STOCHASTIC AND DETERMINISTIC PROCESSES IN
FRAGMENTATION AND SEDIMENTATION

A Dissertation in
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by
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Abstract

In this work we present results of analysis, experiment and simulation of two phenomena involving stochastic and deterministic aspects. In the first case we present a modeling framework for 1D fragmentation in brittle rods, in which the distribution of fragments is written explicitly in terms of the probability of breaks along the length of the rod. This work is motivated by the experimental observation of several preferred lengths in the fragment distribution of shattered brittle rods after dynamic buckling. Our approach allows for non-constant spatial breaking probabilities, which can lead to preferred fragment sizes. The resulting relation is shown to qualitatively match experimentally observed fragment distributions, as well as some other commonly reported distributions such as a power law with a cutoff.

We also present experimental observations of the trajectories and average velocities of solid spheres falling through a curtain of rising bubbles in water. For the quiescent case (no bubbles), the Reynolds numbers are on the order of 1,000, and the average terminal velocity is determined by the form (inertial) drag. The main effect of the introduction of bubbles is to slow down the spheres. In some regimes (smaller or lighter spheres), there is an added random lateral motion to the sphere paths. In this way, a solid sphere sinking in a bubbly fluid and a solid sphere falling through a crowded bed of rigid obstacles (in air) share two common traits: the settling speed is slowed by the obstructions, and the sphere exhibits random lateral motion. We present a mathematical model which begins as an adaptation of Galton’s board to the sedimenting sphere. This allows us to introduce various physical effects of the bubbly fluid, and test their importance, particularly that of bubble collisions. Comparison is made with experimental results.
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Newton attributed his achievements to “standing on the shoulders of giants.” The metaphor is rather grand for my own work, but I have certainly been supported by a number of people – perhaps a human pyramid would be more apt.

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Time and chance happeneth to them all.
–Ecclesiastes 9:11
Chapter 1

Fragmentation: introduction

The fragmentation of a brittle solid is an everyday example of randomness in a physical system: a glass or plate dropped on the floor may shatter “into a million pieces” - a few large ones and many small shards. The process of fragmentation is dramatic and captured imaginations well before the beginning of modern scientific investigation. Note how this passage from the Iliad [40] uses fragmentation to heighten a dramatic scene:

But fierce Atrides wav’d his sword and strook
Full on his casque: the crested helmet shook;
The brittle steel, unfaithful to his hand,
Broke short: the fragments glitter’d on the sand.

Modern investigations into the phenomenon of the spreading of a single crack have matured in many aspects to the stage of a general agreement between a well-developed mathematical theory and many careful experimental studies (see e.g. [55, 25]). On the other hand, the mathematical study of fragmentation – the net effect of many fractures – is still at a developmental stage, with no universally accepted theoretical approach to the wealth of empirical information [27, 15, 3, 35]. The fundamental question of fragmentation is this: how does a single solid object break into many pieces?

To complicate this question, there are a number of different but interrelated physical mechanisms that may produce different kinds of fragmentation [27, 35]. For instance, there is ductile vs. brittle [33, 38], kinetic energy-dominated vs. static stress-dominated [84], or even the fragmentation of a thin brittle coating attached to an easily deformed unbroken substrate [36]. To obtain observed quantities such as the mean fragment size or the fragment distribution, mathematical approaches to modeling fragmentation have been developed in two broadly defined categories: those starting from a mechanical perspective of the stressed material, and those aiming to derive a specific functional form for the distribution (typically a power law) in a post hoc manner.

These latter approaches are usually motivated by the many experimental studies which have reported power laws in measured fragment distributions, such as those observed in the impact fragmentation of brittle glass rods [15, 42], disks, or spheres [72]. A power law dependence, with an exponential cutoff at larger sizes, is also seen in explosively fragmented distributions [47]. Some of the other physical situations in which power law particle distributions have been reported include ice floe size distribution in the Arctic [52], meteor shower mass distributions [73], and the size distribution of mercury drops which break into many pieces upon impact [86]. Post hoc models used to produce power laws in fragmentation include many suggesting the iterative breaking of a body [37, 86, 63]. The models range from simple to complex, but do not necessarily include a physical motivation. In contrast, Astrom described iterative breaking in a
model motivated by the branching and merging of cracks along a fracture surface [3, 2]; some similar results were given earlier by Gilvarry, but based a probabilistic argument concerning the independent formation of single fragments [29].

One early approach to modeling fragmentation was the work of Kolmogorov [51], inspired by the measurement of a log-normal distribution of fragment sizes produced by grinding. That is, the ratio of the number of fragments of size less than or equal to $x$ at time $t$ to the total number of fragments (also at time $t$) is given by [51]

$$\frac{1}{\sqrt{2\pi tB}} \int_{-\infty}^{x} \exp \left( -\frac{(s - At)^2}{2B^2t} \right) ds.$$ 

Kolmogorov used a few mathematical parameters to describe the continual grinding of larger particles into smaller particles. The primary requirement in his approach is that the fragmentation process reaches a condition where it is independent of particle size, independent of the fragmentation of other particles, and independent of the starting time - this last point implies that the probabilities are independent of the history of the particle in question. Under these assumptions, and two others involving the size and integrability of the expected number of particles resulting from a single particle per unit time, Kolmogorov deduced that the long time limit of the fragment distribution was log-normal.

Another historical strand goes back to the 1947 paper by Mott, motivated by military questions on the fragmentation of shell cases [65]. This approach originates from more physical constraints, treating local deformations and stress release after a break occurs. The literature in this area includes energy-based models in the dynamic regime (due to impact or stress-wave loading) [35, 64], as compared to more of a flaw-dominated approach [34]. Many developments have been made in the geophysical community, particularly regarding the fragmentation of rocks due to geological or blasting processes (for an overview see [35]).

More recently, another approach to fragmentation was taken by Audoly & Neukirch [4], in which the dynamics of curvature after an initial break in a 1D brittle solid (in this case spaghetti) is investigated. In their model, the deformation of the spaghetti after an initial break produces areas of increased curvature, leading to subsequent breaking events. The dynamic spreading of fragmentation probability was confirmed by their experiments, in which a first break leads to a second.

In contrast to the focus on scale-invariant distributions, a recent experiment (performed in the Pritchard Lab at Penn State) on the dynamic buckling and fragmentation of thin brittle rods found fragment distributions with two peaks, indicating preferred fragment sizes [30]. These lengths apparently originated with the initial sinusoidal buckling of the rod, leading to local maxima in the fragment distribution near 1/2 and 1/4 of the buckling wavelength, see Fig. 1.1. An explanation was proposed based on the assumption that breaks in the rod were more likely to occur around the points of maximum curvature, although there were many observations of breaking when the spaghetti did not break at every maximum (Fig. 1.1, inset). The speculative conclusion was that the distribution of fragments was being determined primarily by the initial stress distribution rather than by a sequential, multiplicative process [30]. This indication that coherent
Fig. 1.1. Measured number density of fragments $n(\ell)$ from one fragmentation data set for the dynamic buckling of a brittle rod (here spaghetti) [30]. The inset shows an experimental image of an event representing the observation that the rod tends to break either near the top or near the bottom [J. Gladden, unpublished].

patterns in the deformation can play a role in determining the fragment distribution provided the impetus for the present work.

The challenge posed by these observations to mathematical modeling was the existence of multiple peaks in the fragment distribution - as opposed to a self-similar, scaling law or a single preferred fragment size. While it is clear that the coherent pattern comes in some way from the spatial distribution of stress, deformation, and perhaps other fields in the original object [30], none of the existing modeling approaches allowed for this possibility. The mathematics detailed in the following chapters was developed to provide a framework for analyzing these patterns. This will provide a foundation for further study of the relation between stress dynamics and the sizes of the fragments produced.

In Chapters 2 and 3, we will examine for two general cases the mathematics of a rod breaking probabilistically. In Chapter 2, the event space (set of all possible states of the rod after breaking) will be finite – a more intuitive case for probability. In Chapter 3, the probabilities of encountering a break in the rod will be taken to the continuum limit, producing a more complicated event space. In both cases we will develop a relation between breaking probabilities and the distribution of fragment sizes expected from the fragmentation event. This relation will be examined experimentally in Chapter 4. The bulk of the material in Chapters 1 - 4 is taken (by copyright agreement) from the author’s article with Andrew Belmonte, “Fragment distributions for brittle rods with patterned breaking probabilities” [39].
Chapter 2

Fragmentation: modeling a segmented rod

2.1 Introduction

One of the basic modeling assumptions in the study of fragmentation involves the question: in what order are the pieces created? all at once, or sequentially? Few experiments actually measure the temporal sequence of the breaks [4]. Starting with Kolmogorov, many models are built on the conception of multiplicative breaking [76]. On the other hand there is a tradition of simplifying the treatment in terms of independently occurring fractures, which can then be thought of as a “single fragmentation event”. This is for example the approach used by Gilvarry to find a generalized fragment size distribution based on the random distribution and activation of flaws in a material [29]. Here we will take the latter approach.

Experimentally, we are interested in the fragmentation of slender rods, i.e. rods whose diameter is much smaller than their length (in the case of Gladden [30] for example $d/L \approx 0.005$ is typical). We will therefore limit our investigations to the case of fragmentation in one dimension. This is by far the most tractable case: In higher dimensions one has to consider the geometry and intersections of cracks. In one dimension breaks in the rod form at given locations that simply divide the length of the rod. Thinking of the fragmentation as a random event, the sample space consists of all ordered sequences of positive lengths summing to the total length of the rod – a list of fragment sizes in the order in which they occur from one end to the other.

The simplest case of division of a line into fragments occurs when the breaks in the line can only occur at a finite number of predetermined locations. This is (as we shall discuss later) equivalent to dividing a finite sequence into contiguous subsequences. For purposes of developing the mathematics, we shall think in terms of a segmented rod similar to the one portrayed in 2.1.

Fig. 2.1. A rod (above) goes through an event in which it may break at any of a finite number of locations (joints). An outcome of the event is an ordered collection of fragments, or equivalently a list of which joints have broken in the event. In this case we show an outcome in which the rod breaks at the $5^{th}$, $6^{th}$ and $9^{th}$ joints.
We begin with a rod of unit length, comprising $J + 1$ indestructible segments joined by $J$ joints. We consider an event in which every joint breaks with probability $p$ (independently of the others). For convenience we will discuss fragment lengths in units of segments, so that the length of the shortest fragment is 1, instead of $(J + 1)^{-1}$. The event space, adapted from the previous description, is the set of all ordered lists of positive integers summing to $J + 1$ – representing the fragment lengths in order from left to right. As an example, the event space for a rod of 7 segments is shown in Fig. 2.2.

Note that the probability of a given outcome depends only on the number of breaks that occur. If there are $k$ breaks in an outcome, then the probability of the outcome is $p^k(1 - p)^{J - k}$.

Our goal is to find $E_{J+1}(s)$, the expected number of fragments of a specified (segment) length $s$. This will depend among other things on the expected number of breaks in the rod, which is (in this equiprobable case) denoted $C = Jp$.

### 2.2 The combinatorics of breaking

$E_{J+1}(s)$ is given by the sum of the number of fragments of length $s$ in each possible configuration of the broken rod weighted by the probability of achieving that particular configuration. Configurations in which there are no fragments of length $s$ can thus be ignored. For example, to calculate $E_7(2)$ we can count the fragments of length 2 seen in Fig. 2.3 and sum (proceeding down the first column)

$$E_7(2) = 1 \cdot p(1 - p)^5 + 1 \cdot p^2(1 - p)^4 + 1 \cdot p^2(1 - p)^4 + 2 \cdot p^2(1 - p)^4 + 1 \cdot p^3(1 - p)^3 + \cdots \quad (2.1)$$

The case where $s = J + 1$ is easily dealt with, since there is only one such fragment (the unbroken rod) which occurs with probability $(1 - p)^J$. Then $E_{J+1}(J + 1) = (1 - p)^J$.

For the general case where $1 \leq s \leq J$, we first consider the set $S_k$ of all configurations with exactly $k$ breaks. Because we treat all breaks as independent, each configuration with $k$ breaks has the same probability of occurring, and will produce $k + 1$ fragments. For one of the fragments to have a length $s$ requires at least $s + k$ total segments in the rod, thus we will stipulate that $k \leq J + 1 - s$. We now consider the subset of $S_k$ for which the first fragment on the left is $s$ segments long. This leaves $J - s$ joints among which to distribute the remaining $k - 1$ breaks; there are exactly $\binom{J-s}{k-1}$ unique ways in which this can occur. For example, if we consider the event space of Fig. 2.2 (reordered in Fig. 2.4 according to the number of breaks in each outcome) we see that among the outcomes with 3 breaks, there are $\binom{4}{2} = 6$ configurations where this occurs.

Consider next the subset of $S_k$ for which the second fragment on the left is $s$ segments long. Each element of this subset can be obtained by permuting the order of the fragments from the previous subset in a one-to-one map, thus there are also $\binom{J-s}{k-1}$ configurations. In fact there are the same number of configurations for each of the $k + 1$
Fig. 2.2. The event space showing all $2^6 = 64$ possible divisions of a rod of $J + 1 = 7$ segments in length.
Fig. 2.3. Segments of length 2 occurring in the event space of Fig. 2.2.
Fig. 2.4. Segments of length 2 within outcomes of Fig. 2.2 having exactly 3 breaks.
positions of the $s$-length segment, each being a simple rearrangement of the others. Thus

$$E_{J+1}(s) = \sum_{k=1}^{J+1-s} p^k (1-p)^{J-k} (k+1) \binom{J-s}{k-1}. \quad (2.2)$$

Note that this method of enumerating by the number of breaks allows us to sum neatly over $k$, sidestepping the need to treat explicitly the configurations which have more than one piece of length $s$, as these cases were naturally included in the enumeration.

To evaluate this sum, we define the function

$$h(\mu, \eta) = \mu^2 (1 + \mu)^\eta = \mu^2 \sum_{m=0}^{\eta} \mu^m \binom{\eta}{m} \quad (2.3)$$

so that

$$\frac{\partial h}{\partial \mu} = 2\mu(1 + \mu)^\eta + \eta\mu^2 (1 + \mu)^{\eta-1} = \sum_{m=0}^{\eta} (m+2)\mu^{m+1} \binom{\eta}{m}. \quad (2.4)$$

Using this we set $m = k - 1$ and rewrite Eq. 2.2 as

$$E_{J+1}(s) = (1-p)^J \sum_{m=0}^{J-s} \left( \frac{p}{1-p} \right)^{m+1} (m+2) \binom{J-s}{m} = (1-p)^J \left| \frac{\partial h(\mu,J-s)}{\partial \mu} \right|_{\mu=\left(\frac{p}{1-p}\right)}$$

$$= (1-p)^J \left[ 2 \left( \frac{p}{1-p} \right) \left( 1 + \frac{p}{1-p} \right)^{J-s} + (J-s) \left( \frac{p}{1-p} \right)^2 \left( 1 + \frac{p}{1-p} \right)^{(J-s)-1} \right]$$

$$= p(1-p)^{(s-1)\cdot [2 + (J-s)p]} \quad (2.5)$$

It was mentioned to the author that the sum in Eq. 2.2 can also be evaluated by noting that $k + 1 = (k-1) + 2$. Based on this observation, we can alternately rewrite Eq. 2.2 as

$$E_{J+1}(s) = 2 \sum_{k=1}^{J+1-s} p^k (1-p)^{J-k} \binom{J-s}{k-1} + \sum_{k=2}^{J+1-s} p^k (1-p)^{J-k} (k-1) \binom{J-s}{k-1}$$

$$= 2 \sum_{k=1}^{J+1-s} p^k (1-p)^{J-k} \binom{J-s}{k-1} + \sum_{k=2}^{J+1-s} p^k (1-p)^{J-k} (J-s) \binom{J-s-1}{k-2}$$

$$= 2p(1-p)^{(s-1)} \sum_{m=0}^{J-s} p^m (1-p)^{J-s-m} \binom{J-s}{m} +$$

$$p^2(1-p)^{(s-1)} (J-s) \sum_{m=0}^{J-s-1} p^m (1-p)^{J-s-1-m} \binom{J-s-1}{m}$$

$$= p(1-p)^{(s-1)\cdot [2 + (J-s)p]} \quad (2.6)$$

### 2.3 Application of Eq. 2.5 to division of a sequence

An immediate application of this noted by A. Belmonte is that Eq. 2.5 will give the number of runs of heads and tails in a sequence of coin tosses. The connection is made by comparing runs of heads or tails to fragments of the broken rod, with a break in the rod corresponding to a change from heads to tails or vice-versa (e.g. a break in

---

*We thank Professor Qiang Du of Penn State for suggesting this line of approach.
†We thank Professors George Andrews and James Sellers of Penn State for this observation.
the run). For example, the rod in Fig. 2.5 has segments of length 2, 1 and 3, which would correspond to a series of tosses with a run of 2 (HH), a run of 1 (T), and a run of 3 (HHH). (Note that since we are not distinguishing between runs of heads or tails, the broken rod also represents the sequence $TTHTTT$.)

![Fig. 2.5. The mapping from a broken rod to a sequence of coin tosses. See text for details.](image)

To verify this observation, we need to show that if the coin tosses are independent, that breaks in runs are also independent. We first note that a break in a run happens whenever the pattern HT or TH appears in the sequence of tosses. So if $p_h$ is the probability of heads, the total probability of having a break is $p = p_h(1-p_h) + (1-p_h)p_h = 2p_h(1 - p_h)$. With a fair coin this is $2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$. If a coin is tossed $n$ times, there are $n - 1$ opportunities for a break in the run.

Because the tosses are assumed to be independent, and a break depends only on the two tosses to either side of it, we know that breaks that do not occur consecutively occur independently. For example, in Fig. 2.6 the probability of a break at A depends only on the tosses 1 and 2, while the probability of a break at C only depends on the tosses 3 and 4. Because the first and second tosses are independent of the third and fourth tosses, so are breaks at A and C.

However, consecutive breaks do not necessarily occur independently because they both depend on the middle toss. For example, in Fig. 2 breaks at A and B both depend on what happens on the second toss. We here calculate the probability of having a break at B given that there is a break at A. A break happens at A for the patterns HTH, THH, HTT, and THH. The probability of one of these patterns occurring is $2p_h^2(1 - p_h) + 2p_h(1 - p_h)^2$. The first two patterns are the ones for which there is also a break at B, and these occur with probability $p_h^2(1 - p_h) + p_h(1 - p_h)^2$. So the probability of a break at B given that there is a break at A is $P(B|A) = \frac{P(B \cap A)}{P(A)} = 1/2$. Because we have a fair coin, $P(B) = P(B|A) = 1/2$. Similarly, $P(A|B) = 1/2 = P(A)$. In no other case do consecutive breaks occur independently. This applies generally to other pairs of consecutive breaks, so that in the case of a fair coin the breaks in runs occur independently and we can apply Eq. 2.5. The expected number of runs of length $k$ in $n$ tosses is then

$$E_n(k) = (1/2)^k [2 + (n - k - 1)/2].$$  \hfill (2.7)
A few values for the case of $n = 100$ are shown in Table 2.1. In a series of Bernoulli trials (such as coin tossing), the focus is usually confined to runs of successes. In this context, $E_n(k)$ would count only the runs of heads. From symmetry this can easily be obtained from Eq. 2.7 by dividing by two.

This equation (Eq. 2.7) is useful in that the quantity it describes is counterintuitive in some respects. An intuitive approach to coin tossing is considered in Stoppard’s play *Rosencrantz & Guildenstern Are Dead* [90]. As the play begins, Guildenstern has just had heads come up on 92 consecutive tosses of various coins. He observes that

The equanimity of your average tosser of coins depends upon a law, or rather a tendency, or let us say a probability, or at any rate a mathematically calculable chance, which ensures that he will not upset himself by losing too much nor upset his opponent by winning too often. It related the fortuitous and the ordained into a reassuring union which we recognized as nature.

Rozencrantz (who loses a coin on each head and wins a coin on each tail) complains that

We have been spinning coins together since I don’t know when, and in all that time... I don’t suppose either of us was more than a couple of gold pieces up or down.

These two characters express an intuitive sense that the law of large numbers – which we (correctly) understand to mean that about half of a large number of coin tosses will come up heads – indicates that any consecutive subsequence of the series of tosses will also consist of roughly equal numbers of heads and tails, preventing the formation of runs of heads (or tails) that are more than a “couple” of heads longer or shorter than any other.

This intuition has been demonstrated in classroom experiments (see e.g. Schilling [82]) where groups of students have been divided into two groups. One group is assigned to toss a coin 200 times, while the other group is told to write down a representative random result of 200 tosses. Sequences from the two groups are generally identifiable as to their origins (Schilling [82] claims to be able to identify sequences with the correct source with a success rate of 85%) in that students making up a “random” sequence tend to write only shorter runs of heads and tails. According to [11], in sets of 100 tosses
students who made up a sequence could generally be distinguished in that they produced runs of length four or less. This can be compared with the expected values shown in Table 2.1 that show a run of length five is to be expected; indeed, we should expect a run of length six or more.

There are two problems with this intuition: First, five is not really a “large number” so the law of large numbers should not be applied. Also, even if a sequence of, say, five heads is unlikely, there are many subsequences of length five in a sequence of 100 tosses of a coin. To think that the probability of not having five heads in a single subsequence of five tosses also represents the likelihood of having no pattern of five tosses is any subsequence is incorrect. ‡ This mistake in intuition has been referred to as a belief in a “law of small numbers” [94]: in this case, a belief that the results in any small subsequence occur roughly in proportion to their probabilities.

2.4 Application of Eq. 2.5 to a particular recursive function

In this section we will provide a recursive formulation for

$$f_k(J + 1, s) \equiv (k + 1) \binom{J - s}{k - 1}.$$  \hfill (2.8)

This is the quantity seen in Eq. 2.2 representing the number of segments of length $s$ in all configurations of a rod of $J + 1$ segments in which there are exactly $k$ breaks. It is possible to define $f_k$ recursively in the following manner. First we have

$$f_0(J + 1, s) = \delta_{J+1,s}$$  \hfill (2.9)

(where $\delta_{J+1,s}$ is the Kronecker delta) and for $1 \leq s \leq J$

$$f_1(J + 1, s) = 2$$  \hfill (2.10)

(since the fragment must occur at one of the two ends).

The relation between $f_k$ and $f_{k-1}$ is developed by dealing separately with the leftmost fragment and the remaining length, $L$, of the rod. If we look at the set of outcomes with $k$ breaks where the leftmost fragment is of length $J + 1 - L$, then the remaining $k - 1$ breaks occur as in a rod of length $L$, and the number of fragments

‡This is similar to the often-cited birthday problem: in a room with at least 23 people, there is a better-than-even chance that at least two share a birthday. It is of course, unlikely that any two people picked out of a crowd will share a birthday. However, there are many possible pairs of people in the room (253 pairs for 23 people). Given such a larger number of pairs it is likely that at least one pair will have a birthday in common.
of length $s$ among them is $f_{k-1}(L, s)$. As previously noted, there are $\binom{J-s}{k-1}$ leftmost fragments of length $s$. With this information we can calculate the recursive sum

$$f_k(J + 1, s) = \binom{J - s}{k - 1} + \sum_{L=s+k-1}^{J} f_{k-1}(L, s).$$

(2.11)

This was indeed our first description of $f_k$. We quickly discovered that 2.11 was not convenient for calculation as demonstrated by the following code written for Matlab:

```matlab
function rval = f_recursive(k, J_plus, s)
% rval = f_k(J+1, s)

J = J_plus - 1;
if (J < 0) || (s <= 0) || (s > J + 1 - k) || (k < 0) || (k > J)
    rval = 0;
elseif k == 0
    if s == J+1
        rval = 1;
    else
        rval = 0;
    end
elseif k == 1
    rval = 2;
else
    rval = nchoosek(J-s, k-1); %accurate to 15 digits
    for L = s+k-1 : J
        rval = rval + f_recursive(k-1, L, s);
    end
end
```

We found that for even moderate values of $J$, the calculations effectively froze the computer for long periods. To understand this, we analyzed the functional dependencies, and by enumerating them discovered not only how often the recursive function was called, but the relation of the recursive sequence $f_k$ to the expression seen in Eq. 2.2. The function $f_k(J + 1, s)$ depends directly on $f_{k-1}(J, s)$, ..., $f_{k-1}(s + k - 1, s)$. These in turn depend on the other functions shown in Fig. 2.7.

To calculate how often $f_k(J + 1, s)$ calls any function in Fig. 2.7 we begin with the function $f_{k-N}(J + 1 - (M + N - 1), s)$. This function is called once by $f_{k-(N-1)}(J + 1 - (M + N - 2), s)$, ..., $f_{k-(N-1)}(J + 1 - (N - 1), s)$. These functions are in turn called by other functions so that, for example, $f_{k-N}(J + 1 - (M + N - 1), s)$ is called twice by the function $f_{k-(N-2)}(J + 1 - (M + N - 4))$ once through $f_{k-(N-1)}(J + 1 - (N + M - 3))$ and once through $f_{k-(N-1)}(J + 1 - (N + M - 2))$. The number of calls to $f_{k-N}(J + 1 - (M + N - 1), s)$ by each contributing function in Fig. 2.7 is shown in Fig. 2.8. The emergent pattern is that of a Pascal’s triangle. This is confirmed by noting that since each $f_{k-1}$ in the recursion relation Eq. 2.11 is called once, the number of contributions
\[ f_k(J + 1, s) \]

\[
\begin{array}{cccc}
  f_{k-1}(J, s) & f_{k-1}(J - 1, s) & \cdots & f_{k-1}(s + k - 1, s) \\
  f_{k-2}(J - 1, s) & f_{k-2}(J - 2, s) & \cdots & f_{k-2}(s + k - 2, s) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1(J - k + 2, s) & f_1(J - k + 1, s) & \cdots & f_1(s, s) \\
\end{array}
\]

Fig. 2.7. Functional dependencies of \( f_k(J + 1, s) \). Each function depends on all those below and not to the left of itself.

of \( f_{k-N}(J + 1 - (M + N - 1), s) \) to any other \( f_{k-i}(J + 1 - (i + j - 1), s) \) is simply the number of functional paths between them.

