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SYMBOLIC DYNAMICS OF THE WEYL CHAMBER FLOW

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Abstract

This thesis studies codings of orbits of Weyl chamber flows on symmetric spaces of non-compact type.

Let $\mathcal{H}$ be the hyperbolic plane with constant curvature -1 and Γ be a Fuchsian group of finite covolume. Let $\mathcal{D}$ be a Dirichlet domain of Γ on $\mathcal{H}$. The main result shows that the set of cutting sequences of all geodesics in the sense of Morse with respect to the tessellation of $\mathcal{H}$, formed by the sets $g\mathcal{D}$, $g \in \Gamma$, is a topological Markov chain if and only if $\mathcal{D}$ does not have vertices in $\mathcal{H}$.

Also, a background is provided for the study of generalization of continued fractions to higher dimensions. So-called arithmetic Gauss coding of geodesics on $\mathcal{H}$ is described along with its relation with the minus continued fractions. $\mathcal{H}$ is a particular case of a symmetric space of non-compact type, $\mathcal{H} \cong SL_2\mathbb{R}/SO_2\mathbb{R}$, and the geodesic flow on $\mathcal{H}$ implements the Weyl chamber flow on it. A generalization of the minus continued fractions was suspected by S. Katok and A. Katok to exist, which involves orbits of Weyl chamber flows on symmetric spaces of non-compact type $SL_n\mathbb{R}/SO_n\mathbb{R}$ and their compactifications.

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$\mathbb{Z}$ the ring of integer numbers;

$\mathbb{R}$ the field of real numbers;

$\mathbb{C}$ the field of complex numbers;

$\Re(z)$ the real part of the complex number z;

$\Im(z)$ the imaginary part of the complex number z;

B^t the transpose of the matrix B;

$M_n\mathbb{K}$ the ring of $n \times n$ matrices over the ring $\mathbb{K}$;

I the identity matrix in $M_n\mathbb{K}$;

$SL_n\mathbb{K}$ the special linear group over the ring $\mathbb{K}$;

$SO_n\mathbb{K}$ the special orthogonal group over the ring $\mathbb{K}$;

$\mathbf{K}$ the group $SO_n\mathbb{R}$;

$\mathbf{A}$ the subgroup of $SL_n\mathbb{R}$ of diagonal matrices with positive elements;

$\mathcal{Z}_G(H)$ the centralizer of the subgroup H in the group G;

$\mathbf{M}$ the group of integer diagonal matrices with determinant 1, $\mathbf{M} = \mathbf{K} \cap \mathcal{Z}_{SL_n\mathbb{R}}(\mathbf{A})$;

$\mathbf{P}$ the group of real upper triangular matrices with determinant 1;

$\mathbf{P}_{>0}$ the subgroup of $\mathbf{P}$ of matrices with positive diagonal elements, $\mathbf{P} = \mathbf{P}_{>0}\mathbf{M}$;

$\mathbf{N}$ the subgroup of $\mathbf{P}_{>0}$ of matrices with 1's on the diagonal, $\mathbf{P}_{>0} = \mathbf{N}\mathbf{A}$;

X_n the homogeneous symmetric space $SL_n\mathbb{R}/SO_n\mathbb{R}$;

d_n the dimension of the symmetric space X_n;

$\partial_\infty X_n$ the boundary of X_n at infinity;

$WCF_{\mathbf{t}}$ the Weyl chamber flow with parameters $\mathbf{t} \in \mathbb{R}^n$;

$\mathcal{H}$ the hyperbolic plane;

$PSL_2\mathbb{R}$	the real projective special linear group, $SL_2\mathbb{R}/\{\pm I\}$;
$PSL_2\mathbb{Z}$	the modular group, $SL_2\mathbb{Z}/\{\pm I\}$;
$\mathcal{F}$	the standard fundamental domain of the group $PSL_2\mathbb{Z}$ on $\mathcal{H}$;
S	the element of $PSL_2\mathbb{Z}$, given by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$;
T	the element of $PSL_2\mathbb{Z}$, given by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$;
S^n	the unit sphere in $\mathbb{R}^{n+1}$;
$\mathbb{R}P^n$	the real projective space of dimension n;
Θ^n	the set of linearly independent n-tuples in $\mathbb{R}P^{n-1}$;
S^n_+	the hemisphere of S^n, where the $(n+1)$-st coordinate is positive;
$\mathbf{v} \cdot \mathbf{w}$	the inner product of the vectors $\mathbf{v}$ and $\mathbf{w}$.

Acknowledgments

Dedication

To my parents Olga and Andrey and my grandparents Rimma, Vladimir, Ekaterina, and Valeriy.

Chapter 1

Introduction

1.1   History of the question

There have been several approaches to coding of geodesics on the hyperbolic plane
with respect to fundamental domains of Fuchsian groups. One approach was intro-
duced by Morse [Mor21], in which the plane is tesselated according to the chosen
domain and the action of the group, the sides of the tesselation are labeled in a
certain way, and the geodesics are then coded using their intersections with the
sides of the tesselation. Another approach was developed by Artin [Art24] and for
coding it uses the endpoints of geodesics on the absolute, which are identified with
real numbers and are expanded into continued fractions.

Morse's approach utilizes only the way the hyperbolic plane is tiled by funda-
mental domains of the group, which can be reproduced "internally" on the factor
surface $\mathcal{H}/\Gamma$ by marking curves on the surface along which it needs to be "cut"
to obtain one of the fundamental domains on the plane. And Morse's approach is
defined uniquely once a tessellation is chosen. So it is somewhat more natural and
canonic than Artin's. But Artin's approach is known to produce a set of codes
which is a Markov topological chain. We became interested, under what conditions
Morse's coding produces a set of sequences, which is a topological Markov chain
too. In [KU05b] the case of the modular group $PSL(2, \mathbb{Z})$ is considered along with
a slightly modified version of the coding, which uses integer numbers instead of
the group's elements. Their result implies the results of Chapter 4 in the case
of the modular group and its standard fundamental domain, and is the primary

motivation for the work presented in Chapter 4.

Artin's approach was broadened to different types of continued fractions, e.g. by Katok [Kat96], Mayer and Strömberg [MS08], and different Fuchsian groups by Bowen and Series [BS79], Mayer and Strömberg [MS08], Series [Ser81, Ser86], and others (Artin's original approach applied to the modular group $PSL_2\mathbb{R}/PSL_2\mathbb{Z}$). It requires some reduction process (i.e. finding an appropriate geodesic among all Γ-equivalent ones) before actual encoding. Moreover, the coding process may be defined differently for the same tessellation of the plane, which corresponds to different types of continued fractions. Being less visual than Morse's, this approach produces a set of sequences of integers, which is a topological Markov chain and is much easier to structurize than just a "random" set of sequences.

On the other hand, we could reverse Artin's approach and use it to define a continued fraction expansion of the endpoints of a geodesic. It was understood that geodesics on the hyperbolic plane are a particular case of orbits of the Weyl chamber flow (WCF) on a homogeneous symmetric space of non-compact type. S. Katok and A. Katok realized that the generalization of the geodesic flow up to the WCF could lead to a generalization of the notion of continued fractions to higher dimensions. Numerous attempts to generalize continued fractions can be found in the literature. Klein [Kle96] introduced multidimensional continued fractions in the form of geometric objects in $\mathbb{R}^{n+1}$ called sails. Korkina [Kor96] has studied this type of continued fractions for $n = 2$. Karpenkov [Kar06] displayed some examples for $n = 3$. Unfortunately, their notion of continued fractions is difficult to operate in some situations. Chapter 5 summurizes the observations made during an attempt to implement the Katoks' idea.

1.2   Summary of results

Let $\mathcal{H}$ be the hyperbolic plane of constant curvature -1 and let Γ be a Fuchsian group (i.e. a discrete group of orientation preserving isometries of $\mathcal{H}$) of finite covolume. Morse's method of coding of geodesics on $\mathcal{H}$ utilizes a fundamental domain $\mathcal{D}$ of the group Γ. The method assumes that $\mathcal{D}$ is a polygonal set, whose sides are labeled by the generators of Γ, which translate $\mathcal{D}$ to its neighbors. The labeling is also translated to all copies of $\mathcal{D}$ by the group Γ. Since the area of

$\mathcal{D}$ is finite, almost every geodesic on $\mathcal{D}$ does not pass through the vertices of the tessellation. Thus, for a generic directed geodesic γ one can find the unique sequence of sides of the tessellation, which γ intersects. Since each side of the tessellation was previously labeled, one obtains a unique sequence of generators of Γ for γ. This sequence is called *the cutting sequence* of γ or Morse's code for γ.

The Morse's coding method provides a correspondence between the smooth dynamics of the geodesic flow on $\mathcal{H}$ or $\Gamma \backslash \mathcal{H}$ and the symbolic dynamics on the set of sequences in the alphabet of generators of Γ, that are realizable as cutting sequences. When one deals with symbolic dynamics, the question arises of which sequences are realizable. It may or may not have a simple answer. In particular, the answer to this question is quite short, when the set of realizable sequences is a *topological Markov chain*. So it is natural to ask, whether the set of cutting sequences with respect to a given domain $\mathcal{D}$ is a topological Markov chain.

The first and the main result of this work answers this question. The polygonal set $\mathcal{D}$ may or may not have vertices in $\mathcal{H}$. Depending on whether it does have such vertices or not, the set of the cutting sequences is a topological Markov chain or not. In Chapter 4 we prove the following statement:

Theorem 1. *Let Γ be a Fuchsian group with a finite covolume and a Dirichlet domain $\mathcal{D}$. Then the set of Morse codes of generic geodesics on $\mathcal{H}$ with respect to $\mathcal{D}$ is a topological Markov chain, if and only if $\mathcal{D}$ does not have finite vertices.*

This answer implies that it is hard to describe in finite terms the set of realizable cutting sequences, given by an arbitrary fundamental domain of an arbitrary Fuchsian group.

The Morse's method is usually referred to as a geometric coding, since it relies on the intrinsic geometry of the hyperbolic plane. Another class of codes of geodesics proves to be better behaving in more situations. It is the class of arithmetic codes, many classical examples of which have been shown to produce topological Markov chains. And the number of arithmetic codes known to yield topological Markov chains keeps growing due to works of Bowen and Series [BS79], Katok and Ugarcovici [Kat96, KU05b, KU11], who use (a, b)-continued fractions to generalize the coding named after Gauss for its close relation with Gauss's theory of reduction of quadratic forms.

As Gauss coding is very simple, has a clear connection with minus continued fractions, and yields a topological Markov chain, it has been suggested that it may be possible to generalize the coding in terms of the Weyl chamber flow, defined on the spaces $X_n = SL_n\mathbb{R}/SO_n\mathbb{R}$. Original Gauss coding employs certain reduction of geodesics, so one of the goals was to give a simlar notion of reduced orbits of the Weyl chamber flow. In order to make the work with the Weyl chamber flow and its orbits easier, we first describe its relation to the structure at the boundary of a compactification of X_n, known as a spherical Tits building.

It turns out that the orbits of the Weyl chamber flow are in one-to-one correspondence with n-tuples of linearly independent points of $\mathbb{R}P^{n-1}$. In Chapter 5 we describe how this correspondence is obtained. We use the correspondence to introduce a candidate for definition of reduced orbits.

We also introduce a set of coordinates on X_n, which generalize the coordinates on the upper half-plane Poincaré model for the hyperbolic plane. The orbits of the Weyl chamber flow appear to be subsets of spheres in these coordinates, which might prove convenient for some calculations.

In addition to obtaining a way of using symbolic dynamics on the orbits of the Weyl chamber flow, a coding would help generalize continued fractions to multiple dimensions. We outline our plan of constructing such a code, provided a certain class of transformations of $\mathbb{R}P^1$ is generalized first.

In Chapters 2 and 3 we provide necessary background used in later Chapters.

The symmetric spaces X_n and the Weyl chamber flow

In this chapter we describe the exact type of symmetric spaces we consider, namely $SL_n\mathbb{R}/SO_n\mathbb{R}$, and the Weyl chamber flow on each of these spaces. In particular, the hyperbolic plane of constant curvature -1 is such a space for $n = 2$ and the Weyl chamber flow on it is just the geodesic flow. The Weyl chamber flow is an action of $\mathbb{R}^{n-1}$ on $SL_n\mathbb{R}$, which has an elegant representation in matrix terms. Let $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$, so that

$$\sum_{j=1}^{n} t_j = 0,$$

then for $B \in SL_n\mathbb{R}$

$$WCF_{\mathbf{t}}(B) = B \cdot \text{diag}(e^{t_1}, \ldots, e^{t_n}),$$

where $\text{diag}(a_1, \ldots, a_n)$ stands for the diagonal matrix with the diagonal entries $a_1, \ldots, a_n$. If one identifies the rows of B with a basis in $\mathbb{R}^n$ whose frame's volume is 1, then the Weyl chamber flow with parameters $\mathbf{t}$ scales the j-th coordinates of all the vectors by the factor of e^{t_j}. Since $\sum_{j=1}^{n} t_j = 0$, the volume of the frame is preserved.

2.1   The hyperbolic plane as $SL_2\mathbb{R}/SO_2\mathbb{R}$

To begin, we recall a classical construction showing that the unit tangent bundle over the hyperbolic plane $\mathcal{H}$ can be thought of as the group $PSL_2\mathbb{R} = SL_2\mathbb{R}/\{\pm I\}$. At the same time, it can be thought of as the group of orientation preserving isometries of $\mathcal{H}$.

Consider the model of $\mathcal{H}$ in the upper complex half-plane with the hyperbolic metric on it:

$$\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}, \quad dz^2 = \frac{dx^2 + dy^2}{y^2},$$

then every orientation preserving isometry of $\mathcal{H}$ can be written in the form of a Möbius transformation, i.e.

$$z \mapsto \frac{az + b}{cz + d},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2\mathbb{R}$. Denote this correspondence by ϕ, then $\phi : SL_2\mathbb{R} \to Iso(\mathcal{H})$ is a homomorphism of groups, such that $\phi(gf) = \phi(g) \circ \phi(f)$ and its kernel is the set $\{\pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$.

If one is given a vector $\vec{v}$ in the unit tangent bundle $S\mathcal{H}$, then there is a unique orientation preserving isometry of $\mathcal{H}$ which sends $\vec{j}$ to $\vec{v}$, where $\vec{j}$ is the unit vector at i pointing vertically upward. This clearly gives a one-to-one correspondence between $S\mathcal{H}$ and the orientation preserving isometries and, thus, with the group $PSL_2\mathbb{R}$.

Since any unit tangent vector over $\mathcal{H}$ can be moved to any other one using an isometry, the group $SL_2\mathbb{R}$ acts on $\mathcal{H}$ transitively. So $\mathcal{H}$ is the orbit of any of its points under this action. In particular, it is the orbit of i. The subgroup that fixes this point is $SO_2\mathbb{R}$ (see the next paragraph) and, therefore, it corresponds to the set of unit tangent vectors at i. It follows that for a fixed $B \in SL_2\mathbb{R}$ the sets of the form $B \cdot SO_2\mathbb{R}$ correspond to the set of unit vectors, tangent at the point $\phi(B)(i)$, so $\mathcal{H}$ is homeomorphic to the set of cosets of $SO_2\mathbb{R}$, or a homogeneous space, $SL_2\mathbb{R}/SO_2\mathbb{R}$.

To check that $SO_2\mathbb{R}$ is the stabilizer of the point i, we need to determine all real $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, such that $(ai + b) = i(ci + d)$ and $ad - bc = 1$. So we get $c = -b$ and $a = d$, thus $a^2 + b^2 = 1$. The matrix $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ corresponds to a rotation of the Euclidean plane. Moreover, it can be shown, that the matrix $\left(\begin{smallmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{smallmatrix}\right)$ corresponds to the

rotation of the hyperbolic plane around the point i by the angle 2φ in the counter clockwise direction.

We will abuse the notation throughout the rest of the paper by writing

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R}$ and the vector $\vec{v} = D_i[\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix}](\vec{j})$. Here $D_i[f]$ means the differential of the map f at i.

2.2   The geodesic flow over the hyperbolic plane

Definition 1. A map $\theta : \mathbb{R} \times X \to X$ is called a flow on the topological space X, if it is a continuous $\mathbb{R}$-action on X. That is for all $x \in X$

1. The map $t \mapsto \theta_t(x) = \theta(t, x)$ is continuous;

2. $\theta_0(x) = x$;

3. $\theta_t(\theta_s(x)) = \theta_{t+s}(x)$.

