THREE APPLICATIONS OF RANDOM MATCHING MODELS OF MONEY

A Thesis in Economics

by

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Abstract

Chapter 1: Previous work on the denomination structure of currency treats as exogenous the distribution of transactions and the denominations held by people. Here, by way of a matching model, both are endogenous. In the model, trades in pairwise meetings alternate in time with the opportunity to freely choose a portfolio of denominations and people trade off the benefits of small-denomination money for transacting against the costliness of carrying a large quantity of small-denomination money. For a given denomination structure, a monetary steady state is shown to exist. The model implies that too small denominations are abandoned.

Chapter 2: There is wide agreement that currency was not available in conveniently small denominations prior to the 19th century. Parameterized versions of a matching model of money are used to provide estimates of the benefit for trade of additional divisibility. That benefit is balanced against the cost of providing additional divisibility, the cost as inferred from historical reports, to find the implied optimal degree of divisibility. Although the optima are sensitive to the specification of the matching model, all display a degree of divisibility that is much lower than what we see in modern economies.

Chapter 3: Previous studies that compare a uniform money with separate monies used models in which money is the only asset and in which individuals hold either zero or one unit of money. Here, the comparison is made using a model in which money coexists with a higher-return asset and in which individuals are permitted to hold richer portfolios of assets. The results show that a general conclusion is not possible. A uniform money has a higher expected utility than do separate monies in many examples. However, when the discount on bonds and uncertainty about the
nationality of the trading partner are sufficiently high, then there are examples in which separate monies are better.
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Chapter I

Modeling Denomination Structures

(Joint with Neil Wallace and Tao Zhu.)

I.1 Introduction

The small literature on denomination structures for currency treats as exogenous both the distribution of transactions and the portfolios of denominations held by transactors. [4] starts from a problem in number theory.

The problem of Bächtet seeks the smallest number of weights capable of weighing any unknown integer. ... The version pertinent to this paper allows the unknown weights to be placed in either pan... .The solution is weights that are powers of three, namely, 1, 3, 9, 27 and so on....

[For the currency version,] the unknown amount to be weighed corresponds to a ... transaction. ...Allowing weights to go in either pan corresponds to making change. Finally, that the weights must be capable of weighing any quantity between one and [some] upper bound corresponds to the assumption that a transaction is equally likely to be anywhere between one and some finite upper bound ([4] pages, 425-26).

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Bâchéts problem is a one-person problem and its extension to multiple trans-
actors, at least a buyer and a seller, is far from straightforward even given the
exogeneity of the distribution of transactions.\footnote{That is, how are portfolios of denomina-
tions determined? One interpretation is that there is a buyer and a seller and that condi-
tional on the transaction to be accomplished, the weights (denominations) that solve Bâchéts problem are distributed (!) between them in a way that allows the transaction to be carried out.} \cite{7} also assume a uniform distri-
bution of transactions. Their criterion is minimization of the average number of
monetary items exchanged.\footnote{They assume that the portfolios of denomina-
tions held by individuals do not limit what can be used in trade. For another discussion of the different criteria used in \cite{4} and \cite{7}, see \cite{6}.} They show that a powers-of-two structure, 1, 2, 4, ..., can result in a lower average number of monetary objects used in transactions than a powers-of-three structure.

The main limitation of the above work is that too much is taken to be exogenous.
As regards portfolios of denominations, people should choose them—presumably,
before they know the transaction size. There even seems to be a strategic consid-
eration; if everyone else has small change, then why should I carry small change?
As regards the distribution of transactions, it should be allowed to adapt to the
denomination structure—through “rounding” or lotteries.

Here, we formulate a model in which both the distribution of transactions and
the portfolios of denominations held are endogenous. The model is simple in that
it contains no more than the ingredients that we think are necessary for a coherent
model: indivisibility of monetary items, a beneficial role for low-value items in trade,
and costs that penalize the holding of many low-value items. We use a matching
model of money—a model in which trades of money for goods occur in pairwise
meetings, between a buyer and a seller. Those meetings alternate in time with
the opportunity to costlessly choose a portfolio of denominations—for example, the
opportunity to exchange two $5’s for a $10, and vice versa. The two-person trade
of money for goods is a natural and simple way to represent the beneficial role of
low-value monetary items in trade.

For a given denomination structure, we show that if the costs are sufficiently
small, then there exists a \textit{nice} monetary steady state—nice in that the implied
value function for (monetary) wealth is strictly increasing and concave.\textsuperscript{4} Then we discuss adding fixed societal costs per denomination to the model and what our model says about denominations that are too small.

I.2 The Model

The background environment is borrowed from [3] and [5]. Time is discrete. There is a unit measure of each of \( N \geq 3 \) types of infinitely lived agents and there are \( N \) distinct produced and perishable types of divisible goods at each date. A type \( n \) agent, \( n \in \{1, 2, ..., N\} \), produces type \( n \) good and consumes only type \( n + 1 \) good (modulo \( N \)). Each person maximizes expected discounted utility with discount factor \( \beta \in (0, 1) \). Trading histories are private so that any kind of borrowing and lending in pairwise meetings is impossible.

Let \( s = (s_1, s_2, ..., s_K) \) be a denomination structure of the currency. Here \( s_k \) is the size of the \( k \)-th smallest denomination. We assume that \( 1/s_1 \) is a positive integer, that \( s_k/s_1 \) is an integer, and that \( s_k > s_{k-1} \). (For example, in the powers-of-three structure, \( s_k = 3s_{k-1} \).) The generic symbol for a person’s portfolio is \( y = (y_1, y_2, ..., y_K) \), where \( y_k \geq 0 \) is the integer quantity of size-\( s_k \) money held. The wealth implied by \( y \) is the inner product of \( s \) and \( y \).

We adopt the following sequence of actions for a period. First, each person gets to choose a portfolio of denominations subject only to a wealth constraint. After the portfolio is chosen, a person meets another person at random. We assume that trading partners see each other’s portfolios and types. There are no-coincidence meetings during which nothing happens. In single-coincidence meetings, we assume that the buyer (consumer) makes a take-it-or-leave-it offer. (This offer can involve a demand for change—for some of the monetary items that are part of the portfolio initially held by the seller.) For a type-\( n \) person with portfolio \( y \) just prior to pairwise meetings, realized utility in a period is

\[
u(q_{n+1}) - q_n - \gamma \sum_k y_k,
\]

\textsuperscript{4}We often apply the term \textit{concave} to functions defined on discrete subsets of \( \mathbb{R}^n \). Suppose \( X \subset I^n \), where \( I \) is an interval. We say that \( f : X \to \mathbb{R} \) is (strictly) concave if there exists \( g : I^n \to \mathbb{R} \) such that \( f \) is the restriction of \( g \) to \( X \) and \( g \) is (strictly) concave.
where $q_{n+1} \in \mathbb{R}_+$ is consumption of good $n+1$ and $q_n \in \mathbb{R}_+$ is production of good $n$, and $\gamma \geq 0$ is the utility cost of carrying money from the portfolio-choice stage to the pairwise-meeting stage per piece of money. (The linear carrying-cost function could be replaced by a more general class of convex functions of $\sum k y_k$.)

The utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly concave, strictly increasing, continuously differentiable and satisfies $u(0) = 0$ and $u'(\infty) = 0$. In addition, $u'(0)$ is sufficiently large, but need not be infinite.

We formally define and prove existence of a steady state for a version in which people can choose lotteries both at the portfolio choice stage and in meetings. The definition is similar and the existence result holds for a version without lotteries (by allowing randomization over multiple deterministic optima for the individual problems).\footnote{Lotteries do not eliminate the benefits for trade of small-denomination money because the value function turns out to be strictly concave.} To save space, the non lottery version is not presented.

I.3 Existence of a Steady State

We begin by defining a symmetric steady state for a given denomination structure, $s$—symmetric across the $N$ types of agents. To achieve compactness, we assume that an individual's wealth $z \in \{0, s_1, 2s_1, 3s_1, ..., Z\} \equiv Z$, where $Z$ is an integer multiple of $s_1$. We let $\pi$ denote a probability measure on $Z$, where $\pi(z)$ is the fraction of each specialization type with wealth $z$ at the start of a date. As a normalization, we impose $\sum z \pi(z) = 1$. (With this normalization, $1/s_1$ is the ratio of mean wealth to the size of the smallest unit of money and $Z$ is the ratio of the bound on wealth to mean wealth.) We let $Y = \{y = (y_1, y_2, ..., y_K) \in \mathbb{Z}_+^K : sy \leq Z\}$ be the set of feasible individual portfolios. (Here, $\mathbb{Z}_+$ denotes the set of non negative integers.)

And we let $\theta$ denote a probability measure on $Y$, where $\theta(y)$ is the fraction of each specialization type with portfolio $y$ just before pairwise meetings. A steady state is a collection of functions $(w_s, \pi_s, \theta_s)$ that satisfies the conditions described below. We suppress the dependence of a steady state on the denomination structure $s$ whenever the context is clear.

We begin with the choice of a portfolio. We let a person with wealth $z$ choose
any lottery over portfolios in $Y$ whose value does not exceed $z$. That is, we let

$$\Gamma_1(z) = \{ \sigma : \sigma(y) = 0 \text{ if } sy > z \}. \quad (I.1)$$

We let $h : Y \to \mathbb{R}$ denote expected discounted utility after the portfolio is chosen and before pairwise meetings. In terms of $h$, the portfolio-choice problem is

$$g(z, h) = \max_{\sigma \in \Gamma_1(z)} \sum_{y \in Y} \sigma(y)h(y). \quad (I.2)$$

We denote the set of maximizers in (I.2) by $\Delta_1(z, h)$. That is, $\Delta_1(z, h)$ is a subset of probability measures on $Y$, where $\delta \in \Delta_1(z, h)$ is an optimal lottery and $\delta(y)$ is the implied probability of holding portfolio $y$.

We now turn to trade in meetings. Let

$$S(y) = \{ v \in \mathbb{Z}_+^K : v_k \leq y_k \}. \quad (I.3)$$

That is, $S(y)$ is the set of portfolios that a person with $y$ could surrender. Now, consider a meeting between a buyer with $y$ and a seller with $y'$. We let

$$X(y, y') = \{ x \in [0, Z - sy'] : x = s(v - v'), v \in S(y), v' \in S(y') \}. \quad (I.4)$$

That is, $X(y, y')$ is the set of feasible wealth transfers from the buyer to the seller taking into account the denominations held and the possibility of making change. Let $W$ be an upper bound on $w$ that is defined in the existence proof. We define

$$\Gamma_2(y, y', w), a \text{ set of probability measures on } [0, W] \times X(y, y'), \text{ by}$$

$$\Gamma_2(y, y', w) = \{ \sigma : E_\sigma[-q + \beta w(x + sy')] \geq \beta w(sy') \}, \quad (I.5)$$

where $E_\sigma$ denotes the expectation with respect to $\sigma$ and where the arguments of $\sigma$ are $(q, x)$. Then we let

$$f(y, y', w) = \max_{\sigma \in \Gamma_2(y, y', w)} E_\sigma[u(q) + \beta w(sy - x)]. \quad (I.6)$$

This description implies the payoffs from trade. That is, the buyer’s payoff is $f(y, y', w)$ and, because the inequality in (I.5) will be binding, the seller’s is $\beta w(sy')$. 
We denote the set of maximizers in (I.6) by $\tilde{\Delta}_2(y, y', w)$. Because it can be shown that each maximizer is degenerate in $q$, in what follows, for a maximizer, we denote the maximizing $q$ by $\hat{q}$. To describe the law of motion, it is convenient to define $z^{-1}(y, y') = \{x : sy - x = z\}$ for $z \in \mathbb{Z}$, the set of asset trades that leave the buyer with wealth $z$. Then we define $\Delta_2(y, y', w)$, a subset of probability measures on $\mathbb{Z}$, by

$$\Delta_2(y, y', w) = \{\delta : \delta(z) = \sum_{z^{-1}(y, y')} \delta(\hat{q}, y)\},$$

for $\delta \in \tilde{\Delta}_2(y, y', w)$. That is, for $\delta \in \Delta_2(y, y', w)$, $\delta(z)$ is the probability that the buyer with $y$ leaves with wealth $z$ after meeting a seller with $y'$.

Now we can complete the conditions for a steady state. The value function $w$ must satisfy

$$w(z) = g(z, h),$$

where the function $h$ is defined by

$$h(y) = -\gamma \sum y_k + \frac{N - 1}{N} \beta w(sy) + \frac{1}{N} \sum \theta(y') f(y, y', w).$$

And the measures $\pi$ and $\theta$ must satisfy

$$\pi \in T_\pi(w, \theta) \text{ and } \theta \in T_\theta(w, \pi, \theta),$$

where

$$T_\pi(w, \theta) = \{\omega : \omega(z) = \sum_{(y', y'')} \theta(y') \theta(y'')[\delta(z) + \delta(sy' - z + sy'')]\},$$

for $\delta \in \Delta_2(y', y'', w)$,

and

$$T_\theta(w, \pi, \theta) = \{\omega : \omega(y) = \sum_z \pi(z) \delta(y) \text{ for } \delta \in \Delta_1(z, h)\}.$$

(Notice that $T_\pi(w, \theta)$ is a set of probability measures on $\mathbb{Z}$ and $T_\theta(w, \pi, \theta)$ is a set of probability measures on $\mathbb{Y}$. Also, notice that the dependence of $T_\theta(w, \pi, \theta)$ on $(w, \theta)$ is through the dependence of $h$ on $(w, \theta)$. Finally, in (I.11), as a convention, $\delta(x) = 0$ if $x \notin \mathbb{Z}$.)

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*See [1], who were the first to introduce lotteries into matching models of money.*
Definition I.3.1 Given a denomination structure $s$, a steady state is $(w, \pi, \theta)$ that satisfies (I.1)-(I.12).

Now we can state an existence result.

Proposition I.3.2 Given a denomination structure $s$, if $u'(0)$, $1/s_1$, and $Z$ are sufficiently large, then there exists $\gamma_s > 0$ such that for all $\gamma \in [0, \gamma_s]$ there exists a steady state $(w, \pi, \theta)$ with $w$ strictly increasing and strictly concave and with $\pi$ having full support.

Proof. See the Appendix.

As is well known, the challenge in models like the one above is to show that there is a reasonable monetary steady state. This challenge is met by the conclusion that $w$ is strictly increasing and concave. The proof shows that there is a neighborhood of $\gamma = 0$ with such a steady state. Notice that the full support property implies that the smallest denomination is held in such a steady state. It says nothing about higher denominations.\footnote{Situations in which an available denomination is not held are familiar; the U.S. $2 bill is one example.}

Given earlier results in [9] and [11], the proof is simple. The proof establishes properties of the mapping used to define a steady state. First, it uses the fact—previously established in [11], which, in turn, rests on the results in [9]—that for $\gamma = 0$ the mapping has fixed-point index 1. The result in [11] can be applied because if $\gamma = 0$, then portfolios that consist entirely of the smallest denomination money are equilibrium portfolios. But given such portfolios, the model is the same as the money-only case of the model in [11]. (For such portfolios, the non lottery version is the model in [9].) Second, the mapping is shown to be upper hemicontinuous in $\gamma$ at $\gamma = 0$. Those two facts give the result.

