NUMERICAL SIMULATION OF MULTI-DIMENSIONAL
ACOUSTIC PROPAGATION IN AIR INCLUDING THE EFFECTS
OF MOLECULAR RELAXATION

A Thesis in
Acoustics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

May 2006
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A computational acoustic propagation model based upon the Navier-Stokes equations is created that is able to simulate the effects of absorption and dispersion due to shear viscosity, bulk viscosity, thermal conductivity and molecular relaxation of nitrogen and oxygen in one or two dimensions. The model uses a fully nonlinear constitutive equation set that is closed using a thermodynamic entropy relation and a van der Waals equation of state. The use of the total variables in the equations rather than the perturbed (acoustical) variables allow for the extension of the model to include wind, temperature profiles, and other frequency independent conditions. The method of including sources in the model also allow for the incorporation of multiple spatially and temporally complex sources.

Two numerical methods are used for the solution of the constitutive equations: a dispersion relation preserving scheme, which is shown to be efficient and accurate but unsuitable for shock propagation; and a weighted essentially non-oscillatory scheme which is shown to be able to stably propagate shocks but at considerable computational cost. Both of these algorithms are utilized in this investigation because their individual strengths are appropriate for different situations.

It is shown that these models are able to accurately recreate many acoustical phenomena. Wave steepening in a lossless and thermoviscous medium is compared to the Fubini solution and Mendousse’s solution to the Burgers equation, respectively, and the Fourier component amplitudes of the first harmonics is shown to differ from these solutions by at most 0.21 %. Nonlinear amplification factors upon rigid boundaries for high incident pressures and its comparisons to the Pfriem solution is shown to differ by at most 0.015 %. Modified classical absorption, nitrogen relaxation absorption, and oxygen relaxation absorption is shown to differ from the analytical solutions by at most 1 %. Finally, the dispersion due to nitrogen relaxation and oxygen relaxation are also shown to differ from the analytical so-
olutions by at most 1%. It is believed that higher resolution grids would decrease the error in all of these simulations.

A number of simulations that do not have explicit analytical solutions are then discussed. To demonstrate the model’s ability to propagate multi-dimensional shocks in two dimensions, the formation of a Mach stem is simulated. The amplification factors determined in the test demonstrate a qualitative similarity with discussions in the literature and explosion data. The ability of the algorithm to propagate jet noise is then investigated using full scale jet noise as the input into the algorithm. The waveforms predicted by the model are compared to a Burgers equation algorithm and using a weak shock theory analysis of the shock propagation speeds, an under-prediction of shock coalescence is noted in the Burgers equation algorithm. To establish if the under-prediction of shock coalescence by the Burgers equation algorithm is the cause of the discrepancy between its predictions and recently measured scale model data, the WENO scheme is also used to propagate the scale model jet noise. The predictions by the two models agree very well for both cold and heat simulated jet cases which is due in part to the relatively small amplitudes and propagation distances. The two models, however, do not agree very well with the experimental data and it is concluded that more work is needed to determine the precise reasons for this discrepancy.
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List of Symbols

\( A \)  Amplitude
\( A_\nu \)  Affinity Associated with the \( \nu \)-type Molecule
\( c \)  Frozen (high frequency limit) Speed of Sound
\( c_0 \)  Equilibrium (low frequency limit) Speed of Sound
\( c_p \)  Specific Heat at Constant Pressure
\( c_{v\nu} \)  Specific Heat at Constant Volume Associated with the \( \nu \)-type Molecule
\( e \)  Unit Vector
\( \mathbf{F} \)  Flux Terms in the \( x \)-direction in Vector Form
\( f_{N_2} \)  Relaxation Frequency of Nitrogen
\( f_{O_2} \)  Relaxation Frequency of Oxygen
\( \mathbf{G} \)  Flux Terms in the \( y \)-direction in Vector Form
\( \mathbf{H} \)  Source Terms in Vector Form
\( h \)  Fraction of Molecules in Air that is Water Vapor
\( k \)  Wavenumber
\( M \)  Mach number
\(n_\nu/n\)  Fraction of all Molecules that is of Species \(\nu\)

\(p\)  Total Pressure

\(p_0\)  Ambient Pressure

\(p_{vp}\)  Saturation Vapor Pressure

\(R\)  Gas Constant

\(RH\)  Relative Humidity, in Percentage

\(s_{fr}\)  Total Frozen Entropy

\(s_{fr0}\)  Ambient Frozen Entropy

\(T\)  Total Temperature

\(T_0\)  Ambient Temperature

\(T_\nu\)  Apparent Vibration Temperature associated with the \(\nu\)-type Molecule

\(T^{*}_\nu\)  Molecular Constant of Molecule \(\nu\).

\(t\)  Time

\(t'\)  Retarded Time

\(U_{sh}\)  Shock Speed

\(u\)  Total Velocity in the \(x\)-direction

\(\mathbf{v}\)  Total Velocity in Vector form

\(v\)  Total Velocity in the \(y\)-direction

\(v_{ph}\)  Phase Speed

\(\mathbf{w}\)  Solution Vector

\(\bar{x}\)  Shock Formation Distance

\(\alpha\)  Gaussian Half Width

\((\alpha_\nu, \lambda)_m\)  Maximum Absorption per Wavelength associated with \(\nu\)-type molecule

\(\beta\)  Coefficient of Nonlinearity

xix
\( \gamma \) Ratio of Specific Heats
\( \Gamma \) Gol’dberg Number
\( \delta_{ij} \) Kronecker Delta
\( \epsilon \) Acoustic Mach Number
\( \kappa \) Thermal Conductivity
\( \lambda \) Wavelength
\( \mu \) Shear Viscosity
\( \mu_B \) Bulk Viscosity
\( \phi \) Rate of Shear Tensor
\( \phi_e \) Earnshaw Phase Variable
\( \rho \) Total Density
\( \rho_0 \) Ambient Density
\( \sigma \) Distance Relative to the Shock Formation Distance
\( \sigma_s \) Source Terms of the Entropy Equation
\( \tau \) Retarded time
\( \tau_\nu \) Apparent Vibration Temperature associated with the \( \nu \)-type Molecule
\( \omega \) Radial Frequency
I would like to first thank my advisor, Anthony Atchley, for supporting me and giving me considerable freedom to determine the direction of this research. I would also like to thank the members of my committee for their contributions and revisions: Vic Sparrow, Philip Morris, and Thomas Gabrielson. Thanks also to Said Boluriaan Esfahaani, who was a great help to me, especially in the early stages of this research.

This research was supported by the Exploratory and Foundational Program of the Applied Research Laboratory at the Pennsylvania State University.

The scale model jet noise data is courtesy of Benoit Petitjean and Dennis McLaughlin, and their generosity is much appreciated. Thanks to Kent Gee for the F/A-18E jet noise data and his Burgers equation solution algorithm that is used for comparison to the model developed in this work. I would like to acknowledge Henry “Skip” Knoble for assisting me with the parallel computations used for some of the results and also for being the only other person who has yet dared to analyze and improve my code.

Finally, thanks to Nancy Scott for having an inexplicably solid faith in my abilities and for being a wonderful companion throughout. Her encouragement and advice have been invaluable to my professional as well as personal development.
Dedication

This dissertation is dedicated to Allan L. Sorem (1918–2004), my grandfather. The example he set as a man and as a scientist will forever be an influence upon my life.
Introduction

1.1 Introduction

Computational models in acoustics are typically tailored to specific propagation scenarios and hence most require at least a significant revision in order to be useful for other acoustic predictions. The fluid dynamics underlying these acoustic simulations are the same with few exceptions, but the models themselves are unique. The current state of computing has allowed for many advances; the work here focuses on a general acoustic propagation algorithm that can be used for many different applications.

1.2 Motivation

This work originated from a desire to understand the propagation of high amplitude jet noise. Jet noise is very complex and there is much that is still unknown, although considerable progress has been made in the field. Research performed over the past five decades has determined that the noise produced by jet engines has spatial extent, includes moving directional sources of sound, and is the result of small scale turbulence, large scale turbulence, Mach wave radiation, and perhaps
other mechanisms. A thorough review of the progress made in jet noise research can be found in the review by Tam[1].

Predicting the nonlinear evolution of broadband high amplitude noise, and more specifically, jet noise, as it propagates is still a relatively new field, although some notable experimental and theoretical work was done in the 1980’s by Morfey and Howell[2]. The computational models used for the propagation of jet noise in acoustics are usually based on the Burgers equation, a nonlinear equation that describes the evolution of one-dimensional progressive waves. The equation can take account of quadratic nonlinearity, atmospheric absorption and dispersion mechanisms, and geometrical spreading. The purpose of this dissertation is to create a computational model to study the evolution of jet noise that uses the fluid dynamic equations that form the basis of acoustic propagation. To solve these equations, substantially different numerical methods need to be employed as well. The development of an algorithm that propagates broadband high amplitude noise, i.e. jet noise, using significantly different equations and numerical methods is useful for verifying the results obtained using Burgers equation methods. Because the development of a jet noise propagation algorithm is highly demanding and fluid dynamic equations are used, the resulting algorithm will be useful not only for jet noise propagation, but also for many other acoustical situations. In essence, a general acoustic propagation algorithm is developed which can be applied to a large variety of propagation scenarios.

In order to ensure that the model accurately simulates acoustic propagation, many phenomena must be included into the equation set that is used. The requirements include accurate absorption, dispersion, and nonlinear phenomena. A discussion of these phenomena and an explanation as to why they are important for jet noise propagation will be the focus of the next section.
1.3 Salient Acoustical Phenomena

1.3.1 Nonlinearity

One of the two most important nonlinear phenomena in finite amplitude wave propagation is wave steepening. Wave steepening is a consequence of the fact that high amplitude disturbances travel faster than low amplitude disturbances, which can be represented using

\[ c = c_0 + \beta u, \]  

(1.1)

where \( c_0 \) is the small signal sound speed, \( u \) the particle velocity and \( \beta \) the coefficient of nonlinearity. Since the high amplitude disturbances travel at a higher velocity than the ambient portion of the wave, the peak positive pressure eventually catches up to the peak negative pressure, causing the wave, in theory, to become multivalued. It is impossible for a wave to have more than one value of pressure at a particular point in space so instead a shock forms, which is defined in the ideal case as the point at which there is infinite slope in the waveform. In a lossless medium the shock formation distance, given as \( \bar{x} \), for planar propagation is a function of the amplitude of the source as well as the frequency of the disturbance. Figure 1.1 shows the time waveform of an initially sinusoidal wave as it propagates to various values of \( \sigma \), which is the distance traveled normalized by the shock formation distance.

The consequence of wave steepening is that energy is taken from the fundamental frequency and redistributed into higher harmonics. This reallocation of energy has a significant impact upon the power spectrum of the propagated wave as well as the general shape of the time waveform. Consequently, wave steepening must be included for the propagation of jet noise, which can be very high amplitude and may presumably demonstrate a considerable amount of steepening.

Shock coalescence is another important phenomenon associated with finite amplitude wave propagation. When shocks of sufficiently low amplitude, such as those
Figure 1.1. Demonstration of wave steepening. At $\sigma = 0$ the wave is a pure sine wave and by the time the wave has traveled to $\sigma = 1$, the wave has formed a shock.

of interest in this dissertation, have formed due to wave steepening, its speed can be approximated using weak shock theory$[3]$: 

$$U_{sh} = c_0 + \beta \frac{p_a + p_b}{2\rho_0 c_0},$$

(1.2)

where $U_{sh}$ is the shock speed, $\rho_0$ the ambient density, $c_0$ the low frequency sound speed, and $p_a$ and $p_b$ the pressure ahead of and behind the shock, respectively. Equation 1.2 says that a shock with a greater value of $p_a + p_b$ travels faster than a shock with a lesser value. Hence, given a sufficient propagation distance a stronger asymmetric shock will overtake a weaker shock, which is a process called shock coalescence. When shocks coalesce, the number zero-crossings in the waveform typically decreases. This has the effect of lowering the characteristic time scales of the waveform. An example of shock coalescence is given in Fig. 1.2, which shows a wave packet propagating in a domain including spherical spreading and atmospheric absorption to three points in space. The initial state is given in the first waveform at point $x_1$ in space. At point $x_2$, the larger shocks, which are
Figure 1.2. Demonstration of shock coalescence. After propagating to point $x_3 > x_2 > x_1$, the stronger shocks have overtaken the weaker shocks and a number of zero-crossings have been lost in the circled areas.

traveling at higher velocities than the smaller shocks, has begun to overtake the smaller shocks, and at point $x_3$ the loss of many zero-crossings can be seen in the circled portions.

It is often difficult to determine whether shock coalescence has occurred simply by investigating the time waveforms of real jet noise measurements, but an energy shift to lower frequencies in the spectrum of the data may also suggest that shock coalescence has occurred. It is expected that given a sufficiently large amplitude asymmetric shock and a long enough propagation distance, shock coalescence will occur.

The combined effect of wave steepening and shock coalescence is a shift of energy away from the original frequency of peak energy and into higher frequencies due to steepening and also into lower frequencies as a result of shock coalescence. The energy shift to higher frequencies associated with higher harmonic generation has been measured experimentally by Morfey and Howell[2], who used fly-over data of the Concorde to demonstrate that a great deal more high frequency energy
was contained in the waveforms than would be expected from linear propagation. Webster and Blackstock[4] used an array of loudspeakers to demonstrate similar shifts in energy as did Gee et al.[5][6], who used F/A-18E run-up data.

Gallagher and McLaughlin[7] were the first to use scale model jets to demonstrate wave steepening and a downward shift in peak frequency which may have been a result of shock coalescence. Similar energy shifts to higher and lower frequencies using scale model jets were noted by Petitjean and McLaughlin[8] and Gee et al.[9]. While the shift to high frequencies is most likely due to wave steepening, the observed downward shift in low frequency energy may be because of the directionality of the noise source or to near field effects rather than evidence of shock coalescence. This perspective is bolstered by measurements made at the Boeing low speed aeroacoustic facility (LSAF)[10] and static engine run-up tests of the F/A-22 Raptor which do not show this downward shift[11].

1.3.2 Absorption and Dispersion

Accurate absorption and dispersion is also important for realistic simulation of acoustic propagation. The absorption normally included in most fluid dynamic and acoustic models is limited to thermoviscous losses. Thermoviscous absorption, also known as classical absorption, includes losses that are due to two processes: the diffusion of momentum through molecular collisions between fluid regions possessing different velocities, known as shear viscous losses; and the conduction of heat between fluid regions of differing temperatures, known as thermal conduction losses[12]. Modified classical absorption contains losses caused by a third loss mechanism: bulk viscosity, also known as second viscosity. Bulk viscosity accounts for the loss due to the transfer of momentum into translational and rotational degrees of freedom of molecules in the air[13]. Because modified classical absorption increases as frequency squared, it becomes especially important at frequencies above 100 kHz.
For audible frequencies, molecular relaxation of diatomic nitrogen and especially diatomic oxygen dominates the absorption and dispersion in air. Molecular relaxation is a phenomenon that results from internal vibrations that occur upon the collision molecules. In reality, there are at least 24 vibrational energy-transfer reactions that occur in air between diatomic nitrogen, diatomic oxygen, carbon dioxide, and water vapor, which are four prevalent molecular components in air. The most important energy-transfer reactions can be satisfactorily combined into two processes based upon the time scale of the reactions. The process that corresponds to a combination of energy-transfer reactions related to diatomic oxygen is thus named “oxygen relaxation” and similarly the process that corresponds to a combination of reactions involving diatomic nitrogen processes is named “nitrogen relaxation”. For a more detailed discussion of molecular relaxation, the reader is referred to the work of Bass et al.[14].

The characteristic time required for a molecule to return to its previous energy state is known as the relaxation time of the molecule, often denoted as $\tau_\nu$. The relaxation time varies depending upon the types of molecule involved in the collisions. The relaxation time is particularly sensitive to the relative humidity, and to demonstrate this, a plot of the relaxation frequencies of nitrogen and oxygen as a function of relative humidity for 293 K is given in Fig. 1.3. The characteristic vibration frequency of a molecule, known as the relaxation frequency is related to the relaxation time by:

$$f_\nu = \frac{1}{2\pi \tau_\nu},$$

(1.3)

where $f_\nu$ is the relaxation frequency and $\tau_\nu$ the relaxation time.

The transfer of acoustical energy into the internal vibrational energy of molecules causes both absorption and dispersion. The relaxation frequencies of a molecule are important because this is the frequency at which the absorption per wavelength is the greatest. To explain the frequency-dependent nature of molecular relaxation, the asymptotically low and high frequency cases will first be explained.
Figure 1.3. Relaxation frequencies of nitrogen and oxygen as a function of relative humidity for 293 K. Nitrogen’s relaxation frequency is given as the dashed line and oxygen’s is given as the solid line.

At very low frequencies, the acoustic disturbance can cause an internal vibration at frequency \( f_\nu \). The vibrational energy eventually decays through molecular collisions and the molecule returns to its previous energy state before another cycle of the acoustic wave disturbs it. This is known as an *equilibrium state*, which refers to the fact that the acoustical frequency is so low that the molecules have time to return to their equilibrium state between cycles. This state causes absorption qualitatively similar to thermoviscous absorption, in which the absorption scales as frequency squared.

At very high frequencies, an acoustical wave can disturb the molecules with such rapidity that the molecules do not have time to transfer energy in the first place. In this case the molecules are in what is called a *frozen state*, referring to the fact that the molecules cannot enter a higher energy state. In the frozen state the absorption is frequency independent since once the frequency is sufficient to produce a frozen state, increasing the frequency further does not change the behavior of the air.
Figure 1.4. Absorption curves for 20 % relative humidity and 293 K. Nitrogen relaxation absorption is the dashed line, oxygen relaxation absorption the dash-dot, classical absorption the dotted line, and the linear combination the solid line.

At frequencies near the relaxation frequency, the system is transitioning between the equilibrium and the frozen state. This corresponds to acoustic disturbances that can cause the molecules to take energy from the acoustic wave and return to equilibrium at around the same frequency as the disturbance. The resulting absorption coefficients for air at 293 K and 20 % relative humidity is given in Fig. 1.4 and Fig. 1.5 which shows the absorption per meter and absorption per acoustic wavelength, respectively. The total absorption at these frequencies is assumed to be a linear combination of the nitrogen relaxation, oxygen relaxation, and modified classical absorption coefficients.

The Kramers-Kronig relations[15][16] dictate that the absorption owing to nitrogen and oxygen relaxation will also introduce significant dispersion. Dispersion occurs because the sound speed, $c$, is a function of the bulk modulus, $\beta$, in the following way:

$$c = \sqrt{\frac{\beta}{\rho_0}},$$  \hspace{1cm} (1.4)
Figure 1.5. Absorption coefficient curves per wavelength for 20 % relative humidity and 293 K. Nitrogen relaxation absorption is the dashed line, oxygen relaxation absorption the dash-dot, classical absorption the dotted line, and the linear combination the solid line.

and the bulk modulus is different at low and high frequencies based upon the equation of state. An example of a dispersion curve caused by molecular relaxation is given in Fig. 1.6, where $c_0$ is the low frequency sound speed.

The difference between the two asymptotic sound speeds is unique for nitrogen and oxygen, but the curves are qualitatively similar: in the low frequency limit the system is in the equilibrium state, and hence the low frequency sound speed is known as the *equilibrium sound speed*. In the high frequency limit the system is in the frozen state, and the sound speed is known as the *frozen sound speed*.

When the medium has two relaxation processes, the two dispersion processes are assumed to work independently upon the sound speed. Figure 1.7 demonstrates the combined effects of nitrogen and oxygen relaxation upon the sound speed in air. The equilibrium and frozen sound speeds for the medium are given at two extremes of the frequency spectrum, and at approximately 1 kHz the frozen sound speed for nitrogen is simultaneously the equilibrium sound speed for oxygen.
Figure 1.6. Dispersion curve for a monorelaxing fluid as a function of frequency normalized by the relaxation frequency.

Figure 1.7. Dispersion curve for air at 293 K and 20 % relative humidity including the effects of nitrogen (transition at approximately 100 Hz) and oxygen relaxation (transition at approximately 10000 Hz).
1.4 Previous Work

To the best of the author’s knowledge there have not been any previous attempts to include molecular relaxation effects in a computational fluid dynamics algorithm. Algorithms that do include molecular relaxation effects are typically based on the Burgers equation, with the exception of particle methods which model acoustic propagation at a molecular level. These three types of propagation algorithms have progressed for the most part independently of one another, and consequently this section will be divided into a discussion of the development of CFD algorithms, particle methods, and Burgers equation algorithms.

1.4.1 CFD-type Algorithms

Sparrow and Raspet[17] developed an algorithm that is most similar to the one proposed, so it will be described in detail. Their algorithm solved nonlinear acoustics equations similar to the Navier-Stokes equations in two dimensions and applied it to the propagation of spark pulses. The equations used included second order nonlinear effects and modified classical absorption. This system of equations was solved using a split step MacCormack technique that is second-order accurate in time and fourth-order accurate in space. The result of this work was a model that allowed for the propagation and visualization of multi-dimensional waves of complicated geometry as they propagated and evolved. The authors were able to demonstrate that the model predicted amplification factors caused by the reflection of a spark pulse at a rigid boundary that were consistent with explosion data[18],[19]. Due to the computational limitations existing at the time, some room for improvement in the model exists. The use of a moving domain was necessary, but this in effect limited the general utility of the model since a moving domain assumes prior knowledge of the state of the system. Additionally, the system of equations neglected third-order and higher nonlinear terms, used a nu-
numerical method that had difficulty propagating highly steepened waves, and lacked the effects of molecular relaxation.

The aerospace community has been developing its own computational fluid dynamic (CFD) models for at least the last thirty years. Not all of these models are applicable to acoustics, however, since any CFD algorithm that assumes an incompressible fluid will obviously not be applicable to this investigation. Also, many algorithms are only suitable for steady flow, a situation that is inappropriate for jet noise propagation and near field acoustic propagation investigations.

Computational fluid dynamic algorithms solving compressible equations that are suitable for unsteady flow can be extremely useful for acoustic propagation, although the research is usually more concentrated upon the fluid dynamics of the situation than the acoustics. For the computational fluid dynamics and computational aeroacoustics (CAA) communities, the research is often focused on how the fluid dynamics create acoustic disturbances. For example, Morris et al.[20] investigated how various flow conditions affect the acoustic radiation properties of jets using a three-dimensional Reynolds-averaged Navier-Stokes equation set that was computed on a parallel computing system. Their results demonstrated spectral qualities and directionality similar to that recorded from real jets. Shen and Tam[21] numerically simulated jet screech tones using an algorithm employing multiple meshes of various fineness to resolve not only the fine scale structures in the near field of the jet where instability waves are found but also propagated to an approximate far field. The results showed that screech tones simulated matched experimental data not only for fundamental screech tones but also for secondary ones. More detailed explanations of the state of CFD and CAA algorithms can be found in the literature by Tam[22], Long et al.[23], and Kurbatskii and Mankbadi[24]. An extremely accessible text for a newcomer to CFD is authored by Anderson[25] and for a more advanced discussion, the text by Blazek[26] is recommended.
The focus of the present research is specifically on acoustical waveform propagation and evolution; consequently, a somewhat different approach is required. There is less of an interest in simulation of an actual jet and investigations into the mechanisms that create jet noise, and more of an interest in the fine scale propagation characteristics of jet noise itself. Typical CFD and CAA models lack the capability to include molecular relaxation, which is the dominant absorption and dispersion mechanism for the audible frequency range under standard atmospheric conditions, and this should be included in an acoustic propagation algorithm.

### 1.4.2 Direct Simulation Monte Carlo

An interesting development in acoustic propagation algorithms has come from Danforth and Long[27], who have employed a direct-simulation Monte Carlo method (DSMC) based upon the Boltzmann equation to simulate acoustic propagation at the molecular level. This method grants many advantages, among them is the ability to work at very high Knudsen numbers, given as the ratio of the free mean path to a length scale. The ability to propagate at high Knudsen numbers allows for the simulation of extremely high frequency acoustic systems as well as acoustic propagation in rarefied gasses[28]. The DSMC method also inherently includes molecular relaxation effects, assuming that nitrogen, oxygen and water molecules are included in the calculation, and can naturally propagate shocks. The combined abilities of the DSMC method allows for the simulation of molecular dynamics[29] and detonation waves[30] without numerical instabilities. As promising as this method is, it is impractical for acoustic propagation at this time because of the immense amount of calculations that are required to perform even simple calculations. For example, a DSMC algorithm running on a parallel computing cluster is only able to compute scenarios in which the domains are incredibly small—usually on the order of 50 µm for a one-dimensional propagation scenario at standard atmospheric conditions. Clearly, long range multi-dimensional propagation is currently
impossible using the DSMC method.

1.4.3 Burgers Equation Solution Algorithms

All of the propagation algorithms devised in the acoustics community that include molecular relaxation (with the exception of the DSMC method) use some form of the Burgers equation solved using a time domain or hybrid time/frequency domain method. This section will discuss the history of the development of these algorithms and largely follows the chronology of research given in Kent Gee’s doctoral dissertation[11].

The fundamental development of this algorithm was performed by Pestorius and Blackstock[31][32], who created an algorithm to solve the generalized Mendousse-Burgers equation (GMBE) including the effects of boundary layer absorption and applied it to the simulation of plane wave propagation of noise in a tube. The algorithm works by first propagating the wave nonlinearly in the time domain and afterward applying boundary layer absorption and dispersion to the wave in the frequency domain. To reduce computation times, the algorithm used a large propagation step size and weak shock theory to avoid multi-valued waveforms. The transformation to the frequency domain was performed because of the ease with which the frequency dependent absorption and dispersion could be applied.

A great deal of research in acoustics has based its work on this algorithm. Pierce[33] was able to prove that this algorithm reduced to a form of the GMBE which was appropriate for plane wave propagation in a tube. A number of modifications were subsequently made to the Pestorius algorithm for other propagation scenarios. One of the major revisions was performed by Anderson[34] for the study of N-wave propagation. Anderson created what is now known as the Anderson algorithm by removing weak shock theory from the Pestorius algorithm and instead using thermoviscous losses and an adaptive propagation step size to keep the wave from becoming multi-valued. Other modifications to the original Pestorius
algorithm were performed by Blackstock[35], who added spherical spreading into
the domain for the investigation of jet noise propagation, and Theobald[36] and
Blackstock further modified the Pestorius algorithm to include molecular relax-
ation absorption for the investigation of finite amplitude propagation of periodic
signals. Bass and Raspet[37] modified the Anderson algorithm to include relax-
ation absorption of nitrogen and oxygen and applied it to the propagation of blast
waveforms. Orenstein[38] modified the Anderson algorithm to include relaxation
absorption of nitrogen and oxygen as well as dispersion due to oxygen relaxation
and applied it to the propagation of spark pulses. Bass et al.[39] included ab-
sorption and dispersion caused by molecular relaxation of nitrogen and oxygen
to investigate shock wave propagation in the atmosphere. Additionally, Lee and
Hamilton[40] created a hybrid time/frequency domain code similar to that used by
Anderson to solve the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation[41] for
the simulation of directive sound beams created by circular pistons in a thermo-
viscous medium. Cleveland et al.[42] built upon this work by including molecular
relaxation losses in the time domain, altered the algorithm to solve either the
KZK or the Burgers equation, and applied it to study the effect of relaxation on
the propagation of finite amplitude sound beams. For more information and a
comparison of the results obtained by some of these schemes, the reader is referred
to the work of Cleveland[43].