In particular, the function \( f_{k-N}(J + 1 - (M + N - 1), s) \) is called by \( f_k(J + 1, s) \) a total of \( \binom{N+M-2}{M-1} \) times. This means that a single call to \( f_k(J + 1, s) \) results in a total of

\[
\sum_{i=1}^{k-1} \sum_{j=1}^{(J+1-(s+k-1))} \binom{i+j-2}{j-1} = \sum_{i=1}^{k-1} \binom{J+1-s-k+i}{i} = \binom{J+1-s}{k-1} - 1 \quad (2.12)
\]

additional calls to the function, growing like \( J^k \) for fixed \( s \). So, for example, calling \( f_7(30,1) \) results in a total of 475,020 function calls, while \( f_7(100,1) \) results in over a billion. This explains our trouble trying to process values of \( f_k(J + 1, s) \) using our original recursion function.

We can also make use of our alternate derivation of \( f_k(J + 1, s) \) to simplify an awkward double sum. Each time \( f_{k-N}(J + 1 - (M + N - 1), s) \) is called, it adds \( \binom{J+1-M-N-s}{k-1-N} \) (or 2 for \( N = k - 1 \)). Then

\[
f_k(J + 1, s) = \binom{J-s}{k-1} + \sum_{j=1}^{(J+1-(s+k-1))} 2 \binom{k+j-3}{j-1} \\
+ \sum_{i=1}^{k-2} \sum_{j=1}^{(J+1-(s+k-1))} \binom{i+j-2}{j-1} \binom{J+1-i-j-s}{k-1-i} \quad (2.13)
\]

All of this allows us to calculate the sum (for \( k \geq 3 \)) of

\[
\sum_{i=1}^{k-2} \sum_{j=1}^{(J+1-(s+k-1))} \binom{i+j-2}{j-1} \binom{J+1-i-j-s}{k-1-i} = k \binom{J-s}{k-1} - \sum_{j=1}^{(J+1-(s+k-1))} 2 \binom{k+j-3}{j-1} \quad (2.14)
\]
Fig. 2.8. Here we represent the number of times the function $f_{k-N}(J+1-(M+N-1), s)$ is called by the function $f_{k-i}(J+1-(i+j-1))$. These represent the number of times $f_{k-N}(J+1-(M+N-1), s)$ contributes to each function contributing to $f_k(J+1, s)$. 

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Chapter 3

Fragmentation: modeling a continuum rod

3.1 Introduction

We will now consider the case of fragmentation of a rod in a single event in which the rod may fracture at any point along its length according to a nonhomogenous Poisson process. Specifically, we assume that a unit rod breaks in such a manner that the probability of having one or more breaks in the interval \((x_0, x_0 + s)\) is

\[
1 - \exp \left( - \int_{x_0}^{x_0+s} \lambda(q) \, dq \right)
\]

i.e. with the probability density

\[
\lambda(x_0 + s) \exp \left( - \int_{x_0}^{x_0+s} \lambda(q) \, dq \right).
\]

Here \(\lambda\) is the Poisson process parameter [80]. Breaking probabilities in disjoint intervals are assumed to be independent.

The quantity of interest in this case will be \(n(\ell)\) – the density function for the expected number of breaks per unit length. Our approach to define \(n(\ell)\) will be to calculate \(N(\ell + \delta) - N(\ell)\), the expected number of fragments of length ranging from \(\ell\) to \(\ell + \delta\) (we will assume that \(\delta << \ell, 1 - \ell\)), and then take the limit as \(\delta \to 0\)

\[
n(\ell) = \lim_{\delta \to 0} \frac{N(\ell + \delta) - N(\ell)}{\delta}
\]

3.2 Breaking according to a nonhomogenous Poisson process

First note that the left end of a fragment of length \((\ell, \ell + \delta)\) must have its left edge in the interval \((0, 1 - \ell)\). We will partition this into sections of length \(\delta\), as shown in Fig. 3.1. Because we do not require that \((1 - \ell)\) is an integer multiple of \(\delta\), there will be \(M\) such sections, where \(M\) is the integer part of \((1 - \ell)/\delta\), and possibly an extra section \((M\delta, 1 - \ell)\) remaining on the right. With probability one, the left end of any interior fragment lies in one of these \(M\) (or \(M + 1\)) sections, and there can only be one such fragment originating in any of these sections. Then for the \(j\)th segment, if \(P(x_0, \ell; \delta)\) is the probability of having a fragment of length \((\ell, \ell + \delta)\) with its left edge in \((x_0, x_0 + \delta)\), then

\[
N(\ell + \delta) - N(\ell) = P_L + P_R + P_E + \sum_{j=0}^{M-1} P(j\delta, \ell; \delta)
\]
where $P_L$ and $P_R$ are the probabilities that the fragments from the left and right ends of the rod are of length $(\ell, \ell + \delta)$ and $P_E = P(M\delta, \ell; 1 - \ell - M\delta)$ is the probability of having such a fragment begin in the extra section.

From (3.2) we can calculate

$$P_L = \exp \left( - \int_0^\ell \lambda(t) \, dt \right) \left( 1 - \exp \left( - \int_\ell^{\ell+\delta} \lambda(t) \, dt \right) \right) \quad (3.5)$$

and

$$P_R = \exp \left( - \int_{1-\ell}^1 \lambda(t) \, dt \right) \left( 1 - \exp \left( - \int_{1-\ell-\delta}^{1-\ell} \lambda(t) \, dt \right) \right). \quad (3.6)$$

We will make use of the limits

$$\lim_{\delta \to 0} \frac{1}{\delta} P_L = \lambda(\ell) \exp \left( - \int_0^\ell \lambda(t) \, dt \right) \quad (3.7)$$

and

$$\lim_{\delta \to 0} \frac{1}{\delta} P_R = \lambda(1-\ell) \exp \left( - \int_{1-\ell}^1 \lambda(t) \, dt \right). \quad (3.8)$$

To calculate $P(x_0, \ell; \delta)$ note that the probability of the fragment having its left edge in $(x_0 + s, x_0 + s + dx)$ is $\lambda(x_0 + s)dx + o(dx)$. Similarly, the probability of having the right edge of the fragment in $(x_0 + s + \ell + w, x_0 + s + \ell + w + dx)$ is $\lambda(x_0 + s + \ell + w)dx + o(dx)$. Also, the probability of having the intermediate section unbroken is

$$\exp \left( - \int_{x_0+s}^{x_0+s+\ell+w} \lambda(t) \, dt \right).$$

With $0 < s < \delta$ and $0 < w < \delta$, this describes (within $dx$) all possible fragments of length $(\ell, \ell + \delta)$ originating in $(x_0, x_0 + \delta)$. Then the probability density for having a fragment with ends at $x_0 + s$ and $x_0 + s + \ell + w$ is

$$\lambda(x_0 + s)\lambda(x_0 + s + \ell + w) \exp \left( - \int_{x_0+s}^{x_0+s+\ell+w} \lambda(t) \, dt \right) \quad (3.9)$$
Fig. 3.2. A fragment with left edge in \((x_0 + s, x_0 + s + dx)\) and right edge in \((x_0 + s + \ell + w, x_0 + s + \ell + w + dx)\).

and

\[
P(x_0, \ell; \delta) = \int_0^\delta \int_0^\delta \lambda(x_0 + s)\lambda(x_0 + s + \ell + w) \exp \left( - \int_{x_0 + s}^{x_0 + s + \ell + w} \lambda(t) \, dt \right) \, dw \, ds. \tag{3.10}
\]

We will be interested in the quantity \(\lim_{\delta \to 0^+} P(x_0, \ell, \delta)\) which can be easily obtained by noting that the domain of integration in Eq. 3.10 is a square, so that

\[
\lim_{\delta \to 0^+} \frac{P(x_0, \ell; \delta)}{\delta^2} = \lambda(x_0)\lambda(x_0 + \ell) \exp \left( - \int_{x_0}^{x_0 + \ell} \lambda(t) \, dt \right) . \tag{3.11}
\]

Note that this is not applicable when \(x_0 \in (1 - \ell - 2\delta, 1 - \ell)\) (as it envisions breaks past the end of the rod), but the error from all affected terms is \(o(\delta)\). Moreover \(\lim_{\delta \to 0^+} \frac{P_L}{\delta} = 0\).

We can now derive the fragment distribution in Eq. 3.12:

\[
n(\ell) = \lim_{\delta \to 0^+} \frac{N(\ell + \delta) - N(\ell)}{\delta} = \frac{1}{J} \sum_{i=0}^{M-2} P(i\delta, \ell; \delta) + \frac{P_L}{\delta} + \frac{P_R}{\delta} \\
= \frac{1}{J} \sum_{i=0}^{M-2} P(i\delta, \ell; \delta) + \frac{P_L}{\delta} + \frac{P_R}{\delta} \\
= \int_0^{1-\ell} \lambda(s)\lambda(s + \ell) \exp \left( - \int_{s}^{s + \ell} \lambda(t) \, dt \right) \, ds \\
+ \lambda(\ell) \exp \left( - \int_0^\ell \lambda(t) \, dt \right) + \lambda(1 - \ell) \exp \left( - \int_1^{1-\ell} \lambda(t) \, dt \right) \tag{3.12}
\]

3.3 Relation to the segmented rod

We now have expressions for the expected number of fragments from a segmented rod \(E_{J+1}(s)\) (Eq. 2.5) and the density of those expected from a continuum rod \(n(\ell)\) (Eq. 3.12). In this section we will compare the two quantities. We first fix \(\lambda(x) = C\) so that the continuum rod breaks homogeneously as does the segmented rod. We also need to specify how the probability of breaking at a joint relates to the probability density of breaks in the continuum rod. We will do this by setting the expected number of breaks in each to be \(C\), and calculating \(p = C/J\) for the discrete rod.

Because \(\ell \in (0, 1)\) and \(s\) is an integer number of segments, we also need to provide an appropriate map between \((0, 1)\) and \(\mathbb{Z}\). We define \(s(\ell, J)\) to be the integer part of \(\ell\),
such that \( s(\ell, J) \leq (J + 1)\ell \leq s(\ell, J) + 1 \). Note that
\[
\frac{s(\ell, J)}{J} = \frac{s(\ell, J)}{J + 1} \cdot \frac{J + 1}{J} \to \ell \tag{3.13}
\]
as \( J \to \infty \). Also, the difference in the lengths of fragments contemplated by \( n(\ell) \) and \( E_{J+1}(s(\ell, J)) \) is less than \((J + 1)^{-1}\). Thus it is natural to interpret fragments of \( s(\ell, J) \) segments to represent all fragments of lengths \( \left[ \frac{s(\ell, J)}{J+1}, \frac{s(\ell, J) + 1}{J+1} \right) \), and
\[
E_{J+1}(s(\ell, J)) = \int_{s(\ell, J)/J}^{s(\ell, J)/J + 1} n(t) \, dt. \tag{3.14}
\]
In this way we can calculate \( n(\ell) \) as \( J \to \infty \) to be
\[
(J + 1)E_{J+1}(s(\ell, J)) = (J + 1) \int_{s(\ell, J)/J}^{s(\ell, J)/J + 1} n(t) \, dt \to n(\ell). \tag{3.15}
\]
We now calculate
\[
(J + 1)E_{J+1}(s(\ell, J)) = p(1 - p)^{(s(\ell, J) - 1)} \cdot [2 + (J - s(\ell, J))p] = \left( (1 - p)^{-\frac{1}{J}} \right)^{ps(\ell, J)} (1 - p)^{-1}(J + 1)p \cdot [2 + Jp - np] = \left( (1 - p)^{-\frac{1}{J}} \right)^{-C\frac{s(\ell, J)}{J}} \frac{C + p}{(1 - p)} \cdot [2 + C - C\frac{s(\ell, J)}{J}], \tag{3.16}
\]
so that
\[
n(\ell) = \lim_{J \to \infty} (J + 1)E_{J+1}(s(\ell, J)) = C(2 + C - C\ell)e^{-C\ell} \tag{3.17}
\]
for \( \ell \in (0, 1) \). This is precisely Eq. 3.12 with \( \lambda(x) = C \).

There is a small discrepancy in Eq. 3.17 which is easily resolvable at this point: the expected number of fragments should be \( C + 1 \). If we integrate by parts we find
\[
\int_0^1 n(\ell) \, d\ell = -(2 + C - C\ell)e^{-C\ell} \bigg|_0^1 - C \int_0^1 e^{-C\ell} \, d\ell = C + 1 - e^{-C}. \tag{3.18}
\]
The discrepancy is resolved by including the pieces which remain \textit{unbroken}, for which in the discrete case \( E_{J+1}(J + 1) = (1 - p)^J \). Since \( (1 - p)^J = \left( (1 - p)^{-\frac{1}{J}} \right)^{-C} \to e^{-C} \) as \( J \to \infty \), the expected number of pieces of length 1 is \( e^{-C} \). The correct distribution is
\[
n(\ell) = C(2 + C - C\ell)e^{-C\ell} + \delta(\ell, 1)e^{-C}, \quad \ell \in (0, 1] \tag{3.19}
\]
where \( \delta \) is the Kronecker delta for the contribution of the unbroken rods.

Interestingly, there is a good deal of existing literature on fragmentation by a homogenous Poisson process \[35, 63, 59\]. In each case, however, the derived density is given as \( n(\ell) = C^2e^{-C\ell} \). The derivation starts from a probability density \( p(\ell) \) for the fragment size calculated by considering a fragment of length between \( \ell \) and \( \ell + \Delta\ell \). For
this to exist, there must be an unbroken section of length \( \ell \) starting from the left tip of the fragment, which occurs with probability \( P(0; C\ell) = e^{-C\ell} \), followed by one or more breaks within \( \Delta \ell \) of the right tip, which occurs with probability \( 1 - P(0; C\Delta \ell) = 1 - e^{-C\Delta \ell} \). This implies that

\[
\int_{\ell}^{\ell+\Delta \ell} p(s) \, ds = e^{-C\ell} \left( 1 - e^{-C\Delta \ell} \right).
\]  

(3.20)

Dividing both sides of this equation by \( \Delta \ell \) and letting \( \Delta \ell \to 0 \) leads to \( p(\ell) = Ce^{-C\ell} \). For this distribution, the expected number of breaks is \( C \), yielding \( C + 1 \) expected fragments. However the unit rod is assumed to be a section of an infinite rod, so the end fragments are counted only as half fragments. In this way the expected number of fragments is also \( C \).

The density of the expected number of fragments of length \( \ell \) is obtained in this approach by multiplying the expected number of fragments \( C \) by the probability \( p(\ell) \) for fragments of length \( \ell \)

\[
n(\ell) = C^2 e^{-C\ell}.
\]  

(3.21)

This formula would be correct if the sizes of fragments were independent of the number of fragments produced in an event, but in fact it is not at all independent. For example, consider an event with only two fragments; it is certain that exactly one of them is of length less than \( \frac{1}{2} \) (since the probability of a break in the exact center is zero). This would mean that \( \int_0^{1/2} p(\ell) \, d\ell = 1/2 \). However the actual distribution gives

\[
\int_0^{\frac{1}{2}} p(\ell) \, d\ell = 1 - e^{-C/2},
\]  

(3.22)

which depends on the value of \( C \). The contradiction indicates that \( p(\ell) \) should depend on the number of fragments; so while Eq. 3.21 appears to be commonly accepted, it is incorrect. Concerns over this method of derivation have been noted [3, 34], but left unaddressed. Our derivation of the correct version of Eq. 3.21 proceeds along different lines, and avoids this issue altogether.

### 3.4 Properties of the fragment distribution \( n(\ell) \)

#### 3.4.1 Fragment distributions for several basic \( \lambda(x) \)

We will discuss here a few examples of how patterns in \( \lambda(x) \) produce different distributions \( n(\ell) \). Let \( \lambda(x) = Af(x) \) and consider \( n(\ell) \) for very small values of \( A \). In this case, the first integral in Eq. 3.12 is \( O(A^2) \) compared to \( O(A) \) for the last two terms. The dominant terms

\[
Af(\ell) \exp \left( \int_0^{1-\ell} Af(x) \, dx \right) + Af(1 - \ell) \exp \left( \int_{1-\ell}^1 Af(x) \, dx \right)
\]  

(3.23)

are symmetric about \( \ell = \frac{1}{2} \), whether or not \( f \) has any such symmetry, as shown in Fig. 3.3. In the special case where \( f(x) \) is itself symmetric about \( x = \frac{1}{2} \), Eq. 3.23 reduces to \( 2Af(\ell) \), see Fig. 3.3(a-b). It is interesting to note in Fig. 3.3(b) that \( n(0.5) \approx 0.\)
This is because the remaining terms are end fragments, and breaks near the center are unlikely. Since this is the only way to produce end fragments of length $1/2$, $n(\ell)$ is close to zero. Contrast this to Fig. 3.3(d), where fragments of any length can be produced from at least one of the ends. For example a fragment of length $3/4$ cannot be produced from the left-end fragment, as $\lambda(3/4) = 0$. It can however be produced from the right-end fragment.

Fig. 3.3. Examples of fragment distributions $n(\ell)$ produced by simple parabolic functions: a)-b) $\lambda(x) = \frac{1}{2}(x - \frac{1}{2})^2$; c)-d) $\lambda(x) = \frac{1}{2}(x - \frac{3}{4})^2$. The distributions $n(\ell)$ are approximately symmetric about $\ell = 0.5$.

These arguments made for small $A$ may not be realized in practice. As $A$ increases, symmetry is quickly lost as the two dominant terms discussed above lose their significance. The overall mass of the graph shifts to the left as the increased likelihood of breaks creates a greater number of shorter pieces, see Figures 3.4 and 3.5.

Of particular relevance to the fragmentation data obtained previously from dynamic buckling in the Pritchard Lab [30] is the effect illustrated in Fig. 3.5. The fragment distributions shown are obtained from a Gaussian breaking probability with a sharp maximum at the center of the rod:

$$\lambda(x) = A \exp \left( -100 \left( x - \frac{1}{2} \right)^2 \right)$$

(3.24)
Fig. 3.4. Fragment distributions $n(\ell)$ produced by $\lambda(x) = A\sin^2(5\pi x)$ for $A = 1, 5, 10, 15$ (lowest to highest). Inset in the upper right hand corner is a log-log plot for $A = 1, 10, 10^2, 10^3$ (lowest to highest).

Not surprisingly, this leads to a significant number of fragments of length $\ell \approx 1/2$. However as the overall probability of breaking increases (increasing $A$), it becomes more likely to have multiple breaks near the center. Multiple breaks near the regions of high curvature were clearly evident in the images shown in Fig. 1 of [30]. The resulting $n(\ell)$ shown in Fig. 3.5 shows that this results in a downward shift in the fragment peak at $1/2$. The inset to the figure traces this downward shift in the location of the maximum as a function of $A$; this shift was also seen experimentally [30].

Another aspect of our formalism is the delineation of contributions to the fragment distribution $n(\ell)$ from the end pieces and the central pieces. This is illustrated in Fig. 3.6 for $\lambda(x) = 1 + \sin^2(4\pi x)$, which shows the contributions to the total distribution from end fragments (dashed line) and center fragments (dashed and dotted line).

Note that for this figure $\lambda(x)$ would have four peaks along the length of the rod, which produces an $n(\ell)$ with four peaks (Fig. 3.6). However, in Fig. 3.5 there are two peaks visible in the fragment distribution $n(\ell)$, generated by a $\lambda(x)$ consisting of a single peak. As our original concern was with fragment distribution data exhibiting two distinct peaks (see Fig. 1.1), it is natural to inquire in general how peaks in $\lambda(x)$ are expressed in $n(\ell)$. We answer this question when $\lambda(x)$ takes the form of $N$ individual pulses in $(0, 1)$ and is zero elsewhere.

The ends of a fragment produced in such a case must either be located within one of the pulses or coincide with one of the ends of the rods. Fragments produced by breaks at different pulses (or that go to the end of the rod) will produce a peak in $n(\ell)$ at about
Fig. 3.5. The distributions $n(\ell)$ produced by $\lambda(x) = A \exp(-100(x - \frac{1}{2})^2)$, shown for $A = 1, 5, 10, 15$ (with increasing height). The inset shows the shift in the peak location with increasing $A$ (linear-log scale).
the distance between the two. By counting the pairs of possible fragment-end locations, we find that these produce at most \((\binom{N+2}{2} - 1)\) possible peaks in \(n(\ell)\); we exclude the case of the unbroken rod, which is not shown in plots of \(n(\ell)\). Fragments produced by multiple breaks within one pulse (assuming it is relatively narrow) will show up as a peak in \(n\) near \(\ell = 0\). Including this peak at \(\ell = 0\) gives a total of \((\binom{N+2}{2})\) possible peaks in \(n(\ell)\).

Not all of these peaks will necessarily be seen: some may be relatively small. Alternately, when pairs of pulses in \(\lambda(x)\) have similar spacing, the peaks in \(n(\ell)\) may occur at about the same value of \(\ell\) and produce a single wider or higher peak. This can be seen in Fig. 3.7 where \(\lambda(x)\) consists of three pulses centered on \(x = 0.1\), \(x = 0.2\) and \(x = 0.3\). This means the distance from the left end to the first pulse is the same as the distance from the first pulse to the second, and the second to the third. This combines three peaks into one in \(n(\ell)\). Also the distance from the left end to the second pulse is the same as the distance from the first pulse to the third, combining these two peaks in \(n(\ell)\). Hence we see only seven peaks in \(n(\ell)\) instead of the full complement of \((\binom{5}{2}) = 10\) peaks. In contrast, we see in Fig. 3.8 a \(\lambda(x)\) with three staggered pulses at incommensurate distances producing all 10 possible peaks in \(n(\ell)\).

### 3.4.2 Distributions with Power Law Scaling

Many experimental studies (see Chapter 1) have reported a power law or scaling region of \(n(\ell)\) over several decades, especially in the regime where \(\ell\) is small and the total number of expected fragments is large [15, 42]. In terms of our approach, we ask if
Fig. 3.7. Three evenly spaced pulses in $\lambda(x)$ produce fewer than 10 peaks in $n(\ell)$, as calculated by Eq. 3.12.

There is a distribution $\lambda(x)$ which would lead to a power law when $\ell$ is small in Eq. 3.12. This need not be a unique solution. In fact the inverse problem – finding $\lambda(x)$ from a given $n(\ell)$ – does not generally have a unique solution; for example, $\lambda(x)$ and $\lambda(1 - x)$ produce the same number density $n(\ell)$.

Although we developed Eq. 3.12 for integrable functions ($\lambda$ itself is not a probability density, so it may be very large as a whole, but it must be integrable), the transformation itself works for many functions where $\int_0^1 \lambda(x) = \infty$. Because this quantity appears only as a negative power for an exponential, many functions which cause the integral to diverge can still be used in the transformation $\lambda \rightarrow n$. For example, consider $\lambda(x) = x^{-1}$. We can now ignore the term

$$\lambda(\ell) \exp \left( - \int_0^\ell \lambda(t) \, dt \right)$$

since the integral diverges. That is, the left edge breaks into infinitesimal pieces so that we cannot find a left-end fragment.

For the last term in Eq. 3.12, we have

$$\frac{1}{1 - \ell} \exp \left( - \int_{1-\ell}^1 1/t \, dt \right) = \frac{1}{1 - \ell} \ell^{\ln(1-\ell)} = 1.$$
Fig. 3.8. Three distinct but incommensurately spaced pulses in $\lambda(x)$ can produce 10 peaks in $n(\ell)$, as calculated by Eq. 3.12.

Similarly the first integrand in Eq. 3.12 simplifies to

$$\frac{1}{s(s + \ell)} \exp \left( \ln \left( \frac{s}{s + \ell} \right) \right) = \frac{1}{(s + \ell)^2}$$

so that we obtain the power law $n(\ell) = 1 + \int_0^{1-\ell} \frac{1}{(s+\ell)^2} \, ds = \frac{1}{\ell}$. Interestingly, $1/\ell$ is a fixed point for the transformation $\lambda(x) \rightarrow n(\ell)$.

Although we have analytically obtained a power law function $n(\ell)$, the calculation seems to rely on specific aspects of the function $1/x$. In fact the transform becomes much more difficult to solve analytically even with a simple $\lambda(x) = Ax^\beta$ when either $A$ or $\beta$ is not one. We would like to know if these $\lambda$ also produce scaling. Numerical calculations can offer some insight. Fig. 3.9a shows $n(\ell)$ for increasing values of $A$. The graphs appear to show some scaling for small $\ell$ parallel to the graph of $\frac{1}{\ell}$. The point where the scaling behavior ends decreases as $A$ increases.

We also show some examples for varying $\beta$. In Fig. 3.9b, we decrease $\beta$ from 1 to $-1$ by steps of 0.5. The graph flattens out for small $\ell$ as we cross the threshold from $\beta > 0$ to $\beta < 0$ into the regime where there are a finite number of fragments expected. In this regime we can calculate

$$\lim_{\ell \to 0} n(\ell) = \int_0^1 \lambda^2(s) \, ds.$$
In Fig. 3.10 we increase $\beta$ from 1 to 3 in increments of 0.5. The slopes of the tails of $n(\ell)$ steepen as we increase $\beta$. Observing that $\int_0^1 \ell n(\ell) \, d\ell = 1$, if $n(\ell) \propto \ell^{-\beta}$ ($\beta > 0$) for small $\ell$, then we must have $\beta < 2$ for the previous integral to converge. So the slopes in Fig. 3.10 presumably have a limit of $-2$, although we have not attempted to prove this analytically. In an actual experiment, there is a lower bound on the lengths of producible fragments, so that a fragment size distribution may appear to have a power law with $\beta > 2$ over a number of decades. However, many experiments fall within a range that we can reproduce (e.g. $\beta \approx 1.5$ in [15]).

The near ubiquity of experimental observations of power laws would suggest a more general derivation than our singularity-dependent calculation; there may be some experiments reporting power laws where some areas of the rod almost always fragment, and produce many small fragments (i.e. where $\lambda(x)$ is approximately singular). However our assumption of a breaking profile $\lambda(x)$ for a rod breaking predominantly at one end seems too restrictive, so we examined a symmetric class of functions of the form $\lambda(x) = A|x-0.5|^{-\beta}$. This class of functions describes a rod that breaks throughout its length, but is much more likely to break near the center. This seems to us to be a much more plausible scenario. There is some evidence that rods broken in experiments break uniformly along their lengths, but this is mentioned only briefly in one of the papers we have found [42]. As a matter of practicality, we capped the maximum value of $\lambda(x)$ (which also allows for the possibility of fragments of lengths greater than 0.5), as can be seen in Fig. 3.11. We fit parameters $A$ and $\beta$ to the curve $2\ell^{-1.5}$. The resulting functions $\lambda(x) = 5.4926|x-0.5|^{-1.9564}$ and $n(\ell)$ are shown in Fig. 3.11.
Fig. 3.10. Fragment distribution $n(\ell)$ for $\lambda(x) = x^{-\beta}$, $\beta = 1, 1.5, 2, 3$.