Notice, by the way, that the mapping is not continuous in $\gamma$ at $\gamma = 0$; that is, a portfolio that is an equilibrium portfolio when $\gamma = 0$ is not, in general, close to those that are equilibrium portfolios when $\gamma > 0$ and small. That is why the model can have rich implications for portfolios even if $\gamma$ is small.
In accord with such discontinuity, there is a general conclusion about portfolios that is related to Telser’s conclusions about an optimal denomination structure; namely, portfolios do not have surplus divisibility.

**Lemma I.3.3** Let $V(y) = \{ x \in \mathbb{R}^+ : x = sv, v \in S(y) \}$, the set of wealth transfers that is feasible for a person with portfolio $y$. If $\delta$ solves the problem in (I.2) and $y$ is in the support of $\delta$, then $\nexists \tilde{y}$ satisfying (i) $s\tilde{y} = sy$, (ii) $V(y) \subseteq V(\tilde{y})$, and (iii) $\sum_k \tilde{y}_k < \sum_k y_k$.

**Proof.** Suppose, by contradiction, that $\tilde{y}$ exists. It suffices to show that $h(\tilde{y}) > h(y)$. By (I.9) and properties (i) and (iii), $h(\tilde{y}) > h(y)$ if $f(\tilde{y}, y', w) \geq f(y, y', w)$. To establish the latter, suppose that $x \in X(y, y')$ (see (I.4)). That is, there exist $v \in S(y)$ and $v' \in S(y')$ with $x = sv - sv'$. But by property (ii), there exists $\tilde{v} \in S(\tilde{y})$ with $s\tilde{v} = sv$. Therefore $x \in X(\tilde{y}, y')$, which implies that any offer that is feasible when the buyer has $y$ is also feasible when the buyer has $\tilde{y}$. This and property (i) imply that $f(\tilde{y}, y', w) \geq f(y, y', w)$. □

To see that this lemma rules out some portfolios, suppose that $s$ is the powers-of-two structure and consider $y = (3, 0, ..., 0)$ and $\tilde{y} = (1, 1, 0, ...0)$. If $\delta$ solves the problem in (I.2), then $y$ cannot be in the support of $\delta$ because $\tilde{y}$ satisfies the three conditions in the lemma.

And, using strict concavity of $w$, we can say a bit more about offers in meetings. The following lemma shows how to use the best trade when the buyer’s portfolio consists entirely of the smallest denomination to construct the best trade for all portfolios with the same wealth. The best trade is described in terms of a lottery, possibly degenerate, over the amount of wealth transferred. The corresponding implied (deterministic) output is that which satisfies the producer’s participation constraint in (I.5) with equality.

**Lemma I.3.4** Consider a buyer with $y$ and a seller with $y'$. (i) If the buyer’s portfolio is $(sy/s_1, 0, 0, ..., 0)$, then the solution for the wealth transfer from the buyer is a lottery over $\{m, m + s_1\}$ for some $m \in \mathbb{Z}$. (ii) Let $m$ be as described in part (i).
In general, the solution for the wealth transfer from the buyer is a lottery over
\[\{m_-, m_+\} = \{\max_{x \leq m} X(y, y'), \min_{x \geq m + s_1} X(y, y')\},\]
where \(X(y, y')\) is defined in (I.4) and where the lottery is degenerate on \(m_-\) if \(m_+\) does not exist.

**Proof.** Both (i) and (ii) are obvious consequences of strict concavity of \(w\). ■

### I.4 “Too Many” Denominations

If \((w_s, \pi_s, \theta_s)\) is a steady state associated with the denomination structure \(s\), then we view the inner product \(w_s\pi_s\) as the welfare associated with that steady state. That is, \(w_s\pi_s\) is expected utility prior to the assignment of initial wealth, where the assignment is made in accord with the steady-state distribution.\(^8\) As the model stands and from the point of view of that welfare criterion, nothing seems to penalize the availability of many denominations. After all, people are not forced to hold all the denominations offered. Thus, it may well be that for a given \(s_1\), \(s = (s_1, 2s_1, 3s_1, ..., Z)\) is a best structure, and that \(s_1\) should be “small.”

One way to penalize denomination structures with “too many” available denominations is to add a fixed cost that depends on the number of available denominations. For example, there could be a design cost for the government of making a distinct denomination available, a cost that does not depend on the stock of that denomination outstanding. If we assume that any such cost is financed by equal per capita lump-sum taxes, then adding it to the model does not change behavior and does not necessitate any substantive change in the definition of a steady state. In particular, proposition I.3.2 is unaffected.

In order to obtain detailed results on good denomination structures in our model or in variants of it, we suspect that numerical searches have to be carried out. From what we have just said, that should be done for a version with fixed costs per

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\(^8\)As this suggests, steady states are not unique. Autarky is a steady state and there are other steady states with trade that have step-function value functions, some of which are described below. Finally, for all we know, there may be multiple proposition-I.3.2 steady states.
denomination. To find a proposition-I.3.2 steady state for a given denomination structure, one iterates on the mapping used to define a steady state—the standard procedure for finding steady states in heterogeneous-agent models. Doing that is feasible provided that the set of possible individual wealth levels is not too large.

We do, however, have one general result about best denomination structures. It is based on the following conjecture: the linear interpolation of \( w \) is uniformly bounded above by a continuous function \( v \) with \( v(0) = 0 \).\(^9\) If the conjecture holds, then for a fixed carrying cost \( \gamma \), sufficiently small denominations are not valued or held because the value in trade of a sufficiently small denomination cannot exceed a fixed \( \gamma \). Therefore, for any positive fixed costs per denomination, sufficiently small denominations are not part of an optimal denomination structure.

Suppose, however, that such small denominations are offered. If the conjecture holds, then, in accord with the discussion above, there is no steady state that satisfies the conclusions of proposition I.3.2. (As noted above, the smallest denomination is held in a proposition-I.3.2 steady state.) The monetary steady states that do exist in such cases are best described in the context of a discussion of neutrality.

Let \( A = \{M, s, Z\} \) denote the set of exogenous nominal objects, where \( M \) is average money holdings and \( s = \{s_1, s_2, ..., s_K\} \). Now fix \( \gamma \) such that there is a proposition-I.3.2 steady state for \( A \). If \( \lambda \) is a positive integer and \( \lambda A = \{\lambda M, \lambda s, \lambda Z\} \), where \( \lambda s = \{\lambda s_1, \lambda s_2, ..., \lambda s_K\} \), then \( A \) and \( \lambda A \) support the same allocations. This is neutrality. But suppose we compare not \( A \) and \( \lambda A \), but \( A \) and \( A' = \{\lambda M, s', \lambda Z\} \), where \( s' = \lambda s \cup \{s_1\} \). (That is, \( A' \) is \( \lambda A \) but with the addition of the smallest denomination from \( A \).) For the fixed \( \gamma \) and for sufficiently large \( \lambda \), although there may be no steady state that satisfies the conclusions of proposition I.3.2, other steady states with trade do exist. In particular, any steady state for \( \lambda A \) (including a proposition-I.3.2 steady state) extended to the larger domain corresponding to \( s' \) through step-function value functions and distributions is a steady state for \( A' \). Let \( Z_{s'} = \{0, s_1, 2s_1, ..., \lambda Z\} \) with generic element \( z' \) and

\(^9\)Under the assumption that \( u'(0) \) is bounded, the conclusion is proved in [10] for a one-denomination model with \( \gamma = 0 \). We suspect that the result holds for any denomination structure and any fixed \( \gamma \). It is not known whether boundedness of \( u'(0) \) is necessary for the result.
let $Z_{\lambda s} = \{0, \lambda s_1, 2\lambda s_1, ..., \lambda Z\}$ with generic element $z$. That is, if $(w_{\lambda s}, \pi_{\lambda s}, \theta_{\lambda s})$ is a steady state for $\gamma$ and $\lambda A$, then the following is a steady state for $\gamma$ and $A'$: 

$$w'_{s'}(z') = w_{\lambda s}(z) \text{ if } z' \in \{z, z + s_1, z + 2s_1, ..., z + (\lambda - 1)s_1\} \text{ where } z \in Z_{\lambda s};$$

$$\pi'_{s'}(z') = \pi_{\lambda s}(z) \text{ if } z' = z \in Z_{\lambda s} \text{ and } \pi'_{s'}(z') = 0 \text{ otherwise; and analogously for } \theta'_{s'}.$$ 

In such a steady state for $A'$, no one holds the denomination $s_1$.

To get people to actually discard units of the denomination $s_1$, non steady-state initial conditions have to be considered. For example, if there is a zero-measure departure from the above steady state for $A'$ in which one person begins a date with one unit of the denomination $s_1$, then this person wants to discard that unit. It cannot be used at the portfolio-choice stage to get a larger unit and its value in trade is 0, and, hence, does not overcome the cost of carrying it into trade. This is the model’s depiction of the public’s abandonment of the penny.\(^\text{10}\) While the abandonment could arise as a steady state without the carrying cost, here—if the conjecture holds—it is dictated by the carrying cost.

### I.5 Concluding Remarks

We have set out a very simple model. With our model as a base, different complications could be added. Replacement costs for different kinds of currency items could be added. The role of denomination structures in encouraging or discouraging illegal transactions could be considered. And idiosyncratic taste shocks could be added—shocks which, perhaps, would bring the model closer to the specification of exogenous transaction distributions. Given a specification that is “sufficiently realistic,” numerical searches could be carried out to obtain conclusions about the best denomination structure.

\(^{10}\)Matters would be more complicated if the model contained different technologies for storing pennies—in a pocket versus in a jar at home.
I.6 Appendix: Proof of Proposition I.3.2

As a prelude to the proof, we begin with some notation and assumptions. Let \( \gamma \in [0,1] \). Let \( R \equiv [N - (N - 1)\beta]^{-1} < 1 \). Also, let \( D \) be the unique solution to \( u'(D) = [2/(R\beta)]^2 \). Notice that \( D > 0 \) provided that \( u'(0) > [2/(R\beta)]^2 \), one of our assumptions. Let \( W \) be the unique solution of \( N(1 - \beta)W = u(\beta W) \). We assume that \( 1/s_1 \geq \beta W/D \) and that \( Z \geq 4 \).

Let \( W \) be the set of non-decreasing and concave functions \( w : Z \to [0,W] \) with \( w(4) \geq D/\beta \). Let \( K \supset W \) be the set of non-decreasing functions from \( Z \) to \([0,W]\). Notice that the interior of \( W \) (relative to \( K \)) is non-empty and that any element of the interior is strictly increasing, strictly concave, and satisfies \( w(4) > D/\beta \). Let \( \Pi \) be the set of probability measures \( \pi \) defined on \( Z \) satisfying \( \sum \pi(z)z = 1 \). Let \( \Theta \) be the set of probability measures \( \theta \) on \( Y \) satisfying \( \sum \theta(y)sy = 1 \).

Now we can formally define the mapping to be studied. Let the mapping \( T_w : W \times \Theta \times [0,1] \to K \) be defined by

\[
T_w(w,\theta,\gamma)(z) = g(z,h),
\]

where \( g(z,h) \) is given in (I.2) and \( h \) is given by (I.9). (Here, the dependence of \( T_w(w,\theta,\gamma) \) on \( (w,\theta,\gamma) \) is through the dependence of \( h \) on \( (w,\theta,\gamma) \) which is given by (I.9) with \( f \) given by (I.6).) We let \( T : W \times \Pi \times \Theta \times [0,1] \to K \times \Pi \times \Theta \) be defined by

\[
T(w,\pi,\theta,\gamma) = (T_w(w,\theta,\gamma),T_\pi(w,\theta),T_\theta(w,\pi,\theta)),
\]

where \( T_w(w,\theta,\gamma) \) is given by (I.13), \( T_\pi(w,\theta) \) is given by (I.11), and \( T_\theta(w,\pi,\theta) \) is given by (I.12).

Lemma I.6.1 A fixed point of \( T \) is a steady state.

Proof. Obvious. ■

The next four lemmas constitute a proof of proposition I.3.2 through appeal to results based on fixed-point index theory.
Lemma I.6.2 \( T \) is upper hemicontinuous (u.h.c.), compact-valued, and convex-valued.

Proof. This follows from the Theorem of Maximum and convexification by lotteries. (For the non lottery version, one allows for all possible randomizations over the elements of the sets corresponding to \( \Delta_1 \) and \( \tilde{\Delta}_2 \).) ■

Lemma I.6.3 Let \( \Lambda \equiv \Pi \times \Theta \) and let \( \partial W \) denote the boundary of \( W \) (with respect to \( K \)). Let \( \mathcal{G} \) denote the set of u.h.c., compact-valued, and convex-valued mappings \( g : W \times \Lambda \rightarrow K \times \Lambda \) satisfying \( (w, \lambda) \notin g(w, \lambda) \) for all \( (w, \lambda) \in \partial W \times \Lambda \). There exists a fixed-point index defined on \( \mathcal{G} \), denoted \( \text{ind} \), satisfying: (A1) If \( g \) is constant on \( W \times \Lambda \) with the value \( (w_0, \lambda_0) \) where \( w_0 \in W - \partial W \), then \( \text{ind}(g) = 1 \); (A2) if \( \text{ind}(g) \neq 0 \), then \( g \) has a fixed point \( (w, \lambda) \) with \( w \in W - \partial W \); and (A3) if \( g_0, g_1 \in \mathcal{G} \) are homotopic on \( \partial W \times \Lambda \) (that is, there exists an u.h.c., compact-valued, and convex-valued \( G : W \times \Lambda \times [0, 1] \rightarrow K \times \Lambda \) such that \( (w, \lambda, \alpha) \notin G(w, \lambda, \alpha) \) for \( (w, \lambda, \alpha) \in \partial W \times \Lambda \times [0, 1] \) and \( G(\cdot, \alpha) = g_\alpha \) for \( \alpha = 0, 1 \)), then \( \text{ind}(g_0) = \text{ind}(g_1) \).

Proof. For continuous (single-valued) mappings from the closure of an open set \( A \subset \mathbb{R}^n \) to \( \mathbb{R}^n \), existence of a fixed-point index with the relevant properties can be found in Zeidler ([8], Theorem 12.1, p. 535). If \( B \subset \mathbb{R}^n \) is closed and convex and \( A \) is open in \( B \), then by Zeidler ([8], 13.6a, p. 604) that existence result can be generalized to continuous (single-valued) mappings from the closure of \( A \) to \( B \). And, by Krasnoel’skiî and Zabreîko ([2], Theorem 36.1, p. 218), a further generalization can be made to u.h.c., compact-valued, and convex-valued mappings. In particular, the hypotheses required by Krasnoel’skiî and Zabreîko ([2], Theorem 36.1, p. 218) are satisfied by \( \mathcal{G} \) and \( G \). ■

Lemma I.6.4 Denote \( T(\cdot, \gamma) \) by \( T_\gamma(\cdot) \). Then \( T_0 \in \mathcal{G} \) and \( \text{ind}(T_0) = 1 \).