Most recently, Gee[11] modified the Anderson algorithm and applied it to the
propagation of jet noise waveforms in a medium including thermoviscous losses as
well as absorption and dispersion as a result of molecular relaxation of nitrogen
and oxygen. This is not the first attempt to use a hybrid time/frequency domain
solution of the Burgers equation for the simulation of jet noise propagation. As
stated previously, the Pestorius algorithm was used by Blackstock[35] to propagate
jet noise waveforms, but the algorithm used weak shock theory and did not include
the effects of molecular relaxation. Using his modified Anderson algorithm, Gee
was able to demonstrate the model’s ability to predict the spectral properties of propagated jet noise under various conditions. This algorithm is currently the accepted method of predicting jet noise evolution and hence will be used as a comparison to the algorithm developed in this work for jet noise propagation. For completeness, Appendix A includes a more thorough explanation of Gee’s algorithm.

1.5 Proposed Research

As stated previously, the objective of this work is to create a CFD-type algorithm that includes the effects of molecular relaxation and stably propagates shocks. To accomplish these goals, specific attention must be paid to not only the constitutive equations used but also the way in which they are solved numerically.

The fluid dynamics equation set used is based on the Navier-Stokes equations. However, rather than using the typical energy equation, an entropy equation is used, because molecular relaxation effects are easier to include in an entropy equation. The constitutive equation set consists of the standard continuity and momentum equations, the entropy equation, and two relaxation equations corresponding to nitrogen and oxygen effects. Including additional source terms in the entropy equation as well as a relaxation equation corresponding to each relaxation process into the Navier-Stokes equations results in an equation set that allows the numerical model to include molecular relaxation effects.

The equation set is naturally nonlinear and can allow for the simulation of wave steepening, shock formation, and shock propagation. Not all numerical methods are capable of stable shock propagation, however, so the numerical methods used to solve the Navier-Stokes equation set must be carefully chosen. Although acoustic shocks are in reality continuous waveforms, they contain very high frequency components that are difficult to directly resolve numerically while maintaining
the ability to propagate over large distances. The shocks are thus treated as discontinuities in this investigation. A weighted essentially non-oscillatory (WENO) scheme following the development by Shu[44] is proposed to solve the system of equations and allow the propagation of discontinuities. WENO allows for an accurate solution of the Navier-Stokes equations in smooth regions and it also allows for stable propagation of multi-dimensional shocks of complex geometry without prior knowledge of the system.

The WENO scheme is based upon the essentially non-oscillatory (ENO) scheme by Harten et al.[45], which uses polynomial interpolations of a discretized medium. What makes the ENO method unique to standard finite-difference techniques is how it develops the stencils used in these interpolations. The ENO method basically works by starting with a simple interpolation solution using a one or two point stencil whose smoothness is analyzed using Newton divided differences. If the solution is not smooth enough, points are added to the stencil sequentially until the desired accuracy is achieved. The advantage of the ENO scheme is that it is capable of producing uniformly high-order accurate solutions, even over discontinuities. The method was improved with the introduction the weighted essentially non-oscillatory (WENO) scheme by Liu et al.[46]. The scheme uses a fixed number of candidate stencils, each of which has a corresponding smoothness indicator. The smoothness indicators are used to assign weights to the stencils based upon the stability of the solution derived from each candidate stencil. If the solution from a stencil is unstable, it is given a weight near zero which causes the stencil to be essentially omitted from the calculation at that point. The WENO scheme is superior to ENO for two reasons: first, the WENO scheme does not require the use of logical statements which can considerably slow down the calculations for certain types of workstations; and second, for the case of a completely continuous system, the WENO scheme uses an optimized combination of the candidate stencils rather than just one in the case of the ENO, which gives higher accuracy.
The WENO scheme has been shown to be appropriate for a wide variety of applications. To give just a few examples, the WENO method has been used to solve problems related to reacting flow convection\cite{47}, crystallization processes\cite{48}, radiative transfer in scattering media\cite{49}, magnetodyrodyamics\cite{50}, traffic flow modeling\cite{51}, and even cosmology\cite{52}. It has not to the best of the author’s knowledge been used for this type of acoustic propagation, however.

The WENO scheme will prove itself to be suitable for general acoustic propagation, but its implementation is also quite complex and the run times are relatively long. For faster simulation of continuous wave propagation, a dispersion-relation-preserving (DRP) algorithm based upon the work of Tam and Webb\cite{53} is used rather than the WENO scheme to solve the constitutive equations. The original DRP algorithm was optimized to have the minimum possible amount of numerical dissipation and dispersion when solving the Euler equations. The stencil coefficients were derived using a technique to minimize the error between the dispersion relations of the original partial differential equations and the finite-difference scheme. This work has been extremely influential in the development of other numerical methods. For example, Hixon applied the same technique to MacCormack schemes\cite{54}, Wang and Chen applied it to the WENO scheme for the propagation of linear waves with discontinuities\cite{55}, and Ponziani et al. applied it to multiscale compressible flow problems using the Euler equations\cite{56}. The DRP scheme used here will prove itself to be capable of capturing the dissipation and dispersion of this Navier-Stokes equation set and also of being able to do so much faster than the WENO scheme for continuous wave propagation.

1.6 Outline

This dissertation is divided into six chapters and two appendices. Chapter 2 will discuss the Navier-Stokes equation set which allows for nonlinearities, modified
classical absorption, and molecular relaxation effects to be included in the model. The system of equations will then be assembled into a form that is suitable for solution using numerical methods. Chapter 3 will discuss the numerical methods used and which includes explanations of the DRP scheme, the WENO scheme, and also the techniques used to simulate acoustic sources, turn on and off various absorption and dispersion mechanisms, and implement various boundary conditions. Chapter 4 will verify that both schemes are able to reproduce the salient acoustical phenomena with a high degree of accuracy. Once it is determined that the model is able to reliably reproduce these physical phenomena, Chapter 5 will investigate the simulation of propagation scenarios that do not have analytical solutions. The chapter will demonstrate the simulation of Mach stem formation, comparison of full scale jet noise propagation results to that predicted by Gee’s Anderson algorithm, and comparison of scale model jet noise propagation to Gee’s algorithm as well as to measured data. Chapter 6 will summarize the results of this work, discuss other calculations that could be performed, and possible improvements or modifications of the model.

There are two appendices included for completeness. The first appendix will discuss the Gee algorithm that is used as a comparison for the jet noise predictions presented in Chapter 5. The second appendix is a compilation of example FORTRAN code that may be helpful for someone interested in developing a DRP or WENO algorithm similar to those developed here.
Derivation of the Equation Set

2.1 Introduction

In this chapter, a constitutive equation set is constructed which is based upon the Navier-Stokes equations and includes two additional equations to include the effects of molecular relaxation. If solved using a proper numerical scheme, the equations will allow for the simulation of nonlinear two-dimensional acoustic propagation in a medium including the effects of modified classical absorption and molecular relaxation of nitrogen and oxygen. Fluid dynamic equations are used because of their generality and nonlinear characteristics, which means that the variables in this development use the total variables rather than the perturbed values.

2.2 Constitutive Equations in Two Dimensions

The equations in this section are for the most part taken from Pierce[13], and for more detail, the reader is referred to this text. The constitutive equation set that is solved by the algorithm includes the continuity equation[57],

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0,
\]

(2.1)
the Navier-Stokes equation (also known as the momentum equation), 

\[ \rho \frac{Dv}{Dt} = -\nabla p + \nabla(\mu_B \nabla \cdot v) + \mu \sum_{ij} e_i \frac{\partial \phi_{ij}}{\partial x_j}, \tag{2.2} \]

the entropy-balance equation, 

\[ \rho \frac{Ds_{fr}}{Dt} + \sum_{\nu} \frac{\rho}{T_{\nu}} c_{\nu\nu} \frac{DT_{\nu}}{Dt} - \nabla \cdot \left( \frac{\kappa}{T} \nabla T \right) = \sigma_s, \tag{2.3} \]

and the relaxation equation, 

\[ \frac{DT_{\nu}}{Dt} = \frac{1}{\tau_{\nu}} (T - T_{\nu}). \tag{2.4} \]

In the above equations, \( \rho \) is the density, \( v \) is the velocity, \( p \) is the total pressure, \( \mu \) and \( \mu_B \) are the shear and bulk viscosities, \( e_i \) is the unit vector in the \( i \)th dimension, \( \phi_{ij} \) is the rate of shear tensor, \( s_{fr} \) is the frozen entropy, \( T \) is the absolute temperature, \( \kappa \) is the coefficient of thermal conduction, \( c_{\nu\nu} \) is the specific heat at constant volume associated with the \( \nu \)-type molecule, \( \sigma_s \) is a variable used to represent the source terms of the entropy equation, Eq. 2.3, and \( T_{\nu} \) and \( \tau_{\nu} \) are the apparent vibration temperature and relaxation time of the \( \nu \)-type molecule, respectively. In this investigation, the molecules whose effects are taken account of include nitrogen and oxygen, although more could be added to the calculation at considerable computational cost.

Equations 2.1–2.4 are called the constitutive equations. To obtain a closed set of equations, relations for the variables contained on the right-hand sides of the constitutive equations need to be determined. They are given as follows[58]: 

\[ \sigma_s = \frac{\mu_B}{T} (\nabla \cdot v)^2 + \frac{\mu}{2T} \sum_{ij} \phi_{ij}^2 + \frac{\kappa}{T^2} (\nabla T)^2 + \frac{\rho}{T} \sum_{\nu} A_{\nu} \frac{DT_{\nu}}{Dt}, \tag{2.5} \]
\[
\phi_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \nabla \cdot \mathbf{v} \delta_{ij},
\]
(2.6)

\[
A_\nu = \left( \frac{T}{T_\nu} - 1 \right) c_{\nu\nu},
\]
(2.7)

and

\[
c_{\nu\nu} = \frac{n_\nu}{n} R \left( \frac{T_\nu}{T_\nu} \right)^2 e^{-\frac{T}{T_\nu}}.
\]
(2.8)

Here, \( A_\nu \) is the affinity associated with the \( \nu \)-type molecule, \( v_i \) the \( i \)th component of velocity, \( \delta_{ij} \) the Kronecker delta, \( n_\nu/n \) the fraction of all molecules that is of species \( \nu \), \( R \) the gas constant given as 287.06 J/(kg K), and \( T_\nu^* \) the molecular constant of molecule \( \nu \).

In addition to the constitutive equations, the van der Walls form of the state equation and the entropy relation are used[59]:

\[
p = \frac{\rho \tilde{R} T}{M} + \frac{\rho^2}{M^2} \left[ \tilde{R} T b - a \right] + \frac{\rho^3}{M^3} \left[ \tilde{R} T b^2 \right],
\]
(2.9)

and

\[
T = T_0 e^{(s_{fr} - s_{fr_0} - R \ln[\rho_0/\rho]) / c_v},
\]
(2.10)

where \( \tilde{R} \) is the gas constant given as \( 8.314 \times 10^3 \) J/K, \( M \) is the molar mass of air, given as 28.96 kg kmol\(^{-1} \), \( c_v \) the specific heat of air at constant volume, given as 720.4 J kg\(^{-1} \) K\(^{-1} \), \( s_{fr_0} \) the ambient entropy, \( T_0 \) the ambient temperature, given as 293.15 K, and the values of \( a \) and \( b \) used are \( 1.361 \times 10^5 \) J m\(^3\) kmol\(^{-2} \) and 0.03649 m\(^3\) kmol\(^{-1} \), respectively[60].

An alternative form of Eq. 2.9, taken from both Hamilton and Blackstock[61] and Pierce[13] is given as:

\[
p = c^2 \left[ (\rho - \rho_0) + \left( \frac{\gamma - 1}{2\rho_0} \right) (\rho - \rho_0)^2 + \left( \frac{\rho^2 T}{c_p} \right)_0 \left( s_{fr} - s_{fr_0} \right) \right] + p_0.
\]
(2.11)
and an alternative form of Eq. 2.10 is given by Pierce[13]:

\[
T = \left( \frac{T}{c_p} \right)_0 (s_{fr} - s_{fr0}) + \left( \frac{T\beta}{\rho c_p} \right)_0 (p - p_0) + T_0,
\]

(2.12)

where the value used for \( c_p \) is \( 1.01 \times 10^3 \) J kg\(^{-1}\) K\(^{-1}\). Either of these equations of state may be used, although the accuracy obtained by Eq. 2.9–2.10 is typically higher.

The constitutive equations need to be recast in conservative form so that they can be solved by an explicit time-domain finite-difference scheme. An equation is in conservative form if there are no dependent variables in front of the partial derivatives. Usually, one can reduce the nonconservative form to conservative form by inserting a form of the continuity equation, which is by definition equal to zero, on the left-hand side of the nonconservative equation. This addition allows all of the terms on the left-hand side to be regrouped within the partial derivatives, leaving the equation in conservative form. A clear demonstration of this approach is given in Anderson’s text[25]. In a two-dimensional Cartesian coordinate system the conservative forms of the constitutive equations are

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0,
\]

(2.13)

\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} = -\frac{\partial p}{\partial x} + \mu_B \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + \mu \left( \frac{\partial \phi_{xx}}{\partial x} + \frac{\partial \phi_{xy}}{\partial y} \right),
\]

(2.14)

\[
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho vu)}{\partial x} + \frac{\partial (\rho v^2)}{\partial y} = -\frac{\partial p}{\partial y} + \mu_B \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y \partial x} \right) + \mu \left( \frac{\partial \phi_{yx}}{\partial x} + \frac{\partial \phi_{yy}}{\partial y} \right),
\]

(2.15)
\[
\frac{\partial (\rho s f_r)}{\partial t} + \nabla \cdot (\rho s f_r \mathbf{v}) = \sigma_s - \sum_{\nu} \frac{\rho}{T_{\nu}} c_{\nu} \frac{D T_{\nu}}{D t} + \nabla \cdot \left( \frac{\kappa}{T} \nabla T \right), \tag{2.16}
\]

and

\[
\frac{\partial (\rho T_{\nu})}{\partial t} + \nabla \cdot (\rho T_{\nu} \mathbf{v}) = \frac{\rho}{\tau_{\nu}} (T - T_{\nu}). \tag{2.17}
\]

Because the calculations are two-dimensional, the Navier-Stokes equation, Eq. 2.2, has only \( x \) and \( y \) components, leading to Eq. 2.14 and Eq. 2.15. The computational domain resulting from these equations is essentially a slice of a three-dimensional field that is invariant in the third dimension. Therefore, the units used in the calculation are the physical, three-dimensional units, e.g. the units of pressure are pascals as opposed to pascals/meter. An extension to three dimensions is straightforward. It requires the inclusion of the \( z \) component of the Navier-Stokes equation and the reworking of some of the source terms in the constitutive equations to account for the third dimension. It should also be noted that although the equations are in a Cartesian coordinate system, they are equally applicable to other coordinate systems.

The last parameter that needs to be included is the relative humidity. The humidity of the air can change the relaxation frequencies of nitrogen and oxygen by several orders of magnitude and hence it needs to be included in the model. The formulae for the relaxation frequencies of nitrogen and oxygen are based upon the semi-empirical relations given by Bass et al.\[62\]:

\[
f_{N_2} = \frac{1}{2\pi \tau_{N_2}} = \frac{p}{p_{\text{ref}}} \left( 24 + 4.04 \times 10^6 h \frac{0.02 + 100h}{0.391 + 100h} \right), \tag{2.18}
\]

and

\[
f_{O_2} = \frac{1}{2\pi \tau_{O_2}} = \frac{p}{p_{\text{ref}}} \left[ \left( \frac{T_{\text{ref}}}{T} \right)^{1/2} (9 + 2.8 \times 10^4 he^{-\eta}) \right]. \tag{2.19}
\]
where

\[ h = \frac{10^{-2}(RH)p_{vp}(T)}{p}, \]  

(2.20)

\[ \eta = 4.17 \left[ \left( \frac{T_{ref}}{T} \right)^{1/3} - 1 \right], \]  

(2.21)

and \( p_{vp} \) is the saturation vapor pressure of air, calculated using

\[ p_{vp} = p_{ref}10^\zeta, \]  

(2.22)

where

\[ \zeta = 10.79586 \left( 1 - \left( \frac{273.16}{T_0} \right) \right) - 5.02808 \log_{10} \frac{T_0}{273.16} \]

\[ + 1.50474 \times 10^{-4} \left( 1 - 10^{-8.29692 \left( \frac{T_0}{273.16} - 1 \right)} \right) \]  

(2.23)

\[ - 4.2873 \times 10^{-4} \left( 1 - 10^{-4.76955 \left( \frac{273.16}{T_0} - 1 \right)} \right) - 2.2195983. \]

Here, \( h \) is the fraction of molecules in air that is \( H_2O \), \( RH \) is the relative humidity percentage, and the values used for \( T_{ref} \) and \( p_{ref} \) are 293.16 K and 1.01325 \times 10^5 \text{ Pa}, respectively. Equations 2.18–2.21 are solved using the ambient values of pressure and temperature to determine the values of \( f_{N_2} \) and \( f_{O_2} \). The relaxation frequencies are assumed to be constants thereafter.

### 2.3 Recasting of the Constitutive Equations in Two Dimensions

Upon examination of the constitutive equations in conservative form, Eqs. 2.13–2.17, it is apparent that they are all of a similar form and can be written together
in the following vector equation:

\[
\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{H},
\]  

(2.24)

where the time derivatives of the constitutive equations are rows in the \( \mathbf{w} \) vector, the \( x \) derivatives are rows in the \( \mathbf{F} \) vector, the \( y \) derivatives are rows in the \( \mathbf{G} \) vector, and the source terms are rows in \( \mathbf{H} \). This gives the following vectors for \( \mathbf{w} \), \( \mathbf{F} \), \( \mathbf{G} \), and \( \mathbf{H} \):

\[
\mathbf{w} = \begin{pmatrix}
\rho \\
\rho u \\
\rho v \\
\rho s_{fr} \\
\rho T_{N_2} \\
\rho T_{O_2}
\end{pmatrix},
\]  

(2.25)

\[
\mathbf{F} = \begin{pmatrix}
\rho u \\
\rho u^2 \\
\rho uv \\
\rho u s_{fr} \\
\rho u T_{N_2} \\
\rho u T_{O_2}
\end{pmatrix},
\]  

(2.26)

\[
\mathbf{G} = \begin{pmatrix}
\rho v \\
\rho vu \\
\rho v^2 \\
\rho v s_{fr} \\
\rho v T_{N_2} \\
\rho v T_{O_2}
\end{pmatrix},
\]  

(2.27)
and

\[
H = \begin{pmatrix}
0 \\
-\frac{\partial p}{\partial x} + \mu_B \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + \mu \left( \frac{\partial \phi_{xx}}{\partial x} + \frac{\partial \phi_{xy}}{\partial y} \right) \\
-\frac{\partial p}{\partial y} + \mu_B \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y \partial x} \right) + \mu \left( \frac{\partial \phi_{yx}}{\partial x} + \frac{\partial \phi_{yy}}{\partial y} \right) \\
\sigma_s - \sum_{\nu} \frac{\rho}{T_{\nu}} c_{\nu} \frac{D T_{\nu}}{D t} + \nabla \cdot \left( \frac{\kappa}{T} \nabla T \right) \\
\frac{\rho}{\tau_{N_2}} (T - T_{N_2}) \\
\frac{\rho}{\tau_{O_2}} (T - T_{O_2})
\end{pmatrix}.
\] (2.28)

Equations in this form, in which all of the variables on the left-hand sides of the equations are within the derivative operators of \(t, x,\) or \(y,\) can be solved by a large variety of numerical methods. The vector equation given above is one of the forms that is solved by two different numerical methods. A one-dimensional form of these equations will be derived in the next section.

### 2.4 One-Dimensional Forms of the Constitutive Equations

Ideally, all of the calculations would be able to be performed in two or more dimensions, but computer hardware limitations necessitate a one-dimensional model that can specialize in high resolution long range propagation. For the case of jet noise propagation, the model does not need to be more than one-dimensional unless temperature profiles, wind, etc., are desired in the domain, so the propagation is run in one dimension with the effects of spreading included in the constitutive equations.

To include various spreading losses into the one-dimensional constitutive equations, some additional work is required. The one-dimensional form of the consti-
tutive equations, Eqs. 2.1–2.4, in conservative form is given as:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \tag{2.29}
\]

\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} = -\frac{\partial p}{\partial x} + \mu_B \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial \phi_x}{\partial x}, \tag{2.30}
\]

\[
\frac{\partial (\rho s_{fr})}{\partial t} + \frac{\partial (\rho u s_{fr})}{\partial x} = \sigma_s - \sum_{\nu} \frac{\rho}{T_{\nu}} c_{vw} \frac{DT_{\nu}}{Dt} + \frac{\partial}{\partial x} \left( \frac{\kappa}{T} \frac{\partial T}{\partial x} \right), \tag{2.31}
\]

and

\[
\frac{\partial (\rho T_{\nu})}{\partial t} + \frac{\partial (\rho u T_{\nu})}{\partial x} = \frac{\rho}{\tau_{\nu}} (T - T_{\nu}). \tag{2.32}
\]

This equation set allows for one-dimensional plane wave propagation. If cylindrical or spherical spreading is required, it may be included in the form of an additional source term. To illustrate this, the various forms of the continuity equation will be explicitly derived. The one-dimensional continuity equation in spherical coordinates is given as:

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial (\rho u r^2)}{\partial r} = 0. \tag{2.33}
\]

This form of the continuity equation is not suitable for the explicit finite-difference approximations that are going to be used, so this equation must be recast in another form. This is accomplished by expanding the second term of the left-hand side of Eq. 2.33 into its three parts:

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \left( ur^2 \frac{\partial \rho}{\partial r} + \rho r^2 \frac{\partial u}{\partial r} + 2 \rho u r \right) = 0. \tag{2.34}
\]

Multiplying through by the \(1/r^2\) term and grouping the first two terms of the expansion yields

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial r} + \frac{2 \rho u}{r} = 0. \tag{2.35}
\]

This is a form of the continuity equation that can be used by the numerical schemes.
Transforming the continuity equation into a one-dimensional cylindrically spreading form is also very similar and results in the following:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial r} + \frac{\rho u}{r} = 0. \quad (2.36)$$

The similarities between Eq. 2.29, Eq. 2.35, and Eq. 2.36 suggest a generalized one-dimensional continuity equation that is of the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = -\frac{m \rho u}{x}. \quad (2.37)$$

The multiplication factor $m$ has been introduced in the above equation, which is equal to 0 for planar propagation, 1 for cylindrical spreading, and 2 for spherical spreading. Using similar techniques, the remaining constitutive equations in this generalized form are given as:

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} = -\frac{\partial p}{\partial x} + \mu_B \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial \phi}{\partial x} - \frac{m \rho u^2}{x}, \quad (2.38)$$

$$\frac{\partial (\rho s_f)}{\partial t} + \frac{\partial (\rho u s_f)}{\partial x} = \sigma_s - \sum_{\nu} \frac{\rho}{T_{\nu}} \frac{c_{v\nu}}{\rho} \frac{DT_{\nu}}{Dt} + \frac{\partial}{\partial x} \left( \frac{\kappa}{T} \frac{\partial T}{\partial x} \right) - \frac{m \rho u s}{x}, \quad (2.39)$$

and

$$\frac{\partial (\rho T)}{\partial t} + \frac{\partial (\rho u T)}{\partial x} = \frac{\rho}{\tau_{\nu}} (T - T_{\nu}) - \frac{m \rho u T}{x}. \quad (2.40)$$

The numerical schemes use this form of the constitutive equations for the one-dimensional propagation simulations presented later.
Chapter 3

Numerical Methods

3.1 Introduction

Two computational methods are discussed in this chapter: a dispersion-relation-preserving (DRP) scheme in space[53] with a fourth-order Runge-Kutta scheme in time and a weighted essentially non-oscillatory (WENO) algorithm[44] in space and a third-order total variation diminishing (TVD) Runge-Kutta scheme in time[63]. The WENO algorithm has many different possible implementations and variants, so two different methods applicable to the system of equations given in the previous chapter will be discussed and their unique qualities will be highlighted.

As this chapter progresses, it will become clear that the DRP and WENO schemes have distinct strengths and weaknesses. Put simply, the DRP scheme is a fourth-order accurate scheme that has a less complex theoretical background than the WENO and whose implementation in a numerical model is straightforward. That is not to say that the DRP scheme is ideal for all situations. In fact, the DRP implementation used here does very poorly for the propagation of discontinuities. The WENO scheme, on the other hand, is capable of stably propagating shock waves and has fifth-order higher accuracy. What hinders WENO’s general utility is that it is a method whose theoretical basis, and more importantly, whose imple-
mentation is much more complex. This leads to longer computational run-times and larger programs that have greater memory requirements.

The objective of this chapter is to describe these two schemes in a way such that they may be immediately implemented by the reader. The sections will end with summaries that are intended to facilitate execution by one who may still be somewhat uncomfortable with all of the theory presented. The DRP scheme will be described in detail first. The theory behind the WENO scheme will come next, followed by a description of the specific implementation of the WENO scheme used. The final sections of this chapter will describe acoustic sources and how they are inserted into the domain, how the model is able to turn on and off absorption and dispersion mechanisms, and the boundary conditions implemented.

### 3.2 The Dispersion-Relation-Preserving Algorithm

#### 3.2.1 Introduction

The dispersion-relation-preserving scheme was introduced by C. K. W. Tam and J. C. Webb in 1993[53], and it has been widely used since. The scheme is popular because of its relative simplicity in execution, efficiency, and high accuracy.

The scheme was derived and optimized to have the minimum possible amount of numerical dissipation and dispersion when solving the Euler equations. The DRP scheme is chosen because of its ability to accurately capture dispersion, which is one of the central objectives of this investigation.
3.2.2 Theory

The development here follows the more in-depth discussion by Tam and Webb[53]. Starting with the general finite-difference approximation of a spatial derivative:

\[
\frac{\partial f(x)}{\partial x} \approx \frac{1}{\Delta x} \sum_{j=-N}^{M} a_j f(x + j\Delta x), \tag{3.1}
\]

where \( f \) is an arbitrary function of \( x \), \( N \) equal to the number of stencil points to the left of \( f \), \( M \) equal to the number of stencil points to the right of \( f \), \( \Delta x \) the spatial step size, and \( a_j \) the coefficient for stencil point \( j \). The discrete Fourier transform of both sides of Eq. 3.1 is given as

\[
i\alpha \hat{f} \approx \left( \frac{1}{\Delta x} \sum_{j=-N}^{M} a_j e^{i\alpha j \Delta x} \right) \hat{f}. \tag{3.2}
\]

Tam and Webb note that \( \alpha \) is the wave number of the \( x \)-derivative given in Eq. 3.1, and

\[
\bar{\alpha} = \frac{-i}{\Delta x} \sum_{j=-N}^{M} a_j e^{i\alpha j \Delta x}, \tag{3.3}
\]

is effectively the wave number of the Fourier transform of the finite-difference approximation. If \( N = M \), the series is simply a Fourier sine series,

\[
\bar{\alpha} = \frac{-i}{\Delta x} \sum_{j=-N}^{N} a_j \sin(j\alpha \Delta x). \tag{3.4}
\]

If it is imposed that \( N = 3 \), the scheme uses a seven point stencil which can be imposed to be fourth-order accurate, which provides a constrain in the relationship between the values \( a_j \) and leaves just one of the stencil coefficients as a free parameter. To determine the coefficients that minimize the error between the wave number of the \( x \)-derivative and the finite-difference approximation, the integrated
error,
\[ E = \int_{-\pi/2}^{\pi/2} |\alpha \Delta x - \bar{\alpha} \Delta x|^2 d(\alpha \Delta x), \quad (3.5) \]

must be minimized, and using the following:
\[ \frac{\partial E}{\partial a_j} = 0, j = -N \rightarrow M, \quad (3.6) \]

the remaining coefficients can be determined. The coefficients determined by Tam and Webb[53] for \( N = 3 \) are the following:
\[ a_0 = 0, \]
\[ a_1 = -a_{-1} = 0.79926643, \quad (3.7) \]
\[ a_2 = -a_{-2} = -0.18941314, \]
\[ a_3 = -a_{-3} = 0.02651995. \]

In this investigation, since uniform grids are used and the initial condition is always well-resolved, artificial viscosity is not required. If the boundary condition is not periodic, the scheme should eventually become unstable without artificial viscosity if the calculation allows the waves to impact the boundary[65].