Definition 2. The geodesic flow on the unit tangent bundle SM of a smooth geodesically complete manifold M is the flow, which moves every unit tangent vector $\vec{v}$ with unit speed along the geodesic to which it is tangent in the direction the vector $\vec{v}$ points.

We will denote the geodesic flow by g_t, so $g_t(\vec{v})$ is the image of the vector $\vec{v}$ under the flow after time t.

In case of the hyperbolic plane $\mathcal{H}$ one can express $g_t(\vec{v})$ for an arbitrary unit vector $\vec{v}$. Let us find $g_t(\vec{j})$ first and then apply the isometry which sends $\vec{j}$ to $\vec{v}$ (remember that this is the isometry, which corresponds to $\vec{v}$). Under this isometry the entire geodesic following $\vec{j}$ will be sent to the geodesic following $\vec{v}$, so $g_t(\vec{j})$ will be sent to $g_t(\vec{v})$.

Simple calculation shows that $g_t(\vec{j})$ is the unit tangent vector pointing vertically upward at the point $e^t i$. The isometry of the hyperbolic plane that corresponds to
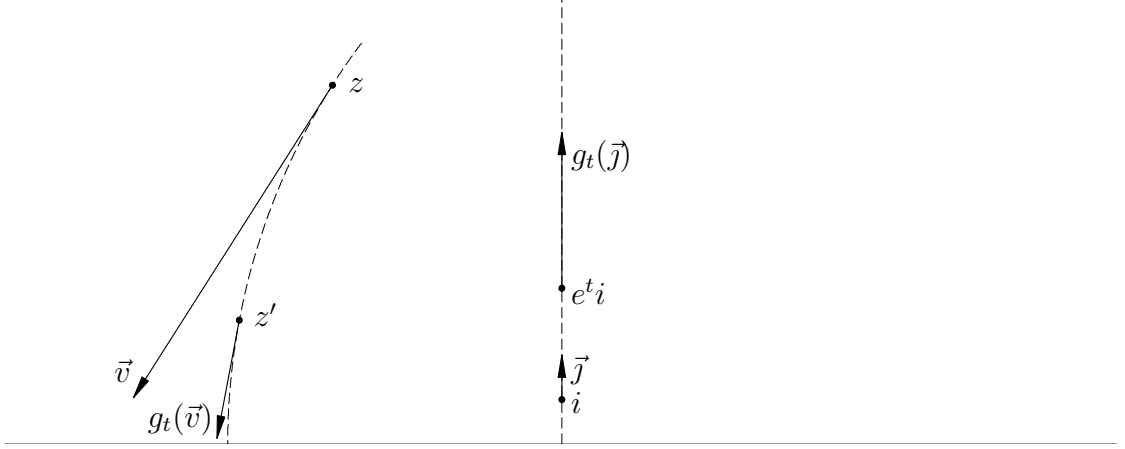
Figure 2.1. The result of geodesic flow of $\vec{j}$ and $\vec{v}$ after time t.

this vector is given by the matrix $\left(\begin{smallmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{smallmatrix}\right)$. It is simply the Euclidean dilation of the upper half-plane by the factor of e^t centered at 0. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ correspond to the vector $\vec{v}$. So to get $g_t(\vec{v})$ we first need to apply the dilation to $\vec{j}$ and then apply $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to the result, or

$$g_t(\vec{v}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = \vec{v} \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

As one can see, this is a particular case of the Weyl chamber flow for $n = 2$.

2.3 The symmetric space $X_n = SL_n\mathbb{R}/SO_n\mathbb{R}$ and its boundary at infinity $\partial_\infty X_n$

In case of groups of higher rank there will be less intuition due to their high dimension. We are going to follow a construction of the symmetric space $X_n = SL_n\mathbb{R}/SO_n\mathbb{R}$, found e.g. in [Gol03, GGT87, Gre88]. Consider the space of symmetric positive definite matrices in $SL_n\mathbb{R}$:

$$\mathcal{P}_n = \{Y \in SL_n\mathbb{R} : Y^t = Y, Y > 0\}.$$

$SL_n\mathbb{R}$ acts on $\mathcal{P}_n$ as follows: for $B \in SL_n\mathbb{R}$, $B(Y) = BYB^t$. Since for every symmetric $Y > 0$ there is a unique positive definite square root $\sqrt{Y} \in SL_n\mathbb{R}$, which is also symmetric, this action can be written as $B(Y) = B\sqrt{Y}\sqrt{Y}^t B^t$. It immediately follows that the action of $SL_n\mathbb{R}$ on $\mathcal{P}_n$ is transitive, since $Y' = B(Y)$ for any $Y' \in \mathcal{P}_n$ and $B = \sqrt{Y'}\sqrt{Y^{-1}}$. Note that $B(I) = Y$ if and only if B is in the coset $\sqrt{Y} \cdot SO_n\mathbb{R}$, which is also equivalent to $B \cdot SO_n\mathbb{R} = \sqrt{Y} \cdot SO_n\mathbb{R}$. Therefore, we can identify all cosets with the elements of $\mathcal{P}_n$ via

$$Y \mapsto \sqrt{Y} \cdot SO_n\mathbb{R}.$$

Clearly, it is a homeomorphism between $\mathcal{P}_n$ and the space of cosets

$$X_n = SL_n\mathbb{R}/SO_n\mathbb{R}.$$

We automatically get the action of $SL_n\mathbb{R}$ on X_n by the multiplication from the left, which should be compatible with its action on $\mathcal{P}_n$.

The dimension of $\mathcal{P}_n$ is
$$d_n = \frac{n(n+1)}{2} - 1$$

and it is homeomorphic to the Euclidean space of the same dimension (since that is the number of entries in a symmetric matrix one can freely choose keeping in mind that $\det(Y) = 1$, while the rest are uniquely determined by this choice). A homeomorphism can be realized via the exponential map at I at which the tangent space is

$$T_I\mathcal{P}_n = \{y \in M_n\mathbb{R} : y^t = y, \operatorname{tr}(y) = 0\}.$$

As $SL_n\mathbb{R}$ acts transitively on $\mathcal{P}_n$, it can be endowed with an $SL_n\mathbb{R}$-invariant metric. The space is a symmetric space of non-compact type and it can be compactified in a number of ways. We are going to consider the compactification, in which the points at infinity are the classes of equivalence of asymptotic geodesics. Each class is considered as a point at infinity of $\mathcal{P}_n$ and the set of these points is denoted by $\partial_\infty \mathcal{P}_n$. For every point $Y \in \mathcal{P}_n$ and every point η at infinity we can find exactly one geodesic passing through Y and going to η (i.e. belonging to η).

So $\partial_\infty \mathcal{P}_n$ can be thought of as the unit sphere in the tangent space of $\mathcal{P}_n$ at I:

$$\partial_\infty \mathcal{P}_n \cong \{y \in M_n\mathbb{R} : y^t = y, \operatorname{tr}(y) = 0, ||y||^2 = \operatorname{tr}(y^2) = 1\} \cong S^{d_n - 1}.$$

Since we showed that X_n and $\mathcal{P}_n$ are homeomorphic and the actions of $SL_n\mathbb{R}$ on them are equivalent, we will use X_n elsewhere in the paper with the metric corresponding to that on $\mathcal{P}_n$.

2.4 Iwasawa (NAK) decomposition of $SL_n\mathbb{R}$ and the horospheric decomposition of X_n

A connected semisimple Lie group G admits so called Iwasawa decomposition (see [BJ06] for the general theory and Chapter 1, §I.1.7 for the example of $SL_n\mathbb{R}$):

$$G = \mathbf{NAK},$$

and its Lie algebra $\mathfrak{g}$ has the corresponding decomposition:

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k},$$

where

$$\mathfrak{n} \oplus \mathfrak{a} = \mathfrak{p},$$

such that $\mathfrak{p} \oplus \mathfrak{k}$ is a Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{a}$ is a maximal Abelian subalgebra of $\mathfrak{p}$. $\mathbf{N}$, $\mathbf{A}$, and $\mathbf{K}$ are then the Lie subgroups of G generated by $\mathfrak{n}$, $\mathfrak{a}$, and $\mathfrak{k}$, respectively.

In particular, $SL_n\mathbb{R}$ decomposes into

$$\mathbf{K} = SO_n\mathbb{R}, \quad \mathbf{A} = \{\operatorname{diag}(a_1, \ldots, a_n) \in SL_n\mathbb{R} : a_1, \ldots, a_n > 0\},$$
$$\mathbf{N} = \text{the upper triangular matrices with 1's on the diagonal.}$$

Any other choice of the subgroups will be out of conjugates of these ones.

It will be convenient for us to use the following groups as well:

$$\mathcal{Z}(\mathbf{A}) = \{\operatorname{diag}(a_1, \ldots, a_n) \in SL_n\mathbb{R}\}, \quad \mathbf{M} = \mathcal{Z}(\mathbf{A}) \cap SO_n\mathbb{R},$$

$$\mathbf{P} = \text{the subgroup of upper triangular matrices in } SL_n\mathbb{R},$$

$$\mathbf{P}_{>0} = \{\text{matrices from } \mathbf{P} \text{ with positive diagonal elements}\}.$$

Here $\mathcal{Z}(\mathbf{A})$ is the centralizer of $\mathbf{A}$, the group of all diagonal matrices. $\mathbf{M}$ is the group of diagonal matrices with ± 1 on the diagonal and determinant 1. Given these groups, we can write the following identities:

$$\mathbf{P}_{>0} = \mathbf{NA}, \quad \mathbf{P} = \mathbf{NAM}, \quad \mathcal{Z}(\mathbf{A}) = \mathbf{AM}.$$

In case of $SL_2\mathbb{R}$ this decomposition allows us to write every orientation preserving isometry of the hyperbolic space $\mathcal{H}$ as the composition of specific parabolic, hyperbolic, and elliptic ones. Indeed, every element B of $SL_2\mathbb{R}$ becomes the product:

$$B = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

Consider the set

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \right\}_{x,t\in\mathbb{R}} = \mathbf{NA} = \mathbf{P}_{>0}.$$

Since $\mathbf{P}_{>0} \cap \mathbf{K} = \{I\}$, all elements of $\mathbf{P}_{>0}$ are in different cosets of $\mathbf{K}$. Thus, for a fixed $R = \left(\begin{smallmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{smallmatrix} \right) \in SO_2\mathbb{R}$, the matrices of the form

$$R \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \tag{2.1}$$

all are in different cosets, as well ($R \in SL_n\mathbb{R}$, which acts on X_n by the multiplication from the left side), and represent all cosets, since $\mathbf{P}_{>0}$ visits all cosets.

As for the hyperbolic plane itself, for a fixed element $R = \left(\begin{smallmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{smallmatrix} \right) \in SO_2\mathbb{R}$ we obtain two foliations of $\mathcal{H}$: into a family of parallel geodesics and into the horocycles orthogonal to these geodesics. The element R corresponds to the vector $\vec{v}$ at the point i making angle 2φ with the vertical. The members of the geodesic
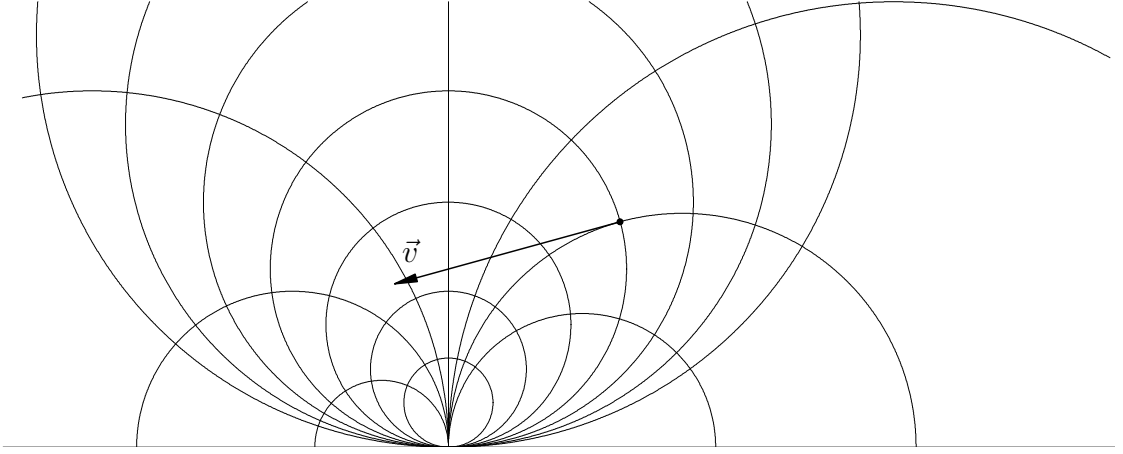
Figure 2.2. The horocyclic decomposition of $\mathcal{H}$, corresponding to $\vec{v}$.

family are obtained by fixing the parameter x and letting t change in (2.1) and taking the corresponding points on $\mathcal{H}$ (then the geodesics converge to the same point at infinity, to which the geodesic given by $\vec{v}$ goes). The members of the horospheric family are obtained by fixing the parameter t and changing x. These two families give us so called horocyclic decomposition of the hyperbolic plane.

If $R = I$, the identity matrix, then one gets the geodesics converging to ∞ and the horocycles given by the equations $\Im(z) = \text{const}$. The point ∞ and the two families are preserved by the group $\mathbf{P}$, which consists of parabolic isometries of the plane and is, therefore, called a parabolic subgroup of $SL_2\mathbb{R}$. For an arbitrary element $R \in SO_2\mathbb{R}$ the corresponding families of geodesics, horocycles, and the point at infinity are preserved by the parabolic subgroup $R \cdot \mathbf{P} \cdot R^t$. In addition, the elements of $R \cdot \mathbf{M} \cdot R^t$ do not move points on the plane, since $\mathbf{M} = \{\pm I\}$.

In the higher dimensional case one gets a similar horospheric decomposition of $SL_n\mathbb{R}/SO_n\mathbb{R}$. For each element $R \in SO_n\mathbb{R}$ one gets two families of submanifolds foliating the symmetric space. To simplify the notation we will explain the construction for X_3, but it extends to every dimension without changes.

Definition 3. A flat $\mathbf{\Phi}$ in a Riemannian manifold M is totally geodesic submanifold (i.e. every intrinsic geodesic in $\mathbf{\Phi}$ is also a geodesic in M), which is isometric to a Euclidean space.

The elements of the form

$$R \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s-t} \end{pmatrix}$$

are in one-to-one correspondence with points of the symmetric space X_3 for the same reason as for $n = 2$ (i.e. $\mathbf{P}_{>0} \cap \mathbf{K} = \{I\}$ and $\mathbf{P}_{>0}$ visits all cosets of $\mathbf{K}$). The first family consists of 2-dimensional (in general, $(n-1)$-dimensional) flats, obtained by fixing the x, y, and z parameters and changing t and s. The second family consists of the 3-dimensional (in general, $(d_n - n + 1)$-dimensional) horospheres, obtained by fixing the parameters t and s and changing x, y, and z. The members of these two families are orthogonal to each other.

The horospheric decomposition for $r = I$ gives a representation of X_n as the group $\mathbf{P}_{>0} = \mathbf{NA}$. So the decomposition is preserved by the action of the group $\mathbf{P}$. The elements of $\mathbf{M} < \mathbf{P}$ do not move the points of X_n at all. The decomposition corresponding to a different $R \in SO_n\mathbb{R}$ is preserved by the conjugate group $R \cdot \mathbf{P} \cdot R^t$.

2.5 Parabolic subgroups of $SL_n\mathbb{R}$, Weyl chambers, and the Tits building at infinity

We will try to give a "visual" idea of the Weyl chambers first. Then we will give more rigorous definitions, following [Ji06].

As described just above, in each horospheric decomposition of X_n one of the two orthogonal families consists of $(n-1)$-dimensional flats. The flats of the same decomposition are asymptotic to each other in *some* directions. Here "some" means that there is a set of points at infinity $\partial_\infty X_n$, that are common to the boundaries at infinity of all the flats in the family, but this set is not the entire boundary of a flat.