Proof. Notice that \( T_0 \) is identical to the mapping \( \Phi \) studied in [9] and is also identical to the mapping \( \Phi_1 \) studied in [11]. Then, by the exact logic used to show that \( \Phi_1 \in \mathcal{G} \) and \( \text{ind}(\Phi_1) = 1 \) in Zhu and Wallace ([11], Lemma 2), we have \( T_0 \in \mathcal{G} \) and \( \text{ind}(T_0) = 1 \). ■
Lemma I.6.5 There exists \( \gamma_s > 0 \) such that if \( \gamma \leq \gamma_s \), then \( T_\gamma \) has a fixed point.

Proof. By property (A2) of lemma I.6.3 and by lemma I.6.4, \( T_0 \) has a fixed point \((w, \lambda)\) with \( w \in W - \partial W \). This and lemma I.6.2 imply that there exists \( \gamma_s > 0 \) such that if \( \gamma \leq \gamma_s \), then \( T_\gamma \) does not have a fixed point \((w, \lambda)\) with \( w \in \partial W \). Then, by property (A3) of lemma I.6.3 and by lemma I.6.4, \( ind(T_\gamma) = 1 \) for \( \gamma \leq \gamma_s \). Finally, by property (A2) of lemma I.6.3, \( T_\gamma \) has a fixed point. \( \blacksquare \)
References


Chapter II

Optimal divisibility of money when money is costly to produce

(Joint with Neil Wallace.)

II.1 Introduction

Historians of monetary systems and others have often noted that currency was not available in conveniently small denominations in the late middle ages and for a long time thereafter (see, for example, Sargent and Velde [6], pages 131-138, and Redish [5], pages 107-135). Previous work contains descriptions of the degree of divisibility of money, the costs of producing monetary items, and the rate at which such items depreciate—all of which is relevant for estimating the costs of maintaining stocks of monetary items of different degrees of divisibility. However, previous work has not asked what seems to be an obvious question: given the costs, what degree of divisibility would be optimal? We ask that question using parameterized versions of a recently formulated matching model of money to describe the benefits for trade of different degrees of divisibility of money. We balance the benefits against the costs of providing different degrees of divisibility, as inferred from the historical estimates of the costs of producing coins in 1400 and the rate at which coins depreciated, and find the implied optimal degree of divisibility.

Our analysis is carried out using a simplified version of the model in Lee, Wallace,
and Zhu [2]. Wealth consists solely of money. Society starts with a given stock of
the object from which money is to be produced and is constrained to have a powers-
of-two denomination structure: monetary items in denominations $s, 2s, 4s, 8s, \ldots$, where $s$ is the size of the smallest monetary item and where the size of any other item is proportional to its denomination. Society's only choice is $s$ and it faces a trade-off regarding that choice.\footnote{The powers-of-two structure, at least its first few terms, was common. In principle, society could, as in [2], choose an entire denomination structure.} Trade of money for goods occurs in pairwise meetings and, as might be surmised, is facilitated by having $s$ be small. However, the production of monetary items is costly and the cost, as suggested by the historical record, is assumed to be proportional to the number of monetary items. That tends to make the cost a decreasing function of $s$. The cost has two components: the cost of providing the initial stock of currency and the per-period cost of maintaining the stock, which would otherwise depreciate (with use). As a modeling convenience, we assume that the cost is financed by lump-sum taxes.

For a given $s$, the sequence of actions at a date is as follows. Each person starts with a portfolio of denominations. Then the person costlessly adjusts denominations—for example, exchanges 2 units of the smallest $s$ denomination for one unit of the $2s$ denomination, and vice versa. (This implies that a person's discounted expected utility at the start of a date depends only on the total nominal value of the money held.) Then each person meets one other person at random. In single-coincidence meetings, there is trade of perishable goods for money. Then the person goes on to the next date with the money the person has after trading.

Given a parameterized version of the model, including a description of how trade (bargaining) occurs in pairwise meetings, for each $s$ we compute a steady state that determines a distribution, $\pi$, and a value function, $w$, (each a function of nominal wealth at the start of a date), and an aggregate distribution of denominations. (Uniqueness is discussed below.) Associated with that steady state is ex ante welfare given by expected discounted utility—the inner product, $\pi w$, minus the present value of per capita resource costs needed to produce and maintain the steady-state distribution of denominations. The best $s$ is that which maximizes expected dis-
counted utility defined in that way.

Several comments are in order about the discrepancies between the model and most pre-19th century monetary systems. First, those monetary systems were commodity-money systems in which the stock of money was not fixed. Our model can be viewed as an extreme version of the commodity-money specification in Sargent and Wallace [7]. They assume that the commodity money can be transformed into the consumption good at one constant rate and that the consumption good can be transformed into the commodity money at another constant rate. So long as those rates and the value in trade of the commodity money make neither kind of transformation profitable, the stock of money stays constant. Our model is an extreme version in which those rates are consistent with a large interval of values of money consistent with making neither transformation profitable. Second, we do not permit monetary items to actually depreciate. Instead, we assume that sufficient resources are expended at each date to maintain the stock in undepreciated form. This allows us to avoid having the distribution of money by wear be a state variable of the money and is analogous to treating capital as uniform when a more realistic specification would keep track of vintages. Third, in most pre-19th century monetary systems, users of money bore the costs of production of money and may have borne even higher costs resulting in seignorage. Replacing such user costs by financing by lump-sum taxation is a huge simplification. It allows us to describe the benefits (for trade) of additional divisibility of currency separately from the costs of providing additional divisibility.

Two pieces of historical information, described in more detail below, are used to estimate the cost of producing and maintaining the stock of money of a given divisibility. One is the estimate that a person working at a mint in 1400 in France could produce approximately 14 monetary items per day. The other is that money depreciated at about 1% per year. These estimates combined with features of the model allow us to describe the cost. Although the cost at the optimum, expressed as a flow, turns out to be small relative to the model’s implied total output, on the order of .01 to .10 of one percent, the optimal degree of divisibility is low relative
to what we see in modern economies.

The simplest measure of divisibility, simple because it is exogenous in the model, is the ratio of the per capita stock of money to the size of the smallest unit, $M/s$ in our notation. Depending on the particular version of the matching model, the optimal magnitude of $M/s$ ranges from about 10 to 70. Although simple in terms of the model, it is not obvious how to construct a comparable statistic for a modern economy in which currency is a tiny fraction of wealth—not all of wealth as in the model. However, if we take currency holdings to be the relevant comparison and if we say that $100$ is the per capita stock of currency, then the comparable ratio in the U.S. is $10^4$, $100$ in pennies. If we replace currency by wealth, then the comparable current is huge. A better measure of divisibility is the ratio of annual per capita nominal GDP to the size of the smallest unit of currency. In the model, at the optimal $s$, that ratio turns out to range between 30 and 600. For the U.S. in 2003, the ratio of per capita GDP to the penny is roughly $4 \times 10^6$. Obviously, we are finding a low degree of divisibility relative to that in the U.S. economy in 2003.

II.2 The model

Aside from costs of maintaining the stock of money and the existence of different denominations, the model is the version of the Shi [8] and Trejos and Wright [14] models studied by Zhu [16]: indivisible money with a sufficiently large upper bound on individual money holdings.

Time is discrete. There is a unit measure of each of $N \geq 3$ types of infinitely lived agents and there are $N$ distinct produced and perishable types of divisible goods at each date. A type $i$ agent, $i \in \{1, 2, ..., N\}$, produces type $i$ good and consumes only type $i + 1$ good (modulo $N$). Each agent maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. Period utility for a type $i$ agent is $u(q_{i+1}) - q_i$, where $q_{i+1} \in \mathbb{R}_+$ is consumption of good $i + 1$ and $q_i \in \mathbb{R}_+$ is production of good $i$. The function $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly concave, strictly increasing, continuously differentiable, and satisfies $u(0) = 0$, $u'(\infty) = 0$, and $u'(0)$ sufficiently large.

There are three exogenous nominal quantities: $(s, M, B)$, where $s$ is the size of
the smallest unit of money, $M$ is the average (per type) quantity of money (later normalized to be unity), and $MB$ is the bound on individual holdings, a bound that is needed to achieve compactness. We assume that $M/s$ and $B$ are integers. The set of possible individual wealth holdings is $\mathbb{Z} = \{0, s, 2s, ..., MB\}$.

At each date, each person meets one other person at random. In meetings, people see each other’s portfolio of denominations. In single-coincidence meetings, the only relevant kind of meeting, the buyer makes a take-it-or-leave-it deterministic offer. (One of our robustness checks is to replace such offers by buyer take-it-or-leave-it offers with probability $3/4$ and seller take-it-or-leave-it offers with probability $1/4$; another robustness check is to have the buyer make a take-it-or-leave-it lottery offer in meetings.\footnote{Lotteries were first introduced into matching models of money in \cite{1}.})

If $x \in \mathbb{Z}$ is the nominal value of money traded in a meeting, then that meeting contributes a loss to nature of $\theta x$ depreciation of money, where $\theta \in \mathbb{R}_+$. However, under financing of depreciation by lump-sum taxes, the traders do not, of course, associate that cost with their trade. Our depreciation assumption assumes that depreciation is proportional to denomination so that, for example, a trade of a $4s$ item implies the same depreciation as a trade of four $s$ units. (For the kind of paper currency we have in the U.S. currently, where size does not vary with denomination, a better assumption would be that depreciation is proportional to the number of monetary items traded.)

In \cite{2}, it is assumed that there is a cost per monetary item of carrying a monetary item from the portfolio choice stage to the pairwise meetings and that agents choose a portfolio of denominations optimally given that cost. That cost gives agents an inducement to economize on their holdings of small denomination items. Here, to simplify the model, we examine the consequences of an alternative and simple way of choosing the portfolio of denominations.

**Problem II.2.1** An agent with wealth $z \in \mathbb{Z}$ chooses a portfolio to minimize the number of monetary items held subject to being able to surrender any amount of wealth in the set $\{0, s, 2s, ..., z\}$.
This problem has a well-known and simple solution for a powers-of-two denomination structure (see [13]): starting with the smallest denomination, choose one of each denomination up to the point at which the next largest would exceed \( z \); then, with the remainder, minimize the number of additional items held. For example, if \( z = 4s \), then the person chooses the portfolio: two units of \( s \) and one unit of \( 2s \). In Figure II.1, we show how the number of monetary items varies with wealth for a given \( s \). The non montonicities aside (a person with \( z = 6s \) holds 4 monetary items, but a person \( z = 7s \) holds only 3), the number of monetary items held increases slowly as a function of wealth.

Figure II.1: The number of monetary items as a function of \( z/s \)

As a robustness check, we also examine the consequences of having people hold only the smallest denomination. The implied total number of monetary items held is also shown in Figure II.1. Imposing the alternative is a crude way to get at the consequences of removing the option to adjust portfolios of denominations. It implies a higher cost of providing a stock of money with a given \( s \) because more monetary items have to be produced.
II.3 Steady states and welfare

For a given $s$, a symmetric steady state, symmetric across the $N$ specialization types, consists of a distribution of wealth, $\pi : \mathbb{Z} \to [0, 1]$, and a value function, $w : \mathbb{Z} \to \mathbb{R}$, where $\pi(z)$ is the fraction of people with wealth $z$ at the start of a period and $w(z)$ is the expected discounted value of beginning a period with wealth $z$. Our assumptions about portfolios imply that any trade in a pairwise meeting that is feasible for the model in which only the smallest denomination is available is also feasible when the problem-II.2.1 portfolio is held. Therefore, as regards the conditions that define a steady-state $(\pi, w)$, this model is identical to the single-denomination model in Zhu [16].

As in Zhu [16], a steady state $(\pi, w)$ is a fixed point of the usual mapping studied in heterogeneous-agent models—a mapping that via the trades in meetings takes a next-period value function and a current-period distribution and maps into a set of current-period value functions and next-period distributions. Zhu proves that if $M/s$, $B$, and $u'(0)$ are large enough, then there exists a steady state with $w$ strictly increasing and strictly concave and with $\pi$ having full-support, a steady state that we label a nice steady state. Zhu’s proof depends on several features of the above mapping, the main one being that it preserves concavity of value functions. That property is implied by buyer take-it-or-leave-it offers. Concavity is also preserved by having an exogenous probability mixture of such offers and take-it-or-leave-it offers by sellers (one of our robustness checks). Therefore, Zhu’s result almost certainly extends to trades given by such mixtures.

Even within the class of nice steady states, there is no uniqueness result. Nor has it been shown that there exists a nice steady state that is locally stable. Despite the absence of such results, our computational procedure converges to a nice steady state that does not seem sensitive to the initial condition we choose and that is locally

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3It is trivial to construct monetary steady states that are not nice. For example, there is one with support $\{0, MB\}$ and a value function that has one value at all $z < MB$ and a higher value at $z = MB$, a steady state that is a relabeling of the steady state that exists when money holdings are restricted to be in the set $\{0, s\}$. Wallace and Zhu [15] show that steady states with such step-function value functions are not limits of commodity-money steady states as the commodity value (in the sense of the terms on which money can be converted into the consumption good) tends to zero. That is one reason for preferring nice steady states.
stable. The computational procedure, which is described in detail in the appendix, consists, with standard adjustments, of iterating on the mapping described above. The notion of local stability used and the local stability results are also described in the appendix.

Now, suppose \((w_s, \pi_s)\) is a steady state associated with \(s\). We associate with this steady state welfare as measured by the inner product \(\pi_s w_s\) minus the present value of the costs of producing the initial stock of currency and the cost of maintaining the stock. This criterion is an ex ante criterion, prior to the assignment of money holdings to people. It also does not deal with the transition from an economy with one magnitude of \(s\) to an economy with a different magnitude of \(s\).

Our specifications of portfolios and \(\pi\) imply a total number of monetary items held. Let \(\bar{n}_s = \sum z \pi_s(z) n_s(z)\) be the average number of monetary items held per type, where \(n_s(z)\) is the number of monetary items held by a person with wealth \(z\) (see Figure II.1 above). The quantity \(\bar{n}_s\) is the number of monetary items to be produced initially and, given the depreciation rate, implies the number to be produced per period to maintain the stock.

The percentage depreciation rate is \(\theta \bar{x}_s/N M\), where \(\bar{x}_s\) is the average nominal value of money traded per period in a single-coincidence meeting in the steady state under consideration. (The quantity \(\bar{x}_s/N M\) is the velocity of money in the sense of the amount traded per period relative to the stock; the factor \(1/N\) appears because only the fraction \(2/N\) of meetings are single-coincidence meetings, and in those meetings the producer’s money is not traded.) Thus, \(\bar{n}_s \theta \bar{x}_s/N M\) is the number of monetary items to be produced per period.