### 3.2.3 Time discretization

The DRP scheme described in the previous section is used for the spatial derivatives. For the time derivatives, a standard fourth-order accurate Runge-Kutta scheme is used. Using the previously derived vector form of our constitutive equations,
\[ \mathbf{w}_t + \mathbf{F}_x + \mathbf{G}_y = \mathbf{H}, \quad (3.8) \]
the fourth-order Runge-Kutta scheme used is of the form:

\begin{equation}
\mathbf{w}^{(0)} = \mathbf{w}^n, \tag{3.9}
\end{equation}

\begin{equation}
\mathbf{w}^{(1)}_{l,m} = \mathbf{w}^{(0)}_{l,m} - \frac{1}{2} \Delta t \mathbf{K}^{(0)}_{l,m}, \tag{3.10}
\end{equation}

\begin{equation}
\mathbf{w}^{(2)}_{l,m} = \mathbf{w}^{(0)}_{l,m} - \frac{1}{2} \Delta t \mathbf{K}^{(1)}_{l,m}, \tag{3.11}
\end{equation}

\begin{equation}
\mathbf{w}^{(3)}_{l,m} = \mathbf{w}^{(0)}_{l,m} - \Delta t \mathbf{K}^{(2)}_{l,m}, \tag{3.12}
\end{equation}

\begin{equation}
\mathbf{w}^{n+1}_{l,m} = \mathbf{w}^{(0)}_{l,m} - \Delta t \sum_{q=0}^{3} b_q \mathbf{K}^{(q)}_{l,m}, \tag{3.13}
\end{equation}

where \( b = [1/6, 1/3, 1/3, 1/6] \) and

\begin{equation}
\mathbf{K}_{l,m} = \frac{1}{\Delta x} \sum_{j=-3}^{3} a_j \mathbf{F}_{l+j,m} + \frac{1}{\Delta y} \sum_{j=-3}^{3} a_j \mathbf{G}_{l,m+j} - \mathbf{H}_{l,m}. \tag{3.14}
\end{equation}

Here, the vectors \( \mathbf{w}, \mathbf{F}, \mathbf{G}, \) and \( \mathbf{H} \) are given previously, and the values for \( a_j \) are given in Eq. 3.7. This time discretization scheme determines values for \( \mathbf{w} \) at temporary times, which in effect adds more points to the time stencil in order to increase the accuracy of the method, much in the same way that spatial discretization schemes use many points in space to increase their accuracy. The variables at temporary times are only used to get from one stage of the Runge-Kutta to the next. They should not be taken literally as derived values of density, velocity, etc.; only the \( \mathbf{w}^n \) and \( \mathbf{w}^{n+1} \) values should be taken literally.

Although Eqs. 3.9–3.13 are acceptable for use in the computational model, these operations can be improved in order to save memory. In Eq. 3.13, a summation must be made of all of the temporary values of \( \mathbf{K} \). Since the computational model is in two dimensions and there are six constitutive equations, that would mean that the computer would have to store four more arrays of size \( N \times M \times 6 \), where \( N \) and \( M \) are the number of points in the \( x \) and \( y \) dimensions, respectively. In
order to keep from having to save all of these temporary values of the vector $K$, a running total of the summation can be performed such that instead of having to save five vectors, namely $w^{(q)}$, $K^{(0)}$, $K^{(1)}$, $K^{(2)}$, and $K^{(3)}$, only two vectors, $w^{(q)}$ and a new variable, $\tilde{w}^{(q)}$, need to be saved. This is accomplished by calculating and using Eqs. 3.9–3.12 as normal, and also calculating $\tilde{w}^{(q)}$, which works as a running total of the vectors $K^{(0,1,2)}$ in the following way:

$$\tilde{w}_{l,m}^{(1)} = w_{l,m}^{n} + \frac{\Delta t}{6} (K_{l,m}^{(0)}),$$  (3.15)

$$\tilde{w}_{l,m}^{(2)} = \tilde{w}_{l,m}^{(1)} + \frac{\Delta t}{3} (K_{l,m}^{(1)}),$$  (3.16)

$$\tilde{w}_{l,m}^{(3)} = \tilde{w}_{l,m}^{(2)} + \frac{\Delta t}{3} (K_{l,m}^{(2)}),$$  (3.17)

and instead of using Eq. 3.13 to determine $w^{n+1}$, the following is used:

$$w^{n+1} = \tilde{w}^{(3)} + \frac{\Delta t}{6} (K^{(3)}).$$  (3.18)

It may seem unclear why both $w^{(q)}$ and $\tilde{w}^{(q)}$ need to be determined. Equation 3.18 indicates that $\tilde{w}^{(q)}$ needs to be determined in order to calculate the next time step of $w$. However, the variable $w^{(q)}$ needs to be determined in order to calculate the values of $K^{(q)}$ at every temporary time step $q$. To further illustrate this, the first few stages of the above Runge-Kutta scheme will be demonstrated.

Assuming that the initial conditions are defined, the values $w^{(1)}$ and $\tilde{w}^{(1)}$ are determined using $w^{n}$ and $K^{(0)}$, which are initially defined. To calculate $w^{(2)}$ and $\tilde{w}^{(2)}$, the value of $K^{(1)}$ must be known. The vector $K$ is a function of three other vectors: $F$, $G$, and $H$, each of which is a function of the model’s primitive variables—that is, the variables of density, velocity, temperature, etc. These primitive variables are easily accessible. When $w^{(1)}$ is known, the primitive variables $\rho$, $u$, $v$, $s_{fr}$, $T_{N_2}$, and $T_{O_2}$ at this intermediate time step can be determined, and using the equations of state derived previously, the variables $p$ and $T$ can be de-
rived as well. These primitive variables taken from $\mathbf{w}^{(1)}$ are then fed into $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, and finally the vector $\mathbf{K}^{(1)}$. This cycle is repeated through the first three steps of the Runge-Kutta scheme, but these $\mathbf{w}^{(q)}$ vectors are not used to determine $\mathbf{w}^{n+1}$; Eqs. 3.15–3.18 are used instead.

To summarize, the complete DRP scheme used in this investigation works in the following way: assuming that all variables are known from a previous time step or from the initial conditions,

- Determine $\mathbf{K}^{(0)}$ using Eq. 3.14 and the variables based on $\mathbf{w}^{(0)} = \mathbf{w}^{n}$.
- Insert $\mathbf{K}^{(0)}$ into Eq. 3.10 and Eq. 3.15 to calculate $\mathbf{w}^{(1)}$ and $\tilde{\mathbf{w}}^{(1)}$.
- Update the vectors $\mathbf{F}^{(0)}$, $\mathbf{G}^{(0)}$, and $\mathbf{H}^{(0)}$ to $\mathbf{F}^{(1)}$, $\mathbf{G}^{(1)}$, and $\mathbf{H}^{(1)}$ using the primitive variables contained within $\mathbf{w}^{(1)}$.
- Determine $\mathbf{K}^{(1)}$ using Eq. 3.14 and the variables based on $\mathbf{w}^{(1)}$.
- Insert $\mathbf{K}^{(1)}$ into Eq. 3.11 and Eq. 3.16 to calculate $\mathbf{w}^{(2)}$ and $\tilde{\mathbf{w}}^{(2)}$.
- Update the vectors $\mathbf{F}^{(1)}$, $\mathbf{G}^{(1)}$, and $\mathbf{H}^{(1)}$ to $\mathbf{F}^{(2)}$, $\mathbf{G}^{(2)}$, and $\mathbf{H}^{(2)}$ using the primitive variables contained within $\mathbf{w}^{(2)}$.
- Determine $\mathbf{K}^{(2)}$ using Eq. 3.14 and the variables based on $\mathbf{w}^{(2)}$.
- Insert $\mathbf{K}^{(2)}$ into Eq. 3.12 and Eq. 3.17 to calculate $\mathbf{w}^{(3)}$ and $\tilde{\mathbf{w}}^{(3)}$.
- Update the vectors $\mathbf{F}^{(2)}$, $\mathbf{G}^{(2)}$, and $\mathbf{H}^{(2)}$ to $\mathbf{F}^{(3)}$, $\mathbf{G}^{(3)}$, and $\mathbf{H}^{(3)}$ using the primitive variables contained within $\mathbf{w}^{(3)}$.
- Determine $\mathbf{K}^{(3)}$ using Eq. 3.14 and the variables based on $\mathbf{w}^{(3)}$.
- Finally, insert $\mathbf{K}^{(3)}$ into Eq. 3.18 to determine $\mathbf{w}^{n+1}$. 
3.3 The Weighted Essentially Non-Oscillatory Scheme

3.3.1 Introduction

The scheme described below is considerably different from the previous method. Rather than simply having one stencil to approximate derivatives, the weighted essentially non-oscillatory scheme (WENO) has multiple stencils available and uses a weighted combination of all stencils to achieve high accuracy and something that the previous scheme lacked: an ability to stably propagate discontinuities. The DRP scheme will prove itself to be a reliable, stable, and highly accurate scheme for all circumstances, provided that the waveform is at all times continuous. The introduction of a discontinuity in a DRP scheme, or most any time-domain scheme, causes a numerical oscillation because the DRP stencil calculates derivatives the same way regardless of the state of the fluid. In a fixed stencil scheme, as these types of schemes are called, if there is a discontinuity in the domain it is included in the derivation, which eventually causes considerable instability unless some form of limiter function and/or selective filter is used to remove grid to grid oscillations[64]. For the purposes of this dissertation, the DRP scheme is used in its current form for reasons of efficiency, and for the propagation of shocks the WENO scheme is used. The WENO scheme does not suffer from this numerical instability because it first determines where discontinuities exist and then uses stencils that do not contain any discontinuities to determine derivatives.

The development below follows the work of Shu[44] and the methods that will be described below are those that are suitable for the system of equations that are used. As stated previously, there are many versions of the WENO algorithm available and the one presented here is more simple than those that have been developed in recent years. For example, Titarev and Toro[66] proposed a variation of the WENO scheme using a second-order total variation diminishing flux which
improves the accuracy and shock capturing capabilities of the original. Levy et al. [67] suggested the use of centered WENO schemes that solve simultaneously in both dimensions, as opposed to most which work in one dimension and then the other.

### 3.3.2 Theoretical Background

One of the many things that distinguishes the WENO from the DRP scheme is that the WENO scheme’s objective is to determine interpolated values of a function at their so-called cell boundaries. In order to describe this, a number of concepts need to be introduced. Cells, cell centers, and cell sizes are defined as

\[
I_i \equiv [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}],
\]

(3.19)

\[
x_i \equiv \frac{1}{2}(x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}}),
\]

(3.20)

\[
\Delta x_i \equiv x_{i-\frac{1}{2}} - x_{i+\frac{1}{2}}, i = 1, \ldots, N.
\]

(3.21)

In this investigation there is a fixed grid spacing \(\Delta x\), so the size of each cell is the same.

The WENO method interprets a computational grid as the center points of cells that extend from \(x_{i-\frac{1}{2}}\) to \(x_{i+\frac{1}{2}}\), so the approach to determine derivatives is different than the DRP scheme. As stated above, the WENO scheme determines the interpolated values at the cell boundaries, and using those interpolated values it determines the derivatives at the cell centers. This is accomplished in the following way: given a function \(v\), the derivative \(v'\) can be given as

\[
\frac{1}{\Delta x_i} (\hat{v}_{i+1/2} - \hat{v}_{i-1/2}) = v'(x_i) + O(\Delta x^k),
\]

(3.22)

where

\[
\hat{v}_{i+1/2} \equiv \hat{v}(v_{i-r}, \ldots, v_{i-s}),
\]

(3.23)
Figure 3.1. Graphical depiction of the WENO stencils. In this case, three three-point stencils are used in order to determine the value at point \( i + 1/2 \).

\[
i = 0, 1, \ldots, N,
\]

\( r \) is defined as the number of points in the stencil to the left of \( x_i \), \( s \) the number of points to the right, and \( k = r + s + 1 \) the total number of points in the stencil. Given Eq. 3.22, the objective now is to determine the values of \( \hat{v}_{i+1/2} \) at all \( i \) so that \( v'(x_i) \) may be calculated.

The first step is to choose a value of \( k \). In this case, \( k = 3 \) is chosen to balance accuracy and computational efficiency. For \( k = 3 \), the scheme is fifth-order accurate in space, which has proven to be sufficient for the purposes of this work. A higher order scheme has not been chosen because of the greatly increased calculations required as \( k \) increases, which may become apparent as this discussion continues.

Most of the following discussion are specific for \( k = 3 \), although Shu[44] describes systems up to \( k = 7 \), corresponding to a 15th-order accurate scheme. The stencils for \( k = 3 \) are illustrated in Fig. 3.1. Mathematically, the stencils are defined as

\[
S_r(i) = \{x_{i-r}, \ldots, x_{i-r+k-1}\},
\]

where \( r = 0, \ldots, k - 1 \). For \( k = 3 \), this corresponds to three stencils that are able
Table 3.1. Table of constants $c_{rj}$ for the 5th-order WENO scheme

to approximate $\hat{v}_{i+1/2}$ for all $i$, given as

$$v_{i+1/2}^{(r)} = \sum_{j=0}^{k-1} c_{rj} v_{i-r+j}.$$  \hfill (3.26)

where $c_{rj}$ are constants. The derivation performed to determine $c_{rj}$ is lengthy and is omitted, but the result from Shu[44] is given below:

$$c_{rj} = \sum_{m=j+1}^{k} \frac{\prod_{l=0;l\neq m}^{k}(r - q + 1)}{\prod_{l=0;l\neq m}^{k}(k - l)}.$$  \hfill (3.27)

For the case of $k = 3$, the values of $c_{rj}$ are given in Table 3.1.

When the medium is continuous, it is possible to use a combination of the three stencils that maximizes the order of accuracy of the scheme. Mathematically, the combination of the stencils is written as

$$\hat{v}_{i+1/2} = \sum_{j=0}^{k-1} w_r v_{i+1/2}^{(r)},$$  \hfill (3.28)

and the weights, $w_r$, that control how the three stencils are combined are essential to the scheme’s stability and accuracy. Maximizing the scheme’s accuracy when it is working in a continuous medium is relatively simple. The weights that the stencils are given when the medium does not contain a discontinuity, known as the linear weights, are given by Shu and make the scheme fifth-order accurate:

<table>
<thead>
<tr>
<th>$r$</th>
<th>$j=0$</th>
<th>$j=1$</th>
<th>$j=2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/3</td>
<td>5/6</td>
<td>-1/6</td>
</tr>
<tr>
<td>1</td>
<td>-1/6</td>
<td>5/6</td>
<td>1/3</td>
</tr>
<tr>
<td>2</td>
<td>1/3</td>
<td>-7/6</td>
<td>11/6</td>
</tr>
</tbody>
</table>
\begin{align*}
  d_0 &= \frac{3}{10}, \\
  d_1 &= \frac{6}{10}, \\
  d_2 &= \frac{1}{10}.
\end{align*} 

(3.29)

Intuitively, the weights make sense: the two stencils that encompass the point \( i + \frac{1}{2} \) (\( d_0 \) and \( d_1 \)) are more heavily weighted than the one that does not (\( d_2 \)), and the stencil that contains more points behind point \( i + \frac{1}{2} \) while still straddling it, is the most heavily weighted of all.

The key to the success of the WENO scheme is its ability to leave out stencils that contain discontinuities. In order to determine the weights for each stencil when the medium is not continuous, the linear weights should change based on the smoothness of the solution. That is, there should be a way to make the scheme’s weights be the linear weights when it is continuous, and if one or more of the stencils contains a discontinuity, it should be left out of the calculation by making the weight approach zero for that stencil. Fig. 3.2 demonstrates this concept for the case of a function of the form

\[ f(x) = \sin(x) \quad \text{if} \quad 0 \leq x \leq 3\pi/2 \]  

(3.30)

and

\[ f(x) = -\sin(x) \quad \text{otherwise}. \]  

(3.31)

In this case, everywhere other than the area around \( 3\pi/2 \) is continuous, so here the linear weights are ideal. A closer view of the discontinuity is given in Fig. 3.3 with the stencils included for clarity. At the point to the left of the discontinuity, the only stencil that should be used is stencil 3, which uses points \( i - 2, i - 1, \) and \( i \) to determine the cell boundary value at \( i + \frac{1}{2} \), since the other two contain the discontinuity. Similarly, at the point to the right of the discontinuity the only
stencil that should be used is the stencil that uses \( i - 2, i - 1, \) and \( i \) to determine the cell boundary value at \( i + 1/2 \).

The way that the scheme quantifies the amount of discontinuity in each stencil is through the use of smoothness indicators. These variables are required to become very large in the presence of a discontinuity, causing the stencil weight which is inversely proportional to the smoothness to approach zero. This means that it is specifically the smoothness indicators that are essential to the inclusion of a stencil that should be used or the exclusion of a stencil that contains a discontinuity and should not be used, and this ability to properly make these decisions is crucial to the stability and accuracy of the scheme. The smoothness indicators given by Jiang and Shu[63] are

\[
\beta_0 = \frac{13}{12} (v_i - 2v_{i+1} + v_{i+2})^2 + \frac{1}{4} (3v_i - 4v_{i+1} + v_{i+2})^2, \quad (3.32)
\]

\[
\beta_1 = \frac{13}{12} (v_{i-1} - 2v_i + v_{i+1})^2 + \frac{1}{4} (v_{i-1} - v_{i+1})^2, \quad (3.33)
\]
Figure 3.3. Demonstration of stencil selection. To determine the function’s value at point $i + 1/2$, it is necessary to use only stencil 3, which uses points $i - 2$, $i - 1$, and $i$.

and

$$\beta_2 = \frac{13}{12} (v_{i-2} - 2v_{i-1} + v_i)^2 + \frac{1}{4} (v_{i-2} - 4v_{i-1} + 3v_i)^2.$$  \hspace{1cm} (3.34)

Using these values, the following equations turn the smoothness indicators into stencil weights:

$$\alpha_r = \frac{d_r}{\left(\epsilon + \beta_r\right)^2},$$  \hspace{1cm} (3.35)

and

$$w_r = \frac{\alpha_r}{\sum_{s=0}^{k-1} \alpha_s}.$$  \hspace{1cm} (3.36)

where $\epsilon$ is a constant, used to keep the denominator of Eq. 3.35 from going to zero. The value chosen for $\epsilon$ in this work is $10^{-6}$. The variable $\alpha_r$ is a temporary variable used by Eq. 3.36, which is included to make sure that

$$\sum_{r=0}^{k-1} w_r = 1.$$  \hspace{1cm} (3.37)

In the presence of a discontinuity, a smoothness indicator becomes very large. This
causes the weight $w_r$ associated with that smoothness indicator to approach zero.

This completes the basic theory of the WENO scheme. To summarize: the scheme determines the smoothness of the medium for each of the three stencils at all points in space and weighs the combination of the three based upon the smoothness of the medium contained within each. If the medium is smooth over all three stencils then the optimized linear weights are used that give the scheme fifth-order accurate results. After the stencil results are added, the values of $v_{i+\frac{1}{2}}$ is known for all $i$, allowing the use of Eq. 3.22.

The above discussion is, unfortunately, only the beginning of what is required to implement a WENO scheme for the solution a system of equations of this type. To solve the system of constitutive equations given in Ch. 2, the component-wise WENO method is used.

### 3.3.3 The Component-wise Solution

This section describes the component-wise WENO solution in only one dimension for simplicity. Extending it to two dimensions is straightforward. In this case, all that is required is to first perform the one-dimensional scheme described below for the $x$-dimension, and then repeat the same process for the $y$-dimension.

For long term stability when propagating high amplitude shocks, upwinding should be used to determine the flux. The use of central WENO schemes has proven to be effective in some situations, but for the case of discontinuities, an upwind-biased method has been determined to be most suitable to ensure that the waves propagate without numerical oscillations and without a need for artificial dissipation[68]. Upwinding, also known as upwind biasing, is a method in which more points in the spatial stencil are placed behind the wave that is being propagated in the domain. This is known to keep the scheme accurate and stable for shock propagation. In order to ensure that the WENO solution is an upwind
biased scheme, flux-splitting is used. This means that for a system of the form

\[ \frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0, \]  

which is a simplified form of the system used in this work, the requirement is that

\[ \frac{\partial \mathbf{F}^+}{\partial \mathbf{w}} \geq 0, \]  

and

\[ \frac{\partial \mathbf{F}^-}{\partial \mathbf{w}} \leq 0. \]

The vectors \( \mathbf{F}^+ \) and \( \mathbf{F}^- \) can be determined using many flux-splitting methods. One of the simplest is known as Lax-Friedrichs splitting[44] and is given by

\[ \mathbf{F}^\pm(\mathbf{w}) = \frac{1}{2}(\mathbf{F} \pm \alpha \mathbf{w}). \]

For the determination of \( \alpha \), the condition that must be satisfied is

\[ \alpha \geq \max \left| \frac{\partial \mathbf{F}}{\partial \mathbf{w}} \right|. \]  

In the calculations provided in this work, \( \alpha \) is chosen to be between 300 and 400.

The WENO scheme is slightly different now that flux-splitting has been implemented. As demonstrated in Fig. 3.1, the three stencils are not centered around point \( i + \frac{1}{2} \) and consequently there is an extra point on the left or the right of the cell boundary whose value is to be determined. This allows for two ways to determine the same cell boundary value: one which uses three points on the left and two on the right as described before, and one which is the mirror image of that with respect to the stencil point coefficients, as shown in Fig. 3.4. The stencil point weights as well as the linear weights for the entire stencil are given for both \( c_{rj} \) and \( d_r \) as well as \( \tilde{c}_{rj} \) and \( \tilde{d}_r \).
Figure 3.4. Graphical depiction of the two ways in which the WENO scheme can determine the interpolated value at point \( i + 1/2 \) (solid line). The values stencil point weights are given for each of the three candidate stencils as well as the linear weights for the entire stencil.

These two methods are given mathematically as

\[
v_{i+1/2}^- = \sum_{j=0}^{k-1} c_{rj} v_{i-r+j}, \tag{3.43}
\]

and

\[
v_{i+1/2}^+ = \sum_{j=0}^{k-1} \tilde{c}_{rj} v_{i-r+j+1}, \tag{3.44}
\]

where

\[
\tilde{c}_{rj} = c_{2-r,2-j}, \tag{3.45}
\]

for \( k = 3 \) and the “+” and “−” signs denote if there are three points on the left or right side of the point \( i + \frac{1}{2} \).

For the case of Eq. 3.44, a different version of Eq. 3.28 is required to keep the mirror image stencils in Eq. 3.44 consistent with the stencils in Eq. 3.43. This is
accomplished through the following:

\[
\hat{v}_{i+1/2} = \sum_{j=0}^{k-1} \tilde{w}_r v_{i+1/2}^{(r)},
\]

where

\[
\tilde{w}_r = \frac{\tilde{\alpha}_r}{\sum_{s=0}^{k-1} \tilde{\alpha}_s},
\]

\[
\tilde{\alpha}_r = \frac{\tilde{d}_r}{(\epsilon + \tilde{\beta}_r)^2},
\]

\[
\tilde{d}_r = d_{2-r},
\]

and

\[
\tilde{\beta}_r = \beta_{r+1}.
\]

As stated before, upwind-biasing is a method in which there are more points behind the propagating wave than in front of it, and Eqs. 3.43 and 3.44 allow this. It is for this reason that in the WENO reconstruction of the vector \( \mathbf{F}^+ \), which satisfies Eq. 3.39, Eq. 3.43 is used and in the WENO reconstruction of the vector \( \mathbf{F}^- \), which satisfies Eq. 3.40, Eq. 3.44 is used. If these rules are followed the scheme becomes upwind-biased.

Once the values of \( \mathbf{F}^+ \) and \( \mathbf{F}^- \) at the cell boundaries have been determined, the fluxes are recombined to determine the numerical flux using

\[
\hat{F}_{i+1/2} = \mathbf{F}^+_{i+1/2} + \mathbf{F}^-_{i+1/2}.
\]

The result of Eq. 3.51 is then used as \( \hat{v}_{i+1} \) in Eq. 3.22 to determine the value \( \partial \mathbf{F} / \partial x \).

To summarize, the 1D component-wise WENO solution works in the following way: assuming that all variables are known from a previous time step or from a set of initial conditions,
• Perform the Lax-Friedrichs flux-splitting scheme, given in Eqs. 3.41, using a value of $\alpha$ that is at least equal to that obtained from Eq. 3.42 to obtain $F^+$ and $F^-$.

• Define $F^+ = v$ and $F^- = v$ to determine the smoothness indicators using Eq. 3.32–3.34 for both $F^+$ and $F^-$.

• Using these smoothness indicators, use Eq. 3.35 and Eq. 3.36 to determine the weights $w_r$ of the three stencils for both $F^+$ and $F^-$.

• As before, define $F^+ = v$ and use Eq. 3.43 and Eq. 3.28 to perform the WENO reconstruction of $F^+$. Then define $F^- = v$ and use Eq. 3.44 and Eq. 3.46 to perform the WENO reconstruction of $F^-$. 

• Combine the reconstructed $F^+$ and $F^-$ using Eq. 3.51 to obtain $\hat{F}_{i+\frac{1}{2}}$.

• Use $\hat{F}_{i+\frac{1}{2}} = \hat{v}_{i+\frac{1}{2}}$ in Eq. 3.22 to determine the value $\partial F / \partial x$.

• For a two-dimensional calculation, repeat all previous steps in the $y$-dimension using the vector $G$ instead of $F$.

### 3.3.4 Time Discretization

An optimal third-order total variation diminishing (TVD) Runge-Kutta scheme is used for the time discretization. The scheme is of the form

$$w^{(1)} = w^n + \Delta t(K^{(0)}),$$  \hspace{1cm} (3.52)

$$w^{(2)} = \frac{3}{4}w^n + \frac{1}{4}w^{(1)} + \frac{1}{4}\Delta t(K^{(1)}),$$  \hspace{1cm} (3.53)

$$w^{n+1} = \frac{1}{3}w^n + \frac{2}{3}w^{(2)} + \frac{2}{3}\Delta t(K^{(2)}),$$ \hspace{1cm} (3.54)

where

$$K = -\frac{\partial F}{\partial x} - \frac{\partial G}{\partial y} + H.$$ \hspace{1cm} (3.55)

These spatial derivatives are determined by approximating the $x$- and $y$-derivatives of the $F$ and $G$ vectors, respectively, using the WENO scheme described in the
previous section. Again, the vector $H$ is the vector containing the source terms of the equations.

