_g : M \to M$ is the right action of the element $g \in G$). In fact, stable ergodicity is generic in this class of maps [B1].

THEOREM 1.2. *Let f be a G-extension of class C^r, $r \geq 1$ of some Anosov diffeomorphism h. Then for any $\delta > 0$ there exists a G- extension f_1 of diffeomorphism h for which $\rho_{C^r}(f, f_1) < \delta$ and f_1 is accessible.*

As we mentioned, ergodicity holds for uniformly hyperbolic systems. Although uniformly hyperbolic systems are not generic in $Diff^r(M)$, they are not even known to exist on any manofold. There are several different approaches to relax the strong hyperbolicity condition. We are going to discuss two of them.

1) The notion of partially hyperbolic systems was first introduced by Pesin and Brin where along with the strong expanding and strong contracting directions there is a "neutral" direction in which expansion or contraction is smaller than in the strong unstable or stable directions or does not exist at all. C.Pugh and M.Shub studied partially hyperbolic systems quite intensively over the last decade. The main idea of their work was: *a little hyperbolicity goes a long way in guaranteeing ergodicity*. We say that a map g is stably ergodic if any map close to g is ergodic. Sometimes the notion of stable ergodicity is used in a more general sense, (see chapter 2).

Pugh and Shub conjectured [PS]

CONJECTURE 1. *Stable ergodicity is an open and dense property among C^2 volume preserving partially hyperbolic diffeomorphisms.*

The main tool in proving ergodicity is the notion of accessibility which was introduced by Pesin and Brin in [BrP] and was called transitivity of foliations. A map possesses a property of essential accessibility if accessibility holds for a set of complete measure. It was conjectured by Pugh and Shub that

CONJECTURE 2. *Stable accessibility is an open and dense property among C^2 partially hyperbolic diffeomorphisms, volume preserving or not.*

The connection between ergodicity and accessibility is clear from the following remarkable result of Pugh and Shub. It involves two technical conditions known as center bunching and dynamical coherence.

THEOREM 1.3. *If $f \in Diff^r_\mu(M)$ for some smooth measure μ is a center bunched and dynamically coherent partially hyperbolic diffeomorphism with the essential accessibility property, then f is ergodic.*

The conditions of center bunching and dynamical coherence are believed to be technical ones and that in fact the following is true.

CONJECTURE 3. *Let $f \in Diff^r_\mu(M)$ where M is compact. If f is partially hyperbolic and essentially accessible then f is ergodic.*

Suspension flows over an Anosov map with a non-constant roof function are a special case of application of theorem 1.3. The conditions of center bunching and dynamical coherence are automatically satisfied. It follows from the theorem 1.3 that if a suspension flow with the roof function f is stably accessible then it is stably ergodic.

In section 5 we prove that the stable accessibility condition is generic for the suspension flows with the roof function f in the class of $r-$smooth positive functions $C^r_+(M)$.

THEOREM 1.4. *Let M be a compact manifold and h be a measure preserving Anosov diffeomorphism on M. Then there is an open and dense set $U \subset C^r_+(M)$ such that for any $f \in U$ the suspension flow over h with the roof function f is stably accessible and stably ergodic.*

What are conditions to be satisfied for a map to be stably ergodic? Obviously, the Theorem 1.3 gives stable ergodicity in a situation when all its hypotheses are stably satisfied.

It is well known that partial hyperbolicity is an open property in the C^1 topology, that is any diffeomorphism g that is close enough in the C^1 topology to a partially hyperbolic map is again partially hyperbolic. The property of center bunching is also C^1 open.

It is not yet known though if in general the dynamical coherence is an open property. Hirsch, Pugh and Shub proved the last result for the case of C^1 center foliation, [HPS].

THEOREM 1.5. *If the center foliation W^c exists and is of a class C^1 then f is dynamically coherent as is any g close enough to f in the C^1 topology.*

The immediate corollary of Theorems 1.3 and 1.5 is

THEOREM 1.6. *If $f \in Diff^2_\mu(M)$ where M is compact. Suppose*

1. *f is partially hyperbolic*
2. *f is center bunched*
3. *f has a central foliation and this foliation is C^1 and*
4. *f is stably essentially accessible*

Then f is stably ergodic.

Since ergodicity has an intimate relation with accessibility it makes sense to study accessibility and, in particular, stable accessibility. Pugh and Shub conjectured

CONJECTURE 4. *Stable accessibility holds for a dense set of C^2 partially hyperbolic diffeomorphisms, volume preserving or not.*

Conjecture 4 was partially justified in 2001 by Dolgopyat and Wilkinson who proved

THEOREM 1.7. *In the space of all C^r partially hyperbolic diffeomorphisms there is a C^1 open and dense set of accessible diffeomorphisms.*

Using this result as well as the results of Burns, Dolgopyat and Pesin, [BDP], C. Bonatti, C. Matheus, M. Viana and A. Wilkinson obtained the answer about stability of ergodicity in the case of 1-dimensional center foliation.

THEOREM 1.8. *Let M be a compact manifold endowed with a smooth measure μ and $PH_\mu(M)$ be the set of all partially hyperbolic diffeomorphisms having 1-dimensional center bundle and preserving the volume form. Then the volume measure μ is ergodic, and even Bernoulli, for any C^2 diffeomorphisms in a C^1 open and dense subset of $PH_\mu(M)$.*

2) The second approach to the ergodicity problem is to relax the conditions of uniform hyperbolicity and consider non-uniform hyperbolicity. Non-uniform hyperbolic maps have asymptotically hyperbolic behavior, namely Lyapunov exponents are non-zero almost everywhere. This approach is due to Pesin.

CONJECTURE 5. *The ergodicity is a generic property among the nonuniformly hyperbolic maps.*

Until recently it was unknown if any manifold had a hyperbolic ergodic map. This question was answered positively by Dolgopyat and Pesin, [DP].

The first big result in this direction was achieved by Katok in 1979, [K]. He proved the following

THEOREM 1.9. *On any surface M there exists a map $f : M \to M$ such that*

1. *f is a C^∞ diffeomorphism*

2. *f is measure-preserving*

3. *f is hyperbolic in the sense that its Lyapunov exponens are non-zero*

4. *f is Bernoulli (and, in particular, ergodic)*

To get such a map, Katok constructed a map $g : D^2 \to D^2$ which possessed properties $(1)-(4)$ in the Theorem 1 and, moreover, was very "flat" at the boundary ∂D^2.

(5) $g_1(x,y) - x \in C^\infty_\rho(D^2)$

and $g_2(x,y) - y \in C^\infty_\rho(D^2)$

where $C^\infty_\rho(D^2)$ is the space of functions $\phi : D^2 \to R$ such that

$$|\frac{\partial^n \phi}{\partial^{n_1} x \partial^{n_2} y}| < \rho_n(x,y)$$

for $(x,y) \in U$ for some neighborhood U of ∂D^2 and $n_1 + n_2 = n$.

Katok then proved the existence of a map D^2 onto a surface M with the following properties:

THEOREM 1.10. *There is a map $F : D^n \to M$ such that:*

1. *$F(D^n) = M$*

2. $F|_{intD^n}$ *is a diffeomorphic embedding*

3. $\mu(M \setminus F(D^n)) = 0$

4. $F_\star\lambda = \mu$

where λ *is a smooth measure on* D^n *and* μ *is a smooth measure on* M.

Then the map $\tilde{g} = FgF^{-1}$ is a well defined map of the manifold M with the desired properties. Map $\tilde{g}$ is well defined on M due to the property (5).

The Katok's map was used many times later in constructions of different maps.

In 1982 M.Brin generalized the Katok's result to the case of all manifolds of dimension greater than 5,[B]. He proved

THEOREM 1.11. *For every manifold of dimension* $n \geq 5$ *there is a diffeomorphism* f *which has* $n-1$ *non-zero Lyapunov exponents almost everywhere. Moreover,* f *preserves the Lebesgue measure and is Bernoulli.*

To get the map f Brin considered an Anosov map $A : T^{n-3} \to T^{n-3}$. By Theorem 1.4 there is a suspension flow $h : L \to L$ where L is the suspension manifold of the map A with the roof function $H : T^{n-3} \to \mathbb{R}_+$ such that h is stably accessible.

Then define the map f as a skew product, $f(x,y) = (g(x), h^{\alpha(x)}y)$ where $\alpha(x)$ is a smooth nonnegative function $\alpha : D^2 \to \mathbb{R}$, $\alpha \not\equiv 0$, $\alpha(x) = 0$ in a neighborhood U, $\partial D^2 \subset U$ and g is the Katok's map.

Let B^n be the unit ball in $\mathbb{R}^n$. Since L can be embedded into B^n [B] and L has a trivial tangent bundle, the manifold $M = D^2 \times L$ can be embedded into B^n.

Next B^n can be embedded into any manifold N of dimension n so that the measure of the image of B^n coincides with the whole measure of N (see Theorem 1.10). It follows that the map f "induces" a map on N with all the necessary properties.

The question about existence of a hyperbolic map on a manifold of dimension 3 and 4 was still open as well as the question of whether it was possible to get rid of the remaining zero Lyapunov exponents.

In 2000, M. Shub and A. Wilkinson published their article "Pathological foliations and removable zero exponents" where they have shown a way to make a non-zero Lyapunov exponents by a C^r perturbation.

The latter result was widely used in many studies that followed, in particular in the construction of a hyperbolic Bernoulli map on any manifold of dimension greater than 2. This result is due to D. Dolgopyat and Ya. Pesin, [DP].

THEOREM 1.12. *On any manifold M of dimension $n \geq 2$ there exists a C^∞ map $f : M \to M$ with the following properties*

1. *f is measure preserving*
2. *f has nonzero Lyapunov exponents almost everywhere*
3. *f is Bernoulli*

We prove the continuous time version of the same result.This is a joint work with H. Hu and Ya. Pesin.

THEOREM 1.13. *On any manifold M of dimension $n \geq 3$ there exists a flow $F^t : M \to M$ such that for any $t \neq 0$*

1. *F^t is measure preserving*

2. *F^t has all but one nonzero Lyapunov exponents almost everywhere*

3. *F^t is Bernoulli*

The map F^t, $t \neq 0$ is not completely hyperbolic, the Lyapunov exponent in the direction of the flow is zero.

Very little is still known about ergodic components of maps. In particular it is not known when its ergodic components are open (mod 0). The following result is due to Pesin, [PP]

THEOREM 1.14. *Let f be a map with nonzero Lyapunov exponents on an invariant set L of positive measure. Then the ergodic components of $f|_L$ are all of positive measure.*

COROLLARY 1.1. *If f has nonzero Lyapunov exponents on a set of positive measure then there are only countably many such components.*

In 1973 Pesin constructed an example of a map with more than one ergodic component. An example of a map with infinitely many ergodic components was constructed by Dolgopyat, Hu and Pesin, [DHP].

THEOREM 1.15. *There exists a diffeomorphism $F : T^3 \to T^3$ such that*

1. *F is measure-preserving*

2. *F has nonzero Lyapunov exponents a.e.*

3. *F has countably (non finitely) many ergodic components which are open (mod 0)*

The construction of such a map was based on an Anosov map $h : T^2 \to T^2$ which has nonzero Lyapunov exponents and is ergodic. The map $H : T^3 \to T^3$, $H(x,y) = (h(x),y)$ where $x \in T^2$, $y \in S^1$ has two nonzero Lyapunov exponents and one zero Lyapunov exponent. Since $T^3 = T^2 \times S^1$, the second factor, S^1, can be divided into infinitely many intervals I_n. The map H is then perturbed inside $T^2 \times I_n$ to make it ergodic and hyperbolic. The resulting map F inherits two nonzero Lyapunov exponents from the original Anosov map h and gets the third nonzero Lyapunov exponent as a result of the perturbation.

It is clear from the construction that the map F is away from the identity map.

Rearranging the intervals I_n we can construct a map F with the following properties:

THEOREM 1.16. *There exists a diffeomorphism $F : T^3 \to T^3$ such that*

1. *F is measure preserving*

2. *F has non-zero Lyapunov exponents on an invariant everywhere dense set A of positive measure*

3. *F has one zero Lyapunov exponent on the set $B = M \backslash A$ of positive measure.*

Note that all the ergodic components in the mentioned theorems are open (mod 0).

THEOREM 1.17. *[BDP] Let f be a C^2 diffeomorphism of a compact smooth Riemannian manifold M preserving a smooth measure μ. Assume that there exists an invariant subset $A \subset M$ with $\mu(A) > 0$ on which f has negative central exponents. Then every ergodic component of $f|_A$ is open (mod 0) and so is the set A.*

If the map is topologically transitive, then A is dense and $f|_A$ is ergodic. Topological transitivity itself does not guarantee ergodicity. The stronger requirement is accessibility. If f is essentially accessible then the orbit of almost every point is dense in M. It follows

THEOREM 1.18. *[BDP] Let f be a C^2 partially hyperbolic diffeomorphism of a compact Riemannian manifold M preserving a smooth measure μ. Assume that f has negative central exponents on an invariant set A of positive measure and is essentially accessible. Then f has negative central exponents on the whole of M, the set A has full measure and f is ergodic.*

Chapter 2

Preliminaries

DEFINITION 1. *Let f be a diffeomorphism of a manifold M. We say that f is ergodic if for any invariant set $A \in M$, $\mu(A) = 1$ or 0. A set $A \subset M$ is an ergodic component if $f|_A$ is ergodic.*

We say that f is stably ergodic if any map g close to f is ergodic. The notion of ergodicity can even be used more generally, for example, $f \in Diff^2(M)$ is stably ergodic if there is a C^1 neighborhood U of f such that for any $g \in U$, g is ergodic if $g \in Diff^2(M)$.

DEFINITION 2. *A diffeomorphism f is partially hyperbolic if the tangent bundle of M splits as a Tf-invariant sum*

$$TM = E^u \oplus E^s \oplus E^c$$

with at least two of the subbundles in the sum nontrivial, and there exist constants $a < b < 1 < c < d$ and a Finsler structure $||\cdot||$ on M such that for all $p \in M$ and all $v \in T_pM$

$$v \in E^u(p) \Rightarrow d||v|| \leq ||dfv||$$

$$v \in E^c(p) \Rightarrow b||v|| \leq ||dfv|| \leq c||v||$$

$$v \in E^s(p) \Rightarrow ||dfv|| \leq a||v||$$

The bundles E^u and E^s are called the stable and unstable bundles respectively, and E^c is called the center bundle of f. These bundles are in fact continuous in p.

An Anosov diffeomorphism is a partially hyperbolic diffeomorphism with a trivial center bundle. Every Anosov map can be considered as partially hyperbolic in three different ways: the obvious way, with trivial E^c, or as a map with trivial E^s and contracting direction E^c or as a map with trivial E^u and contracting central direction.

Examples of partially hyperbolic maps include the Anosov diffeomorphisms themselves, time-t maps of Anosov flows, such as suspension flows over Anosov maps or geodesic flows on a manifold of negative curvature.

DEFINITION 3. *Two points $p,q \in M$ are called accessible if there are points $p = z_0, z_1, \ldots, z_{l-1}, z_l = q$, $z_i \in M$ such that $z_i \in W^s(z_{i-1})$ for $i = 1, \ldots, l$. The collection of points $z_0, z_1, \ldots, z_l$ is called a us-path connecting p and q by $[p,q] = [z_0, z_1, \ldots, z_l]$.*

Accessibility is an equivalence relation. Map f has an accessibility property if there is only one equivalence class of accessible points, in other words any two points $x, y \in M$ are accessible from each other. If every measurable set which is a union of equivalence classes has either full measure or measure 0 then f has the essential accessibility property. Obviously, accessibility implies essential accessibility.

The map f is stably accessible if f is accessible and any map g which is close enough to f is accessible. Stable accessibility can be used in a more generac case, namely $f \in Diff^2(M)$ is stable accessible if any C^1 close to f map g, $g \in Diff^2(M)$ is accessible.

DEFINITION 4. *A partially hyperbolic diffeomorphism f is dynamically coherent if*

1. *the distributions E^c, $E^c \oplus E^u$ and $E^c \oplus E^s$ are integrable and everywhere tangent to foliations W^c, W^{cu} and W^{cs} called the center, center-unstable and center-stable foliations.*

2. *W^c and W^u subfoliate W^{cu} while W^c and W^s subfoliate W^{cs}.*

In fact, W^u and W^s always subfoliate W^{uc} and W^{sc} but since the center foliation may not be uniquely integrable, we can not avoid assuming that W^c subfoliate W^{uc} and W^{uc}.

DEFINITION 5. *A partially hyperbolic diffeomorphism f is center bunched if the action of df on the center bundle E^c is close enough to isometric, that is $||df|_{E^c}||$ and $m(df|_{E^c})$ are close enough to 1. Here $m(A) = ||A^{-1}||^{-1}$ is the conorm of A.*

DEFINITION 6. *A real nymber λ is a Lyapunov exponent of the diffeomorphism $g : M \to M$ if there is a nonzero vector $v \in TM$ such that* $\limsup_{n\to\infty} \frac{1}{n} \log ||dg^n(v)|| = \lambda$

By Oseledets' theorem if M is compact then there is a set $L \subset M$ which has full measure with respect to any g-invariant probability measure and such that the limit in the previous definition exists for all $v \in T_xM$ with $x \in L$.

For a given $x \in L$ there are at most $n = \dim(M)$ different exponents some or all of which may be zero.

DEFINITION 7. *A diffeomorphism of a manifold M is called nonuniformly partially hyperbolic on a set $X \subset M$ if for every $x \in X$ the tangent space at x admits an invariant splitting*

$$T_xM = E^s_F(x) \oplus E^c_F(x) \oplus E^u_F(x)$$

of T_xM into stable, central and unstable subspaces. There exist numbers $0 < \lambda^s < \lambda^c_1 \leq 1 \leq \lambda^c_2 \leq \lambda^u$ and Borel functions $c(x) > 0$ and $k(x) > 0$ such that

1. *for $n > 0$*

$$||d_xF^n v|| \leq c(x)(\lambda^s)^n e^{\varepsilon n}||v||, \quad v \in E^s(x)$$

$$||d_xF^{-n}v|| \leq c(x)(\lambda^s)^{-n}e^{-\varepsilon n}||v||, \quad v \in E^u(x)$$

$$c(x)^{-1}(\lambda_1^c)^n e^{-\varepsilon n}||v|| \leq ||d_xF^n v|| \leq c(x)(\lambda_2^c)^n e^{-\varepsilon n}||v||, \quad v \in E^c(x)$$

2. $\angle(E^s(x), E^u(x)) \geq k(x),\ \angle(E^s(x), E^c(x)) \geq k(x),\ \angle(E^u(x), E^c(x)) \geq k(x)$

3. *for* $m \in \mathbb{Z}$ $c(F^m(x)) \leq c(x)e^{\varepsilon|m|}$, $k(F^m(x)) \geq k(x)e^{-\varepsilon|m|}$.