As described below, we have an estimate of the number of monetary items produced per period by a person employed in a mint at the beginning of the 15th century. Let \(r\) denote that number per (model) period and let \(\bar{y}_s\) be the average amount produced per period in a single-coincidence meeting in the steady state under consideration. Then, we take \(\bar{y}_s/N r\) to be the cost of producing a monetary item, where \(\bar{y}_s/N\) is average output that someone steadily employed in producing money would, according to the model, have otherwise produced. (The factor \(1/N\)
appears for the same reason it appears in the velocity expression.) Therefore, the welfare associated with the steady state \((w_s, \pi_s)\) is

\[
V(M/s) = \pi_s w_s - \frac{\bar{y}_s}{N_F} \bar{n}_s \left(1 + \frac{\theta \bar{x}_s/NM}{1 - \beta}\right).
\]  

(II.1)

Before we describe how we choose parameters, we give some preliminary discussion of how the components of welfare as given in (II.1) behave as a function of \(M/s\).

As is well-known and easy to demonstrate, \(\pi_s w_s\) is proportional to the expected value of the gains from trade, \(g(y) \equiv u(y) - y\), over all single-coincidence meetings. Because \(g\) is bounded, \(\pi_s w_s\) is bounded. And, although there are no analytical results of this sort, viewed as a function of \(M/s\), we always find that \(\pi_s w_s\) is increasing and strictly concave. If it is increasing, then it approaches a limit.

The cost, the second term on the right-hand side of (II.1), contains three objects that depend on \(M/s\): \(\bar{y}_s\), \(\bar{n}_s\), and \(\bar{x}_s\). Although both \(\bar{y}_s\) and \(\bar{x}_s\) are quite sensitive to \(M/s\) at very low values of \(M/s\), as suggested by our remarks about a limit, they do not vary much when \(M/s\) is larger. That leaves \(\bar{n}_s\) to determine the behavior of the cost for sufficiently large \(M/s\). For the problem-II.2.1 determination of \(n_s(z)\), \(\bar{n}_s\) can be expected to increase slowly with \(M/s\). And, although the cost is not necessarily convex in \(M/s\), \(V\) always seems to have an internal maximum.

### II.4 The choice of parameters

As regards the background matching model, we assume \(N = 3\), the smallest magnitude consistent with no double-coincidence meetings, and \(u(y) = y^{1-\alpha}/(1 - \alpha)\) for \(\alpha \in (0,1)\). This choice of \(u\) implies that \(\max_y[u(y) - y]\) is attained at \(y = 1\). (It also implies that \([\alpha/(1-\alpha)]/[N(1-\beta)]\) is the upper bound on \(\pi_s w_s\) implied by production and consumption equal to unity in every single-coincidence meeting.) We primarily study \(\alpha = 1/2\) and as robustness checks report some results for \(\alpha = 1/4\) and \(\alpha = 3/4\). We study \(\beta = .9^{(1/F)}\) and also report some results for \(\beta = .8^{(1/F)}\) and \(\beta = .95^{(1/F)}\), where \(F\) is the number of model periods per year. We primarily study \(F = 12\) and as robustness checks consider \(F = 6\) and \(F = 24\). (For the determination of \(\pi w\), \(F\) plays no role given \(\beta\). However, that is not the case regarding costs,
as will be discussed below.) We set $B = 4$, which, as we will see, is large enough so that the results would not be affected if it were larger. And, as noted above, $M = 1$ is a normalization.

We next describe how we choose the depreciation rate $\theta$. Patterson ([4], page 220) provides an estimate of an annual loss rate of 1.0%. Call his annual depreciation rate $\delta$. We choose $\theta$ so that a steady state of the model for magnitudes of $M/s$ close to those actually observed and for our main specification—buyer deterministic take-it-or-leave-it offers, the problem-II.2.1 portfolio of denominations, and $(\alpha, \beta) = (0.5, 0.9^{(1/F)})$—gives rise to that annual depreciation rate. That is, according to our model, $\theta$ satisfies

$$(1 - \delta)^{(1/F)} = 1 - \frac{\theta \bar{x}(s)}{NM},$$

(II.2)

where, as above, $\bar{x}/NM$ is velocity. The implied values of $\theta$ are shown in Table II.1.

<table>
<thead>
<tr>
<th>$M/s$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.050</td>
<td>0.065</td>
<td>0.061</td>
<td>0.063</td>
</tr>
<tr>
<td>12</td>
<td>0.026</td>
<td>0.050</td>
<td>0.070</td>
<td>0.062</td>
</tr>
<tr>
<td>24</td>
<td>0.013</td>
<td>0.025</td>
<td>0.038</td>
<td>0.050</td>
</tr>
</tbody>
</table>

In our calculations of costs, we set $\theta = .03$ and as robustness checks also examine $\theta = .01$ and $\theta = .05$. Although these seem high—perhaps because Patterson’s estimate includes wear produced by intentional clipping—we will see that $\theta$ is not very important.

Finally, we describe how we choose $r$, the number of monetary items that a person could produce per model period. As noted above, we assume that this number is independent of the size of the coins, which is roughly in accord with the description in Sargent and Velde ([6], pages 50-52): “Per unit of value, the production process made small coins more expensive to produce than larger ones, since the same effort was required to strike a coin of any size, and not much less to prepare smaller blanks than larger blanks.” We choose $r$ on the basis of descriptions of the number of people employed in the Paris mint at the beginning of the 15th
century and the output of that mint.

As regards coin production, the first column of Table II.2 is annual average silver and gold coin production at the Paris mint in the years 1405 and 1410 (measured in the unit of account), as described by Miskimin ([3], pages 167 and 245).

<table>
<thead>
<tr>
<th></th>
<th>annual production</th>
<th>denomination</th>
<th>number of coins</th>
</tr>
</thead>
<tbody>
<tr>
<td>silver</td>
<td>10,167</td>
<td>0.042 (blanc)</td>
<td>242,071</td>
</tr>
<tr>
<td>gold</td>
<td>14,168</td>
<td>1.130 (ecu)</td>
<td>12,538</td>
</tr>
<tr>
<td>total</td>
<td>254,609</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To convert this into a number of coins, we would like to know the denominational composition of the coins produced. We have not found that information. As a crude substitute, we assume that the average denomination of silver coins produced was that of the most common type of silver coin, the blanc, and that the average denomination of the gold coins was that of the most common type of gold coin, the ecu (see Spufford [11], pages 323-324 and pages 408-409). The second column is the legal value of those coins in terms of the unit of account (Redish [5], page 98, and Spufford [11], page 409). The result is that Paris mint produced about 0.25 million coins per year in those years. Approximately 60 workers were regularly employed in the Paris mint in the early 15th century (see Spufford [10], page 15). Therefore, \( r = \frac{254,609}{60}F = \frac{4243}{F} \), about 14 coins per day per person working at the mint based on a 300 day work-year. In our calculations, we take \( r \) to be \( 4,500/F \) (15 coins per day), and as robustness checks also present some results for \( r = 3,000/F \) (10 coins per day) and \( r = 6,000/F \) (20 coins per day).

Notice that the choice of \( F \) affects the term \( \bar{y}_s/Nr \) independent of its effect on \( \beta \). If we double \( F \), then we halve \( r \). However, the model does not imply a halving of \( \bar{y}_s \). There is an effect on \( \bar{y}_s \) coming from the approximate halving of the discount factor as we double \( F \), but it tends to increase \( \bar{y}_s \). The effect of \( F \) on cost is consistent with the background matching model. According to that model, an economy with more frequent meetings is a richer economy; it has more opportunities for production and
trade. But, because we do not assume that it has a better technology for producing monetary items, the greater frequency of trade increases the opportunity cost of using labor to produce monetary items.

II.5 The benefit of additional divisibility

With lump-sum financing of the costs of providing and maintaining the stock of money, the cost parameters do not influence the behavior of agents in the model. Therefore, we can describe $\pi$ and $w$ as functions of divisibility without committing ourselves to cost parameters.

Figures II.2-II.4 show the inner product $\pi w$ as a function of $M/s$ for various specifications of the matching model. We express $\pi w$ relative to its upper bound implied by production and consumption of unity in every single-coincidence meeting. (The upper bound depends on the parameters of the matching model, but does not depend on $M/s$. Unless otherwise noted, the results depicted are for buyer (take-it-or-leave-it) deterministic offers.) In all three figures, $\pi w$ is an increasing and concave function of $M/s$. Although it seems to be converging to unity (the upper bound), as explained below, that is not the case.

Figure II.2: $\pi w$ as a function of $\alpha$: $\beta, F = (0.90^{(1/F)}, 12)$

Figure II.3: $\pi w$ as a function of $F$: $\alpha, \beta = (0.5, 0.90^{(1/F)})$
Figure II.4: $\pi w$ and different trading rules:

\[(\alpha, \beta, F) = (0.5, 0.90^{1/F}, 12)\]

role of discounting or, equivalently, the frequency of meetings. (In each figure, each function is relative to its own upper bound.) Notice that the domains differ for the different parameters. That is because we did not compute steady states for magnitudes of $M/s$ that far exceed the optimal $M/s$. Figure II.4 shows the role of different trading rules, where the label “buyer-seller offer” means the $3/4$ and $1/4$ mixture over who makes the deterministic take-it-or-leave-it offer. As might be expected, both in terms of total welfare and marginal welfare, divisibility is less important under lotteries.

Figures II.5 and II.6 show a selection of the underlying distributions. Figure II.5 shows $\pi$. Although not evident from the figure, all these distributions are full support distributions. Nevertheless, the measure at wealth levels near the upper bound of 4 is evidently very small. As a function of $M/s$, the distributions seem to be converging. However, they are not converging to a degenerate distribution, which cannot be a steady state with trade. Figure II.6 shows the distribution of consumption (and production) across single-coincidence meetings (the area under each curve is unity). The upper bound $\pi w$ corresponds to a distribution in Figure II.6 that is degenerate at unity. As in Figure II.5, the distribution seems to be converging as a function of $M/s$. Notice that although the distributions in Figure II.6 for $M/s = 30$ and $M/s = 35$ are not all that close to a degenerate distribution
at unity, the corresponding $\pi w$ in Figure II.2 is close to the upper bound. The explanation is that those distributions are centered near unity and that the function $u(y) - y$ is quite flat in the neighborhood of unity.

II.6 The cost of additional divisibility

Now we show how the cost, the second term on the right-hand side of (II.1), varies with $M/s$ and with the other parameters. The cost depends on all the parameters—those of the matching model through average output and the average amount of money traded, the cost parameters $r$ and $\theta$, and the rule for choosing a portfolio of denominations.

As we did with $\pi w$, we describe the cost term relative to the upper bound for $\pi w$, here as a percentage of the upper bound for $\pi w$. We show these in figures II.7-II.11. (Unless otherwise noted, the results depicted are for buyer (take-it-or-leave-it) deterministic offers and for the problem-II.2.1 portfolio of denominations.) In Figure II.7, we display the role of the rule for choosing a portfolio of coins and the rate at which money wears out with use, $\theta$. As shown there, $\theta$ does not matter much. In fact, for our parameters, the cost of producing the initial stock is far more
significant than the present value of maintaining the stock.

Figure II.7: Cost as a function of $\theta$:
$(\alpha, \beta, F, rF) = (0.5, 0.90^{(1/F)}, 12, 4500)$

Figure II.8: Cost as a function of $\alpha$:
$(\beta, F, \theta, rF) = (0.90^{(1/F)}, 12, 0.03, 4500)$

Figure II.9: Cost as a function of $F$:
$(\alpha, \beta, \theta, rF) = (0.5, 0.90^{(1/F)}, 0.03, 4500)$

Figure II.10: Cost and different trading rules:
$(\alpha, \beta, F, \theta, rF) = (0.5, 0.90^{(1/F)}, 12, 0.03, 4500)$

In Figure II.8, we show the cost as function of the $u$ function and in Figure II.9 as a function of the frequency of trade, $F$. At low levels of divisibility, the cost is roughly proportional to $F$. Figure II.10 shows how the cost varies over the different trading rules. Finally, figure II.11 shows how the cost varies with $r$, the number of coins that a person could produce per model period.
II.7 The optimal degree of divisibility

Table II.3 contains the results for the optimal degree of divisibility expressed in two ways: the ratio of the average per capita amount of money to the smallest unit, denoted $M/s^*$ in the table; and the ratio of the model’s annual nominal GDP per capita to the smallest unit, denoted $F\bar{x}/Ns^*$ in the table. While the results vary with the specification, as noted above, the optima display a very high degree of indivisibility relative to what we see in most modern economies.

In order to get a sense of the magnitude of the cost of currency provision, Table II.4 contains the flow cost of currency relative to the model’s real GDP at the optimum. To get this relative measure of cost, we first convert the per capita present value of the cost, as it appears in (II.1), to a flow by multiplying it by $(1-\beta)$. Then, we divide it by $\bar{y}s/N$, the model’s per capita GDP per model period. The result is $(\bar{n}s/r)(1-\beta + \theta\bar{x}s/\bar{M})$. When the portfolio of denominations is determined by problem II.2.1, that relative cost is about .01 to .02 percent. When only the smallest monetary unit is held, it tends to be about .1 percent.