This third-order scheme does not need to be altered in the way the fourth-order scheme was in the previous discussion, so it is used in its present form. In summary, the complete WENO solution works in the following way:

- Determine the values of $\partial F/\partial x$ and $\partial G/\partial y$ using the component-wise WENO scheme described and summarized in Section 3.3.3 using the variables $w^{(0)} = w$, $F^{(0)} = F$, $G^{(0)} = G$, and $H^{(0)} = H$.
  - Use $F^{(0)}$, $G^{(0)}$, and $H^{(0)}$ to calculate $K^{(0)}$ in Eq. 3.55.
  - Use $w^{(0)}$ and $K^{(0)}$ in Eq. 3.52 to determine $w^{(1)}$.
  - Update the vectors $F^{(0)}$, $G^{(0)}$, and $H^{(0)}$ to $F^{(1)}$, $G^{(1)}$, and $H^{(1)}$ using the primitive variables contained within $w^{(1)}$.
- Determine the values of $\partial F/\partial x$ and $\partial G/\partial y$ again, using the variables $w^{(1)}$, $F^{(1)}$, $G^{(1)}$, and $H^{(1)}$.
  - Use $F^{(1)}$, $G^{(1)}$, and $H^{(1)}$ to calculate $K^{(1)}$ in Eq. 3.55.
  - Use $w^{(0)}$, $w^{(1)}$, and $K^{(1)}$ in Eq. 3.53 to determine $w^{(2)}$.
  - Update the vectors $F^{(1)}$, $G^{(1)}$, and $H^{(1)}$ to $F^{(2)}$, $G^{(2)}$, and $H^{(2)}$ using the primitive variables contained within $w^{(2)}$.
- Determine the values of $\partial F/\partial x$ and $\partial G/\partial y$ again, using the variables $w^{(2)}$, $F^{(2)}$, $G^{(2)}$ and $H^{(2)}$.
  - Use $F^{(2)}$, $G^{(2)}$, and $H^{(2)}$ to calculate $K^{(2)}$ in Eq. 3.55.
  - Use $w^{(0)}$, $w^{(2)}$, and $K^{(2)}$ in Eq. 3.54 to determine $w^{n+1}$.

### 3.4 Source Simulations

The objective of this section is to describe a method of approximating acoustic sources. For the purposes of comparison to known analytical solutions, simulated
point and line sources are used. In the interest of giving the model the maximum amount of generality and flexibility, it will also be shown that this method allows for the addition of acoustic sources of essentially arbitrary spatial and temporal dependence.

This section first describes the concept of adding sources into the model and its application to the simulation of point and line sources. The concepts are then extended to more complicated geometry and temporal dependence.

The point source is a very simple example that is used in the development of the more general theory of source addition. An ideal point source has an infinitely small spatial distribution, but this type of source is perfectly discontinuous because it is exactly zero everywhere in space except for at its location, where it has jumped with infinite slope to a finite amplitude. A source with infinite slope is unstable for numerical methods and instead a spatially distributed source must be used. A Gaussian distribution is frequently used, which is of the form

\[ p(x,t) = Ae^{-\frac{\ln(2)(x-x_0)^2}{\alpha^2}}e^{j\omega t}, \quad (3.56) \]

where \( \alpha \) is the Gaussian half-width, which controls the amount of spatial extent of the source. The objective is to have a source that is as near a point source as possible while making sure that the source remains stable.

Although other sources types are possible, in this work a density source is used which is inserted into the domain simply through adding a source term to the left-hand side of Eq. 2.1 as shown below:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = Ae^{-\frac{\ln(2)(x-x_0)^2}{\alpha^2}}e^{j\omega t} \sin(\omega t). \quad (3.57) \]

For spatially complex sources, this method also works, granted that the shape of the source can be explained mathematically and inserted into the right hand side of the continuity equation.
The insertion of broadband waveforms is desirable as this greatly increases the model’s utility. For this algorithm, a subroutine has been written that reads in a text file of density perturbations as a function of time and assigns the values to an array. This array is used instead of the sinusoid function in Eq. 3.57 and a Gaussian spatial distribution is used as usual. For more information, a sample FORTRAN code capable of accomplishing this is given in Appendix B.

The Gaussian distribution in space can affect the frequency content of the wave that propagates into the medium. If the source is spatially too large, it will certainly have trouble creating a frequency with a small wave number. It has been noted in this investigation that for a monofrequency source, this can be easily avoided if the Gaussian half-width is less than approximately 40% of the wavelength.

The spatial distribution of the source can have a considerable impact upon broadband waveforms. To give a simple demonstration of this phenomenon, a lossless spatial domain of fixed resolution was disturbed using a broadband source of varying Gaussian half-width. The time waveform resulting from this calculation, recorded $4.02 \times 10^{-3}$ m from the source, is shown in Fig. 3.5 for different values of the Gaussian half-width, $\alpha$. It can be seen that for $\alpha = 64\Delta x$, a considerable amount of the high frequencies have been lost. This is a factor that needs to be kept in mind when inserting broadband sources.

Another factor remains to be discussed as well. Once the source has disturbed the medium at a particular frequency, it is not guaranteed that the frequency is propagated accurately. If the wave is not properly resolved spatially then the wave will not be propagated; this typically occurs when the spatial resolution drops somewhere around 10 points per wavelength. The difficulties associated with attempting to simulate broadband acoustic propagation can be demonstrated. A one second jet noise waveform with a center frequency of approximately 300 Hz sampled at 44.1 kHz was used as the input into the domain. The time waveform
Figure 3.5. Demonstration of the effect that the spatial distribution of a Gaussian-distributed source has upon the propagated waveform. The $\alpha=64 \Delta x$ case clearly has the least high frequency content.

of the recording is given in Fig. 3.6 and the power spectral density of the jet noise recording is given in Fig. 3.7, which indicates a large amount of high frequency components in the waveform.

To illustrate the effects of wave resolution upon broadband acoustic signals, the time waveform given in Fig. 3.6 was used as the source input and the resulting waveform propagating in the medium very near the source was recorded. The comparison of the results of this calculation and the original waveform are given in Fig. 3.8 and the power spectral density (PSD) of the calculated waveform is given Fig. 3.9. The time waveform in the computational domain shows a significant amount of smoothing, which is to be expected considering the relatively low spatial resolution. The PSD of the result confirms that high frequencies over 3 kHz, corresponding to waves resolved with less than approximately seven points per wavelength, have not been propagated accurately.

To better understand the impact of smoothing, the spatial resolution was increased to varying degrees. In order to increase the spatial resolution while keeping
the model stable and without altering the input waveform’s frequency content, the waveform was interpolated to varying degrees. This allows for smaller temporal steps and hence smaller spatial steps, which should allow for higher frequency content in the medium. The interpolation was performed using an algorithm given as Algorithm 8.1 in Ref. [69], which first inserts zeros between the original data points, designs a symmetric finite impulse response filter that does not affect the original data points and minimizes the mean-square of the interpolated points, and finally applies the filter to the data to obtain the interpolated result. To ensure that the interpolation scheme is not affecting the frequency content of the data, Fig. 3.10 compares the frequency content of the original waveform to the waveforms resulting from interpolation factors of two, four and eight. The agreement is very good for all frequencies, although there is a small discrepancy as the Nyquist frequency of 149 kHz for the original data is approached. This interpolation scheme will be used for the calculations presented in later sections of this dissertation.

Using the described algorithm, waveforms with interpolation factors of two, six, and ten were created and propagated in the model using spatial discretizations
Figure 3.7. Power spectral density of the input jet noise recording. A considerable amount of energy is contained in the high frequencies of this waveform.

Figure 3.8. Time waveform recording resulting from the sample jet noise waveform input used as is. There has been a large amount of smoothing because of the model's inability to resolve all frequencies.
Figure 3.9. Fourier transform of the resulting time waveform using the sample jet noise waveform input used as is. The high frequency content has been sacrificed, which is because of the model’s inability to capture these high frequencies.

Figure 3.10. Comparison of the frequency content of a noise waveform (dotted blue line) to the frequency content of the waveforms resulting from interpolations of two (dashed red line), four (solid green line), and eight (dash dot black line). A small deviation exists as the frequency approaches the Nyquist frequency of the original waveform.
Figure 3.11. Comparison of the original recording (dashed green line) to the time waveforms recorded in the computational domains resulting from interpolation factors of two (dotted blue line), six (dash dot red line), and ten (solid black line).

equal to 1/2, 1/6, and 1/10 that of the original data given in Fig. 3.6, respectively, and the results are compared in Fig. 3.11 and Fig. 3.12.

As expected, the case with the highest amount of interpolation, which has the finest spatial resolution, allows for the highest frequencies to be recreated in the domain. It should be stated, however, that if the new time waveform has ten times the number of temporal points, it will take ten times longer to propagate a signal of the same length in time, which gives reason to carefully weigh these two factors.

The conclusion of this discussion is simply that the user be aware of these two factors: the impact that the spatial distribution of the source and the grid resolution have upon the frequency content of the actual propagated wave. In Chapter 5, when scale model data are inserted into the computational domain, the frequency content is investigated for every calculation to ensure that the desired input has been suitably simulated.

A final note for this section: the application of spatially complex sources has not been investigated, although it is certainly possible. Providing that the source
is continuous enough and the shape can be described mathematically, the model can theoretically simulate these acoustic sources as well.

3.5 Turning on or off Absorption and Dispersion Mechanisms

To give the model greater utility, a simple method of turning on or off absorption and/or dispersion mechanisms is implemented. There are five “on/off switches” inserted in the model, corresponding to the inclusion or exclusion of the effects of thermal conductivity, shear viscosity, bulk viscosity, nitrogen relaxation, or oxygen relaxation.

Implementing these switches is very simple. First, revisit the momentum equation in two dimensions, the entropy equation, and the relaxation equations given
in Ch. 2 and repeated here for convenience,

$$
\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} = -\frac{\partial p}{\partial x} + \mu_B \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + \mu \left( \frac{\partial \phi_{xx}}{\partial x} + \frac{\partial \phi_{xy}}{\partial y} \right), \quad (3.58)
$$

$$
\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho vu)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} = -\frac{\partial p}{\partial y} + \mu_B \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y \partial x} \right) + \mu \left( \frac{\partial \phi_{yx}}{\partial x} + \frac{\partial \phi_{yy}}{\partial y} \right), \quad (3.59)
$$

$$
\frac{\partial(\rho s_{fr})}{\partial t} + \nabla \cdot (\rho s_{fr} \mathbf{v}) = \frac{\mu_B}{T} (\nabla \cdot \mathbf{v})^2 + \frac{\mu}{2T} \sum_{ij} \phi_{ij}^2 + \frac{\kappa}{T^2} (\nabla T)^2
$$

$$
+ \frac{\rho}{T} \sum_{\nu} A_{\nu} \frac{DT_{\nu}}{Dt} - \sum_{\nu} \frac{\rho}{T_{\nu}} c_{v\nu} \frac{DT_{\nu}}{Dt} + \nabla \cdot \left( \frac{\kappa}{T} \nabla T \right), \quad (3.60)
$$

$$
\frac{\partial(\rho T_{N_2})}{\partial t} + \nabla \cdot (\rho T_{N_2} \mathbf{v}) = \frac{\rho}{\tau_{N_2}} (T - T_{N_2}), \quad (3.61)
$$

and

$$
\frac{\partial(\rho T_{O_2})}{\partial t} + \nabla \cdot (\rho T_{O_2} \mathbf{v}) = \frac{\rho}{\tau_{O_2}} (T - T_{O_2}), \quad (3.62)
$$

where the relaxation equation given previously is expanded into two equations, Eq. 3.61 for nitrogen and Eq. 3.62 for oxygen relaxation.

If the effects of nitrogen relaxation are not wanted, they are eliminated by imposing that the specific heat at constant of volume of nitrogen, $c_{vN_2}$, equals zero. Investigation of Eq. 3.60 and Eq. 3.61 reveals that this indeed removes any terms related to nitrogen relaxation that affect the calculation. The same is true for oxygen relaxation; if $c_{vO_2}$ is set to equal zero, then referring to Eq. 3.60 and Eq. 3.62, oxygen relaxation effects are removed.

Looking instead to Eq. 3.58, Eq. 3.59, and Eq. 3.60, it is clear that if the bulk
viscosity, shear viscosity, or the coefficient of thermal conductivity are individually forced to equal zero, then the absorption effects of these mechanisms are removed. This feature is especially helpful when investigating the effect that each specific absorption mechanism has upon the propagation characteristics of the medium and allows for potential comparison of the computational results to a variety of data which may not include nitrogen or oxygen.

3.6 Boundary Conditions

3.6.1 Introduction

Three different types of boundary conditions have been implemented in the model. Two of these, the periodic and rigid conditions, are relatively simple while the third, the absorbing condition, can be achieved using various methods and consequently with various levels of success. Another system of on/off switches is implemented into the algorithm which allows the user of the program to be able to apply any of the boundary conditions to any of the domain boundaries.

3.6.2 Periodic Boundary Conditions

The periodic boundary condition is a very useful tool for computations in one dimension. Although it is a non-physical boundary condition it allows the computational domain to remain very small while still permitting propagation over long distances. The boundary condition is particularly useful for the simulation of one-way propagation of finite length disturbances, and it should be noted that spherical or cylindrical spreading should not be included if a periodic boundary condition is applied.

The periodic condition works as follows: if, for example on a one-dimensional domain of length $N$, the points $1, \ldots, 5$ are forced to be equal to the points $N -$
Figure 3.13. Graphical representation of the periodic boundary condition implemented. The wave is simply reinserted into the other side of the domain by equating N-4, N-3, N-2, N-1, and N to the points 1, 2, 3, 4, and 5.

9, ..., N−5 and points N−4, ..., N are forced to equal 6, ..., 10, then as a right-going wave reaches the end of the domain it simply is wrapped around to the left side, where it is free to propagate through again, and vice versa for a left-going wave. This boundary condition is demonstrated in Fig. 3.13. It should be noted that this condition is usually only useful for plane propagation in one dimension since, for example, a cylindrically spreading wave would not be able to wrap around a two-dimensional domain in the same way as a plane wave.

3.6.3 Rigid Boundary Conditions

The rigid boundary condition is explained next. In linear acoustics, a reflection at a rigid boundary can be equated to a linear superposition of the incident wave and an incoming wave of equal amplitude propagating in the opposite direction. The graphical depiction of this boundary condition is given in Fig. 3.14. The computational analog of this is equating the last two points of \( w \) in the domain to each other with the exception of the sign of the velocity, which is reversed. That
Figure 3.14. Graphical representation of the reflecting boundary condition, which is created by introducing an incoming wave of equal magnitude traveling in the opposite direction.

is to say, for the simulation of a rigid boundary at $x = N$ and for all values of $y$, the following relations are set up:

$$\rho(N, j) = \rho(N - 1, j), \quad (3.63)$$

$$u(N, j) = -u(N - 1, j), \quad (3.64)$$

$$v(N, j) = v(N - 1, j), \quad (3.65)$$

$$s_{fr}(N, j) = s_{fr}(N - 1, j), \quad (3.66)$$

$$T_{N_2}(N, j) = T_{N_2}(N - 1, j), \quad (3.67)$$

and

$$T_{O_2}(N, j) = T_{O_2}(N - 1, j). \quad (3.68)$$

This in essence creates an incoming wave with exactly the same values of pressure, temperature, density, etc., that is traveling in the opposite direction and acts to
3.6.4 Absorbing Boundary Conditions

The final boundary condition considered is also the most difficult. The scheme used here is of the simplest sort and uses a method that essentially mimics an anechoic termination, which is similar in concept to the work of Wasistho et al.[70]. To simulate an anechoic termination, the model uses a Gaussian envelope of the form:

\[ Q(\xi) = e^{-\ln(2)(\xi - \xi_0)^2 / \alpha_a^2} , \]

where \( Q \) is a generalized variable, \( \xi \) is a spatial variable which could be \( x \) or \( y \), \( \xi_0 \) is the position of the center of the Gaussian, and \( \alpha_a \) is the half-width. This envelope is imposed upon all of the primitive variables as well as pressure and temperature. The condition is illustrated in Fig. 3.15, in which a Gaussian decay is imposed on the right hand side of the boundary.
This method, as simple as it is, will prove itself to be quite effective. The major drawback of using this method is that the number of points in the envelope may become too computationally expensive in two- or three-dimensions. When this drawback becomes a serious problem, another method should be used such as the radiation boundary condition introduced by Tam and Webb[53].

3.7 Stability Requirements

For explicit schemes like the DRP and WENO schemes, there are stability requirements that the model must meet. The most general requirement is the Courant-Friedrichs-Levy (CFL) condition, which dictates the maximum time step that can be chosen based upon the spatial discretization. The CFL number is given as:

\[
CFL = \frac{c\Delta t}{\Delta x},
\]

where \(c\) is the sound speed, \(\Delta x\) is the spatial grid size, and \(\Delta t\) is the temporal grid size. For the implementation of the DRP scheme used in this work, it has been noted that \(CFL < 1\) is required for stability; for the WENO scheme used in this work, \(CFL < 0.5\) is necessary for stability.

In many cases, the CFL condition is that \(CFL < 1\), and a short explanation of the physical meaning behind this condition may help explain why this is so. Solving for \(\Delta t\) in Eq. 3.70:

\[
\Delta t = CFL \frac{\Delta x}{c},
\]

The last two terms on right-hand side of the Eq. 3.71, \(\Delta x/c\), determine the propagation time required to span a distance \(\Delta x\). If the CFL number were greater than one then consequently the value of \(\Delta t\) would also be greater than the time required for an acoustic wave to span a distance \(\Delta x\). The impact of having a \(\Delta t\) greater than \(\Delta x/c\) will be discussed graphically on a computational grid. Figure
Figure 3.16. Physical demonstration of a stable CFL condition. In this case, the domain of influence for point $i$ contains points $i-1$, $i$, and $i+1$, which all are able to reach point $i$ within the chosen $\Delta t$, which represents a stable scenario.

3.16 shows how point $i$ is influenced by its neighboring points after a propagation time $\Delta t$. The area of space that is able to affect the results for point $i$ after a propagation time $\Delta t$ is called the domain of influence. Assuming that points $i-1$, $i$, and $i+1$ are the only points within the domain of influence, the situation depicted in Fig. 3.16 represents a stable condition. In essence, to have a stable solution for point $i$ at time $t + \Delta t$, the time spacing $\Delta t$ must be small enough that the entire domain of influence has time to reach it. An unstable case is given in Fig. 3.17, where the domain of influence does not have adequate time to propagate to point $i$.

The final note on stability regards the choice of $\Delta t$ when molecular relaxation is turned on. The two algorithms have demonstrated themselves to be unconditionally unstable unless $\Delta t$ is chosen to be at most $1/4$ of the shortest relaxation time included in the calculation. This means that if the calculation includes only
Figure 3.17. Physical demonstration of an unstable CFL condition. In this case, the domain of influence for point \( i \) contains points \( i - 1, i, \) and \( i + 1 \), none of which are able to reach point \( i \) within the chosen \( \Delta t \), which represents an unstable scenario.

nitrogen relaxation and not oxygen relaxation, the stability guideline is that

\[
\Delta t \leq \frac{\tau_{N_2}}{4},
\]  

(3.72)

and if oxygen relaxation is included, regardless of whether nitrogen relaxation is included, the stability guideline is that

\[
\Delta t \leq \frac{\tau_{O_2}}{4}.
\]  

(3.73)

If both the CFL and relaxation time stability requirements are met, then the model should propagate the waves stably.

### 3.8 Conclusions

This chapter has discussed the computational foundations of the model as well as the additional features that have been added to increase the versatility of the
model. In order to further clarify the concepts in this section, Appendix B includes sample FORTRAN 90 code demonstrating the DRP scheme, the WENO scheme in component form, source addition, absorption/dispersion on/off switches, the inclusion of periodic, reflecting, and absorbing boundary conditions, and a graphical user interface that has been constructed for user convenience. It is hoped that this chapter coupled with the sample code included at the end, has given the reader enough of an understanding of the workings of this numerical model that they may immediately begin their own development of a numerical scheme suitable for their needs.
Numerical Verification

4.1 Introduction

In this chapter, the two numerical schemes introduced previously will be tested to confirm that they produce accurate results. The models will simulate simple acoustic situations in which the results are known through analytical expressions. This chapter includes verification of absorption and dispersion, as well as nonlinear phenomena such as wave steepening and nonlinear amplification factors for normal incidence upon a rigid boundary. The two schemes have varying amounts of inherent numerical dissipation and dispersion, and it will be shown that the accuracy of the solution is influenced by their inherent error when the computational parameters are kept constant.

4.2 Wave Steepening

4.2.1 Introduction

The steepening of finite amplitude waves as a result of nonlinearity is a well understood phenomenon. Simply stated, in a nonlinear system of sufficiently high amplitude, pressure disturbances of higher amplitude travel considerably faster
Figure 4.1. Example of wave steepening for an initially sinusoidal waveform. By $\sigma=1$, the waveform has an infinite slope.

than lower pressure disturbances, causing a steepening of the waveform as the high pressures begin to overtake the low pressures. As this steepening progresses the wave reaches a point at which the slope of the waveform goes to infinity. At this point, it is said that a shock has formed. The distance at which this occurs is called the shock formation distance, denoted by $\bar{x}$. For convenience, the distance that the wave has traveled is normalized by the shock formation distance and renamed $\sigma$:

$$\sigma = \frac{x}{\bar{x}}, \quad (4.1)$$

where $x$ is the propagation distance. The phenomenon of wave steepening over a variety of $\sigma$ is demonstrated in Fig. 4.1 and the general theory is more fully explained by Hamilton & Blackstock[3].

Wave steepening in a lossless and a thermoviscous medium is verified below. The first verification compares the computational model’s steepening to that expected using analytical methods. This is accomplished through an investigation of the harmonic component amplitudes of the propagated wave rather than through
an investigation of the time waveform itself. For a purely sinusoidal source propagating in one dimension no more than one shock formation distance in a lossless medium, the solution for the Fourier component amplitudes normalized by the fundamental is given by the Fubini solution [72],

\[ B_n = \frac{2}{n\sigma} J_n(n\sigma), \]  

where \( B_n \) is the Fourier component amplitude of harmonic number \( n \). For the case of lossless plane wave propagation from a sinusoidal source, the shock formation distance is given by

\[ \bar{x} = \frac{1}{\beta\epsilon k}, \]  

where \( \beta \) is the coefficient of nonlinearity for a perfect gas, \( \epsilon \) is the acoustic Mach number and \( k \) is the wave number of the fundamental frequency. Using this analytical solution, comparison to the computed results is relatively simple and requires only a Fourier transform of the time waveform calculated by the model and normalization of the amplitudes to match the analytical results.

For the case of plane wave propagation in a thermoviscous medium, the analytical solution is based on Mendousse’s solution of Burgers’ equation given by Pierce[71]:

\[ \frac{p}{p_0} = \frac{4}{I_0\left(\frac{\Gamma}{2}\right)} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n I_n\left(\frac{\Gamma}{2}\right) ne^{-n^2\sigma/\Gamma} \sin(n\omega t')}{\Gamma^2 ne^{-n^2\sigma/\Gamma} \cos(n\omega t')} \]  

where \( \Gamma \) is the Gol’dberg number and \( \delta \) is the thermoviscous diffusivity of sound, calculated using

\[ \Gamma = \frac{c^3}{\delta \omega^2 \bar{x}}, \]  

and

\[ \delta = \frac{\mu}{2\rho} \left[ \frac{4}{3} + \frac{\mu_B}{\mu} + \frac{(\gamma - 1)\kappa}{c_p\mu} \right], \]  

t’ is the retarded time, and \( I_n \) are the modified Bessel functions. This solution outputs a time waveform, so the solution and the computed solution will need
to be broken down into their Fourier component amplitudes for comparison. It should also be noted that the argument \( \Gamma \) in the modified Bessel functions must be small, as the function quickly grows to infinity with increasing \( \Gamma \), so an asymptotic solution is necessary. The results here use a small value of \( \Gamma \) for this reason, but more because of the necessary thermoviscous attenuation that is required in order to keep the waveform from becoming too steep for the DRP algorithm. This is why in the following sections the frequency of the waves are as high as they are: a high frequency wave has a large value of \( \delta \) and \( \omega \), thus keeping \( \Gamma \) small.

In order to capture the time waveforms that are needed to compare the model’s results with the analytical solution, virtual microphones need to be inserted into the medium. A microphone is inserted into the domain simply by creating an additional array in the algorithm whose purpose is to save the values of pressure at each time at a particular point in space. The length of the array is usually equal to the total number of time steps that is calculated, and each sequential element of the array represents the value of some recorded parameter at sequential time samples. A sample of FORTRAN code is given below to demonstrate how a virtual microphone may be used to record pressure values at one point in space over the entire time of the computation:

\[
\begin{align*}
do & \text{time}=\text{timezero},\text{timemax} \\
& \text{mic}(\text{time})=\text{pressure(point-in-space)} \\
& \text{end do}
\end{align*}
\]

In the following calculations, fifty-three virtual microphones are placed in regular intervals along the propagation path, and the time waveforms of the propagating wave are recorded. The computational setup is given in Fig. 4.2.

### 4.2.2 Computed Results

The results in this section will be compared to the analytical solutions in two ways: first, the time waveforms that are recorded by the microphones will be compared to the time waveforms generated by the analytical solutions; and second, the normalized Fourier component amplitudes of the time waveforms will be
Figure 4.2. Graphical depiction of the insertion of virtual microphones into the computational domain. For this case the points are evenly distributed throughout the domain.

compared to those determined using analytical solutions. The results for both
the two-dimensional DRP and the two-dimensional WENO schemes will be given
sequentially. The test cases for these two numerical methods were run with essentially
the same computational parameters, although at times the grid sizes had to
be varied slightly due to the specific stability requirements of each scheme. At the
end of this section, the specific computational parameters used will be summarized
in tabular form along with the obtained results.

The computational parameters used for both the lossless and thermoviscous
calculations for the DRP include a computational domain of $20,001 \times 7$ points,
800 points per wavelength, a frequency of 1 MHz with a source amplitude of 157
dB re $20 \mu$Pa, a Gaussian half-width of 10% of a wavelength, and a CFL number
of 0.125. The parameters for the WENO scheme are exactly the same with the
exception of a domain size of $20,001 \times 3$. The time waveforms of the calculated
results for a lossless medium, showing only a few of the 800 calculated points
for clarity, and its comparison to the time waveforms given by Fubini’s solution,
Eq. 4.2, are shown in Fig. 4.3 and Fig. 4.4 for the DRP and WENO schemes,
respectively.

The solution appears to match up well with the computational solution in
the time domain, but in order to gain more insight into the results, the Fourier
component amplitudes from this computation were then compared to those given
by the Fubini solution. The results of this comparison for the DRP and the WENO
scheme are given in Fig. 4.5 and Fig. 4.6, respectively. The results match the
Figure 4.3. Graph showing a comparison between the time waveforms resulting from the analytical solution (solid line) and the computed solution using the DRP scheme (circles) at four different propagation distances in a lossless medium. The vertical axis is the acoustic density (kg/m$^3$) and the horizontal axis is the temporal grid point number. The number of points plotted in the computed time waveform has been decreased significantly for clarity.

Figure 4.4. Graph showing a comparison between the time waveforms resulting from the analytical solution (solid line) and the computed solution using the WENO scheme (circles) at four different propagation distances in a lossless medium. The vertical axis is the acoustic density (kg/m$^3$) and the horizontal axis is the temporal grid point number. The number of points in the computed time waveform has been decreased significantly for clarity.
Figure 4.5. Graph of Fourier component amplitudes as a function of shock formation distance showing the comparison between the Fubini solution (solid line) and the computed solution using the DRP scheme (circles) for the first three harmonic components of a 1 MHz plane wave traveling in a lossless medium. For clarity, every other data point for the computed solution has been omitted. The percent difference between the first harmonic amplitudes is at most 0.2 % throughout the entire calculation.

Figure 4.6. Graph of Fourier component amplitudes as a function of shock formation distance showing the comparison between the Fubini solution (solid line) and the computed solution using the WENO scheme (circles) for the first three harmonic components of a 1 MHz plane wave traveling in a lossless medium. For clarity, every other data point for the computed solution has been omitted. The percent difference between the first harmonic amplitudes is at most 0.07 % throughout the entire calculation.
analytical solution and to emphasize this, the percent difference between the two solutions for the first harmonic is calculated for the DRP and the WENO schemes and is shown in Fig. 4.7 and Fig. 4.8, respectively.