In case of the hyperbolic plane ($n = 2$) the set at infinity, to which all the geodesics in one family converge, consists of just one point, whereas every geodesic has two ends. In higher dimensional cases the set at $\partial_\infty X_n$ common to a single family of flats is homeomorphic to a closed $(n-2)$-dimensional simplex. This set

at infinity is what is called a *Weyl chamber at infinity.* A Weyl chamber then is the cone in a flat with the vertex at any point and the base being a Weyl chamber at infinity. We are seldom, if at all, going to use Weyl chambers in the flats, so we will refer to the Weyl chambers at infinity as simply Weyl chambers and to the Weyl chambers in the flats as we just did.

The group $SL_n\mathbb{R}$ acts on X_n by isometries by the definition of the metric on X_n. Due to our definition of the compactification this action can be extended to $\partial_\infty X_n$. Since every horospheric decomposition is invariant under the action of one of the groups $R \cdot \mathbf{P} \cdot R^t$, the Weyl chambers are also preserved by these groups. In fact, each Weyl chamber is the entire subset of points of $\partial_\infty X_n$ preserved by one of these groups, which are also called parabolic subgroups of $SL_n\mathbb{R}$.

Now we will give a precise definition of a parabolic subgroup. Each parabolic group is a subgroup of $SL_n\mathbb{R}$, so it acts on $\mathbb{R}^n$ (the space of columns) by multiplications from the left.

Definition 4. A subgroup of $SL_n\mathbb{R}$ is called parabolic, if it consists of all the elements preserving a given (not necessarily full) flag in $\mathbb{R}^n$.

For instance, the group $\mathbf{P}$ preserves the full flag f containing the x_1-axis, the x_1x_2-plane, the $x_1x_2x_3$-subspace, etc. A conjugate group of the form $B \cdot \mathbf{P} \cdot B^{-1}$, $B \in SL_n\mathbb{R}$ preserve the flag $B \cdot f$.

Since some of incomplete flags are contained in other flags, some of the parabolic subgroups are contained in others. With respect to the relation of containment, the group $\mathbf{P}$ and its conjugates are the minimal ones, since they preserve the maximal flags. It is easy to see that every minimal parabolic subgroup is a conjugate of $\mathbf{P}$ (since every maximal flag is obtained from f using the multiplication by an element of $SL_n\mathbb{R}$).

As mentioned before, every Weyl chamber is an $(n - 2)$-dimensional simplex, preserved by the action of a minimal parabolic subgroup $\mathbf{P}'$ on $\partial_\infty X_n$. It appears that the faces of the Weyl chamber are preserved by the parabolic subgroups containing $\mathbf{P}'$ (naturally, the smaller-dimensional faces are preserved by the larger subgroups) and each parabolic subgroup preserves some face of some Weyl chamber. This property can be used to give a proper definition of a Weyl chamber:

Definition 5. A Weyl chamber at infinity is the subset of points of $\partial_\infty X_n$ preserved by a minimal parabolic subgroup of $SL_n\mathbb{R}$.

Every flat has its boundary at infinity divided into a number of Weyl chambers. The subdivision is disjoint except for the faces of the Weyl chambers and the Weyl chambers cover the boundary completely. Since the boundary at infinity of a flat is homeomorphic to an $(n-2)$-dimensional sphere, this forms a simplicial complex (a triangulation of S^{n-2}) in which the Weyl chambers are the cells of maximal dimension. The triangulations for the boundaries of different flats in X_n are isomorphic via the action of $SL_n\mathbb{R}$.

Definition 6. A Weyl group is a finite group of isometries of $\mathbb{R}^k$ generated by a number of reflections about linear subspaces of codimension 1. Naturally, it is also a group of isometries of S^{k-1}. An abstract group isomorphic to a Weyl group is called a Coxeter group.

The structure of a single triangulation is invariant under the action of a certain Weyl group. In addition, two cells of the same dimension of triangulations of two boundaries of flats are either disjoint or coincide. Since every point of $\partial_\infty X_n$ is at the boundary of some flat, $\partial_\infty X_n$ itself becomes an infinite simplicial complex. This complex is a so called *(spherical) Tits building* (see [Ji06] and the definitions below) in which *chambers* are the Weyl chambers and the *apartments* are the whole spheres at the boundaries of flats.

Definition 7. A chamber in a simplicial complex is a simplex of the maximal dimension.

Definition 8. A simplicial complex is called a spherical Tits building, if it contains a family of closed subsets called apartments, such that:

1. Every apartment is a finite Coxeter complex (a triangulation of a sphere of some dimension, whose group of automorphisms is a Weyl (or Coxeter) group of isometries of the sphere);

2. Any two simplices are contained in some apartment;

3. Given two apartments Σ and Σ' and two simplices $\sigma, \sigma' \in \Sigma \cap \Sigma'$, there exists an isomorphism $\Sigma \to \Sigma'$ keeping σ and σ' pointwise fixed.

The vertices (0-dimensional simplices) of the Tits building correspond to the maximal proper parabolic subgroups of $SL_n\mathbb{R}$. Each of the maximal parabolic subgroups fixes a minimal flag of $\mathbb{R}^n$, i.e. a flag consisting of one linear subspace. There are $n-1$ types of nonconjugate maximal parabolic subgroups, which correspond to linear subspaces of the $n-1$ different dimensions. Each maximal parabolic subgroup is conjugate to one of the following groups:

$$\mathbf{P}_i = \left\{ \left(\begin{array}{c|c} *_{i \times i} & *_{i \times n-i} \\ \hline 0 & *_{n-i \times n-i} \end{array} \right) \right\} \bigcap SL_n\mathbb{R}, \qquad i = 1, \ldots, n-1.$$

An arbitrary simplex in the Tits building corresponds to the parabolic group, which is the intersection of the maximal parabolic subgroups corresponding to the vertices of the simplex. It also corresponds to the flag, which consists of the subspaces preserved by the mentioned maximal parabolic subgroups.

2.6 The Weyl chamber flow on $SL_n\mathbb{R}$

As mentioned before, the Weyl chamber flow (WCF) is the action on $SL_n\mathbb{R}$ by its maximal Abelian subgroup consisting of diagonal matrices:

$$WCF_{\mathbf{t}}(B) = B \cdot \mathrm{diag}(e^{t_1}, \ldots, e^{t_n}), \quad \sum_{j=1}^{n} t_j = 0.$$

Recall that $\mathbf{P}$ is the subgroup of the upper triangular matrices in $SL_n\mathbb{R}$, $\mathbf{K} = SO_n\mathbb{R}$, $\mathbf{A}$ is the subgroup of diagonal matrices with positive entries, and $\mathbf{N}$ is the subgroup of upper triangular matrices with ones on the diagonal. Then $SL_n\mathbb{R} = \mathbf{NAK} = \mathbf{P}_{>0}\mathbf{K}$, where $\mathbf{P}_{>0} = \mathbf{NA}$ is the group of upper triangular matrices with positive diagonal elements. Points of X_n are in one-to-one correspondence with elements of $\mathbf{P}_{>0}$. The cosets of $\mathbf{A}$ and $\mathbf{N}$ in $\mathbf{P}_{>0}$ give the horospherical decomposition of X_n, in which a coset $N \cdot \mathbf{A}$, $N \in \mathbf{N}$, is a flat and a coset $\mathbf{N} \cdot A$, $A \in \mathbf{A}$, is a horosphere.

Since every horospheric decomposition of X_n comes from the elements of $SL_n\mathbb{R}$ of the form $R' \cdot \mathbf{NA}$, where $R' \in SO_n\mathbb{R}$ is fixed, then for every flat $\mathbf{\Phi}$ in X_n we can

find an element $R \in SO_n\mathbb{R}$ and an element $N \in \mathbf{N}$ such that

$$\mathbf{\Phi} = RN \cdot \mathbf{A}.$$

Let $B \in \mathbf{\Phi}$, so that $B = RNA$, $A \in \mathbf{A}$. When we apply the Weyl chamber flow to the element B, we get an element

$$B' = WCF_{\mathbf{t}}(B) = RNA',$$

where

$$A' = A \cdot \mathrm{diag}(e^{t_1}, \ldots, e^{t_n}), \qquad \sum_{j=1}^{n} t_j = 0.$$

Since the elements R and N in the $\mathbf{KNA}$ decomposition for B' are the same as those for B, the point of X_n, corresponding to B', is on the flat $\mathbf{\Phi}$, too. It is also clear that one could get any point on the same flat as B by applying the WCF with appropriate parameters.

Remark. Since elements of $SL_n\mathbb{R}$ determine points of X_n, we will say that the WCF is also *over* X_n. In order to determine the trajectory of a point $x \in X_n$ under the WCF one needs to specify an element $R \in SO_n\mathbb{R}$. If the element R is known, then the trajectory of x is the flat in X_n, containing x in the horospheric decomposition of X_n, corresponding to R.

The above paragraphs prove the following fact.

Proposition 1. *The Weyl chamber flow over $X_n = SL_n\mathbb{R}/SO_n\mathbb{R}$ preserves the flats in X_n. In addition, since the asymptotic families of flats are also preserved, the Weyl chambers of $\partial_\infty X_n$ are preserved by the WCF.* $\qquad\square$

2.7 The structure of the apartments of the Tits building at infinity

Recall, that an apartment of the spherical Tits building is a triangulation of the boundary at infinity of one of the flats in X_n. Each point of the apartment is fixed by the elements of a proper parabolic subgroup of $SL_n\mathbb{R}$. The set of all points of

the apartment, which is fixed by a given parabolic subgroup is a closed simplex used in the triangulation.

A parabolic subgroup $\mathbf{P}_1$ is contained in a parabolic subgroup $\mathbf{P}_2$, if the simplex σ_2, corresponding to $\mathbf{P}_2$, is a face of the simplex σ_1, corresponding to $\mathbf{P}_1$. Each parabolic subgroup also fixes a flag in $\mathbb{R}^n$. Then smaller subgroups also preserve longer flags, with the relations "smaller" and "longer" given by the usual hierarchy of groups and flags.

The vertices of the triangulation correspond to maximal parabolic subgroups of $SL_n\mathbb{R}$. Each maximal parabolic subgroup preserves a minimal flag in $\mathbb{R}^n$, consisting of a single linear subspace. Let us say that a maximal parabolic subgroup $\mathbf{P}$ is of the "line" type, if it preserves a flag, consisting of a line, of the "plane" type, if it preserves a flag, consisting of one plane, etc. So all the maximal parabolic subgroups fall into one of the $n-1$ types.

We call a vertex, corresponding to a subgroup of the line type a "line"-vertex or $\mathbb{R}^1$-vertex, a vertex, corresponding to a subgroup of the plane type a "plane"-vertex or $\mathbb{R}^2$-vertex, etc. Let us also say that an $\mathbb{R}^k$-vertex corresponds to the k-dimensional subspace of $\mathbb{R}^n$, which forms the flag, preserved by the corresponding maximal parabolic subgroup.

Now we can describe the structure of the triangulation. Since an $(n-2)$-sphere is homeomorphic to the hypersurface of the regular $(n-1)$-simplex Σ, we will describe the isomorphic triangulation of the hypersurface of Σ.
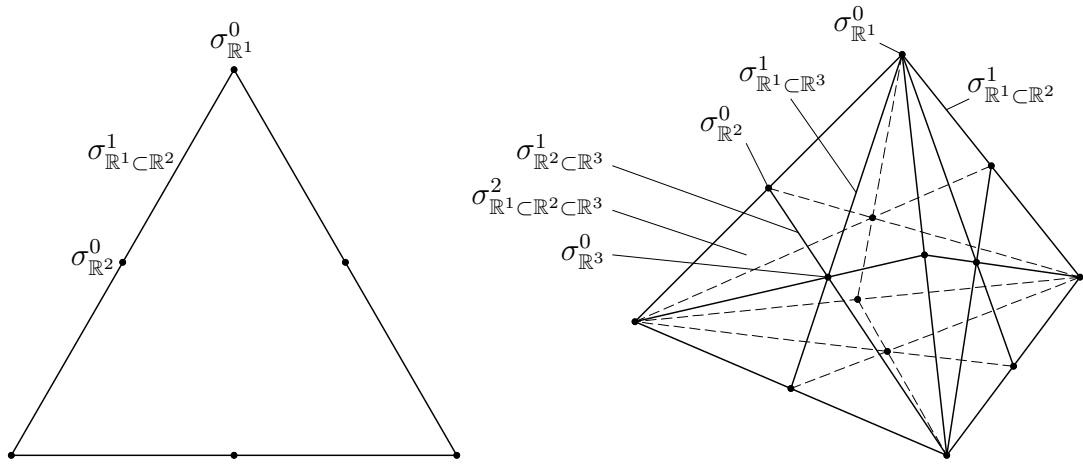

Figure 2.3. Triangulations of the apartments in $\partial_\infty X_3$ (left) and $\partial_\infty X_4$.

There are n vertices (0-simplices) of the apartment coinciding with the vertices of Σ. Each of these vertices is a "line"-vertex. In the middle of each edge of Σ sits a "plane"-vertex, which splits the edge into two 1-simplices. In the center of each 2-simplex of Σ there is an $\mathbb{R}^3$-vertex, which is connected by 1-simplices to all the "line"- and "plane"-vertices on the boundary of this 2-simplex. Then every $\mathbb{R}^k$-vertex is located in the center of a $(k-1)$-face of Σ and is connected by a 1-simplex to every vertex on the boundary of this face of Σ.

An $\mathbb{R}^k$-vertex (the type denoted by $\sigma^0_{\mathbb{R}^k}$ on Figure 2.3) corresponds to the minimal flag, which only contains the k-dimensional subspace of $\mathbb{R}^n$, spanned by the k lines, which correspond to the k "line"-vertices ($\sigma^0_{\mathbb{R}^1}$) connected to this $\mathbb{R}^k$-vertex. The 1-simplices of the triangulation correspond to the flags, consisting of 2 linear subspaces of $\mathbb{R}^n$ ($\sigma^1_{\mathbb{R}^k \subset \mathbb{R}^l}$), which correspond to the vertices connected by this 1-simplex. The 2-simplices of the triangulation correspond to flags of 3 subspaces, etc. All the subspaces, forming the flags, are the spans of one or more of the lines, which give the start to the construction. The longest flags in $\mathbb{R}^n$ consist of $n-1$ subspaces and correspond to the $(n-2)$-simplices of the triangulation. There are $n!$ full flags one can form using subspaces spanned by n lines in general position.

Coding of geodesics on the hyperbolic plane

This chapter summarizes known facts about different methods of coding of geodesics on the hyperbolic plane. Two classes of coding are considered: geometric and arithmetic. The geometric coding relies on a tessellation of the plane by congruent tiles and tracing of the intersection of the geodesics with the tiles. The arithmetic coding works mainly with the ends of the geodesics at infinity. A good overview of works in the area is contained in the Introduction to the article by Katok and Ugarcovici [KU07].

3.1 Fuchsian groups and coding of geodesics according to Morse

The first example is due to Morse [Mor21]. But first we need the following definitions, which can be found in the book by Katok [Kat92], Chapter 3.

Definition 9. A group Γ acting on a topological space X is called discrete, if the orbits $\Gamma(x)$, are discrete for all $x \in X$.

Definition 10. Assume that X is a metric space and Γ is a discrete group of isometries of X. A domain $\mathcal{D} \subset X$ (the closure of a nonempty simply connected open set) is called a fundamental domain for Γ, if

- $\bigcup_{g \in \Gamma} g(\mathcal{D}) = X$ and

- if $g \neq g'$, then $g(\mathring{\mathcal{D}}) \cap g'(\mathring{\mathcal{D}}) = \varnothing$, where $\mathring{\mathcal{D}}$ is the interior of $\mathcal{D}$.

Finding a fundamental domain may be useful and there is a well known way to find one.

Definition 11. Let $x \in X$ be a point fixed only by the identity of Γ. The fundamental domain given by

$$\mathcal{D}(x) = \{y \in X : \text{dist}(y, x) \leq \text{dist}(y, g(x)), g \in \Gamma\}$$

is called the Dirichlet domain of Γ at point x.

Definition 12. A discrete subgroup of orientation preserving isometries of the standard hyperbolic plane is called a Fuchsian group.

Consider a Dirichlet domain $\mathcal{D}$ for a Fuchsian group Γ. One can show that such a domain must be a polygonal set on the hyperbolic plane. The images of $\mathcal{D}$ under the action of Γ cover the hyperbolic plane and intersect only at the boundaries. So they form a tessellation of the plane.