Systems with nonzero Lyapunov exponents are nonuniformly hyperbolic.

If f is nonuniformly hyperbolic with $c(x) = \text{const}$ then f is uniformly partially hyperbolic or partially hyperbolic.

DEFINITION 8. *A map f of a manifold M is Bernoulli if it is isometric to a Bernoulli shift.*

In particular, any Bernoulli map is ergodic.

Chapter 3

Hyperbolic Flow on any Manifold

3.1 Introduction

In [DP], Dolgopyat and Pesin obtained an affirmative solution to the long-standing problem of whether a compact smooth Riemannian manifold admits a volume-preserving ergodic (Bernoulli) diffeomorphism with non-zero Lyapunov exponents. In this paper we discuss a continuous time version of this problem and we prove the following result.

THEOREM 3.1. *Given a compact smooth Riemannian manifold M of* $\dim M \geq 3$*, there exists a* C^∞ *flow* f^t *such that for each* $t \neq 0$*,*

1. *f^t preserves the Riemannian volume μ on M;*
2. *f^t has non-zero Lyapunov exponents (except for the exponent along the flow direction) at m-almost every point $x \in M$;*
3. *f^t is a Bernoulli diffeomorphism.*

3.2 Construction of diffeomorphisms by Katok and Brin

In our construction of the flow we use a special diffeomorphism of the two dimensional unit disc $\mathcal{D}2$ constructed by Katok in [K]. We summarise the description and properties of this diffeomorphism in the following proposition.

PROPOSITION 3.2. *There exists a C^∞ diffeomorphism $g : \mathcal{D}2 \to \mathcal{D}2$ with the following properties:*

1. *g preserves area on $\mathcal{D}2$;*

2. *g has non-zero Lyapunov exponents almost everywhere;*

3. *g is a Bernoulli map;*

4. *$d^k(g-\mathrm{id})|\partial\mathcal{D}2 = 0$ for any $k \geq 0$, i.e., on the boundary of the disk g is the identity map and has all its derivatives zero.*

Let us sketch the construction of the diffeomorphism g. We begin with the automorphism g_0 of the torus $\mathcal{T}^2$ given by the matrix $\begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$. The map g_0 has four fixed points

$$q_1 = (0,0)\,, \;\; q_2 = \left(\frac{1}{2},0\right), \;\; q_3 = \left(0,\frac{1}{2}\right), \;\; q_4 = \left(\frac{1}{2},\frac{1}{2}\right).$$

In a small neighborhood $D_r^i = \{(s_1,s_2) : s_1^2 + s_2^2 \leq r\}$ of each q_i, $0 < r < 1$, the map g_0 is the time-1 map of the flow given by

$$\dot{s}_1 = -(\log\alpha)s_1, \quad \dot{s}_2 = (\log\alpha)s_2,$$

where $\alpha > 1$ is the larger eigenvalue of g_0 and $\{s_1,s_2\}$ is the coordinate system in a neighborhoods of each q_i generated by the eigenvectors of g_0.

Then we consider the map g_1 that is conjugate to g_0 via a conjugacy ϕ_0 that slows down the motion near q_i. More precisely, g_1 is the time-1 map of the flow given by

$$\dot{s}_1 = -(\log\alpha)s_1\psi(s_1^2+s_2^2), \quad \dot{s}_2 = (\log\alpha)s_2\psi(s_1^2+s_2^2) \tag{3.1}$$

in D_r^i, and $g_1 = g_0$ otherwise, where ψ is a C^∞ function except at zero and such that $\psi(0) = 0$, $\psi(\xi) \geq 0$ for $\xi \geq 0$, $\psi(\xi) = 1$ for $\xi \geq r$ and

$$\int_0^r \sqrt{\frac{1}{\psi(\xi)}}\,d\xi < \infty.$$

The map g_1 preserves a probability measure $d\nu = \kappa_0^{-1}\kappa\,dm$, where m is area and the density κ is a positive C^∞ function defined by the formula

$$\kappa(s_1,s_2) = \begin{cases} (\psi({s_1}^2+{s_2}^2))^{-1} & \text{if } (s_1,s_2)\in D_r^i. \\ 1 & \text{otherwise} \end{cases}$$

Here

$$\kappa_0 = \int_{\mathcal{T}^2} \kappa\,dm.$$

Note that κ is infinite at q_i.

Define the map ϕ_1 by the formula

$$\phi_1(s_1,s_2) = \frac{1}{\sqrt{\kappa_0({s_1}^2+{s_2}^2)}}\left(\int_0^{{s_1}^2+{s_2}^2}\frac{du}{\psi(u)}\right)^{1/2}(s_1,s_2)$$

near each q_i and then extend the map to $\mathcal{T}^2$ in such a way that ϕ_1 is C^∞, commutes with the involution $J(t_1,t_2) = (1-t_1, 1-t_2)$ on $\mathcal{T}^2$, and satisfies $(\phi_1)_*\nu = m$. Hence, $g_2 = \phi_1 \circ g_1 \circ \phi_1^{-1}$ is a C^∞ area preserving map.

Let $\phi_2 \colon \mathcal{T}^2 \to S^2$ be a double branched covering that satisfies $\phi_2 \circ J = \phi_2$, $(\phi_1)_* m = m$, and C^∞ everywhere except for the points q_i, where it branches and near q_i,

$$\phi_2(s_1,s_2) = \frac{1}{\sqrt{{s_1}^2 + {s_2}^2}}(s_1^2 - s_2^2,\ 2s_1s_2).$$

The map $g_3 = \phi_2 \circ g_2 \circ \phi_2^{-1}$ is a C^∞ diffeomorphism of the sphere $\mathcal{S}^2$.

Finally, let ϕ_3 be a C^∞ map that blows up the point q_4 into a circle and makes $g = \phi_3 \circ g_3 \circ \phi_3^{-1}$ to be the desired map of the disk. We refer the reader to [K] (see also [DP]) for more details.

Note that the map g_1 preserves measure ν and the maps g_2, g_3 and g preserve area. The desired result follows.

We call the map g *Katok's map*. We need some additional properties of this map.

Since g has non-zero Lyapunov exponents, for almost every x there are global stable and unstable manifolds, $\mathcal{W}_g^s(x)$ and $\mathcal{W}_g^u(x)$, at x.

Let $Q = \{q_1, q_2, q_3\} \cup \partial\mathcal{D}2$ be the *discontinuity set* of g.

PROPOSITION 3.3. *The following properties hold:*

1. *periodic points of g are everywhere dense;*

2. *g possesses two one-dimensional continuous foliations which are extensions of the stable and unstable global foliations $W_g^s(x)$ and $W_g^u(x)$; we will use the same notations for these foliations;*

3. *there exist neighborhoods $U \subset U_1$ of $\partial\mathcal{D}2$ and a vector field V in U_1 which generates an area-preserving flow $g^t : U \to \mathcal{D}2$, $-2 < t < 2$ for which $g|U = g^1$.*

Proof: Note that the map g_1 is topologically conjugate to g_0 and Statement 1 follows. Statements 2 is proved in [K] (see Lemma 4.1). We prove the last statement. By construction, near each point q_i, the map g_1 is the time-1 map of a vector field V_1. Moreover, near each q_i we have $V_1(-x) = -V_1(x)$ for any x, see (3.1). It follows that the maps g_2 and g_3 near q_i are the time-1 maps of the vector fields given by $V_2 = d\phi_1 \circ V_1 \circ \phi_1^{-1}$ and $V_3 = d\phi_2 \circ V_2 \circ \phi_2^{-1}$ respectively. Here we should stress that V_3 is well defined even though ϕ_2 is a two to one map. In fact, near q_i we have $\phi_1(-x) = -\phi_1(x)$ and $\phi_2(-x) = \phi_2(x)$. This gives $V_2(-x) = -V_2(x)$ and therefore, for any y near q_i, $\phi_2^{-1}(y)$ has two preimages x and $-x$ at which

$$(d\phi_2)_{-x}(V_2(-x)) = (d\phi_2)_{-x}(-V_2(x)) = (d\phi_2)_x(V_2(x)).$$

Now we see that g is the time-1 map of the vector field $V = d\phi_3 \circ V_2 \circ \phi_3^{-1}$.

The next result shows that the map g is diffeotopic to the identity map.

PROPOSITION 3.4. *There exists a map $G : \mathcal{D}2 \times [0,1] \to \mathcal{D}2$ with the following properties:*

1. *$G(x,t)$ is C^∞ in (x,t);*

2. *$G(\cdot,0) = \mathrm{id}$ and $G(\cdot,1) = g$;*

3. *$G(x,t) = g^t(x)$ for any $x \in U$ and $t \in [0,1]$, where $g^t(x)$ is the flow in Proposition 3.3;*

4. *for any $t \in [0,1]$ the map $G(\cdot,t) : \mathcal{D}2 \to \mathcal{D}2$ is an area-preserving diffeomorphism;*

5. *$d^kG(x,1) = d^kG(g(x),0)$ for any $k \geq 0$.*

Proof: Recall that in the neighborhood U of the boundary of $\mathcal{D}2$ the map g is the time-1 map of the flow generated by the vector field V. We extend V to a smooth vector field $\widehat{V}$ on the whole $\mathcal{D}2$, and let $\widehat{g}^t$ be the flow generated by $\widehat{V}$. Note that $g|U = \widehat{g}^1|U$. We need the following result of Smale (see [S], Theorem B):

LEMMA 3.1. *Let $\mathcal{A}$ be the space of C^∞ diffeomorphisms of the unit square which are equal to the identity in some neighborhood of the boundary. Endow $\mathcal{A}$ with the C^r topology, $1 < r \leq \infty$. Then $\mathcal{A}$ is contractible to a point.*

The statement also holds if the unit square is replaced by the unit disk. Applying the result to the diffeomorphism $g \circ \widehat{g}^{-1}$, which is equal to the identity on U, we obtain a homotopy $\widetilde{G} : \mathcal{D}2 \times [0,1] \to \mathcal{D}2$ such that $\widetilde{G}(\cdot,0) = \mathrm{id}\,|\mathcal{D}2$ and $\widetilde{G}(\cdot,1) = \widehat{g}^{-1} \circ g$. Moreover, $\widetilde{G}$ is C^∞ in (x,t), i.e., $\widetilde{G}$ is a diffeotopy in $\mathcal{A}$ (see [S], Theorem 4). Therefore, for each $t \in [0,1]$, there is a neighborhood U_t of $\partial\mathcal{D}2$ such that $\widetilde{G}(\cdot,t)|U_t = \mathrm{id}\,|U_t$. One can show that the set

$$U = \mathrm{int} \bigcap_{t \in [0,1]} U_t$$

is not empty and is a neighborhood of $\partial\mathcal{D}2$. Denote $\widetilde{g}^t = \widetilde{G}(\cdot,t)$. It follows that the map $G(\cdot,t) = \widetilde{g}^t \circ \widehat{g}^t$ satisfies Statements 1- 3 of the proposition. To prove Statements 4 and 5, we need the following lemma.

LEMMA 3.2. *Let* $\{\Omega_0^t\}$ *and* $\{\Omega_1^t\}$ *be two families of volume forms on* $\mathcal{D}2$ *that are* C^∞ *in* (x,t)*. Assume that* $\Omega_0^t|U = \Omega_1^t|U$ *for any* t *and* $\Omega_0^t = \Omega_1^t$ *for* $t \in [0,\varepsilon) \cup (1-\varepsilon, 1]$*. Then there exists a map* $\bar{G} : \mathcal{D}2 \times [0,1] \to \mathcal{D}2$ *with the following properties:*

1. $\bar{G}(x,t)$ *is* C^∞ *in* (x,t)*;*

2. $\bar{G}(\cdot,0) = \bar{G}(\cdot,1) = \mathrm{id}$*;*

3. *for any* $t \in [0,1]$ *the map* $\bar{G}(\cdot,t) : \mathcal{D}2 \to \mathcal{D}2$ *is a diffeomorphism with* $\bar{G}(\cdot,t)^*\Omega_1^t = \Omega_0^t$*;*

4. $\bar{G}(x,t) = x$ *for any* $t \in [0,1]$ *and* x *in some neighborhood* $U' \subset U$ *of* $\partial\mathcal{D}2$*.*

Proof: The argument is a modification of the proof of Moser's theorem ([M]). We follow here the approach in [KH] (see Theorem 5.1.27). Let $\Omega^t = \Omega_1^t - \Omega_0^t$ and $\Omega_s^t = \Omega_0^t + s\Omega^t$ for $s \in [0,1]$. We know that $\Omega^t|U = 0$. We construct a family of one forms $\mathcal{H}^t$ such that $\mathcal{H}^t$ is C^∞ in (x,t), $d\mathcal{H}^t = \Omega^t$ and $\mathcal{H}^t|U' = 0$ for some neighborhood $U' \subset U$ of $\partial\mathcal{D}2$. Consider a Euclidean coordinate system (x_1,x_2) such that $\mathcal{D}2 = \{(x_1,x_2) : x_1^2 + x_2^2 \le 1\}$. We have

$$\Omega^t = \rho^t(x)dx_1 \wedge dx_2$$

and

$$\Omega_s^t = \rho_s^t(x)dx_1 \wedge dx_2$$

with $\rho_s^t > 0$. Note that $\rho^t|_U = 0$ and $\int \rho^t\, dx = 0$ for any $t \in [0,1]$. Given a C^∞ function $\theta^t = \theta^t(x)$, $x \in \mathcal{D}2$, $t \in [0,1]$, let

$$E_1(\theta^t)(a) = \int_{\mathcal{D}2 \cap \{x_2 = a\}} \theta^t(x_1,a)dx_1$$

and

$$E_2(\theta^t)(a) = \int_{\mathcal{D}2\cap\{x_1=a\}} \theta^t(a,x_2)dx_2$$

be the expectation of θ^t along the lines $x_2 = \text{constant}$ and $x_1 = \text{constant}$. We choose the function θ^t such that

$$E_1(\theta^t) = E_1(\rho^t), \quad E_2(\theta^t) = 0,$$

and $\theta^t|U' = 0$ where $U' \subset U$ is a neighborhood of $\partial\mathcal{D}2$. Such θ^t exists. Indeed, choose any positive C^∞ function δ such that $\int \delta dx_1 = 1$, the support $\operatorname{supp}\delta \subset (-\varepsilon,\varepsilon)$ for some small $\varepsilon > 0$, and then set $\theta^t(x_1,x_2) = \delta(x_1)E_1(\rho^t)(x_2)$. Now let

$$\begin{aligned} a^t(x_1,x_2) &= -\int_{\mathcal{D}2\cap\{(x_1,y):y<x_2\}} \theta^t(x_1,y)\,dy, \\ b^t(x_1,x_2) &= \int_{\mathcal{D}2\cap\{(y,x_2):y<x_1\}} \left[\rho^t(y,x_2) - \theta^t(y,x_2)\right] dy. \end{aligned}$$

The 1-forms

$$\mathcal{H}^t(x_1,x_2) = a^t(x_1,x_2)dx_1 + b^t(x_1,x_2)dx_2$$

satisfy the desired requirements.

For each $s \in [0,1]$, consider the vector field

$$V_s^t = -\frac{b^t}{\rho_s^t}\frac{\partial}{\partial x_1} + \frac{a^t}{\rho_s^t}\frac{\partial}{\partial x_2}.$$

It is well defined since $\rho_s^t > 0$ on $\mathcal{D}2$ and it is C^∞ in (x,t,s). We also have $V_s^t|U' = 0$ and $\Omega_s^t(V_s^t,\cdot) = -\mathcal{H}^t$. Let $(\bar{G}^t)^s\colon \mathcal{D}2 \to \mathcal{D}2$ be the solution of the differential equation $\frac{dx}{ds} = V_s^t(x)$

satisfying the initial condition $(\bar{G}^t)^0 = \mathrm{id}$. We have that

$$(d\bar{G}^t)^{1*}\Omega_1^t = (d\bar{G}^t)^{0*}\Omega_0^t = \Omega_0^t$$

(see [KH], Theorem 5.1.27 for more details). So $\bar{G}^t = (\bar{G}^t)^1$ is the desired map. We proceed with the proof of the proposition. Consider the map $\bar{G}$ as in the lemma with $\Omega_0^t = dx_1 \wedge dx_2$ on $\mathcal{D}2$ for any $t \in [0,1]$ and $\Omega_1^t = d\tilde{g}^{t*} d\hat{g}^{t*}\Omega_0^t$. Let $\bar{g} = \bar{G}(\cdot,t)$. Then the map $G : \mathcal{D}2 \times [0,1] \to \mathcal{D}2$,

$$G(x,t) = \bar{g} \circ \hat{g}^t \circ \tilde{g}^t$$

satisfies Statements 1 – 4 of the proposition. If necessary one can change the map $\tilde{G}(\cdot,t)$ in a small neighborhood of the set $\mathcal{D}2 \times 0$ and $\mathcal{D}2 \times 1$ so that it will also satisfy Statement 5. For example, one can choose $\widetilde{G}$ in such a way that

$$\widetilde{G}\big(\widehat{G}(x,t),t\big) = \widetilde{G}\big(\widehat{G}(x,0),0\big),\ t \in [0,\varepsilon), \quad \widetilde{G}\big(\widehat{G}(x,t),t\big) = \widetilde{G}\big(\widehat{G}(x,1),1\big),\ t \in (1-\varepsilon,1].$$

Hence, $\hat{g}^t \circ \tilde{g}^t$ is area-preserving for any $t \in [0,\varepsilon) \cup (1-\varepsilon,1]$. Therefore, by Lemma 3.2, $\Omega_1^t = \Omega_0^t$ and $\bar{G}(\cdot,t) = \mathrm{id}$ for all $t \in [0,\varepsilon) \cup (1-\varepsilon,1]$, because in this case $\Omega^t = 0$, $\rho^t = 0$, $a^t = b^t = 0$ and $V_s^t = 0$ for every s.

We also need Brin's construction from [B]. Given $n \geq 5$, let $A\colon \mathcal{T}^{n-3} \to \mathcal{T}^{n-3}$ be a hyperbolic automorphism of the $(n-3)$-dimensional torus and $h^t\colon L \to L$ the suspension flow over A with the roof function $H = 1$, $y \in \mathcal{T}^{n-3}$. The suspension manifold L is diffeomorphic to $\mathcal{T}^{n-3} \times [0,1]/\sim$, where $\sim$ is the identification $(y,1) = (Ay,0)$. The flow h^t preserves volume on L and one can choose A so that L can be embedded into $\mathcal{R}^{n-1} \times \mathcal{S}^1$ with trivial normal bundle.