Fig. 3.11. $\lambda(x) = 5.4926|x - 0.5|^{-1.9564}$ (above) and the resulting $n(\ell)$ plotted alongside $2\ell^{-1.5}$ (dotted line).
Chapter 4

Fragmentation: experiment and model comparison

4.1 Introduction

As the main impetus for our study was the observation of peaks in the fragment distribution of brittle rods [30], we return to this experiment and the proposed theory that the source of breaking probability is the bending moment, proportional to the local curvature $\kappa$ of the buckled rod [4]. Following this theory, we assume that the breaks in different intervals of the pasta occur independently of one another, but proportional to some function $\lambda(x)$ (envisioned as a function of curvature). Note that because we will interpret this probability in the sense of chapter 3, $\lambda(x)$ can be thought of as either a probability density for breaking, or as the density of the expected number of breaks per unit length.* The results of chapter 3 can then be used to relate $\lambda$ to the resulting fragment distribution.

4.2 Experimental Comparisons

4.2.1 Fitting the previously measured distribution

We first consider the data previously collected in the Pritchard Lab [30] by breaking San Giorgio #8 spaghetti. The exact form of the buckled pasta is not known. However, the buckled pasta had a wavelength of about 0.31 (normalized by the length of the pasta – 22.5 cm with a wavelength of 7 cm). The amplitudes of the peaks decreased along the length of the pasta, often vanishing after the first few. Based on this, we conjecture that the displacement of the buckled pasta is of the form $Ae^{-bx}\sin(6\pi x)$. The curvature of this we will call $\kappa(x; A, b)$. The simplest possibility is that $\lambda(x)$ is proportional to $\kappa(x; A, b)$.

Unfortunately, this is not simply $\lambda(x) = c\kappa(x; A, b)$, since the argument $x$ in $\lambda$ represents length along the curve, and here the argument $x$ in $\kappa$ represents a vertical distance $y$ from the top of the pasta. To be clear, we will write $\lambda(x) = c\kappa(y(x; A, b); A, b)$. We used MATLAB’s \texttt{lsqnonlin} function to optimize the parameters $A, b, c$ in fitting $n(\ell)$ to the experimental data for brittle fragmentation reported in [30]. The results are shown in Fig. 4.1 with $A = 0.48$, $b = 18.52$ and $c = 4.35$. The discrepancy between curvature by distance $y$ as opposed to arc length $x$ is visible in Fig. 4.1a,b where the curvatures appear to be shifted to the left of the actual peaks in the rod. A similar effect appears in the experimental data presented in Fig. 4.5 (where the x-axis in (a) is arc length, and

*That is, we assume the pasta breaks in such a way that the probability of having exactly one break in the interval $(x, x + \delta x)$ is $\lambda(x)\delta x + o(x)$ and the probability of having more than one break is $o(x)$.
the x-axis in (b) is distance). This fit does show some sensitivity to initial parameters, and should not be assumed to be identical to the actual probability density of breaking along the rod; however, it is a plausible approximation.

\[ \sum_m A_m \cos^2 (B_m(x - C_m)) I_{[-\pi/2, \pi/2]}(B_m(x - C_m)) \]  

where \( I \) is the indicator or characteristic function on the interval \([-\pi/2, \pi/2]\), and \( C_m \) is the center of the pulse. We expected a contribution from each of the six peaks in the buckled pasta of the experiment, and indeed found that a good fit could be obtained with only six pulses. After deciding on this functional form and free parameters for \( \lambda(x) \), we again optimized these parameters in fitting \( n(\ell) \). An example of this is shown in Fig. 4.2, with the parameters listed in Table 4.1. This ‘fit’ shows possible breaks near both ends.

For comparison we analyzed a part of the data set shown in Fig. 1.1 (because the length of the spaghetti varied in some of the experiments, only a part of this data set was used to ensure correct normalization of lengths). In these experiments, the shorter buckling wavelength allows for nine peaks. We therefore used nine pulses in fitting this data. In the fit shown in Fig. 4.3, however, only five of these were identified as significant.

Fig. 4.1. Comparison of the probability model (lines) and experimental data (represented by open circles): (a) the displacement \( ae^{-bx} \sin(6\pi x) \); (b) \( \lambda(x) \) as a multiple of curvature, arc length dependent; (c),(d) the calculated fragment probability function \( n(\ell) \), with data points for San Giorgio #8 spaghetti, from [30]
Fig. 4.2. Comparison of the probability model (lines) and experimental data (represented by open circles): (a) the input to the model, represented by the breaking probability function $\lambda(x)$ plotted on a log scale; (b) the calculated fragment probability function $n(\ell)$, with data points for San Giorgio #8 spaghetti, from [30]; (c) same plot of $n(\ell)$, shown as log-log.
Fig. 4.3. Comparison of the probability model (lines) and experimental data (represented by open circles): (a) the input to the model, represented by the breaking probability function $\lambda(x)$; (b) the calculated fragment probability function $n(\ell)$, with data points for angel hair spaghetti, from [30]; (c) same plot of $n(\ell)$, shown as log-log.
### Table 4.1. Fitted parameters for the pulses in Eq. 4.1, used for the curves shown in Fig. 4.2

<table>
<thead>
<tr>
<th>( A_m ) (amplitude)</th>
<th>( m = 1 )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>46.19</td>
<td>26.16</td>
<td>1095.20</td>
<td>9.77</td>
<td>1.76</td>
<td>30.77</td>
</tr>
<tr>
<td>( \pi/B_m ) (width)</td>
<td>0.06</td>
<td>0.07</td>
<td>0.07</td>
<td>0.24</td>
<td>0.75</td>
<td>0.03</td>
</tr>
<tr>
<td>( C_m ) (center)</td>
<td>0.05</td>
<td>0.11</td>
<td>0.22</td>
<td>0.37</td>
<td>0.60</td>
<td>0.87</td>
</tr>
</tbody>
</table>

For both experimental data sets we see a similar general pattern in \( \lambda \): several pulses clustered near the impacted end \((x = 0)\), with a low, wide pulse further down, and a narrow pulse near \( x = 0.9 \). The differences in the \( \lambda(x) \) shown for the different experiments confirm that it is not possible to determine the form \( \lambda(x) \) takes by merely fitting parameters. To actually measure \( \lambda(x) \) we performed a new experiment in which we measured the positions of breaks as well as the sizes of fragments obtained.

#### 4.2.2 Measuring \( \lambda(x) \) experimentally

In many of the fragmentation problems studied in engineering or geophysics, data consists of only the fragment distributions \( n(\ell) \). However, the experiments described in [30] allow in principle for the measurement of both the probability of breaking along the rod and the resultant fragment distribution. Here we report a preliminary measurement \(^\dagger\) of \( \lambda(x) \) for brittle rods in a similar experimental setup, from which we make a comparison between the fragment distribution \( n(\ell) \) as calculated from Eq. 3.12 and the directly measured distribution.

The experiments were performed with spaghetti (San Giorgio angel hair) of length \( L = 17 \pm 0.1 \) cm diameter \( d \approx 0.85 \) mm, kept upright by holding one end in an aluminum base. A cylindrical brass weight of 51 g was dropped onto one end of the pasta, impacting with a velocity of about 4 m/s; an example of the resulting fragmentation is shown in Fig. 4.4. For our experimental parameters, the predicted buckling wavelength is 3.6 cm [30], while observation of the images indicate a wavelength of 3.2 cm (see Fig. 4.5b).

The experiments were filmed at 11,111 fps using a Phantom v5.0 high-speed digital video camera. In each image the pasta is directly visible, as is a mirror image (see Fig. 4.4). The angle of the mirror was controlled so that on the line where the pasta sat the images were orthogonal. This was accomplished by sighting along pins set through a piece of graph paper into a block of wood that was milled flat. Only about 8 cm of the spaghetti is visible in all of the photos, so data collection was restricted to this range. The independence assumption of our model allows us to treat the breaks in this section of the rod as if it was the whole rod: we simply normalize our data to the length of the section (8 cm) instead of by the total length \( L \).

\(^\dagger\) A major limiting factor has been the frame size captured by our camera at higher frame rates. Advances in new camera technology are already sufficient to allow for more detailed measurements. The Phantom V710 for example is advertised with a maximum frame rate of 1,400,000 fps.
We used images of the pasta just after it broke to record the locations of breaks along the length of the spaghetti. This provided us with experimental values of both $\lambda(x)$ and $n(\ell)$ (see Figures 4.5 and 4.7). Note that the measured $n(\ell)$ is somewhat different than what would be measured by picking up the pieces afterwards - it does not include any secondary breaks due to fragment collisions after the primary event (including the brass re-impacting the unbroken end of the pasta, or curvature dynamics leading to further breaking [4]).

We compared the measured values of $n(\ell)$ with values calculated from the experimental values of $\lambda(x)$ to see if the relationship is well described by Eq. 3.12. Moreover, we expanded this comparison to all possible values for $\lambda(x)$ that lie within the 95% confidence interval of the experimental values. Unfortunately, even the closest fit - shown in Figures 4.6 and 4.7 - has obvious differences with measured values of $n(\ell)$. The differences do not necessarily indicate that the hypotheses leading to Eq. 3.12 are incorrect: There are many sources of error, including low statistical numbers and limited optical resolution. What is important is that the fragment distributions compared in Fig. 4.7 come from the same set of measurements, and that - with the exception of an apparent experimental peak at $\ell \simeq 0.3$ - the features of the experimentally measured fragment distribution are qualitatively captured by Eq. 3.12.

4.3 Discussion and Conclusions

The mathematical framework developed here provides an explicit answer to the primary hypothesis put forward to explain the non-power law distribution observed in
Fig. 4.5. Observed patterns in the probability of breaking: a) measured probability of breaking $\lambda(x)$ over 175 events (see text); b) several cases of the initial buckling contributing to $\lambda(x)$, where the buckling plane was almost entirely in a single plane of view. The error bars are produced by considering the measurements for $\lambda(x)$ as independent experiments for different values of $x$. 
Fig. 4.6. The open points are the measured distribution for $\lambda(x)$, while the solid points show values within the 95% confidence interval of these measurements that provide a best approximation to measured values of $n(\ell)$.

Fig. 4.7. Calculated and measured values of $n(\ell)$ for the experiment described in Fig. 4.4. The open points are the measured distribution, and the solid points show the calculated $n$ based on values shown in Fig. 4.6. Note that the discretization of the calculated $n$ comes from the discrete measurements of $\lambda(x)$. 
it also goes much further, allowing us to map any 1D probability of breaking along the length of a rod to its resultant fragment distribution. Often in fragmentation studies in engineering or geophysical applications, one does not have access to this original probability of breaking, usually because the fragmentation process itself was not observed - this is typically true for accidents, for which determining the cause of the failure, and therefore the original stress distribution, is often an important goal. While the inverse problem is evidently complicated, it can be done numerically, as we have shown in attempting to fit the data shown in Fig. 1.1. Note however that while the expected total number of fragments is easily obtained in our framework, it remains to connect this quantity to the energy input into the object, via impact or other mechanisms [33].

Our derivation relies on a generalization of the Poisson process, which underlies several complex and accepted models for fragment distributions [65, 3, 29]; however, its utility for describing fragment distributions in one dimension is often dismissed [63, 1] since it is thought to produce only exponential fragment distributions. In modeling our own one dimensional experiment, we have shown the fragment distributions produced by such a Poisson process to be much more rich and complex.
Chapter 5

A sphere sinking in bubbles: introduction

Visual perception is tuned to motion, indeed objects that are motionless on the retina, or even move slowly compared to other objects can be subject to ‘visual disappearance’ [12]. Little wonder then that motion of terrestrial and celestial bodies forms one of the oldest and most basic branches of physics. Modern treatment of the subject is formalized with Newton’s (1642 - 1727) second law

\[ \text{the alteration of motion is ever proportional to the motive force impressed;} \]
\[ \text{and is made in the direction of the right line in which that force is impressed} \] [69]

or, more familiarly the net force on a body \( F \) is related to its acceleration \( a \) by \( F = ma \) or to its momentum \( P \) by \( F = dP/dt \). With this simple formula, the motion of everyday objects can be described. Experience teaches us, however, that the motion of objects can be complex and unexpected. Indeed, the simple form of Newton’s second law belies the complexity of the assorted phenomena that can create these forces.

The simplest experiments observe motion under the influence of gravity using the simplest of shapes - a sphere. One early and famous experiment was one attributed (posthumously by Vivandi) to Galileo (1564 - 1642) in which he dropped spheres of different weights from the leaning tower of Pisa to show that they fell at the same speed and not (as supposed from Aristotelean mechanics) in inverse proportion to their weights [17].

Contemporaries such as Simon Stevin (1548 - 1620) and Jan Cornet de Groot (1554 - 1640) were performing similar experiments. Stevin, for example, records dropping two lead spheres from a height of thirty feet and observing that the sounds of their impacts were simultaneous [87].

Stevin also briefly describes two other factors that can be seen in the gravity-driven motion of experiments dropping spheres in air or water – buoyancy and resistance from the surrounding fluid [87]. The principle of buoyancy has been known, of course, from the time of Archimedes: there is an upward force on an object equal to the weight of the fluid it displaces. The effect of other forces on a sphere moving in a fluid, on the other hand, is complicated by the need to know how the fluid is moving about the sphere.

*Whether this experiment was actually performed has been questioned [18, 17]. Other experiments recorded during Galileo’s lifetime reported somewhat different results – a wooden and iron ball were released from a height. Initially the wooden ball was seen to lead the iron ball, which overtook it and struck faster. This appears to be a combination of air resistance slowing the wooden ball to a greater extent, and muscle fatigue which causes the iron ball to be released more slowly [18].
Our own experiments will follow the grand tradition of sphere dropping by observing the motion of a sphere settling in a bubbly fluid. The effect of bubbles on a sinking sphere depends largely on the relative size of the sphere and bubbles. For example, the effect on a large sphere may be to reduce buoyancy by making the surrounding fluid less dense on the average, causing the sphere to sink more quickly or even to sink when it may have been buoyant in quiescent water. The effect is exacerbated by a bubble “shadow” above the sphere where the fluid is at its normal density.[22]

On the other hand, a small, hydrophobic particle can attach to a bubble and float free of a suspension. This process can be used, for example, in ink recycling. The paper is pulped and put into a dilute suspension where ink particles are separated chemically and mechanically from paper fibers. The suspension is also chemically treated to make ink particles more hydrophobic. After this, bubbles are run through the suspension, carrying ink particles to the top of the fluid where it is removed as a foam or scum (see e.g. [48]).

Our experiments will focus on a middle regime where the bubbles and the sphere are the same size. This interest was originally sparked by Professor Eduardo Ramos of UNAM, with whom we have discussed the use of bubbles for suspending and mixing particles. We are interested in spheres that do not necessarily fall or float in the fluid. We will in fact see a transition from floating to falling based on the size and mass of the sphere and the flow rate of the rising bubbles. Our first experiment will focus mostly on the settling (terminal) velocity of the spheres. Aside from the effect of direct interactions between the bubbles and spheres, we will need to take some account of the fluid-sphere interaction. We devote the remainder of this introduction to a brief review of the equations governing fluid motion and its interaction with a sphere. We will return to these in later chapters while modeling the motion of the spheres.

The motion of many fluids can be described by the incompressible Navier-Stokes equations (even for such fluids as air, which are compressible) where the fluid velocity $\mathbf{u}$ is given by

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla p + \mu \Delta \mathbf{u} \quad (5.1)$$

with the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0. \quad (5.2)$$

Here $p$ is pressure, $\rho$ the fluid density and $\mu$ the viscosity of the fluid (see e.g. [54]).

---

†The author also follows a more personal history of his grandfather Dee H. Barker’s dissertation “The effect of shape and density on the free settling rates of particles at high Reynolds numbers”. This tradition is a little less grand, as Dr. Barker contracted arsenic poisoning (painful but fortunately not fatal) by dropping spheres down smoke-stacks.
The total force on a sphere of mass \( m \) comes from body forces such as gravity and from surface forces acting on the surface \( \partial S \) of the sphere. That is,

\[
F = mg + \int_{\partial S} T \hat{n} \, dS
\]  

(5.3)

where \( \hat{n} \) is the outward unit normal vector to the sphere, and the components \( \sigma_{ij} \) of the stress tensor \( T \) are related to Eq. 5.1 by

\[
\sigma_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]  

(5.4)

(see e.g. [54]).

The forces on the sphere can be rewritten to highlight various effects of the fluid. G. Mougin & J. Magnaudet [67] for instance, detail generalized equations – originally due to Kirchoff – for the motion of a submerged rigid body of volume \( V \). In the lab inertial frame, the center of mass of the body has velocity \( U \) while the rotation rate of the body is \( \Omega \). The equations of motion are framed in body-centered coordinates (coordinates that center on the center of mass of the body and rotate at the rate \( \Omega \).) This allows the components of the inertia tensor, among others, to remain constant in the equations. In these coordinates

\[
(mI + A) \frac{dU}{dt} + \Omega \times ((mI + A)U) = F_\omega + (m - \rho V)g
\]  

(5.5)

and

\[
(J + D) \frac{d\Omega}{dt} + \Omega \times ((J + D)\Omega) + U \times (A\Omega) = \Gamma_\omega.
\]  

(5.6)

The quantities \( I, A, J, D \) are the identity tensor, added mass tensor, inertia tensor and added inertia tensor. The quantities \( F_\omega \) and \( \Gamma_\omega \) represent force and moment about the center of mass due to vorticity in the fluid.[67]

The added mass and rotational inertia represent momentum and angular momentum in the fluid corresponding to the velocity and angular rotation of the body. These quantities are calculated as if the body moved in an unbounded ideal fluid.[67]

For a sphere the equations simplify considerably as \( A = \frac{1}{2} \rho V \), \( J = \frac{2}{5} m r^2 I \) and \( D = 0 \). This allows us to rewrite Eq. 5.5 as

\[
\left( m + \frac{1}{2} \rho V \right) \left( \frac{dU}{dt} + \Omega \times U \right) = F_\omega + (m - \rho V)g.
\]  

(5.7)

Here the quantity \( \frac{dU}{dt} + \Omega \times U \) can be replaced by \( \frac{dU}{dt} \) by fixing the coordinate system in the lab frame rather than allowing it to move and rotate with the sphere [24]. This leaves us with the equation of motion

\[
\left( m + \frac{1}{2} \rho V \right) \frac{dU}{dt} = (m - \rho V)g + F_\omega.
\]  

(5.8)
Some examples of approximations to Eq. 5.1 which follow the form of Eq. 5.8 (see e.g. an overview in [49]) are those derived by Basset, Boussinesq and Oseen for creeping flow (i.e. where \( \nabla p = \mu \Delta u \))

\[
\left( m + \frac{1}{2} \rho \nu \right) \frac{dU}{dt} = (m - \rho \nu) g - 6\pi a \mu U - 6a^2 \sqrt{\pi \mu \rho} \int_0^t \frac{dU/d\tau}{\sqrt{t - \tau}} d\tau \tag{5.9}
\]

and by Maxey and Riley for unsteady creeping flow [49]

\[
\left( m + \frac{1}{2} \rho \nu \right) \frac{dU}{dt} = (m - \rho \nu) g + \frac{3}{2} \rho \nu \frac{du}{dt} + \frac{1}{2} \rho \nu (u \cdot \nabla) u + 6a^2 \sqrt{\pi \mu \rho} \int_0^t \frac{d(u - U)/d\tau}{\sqrt{t - \tau}} d\tau \tag{5.10}
\]

The latter equation has been adapted (Berlemont, Desjoqueres and Gouesbet [49]) to approximate the effect of more general flows as

\[
\left( m + \frac{1}{2} \rho \nu \right) \frac{dU}{dt} = (m - \rho \nu) g + \frac{1}{2} \pi a^2 \rho C_D |u - U|(u - U) + \frac{3}{2} \rho \nu \frac{du}{dt} + \frac{1}{2} \rho \nu (u \cdot \nabla) u + 6a^2 \sqrt{\pi \mu \rho} \int_0^t \frac{d(u - U)/d\tau}{\sqrt{t - \tau}} d\tau \tag{5.11}
\]

using the form drag approximation \( \frac{1}{2} \pi a^2 \rho C_D |u - U|(u - U) \) that will be described later in this section.

In 1851, G. G. Stokes established one case in which solutions for the effect of a fluid on a sphere of radius \( a \) can be calculated. He examined the case of a sphere in steady translational motion \( U \) in an unbounded fluid in which the inertia terms on the right hand side of Eq. 5.1 are negligible. In this case the simplified equation is

\[
\nabla p = \mu \Delta u. \tag{5.12}
\]

Additional condition are that \( u \to 0, p \to p_0 \) as \( |x| \to \infty \), and on the surface of the sphere \( u = U \) (impermeability and no-slip boundary conditions). In this case the movement of the fluid is symmetric about the axis along which the sphere moves. In polar coordinates with \( \theta = 0 \) in the direction of \( U \), the components of velocity are taken from the stream function

\[
\Psi = Ur^2 \sin^2 \theta \left( \frac{3a}{4r} - \frac{1}{4r^3} \right) \tag{5.13}
\]

with

\[
u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \Psi}{\partial \theta} = U \cos \theta \left( 1 - \frac{3R}{2r} + \frac{R^3}{2r^3} \right) \tag{5.14}
\]

and

\[
u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial r} = U \sin \theta \left( -1 + \frac{3R}{4r} + \frac{R^3}{4r^3} \right) \tag{5.15}
\]
(see e.g. [10, 54, 71]).

By symmetry, any force must be directed parallel to the line of motion. On the surface of the sphere, this component of force is given [54] from Eq. 5.4 by

\[ F = \int_{r=a}^{r=a} -p_0 \cos \theta + \frac{3\mu}{2a} U \cos^2 \theta + \frac{3\mu}{2a} U \sin^2 \theta \, dS = 6\pi \mu a U. \tag{5.16} \]

It is also possible to calculate the drag on a sphere in steady motion in circumstances where the inertial terms of Eq. 5.1 are important, but the dissipative term \( \mu \Delta u \) is negligible. In the case of an unbounded ideal fluid \( (\mu = 0) \) in irrotational motion, the flow around a sphere of radius \( a \) rising at a speed of \( U \) is given by

\[ u = -\frac{1}{2} U a \frac{3\nabla z}{(x^2 + y^2 + z^2)^{3/2}} \tag{5.17} \]

where the center of the sphere is located at the origin [10]. Because the fluid flow and sphere surface are symmetric before and behind the sphere, the net force from Eq. 5.4 must be 0, giving an example of d’Alembert’s paradox that there is no drag on a body in steady motion through an ideal fluid [16].

In most circumstances however, one is not able to analytically determine the effect of the fluid on a submerged sphere and must rely on either numerical or ad-hoc methods. One rule of thumb usable at higher Reynolds numbers – where the flow separates from around the back of the sphere – is the concept of form drag. The idea is that the fluid upstream moving relative to the body at speed \( U \) stagnates at the surface over a short distance. From Bernoulli’s equation [54]

\[ \frac{1}{2} u^2 + \frac{p}{\rho} + g\Delta z = \text{constant} \tag{5.18} \]

we see that (since \( \Delta z \approx 0 \) as the fluid decelerates)

\[ \Delta p \approx -\frac{\rho}{2} \Delta u^2 = \frac{\rho}{2} U^2. \tag{5.19} \]

With a forward facing surface with projected area \( A = \pi r^2 \), this pressure difference creates a force of

\[ F \approx \frac{\rho \pi r^2 U^2}{2} \tag{5.20} \]

downstream (i.e. opposing the motion of the sphere). There is not a corresponding pressure change behind the object where the fluid inside the separated streamlines moves at about the same velocity as the sphere [10]. However, because not all the streamlines in the area in front of the sphere stagnate at the surface Eq. 5.20 is at best an argument for the order of magnitude of the drag, and one looks for a drag coefficient \( C_D \) so that

\[ F = \frac{\rho C_D \pi r^2}{2} U^2 \tag{5.21} \]
The drag coefficient on a sphere has been measured by many experiments and compiled as a function of Reynolds number \( \text{Re} = \frac{\rho d U}{\mu} \) into the standard drag curve [16] shown in Fig. 5.1.

\[ C_D \]

\[ \text{Re} \]

**Fig. 5.1.** The standard drag curve from [16] gives the coefficient of drag for Eq. 5.21 as a function of a sphere’s Reynolds number.

In the following chapters we will present an experiment of a sphere settling in a bubbly fluid. We will show that in our experimental setup, the settling velocity of the spheres in the bubbly fluid is dominated by the form drag of Eq. 5.21. We will afterwards examine the effect of collisions between bubbles and the settling sphere seeing how this affects not only the settling velocity, but the lateral dispersion of spheres falling in a bubbly fluid.
Chapter 6

A sphere sinking in bubbles: settling velocity

6.1 Experimental setup

6.1.1 The container

To observe the motion of a sphere sinking in a bubbly fluid, we constructed a quasi-2D experiment in which the bubbles and the sphere were confined to a narrow channel with inner dimensions of 2 cm \( \times \) 18.4 cm \( \times \) 30.6 cm (see Fig. 6.1). This acrylic cell was supported by four stainless steel bolts used to raise lower and level the cell. Performing the experiment in a narrow cell had two significant advantages. First, we had found that in a more fully 3D experiment (as seen in Fig. 6.2) the spheres (even the largest we used) were hard to observe. Secondly, we wished to capture as much of the motion as possible when we recorded the experiment. The cell was placed in an 18 gallon tank which was filled with tap water. Bubbles were generated below the cell, while spheres were released slight inside the top of the cell (or in some cases at the bottom.) The experiment was lit from behind using halogen lights with a translucent plastic sheet acting as diffuser, and the progress of the sphere through the cell was captured on video.

6.1.2 The bubble field

The bubbles in this experiment were produced by an Aquatic Gardens Bubble Bar, an “air stone” commercially available for aquariums. The bar was centered under the cell and held in place with brass weights during the experiments. Air to the stone came from the building’s compressed air source through a valve monitored by an Omega 1200 series digital pressure gauge. To prevent bubbles from coalescing on the underside of the cell and rising inside the cell, the walls at the bottom were tapered to direct such bubbles towards the outside of the cell.

To measure the properties of the bubble field, we first measured the gas flow rate as a function of the applied overpressure (gas pressure above the ambient air pressure in the lab). Bubbles were funneled from 9.5 cm of the air-stone into an upside-down beaker. The amount of air in the funnel was constant, so the inflow and outflow were the same. The collection time and volume of the collected gas were measured to find the volume flux per centimeter along the stone. The results are shown in Fig 6.3. As a function of applied overpressure \( p_a \) [psi], the flux per centimeter, \( f \) is approximately \( f = (1.0p_a[psi] - 1.5) \) cm\(^2\)/s.