Consider the images of $\mathcal{D}$ which are adjacent to $\mathcal{D}$ itself. Since the sets are polygonal, their intersections are geodesic pieces. For every image $g(\mathcal{D})$, $g \in \Gamma$, that intersects with $\mathcal{D}$ we can label the corresponding piece of a geodesic by the element g. After we label the sides of $\mathcal{D}$ we can spread the labeling to all the images of $\mathcal{D}$ by transferring the labels from $\mathcal{D}$ to the respective sides of the images.

It is worth mentioning that in such labeling every geodesic piece which separates two images of $\mathcal{D}$ is labeled in both images. Moreover, the labels on the two sides of the piece are the inverse elements of the group Γ (this is not difficult to see, when one looks at the original labeling of $\mathcal{D}$ and the labeling of an image $g\mathcal{D}$, adjacent to $\mathcal{D}$. The label outside of $\mathcal{D}$ should be g^{-1}).

In addition to that, the elements of Γ which appear on the sides of $\mathcal{D}$ generate the group Γ (since the images of $\mathcal{D}$ are in one-to-one correspondence with the elements of Γ and to get from $\mathcal{D}$ to any other of its images $g\mathcal{D}$ one has to cross only a finite number of sides of other images. Then the labels on the sides written in the reverse order give the word for the element g). Now we can describe Morse's construction.

Morse's method.

Let Γ be a Fuchsian group and $\mathcal{D}$ be a Dirichlet domain. For every geodesic γ on the hyperbolic plane in a general position (i.e. not passing through the vertices of any of the images of $\mathcal{D}$) one can write the sequence of labels $\{g_i\}_{i \in \mathbb{Z}}$ from the sides of the images of $\mathcal{D}$ in the order the geodesic intersects the sides (one should take only one label from each intersection, say the front one).

Morse's method is also called a *geometric* coding. It solely relies on the geometric properties of the fundamental domains.
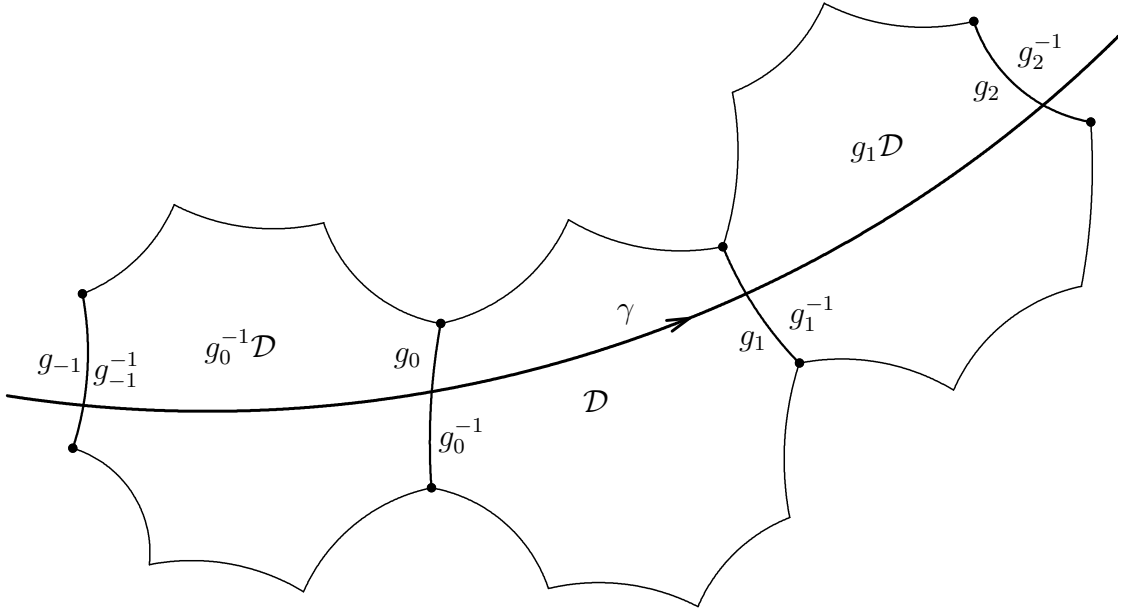
Figure 3.1. A geodesic γ with code $(\dots, g_{-1}, g_0, g_1, g_2, \dots)$.

3.2 Arithmetic coding of geodesics on the hyperbolic plane

Another type of codings of geodesics on the hyperbolic plane $\mathcal{H}$ is called *arithmetic*. An example of such coding was first given by Artin [Art24] and later modified by Series [Ser80] in order to classify geodesics on the modular surface $PSL_2\mathbb{Z} \backslash PSL_2\mathbb{R}$. Then it was realized by Katok [Kat96] that it is more suitable to use minus continued fractions. In general, the coding uses a minus continued fraction representation

of the ends of a geodesics in the upper half-plane model. Each of the two ends corresponds to an infinite sequence of integers used in the fraction, the two sequences are then juxtaposed to form a bi-infinite one. The different methods of coding vary in the choice of the minus continued fraction expansion and sometimes use different expansions for the terminal and initial ends of the geodesic.

The construction of so called Gauss arithmetic code is explained below, as this is the type of coding we are trying to generalize for symmetric spaces of higher dimensions.

Every real irrational number x has a representation as a minus continued fraction:

$$x = n_0 - \cfrac{1}{n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots}}}.$$

One can also write $x = [n_0; n_1, n_2, \dots]$ for short. The expansion is not unique but we can specify the exact way to find it. For the Gauss method we choose

$$n_0 = \lfloor x \rfloor + 1, \quad x_1 = -\frac{1}{x - n_0},$$
$$n_i = \lfloor x_i \rfloor + 1, \quad x_{i+1} = -\frac{1}{x_i - n_i},$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Note that $-1 \leq x_{i-1} - n_{i-1} < 0$, so $x_i \geq 1$ and $n_i \geq 2$, for $i \geq 1$.

Now we look at the complex upper half-plane model of the hyperbolic plane. A generic geodesic then has two ends on the absolute, which are two real numbers.

Initially, the coding is applied only to *reduced geodesics*, which in case of Gauss coding are the ones for which the initial end u and the terminal end w satisfy

$$0 < u < 1, \quad 1 < w. \tag{3.1}$$

The code of such geodesic $(\dots, n_{-2}, n_{-1}, n_0, n_1, n_2, \dots)$ is given by the expansions of $\frac{1}{u} = [n_{-1}; n_{-2}, \dots]$ and $w = [n_0; n_1, n_2, \dots]$.

As explained in Chapter 2, the group of orientation preserving isometries of the hyperbolic plane $\mathcal{H}$ is isomorphic to $PSL_2\mathbb{R}$ via the two-to-one correspondence of

matrices from $SL_2\mathbb{R}$ to the Möbius transformations of the upper half-plane:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R} \quad \mapsto \quad (z \mapsto \frac{az+b}{cz+d}).$$

Definition 13. Let Γ be a discrete subgroup of $PSL_2\mathbb{R}$. We will say that two geodesics are Γ-equivalent, if they are images of one another under the action of some elements of Γ.

If a geodesic is not reduced (i.e. its ends u and w do not meet the conditions (3.1)), then one can find a $PSL_2\mathbb{Z}$-equivalent geodesic, which is reduced. The existence of a reduced geodesic $PSL_2\mathbb{Z}$-equivalent to any given one is shown by Gurevich and Katok in [GK01]. The equivalent reduced geodesic is found by application of the appropriate *reduction algorithm*, described below. It is also shown there, that if two geodesics are equivalent and both are reduced, then their codes differ only by a shift by a finite number of positions. Thus, we can simply disregard the information about the initial position in the code and obtain a coding in which two $PSL_2\mathbb{Z}$-equivalent geodesics have the same code. It can also be used as the code of the geodesic on the modular surface $PSL_2\mathbb{Z}\backslash PSL_2\mathbb{R}$, obtained by the projection of any of the equivalent reduced geodesics.

Reduction algorithm.

The algorithm from [GK01] works like follows. If $w = [n_0; n_1, n_2, \dots]$ is the terminal end of an arbitrary geodesic, u is the initial end of the geodesic, and $S : z \mapsto -\dfrac{1}{z}$ and $T : z \mapsto z + 1$ are the standard generators of $PSL_2\mathbb{Z}$, then one constructs a sequence of $PSL_2\mathbb{Z}$-equivalent geodesics by subsequently applying the transformations ST^{-n_i}, $i = 0, 1, \dots$, until the requirements (3.1) for u and w are met. It's shown that the algorithm ends after a finite number of geodesics has been constructed.

Remark. One may consider different continued fraction expansions, and thus obtain different codings. In [KU05a, KU07] two of such other codings using so-called (a, b)-continued fractions are mentioned, named after Artin and Hurwitz.

3.3 The fundamental domain of $SL_2\mathbb{Z}\backslash SL_2\mathbb{R}$

Since the codings described in the previous sections are invariant with respect to the action of a specific Fuchsian group, we may assume that the coding is performed on the geodesics on the corresponding factor surface. Let $\mathcal{M}$ be the factor $\Gamma\backslash\mathcal{H}$ with a finite volume. This space is a so-called orbifold, which is a surface smooth everywhere except for a finite set of points.

The Morse's method is suitable for an arbitrary Fuchsian group, whereas the Gauss coding is specific (and so are the Artin and Hurwitz codings) for the group $PSL_2\mathbb{Z}$. Therefore, the relevant surface for the arithmetic codes is the modular surface ($PSL_2\mathbb{Z}\backslash PSL_2\mathbb{R}$, or $SL_2\mathbb{Z}\backslash SL_2\mathbb{R}$).

When one needs to study a factor space, it is convenient to consider a fundamental domain instead. For the modular surface the standard domain is

$$\mathcal{F} = \{z \in \mathbb{C} : |z| \geq 1, -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2}\}.$$

The group $PSL_2\mathbb{R}$ is generated by two elements S and T:

$$S : z \mapsto -\frac{1}{z}, \quad T : z \mapsto z + 1,$$

represented in $SL_2\mathbb{Z}$ by the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, respectively.
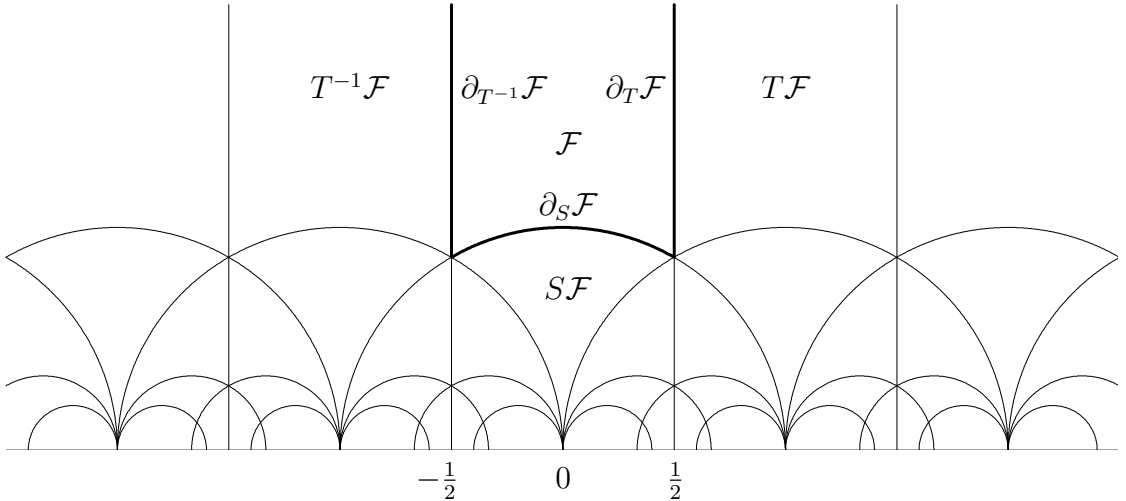

Figure 3.2. The tessellation of the upper half-plane by images of $\mathcal{F}$.

The boundary of $\mathcal{F}$ can be divided into three pieces, $\partial_{T^{-1}}\mathcal{F}$, $\partial_T\mathcal{F}$, and $\partial_S\mathcal{F}$, separating $\mathcal{F}$ from $T^{-1}\mathcal{F}$, $T\mathcal{F}$, and $S\mathcal{F}$, respectively, as shown on Figure 3.2.

3.4   Special flows

Definition 14. Consider a flow φ_t (a continuous action of $\mathbb{R}$) on a manifold X. Suppose there exists a closed subset $\mathcal{B}$ of X, such that every orbit visits $\mathcal{B}$ after any given time t and such that every orbit is transverse to $\mathcal{B}$. Then $\mathcal{B}$ is called a cross section of φ_t.

Definition 15. If $\mathcal{B}$ is a cross section of a continuous flow φ_t, then every orbit of φ_t is the orbit of some $x \in \mathcal{B}$. For every $x \in \mathcal{B}$ one can consider the time of first return to $\mathcal{B}$:

$$\theta(x) = \min_{t>0}\{\varphi_t(x) \in \mathcal{B}\}$$

and the first recurrence map (or the Poincaré map), which gives the point of the first return:

$$\Phi(x) = \varphi_{\theta(x)}(x).$$

Then $(\mathcal{B}, \Phi)$ is a discrete dynamical system and φ_t is called a special flow over $\mathcal{B}$ with the roof function θ and the first recurrence map Φ.

If a continuous flow φ_t on X is represented as a special flow over a cross section $\mathcal{B}$, then X itself could be thought of as a fibration over $\mathcal{B}$ with the fibers being the pieces of orbits of φ_t between their visits to $\mathcal{B}$ and can be visualized as the subgraph of the roof function θ fibered into the vertical segments over the points of $\mathcal{B}$.

Coding of orbits of a special flow.

Consider a special flow φ_t over a dynamical system $(\mathcal{B}, \Phi)$. Assume also that the space $\mathcal{B}$ is partitioned and the elements of the partition are labeled by an alphabet Σ. Then the orbit of a point $x \in \mathcal{B}$ under the flow φ_t could be coded by the sequence

$$(\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \ldots): \quad \lambda_n \in \Sigma, \quad \Phi^n(x) \in \lambda_n,$$

which lists the elements of the partition that the orbit visited in the past and will visit in the future. We then say that $\mathcal{B}$ captures the coding of the orbits.

3.5 Capturing the codes for the orbits of the geodesic flow

Here we will describe the cross sections which make the geodesic flow over the hyperbolic plane $\mathcal{H}$ a special flow and the partitions of these cross sections, which capture the codings of its orbits (i.e. geodesics) described earlier in this chapter.

Morse's method.

The geodesic flow acts on the unit tangent vectors at points of $\mathcal{H}$, so a cross section should be a subset of the unit tangent bundle $S\mathcal{H}$. In the Morse's method we have a tessellation of $\mathcal{H}$. Consider the set of vectors, whose base points are on the edges of the tiles. If Γ is a Fuchsian group of the first kind, this set is a cross section, since almost every orbit (geodesic) returns to the boundaries and we will ignore the set of orbits, which do not return to the cross section, because the set of these orbits has zero measure.

The partition of the cross section is then given by the labels on the edges of the tiles. All vectors, which are based at an edge are marked by the label on the inner side of the edge. It is almost immediate that when one considers the code, captured by this cross section for a given geodesic, one gets the same sequence as the one provided by the Morse's method.

Remark. Two geodesics which are Γ-equivalent, produce the same codes. If the group is of a finite covolume, the converse is also true, two geodesics with the same code are Γ-equivalent. As pointed out before, instead of considering the geodesics on $\mathcal{H}$ one could consider the geodesics on the factor surface $\mathcal{M} = \Gamma\backslash\mathcal{H}$ (more precisely, an orbifold, since the metric on it may not be smooth anymore). Γ-equivalent geodesics on $\mathcal{H}$ become a single one on $\mathcal{M}$, so the codes of geodesics become unique.

If we consider $\mathcal{M} = \Gamma\backslash\mathcal{H}$ instead of $\mathcal{H}$, we can still construct a cross section for the geodesic flow on $S\mathcal{M}$. It should be the factor over Γ of the cross section in

$S\mathcal{H}$, as the cross section in $S\mathcal{H}$ is Γ-invariant (the partitioning is also Γ-invariant, so it factors as well).

Gauss coding.