3.3 Proof of the theorem: the case $\dim M \geq 5$

Consider the map

$$R = g \times A : \mathcal{D}2 \times \mathcal{T}^{n-3} \to \mathcal{D}2 \times \mathcal{T}^{n-3},$$

where $g\colon \mathcal{D}2 \to \mathcal{D}2$ is Katok's diffeomorphism and $A\colon \mathcal{T}^{n-3} \to \mathcal{T}^{n-3}$ is the automorphism from Brin's construction. Consider the suspension flow over R with roof function $H = 1$ and the suspension manifold $K = \mathcal{D}2 \times \mathcal{T}^{n-3} \times [0,1]/\sim$, where $\sim$ is the identification $(x,y,1) = (g(x), A(y), 0)$. In other words, K is the manifold $\mathcal{D}2 \times \mathcal{T}^{n-3} \times [0,1]$ where the points $(x,y,1)$ are identified with the points $(g(x), A(y), 0)$. Denote by Z the vector field of the suspension flow and by φ_Z^t the suspension flow itself.

For each point $(x,y,t) \in K$ consider the coordinate system

$$(x_1, x_2, y_1, \ldots, y_{n-3}, t) = (x,y,t) \tag{3.2}$$

in its neighborhood where $x = (x_1, x_2) \in \mathcal{D}2$, $y = (y_1, \ldots y_{n-3}) \in \mathcal{T}^{n-3}$ and $t \in [0,1]$. In this coordinate system, $Z = (0,0,1)$.

Set $N = \mathcal{D}2 \times L$, where L is the suspension manifold in Brin's construction (see the previous section). Write $N = \mathcal{D}2 \times (\mathcal{T}^{n-3} \times [0,1]/\sim)$ where $\sim$ is the identification $(y,1) = (A(y), 0)$ for all $y \in \mathcal{T}^{n-3}$.

Consider the map $F\colon K \to N$ given by

$$F(x,y,t) = (G(x,t), y, t),$$

where $G : \mathcal{D}2 \times [0,1] \to \mathcal{D}2$ is the diffeotopy constructed in Proposition 3.4. We have

$$F(x,y,1) = (g(x),y,1) = (g(x),A(y),0) = F(g(x),A(y),0).$$

Therefore, the map F is well-defined. It is easy to see that F preserves volume, is one-to-one and continuous. Hence, it is a homeomorphism. Formal differentiation yields

$$dF(x,y,t) = \begin{pmatrix} G_x(x,t) & 0 & G_t(x,t) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.3}$$

It follows from Statement 5 of Proposition 3.4 that for every $k \geq 1$,

$$d^kF(x,y,1) = d^kF(g(x),Ay,0)$$

and hence, F is a C^∞ diffeomorphism.

Consider the vector field $Y = dFZ$ on N and let φ_Y^t be the corresponding flow. In the coordinate system (3.2), we have

$$Y(G(x,t),y,t) = \left(\frac{\partial G}{\partial t}(x,t),0,1\right), \qquad (x,y,t) \in K. \tag{3.4}$$

The vector field Y is divergence free since it is the image of the divergence free vector field Z under the volume-preserving map F.

Let us choose a C^∞ function $\alpha : \mathcal{D}2 \to [0,1]$ such that

(A1) α and all its partial derivatives of any order are equal to zero on $\partial\mathcal{D}2$;

(A2) $\alpha(x) > 0$ for $x \in \text{int}\mathcal{D}2$, $\alpha(x) = 1$ for $x \in \mathcal{D}2 \setminus U$.

Define the vector field X on N by

$$X(G(x,t),y,t) = \left(\frac{\partial G}{\partial t}(x,t), 0, \alpha(G(x,t))\right), \qquad (x,y,t) \in K. \tag{3.5}$$

Note that by Statement 3 of Proposition 3.4, $\frac{\partial G}{\partial t}(x,t) = V(G(x,t))$ for $x \in U$. Therefore, Equalities (3.4) and (3.5) imply that for $(x,y,t) \in N$ with $x \in U$,

$$Y(x,y,t) = (V(x), 0, 1), \qquad X(x,y,t) = (V(x), 0, \alpha(x)). \tag{3.6}$$

Let $\varphi^t = \varphi^t_X$ be the flow on N generated by the vector field X. It is easy to see that X is divergence free and hence, φ^t is volume-preserving.

LEMMA 3.3. *All but one Lyapunov exponents of the flow φ^t_X are non-zero almost everywhere.*

Proof:

We begin with a construction of a map which is similar to Katok's map: it has the same topological and hyperbolic properties but does not preserve area. It is better adopted to the flow. We will also use this map in the proof of the ergodicity of the flow.

We assume that the function $\alpha(x)$ is chosen such that the following additional condition holds:

(A3) $\alpha(x)^{-1}V(x) \to 0$ as $x \to \partial\mathcal{D}2$, where V is the vector field defined in Statement 3 of Proposition 3.3.

Consider the map $g^* : \mathcal{D}2 \to \mathcal{D}2$ such that $g^* = g$ on $\mathcal{D}2\backslash U$ and g^* is the time-1 map of the flow g^{*t} generated by the vector field $V^* = \alpha^{-1}V$ on U_1 (see Statement 3 of Proposition 3.3). By $(A1)-(A3)$, the map g^* is well defined and is a diffeomorphism. It also satisfies Statements 2 – 4 of Propositions 3.2 and 3.3. Note that the map g^* preserves a measure μ^* which is absolutely continuous with respect to area with positive density $\rho^*(x)$; the latter is unbounded as x approaches $\partial\mathcal{D}2$.

We now proceed as before replacing g by g^*. Namely, define $G^* : \mathcal{D}2 \times [0,1] \to \mathcal{D}2$ by $G^*(x,t) = G(x,t)$ if $x \in X \setminus U$, and $G^*(x,t) = g^{*t}(x)$ otherwise. Let $\phi^t_{Z^*}$ be the suspension flow over $g^* \times A$ with the suspension manifold $K^* = X \times \mathcal{T}^{n-3} \times [0,1]/\sim$, where $\sim$ is the identification $(x,y,1) = (g^*(x), A(y), 0)$ and Z^* is the vector field of the suspension flow. Define the map $F^* : K^* \to N$ by

$$F^*(x,y,t) = (G^*(x,t),y,t)$$

and let $Y^* = dF^*Z^*$. We have for $x \in U$,

$$\frac{\partial G^*}{\partial t}(x,t) = \alpha(g^t(x))^{-1}V(g^t(x))$$

and hence,

$$Y^*(G^*(x,t),y,t)) = (\alpha(g^{*t}(x))^{-1}V(g^{*t}(x)),0,t)$$

or equivalently,

$$Y^*(x,y,t) = (\alpha(x)^{-1}V(x),0,t).$$

Since $Y^*(x,y,t) = Y(x,y,t)$ for $x \notin U$, we obtain by (3.6) that $X = \alpha Y^*$.

Define the vector field $\tilde{Z}$ on K^* by

$$\tilde{Z}(x,y,t) = (dF^*)^{-1}X(F^*(x,y,t)) \qquad (x,y,t) \in K^*.$$

It is easy to see that $\phi_X^t = F^* \circ \phi_{\tilde{Z}}^t \circ F^{*-1}$ and the vector fields Z^* and $\tilde{Z}$ have the same orbits. In other words, there is a function $\beta\colon K^* \to \mathcal{R}^+$ such that for every $(x,y,t) \in K^*$,

(B1) $\tilde{Z}(x,y,t) = \beta(x,y,t)Z^*(x,y,t)$;

(B2) $\beta(x,y,t) = 0$ if $x \in \partial\mathcal{D}2$;

(B3) $\beta(x,y,t) = 1$ if $x \notin U$.

By construction, the flow $\phi_{Z^*}^t$ has non-zero Lyapunov exponents almost everywhere. Denote by $E_{Z^*}^s(x,y,t)$, $E_{Z^*}^u(x,y,t)$, $E_{Z^*}^{cs}(x,y,t)$ and $E_{Z^*}^{cu}(x,y,t)$ the stable, unstable, center-stable and center-unstable invariant subspaces at the point $(x,y,t) \in K^*$ repectively. Observe that the subspaces $E_{Z^*}^{cs}(x,y,t)$ and $E_{Z^*}^{cu}(x,y,t)$ are also invariant under the flow $\phi_{\tilde{Z}}^t$. Chose a point (x_0,y_0,t_0) and a vector $v \in E_{Z^*}^u(x_0,y_0,t_0)$. Note that for almost every (x_0,y_0,t_0) (with respect to the Riemannian volume) the proportion of time the trajectory $\{\phi_{\tilde{Z}}^t(x_0,y_0,t_0)\}$ spends in the set $\{(x,y,t)\colon x \notin U\}$ is strictly positive. It follows that the Lyapunov exponent $\chi(v)$ at (x_0,y_0,t_0) with respect to the flow $\phi_{\tilde{Z}}^t$ is positive.

Almost every point $(x_0,y_0,t_0) \in N$ has stable, unstable, center-stable and center-unstable global manifolds $W_X^s(x_0,y_0,t_0)$, $W_X^u(x_0,y_0,t_0)$, $W_X^{cs}(x_0,y_0,t_0)$ and $W_X^{cu}(x_0,y_0,t_0)$ with respect to the flow φ_X^t. Similarly, almost every point $(x_0,y_0,t_0) \in K^*$ has stable, unstable, center-stable and center-unstable global manifolds $W_{\tilde{Z}}^s(x_0,y_0,t_0)$, $W_{\tilde{Z}}^u(x_0,y_0,t_0)$, $W_{\tilde{Z}}^{cs}(x_0,y_0,t_0)$ and $W_{\tilde{Z}}^{cu}(x_0,y_0,t_0)$ with respect to the flow $\varphi_{\tilde{Z}}^t$. By Proposition 3.2 these foliations can be extended to foliations which are continuous everywhere except for the discontinuity set $Q \times \mathcal{T}^{n-3} \times [0,1]/\sim$. We will use the same notations for these foliations. Observe that

$$\pi_x(W^{cs}_{\tilde{Z}}(x_0,y_0,t_0)) = W^s_{g^*}(x_0), \quad \pi_x(W^{cu}_{\tilde{Z}}(x_0,y_0,t_0)) = W^u_{g^*}(x_0) \tag{3.7}$$

and

$$\pi_y(W^{cs}_{\tilde{Z}}(x_0,y_0,t_0)) = W^s_A(y_0), \quad \pi_y(W^{cu}_{\tilde{Z}}(x_0,y_0,t_0)) = W^u_A(y_0), \tag{3.8}$$

where $\pi_x\colon K^* \to \mathcal{D}2$ and $\pi_y\colon K^* \to \mathcal{T}^{n-3}$ are natural projections.

We say that two points $z,z' \in N$ are *accessible* if there are points $z = z_0, z_1, \ldots, z_{\ell-1}, z_\ell = z', z_i \in N$ such that $z_i \in W^u_X(z_{i-1})$ or $z_i \in W^s_X(z_{i-1})$ for $i = 1,\ldots,\ell$. The collection of points $z_0,z_1,\ldots,z_\ell$ is called a *path* connecting z and z' and is denoted by $[z,z'] = [z_0,z_1,\ldots,z_\ell]$. Accessibility is an equivalence relation. We say that the time-t map of the flow φ^t_X has *accessibility property* if the partition into accessibility classes is trivial (i.e. any two points z,z' are accessible) and to have *essential accessibility property* if the partition into accessibility classes is ergodic (i.e. a measurable union of equivalence classes must have zero or full measure). Similarly, one defines (essential) accessibility of the time-t map of the flow $\varphi^t_{\tilde{Z}}$ using its global stable and unstable foliations.

LEMMA 3.4. *For every t the time-t map of the flow φ^t_X has essentially accessibility property. Moreover, for any set $\mathcal{E}$ of zero measure and any two points $z,z' \notin \mathcal{E}$ one can find a path $[z,z'] = [z_0,z_1,\ldots,z_\ell]$ such that each $z_i \notin \mathcal{E}$.*

Proof: It suffices to establish essential accessibility property of the time-t map of the flow $\varphi^t_{\tilde{Z}}$. Note that the map g^* has essential accessibility property, indeed, any two points outside of the discontinuity set Q are accessible. Note also that the automorphism A in Brin's construction is accessible. Denote by $Q^* = \{(x,y,t) \in K^* : x \in Q, y \in \mathcal{T}^{n-3}, t \in [0,1]\}$. Fix t and consider the time-t map of the flow $\varphi^t_{\tilde{Z}}$. Relations (3.7) and (3.8) imply that for any two points (x,y) and $(x',y') \in \mathcal{D}2 \times \mathcal{T}^{n-3}$ and any $t \in [0,1]$ the point (x,y,t) is accessible to a point (x',y',t'). In particular, any point $(x,y,t) \in K^* \setminus Q^*$ is accessible to a point in $\Pi_{p,q} = \{(p,q,t') \in K^* : \ t' \in [0,1]\}$ for some $p \in \mathcal{D}2$ and $q \in \mathcal{T}^{n-3}$. It remains to show that any two points in $\Pi_{p,q}$ are accessible.

Let $q \in \mathcal{T}^{n-3}$ be a periodic point of the automorphism A in Brin's construction and p, p' be two periodic points of g^* such that the orbit of p and the orbit of p' except for p' itself (under g^*) lie outside the neighborhood U. Consider local stable and unstable leaves (curves) at p and p', $V^s_{g^*}(p)$, $V^u_{g^*}(p)$, $V^s_{g^*}(p')$, and $V^u_{g^*}(p')$. One can choose points p and p' such that the local unstable leaf from p to $V^u_{g^*}(p) \cap V^s_{g^*}(p')$, and the local stable and unstable leaves from p to $V^s_{g^*}(p) \cap V^u_{g^*}(p')$ also lie outside the neighborhood U.

For a point $(p,q,t) \in \Pi_{p,q}$, let $\gamma^u_p \subset W^u_{\tilde{Z}}(p,q,t)$ and $\gamma^s_p \subset W^s_{\tilde{Z}}(p,q,t)$ be the curves such that $\pi_x(\gamma^u_p) \subset V^u_{g^*}(p)$, $\pi_y(\gamma^u_p) = q$ and $\pi_x(\gamma^s_p) \subset V^s_{g^*}(p)$, $\pi_y(\gamma^s_p) = q$. Since γ^u and γ^s lie outside U and the point (p,q) is periodic, the t coordinate of a point remains unchanged when this point moves along γ^u_p from (p,q,t) toward $\gamma^u_p \cap \pi_x^{-1}(V^u_{g^*}(p) \cap V^s_{g^*}(p')) \cap \pi_y^{-1}(q)$. So does the t coordinate of a point moving along γ^s_p from (p,q,t) toward $\gamma^s_p \cap \pi_x^{-1}(V^u_{g^*}(p') \cap V^s_{g^*}(p)) \cap \pi_y^{-1}(q)$. On the other hand, for a point $(p',q,\tau) \in \Pi_{p',q}$, if we choose $\gamma^u_{p'}$ and $\gamma^s_{p'}$ in a similar way, then the third coordinate τ increases when a point moves along $\gamma^u_{p'}$ from (p,q,τ) toward $\gamma^u_p \cap \pi_x^{-1}(V^u_{g^*}(p') \cap V^s_{g^*}(p)) \cap \pi_y^{-1}(q)$ and the third coordinate τ decreases when a point moves along $\gamma^s_{p'}$ from (p,q,τ) toward $\gamma^u_p \cap \pi_x^{-1}(V^u_{g^*}(p) \cap V^s_{g^*}(p')) \cap \pi_y^{-1}(q)$ (we assume that $\alpha(p') < 1$). We can now use the argument in [DHP] (see the proof of Lemma B.4) to obtain that the point (p,q,t) is accessible to some point $(p,q,t') \in \Pi_p$ with $t' < t$, and then any two points in Π_p are accessible. This show that for every t the time-t map of the flow φ_X^t has essentially accessibility property.

To complete the proof consider a set $\mathcal{E}$ of zero measure and two points $z = (x,y,t), z' = (x',y',t') \notin \mathcal{E}$. Note that $x, x' \notin \pi_x(\mathcal{E})$ and $y, y' \notin \pi_y(\mathcal{E})$ and that both sets $\pi_x(\mathcal{E})$ and $\pi_y(\mathcal{E})$ have zero measure. Moreover, the points x and x' can be connected by a path $[x,x'] = [x_0, x_1, \ldots, x_\ell]$ such that each $x_i \notin \pi_x(\mathcal{E})$ and the points y and y' can be connected by a path $[y,y'] = [y_0, y_1, \ldots, y_k]$ such that each $y_i \notin \pi_y(\mathcal{E})$. Finally, we note that the quadrilateral path from (p,q,t) to (p,q,t') in the above argument can be replaced by a nearby path from (x,y,s) to (x',y',s') with $s' < s$ such that both (x,y,s) and (x',y',s') and all other points in the path do not belong to $\mathcal{E}$.

LEMMA 3.5. *The flow φ_X^t on N is Bernoulli.*

Proof: By results in [P3], a flow with non-zero Lyapunov exponents is Bernoulli if it is a K-flow, i.e., the Pinsker algebra ¶, (the largest subalgebra for which $\phi^t|$¶ has zero entropy), is trivial.

Let $\mathcal{B}$ be the σ-algebra of Borel subsets in N and $\mathcal{A} \subset \mathcal{B}$ a subalgebra. Denote by $\mathrm{Sat}_0(\mathcal{A})$ the *saturation of* $\mathcal{A}$ by sets of measure zero, that is,

$$\mathrm{Sat}_0(\mathcal{A}) = \{B \in \mathcal{B} : \text{ there exists } A \in \mathcal{A} \text{ such that } \mu(A \triangle B) = 0\}$$

(where μ is volume). Let $\mathcal{S}^s$ (respectively, $\mathcal{S}^u$) be the subalgebra of Borel sets that consist of whole stable (respectively, unstable) leaves, that is, for $E \in \mathcal{S}^s$ (respectively, $E \in \mathcal{S}^u$) and $x \in E$ we have $W_X^s(x) \subset E$ (respectively, $W_X^u(x) \subset E$). By [P2] (see Theorem 2), the Pinsker algebra ¶ is contained in $\mathrm{Sat}_0(\mathcal{S}^s)$. Similarly, ¶ is contained in $\mathrm{Sat}_0(\mathcal{S}^u)$. Therefore, ¶ is contained in $\mathrm{Sat}_0(\mathcal{S}^u) \cap \mathrm{Sat}_0(\mathcal{S}^u)$ and we wish to show that this intersection is the trivial algebra.

Let $\mathcal{A} \subset \mathrm{Sat}_0(\mathcal{S}^u) \cap \mathrm{Sat}_0(\mathcal{S}^u)$ with $\mu(\mathcal{A}) > 0$. We shall show that $\mu(\mathcal{A}) = 1$. Recall that a point $x \in N$ is a *density point* of $\mathcal{A}$ if

$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap \mathcal{A})}{\mu(B(x,r))} = 1.$$

Denote by $D(\mathcal{A})$ the set of density points of $\mathcal{A}$. By the Lebesgue-Vitali theorem $D(\mathcal{A}) = \mathcal{A}$ (mod 0) and hence, it suffices to show that $\mu(D(\mathcal{A})) = 1$. Let $\mathcal{E}_1$ the complement of $D(\mathcal{A}) \cup D(N \backslash \mathcal{A})$. Applying again the Lebesgue-Vitali theorem we obtain that $\mu(\mathcal{E}_1) = 0$.