Next we measured the number density of bubbles per unit volume. We counted for several pressures the number of bubbles in sections of images 184 pixels \( \times \) 196 pixels. Because bubbles coalesce as they rise, the number density near the bottom of the tank is higher than that near the top of the tank. Results are shown in Fig. 6.4. The average
Fig. 6.1. The first experimental setup for observing a sphere settling in a bubbly fluid. Fabrication was by R. Geist of the Pritchard Lab. See text for details.
Fig. 6.2. A fully 3D bubble field produced by an array of commercial air stones.
number of bubbles per cubic centimeter is approximately $N = (0.098p_a[\text{psi}] + 2.9) \ cm^{-3}$.

Finally, we measured the rise velocity of the bubbles as a function of the applied overpressure. This was done by picking bubbles that appeared to be moving at the average speed of the bubble field and recording their rise times over as long a distance as possible. These results are shown in Fig. 6.5. We find that the average rise velocity is approximately $U_b \simeq (3.5p_a[\text{psi}] + 37) \ cm/s$.

From these measured quantities, we can derive the void fraction: In one second the air released from one centimeter’s length of the stone rises $U_b$ cm into a channel of 2 cm width. Thus in $2U_b \ cm^3$ of water we have $f \ cm^3$ of air for a void fraction of $f/[2U_b]$, which averages to 3.2% in the range of pressures shown. We can also calculate the average volume equivalent diameter $d_e$ of the bubbles. That is, as the bubbles are ellipsoidal rather than spherical, we use the diameter of a sphere of equivalent volume. In our case we will calculate the volume of a bubble based on the volume of air in a cubic centimeter $f/[2U_b]$ divided by the number of bubbles per cubic centimeter $N$, giving us

$$d_e = \frac{f}{2U_b \cdot N} \tag{6.1}$$

the results of which are shown in Fig. 6.6. The average of the values shown from $3 \leq p_a \leq 7 \ psi$ is $d_e = 0.26 \ cm$.

These measurements of the diameter are similar to visual measurements made of ellipsoidal bubbles produced by the air-stones in Fig. 6.2. The major and minor axis ($d_M$ and $d_m$) of 45 bubbles were measured, the sample evenly distributed among higher

Fig. 6.3. Bubble volume flux per centimeter $f \ [\text{cm}^2/\text{s}]$ as a function of applied overpressure $p_a \ [\text{psi}]$. The dashed line shows the approximation $f = (1.0p_a[\text{psi}] - 1.5) \ cm^2/\text{s}$. 

![Graph showing bubble volume flux per centimeter as a function of applied overpressure.](image-url)
Fig. 6.4. Bubble number density $N \, [\text{cm}^{-3}]$ as a function of applied overpressure $p_a \, [\text{psi}]$. Measurements shown in blue indicate values in the lower part of the cell. Measurements in red show values in the upper part of the cell. The dashed lines approximate their values as $N = (0.049p_a \, [\text{psi}] + 3.5) \, \text{cm}^{-3}$ and $N = (0.15p_a \, [\text{psi}] + 2.2) \, \text{cm}^{-3}$ respectively. The black line shows the average for the cell approximated as $N = (0.098p_a \, [\text{psi}] + 2.9) \, \text{cm}^{-3}$.

Fig. 6.5. Bubble rise velocity $U_b \, [\text{cm/s}]$ as a function of applied overpressure $p_a \, [\text{psi}]$. 
Fig. 6.6. Plot of equation 6.1: bubble volume equivalent diameter $d_e$ [cm] as a function of applied overpressure $p_a$ [psi].

and lower bubbles. We then calculated $d_e = \sqrt[3]{d^2 M d_m}$. The average diameter in this case was found to be $d_e = 0.2$ cm.

Some of the average bubble properties are listed in Table 6.1. These include the bubble Reynolds number (\(Re = \frac{\rho U_b d_e}{\mu}\) - used to characterize flow in the non-dimensionalized Navier-Stokes equations) and the Eotvos number (\(Eo = |g| \frac{\Delta \rho d^2}{\sigma}\) where $\sigma$ is the surface tension of the air-water interface, which we take to be 72 dyne/cm.) The Reynolds number and Eotvos number can be used, for example, to characterize the shapes of bubbles (e.g. spherical, ellipsoidal, wobbly, spherical cap, etc.) as in [16]. In our case, the numbers indicate bubbles that are ellipsoidal with a tendency to become wobbly as they rise. This seems to match experimental observations of both ellipsoidal bubbles and bubbles that flattened and changed in shape as they rose.

6.1.3 The spheres

Apart from the applied overpressure, the controllable variables in the experiment were the size and density of the spheres being dropped (since the cell geometry, bubble generator and fluid were not changed during the course of the experiment). Two main materials, glass and Teflon, were used in a variety of sizes from about a half a centimeter to a centimeter. One series of a fixed diameter was also tested for a number of other materials as shown in Table 6.2.

The spheres were initially inserted into the bubble field by hand, but we discovered that the presence of fingers in the stream noticeably altered our results. For example, a sphere dropped in quiescent water would often move obliquely to the side in its descent,
Table 6.1. Average properties of the bubble field for $3 \leq p_a \leq 7$ [psi].

<table>
<thead>
<tr>
<th>Property</th>
<th>Average Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$ (gas flow rate)</td>
<td>$3.5 \text{ cm}^2/\text{s}$</td>
</tr>
<tr>
<td>$N$ (number density)</td>
<td>$3.4 \text{ cm}^{-3}$</td>
</tr>
<tr>
<td>$U_b$ (rise velocity)</td>
<td>$54 \text{ cm/s}$</td>
</tr>
<tr>
<td>$d_e$ (volume-equivalent diameter)</td>
<td>$0.26 \text{ cm}$</td>
</tr>
<tr>
<td>void fraction</td>
<td>$3.2%$</td>
</tr>
<tr>
<td>Re (Reynolds number)</td>
<td>$620$</td>
</tr>
<tr>
<td>Eo (Eotvos number)</td>
<td>$0.92$</td>
</tr>
</tbody>
</table>

Table 6.2. Properties of the spheres used in the first experiment.

<table>
<thead>
<tr>
<th>Material</th>
<th>Diameter (cm)</th>
<th>Density (g/cm$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black Glass</td>
<td>0.48, 0.64, 0.79, 0.95</td>
<td>2.61</td>
</tr>
<tr>
<td>Delrin</td>
<td>0.79</td>
<td>1.35</td>
</tr>
<tr>
<td>Lucite</td>
<td>0.79</td>
<td>1.19</td>
</tr>
<tr>
<td>Nylon</td>
<td>0.79</td>
<td>1.14</td>
</tr>
<tr>
<td>Teflon</td>
<td>0.48, 0.64, 0.79, 0.95</td>
<td>2.17</td>
</tr>
<tr>
<td>Viton</td>
<td>0.79</td>
<td>1.8</td>
</tr>
</tbody>
</table>

even though its initial movement was downward. As the wakes of spheres are known to produce lift (see e.g. [43]), we speculate that the presence and movement of our fingers created an asymmetry in the developing wake of the spheres.* After this, the experiments were repeated using tweezers to place the spheres into the tank. Care was taken to make sure that the spheres were thoroughly wetted and free from any attached bubbles before being inserted into the stream. Spheres light enough to rise in the stream were released at the bottom of the cell rather than near the top.

6.1.4 Extracting data

Experiments were recorded using the lab’s Phantom V5 high-speed digital video camera. A typical video had a frame size of 1024pixels$\times$1024pixels, recorded at 300 fps. The videos were cut at the time of recording so that the sphere was present in image. The videos were transferred from a proprietary format to AVI files for analysis. An application for extracting location data from the AVI files using was created in Microsoft’s C# because of useful and publicly available libraries for reading and writing to AVI files.

*There is a parallel used in experiments observing the wake-induced zigzag motion of bubbles: the plane of the zigzagging motion can be controlled by producing the bubbles using an angled hypodermic needle. The (vertical) plane of the zigzagging motion is perpendicular to the angled face of the needle (see e.g. [85]).
The process of algorithmically locating the sphere within the images contained in the AVI files evolved over time. For the data presented in the first experiment, the size of the sphere and its location for one frame in the image were set by the user, along with a maximum search width. Beginning at the known location, the program would advance a frame and check the pixels in a box extending a distance of the search width from the last known location. The pixel with the darkest set of pixels within a sphere radius was determined to be the center of the sphere, and the process was repeated for the whole sequence of images.

After processing, the video was played back with the position and path of the sphere superimposed. It did happen at times that, because the sphere was often obscured for a short time, a bubble might be identified and tracked as the sphere. In cases where this happened, the location was manually adjusted, and the video re-analyzed from that point. A picture of a sphere in the bubble field shown in Fig. 6.7 gives some indication of the difficulty of the process. A path showing a sphere trajectory over the course of an experiment is shown in Fig. 6.8.

The location data were analyzed over about 15 cm of the height of the cell. This was done to give the sphere time to accelerate, and to avoid the turbulent region near the bottom of the cell where the flow and sphere behavior were noticeable different. (Lighter spheres would often be trapped for a short time in this region.) The average velocity over the measured distance was taken to be the terminal velocity of the sphere.†

For later experiments we became interested in having more detailed information about a sphere’s position and progress through the tank. We therefore adjusted the location algorithm as follows: first a mask was made by making a scale measurement (px/cm) in an image, and entering the known diameter of the sphere. A mask was made using the sphere diameter with a Gaussian drop in intensity from interior to exterior pixels. The actual foreground and background brightness values were left variable, as there is sometimes a change in the lighting intensity.

A foreground and background brightness for the mask were chosen by looking for the brightest and darkest pixels in the search area. The pixel location of the sphere was determined as before by looking for the minimum error between the masked and actual brightness values. This constituted a rough search and gave a preliminary position for the location of the center.

To better determine the center of the sphere, we used the following algorithm. The rough location of the center was at \((i, j)\). We define the function \(b(m, n)\) to be the brightness value at pixel \((m, n)\). Let \(\Delta x = x - \lfloor x \rfloor\) and \(\Delta z = z - \lfloor z \rfloor\) (where \(\lfloor z \rfloor\) is the

†The averaging scheme for measuring terminal velocities was adopted because of fluctuations in the measured velocity of the spheres over smaller time scales. The largest, heaviest spheres take longer to reach their terminal velocities, and the averaging method may underestimate those numbers. This showed a need to be able to more accurately measure sphere positions and velocities. Some of the fluctuations in speed were because of sphere-bubble interactions, but some came from limited resolution of the sphere locations. This was part of the motivation for the more accurate location algorithms developed to track spheres in later experiments.
Fig. 6.7. Images from an experimental video. In this image the sphere (silicon nitride $d = 3/32''$) is indicated by a red arrow.
Fig. 6.8. Image from an experimental video. In this image the sphere (silicon nitride $d = 3/32''$) is highlighted in red, and its path to that point is shown in yellow.
greatest integer less than or equal to \( z \). We further define
\[
B(m, n, x, z) = (1 - \Delta x)(1 - \Delta y) b(m - \lfloor x \rfloor, bn - \lfloor z \rfloor) \\
+ \Delta x(1 - \Delta y) b(1 + m - \lfloor x \rfloor, bn - \lfloor z \rfloor) \\
+ \Delta y(1 - \Delta x) b(m - \lfloor x \rfloor, 1 + bn - \lfloor z \rfloor) \\
+ \Delta x\Delta y b(1 + m - \lfloor x \rfloor, 1 + bn - \lfloor z \rfloor)
\] (6.2)
that is, as if a square representing one pixel were shifted to cover adjacent pixels also, and the brightnesses summed proportionally to the area covered.

The quantity \( B(m, n, x, z) \) was now compared to the brightness values in the mask, and the difference measured. To minimize the error, we implemented a steepest descent line search, with the step lengths chosen to satisfy the Strong Wolfe Conditions using algorithms from [70]. To observe the effect of the sub-pixel location algorithm, we chose videos of spheres in quiescent water (where the motion is naturally smoother than in the bubbly fluid). Videos extracted from the larger image and centered on the integer pixel location were compared with videos extracted using the sub-pixel location and averaging schemes. The sub-pixel algorithm provided a noticeable stabilization of the sphere at the center of the images.

6.2 Findings: terminal velocity in quiescent and bubbly water

For a sphere settling downward at a terminal velocity \( V_t \) the form drag of Eq. 5.21 balances with buoyancy to give
\[
\frac{4}{3}\pi r_s^3(\rho_s - \rho)|\mathbf{g}| = \frac{1}{2}C_D \rho V_t^2 \pi r_s^2
\] (6.3)
so that
\[
V_t = \sqrt{\frac{8r_s(\rho_s - \rho)|\mathbf{g}|}{3C_D \rho}} \propto \sqrt{\frac{r_s(\rho_s - \rho)|\mathbf{g}|}{\rho}}.
\] (6.4)
Measured velocities of \( V_t \) are shown in Fig. 6.9. The square-root dependence is also shown, best fit by \( C_D = 0.56 \) — a value comparable to that expected without the cell. For convenience, we will refer to the quantity determining form drag \( \frac{r_s(\rho_s - \rho)|\mathbf{g}|}{\rho} \) from Eq. 6.4 as
\[
\Delta_{fd} \equiv \frac{r_s(\rho_s - \rho)|\mathbf{g}|}{\rho}.
\]

In Fig. 6.10 are shown measurements of the terminal velocities of spheres both in quiescent water and in bubble fields at a variety of applied overpressures. Note that the downward velocity is no longer as steady in these cases, and represents an average settling speed. One major effect of the bubbles is to decrease the terminal velocity of the spheres. In fact, there is a distinct transition where the spheres move from sinking to rising in the water. A picture of this transitional behavior is shown in Fig. 6.11. In addition to the slowing of the spheres’ terminal velocities we also see that the in the bubbly fluid there is some random lateral motion. Although our motivation in performing this experiment
was to control the settling speed of the spheres in order to suspend them, this second effect is of interest and will be considered later in this work.

### 6.3 Observations: bubble-sphere interactions

There are at least three types of interactions we observed between the bubbles and the spheres. The first is that bubbles can attach to spheres. We found this to happen most often when using Viton or one of the plastics. Depending on the number of bubbles attached to the spheres, the sphere was slowed in its descent or even floated. This is visible in Fig. 6.12 where the sphere on the left has one attached bubble, and an identical sphere on the right has three bubbles attached to it.

The second interaction is the entrapment of air bubbles in the wake of a sphere. We show two typical examples. In Fig. 6.13 a bubble is trapped in the wake of a smaller sphere \((d = 0.48 \text{ cm})\). It stays at a fixed distance from the sphere for a short time, until another bubble collides with it and carries it away. In Fig. 6.14 a larger sphere \((d = 1.27 \text{ cm})\) captures many bubbles in its wake. These bubbles can be seen being shed in groups during the earlier stages of the sphere’s acceleration.

The third interaction is the collision of spheres and bubbles. These are most often inferred by us rather than observed: In a video recorded at, say, 100 fps a bubble will travel about a sphere diameter every frame, making collisions hard to observe. We have, however, isolated images of a collision which can be seen in Fig. 6.15.
6.4 Bubble-flow interactions

Aside from the bubble-sphere interactions, there is one more factor to be considered. Because the cell is open at the top and the bottom, bubbles rising in the cell entrain water and cause a net flow of water up through the cell. By measuring the rise time of buoyancy-neutral particles \(^\ddagger\) (see Fig 6.16), we discovered that water circulates up through the cell at a velocity of \(U_w = (3.4p_a[\text{psi}] + 13) \text{ cm/s}\). Note that in this case, the return flow is outside the cell, a major difference from a closed geometry such as that in [100] where the net circulation of water through the cell is zero.

The difference of 24 cm/s between the circulation velocity and bubble velocity is about the speed of a single similarly-sized bubble in quiescent water [16]. Because the bubble velocities are consistent across the width of cell, we infer that the circulation velocity is also uniform (except of course at the very edge of the cell). The observed relation between the bubble velocity and circulation is supported by experiments comparing bubbly flow in a closed geometry with flow in which circulation was introduced [68]. It was discovered that difference between the bubbly flows with and without circulation was approximately the circulation rate.

Because the water is circulating through the cell, it is appropriate to measure drag on the spheres (and hence their terminal velocities) in the rest frame of the circulation. We see in Fig. 6.17 that by adjusting the terminal velocities of the spheres by

\(^\ddagger\) After several attempts to construct a buoyancy-neutral sphere, we decided to use small clumps of towel lint which was observed floating at a constant depths in the tank.
Fig. 6.11. Left to right: spheres of glass, Teflon, Delrin, and Nylon (with diameters of 0.79 cm). The last three spheres are in bubble fields at $p_a = 4$ [psi].
the appropriate circulation velocity, that the difference between the terminal velocities in quiescent and bubbly fluids is almost entirely explained. Hence, if $V_t$ is terminal velocity in quiescent fluid, the sphere is suspended between falling and floating when

$$p_a[\text{psi}] \approx \frac{V_t - 13}{3.4}.$$  \hspace{1cm} (6.5)$$

If the relation of the rising bubbles to applied overpressure is unknown, it should also suffice to adjust the field until $U_b - V_t$ is the same as the speed of a single bubble rising in quiescent water.

Although this satisfactorily answers the question of how to suspend a sphere (at least in this experimental setup), it is very dependent on the geometry of the setup. It also fails to address the nature of the bubble-sphere interactions that were observed. Of the three described bubble-sphere interactions, we chose to focus on the effect of collisions. The next chapters will detail a model for the effect of collisions on the sphere, and a refined experiment used to observe these collisions and their effects.
Fig. 6.13. A bubble (colored for visibility) is trapped in the wake of a sphere (Teflon, $d = 0.48$ cm, $\Delta t = 5$ ms).
Fig. 6.14. Bubbles are trapped in the wake of a sphere (glass, $d = 1.27$ cm, elapsed time 107 ms) and periodically released in clusters. The cluster about to be released in the second frame, for example, can be seen slightly above and to the right of the sphere in the third frame. The group forming in the third frame can be seen in the fourth frame at a distance of about a sphere diameter above the sphere.

Fig. 6.15. A bubble (colored for visibility) collides with a sphere.
Fig. 6.16. Bubble rise velocity $U_b$ (in red) and water circulation velocity $U_w$ (in blue) as a function of applied overpressure $p_a$ [psi].

Fig. 6.17. Measured values of the terminal velocities of spheres in quiescent water (black circles) and in bubbles (blue circles) in the rest frame of water circulating in the tank.
Chapter 7

A sphere sinking in bubbles: collision model

7.1 Introduction

Observing the erratic motion of the spheres in the experiments of the preceding chapter reminded the author of a game that used to sit in his grandparents’ basement – a “pachinko” machine brought back from their travels in east Asia (see Fig. 7.1). The game is intended for gambling: Small metal balls are purchased and inserted in the blue-and-white tray near the middle of the machine. They are then launched through a track circling the left of the machine by use of the lever seen at the bottom right of the machine. The balls are released into free-fall near the top of the machine and bounce unpredictably among the pins as they fall. This erratic, collision-driven motion bears some similarity to the motion of the falling spheres (particularly the smaller spheres) during repeated collisions with bubbles.

Fig. 7.1. The Barker family pachinko machine (photos by DeeAnne Higley). A payoff in the game comes when a ball enters one of the receptacles shown in detail to the right.
The mechanics of the pachinko machine are also reminiscent of the Galton board or *quincunx* shown in Fig. 7.2. In this chapter we will examine the Galton board and its mathematics as presented by various authors. We will then discuss a modification of the Galton board to represent the bubbly fluid discussed in the previous chapter. Finally we will discuss a computer simulation of the fluid Galton model and highlight some of the results.

### 7.2 The Galton board

The Galton board described in Galton’s 1889 work *Natural Inheritance* consisted of a wooden back and a glass front “about a quarter of an inch” apart. Pins were fixed in the board in quincunx fashion (that is, like the five spots on a dice, forming a triangular or hexagonal lattice). Small pellets poured into the top were funneled to the central part of the board where they fell among the pegs scampering deviously down through the pins in a curious and interesting way; each of them darting a step to the right or left, as the case may be, every time it [struck] a pin. [28]

The pellets were collected at the bottom of the device in compartments. The height of the collected pellets was found to be consistent over multiple experiments and to closely approximate the normal distribution. The experiment was repeated by turning the frame “topsy-turvey”, with the pellets being collected by the inverse funnel at the top of the frame in position for another experiment.

Galton’s explanation for the distribution of the pellets is that as each drops from row to row it strikes a pin and is randomly shunted to the left or the right of it, independently of the history of its path. That is, in his own words,

> the principle on which the action of the apparatus depends is, that a number of small and independent accidents befall each shot in its career. In rare cases, a long run of luck continues to favour the course of a particular shot towards either outside place, but in the large majority of instances the number of accidents that cause Deviation to the right, balance in a greater or less degree those that cause Deviation to the left. Therefore most of the shot finds its way into the compartments that are situated near to a perpendicular line drawn from the outlet of the funnel, and the Frequency with which shots stray to different distances to the right or left of that line diminishes in a much faster ratio than those distances increase.[28]

We can be more precise about this process, as the mathematics of Galton’s explanation are straightforward. We will number the rows and columns of the lattice \((i, j)\), beginning at the top row with \(i = 0\) and with the column directly under the funnel designated as \(j = 0\). For example, the path of a pellet will always begin at \((0, 0)\). Upon reaching the pin at \((1, 0)\), a falling pellet – hereafter referred to as the tracer – is forced into one of the positions \{\((1, -1), (1, 1)\)\}. It is assumed that these outcomes are equally likely. More generally, a tracer at position \((i, j)\) is forced by the next pin into one of
positions \((i + 1, j - 1)\) or \((i + 1, j + 1)\). Either of these events is assumed to be equally likely, and thus independent of the history of tracer’s position.

The map from \(j \to j + 1\) has several consequences. As the tracer moves from row to row, the maximum allowable position of the tracer grows linearly, being \(i\) in the \(i^{th}\) row. Also, a tracer in an even(odd) numbered row can only occupy an even(odd) numbered column. The allowable positions of the tracer in the \(i^{th}\) row are then \(\{(i, -i), (i, -i + 1), \ldots, (i, i)\}\).

There are \(2^i\) possible paths from \((0, 0)\) into the \(i^{th}\) row, each being equally likely. The number of paths leading to \((i, -i + 2j)\) is \(\binom{i}{j}\), for \(0 \leq j \leq i\). Then the probability that a sphere released at \((0, 0)\) will be found at \((i, -i + 2j)\) is \(2^{-i} \binom{i}{j}\). This is a binomial distribution, which for large values of \(i\) closely follows the normal distribution claimed by Galton (see e.g. [26]). A diagram of a few of these probabilities is shown in Fig. 7.3.

Due to the symmetry of the board, the expected value of \(j\) on the \(i^{th}\) row is 0 (even in odd rows where \((i, 0)\) is occupied by a pin). The variance of the binomial distribution is \(\sigma^2 = \frac{1}{4}i\), so that the variance in positions is \(\sigma^2 = i\) (see e.g. [26]). The lateral motion of the tracer is the familiar random (drunkard’s) walk. One important aspect of the lateral motion is that if we assume that the tracer descends from row to row at a constant rate, then \(i = \alpha t\) and \(\sigma^2 = \alpha t\). The linear growth of the variance over time is the hallmark of diffusive motion (see e.g. [23]). Hence if we model the movement of tracers through the board using Galton’s explanation, we can characterize the motion of the tracers as falling vertically and diffusing horizontally.
7.3 Examinations and adaptations of the Galton board

In this section we will discuss some recent investigations into the Galton board and similar systems. In subsection 7.3.1 we will discuss treatments using experiments and simulations, most often with statistical analysis or results in mind. In subsection 7.3.2 we will review some treatment of the Galton board as a dynamical system. This will preface section 7.4 in which we will make our own adaptation of the Galton board for modeling sphere-bubble collisions in a fluid.

7.3.1 Deterministic randomness

We begin by discussing an investigation motivated by interparticle percolation [62]. In a material containing mixed smaller and larger particles, the smaller can move down through the larger when the material is under stress. Experimentally, Masliyah and Bridgewater used a bed of packed spheres into which smaller spheres were dropped [62]. If the ratio of sphere diameters is less than 0.1547, then the smaller spheres will settle among the larger without becoming trapped. They found that the smaller spheres settled among the larger with the time of passage through the bed being proportional to the height of the bed, i.e., with a constant average speed. Radially, the particles moved out diffusively from the center.

To be able to investigate the effect of physical parameters such as the interstice between the scatterers (packed spheres) they created a computer simulation. In this simulation, the tracer moves from scatterer to scatterer under the influence of gravity with the effects of air resistance neglected, that is, in parabolic paths with

$$\frac{\delta^2}{\delta t^2} (\hat{x}, \hat{z}) = (0, -|g|).$$

When the tracer reaches the surface of a scatterer, the interaction is modeled as an instantaneous collision using the coefficient of restitution (see Appendix A for details). That is if $\mathbf{v}_I$ is the incoming velocity and $\mathbf{v}_R$ is the rebound velocity,

$$\mathbf{v}_R \cdot \hat{n} = -\epsilon \mathbf{v}_I \cdot \hat{n}$$

where $\hat{n}$ is the unit normal to the spheres at the point of contact and $\epsilon$ is the coefficient of restitution.

We wish to note here that one complication of the instantaneous collision model is that it is possible for a large (or indeed, infinite) number of collisions to occur between
the tracer and a scatterer in finite time (with bounces that decay exponentially). In such a case, the progress of any simulation based on this model will be stopped. They dealt with this problem by fixing the coefficient of restitution at $e = 0.8$. They found (empirically) that this was sufficient to avoid situations in which the tracer bounced on the same scatterer many times in succession.\[62\] We will detail in subsection 7.5.2 how we dealt with the possibility of successive collisions.

We pause here to observe that there are two differences in this more detailed collision model of the Galton board from the model described in section 7.2. The first is a question of geometry. In the Galton board we have a sphere bouncing among small cylinders while in [62] a small sphere is bouncing among larger spheres. Because the movement that will be considered is planar and the tracer and scatterer both have a circular cross-section, a 2D model of a circular tracer moving among circular scatterers also describes 3D models in which a sphere moves among fixed spheres, a cylinder (or disk) moves among smaller cylinders, or any combination of these. Furthermore, because the position of each is described by the center of mass, the interaction between the two always occurs at a distance of the sum of their radii, and the normal vector at the point of collision points between the centers. Hence the actual sizes of the two spheres are immaterial and only the sum of the radii is important. It is thus natural (as in [53, 62, 46]) to replace the tracer with a point particle as in Fig. 7.4.