From the explanation of the coding process it is clear that the code is $PSL_2\mathbb{Z}$-invariant up to a shift by a finite number of positions. So we can find a closed subset $\mathcal{B}_0 \subset S\mathcal{H}$, to which all reduced geodesics return, and then translate it by all the elements of $PSL_2\mathbb{Z}$ to get

$$\mathcal{B} = \bigsqcup_{g \in PSL_2\mathbb{Z}} g\mathcal{B}_0,$$

a set to which all the geodesics return (since every geodesic is $PSL_2\mathbb{Z}$-equivalent to a reduced one). The set $\mathcal{B}_0$ is given by all the vectors tangent to the reduced geodesics at their intersection with the circle $|z| = 1$ (every reduced geodesic has to cross the circle, since it starts inside the circle and ends outside of it).
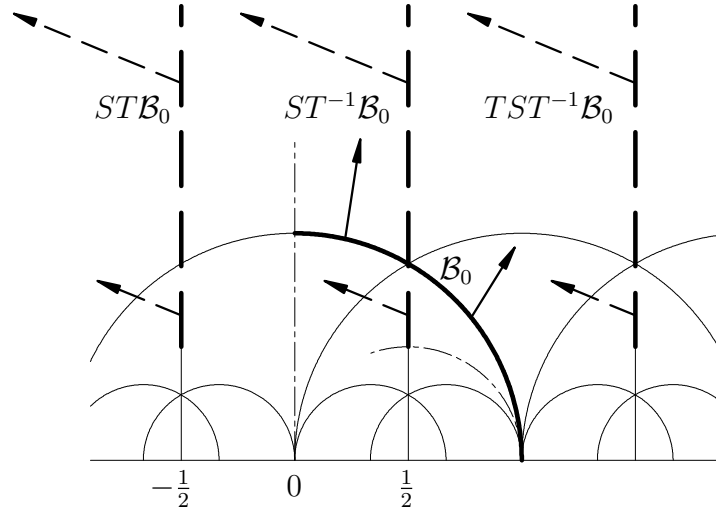
Figure 3.3. Vectors in the set $\mathcal{B}_0$ and the cross section $\mathcal{B}$.

We would like to show that every reduced geodesic γ with the base vector in $\mathcal{B}_0$ and code $(\ldots, n_{-1}, n_0, n_1, n_2, \ldots)$ will return to the set $\mathcal{B}$ again after some time. We can apply a step of the reduction algorithm, described in Section 3.2. We will get a new *reduced* geodesic, $ST^{-n_0}\gamma$, with the base point inside the circle $|z| = 1$, so it will have to intersect $\mathcal{B}_0$ at a later time, so γ will have to intersect $\mathcal{B}$ somewhere

else at the same time. Therefore any reduced geodesic, and thus any at all, will return to the set $\mathcal{B}$ infinitely many times.

Now we can partition of the set $\mathcal{B}$ by first partitioning the set $\mathcal{B}_0$ and then copying the labels of the vectors in $\mathcal{B}_0$ to the $PSL_2\mathbb{Z}$-equivalent vectors in $\mathcal{B}$. The set $\mathcal{B}_0$ is partitioned as follows: the vectors tangent to the geodesics whose terminal end is between $n-1$ and n receive label n.

It is not difficult to show that the cross section $\mathcal{B}$ captures the arithmetic Gauss code for geodesics. Consider a reduced geodesic γ. For the entries of the Gauss code $(\ldots, n_{-2}, n_{-1}, n_0, n_1, n_2, \ldots)$ with non-negative subscripts, look at the terminal end w of γ. From the construction of the Gauss code it follows that w should be between $n_0 - 1$ and n_0 and the cross section captures code n_0 for such w. Now we need to see when γ intersects $\mathcal{B}$ next time.

Note that for some time after g crosses the unit circle $|z| = 1$, it travels above the unit circles $|z - k| = 1$, $0 \leq k \leq n_0$. In fact, it may go below or intersect the circle $|z - 1| = 1$, but it will not enter the cross section $\mathcal{B}$ at the intersection point. The base points of vectors from $\mathcal{B}$ that are above these circles are located on the lines $\Re(z - k) = \dfrac{1}{2}$, $0 \leq k < n_0$. The vectors themselves are directed so that their horizontal component is negative, so none of them could be tangent to γ, which travels from the left to the right. But when γ intersects the unit circle $|z - n_0| = 1$, its tangent vector enters $\mathcal{B}$. It can be seen by looking at the $PSL_2\mathbb{Z}$-equivalent geodesic $ST^{-n_0}\gamma$, which is also reduced, and the image of the circle $|z - n_0| = 1$ under the transformation ST^{-n_0} is the circle $|z| = 1$, which the new geodesic intersects and its tangent vector at the intersection is in $\mathcal{B}_0$. Now the terminal end of the equivalent geodesic is $ST^{-n_0}w$, which is between $n_1 - 1$ and n_1. This shows that the next entry of the Gauss code is also captured by the cross section. One can continue producing geodesics that are $PSL_2\mathbb{Z}$-equivalent to γ in a similar fashion for the further entries and see that the code captured by the cross section $\mathcal{B}$ is the same as the Gauss code.

For the entries of the sequence with negative subscripts one needs to produce $PSL_2\mathbb{Z}$-equivalent geodesics of γ, subsequently applying the transformations $T^{n-i}S$, $i > 0$, and make observation about $\dfrac{1}{u}$, where u is the initial end of γ, similar to those made about its terminal end w in the previous paragraph.

Remark. Similar to the Morse's method, the Gauss coding could be used for the

geodesics on the modular surface $PSL_2\mathbb{Z}\backslash\mathcal{H}$ and be captured by the respective factor of the cross section $\mathcal{B}$ over $PSL_2\mathbb{Z}$, since $\mathcal{B}$, as well as its partition, is $PSL_2\mathbb{Z}$-invariant.

Chapter 4

Topological Markov chains

The first three sections of this chapter explain the result proven by the author in [Ego09] for Morse's method of coding of geodesics on $\mathcal{H} = X_2$. The last section reviews the results in the same area obtained by other authors for arithmetic codings.

Conventions used in this chapter

- Unit tangent vectors are referred to as "directions".

- Given a Fuchsian group Γ and its Dirichlet domain $\mathcal{D}$, we will refer to the edges and vertices of the tessellation $\Gamma\mathcal{D}$ as just "edges" and "vertices", because there are no edges or vertices considered in the proofs other than those.

- We will call the vertices on $\partial_\infty \mathcal{H}$ "infinite" and all the others – "finite".

- The geodesics are always directed. Then referring to the left and right hand side of the plane with respect to a given geodesic makes sense. If one geodesic crosses another one, we will say it is on the right from the other one, if its direction is from the left to the right, otherwise we will say it is on the left from the other one. Clearly, if one geodesic is on the right from another one, then the latter one is on the left from the former one and vice versa.

- (x, y) denotes the open geodesic segment between x and y, $[x, y]$ the corresponding closed segment. We write "$\langle x \dots$" or "$\dots y \rangle$" whenever it is immaterial or indefinite whether the corresponding endpoint is included or not.

4.1   Main result

Definition 16. Let Σ be a finite or countable alphabet. Let $\Lambda \subseteq \Sigma^{\mathbb{Z}}$ be a set of bi-infinite sequences of elements of Σ. Assume there are $k \in \mathbb{N}$ and $\Lambda_k \subseteq \Sigma^{k+1}$, such that $\Lambda = \{\lambda \in \Sigma^{\mathbb{Z}} : (\lambda_n, \dots, \lambda_{n+k}) \in \Lambda_k, \forall\, n \in \mathbb{Z}\}$. Then Λ is called a k-step topological Markov chain.

We will say that Λ is a topological Markov chain, if it is a k-step topological Markov chain for some $k \in \mathbb{N}$.

The elements of Λ_k are called "allowed" $(k + 1)$-tuples.

Definition 17. Let Γ be a Fuchsian group. If the volume of the factor surface $\Gamma \backslash \mathcal{H}$ is finite, the group is said to have a finite covolume.

Theorem 1. *Let Γ be a Fuchsian group with a finite covolume and a Dirichlet domain $\mathcal{D}$. Then the set of Morse codes of generic geodesics on $\mathcal{H}$ with respect to $\mathcal{D}$ is a topological Markov chain, if and only if $\mathcal{D}$ does not have finite vertices.*

Remark. The proof of sufficiency is essentially contained in [Ser86, §1, pp. 603–604]: one needs to consider the tree, which connects the elements of the Γ-orbit of the point, defining $\mathcal{D}$, if and only if they are contained in copies of $\mathcal{D}$, sharing a side. Every infinite sequence consisting of g_1, $\dots$, g_n will then correspond to a branch of the tree, starting at the root, and a point on $\partial_\infty \mathcal{H}$. Every bi-infinite sequence will correspond to two points on $\partial_\infty \mathcal{H}$ and, thus, to a geodesic. One only needs to check this geodesic's cutting sequence coincides with the original bi-infinite sequence.

Although this scheme produces a good proof, we present our own proof of sufficiency, which seems to be slightly shorter and doesn't need too many details to be checked.

4.2 Auxiliary statements

Throughout this section Γ is a Fuchsian group of a finite covolume. $\mathcal{D}$ is a Dirichlet domain for Γ.

Lemma 1 (There are plenty of finite vertices on the plane). *Assume $\mathcal{D}$ has a finite vertex. Let $y \in \mathcal{H}$ be some point. Consider the set Ξ of directions leading from y to all the finite vertices on the plane. Then the set is dense in the set of all directions at y (or, equivalently, between any two directions at y there is a direction from Ξ).*

Proof. There are two cases: $\mathcal{D}$ has no infinite vertices and otherwise.

Case 1. Suppose $\mathcal{D}$ has no infinite vertices, that is $\mathcal{D}$ is bounded. Consider η_1 and η_2, two arbitrary directions at y. Let us prove there is a direction between η_1 and η_2 leading from y to a finite vertex.

Let γ_1 and γ_2 be the two geodesics, passing through y and following η_1 and η_2, respectively. Since the geodesics are not parallel, there should be a point z inside the angle formed by the two geodesics, such that $\text{dist}(z, \gamma_1 \cup \gamma_2) > \text{diam}(\mathcal{D})$. Since z is contained in a copy of $\mathcal{D}$, that copy is entirely contained inside the angle as well. Since $\mathcal{D}$ has a finite vertex, so does the copy, containing z. Let V be a finite vertex of the copy. Clearly, V is inside the angle between γ_1 and γ_2, too. Thus, the direction, leading from y to V, is between η_1 and η_2.

Case 2. Assume $\mathcal{D}$ has an infinite vertex. Since $\mathcal{D}$ has a finite vertex too, $\mathcal{D}$ should have a side with both a finite and an infinite vertex. Let us denote the infinite vertex of the side by ξ' and the finite vertex by X'. By [Kat92, Theorem 4.2.5], every infinite vertex is the fixed point of some parabolic element of Γ, therefore so is ξ'.

Γ has a finite covolume, so by [Kat92, Lemma 4.5.3], in the unit circle Poincaré model the Euclidean diameters of sets $g_n\mathcal{D}$, $g_n \in \Gamma$, go to 0, as $n \to \infty$. So if one considers any point η on $\partial_\infty\mathcal{H}$ and a circle L of small radius $\varepsilon/2 > 0$ around it, there will always be infinitely many sets of the form $g\mathcal{D}$, $g \in \Gamma$, intersecting the interior of L (since the interior of L has an infinite hyperbolic volume, and $\mathcal{D}$ has a finite one). Therefore there should be an element $g_\eta(\varepsilon) \in \Gamma$, such that $g_\eta(\varepsilon)\mathcal{D}$ intersects the interior of L and the Euclidean diameter of $g_\eta(\varepsilon)$ is less than $\varepsilon/2$, and thus it is completely contained in the circle of Euclidean radius ε around η.

Since η and ε are arbitrary, we showed, that infinite vertices corresponding to ξ' in all copies of $\mathcal{D}$ are dense on $\partial_\infty \mathcal{H}$.

Consider a parabolic element $g' \in \Gamma$, fixing ξ'. The points $g'^m X'$, $m \in \mathbb{Z}$, lie on an horocycle, passing through ξ', and form a regular polygon with infinite number of sides. Each of these points is a finite vertex. As $m \to \pm\infty$, the geodesics, going from y to $g'^m X'$, converge to the geodesic, going from y to ξ'. So the direction from y to ξ' is an accumulation point of Ξ.

A similar argument can be used for any infinite vertex of the form $g\xi'$, $g \in \Gamma$. Thus Ξ has accumulation points at every direction from y to $g\xi'$. As mentioned above, the set $\{g\xi' : g \in \Gamma\}$ is dense in $\partial_\infty \mathcal{H}$. But it is the same as saying that the directions from y to $g\xi'$, are dense in the set of all directions at y. Hence Ξ is dense in the set of all directions as well. $\qquad\square$

Lemma 2. *If a geodesic passes near enough to a finite vertex, it intersects an edge, ending at the vertex. (There exists $\varepsilon > 0$, depending only on $\mathcal{D}$, such that if a geodesic γ passes at a distance less than ε from a finite vertex X, then γ intersects one of the edges, ending at X.)*
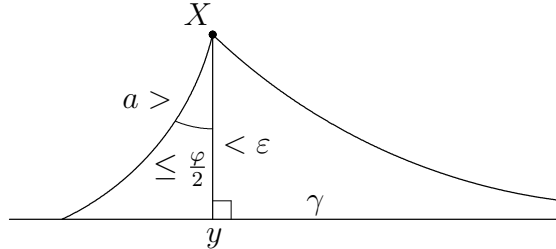

Figure 4.1. The perpendicular from X to a close geodesic γ.

Proof. Since $\mathcal{D}$ has a finite volume, it has a finite number of edges. Let φ be the greatest angle of $\mathcal{D}$, and let a be the length of the shortest edge of $\mathcal{D}$. Choose ε to be such, that a right triangle with the hypothenuse of length a and an acute angle $\frac{\varphi}{2}$ has the leg at that angle equal to ε.

Suppose a geodesic γ passes closer than ε to a finite vertex X, consider the perpendicular from X to γ, let the base of it be point y. The minimal angle between $[X, y]$ and an edge ending at X is at most $\frac{\varphi}{2}$, the length of $[X, y]$ is less than ε, and the length of the edge forming the minimal angle with $[X, y]$ is at least a, so the edge has to cross γ. $\qquad\square$

Lemma 3. *Suppose $\mathcal{D}$ has at least one finite vertex. Assume a geodesic γ passes through a point y and then crosses edges $\langle A_1, B_1 \rangle$, ..., $\langle A_n, B_n \rangle$. Then there are geodesics, γ^l and γ^r, which pass through y, cross the same edges and at least one more after that, such that the new edge for γ^l has a finite vertex on the right from γ^l, and the new edge for γ^r has a finite vertex on the left from γ^r.*

Proof. It is enough to prove the statement for γ^r. The proof for γ^l is absolutely the same.

It is clear that we can rotate γ around y clockwise by a sufficiently small angle, so that it still intersects all of $\langle A_i, B_i \rangle$. Consider such a rotation of γ and call the new geodesic $\hat{\gamma}^r$.

According to Lemma 1, there is a direction from y between γ and $\hat{\gamma}^r$, which leads to a finite vertex. Consider such a vertex X. Without loss of generality we may assume, that X is located behind $\langle A_n, B_n \rangle$ from y (since finite vertices constitute a discrete set on $\mathcal{H}$, there is only a finite number of finite vertices in the triangle bounded by γ, $\hat{\gamma}^r$, and $\langle A_n, B_n \rangle$, while there is an infinite number of finite vertices in the angle between γ and $\hat{\gamma}^r$).

Now rotate $\hat{\gamma}^r$ around y counterclockwise, so that the distance between X and the new geodesic becomes less than ε (cf. Lemma 2), but X is still on the left. Call the new geodesic γ^r. Clearly, it passes through y and crosses all of $\langle A_i, B_i \rangle$. Since it is closer than ε to X, it has to cross an edge ending at X, and the edge cannot be any of $\langle A_i, B_i \rangle$, because X is behind all of them from y. $\qquad\square$

4.3   Proof of Theorem 1

To prove that a set $\Lambda \subset \Sigma^{\mathbb{Z}}$ is a topological Markov chain, we only need to present a number $k \in \mathbb{N}$ and a set of allowed $(k+1)$-tuples Λ_k.