Recall that the diffeomorphism $g^* \colon \mathcal{D}2 \to \mathcal{D}2$ has non-zero Lyapunov exponents and results of smooth ergodic theory applies (see [BP] for relevant notions and details). Let $\mathcal{R}^\ell \subset \mathcal{D}2$ be the regular set (of level ℓ) for g^*. The set $\bigcup_\ell \mathcal{R}^\ell$ has full measure in $\mathcal{D}2$ and so does the set $\overline{\mathcal{R}} = \bigcup_\ell D(\mathcal{R}^\ell)$.

For $x \in \mathcal{D}2$ and outside the discontinuity set let $\gamma^s_{g^*}(x,r)$ and $\gamma^u_{g^*}(x,r)$ be arcs in $V^s_{g^*}(x)$ and $V^u_{g^*}(x)$ centered at x of length r.

Choose two points x and x' outside the discontinuity set which are sufficiently close to each other. Consider the holonomy map $\theta : V^s_{g^*}(x) \to W^s_{g^*}(x')$ generated by the family of local leaves $V^u_{g^*}(y), y \in V^s_{g^*}(x)$. Since the foliation $W^s_{g^*}$ is absolutely continuous the map θ moves the conditional measure (length) μ^s_x on $V^s_{g^*}(x)$ to a measure on $V^s_{g^*}(x')$ which is absolutely continuous with respect to $\mu^s_{x'}$. The Jacobian $\mathrm{Jac}(\theta)(y)$ is not bounded. It is however bounded if we allow y to run over the set $V^s_{g^*}(x) \cap \mathcal{R}^\ell$. More precisely, for every $x \in \mathcal{R}^\ell$ there is $J = J(\ell)$ such that

$$\mu^s_{x'}(\theta(V^s_{g^*}(x) \cap \mathcal{R}^\ell)) \leq J \mu^s_x(V^s_{g^*}(x) \cap \mathcal{R}^\ell). \tag{3.9}$$

Since $x' \in D(\mathcal{R}^{\ell})$ we obtain that for sufficiently small r,

$$\mu^s_{x'}(\gamma^s(x',3Jr)\cap\mathcal{R}^{\ell}) \geq 0.9\mu^s_{x'}(\gamma^s(x',3Jr)). \tag{3.10}$$

We claim that

$$\theta(\gamma^s(x,r)) \subset \gamma^s(x',3Jr) \tag{3.11}$$

for all sufficiently small r. Indeed, if (3.11) does not hold then $\mu^s_{x'}(\gamma') \geq 3Jr$ where $\gamma' = \theta(\gamma)$ is the longer component of $\theta(\gamma^s(x,r))\setminus\{0\}$ and $\gamma = \theta^{-1}(\gamma') \subset \gamma^s(x,r)$. By (3.10), we have

$$\mu^s_{x'}(\gamma^s(x',3Jr)\cap\gamma'\cap(N\setminus\mathcal{R}^{\ell})) \leq 0.2\mu^s_{x'}(\gamma^s(x',3Jr)\cap\gamma').$$

It follows that

$$\mu^s_{x'}(\theta(\gamma^s(x,r))\cap\mathcal{R}^{\ell}) \geq \mu^s_{x'}(\gamma^s(x',3Jr)\cap\gamma'\cap\mathcal{R}^{\ell}) \geq 0.8\mu^s_{x'}(\gamma^s(x',3Jr)\cap\gamma')$$

$$\geq 0.8\cdot 0.5\,\mu^s_{x'}(\gamma^s(x',3Jr)) \geq 0.4\cdot 3J\mu^s_{x'}(\gamma^s(x',3Jr)) > J\mu^s_x(\gamma^s(x,r)),$$

contradicting to (3.9).

Let $\mathcal{E}_2$ be the set of $(x,y,t)\in N$, where $x\in\mathcal{D}2$ with $x\notin D(\mathcal{R}^{\ell})$ for any $\ell>0$, $y\in\mathcal{T}^{n-3}$ and $t\in\mathbb{R}$. Clearly, $\mu(\mathcal{E}_2)=0$. Let $\mathcal{E}=\mathcal{E}_1\cup\mathcal{E}_2$. We have $\mu(\mathcal{E})=0$.

Choose a point $z\in D(\mathcal{A})\setminus\mathcal{E}$ and a point $z'\in V^u_X(z)\setminus\mathcal{E}$ (recall that $V^u_X(z)$ is a local unstable manifold at z for the flow ϕ^t_X). Consider the holonomy map $\theta : V^{cs}_X(z)\to W^{cs}_X(z')$ generated by the family of local unstable manifolds of ϕ^t_X (here $V^{cs}_X(z)$ is a local center-stable manifold at z and $W^{cs}_X(z')$ is the global center-stable manifold at z' for the flow ϕ^t_X). Consider a small ball

$B^{cs}(z,r) \subset V_X^{cs}(z)$ centered at z. By (3.11), the size of $\theta(B^{cs}(z,r))$ in the stable direction $E^s_{g^*}(z)$ of the disc is bounded by $C_1 r$ for some $C_1 > 3J$ independent of r. Since the stable foliation W^s_A of the torus is smooth, the size of $\theta(B^{cs}(z,r))$ in the stable direction $E^s_A(z)$ is bounded by $C_2 r$ for some $C_2 > 0$. Also, the size of $\theta(B^{cs}(z,r))$ in the central direction is bounded by $C_3 r$ for some $C_3 > 0$. Hence, the point z' cannot belong to $D(N \setminus \mathcal{A})$. Note that $z \notin \mathcal{E}$ implies $z' \in D(\mathcal{A}) \cup D(N \setminus \mathcal{A})$ and hence, $z' \in D(\mathcal{A})$.

We shall now show that $\mu(\mathcal{A}) = 1$. By Lemma 3.4, for almost every point $z' \in N \setminus \mathcal{E}$ one can find a point $z \in D(A) \setminus \mathcal{E}$ such that z and z' are accessible through a path $z_0, \dots, z_\ell$ such that $z_i \notin \mathcal{E}$. Repeating the above arguments we obtain that $z_1, \dots, z_\ell = z' \in D(\mathcal{A})$. Hence, $\mu(D(\mathcal{A})) = 1$. The desired result follows.

By identifying some boundary points, it is easy to see that the manifold N can be mapped onto the n-dimensional disc $\mathcal{D}^n$ via a map $\phi : N \to \mathcal{D}^n$ such that $\phi(N) = \mathcal{D}^n$ and $\phi|\text{int}(N)$ is a diffeomorphism. Since $X|\partial N = 0$, $d\phi(X)$ is smooth on $\mathcal{D}^n$. There is also a mapping $\psi : \mathcal{D}^n \to M$ (see [K]), and the vector field $d\psi d\phi(X)$ generates the flow with the desired properties.

3.4 Proof of the theorem: the case $\dim M = 3$ and 4

In the case $\dim M = 3$, the proof is essentially the same. Consider the suspension flow over g with roof function 1. The suspension manifold $K = \mathcal{D}2 \times [0,1]/\sim$ (where $\sim$ is the identification $(x,1) = (g(x),0)$) is diffeomorphic to $N = \mathcal{D}2 \times \mathcal{S}^1 = \{(x,t) : x \in M,\ t \in [0,1]\}/\sim$ (where $\sim$ is the identification $(x,0) = (x,1)$). Let Z be the vector field of the suspension flow. For each $(x,t) \in K$ we have $Z = (0,1)$.

Let $F : K \to N$ be given by $F(x,t) = (G(x,t),t)$ (see Proposition 3.4). We have

$$dF(x,t) = \begin{pmatrix} G_x(x,t) & G_t(x,t) \\ 0 & 1 \end{pmatrix}.$$

Consider the vector field $Y = dFZ$. Note that

$$Y(G(x,t),t) = \Big(\frac{\partial G}{\partial t}(x,t), 1\Big).$$

Define the vector field X on N by

$$X(G(x,t),t) = \left(\frac{\partial G}{\partial t}(x,t), \alpha(G(x,t))\right),$$

where $\alpha(x)$ is a C^∞ function on $\mathcal{D}2$ satisfying Conditions $(A1) - (A3)$. The vector field X is divergence-free and the flow ϕ_X^t has all the desired properties.

In the case $\dim M = 4$ we start with a Bernoulli map with non-zero Lyapunov exponents on a 3-manifold constructed in [DP]. More precisely, define

$$S = g \times \mathrm{id} : \mathcal{D}2 \times \mathcal{S}^1 \to \mathcal{D}2 \times \mathcal{S}^1.$$

Let $T(x,y) = (g(x), T_{\gamma(x)}y) : \mathcal{D}2 \times \mathcal{S}^1 \to \mathcal{D}2 \times \mathcal{S}^1$, where $T_{\gamma(x)}$ is rotation by $\gamma(x)$. Here γ: $\mathcal{D}2 \to \mathbb{R}$ is a nonnegative C^∞ function which is equal to zero in a small neighborhood of the discontinuity set $\overline{Q} = \{q_1, q_2, q_3, \partial\mathcal{D}2\} \times \mathcal{S}^1$ and is positive elsewhere.

It is shown in [DP] that the function γ can be chosen so that the map T is *robustly accessible*, i.e., any C^1 perturbation R of T is accessible provided R coincides with T in a small neighborhood of the discontinuity set $\overline{Q}$. Moreover, there is a perturbation R of T which has non-zero Lyapunov exponents and is of the form $R = \phi \circ T$. Here the map ϕ differs from id in a small neighborhood of a point $z_0 \in \mathcal{D}2 \times \mathcal{S}^1$ which lies outside a small neighborhood $\overline{U}$ of the set $\overline{Q}$. To describe the perturbation ϕ consider the coordinate system $\xi = \{\xi_1, \xi_2, \xi_3\}$ in an open disc $B(z_0, \varepsilon)$ of asufficiently small radius ε centred at z_0 such that

1. $dm = d\xi$;
2. $E^c_T(z_0) = \partial/\partial\xi_1$, $E^s_T(z_0) = \partial/\partial\xi_2$, $E^u_T(z_0) = \partial/\partial\xi_3$.

Let $\psi(t)$ be a C^∞ function with support in $(-\varepsilon, \varepsilon)$. Set $\tau = \|\xi\|^2/\gamma^2$ and

$$\phi^{-1}(\xi) = (\xi_1 \cos(\varepsilon\psi(\tau)) + \xi_2 \sin(\varepsilon\psi(\tau)), -\xi_1 \sin(\varepsilon\psi(\tau)) + \xi_2 \cos(\varepsilon\psi(\tau)), \xi_3).$$

Choose a function $\widehat{\psi}(x,t) : \mathbb{R} \times [0,1] \to \mathbb{R}$ with the following properties

1. $\widehat{\psi}(x,0) = 0$ and $\widehat{\psi}(x,1) = \psi(x)$ for any $x \in \mathbb{R}$;
2. $\widehat{\psi}$ is C^∞ in x and t and $\widehat{\psi}'_t(x,t) = 0$ for $t = 0,1$ and $x \in \mathbb{R}$;
3. $\|\widehat{\psi}(x,t)\|_r < \delta$ for all $t \in [0,1]$.

Define the map $\Psi : \mathcal{D}2 \times \mathcal{S}^1 \times [0,1] \to \mathcal{D}2 \times \mathcal{S}^1$ by the formula

$$\Psi(x,y,t) = \phi^t(x,y),$$

where

$$\phi^t(\xi) = (\xi_1 \cos(\varepsilon\widehat{\psi}(\tau,t)) + \xi_2 \sin(\varepsilon\widehat{\psi}(\tau,t)), -\xi_1 \sin(\varepsilon\widehat{\psi}(\tau,t)) + \xi_2 \cos(\varepsilon\widehat{\psi}(\tau,t)), \xi_3).$$

Note that Ψ has the properties similar to those of the map G in Proposition 3.4. Namely,

1. Ψ is C^∞ in (x,y,t);

2. $\Psi(\cdot,0) = \text{id}$ and $\Psi(\cdot,1) = \phi$;

3. for any $t \in [0,1]$ the map $\Psi(\cdot,t) : \mathcal{D}2 \times \mathcal{S}^1 \to \mathcal{D}2 \times \mathcal{S}^1$ is area-preserving;

4. $\Psi_t(x,y,1) = \Psi_t(\phi(x,y),0)$;

5. $\Psi(x,y,t) = \text{id}$ for $x \in \overline{U}$.

Set

$$H = \{(x,y,t) : x \in \mathcal{D}2, y \in \mathcal{S}^1, t \in [0,1]\}/\sim_1$$

with the identification $\sim_1$: $(x,y,1) = (T(x,y),0) = (g(x), T_{\gamma(x)}(y),0)$ and

$$K = \{(x,y,t) : x \in \mathcal{D}2, y \in \mathcal{S}^1, t \in [0,1]\}/\sim_2$$

with the identification $\sim_2$: $(x,y,1) = (\phi \circ T(x,y),0) = (\phi(g(x), T_{\gamma(x)}(y)),0)$. Consider the diffeomorphism $\tilde{G} : H \to K$ given by

$$\tilde{G}(x,y,t) = (\Psi^{-1}(x,y,t),t),$$

where $\Psi^{-1}(x,y,t)$ for each t is the inverse of Ψ. Note that T itself is diffeotopic to S. Let $\bar{G}$ be the diffeomorphism $K \to K'$ where K' is the suspension manifold of the suspension flow over S. The manifold K' is diffeomorphic to $N = \mathcal{D}2 \times S^1 \times S^1$. If $\widehat{G} : K' \to N$ then $F = \tilde{G} \circ \bar{G} \circ \widehat{G} : H \to N$ is a diffeomorphism.

Let Z be the vector field on H of the suspension flow over R . Obviously, $Z = (0,0,1)$ is divergence free. Set $Y = dFZ$. Since $F(x,y,t) = (G(x,t),y,t)$ for (x,y,t) in a neighborhood $\tilde{U}$ of the boundary of H then for $(x,y,t) \in \tilde{U}$,

$$Y(F(x,y,t)) = Y(G(x,t),y,t) = \left(\frac{\partial G}{\partial t}(x,t),0,1\right).$$

Let X be the vector field on N defined by the formula

$$X = \left(\frac{\partial G}{\partial t}(x,t),0,\alpha(x)\right),$$

where $\alpha(x)$ satisfies Conditions $(A1)$ – $(A3)$. Clearly, X is divergence free and the flow generated by X has all the desired properties.

Chapter 4

A hyperbolic map in a vicinity of the Identity map

4.1 Introduction

It is generally beleived that there is no Anosov map close to the identity map. The natural question arises: does it hold for a nonuniformly hyperbolic map?

Shub and Wilkinson developed a technique to eliminate a zero Lyapunov exponent in some special cases of partially hyperbolic maps. It is acheived by a perturbation of a product of some Anosov map A and the identity map on a circle, $A \times \mathrm{id} : T^2 \times S^1 \to T^2 \times S^1$. In fact two perturbations P_1 and P_2 are made. P_1 takes care of ergodicity creating accessibility. The perturbation P_2 eliminates the zero Lyapunov exponent in the direction of the circle. It is acheived by tilting the unstable subspace into the direction of the central subspace. As a result, the central foliation borrows some hyperbolicity from the unstable foliation. The resulting map (though not uniformly hyperbolic) has all nonzero Lyapunov exponents. Note that since the original map A was away from the identity map so is P.

To construct a nonuniformly hyperbolic map close to the identity we use a similar idea, but start with a time δ map of the geodesic flow on a surface of constant negative curvature instead of a toral automorphism. We also need three perturbations since there are two zero Lyapunov exponents to remove.

There are two new problems in this construction. First, to obtain ergodicity we need accessibility. For this purpose we need to show that almost every point is accessible to a given central leaf along the interval direction. Since both strong stable and unstable leaves are one dimensional, the accessibility is easy to achieve in a three dimensional space as in [DHP] and extra work is necessary in our case because the space we work with is four dimensional. Secondly , the technique to remove the second zero Lyapunov exponent is more delicate. While we change the last Lyapunov exponent, we need to keep all other exponents nonzero at the same time. But the direction corresponding to the ”small” nonzero exponent obtained by the earlier perturbation is not stable under perturbation.

Our results show that systems which are hyperbolic, volume preserving, and ergodic exist in any neighborhood of the identity.

4.2 Statement of Results

We prove the following result.

THEOREM 4.1. *There exists a four dimensional compact Riemannian manifold M such that for any $\delta_0 > 0$, we can find a C^∞ diffeomorphism f of M with the following properties:*

1. $\|f - \mathrm{id}\|_{C^1} \leq \delta_0$;
2. *f preserves the Riemannian volume μ on M;*
3. *μ is a hyperbolic measure;*
4. *f is ergodic.*

4.3 Construction

Let $g^t : M_0 \to M_0$ be a geodesic flow on a compact surface of a negative constant curvature.

Choose a closed orbit $\mathcal{C}$.

Take $0 < \delta_1 \leq \delta_0/2$ such that all points in $\mathcal{C}$ are periodic points of g^{δ_1}. Take $G = g^{\delta_1}$.

Let $F = G \times \mathrm{id}$ be the map from $M = M_0 \times \mathbb{S}^1$ to itself. We will perturb F to obtain the desired map f.

Let P be the map on $M_0 \times I$ defined in theorem 4.2 for $\delta < \delta_0/2$. Denote by $\bar{P}$ the map on $M = M_0 \times S^1$ such that $\bar{P} = P|_{M_0 \times I}$ and $\bar{P} = g^{\delta_1}|_{M_0 \times \{0\}}$. By property 4 of theorem 4.2 $\bar{P}$ is a C^∞ map. $\bar{P}$ has all the desired properties.

4.4 Main Proposition

The goal of this section is to prove the following statement.

THEOREM 4.2. *Let $S = G \times \mathrm{id}$ be the diffeomorphism from $N = M_0 \times I$ to itself. For any $\delta > 0$, there exists a map P such that:*

1. *P is a C^∞ volume preserving diffeomorphism of N;*
2. *$\|S - P\|_{C_1} \leq \delta$;*
3. *for all $0 \leq n < \infty$ $D^n P|M_0 \times \{s\} = D^n S|M_0 \times \{s\}$ for $s = 0$ and 1;*
4. *P is ergodic with respect to the Riemannian volume and has non-zero Lyapunov exponents almost everywhere.*

Note that S is not ergodic, and has two zero Lyapunov exponents. We perturb S by three small perturbations $h^{(i)} : \Omega_i \to \Omega_i$, $i = 1,2,3$, where $\Omega_i \subset N$, to get ergodicity and to remove zero Lyapunov exponents.