In Table II.5, we report the welfare cost of having a degree of divisibility that is half the optimum (an $s$ that is twice the optimal $s$). This welfare cost is computed by asking how much additional consumption must be given to the consumer in every
Table II.3: Optimal divisibility

<table>
<thead>
<tr>
<th>(α, β^F, F, θ, rF)</th>
<th>problem-II.2.1 portfolio</th>
<th>smallest coin only</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>M/s^* F\bar{\pi}/Ns^*</td>
<td>M/s^* F\bar{\pi}/Ns^*</td>
</tr>
<tr>
<td>buyer deterministic offer</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.50, 0.90, 6, 0, 0, 0.03, 4500)</td>
<td>17</td>
<td>43</td>
</tr>
<tr>
<td>(0.50, 0.90, 12, 0, 0, 0.03, 4500)</td>
<td>33</td>
<td>153</td>
</tr>
<tr>
<td>(0.50, 0.90, 24, 0, 0, 0.03, 4500)</td>
<td>69</td>
<td>612</td>
</tr>
<tr>
<td>(0.75, 0.90, 12, 0, 0, 0.03, 4500)</td>
<td>63</td>
<td>294</td>
</tr>
<tr>
<td>(0.25, 0.90, 12, 0, 0, 0.03, 4500)</td>
<td>14</td>
<td>66</td>
</tr>
<tr>
<td>(0.50, 0.80, 12, 0, 0, 0.03, 4500)</td>
<td>16</td>
<td>80</td>
</tr>
<tr>
<td>(0.50, 0.95, 12, 0, 0, 0.03, 4500)</td>
<td>70</td>
<td>305</td>
</tr>
<tr>
<td>(0.50, 0.90, 12, 0, 0.01, 4500)</td>
<td>33</td>
<td>153</td>
</tr>
<tr>
<td>(0.50, 0.90, 12, 0, 0.05, 4500)</td>
<td>33</td>
<td>153</td>
</tr>
<tr>
<td>(0.50, 0.90, 12, 0, 0.03, 3000)</td>
<td>33</td>
<td>153</td>
</tr>
<tr>
<td>(0.50, 0.90, 12, 0, 0.03, 6000)</td>
<td>33</td>
<td>153</td>
</tr>
<tr>
<td>lottery offer</td>
<td>(0.50, 0.90, 12, 0, 0, 0.03, 4500)</td>
<td>20</td>
</tr>
<tr>
<td>buyer-seller offer</td>
<td>(0.50, 0.90, 12, 0, 0, 0.03, 4500)</td>
<td>10</td>
</tr>
</tbody>
</table>

Table II.4: Flow cost as a percentage of output at the optimum

<table>
<thead>
<tr>
<th>(α, β^F, F, θ, rF)</th>
<th>problem-II.2.1 portfolio</th>
<th>smallest coin only</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>M/s^* F\bar{\pi}/Ns^*</td>
<td>M/s^* F\bar{\pi}/Ns^*</td>
</tr>
<tr>
<td>buyer deterministic offer</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.50, 0.90, 6, 0, 0, 0.03, 4500)</td>
<td>0.013</td>
<td>0.034</td>
</tr>
<tr>
<td>(0.50, 0.90, 12, 0, 0, 0.03, 4500)</td>
<td>0.016</td>
<td>0.066</td>
</tr>
<tr>
<td>(0.50, 0.90, 24, 0, 0, 0.03, 4500)</td>
<td>0.020</td>
<td>0.150</td>
</tr>
<tr>
<td>(0.75, 0.90, 12, 0, 0, 0.03, 4500)</td>
<td>0.019</td>
<td>0.138</td>
</tr>
<tr>
<td>(0.25, 0.90, 12, 0, 0, 0.03, 4500)</td>
<td>0.013</td>
<td>0.029</td>
</tr>
<tr>
<td>(0.50, 0.80, 12, 0, 0, 0.03, 4500)</td>
<td>0.026</td>
<td>0.067</td>
</tr>
<tr>
<td>(0.50, 0.95, 12, 0, 0, 0.03, 4500)</td>
<td>0.010</td>
<td>0.076</td>
</tr>
<tr>
<td>(0.50, 0.90, 12, 0, 0.01, 4500)</td>
<td>0.016</td>
<td>0.064</td>
</tr>
<tr>
<td>(0.50, 0.90, 12, 0, 0.05, 4500)</td>
<td>0.017</td>
<td>0.068</td>
</tr>
<tr>
<td>(0.50, 0.90, 12, 0, 0.03, 3000)</td>
<td>0.024</td>
<td>0.099</td>
</tr>
<tr>
<td>(0.50, 0.90, 12, 0, 0.03, 6000)</td>
<td>0.012</td>
<td>0.049</td>
</tr>
<tr>
<td>lottery offer</td>
<td>(0.50, 0.90, 12, 0, 0, 0.03, 4500)</td>
<td>0.014</td>
</tr>
<tr>
<td>buyer-seller offer</td>
<td>(0.50, 0.90, 12, 0, 0, 0.03, 4500)</td>
<td>0.012</td>
</tr>
</tbody>
</table>
single-coincidence meeting to bring welfare up to what it is at the optimum and expressing that addition as a fraction of average consumption in single-coincidence meetings.\textsuperscript{4} For example, the last entry in the first row says that the compensating consumption for having $M/s$ equal to 7 instead of its optimal value, 14, is 9.5% of average consumption when people hold only the smallest monetary unit. This cost varies a great deal with the specification of the matching model and is tiny when lotteries are used in trade.

\begin{table}[h]
\centering
\caption{The welfare cost of divisibility equal to 50\% of the optimum}
\begin{tabular}{|c|c|c|}
\hline
 & $(\alpha, \beta^F, F, \theta, rF)$ & problem-1 portfolio smallest coin only \\
\hline
buyer deterministic offer & (0.50, 0.90, 6, 0.03, 4500) & 0.0642 0.0954 \\
 & (0.50, 0.90, 12, 0.03, 4500) & 0.0532 0.1161 \\
 & (0.50, 0.90, 24, 0.03, 4500) & 0.0524 0.0837 \\
 & (0.75, 0.90, 12, 0.03, 4500) & 0.2733 0.3588 \\
 & (0.25, 0.90, 12, 0.03, 4500) & 0.0108 0.0424 \\
 & (0.50, 0.80, 12, 0.03, 4500) & 0.0550 0.1121 \\
 & (0.50, 0.95, 12, 0.03, 4500) & 0.0547 0.0838 \\
 & (0.50, 0.90, 12, 0.01, 4500) & 0.0532 0.1161 \\
 & (0.50, 0.90, 12, 0.05, 4500) & 0.0532 0.1161 \\
 & (0.50, 0.90, 12, 0.03, 3000) & 0.0532 0.1160 \\
 & (0.50, 0.90, 12, 0.03, 6000) & 0.0532 0.1162 \\
lottery offer & (0.50, 0.90, 12, 0.03, 4500) & 0.0015 0.0013 \\
buyer-seller offer & (0.50, 0.90, 12, 0.03, 4500) & 0.1092 0.1092 \\
\hline
\end{tabular}
\end{table}

\section*{II.8 Concluding remarks}

Although the optima we find are sensitive to the specification of the matching model, they all display a degree of divisibility that is much lower than what we see in modern economies. This holds whether we express the degree of divisibility as a ratio of the per capita stock of money relative to the size of the smallest unit or as the ratio of nominal GDP to the size of the smallest unit.

\textsuperscript{4}We compute compensating consumption as an addition to consumption in every single-coincidence meeting rather than as a multiple of it because consumption is zero in some single-coincidence meetings.
How seriously should we take this conclusion? One reason to take it seriously is that we did not do a specification search to get a particular answer. We took an off-the-shelf matching model and parameter values for it to cover a range of possibilities. A reason not to take the result seriously is that a myriad of other matching models could have been chosen. Among the possible alternatives are models with some double-coincidence trade, with some trade supported by credit, and with idiosyncratic preference shocks in meetings. Because all trade in our model requires money, the model seems, on balance, to overstate the importance of divisibility of money in actual economies.
II.9 Appendix

Here we describe in more detail the computational procedure used to find steady states and the way we checked some of the computed steady states for local stability.

II.9.1 Computing nice steady states

The computational procedure is essentially to iterate on the mapping from an end-of-period value function and a beginning-of-period distribution into a set of beginning-of-period value functions and end-of-period distributions.\(^5\) Because the mapping preserves concavity of value functions and full support distributions are, with some provisos about a minimal degree of divisibility, a consequence of concavity of value functions, if we start with a concave value function and the procedure gives rise to a limit, then the limit is a nice steady state.

We start with an initial guess, denoted \(g^{(0)} = (w^{(0)}, \pi^{(0)})\), where \(w^{(0)}\) is interpreted as an end-of-period value function and \(\pi^{(0)}\) as a beginning-of-period distribution. For the given \(w^{(0)}\), we find the best deterministic offer of money for each single-coincidence meeting by way of global search over all feasible offers. (Although, in principle, the solution need not be unique, not surprisingly, we never find multiplicity.) Call the best offer \(x(z, z')\), where \(z\) is the wealth of the buyer and \(z'\) is that of the seller. Then, the function \(x(z, z')\) can be used to generate an (end-of-period) distribution, \(\pi^{(1)}\). In addition, the implied payoff for the buyer and \(\pi^{(0)}\) give a (beginning-of-period) value function, \(w^{(1)}\).

The values for the \((i+1)\)-th iteration, denoted \(g^{(i+1)} = (w^{(i+1)}, \pi^{(i+1)})\), are formed by a weighted average of the elements \(g^{(i)}\) and \(g^{(i-1)}\), where different weights are used for the different components. This updating process is repeated until the components of \(g\) satisfy the convergence criterion:

\[
\max_j \left| \frac{w_j^{(i+1)} - w_j^{(i)}}{w_j^{(i)}} \right| < 10^{-4}
\]

for \(w\) and

\[
\left( \sum_j \left( \pi_j^{(i+1)} - \pi_j^{(i)} \right)^2 \right)^{1/2} < 10^{-4}
\]

\(^5\) A copy of the computer code, which is written in MatLab, is available on request.

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for \( \pi \).

For the lottery offers, we start, as above, by finding the best deterministic offer, \( x(z, z') \). Then, by concavity of the value function, the best lottery is either a weighted average of \( x - s \) and \( x \) or a weighted average of \( x \) and \( x + s \). The weighted average can be found by treating the value function as linear in each interval and solving the implied first-order condition. Indeed, for \( \alpha = 0.5 \), the first order-condition is linear.

As is standard, convergence is slower the larger is \( \beta \).

II.9.2 Local stability

Our local stability analysis is based on the discussion in [12]. The local stability requirement is that if the initial \( \pi \) is sufficiently close to the steady state, then there is an equilibrium path that implies convergence to the steady state \( \pi \). (This is a weaker notion than requiring that every equilibrium converge to the steady state. Requiring that every equilibrium converge is too strong for a model of fiat money because there are other equilibria for any initial condition—in particular, no trade and valueless money.)

To check deterministic steady states is simple because the trades that are optimal at the steady state are also optimal in a neighborhood of it. Therefore, for such steady states, it is enough to check that the law of motion implied by the steady state trades is locally stable. Suppose, then, that \( (w, \pi) \) is a steady state. In the model, an initial condition is a vector of \( \pi \in [0, 1]^{B/s-1} \), where \( \pi(0) \) and \( \pi(1) \) are eliminated by the adding-up conditions: \( \sum z \pi(z) = 1 \) and \( M = \sum z z \pi(z) \). Then, the steady state is locally stable if a linear approximation of the implied law of motion has all its characteristic roots with modulus less than one. We checked local stability for every steady state for the case of buyer deterministic (take-it-or-leave-it) offers and \( (\alpha, \beta, F) = (0.5, 0.9^{(1/F)}, 12) \). Every such steady state is locally stable.

Checking for local stability for the version with lottery offers requires that we use the entire system, not just the law of motion (again, see [12]). We checked and confirmed local stability for two lottery-offer steady states: \( (\alpha, \beta, F) = (0.5, 0.9^{(1/F)}, 12) \), and \( M/s = 10 \) and \( M/s = 20 \).
References


Chapter III

Is uniform money always better than separate monies?

III.1 Introduction

In the actual economy, there is, at least on the surface, a distinction between a uniform money and fixed exchange rates. However, as noted by Kehoe [2], most models do not distinguish between the two. There is, though, a small literature that distinguishes between a uniform money and fixed exchange rates. This paper fits into and is related to that literature.

The background environments used in that literature are random matching models of money. The general features of the models are: trade occurs in pairs, money holdings are in the set \( \{0, 1\} \), monies are distinguished by an irrelevant characteristic like color, and monies are the only assets.

In Matsuyama, Kiyotaki, and Matsui [7], Ravikumar and Wallace [8], Trejos and Wright [11], and Zhou [12], the feasible allocations in which people distinguish between monies are worse in terms of welfare than one in which they do not. There are a couple of counter-examples to that result. In Kiyotaki and Moore [3], sellers can produce either a local good or a generic good. The generic good can be consumed universally but gives strictly less utility than the local good. If money is uniform, then sellers produce the generic good, while if monies are distinguished, then sellers produce the local good. For some parameters, the latter Pareto dominates the former. In Kocherlakota and Krueger [4], a buyer’s preferences concerning the origin
of goods (either local or foreign) is private information and separate monies can signal such private information credibly. As a result, production is differentiated based on the nationality of a buyer, which can be optimal because buyers value home and foreign goods differently. It is not known whether these counter-examples are robust to more general money holdings and to trade in lotteries.

In this paper, we compare separate and uniform monies, but we do so in a model in which money holdings, though still indivisible, are permitted to be in a larger set, in which lotteries are allowed, and, most important, in which money is dominated in rate of return by government bonds as it is in most modern economies. As a theory of the coexistence of money and such government bonds, we adopt the trading procedure in a recent paper by Zhu and Wallace [13]. Under a uniform money, our model is the same as theirs. Under separate monies, it is a generalization of it.

The trading procedure is a two-step maximization problem. In the first step, buyer utility is maximized subject to no trade as a lower bound on seller utility and to a country-specific cash constraint; that is, the favored asset in this step is the seller’s home money. In the second step, seller utility is maximized subject to a lower bound on buyer utility given by the outcome of the first step. Under a uniform money, the country-specific cash is, of course, the same money for sellers in both countries.

The model is one of two symmetric countries. The sequence of actions at a date is as follows. For a given price of a one-period discount government bond and a fixed exchange rate between the monies, a person chooses a portfolio consisting of home money, foreign money, and bonds. Having chosen a portfolio, a person is randomly matched with one other person, who may be from the same country or from a foreign country. If the meeting is a single-coincidence meeting, then trade occurs according to the procedure described above. After such trade, each bond turns into a unit of money, and the implied interest payment on bonds is financed by a tax that is equivalent to inflation. Then a person goes on to the next date.

In this model, the assumed timing of the shock realization is necessary for a distinction between separate monies and a uniform money; that is, if people knew their
trading partner’s nationality before choosing a portfolio, then there is no difference between the two in this model. One limited interpretation of the assumed timing is that it is pertinent to trades that occur around border areas.

We first compare a uniform money and separate monies in partial equilibrium—for a given pairwise meeting and a given way of valuing end-of-trade wealth. It is shown that a uniform money has a (weakly) larger output and a (weakly) lower price than separate monies. Then we turn to general equilibrium. We make comparisons numerically across steady states using as a welfare criterion the expected utility of a representative agent prior to the assignment of wealth, where that assignment is made in accord with the steady-state distribution of wealth. We produce examples that show that no general conclusion is possible. While a uniform money is better in most of the examples, when the bond discount is sufficiently high and the probability of meeting a seller from the other country is one half, then it can happen separate monies is better.

However, this last result is in the nature of a second-best result in at least two senses. First, under our financing scheme for interest payments, the larger bond holdings with a uniform money imply a higher inflation tax rate; we do not solve for a best feasible financing scheme. Second, the welfare comparison is made for a given nominal interest rate on bonds; we do not solve jointly for the best such rate and the best monetary regime in the sense of a uniform money or separate monies.