![Fig. 7.4. The Galton board arrays at the left and the right are equivalent when modeling the system as a tracer in free fall without air resistance or friction. A scatterer and tracer from the left array have been superimposed to the right for comparison.](image)

When the tracer is replaced with a point particle, it is natural to choose a length scale to non-dimensionalize the measured parameters $\left[ \frac{\hat{x}}{\hat{z}} \right]$ and $\hat{t}$ of the Galton board. This is generally done in terms of the joined diameters $D$ so that

$$X = \left[ \begin{array}{c} x \\ z \end{array} \right] = \frac{1}{D} \left[ \begin{array}{c} \hat{x} \\ \hat{z} \end{array} \right]. \quad (7.1)$$
The time is normalized by the free-fall time through a scatterer diameter so that
\[ t = \hat{t}/\sqrt{2D/|g|}. \quad (7.2) \]

The equation of motion in free fall then becomes
\[ \ddot{X} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}. \quad (7.3) \]

This also reduces the variable parameters of the Galton board model to two: the interstice from scatterer to scatterer and the coefficient of restitution. We note that in the study of [62], the distance was normalized by the distance between scatterer centers. This left the free distance parameter to be the diameter of the scatterer.

The second and more troubling difference between the collision model of [62] and the random walk model of [28] is that the motion of the tracer in the collision model is fully specified: parabolic paths between collisions with a rebound law based on the coefficient of restitution. The argument for diffusive behavior is no longer evident when successive bounces no longer constitute a random walk. The system may now be treated as a dynamical system for purposes of analysis, and indeed we will discuss at a later point some results that may be obtained this way.

Masliyah and Bridgewater [62] make a case for treating successive jumps as if their lateral components were random (for purposes of analysis, not simulation). This creates an effective random walk, explaining the diffusive behavior. We will follow their arguments here. To analyze the path of a particle and decide whether its motions were random, they defined the net displacement of the tracer after \( n \) jumps as
\[ R_n = \sum_{i=1}^{n} r_i \]
where \( r_i \) is the displacement from the \( i \)th jump. They also let \( R = \lim_{n \to \infty} R_n/n \) and defined the local fluctuations \( s_i = r_i - \overline{r} \) and the mean fluctuation
\[ S_n = \sum_{i=1}^{n} s_i = R_n - n\overline{r}. \]

They calculated the magnitude squared of the mean fluctuation to be
\[ |S_n|^2 = \sum_{i=1}^{n} |s_i|^2 + 2 \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} |s_i||s_{i+j}| \cos \theta_{i,i+j} \quad (7.4) \]
where \( \cos \theta_{i,i+j} \) is the angle between \( s_i \) and \( s_{i+j} \). The average of \( |S_n|^2 \) for many paths of \( n \) bounces is
\[ |S_n|^2 = n|\overline{s}|^2 + 2\Pi \]
where \( \Pi \) is the average of the double sum in Eq. 7.4, and the averages of \( |s_i| \) are assumed to be independent of \( i \).

They argue that if \( s_i, s_{i+1} \) are independent, then \( |s_i||s_{i+j}| \) is independent of \( \theta_{i,i+j} \). Then because the average of \( \cos \) is zero, \( \Pi = 0 \) also and
\[ |S_n|^2 = n|\overline{s}|^2. \quad (7.5) \]

This analysis is also applicable to the motion in a single coordinate direction with the simplification that \( \theta_{i,i+j} \in \{0, \pi\} \).

They examined \( |S_{nx}|^2 \) and \( |S_{ny}|^2 \) in their computer simulations for scatterer diameters of 0.826, 0.866 and 0.906. They found that \( |S_{nx}|^2 \) satisfied 7.5. They concluded
that local fluctuations of the tracer’s lateral motions could be considered to be independent. Data for $|S_{ny}|^2$ followed a linear relationship for $n > 4$, indicating that there was a short term dependence but independence for fluctuations at a sufficient remove (at least four bounces) from each other.

Masliyah’s simulations in [62] mirrored experimental results in that the tracer would quickly reach an average settling speed, and had diffusive lateral motion. These results have been borne out in other experiments in which one or more spheres fall through a packed bed of spheres [41, 60], a sphere rolls down an inclined board with pins fixed in it [79], or one or more disks falls through an array of disks [13]. The effects of lateral diffusion and terminal velocity have also been seen in experiments in which a sphere rolls down a bumpy plane [81]. In tandem with these experiments, many simulations have been performed. For example, the experiments in [13, 79, 60] all have accompanying simulations. Other simulations such as those in [53, 75] are stand-alone works meant to examine the effects of changing parameters. The settling speed is a point of interest to us that makes these more detailed models better suited to our purposes than the Galton random walk model.

7.3.2 Dynamical system

Another approach to the Galton board is to model it not as a random system, but as a deterministic dynamical system. One champion of this view is K. Judd [46], who in a 1985 article attacks the independence of successive lateral motions assumed by Galton [28]. Judd states his preference in the following way:

Some readers might, as we do, question the assumption of “independent accidents” and view the mechanical device as a deterministic dynamical system.

Judd uses a computer simulation based on similar assumptions to those in [62]. He does have some noticeable differences. First, he does not confine himself to a higher coefficient of restitution to avoid the problem of many successive bounces on a given scatterer. Instead he creates a cutoff velocity for the rebound after which the tracer simply rolls off of the scatterer (although the rotation of the tracer is generally ignored and the diameter joined to that of the scatterers). He also assumes that the coefficient of restitution applies equally to tangential and normal components of velocity so that the angle of incidence is the same as the angle of reflection. In addition, much of his investigation was done on a section of map containing one cell of the quincunx lattice – a rectangle with a central scatterer and four quarter-scatterers in the corners – with periodic boundary conditions.

For each time the tracer passes vertically through the rectangle, an integer number is recorded: beginning at 0, 1 is added every time the tracer passes through the right wall, and $-1$ for each passage through the left. He argues that if the tracer randomly moves to the left or right at each row of the board (as in the classic model) that $\frac{1}{2}$ of all numbers in the sequence defined by a tracer’s path should be 0. If the fraction is greater than $\frac{1}{2}$, a series of balls moving through a Galton board would be expected to collect in the compartments beneath in a distribution that was broader than Gaussian (platykurtic).
In the case where the fraction of zeros is less than $\frac{1}{2}$, a narrower distribution would be obtained.

His findings were that for most of the tested parameters he felt were physically realistic, the fraction of zeros was less than $\frac{1}{2}$ (most often between 0.3 and 0.4). He claims that this is borne out by photographs of Galton’s board and similar devices in which he reports that the distribution appears to be platykurtic. (An observation not in harmony with the experimentalist in [62, 41], for example.)

His conclusion is that

If one’s goal is to make predictions and forecasts, then the dynamical model is surely more informative and accurate, and a statistical model may even be misleading.

More broadly, he feels that

Whenever aiming to forecast or predict nonlinear systems one needs to consider carefully the role of nonlinear dynamics and not immediately adopt a statistical framework. . . . Our continuing work indicates that where dynamics plays a significant role, then there is advantage in exploiting it.

In our own simulations and experiments (which we will describe later) we have seen something of the effect he describes – a short term continuation of the tracer’s horizontal direction as it descends from row to row of scatterers, an effect which we will refer to as “staircasing” (a movement reminiscent of walking down a staircase.) In most cases we have observed, however, these staircasing events create a short-term order which produces random lateral direction changes and diffusive behavior on a somewhat longer time scale.

Analysis of the Galton board system does yield many interesting results. For example, Chernov and Dolgopyat [14] consider an infinite Galton board system with perfectly elastic collisions. They show the limiting distribution of $ct^{-1/3}v(t)$ is

$$\frac{3z}{\Gamma(2/3)} \exp\left(-z^3\right)$$

for $z \geq 0$. Because energy is conserved, this means that the depth $-2|g|c^2t^{-2/3}y(t)$ has the limiting distribution

$$\frac{3}{2\Gamma(2/3)} \exp\left(-z^{3/2}\right).$$

Interestingly, the analysis of Chernov and Dolgopyat does not appear to require that the scatterers are circular or monodisperse in size – only that they are convex and periodically arranged (with the finite horizon condition met). This result and a few others are compared with our simulations in appendix C.

### 7.3.3 An argument for randomness

To provide some insight into why successive jumps may be treated as independent, we made some calculations of the time of a single jump – that is, the time a tracer
spends in free fall between two successive bounces. The tracer will begin its jump at the point \((x_0, y_0)\) with initial velocity \((v_{x0}, v_{y0})\). Until it collides with another scatterer, the equation of motion is
\[
x(t) = (x_0 + v_{x0}t, y_0 + v_{x0}t + t^2).
\]  (7.8)

It will collide with a scatterer with center \((x_c, y_c)\) (assuming it is not impeded in its flight) when
\[
\frac{1}{4} = |x(t) - (x_c, y_c)|^2.
\]  (7.9)

This is a fourth degree polynomial in \(t\) and can be solved analytically or (more often) numerically for scatterers in the vicinity of \((x_0, y_0)\) with the minimum positive time of arrival taken to be the jump time.

We present here some sample jump times. The interstice \(I\) between gaps will be fixed (arbitrarily) at \(I = 0.5\). The starting position will be on a scatterer with center at \((0,0)\), and measured in terms of the angle \(\theta\) with the horizontal, so that \((x_0, y_0) = \left(\frac{1}{2} \cos \theta, \frac{1}{2} \sin \theta\right)\). We will define the initial velocity by the speed \(v\) and the angle \(\phi\) from the normal to the sphere at \((x_0, y_0)\) (requiring \(-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\)). so that \((v_{x0}, v_{y0}) = (v \cos(\theta + \phi), v \sin(\theta + \phi))\) as shown in Fig. 7.5.

![Fig. 7.5. A single jump on the Galton board.](image)

In Fig. 7.6 we show jump times from \(\theta = \frac{\pi}{4}\) for \(0 < v < 2\) and \(-\frac{\pi}{2} < \phi < \frac{\pi}{2}\). Note that the jump times are not a smooth function of \(v\) and \(\phi\). Rather there are sharp lines of discontinuity along certain curves. An explanation for this is seen in Fig. 7.7 in which the paths of two jumps beginning at \(\theta = \frac{\pi}{4}\) and \(\phi = 0.69\). The starting speed of the shorter jump is 1.46 and the jump time is 2. As the starting velocity increases slightly to 1.47, the tracer goes high enough to collide with an entirely different scatterer, reducing the jump time sharply to 0.72.

This sort of sensitivity makes paths consisting of many jumps extremely unpredictable. For example, Masliyah [62] found that by changing the accuracy of the point
Fig. 7.6. Jump times from $\theta = \frac{\pi}{4}$.

Fig. 7.7. A small change in parameters results in a much different jump.
of intersection of a jump from $10^{-9}$ to $10^{-6}$ or $10^{-13}$, that tracer paths would be similar for about 10-20 bounces then diverge rapidly. The authors have also observed this phenomenon in their own simulations. The unpredictability of the behavior of the system is what we mean by randomness.

7.4 The fluid Galton model

We envision the experiment in which a sphere settles in a bubbly fluid as a type of Galton system in which the sphere is the tracer (which, despite having finite radius, is treated as a particle, not a rigid body) and the bubbles represent the scatterers. We will make the following adaptations to the model discussed in section 7.3.1. First, the tracer will be moving in water where the fluid effects are no longer negligible. Therefore, we will modify the equation of motion for the tracer to approximate fluid effects as written in Eq. 5.8. In particular, we will use a form similar to Eq. 5.11 (because of the dependence of drag and buoyancy. As a starting point, we will let $u = 0$ and omit the history term. This leaves us with.

\[
(m + m_A)\ddot{x} = \frac{4}{3}\pi r_s^3 (\rho - \rho_s) g |\dot{k}| \hat{k} - \frac{1}{2} C_D \rho \pi r_s^2 |\dot{x}| \dot{x}. \tag{7.10}
\]

The scatterers (bubbles) will also move subject to Eq. 7.10, although we will artificially fix the coefficient of drag in this case to allow for bubbles rising with a predetermined terminal velocity.

The effect of collisions will be determined by Eqs. A.16, A.17 with the modification that instead of mass alone, we will use both mass and added mass so that the normal component of the tracer rebound velocity will be

\[
\dot{x}_R \cdot \hat{n} = \frac{M \ddot{x}_I \cdot \hat{n} + M_B v_{BI} \cdot \hat{n} + e M_B (v_{BI} \cdot \hat{n} - \dot{x}_I \cdot \hat{n})}{M + M_B}. \tag{7.11}
\]

Where $M = m + m_A$ is the mass and added mass of the tracer and $M_B$ is the mass and added mass of the bubble. The scatterer rebound velocity will be similarly calculated.

7.5 Simulating the fluid Galton model

7.5.1 Adjustable parameters

For our simulation we wished to have as many free parameters as possible. The following parameters are adjustable in the model:

**Floating** This parameter specifies whether the scatterers are floating or fixed as in the classic model.

**Drag** This parameter specifies whether the equations of motion should include the effects of drag, buoyancy and added mass for the tracer. The classical Galton board can be recovered by setting Floating and Drag to false.

**Collisions** This parameter prevents the tracer from passing through the particles. This is useful for example to compare terminal velocities to those in quiescent fluid.
Tracer Density, Tracer Radius, Scatterer Density and Scatterer Radius These properties of the sphere and bubbles.

Scatterer Speed The terminal rise velocity of the bubbles, used to calculate a drag coefficient for use with the bubbles. All bubbles will rise at this speed until colliding with the tracer.

Coefficient of Restitution This is $e$ as described in appendix A. We discuss the applicability of this for collisions in a fluid in appendix B.

Interstitial The distance between scatterers.

$x_0$ Min, Max The starting position of the tracer has an $x$-coordinate with a uniform random distribution in the interval $(x_0 \text{ Min}, x_0 \text{ Max})$. Kozlov [53] states that in his simulations he found that positioning particles with an initially uniform random distribution led to an arcsin distribution of lateral positions, rather than Gaussian. He was able to recover a Gaussian distribution of end positions only by using a Gaussian distribution of initial positions. The authors have not noticed this to be the case in their own experiments. Perhaps this is a case in which some staircasing occurred.

The vertical starting coordinate and initial velocities are also free parameters in our simulation, but we have not as yet made use of them.
7.5.2 Tracer motion

The equations of motion are integrated using the fourth order Runge-Kutta method (see e.g. [89]) for the tracer and each scatterer (the latter when the Floating parameter is set to true). The time step is kept lower than it would typically be (using an adaptive time step) so that the movement between steps can be interpolated linearly to check for collisions. To see if a collision has occurred between time \( t_0 \) and \( t_1 = t_0 + \Delta t \), we evaluate the motions as if the path between endpoints were linear. That is, we have a bubble whose position at time \( t_0 \) is \( \mathbf{x}(t_0) = \mathbf{x}_0 \) and at \( t_1 \) is \( \mathbf{x}(t_1) = \mathbf{x}_1 \). Similarly, the scatterer beginning and end positions are \( \tilde{\mathbf{x}}_0 \) and \( \tilde{\mathbf{x}}_1 \) respectively. We take the difference in initial positions \( \mathbf{D} = \tilde{\mathbf{x}}_0 - \mathbf{x}_0 \) and relative velocity \( \mathbf{V} = (\mathbf{x}_1 - \mathbf{x}_0) - (\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_0) / \Delta t \).

If \( |\mathbf{D}| > |\mathbf{x}_1 - \mathbf{x}_0| + |\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_0| + r_t + r_s \), then there is no collision as the separation is too great. If \( \mathbf{V} \cdot \mathbf{D} \leq 0 \) there is also no collision, as the relative motion of the tracer has no component in the direction of the scatterer. In all other cases we will check for a collision at positive time by setting \( |\mathbf{V} t - \mathbf{D}|^2 = (r_s + r_B)^2 \) and solving for \( t \) using the quadratic equation. The minimum positive value of \( t \) for all scatterers is checked, and if \( t \leq \Delta t \), a collision is determined to have occurred. In this case a linear interpolation is made for the position of the tracer and all particles.\(^*\)

To avoid the possibility of the simulation being stopped by many sequential and increasingly microscopic collisions with one scatterer, we check for situations in which collisions occur at two consecutive time steps, and in which the time between collisions is not more than a tenth of a percent of the attempted time step. In such a case we reason that the two spheres are essentially moving together as to the normal component of their velocities. Hence we combine the normal components of their momenta for the next time step as if they collided inelastically. During the next time step we evaluate normal forces to the spheres to accelerate the spheres together as one body along the line between their centers. This slide step advances the motion through times when successive collisions may have otherwise halted the simulation.

7.5.3 Tracking scatterers

Because of the need to update positions and check for collisions at each time step, it is desirable to keep track of as few scatterers as possible. Judd’s simulations, for example, make use of periodicity of the array to use just five (partial) scatterers [46]. Our approach was to use a movable frame around the tracer, and restrict our attention to the scatterers within the frame.

At the beginning of the simulation the frame is created and populated with scatterers. The height and width of the frame are multiples of the row and column spacing. The tracer is initialized near the center of the frame. At each time step the position of the frame rises at the Scatterer Speed. If the combined movement of the frame and tracer put the tracer farther than a row height below the center of the frame, the frame height is decreased by a row height and the scatterers that formed the top of the frame

\(^*\)A modification of the simulation using a continuous Runge-Kutta method to avoid linear interpolations is presented in Appendix C.
are placed at the bottom. The positions and velocities of the cycled scatterers are adjusted to match a rising section of the array in which there has been no collision with the tracer. A similar shift occurs if the tracer moves more than a row height above the center of the frame, or more than a column distance to the left or the right of center. In this way, while keeping track of relatively few scatterers we create the effect of an infinite Galton board.

In the case where the Floating parameter is set to false, there is no change to the scatterers' position or velocities because of a collision. In any other case, some information may be lost when the frame is cycled. In such a case, we wish to keep track of any scatterer likely to interact with the tracer in the future. So for the case where the Floating parameter is set to false, we use a frame of 8 columns by 8 rows. (Two or three would suffice, but we sometimes plot positions of tracers and scatterers to make a video clip of the simulations, and use the extra scatterers for the sake of appearance.)

In the case where scatterers are mobile, we use more caution. In the range of parameters that we have visualized; however, we do not see the tracer interact with the same row of scatterers once it has fallen below them. So for our purposes, the same 8 columns by 8 rows should be more than sufficient to avoid losing any information affecting the future path of the tracer. To allow an extra margin of error (and allow for parameter regimes we have not yet explored) we double the dimensions to 16 columns by 16 rows.

7.6 Simulation results

The tracer particle in the classic Galton board setting had two features in which we were primarily interested: a settling velocity and lateral diffusive motion. In most cases these features are preserved in the fluid Galton model. The exception, which we will discuss later, is a staircasing effect resulting in a v-shaped arrangement of paths. We will discuss first the parameters in terms of which results are presented for the fluid Galton model. We will then proceed to discuss the settling velocity and the diffusive (or staircasing) lateral motion of the tracers.

7.6.1 Non-dimensionalization of parameters for floating scatterers

Because the dimensions of the tracer are of vital importance in calculating the drag, we can no longer simply join the diameters of tracer and scatterer to describe the model. Instead, we will non-dimensionalize distances by the size of the scatterer, so that

\[ x = x' / D_s. \]  

(7.12)

The time is normalized by the transit time through the scatterer diameter at the scatterer rise velocity, so that

\[ t = t' v_0 / D_s. \]  

(7.13)

In other words, the velocity of the tracer will be presented in terms of \( v_0 \) (we will use \( v_0 = 26.0 \text{ cm/s} \).)
7.6.2 Average settling velocity

For the tracer falling subject to Eq. 7.10, we see an average settling speed $V_T$ much lower than that predicted by the form drag alone. This is consistent with the classical model. Beyond that however, the tracer velocity can be reversed by collisions so that there is a sinking-to-rising transition. This transition is the result of collisions and not fluid circulation. We present here (Figs. 7.9 and 7.10) some sample paths for tracers of $D_t = 0.3$, $I = 0.75 + 2D_t$. The sinking-rising transition is visible at a density of about $13 \text{ g/cm}^3$.

![Fig. 7.9. Sample paths for a tracer of $D_t = 0.3$ and $\rho = 7.2 \text{ g/cm}^3$. Fifty paths are shown. Scale is shown in non-dimensionalized units on the horizontal and vertical axes.](image)

The decrease in settling speed in this case follows a mostly linear trend as seen in Fig. 7.11. For many diameters and density ranges $V_t$ decreases with density, as seen in Fig. 7.12. Also in Fig. 7.12 we see that the transition point from floating to falling shifts downward with increasing diameter. A plot of this movement is shown in Fig. 7.13.

We also see a notable difference (shown separately in Fig. 7.14) for $D_t = 0.4$ where the dependence of $V_t$ on $\rho$ is not monotonic. The major component of the non-monotonicity appears to be staircasing behavior. Staircasing can be seen in Fig. 7.15 for moderately low to moderate densities. The staircasing ends at about $\rho = 16 \text{ gm/cm}^3$.

7.6.3 Diffusivity

In a regime where staircasing dominates, the lateral motion is linear in time and so the variance grows as a second power. For example, the growth of variance over time for the v-shaped path seen in Fig. 7.15 in the case of $\rho = 14.8 \text{ gm/cm}^3$ grows faster than
Fig. 7.10. Sample paths for a tracer of $D_z = 0.3$. Fifty paths are shown (moving left-to-right, top-to-bottom) for densities 7.2, 9.1, 11.0, 12.9, 14.8 and 16.7 g/cm$^3$. Scale is shown in non-dimensionalized units on the horizontal and vertical axes of each plot.
Fig. 7.11. Average non-dimensionalized settling velocity $V_t$ for a tracer with diameter $D_t = 0.3$ as a function of density $\rho$ (gm/cm$^3$).

Fig. 7.12. Average non-dimensionalized settling velocity $V_t$ as a function of density $\rho$ [gm/cm$^3$].
Fig. 7.13. Transition density $\rho_t \, [gm/cm^3]$ as a function of non-dimensionalized diameter $D_t$.

Fig. 7.14. Average settling velocity $V_t$ for a tracer with $D_t = 0.4$ as a function of density $\rho \, (gm/cm^3)$. 
Fig. 7.15. Sample paths for a tracer of $D_z = 0.4$. Fifty paths are shown (moving left-to-right, top-to-bottom) for densities 5.3, 6.2, 8.1, 14.8, 15.7 and 16.7 g/cm$^3$. Scale is shown in non-dimesionalized units on the horizontal and vertical axes of each plot.
linear. This can be seen in Fig. 7.16. In the case of $\rho = 15.7 \text{ gm/cm}^3$, the growth of variance over time is evidently linear (after a short initial period) as seen in Fig. 7.17.

A phase plot (Fig. 7.18) indicates regimes in which staircasing and diffusive behavior occur. There is a third regime where the variance in position remains fairly constant over time. This behavior (usually for higher values of $\rho$ and $D_t$) can be understood by plotting sample paths which head down in a fairly straight line – the scatterers simply lack the momentum to move the tracer appreciably to the side. In the smaller, lighter end of the spectrum, we also see this direct rise behavior. Here, the tracers are presumably light enough to be quickly slowed by the fluid. It is the middle regime where staircasing and diffusion can occur – where the tracer is dense enough to move some distance through the fluid, but light enough to rebound noticeably from the scatterers. It would be interesting to know whether a transition (by varying the density or radius of the tracer) from staircasing to sinking in a straight line always passes through a regime of diffusive behavior.

![Graph showing variance of x-coordinates](image)

Fig. 7.16. Variance of $x$-coordinates for 50 sample paths shown as a function of time for spheres of $D_t = 0.4$ and $\rho = 14.8 \text{ gm/cm}^3$.

### 7.6.4 Sinking of a heavy sphere in bubbles

In the case of the heavier spheres the simulations show an apparent relationship (see Figs. 7.19, 7.20) of $V_T - \hat{V}_T \propto D_t^{-1/2} \rho^{1/3}$, where $\hat{V}_T$ is the terminal speed of the sphere in quiescent fluid. An estimate of $V_T$ can be based on the observation (seen in simulations and some experiments) that a heavy sphere can push aside bubbles to follow a fairly straight path down with small side-to-side motions as it is pushed from
Fig. 7.17. Variance of $x$-coordinates for 50 sample paths shown as a function of time for spheres of $D_t = 0.4$ and $\rho = 15.7$ gm/cm$^3$.

Fig. 7.18. Points marked with white squares show parameters where diffusive lateral motion was observed, while black circles indicate tracers that rise or fall in fairly straight paths. Gray triangles are used to show parameters where staircasing occurs.
Fig. 7.19. $V_t - \hat{V}_T$ grows as $D_t^{-1/2}$ for $\rho = 20 \text{ gm/cm}^3$ (blue) and $\rho = 19 \text{ gm/cm}^3$ (red). A line of slope $-1/2$ is provided for reference.

Fig. 7.20. $V_t - \hat{V}_T$ grows as $\rho^{1/3}$ for $D_t = 0.8$ (blue) and $D_t = 0.9$ (red). Lines of slope $1/3$ are provided for reference.
one side and then the other by rising bubbles. We will then adapt Eq. 7.11 to represent the collision of a tracer with velocity $\dot{x} = (0, -V_t)$ and a scatterer with velocity $(0, v_B)$. This gives us

$$\dot{x}_R \cdot \hat{n} = -\frac{M V_t + M_B v_B + e M_B (v_B + V_t)}{M + M_B} \sin \theta \quad (7.14)$$

where $\theta$ is the angle between the horizontal and the normal vector at the point of contact. Also we have

$$\dot{x}_R \cdot \hat{t} = V_t \cos \theta. \quad (7.15)$$

The vertical component of the velocity after collision is then

$$-\frac{M V_t + M_B v_B + e M_B (v_B + V_t)}{M + M_B} \sin^2 \theta - V_t \cos^2 \theta. \quad (7.16)$$

The change in the vertical tracer speed of

$$-\frac{M V_t + M_B v_B + e M_B (v_B + V_t)}{M + M_B} \sin^2 \theta - V_t \cos^2 \theta + V_t \quad (7.17)$$

creates a vertical impulse of magnitude

$$M \left(-\frac{M V_t + M_B v_B + e M_B (v_B + V_t)}{M + M_B} \sin^2 \theta - V_t \cos^2 \theta + V_t \right) \quad (7.18)$$

and, in a longer time scale, an effective acceleration of magnitude

$$\frac{M}{\tau} \left(-\frac{M V_t + M_B v_B + e M_B (v_B + V_t)}{M + M_B} \sin^2 \theta - V_t \cos^2 \theta + V_t \right) \quad (7.19)$$

where $\tau$ is the average time between collisions.