Proof of Main Proposition: Note that the tangent bundle of N can be written as a direct sum of four one-dimensional S-invariant subbundles:

$$TN = E^u(S) \oplus E^s(S) \oplus E^c(S) \oplus E^n(S),$$

where $E^u(S)$, $E^s(S)$ and $E^c(S)$ are the unstable, stable and flow directions of the geodesic flow $g^t : M_0 \to M_0$, and $E^n(S)$ is the tangent space of I. The corresponding Lyapunov exponents of S at $w \in N$ are denoted by $\lambda^u(w,S)$, $\lambda^s(w,S)$, $\lambda^c(w,S)$ and $\lambda^n(w,S)$ respectively. It is clear that the Lyapunov exponents are constants at almost every point $w \in N$, though S is not ergodic. We simply denote them by $\lambda^u(S)$ etc. Also, we know that $\lambda^u(S) > 0$, $\lambda^s(S) < 0$ and $\lambda^c(S) = \lambda^n(S) = 0$. We will often take local coordinate system $w = (x,y,t,z)$ in N in such a way that

$$d\mu = dxdydtdz, \text{ where } \frac{\partial}{\partial x} = E^u(S), \ \frac{\partial}{\partial t} = E^c(S), \ \frac{\partial}{\partial z} = E^n(S). \tag{4.1}$$

Denote $\eta = \lambda^u(S)$.

Fix $\tau \in (0,1)$.

Take $k \in \mathbb{N}$, $k_0 > 0$ such that for $\gamma \leq \delta$, $\tau_2 = 1/4$ and $\theta = \pi/k_0$ the properties stated in Lemma 4.6 are satisfied.

Recall that $\mathcal{C}$ is the closed orbit taken in Section 2. Take another closed orbit $\mathcal{C}' \subset M_0$ of g^t.

Choose a set Ω_0 and constant ε_0 according to Lemma 4.5 with $\tau_1 = (1-1.3\tau)k_0^{-1}$. Hence

$$\mu\Big(\bigcup_{0\le i\le k_0} G^i\Omega_0\Big) \le 1-1.3\tau. \tag{4.2}$$

We also assume that ε_0 is small enough such that $\mathcal{C}$ and $\mathcal{C}'$ are at least $3\varepsilon_0$ separated.

Construction of $h^{(1)} : \Omega_1 \to \Omega_1$.

Fix $p \in \mathcal{C}$, a periodic point of G with period $m \in \mathbb{N}$.

We assume further that $\varepsilon_0 > 0$ is small such that for any $w \in N$,

$$\mu(B(w,\varepsilon_0)) < \frac{0.1\tau}{k_0 m} \tag{4.3}$$

Here $B(w,\varepsilon_0)$ denotes the ball in M_0 of radius ε_0 centered at w.

We assume that $G^i(B(p,\varepsilon_0)) \cap B(p,\varepsilon_0) = \emptyset$ for $i = 1,\cdots m-1$.

For any $z \in M_0$, let $V^u(z)$ and $V^s(z)$ be the local unstable and stable one dimensional manifold at z for G of "size" ε_0.

Since both $W^u(\mathcal{C}')$ and $W^s(\mathcal{C}')$ are dense in M_0, we can choose $p_1, p_2 \in \mathcal{C}'$, and the smallest integers $n_1, n_2 > 0$ such that each intersection

$$G^{-n_1}V^s(G^{n_1}p_1) \cap V^u(p) \cap B(p,\varepsilon_0) \quad \text{and} \quad G^{n_2}V^u(G^{-n_2}p_2) \cap V^s(p) \cap B(p,\varepsilon_0)$$

consists of a single point q_1 and q_2 respectively.

Take $\varepsilon_1 \le \min\{\delta, d(p,q_1)/2, d(p,q_2)/2\}$. Take $\ell \ge 2$ such that

$$G^{-\ell m}(q_1) \notin B(p,\varepsilon_1), \quad G^{-(\ell+1)m}(q_1) \in B(p,\varepsilon_1). \tag{4.4}$$

Then we take $\varepsilon_2 \in (0, \varepsilon_1)$ such that $G^{-(\ell+1)m}(q_1) \in B(p, \varepsilon_2)$.

Let $\Omega_1 = B(p, \varepsilon_0) \times I$. Denote

$$\tilde{\Omega}_1 = \Omega_1 \bigcup \big(B(\mathcal{C}, \varepsilon_3) \times I\big) \bigcup \big(\cup_{i=1}^{m-1} G^i(B(p, \varepsilon_0)) \times I\big) \bigcup \Omega_1' \bigcup \Omega_2'', \tag{4.5}$$

where

$$\Omega_1' = B(\cup_{i=0}^{\infty} G^{-n_1+i}(V^s(G^{n_1}(p_1))), \varepsilon_3) \times I$$

$$\Omega_1'' = B(\cup_{i=0}^{\infty} G^{n_2-i}(V^u(G^{-n_2}(p_2))), \varepsilon_3) \times I$$

and $B(\Omega, \varepsilon)$ is the ε-neighborhood of the set Ω and ε_3 is chosen such that

$$\mu \tilde{\Omega}_1 \leq 0.1\tau / k_0. \tag{4.6}$$

Take a coordinate system $w = (x, y, t, z)$ in Ω_1 satisfying (4.1). Take the y-coordinate in such a way that $\frac{\partial}{\partial y} = E^s(S)$ along the path $V^s(p)$.

Choose a C^∞ function $\phi = \phi(r) : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies

1. $\phi(r) = \phi_0$ if $r \in [0, \varepsilon_2]$, where ϕ_0 is a positive constants;
2. $\phi(x) = 0$ if $x \geq \varepsilon_1$;
3. $\phi'(x) \leq 0$ for any x;
4. $\|\phi\|_{C^1} \leq \delta$.

Choose a C^∞ function $\psi = \psi(y) : \mathbb{R} \to \mathbb{R}^+$ which satisfies

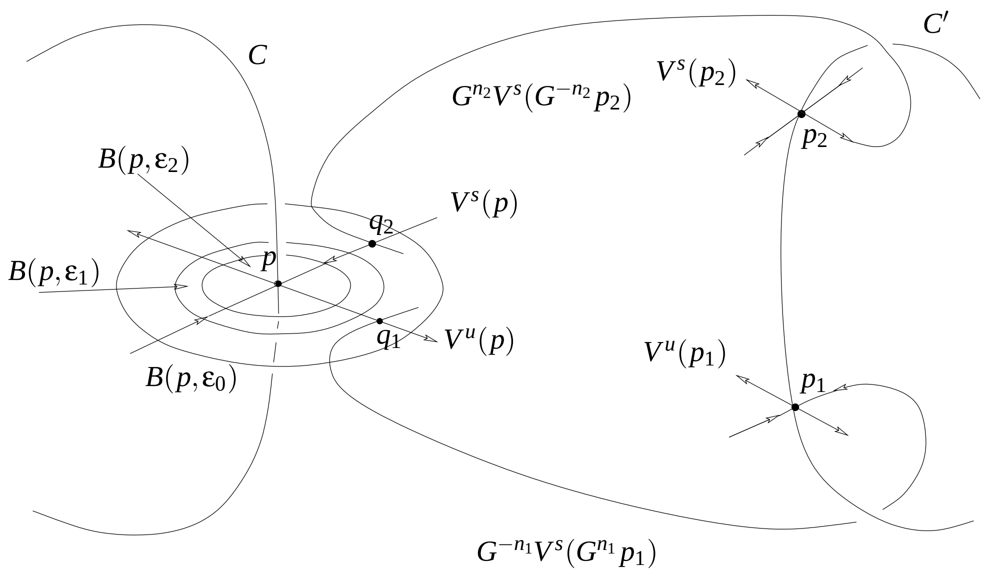

Fig. 4.1. Construction of $h^{(1)}$.

5. $\psi(x) = \psi_0$ if $x \in (-\varepsilon_2, \varepsilon_2)$, where ψ_0 is a positive constant;

6. $\psi(x) = 0$ if $|x| \geq \varepsilon_1$;

7. $\|\psi\|_{C^1} \leq \delta$;

8. $\int_0^{\pm\varepsilon_1} \psi(s)ds = 0$.

We also choose a C^∞ functions $\xi : I \to \mathbb{R}^+$ satisfying:

9. $\xi(s) > 0$ on $(0,1)$;

10. $\xi^{(i)}(0) = \xi^{(i)}(1) = 0$ for $i = 0,1,2,\cdots,$;

11. $\|\xi\|_{C^1} \leq \delta$.

Then we define the vector field X on Ω_1 by

$$X(x,y,t,z) = \left(-\phi(\sqrt{y^2+t^2})\xi'(z)\int_0^x \psi(u)du,\ \ 0,\ \ 0,\ \ \phi(\sqrt{y^2+t^2})\xi(z)\psi(x)\ \ \right).$$

It is easy to check that X is a divergence free vector field supported in $(-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_1, \varepsilon_1) \times I \in \Omega_1$. We define the map $h^{(1)} = h^{(1)}_\beta$ on Ω_1 to be the time β map of the flow generated by X and we set $h^{(1)} = \mathrm{id}$ on the complement of Ω_1. It is easy to see that $h^{(1)}$ is a C^∞ volume preserving diffeomorphism. Also, we assume that β is small enough such that $\|h^{(1)} - \mathrm{id}\| \leq \delta$.

Let $R = R_\beta = h^{(1)}_\beta \circ S$ for some small $\beta > 0$. By Proposition 4.3, for any small $\beta > 0$, $R : N \to N$ is ergodic.

Construction of $h^{(2)} : \Omega_2 \to \Omega_2$.

Choose a point $w^* = (w_0^*, z_0) \in N$, where $w_0^* \in M_0$ and z_0 is the midpoint of I. Choose $0 < \varepsilon_4 \leq \varepsilon_0$. Let $\Omega_2 = B(w^*, \varepsilon_4)$. We assume that w_0^* and ε_4 is chosen in such a way that $\Omega_2 \cap (\tilde{\Omega}_1 \cup (\Omega_0 \times I)) = \emptyset$ and $S^{-i}\Omega_2 \cap (\Omega_2 \cup \Omega_1) = \emptyset$ for $i = 1, \cdots, N_0$, where N_0 is an integer large enough such that (B. 19) in [DHP] holds for N_2.

Let $\rho(r) = (\varepsilon_4/\varepsilon_1)\phi(r\varepsilon_1/\varepsilon_4)$. To define $h^{(2)}$, we take the cylindrical coordinate system (r, θ, y, z) in Ω_2, where $x = r\cos\theta$, $y = y$, $t = t$ and $z = r\sin\theta$. Define $h^{(2)} = h_\alpha^{(2)}$ on N by

$$h^{(2)}(r, \theta, y, t) = (r,\ \theta + \alpha\rho(\sqrt{y^2 + t^2})\rho(r),\ y, t). \tag{4.7}$$

Then we extend $h^{(2)}$ to N be letting $h^{(2)} = \text{id}$ on Ω_2^c. Note that due to (4.3)

$$\mu(\Omega_2) \leq \frac{0.1\tau}{k_0}. \tag{4.8}$$

Let $Q = Q_{\alpha\beta} = R_\beta \circ h_\alpha^{(2)} = h_\beta^{(1)} \circ S \circ h_\alpha^{(2)}$.

Denote by $\eta_{\alpha\beta}(w)$ the expanding rate of $Q_{\alpha\beta}$ along its unstable direction $E_w^u(Q_{\alpha\beta})$. By Proposition 4.4, $\int_N \log\eta_{\alpha 0}(w)dw < \lambda^u(S)$. Since $\eta_{\alpha\beta}$ change smoothly with β we conclude that for some small $\beta > 0$

$$\int_N \log\eta_{\alpha\beta}(w)dw < \lambda^u(S).$$

By Proposition 4.3, $Q_{\alpha\beta}$ is ergodic. So we denote by $\lambda^u(Q_{\alpha\beta})$ the largest Lyapunov exponent of $Q_{\alpha\beta}$. By the above inequality, $\lambda^u(Q) \leq \lambda^u(S)$.

Note that both $Dh^{(1)}$ and $Dh^{(2)}$ preserve $E^{un}(S)$ bundle, and for any w, $|\det Dh_w^{(1)}|E^{un}(S)| = |\det Dh_w^{(2)}|E^{un}(S)| = 1$. So DQ preserves $E^{un}(S)$ bundle, and $|\det DQ_w|E^{un}(S)| = |\det DS_w|E^{un}(S)|$.

Let $\lambda^n(Q)$ denote the other Lyapunov exponent on $E^{un}(S)$. Then we have

$$\lambda^u(Q)+\lambda^n(Q)=\lambda^u(S)+\lambda^n(S).$$

Since $\lambda^u(Q)<\lambda^u(S)$ and $\lambda^n(S)=0$, we have $\lambda^n(Q)>0$.

Note that the perturbations also preserve $E^{ucn}(S)$ bundle. So we have

$$\lambda^u(Q)+\lambda^c(Q)+\lambda^n(Q)=\lambda^u(S)+\lambda^c(S)+\lambda^n(S)$$

and therefore $\lambda^c(Q)=0$, where $\lambda^c(Q)$ is the third Lyapunov exponent of Q on $E^{ucn}(S)$. Apply the same arguments, we also get $\lambda^s(Q)=\lambda^s(S)<0$, though the stable bundle $E^s(Q)$ may not be equal to $E^s(S)$.

Construction of $h^{(3)}:\Omega_3\to\Omega_3$.

Denote $\lambda=\lambda^n(Q)$. We assume that λ is small in comparison with $\lambda^u(Q)$.

Denote

$$\Lambda'=\Lambda'(K)=\{w\in N:|\log|DQ^k_w|E^n(w,Q)|-k\lambda|\leq 0.1k\lambda,\forall|k|\geq 0.5K\},$$

and $\Lambda=\bigcap_{i=0}^{k_0}Q^{-i}\Lambda'$. Note that $\mu\Lambda'\to 1$ as $K\to\infty$. We assume that K satisfies

$$K\lambda\geq\max\{2k_0\lambda,\ 1.25\log 2,\ -10k_0\log(1-\delta)\},\tag{4.9}$$

$$0.001\tau^2\lambda+\mu\Lambda^c\log(1-\delta)>0,\tag{4.10}$$

$$\mu\Lambda^c\leq 0.1\tau,\tag{4.11}$$

where Λ^c is the complement of $\Lambda(K)$ in N.

Note that if $w \in \Lambda'$ then $\|DQ^n_w(v)|E^{un}\| \geq e^{n\lambda 0.9K}\|v\|$ for $v \in E^{un}_w(Q)$.

Denote

$$\Omega = \Lambda^c \bigcup \Big(\bigcup_{i=1}^{k_0} Q^{-i}\big(\Omega_0 \cup \tilde{\Omega}_1 \cup \Omega_2\big)\Big).$$

By (4.11), (4.2), (4.6) amd (4.8),

$$\mu\Omega^c \geq 1 - 0.1\tau - (1 - 1.3\tau) - 0.1\tau - 0.1\tau = \tau. \tag{4.12}$$

Choose a set $\Gamma' \in N$ such that $Q^i\Gamma' \cap \Gamma' = \emptyset$, $-K \leq i \leq 5\tau^{-1}K - 2K - 1$, $i \neq 0$. Here we assume that τ^{-1} is an integer, otherwise we can use a smaller τ instead. Denote

$$\overline{\Gamma}' = \bigcup_{i=-K}^{5\tau^{-1}K-2K-1} Q^i\Gamma'.$$

We also require that $\mu\overline{\Gamma}'$ is close to 1 such that

$$(1 - 0.5\tau) \cdot \mu\overline{\Gamma}' \geq 1 - 0.6\tau. \tag{4.13}$$

The choice of such Γ' is possible because of the Rokhlin-Halmos Lemma.

We define

$$\Gamma_0 = \{Q^j w : \ w \in \Gamma', 0 \leq j \leq 4\tau^{-1}K - K - k_0, Q^j w \in \Omega^c, Q^i w \notin \Omega^c \text{ for } i \leq j\}.$$

In other words, Γ_0 is the set of points from each trajectory $\{Q^i w\}_{i=0}^{4\tau^{-1}K-K-k_0}$ that enter the set Ω^c the first time.

Clearly $Q^i\Gamma$, $i=-K,\cdots,K+k_0$, are pairwise disjoint. Let $\Gamma_i = Q^i\Gamma$ for $i=-K,\cdots,K+k_0$, $\Gamma_{j,k} = \cup_{i=j}^{k}\Gamma_i$ for $j \le k$, and in particular, $\overline{\Gamma} = \Gamma_{-K,K+k_0}$. Since Γ is disjoint with Ω, it is clear that $\Gamma_i \cap (\Omega_0 \cup \tilde{\Omega}_1 \cup \Omega_2) = \emptyset$ for $i=1,\cdots,k_0$. Since we are going to make perturbation $h^{(3)}$ around the set Γ_{0,k_0-1}, this condition guarantees that the properties of Q we mentioned above still remain.

Approximate $\Gamma = \Gamma_0$ by finitely many number of disjoint sets of the form

$$\Delta_{0j} = B^u(x_j, r'_j) \times B^s(y_j, r''_j) \times B^{cn}((t_j, z_j), r_j),$$

where $w_i = (x_j, y_j, t_j, z_j) \in N$, $r'_j, r''_j \ge r_j$ for $j=1,\cdots,J$ and B^u, B^s and B^{cn} are the balls that correspond to the x, y and zt- coordinates. Denote $\Delta_{ij} = Q^i\Delta_{0j}$, $\Delta_i = \cup_{j=1}^{J}\Delta_{ij}$ for $i=-K,\cdots,K+k_0$. We can pick Δ_{0j} in such a way that $\Delta_{ij} \cap \Delta_{kl} = \emptyset$ for $(i,j) \ne (k,l)$, $-K \le i,k \le K+k_0$, $1 \le j,l \le J$ and $\Delta_{ij} \cap (\Omega_0 \cup \tilde{\Omega}_1 \cup \Omega_2) = \emptyset$ for $0 \le i \le k_0$, $0 \le j \le J$.

Also, denote $\overline{\Delta} = \cup_{i=0}^{k_0-1}\Delta_i$. Clearly, Δ_i is an approximation of Γ_i for $i=1,\cdots,k_0$. We may assume that for each $i=0,\cdots,k_0$,

$$\mu(\Gamma_i \triangle \Delta_i) \le 0.05\max\{\mu\Gamma_i, \mu\Delta_i\}. \tag{4.14}$$

Take $\Omega_3 = \overline{\Delta}$. On each Δ_{ij} we apply Lemma 4.6 to get a map $h = h_{ij}$ and a subset $\Delta'_{ij} \subset \Delta_{ij}$ such that $\|h_{ij} - \mathrm{id}\| \le \delta$, $\mu\Delta'_{ij}/\mu\Delta_{ij} \ge 3/4$ and restricted on Δ'_{ij}, h is a rotation of angle $\pi/2k_0$ along the cn plane. Note that $DS|E^{cn} = \mathrm{id}$, we can require that $Q\Delta'_{ij} = \Delta'_{i+1,j}$ for $j=0,\cdots,k_0-1$.