The rest of this paper is organized as follows. Section 2 describes the model. Section 3 explores the characteristics of the pairwise-trade outcome in partial equilibrium. Section 4 defines a steady state in general equilibrium and describes an existence result. In section 5, numerical comparisons are made across steady states for a uniform money and separate monies. The paper concludes in section 6. Appendix A formally defines a steady state. Appendix B contains some proofs of partial equilibrium results. Finally, Appendix C presents some details of numerical results.
III.2 The Model

The background environment is that of Shi [9] and Trejos and Wright [10]. Time is discrete. The world is composed of two symmetric countries. There is a non-atomic, unit measure set of each of $N > 2$ types of infinitely lived agents and $N$ distinct produced and perishable types of divisible goods at each date in each country. A type $n$ agent, $n \in \{1, 2, 3, \ldots, N\}$, produces only good $n$ and consumes only good $n + 1$ (modulo $N$). Each agent maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. For a type $n$ person, realized utility in a period is $u(y_{n+1}) - y_n$, where $y_{n+1} \in \mathbb{R}_+$ is consumption of good $n + 1$ and $y_n \in \mathbb{R}_+$ is production of good $n$. The function $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave, continuously differentiable, and satisfies $u(0) = 0$ and $u'(\infty) = 0$. In addition, $u'(0)$ is sufficiently large.

Each country has its own money which is intrinsically useless and indivisible. We normalize by letting the smallest unit be unity so that the elements of the set of possible individual wealth holdings always consist of integer numbers; namely, $\mathbb{Z} = \{0, 1, 2, \ldots, Z\}$, where $Z$ is the upper bound on individual wealth holdings, a bound that is needed to achieve compactness. In each country, the average stock of money at the start of each date per specialization type is $\bar{z}$.

At each date, a person enters a period with some wealth comprised of monies, $z = m_h + m_f \in \mathbb{Z}$, where $m_h$ and $m_f$ denote the holdings of home and foreign monies, respectively, and where the nominal exchange rate has been normalized to be unity. Then a person meets the central bank (a centralized meeting) to adjust his portfolio. The central bank operates a free money-exchange window and also sells one-period risk-free bonds at an exogenous price $\gamma \in (0, 1)$—each bond being a title to a unit of money at the end of the period. We assume that an inflation tax is levied by way of a proportional tax on after-trade wealth to finance interest payment on bonds. (It is such as to keep average wealth constant.) After a portfolio is chosen, each person is randomly matched with another person. With probability $\phi$, the meeting partner comes from the same country; and with the probability $1 - \phi \in [0, 0.5]$, the partner comes from a foreign country. Trades occur only in single-coincidence meetings in
which specialization type \( n \) person meets specialization type \( n + 1 \) person. Other meetings are not relevant. People in a meeting know each other’s specialization type, portfolio, and nationality. However, people cannot commit to future actions, and trading histories are private.

We now spell out the trading procedure, a two-step maximization problem which is borrowed from [13]. To facilitate the description, we first introduce some notation. We let

\[
\Omega = \{ \omega = (m_h, m_f, b) \in \mathbb{Z}^3 : m_h + m_f + b \leq Z \}
\]

be the set of feasible individual portfolios after bond purchases and before the pairwise meeting, where \( b \) denotes holding of bonds measured at maturity value. We let \( \omega = (m_h, m_f, b) \in \Omega \) denote the pre-trade portfolio of the consumer and let \( \omega' = (m'_h, m'_f, b') \in \Omega \) denote that of the producer in a single-coincidence meeting. For \( \omega \in \Omega \), we let \( \omega_z = m_h + m_f + b \in \mathbb{Z} \) denote the total wealth implied by the portfolio \( \omega \) and similarly for \( \omega'_z \). We let \( W : \mathbb{Z} \rightarrow \mathbb{R} \) be the function that values end-of-trade wealth. Finally, let \((y, d)\) denote a trade outcome, where \( y \) is the quantity of the goods transferred from the producer to the consumer and \( d \) is the wealth transferred from the consumer to the producer. The trade that occurs is the outcome of the following two-step maximization problem.

**Problem III.2.1** Step-1: choose \((y_1, d_1)\) to \( \max \{ u(y_1) + W(\omega_z - d_1) \} \) subject to

\[
d_1 \leq m_i \quad \text{and} \quad W(\omega'_z + d_1) - y_1 \geq W(\omega'_z),
\]

where in separate monies, \( m_i = m_h \) (\( m_f \)) if the consumer meets a home (foreign) producer and in a uniform money, \( m_i = m_h + m_f \).

Step-2: choose \((y, d)\), the final trade, to \( \max \{ W(\omega'_z + d) - y \} \) subject to

\[
u(y) + W(\omega_z - d) \geq u(y_1) + W(\omega_z - d_1).
\]

Notice that there is no cash constraint in the second step. As noted below, this trading procedure always achieves an outcome in the pairwise core for the meeting.
III.3 A Uniform Money and Separate Monies in Partial Equilibrium

We first compare pairwise trading outcomes between a uniform money and separate monies for a given pairwise meeting with a given valuation of end-of-trade wealth.

Consider a generic pairwise meeting in which the consumer holds portfolio $\omega$ and the producer holds portfolio $\omega'$ with a given $W : Z \to \mathbb{R}$, a strictly increasing and concave function. Here, to simplify the argument, we assume that assets are divisible and we let $\tilde{W} : [0, Z] \to \mathbb{R}$ be defined by the extension of $W$ via linear interpolation.\(^1\) Let $[y(\omega, \omega'), d(\omega, \omega')]$ be the final trade and $[y_1(\omega, \omega'), d_1(\omega, \omega')]$ be the step-1 trade of problem 1. Then proposition 1 in [13] still holds in our model: the unique trading outcome $[y(\omega, \omega'), d(\omega, \omega')]$ is in the consumer-producer core and $[y(\omega, \omega') - y_1(\omega, \omega')] \geq 0$ and $[d(\omega, \omega') - d_1(\omega, \omega')] \geq 0$. Moreover, the two-step maximization procedure implies that the price of final trade is (weakly) higher than that of the first step.

**Lemma III.3.1** If $d_1 > 0$, then the solution to problem 1 satisfies

$$\frac{d(\omega, \omega')}{y(\omega, \omega')} \geq \frac{d_1(\omega, \omega')}{y_1(\omega, \omega')}.$$  

**Proof.** See Appendix B. \(\blacksquare\)

Now let $[y^U(\omega, \omega'), d^U(\omega, \omega')]$ and $[y^S(\omega, \omega'), d^S(\omega, \omega')]$ be the final trade for a uniform money and separate monies, respectively. Then, in a given pairwise meeting, the consumer’s final consumption in a uniform money is (weakly) larger than that in separate monies. Moreover, the nominal price level in a uniform money is (weakly) lower than in separate monies. Notice that step-1 only trades, one version of a cash-in-advance model, gives a different result: a uniform money has a (weakly) higher price level than do separate monies.

---

\(^1\)The trade outcomes implied by $\tilde{W}$ with divisible assets can be regarded as trade outcomes of lotteries with indivisible assets, which give the same payoff. Hence, we do not need to worry about the disparity between this divisibility assumption and the indivisibility assumption in the model in section 2.
Lemma III.3.2 If \( d^U > 0 \) and \( d^S_1 > 0 \), then for a given \( W \), the solutions to problem 1 satisfy

\[
\begin{align*}
(i) & \quad y^U(\omega, \omega') \geq y^S(\omega, \omega'), \\
(ii) & \quad \frac{d^S(\omega, \omega')}{y^S(\omega, \omega')} \geq \frac{d^U(\omega, \omega')}{y^U(\omega, \omega')}, \\
(iii) & \quad y^U_1(\omega, \omega') \geq y^S_1(\omega, \omega'), \\
(iv) & \quad \frac{d^S(\omega, \omega')}{y^S_1(\omega, \omega')} \leq \frac{d^U_1(\omega, \omega')}{y^U_1(\omega, \omega')}.
\end{align*}
\]

Proof. See Appendix B. □

Of course, this partial equilibrium analysis does not take into account interdependence between the monetary regimes, on the one hand, and the valuations of end-of-trade wealth and the distribution of portfolios, on the other hand.

### III.4 A Steady State in General Equilibrium

Now we extend our analysis to general equilibrium; that is, value functions, portfolio choices, payoffs from trades and wealth distributions are determined endogenously.

For given \( \phi \) and \( \gamma \), a symmetric steady state, symmetric across specialization types and across countries, is a collection of functions \( (v, f, \pi, \lambda) \) which satisfies the conditions described in Appendix A, where \( v : Z \to \mathbb{R} \) and \( v(z) \) is the expected discounted value of beginning a period with wealth \( z \), \( f : \Omega \to \mathbb{R} \) and \( f(\omega) \) is the expected discounted value of having the portfolio \( \omega \) before pairwise meetings, \( \pi : Z \to [0, 1] \) and \( \pi(z) \) is the fraction of each specialization type with wealth \( z \) at the start of a period, and \( \lambda : \Omega \to [0, 1] \) and \( \lambda(\omega) \) is the fraction of each specialization type with a portfolio \( \omega \) after bond purchases and before pairwise meetings.

The definition of a steady state is an obvious generalization of that in [13]. One crucial difference between them is in the set of feasible portfolios in a portfolio-choice stage. In [13], they allow a person with \( z \in Z \) to buy any lottery over portfolios in \( \Omega \) whose expected cost does not exceed \( z \). This formulation makes the numerical choice of a lottery over portfolios a high-dimensional linear programming problem. An alternative is to allow lotteries only over portfolios whose cost does not exceed
\(z, m_h + m_f + \gamma b \leq z\), as in [6]. This is relatively easy to deal with, but seems to give too much scope to the indivisibility of assets. For example, if \(z = 2\), there is a uniform money, and \(\gamma \in (0.5, 1]\), then a portfolio with two units of money and no bonds is optimal. It is also possible that a portfolio with one unit of money and one bond is optimal, a portfolio whose cost falls short of \(z\). We compromise between the above two definitions. We let a person with \(z \in \mathbb{Z}\) buy any lottery over portfolios in \(\Omega_z = \{\omega = (m_h, m_f, b) : m_h + m_f + \gamma(b - 1) \leq z\}\) whose expected cost does not exceed \(z\); namely, we let \(\Gamma(z, \gamma)\), a set of probability measures defined on \(\Omega_z\), be defined by

\[
\Gamma(z, \gamma) = \{\sigma : \mathbb{E}_\sigma(m_h + m_f + \gamma b) \leq z\},
\]

where, for \(\omega \in \Omega_z\), \(\sigma(\omega)\) is the probability of purchasing the portfolio \(\omega\), and \(\mathbb{E}_\sigma\) is the expectation with respect to \(\sigma\). That is, for a person with wealth \(z\), a portfolio whose value exceeds \(z\) by \(\gamma\), the unit cost of bond purchasing, is available. By doing so, we minimize the dimension of the portfolio-choice problem, subject to insuring that the optimal lottery satisfies the constraint in (III.1) with equality.

Then, a portfolio-choice problem is the following linear programming problem: choose \(\sigma \in \Gamma(z, \gamma)\) to maximize \(\sum \sigma(\omega)f(\omega)\), where \(f(\omega)\) is determined by the two-step maximization problem and \(\lambda\). (See Appendix A.) Let the set of maximizers of this problem be \(\Sigma(z, f, \gamma)\). Then \(\delta \in \Sigma(z, f, \gamma)\) is an optimal lottery, and \(\delta(\omega)\) is the probability of holding the portfolio \(\omega\).

For the uniform-money version, [13] proves the following existence result: if \(u'(0), \bar{z}, \text{ and } Z/\bar{z}\) are large enough, and \(\gamma\) is sufficiently close to unity, then there exists a steady state \((v, f, \pi, \lambda)\) with \(v\) strictly increasing and strictly concave, and with \(\pi\) having full support, a steady state that we label a nice steady state. The existence of a nice steady state for separate monies for some parameters is a straightforward extension of that result; if \(u'(0), \bar{z}, \text{ and } Z/\bar{z}\) are large enough, and \(\gamma\) is sufficiently close to unity, and if \(\phi\) is in the neighborhood of 1, then there exists a steady state \((v, f, \pi, \lambda)\) with \(v\) strictly increasing and strictly concave, and with \(\pi\) having full support. The proof rests on the previously established results in [6] and [14]. Call the mapping defined in [14] as \(T_0\) and let \(T_1\) be the corresponding mapping used to
define a steady state in our model. If $\phi = 1$, then $T_0$ and $T_1$ are the same. From [14], we know that $T_0$ has a fixed point index 1, and it can be shown that $T_1$ is upper hemicontinuous in $\phi$ at $\phi = 1$. Those two facts imply existence for $\phi$ near 1 via an argument that is similar to the implicit function theorem. We do not present the detailed proof because it is very similar to that in [6].

### III.5 Comparison in General Equilibrium: Numerical Examples

In this section, we make comparisons across steady states for a uniform money and for separate monies in terms of a representative agent welfare criterion, the inner product of $\pi v$. Mainly due to the endogeneity of the full-support steady-state distributions, it does not seem possible to derive analytical results for welfare. Hence, we rely on numerical examples.

A standard computational procedure to find a steady state involves iterating on the mapping that defines it. That is, the main numerical algorithm is as follows. We first start with an initial guess, $(v^0, \pi^0, \lambda^0)$. We let $\tilde{W} : [0, Z] \rightarrow \mathbb{R}$ be defined by $\tilde{W}(z) = \beta \tilde{v}[(1 - \tau(\lambda))z]$, where $\tilde{v}$ is the extension of $v$ to $[0, Z]$ by linear interpolation, $z$ is wealth after pairwise meeting but before tax, and $\tau(\lambda)$ is the inflation tax rate implied by $\lambda$. Now for each $\omega \in \Omega_z$, we can solve problem 1. Then implied payoff functions for consumers and producers, and $\lambda^0$ give the function $f(\omega)$. The next step is to solve the portfolio-choice problem. The solutions of this step, solutions of a linear programming problem, and the associated $f(\omega)$ are used to update the value function, giving us a $v^1$. In addition, those solutions and $\pi^0$ give $\lambda^1$. Finally, $\pi^1$ is obtained from wealth transfer functions implied by optimal portfolios and $\lambda^0$. This updating process is repeated until the convergence criterion is satisfied.\(^2\)

Several subtle issues are associated with this computational procedure. First, there are no theorems that guarantee that the iterations of such a mapping converge. Actually, a convergence problem arises in our examples if we iterate on the mapping

\(^2\)The convergence criterion is set to be $\max_j \left[ \frac{(v_j^{(n+1)} - v_j^{(n)})}{v_j^{(n)}} \right] < 10^{-4}$ for $v$, and $\sum_j (\theta_j^{(n+1)} - \theta_j^{(n)})^2/2 < 10^{-4}$ for $\pi$ and $\lambda$. 

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directly. Hence, we use a “relaxation method”; the initial values for the \((n + 1)\)-th iteration are a weighted average of outcomes of \((n)\)-th and \((n - 1)\)-th iterations.

Second, there are no uniqueness results even within the class of nice steady states. However, our computational procedure always converges to a steady state which does not seem sensitive to the initial condition we choose.