It is natural to suppose that the average time between collisions is the time it takes the tracer to move from one row of scatterers to the next. That is, we assume

$$\tau = \frac{\sqrt{3}}{2} \frac{I + D_s}{V_t + v_B}. \quad (7.20)$$

We can support this conclusion by examination of the autocorrelation $C(s)$ of the horizontal velocities $V_x$ of the tracer as it settles. This is

$$C(s) = \frac{1}{M - s} \sum_{i=1}^{M-s} V_x(t_i) V_x(t_i + s) / C_0(s) \quad (7.21)$$

where $C_0(s) = \frac{1}{M - s} \sum_{i=1}^{M-s} V_x(t_i) V_x(t_i)$. If the tracer is being jostled from side to side, the correlation should change sign every $s = \tau$. In Fig. 7.21 we show a small section of $C(s)$ for the tracer of diameter 0.9 and density 10 g/cm$^3$ averaged over 50 runs. The sign of the function changes every $\tau$ units, confirming our assumption.
Replacing the $\sin^2 \theta$ and $\cos^2 \theta$ terms of Eq. 7.19 with their average values of $\frac{1}{2}$, we modify Eq. 6.3 to see that

$$\frac{M(V_t + v_B)}{\sqrt{3(I + D_s)}} \left( -MV_t + M_Bv_B + \epsilon M_B(v_B + V_t) + V_t \right) = \frac{4}{3} \pi r_s^3 (\rho_s - \rho) |g| - \frac{1}{2} C_D \rho V_t^2 \pi r_s^2. \quad (7.22)$$

We present solutions for Eq 7.22 for tracers of $D_t = 0.9, 0.8$ in Fig. 7.22. The results match qualitatively with the simulations at higher densities.

### 7.6.5 Autocorrelation

It is worth pausing here to note that the autocorrelation function for the horizontal velocity of a tracer varies greatly depending on which phase of motion a tracer is in. We have seen in Fig. 7.21 the periodic autocorrelation of a sphere sinking mostly downwards with a small back-and-forth motion. For a tracer that is staircasing, the horizontal velocity remains (on average) fairly constant. This produces an autocorrelation that does not die off with time as seen in Fig. 7.23. For a tracer in diffusive motion, the correlation dies off quickly. In the case of Fig. 7.24 we see that this take about the passage time for 10 scatterers, so presumably about 10 collisions.

### 7.7 Simulation results: added velocity

In the fluid Galton model and simulation of sections 7.4 and 7.5 one can think of a small, light tracer that, instead of colliding with the scatterers is advected around them.
Fig. 7.22. \( V_t - \hat{V}_T \) simulated (circles) and estimated (+) by Eq 7.22 for \( D_t = 0.8 \) (blue) and \( D_t = 0.9 \) (red).

Fig. 7.23. The (average) autocorrelation \( C(s) \) of the horizontal velocities of a tracer (\( D_t = 0.4, \rho \approx 10\text{g/cm}^3 \)) in a staircasing regime reflects its continued linear path by remaining near 1.
Fig. 7.24. The (average) autocorrelation $C(s)$ of the horizontal velocities of a tracer ($D_t = 0.4, \rho \approx 2.4g/cm^3$) in a diffusive regime dies off quickly.

In the model and simulation as described, there is no fluid motion taken into account that is not uniform (i.e. an overall circulation velocity) and this effect is not seen. We wished to find some approximation for the non-uniform component of fluid flow directly around the scatterers and observe its effect on the simulations. To this end we created an added velocity in the fluid about each scatterer.

The flow induced by a single rising spherical bubble is close to potential flow around a sphere – especially on the upstream side of the bubble where collisions tend to occur. To create an added velocity for the multiple scatterers, we simply sum the dipoles shown in Eq. 5.17. This is, of course, not the same as the potential flow calculated for multiple spheres simultaneously, in that it does not satisfy the appropriate boundary conditions at either the scatterer or tracer surfaces. However, it seems appropriate to the simplicity of our model.

Now we will modify Eq. 7.10 for the tracer for a situation in which $u \neq 0$ to

\[
(m + m_A)\ddot{x} = \frac{4}{3} \pi r^3_s (\rho - \rho_s) g |\hat{k}| + \frac{1}{2} C_D \rho \pi r^2_s |u - \dot{x}| (u - \dot{x}).
\]  

(7.23)

It is not immediately clear, however, what value of fluid velocity $u$ should be used in Eq. 7.23. For instance, using the value at the center of the tracer is undesirable because the velocity given by Eq. 5.17 drops off quickly (at slowest, on the order of $\frac{1}{|x|}$) away from the surface of a scatterer. So to measure $u$ at the center of a tracer of larger radius might have very little effect. Also because the field drops off rapidly with distance (and can also change rapidly by changing the angle between the tracer and the axis along which
the scatter moves) the effect on the tracer would become sensitive to small changes in location.

We determined to use for $\mathbf{u}$ an average velocity over the volume of the sphere. This is similar in some sense to the case where a tracer moves in a uniform flow, and the value of $\mathbf{u}$ is the same as the average over the volume of the sphere of the flow as it would have been without the presence of the tracer. This is a very rough estimate, but perhaps commensurate with the nature of the velocity field we have chosen.

The results are qualitatively the same as without the added velocity terms. For example, in Fig. 7.25 we see the same trend of a floating to falling transition with increasing density. We also see the transition from staircasing to diffusive lateral motion. An example of this is shown in Fig. 7.26. This does indicate that the staircasing behavior seen in the previous section has some insensitivity to small perturbations.

It is interesting to note that the transition from staircasing to diffusive motion is not a sudden one, but characterized by staircasing with a small number of tracers in apparently diffusive motion. It appears that the beginning motions of the tracers in the staircasing regime are much the same as those after the transition. However, the tracers appear to have a chance of entering a steady staircasing state over time. This is not unique to simulations with the added velocity effect, but is seen also in the previous section. This helps explain, for instance, the small amount of time in Fig. 7.16 where the variance appears to grow linearly – most of the tracers may still be moving diffusively at this point. A detail of these paths changing from diffusive to staircasing behavior is shown in Fig. 7.27.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig725.png}
\caption{Average settling velocity $V_t$ as a function of density $\rho$ [gm/cm$^3$]. The trends are similar to those seen in Fig. 7.12.}
\end{figure}
Fig. 7.26. Average settling velocity $V_t$ for a tracer of diameter $D_t = 0.45$ as a function of density $\rho$ [gm/cm$^3$]. Tracer paths still transition from staircasing to diffusive behavior.

Fig. 7.27. A detail of 50 sample paths for a tracer of $D_t = 0.4$ and density $\rho = 14.8$ g/cm$^3$. An image of these paths at a larger scale is found in Fig. 7.15.
Chapter 8

A sphere sinking in bubbles:
experimental measurement of collision effects

8.1 Experimental setup

To be able to observe the effects of collisions and do experimental comparison with the simulations described in chapter 7, we decided to perform another experiment in which collisions would be highlighted. The primary goal is to measure the lateral motion and test for diffusive behavior. We refined the experimental setup described in chapter 6 as follows. First we narrowed the cell in which the spheres were to fall. The purpose to this was to create a cell slightly larger than the ascending bubbles. This creates a single layer of bubbles in which the risk of obscuring the sphere is much smaller. Conversely, because the sphere cannot easily pass before or behind a bubble, the chance of a collision is higher.

In the new experiment, the cell is constructed of two acrylic plates of dimensions 0.64 cm×23.2 cm×30.5 cm (\(\frac{1}{4}''\times 9\frac{1}{8}''\times 12''\)). The front sheet is transparent, while the back sheet is translucent to help diffuse light. The acrylic plates slot into aluminum plates in channels separated by 0.32 cm (\(\frac{1}{8}''\)). Because the cell is wider than the bubbler, weighted inserts were made to insert into the cell to close the ends of the experiments. (If the edges of the cell are not closed, the overall flow becomes irregular.)

We decided to use a small, light sphere so that the collisions with bubbles would be more evident. This also has the benefit of avoiding situations in which bubbles are likely to be trapped in the wake of the spheres. For the material of the sphere we determined to avoid plastics and rubber, which were materials we had seen attach to bubbles in our first experiment. We chose to use aluminum spheres of \(\frac{3}{32}''\) diameter (0.24 cm, McMaster-Carr aluminum alloy 2017, density 2.8 g/cm\(^3\)).

One area of great interest to us was to make a uniform field of bubbles; that is, a bubble field in which the bubbles sources were evenly distributed and the bubble sizes were monodisperse. In this we decide to follow the work of R. Zenit who produces monodisperse bubble fields using banks of capillaries (see e.g. [100]).\(^*\) To prevent polydispersity through the coalescence of bubbles, Zenit also adds MgSO\(_4\) to the fluid (both water and water-glycerol mixes). At a concentration of 0.05mol/l, this inorganic salt helps prevent bubbles form coalescing (a common phenomenon in our previous experiment) by creating a short-range (hydration) force between the bubbles. However, it does not at these concentrations substantially alter the surface tension of the bubbles, or change the density or viscosity of the fluid [100]. At his advice, we also began to supply

\(^*\)As we experimented with capillary banks, we noticed an interestingly-shaped bubble plume which we describe in appendix D.
gas from a Nitrogen tank to avoid the possibility of introducing oil from the compressor into the system.

To create the capillary bank, we initially used glass capillaries (Drummond Microcaps, length 5.5 cm, inner diameter 0.015 cm), which produced uniform bubbles of $D_t = 0.21$ cm. The capillaries were put in arrays in acrylic sheets, and held in place with casting acrylic. An example of the bubble field generated by such an array is shown in Fig. 8.1. While experimenting with these capillaries, we came across the interesting phenomenon of hourglass-shaped bubble plumes which will be described in appendix D. Unfortunately, despite much care, the capillaries were very prone to breaking and impossible to replace in the array once broken.

![Fig. 8.1. Bubbles produced uniformly by a bank of nitrogen-fed glass capillaries.](image)

We replaced the glass capillaries with stainless steel needle-tubing that had been cut flat and deburred (Vita Needle, length 3.8 cm, inner diameter 0.015 cm). Forty-five needles were spaced evenly in an acrylic sheet with a separation of 0.37 cm. The heights were staggered by 0.32 cm to prevent the possibility of coalescence of the bubbles while they were forming on the needles. Also, the needles were glued on the top with super-glue so that they could be removed and replaced if necessary. The new arrangement is shown in Fig. 8.2.

Our initial reaction was disappointment: although the needles had the same inner diameter as the glass capillaries, the bubbles produced had much smaller diameters (a factor of about two). To increase the size, we treated the surface of the needles to make them hydrophobic. Our first attempts with rain-repellent for cars produced larger bubbles, but quickly wore off of the needles. We found success with Dow Corning high
vacuum grease (a silicon lubricant). Larger bubbles were produced and the product is stable for the course of experiments, the results are shown in Fig. 8.3. Before a run of experiments the needles are cleaned and re-greased. The grease is thick enough to clog some of the needles, but this is remedied by suctioning individual needles. Care is taken to clean the spheres after an experiment, as there is some transfer of grease from the needles when the reach the bottom of the cell. If not removed, this leaves them susceptible to having bubbles attach to them regardless of the material.

8.2 Experimental Results

8.2.1 Settling Velocity

The average \( y \) position of 50 experiments shows a linear trend as shown in Fig. 8.4, along with a fit of the data. The settling velocity \( V_T \) is 3.9 cm/s. Note that the length of the runs varied from a little over three seconds to more than the camera's buffer length of about 13 seconds. The average recorded length was 6.5 s. In this analysis, we have used only 3.6 seconds of data as this was the length of the shortest run. A set of 25 experiments on spheres of \( D_s = 5/64" \) similarly yields a settling speed of 1.8352 cm/s over a sample period of about 6 s.

We wished to determine the circulation velocity in the cell, so that this could be subtracted from the terminal velocity of spheres sinking inside the cell but in quiescent fluid. The difference between this and the settling speed is likely caused by collision between the sphere and the bubbles. We can infer - based on our experiment described in Chapter 6 that the circulation velocity is the difference between the speed of a single bubble rising in the cell and the rise velocity of the bubbles in the bubbly fluid.

We measured the average rise velocity of bubbles rising singly through the cell and found a velocity of 18.8 cm/s. For bubbles rising in the array we measure an average rise velocity of 34.4 cm/s. We therefore estimate a circulation velocity of 15.6 cm/s.
The settling velocities of the larger and smaller spheres in quiescent water in the cell we measured respectively at 29.4 cm/s and 26.8 cm/s. The effect of circulation alone should bring the settling velocities to 13.8 cm/s and 11.2 cm/s. Thus we estimate the slowing effect of the collisions to be 9.9 cm/s and 9.4 cm/s respectively.

8.2.2 Diffusivity

Paths from the first 3.6 s of the 50 runs for $D_s = 3/32"$ are shown in Fig. 8.5. The starting positions of the recorded paths are placed at the origin. The single run rising up above the origin was unusually long and the path recording began well below the top of the cell. There is a noticeable outward spread and the lateral motion does appear to be diffusive. The growth of variance over time is shown in 8.6 with a fit of $y = 0.59x + 0.02$. This translates to a coefficient of diffusion of $D = 0.30$. Note that the variance of the sample variance is proportional to the square of the variance [6] which may explain the growing fluctuations in the variance about the linear fit. The spheres of $D_s = 5/64"$ have a coefficient of diffusion of $D = 0.35$.

We can visually compare sample paths for the larger spheres to solutions of $\Phi_t = 0.3\Phi_{xx}$ with initial condition $\Phi(x,0) = 50\delta(x)$. The solution is $\Phi(x,t) = \frac{50}{\sqrt{1.2\pi t}} \exp\left(-\frac{x^2}{1.2t}\right)$. Propagated downward at velocity 3.9 this becomes

$$\Phi(x,-y) = \frac{50}{\sqrt{1.2\pi y/3.9}} \exp\left(-\frac{x^2}{1.2y/3.9}\right)$$

(8.1)

as shown in Fig. 8.7.

8.2.3 Staircasing

As staircasing was a prominent feature of the model results, we wished to know whether the phenomenon was seen experimentally. The answer is that staircasing does occur, but on a much smaller scale. For example, the sphere in 8.8 shows short instances of staircasing. The instance ends when the sphere runs into a larger gap between bubbles.
Fig. 8.4. Average vertical position $\langle y \rangle$ [cm] as a function of time [s] for spheres of diameter $D_t = 3/32"$. This is shown along with a fit of $y = -3.9t - 0.18$.

The staircasing behavior in simulations is very dependent on the regularity of the arrays and such short instances are as much as we might reasonably expect.
Fig. 8.5. Experimentally observed paths. Data are shown for 50 paths over about 3.6 seconds for spheres of diameter $D_s = 3/32^\circ$. 
Fig. 8.6. Variance in the $x$-coordinate of position as a function of time. Data are shown for 50 paths over about 3.6 seconds for spheres of diameter $D_s = 3/32"$. Also shown is a fit of $y = 0.59x + 0.02$.

Fig. 8.7. Comparison of sample paths with a solution to diffusion. See text for details.
Fig. 8.8. The path of a sphere \( (D_s = 3/32\) ) (shown in yellow) shows short instances of staircasing.
Chapter 9

Conclusion and future directions

9.0.4 Fragmentation of a 1-D rod

In the beginning of this work, we examined the mathematics describing the fragments produced by breaking a one-dimensional rod. We were able to provide models in the case of fragmentation of either a discrete or continuum rod. The continuum model was found to be capable of producing scaling, which is important as power laws describe the distribution of fragment sizes in many different circumstances. We also found that this model provided satisfactory qualitative results when compared to experiment.

The continuum model corrects a commonly accepted model for fragment sizes arising from a homogenous Poisson process. The misunderstanding seen in the presentation of the extant model seem to typify errors made in more complex analyses.

In the future there are two important additions that could be made to this work. The first would be to relate the physical state of a rod to its probability of breaking. The second would be to extend the one dimensional model to two or three dimensions. Of these two, we became interested in the fragmentation in higher dimensions. Although the question of geometry and crack intersection becomes very difficult in two dimensions, there may be some promise in pursuing fragments distributions that are created as Voronoi tessellations.

9.0.5 Spheres sinking in bubbles – a proposed continuum model for path-clearing

In the latter part of this work we examined the motion of a sphere settling in a bubbly fluid. We found that water circulating with the bubbles must be taken into account when trying to determine the effects of the bubbles on the spheres. By refining our experiment we were able to examine the effect of bubble collisions with small, light spheres. Two effects of these are a further slowing of a sphere’s descent in the liquid, and lateral diffusive motion of the spheres.

Based on similarities with the classic Galton board, we created a model for a fluid Galton board and demonstrated the occurrence of these two effects (slowing and diffusion) in simulations. We also found parameter ranges in which staircasing – the diagonal descent of a sphere as if down a staircase – may occur.

One useful extension of this model would be to determine what the effect of multiple tracers would be on the model. Heuristically, one might expect that many tracers released over time from a central source might clear a path through the rising bubbles so that those following experienced fewer collisions. Perhaps this sort of path-clearing can come, not from the fluid Galton model, but from an expanded version of
Galton’s model of random left-right steps. We present a model of this sort, combining a random lateral movement of the tracer with partially mobile scatterers.

We assume that there is a population of tracers with (number) density $T(x, z, t) \text{cm}^{-2}$ and a population of scatterers with density $S(x, z, t) \text{cm}^{-2}$. We will reduce the scatterers once more to point particles for convenience and let the radius of the tracer be $r_t$. We will furthermore allow the tracers to interact only with scatterers, and likewise scatterers to interact only with tracers.

The tracers will move down at a steady speed $v$. In principle we might let $v = v(S)$, but for now we will assume this is constant. At each collision of a tracer with a scatterer the tracer will move to the left or right a distance $L_T$, each with probability $\frac{1}{2}$. The scatterer will be moved in the opposite direction a distance of $L_S$. Thus, after $2N$ collisions, the probability that the particle has moved to the right a distance of $2ML_T$ is

$$\left(\frac{1}{2}\right)^{2N} \binom{2N}{M+N}. \quad (9.1)$$

A similar expression is available for an odd number of collisions. In either case, the standard deviation of the distance moved after $N$ collisions is

$$\sigma = \frac{L_T}{2} \sqrt{N}. \quad (9.2)$$

A tracer moving with velocity $v$ will sweep out new area at a rate of $2r_T|v|$ per unit time. We take this to be approximately $2r_Tv$, and take the expected number of collisions in time $\Delta t$ to be

$$E(N) = 2r_Tv\Delta t S. \quad (9.3)$$

The rms lateral displacement in time $\Delta t$ of a tracer based on this number is

$$\sigma = \frac{L_T}{2} \sqrt{2r_Tv\Delta t S}. \quad (9.4)$$

This can also be derived more rigorously by saying that we take the scatterers to be distributed according to a Poisson process with density $S$ and finding the variance of the displacement for $N$ and $M$.

We will now parallel a derivation for the flux of tracers per unit time through a vertical line following the derivation of Einstein (1907) [23] – the main differences in our argument being that the particles with random lateral motion are also moving downward at a constant rate, and are being scattered by a non-uniform population. To begin with, we will make our calculations as if all the tracers moved a distance $\sigma$ in time $\Delta t$. Now fix a vertical line going through $x_0$. We replace the quantities $T$ and $S$ with the quantities

$$T(x) = \frac{1}{\Delta z} \int_{z_0}^{z_0+\Delta z} T \, dz$$

and

$$S(x) = \frac{1}{\Delta z + v\Delta t} \int_{z_0}^{z_0+\Delta z + v\Delta t} S \, dz.$$
We establish rectangles to each side of the line with width $\sigma$ and height $\Delta z$, centered at $(x_0 \pm \sigma/2, z_0 + \Delta z/2 - vt, t_0 + t)$. In time $\Delta t$, half of the tracers in the box to the left of the line will move across the line to the right. Similarly half of the tracers in the box to the right will cross to the left. With $T^\pm = T(x_0 \pm \sigma/2)$, this gives a net flux of approximately

$$
\frac{1}{2} \left( T^+ - T^- \right) \sigma \Delta z = \frac{1}{2} \left( \frac{T^+ - T^-}{\sigma} \right) \sigma^2 \Delta z \quad (9.5)
$$

across a segment of the line of height $\Delta z$. In other words, the rate of flux per unit height per unit time is

$$
\frac{1}{2} \frac{\sigma^2}{\Delta t} \left( \frac{T^+ - T^-}{\sigma} \right). \quad (9.6)
$$

Letting $\Delta z$ and then $\Delta t$ go to zero we see that the flux across the line is

$$
J(x, z, t) = -\frac{1}{4} L_T^2 r_T v S(x, z, t) \frac{\delta T(x, z, t)}{\delta x} \quad (9.7)
$$

per unit time per unit distance.

We proceed with a standard conservation of mass approach to find that

$$
T(x, z - v dt, t + dt) - T(x, z, t) \approx -dt \frac{\partial J}{\partial x} = dt \frac{1}{4} L_T^2 r_T v \frac{\delta}{\delta x} \left( S \frac{\delta T}{\delta x} \right). \quad (9.8)
$$

so that

$$
\frac{\delta T}{\delta t} - v \frac{\delta T}{\delta z} = \frac{1}{4} L_T^2 r_T v \frac{\delta}{\delta x} \left( S \frac{\delta T}{\delta x} \right). \quad (9.9)
$$

Note that if $S(x, z, t) = S_0$, then this describes (in the rest frame of the tracers) the lateral diffusion

$$
\frac{\delta T}{\delta t} = \frac{1}{4} L_T^2 r_T v S_0 T_{xx}.
$$

For the evolution of $S$, we likewise examine the flux through a vertical line through $x_0$. Take a small rectangle of width $dx$ and height $v dt$. In time $dt$ there are $vT dx dt$ tracers traveling (in a cumulative sense) the height of the box. This produces a total of $2r_T v^2 ST dx dt$ collisions. To establish (as we did for the tracers) a common distance traveled by each scatterer we assume that each scatterer is hit an equal number of times. This is then

$$
N = 2r_T v^2 ST dx dt / (vS dx dt) = 2r_T v T
$$

so that for the scatterers, the rms distance traveled per unit time is

$$
\sigma_S = \frac{L_S}{2} \sqrt{2r_T v T}. \quad (9.10)
$$

Similarly to the above,

$$
J_S(x, z, t) = -\frac{1}{2} \sigma_S^2 \frac{\delta S}{\delta x} = -\frac{1}{4} L_S^2 r_T v T \frac{\delta S}{\delta x} \quad (9.12)
$$
$\frac{\delta S}{\delta t} = \frac{1}{4} L^2 T \frac{\delta}{\delta x} \left( T \frac{\delta S}{\delta x} \right). \tag{9.13}$

This is similar to 9.9 with the exception that there is no vertical transfer. As before, if $T$ is constant in $x$, this simply becomes the diffusion

$\frac{\delta S}{\delta t} = \frac{1}{4} L^2 T \frac{\delta}{\delta x} T \frac{\delta S}{\delta x}. \tag{9.14}$

This, in fact, is what leads us to speculate that path-clearing may occur: if we have a source of tracers, $T$ will be fixed along the line of insertion, positive at the place of insertion and zero elsewhere. The scatterers below the source will be mobile, while those elsewhere will not be as mobile. This may lead to a decrease of the scatterers below the source, and allow the concentration of tracers to propagate downwards.

We have derived a coupled PDE model

$$\begin{cases}
\frac{\delta T}{\delta t} - v \frac{\delta T}{\delta z} = \frac{1}{4} L^2 T \frac{\delta}{\delta x} \left( S \frac{\delta T}{\delta x} \right) \\
\frac{\delta S}{\delta t} = \frac{1}{4} L^2 T \frac{\delta}{\delta x} \left( T \frac{\delta S}{\delta x} \right). 
\end{cases} \tag{9.15}$$

Individually these equations are those of heat conduction with a variable thermal conductivity. It remains to find solutions either analytically or numerically and see if path-clearing occurs and if this resembles the fluid Galton model with many successive tracers.
Appendix A

Coefficient of restitution

In this appendix we will closely follow the presentation of Stronge [91] on rigid body impact, although our work will be much simplified in that we will consider only collisions between smooth spherical particles. We have spheres one and two with velocities $V_1$, $V_2$ and masses $m_1$, $m_2$. First consider a direct impact in which the relative velocities are parallel to the normal vector at the point of collision as in Fig. A.1. Because the components of motion and acceleration are along one axis, we shall write them as scalars.

![Fig. A.1. A direct collision.](image)

From the conservation of momentum, the impulse imparted to each particle is equal in magnitude and opposite in direction. Then we can write the velocities from the time of impact $t_0$ as

$$V_1(t) = V_1(t_0) + \frac{1}{m_1}P(t)$$  \hfill (A.1)

and

$$V_2(t) = V_2(t_0) - \frac{1}{m_2}P(t).$$  \hfill (A.2)

(Note that this implies $P(0) = 0$.)