Let $\Delta'_i = \cup_{j=1}^{J}\Delta'_{ij}$.

Let $h^{(3)} = h_{i,j}$ on each $\Delta_{i,j}$ and $h^{(3)} = \text{id}$ otherwise. Take $P = Q \circ h^{(3)}$. By Proposition 4.3, P is ergodic. By Proposition 6.1, we have that for almost every $w \in N$, for all $v \in E^{ucn}(w,S)$, $\lambda(w,v,P) > 0$. So P has three positive Lyapunov exponents. Since the foliation $W^{ucn}(S)$ is preserved, we have $E^{ucn}(w,S) = E^{ucn}(w,P)$. This implies $\lambda^s(P) = \lambda^s(S) < 0$. □

4.5 Ergodicity

Recall that for a smooth system $f : M \to M$, two points $w_1, w_2 \in M$ are called *accessible (with respect to f)* if they can be joined by a piecewise differentiable piecewise nonsingular path which consists of segments tangent to either $E^u(f)$ or $E^s(f)$. The diffeomorphism f is *essentially accessible* if almost any two points in M (with respect to the Riemannian volume) are accessible.

PROPOSITION 4.3. *For any $\alpha, \beta, \gamma > 0$ sufficiently small, the diffeomorphisms $R = R_\alpha$, $Q = Q_{\alpha\beta}$ and $P = P_{\alpha\beta\gamma}$ of N are ergodic with respect to the Riemannian volume.*

Proof: By a result of Pugh and Shub, if a C^2 diffeomorphism is partially hyperbolic, center bunched, dynamically coherent and essentially accessible, then the diffeomorphism is ergodic. (See [PS]).

Clearly, all S, R, Q and P are partially hyperbolic and center bunched if α, β and γ are small.

Note that S is dynamically coherent and its center foliation is plague expansive. Since R, Q and P are C^1 close to S, they are all dynamically coherent if α, β and γ are small by a theorem of Pugh and Shub ([PS], Theorem 2.3).

By the lemma below, R, Q and P are essentially accessible. So they are ergodic. □

LEMMA 4.1. *Any two points $w, w' \in \operatorname{int} N$ are accessible with respect R, Q or P. Therefore, R, Q or P are essentially accessible.*

Proof: Recall that $p \in \mathcal{C}$ is chosen to construct $h^{(1)}$. Denote $I_p = \{p\} \times (0,1)$, and $\overline{I_p} = \{p\} \times [0,1]$. For $T = R, Q$ or P, let $\mathcal{A}_T$ be the set of points that are accessible to some point in $\overline{I_p}$ with respect to T. Since $\mathcal{A}_T$ is both open and closed in N, we get $\mathcal{A}_T = N$. Further, since ∂N is unperturbed by any of $h^{(i)}$, $i = 1, 2, 3$, any point in $\operatorname{int} N$ is accessible to a point in I_p.

By the Sublemma below, we know that any two points in I_p are accessible. Since the accessibility property is symmetric and transitive, we get that any two points in $\operatorname{int} N$ is accessible.

□

LEMMA 4.2. *Let $T = R, Q$ or P. For any $s \in (0,1)$,*

$$\mathcal{A}_T(p,s) \supset I_p, \tag{4.1}$$

where $\mathcal{A}_T(p,s)$ is the set of points accessible to $(p,s) \in N$ with respect to T.

Proof: In the arguments below we only use the sets $\tilde{\Omega}_1$ defined in (4.5) Ω_0 chosen in Lemma 4.5, and the fact that the strong stable and unstable foliations are continuous with the maps. Since Ω_2 and Ω_3 are disjoint with Ω_0 and $\tilde{\Omega}_1$, and $h^{(i)}$ are identity outside Ω_i for $i = 2, 3$, the arguments work for all R, Q and P. So we drop the subscript T in $\mathcal{A}_T(p,s)$ and simply write $\mathcal{A}(p,s)$ instead.

We use the coordinate system (x, y, t, z) in Ω_1 described as we construct $h^{(1)}$. Denote $h = h^{(1)}_\alpha$. Since the map h preserves the leaf I_p, we have that

$$h(\overline{0}, z) = (h^1(\overline{0}, z), h^2(\overline{0}, z), h^3(\overline{0}, z), h^4(\overline{0}, z)) = (\overline{0}, h^4(\overline{0}, z))$$

for all $z \in (0,1)$, where $\overline{0} = (0,0,0)$. It suffices to show that for every $z \in (0,1)$,

$$\mathcal{A}(p,z) \supset \{(p,z') : z' \in [(h_t^{-\ell})^4(p,z),z]\}, \tag{4.2}$$

where ℓ is chosen by (4.4). In fact, since accessibility is a transitive relation and $h_t^{-n}(p,z) \to (p,0)$ for any $z \in (0,1)$ as $n \to \infty$, (4.2) implies that $\mathcal{A}(p,z) \supset \{(p,z') : z' \in (0,z]\}$. Since this holds true for all $z \in (0,1)$ and accessibility is a reflective relation, we obtain (4.1).

Now we proceed with the proof of (4.2).

Recall that $p_1, p_2 \in \mathcal{C}'$, $q_1 \in G^{-n_1}V^s(p_1) \cap V^u(p)$ and $q_2 \in G^{n_2}V^u(p_2) \cap V^s(p)$. Also, recall that $\pi V^s(w,T) = V^s(\pi w, G)$ for $w \in V^s(p) \times I$ and $\pi V^u(w,T) = V^u(\pi w, G)$ for any $w \in N$, where $\pi : N = M_0 \times I \to M_0$ is a projection. In other words, $V^s(\pi w, T) \in V^s(w,G) \times I$ and $V^u(\pi w, T) \in V^u(w,G) \times I$. Let $z_0 = z$. We can choose z_i, $i = 1, \cdots, 5$, such that

$$(q_1,z_1) \in V^u((p,z_0),T), \qquad G^{-n_1}V^s((p_1,z_2),T) \ni (q_1,z_1),$$

$$(p_2,z_3) \text{ is accessible to } (p_1,z_2),$$

$$(q_2,z_4) \in G^{n_2}V^u((p_2,z_3),T), \qquad V^s((p,z_5),T) \ni (q_2,z_4).$$

This means that $(p,z_5) \in \mathcal{A}(p,z_0)$.

Note that z_1,z_2,z_4,z_5 are uniquely determined by z_0,z_1,z_3,z_4 respectively. We will show that

1. $z_1 = (h^{-\ell})^4(p,z_0)$;

2. $z_2 = z_1$;

3. z_3 can be chosen arbitrarily close to z_2;

4. $z_4 = z_3$;

5. $z_5 \leq z_4$.

Hence we get

$$z_5 \leq (h_t^{-\ell})^4(p, z_0). \tag{4.3}$$

By continuity, we conclude that

$$\{(p, z') : z' \in [z_5, z_0]\} \subset \mathcal{A}(p, z_0)$$

and (4.2) follows.

By the construction of $h = h_\alpha^{(1)}$, we know that for any $q \in B(p, \varepsilon_2)$,

$$h^4(q, z) = h^4(p, z). \tag{4.4}$$

Recall that p is a periodic point of G with period m. We have

$$T^{-m}(p, z_0) = h^{-1}S^{-m}(p, z_0) = h^{-1}(G^{-m}p, z_0) = (p, (h^{-1})^4(p, z_0)),$$

and therefore for any $k \geq 1$,

$$T^{-km}(p, z_0) = (p, (h^{-k})^4(p, z_0)). \tag{4.5}$$

On the other hand, $G^{-km}q_1 \notin B(p,\varepsilon)$ if $k \le \ell$. It follows that $T^{-km}(q_1,z_1) = S^{-km}(q_1,z_1) = (G^{-km}q_1,z_1)$. Hence, by (4.4),

$$\begin{aligned} T^{-(\ell+1)m}(q_1,z_1) &= h^{-1}S^{-(\ell+1)m}(q_1,z_1) \\ &= h^{-1}(G^{-(\ell+1)m}q_1,z_1) = (q^{(1)},(h^{-1})^4(p,z_1)) \end{aligned}$$

for some $q^{(1)} \in B(p,\varepsilon_2)$ and therefore, for any $k > 0$,

$$T^{-(\ell+k)m}(q_1,z_1) = (q^{(k)},(h^{-k})^4(p,z_1)) \tag{4.6}$$

for some $q^{(k)} \in B(p,\varepsilon_2)$ Since $(q_1,z_1) \in V^u((p,z_0),T)$, by (4.5) and (4.6),

$$d(T^{-km}(p,z_0),T^{-km}(q_1,z_1)) = d((p,(h^{-k})^4(p,z_0)),(q^{(k)},(h^{-k})^4(p,z_1)) \to 0$$

as $k \to \infty$, and the convergence is exponentially fast. So by taking the z component, we have that $|(h^{-k})^4(p,z_0) - (h^{-k+\ell})^4(p,z_1)| \to 0$ converges exponentially fast as $k \to \infty$. On the other hand, if $z' \neq z''$, then $|(h^{-k})^4(p,z') - (h^{-k})^4(p,z'')| \to 0$ with at most at a polynomial rate because both $h^{-k})^4(p,z')$, $(h^{-k})^4(p,z'')| \to (p,0)$, $Dh = \mathrm{id}$ at $(p,0)$ and h is a C^∞ diffeomorphism. So we get $z_1 = (h^{-\ell})^4(p,z_0)$. This proves (1).

By using the same arguments, and the fact that $h^4(0,y,0,z) \ge z$ for any $z \in (0,1)$, we can show that the z-coordinate of h is non-increasing from (p,z_5) to (q_2,z_4) along $V^s((p,z_5),T)$. That is, $z_4 \ge z_5$. This is (5).

Since the periodic orbit $\mathcal{C}'$ and the sets $\cup_{i=0}^{\infty} G^{-n_1+i}V^s(G^{n_1}(p_1),G) \times (0,1)$ and $\cup_{i=0}^{\infty} G^{n_2-i}V^u(G^{-n_2}(p_2),G) \times (0,1)$ are unperturbed, we know that the z-coordinates are constant along the stable leaves

$T^{-n_1}V^s((G^{n_1}(p_1),z),T)$ and the unstable leaves $T^{n_2}V^u((G^{-n_2}(p_1),z),T)$. So we get $z_1 = z_2$ and $z_3 = z_4$, which are (2) and (4).

Now we prove (3).

Denote by $\mathcal{C}'(p_1p_2)$ the part of closed orbit $\mathcal{C}'$ from p_1 to p_2. By Lemma 4.5, for any small $\varepsilon > 0$, we can choose a closed orbit $\mathcal{C}'_\varepsilon \subset \Omega_0$ such that a part of $\mathcal{C}'_\varepsilon$ is in the ε-neighborhood of $\mathcal{C}'(p_1p_2)$ in the sense that for any $p' \in \mathcal{C}'(p_1p_2)$, there is $p'' \in \mathcal{C}'_\varepsilon$ with $d(p',p'') \leq \varepsilon$.

Recall that $H = H_{\mathcal{C}'\mathcal{C}'_\varepsilon}$ is a map from $\mathcal{C}'(p_1p_2)$ to $\mathcal{C}'$ given in Section 7. Starting from p_1, we can apply the map consequently to get a sequence of points $H^{(1)}(p_1), H^{(2)}(p_1), \cdots$, where $H^{(i+1)}(p_1) = H(H^{(i)}(p_1))$. Let k' be such a number that $H^{(k')}(p_1) \in \mathcal{C}'(p_1p_2)$ and $H^{(k'+1)}(p_1) \notin \mathcal{C}'(p_1p_2)$. Clearly, p_1 and $p^{(k')}$ can be joint by $4k'$ local stable and unstable manifolds of G at points on either $\mathcal{C}'$ or $\mathcal{C}'_\varepsilon$. Recall that the neighborhoods of both $\mathcal{C}' \times I$ or $\mathcal{C}'_\varepsilon \times I$ are unperturbed, and so are the $4k'$ local stable and unstable manifolds. We know then that (p_1,z_2) is accessible to $(H^{(k')}(p_1),z_2)$ with respect to T.

Note that $H^{(k')}(p_1) \in \mathcal{C}'(p_1p_2)$ and $H^{(k'+1)}(p_1) \notin \mathcal{C}'(p_1p_2)$. By continuity we can find another closer orbit $\mathcal{C}'_\varepsilon$ such that $H_{\mathcal{C}'\mathcal{C}'_\varepsilon}(H^{(k')}(p_1)) = p_2$. Hence there is $z_3 \in (0,1)$ such that $(H^{(k')}(p_1),z_2)$ and (p_2,z_3) are accessible. We can make $H^{(k')}(p_1)$ arbitrarily close to p_2 by taking ε sufficiently small. Also note that the strong stable and unstable manifolds change continuously with respect to the diffeomorphism in the partially hyperbolic systems. If $H^{(k')}(p_1)$ is sufficiently close to p_2, then the 4 strong stable and unstable leaves to construct $H_{\mathcal{C}'\mathcal{C}'_\varepsilon}$ are sufficiently short, and therefore z_3 can be chosen arbitrarily close to z_2. $\square$

4.6 Hyperbolicity of the Map Q

PROPOSITION 4.4. *There exists* $\alpha_0 > 0$ *such that for any* $\alpha \in (0, \alpha_0)$,

$$\int_N \log \eta_{\alpha 0}(w) dw < \lambda^u(S).$$

Proof: Since $h^{(1)}|_{E^{un}(S)} = E^{un}(S)$ for any $w \in N$, given $\alpha > 0$, there exists a unique number $a_\alpha(w)$ such that the vector $v_\alpha(w) = (1, 0, 0, a_\alpha(w)) \in E^u(Q_{\alpha 0})$. Hence $DQ_{\alpha 0}(w) v_\alpha(w) = (\eta_\alpha(w), 0, 0, \eta_\alpha a_\alpha(w))$ for some $\eta_\alpha > 1$. The expanding rate of $DQ_{\alpha 0}$ along its unstable direction is

$$\eta_{\alpha 0}(w) = \eta_\alpha(w) \frac{\sqrt{1 + a_\alpha(Q_{\alpha 0}(w))^2}}{\sqrt{1 + a_\alpha(w)^2}}$$

Let $L_\alpha = \int_N \log \eta_{\alpha 0}(w) dw = \int_N \log \eta_\alpha(w) dw$. The second inequality is true because that the map $Q_{\alpha 0}$ preserves the Riemannian volume and therefore $\int_N \log\big(1 + a_\alpha(Q_{\alpha 0} w)^2\big) dw = \int_N \log\big(1 + a_\alpha(w)^2\big) dw$.

$$Dh_\alpha^{(2)}|_{E^{un}} = \begin{pmatrix} A(w,\alpha) & B(w,\alpha) \\ C(w,\alpha) & D(w,\alpha) \end{pmatrix}$$

$$= \begin{pmatrix} r_x \cos\sigma - r\sigma_x \sin\sigma & r_t \cos\sigma - r\sigma_t \sin\sigma \\ r_x \sin\sigma + r\sigma_x \cos\sigma & r_t \sin\sigma + r\sigma_t \cos\sigma \end{pmatrix}$$

where $r_x = \cos\theta$, $r_t = \sin\theta$ and $\sigma = \theta + \alpha\rho(\sqrt{y^2+z^2})\rho(r)$ and then $\sigma_x = \frac{\sin\theta}{r} + \alpha\rho(\sqrt{y^2+z^2})\rho_r'(r)\cos\theta$ and $\sigma_t = \frac{\cos\theta}{r} + \alpha\rho(\sqrt{y^2+z^2})\rho_r'(r)\sin\theta$.

By the same proof as in the proof of Proposition B.6 in [DHP], we can get that

$$\frac{dL_\tau}{d\tau}\Big|_{\tau=0} = 0, \qquad \frac{d^2L_\tau}{d\tau^2}\Big|_{\tau=0} < 0.$$

So we can choose $\alpha_0 > 0$ so small that $L_\alpha < \log\eta$ for any $\alpha \in (0,\alpha_0)$. □

4.7 Hyperbolicity of the Map P

PROPOSITION 4.5. *For almost every $w \in N$, $v \in E^{ucn}(x,S)$, $\chi(x,v,P) > 0$.*

Proof: Denote $\Delta_0^* = \Delta_0' \cap \Lambda$. Then set

$$U_1 = P^{-K}\Delta_0^*, \quad U_2 = \Delta_0 \backslash \Delta_0^*, \quad U_3 = \Delta_{k_0} \backslash P^{k_0}\Delta_0^*, \quad U_4 = P^{-K}(\Delta_0 \backslash \Delta_0^*).$$

Clearly, $\{U_i, i = 1,2,3,4\}$ are pairwise disjoint. Let $U = U_1 \cup U_2 \cup U_3 \cup U_4$. Note that $\Delta_0^* \supset \Gamma \cap \Delta_0'$. Let $\bar{P} = P^\tau : U \to U$ be the first return map of P, where $\tau = \tau(w)$ is the first return time of $w \in U$.

In the proof below, for any $w \in U$, we always assume $v \in E^{ucn}(w)$.

If $w \in U_1$, then $\tau(w) \geq 2K + k_0$. By Lemma 4.3 and (4.9),

$$\log \|D\bar{P}_w(v)\| \geq 0.9K\lambda - 0.5\log 2 + \log\|v\| \geq 0.5K\lambda + \log\|v\|. \tag{4.1}$$

By (4.14),

$$\mu U_1 \geq \mu(\Delta_0^*) \geq \mu(\Gamma \cap \Delta_0') \geq \mu(\Delta_0") - \mu(\Gamma\Delta\Delta_0')$$

$$\geq \tfrac{3}{4}\mu\Delta_0 - \mu(\Gamma_0 \triangle \Delta_0) \geq (0.75 - 0.05)\mu\Delta_0 = 0.7\mu\Delta_0. \tag{4.2}$$

Note that $\|DP - \mathrm{id}\| \leq \delta$. So if $w \in U_2$, then $\bar{P} = P^{k_0}$ and

$$\log \|D\bar{P}_w(v)\| \geq k_0 \log(1-\delta) + \log \|v\|. \tag{4.3}$$

Also,

$$\begin{aligned}
\mu U_2 &= \mu(\Delta_0 \backslash \Delta_0^*) \leq \mu(\Delta_0 \backslash \Delta_0') + \mu(\Delta_0' \backslash \Delta_0^*) \leq \frac{1}{4}\mu(\Delta_0) + \mu(\Delta_0' \backslash (\Gamma \cap \Delta_0')) \\
&\leq \frac{1}{4}\mu\Delta_0 + \mu(\Gamma_0 \triangle \Delta_0) \leq (0.25 + 0.05)\mu\Delta_0 = 0.3\mu\Delta_0.
\end{aligned} \tag{4.4}$$

Consider the case that $w \in U_3$. We have that $\tau(w) \geq K$ and the orbit $\{w, \cdots, P^{\tau(w)}w\}$ does not intersect Ω_3. It is obvious that $P^i(w) \notin \Omega_3$ for $0 \leq i \leq K$. So if $P^i w \in \Delta_j$ for $0 \leq j \leq k_0 - 1$, then $P^{i-j-K}w \in P^{-K}(\Delta_0 \backslash \Delta_0^*) \cup P^{-K}\Delta_0^*$ and $\tau \leq i$ what is not possible. Then the orbit does not intersect Ω_3 and $\bar{P}_w(v) = Q_w^{\tau(w)}(v)$. Let $\tau'(w)$ be the smallest positive integer such that $P^{\tau'(w)}w \in \Lambda$ for some $0 \leq \tau'(w) \leq \tau(w)$, and let $\tau'(w) = \tau(w)$ if there is no such integer. Denote

$$U_3' = U_3 \cap \{w : \tau(w) - \tau'(w) \geq 0.5K\}, U_3'' = U_3 \cap \{w : \tau(w) - \tau'(w) < 0.5K\},$$

and

$$\hat{U}_3' = \{P^i w : w \in U_3', 0 \leq i < \tau'(w)\}, \hat{U}_3'' = \{P^i w : w \in U_3'', 0 \leq i < \tau'(w)\}.$$

Note that if $n \geq 0.5K$ and $w \in \Lambda$, then $\|DQ_w^n(v)\| \geq \|v\|$ for any $v \in E^{ucn}(w)$. Also note that $P = Q$ on $N \backslash \Omega_3$.