As noted above, people are allowed to trade lotteries at the portfolio-choice stage and in pairwise meetings. Our numerical examples with \(\gamma \in (0, 1)\) always give unique outcomes for both choice problems. For these outcomes, let \(\sigma^z\) be the best lotteries and let \(\omega^z\) be the associated optimal portfolios for the person whose wealth level is \(z\). Then it can be easily shown that

\[
[(1 - \beta)N]\pi v = \pi[\phi G_h + (1 - \phi)G_f]\pi'.
\]

Here, the element in row \(z\) and column \(z'\) of the matrix \(G_h\) (\(G_f\)) is

\[
\sum_{i,j} \sigma^z_i \sigma^z_j' \{u[y(\omega^z_i, \omega^z_j')] - y(\omega^z_i, \omega^z_j')\},
\]

where \(y(\omega^z_i, \omega^z_j')\) is the output when the consumer with a portfolio \(\omega^z_i\) meets the home producer (foreign producer) holding a portfolio \(\omega^z_j'\). Hence, an upper bound on \(\pi v\) would be obtained if \(y^*\), the maximizer of \(u(y) - y\), is produced and consumed in every single-coincidence meeting.

### III.5.1 Parameterizations

We need to specify \(u, N, Z, \bar{z}, \beta, \phi\) and \(\gamma\). We let \(u(y) = y^{1/2}/(1/2)\). This satisfies all our assumptions and has the virtue of implying linear first-order conditions for the choice of lotteries in meetings. It implies \(y^* = 1\) and \(u(y^*) - y^* = 1\). We let \(N = 3\), the smallest magnitude consistent with our assumption. We choose \(Z = 4\bar{z}\), which is large enough so that almost no one is at the upper bound of wealth in a steady state. We set \(\bar{z} = 10\). This choice of \(\bar{z}\) and normalization of the smallest unit of money imply that there are 41 elements in the set \(Z\), a conveniently small number. We set \(\beta = 0.9^{(1/12)}\), which we regard as arising from an annual discount factor of 0.9 with a monthly meeting frequency. Regarding \(\phi\), we study four different magnitudes
of it: $\phi = 0.95, \phi = 0.90, \phi = 0.70, \text{ and } \phi = 0.50$. The higher $\phi$ implies a lower uncertainty, and $\phi = 0.50$ is the highest degree of uncertainty in our model. We surmise that the money balance in separate monies would increase as $\phi$ approaches one half. Finally, we study four different discounts on bonds: $\gamma = 1.00, \gamma = 0.99, \gamma = 0.98, \text{ and } \gamma = 0.97$. The economy with $\gamma = 1.00$ is equivalent to a no-bond economy, and $\gamma = 0.98 \text{ and } \gamma = 0.97$ imply high nominal interest rates on bonds; for example, $\gamma = 0.98$ corresponds to an annual nominal interest rate of about 27%. In either a uniform money or separate monies, we expect that the total money balance would decrease as $\gamma$ decreases.

### III.5.2 Results

For the above numerical environment, we find steady states for all combinations of $(\phi, \gamma)$. We first present the results for $\phi = 0.95$. Table III.1 contains some summary statistics and welfare results.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>uniform money</th>
<th>separate monies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.00 0.99 0.98 0.97</td>
<td>1.00 0.99 0.98 0.97</td>
</tr>
<tr>
<td>$E_y$</td>
<td>0.984 0.976 0.967 0.958</td>
<td>0.984 0.965 0.952 0.944</td>
</tr>
<tr>
<td>$E_\sigma(d/y)$</td>
<td>0.525 0.498 0.459 0.436</td>
<td>0.525 0.507 0.569 0.570</td>
</tr>
<tr>
<td>$E_\sigma(d-d_1/d) \times 10^2$</td>
<td>0.000 0.001 0.002 0.002</td>
<td>0.000 2.099 8.930 9.194</td>
</tr>
<tr>
<td>$\tau(\lambda) \times 10^2$</td>
<td>– 0.901 1.806 2.708</td>
<td>– 0.832 1.805 2.709</td>
</tr>
<tr>
<td>$E_\sigma(m_h + m_f)$</td>
<td>1.006 0.999 0.998 0.998</td>
<td>2.010 1.693 1.016 1.001</td>
</tr>
<tr>
<td>$\pi v$</td>
<td>38.312 38.309 38.303 38.296</td>
<td>38.312 38.302 38.291 38.284</td>
</tr>
</tbody>
</table>

The first row and the second row describe average consumption and average price over single-coincidence meetings, respectively. When $\phi$ is high enough, the properties on trading outcomes shown in partial equilibrium are preserved. The higher average price in separate monies is largely attributed to high frequency of step-2 trades. The fraction of step-2 trades in terms of average wealth transfer over single-coincidence meetings is in the third row. (See also Figure III.3 and III.4 in Appendix C.) Notice that when $\gamma = 1$, the case analogous to that in the existing literatures in the sense that it is equivalent to a no-bond economy, there is no sharp
distinction in trading outcomes between a uniform money and separate monies. This is mainly because in almost all single-coincidence meetings in separate monies, cash constraints are not binding. The one exception occurs in a single-coincidence meeting where the consumer has only a unit of wealth, but the measure of people with one unit is almost zero in the steady state. The fourth row shows the inflation tax rate.

The fifth row describes average money holdings. The average money holdings for $\gamma = 1$ is computed by selecting a particular portfolio among optimal portfolios. That is, when $\gamma = 1$, optimal lotteries are not unique. To see this, consider a person holding $z$ amount of wealth. Let $m^z$ be the minimal money holdings that guarantees null step-2 trade in all possible meetings with a domestic producer. Then, for any $\omega$ such that $\omega \in \bar{\omega}^z = \{(m_h, m_f, b) : m_i \geq m^z \text{ and } b = z - m_h - m_f\}$, $f(\omega)$ is the same. (Because a foreign producer is identical to a domestic producer if their wealth holdings are the same, $m^z$ also guarantees null step-2 trade in all possible meetings with a foreign producer.) If the set $\bar{\omega}^z$ contains multiple elements, then we choose a portfolio which minimizes money holdings, say $(m^z, m^z, z - 2m^z)$. We make that choice, because it is the limit of money holdings when $\gamma$ approaches one.

The average money balance for separate monies varies more with respect to $\gamma$ than it does for a uniform money. In addition, for a given $\gamma$, money balance varies hardly at all with wealth both in a uniform money and separate monies (see Table III.4 and III.6 in Appendix C). Although lotteries are allowed in our examples, indivisibility still matters to some extent with $\bar{z} = 10$ (see Table III.5 in Appendix C). The small discounting of future consumption seems to be another source of such inelasticity (see Table III.4 in Appendix C).

The last row describes expected utility. For this magnitude of $\phi$, a uniform money has a (weakly) higher welfare than do separate monies. Underlying these welfare consequences are outcomes of consumption distributions or equivalently, outcomes for $v$ and $\pi$. Figure III.1 shows consumption distributions for $\gamma = 0.98$. Both distributions are concentrated around the first-best consumption distribution, but that for a uniform money seems to be slightly closer to the first-best.
Figure III.1: Consumption distributions: $\phi = 0.95$ and $\gamma = 0.98$.

Figure III.2: $v$ and $\pi$: $\phi = 0.95$ and $\gamma = 0.98$.

Figure III.2 shows $v$ and $\pi$ for $\gamma = 0.98$. The poor have a higher expected discounted value under separate monies for a given wealth than they do under a uniform money. The distributions of wealth for the two regimes show a very similar shape, but that for separate monies seems to be slightly more bunched around
average wealth. Notice that although these distributions have full support, almost no one has as much as 3 times average wealth.

Now we vary the magnitude of $\phi$. Table III.2 compares the trading outcomes for $\phi = 0.95$ and $\phi = 0.50$ in separate monies.\(^3\) (With a uniform money, $\phi$ does not matter.) In economies for $\phi = 0.50$, the cost of abandoning a foreign money to hold more bonds would be very high relative to that for $\phi = 0.95$ because people meet foreign- or home-country people with the same probability. This effect seems to be embedded in average money balances; the average money balance for $\phi = 0.50$ is much larger than that for $\phi = 0.95$. As a consequence, there are fewer step-2 trades for $\phi = 0.50$, and, therefore, a higher average output and a lower average price.

Table III.2: Trading outcomes for different $\phi$ in separate monies

<table>
<thead>
<tr>
<th></th>
<th>$\phi = 0.95$</th>
<th>$\phi = 0.50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>0.99 0.98 0.97</td>
<td>0.99 0.98 0.97</td>
</tr>
<tr>
<td>$\mathbb{E}y$</td>
<td>0.965 0.952 0.944</td>
<td>0.975 0.972 0.960</td>
</tr>
<tr>
<td>$\mathbb{E}_\sigma(d/y)$</td>
<td>0.507 0.569 0.570</td>
<td>0.476 0.405 0.360</td>
</tr>
<tr>
<td>$\mathbb{E}<em>\sigma(d</em>{d_1}/d) \times 10^2$</td>
<td>2.099 8.930 9.194</td>
<td>0.033 0.054 0.127</td>
</tr>
<tr>
<td>$\tau(\lambda) \times 10^2$</td>
<td>0.832 1.805 2.709</td>
<td>0.802 1.607 2.416</td>
</tr>
<tr>
<td>$\mathbb{E}_\sigma(m_h + m_f)$</td>
<td>1.693 1.016 1.001</td>
<td>1.999 1.999 1.998</td>
</tr>
<tr>
<td>$\pi v$</td>
<td>38.302 38.291 38.284</td>
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</tbody>
</table>

Table III.3 contains consumption equivalent welfare costs relative to the first-best consumption distribution for all combinations of $(\phi, \gamma)$.\(^4\) Primarily because we allow people to trade lotteries, consumptions in all examples are bunched around the first-best output level, $y^*$. Hence, the welfare cost in each economy is very small. Our results show that when the bond discount is sufficiently high and $\phi$ is close to one half, a uniform money has a higher welfare cost than do separate monies. We suspect that this is due to the financing scheme for interest payment on bonds: when $\phi$ is one half, the inflation-tax rate with a uniform money is much higher than

---

\(^3\)The steady state for $\phi = 0.50$ is not a nice steady state because $v$ is not strictly concave when the wealth level is very low.

\(^4\)We first find an additive compensating variation of consumption for each economy which makes its welfare level equal to that at the upper bound on welfare, $[(1-\beta)N]^{-1}$. Then the welfare cost of that economy is a ratio of that additive compensating variation to the average consumption of that economy.
with separate monies. Such higher inflation would tend to have an adverse effect on welfare through its effect on output.

Table III.3: Welfare costs (uniform, separate, %) relative to first best

<table>
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<tr>
<th>γ</th>
<th>φ</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>0.95</th>
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<td>(0.020, 0.021)</td>
<td>(0.020, 0.021)</td>
<td>(0.020, 0.020)</td>
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<td>(0.031, 0.033)</td>
<td>(0.031, 0.033)</td>
<td>(0.031, 0.041)</td>
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<tr>
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<td>0.9</td>
<td>(0.041, 0.030)</td>
<td>(0.041, 0.031)</td>
<td>(0.041, 0.043)</td>
<td>(0.041, 0.074)</td>
</tr>
<tr>
<td>0.97</td>
<td>0.95</td>
<td>(0.063, 0.050)</td>
<td>(0.063, 0.052)</td>
<td>(0.063, 0.072)</td>
<td>(0.063, 0.095)</td>
</tr>
</tbody>
</table>

III.6 Summary and Conclusions

In this paper, we generalize the model in Zhu and Wallace [13] and use it to compare a uniform money with separate monies. Aside from the trading procedure in pairwise meetings and a general upper bound on individual money holdings, our model differs from the previous literature by permitting people to hold money and a higher-return asset. Our results show that the welfare comparison between a uniform money and separate monies can go either way.

Although our model contains, we think, only indispensable ingredients, it is very complicated, and does not lend itself to analytical results. The second-best nature of the welfare comparison could be reduced by replacing the government bonds with a real asset whose return in the form of a real dividend is realized as part of the centralized market in which people in the current model buy and sell monies and government bonds. The resulting model would be a version of that in Lagos and Wright [5], but need not have quasi-linear preferences. In such a model, there would be no need to finance interest payments.

Finally, it should be noted that our comparison of welfare is a comparison across steady states. It does not deal with the transition from one monetary regime to another. To deal with such transitions, we would have to compute approximations to equilibrium paths.
III.7 Appendix A: Definition of a steady state

We first start with the portfolio-choice stage. We let $\Gamma_1(z, \gamma)$, a subset of probability measures defined on $\Omega_z = \{\omega = (m_h, m_f, b) \in \Omega : m_h + m_f + \gamma(b - 1) \leq z\}$, be defined by

$$\Gamma_1(z, \gamma) = \{\sigma : \mathbb{E}_{\sigma}(m_h + m_f + \gamma b) \leq z\}.$$ 

It follows that $v_t$ and $f_t$ satisfy

$$v_t(z) = \max_{\sigma \in \Gamma_1(z, \gamma)} \sum \sigma(\omega) f_t(\omega). \quad (III.2)$$

We denote the set of maximizers in (III.2) by $\Sigma_1(z, f_t, \gamma)$. Then, we define a set of distributions on $\Omega$, $\Lambda(f_t, \pi_t, \gamma)$, by

$$\Lambda(f_t, \pi_t, \gamma) = \{\lambda_t : \lambda_t(\omega) = \sum_z \pi_t(z) \varphi(\omega) \text{ for } \varphi \in \Sigma_1(z, f_t, \gamma)\}.$$ 

For pairwise trades, first, we let $W : Z \to \mathbb{R}$ denote expected discounted utility after the date-$t$ pairwise meetings but before people are taxed. Let $I$ be an indicator that takes the value 0 if the producer and consumer are from different countries and takes the value 1 otherwise. For step-1 of problem 1, we let $\Gamma^i_{21}(\omega, \omega'; W)$, a set of probability measures on $\mathbb{R}_+ \times \{0, 1, \ldots, \min\{Im_h + (1 - I)m_f, Z - \omega'\}\}$, be

$$\Gamma^i_{21}(\omega, \omega'; W) = \{\sigma : \mathbb{E}_{\sigma}[-y^i_1 + W(\omega'_z + d^i_1)] \geq W(\omega'_z)\}, \quad (III.3)$$

where superscript $i$ follows the value of $I$. Then the consumer’s payoff from step-1 trade is

$$g^i_c(\omega, \omega'; W) = \max_{\sigma \in \Gamma^i_{21}(\omega, \omega'; W)} \mathbb{E}_{\sigma}[u(y^i_1) + W(\omega_z - d^i_1)],$$

where $g^0_c$ is the consumer’s payoff when he meets a foreign producer and $g^1_c$ is the consumer’s payoff when he meets a home producer. For step-2 of problem 1, we let $\Gamma^i_{22}(\omega, \omega'; W)$, a set of probability measures on $\mathbb{R}_+ \times \{0, 1, \ldots, \min\{\omega_z, Z - \omega'\}\}$, be

$$\Gamma^i_{22}(\omega, \omega'; W) = \{\sigma : \mathbb{E}_{\sigma}[u(y^i) + W(\omega_z - d^i)] \geq g^i_c(\omega, \omega'; W)\}.$$ 

Then the producer’s payoff is

$$g^i_p(\omega, \omega'; W) = \max_{\sigma \in \Gamma^i_{22}(\omega, \omega'; W)} \mathbb{E}_{\sigma}[-y^i + W(\omega'_z + d^i)]. \quad (III.4)$$
It follows that the expected payoff from holing the portfolio $\omega$ before meetings is

$$f_t(\omega) = N^{-1}\{\phi\sum\lambda_t(\omega')[(g_c^1(\omega,\omega';W) + g_p^1(\omega',\omega;W)] + (1 - \phi)\sum\lambda_t(\omega')\{g_c^0(\omega,\omega';W) + g_p^0(\omega',\omega;W)\} + (N - 2)W(\omega_z)\}. \quad (III.5)$$

Let the set of maximizers in (III.4) be $\Sigma_2^i(\omega,\omega';W)$ and, for a maximizer, we can denote the maximizing $y$ by $\tilde{y}$.\(^5\) Let $\tilde{z}_i(\omega,\omega') = \{d^i : \omega_z - d^i = z\}$ for $z \in Z$, the set of wealth transfers that leaves the consumer with wealth $z$ before imposing the tax.