The centers of the particles move closer together until the time $t_c$ when the velocities are equal, i.e.

$$V_1(t_0) + \frac{1}{m_1}P(t_c) = V_2(t_0) - \frac{1}{m_2}P(t_c)$$  \hfill (A.3)
so that

\[
\left( \frac{1}{m_1} + \frac{1}{m_2} \right) P(t_c) = - (V_1(t_0) - V_2(t_0)). \tag{A.4}
\]

The work done during this time is

\[
W_c = \int_0^{t_c} F V_1 - F V_2 \, dt = \int_0^{t_c} (V_1 - V_2) (F dP) = \int_0^{t_c} (V_1(t) - V_2(t)) P(t_c) \, dP = (V_1(t_0) - V_2(t_0)) P(t_c) + \frac{1}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) P(t_c)^2
\]

\[
= - \frac{m_1 m_2}{m_1 + m_2} (V_1(t_0) - V_2(t_0))^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (V_1(t_0) - V_2(t_0))^2 \]

\[
= - \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (V_1(t_0) - V_2(t_0))^2. \tag{A.5}
\]

During this time, a small volume of the colliding particles is deformed and some of the energy from the deformation may be stored elastically, causing the spheres to rebound from each other. This process will take place from time \( t_c \) to time \( t_f \) doing the work

\[
W_r = \int_{P(t_c)}^{P(t_f)} (V_1(t_0) - V_2(t_0)) + \left( \frac{1}{m_1} + \frac{1}{m_2} \right) P \, dP = (V_1(t_0) - V_2(t_0)) (P(t_f) - P(t_c)) + \frac{1}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) (P(t_f)^2 - P(t_c)^2)
\]

\[
= - \frac{m_1 m_2}{m_1 + m_2} (V_1(t_0) - V_2(t_0))^2 \left( \frac{P(t_f)}{P(t_c)} - 1 \right) + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (V_1(t_0) - V_2(t_0))^2 \left( \frac{P(t_f)}{P(t_c)} - 1 \right)
\]

\[
= - \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (V_1(t_0) - V_2(t_0))^2 \left( \frac{P(t_f)}{P(t_c)} - 1 \right)^2. \tag{A.6}
\]

The coefficient of restitution is the root of the proportion of the restoring work to the compressing work, i.e.

\[
e^2 = \frac{W_r}{W_c} = \left( \frac{P(t_f)}{P(t_c)} - 1 \right)^2. \tag{A.7}
\]

The coefficient of restitution defined as

\[
e = \frac{P(t_f) - P(t_c)}{P(t_c)} \tag{A.8}
\]

is called the kinetic coefficient of restitution, and dates to Poisson in 1811. We can also use Eqs. A.1, A.2, A.4 to obtain

\[
e = \frac{V_1(t_f) - V_2(t_f)}{V_1(t_0) - V_2(t_0)} \tag{A.9}
\]

which is the kinematic coefficient of restitution proposed by Newton in 1686. In the situations we are considering, these are equivalent.
From Eq. A.9 we have
\[ V_1(t_f) - V_2(t_f) = eV_2(t_0) - eV_1(t_0). \]  
(A.10)

From conservation of momentum we have
\[ V_2(t_f) = V_2(t_0) + \frac{m_1}{m_2} (V_1(t_0) - V_1(t_f)). \]  
(A.11)

Together these give us
\[ V_1(t_f) - V_2(t_0) + \frac{m_1}{m_2} (V_1(t_f) - V_1(t_0)) = eV_2(t_0) - eV_1(t_0) \]  
(A.12)

so that
\[ \frac{m_2 + m_1}{m_2} V_1(t_f) = (1 + e)V_2(t_0) - eV_1(t_0) + \frac{m_1}{m_2} V_1(t_0) \]  
(A.13)

and
\[ V_1(t_f) = \frac{m_2}{m_2 + m_1} ((1 + e)V_2(t_0) - eV_1(t_0)) + \frac{m_1}{m_2 + m_1} V_1(t_0). \]  
(A.14)

Because \( t_f - t_0 \) is very small, we will consider the collisions to be instantaneous and write
\[ V_1(t_0) = v_I, \quad V_1(t_f) = v_R \quad \text{and} \quad V_2(t_0) = u_I, \quad V_2(t_f) = u_R. \]

So that
\[ v_R = \frac{m_1 v_I + m_2 u_I + em_2 (u_I - v_I)}{m_1 + m_2}. \]  
(A.15)

For an equation in which there are normal components to the velocity as in Fig. A.2 the situation is generally more complicated. However, because we are supposing the particles to be smooth (in this sense, to pass without sticking or friction) the tangential components of forces between the spheres are zero. In this case, the tangential velocities remain unchanged. In this case (with \( \hat{t} \) in the direction of the tangential component of \( v_I \))
\[ v_R \cdot \hat{t} = v_I \cdot \hat{t} \]  
(A.16)

and
\[ v_R \cdot \hat{n} = \frac{m_1 v_I \cdot \hat{n} + m_2 u_I \cdot \hat{n} + em_2 (u_I \cdot \hat{n} - v_I \cdot \hat{n})}{m_1 + m_2}. \]  
(A.17)

If the second sphere is fixed in position, it has no velocity and can be treated as a particle of infinite mass. In this case Eq. A.17 becomes
\[ v_R \cdot \hat{n} = -e v_I \cdot \hat{n}. \]  
(A.18)
Fig. A.2. An oblique collision.
Appendix B

A coefficient of restitution
for bubbles, drops, and particles

In our simulations a simple coefficient of restitution was used for the rebound of the spheres and bubbles. A more realistic coefficient of restitution would depend on characteristics of the collision such as the Stokes number (provided later in this section) and the capillary number \( Ca = \mu U/\sigma \) (where \( \sigma \) is the surface tension of the bubble). Although the nature of a coefficient of restitution for a bubble and sphere is unknown at this time, we provide here a review of literature discussing what is known about using a coefficient of restitution to describe the rebound of submerged solid spheres, drops and bubbles.

B.1 Approach and rebound of two spheres

Ironically, the first predictions for a submerged coefficient of restitution were developed by researchers interested in criteria for particles to remain together after collision in a fluid. In 1986, Davis, Serayssol and Hinch [21] modeled the collision of two smooth elastic spheres in a fluid of constant viscosity and density. Their analysis shows how high pressure in the inter-particle gap may cause deformation of the spheres, leading to a rebound.

Much of the work in their paper and in subsequent research from members of this group and their colleagues [83, 7, 8, 9, 20] focuses on the forces between two approaching spheres in a viscous fluid and the corresponding deformation of the spheres. Although we will not expand our review to the more general question of the approach of two spheres, we will consider these papers as they lay a foundation that provides important results in describing a submerged coefficient of restitution. The theoretical work by the preceding authors has been examined in experiments with collisions that are not fully submerged.

The initial analysis of Davis [21] was used and expanded in subsequent papers [83, 7, 8, 9, 20]. In their analysis, two spheres of radii \( a_1, a_2 \) approach each other in a viscous fluid. It is assumed that very little deformation occurs until the inter-particle gap is small enough that the lubrication equation

\[
\frac{\partial h}{\partial t} = \frac{1}{12\mu r} \frac{\partial}{\partial r} \left[ rh^3 \frac{\partial p}{\partial r} \right]
\]

(B.1)

holds. Here \( h \) is the inter-particle gap parallel to the axis of the two spheres, and measured at a radius \( r \) from the axis. The profile of the spheres are approximated by
parabolas, so that the gap $h$ can be written (see Fig. B.1)

$$h(r, t) = x(t) + w(r, t) + \frac{r^2}{2a} \tag{B.2}$$

where $x$ is the distance that would have existed between the undeformed surfaces, $w$ is the sum of the deformations of the two surfaces and $a$ is the reduced radius of the spheres (i.e. $a = \frac{a_1 a_2}{a_1 + a_2}$). Note that one of the radii may be set to $\infty$ to describe the collision of a sphere with a wall.

![Diagram](image)

Fig. B.1. The undisturbed (dotted lines) and physical profiles (solid lines) of two spheres show the centerline distance $x(t)$ and the deformation $w(r, t) = w_1(r, t) + w_2(r, t)$.

Describing the gap in this manner allows a coupling with the motion of the spheres

$$\frac{dx}{dt} = -v(t), \quad m\frac{dv}{dt} = -F(t) \tag{B.3}$$

where $m$ is the reduced mass of the spheres, and the force $F$ is found by integrating surface forces (assumed to act equally on the two surfaces.) In the initial analysis, this is the hydrodynamic pressure. (Gravity is ignored as its effect is of a smaller order of magnitude based on typical Froude numbers.)

Later papers considered the contribution of other sources of interparticle force. Serraysol and Davis [83] (with an eye to colloids and aerosols) considered the addition of London-van der Waals and electrical double layer forces. However, in the case of rebound leading to separation of the particles (the case of interest to us) they conclude that

For particles that have sufficient inertia to deform and rebound, the electrical double layer repulsive forces, and the potential barrier between the primary
and secondary minimum, are negligible during the collision process for all cases of practical interest.

Barnocky and Davis [7] considered effects based on the observation that in air, the mean separation between the spheres considered by Davis [21] could be on the order of several nanometers – smaller than the mean free path of the air molecules. They account for non-continuum effects by adjusting the no-slip boundary conditions of the continuum equations. This reduces the magnitude of interparticle forces, allowing them to draw much closer or even collide. However the results are again not applicable to a liquid, whose molecules have a much shorter mean free path. Davis [19] discusses both of these effects with a small section on the effect of surface roughness, modeled as small hemispheres attached to the larger, colliding spheres. He concludes that for an example case, the hydrodynamic effect of the hemispheres is unimportant up to the point where they begin to touch and support the other sphere.

Returning to the initial analysis of Davis [21], the deformation of the spheres was coupled to the interparticle forces through the experimental parameters of Young’s modulii and Poisson ratios. By non-dimensionalizing, two parameters governing the rebound of the sphere were identified: the particle Stokes number

\[ St = \frac{mv_0}{6\pi\mu a^2} \]  

(B.4)

– a measure of the relative effect of particle inertia and viscous dissipation in the fluid (as opposed to the Reynolds number which considers the fluid inertia) – and the elasticity parameter

\[ \epsilon = \frac{4\theta \mu v_0 a^{3/2}}{x_0^{5/2}} \]  

(B.5)

where

\[ \theta = \frac{1 - \nu_1^2}{\pi E_1} + \frac{1 - \nu_2^2}{\pi E_2} \]  

(B.6)

is derived from the Young’s modulus \(E_i\) and Poisson’s ratio \(\nu_i\) for each sphere, and \(x_0, v_0\) are initial separation and velocity at which deformation is considered to begin. In the initial consideration a typical value for \(\epsilon\) is given as \(10^{-7}\) to \(10^{-5}\) (based on \(10 - 100\ \mu m\) sandstone particles in water.) For comparison, later experiments in which steel and nylon balls of \(a = 0.32\) to \(0.64\ cm\) were dropped onto a quartz disk have elasticity parameters on the order of \(10^{-6}\) to \(10^{-4}\) and \(10^{-3}\) to \(10^{-2}\) respectively [20].

There are two critical findings derived from their analysis (based on asymptotic analysis for small deformations and numerical analysis for larger). The first is that there is a critical Stokes number below which no rebound occurs. This number is based off of the elasticity parameter and can be taken to be [20]

\[ St_c = -0.4 \log(\epsilon) - 0.2. \]  

(B.7)

For the values of \(\epsilon\) mentioned here, this ranges from about 1 to 7.

Barnocky and Davis [8] verified the existence of the critical Stokes number experimentally by dropping stainless steel and lucite spheres \((a = 0.8\ to\ 3.2\ mm)\) onto a
quartz disk covered with a thin layer of oil. By measuring the dropping height at which
the spheres rebounded for varying oil depths and viscosities, they measured the critical
Stokes number and found favorable comparison to their (slightly different) prediction of

\[ St_c = -0.5 \log(\epsilon) - 1.3. \]  

(B.8)

Davis, Rager and Good [20] found similar agreement with experiments using stainless
steel and nylon spheres. They noted that their measured values of \( St_c \) were generally too
low for small values of \( 1/\epsilon \) (for which the spheres do not penetrate very far into the oil
layer), but that in this case the analysis does not hold as the requirement that \( \epsilon << 1 \)
is not met.

Alternately, for large values of \( 1/\epsilon \) the measured values of \( St_c \) were higher than
predicted. This could be [8] because the analysis predicts a rebound, not necessarily
enough of a rebound to visibly exit the oil layer. The rebounding sphere is also subject
to tensile stress as it moves away from the wall. However, a typical pressure drop of
several hundred atmospheres may cause cavitation and mitigate this effect. This could
also be [20] from an initial deformation of the sphere as it first penetrates the oil, before
the point at which the original analysis assumes the beginning of deformation.

The second finding of Davis [21] is that the non-dimensionalized (maximum) re-
bound velocity \( \frac{v_r}{v_0} \) may be thought of as a coefficient of restitution. This coefficient
of restitution was explored numerically for a variety of \( \epsilon \) through Stokes numbers of about
\( St = 30 \). The coefficient of restitution is seen to remain near zero until the critical Stokes
number, after which it increases rapidly. In their experiments with stainless steel and
nylon spheres, Davis [20] measured the coefficient of restitution. To account for energy
dissipated by the spheres and quartz, the coefficient of restitution is multiplied by the
“dry” coefficient of restitution to get

\[ e = e_{dry} \left( 1 - \frac{St_c}{St} \right). \]  

(B.9)

The data collapse convincingly to this master curve with no free parameters. This curve
is shown in Fig. B.2.

Another experimental study on a wet coefficient of restitution using a thin oil
coating is due to Lundberg and Shen [61]. In this experiment, a roller attached to the
end of a pendulum arm was allowed to strike a sphere resting on a piezoelectric sensor.
The coefficient of restitution was measured both by comparing incoming and outgoing
velocities using an accelerometer in the pendulum, and by integrating the forces before
and after the instant of rebound. A drop of oil placed on the sphere provided the lu-
brication layer. It is difficult to compare results, as Lundberg presented the normalized
coefficient of restitution \( (e/e_{dry}) \) for several impact velocities as a function of the viscos-
ity of the applied oil rather than using the Stokes number. Lundberg showed that the
coefficient of restitution generally decreases with increasing viscosity (as should be ex-
pected for a decreasing Stokes number) and that the effect is more pronounced for nylon
than for steel (also expected from the theory of Davis, as the coefficient of restitution
increases more rapidly at smaller Stokes numbers). An interesting note from Lundberg’s
study is a measurement of the wet coefficient of restitution above that of the dry. They also show trends where the coefficient of restitution increases with increasing viscosity.

Returning once more to the study of Davis, Serayssol and Hinch [21], another quantity that was calculated was the minimum distance of approach for two spheres. They noted (correctly as will be seen in later experiments) that their analysis may be problematic in cases where the surface roughness is of a comparable length to the minimum distance of approach. Later experiments, both those of Barnocky and Davis as well as experiments involving fully submerged collisions showed the effects of surface roughness.

Barnocky and Davis [9] extended the results of Davis by allowing viscosity and density of the interstitial fluid to vary with pressure - which in other experiments had been found to be as high as several hundred atmospheres. In their approach

$$\frac{\rho}{\rho_0} = 1 + \frac{\alpha p}{1 + 3\alpha|p|} \quad (B.10)$$

where $\rho_0$ is the fluid density at ambient pressure, and $\alpha$ is an experimental compressibility parameter (typically $O(10^{-4})$ atm$^{-1}$). Viscosity is allowed to vary as

$$\mu = \mu_0(1 + \beta p)^{16} \quad (B.11)$$

where $\mu_0$ and $\beta$ are similarly viscosity at ambient pressure and an experimental parameter.
Their analysis showed that one effect of increased density with pressure is to store a small amount of energy. They noted that this might make a rebound possible even for rigid spheres. In the case of elastic spheres, the effect is to increase the rebound of the spheres and enhance the rebound. Increased viscosity does not store energy in the same way, but does also enhance the rebound by making the fluid harder to evacuate for the interstice – essentially making the fluid act more like a solid.

The general predictions for the coefficient of restitution are qualitatively similar to the ones mentioned before – a critical Stokes number below which no rebound occurs but above which the coefficient of restitution increases rapidly.

Attempts were made to simplify the equations of motion presented by Davis [21]. Wells [95] presented an approximation to sphere deformation designed to preserve the normal force on the spheres. In an inner region, the profile of the sphere is considered to be flat, while outside this region the deformation decreases to zero as the radius increases. This loses some features of the deformation profile as described by Davis, such as a dimple on the axis of approach. However, in this way he reduces the equations of motion for the sphere to a fourth order system of ODEs. The calculated coefficient of restitution is similar to that of Davis for small values of St, but increases more rapidly beyond Stokes numbers of about \( St = 15 \).

In 1996, Lian, Adams and Thornton [58] proposed a further simplification to the theory of Davis [21]. In their model, the profile of the approaching spheres was assumed to be flat within a central region similarly to Wells [95]. Outside of this region, however, the deformation of the sphere profile was taken to be constant. This allowed for an analytic evaluation of the force on the spheres, which for Davis and Wells was only available numerically for all but small deformations. Lian showed that for purposes of measuring the coefficient of restitution, their approximation fits the calculations of Davis very well. Note, however, that this deals with Stokes numbers of up to about 30 – two orders of magnitude smaller than much of the experimental data discussed previously and hereafter.

B.2 Submersed collisions

Experimental work on a fully submerged coefficient of restitution began to be published in 1999 in a brief paper by Gondret, Hallouin, Lance and Petit [31]. Their experiment consisted of glass and steel spheres ranging from 0.05 to 2.5 mm radius falling onto a glass plate in a tank of fluid - variously water, glycerol and silicon oil. The motion of the sphere was recorded on a high speed camera to allow measurement of velocities before and after collision to determine a coefficient of restitution. They observed a qualitative similarity to the experiments of Barnocky and Davis [8] in that there was a critical Stokes number after which the coefficient of restitution rapidly increased. Their measurement of this critical number, however was about 20, over twice as high as those measured previously. For Stokes numbers greater than the critical value they observed a rapid increase of the coefficient of restitution. These values they measured also at high Stokes numbers up to \( O(10^4) \). They found that the coefficient of restitution plateaued at about \( e = 0.65 \) (compared to dry coefficients of restitution of about \( e = 0.99 \) for the steel spheres).
This is somewhat at odds with contemporary work by Joseph, Zenit, Hunt, and Rosenwinkel [45] in which the coefficient of restitution for a submerged collision was measured by using spherical pendula. The spheres used were of glass, steel and plastic with diameters ranging from 3 to 12.7 mm attached to a nylon line of 0.075 mm in diameter. A glass or lucite wall was placed vertically so that the pendulum would strike it as it reached its nadir. The surrounding fluid was variously water or glycerol mixtures. By tracking the particle with a high speed camera, measurements of the sphere velocity before and after the collision were obtained. The critical Stokes number was measured at about 15, and the increase in the coefficient of restitution was found to match well with predictions by Davis [21] for low Stokes numbers as long as the submerged coefficient of restitution was scaled by the dry coefficient of restitution. For higher Stokes numbers, experimental values matched theory based on a model by Barnocky [8] in which the sphere motion follows the model of Davis [21] until contact is made between asperities on the surface of the spheres. At this point the rebound occurs with the same coefficient of restitution as a dry sphere [45].

Joseph found that as Stokes number increased, the value of $e/e_{\text{dry}}$ approached unity. They speculated that the low values of the coefficient of restitution obtained by Gondret were due to the glass plate with which the spheres collided being too thin, allowing more dispersion of energy through the plate. They also observed that the dependence of the critical Stokes number on the elastic properties of the spheres is not apparent in the fully submerged experiment.

Finally, they discussed the importance of surface roughness in the experiments. It has been noted before that the distance between the colliding spheres (or sphere and wall in this case) may be on the same order as the surface roughness, allowing for direct contact between the two. They found evidence for this in that the dispersion of their measurements for the coefficient of restitution was larger than experimental error for Stokes numbers of less than 80 for glass and nylon spheres, but not for steel spheres.

They calculated the minimum distance of approach $h_m$ for the various particles compared to $\sigma_s$, the rms surface roughness. In the case of steel spheres at Stokes numbers of less than 80, the ratio $\sigma_s/h_m$ was typically less than unity (0.5 – 0.6) while those for glass and nylon were anywhere from 5 to 70. This indicates that there may be significant interaction with a random set of surface projections. They also note that the unevenness of the surface at this level may also alter the hydrodynamic interactions by allowing for pockets of high pressure.

Another relevant parameter is the ratio of the contact area $A_h$ to the correlation length of the surface roughness $\lambda_s$. In the case of some of the glass particles $A_h/\lambda_s$ was less than unity, indicating that the sphere does not interact with enough protrusions to hope for some statistical evening of their effects on collisions. Increasing the Stokes number of the collisions lessens both of these effects by increasing the minimum distance of separation and increasing the contact area.

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*This idea was expanded into a more complete model Yang and Hunt in a 2008 paper [99]. In their model, the spheres slow and deform due to lubrication forces until projections from the surfaces can interact. The “dry” collision at this point is damped by viscous forces from the fluid still trapped between the surfaces.
Gondret, Lance and Petit [32] published a subsequent report whose findings about the coefficient of restitution were much more in line with those of Joseph, et. al. In this paper experiments are described in which glass, metal and plastic spheres were dropped onto a horizontal glass plate in a tank filled with air, water or a variety of silicon oils. The release of the spheres was accomplished by means of a magnet or suction with the sphere initially submerged so that air entrainment would not be a factor. Experiments were filmed and the images analyzed to find the paths of motion for the spheres. The collision times were measured with a piezo-electric sensor to be 0.01 ms, whereas the frame rate for recording was 500 fps. Because of the difference in time scales the collisions do appear to be instantaneous in the data.

The effect of the side walls on the measurements was found to be insignificant. The effect of the thickness of the bottom plate of glass was also measured— for thin plates (relative to the diameter of the sphere) more energy is dissipated through elastic waves in the plate. Indeed, the 1999 experiments show much lower coefficients of restitution than those reported in 2002 (Gondret attributed this to a “crude” data analysis in their previous paper.)

Gondret found that by normalizing the coefficient of restitution by the dry coefficient of restitution, they obtain a “master curve” to which all of the measured coefficients of restitution collapse as a function of the particle Stokes number. In particular, the critical Stokes number (10) was not found to vary with the material properties of the spheres. Also, they measured the coefficient of restitution not only for the first bounce of a sphere, but for subsequent bounces, showing that the coefficient of restitution they measured is not affected by the history of the sphere’s path.

In addition to the coefficient of restitution, they gave some consideration to the path of the sphere between bounces. They determine that the path of the sphere was well explained by the forces of form drag, gravity, added mass and a history term scaling with 1/t. They emphasize the importance of the history term because the sphere is rebounding into its own wake.

By 2004 two articles appeared considering oblique collisions. One was by Joseph and Hunt [44] and was a continuation of their earlier work, but with a pendulum striking a vertical wall set obliquely to the line of impact. They found that the coefficient of restitution for the normal impact fit their curve of coefficients of restitution for normal collisions [45] as long as the normal velocity was used to measure the particle Stokes number for the collision. In addition to this they measured a rotational coefficient of restitution and sliding coefficient of friction for describing the effect of oblique impacts. These values were shown to vary greatly depending on the surface roughness of the materials and the angle of impact. A similar experiment by Yang and Hunt was detailed in a 2006 paper [98]. In this experiment the wall was replaced with a second pendulum. The functional nature of the coefficient of restitution, and its independence of tangential interaction between the particles was confirmed in this case as well. A model for tangential and angular restitution was also provided based on the coefficient of restitution in the normal direction.

The other article, by Kantak and Davis, is likewise a continuation of their 2002 experiment [20]: spheres dropped on a quartz disk with a thin layer of silicon oil brushed on. As with Joseph and Hunt, one conclusion was that the results for the normal
coefficient of restitution held as long as parameters were adjusted to the normal velocity of the sphere to the disk. Joseph and Hunt also describe a tangential coefficient of restitution and imparted rotational velocity based on tangential viscous stress in the lubrication layer. Their data fit the trend of their predictions, although the data are clustered in such a way that they represent only very small portion of the functional domain. This makes it hard to see if the functional relationship holds in general.

In 2005 an experiment was reported by Stocchino and Guala [88] for the collision of spheres against a wall in a shear thinning fluid (carboxymethyl cellulose - a commercial product used in detergents and foods.) In their experiments a nickel-steel sphere was dropped from an electromagnetic holder about 14 cm (depending on the sphere size, from 11 to 16 mm) onto the glass bottom of the tank. Measurements of the coefficient of restitution were made in air, water-glycerol mixtures and in CMC solution. Measurements of the coefficient of restitution were consistent with those of Gondret [32]. Measurements of the coefficient of restitution in the shear thinning fluid were higher than those in the Newtonian ones. This was explained as a visco-elastic effect of the fluid in the gap.

It is important to note that in this case the choice of Stokes number is more arbitrary, as the viscosity is not a constant. They chose to use viscosity based on \( v_0, x_0 \) (presented earlier in this appendix) as described Joseph [45]. Also, the measurements were made for a limited range of Stokes numbers. The critical stokes number was not apparent from their experiments, although one can see that it is lower than the value of about 10 given by other researchers. By adjusting these parameters it may be the case that the coefficient of restitution described in this paper is functionally the same as that measured in other papers.

### B.3 A coefficient of restitution for bubbles and drops

It is not immediately clear that a coefficient of restitution is immediately applicable to the collision of a drop or a bubble. The coefficient of restitution models - as an instantaneous event - the compression and restoration of an elastic material over a very short time. We mentioned previously the time scale of 0.01 ms for a submerged collision of a solid sphere with a wall. In contrast, a bubble colliding with a wall undergoes significant deformation over a period on about 10 ms [93]. Not only is the time scale significantly larger, but the center of the bubble remains in motion after hitting the wall, until the bubble is sufficiently flattened that its energy has been dissipated or stored in the increased surface area [93].

Tsao and Koch released in 1997 a study of bubbles bouncing from (or sliding on) solid walls [93]. In their study, bubbles of 0.5 to 0.7 mm \( d_e \) (\( Re \) from 40 to 300) were released beneath a horizontal wall, or a wall inclined from 10° to 85°. Bubbles striking the horizontal wall bounced several times before coming to rest against the wall. To about 55° the bubbles were seen to slide along the wall. For more steeply inclined planes they bounced up the plane with a series of bounces of similar amplitudes. They argue that a coefficient of restitution is inapplicable to the collision of a bubble because of
the large deformation. They also base their argument on the complicated dynamics that cause the bubble to bounce up the inclined plane without losing any amplitude in successive bounces. (They argued that this is impossible for a coefficient of restitution of less than one, although this argument ignores the fact that the bubble also recovers kinetic energy as it moves upwards in the tank.) They did, however, measure enough data to allow later authors to measure the coefficient of restitution for their experiment as \( e = 0.74 \) [101].

There are many studies related to film drainage and collisions of bubbles or fluid drops. For instance Klaseboer, Chevaillier, Mate, Masbernat and Gourdon released experimental work on droplet-wall impact and rebound with an accompanying analysis reminiscent of the one given by Davis for solid particles [50]. However, to our knowledge, the discussion of a coefficient of restitution for drops and bubbles dates to a 2000 report by Richard and Quere [78] (see also Richard, Clanet and Quere [77] or Okumura, Chevy, Richard, Quere and Clanet [74].)

Richard reports experiments dropping 0.4 to 1 mm drops of water and water-glycerol onto a super-hydrophobic surface. The coefficient of restitution was measured at about 0.9 for the smaller drop for a range of impact velocities from about 0.1 to 2.5 m/s. The coefficient of restitution drops slightly for higher or lower velocities of impact, or for the larger spheres.