To prove otherwise, we need to show that given any $k \in \mathbb{N}$ one can find a $(k+l+1)$-tuple λ, where $l > 0$, such that no infinite sequence, containing λ, is in Λ, but every subsequence of λ of length $k+1$ is contained in some infinite sequence from Λ.

Proof. In our case the alphabet consists of the labels put on the edges of $\mathcal{D}$: $\Sigma = \{g_1, \ldots, g_n\}$, where n is the number of edges of D, and the set of infinite

sequences Λ is the set of Morse codes of generic geodesics on $\mathcal{H}$ with respect to $\mathcal{D}$.

First we prove that if $\mathcal{D}$ has a finite vertex, the set of Morse codes, generated by $\mathcal{D}$, is not a topological Markov chain.

We want to prove that given any $k \in \mathbb{N}$, there can be found a sequence $\lambda_{k+l} = (h_0, h_1, \ldots, h_k, h_{k+1}, \ldots, h_{k+l})$ of elements of Σ, for some $l \in \mathbb{N}$, such that there exist generic geodesics $\gamma^{(i)}$, for $0 \leq i \leq l$, so that Morse code of $\gamma^{(i)}$ contains $(h_i, h_{1+i}, \ldots, h_{k+i})$, $0 \leq i \leq l$, but no geodesic has its Morse code containing λ_{k+l} as a subsequence.

Fix some $k \in \mathbb{N}$. Consider a finite vertex V. Consequently applying Lemma 3, we can find a geodesic γ_0, passing through V and crossing at least k edges, with the last edge having a finite vertex on the right from γ_0. Without loss of generality we may assume that the k-th edge that γ_0 crosses after passing V has a finite vertex (otherwise we can consider a greater k, that will cover all the smaller ones) and the vertex is on the right from γ_0. Since there are only countably many finite vertices on the plane, we can choose γ_0 in such a way, that it doesn't pass through any finite vertex other than V.

Denote the endpoints of the first k edges γ_0 crosses after passing through V, finite or infinite, by $A_1, B_1; \ldots; A_k, B_k$ with A_i's being on the left from γ_0.

Let $\{\gamma_\varphi\}_{0 \leq \varphi < 2\pi}$ be the family of geodesics, passing through V, such that the angle between γ_0 and γ_φ in the clockwise direction is φ. It is clear that for all sufficiently small $\varphi > 0$ the first k edges γ_φ crosses after it passes through V, are the same as for γ_0.

Let $\varphi' = \inf\{\varphi > 0$, such that the first k edges, crossed by γ_φ after it passes through V, are different from those of $\gamma_0\}$. Obviously, $\varphi' > 0$. It is also clear that $\gamma_{\varphi'}$ should be a geodesic, intersecting all of $\langle A_1, B_1 \rangle$, $\langle A_2, B_2 \rangle$, ..., $\langle A_k, B_k]$ within their relative interior or at the right endpoint and passing through at least one of the right endpoints.

The above construction of $\gamma_{\varphi'}$ can be thought of as a clockwise rotation of γ_0 around V until it meets the first of B_i's.

Now consider the fundamental domain, which γ_0 enters immediately after it passes through V, name it $\mathcal{D}_0$. Let the edges of $\mathcal{D}_0$, that end at V, have the other endpoints denoted A_0 and B_0, so that A_0 is on the left from γ_0.

It can be easily seen, that by another small rotation of γ_0 around one of its
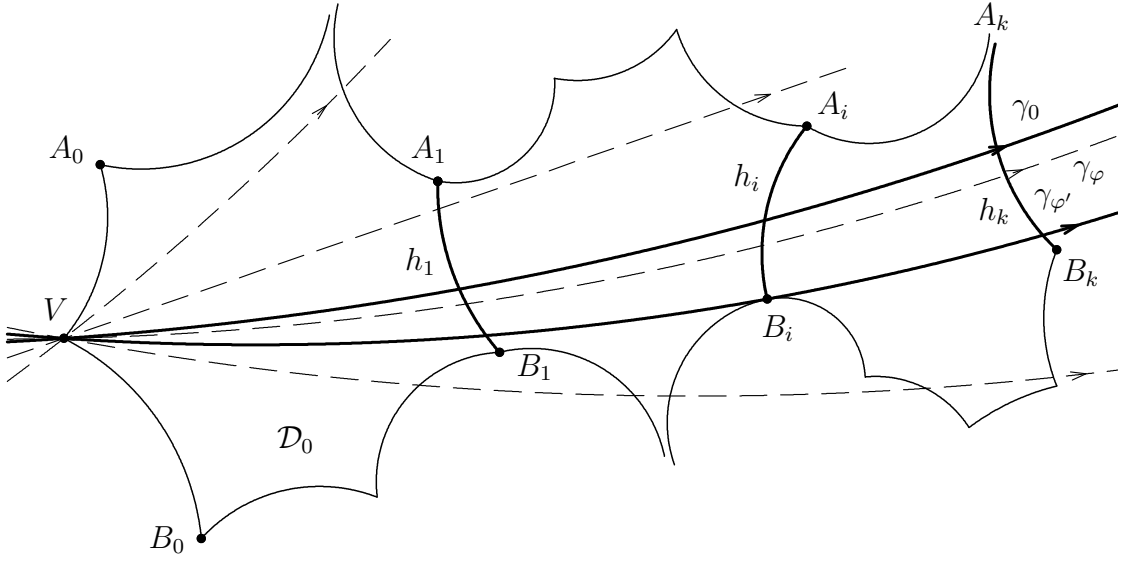
Figure 4.2. The family of geodesics passing through V.

points other than V, one can find a generic geodesic γ^b, which intersects $[V, B_0\rangle$, $\langle A_1, B_1\rangle$, ..., $\langle A_k, B_k]$. Let the label at $[V, B_0\rangle$ outside $\mathcal{D}_0$ be h_0, and the labels on $\langle A_i, B_i\rangle$ on the side facing V be h_i, $1 \le i \le k$. Then the sequence $(h_0, h_1, \ldots, h_k)$ is a part of Morse code of γ^b. Denote this sequence by $\lambda^{(0)}$.

On the other hand, rotating $\gamma_{\varphi'}$ clockwise around a point w behind $\langle A_k, B_k]$, it is possible to find another generic geodesic γ^a, which intersects $\langle A_0, V]$, $\langle A_1, B_1\rangle$, ..., $\langle A_k, B_k]$.

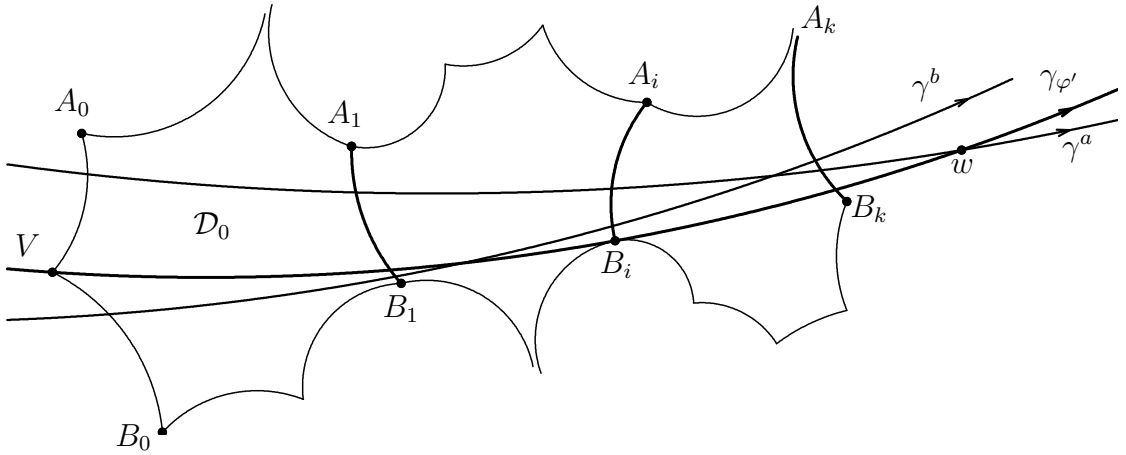**Figure 4.3.** Geodesics $\gamma_{\varphi'}$, γ^b, and γ^a.

Due to Lemma 1, between γ^a and $\gamma_{\varphi'}$ after they cross there should be an infinite number of finite vertices. We could have chosen γ^a so that one of those finite vertices is close enough to γ^a, so we can apply Lemma 2. Let Z be such a finite vertex between the two geodesics, that there is an edge, ending at Z and crossing γ^a.

Let the edges, crossed by γ^a after it crosses $\langle A_k, B_k]$ and up to the moment it crosses the edge, ending at Z, be labeled by $h_{k+1}, \ldots, h_{k+l}$, and have endpoints $A_{k+1}, B_{k+1}; \ldots; Z, B_{k+l}$. This automatically means that sequences $\lambda^{(i)} = (h_i, h_{1+i}, \ldots, h_{k+i})$, $0 < i \le l$, all are parts of Morse code of γ^a.
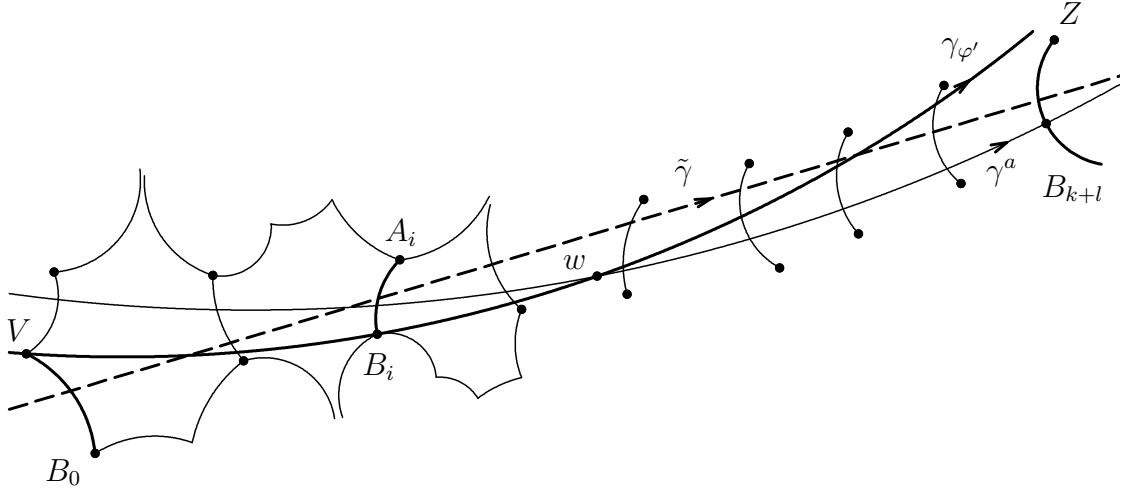

Figure 4.4. It is impossible for a geodesic to cross $[V, B_0\rangle$, $\langle A_1, B_1 \rangle$, $\ldots$, $\langle A_k, B_k]$, and $[Z, B_{k+l}\rangle$ all at the same time.

Let us show that the sequence $(h_0, h_1, \ldots, h_k, h_{k+1}, \ldots, h_{k+l})$ cannot be a part of Morse code of any geodesic. Assume this sequence is Morse code of some geodesic. Then there should be such geodesic $\tilde{\gamma}$ that crosses $[V, B_0\rangle$, $\langle A_1, B_1 \rangle$, $\ldots$, $[Z, B_{k+l}\rangle$. But this is impossible, because $[V, B_0\rangle$ and $[Z, B_{k+l}\rangle$ lie on the same side from $\gamma_{\varphi'}$, and $[Z, B_{k+l}\rangle$ does not intersect $\gamma_{\varphi'}$, while at least one of $\langle A_i, B_i \rangle$, $1 \le i \le k$, lies on the other side of $\gamma_{\varphi'}$.

So we proved, that $(h_0, h_1, \ldots, h_{k+l})$ is not a part of a geodesic's Morse code, while $(h_0, \ldots, h_k)$ is a part of Morse code of γ^b, and $(h_i, \ldots, h_{k+i})$, $0 < i \le l$, are parts of Morse code of γ^a. This ends the proof of necessity.

Now assume that $\mathcal{D}$ does not have finite vertices. Then every edge of $\mathcal{D}$ is a complete geodesic on $\mathcal{H}$.

Recall that set Σ consists of the labels put on the edges of $\mathcal{D}$, which are some elements of Γ. Since any edge is labeled by inverse elements from the opposite sides, and the same label does not appear on different edges inside a single domain, two edges, crossed by a geodesic subsequently, cannot be labeled by inverse elements. Consider set $\Lambda = \{\lambda \in \Sigma^{\mathbb{Z}} : \lambda_{n+1} \neq \lambda_n^{-1}, \forall n \in \mathbb{Z}\}$. If we introduce set $\Lambda_1 = \{(g, h) \in \Sigma^2 : g \neq h^{-1}\}$, set Λ can be written as $\{\lambda \in \Sigma^{\mathbb{Z}} : (\lambda_n, \lambda_{n+1}) \in \Lambda_1, \forall n \in \mathbb{Z}\}$. It is clear, that Morse code of any geodesic must be a subset of Λ. In order to prove that Morse coding of geodesics with respect to $\mathcal{D}$ is a topological Markov chain, we will show, that every sequence from Λ can be realized as Morse code of some geodesic.

Consider a sequence $\lambda' \in \Lambda$. Note that we can always find a sequence of copies of $\mathcal{D}$, $(\ldots, \mathcal{D}_{-1}, \mathcal{D}_0, \mathcal{D}_1, \ldots)$, such that $\mathcal{D}_n$ and $\mathcal{D}_{n+1}$ share an edge, which is labeled by λ'_n inside $\mathcal{D}_n$. To end the proof, we only need to find a geodesic, which crosses all the edges in the same order. If such a geodesic exists, it cannot cross any other edges in between of the ones just mentioned, since all domains are convex sets, and any two consecutive edges in the sequence belong to a same domain.

Let the edge, shared by $\mathcal{D}_n$ and $\mathcal{D}_{n+1}$, have endpoints $\xi_n, \xi'_n \in \partial_\infty \mathcal{H}$. It is clear, that ξ_{n+l} and ξ'_{n+l} lie on the same side from geodesic (ξ_n, ξ'_n), for any $n \in \mathbb{Z}$ and $l \neq 0 \in \mathbb{Z}$. Consider intervals $[\xi_n, \xi'_n]$ on $\partial_\infty \mathcal{H}$, for all $n \neq 0$, chosen in such a way, that they do not contain ξ_0 and ξ'_0. Then $[\xi_n, \xi'_n]$, $n \neq 0$, form two nested sequences as $n \to \infty$ and as $n \to -\infty$, one on each side from geodesic (ξ_0, ξ'_0). Thus, there should be points $\xi_\infty, \xi_{-\infty} \in \partial_\infty \mathcal{H}$, such that ξ_∞ is on the other from (ξ_0, ξ'_0) side of (ξ_n, ξ'_n), for all $n > 0$, and $\xi_{-\infty}$ is on the other from (ξ_0, ξ'_0) side of (ξ_n, ξ'_n), for all $n < 0$. Obviously, ξ_∞ and $\xi_{-\infty}$ are two different points. Consider the geodesic $(\xi_{-\infty}, \xi_\infty)$. It has to cross all of (ξ_n, ξ'_n) and they can only be crossed in the order of increasing n. $\qquad\square$

The proof of the sufficiency implies the following corollary.

Corollary 1. *If Morse coding, given by a Dirichlet domain of a Fuchsian group, produces a topological Markov chain, it produces a 1-step one.*

4.4 Arithmetic codings

Here we again are following the paper by Katok and Ugarcovici [KU07, §3]. The Gauss coding and two other arithmetic codings (named after Artin and Hurwitz) are considered. The difference between Gauss and the other two codings is in the definition of the minus continued fraction expansion of the endpoints of the geodesics as well as in the set of reduced geodesics.

The reduced geodesics for each coding intersect the circle $|z| = 1$. The cross section for each method consists of all the vectors tangent to reduced geodesics at the intersection with this circle. Then the cross section is partitioned according to the n_0 entry of the code of the geodesics passing the cross section.