The test was then rerun for a medium that includes thermoviscous losses. The results for the comparison of the time waveforms calculated by the two schemes is given in Fig. 4.9 and Fig. 4.10. The normalized Fourier component amplitudes based on the time waveforms calculated by the two computational models are compared to the solution given in Eq. 4.4 and is shown in Fig. 4.11 and Fig. 4.12 for the DRP and the WENO schemes, respectively. Again, the percent difference between the computational and analytical solutions is small under these conditions. To illustrate this, the percent difference between the analytical and the DRP and WENO schemes is given in Fig. 4.13 and Fig. 4.14, respectively.

The results given in this section are summarized in Table 4.1 for clarity. These results confirm that both schemes can accurately simulate wave steepening, but they also demonstrate that the WENO scheme, which is 5th-order accurate in smooth regions, demonstrates higher accuracy than the 4th-order DRP scheme under similar computational conditions.

<table>
<thead>
<tr>
<th>Test Case</th>
<th>DRP Lossless</th>
<th>DRP Lossy</th>
<th>WENO Lossless</th>
<th>WENO Lossy</th>
</tr>
</thead>
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<tr>
<td>Frequency</td>
<td>1 MHz</td>
<td>1 MHz</td>
<td>1 MHz</td>
<td>1 MHz</td>
</tr>
<tr>
<td>Amplitude</td>
<td>157 dB</td>
<td>157 dB</td>
<td>157 dB</td>
<td>157 dB</td>
</tr>
<tr>
<td>Grid Size</td>
<td>20001×7</td>
<td>20001×7</td>
<td>20001×3</td>
<td>20001×3</td>
</tr>
<tr>
<td>Points/λ</td>
<td>800</td>
<td>800</td>
<td>800</td>
<td>800</td>
</tr>
<tr>
<td>CFL</td>
<td>0.125</td>
<td>0.125</td>
<td>0.125</td>
<td>0.125</td>
</tr>
<tr>
<td>Half-Width</td>
<td>λ/10</td>
<td>λ/10</td>
<td>λ/10</td>
<td>λ/10</td>
</tr>
<tr>
<td>% Error</td>
<td>&lt; 0.21 %</td>
<td>&lt; 0.068%</td>
<td>&lt; 0.18 %</td>
<td>&lt; 0.00013%</td>
</tr>
</tbody>
</table>

Table 4.1. Summary of DRP/WENO wave steepening results
Figure 4.7. Graph of the percent difference between the first harmonic amplitudes calculated using the Fubini solution and the DRP scheme as a function of propagation distance.

Figure 4.8. Graph of the percent difference between the first harmonic amplitudes calculated using the Fubini solution and the WENO scheme as a function of propagation distance.
Figure 4.9. Graph showing the comparison between the time waveforms resulting from Mendousse’s solution of the Burgers equation (solid line) and the computed solution (circles) using the DRP scheme at four different propagation distances at 1 MHz and in a medium including thermoviscous losses. The vertical axis is the acoustic density (kg/m$^3$) and the horizontal axis is the grid point number. The number of points in the computed time waveform has been decreased significantly for clarity.

Figure 4.10. Graph showing the comparison between the time waveforms resulting from Mendousse’s solution of the Burgers equation (solid line) and the computed solution (circles) using the WENO scheme at four different propagation distances at 1 MHz and in a medium including thermoviscous losses. The vertical axis is the acoustic density (kg/m$^3$) and the horizontal axis is the grid point number. The number of points in the computed time waveform has been decreased significantly for clarity.
Figure 4.11. Graph of Fourier component amplitudes as a function of shock formation distance showing the comparison between the Fubini solution (dashed line) for a lossless medium, the Burgers equation solution (solid line), and the DRP scheme’s solution (circles) for a medium including thermoviscous losses. For clarity, every other data point for the computed solution has been omitted. The percent difference between the first harmonic amplitudes is at most 0.16% throughout the entire calculation.

Figure 4.12. Graph of Fourier component amplitudes as a function of shock formation distance showing the comparison between the Fubini solution (dashed line) for a lossless medium, the Burgers equation solution (solid line), and the WENO scheme’s solution (circles) for a medium including thermoviscous losses. For clarity, every other data point for the computed solution has been omitted. The percent difference between the first harmonic amplitudes is at most $1.4 \times 10^{-3}$% throughout the entire calculation.
**Figure 4.13.** Graph of the percent difference between the first harmonic amplitudes calculated using Mendousse’s solution of the Burgers equation and the DRP scheme’s calculated amplitudes as a function of propagation distance.

**Figure 4.14.** Graph of the percent difference between the first harmonic amplitudes calculated using Mendousse’s solution of the Burgers equation and the WENO scheme’s calculated amplitudes as a function of propagation distance.
4.3 Nonlinear Amplification Factors

4.3.1 Introduction

This test demonstrates the ability of the code to predict nonlinear amplification factors that occur when a suitably high amplitude wave encounters a perfectly rigid boundary. In the linear acoustic regime when the incident pressure amplitude is low, simple pressure doubling will occur. In the nonlinear regime, however, the amplification factor is greater than simple doubling. In the case of a normally incident plane wave that is still completely continuous, the amplification factor at the wall may be determined using the Pfriem solution[72]:

\[
a = \frac{[2(p_{inc}/p_0)^{1/x} - 1]^x - 1}{p_{inc}/p_0 - 1},
\]

where

\[
\chi = \frac{2\gamma}{\gamma - 1}.
\]

In the above equations, \(p_{inc}\) is the total incident pressure, \(\gamma\) is the ratio of specific heats for a perfect gas, and \(a\) is the amplification factor which again is equal to 2 in the linear acoustic regime. This relationship was also independently derived by Blackstock using an analysis of lossless plane waves in a tube reflecting off of a rigid boundary[73].

To test for this phenomenon in the computational model, Gaussian-distributed plane wave pulses of varying amplitude was propagated through a lossless domain and onto a rigid boundary. The rigid boundary was created by imposing that on a \(N \times M\) sized grid that

\[
\rho(N,j) = \rho(N-1,j),
\]

\[
u(N,j) = v(N-1,j),
\]

\[
\chi = \frac{2\gamma}{\gamma - 1}.
\]
\[ s_{fr}(N,j) = s_{fr}(N - 1, j), \]  
\[ T_{N_2}(N,j) = T_{N_2}(N - 1, j), \]  
\[ T_{O_2}(N,j) = T_{O_2}(N - 1, j). \]

In each of the tests, a virtual microphone was placed on the boundary and a time history of the pressure at the rigid boundary was recorded as the plane wave pulse approached and reflected off of the rigid boundary. The peak pressure of this recorded waveform was then taken and used in the calculation of the model’s amplification factor, given as

\[ a = \frac{p_{\text{max}}}{p_{\text{inc}}}. \]

Using the above parameters, the ability of the two schemes to reproduce nonlinear amplification factors was investigated.

### 4.3.2 Computed Results

In both tests, a pulse with a Gaussian half-width of \(2.2878 \times 10^{-5}\) m were propagated in a lossless domain of 4,001 points in the direction of propagation and using a value of \(4.29 \times 10^{-7}\) m for \(\Delta x\) and \(1.56 \times 10^{-10}\) s for \(\Delta t\). The comparison between the amplification factor predicted by the Pfriem solution and the model’s results are shown in Fig. 4.15 and Fig. 4.17 for the DRP and WENO schemes, respectively. The computational results match the analytical results very closely. The percent difference between the two results over the entire range of incident pressure amplitudes is within 0.015 % for the DRP scheme and 0.011 % for the WENO scheme. The percent difference between the DRP results and the Pfriem solution is given in Fig. 4.16, the percent difference between the WENO scheme results and the Pfriem results is shown in Fig. 4.18.
Figure 4.15. Graph of the amplification factor as a function of incident pressure comparing the Pfriem solution (solid line) and the computed solution using the DRP scheme (circles) for a plane-wave pulse incident upon a perfectly rigid boundary in a lossless medium. The percent error between the two solutions is at most 0.015%.

Figure 4.16. Graph of the percent difference between the amplitude factors calculated using the Pfriem solution and the DRP scheme’s calculated amplification factors as a function of incident amplitude.
Figure 4.17. Graph of amplification factor as a function of incident pressure comparing the Pfriem solution (solid line) and the computed solution using the WENO scheme (circles) for a plane-wave pulse incident upon a perfectly rigid boundary in a lossless medium. The percent error between the two solutions is at most 0.011 %.

Figure 4.18. Graph of the percent difference between the amplitude factors calculated using the Pfriem solution and the WENO scheme’s calculated amplification factors as a function of incident amplitude.
4.4 Calculated Absorption

4.4.1 Introduction

This section tests the model’s ability to simulate various acoustical absorption mechanisms, which will be accomplished by propagating sinusoidal plane waves of various frequencies through a medium that includes one or all of the absorption mechanisms. Line sources are used so that geometrical spreading is avoided and the only non-numerical loss observed is a result of the particular mechanism that was turned on.

In the following investigations, classical absorption is verified, followed by nitrogen and then oxygen relaxation absorption, and finally all three absorption mechanisms will be included. For all of the tests, the decay envelope of the propagated wave was used to determine the absorption values of the model.

4.4.2 Classical Absorption

Classical absorption includes losses that are attributable to two processes: the diffusion of momentum through molecular collisions between fluid regions possessing different velocities, known as shear viscous losses; and the conduction of heat between fluid regions of differing temperatures, known as thermal conduction losses[12]. Modified classical absorption contains losses caused by another loss mechanism: bulk viscosity, also known as second viscosity. Bulk viscosity accounts
for the loss due to the transfer of momentum into translational and rotational degrees of freedom of molecules in the air and these losses can be significant so they are included in the model. Modified classical absorption is especially important in the high frequency range since the modified classical absorption coefficient scales as frequency squared.

To test the model’s ability to simulate modified classical absorption, Gaussian-distributed line sources were propagated over a frequency range of 10 Hz to 1 MHz with only the modified classical absorption mechanisms (shear viscosity, bulk viscosity and thermal conductivity) included in the calculation. The analytical, modified classical absorption equation is given by [13]:

\[
\alpha'_{cl} = \frac{\omega^2 \mu}{2 \rho_0 c^3} \left[ \frac{4}{3} + \frac{\mu_B}{\mu} + \frac{(\gamma - 1)\kappa}{c_p\mu} \right].
\]

(4.16)

For these results, the computational parameters used included a grid size of 10,001 points in the direction of propagation with 200 points per wavelength and a CFL number of 0.99 for the DRP scheme and a CFL number of 0.343 for the WENO scheme. The results from the DRP scheme are given in Fig. 4.19 and the results from the WENO scheme are given in Fig. 4.20. The results for both schemes are quite close to the analytical results, usually differing by less than 1 %, although at low frequencies it becomes quite difficult to capture the absorption. This is because, as shown in Eq. 4.16, the absorption increases with the square of frequency, so at low frequencies the absorption per wavelength is very small. This small absorption requires large propagation distances and/or very fine waveform resolution—otherwise, the absorption is not captured. For this reason, the percent difference between the computational results and the analytical results is greatest at low frequencies.
4.4.3 Absorption Due to Nitrogen and Oxygen Relaxation

Absorption tests, also using a Gaussian-distributed line source, were run with only nitrogen relaxation and then only oxygen relaxation turned on. The analytical relaxation absorption equation is given by\cite{13}:

\[ \alpha_\nu = \frac{1}{\lambda} (\alpha_\nu \lambda)_m \frac{2\omega \tau_\nu}{1 + (\omega \tau_\nu)^2}, \]  

(4.17)

where

\[ (\alpha_\nu \lambda)_m = \frac{\pi (\gamma - 1)c_{\nu\nu}}{2c_p}. \]  

(4.18)

Here $\nu$ denotes either nitrogen or oxygen, $\alpha$ is the absorption, $\tau$ is the relaxation time, $\omega$ is the angular frequency, $\lambda$ is the wavelength of the propagated wave, $c_{\nu\nu}$ is the specific heat at constant volume, and $c_p$ is the specific heat at constant pressure.

The absorption tests were run in the same way as in the modified classical absorption tests. Sinusoidal plane waves are propagated in a domain that includes

Figure 4.19. Comparison of calculated classical absorption using the analytical solution (solid line) and the DRP scheme (circles).
only nitrogen relaxation effects, and the decay envelope was used to determine the absorption values. The results of the nitrogen relaxation absorption tests are given in Fig. 4.21 and Fig 4.22. The tests were rerun with only oxygen relaxation effects and the computed results compared to the analytical results are given in Fig. 4.23 and Fig 4.24.

The percent difference between the computed and analytical solutions for these tests is typically less than 1 %. As in the classical absorption case, since the absorption at frequencies suitably below the relaxation frequency scale as frequency squared, the absorption becomes increasingly difficult to capture. Another difficulty in capturing the absorption caused by molecular relaxation of nitrogen or oxygen is the fact that the absorption at high frequencies is a constant. The impact of this is that although the absorption is frequency independent as the frequencies increase, the absorption per wavelength decreases dramatically. This means that at high frequencies the wave must be propagated many more wavelengths in order to observe the same absorption that is present at lower frequencies. It is for this reason that the results lying at the extremities of the frequency spectrum have
the greatest percent difference. It is believed that given a very large domain and fine spatial grids, the error for these tests would also approach zero. However, the results already obtained are sufficient to show that the models can accurately recreate atmospheric absorption, and any further tests are left for future work.
Figure 4.21. Comparison of calculated nitrogen relaxation absorption using the analytical solution (solid line) and the DRP scheme (circles).

Figure 4.22. Comparison of calculated nitrogen relaxation absorption using the analytical solution (solid line) and the WENO scheme (circles).
Figure 4.23. Comparison of calculated oxygen relaxation absorption using the analytical solution (solid line) and the DRP scheme (circles).

Figure 4.24. Comparison of calculated oxygen relaxation absorption using the analytical solution (solid line) and the WENO scheme (circles).
4.4.4 Total Calculated Absorption

The analytical absorption curve for the combination of classical absorption and nitrogen and oxygen relaxation has been shown analytically[74] and experimentally [75] to be equal to a sum of the three coefficients for frequencies below 10 MHz; thus, the following analytical form is used in the comparison:

$$\alpha = \alpha_{cl}^\prime + \alpha_{N_2} + \alpha_{O_2}. \quad (4.19)$$

This test had the same computational parameters as in the classical and molecular relaxation absorption tests. The results are given in Fig. 4.25 and Fig. 4.26 for the DRP and WENO schemes, respectively. The percent error is typically within 1 % for all of the tests. To summarize the results from this section, the results for all four conditions are plotted together and given in Fig. 4.27 and Fig. 4.28 and all of the computational parameters used for both the DRP and the WENO schemes are given in Table 4.3 and Table 4.4.
Figure 4.26. Comparison of total calculated absorption using the analytical solution (solid line) and the WENO scheme (circles).

Figure 4.27. Summary of the absorption calculated using the DRP scheme for four different conditions: analytical modified classical absorption (dotted line) and calculated modified classical absorption (open circles); analytical nitrogen relaxation absorption (dashed line) and calculated nitrogen relaxation absorption (squares); analytical oxygen relaxation absorption (dash-dot line) and calculated oxygen relaxation absorption (diamonds); and analytical total atmospheric absorption (solid line) and calculated total atmospheric absorption (filled circles).
Figure 4.28. Summary of the absorption calculated using the WENO scheme for four different conditions: analytical modified classical absorption (dotted line) and calculated modified classical absorption (open circles); analytical nitrogen relaxation absorption (dashed line) and calculated nitrogen relaxation absorption (squares); analytical oxygen relaxation absorption (dash-dot line) and calculated oxygen relaxation absorption (diamonds); and analytical total atmospheric absorption (solid line) and calculated total atmospheric absorption (filled circles).

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<th>O(_2)</th>
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<td>0.99</td>
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<td>(\lambda/10)</td>
<td>(\lambda/10)</td>
</tr>
</tbody>
</table>

Table 4.3. Summary of absorption calculations using the DRP scheme

<table>
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<th>Absorption Test</th>
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<th>N(_2)</th>
<th>O(_2)</th>
<th>Total</th>
</tr>
</thead>
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<td>1×10(^{-4})</td>
<td>1×10(^{-4})</td>
<td>1×10(^{-4})</td>
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<td>10001×3</td>
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<td>(\lambda/10)</td>
<td>(\lambda/10)</td>
</tr>
</tbody>
</table>

Table 4.4. Summary of absorption calculations using the WENO scheme
4.5 Dispersion Due to Molecular Relaxation

4.5.1 Introduction

The next results show the effects of molecular relaxation upon the propagation speed of waves of different frequencies. Again, using a Gaussian-distributed line source, the speed of propagation of monofrequency waves was determined over a variety of frequencies and including nitrogen and/or oxygen relaxation in the calculations.

The analytical result for the variation of sound speed as a function of frequency is given as [13]:

\[ v_{ph} = c - \frac{c}{\pi} \sum_{\nu} \frac{(\alpha_{\nu}\lambda)_m}{1 + \omega^2 \tau^2_{\nu}}, \]  \hspace{1cm} (4.20)

where \( v_{ph} \) is the phase velocity. The difference between the equilibrium (low-frequency limit) sound speed and the frozen (high-frequency limit) sound speed [76], another important parameter in molecular dispersion, is determined. The analytical result for this is given by [77]:

\[ \Delta c_{\nu} = \frac{c}{\pi} (\alpha_{\nu}\lambda)_m, \]  \hspace{1cm} (4.21)

and this parameter is used to determine the accuracy of the model. The analytical values determined using Eq. 4.18 and Eq. 2.8 lead to a value of \( \Delta c_{N_2} = 0.0210 \text{ m/s} \) and \( \Delta c_{O_2} = 0.114 \text{ m/s} \).

In these tests, waves are propagated with a low CFL number, meaning that the time steps are very small. The result of this is that the virtual microphone output will have a greater number of points, allowing easier determination of the phase speed. It should be noted that the differences expected in the wave speed are very small, changing only by about 0.006 % for nitrogen and 0.03 % for oxygen, so the measurement of such small variations can be difficult. That is not to say that such small variations in sound speed are unimportant, because after the propagation of
many wavelengths this dispersion becomes significant. For accurate calculation of the dispersion of the medium, the microphone output is interpolated. Using this interpolated waveform, the propagation speed of the zero-crossings is measured using a simple formula for determining velocity:

\[ v_{ph} = \frac{\delta x}{\delta t}, \]  

(4.22)

where \( \delta x \) is the distance propagated and \( \delta t \) is the propagation time.

### 4.5.2 Computed Nitrogen Dispersion

The model used 200 points per wavelength, 7,501 points in the direction of propagation, and a Gaussian half-width equal to 10% of a wavelength. Using only nitrogen relaxation, the wave speed of each computational model was determined and is compared to the analytical result in Fig. 4.29 and Fig. 4.30.

The difference between the equilibrium sound speed and the frozen sound speed using Eq. 4.21 was determined to be approximately 0.02107 m/s for the frequency range given in these calculations. For the DRP scheme, the difference between the analytical solution for the sound speed given in Eq. 4.20 and computational results for the sound speed is at most \( 5.539 \times 10^{-5} \)%, and the dispersion was calculated to be 0.02091 m/s, a difference of 0.7257%. For the WENO scheme the difference between the analytical and computational solutions for the sound speed is at most \( 5.509 \times 10^{-5} \)%, and the dispersion for the computational model was calculated to be 0.02118 m/s, a difference of 0.5105%.

### 4.5.3 Computed Oxygen Dispersion

The computational parameters used in these calculations include 7501 points in the direction of propagation, 200 points per wavelength, and a Gaussian half-width equal to 10% of a wavelength. Using only oxygen relaxation, the wave speed of
the computational model was determined and is compared to the analytical result in Fig. 4.31 and Fig. 4.32 for the DRP and WENO schemes, respectively.

The difference between the equilibrium sound speed and the frozen sound speed using Eq. 4.21 was determined to be approximately 0.1114 m/s for the frequency range given in these calculations. For the DRP scheme, the difference between the analytical solution for the sound speed given in Eq. 4.20 and computational results for the sound speed is at most $2.811 \times 10^{-4}$ %, and the dispersion was calculated to be 0.1120 m/s, a difference of 0.5026 %. For the WENO scheme the difference between the analytical and computational solutions for the sound speed is at most $2.708 \times 10^{-4}$ %, and the dispersion for the computational model was calculated to be 0.1123 m/s, a difference of 0.8243 % from the analytical solution.

### 4.5.4 Total Calculated Dispersion

In the final dispersion tests, both nitrogen and oxygen relaxation were included in the calculation of the wave speed. The tests used 7501 points in the direction of

<table>
<thead>
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<th>Dispersion Case</th>
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<th>O\textsubscript{2}</th>
<th>N\textsubscript{2} &amp; O\textsubscript{2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude (kg/m\textsuperscript{3})</td>
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<td>7501×3</td>
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</tr>
<tr>
<td>Half-Width</td>
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<tr>
<td>Dispersion % Error</td>
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<td>0.5026 %</td>
<td>0.2797 %</td>
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</table>

**Table 4.5.** Summary of calculated dispersion results using the DRP scheme

<table>
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<tr>
<th>Dispersion Case</th>
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<th>O\textsubscript{2}</th>
<th>N\textsubscript{2} &amp; O\textsubscript{2}</th>
</tr>
</thead>
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<tr>
<td>Amplitude (kg/m\textsuperscript{3})</td>
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<td>$1 \times 10^{-4}$</td>
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<tr>
<td>Points/λ</td>
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<tr>
<td>CFL</td>
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<tr>
<td>Half-Width</td>
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<td>$\lambda/10$</td>
<td>$\lambda/10$</td>
</tr>
<tr>
<td>Dispersion % Error</td>
<td>0.5105 %</td>
<td>0.8243 %</td>
<td>0.4732 %</td>
</tr>
</tbody>
</table>

**Table 4.6.** Summary of calculated dispersion results using the WENO scheme
propagation, 200 points per wavelength, and a Gaussian half-width equal to 10% of a wavelength. Again, the waveforms recorded were interpolated and the wave speed was determined using $\delta x/\delta t$, and the results are given in Fig. 4.33 and Fig. 4.34. The difference between the equilibrium sound speed and the frozen sound speed using Eq. 4.21 was determined to be approximately 0.1338 m/s for the frequency range given in these calculations. For the DRP scheme, the difference between the analytical solution for the sound speed given in Eq. 4.20 and computational results for the sound speed is at most $2.961 \times 10^{-4}$ %, and the dispersion was calculated to be 0.1342 m/s, a difference of 0.2797 %. For the WENO scheme the difference between the analytical and computational solutions for the sound speed is at most $2.058 \times 10^{-4}$ %, and the dispersion for the computational model was calculated to be 0.1345 m/s, a difference of 0.4732 %.

Finally, the dispersion results are summarized and given in Table 4.5 and Table 4.6 for the DRP and WENO schemes, respectively.
Figure 4.29. Graph of the calculated sound speed minus the equilibrium sound speed as a function of frequency normalized by the relaxation frequency showing the comparison between the analytical dispersion curve (solid line) and the computed solution using the DRP scheme (circles) for a medium including nitrogen relaxation only.

Figure 4.30. Graph of the calculated sound speed minus the equilibrium sound speed as a function of frequency normalized by the relaxation frequency showing the comparison between the analytical dispersion curve (solid line) and the computed solution using the WENO scheme (circles) for a medium including nitrogen relaxation only.
Figure 4.31. Graph of the calculated sound speed minus the equilibrium sound speed as a function of frequency normalized by the relaxation frequency showing the comparison between the analytical dispersion curve (solid line) and the computed solution using the DRP scheme (circles) for a medium including oxygen relaxation only.

Figure 4.32. Graph of the calculated sound speed minus the equilibrium sound speed as a function of frequency normalized by the relaxation frequency showing the comparison between the analytical dispersion curve (solid line) and the computed solution using the WENO scheme (circles) for a medium including oxygen relaxation only.
Figure 4.33. Graph of the calculated sound speed minus the equilibrium sound speed as a function of frequency showing the comparison between the analytical dispersion curve (solid line) and the computed solution using the DRP scheme (circles) for a medium including nitrogen and oxygen relaxation.

Figure 4.34. Graph of the calculated sound speed minus the equilibrium sound speed as a function of frequency showing the comparison between the analytical dispersion curve (solid line) and the computed solution using the WENO scheme (circles) for a medium including nitrogen and oxygen relaxation.
4.6 Boundary Condition Verifications

4.6.1 Periodic Boundary Condition

This section shows the results of implementing a periodic boundary condition in the model as described in Section 3.6.2. In Fig. 4.35, the state of the system is given for three points in time. The original source pulse was centered at grid point 250 and at time $t = 290$ the two resulting waves have traveled to their current positions. At time $t = 390$ the left-going wave has approached the left edge of the boundary and it can be seen entering into the domain again on the right edge of the boundary. Finally, at time $t = 490$ the left-going wave has completely exited the left side of the boundary and is now completely on the right side of the boundary. This periodic boundary condition allows the waves to wrap around the boundary as many times as is desired in order to propagate the wave as far as is needed.

Figure 4.35. Demonstration of the periodic boundary condition. The waveform resulting from a pulse originally located at point 250 is given for three propagation distances. At time $t = 290$, the left going wave is about to impact the boundary. At time $t = 390$, the left going wave is in the process of being wrapped from the left-hand-side of the boundary to the right. At time $t = 490$, the left going wave has completely wrapped back into the domain.
4.6.2 Reflecting Boundary Condition

The result of implementing the reflecting boundary condition, described in Section 3.6.3, is shown in Fig. 4.36 for three points in time. The incident wave is depicted at time $t = 400$, whose amplitude is approximately 5.9 Pa. The pressure on the boundary has doubled at time $t = 503$, whose amplitude is approximately 11.8 Pa. Finally, at time $t = 700$ the wave has completely reflected, is virtually identical to the incident, and traveling away from the boundary. The percent difference of the amplitude between the incident and reflected waveforms is calculated to be 0.2125 %, although discretization effects are likely also contributing to this value.

4.6.3 Absorbing Boundary Conditions

This section demonstrates the results of implementing the absorbing boundary condition, as described in Section 3.6.4. This absorbing boundary condition has proven to be quite effective, although again this method is certainly not optimally

![Diagram](image)

**Figure 4.36.** Demonstration of the reflecting boundary condition. At time $t = 400$, the wave has yet to impact the boundary. At time $t = 503$, the wave has fully impacted the boundary and pressure doubling has occurred. At time $t = 700$, the reflected waveform is shown, virtually identical to the incident waveform but traveling in the opposite direction.
efficient. A demonstration of the efficacy of the Gaussian method is given in Fig. 4.37 for the one-dimensional case. Here, the center of the Gaussian is placed 200 grid points away from the boundary and the half-width was equal to 600 points. The incident waves, given by the dashed line, have an initial amplitude of approximately 6 Pa. The waveform resulting from the incident pulse impacting the absorbing boundaries is given as the solid line. On this scale, there does not appear to be any reflection, although in reality there is a small reflection whose amplitude is $3 \times 10^{-6}$ Pa, equal to $5 \times 10^{-5}$ % of the original amplitude.

The two-dimensional case is very similar. In this case, the Gaussian decay is imposed in not only $x$ but also $y$. As before, a Gaussian envelope is centered 200 grid points away from the boundary with a half-width of 600 points. Figure 4.38 shows the initial cylindrically spreading incident wave as the solid line and the reflected wave as the dashed line. The upper plot shows the propagation in the $x$-direction while the lower plot shows the propagation in the $y$-direction.
Figure 4.38. Demonstration of absorbing boundary condition for the two-dimensional case. The top plot shows the propagation in the $x$-direction and the bottom plot shows the propagation in the $y$-direction. In this case, the cylindrically spreading pulse of 1881 Pa initial amplitude (dashed line) reflects from the boundary with an amplitude of 0.35 Pa (solid line).