Then we define $\Upsilon(\omega,\omega';W)$, a subset of probability measures on $Z$, by

$$\Upsilon(\omega,\omega';W) = \{\varsigma : \varsigma(z) = \sum_{\tilde{z}_i(\omega,\omega')} \phi\tilde{z}(\tilde{y}, d^1) + \sum_{\tilde{z}_0(\omega,\omega')} (1 - \phi)\tilde{z}(\tilde{y}, d^0) \text{ for } \tilde{\varsigma} \in \Sigma_2^i(\omega,\omega';W)\}.$$

Then we define a set of post-trade and pre-tax distributions on $Z$ by

$$\Psi(v_{t+1}, \lambda_t) = \{\kappa : \kappa(z) = N^{-1}\sum_{(\omega,\omega')} \lambda_t(\omega)\lambda_t(\omega')[\varsigma(z) + \varsigma(\omega_z - z + \omega_z')] + (N - 2)N^{-1}\sum_{\omega} I(z;\omega)\lambda_t(\omega) \text{ for } \varsigma \in \Upsilon(\omega,\omega';W)\},$$

where $I(z;\omega) = 1$ if $\omega_z = z$ and is 0 otherwise. The dependence of $\Psi$ on $(v_{t+1}, \lambda_t)$ is through the dependence of $W$ on $(v_{t+1}, \lambda_t)$, which involves the following description of the taxing of end-of-trading wealth.

We let $\tilde{z}_\lambda = \sum \lambda_t(\omega)\omega_z$. Then $\tau(\lambda) = 1 - \tilde{\epsilon}/\tilde{z}_\lambda$ denotes the inflation tax rate needed to finance interest payments on bonds. To ensure that after-tax wealth is in the set $Z$, we let each person choose a lottery subject to having an expected wealth equal to $z'(\tilde{\epsilon}/\tilde{z}_\lambda)$, where $z'$ is end-of-trade wealth. Let $\Gamma_3(z';\lambda_t)$, a set of probability measure on $Z$, be defined by

$$\Gamma_3(z';\lambda_t) = \{\sigma : E_{\sigma}(z) = z'(\tilde{\epsilon}/\tilde{z}_\lambda)\},$$

where $E_{\sigma}(z)$ is the expectation of post tax-wealth implied by $\sigma$. Then we let

$$W(z') = \beta \max_{\sigma \in \Gamma_3(z';\lambda_t)} E_{\sigma}v_{t+1}(z). \quad (III.6)$$

\(^5\)See Berentsen, Molico and Wright [1].
Let $\Sigma_3(z'; v_{t+1}, \lambda_t)$ be the set of maximizers in (III.6). Then a set of post-tax distributions on $\mathbf{Z}$, $\Pi(v_{t+1}, \lambda_t)$, can be defined by

$$
\Pi(v_{t+1}, \lambda_t) = \{ \pi_{t+1} : \pi_{t+1}(z) = \sum_{z'} \kappa(z') \delta(z; z'), \quad \text{for } \kappa \in \Psi(v_{t+1}, \lambda_t) \text{ and } \delta(\cdot; z') \in \Sigma_3(z'; v_{t+1}, \lambda_t) \}.
$$

Now for given $\phi, \gamma$ and an initial distribution of wealth on $\mathbf{Z}$, an equilibrium is a sequence $\{v_t, f_t, \pi_{t+1}, \lambda_t\}_{t=0}^{\infty}$ that satisfies (III.2), (III.5), $\pi_{t+1} \in \Pi(v_{t+1}, \lambda_t)$ and $\lambda_t \in \Lambda(f_t, \pi_t, \gamma)$. A steady state is $(v, f, \pi, \lambda)$ such that $\{v_t, f_t, \pi_{t+1}, \lambda_t\}_{t=0}^{\infty} = (v, f, \pi, \lambda)$ is an equilibrium for $\pi_0 = \pi$. Notice that a steady state for a given $\phi = 1$ is equivalent to a steady state for a uniform money and in that case, our model is the same as the model in [13].
III.8 Appendix B: Proofs

Proof of Lemma III.3.1 Since the producer cannot be worse off after trade

\[ y_1(\omega, \omega') + \Delta y(\omega, \omega') \leq \tilde{W}[\omega'_z + d_1(\omega, \omega') + \Delta d(\omega, \omega')] - \tilde{W}(\omega'_z), \tag{III.7} \]

where \( \Delta y(\omega, \omega') \equiv [y(\omega, \omega') - y_1(\omega, \omega')] \) and \( d_1(\omega, \omega') + \Delta d(\omega, \omega') \) is a wealth transfer corresponding to the output \( y_1(\omega, \omega') + \Delta y(\omega, \omega') \). From here on, we suppress the dependence of a trading outcome on portfolios because the context is clear. We know from Zhu and Wallace [13], proposition 1, that \((\Delta y, \Delta d) \geq (0, 0)\). If \( \Delta y = 0 \), then the claim is obvious. If \( \Delta y \neq 0 \), then

\[ \frac{1}{y_1 + \Delta y} \geq \frac{1}{\tilde{W}(\omega'_z + d_1 + \Delta d) - \tilde{W}(\omega'_z)}, \tag{III.8} \]

because \( y_1 + \Delta y > 0 \) and \( \tilde{W}(\omega'_z + d_1 + \Delta d) - \tilde{W}(\omega'_z) > 0 \). Then, for \( d_1 + \Delta d > 0 \),

\[
\frac{d}{y} = \frac{d_1 + \Delta d}{y_1 + \Delta y} \\
\geq \frac{d_1 + \Delta d}{\tilde{W}(\omega'_z + d_1 + \Delta d) - \tilde{W}(\omega'_z)} \\
\geq \frac{d_1}{\tilde{W}(\omega'_z + d_1) - \tilde{W}(\omega'_z)} = \frac{d_1}{y_1},
\]

where the first inequality follows from (III.8) and the second inequality follows from the concavity of \( \tilde{W} \); that is, \([\tilde{W}(\omega'_z + d_1 + \Delta d) - \tilde{W}(\omega'_z)]/(d_1 + \Delta d) \) is the slope of \( \tilde{W} \) in the interval \((\omega'_z, \omega'_z + d_1 + \Delta d)\), and \([\tilde{W}(\omega'_z + d_1) - \tilde{W}(\omega'_z)]/d_1 \) is the slope in the interval \((\omega'_z, \omega'_z + d_1)\). By the concavity of \( \tilde{W} \), the former is weakly less than the latter.

Proof of Lemma III.3.2 (i) \( y^U(\omega, \omega') \geq y^S(\omega, \omega') \). I do a proof by contradiction. Let \((y^U, d^U)\) and \((y^S, d^S)\) be the unique solutions to problem 1 in a uniform money and separate monies, respectively. Let \( g^U_1 = u(y^U_1) + \tilde{W}(\omega_z - d^U_1) \) and similarly for \( g^S_1 \). Because the cash constraint is (weakly) tighter under separate monies, \( g^U_1 \geq g^S_1 \). (That is, the step-1 objective is weakly higher under a uniform money.) Now,

\[
y^U = u^{-1}[g^U_1 - \tilde{W}(\omega_z - d^U)] \\
\equiv u^{-1}[g^S_1 - \tilde{W}(\omega_z - X)] \\
< y^S = u^{-1}[g^S_1 - \tilde{W}(\omega_z - d^S)], \tag{III.9}
\]

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where the first equality follows from equality of the constraint in step-2, the second equality defines $X$, the third inequality follows from the contradiction, and the last equality follows from the constraint exactly as in the first equality. It follows the $d^U \leq X < d^S$. Now for some arbitrary small $\varepsilon > 0$ satisfying $X < d^S - \varepsilon \equiv \tilde{d}^s$, the concavity of $\tilde{W}$ gives

\[
\Delta \tilde{W}^u \equiv [\tilde{W}(\omega'_z + d^U + \varepsilon) - \tilde{W}(\omega'_z + d^U)] \\
\geq [\tilde{W}(\omega'_z + X + \varepsilon) - \tilde{W}(\omega'_z + X)] \\
\geq [\tilde{W}(\omega'_z + \tilde{d}^s + \varepsilon) - \tilde{W}(\omega'_z + \tilde{d}^s)] \equiv \Delta \tilde{W}.
\] (III.10)

Then, by (III.9), the strict convexity of $u^{-1}$, and the concavity of $\tilde{W}$,

\[
\Delta y^u \equiv u^{-1}[g^U_{1} - \tilde{W}(\omega_z - d^U - \varepsilon)] - u^{-1}[g^U_{1} - \tilde{W}(\omega_z - d^U)] \\
\leq u^{-1}[g^S_{1} - \tilde{W}(\omega_z - X - \varepsilon)] - u^{-1}[g^S_{1} - \tilde{W}(\omega_z - X)] \\
< u^{-1}[g^S_{1} - \tilde{W}(\omega_z - \tilde{d}^s - \varepsilon)] - u^{-1}[g^S_{1} - \tilde{W}(\omega_z - \tilde{d}^s)] \equiv \Delta y. \quad \text{(III.11)}
\]

Then, $\Delta y > \Delta y^u > \Delta \tilde{W}^u \geq \Delta \tilde{W}$, where the first inequality follows from (III.11), the second inequality holds because $(y^U, d^U)$ solves step-2 problem under a uniform money, and the last inequality follows from (III.10). But $\Delta y > \Delta \tilde{W}$ can be written as

\[-u^{-1}[g^S_{1} - \tilde{W}(\omega_z - \tilde{d}^s)] + \tilde{W}(\omega'_z + \tilde{d}^s) > -u^{-1}[g^S_{1} - \tilde{W}(\omega_z - d^S)] + \tilde{W}(\omega'_z + d^S),
\]

a contradiction.

\[\text{(ii)} \quad d^S(\omega, \omega')/y^S(\omega, \omega') \geq d^U(\omega, \omega')/y^U(\omega, \omega').\] If the cash constraint is not binding under a uniform money, then the claim is obvious because $d/y \geq d_1/y_1$ by Lemma 1. Consider then the case in which it is binding. It suffices to show $d^U \leq d^S$ because $y^U \geq y^S$ from (i). Let $(y^U, d^U)$ be the unique solution to problem 1 in a uniform money and let $(y^S, d^S)$ be that for separate monies. By way of a contradiction, suppose $d^U > d^S$. Then there is $\varepsilon > 0$ such that $d^U = d^S + \varepsilon$. Now,

\[
\Delta \tilde{W} \equiv [\tilde{W}(\omega'_z + d^U) - \tilde{W}(\omega'_z + d^U - \varepsilon)] \\
> u^{-1}[g^U_{1} - \tilde{W}(\omega_z - d^U)] - u^{-1}[g^U_{1} - \tilde{W}(\omega_z - d^U + \varepsilon)] \\
\geq u^{-1}[g^S_{1} - \tilde{W}(\omega_z - d^U)] - u^{-1}[g^S_{1} - \tilde{W}(\omega_z - d^U + \varepsilon)] \equiv \Delta y,
\]

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where the first inequality holds because $(y^U, d^U)$ solves the step-2 problem in a uniform money, and the second inequality follows from $g_1^U \geq g_1^S$ and the strict convexity of $u^{-1}$. But $\Delta \tilde{W} > \Delta y$ can be written as
\[
-u^{-1}[g_1^S - \tilde{W}(\omega_z - d^U)] + \tilde{W}(\omega_z' + d^U) > -u^{-1}[g_1^S - \tilde{W}(\omega_z - d^S)] + \tilde{W}(\omega_z' + d^S),
\]
a contradiction.

(iii) $y_1^U(\omega, \omega') \geq y_1^S(\omega, \omega')$. Because the cash constraint is (weakly) tighter under separate monies, $d_1^U \geq d_1^S$. In addition, because the producer’s participation constraint in step-1 holds with equality, $\tilde{W}(\omega_z' + d_1) - \tilde{W}(\omega_z') = y_1$. Therefore, $y_1^U \geq y_1^S$.

(iv) $d_1^S(\omega, \omega')/y_1^S(\omega, \omega') \leq d_1^U(\omega, \omega')/y_1^U(\omega, \omega')$. Because $d_1^U \geq d_1^S$ and $y_1^U \geq y_1^S$ from (iii),
\[
\frac{d_1^U}{y_1^U} - \frac{d_1^S}{y_1^S} = \frac{d_1^S + \Delta d}{\tilde{W}(\omega_z' + d_1^S + \Delta d) - \tilde{W}(\omega_z')} - \frac{d_1^S}{\tilde{W}(\omega_z' + d_1^S) - \tilde{W}(\omega_z')} \geq 0,
\]
where the equality holds because the producer’s participation constraint in step-1 holds with equality and where the inequality follows from the concavity of $\tilde{W}$ exactly as in the last step of the proof of Lemma 1.
### Appendix C: The details of numerical results

Table III.4: Optimal portfolios for a uniform money: $\gamma = 0.98$.

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<th>$z$</th>
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<th>$\beta = 0.90^{1/4}$</th>
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Table III.5: Optimal portfolios for a uniform money: $\bar{z} = 20$, $\gamma = 0.98$ and $\beta = 0.90^{(1/12)}$.

<table>
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<th>b</th>
<th>σ</th>
<th>m</th>
<th>b</th>
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Figure III.3: $E_\phi d_2(z, z')$: $\phi = 0.95$ and $\gamma = 0.99$.

Figure III.4: $E_\phi d_2(z, z')$: $\phi = 0.95$ and $\gamma = 0.98$. 
References


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- Optimal divisibility of money when money is costly to produce (with Neil Wallace)
- Is uniform money always better than separate monies?

Presentation
- Sixteenth-Century Replacement Costs of Coins: Implications for Optimal Financing and Divisibility, at the Cornell–PSU Macro Workshop, Fall 2004
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