Katok and Ugarcovici refer to their own results in [KU05a], stating that the the partitions of the cross section for each of the arithmetic codings satisfy Markov property with respect to the first recurrence map. Then they use Adler's result [Adl98, Theorem 7.9], from which it follows that the set of codes should be a 1-step topological Markov chain. Thus, Katok and Ugarcovici proved the following statement:

Theorem 2. *For each of the three arithmetic codings (Artin, Gauss, and Hurwitz) the set of codes of geodesics on $\mathcal{H}$ is a topological Markov chain.*

Remark. The methods and the result applies to many other methods of arithmetic coding. The main focus of the method is finding a suitable definition for "reduced" geodesics. Each geodesic on $\mathcal{H}$ then has a $PSL_2\mathbb{Z}$-equivalent reduced geodesic, whose code it acquires. Reduced geodesics all must pass through a "pre-cross section", whose $PSL_2\mathbb{Z}$ orbits form an actual cross section. The cross section is then partitioned according to the given arithmetic code.

Chapter 5

Further directions of research

In the view of the simple structure of the codes produced by the Gauss coding (the set of codes is a topological Markov chain) it is tempting to try to generalize the coding procedure to code the orbits of the Weyl chamber flow over X_n. The coding of the geodesics can be reduced to the question of expanding of real numbers into minus continued fractions. Each expansion gives a sequence of integers corresponding to the real number. A geodesic's code comes from the two of its ends.

Recall, that an apartment in the Tits building at infinity of X_n is the infinite boundary of one of the flats in X_n. In the upper half-plane model the infinite boundary of every geodesic consists of two real numbers (one of which may actually be the infinity), so each pair of distinct real numbers is an apartment in the Tits building structure of $\partial_\infty \mathcal{H}$. Each of the two ends of a geodesic is a Weyl chamber in $\partial_\infty \mathcal{H}$. So the coding of geodesics could be considered as coding of ordered pairs of the Weyl chambers (the two directions of a geodesic are most likely to have different codes).

In the attempts to produce such a generalization, the observations described in this chapter were made. Since the Gauss method produces $SL_2\mathbb{Z}$-invariant codes, we are trying to find $SL_n\mathbb{Z}$ invariant constructions. One of the advantages is that this way we can restrict the work to only a fundamental domain for $SL_n\mathbb{Z}$ in X_n, which can be found explicitly according to works of Minkowski [Min67], for small values of n, and Grenier [Gre88], for arbitrary n.

5.1 Coordinates in X_n

Here we would like to introduce one more way to look at the space X_n. Namely, since it is the homogeneous space $SL_n\mathbb{R}/SO_n\mathbb{R}$, it is the space of equivalence classes of positively oriented bases with unit frame volume, such that two bases are equivalent if and only if they differ by a rotation of $\mathbb{R}^n$.

We can choose a canonical representative for each element of X_n in the following way. Consider the Iwasawa decomposition of $SL_n\mathbb{R}$:

$$SL_n\mathbb{R} = \mathbf{NAK},$$

where $\mathbf{N}$ is the group of upper triangular matrices with 1's on the diagonal, $\mathbf{A}$ is the group of the diagonals matrices with positive elements on the diagonal and determinant 1, and $\mathbf{K} = SO_n\mathbb{R}$. Each element $B \in SL_n\mathbb{R}$ admits a unique representation in the form

$$B = NAK.$$

Let us call the element $V = NA \in \mathbf{P}_{>0}$ the *canonical* representative of B. As we have shown before, this is a bijection between $\mathbf{P}_{>0}$ and X_n.

Assume the rows of an $n \times n$ matrix represent vectors in $\mathbb{R}^n$ and the coordinates in $\mathbb{R}^n$ be $x_1, \ldots, x_n$, then for a basis $V = (\mathbf{v_1}, \ldots, \mathbf{v_n})$ to be canonical:

- $\mathbf{v_n}$ must follow the x_n-axis in the positive direction,

- $\mathbf{v_{n-1}}$ must be in the $x_{n-1}x_n$-plane and have positive x_{n-1}-coordinate,

  . . .

- $\mathbf{v_2}$ must be in the $x_2 \ldots x_n$-subspace and have positive x_2-coordinate,

- $\mathbf{v_1}$ must have positive x_1-coordinate.

To parameterize an element of X_n we could use the coordinates of the canonical representative V on or above the diagonal. Then we would use $\dfrac{n(n+1)}{2}$ parameters: $(\mathbf{v_{ij}}), i \leq j$. On the other hand, X_n has $\dfrac{n(n+1)}{2} - 1$ dimensions. But one of the coordinates is not necessary, since we know that frame volume of the basis is 1, so we can always find $\mathbf{v_{nn}}$ from $\mathbf{v_{11}}, \ldots, \mathbf{v_{n-1n-1}}$. So we divide all coordinates by $\mathbf{v_{nn}}$ and disregard the latter one.

We will need to rearrange the indices, too. We will use the following arrangement of the coordinates x_{ij}, $2 \leq i \leq n$, $1 \leq j \leq i$:

$$\frac{1}{\mathbf{v}_{\mathbf{n}n}} \begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} & \cdots & \mathbf{v}_{\mathbf{1}n-1} & \mathbf{v}_{\mathbf{1}n} \\ 0 & \mathbf{v}_{\mathbf{2}2} & \cdots & \mathbf{v}_{\mathbf{2}n-1} & \mathbf{v}_{\mathbf{2}n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{v}_{\mathbf{n-1}n-1} & \mathbf{v}_{\mathbf{n-1}n} \\ 0 & 0 & \cdots & 0 & \mathbf{v}_{\mathbf{n}n} \end{pmatrix}$$

$$= \begin{pmatrix} x_{nn} & x_{n,n-1} & \cdots & x_{n2} & x_{n1} \\ 0 & x_{n-1,n-1} & \cdots & x_{n-1,2} & x_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_{22} & x_{21} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

For points of the hyperbolic plane $\mathcal{H} = X_2$ we obtain two coordinates with the second one being positive. Namely, the element $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$ yields coordinates $x_{21} = x$ and $x_{22} = y > 0$. It also corresponds to the upward vertical unit vector in the Poincaré model in the upper half-plane, tangent at the point (x, y). For X_3 we obtain a correspondence with the part of $\mathbb{R}^5$ where the second and fifth coordinates are positive.

Other coordinate systems.

Other coordinate systems that were considered in the literature are worth mentioning here.

One of the systems, mentioned in the paper by Gordon, Grenier, and Terras [GGT87], is the (partial) Iwasawa coordinates. A symmetric positive representative

of an element of X_n is decomposed into the product of the form $X^t Y X$, such that

$$
Y = \begin{pmatrix} y_1 & 0 & \ldots & 0 \\ 0 & y_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & y_n \end{pmatrix}, \quad X = \begin{pmatrix} 1 & x_1 & \ldots & x_{n-2} & x_{n-1} \\ 0 & 1 & \ldots & x_{2n-4} & x_{2n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & x_{\frac{n(n-1)}{2}} \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix},
$$

where $\Pi_{i=1}^n y_i = 1$, $y_i > 0$ for all i. The parameters from the matrices X and Y are used as the coordinates. Minkowski used these coordinates in proofs supporting his description of the fundamental domains for $GL_n\mathbb{Z}$, but gave explicit inequalities only for a small number of n's. Our coordinates can be easily restored from these by rotating each matrix by 180 degrees and taking the square root of Y.

The second system is also mentioned in [GGT87] and is used in the work of Grenier [Gre88]. Grenier generalizes the Iwasawa coordinates and decomposes an element further, so that the inequalities for the boundaries of a fundamental domains could be obtained inductively. This method allowed to obtain explicit inequalities for the fundamental domain of $GL_n\mathbb{Z}$ for any n, though this fundamental domain does not consist of Minkowski reduced elements. The coordinates of an element of X_n can be obtained recursively by representing an element $V \in X_n$ as

$$
V = \left(\begin{array}{c|c} 1 & 0 \\ \hline \mathbf{x}^t & I_{n-1} \end{array} \right) \left(\begin{array}{c|c} y & 0 \\ \hline 0 & Z \end{array} \right) \left(\begin{array}{c|c} 1 & \mathbf{x} \\ \hline 0 & I_{n-1} \end{array} \right),
$$

where $Z \in X_{n-1}$ and $\mathbf{x} = (x_1, \ldots, x_{n-1})$, using y, $x_1, \ldots x_{n-1}$, and the coordinates of Z in X_{n-1} as the coordinates of V.

5.2   Bases in $\mathbb{R}^n$

Minkowski reduced bases.

Definition 18. Let us call a basis $V = (\mathbf{v_1}, \ldots, \mathbf{v_n})$ in $\mathbb{R}^n$ Minkowski reduced, if

- $\mathbf{v_n}$ is the shortest non-zero vector in the lattice $\langle V \rangle_\mathbb{Z}$ spanned by V and

- $\mathbf{v_{i-1}}$ is the shortest vector in $\langle V \rangle_\mathbb{Z} - \langle \mathbf{v_i}, \ldots, \mathbf{v_n} \rangle_\mathbb{Z}$.

For almost every basis B in $\mathbb{R}^n$ there is a unique, up to an even number of reflections through the origin, Minkowski reduced basis V, which spans the same lattice (i.e. there are several $SL_n\mathbb{Z}$-equivalent to B Minkowski reduced bases, which are all $\mathbf{M}$-equivalent).

In the coordinates we introduced the standard fundamental domain for the $SL_2\mathbb{Z}$ action on the upper half-plane consists of the points corresponding to Minkowski reduced bases in $\mathbb{R}^2$. Almost all lattices in $\mathbb{R}^2$ have two Minkowski reduced bases, which differ by the multiplication by $-I$. The $SL_2\mathbb{Z}$ action by the linear fractional transformations on the upper half-plane is also compatible with the $SL_2\mathbb{Z}$ action on bases in $\mathbb{R}^2$. It follows naturally that the set of Minkowski reduced bases in $\mathbb{R}^n$ corresponds to a fundamental domain for the $SL_n\mathbb{Z}$ action on X_n in these coordinates (a slightly broader notion of a fundamental domain should be used here, since the elements of $\mathbf{M} < SL_n\mathbb{Z}$ are fixing all points of X_n).

Gauss reduced geodesics.

Here we would like to point out the relation between the reduced geodesics (for the purpose of the Gauss coding) and the bases in $\mathbb{R}^2$.

Proposition 2. *An element B of $SL_2\mathbb{R}$ corresponds to a vector whose base point is on the unit circle $|z| = 1$ and which is tangent to a reduced geodesic if and only if the basis $(\mathbf{b_1}, \mathbf{b_2})$, consisting of the rows of B, is such that:*

1. $\|\mathbf{b_1}\| = \|\mathbf{b_2}\|$;

2. $\mathbf{b_1}$ and $\mathbf{b_2}$ are in the first quadrant of $\mathbb{R}^2$.

Proof. Let $B = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \dfrac{1}{\sqrt{y}} \end{pmatrix} R$, where $R \in SO_2\mathbb{R}$. Let B also correspond to the vector $\vec{\zeta}$ with the base point z. We know that $z = x + iy$.

Since R is a rotation of the rows, the ratio of $\|\mathbf{b_1}\|$ and $\|\mathbf{b_2}\|$ is the same as that of the first and second rows of the matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \dfrac{1}{\sqrt{y}} \end{pmatrix} = \begin{pmatrix} \sqrt{y} & \dfrac{x}{\sqrt{y}} \\ 0 & \dfrac{1}{\sqrt{y}} \end{pmatrix}$,

which is $\sqrt{x^2 + y^2} = |z|$. The angle between $\mathbf{b_1}$ and $\mathbf{b_2}$ in the counter clockwise
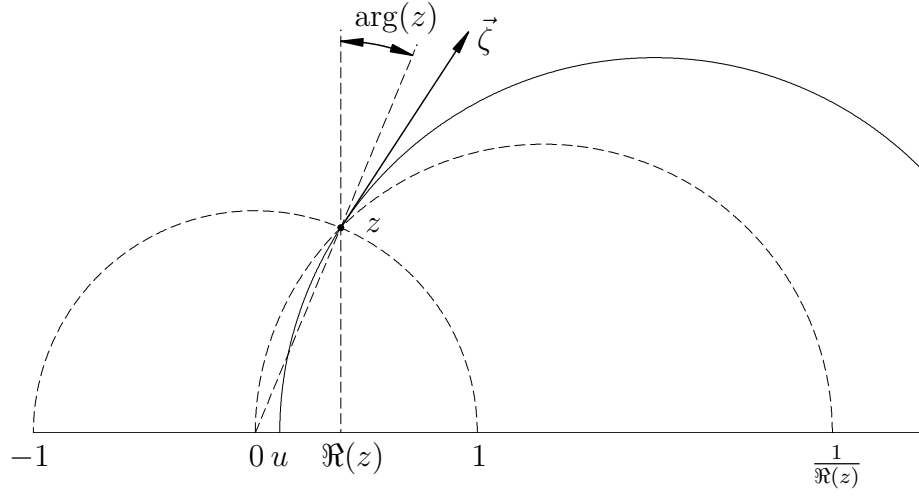
Figure 5.1. Reduced unit vector $\vec{\zeta}$ at z on the unit circle.

direction in $\mathbb{R}^2$ is the same as that for $\begin{pmatrix} \sqrt{y} & \dfrac{x}{\sqrt{y}} \\ 0 & \dfrac{1}{\sqrt{y}} \end{pmatrix}$, for which it is $\arctan(\dfrac{\sqrt{y^2}}{x}) =$ $\arg(z)$.

The angle α between $\vec{\zeta}$ and the vertical is equal to twice the angle between $\mathbf{b_2}$ and x_2-axis in $\mathbb{R}^2$ measured in the same direction. That is $R = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$.

Elementary geometry shows that for a geodesic is reduced if and only if $|z| = 1$, $\arg(z) < \frac{\pi}{2}$, and $0 < \frac{\alpha}{2} < \frac{\pi}{2} - \arg(z)$, which is equivalent to the two conditions. $\square$

This proposition suggests the next definition.

Definition 19. A basis $(\mathbf{b_i})_{i=1}^n$ in $\mathbb{R}^n$ (or an element $B \in SL_n\mathbb{R}$ whose i-th row is $\mathbf{b_i}$) is called Gauss reduced, if

1. $\|\mathbf{b_i}\| = \|\mathbf{b_j}\|$ for all i, j;

2. $\mathbf{b_i}$ are all in the positive orthant of $\mathbb{R}^n$.

5.3   Flats in X_n

Recall that a flat in X_n is the orbit of the base point of an element from $SL_n\mathbb{R}$ which is acted upon by the Weyl chamber flow.

Proposition 3. *Almost all flats in X_n are submanifolds of spheres, when described in the coordinates introduced in §5.1.*

This fact is obvious for the upper half-plane model of $\mathcal{H} = X_2$, but we couldn't find in the literature a generalization of it for larger n.