If $w \in U_3'$, then

$$\|D\bar{P}_w(v)\| = \|DP_w^{\tau(w)}(v)\| = \|DQ_{P^{\tau'(w)}(w)}^{\tau(w)-\tau'(w)}(DP_w^{\tau'(w)}(v))\| \geq \|DP_w^{\tau'(w)}(v))\|.$$

Hence,

$$\log\|D\bar{P}_w(v)\| \geq \log\|P_w^{\tau'(w)}(v)\| \geq \sum_{i=0}^{\tau(w)-1} \chi_{\hat{U}_3'}(P^i w)\log(1-\delta) + \log\|v\|, \tag{4.5}$$

where $\chi_\Omega(\cdot)$ is the characteristic function of the set Ω.

If $w \in U_3''$, we denote

$$\tilde{U}_3'' = \{P^i w : \ w \in U_3'', 0 \leq i < \tau(w)\}.$$

Clearly, we have

$$\log\|D\bar{P}_w(v)\| \geq \sum_{i=0}^{\tau(w)-1} \chi_{\tilde{U}_3''}(P^i w)\log(1-\delta) + \log\|v\|. \tag{4.6}$$

Lastly, we consider the case that $w \in U_4$. We have $\tau(w) = K$ and $\bar{P}(w) \in U_2$. We define τ'' and U_4', U_4'', $\hat{U}_4'$, $\hat{U}_4''$ and $\tilde{U}_4''$ in the same way as in the previous case, and get similar inequalities. That is, if $w \in U_4'$, then

$$\log\|D\bar{P}_w(v)\| \geq \sum_{i=0}^{\tau(w)-1} \chi_{\hat{U}_4'}(P^i w)\log(1-\delta) + \log\|v\|, \tag{4.7}$$

if $w \in U_4''$, then

$$\log \|D\overline{P}_w(v)\| \geq \sum_{i=0}^{\tau(w)-1} \chi_{\tilde{U}_4''}(P^i w) \log(1-\delta) + \log\|v\|, \tag{4.8}$$

Clearly, $\hat{U}_3', \hat{U}_3'', \hat{U}_4', \hat{U}_4'' \in \Lambda^c$. So

$$\mu(\hat{U}_3' \cup \hat{U}_3'' \cup \hat{U}_4' \cup \hat{U}_4'') \leq \mu\Lambda^c. \tag{4.9}$$

By the fact that $\tau'', \tau' \geq 0.5K$ on $\tilde{U}_3''$ and $\tilde{U}_4''$, we have

$$\mu(\tilde{U}_3'' \cup \tilde{U}_4'') \leq 2\mu(\hat{U}_3'' \cup \hat{U}_4'') \leq 2\mu\Lambda^c \tag{4.10}$$

Now we estimate the Lyapunov exponent of v at a typical point w. We may assume $w \in U$. Let $n_i = \tau_i(w)$ be the ith return time of w. Then we have

$$\lambda(v,w,P) = \lim_{n\to\infty} \frac{1}{n} \log \frac{\|DP_w^n(v)\|}{\|v\|} = \lim_{i\to\infty} \frac{1}{n_i} \log \frac{\|DP_w^{n_i}(v)\|}{\|v\|}$$

Using the fact that the frequency of the orbit visiting a set U is equal to μU, and noticing (4.1)-(4.10), we have

$$\begin{aligned}
&\lim_{j\to\infty} \frac{1}{n_j} \log \frac{\|DP_w^{n_j} v\|}{\|v\|} = \lim_{j\to\infty} \frac{1}{n_j} \sum_{i=0}^{j-1} \log \frac{\|D\overline{P}_{\overline{P}^i w}(D\overline{P}_w^i(v))\|}{\|DP_w^i(v)\|} \\
\geq\ & \mu U_1 \cdot 0.5K\lambda + \mu U_2 \cdot k_0 \log(1-\delta) + (\mu\hat{U}_3' + \mu\hat{U}_4' + \mu\tilde{U}_3'' + \mu\tilde{U}_4') \log(1-\delta) \\
\geq\ & 0.7\mu\Delta_0 \cdot 0.5K\lambda + 0.3\mu\Delta_0 \cdot k_0 \log(1-\delta) + 3\mu\Lambda^c \cdot \log(1-\delta)
\end{aligned}$$

Using (4.10) and (4.11) we conclude that the right side of the above inequality is greater than

$$0.33\mu\Delta_0 K\lambda - 0.003\tau^2\lambda.$$

Since $\mu\Delta_0 \geq \mu\Gamma - \mu\{\Gamma_0\Delta\Delta_0\} \geq 0.08\tau^2 K^{-1} - 0.05(0.08\tau^2 K^{-1})$ by Lemma 4.4 and (4.14), we conclude finally that $\lambda(v,w,P) > 0$. □

LEMMA 4.3. *Let* $w \in P^{-K}(\Delta_0' \cap \Lambda)$. *Then for any* $v \in E^{ucn}(w,S)$,

$$\|DP_w^\tau(v)\| \geq \frac{\sqrt{2}}{2}\|v\|e^{0.9K\lambda}.$$

Proof: Note that on $\Gamma_{-K,-1}$, $h^{(3)} = \mathrm{id}$, hence $P^K = Q^K$. Also, note that both $Dh^{(1)}$ and $Dh^{(2)}$ preserve the subbundle $E^{un}(S)$, we have $E^{un}(w,Q) = E^{un}(w,S)$.

Write $v = v^{un} + v^c$, where $v^{un} \in E^{un}(w,Q)$ and $v^c \in E^c(w,Q)$.

Assume $\|v^c\| \leq \frac{\sqrt{2}}{2}\|v\|$. Hence $\|v^{un}\| \geq \frac{\sqrt{2}}{2}\|v\|$. Since $DQ^K(v^{un}) \in E^{un}(Q^K w,Q)$ and $Q^K w \in \Lambda$, we have

$$\|v^{un}\| = \|DQ^{-K}(DQ^K v^{un})\| \leq \|DQ^K v^{un}\|e^{-0.9K\lambda}.$$

Hence,

$$\|DP^K v\| = \|DQ^K v\| \geq \|DQ^K v^{un}\| \geq \|v^{un}\|e^{0.9K\lambda} \geq \frac{\sqrt{2}}{2}\|v\|e^{0.9K\lambda(Q)}.$$

Note that at $P^K w, \cdots, P^{K+k_0-1}w$, $Dh^{(3)}$ is a rotation, and $DQ|E^{ucn}(P^i w) = DS|E^{ucn}(P^i w)$ is non-contracting for $i = K, \cdots, K+k_0-1$. So $DP^{k_0}|E^{ucn}(P^K w)$ is noncontracting. Further, since $\{P^i w\}_{i=K+k_0}^{\tau} \cap \Omega_3 = \emptyset$ and $P^{K+k_0}w \in \Lambda'$, we have that $DP^{\tau-(K+k_0)}|E^{un}(P^{K+k_0}w) = DQ^{\tau-(K+k_0)}|E^{un}(P^{K+k_0}w)$

is expanding, and $DP^{\tau-(K+k_0)}|E^{ucn}(P^{K+k_0}w)$ is noncontracting. So we have

$$\begin{aligned}\|D\bar{P}(v)\| = &\quad \|DP^{\tau-(K+k_0)}_{P^{K+k_0}w}(DP^{K+k_0}_w(v)\| = \|DQ^{\tau-(K+k_0)}_{P^{K+k_0}w}(DP^{K+k_0}_w(v))\| \\ \geq &\quad \|DP^{K+k_0}_w(v)\| \geq \|DP^K_w(v)\| = \|DQ^K_w(v)\| \geq \tfrac{\sqrt{2}}{2}\|v\|e^{0.9K\lambda}.\end{aligned}$$

Now we consider the case that $\|v^c\| \geq \frac{\sqrt{2}}{2}\|v\|$.

Note that $DQ^K(v^c) \in E^c(Q^K w, Q)$. By the construction of $h^{(3)}$, we can see that $DP^{k_0}(Q^K w)$ rotate the vector in $E^{cn}(Q^K w, S)$ by $\pi/2$. It means that $DP^{K+k_0}(v^c) = DP^{k_0}(DQ^K(v^c)) \in E^{un}(P^{K+k_0}w, Q)$. Using the fact that $P^{K+k_0}w \in \Lambda$, we have

$$\|D\bar{P}(v^c)\| = \|DP^{\tau-(K+k_0)}_{P^{K+k_0}w}(P^{K+k_0}_w(v^c)\| \geq \|DP^K(DP^{K+k_0}(v^c))\|$$

$$\geq \|DP^{K+k_0}(v^c)\|e^{0.9K\lambda} \geq \|v^c\|e^{0.9K\lambda} \geq \frac{\sqrt{2}}{2}\|v\|e^{0.9K\lambda}.$$

This is the result. □

LEMMA 4.4. $\mu\Gamma \geq 0.08\tau^2 K^{-1}$.

Proof: Let

$$\hat{\Gamma}' = \bigcup_{i=0}^{5\tau^{-1}K-2K-k_0} Q^i\Gamma'.$$

Since $\tau^{-1} \geq 2$ and $K \geq 2k_0$, we have

$$\frac{\mu\hat{\Gamma}'}{\mu\overline{\Gamma}'} = \frac{5\tau^{-1}K - 2K - k_0 + 1}{5\tau^{-1}K - 2K - 1 + K + 1} = 1 - \frac{K - k_0 + 1}{5\tau^{-1}K - K} \geq 1 - 0.5\tau.$$

By (4.13), $\mu\hat{\Gamma}' = (1-0.5\tau)\mu\overline{\Gamma}' \geq 1-0.6\tau$. Then by (4.12),

$$\mu(\hat{\Gamma}\backslash\Omega) \geq 0.4\tau.$$

For $w \in \Gamma'$, we denote $O(w) = \{Q^i w : i = 0, \cdots, (5\tau^{-1}-2)K - k_0\}$, the piece of orbit that start at w from time 0 to $(5\tau^{-1}-2)K - k_0$. Let

$$\Gamma'_a = \{O(w) : w \in \Gamma', O(w) \cap \Omega^c \neq \emptyset\},\ \Gamma'_b = \{O(w) : w \in \Gamma', O(w) \cap \Omega^c = \emptyset\}.$$

Clearly, $\{\Gamma'_a, \Gamma'_b\}$ forms a partition of $\hat{\Gamma}'$, and $\Gamma'_b \subset \Omega$ and therefore by (4.12),

$$\mu\Gamma'_a = \mu\hat{\Gamma}' - \mu\Gamma'_b \geq \mu\hat{\Gamma}' - \mu\Omega \geq (1-0.6\tau) - (1-\tau) = 0.4\tau.$$

Note that Γ consists of exactly one point from each orbit $O(w)$ in Γ_a. We get

$$\mu\Gamma \geq \frac{\mu\Gamma_a}{(5\tau^{-1}-2)K - k_0 + 1} \geq \frac{0.4\tau}{(5\tau^{-1}-2)K} \geq \frac{0.4\tau}{5\tau^{-1}K} = 0.08\tau^2 K^{-1}$$

This is the result. □

4.8 Properties of geodesic flows

Let $g^t : M_0 \to M_0$ be the geodesic flow on a compact surface of a negative constant curvature. We list some properties of the flow here.

1) $d(g^t x, x) \leq |t|$ for any $t \in \mathbb{R}$ and $x \in M_0$;

2) g^t is a uniformly hyperbolic flow, that is, there is a decomposition of the tangent bundle into

$$TM_0 = E^u \oplus E^s \oplus E^c$$

and a constant $\tilde{\eta} > 1$ such that for any $z \in M_0$,

$$|Dg_z^t(v)| \geq \tilde{\eta}^t |v| \qquad v \in E_z^u,$$

$$|Dg_z^{-t}(v)| \geq \tilde{\eta}^{-t} |v| \qquad v \in E_z^s,$$

and E^c is the one dimensional bundle tangent to the flow.

3) The closed orbits are dense in M_0. Moreover, for any closed orbit $\mathcal{C}$, both $W^u(\mathcal{C}) = \cup_{z\in\mathcal{C}} W^u(z)$ and $W^s(\mathcal{C}) = \cup_{z\in\mathcal{C}} W^s(z)$ are dense in M_0.

4) g^t preserves the Riemannian volume and for any $t \neq 0$, g^t is ergodic with respect to the volume,

5) g^t has the accessibility property. That is, any two points can be joint by a piecewise differentiable piecewise nonsingular path which consists of segments tangent to either E^u or E^s.

6) g^t is topologically conjugate to a symbolic flow that is a suspension of a subshift finite type with a continuous roof function. More precisely, there is a symbolic space Σ_A, two sided left shift $\sigma_A : \Sigma_A \to \Sigma_A$, a continuous function $\iota : \Sigma_A \to \mathbb{R}^+$ and a finite to one map $\pi : \Sigma_A^t \to M_0$ such that

$$\pi \circ g^t = \overline{\sigma}_A^t \circ \pi,$$

where

$$\Sigma_A^t = \{(\underline{w}, t) \in \Sigma_A \times \mathbb{R}\} / \{(\underline{w}, \iota(\underline{w})) = (\sigma_A(\underline{w}), 0)\ \},$$

$$\overline{\sigma}^t_A(\underline{w}, s) = (\underline{w}, s+t).$$

Moreover, $p \in M_0$ is periodic under g^t if and only if $\pi^{-1}(p)$ is periodic under $\overline{\sigma}^t_A$. Also, Σ_A can be chosen in such a way that the size of πR_α can be arbitrarily small, where $R_\alpha = \{w = \cdots w_{-1} w_0 w_1 \cdots : w_0 = a\}$ is a cylinder in Σ_A. (See [B] and [BR] for more details.)

DEFINITION 9. *Let $\mathcal{C}_\varepsilon$ and $\mathcal{C}'$ be two closed orbits, $p \in \mathcal{C}'$, $x \in \mathcal{C}_\varepsilon$ and $d(p,x)$ is small. Define the holonomy $H_{\mathcal{C}',\mathcal{C}_\varepsilon} : B_{\mathcal{C}'}(p,\delta) \to \mathcal{C}'$ where $B_{\mathcal{C}'}(p,\delta)$ is a δ-neighborhood of point p in $\mathcal{C}'$. $H_{\mathcal{C}',\mathcal{C}_\varepsilon}(x_0)$ is constructed the following way (see Figure 2):*

1. $x_1 = W^s(x_0) \cap W^{uc}(x)$
2. $x_2 = W^u(x_1) \cap W^{cs}(x) \subset W^u(x_1) \cap \mathcal{C}_\varepsilon$
3. $x_3 = W^s(x_2) \cap W^{cu}(x_0)$
4. $x_4 = W^u(x_3) \cap W^{cs}(x_0) \subset W^u(x_3) \cap \mathcal{C}'$
5. $H_{\mathcal{C}',\mathcal{C}_\varepsilon}(x_0) = x_4$.

Note that for any point $x_0 \in \mathcal{C}'$, $H_{\mathcal{C}',\mathcal{C}_\varepsilon}(x_0) \neq x_0$ due to the Anosov's theorem (see e.g. [A1]).

LEMMA 4.5. *Let $\mathcal{C}'$ be a given closed orbits of g^t. For any $\tau_1 \in (0,1)$, there is an open subset $\Omega_0 \subset M_0$ containing $\mathcal{C}'$ with $\mu\Omega_0 \leq \tau_1$ and there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there is a closed orbit $\mathcal{C}_\varepsilon$ with $d(\mathcal{C}', \mathcal{C}_\varepsilon) < \varepsilon$ and $B(\mathcal{C}_\varepsilon, \varepsilon) \subset \Omega_0$, where $B(\mathcal{C}_\varepsilon, \varepsilon)$ denotes the ε neighborhood of $\mathcal{C}_\varepsilon$ in M_0.*

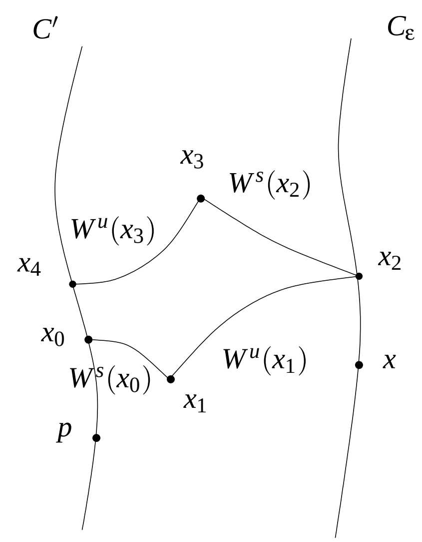

Fig. 4.2. Holonomy on a closed orbit.

Proof: Let $\sigma_A : \Sigma_A \to \Sigma_A$ be a symbolic flow that is conjugate to $g^t : M_0 \to M_0$ with the conjugacy $\pi : \Sigma_A \to M_0$.

For a word $W = s_i \cdots s_j$, $i \leq j$, we denote by $\mathcal{C}(W)$ the periodic orbit of σ_A, and by $\mathcal{R}(W)$ the cylinder determined by W. Abusing notations we denote by $\pi\mathcal{C}(W)$ the corresponding closed orbit of g^t and by $\pi\mathcal{R}(W)$ the corresponding set in M_0. More precisely, the latter means the set

$$\{(\pi\underline{w}, t) \in M_0 : \underline{w} \in \mathcal{R}(W), 1 \leq t \in \iota(\underline{w})\}.$$

Let W' be the word such that $\pi\mathcal{C}(W') = \mathcal{C}'$ and W_a be its subword. Take another word W^* that generates a periodic orbit of σ_A and contains the same subword W_a Such word W^* exists since the periodic orbits of σ_A are dense. We may assume that $W' = W_a W_b$ and $W^* = W_a W_c$ for some word W_b and W_c. It implies that one of the words W' and W^* can be followed by another.