The peak amplitude of the initial pulse was equal to 1881 Pa, and the reflected wave resulting from this calculation had a peak pressure of 0.35 Pa, 0.019% of the initial amplitude.

4.7 Limitations of the WENO Scheme

Although the WENO scheme is able to propagate shocks stably, it is still not able to propagate waves of infinite slope and because of this an investigation of the limitations of the WENO scheme as implemented will be performed.

Ideally, the numerics of the model should not appreciably affect the results. The problem is that if there is no analytical solution for comparison, it is quite difficult to tell if the results are almost purely the result of the physics of the system or if the numerics have significantly affected the results. A simple method of determining the maximum slope allowed by the model will be performed by
Figure 4.39. Comparison of the calculated pressure waveforms resulting from an initially sinusoidal waveform propagating in a lossless medium to a distance of five shock formation distances using 200 (dashed line) and 1000 (solid line) points per wavelength as a function of time. The shape of the 1000 points per wavelength waveform is much closer to a sawtooth shape.

propagating high amplitude plane waves in a lossless medium to the point where the wave should be a perfect sawtooth, which occurs after propagating more than three shock formation distances. For the analysis below, the waves were propagated to at least six shock formation distances in a lossless medium because at this point it is assured that the waveform has become a sawtooth, and it may be safely assumed that any deviation from this sawtooth shape is a result of the numerical method rather than the physics.

In Fig. 4.39, the waveforms resulting from plane wave propagation to five shock formation distances in a lossless medium is compared for calculations using 200 versus 1000 points per wavelength. The 1000 points per wavelength case has a much shorter shock rise time in this comparison.

This is not surprising since finer spatial grids should produce higher resolution results in the time and frequency domain. In Fig. 4.40, a comparison of propagating an initially sinusoidal waveforms to six shock formation distances in a lossless
Figure 4.40. Comparison of the calculated waveforms resulting from an initially sinusoidal waveform propagating in a lossless medium five shock formation distances using 100, 200, 300, 400, 500, 600, 700, 800, 900, and 1000 points per wavelength plotted using equal time grid spacing. The existence of a maximum pressure increase per time grid point can be seen at the steepest portions of the ten waveforms.

medium using 100–1000 points per wavelength is shown not as a function of time as before, but as a function of the temporal grid point. Plotting the waves as a function of the temporal grid point makes the 1000 points per wavelength case ten times longer than the 100 points per wavelength case on this scale because the 1000 points per wavelength case has ten times the number of samples per time. Focusing on the steepest portion of the ten waveforms, it appears that the maximum $\Delta p$ over two grid points approaches a constant value regardless of the points per wavelength. The maximum rise per grid point was then calculated and is shown in Fig. 4.41.

The rise time of shocks are of particular interest to those studying sonic booms, for example, since the specific qualities of the rise time can affect the perceived annoyance of the waveform\cite{78}, and although the results obtained from the model after the grid limitation has been reached may be inaccurate for the determination of shock rise times, the more basic qualities of the waveform are conserved. In
Figure 4.41. Calculated maximum pressure increase per time grid point using 100, 200, 300, 400, 500, 600, 700, 800, 900, and 1000 points per wavelength. This value does vary greatly as a function of points per wavelength or after propagating four shock formation distances.

Fig. 4.42, an initially sinusoidal waveform has propagated to approximately ten shock formation distances using 200 points per wavelength, and it is clearly not a perfect sawtooth. Referring to Fig. 4.41, it can be determined that the 200 points per wavelength case has reached its slope limit after only six shock formation distances, so by the time the waveform has propagated an additional four shock formation distances the model has had ample time to affect the accuracy of the results.

To further investigate the effects of this slope limitation, the resulting Fourier component amplitudes are compared to those given by analytical solutions. For propagation of less than one shock formation distance, the Fubini solution, given in Eq. 4.2, is used. For propagation distances greater than three shock formation distances, the Fay solution is used[3]:

\[ p = \frac{2p_0}{1 + \sigma} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\omega \tau, \] (4.23)
Figure 4.42. The calculated waveform resulting from a 200 points per wavelength sinusoid propagating to a distance of ten shock formation distances. The shape is clearly not a perfect sawtooth after propagating this distance.

where $p_0$ is the acoustic pressure amplitude, $\sigma$ the propagation distance normalized to the shock formation distance, $n$ the harmonic number, $\omega$ the angular frequency, and $\tau$ the retarded time, determined using:

$$\tau = t - x/c.$$  \hspace{1cm} (4.24)

Finally, the Blackstock bridging function is used for propagation distances between one and three $\bar{x}$, which is given below[79]:

$$B_n = \frac{2}{n\pi}P_{sh} + \frac{2}{n\pi\sigma} \int_{\Phi}^{\pi} \cos n(\Phi - \sigma \sin \Phi) d\Phi,$$  \hspace{1cm} (4.25)

where $P_{sh}$ is the amplitude of the shock pressure, $\Phi_{sh} = \omega \phi_e$, and $\phi_e$ is the Earnshaw phase variable, given as:

$$\phi_e = \tau + \frac{\beta x p}{\rho_0 c^3}.$$  \hspace{1cm} (4.26)
The results of the comparison of the combination of the three analytical solutions to the calculated solution are given in Fig. 4.43, which show that the results obtained for the first four harmonics are quite accurate; the percent difference between the first harmonic amplitude calculated using the analytical solutions and the WENO solution is at most 1%.

Finite-difference schemes require a certain number of points per wavelength in order to properly resolve a wave. To determine whether this difficulty in resolving the rise time is due to the wave number resolution alone, the harmonic content of the waves propagated to 5 shock formation distances is investigated. In Fig. 4.44, the percent error of the magnitude of each harmonic is given for calculations ranging from 100–1000 points per wavelength. It is expected that given more spatial resolution, the percent error should decrease for a given harmonic and this general trend can be seen. Figure 4.44 also has a point included on each curve which serves to indicate the harmonic at which ten points per wavelength
Figure 4.44. Comparison of the percent difference of the calculated harmonic content versus that expected from the Fay solution as a function of the harmonic number for a propagation distance of 5 shock formation distances. The curves represent 100–1000 points per wavelength relative to the fundamental and the circles on each curve represents the harmonic for which there is ten points per wavelength spatial resolution.

is achieved. Harmonics that are above this threshold are resolved with fewer than ten points per wavelength. This threshold is given because if the error in the numerical method is purely a function of the wave number resolution, it is expected that the error at ten points per wavelength should be the same for each curve. This investigation suggests that since the error for harmonics at ten points per wavelength actually increases with decreasing resolution of the fundamental, there is some shock smearing that is occurring as a result of the WENO scheme which has nothing to do with simple wave resolution issues.

The conclusion that has been made based upon this analysis is that the user of the model simply must be aware of these limitations in analyzing the results obtained from the model. An investigation specific to the numerical conditions is suggested in preparation of an analysis of results.
4.8 Conclusions

The two numerical methods, DRP and WENO, have been shown to be able to closely approximate the major characteristics of nonlinear acoustic propagation, i.e., wave steepening in a lossless medium and in a medium including thermoviscous losses, nonlinear amplification factors, and absorption and dispersion in a medium including thermoviscous losses and/or the effects of relaxation. These are the fundamental properties that need to be accurate in the model in order to trust the results in the following chapter for which there are no analytical solutions. Since the major focus of the research in the following sections will be in long range high amplitude propagation, the WENO scheme will be used.
Chapter 5

Results

5.1 Introduction

This chapter will discuss the following results: two-dimensional simulation of Mach stem formation created by a high intensity acoustic pulse incident upon a rigid boundary, comparison of a Burgers equation-based prediction model to the WENO scheme for the propagation of full scale jet noise, and comparison of propagated scale model jet noise using the WENO scheme to a Burgers equation prediction and experimental data. The simulations presented in this chapter differ from those in the previous sections in that they do not have exact analytical solutions against which to compare. This chapter will also help to reveal the various capabilities of the model, such as the ability to propagate broadband noise waveforms, demonstrate nonlinear phenomena, and stably propagate complex multidimensional shocks, all in a medium that includes the effects of modified classical absorption and molecular relaxation.
5.2 Mach Stem Formation

In this section the formation of a Mach stem resulting from a high amplitude pulse that is incident upon a rigid boundary will be demonstrated in a medium that includes modified classical absorption losses and the effects of molecular relaxation. It is expected that the incident pulse will initially reflect off the boundary with normal incidence and, as has been shown in the previous chapter, will exhibit a nonlinear amplification factor consistent with that given by the Pfriem solution. After the initial reflection of the normally incident wave, the cylindrical wavefront will reflect off the boundary at various incident angles. As the wave continues to propagate along the boundary it will steepen and form a shock. When a shock has formed along the boundary, the behavior of the wave changes. This is because at sufficiently high incident angles shocks reflect at angles that are slightly more normal to the boundary. The cumulative effect of this differing angle of reflection is the fusion of the incident and reflected waves[81], which is called a Mach stem. Figure 5.1 depicts the formation of a Mach stem. To study Mach stem formation, the amplification factor at the boundary is often investigated, which is calculated by determining the ratio of the acoustic pressure at the boundary to the acoustic pressure of the incident wave. The creation of a Mach stem causes the amplification factor to increase gradually as the wave progresses along the boundary. When the angle of incidence approaches a grazing angle, however, the amplification factor at the boundary with relation to the cylindrically spreading incident wavefront begins to drop off to values below two, which is expected considering that for grazing incidence the reflection coefficient should equal one.

The case of Mach stem formation is particularly illuminating since it demonstrates a number of unique features of the WENO scheme that is used in the model. This simulation demonstrates the WENO scheme’s ability to simulate nonlinear phenomena such as wave steepening, shock formation and nonlinear amplification factors. This test also demonstrates the ability of the model to stably and ac-
Figure 5.1. Depiction of the formation of a Mach stem. A high amplitude acoustic source is located above a rigid boundary, and as the shock from the resulting wave impacts the boundary at sufficiently large incident angles, a Mach stem will form due to the interaction of the incident and reflected waves.

Accurately propagate multi-dimensional shocks of complicated geometry and output multiple state variables throughout the calculation.

In this simulation, an initial condition was set up in which a high amplitude pulse was placed near a perfectly rigid boundary. The boundaries were forced to be rigid and thermally non-conducting (assuming a $N \times M$ grid) using the following:

\begin{align*}
\rho(1,j) &= \rho(2,j), \quad (5.1) \\
\mathbf{u}(1,j) &= -\mathbf{u}(2,j), \quad (5.2) \\
v(1,j) &= v(2,j), \quad (5.3) \\
\mathbf{s}_{fr}(1,j) &= \mathbf{s}_{fr}(2,j), \quad (5.4) \\
T_{N_2}(1,j) &= T_{N_2}(2,j), \quad (5.5)
\end{align*}
\[ T_{O_2}(1, j) = T_{O_2}(2, j), \]

where \( j \) is an index spanning all of the \( y \)-dimension. The computational parameters used in this calculation included a spatial discretization of \( \Delta x = \Delta y = 0.686 \) mm, \( \Delta t = 0.686 \) \( \mu s \), Gaussian half-width of 13.7 mm, and initial pressure amplitude of 61 kPa (187 dB). The grid consisted of 1501\( \times \)751 points, corresponding to a domain of 1.03 m in \( x \) and 0.515 m in \( y \). The initial condition of the model is shown in Fig. 5.2.

The model was then allowed to propagate the initial pulse, and as it was doing so the system’s pressure, temperature, and density were recorded every ten time steps. For clarity, the propagated acoustic pressure is given in Fig. 5.3, which shows the Mach stem and the original position of the pulse after 1.6 ms, and Fig. 5.4, which shows a closer view of the waveform after 1.9 ms computational iterations. Slices of the pressure, temperature, and density along the rigid boundary are given in Fig. 5.5 for a propagation time of 1.98 ms. The relatively low resolution of the waveform is obviously somewhat of a problem, and considering the low number of points in the shock, it is certain that the shock smearing phenomenon discussed in section 4.7 has occurred. This limitation should not be a problem for the purposes of this investigation, however, since the peak pressure and not the specific rise time of the waveform is of interest.

The amplification factor, given as the ratio of the peak pressure at the boundary to the cylindrically spreading incident wave as a function of propagation distance along the boundary, is calculated and given in Fig. 5.6. For normal incidence, where the distance that the wave has traveled along the boundary is zero in Fig. 5.6, the amplification factor was calculated to be 2.055, corresponding to a percent difference of 0.026 % from the Pfriem solution. For oblique incidence, the Pfriem solution is not applicable, but as stated before it is known that at larger angles of incidence a Mach stem should eventually form. Figure 5.7 again shows the ampli-
Figure 5.2. Initial condition for the Mach stem calculation. A Gaussian-distributed pulse of initial amplitude 61 kPa is inserted into the domain above a perfectly rigid, thermally non-conducting boundary in a medium including molecular relaxation effects and modified classical absorption.

Figure 5.3. Overhead view of the calculated Mach stem pressure. A coalescence of the incident and reflected wavefronts can be seen on the bottom right portion of the graph and the position of the original source is shown on the left.
Figure 5.4. Close-up view of the calculated Mach stem. The incident and reflected waveforms have combined into the extended Mach stem, visible at the bottom right of the figure.

Figure 5.5. Boundary slice of the recorded acoustic pressure, temperature, and density. This model allows various state variables to be recorded at any point at any time.
Figure 5.6. Graph of the calculated amplification factor as a function of distance propagated along the boundary.

Figure 5.7. Graph of the calculated amplification factor as a function of incident angle. The amplification factor becomes greater than that for normal incidence at an angle of approximately 52° and continues to grow until it begins to approach grazing incidence, whereupon the amplification factor is expected to decrease to a value of one.
fication factor, which is calculated by the same method as before, but now given as a function of incident angle. At an angle of approximately 52° the amplification factor becomes greater than that calculated using the Pfriem solution for normal incidence, which is to be expected based upon the previous discussion. This amplification factor continues to grow until the angle of incidence begins to approach a grazing angle, upon which the amplification factor begins to decrease and results in a reflection coefficient of one at 90°. The amplification factor curve calculated here is similar qualitatively to that given in previous experimental measurements of Mach stem formation by Heaps et al.[19]. The lowest incident overpressure given by Heaps et al. is approximately 13,800 Pa and the angle at which the maximum amplification factor occurs is approximately 65°. Based on the trend of the data by Heaps et al., it can be concluded that a peak amplification factor located at approximately 78° for a incident overpressure of 6200 Pa is reasonable.
5.3 One-Dimensional Jet Noise Propagation

In this section, a jet noise recording is read by the model and propagated into the domain using a Gaussian distribution in space as described in Section 3.4. The recording used as the input is that of a F/A-18E \( (D \approx 0.5 \text{ m}) \) recorded at 135° from the jet inlet and at a distance of 18.3 m \( (r/D = 36.6) \) from the source. Because of the excessively large amount of time that would be required to propagate the entire waveform, the recording was truncated significantly. The data were also interpolated to various degrees using the algorithm described in Section 3.4 in order to investigate its impact upon the calculated results.

In the first case given, the waveform was interpolated by a factor of five so that the sampling frequency, initially equal to 44.1 kHz, increased to 220.5 kHz. The reason for doing this is that, as shown in Section 3.4, it has been noted that unless the spatial and temporal scales are fine enough, the source is not able to produce sufficiently high frequencies in the domain. The relevant computational parameters for the results using this interpolation factor include temporal steps of \( 4.535 \times 10^{-6} \) s, a noise recording length of one second, spatial steps of \( 3.175 \times 10^{-3} \) m, and a one-dimensional grid of 70,000 points that includes the effects of spherical spreading for a computational source located 18.3 m from the origin corresponding to a total propagation distance of 232.9 m \( (r/D = 466) \). Virtual microphones are placed at regular intervals throughout the domain and absorbing boundary conditions are applied to the ends of the domain in order to suppress wave reflection at the boundary. A depiction of the computational setup is given in Fig. 5.8 which shows the location of the origin for the spherical spreading, virtual microphones, and position of the absorbing boundary conditions.

The resulting waveform calculated by the WENO scheme is compared in Fig. 5.9 and Fig. 5.10 to Gee’s Anderson algorithm predictions[11] for linear and nonlinear propagation in a spherically spreading domain including the effects of molecular relaxation and modified classical absorption. What should be immediately obvious
Figure 5.8. Depiction of the computational setup used for one-dimensional jet noise propagation. Virtual microphones are placed in the domain on one side of the propagation path, absorbing boundary conditions are placed on both ends of the domain, and the geometric origin for the spherical spreading is located outside of the computational domain.

Figure 5.9. Comparison of the predicted waveforms at 232.9 m using Gee’s algorithm with (solid red line) and without (dash dot blue line) nonlinearities to the WENO results (dashed black line) using an interpolation factor of five on the original waveform.
is the impact of nonlinearity upon the results. Neglecting nonlinearities causes the energy in high frequencies to be underestimated considerably and the frequency of peak energy to be overestimated. This is most likely a result of two major phenomena which have been discussed previously. First, wave steepening which has caused a shift of energy to high frequencies and helps to counteract the atmospheric absorption that is much greater at high frequencies. Second, shock coalescence which occurs because stronger shocks travel faster than, and eventually overtake, weaker shocks.

Referring to Fig. 5.10, the differences between the results from Gee’s nonlinear prediction and the WENO prediction are also apparent, especially in the high and low frequency regimes. The reason for this high frequency roll-off is the inability of the WENO scheme to resolve all relevant wavelengths using the given spatial discretization. The increased levels at lower frequencies in the WENO scheme’s predictions may be the result of an increased amount of shock coalescence in the
Figure 5.11. Comparison of the power spectral densities of Gee’s algorithm with (solid red line) and without (dash dot blue line) nonlinearities to the WENO results (dashed black line) for an interpolation factor of ten. The high frequency roll-off has shifted to twice the frequency as that for the interpolation factor of five which is to be expected if the roll-off is because of under-resolved waves.

WENO scheme compared to Gee’s nonlinear prediction. To investigate the discrepancy between the two nonlinear predictions at very high and low frequencies, the tests are re-run using finer spatial and temporal resolution.

The same waveform, truncated further to a length of approximately 0.3 s, was interpolated by a factor of ten, corresponding to a sampling frequency of 441 kHz. The relevant computational parameters used in this calculation include temporal steps of $2.268 \times 10^{-6}$ s, spatial steps of $1.587 \times 10^{-3}$ m, and a one-dimensional grid of 110,000 points that includes the effects of spherical spreading for a source located 18.3 m ($r/D = 36.6$) from the source. A comparison of the power spectral densities of the predicted waveforms at 186.5 m ($r/D = 373$) using the WENO scheme and Gee’s linear and nonlinear prediction is given in Fig. 5.11, and a comparison of just the two nonlinear predictions is given in Fig. 5.12.

Looking at Fig. 5.11, it is apparent that again the linear prediction deviates considerably from the nonlinear predictions, but now since the spatial and temporal
Figure 5.12. Closer view of the power spectral densities of Gee’s nonlinear prediction (solid red line) and the WENO prediction (dashed black line). The results are similar for the frequency range 500 Hz–5000 Hz, but for frequencies above 5000 Hz, the WENO results begin to roll-off as a result of resolution issues. For frequencies below 500 Hz the low frequency power is greater in the WENO prediction, presumably due to the increased amount of shock coalescence.

resolution has been increased by a factor of two, the high frequency roll-off observed in the WENO scheme’s prediction is different. The approximately 15 dB difference between the two nonlinear algorithms that occurred at 10 kHz in Fig. 5.10 has now shifted to 20 kHz in Fig. 5.11, which suggests that the roll-off by the WENO scheme in the high frequency range can be shifted to various frequencies simply by changing the spatial resolution. Having established the frequency limitations of WENO for broadband signals and recognizing the fact that the results obtained using a linear prediction model are quite different from the nonlinear predictions, the frequency content of the two nonlinear propagation models will now be investigated further.

Fig. 5.12 gives a closer look at the differences between the two nonlinear predictions. Looking at this figure it becomes apparent that for a frequency range of approximately 300 Hz–5000 Hz, the power spectral densities calculated by the two algorithms are very similar. Since the difference between the high frequency energy
Figure 5.13. Comparison of the predicted waveforms using Gee’s algorithm with (solid red line) and without (dash dot blue line) nonlinearities to the WENO results (dashed black line) for an interpolation factor of ten. The shock located at $\tau=0.063$ s in Gee’s algorithm is predicted to be located in a considerably different position in time using the WENO solution. The circled portion of the graph indicates the loss of two zero-crossings that has occurred as a result of shock coalescence in the WENO solution but not in Gee’s algorithm. Shock coalescence shifts the energy contained in the waveform to lower frequencies.

The loss of two zero-crossings calculated has been demonstrated to be due to a resolution issue of the WENO scheme, the remaining difference between the two algorithms at low frequencies must be addressed.

Figure 5.13 shows a section of the time waveforms predicted by both nonlinear schemes. Focusing on the shock predicted by Gee’s algorithm at $\tau = 0.063$ s and comparing it to the location of the same shock predicted by the WENO scheme, it is apparent that the difference between the two algorithms can be considerable. The WENO scheme has predicted that the stronger shock will overtake the weaker shock at a faster rate than Gee’s algorithm predicts. To ensure that the difference between the two predictions was not due to one model propagating the waveform further than the other, the continuous portions of the waveform were investigated. The sections of the predicted waveform that are continuous and relatively low
amplitude, such as the region around $\tau = 0.065$ s, do match very closely, indicating that they have propagated the same distance and have hence been subject to the same amount of spherical spreading and absorption.

To determine if the WENO method is over-predicting shock coalescence or if Gee’s algorithm is under-predicting shock coalescence, the ability of each scheme to propagate shocks at their appropriate velocity is now investigated. The WENO scheme was implemented to output a large number of virtual microphone recordings at the end of the calculation and these were used in the investigation of the shock speed. These virtual microphones were placed at intervals of 1.75 m throughout the domain. The speed of the propagating shock was determined by finding the pressure in front of and behind the shock, taking the average, and then determining the wave speed of the averaged pressure. The speed of the averaged pressure was calculated rather than the pressure in front of or behind the shock because of the difficulty in determining the velocity based upon a peak value, which is subject to discretization effects. Interpolating the rise portion of the shock and determining the precise spatial position of the averaged pressure has proven to be much easier in this investigation.

The expected propagation speed of a shock is given by weak shock theory\cite{3}:

$$U_{sh} = c_0 + \beta \frac{p_a + p_b}{2 \rho_0 c_0},$$  \hspace{1cm} (5.7)$$

where $U_{sh}$ is the shock speed, $\rho_0$ the ambient density, $c_0$ the low frequency small signal sound speed, and $p_a$ and $p_b$ the pressure ahead of and behind the shock, respectively. To perform the comparison between the WENO scheme’s results and weak shock theory, the maximum and minimum pressure of the shock is determined at one point in space, and these values are used in Eq. 5.7 to predict the velocity at which the shock will travel to the next point in space 1.75 m away. The actual shock velocity by the model is determined using the averaged pressure of the shock as described above and the new recorded waveform’s values of $p_a$ and $p_a$ are used
Figure 5.14. Comparison of the calculated shock propagation speed in the WENO prediction to the expected shock propagation speed calculated using weak shock theory.

In the weak shock theory prediction of the shock velocity for the next point in space. The process is then repeated for all microphone recordings. This method is not perfect, however, since the waveform in the model is constantly subject to spherical spreading, which continuously decreases the amplitude of the shock and hence should cause the weak shock theory prediction to slightly over-predict the shock velocity in the WENO scheme.

Using the method described above, the velocity of the shock described previously that is located at approximately $\tau = 0.0625$ s in Fig. 5.13 was determined using the microphone recordings located from 64.8 m to 190 m from the source in increments of 1.75 m. A comparison of the calculated shock propagation speed and the shock propagation speed expected using weak shock theory is given in Fig. 5.14 and the agreement is generally very good between the two.

The same investigation that was performed on the WENO scheme, in which the waveform was propagated from 64.8 m to 190 m in equally spaced increments of 1.75 m, was performed using Gee’s algorithm. A comparison of the calculated shock propagation speed in Gee’s algorithm versus the predicted shock propagation
Figure 5.15. Comparison of the calculated shock propagation speed in Gee’s algorithm to the expected shock propagation speed calculated using weak shock theory.

The discrepancy between the WENO solution and Gee’s solution at low frequencies is interesting since in previous investigations Petitjean and McLaughlin[8] have reported a downward shift in peak frequency in their scale model experiments, which was concluded to be due to shock coalescence. However, work reported on scale model jets at the Boeing low speed aeroacoustic facility (LSAF)[10] and static engine run-up tests of the F/A-22 Raptor [11] do not show this downward shift. The reason for this discrepancy is not clear, but it has been suggested that the apparent downward shift in peak frequency in Petitjean’s and McLaughlin’s data is because of the directionality of the source, which keeps the propagation path from being perfectly in line with the assumed direction of propagation. The assertion that perhaps the low frequency energy is due to factors other than shock coales-
cence has been bolstered by the fact that Gee’s algorithm did not demonstrate a considerable amount of shock coalescence for propagation of similar scale model jet noise waveforms. Now that it has been concluded that Gee’s algorithm is not propagating shocks at the appropriate speed and very possibly downplaying the effects of shock coalescence, a potential explanation for the inconsistency between the two is available.

In the next section, scale model data recorded by Petitjean and McLaughlin is input into the WENO scheme and Gee’s algorithm. The resulting predictions given by the two nonlinear algorithms as well as a linear prediction are compared to the noise waveforms recorded at various distances from the jet. The results should be able to demonstrate, for this particular case at least, whether the WENO scheme agrees more with the Gee algorithm’s predictions or the scale model data.

5.4 Comparison to Scale Model Data

5.4.1 Introduction

This section will investigate the discrepancy between scale model data recorded by Petitjean and McLaughlin[8], which was reported to show a downward shift in peak frequency, and Gee’s algorithm, which does not demonstrate this effect. As stated previously, this disagreement may be because Gee’s algorithm does not propagate shocks at the appropriate speed as predicted by weak shock theory. The WENO scheme, which has been shown to propagate shocks at speeds that agree with weak shock theory predictions, will be used to propagate the jet noise data recorded by Petitjean’s 0.5\(^\prime\) (0.5-inch) scale model jet (1 inch = 2.54 cm) in one dimension including spherical spreading. The data used for comparison were recorded at 35\(^\circ\) from the jet outlet and at distances of 7.5\(^\prime\) (r/D = 15), 15\(^\prime\) (r/D = 30), 30\(^\prime\) (r/D = 60), and 50\(^\prime\) (r/D = 100) from the end of the potential core, assumed to be approximately 2.5\(^\prime\) directly behind the jet outlet, under a cold and
2.5 temperature ratio heat simulated condition. To simulate a heated jet, the scale model jet uses a mixture of helium and air. For a complete explanation of the use of helium to simulate heated jets, the reader is referred to Michael J. Doty’s Ph.D. dissertation[83].

There are a total of six simulations in this section. The cold and heat simulated jet noise data at 7.5″, 15″, and 30″ will be used as the input into the WENO scheme and propagated to 50″. The results of the propagation will be compared to the linear and nonlinear predictions given by Gee’s algorithm and also to the measured data at 50″.

The ambient conditions in these experiments included a temperature of 295.5 K, pressure of 97900 Pa, and relative humidity of 41.5 %. The data were recorded at a sampling frequency of 298,507.5 Hz, high-pass filtered at 250 Hz, and low-pass filtered at 120 kHz.

The data were truncated and interpolated by a factor of 35 using the interpolation scheme described in Section 3.4. The spatial resolution associated with this amount of interpolation is high enough so that at frequencies below 120 kHz there is no high frequency roll-off such as that shown in the previous section. The computational parameters included a $\Delta t$ equal to $9.57 \times 10^{-8}$ s, $\Delta x$ equal to $6.70 \times 10^{-5}$ m, and jet noise record length of 520,000 points, corresponding to $4.98 \times 10^{-2}$ s. The spatial domain consisted of 17,000, 15,000, and 8,500 points for the simulations that used the recorded data measured at 7.5″, 15″, and 30″, respectively. Additionally, absorbing boundary conditions were applied to both ends of the domain in all calculations.