During the time of the collision (\( \approx 2.5 \) ms), viscous dissipation within the drop is assumed to be negligible (as Reynolds numbers are on the order of 100). This was confirmed by adding glycerol to the drop and finding the coefficient of restitution unchanged in spite of the increase in viscosity. Contact angle hysteresis (the edge of the contact line for the water being caught and released from surface artifacts) was negligible (less than 5 degrees) and also discounted as a mechanism for energy dissipation. Instead Richard attributes the loss of energy to oscillation within the sphere which is dissipated over the inter-collision times. He provides an estimate for the coefficient of restitution based on a linear velocity profile within the drop of 0.91. While he allows that the very good agreement of this estimate may be coincidental, his report does highlight the importance of oscillation as a pathway for energy loss during collisions. This is known to be very important for the case of bubbles [93].

A 2005 study by Legendre, Daniel and Guiraud [56] provided a description for the coefficient of restitution that described much of the available data for both particles and drops. In their experiment they observed drops of toluene (colored by Sudan Red IV) rising in water to strike a horizontal stainless steel plate. Careful measurements were made of kinetic energy of the droplets by adapting the coefficient of added mass to the ellipsoidal shape of the particle. Similarly the energy of surface deformation was measured. It was found that the transfers of kinetic energy to and from energy of deformation lost about 60% of the energy at each transfer.

More than 80% of the kinetic energy was lost in the initial collision of the bubble with the plate. About 4% of energy loss was estimated to be due to viscous dissipation

\(^{†}\)Tsao and Koch did however measure and report [92] a coefficient of restitution of 0.8 – 0.9 for two bubbles of small Weber numbers prevented from coalescing by the addition of salt to the water. These bubbles, in contrast to their later experiment, remained nearly spherical in profile during their collision (which still spanned about 11ms).
within the drop. The dissipation of energy was thus attributed to the external fluid, mainly in the lubrication layer between the drop and the wall. This was estimated in the manner described for the work of Davis and coupled with a mass-spring-type oscillator model for energy stored in deformation. They found the relationship for the coefficient of restitution to be

\[ e = \exp \left( -\frac{\pi}{4} \sqrt{\frac{5}{6} C_\mu \frac{W e^*}{S t^*}} \right) \]  

(B.12)

where \( W e^* \) and \( S t^* \) are Weber and Stokes numbers based on the velocity at the time of impact with the inertial components modified to include the mass and added mass of the droplet at the time of impact. The coefficient \( C_\mu \) is the square root of the capillary number, here \( C_\mu \approx 30 \). For comparison with solid particle experiments, the relation was reformulated to

\[ e = \exp \left( -\beta \frac{S t^*}{S t^*} \right) \]  

(B.13)

The value of \( \beta \) for Legendre’s experiment was about 14, and this curve does indeed pass through the center of his measured values, which unfortunately cover a fairly small range of (modified) Stokes numbers. However, by changing the value of \( \beta \) to 35, they created a single curve that describes well the values of \( e/e_{dry} \) from many other experiments including those of Joseph [45], Gondret [32], Richard [78] and Tsao [93] (here the modified Stokes numbers use the coefficient of added mass calculated far from the wall [57]). (In the case of fluid drops the value of \( e_{dry} \) is set to the maximum observed coefficient of restitution. In the case of bubbles, it is arbitrarily set to one.) Even Legendre’s own experimental data fit this curve if the incoming velocity is assumed to be the maximum approach velocity and not the velocity at the time of impact. Note also that the bubble data of Tsao is very limited, and the values for their earlier experiment [92], although shown, lie far from the curve.

Legendre, Zenit, Daniel and Guiraud added a cautionary note for modeling collisions using a wet coefficient of restitution [57]. In cases where the contact time for fluid particles is of the same order of magnitude as the relaxation time (e.g. with the toluene drops) the time of contact must be an explicit part of any simulations.

In 2009 a study was published by Zenit and Legendre that most closely addresses our area of interest – a coefficient of restitution for air bubbles colliding with a solid wall [101]. In this experiment, air bubbles from 0.1 to 1 mm were released in water and water-glycerol mixtures to impact a horizontal wall of glass or plexiglas. By using a damped oscillator model similar to that of Legendre [56] they measured the coefficient of restitution (based on the terminal velocity of the bubble for the incoming velocity) for (modified) Stokes numbers of about 1 to 300. they found a good deal of variability in their measurements (up to about 20%).

Their analysis showed that the coefficient of restitution did not vary solely with the (modified) Stokes number, but

\[ e \approx \exp \left( \beta \sqrt{\frac{C a}{S t^*}} \right) \]  

(B.14)
Fig. B.3. The master curve of Legendre for values of the normalized coefficient of restitution [56]. Here \( e/e_{dry} = \exp \left( -\frac{35}{St^*} \right) \).

This model with \( \beta = 30 \) (determined by a best fit to the data) matches the trend of the experimental data for the coefficient of restitution. Note that since the capillary number and Stokes numbers both depend linearly on velocity, the coefficient of restitution in this case is predicted to be independent of the approach velocity. Zenit found it difficult because of the limited range of capillary numbers in the experiment to determine whether this model fit the experimental data more closely than the relation for solid spheres (with the \( \beta \) of Eq. B.13 also fit to the data.)

Finally we note that Zenit considered cases of linear bubble rise. He points out that the complicated wakes of higher Reynolds number bubbles complicate and may substantially change the rebounds described above.
Appendix C

A modified simulation

C.0.1 Modifications

In the original simulations of chapter 7, the equations of motion were integrated using a fourth order Runge-Kutta method using linear interpolation between steps to evaluate collisions. Unfortunately, this turns a fourth order method into a first order method during the times of greatest interest to us! To see what effect this had on the simulations, we replaced the original method with the continuous Runge-Kutta method described in [89]. This method uses simultaneous $4^{th}$ and $5^{th}$ order Runge-Kutta methods to create an adaptive time step, with the advantage that an interpolation of order 4 can be made in between. The interpolation for coordinates is a fourth degree polynomial in time, meaning that intersection times of spheres can be found as the root of an eighth degree polynomial, evaluated using Jenkins-Traub.

In this case, if the distance $D$ between two particles at $x_1, \tilde{x}_1$ is at least twice the linear distance moved by each from their previous positions at $x_0, \tilde{x}_0$, i.e.

$$D > 2.0 \times (|x_1 - x_0| + |\tilde{x}_1 - \tilde{x}_0| + r_t + r_s),$$

then we assume that there is no collision, as the separation is too great. In all other cases we check for a collision at positive time. If the minimum positive collision time $t$ is not greater than the length of the time step, $t \leq \Delta t$, a collision is determined to have occurred. In this case an interpolation is made for the position of the tracer and all particles. There is an added issue that if two particles collide and are positioned so that they overlap slightly, the simulation will not continue. In these cases we arbitrarily move the first particle (because in many cases the second particle will be a fixed scatterer) along the line between particle centers until they are no longer overlapping.

We also modified the evaluation for sliding. In the original version of a slide step, we combine normal forces in acceleration without requiring that the particles are actually remaining in contact during the step. During the next time step we allow the spheres to slide across the surface of each other (without friction). Consider the problem of two spheres, each sliding across the surface of the other. The first has center, (virtual) mass and radius of $(x_1, \tilde{z}_1)$, $M_1$ and $R_1$ and is acted on by a force $f = (f_x, f_z)$. The corresponding quantities for the second sphere are $(x_2, \tilde{z}_2)$, $M_2$, $R_1$ and $F = (F_x, F_z)$. We will define for convenience $X = x_1 - x_2$, $Z = \tilde{z}_1 - \tilde{z}_2$ and $R = R_1 + R_2$.

We wish to determine the equations of motion for the two spheres. We will require conservation of momentum, i.e.

$$f_x + F_x = M_1 \ddot{x}_1 + M_2 \ddot{x}_2, \quad f_z + F_z = M_1 \ddot{z}_1 + M_2 \ddot{z}_2$$

so that

$$\ddot{x}_2 = \frac{f_x + F_x}{M_2} - \frac{M_1}{M_2} \ddot{x}_1, \quad \ddot{z}_2 = \frac{f_x + F_x}{M_2} - \frac{M_1}{M_2} \ddot{z}_1.$$  (C.2)
The accelerations along the tangent line to the point of contact remain unchanged. That is
\[ \frac{1}{R^2}(f_x, f_z) \cdot (Z, -X) = \frac{M_1}{R^2} (\ddot{x}_1, \ddot{z}_1) \cdot (Z, -X), \] (C.3)
\[ \frac{1}{R^2}(F_x, F_z) \cdot (Z, -X) = \frac{M_2}{R^2} (\ddot{x}_2, \ddot{z}_2) \cdot (Z, -X). \]

Taking the difference of these two equations gives
\[ \left( \frac{f_x}{M_1} - \frac{F_x}{M_2} \right) Z - \left( \frac{f_z}{M_1} - \frac{F_z}{M_2} \right) X = (\ddot{x}_1 - \ddot{x}_2) Z - (\ddot{z}_1 - \ddot{z}_2) X \] (C.4)
so that (ignoring for the moment the possibility that \( X = 0 \))
\[ \ddot{Z} = \frac{Z}{X} \ddot{X} + \left( \frac{f_z}{M_1} - \frac{F_z}{M_2} \right) Z - \left( \frac{f_x}{M_1} - \frac{F_x}{M_2} \right) \frac{Z^2}{X}. \] (C.5)

The final requirement is the circling behavior
\[ X^2 + Z^2 = R^2, \] (C.6)
from which we obtain the requirement
\[ X \ddot{X} + Z \ddot{Z} + (\dot{X}^2 + \dot{Z}^2) = 0. \] (C.7)

From Eq. C.5 and Eq. C.7 we obtain
\[ X \ddot{X} + \frac{Z^2}{X} \ddot{X} = \left( \dot{X}^2 + \dot{Z}^2 \right) - \left( \frac{f_z}{M_1} - \frac{F_z}{M_2} \right) Z + \left( \frac{f_x}{M_1} - \frac{F_x}{M_2} \right) \frac{Z^2}{X} \] (C.8)
so that
\[ \frac{R^2}{X} \ddot{X} = \left( \dot{X}^2 + \dot{Z}^2 \right) - \left( \frac{f_z}{M_1} - \frac{F_z}{M_2} \right) Z + \left( \frac{f_x}{M_1} - \frac{F_x}{M_2} \right) \frac{Z^2}{X} \] (C.9)
and
\[ \ddot{X} = - \left( \dot{X}^2 + \dot{Z}^2 \right) \frac{X}{R^2} - \left( \frac{f_z}{M_1} - \frac{F_z}{M_2} \right) \frac{XZ}{R^2} + \left( \frac{f_x}{M_1} - \frac{F_x}{M_2} \right) \frac{Z^2}{R^2}. \] (C.10)
Together with C.2 this gives us
\[ \left( 1 + \frac{M_1}{M_2} \right) \ddot{x}_1 = \frac{f_x + F_x}{M_2} - \left( X^2 + \dot{Z}^2 \right) \frac{X}{R^2} - \left( \frac{f_z}{M_1} - \frac{F_z}{M_2} \right) \frac{XZ}{R^2} + \left( \frac{f_x}{M_1} - \frac{F_x}{M_2} \right) \frac{Z^2}{R^2}. \] (C.11)

The equation for \( \ddot{z}_1 \) is obtained by switching \( x \) and \( z \). This equation can be alternately derived in the case where \( X = 0 \). Equations of motion in the case of one of the spheres being fixed can be derived from similar conditions by fixing \((x_2, z_2)\) and letting \((F_x, F_z) = (0, 0)\), or by letting \((F_x, F_z) = (0, 0)\) and \( M_2 = \infty \) in Eq. C.11. In either case,
\[ \ddot{x}_1 = - \left( \dot{x}_1^2 + \dot{z}_1^2 \right) \frac{X}{R^2} - \frac{f_x}{M_1} \frac{XZ}{R^2} + \frac{f_x}{M_1} \frac{Z^2}{R^2}. \] (C.12)
We also consider the question of how long to continue the sliding for. We do not wish, for example to have a tracer circle the underside of fixed scatterers. The cutoff condition is that there is no attractive force between the spheres. Thus if during any evaluation of \( C.12 \), for example, we discovered that

\[
M_1(\ddot{x}_1, \ddot{z}_1) \cdot (x_2 - x_1, z_2 - z_1) > (f_x, f_z) \cdot (x_2 - x_1, z_2 - z_1)
\]  

(C.13)

the sliding will be judged to have terminated, and the original equations of motion will be restored.

### C.0.2 Comparison of results

A phase diagram similar to Fig. 7.18 is shown in Fig. C.1. Classifications of staircasing, diffusion and nearly vertical ascent or descent are again marked. Data not easily identifiable (for instance, if the variance appeared to grow too fast but no staircasing was seen) is also marked. The boundaries of the different phases closely resemble those shown in Fig. 7.18. We can conclude that staircasing behavior is not an artifact of simplifications used in the original simulations.

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Fig. C.1. Points marked with white squares show parameters where diffusive lateral motion was observed, while black circles indicate tracers that rise or fall in fairly straight paths. Gray triangles are used to show parameters where staircasing occurs, and plus signs mark parameters whose behavior was not easily put into one of these classes.

The settling velocities of the spheres also closely resemble those seen in Fig. 7.12. A few of these settling velocities are shown in Fig. C.2.
Fig. C.2. Average non-dimensionalized settling velocity $V_t$ as a function of density $\rho$ [gm/cm$^3$].

C.0.3 Comparison with theory

In the case of fixed scatterers and no drag, it is possible to make some comparison of the simulation results with theory. For example, in the case of elastic collisions the vertical component of the tracer paths are recurrent – they always return to the height from which they were released [14]. This can be seen in a plot of 100 paths shown in Fig. C.3. The origin of all the paths is close to (0, 0) (enough so that they appear to begin at the same point in the figure). However, because the tracers always return to the level of release, it appears in the plot as if tracers had been released from many locations on the line $y = 0$.

Although the tracers always return to the height at which they were released, the average depth of the tracers increases with time. That is, over time they tend to make longer falls before bouncing back to the top. This allows the average vertical displacement to grow as $t^{2/3}$ and the average speed as $t^{1/3}$ [14]. The average values of displacement and speed are shown in Fig. C.4. Dashed lines show the respective values growing as $t^{2/3}$ and $t^{1/3}$ as expected.

Finally, for nearly elastic collisions Wilkinson [96] calculates that the settling velocity should be proportional to $(1 - e)^{1/4}$. Repeating the above experiment with the number of paths reduced to 50 and the coefficient of restitution reduced to values from 0.95 to 0.995 shows a reasonable agreement with these predictions. See Fig. C.5. The trend does not hold over a wide range of values of $e$. 
Fig. C.3. Paths are shown for 100 tracers colliding elastically ($e = 1$) with the scatterers. Lengths are non-dimensionalized by the sum of tracer and scatterer diameters. E.g. $x = \hat{x}/(D_s + D_t)$. Every path begins near $(0, 0)$.

Fig. C.4. Average displacement (left) and speed (right) for the tracers whose paths are shown in Fig. C.3. Dashed lines of slope $2/3$ and $1/3$ (respectively) have been added to show the rate of growth. Time and distance have been non-dimensionalized in the manner given in section 7.3.1.
Fig. C.5. Average non-dimensionalized settling velocity $V_t$ as a function of $1 - e$. A dashed line of slope $1/4$ is shown for comparison.

C.0.4 Eliminating staircasing

Because the staircasing of tracers is very dependent on the ordered array of scatterers, randomizing the scatterer array should eliminate the effect. We verified this by randomizing the beginning positions of scatterers. The center of each scatterer was altered to be distributed uniformly in a rectangle centered on the quincunx lattice. The rectangle was kept small enough that scatterers would not overlap in their initial positions (although collisions with the tracer may lead to overlap). We found that randomization does eliminate long, stable instances of staircasing. One effect of this is that $V_t$ decreases monotonically with density in the randomized array whereas staircasing causes irregularities in the uniform array. An example of this is shown in Fig. C.6 for tracers of diameter $D_t = 0.4$. 
Fig. C.6. Settling velocity $V_t$ as a function of density $\rho$ (gm/cm$^3$) for tracers of diameter $D_t = 0.4$. The filled circles show results from the randomized array, while the open squares show results from the ordered array as seen in Fig. 7.14.
Appendix D

Hourglass bubble plumes

D.0.5 The plume

While creating the experimental setup described in chapter 8, we evaluated glass pipettes (Drummond Microcaps, length 5.5 cm, inner diameter 0.015 cm) for use as capillary sources of bubbles. We tested an arrangement of 9 capillaries in a hexagonal (triangular) array depicted in D.1. Images of the bubbles were analyzed to find a bubble diameter of $d_e = 0.21$ cm. In the initial images taken we observed a uniform column of bubbles; however, subsequent images showed the plume spreading noticeably as it rose (as seen in Fig. D.2).

![Fig. D.1. Arrangement of capillaries.](image)

We added water to the tank to discover whether the plume would continue to spread or reach a certain width as rise in a columnar fashion. To our surprise, the plume did neither, but narrowed and expanded in turn to form an hourglass profile as seen in Fig. D.3. In Fig. D.4 a (hexagonal) arrangement of six capillaries produces a repeated hourglass plume. As the depth of the water is lowered, the pattern remains fairly constant (except for the a slight change as the pressure lowers with decreasing water depth, making the plume appear fuller).

D.0.6 Spiraling bubbles

The periodic nature of the hourglass seen in D.4 reveals the structure of the plume. It is well known that both solid and fluid spheroids can exhibit a zigzag or spiraling movement as they rise or fall in a fluid (see e.g. [97, 43]). The lift is produced by planar or 3D asymmetries in the wake [66]. We measured the helical paths of bubbles rising singly (as seen in Fig. D.5) and found that in one period of the spiral the bubble rose $h = 6.7$ cm, with a lateral radius of $r = 0.55$ cm. This gives the pitch of the rise
Fig. D.2. The bubble plume to the left rises in a columnar fashion. The plume to the right spreads as it rises. After an initial shot of the columnar pattern, the spreading effect appeared to be stable.

as \( \tan^{-1} \left( \frac{h}{r} \right) = 63^\circ \). This compares with earlier experimental data [5] that can be interpolated for our bubble diameter to find values of \( r = 0.58 \text{ cm} \) and a pitch of 60\(^\circ\).

These numbers \((h \text{ and } r)\) appear to be unchanged when the bubble is part of a plume. This can be seen in Fig. D.6 where a lone bubble path is shown to scale with a plume. Note that because it is easier in the plume to focus only the bright reflection from the right-hand side of the bubble, the scales may appear different at first glance. The most noticeable difference is that the lone bubble takes much longer to begin its spiral motion. One more – and very important difference – is that in the plume there is an initial oblique movement. We cannot reproduce this oblique movement for bubbles moving singly, but can produce this motion with a stream of bubbles from a single source. This is shown in Fig. D.7 for a train of bubbles released from a single capillary at rate of about 17 s\(^{-1}\). The orientation of the planar movement evolves, but over a much longer time scale than that of the bubbles’ release. Hence the oblique line from the capillary looks very stable. The displacement from the capillary to center of the helix we estimate at 2.6 cm, again similar to that in the plume.

It is apparent that the hourglass profile is a superposition of individual helical bubble trajectories separated by an initial oblique movement. The remaining question is how important is the interaction between bubbles released by different capillaries to the overall shape? Are the oblique motions, for example, forced away from the center of the array to expand the profile of the bubble plume into an hourglass? We will show in the next section how individual helices can be expected to randomly organize into the coherent hourglass shape, so that the interaction between bubbles released by different capillaries is not a key factor in creating the hourglass shape.
Fig. D.3. To the left, superposition of images (50 images over 0.1 s) shows the bubble plume forming an hourglass profile. A longer superposition shown to the right fills in the plume.

D.0.7 Random spirals

Using the parameters of $h = 6.7$ cm, $r = 0.55$ cm and the displacement of 2.6 cm we can parameterize the path of a bubble during its helical motion. We set the $x$ and $y$ coordinates of the capillary at $(0, 0)$ but leave the $z$ coordinate unspecified. If the direction of the initial oblique movement is given by the angle $\phi$ (as shown in Fig. D.8) then the helical motion can be parameterized by

$$x(\theta) = 2.6 \cos(\phi) + 0.55 \cos(\theta + \phi), \quad (D.1)$$

$$y(\theta) = 2.6 \sin(\phi) + 0.55 \sin(\theta + \phi), \quad (D.2)$$

$$z(\theta) = 6.7 \frac{\theta}{2\pi} \quad (D.3)$$

where $\theta$ describes the angle around the helix from the furthest point (in the $x$-$y$ plane) from the capillary. Because the parameter $\phi$ is unknown we may treat it as random. By letting $\phi$ and $\theta$ vary over all possible values from $[0, 2\pi]$, we obtain a parameterized surface representing the profile of all possible paths of the bubble. The profile of possible paths is shown in Fig. D.9 with a few sample bubble paths superimposed. The shape is that of an hourglass.

For the array of capillaries seen in Fig. D.1, the envelope of possible bubble paths is much more complex, but still (based on the observation that the bubbles begin their helical motion at about the same heights) retains a general hourglass shape as seen in Fig. D.10. Because the initial direction of bubbles from a capillary evolves slowly, we can
Fig. D.4. The lobes of the hourglass repeat themselves as the water depth increases.

visualize the shape of a plume formed randomly by plotting a single spiral from each of the nine capillaries. The random plume may either fill the profile in a way that appears to have an hourglass or columnar shape as shown in Fig. D.11.

To describe the likelihood of seeing an hourglass or columnar profile, we need to establish a point of view from which the profile is being seen. We will put the observer on the negative $y$ axis. We then first calculate the probability that the maximum extent of a single spiral runs at least $x$ units to the right of its source. This quantity $p(x)$ is given by

$$ p(x) = \begin{cases} 
0 & x > 3.15 \\
\frac{1}{2} \cos^{-1} \left( \frac{x - 0.56}{2.6} \right) & -2.05 < x \leq 3.15 \\
1 & x \leq -2.05 
\end{cases} \quad (D.4) $$

This comes from the simple observation that for an initial angle $\phi$, the maximum $x$ coordinate $x_{\text{max}}$ is simply

$$ x_{\text{max}} = 2.6 \cos(\phi) + 0.55. \quad (D.5) $$

From this we can construct the probability of having a columnar plume - one contained within a given distance $x$ to each side of the center as shown in Fig. D.13. We envision that this distance will be small enough so that the overlapping of the bubble
paths in the line of sight will obscure their sinusoidal profile. We will assume the orientation $\phi$ of each bubble from its source is independent and identically distributed over $[0, 2\pi]$. In this case the probability of containment, $P_{\text{int}}(x)$, is

$$P_{\text{int}}(x) = \prod_{i} 1 - [p(x - x_{i}) + p(x + x_{i})]$$  \hspace{1cm} (D.6)

where $x_{i}$ are the $x$ coordinates of the bubble sources.

We can also construct a probability that the plume spreads to fill the envelope, creating an hourglass profile. This will be the probability that at least one bubble meets or exceeds a distance of $x$ cm to the left and to the right as shown in Fig. D.14. Since we wish to consider values of $x$ large enough to fill most of the envelope, we may ignore cases in which one bubble can exceed a distance of $x$ on both sides at once. In this case, the probability of having at least one bubble $x$ cm to the left and to the right, $P_{\text{ext}}(x)$ is

$$P_{\text{ext}}(x) = \sum_{j} \left[ \prod_{j < i} 1 - [p(x - x_{i}) + p(x + x_{i})] \right] \times \left[ \prod_{j < \ell < k} 1 - p(x + x_{\ell}) \right] + \left[ \prod_{j < \ell < k} 1 - p(x - x_{\ell}) \right].$$  \hspace{1cm} (D.7)

Calculated values for $P_{\text{int}}(x)$ and $P_{\text{ext}}(x)$ are shown in Figs. D.15 and D.16. It can be seen that a columnar profile is unlikely: there is a minimal chance at containing the plume even within 70% of its maximum value. Alternately there is about an 80% chance that the plume will fill out the profile with one or more bubble paths at least 70% of the maximum width on both sides of center.

D.1 Conclusions

It appears that the hourglass profile is a natural result of the random arrangement of bubble trajectories into helical paths. This does not mean that bubbles from different sources do not interact. For example, the same capillaries from Fig. D.4 in vegetable oil form a tight columnar plume as shown in Fig. D.17. The bubbles are not spiraling and are perhaps drawn together by a column of moving oil in the center of the array. However, it may be that interactions between bubbles from different sources are not important to the phenomenon. Indeed, if we increase the interaction between bubbles from different sources by increasing the applied pressure to the capillaries, the hourglass profile is quickly lost.
Fig. D.5. Superimposed images show the helical paths of the bubbles. The height of one period of the spiral is $h = 6.7$ cm and the radius is $r = 0.55$ cm.
Fig. D.6. Superimposed images show the helical paths of the bubbles in a plume and singly. The height $h$ and radius $r$ appear unchanged from case to case.
Fig. D.7. Superimposed images show an initial oblique movement of bubbles before they begin their helical motion.
Fig. D.8. Downward view of the helical motion of a bubble.

Fig. D.9. The possible paths
Fig. D.10. The envelope of possible paths for the capillary array shown in Fig. D.1 shown from the side (left) and top (right). The hourglass profile is still evident.

Fig. D.11. A random arrangement of bubble spirals may fill the plume in hourglass fashion (as shown to the left) or in an apparent column (as shown to the right).
Fig. D.12. The larger circle shows the far point of all possible paths. An example path is represented by the smaller circle. Eq. D.4 describes the probability that the maximum $x$ coordinate on the spiral $x_{\text{max}}$ exceeds a given value $x$. 
Fig. D.13. A necessary condition to create a columnar plume is that all of the bubble paths must be contained within some small distance from the center where their overlapping paths can obscure their sinusoidal profile. The probability of containment is given in Eq. D.6.

Fig. D.14. To create an hourglass plume, we wish to fill the envelope of possible paths to the left and to the right. The probability of filling the profile in this way is given in Eq. D.7.
Fig. D.15. Calculated values from Eq. D.6 show how unlikely it is to contain the plume in a columnar fashion, even at 70% of the maximum possible width.

Fig. D.16. Calculated values from Eq. D.7 show that the hourglass-shaped envelope of possible paths tends to fill to both sides. There is about a fifty-fifty chance, for example, that the plume will fill up to at least 80% of its maximum width to both sides.
Fig. D.17. A bubble plume in vegetable oil forms a narrow column.

Fig. D.18. At higher pressures the hourglass profile disappears from the plume.
References


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