Note that the maximal volume of all sets of the form $\pi\mathcal{R}(W(n))$ converges to 0 exponentially fast, where $W(n)$ is a $(2n+1)$-word of the form $s_{-n} \cdots s_n$. For any $\tau_1 > 0$, we can take $n > 0$ such that $\mu\big(\pi\mathcal{R}(W(n))\big) \leq \tau_1/(4n+2)$ for any $(2n+1)$-word $W(n)$. We may assume $n > |W'|, |W^*|$, where $|W|$ denotes the length of W. Note that for any integer $k \geq n/|W'|$, there are at most $4n+2$ different $(2n+1)$-words of the form $W(n)$ in the orbit of $\underline{w} = \mathcal{C}((W')^k W^*)$, where W^k is the word consists k consecutive W. Let $W_1, \cdots, W_j$, $j \leq 4n+2$, denote these words.

Set

$$\hat{\Omega}_0 = \bigcup_{i=1}^{j} \pi\mathcal{R}(W_i).$$

Clearly, $\mu\hat{\Omega}_0 \leq \tau_1/2$. Choose $\varepsilon_0 > 0$ such that $\mu B(\hat{\Omega}_0, \varepsilon_0) \leq \tau_1$. Then we set $\Omega_0 = B(\hat{\Omega}_0, \varepsilon_0)$.

Let $\varepsilon \in (0, \varepsilon_0)$. We can a word of the form $(W')^k W^*$ for some large enough k, and then take $\mathcal{C}_\varepsilon = \pi(\mathcal{C}((W')^k W^*)$. Clearly, $\mathcal{C}_\varepsilon \subset \hat{\Omega}_0$, and therefore $B(\mathcal{C}_\varepsilon, \varepsilon) \subset \Omega_0$ by the choice of Ω_0.

Also if k is large enough, then the distance between $\mathcal{C}_\varepsilon$ and $\mathcal{C}'$ can be made arbitrarily small. Hence we have $\mathcal{C}' \subset B(\mathcal{C}_\varepsilon, \varepsilon)$. □

LEMMA 4.6. *For any $\gamma > 0$ and $\tau_0 \in (0,1)$, we can find a constant $\theta_0 > 0$ such that for any number $\theta \in [0, \theta_0]$, any set of the form $\Delta = \Delta_{ss's''} = B^u(x,s') \times B^s(y,s'') \times B^{cn}((t,z),s)$, where $s, s'' \geq s$, there exists a set $\Delta^{(0)}$ and a map $h : N \to N$ with the following properties:*

(a) $h = T_\theta$ *on* $\Delta^{(0)}$, *and* $h = \mathrm{id}$ *on* Δ^c*;*

(b) $\mu\Delta^{(0)}/\mu\Delta \geq \tau_2$*;*

(c) $\|h - \mathrm{id}\| \leq \gamma$.

where T_θ is a rotation given by

$$T_\theta(x,y,t,z) = (x, y, t\cos\theta - z\sin\theta, t\sin\theta + z\cos\theta).$$

Proof: Take $\kappa > 0$ such that $\mu\Delta_{1-\kappa,1-\kappa,1-\kappa}/\mu\Delta_{111} \geq \tau_2$. Hence, for any $r > 0$, $r', r'' > r$, we have $\mu\Delta_{r-\kappa r, r'-\kappa r, r''-\kappa r}/\mu\Delta_{rr'r''} \geq \tau_2$, since $r'/(r'-\kappa r)$ and $r''/(r''-\kappa r)$ are increasing.

Take a family of C^∞ functions $\zeta_r = \zeta_r(s) : \mathbb{R}^+ \to \mathbb{R}^+$, for $r \geq 1$ such that

1 $\zeta_1(s) = 1$ if $s \in [0, 1-\kappa]$ and $\zeta_1(s) = 0$ if $s \geq 1$;

2 $\zeta_r(s) = 1$ if $s \in [0, r-1)$ and $\zeta_r(s) = \zeta_1(s-r+1)$ if $s \geq r-1$.

Clearly, ζ_r have that same C^∞ norm for all $r \geq 1$.

Take coordinate system $w = (x,y,t,z)$ as in (4.3). Then we define $h : \Delta \to \Delta$ by

$$h(w) = T_{\theta(s,s',s'')}(w), \qquad \theta(s,s',s'') = \theta\zeta_{s'/s}(x/s)\zeta_{s''/s}(y/s)\zeta_1(\sqrt{t^2+z^2}/s).$$

By the construction, we see that h satisfies (a) and (b). For (c), note that if $\theta = 0$, then $h = \mathrm{id}$, and note that the C^1 norm of h change smoothly with θ, we get the result. □

THEOREM 4.6. *There exists a C^∞ diffeomorphism of the three dimensional torus $\mathbf{T}^3$ such that*

- *f preserves the Riemannian volume on $\mathbf{T}^3$.*
- *f has countably many ergodic components which are open $(\mod 0)$.*
- *$f|_{\mathbf{T}^2\times K} = A \times Id|_{\mathbf{T}^2\times K}$, where A is a linear hyperbolic diffeomorphism $A : \mathbf{T}^2 \longrightarrow \mathbf{T}^2$ and K is the standard middle-third Cantor set.*

Proof. We will use the same argument as in [DHP] in construction of a volume preserving map with infinitely many ergodic components, modifying it to suit our case. Let A be a linear hyperbolic diffeomorphism. We may assume that A has two fixed points (otherwise consider a power of A). We will get the desired map f by countably many small perturbations of $F = A \times Id : \mathbf{T}^3 = \mathbf{T}^2 \times S^1 \longrightarrow \mathbf{T}^3$.

Consider the Cantor set $K = I - \cup_{n=1}^{\infty} \cup_{i=1}^{2^{n-1}} I_{ni}$, where $I = [0,1]$, and I_{ni}'s are the open intervals of length $\frac{1}{3^n}$.

Let $L_{ni} : I_{ni} \longrightarrow [0,1]$ be the affine map and $\pi_{ni} = (Id, L_{ni}) : \mathbf{T^2} \times I_{ni} \longrightarrow \mathbf{T^2} \times [0,1]$. By lemma 1 below, for any n there exists a C^∞ volume preserving ergodic diffeomorphism $K_n : \mathbf{T^2} \times [0,1] \longrightarrow \mathbf{T^2} \times [0,1]$ such that:

1. $||F - K_n||_{C^n} \leq e^{-3n^2}$.
2. for all $0 \leq m \leq \infty$ $D^m K_n|_{\mathbf{T^2}\times\{z\}} = D^m F|_{\mathbf{T^2}\times\{z\}}$ for $z = 0,1$.
3. K_n has nonzero Lyapunov exponents μ-almost everywhere.

Let $F = F_0 = A \times Id$. Let us describe a sequence of maps $\{F_n\}_{n=1}^{\infty}$, which converges to the desired map f. At each step n: $F_n = F_{n-1}|_{I-\{I_{ni}\}_{i=1}^{2i-1}}$ and $F_n|_{I_{ni}} = \pi_{I_{ni}}^{-1} \circ K_n \circ \pi_{Ini}$. Then for every $n > 0$ and $0 \leq m \leq n$

$$||D^m F_n - D^m F_{n-1}|| = ||D^m F_n - D^m F_{n-1}||_{\cup_{i=1}^{2^{n-1}} I_{ni}} =$$

$$= ||D^m(\pi_{ni}^{-1} \circ K_n \circ \pi_{ni}) - D^m F||_{\cup_{i=1}^{2^{n-1}} I_{ni}} = ||D^m(\pi_{ni}^{-1} \circ (K_n - F) \circ \pi_{ni})||_{\cup_{i=1}^{2^{n-1}} I_{ni}} \leq$$

$$\leq e^{-3n^2} \cdot 3^{nm} \leq e^{-n^2} \cdot e^{-2n^2} \cdot 3^{n^2} \leq e^{-n^2}$$

. The series $\sum_{n=1}^{\infty} e^{-n^2}$ converges, so F_n converges uniformly to a C^∞ map $f = \lim_{n\to\infty} F_n$. Notice that for any n, $F_n|_{\mathbf{T}^2 \times K} = f|_{\mathbf{T}^2 \times K} = A \times Id$.

LEMMA 4.7 (DHP). *For any $k \geq 2$ and $\delta > 0$, there exists a map g of $M = \mathbf{T}^2 \times I$ such that:*

- *g is a C^∞ volume preserving diffeomorphism of M.*
- $||F - g||_{C^k} \leq \delta$.
- *for all $0 \leq m \leq \infty$, $D^m g|_{\mathbf{T}^2 \times \{z\}} = D^m F|_{\mathbf{T}^2 \times \{z\}}$ for $z = 0, 1$.*
- *g is ergodic with respect to the Riemannian volume.*
- *g has non-zero Lyapunov exponent μ-almost everywhere.*

Chapter 5

Stable Accessibility of Suspension Flow

Denote by $C^r_+(M)$ the space of $r-$smooth positive functions on a manifold M.

PROPOSITION 5.1. *Let h be an Anosov measure preserving map on a compact manifold M. Then there is an open and dense set $U \subset C^r_+(M)$, such that for any $f \in U$, the suspension flow of h with the roof function f is stably accessible.*

Proof: Since h is an Anosov measure preserving map, we may assume that h is accessible and ergodic. Let $f > 0$, $f \in C^r(M)$.

Let N be the suspension manifold of h, $N = \{(x,t) : x \in M,\ t \in [0, f(x)]\}/\sim$, where $\sim$ is the identification $(x, f(x)) = (h(x), 0)$ for $x \in M$. Let F be the corresponding suspension flow. Denote the projection $\pi : N \to M$, $\pi(x,t) = x$, $x \in M$, $0 \le t < f(x)$. The projection π is discontinuous at the "glued" points of N, namely at points $\{(x,0) : x \in M\}$. Let τ be the projection in the time direction $\tau : N \to M$, $\tau(x,t) = t$ for $x \in M$, $0 \le t < f(x)$.

Note that $\pi(W^{u,s}_F(x_0,t_0)) \subset \cup_k W^{u,s}_h(h^k(x_0))$.

We denote by $V^s_h(x_0)(\varepsilon) = V^s_h(x_0)$ the connected component of $W^s_h(x_0)$ contained in $B(x_0,\varepsilon)$. Similarly, $V^s_F(x_0,t_0)$ is the connected component of $W^s_F(x_0,t_0)$ such that $\pi(V^s_F(x_0,t_0)) \subset B(x_0,\varepsilon)$.

Let $(x_0,t_0) \in N$, $0 < t_0 < f(x_0)$. Then there exists a heighborhood $B(x_0,\varepsilon)$ of x_0 such that $\tau(V^{u,s}_F(x_0,t_0)(\varepsilon)) \subset (0, f(x_0))$. In other words, $V^{u,s}_F(x_0,t_0)$ does not reach top or bottom of N.

When we start at a point (x_0,t_0) and follow the stable foliation, the π projection follows $W_h^s(x_0)$ till $W_F^s(x_0,t_0)$ reaches "top" or "bottom", that is when τ becomes $f(x)$ or 0. At this point the π projection "jumps" to $W_h^s(h(x_0))$ or to $W_h^s(h^{-1}(x_0))$ and so on.

Since h is accessible, one can connect any two points $x,y \in M$ by a *us*-path. It follows that any point $(x,t) \in N$ can be connected with the point (y^k,t') where $y \in M$ for some $k \in \mathbf{Z}$. In other words orbits of F are accessible from each other. For the accessibility of F it suffices to show that the points on one particular orbit are accessible from each other. We will show that the point $(x,t) \in N$ is accessible from any point (x,t'). Since the unstable and stable foliations are invariant under F^t and accessibility is transitive, we get the accessibility of the entire orbit. The global accessibility follows.

Let q_1 be a periodic points of h and $w = (q_1,t_0) \in \pi^{-1}(q_1)$, $0 < t_0 < f(q_1)$. Pick a neighborhood $B(q_1,\varepsilon)$ of q_1 satisfying the properties stated above for w. Since h posesses a dense set of periodic points, it is possible to find a periodic point q_2 which belongs to $B(q_1,\varepsilon)$. We assume that $B(q_1,\varepsilon)$ is small enough so that the stable and unstable foliations form a product and $V_h^u(q_1) \bigcap V_h^s(q_2), V_h^s(q_1) \bigcap V_h^u(q_2) \neq \emptyset$.

Let p be the *us*-path $[x_0,\ldots,x_4]$ such that $x_{0,4} = q_1$, $x_2 = q_2$ and $x_{i+1} \in W_h^{u,s}(x_i)$ for $i = 0,\ldots,4$. Without loss of generality we can assume that $x_1 \in W_h^s(q_1)$.

Denote by $\tilde{p}$ the *us*-path $z_0,\ldots,z_4$ such that $\pi(\tilde{p}) = p$ and $z_0 = w = (x_0,t_0)$. The points $z_i = (x_i,t_i)$ for some t_i.

If $z_4 \neq w$ (say $t_4 > t_0$) then F is accessible. Indeed, by continuity of stable and unstable foliations we obtain that for that for any s, $t_0 < s < t_4$ the point (q_1,s) is accessible from w. The desired result follows.

If $z_4 = w$ then we are going to perturb f to get the necessary accessibility.

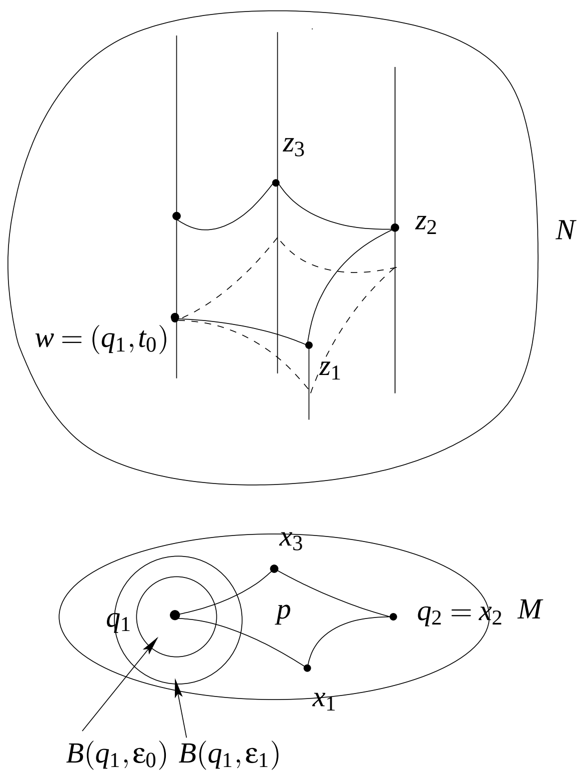

Fig. 5.1. Accessibility of a suspension flow

Let n be the period of q_1. Let $B(q_1,\varepsilon_0)$ be a neighborhood of q_1, such that $B(q_1,\varepsilon_0)\cap h^i(B(q_1,\varepsilon_0))=\emptyset$ for $i=1,\dots,n-1$ and

$$B(q_1,\varepsilon_0)\bigcap h^i(V^s(q_2))=\emptyset \text{ for } i\geq 0$$

$$B(q_1,\varepsilon_0)\bigcap h^{-i}(V^u(q_2))=\emptyset \text{ for } i\geq 0$$

Let nk be the smallest positive number such that $h^{nk}(x_1)\in B(x_0,\varepsilon_0)$ and $h^{n(k-1)}(x_1)\notin B(x_0,\varepsilon_0)$. Pick a number $\varepsilon_1<\varepsilon_0$ such that $h^{nk}(x_1)\in B(x_0,\varepsilon_1)$.

Define function

$f_1=f$ on $M\setminus B(q,\varepsilon_0)$ and $f_1=f-\psi(\rho(q_1,x))$

where $\rho(q,x)$ is the distance between q and x and ψ is a C^∞ function, $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such that

1. $\psi(x)=0$ if $x\geq\varepsilon_0$
2. $\psi(x)>0$ if $x<\varepsilon_0$
3. $\psi(x)=C$ if $x\leq\varepsilon_1$
4. $||\psi(x)||_{C^r}<\delta$

Let $\tilde{p}'$ be the *us* path $[z'_0,\dots,z'_4]$ of the suspension with roof function f_1 such that $\pi(\tilde{p}')=p$ and $z'_0=z_0$.

We claim that $z'_4\neq w$.

Note that if $(x,t)\in W^s_F(x_0,t_0)$ then $(x,t+\xi)\in W^s_F(x_o,t_0+\xi)$. In other words, $W^s_F(x_0,t)$ "slides upwards" with increasing t.

The parts of $\tilde{p}$ that belong to $W_F^u(z_2)$ and $W_F^s(z_2)$ are not disturbed by changing f to f_1. Then $t_3 - t_1 = t_3' - t_1'$.

LEMMA 5.1. $t_1 - t_0 < t_1' - t_0' - C$.

Proof. Denote $W_f^s(w)$ the stable foliation at point w of suspension with roof function f. Since $f(x) = f_1(x) + C$ for $x \in B(x_0, \varepsilon_1)$, it is easy to see that the unstable and stable foliations $V_f^{u,s}(x_0,t)$ and $V_{f_1}^{u,s}(x_0,t)$ coincide as long as $\pi(V_f^{u,s}(x_0,t)) \subset B(x_0,\varepsilon_1)$.

Let T be the return time for a point z_0 for the suspension F_f. Then $F_f^{kT}(z_0) = z_0$ and $F_f^{kT}(z_1) = z' \in W_f^s(z_0)$. By the choice of ε_0 and ε_1, $\pi(z') \in B(x_0,\varepsilon_1)$. Now, after the perturbation, $F_{f_1}^{kT}(z_0)) = (x_0, t_0 + kC)$ and $F_{f_1}^{kT}(z_1) = z'$. Since $V_f^s(x_0,t)$ and $V_{f_1}^s(x_0,t)$ coincide in the proximity of the orbit of z_0, we get $z_1 \in W_{f_1}^s(x_0, t_0 - kC)$. It is clear that $t_1' - t_0' = t_1 - t_0 + kC$.

Similar argument shows that $t_4 - t_3 < t_4' - t_3'$.

By lemma 5.1 and due to the fact that $V^{s,u}(z_2)$ are unperturbed we conclude that

$$t_4' - t_0' = t_4' - t_3' + t_3' - t_1' + t_1' - t_0' >$$

$$t_4 - t_3 + t_3 - t_1 + t_1 - t_0 + C =$$

$$t_4 - t_0 + C = C > 0$$

Since unstable and stable foliations change continuously with f, one can easily see that accessibility holds for any small perturbation of f_1.

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