The remainder of this section will investigate the results in a standardized way. As stated previously, inserting a broadband source into the WENO scheme using the method described in Section 3.4 does not guarantee that all of the desired frequency content is transferred into the domain. To verify that the frequency content of the waveform inserted into the WENO scheme was essentially identical
to that of the recorded data, a comparison of the power spectral densities of the recorded data to a virtual microphone recording located very near, but not within the computational source is first given. Second, a comparison will be shown of the power spectral densities of the recorded data at 50″ to that of those predicted waveforms using the WENO scheme and Gee’s algorithm which will determine a linear and nonlinear prediction. Third, the input PSD will be compared to the PSD of the WENO scheme’s predicted noise waveform to demonstrate the expected evolution of the waveform.

5.4.2 Cold Jet Predictions

The cold jet data recorded at 7.5″, 15″, and 30″ were used as the input to the prediction models for the first three scale model jet noise propagation simulations. Figures 5.16–5.21 compare the power spectral densities and the time waveforms of the recorded data used as the input into the WENO scheme to the waveforms that were actually created in the computational domain.

The magnitude of the input waveforms are in reasonably good agreement with the data in these three cases and the typical high frequency roll-off in the WENO scheme that is a result of the discretization of the wave has been pushed above the highest frequency of interest. The time domain comparison of the two waveforms shows that the phase information has also been preserved. Based upon these investigations it will be assumed for the following analysis that the waveforms in the WENO scheme are essentially the same as the intended input data.

For consistency between the two algorithms, the waveform that was propagated in the WENO scheme was used as the input to Gee’s algorithm. The two algorithms were then used to predict the evolution of the waveforms when propagated to a distance of 50″. A comparison of a representative portion of the time waveforms predicted by the WENO scheme and Gee’s algorithm, given in Fig. 5.22, shows very good agreement between the two, and it should be noted that there is little
Figure 5.16. PSD comparison of the input jet noise waveform recorded at 7.5″ (solid red line) to that used in the WENO scheme (dashed black line) for the cold jet case. The two spectra are considered to be essentially identical.

Figure 5.17. Time waveform comparison of the jet noise recorded at 7.5″ (solid line) to that used in the WENO scheme (circles) for the cold jet case. For clarity, every tenth data point of the waveform propagated by the WENO scheme is shown.
Figure 5.18. PSD comparison of the input jet noise waveform recorded at 15" (solid red line) to that used in the WENO scheme (dashed black line) for the cold jet case. The two spectra are considered to be essentially identical.

Figure 5.19. Time waveform comparison of the jet noise recorded at 15" (solid line) to that used in the WENO scheme (circles) for the cold jet case. For clarity, every tenth data point of the waveform propagated by the WENO scheme is shown.
Figure 5.20. PSD comparison of the input jet noise waveform recorded at 30″ (solid red line) to that used in the WENO scheme (dashed black line) for the cold jet case. The two spectra are considered to be essentially identical.

Figure 5.21. Time waveform comparison of the jet noise recorded at 30″ (solid line) to that used in the WENO scheme (circles) for the cold jet case. For clarity, every tenth data point of the waveform propagated by the WENO scheme is shown.
Figure 5.22. Time waveforms of the jet noise predicted at 50″ by the Gee algorithm (circles) and the WENO scheme (solid line) for the cold jet case using data recorded 7.5″ from the source. The two predictions are in excellent agreement.

visable waveform steepening. A time waveform comparison between the WENO scheme and Gee’s algorithm for the predictions using data recorded at 15″ and 30″ will not be included due to the excellent agreement of the two, which becomes apparent through an analysis of the power spectral densities.

The power spectral densities of the predicted waveforms at 50″ using data recorded at 7.5″, 15″, and 30″ from the source is compared to the measured noise spectrum at 50″ in Figs. 5.23–5.25. Using the data recorded at 7.5″ as the input to the models, it is apparent that the predicted results at 50″ are in relatively good agreement with one another, but there is considerable disagreement between the predictions and the data recorded at 50″. For the data recorded at 15″, the agreement is better than the 7.5″ data when propagated to 50″, and for the data recorded at 30″, the agreement is better still. Another interesting feature of the predictions using data recorded at 7.5″ and 15″ is that the frequency of peak energy is slightly higher than in the data recorded at 50″, which will be discussed in more detail later.
Looking at the 7.5′′ recording given in Fig. 5.23, the agreement at high frequencies between the predictions and the recorded data is noted to begin at approximately 20 kHz. The 15′′ recording case, Fig. 5.24, shows the agreement to begin at approximately 10 kHz, and the 30′′ recording case, Fig. 5.25, shows the agreement to begin at approximately 5 kHz. Near the peak frequency, the PSD of the recorded data tends to agree better with the predictions that use data recorded farthest from the source. However, it must also be kept in mind that the propagation distances also become shorter as the input is moved farther from the jet and closer to 50′′, so the reason for this trend is unclear.

The predicted evolution of the waveforms in these three tests is next compared to their input waveforms. Only the WENO scheme’s results are used for clarity and because of the good agreement between the three prediction methods. A comparison of the PSD of the input waveform to the PSD of the predicted waveform at 50′′ using the WENO scheme for the data recorded at 7.5′′, 15′′, and 30′′ is given in Figs. 5.26–5.28. In these figures the effects of spherical spreading have been removed by adding $20 \log_{10}(50/7.5)$, $20 \log_{10}(50/15)$, and $20 \log_{10}(50/30)$ to the predictions that used the data recorded at 7.5′′, 15′′, and 30′′, respectively. The fact that the waveforms compared in these three figures are essentially the same suggests that the predicted effect of the propagation is simply spherical spreading and little else. Low source amplitudes and small propagation distances have the combined effect of keeping nonlinear properties as well as absorption and dispersion from substantially affecting the results, so this result is not surprising.
Figure 5.23. PSD comparison of the experimental measurement at 50′′ (dash dot blue line) to that predicted by the Gee algorithm for nonlinear (solid red line) and linear (dotted green line) propagation as well as the prediction given by the WENO scheme (dashed black line) for the cold jet case using input data recorded at 7.5′′.

Figure 5.24. PSD comparison of the experimental measurement at 50′′ (dash dot blue line) to that predicted by the Gee algorithm for nonlinear (solid red line) and linear (dotted green line) propagation as well as the prediction given by the WENO scheme (dashed black line) for the cold jet case using input data recorded at 15′′.
Figure 5.25. PSD comparison of the experimental measurement at 50" (dash dot blue line) to that predicted by the Gee algorithm for nonlinear (solid red line) and linear (dotted green line) propagation as well as the prediction given by the WENO scheme (dashed black line) for the cold jet case using input data recorded at 30".

Figure 5.26. PSD comparison of the noise waveform propagated in the WENO scheme at 7.5" (solid red line) to the predicted evolution of that waveform at 50" (with the effects of spherical spreading removed) using the WENO scheme (dashed black line) for the cold jet case.
Figure 5.27. PSD comparison of the noise waveform propagated in the WENO scheme at 15$^\circ$ (solid red line) to the predicted evolution of that waveform at 50$^\circ$ (with the effects of spherical spreading removed) using the WENO scheme (dashed black line) for the cold jet case.

Figure 5.28. PSD comparison of the noise waveform propagated in the WENO scheme at 30$^\circ$ (solid red line) to the predicted evolution of that waveform at 50$^\circ$ (with the effects of spherical spreading removed) using the WENO scheme (dashed black line) for the cold jet case.
5.4.3 Heat Simulated Jet Predictions

The heat simulated jet data were then investigated. As before, the input data were recorded at 7.5\arcsec, 15\arcsec, and 30\arcsec at an angle of 35\degree from the jet outlet. To ensure that the heat simulated jet noise waveforms are being properly inserted into the WENO scheme, comparisons of the recorded data to that propagated in the WENO scheme for the three scenarios in the frequency and time domain is given in Figs. 5.29–5.34. The heated jet case is of higher amplitude than the cold jet case. In all three simulations, the two input waveforms are similar enough to be considered identical. As before, the waveform that was propagated in the WENO scheme was used as the input to Gee’s algorithm.

A representative section of the time waveforms predicted by the two algorithms using the data recorded at 7.5\arcsec is given in Fig. 5.35. The two time waveforms are very similar, although because of the higher source amplitudes there has been more significant wave steepening and there is also some shock smearing in the WENO prediction for the shocks located at approximately $\tau = 4 \times 10^{-5}$ s and $\tau = 6 \times 10^{-5}$ s. For the two other scenarios, the time waveforms are even more similar due to the decreased source amplitudes and shortened propagation distances.

The power spectral densities of the propagated waveforms using the WENO scheme, Gee’s linear and nonlinear prediction, and the recorded data at 50\arcsec were then calculated. The results are given in Fig. 5.36–5.38 using input data recorded at 7.5\arcsec, 15\arcsec, and 30\arcsec, respectively. The PSD comparisons demonstrate some interesting differences between the predictions and the recorded data. The data recorded at 50\arcsec shows the same relative downward shift in peak frequency when compared to the linear and nonlinear predictions using the data recorded at 7.5\arcsec. Also, for the prediction based on the data recorded at 7.5\arcsec, the heated jet data at 50\arcsec has significantly lower high frequency energy than either nonlinear model predict and the data does not even match the linear prediction until approximately 80 kHz. Whether this is a result of the use of a mixture of helium and air is still
unclear.

The agreement between the two nonlinear predictions at 50" using the data recorded at 15" and 30" is much closer to the actual recording at 50" than in the case where the data recorded at 7.5" from the jet is used. The linear predictions vary somewhat from the nonlinear predictions at the highest frequencies due to the wave steepening that has been demonstrated in Fig. 5.35. In the 15" input scenario, it is noted that the high frequency content in the data begins to match the nonlinear predictions at frequencies above approximately 20 kHz, while in the 30" input scenario the high frequency content begins to match the predictions at frequencies above approximately 10 kHz.

The predicted PSD at 50" using the WENO scheme, with the effects of spherical spreading removed as before, is compared to the input PSD used in the calculation in Fig. 5.39–5.41 for the inputs recorded at 7.5", 15", and 30" from the source, respectively. In these three cases, the lower frequency energy is predicted to be subject to spherical spreading for the most part and the high frequency energy is predicted to increase significantly in the 7.5" case, which is likely because of wave steepening. Figures 5.40 and 5.41, based on the recorded data at 15" and 30", does not predict significant wave steepening, which is presumably because the propagation distances are not long enough for wave steepening to noticeably affect the results.
Figure 5.29. PSD comparison of the input jet noise waveform recorded at 7.5" (solid red line) to that used in the WENO scheme (dashed black line) for the heat simulated jet. The two spectra are considered to be essentially identical.

Figure 5.30. Time waveform comparison of the jet noise recorded at 7.5" (solid line) to that used in the WENO scheme (circles) for the heat simulated jet. For clarity, every tenth data point of the waveform propagated by the WENO scheme is shown.
Figure 5.31. PSD comparison of the input jet noise waveform recorded at 15" (solid red line) to that used in the WENO scheme (dashed black line) for the heat simulated jet. The two spectra are considered to be essentially identical.

Figure 5.32. Time waveform comparison of the jet noise recorded at 15" (solid line) to that used in the WENO scheme (circles) for the heat simulated jet. For clarity, every tenth data point of the waveform propagated by the WENO scheme is shown.
Figure 5.33. PSD comparison of the input jet noise waveform recorded at 30" (solid red line) to that used in the WENO scheme (dashed black line) for the heat simulated jet. The two spectra are considered to be essentially identical.

Figure 5.34. Time waveform comparison of the jet noise recorded at 30" (solid line) to that used in the WENO scheme (circles) for the heat simulated jet. For clarity, every tenth data point of the waveform propagated by the WENO scheme is shown.
**Figure 5.35.** Time waveforms of the jet noise predicted at 50″ by the Gee algorithm (circles) and the WENO scheme (solid line) for the heated jet case using data recorded 7.5″ from the source. A significant amount of wave steepening has occurred in these predictions.

**Figure 5.36.** PSD comparison of the experimental measurement at 50″ (dash dot blue line) to that predicted by the Gee algorithm for nonlinear (solid red line) and linear (dotted green line) propagation as well as the prediction given by the WENO scheme (dashed black line) for the heat simulated jet case and using input data recorded at 7.5″.
Figure 5.37. PSD comparison of the experimental measurement at 50′′ (dash dot blue line) to that predicted by the Gee algorithm for nonlinear (solid red line) and linear (dotted green line) propagation as well as the prediction given by the WENO scheme (dashed black line) for the heat simulated jet using input data recorded at 15′′.

Figure 5.38. PSD comparison of the experimental measurement at 50′′ (dash dot blue line) to that predicted by the Gee algorithm for nonlinear (solid red line) and linear (dotted green line) propagation as well as the prediction given by the WENO scheme (dashed black line) for the heat simulated jet and using input data recorded at 30′′.
Figure 5.39. PSD comparison of the noise waveform propagated in the WENO scheme at 7.5" (solid red line) to the predicted evolution of that waveform at 50" (with the effects of spherical spreading removed) using the WENO scheme (dashed black line) for the heat simulated jet.

Figure 5.40. PSD comparison of the noise waveform propagated in the WENO scheme at 15" (solid red line) to the predicted evolution of that waveform at 50" (with the effects of spherical spreading removed) using the WENO scheme (dashed black line) for the heat simulated jet.
Figure 5.41. PSD comparison of the noise waveform propagated in the WENO scheme at 30\('\) (solid red line) to the predicted evolution of that waveform at 50\('\) (with the effects of spherical spreading removed) using the WENO scheme (dashed black line) for the heat simulated jet.

5.4.4 Conclusions

The downward shifts that were noted in the peak frequency were concluded by Petitjean and McLaughlin to be due to shock coalescence. In Section 5.3, evidence was given showing that the Gee algorithm is unable to propagate shocks at their appropriate speeds and consequently under-predicts shock coalescence. To investigate whether the under-prediction of shock coalescence is the cause of the discrepancy between Petitjean’s and McLaughlin’s data and Gee’s nonlinear prediction model, the WENO scheme was used to propagate the scale model jet noise. The WENO scheme is able to act as an independent numerical verification of Gee’s algorithm since it uses substantially different equations and numerical methods and it has been shown to propagate shocks at speeds that agree with weak shock theory.

The results from the investigation show that the shifts in peak frequency interpreted to be because of shock coalescence by Petitjean and McLaughlin are not
predicted to occur by either model in either the cold or heat simulated conditions. Furthermore, the results that the models give are in very good agreement with one another. For the cold jet case, the prediction using the input recorded at 7.5″ begins to match the actual data at frequencies above 20 kHz, which corresponds to propagation distances of at least 11 wavelengths. Similarly, the predictions using the cold jet input recorded at 15″ and 30″ show agreement at frequencies above 10 kHz and 5 kHz, respectively, again corresponding to 11 wavelengths or more of propagation distance. The discrepancy between the data and the predictions for frequencies that have propagated less than approximately 11 wavelengths may be because both models assume that the waveforms travel directly from the location of the source microphone to the microphone at 50″ and assume spherical spreading. Additionally, phase problems associated with the data acquisition and also the extended nature of the jet noise source may account for the discrepancy, but additional simulations are required in order to comment more on the cold jet data.

For the hot jet case, the conclusions that can be drawn are less clear. The comparison between the predicted waveform at 50″ using the data recorded at 7.5″ has no significant agreement with the actual data recorded at 50″. The reason for this may be the directionality of the source, as in the cold jet case, but the high frequencies of the actual data do not appear to match the predictions at any frequency range. The predictions based upon the recorded data at 15″ and 30″ appear to agree with the data recorded at 50″ beginning at approximately 20 kHz and 10 kHz, respectively. At this point, it is still unclear why this discrepancy exists between the predictions and the heat simulated jet noise data and more work is needed to clarify this issue as well.
Chapter 6

Conclusions

6.1 Summary

The purpose of this investigation was to create a unique acoustic propagation algorithm suitable for many scenarios, including the simulation of jet noise propagation. A Navier-Stokes system of six equation was used as the constitutive equation set which allowed highly accurate nonlinear acoustic propagation as well as accurate simulation of absorption and dispersion due to modified classical absorption and molecular relaxation of nitrogen and oxygen. Two state equations, a thermodynamic entropy relation and a van der Waals equation of state, were used to close this set of equations.

Two numerical methods were used to solve the constitutive equations. For general acoustic propagation, a weighted essentially non-oscillatory scheme was used, which allows for the propagation of continuous and discontinuous waves in a time-domain numerical model. The WENO scheme was shown to be highly reliable for stable propagation of shocks and for highly accurate acoustic propagation for continuous waves, but these benefits came at a considerable computational cost. Thus, for continuous wave propagation, a dispersion-relation-preserving scheme was introduced that was nearly as accurate as the WENO scheme and much faster.
Specific aspects of the model were introduced and discussed, such as the ability to turn on or off various absorption and/or dispersion mechanisms, methods of source simulation and the difficulties inherent to simulating broadband acoustic signals in these types of models, implementation of various types of boundary conditions, and inclusion of spherical or cylindrical spreading into a one-dimensional equation set.

The two numerical schemes were then tested to verify that they simulated wave steepening, nonlinear amplification factors at rigid boundaries, classical absorption, molecular relaxation absorption, and molecular relaxation dispersion with good accuracy. As expected, the DRP algorithm, which is fourth-order accurate in space, produced slightly less accurate results than the WENO scheme, which is fifth-order accurate in smooth regions. Nonetheless, it was concluded that both schemes allow for accurate simulation of all of the relevant acoustical phenomena. The WENO scheme’s ability to propagate shocks was finally investigated, demonstrating that although the WENO can stably propagate shocks, the rise time of a shock can be affected substantially by the choice of spatial resolution.

After the models were verified, a number of simulations were performed for which analytical solutions do not exist. To demonstrate two-dimensional acoustic propagation and shock propagation, the formation of a Mach stem was simulated. A high intensity acoustic pulse was inserted into the domain above a rigid boundary and it was demonstrated that the amplification factor at normal incidence was consistent with that determined using the Pfriem solution. At higher angles of incidence a shock formed and the creation of a Mach stem was shown. The amplification factor at the boundary, given as the ratio of the peak pressure to the incident pressure, was shown to be similar to the curves given by experimental explosion data.

To demonstrate high resolution one-dimensional shock propagation using the WENO scheme, a waveform recorded of a F/A-18E at 18.3 m was propagated to
232.9 m and 186.5 m using varying spatial resolution. The results of the propagation showed a significant amount of wave steepening and shock coalescence, which is expected from long distance propagation of a high amplitude source such as jet noise. The results were compared to Gee’s algorithm, a variant of the Anderson algorithm, and some observations were made. First, the WENO scheme has an unavoidable frequency roll-off that is because approximately 10 points per wavelength are required to capture a particular frequency. This roll-off can be shifted upward using finer spatial grids, but doing so limits the spatial domains possible, so a compromise must be attained. Second, a comparison of the jet noise power spectral densities predicted at 186.5 m using the WENO scheme and Gee’s algorithm are very similar throughout the middle of the frequency range, while the WENO scheme’s roll-off causes the high frequencies to differ. At low frequencies, however, there is a considerable discrepancy in the results, and it is proposed that this is because of an under-prediction of shock coalescence by Gee’s algorithm. An investigation of the shock propagation speed observed in both algorithms shows that the WENO scheme closely matches analytical predictions while Gee’s algorithm does not appear to be propagating shocks of different strength at different enough velocities, which further suggest that shock coalescence will be under-predicted by Gee’s algorithm.

The WENO scheme and Gee’s algorithm were then compared to scale model data recorded under cold and heat simulated conditions. Using data recorded at three distances in space as the inputs in the models, the predictions at the farthest microphone location given by both algorithms was compared for the actual data recorded. The results obtained by the two models were essentially identical to one another in all cases, while the comparison to recorded data did not show good agreement with the predictions for either the cold or heat simulated condition until the input used into the algorithms was recorded very near the location of the farthest microphone.
The recorded data shows a downward shift in peak frequency in the cold and heat simulated conditions when the data recorded at 7.5″ is compared to the data recorded at 50″. The reason for this discrepancy is still unclear. The fact that the frequency content of the recorded waveforms at 50″ tends to match the predicted data when the data used as the input is recorded at least 11 wavelengths from the source is noted. It is also noted that the heat simulated jet data also showed a larger discrepancy with the predicted results at 50″.

In this scale model jet noise comparison, it was also noted that the two numerical models did not differ nearly as much as in the F/A-18E propagation case. The better agreement between the results obtained by the two algorithms was mostly likely because there was not a considerable amount of shock formation predicted and the propagation distances were relatively short.

\section*{6.2 Future Work}

\subsection*{6.2.1 Introduction}

The creation of such a general acoustic propagation algorithm allows for a wide variety of further investigations. This section will describe just a few additional investigation that can be performed using the algorithm with only minor alterations to the source code. More in-depth alteration of the source code is discussed in the next section.

\subsection*{6.2.2 Additional Investigation of Jet Noise Predictions}

In order to have a better understanding of the precise reasons why the cold and heat simulated scale model jet data do not agree with the predictions offered by both the WENO scheme and Gee’s algorithm, considerably more work should be done. The observation that eleven wavelengths of propagation distance are required before
the far field condition can be assumed is interesting but this should be further verified using additional experimental measurements and numerical predictions.

If further one-dimensional jet noise propagation investigations are performed, a better interpolation scheme should perhaps be used. Section 3.4 demonstrated that the interpolation scheme used in the current investigation was capable of retaining the frequency content of the original waveform. In future jet noise predictions, however, in order to ensure the frequency content of the original waveform is always maintained when it is interpolated, a fast Fourier transform method should be used instead. The method works by transforming the original waveform into the frequency domain, interpolating it, and transforming it back into the time domain.

Additionally, the unique capabilities of the WENO model also allow for another type of jet noise simulation that is not possible using a Burgers equation algorithm. This model could be utilized to propagate a spatially distributed broadband source. Although the two-dimensional model would spread the waves cylindrically rather than spherically, it would likely still be a useful investigation. The creation of a spatially distributed broadband source is possible but it would probably require significant development to be able to simulate a jet noise source with reasonable accuracy. This may be accomplished using an application of Lighthill’s acoustic analogy[84],[85] which uses quadrupole sources to approximate the noise created by small turbulent eddies.

6.2.3 Sonic Boom Propagation in a Turbulent Atmosphere

The recent interest in the development of supersonic business jets has encouraged research in the design of jet bodies whose sonic booms are less annoying to observers. These body shapes create a “low boom”, meaning that the rise time at the front of the waveform is much longer than the very short rise times associated with standard N-shaped sonic boom waveforms. Sonic booms with longer rise times are frequently perceived to be less annoying[86] and it is hoped that aircraft
that create these types of booms will be allowed to fly over the continental United States. Turbulence in the atmosphere can have the effect of focusing sonic booms and it not known how this will effect the perceived annoyance of a low boom. The model developed here could be used to simulate sonic boom focusing and do so in a medium including the effects of molecular relaxation.

6.2.4 Near Field Streaming

In a previous investigation, Yano and Inoue[87] demonstrated acoustic streaming due to strongly nonlinear waves generated by a harmonically pulsating spherical source. The algorithm used by the authors solved the Euler equations, which do not include any losses, and the propagation of shocks was accomplished using a third-order accurate upwind finite-difference scheme developed by Osher and Chakravarthy[88]. The model created in this dissertation is an unintended extension of their work: the equation set used in this dissertation is essentially the Euler equations but includes the effects of modified classical absorption and the effects of molecular relaxation of nitrogen and oxygen and similarly, the numerical scheme used by Yano and Inoue is a precursor to the WENO scheme. The algorithm created in this work allows for a similar investigation and uses a higher resolution method and includes atmospheric absorption effects.

6.3 Future Improvements of the Models

This model has proven to be very accurate and versatile, but as with most computational schemes, accuracy and versatility come at a high price. The WENO scheme in particular is computationally intensive. To further expand the capabilities of the model some additional work could be done. This section will highlight some of the major directions in which this research could progress.
6.3.1 Grid Improvement

One of the obvious improvements that could be made is to change the type of grid used from a Cartesian to one specifically designed for the problem being studied. The grid refinements given below can all increase the utility of the model, although they also hinder the generality of the algorithm because these grids are often problem-dependent. Some of the discussion given here follows that in Blazek’s text[26] and for additional information and references, the reader is referred there and also to Anderson’s text[25].

Compressed grids have been used in CFD applications for years. These grids put more points in locations where it is expected that relatively small scale structures will occur. Boundary flow problems and turbulence studies are just a few of the problems that can use this type of grid stretching technique[89]. The use of compressed grids complicate the solution techniques slightly because Jacobian transformations become necessary to transform from the computational plane to the physical plane and vice versa. However, stretched grids decrease the total number of points needed, so the net effect is increased computational efficiency.

Boundary-fitted techniques are also commonly used in CFD applications. This is a technique in which the grid is fixed to the physical boundaries of the problem. This is frequently used in airfoil flow computations[90] and allows for high resolution computations on the boundary, although this comes at a higher computational cost than a Cartesian grid. If the properties near a boundary need to be resolved, however, this is often still a more efficient method of doing so. This method was first implemented by Thompson et al.[91] and for more information the reader is referred there.

The multiblock approach to grid generation divides the domain into many pieces and uses Cartesian grids of a different size for each block. This approach requires a sometimes very complex set of relations to transfer data from one grid to another, but as in the stretched case, these additional computations are offset
by the decreased number of points required in the grid. For more information on this technique, references [92]–[96] are suggested by Blazek.

Moving spatial domains have also been implemented in previous Mach stem investigations by Sparrow and Raspet[17]. In their work, a spatial domain that moved along with the propagating waveform was used which allowed the model to propagate much longer distances. A moving domain could be implemented into the WENO scheme and it is very likely that higher resolution results and larger propagation distances for some scenarios, particularly Mach stem formation, could be obtained.

Adaptive grids based on the work of Dwyer et al.[97] have proven to be very effective for steady flow propagation. The grid is implemented to cluster a large amount of grid points in regions where the gradients of pressure, velocity, etc., are very large, thus increasing the resolution of the grid where it is required and conserving grid points in regions where pressure gradients are small. The use of an adaptive grid would be most effective in conjunction with a moving spatial domain since this would help to keep regions of high pressure gradients relatively stationary so that the grid has time to fully adapt to the particular waveform that is being propagated.

6.3.2 Radiation Boundary Conditions

The three boundary conditions currently implemented in this algorithm have been proven to work acceptably well, and the reflecting and periodic boundary conditions do not appear to require revision. The absorbing boundary condition implemented in the model is very simple conceptually and has proven to be adequate for one-dimensional propagation. A one-dimensional model can afford the use of one or two hundred points to absorb the incoming wave, but when it is applied to a two- or three-dimensional domain, the number of points required simply for an absorbing boundary condition becomes intolerably large, so for these cases a
better boundary condition should be developed.

There have been many attempts to create radiation boundary conditions, in which rather than absorbing the waves, they are allowed to leave the domain without reflecting at the boundary. For example, Tam and Webb[53] introduced a set of radiation and outflow boundary conditions that allowed for acoustic, vorticity, and entropy waves to exit the boundary using only three points at the boundary for a seven-point DRP stencil like the one used in this work. Berenger[98] introduced the perfectly matched layer (PML) for the absorption of electromagnetic waves, and the concepts have been applied to a large variety of problems. Chew and Liu[99] applied Berenger’s work to elastodynamics, Hu[100] applied it to the linearized Euler equations and in the acoustics community the PML boundary condition has been used most recently by Cleveland and Sapozhnikov[101] to simulate elastic wave propagation in kidney stones. For a more detailed review of the progress in the development of boundary conditions for computational aeroacoustics, the reader is referred to the article by Hixon[102].

6.3.3 Variations on the WENO Scheme

The weighted essentially non-oscillatory algorithm used in this investigation is based upon the earlier work of Shu[44], which is simpler and allowed for shorter development time but it is not the most advanced implementation available. Research on improving the WENO scheme’s abilities is ongoing, and new versions and enhancements are constantly showing up in the literature. This section will briefly discuss some of the most promising advancements that would be able to improve some aspects of the WENO scheme that was used in this work.

A very promising study was published by Xu and Shu[103], and in their paper they discuss a method to sharpen the resolution for contact discontinuities using anti-diffusive flux corrections. It is based upon the previous work of Bouchut[104] and also Desprès and Lagoutière[105], who both discussed anti-diffusive flux cor-
rections for first-order accurate systems. Xu and Shu introduce similar corrections for fifth-order accurate WENO schemes and this correction appears to be able to help the shock smearing that is apparent in the WENO scheme that is implemented in this work. The question that remains is whether this will help for broadband signals such as jet noise. Although the shocks would be sharper with this method, approximately 10 points per wavelength should still be required to actually resolve a particular frequency, so it is doubtful that this will completely remedy the high frequency roll-off problems that have been demonstrated.

To attempt to decrease the number of required points per wavelength required by the WENO scheme to resolve a wave, a number of investigations have focused on optimization techniques similar to that performed by Tam and Webb[53] in their work on the DRP scheme. For example, Wang and Chen[55] created an optimized scheme for the linearized Euler equations and demonstrated a considerable improvement in the propagation of waves resolved using only six points per wavelength. Ponziani et al.[56] used a similar approach to optimize the WENO scheme for the standard Euler equations and were able to increase the wave resolution as well. This type of optimization could be executed for the correct equation set and would most likely improve the ability of the WENO to capture low resolution waves.

Finally, the work of Lin and Hu[106] investigated how altering the linear weights, i.e., the stencil weights that are employed in smooth regions, affects the ability of the WENO scheme to minimize dispersion or amplitude loss using a fifth order WENO scheme solving the Euler equations. They introduced new linear weights that they believe are an optimal balance between low numerical dispersion and dissipation, and also a slightly altered method of flux-splitting that may further help increase the accuracy of the scheme. A similar development may also help the solution of the modified Navier-Stokes equation set that is used in this investigation.
6.3.4 Other Numerical Methods

Some very interesting progress has been made in the field of computational aeroacoustics (CAA) using a variant of the dispersion-relation-preserving scheme discussed in this work. Tam and Kurbatskii[107] introduced the multi-size-mesh multi-time-scale dispersion-relation-preserving algorithm which is similar to the multiblock method of constructing grids in that it uses multiple grids of various resolutions. The work by Tam and Kurbatskii is unique in that it also utilizes different time scales to calculate the flow on different grids. Multiblock methods use a time step based on the finest grid used in the calculation, which can be very wasteful, so having a time step specific for each grid makes the algorithm much more efficient.

As stated previously, this multi-size-mesh DRP strategy was used by Shen and Tam[21] to simulate the generation of jet screech tones. The method allows for high resolution grids in regions where fine scale flow is expected to occur and efficient low resolution grids in regions where simple propagation is expected to occur. This simulation also required the use of selected filtering that Tam et al. had previously developed[64]. More recently, Bogey and Bailey[108] have worked on additional methods of selective filtering for these types of computational methods. Selective filtering in conjunction with a faster numerical method such as the DRP scheme is certainly worth investigating in the future, but Zhuang and Chen warn that this filtering method is problem-dependent and may require prior knowledge of the problem[68] and hence still advocate the use of the WENO method when the solution is unknown.

6.3.5 High Performance Computing

Another possible improvement would be to parallelize the model using a message passing interface (MPI). A typical run time for a spatial domain of $500 \times 500$ points propagated to a time $t = 500\Delta t$ and run on a Pentium 4 3.2 GHz system
is approximately 64 minutes for the DRP scheme and 148 minutes for the WENO scheme, so additional processing power is desirable. The algorithm has been run on a system of four processors with 8 GBytes of RAM using a compiler with an automatic parallelization routine, and this has been a great improvement over the single processor computer that was used for many of the verification calculations presented in this work, but using MPI would allow for the problem to be split into many more subdomains and calculated on many more processors. Morris et al. [20] have demonstrated an aeroacoustic algorithm that has proven to be almost ideally efficient when split over 128 processors. Although it may not be possible for a parallelized WENO scheme to be as efficient as what Morris accomplished, there is certainly room for improvement and it may be worth investigating.

6.3.6 Graphics Processors For General Purpose Computing

In the last few years, graphics processing units (GPUs) with fully programmable vertex and fragment processors have been introduced into the market. The result of having a fully programmable GPU is that it can be used for tasks other than just processing graphics. The reason GPUs are an attractive option is because they are very fast at performing vector calculations and are also internally parallel—often consisting of 16 or more parallel data pipelines. The GPU works by applying textures to geometrical objects such as points, lines, triangles, etc., known as primitives, and outputting the result into a frame buffer in the form of a two-dimensional set of pixels. It is possible to take the same commands that are used to perform these operations and use them to perform many other tasks. In essence, the GPU thinks that it is processing images and outputting the results to a frame buffer for display, but it is actually performing vector calculations and producing data output into the frame buffer where it is taken and stored.

Hagen et al. [109], have introduced a parallelized WENO scheme that uses a
graphics processor in a way similar to that described above. The authors presented results demonstrating an order of magnitude performance increase using a single nVidia GeForce 6800 graphics card ($500) compared to a dual Intel Xeon 2.8 GHz processor system with 2.0 GB of memory. A rough estimate provided by the authors gave the following (in 2005 dollars):

<table>
<thead>
<tr>
<th>Price of graphics card:</th>
<th>$ 500</th>
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<tbody>
<tr>
<td>Price of 16 computers ($ 1000 each):</td>
<td>$ 16000</td>
</tr>
<tr>
<td>Speedup of graphics card:</td>
<td>10 times</td>
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<tr>
<td>Speedup of cluster:</td>
<td>16 times</td>
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<tr>
<td>Dollar/speedup for graphics card:</td>
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<tr>
<td>Dollar/speedup for cluster:</td>
<td>$ 1000</td>
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</tbody>
</table>

The potential increase in computing power per dollar of using GPUs is almost certainly large enough to draw substantial interest in the scientific community. Indeed, there appears to be a greater and greater number of people that are using GPUs instead of CPUs in their calculations. The work can be seen in varied topics such as shallow water flows[109], real-time cloud simulation[110], and recently even a cluster of GPUs have been employed to solve flow simulation of airborne contaminants through Times Square using a lattice Boltzmann model[111]. Surprisingly, the use of hardware that was originally intended to process visual data could have one of the greatest impacts upon the development of larger and more accurate acoustic propagation models.
Appendix A

Anderson Algorithm Used for Comparison

A.1 Introduction

The specific form of the Anderson algorithm that is given here was devised by Gee[11] for the propagation of jet noise waveforms. This algorithm is used as a comparison against the one-dimensional WENO predictions for jet noise propagation, and consequently a more detailed explanation of the method may be helpful. The discussion that follows is taken mostly from Kent Gee’s thesis[11].

A.2 Method of Solution

The generalized Mendousse-Burgers equation is the foundation of this algorithm. Assuming spherical spreading, the equation is of the form:

\[
\frac{\partial p}{\partial r} = \frac{\beta}{2\rho_0 c_0^3} \frac{\partial p^2}{\partial \tau} + \psi(\tau)p - \frac{p}{r}.
\]  

(A.1)

where \( p \) is the acoustic pressure, \( r \) the distance, \( \beta \) the coefficient of nonlinearity equal to 1.201, \( \rho_0 \) the ambient density, \( c_0 \) the equilibrium sound speed, \( \tau \) the
retarded time, given as
\[ \tau = t - \frac{r - r_0}{c_0}, \]
and \( \psi \) the absorption and dispersion operator upon \( p \).

The assumption in the algorithm is that if the propagation distance is small enough, nonlinearities can be treated separately from absorption and dispersion. This assumption allows for Eq. A.1 to be divided into two equations, both of which have analytical solutions,
\[ \frac{\partial p}{\partial r} = \frac{\beta}{2 \rho_0 c_0^3} \frac{\partial^2 p}{\partial \tau^2}, \]  
(A.3)  
and
\[ \frac{\partial P(f)}{\partial r} = \psi(f)P(f) - \frac{P(f)}{r}, \]  
(A.4)
given that the propagation step size, \( \Delta r \), follows the relation:
\[ \Delta r \leq \frac{\rho_0 c_0^3}{\beta} \left( \frac{1}{\left[ \Delta p(r)/\Delta \tau(r) \right]_{\text{max}}} \right). \]  
(A.5)

Note that Eq. A.4 is transformed into the frequency domain due to the simplicity of applying absorption and dispersion on a frequency by frequency basis. This form of Eq. A.4, however, requires that a Fourier transform and inverse transform of pressure is performed after each nonlinear propagation step.

Using Earnshaw’s solution for lossless planar propagation, the arrival time of a portion of a wave as a function of pressure can be given as
\[ \tau_{r+\Delta r} = \tau_r - \frac{\beta \Delta rp}{\rho_0 c_0^3}, \]  
(A.6)
which dictates that higher pressures arrive earlier than lower pressures, which is consistent with the discussion given previously on wave steepening. This is the basis of the nonlinear propagation step.

Next, the Fourier transform of the waveform is executed and absorption and
dispersion are applied using the following solution to Eq. A.4:

\[ P(r + \Delta r, f) = \frac{r}{r + \Delta r} e^{-\alpha_c(f)\Delta r} P(r, f), \]  

\[ (A.7) \]

where

\[ \alpha_c = (\alpha_{tv} + \alpha_{r,N_2} + \alpha_{r,O_2}) + f \left( \frac{\alpha_{r,N_2}}{f_{r,N_2}} + \frac{\alpha_{r,O_2}}{f_{r,O_2}} \right), \]  

\[ (A.8) \]

\( \alpha_{tv}, \alpha_{r,N_2}, \text{ and } \alpha_{r,O_2} \) are the absorption coefficients due to thermoviscous and bulk absorption, nitrogen relaxation, and oxygen relaxation, respectively, and \( f_{r,N_2} \) and \( f_{r,O_2} \) the relaxation frequencies of nitrogen and oxygen, respectively.

An inverse Fourier transform is next performed to return to the time domain. The waveform is then combined with the nonlinear arrival times and the distorted, absorbed and dispersed waveform is attained. The waveform is no longer sampled at equal intervals, however, so the waveform is linearly resampled in preparation for the next Fourier transform.

In summary, the algorithm is implemented in the following way:

- Choose the nonlinear propagation step size using Eq. A.5.
- Perform the nonlinear arrival times using Eq. A.6.
- Fourier transform the waveform into the frequency domain.
- Apply atmospheric absorption and dispersion to the waveform using Eq. A.7.
- Inverse Fourier transform back into the time domain.
- Combine the resulting waveform with the nonlinear arrival times.
- Perform a linear resampling of the waveform.

These steps are repeated until the wave has propagated to the desired distance.
Example FORTRAN Code

B.1 Introduction

This appendix will provide some of the FORTRAN commands that were used in the implementation of the DRP and WENO schemes. The actual programs used consist of thousands of lines of code so they will not be included in their entirety. Sections of the program that cover things such as declaration statements, memory allocation statements, initialization statements, etc., are not included but are necessary for the algorithms to compile and run properly.

B.2 DRP Schemes with Runge-Kutta Time Integration

This first case will demonstrate how a dispersion-relation-preserving algorithm very similar to that used in this dissertation can be programmed. For the spatial derivatives a seven point, fourth-order-accurate DRP stencil is used while a four stage, fourth-order accurate Runge-Kutta algorithm is used in time. Assuming
again an equation set of the form:

$$w_t + F_x + G_y = H,$$

(B.1)

the scheme works in the following way.

begin program
    call initial_conditions
    do time=t_init,t_final
        call drp_runge_kutta
        call update_variables
        call add_source
    enddo
end program

end program

The subroutine "initial_conditions" will not be defined, although it is simply used to initialize the constants. The subroutine that calculates the spatial derivatives using the DRP scheme combined with the Runge-Kutta time integration is given below:

subroutine drp_runge_kutta
    do i=4,N-3
        do j=4,M-3
            w_temp(i)=(-a_3*F(i-3,j,:)-a_2*F(i-2,j,:)-a_1*F(i-1,j,:)
                +a_1*F(i+1,j,:)+a_2*F(i+2,j,:)+a_3*F(i+3,j,:))/dx
                +(-a_3*G(i,j-3,:)-a_2*G(i,j-2,:)-a_1*G(i,j-1,:)
                +a_1*G(i,j+1,:)+a_2*G(i,j+2,:)+a_3*G(i,j+3,:))/dy
        enddo
    enddo
    w_tilde_n=w+b*dt*(w_temp-H)
    w_n=w-c*dt*(w_temp-H)
end subroutine drp_runge_kutta

Here, the constants $a_1$, $a_2$, and $a_3$ are given in section 3.2.2; $dx$ and $dy$ are the spacing in the $x-$ and $y-$ direction, respectively; $w_{\text{tilde n}}$ represents the variable $\tilde{w}^{(n)}$; $w_n$ represents $w^{(n)}$; $n$ is the number of the Runge-Kutta stage; and
b and c represent the constants specific for each stage of the Runge-Kutta scheme. The other main subroutine updates the variables $F$, $G$, $H$, $p$ and $T$ based upon the new values determined at an intermediate time.

```plaintext
subroutine update_variables
  rho=w_n(:,1)
  u=w_n(:,2)/w_n(:,1)
  v=w_n(:,3)/w_n(:,1)
  s=w_n(:,4)/w_n(:,1)
  T_n=w_n(:,5)/w_n(:,1)/tau_n
  T_o=w_n(:,6)/w_n(:,1)/tau_o
  F(:,:,1)=rho*u
  F(:,:,2)=rho*u*u+p
  F(:,:,3)=rho*u*v
  F(:,:,4)=rho*u*s
  F(:,:,5)=rho*u*T_n
  F(:,:,6)=rho*u*T_o
  G(:,:,1)=rho*v
  G(:,:,2)=rho*v*u
  G(:,:,3)=rho*v*v+p
  G(:,:,4)=rho*v*s
  G(:,:,5)=rho*v*T_n
  G(:,:,6)=rho*v*T_o
  H(:,:,1)=...p=...
  T=...
end subroutine update variables
```

The formulae for the $H$, $p$ and $T$ have been omitted for brevity and may be obtained in Chapter 2. The subroutine “add source” will be described in the next section.
B.3 Source Addition

This subroutine can add a density, temperature, etc., source into the domain with a Gaussian distribution in space. The form given here is of the simplest kind; it will add a Gaussian-distributed density sinusoidal source into the domain. More complicated schemes are certainly possible and the reader is encouraged to modify the time and spatial dependencies to suit their needs.

The form of the subroutine “add_source” introduced in the previous section looks something like the following:

```
subroutine add_source
    do i=1,N
        do j=1,K
            H(i,j,1)=amp*exp(-log(2)/(alpha**2)*((xgrid(i)-x_position)**2+(ygrid(j)-y_position)**2))
               *sin(omega*tt*dt)
        enddo
    enddo
end subroutine add_source
```

where $\alpha$ represents the variable $\alpha$, $\omega$, $\Delta t$, $\text{amp}$ the amplitude, $\text{xgrid}$ and $\text{ygrid}$ the assigned positions in space for each grid point, and $\text{x_position}$ and $\text{y_position}$ the position in $x$ and $y$ where the source is centered.

For the more complex problem of adding a jet noise source into the domain, a subroutine of the following sort can be used:

```
subroutine add_source
    open(UNIT=1000,FILE='data.txt',STATUS='old',FORM='formatted')
    read(UNIT=1000,FMT=*) r_o
    close(1000)
    do i=1,N
        do j=1,K
            H(i,j,1)=amp*exp(-log(2)/(alpha**2))
        enddo
    enddo
end subroutine add_source
```
This subroutine reads in a text file named “data.txt” and assigns the read values to a variable named \( r_o \). This variable is a function of time and consequently has a constant value throughout the two “do” loops in the subroutine. It applies the Gaussian envelope in exactly the same way and allows for the amplitude to be adjusted using the variable \( \text{amp} \).

Using this section and the previous one, a fourth-order dispersion-relation-preserving algorithm with a fourth-order Runge-Kutta scheme in time can be created. All that remain to be completed are subroutines that read in the computational parameters desired, initialize the constants and initial conditions, and write the results to disk. These subroutines are relatively simple in concept, so they are not included.

### B.4 Component-wise WENO Schemes

In this section, the component-wise WENO scheme will be demonstrated. This scheme is much more complicated than the DRP scheme and consequently some less difficult aspects of the scheme will be omitted in the interest of brevity. The main program for a component-wise WENO scheme can be of the form:

```
begin program
    call initial_conditions
    do time=t_init,t_final
        call weno_x
        call weno_y
        call runge_kutta_tvd
        call add_source
    enddo
end program
```
Here it should be apparent that the WENO scheme needs to treat $x$ and $y$ separately. The subroutine “update_variables” is virtually identical to the DRP scheme’s subroutine of the same name. The subroutines “weno_x” and “weno_y” differ only through the fact that they operate in different dimensions so only the subroutine “weno_x” will be described in detail. The subroutine can be of the form:

```
subroutine weno_x
    Fp=(F+alph*w)/2
    do i=3,N-2
        B0(i,:,:)=13/12*((Fp(i,:,:)-2*Fp(i+1,:,:)+Fp(i+2,:,:))**2)
        +((Fp(i+2,:,:)-4*Fp(i+1,:,:)+3*Fp(i,:,:))**2)/4
        B1(i,:,:)=13/12*((Fp(i-1,:,:)-2*Fp(i,:,:)+Fp(i+1,:,:))**2)
        +((Fp(i-1,:,:)-Fp(i+1,:,:))**2)/4
        B2(i,:,:)=13/12*((Fp(i-2,:,:)-2*Fp(i-1,:,:)+Fp(i,:,:))**2)
        +((Fp(i-2,:,:)-4*Fp(i-1,:,:)+3*Fp(i,:,:))**2)/4
    enddo
    a0=.3/((10e-6+B0)**2)
    a1=.6/((10e-6+B1)**2)
    a2=.1/((10e-6+B2)**2)
    w0=a0/(a0+a1+a2)
    w1=a1/(a0+a1+a2)
    w2=a2/(a0+a1+a2)
    do i=3,N-2
        vm(i,:,:)=w0(i,:,:)*(Fp(i,:,:)/3+5*Fp(i+1,:,:)/6-Fp(i+2,:,:)/6)
        +w1(i,:,:)*(-Fp(i-1,:,:)/6+5*Fp(i,:,:)/6+Fp(i+1,:,:)/3)
        +w2(i,:,:)*(Fp(i-2,:,:)/3-7*Fp(i-1,:,:)/6+11*Fp(i,:,:)/6)
    enddo
    Fm=(F-alph*w)/2
    do i=3,N-2
        B0(i,:,:)=13/12*((Fm(i,:,:)-2*Fm(i+1,:,:)+Fm(i+2,:,:))**2)
        +((Fm(i+2,:,:)-4*Fm(i+1,:,:)+3*Fm(i,:,:))**2)/4
    enddo
end subroutine weno_x
```
This may require some explanation. The code is really split into two very similar sections. The first part calculates the flux-split value $F^+$ ($F_p$). Using this, the smoothness indicators $\beta_0$, $\beta_1$, and $\beta_2$ ($B0$, $B1$ and $B2$) are calculated and used to determine the values of $\alpha_0$, $\alpha_1$, and $\alpha_2$ ($a0$, $a1$, and $a2$), which are then used to determine the stencil weights $w_0$, $w_1$, and $w_2$ ($w0$, $w1$, and $w2$). Finally, $\hat{v}^-$ ($\hat{v}^{-}$) is determined using the determined stencil weights. In a similar way, the smoothness indicators for $F^-$ ($F_m$) are determined in order to arrive at the values of $\hat{v}^+$ ($\hat{v}^{-}$). The results of these two steps are finally used to calculate the value of $\hat{F}_{i+1/2}$ ($F$).

Having calculated this quantity, the Runge–Kutta scheme is employed to determine the actual spatial derivatives based upon these reconstructed values $F$. The subroutine "runge_kutta_tvd" is simpler than that given previously and can be calculated using the following:
The first stage alone is shown here but the other two are not substantially different. For details on how to perform the entire Runge-Kutta discussed here, see Section 3.3.4.

B.5 Turning off Absorption and Dispersion Mechanisms

In order to exclude absorption and dispersion mechanisms from the model, a simple set of logical statements are included at the beginning of the program that force certain constants to equal zero. If the bulk viscosity, shear viscosity, or the coefficient of thermal conductivity are individually forced to equal zero, then the absorption effects due to these mechanisms are removed. Similarly, if the specific heat at constant of volume of nitrogen, \( c_v N_2 \), or oxygen, \( c_v O_2 \), is set to zero, then the respective absorption and dispersion effects are removed.

The following sample code will demonstrate how the logical statements should be implemented. There are other logical statements that also could be included in the model in order to increase the computational efficiency of the code. For example, if shear viscosity is excluded from the calculation, then there is no reason to calculate the rate of shear tensor, \( \phi_{ij} \), which is always equal to zero. These
more advanced considerations will not be included in the example, but it should be reasonably simple to extend these concepts.

begin program
  ! set these values to 1 to include them and 0 to exclude them
  shear = 1
  bulk = 1
  thermal = 1
  nitrogen_relaxation = 1
  oxygen_relaxation = 1

  if(shear==1) then
    m=1.846e-5
  elseif(shear == 2) then
    m=0
  endif
  if(bulk == 1) then
    mB=0.6*1.846e-5
  elseif(bulk == 2) then
    mB=0
  endif
  if(thermal == 1) then
    kappa=2.624e-2
  elseif(thermal == 2) then
    kappa=0
  endif
  if(N2 == 2) then
    c_vn=0
  endif
  if(O2 == 2) then
    c_vo=0
  endif

end program
B.6 Boundary Conditions

B.6.1 Reflecting Boundary Conditions

The reflecting boundary condition is relatively simple to implement. The boundary condition is inserted into the subroutine “drp_runge_kutta” or “runge_kutta_tvd”, depending on the numerical scheme used in space. This can be done using the following:

```fortran
subroutine drp_runge_kutta
    ... normal runge_kutta here ...
    if(bc==1) then ! bc=1 gives reflecting bc
        w(1,:,:)=w(2,:,:)
        w(1,:,2)=-w(2,:,2)
        w(:,1,:)=-w(:,2,):
        w(:,1,3)=-w(:,2,3)
        w(N,:,:)=w(N-1,:,:)
        w(N,:,2)=-w(N-1,:,2)
        w(:,K,:)=-w(:,K-1,:)
        w(:,K,3)=-w(:,K-1,3)
    endif
end subroutine drp_runge_kutta
```

B.6.2 Periodic Boundary Conditions

A periodic boundary condition can be useful especially for one-dimensional propagation, although it can used in some two-dimensional cases. It is also relatively simple to implement, as shown in the following example. For this case, all variables are wrapped, not just the primitive variables. Again, this algorithm is added to the end of the Runge-Kutta subroutine.

```fortran
subroutine drp_runge_kutta
    ... normal runge_kutta here ...
    if(bc==2) then ! bc=2 gives periodic bc
        do i=1,10
            w(i,:,:)=w(N-20+i,:,:)
        enddo
    endif
end subroutine drp_runge_kutta
```
B.6.3 Absorbing Boundary Conditions

The absorbing boundary condition are implemented by inserting a Gaussian decay envelope at a boundary. The origin and half-width of the Gaussian can be adjusted and it is straightforward to insert the envelope onto a boundary. Similar to the periodic boundary condition, all variables are acted upon rather than just the primitives and this bit of code is added at the final stage of the Runge-Kutta algorithm. In the following example, an absorbing boundary condition centered 100 points into the domain with a half-width of 300 grid points is applied to one
of the boundaries:

```fortran
subroutine drp_runge_kutta
    ...
    normal runge_kutta here ...
    if(bc==3) then ! bc=3 gives absorbing bc
        numpoints=100 ! Set the origin of the Gaussian
        x_p=numpoints*dx
        al=300*dx ! Set the number of points in the half-width
        pzero=101325 ! Set the ambient pressure
        do i=1,numpoints
            w(i,:,1)=(w(i,:,1)-rho)*exp(-log(2)/(al**2)
            *(x_grid(i)-x_p)**2)+rho
            w(i,:,2)=w(i,:,2)*exp(-log(2)/(al**2)
            *(x_grid(i)-x_p)**2)
            w(i,:,3)=w(i,:,3)*exp(-log(2)/(al**2)
            *(x_grid(i)-x_p)**2)
            w(i,:,4)=w(i,:,4)*exp(-log(2)/(al**2)
            *(x_grid(i)-x_p)**2)
            w(i,:,5)=(w(i,:,5)-Tzero*rho)*exp(-log(2)/(al**2)
            *(x_grid(i)-x_p)**2)+Tzero*rho
            w(i,:,6)=(w(i,:,6)-Tzero*rho)*exp(-log(2)/(al**2)
            *(x_grid(i)-x_p)**2)+Tzero*rho
            p(i,:)=(p(i,:)-pzero)*exp(-log(2)/(al**2)
            *(x_grid(i)-x_p)**2)+pzero
            T(i,:)=(T(i,:)-Tzero)*exp(-log(2)/(al**2)
            *(x_grid(i)-x_p)**2)+Tzero
        enddo
    endif
end subroutine drp_runge_kutta
```

B.7  Graphical User Interface

To increase the program’s accessibility, a graphical user interface (GUI) has been implemented that allows for specification of most of the major parameters of the
Figure B.1. Picture of the graphical user interface used for the Relax algorithm. The major parameters of the code can be adjusted using this window, although for more complex propagation scenarios the source code must be altered.

model, although complex propagation scenarios require alteration of the source code. Fig. B.1 depicts the current layout of the GUI.

The FORTRAN code used to interface with the GUI is lengthy and will be omitted. The interface used to include the GUI essentially sets up default values for the parameters and then reads in the values input by the user. For the check boxes and buttons, logical true/false statements are used to set defaults and read user input.
Bibliography


Equations 2.1–2.4 are given in Pierce[13] as equations (1-2.4), (10-7.19), (10-7.20), and (10-7.22), respectively.

Equations 2.5–2.8 are given in Pierce[13] as equations (10-17.18), (10-1.10), (10-7.14), (10-7.15), and (10-8.1f), respectively.


Ref[41], pp. 52.


Ref[13], pp. 595.


[76] In order to avoid confusion it should be noted that for Figure 10-12(b) in Pierce [13], the labels for equilibrium and frozen sound speed are misprinted in reverse positions.

[77] Ref[13], pp. 561.


Vita

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Mark S. Wochner was born in Rochester, New York in 1979 and lived in Webster, New York until age 18, whereupon he left for undergraduate study at Vassar College in Poughkeepsie, New York. After graduating in August 2001 with a Bachelor of Arts degree in physics, he enrolled for doctoral study in the Graduate Program in Acoustics at the Pennsylvania State University. While at Penn State he had the opportunity to travel abroad for study in 2002: first to the University of Tokyo to study architectural acoustics through the National Science Foundation Summer Program in Japan, and then to the United Kingdom to study fluid dynamics at the Institute for Sound and Vibration Research, University of Southampton through the World University Network. He has been awarded a post-doctoral fellowship at the University of Texas at Austin under the supervision of Mark Hamilton.