First we examine the canonical representative for an element of X_n. Consider an element $B \in SL_n\mathbb{R}$ and its canonical representative $V \in X_n$:

$$
B = \begin{pmatrix} \mathbf{b_1} \\ \vdots \\ \mathbf{b_n} \end{pmatrix}, \quad V = \begin{pmatrix} \mathbf{v_1} \\ \vdots \\ \mathbf{v_n} \end{pmatrix}.
$$

Note that $\mathbf{v_i} \cdot \mathbf{v_j} = \mathbf{b_i} \cdot \mathbf{b_j}$ for all i, j, since the two bases differ only by a rotation. Since $\mathbf{v_1}$ has only the first component non-zero, $\mathbf{v_1} \cdot \mathbf{v_i} = \mathbf{v_{11}} \mathbf{v_{i1}} = \mathbf{b_1} \cdot \mathbf{b_i}$, so

$$
\mathbf{v_{11}} = \|\mathbf{b_1}\|, \quad \mathbf{v_{i1}} = \frac{\mathbf{b_i} \cdot \mathbf{b_1}}{\|\mathbf{b_1}\|}, i \geq 2.
$$

Another immediate observation is that

$$
\sum_{j=1}^{n} \mathbf{v_{ij}^2} = \|\mathbf{v_i}\|^2 = \|\mathbf{b_i}\|^2.
$$

So when we pass from an arbitrary element $B \in SL_n\mathbb{R}$ to its coordinates defined in the previous section, we obtain numbers x_{ij}, $2 \leq i \leq n$, $1 \leq j \leq i$, such that

$$
x_{i1} = \frac{\mathbf{v_{n+1-i}}_n}{\mathbf{v_{nn}}} = \frac{\mathbf{b_{n+1-i}} \cdot \mathbf{b_n}}{\|\mathbf{b_n}\|^2}, \quad 2 \leq i \leq n, \tag{5.1}
$$

$$
\sum_{j=1}^{i} x_{ij}^2 = \sum_{j=1}^{i} \frac{\mathbf{v_{n+1-i}}_{n+1-j}^2}{\mathbf{v_{nn}^2}} = \frac{\|\mathbf{b_{n+1-i}}\|^2}{\|\mathbf{b_n}\|^2}, \quad 2 \leq i \leq n. \tag{5.2}
$$

A point B' of the orbit of B is of the form $B \cdot \mathrm{diag}(e^{t_1}, \dots, e^{t_n})$, $\sum_{j=1}^{n} t_j = 0$. Consider the following limits for all $1 \leq k \leq n$, where x_{ij}' are the coordinates of B':

$$
\lim_{t_k \to +\infty, \ |\frac{t_j}{t_k}| < 1-\varepsilon, j \neq k} x_{i,1}' = \frac{\mathbf{b_{n+1-i}}_k \mathbf{b_{nk}}}{\mathbf{b_{nk}^2}}
$$

$$= \frac{\mathbf{b_{n+1-i}}_k}{\mathbf{b_n}_k}$$

$$= \lim_{t_k \to +\infty, \ |\frac{t_j}{t_k}|<1-\varepsilon, j\neq k} \frac{\|\mathbf{b_{n+1-i}}\|}{\|\mathbf{b_n}\|},$$

then from (5.2) it follows that

$$\lim_{t_k \to +\infty, \ |\frac{t_j}{t_k}|<1-\varepsilon, j\neq k} x'_{ij} = 0, \quad 1 < j \leq i.$$

The point $\xi_k = \lim_{t_k \to +\infty, \ |\frac{t_j}{t_k}|<1-\varepsilon, j\neq k} B'$ is one of the points at infinity of the flat, containing B. For each k all but at most $n-1$ coordinates of ξ_k are zero, namely the first one of every subsequent i-tuple, $2 \leq i \leq n$. So we obtain n points in $\mathbb{R}^{n-1}$. In order for $\det(B) = 1$ these n points should not be in one $(n-2)$-dimensional affine subspace of $\mathbb{R}^{n-1}$. This gives us the circumscribed sphere of the points. Let $\mathbf{C} = (C_2, \ldots, C_n)$ be the center of the sphere and r be its radius.

Proof of Proposition 3. We want to show that the points, given by the coordinates of B in $\mathbb{R}^{d_n}$, lie at the same distance from $\mathbf{C} \in \mathbb{R}^{n-1} \subset \mathbb{R}^{d_n}$. If we show that this distance is r for B, we will prove the distance is r for all the points of the orbit, since $\mathbf{C}$ and r are determined by the orbit alone and hence are the same for each point of the orbit.

Indeed,

$$\text{dist}(\mathbf{C}, (x_{21}, \ldots, x_{nn}))^2 = \sum_{i=2}^{n}(C_i - x_{i1})^2 + \sum_{i=2,j=2}^{n,i} x_{ij}^2$$

$$= \sum_{i=2}^{n}(C_i^2 - 2C_i x_{i1}) + \sum_{i=2,j=1}^{n,i} x_{ij}^2$$

$$= \sum_{i=2}^{n}(C_i^2 - 2C_i \frac{\mathbf{b_{n+1-i}} \cdot \mathbf{b_n}}{\|\mathbf{b_n}\|^2}) + \sum_{i=2}^{n} \frac{\|\mathbf{b_{n+1-i}}\|^2}{\|\mathbf{b_n}\|^2}$$

$$= \sum_{i=2}^{n} \frac{C_i^2 \|\mathbf{b_n}\|^2 - 2C_i \mathbf{b_{n+1-i}} \cdot \mathbf{b_n} + \|\mathbf{b_{n+1-i}}\|^2}{\|\mathbf{b_n}\|^2}$$

$$= \sum_{i=2}^{n} \frac{(C_i \mathbf{b_n} - \mathbf{b_{n+1-i}})^2}{\|\mathbf{b_n}\|^2}$$

$$= r^2,$$

because $\mathbf{C}$ is the center of the sphere around $\{\xi_k = (\frac{\mathbf{b_{n+1-i}}_k}{\mathbf{b_{n}}_k})_{i=2}^n\}_{k=1}^n$ and, thus,

$$\sum_{i=2}^n (C_i - \frac{\mathbf{b_{n+1-i}}_k}{\mathbf{b_{n}}_k})^2 = r^2, \ \forall 1 \le k \le n,$$

$$\sum_{i=2}^n (C_i \mathbf{b_{n}}_k - \mathbf{b_{n+1-i}}_k)^2 = r^2 \mathbf{b_{n}}_k^2, \ \forall 1 \le k \le n,$$

$$\sum_{i=2}^n \sum_{k=1}^n (C_i \mathbf{b_{n}}_k - \mathbf{b_{n+1-i}}_k)^2 = \sum_{k=1}^n r^2 \mathbf{b_{n}}_k^2 = r^2 \|\mathbf{b_{n}}\|^2.$$

These calculations work out only if none of the components of $\mathbf{b_n}$ is zero, which still holds for almost all $B \in SL_n\mathbb{R}$. The set of points with coordinates x'_{ij} is clearly smoothly parameterized by (t_j), so it is a submanifold of sphere in $\mathbb{R}^{d_n}$ with the center in the $(n-1)$-dimensional subspace. $\qquad\square$

It becomes clear that any n points in the $(n-1)$-dimensional subspace of $\mathbb{R}^{d_n}$, which are not in the same $(n-2)$-dimensional affine subspace, determine an orbit of the WCF. The WCF does not change the ratios of the i-th components of the rows in B and these ratios determine the n points in $\mathbb{R}^{n-1}$. It is also possible to restore B from these ratios and the fact that $\det B = 1$, up to the action of the WCF and multiplications from the right by the elements of $\mathbf{M}$.

Since some of the components of $\mathbf{b_n}$ could actually be zero, it makes sense to consider the homogeneous ratios of all the i-th coordinates of the rows of B. Thus, one gets all the orbits of the WCF described by linearly independent n-tuples of points on $\mathbb{R}P^{n-1}$.

5.4 Using n-tuples of points in $\mathbb{R}P^{n-1}$

It is clear that the boundary of a flat at infinity determines the flat uniquely. So we have a one-to-one correspondence between apartments of the Tits building at infinity and the flats. Moreover, an apartment is uniquely specified by its "line"-vertices, as described in §2.7. A Weyl chamber in the apartment is specified further by ordering of the "line"-vertices. Since each "line"-vertex corresponds to a line in $\mathbb{R}^n$ or a point of $\mathbb{R}P^{n-1}$ and any n linearly independent "line"-vertices define an

apartment, the set of all Weyl chambers at infinity is in one-to-one correspondence with ordered n-tuples of linearly independent points of $\mathbb{R}P^{n-1}$.

Since the action of $SL_n\mathbb{R}$ on X_n is isometric, the action extends on $\partial_\infty X_n$. Because $SL_n\mathbb{R}$ preserves linear structures in $\mathbb{R}^n$, the action on the simplices of the Tits building at infinity is compatible with the action on the flags in $\mathbb{R}^n$. In particular, if $B \in SL_n\mathbb{R}$ and $p \in \partial_\infty X$ is a "line"-type vertex, corresponding to a line $l \subset \mathbb{R}^n$, then $B(p)$ is the "line"-type vertex, corresponding to $B(l)$. Similar statement is valid for any simplex of the Tits building. Thus the set of same type simplices is invariant under the action.

Let us denote the set of linearly independent n-tuples of points of $\mathbb{R}P^{n-1}$ by Θ^n. The Gauss coding of geodesics then turns out to be a coding of the set Θ^2. Our goal is to introduce a coding of Θ^n, that generalizes Gauss coding.

It is convenient to think of $\mathbb{R}P^{n-1}$ as a half of the unit sphere in $\mathbb{R}^n$. The boundary points are then identified with the opposite ones. They introduce ambiguity in the coding process, but have zero measure, so we are not concerned with them. To be specific, let us use the "northern" hemisphere,

$$S_+^{n-1} = \{\mathbf{p} \in S^{n-1} \subset \mathbb{R}^n : p_n > 0\}.$$

Thus Θ^n will also refer to the corresponding set of n-tuples of points on S_+^{n-1}.

We would like to describe the Gauss coding using transformations of Θ^2.

The Gauss coding.

The Gauss coding is $SL_2\mathbb{Z}$-invariant and is first defined for the class of so called "reduced" geodesics. It is shown that for almost every geodesic on $\mathcal{H}$ one can find an $SL_2\mathbb{Z}$-invariant reduced geodesic via certain reduction process. So first of all we would like to describe the reduced elements of Θ^2.

The correspondence between the points on the absolute in the upper half-plane model and on S_+^1 is given by the map

$$u \in \mathbb{R} \mapsto \left(\frac{u}{u^2 + 1}, \frac{1}{u^2 + 1}\right) \in S_+^1.$$

The latter is the result of the central projection of the point $(u, 1)$ onto S_+^1.
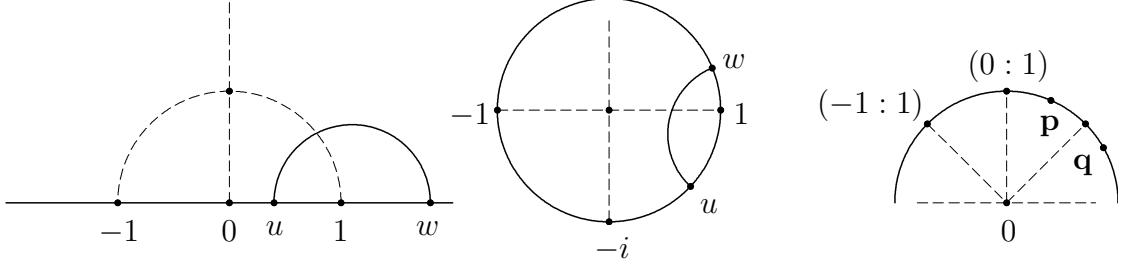
Figure 5.2. A reduced pair of points in the upper half-plane model, the unit disk model, and on S_+^1.

Definition 20. Let us call a pair $(\mathbf{p}, \mathbf{q})$ of distinct points on S_+^1 reduced, if $0 < p_1 < p_2$ and $0 < q_2 < q_1$.

A reduced pair $(\mathbf{p}, \mathbf{q}) \in \Theta^2$ corresponds to a pair of numbers (u, w), which are the initial and terminal ends of a reduced geodesic, respectively, in the upper half-plane model. The meaning of the inequalities in the previous definition is that the points $\mathbf{p}$ and $\mathbf{q}$ are both in the first quadrant of $\mathbb{R}^2$ and are separated by the point $(1 : 1)$.

If a pair $(\mathbf{p}, \mathbf{q})$ is reduced, then we use the transformation $T^{-1} : (x_1 : x_2) \mapsto (x_1 : x_2 - x_1)$ to the pair n_0 times, so that $T^{-n_0}(\mathbf{q})$ is between $(-1 : 1)$ and $(0 : 1)$ (at this moment $T^{-n_0}(\mathbf{p})$ is between $(\infty : 1)$ and $(-1 : 1)$). n_0 is the 0-th entry of the code. We then apply $S : (x_1 : x_2) \mapsto (x_2 : -x_1)$, which is the 90 degree rotation of the circle, to both points, so that the pair becomes reduced again. To get the next entry of the code, we repeat the process. To get the entries at negative positions, n_{-i}, we need to follow the process in the reverse order, i.e. apply $S^{-1} = S$ and then T^{n-i} until the pair is reduced again.

Generalization.

An analogue of this coding would be a coding of the Weyl chambers of the Tits building at infinity, or of the elements of Θ^n. An apartment is given by its "line"-type vertices. We know that in the coordinates, defined in §5.1, the "line"-type vertices of an apartment are given by points in the $(n-1)$-dimensional subspace of $\mathbb{R}^{dn}$, in which the only non-zero coordinates are x_{i1}, $2 \leq i \leq n$. Then the correspondence of these points with points on S_+^{n-1} is given by the map:

$$(x_{21}, \ldots, x_{n1}) \in \mathbb{R}^{n-1}$$

$$\mapsto \left(\frac{x_{21}}{x_{21}^2 + \cdots + x_{n1}^2 + 1}, \ldots, \frac{x_{n1}}{x_{21}^2 + \cdots + x_{n1}^2 + 1}, \frac{1}{x_{21}^2 + \cdots + x_{n1}^2 + 1}\right) \in S_+^{n-1},$$

which is the central projection of the point $(x_{21}, \ldots, x_{n1}, 1)$ onto S_+^{n-1}. For an apartment we need to specify n points on S_+^{n-1}, which are linearly independent in $\mathbb{R}^n$. To specify a Weyl chamber in the apartment we need to order the "line"-vertices of the apartment.

For a generalization we would like to introduce the notion of a "reduced" n-tuple first. Then we would need to encode the reduced n-tuples using transformations of several types. The first type seems more or less clear, the interchanges of coordinates in $\mathbb{R}^n$, analogous to the transformation S for Θ^2:

$$S_k : (x_1 : \cdots : x_{k-1} : x_k : \cdots : x_n) \mapsto (x_1 : \cdots : x_k : -x_{k-1} : \cdots : x_n),$$

where k is taken modulo n. For $n = 2$, $S_1 = S_2 = S$. The other type of transformations is used to map reduced elements of Θ^n to the sets of the form $S_k^{-1}\{$reduced elements of $\Theta^n\}$. We would like those transformations to generalize the maps T and T^{-1}. We would like these maps to be in $SL_n\mathbb{Z}$, so that the codes of bases of the same lattice in $\mathbb{R}^n$ would coincide. The idea is to use elements of the group $\mathbf{N}$, since that is the group to which T belongs in the case $n = 2$.

Identifying the type of transformations analogous to T and T^{-1} is somewhat we could not accomplish yet, since there is usually no unique map from $\mathbf{N}$ that takes a specified reduced element of Θ^n to the given set of the form $S_k^{-1}\{$reduced elements of $\Theta^n\}$. It is assumed that either the group should be narrowed down, or a whole class of elements of $\mathbf{N}$ be considered as an entry of the code.

Once the transformations of type T are identified, each reduced element of Θ^n gets n entries of type T. So we can arrange the code on an infinite uniform tree, in which every vertex has incidence n. An apartment is coded by the entire tree and a Weyl chamber in the apartment is additionally specified by a permutation of n elements.

We suggest the following definition of a reduced n-tuple of points in Θ^n:

Definition 21. An n-tuple $(\mathbf{p_1}, \ldots, \mathbf{p_n}) \in \Theta^n$ is called reduced, if

1. All $\mathbf{p_i}$ correspond to points on S_+^{n-1} in the positive orthant;

2. On S_+^{n-1} the point $(1 : \cdots : 1)$ is in the convex hull of the set $\{\mathbf{p_i^2} = (\mathbf{p_{i1}^2} : \cdots : \mathbf{p_{in}^2})\}_{i=1}^n$.

Remark. For $n = 2$ the requirement 2 is equivalent to saying that $(1 : 1)$ is contained in the convex hull of $\{\mathbf{p_1}, \mathbf{p_2}\}$ or that $(1 : 1)$ is between the two points, so it is consistent with our definition of a reduced pair given before. For greater values of n these two requirements are never equivalent.

Proposition 4. *Let $B \in SL_n\mathbb{R}$ and $(\mathbf{p_i})_{i=1}^n \in \Theta^n$ be such that $\mathbf{p_i} = (\mathbf{b_{1i}} : \cdots : \mathbf{b_{ni}})$. Then the WCF orbit of B contains a Gauss reduced basis if and only if B has only positive elements and $(\mathbf{p_i})_{i=1}^n$ is reduced.*

Proof. Since the WCF does not change the signs of entries of a matrix, the orbit of B can contain a Gauss reduced element, only if B has only positive elements. This is why we need requirement 1 of the definition.

In order for the orbit of B to contain a Gauss reduced element, there should exist $\mathbf{t} \in \mathbb{R}^n$, such that $\sum_{j=1}^n t_j = 0$ and $\sum_{j=1}^n e^{2t_j} \mathbf{b_{ij}^2}$ equal to the same value $\lambda > 0$ for all $i = 1, \ldots, n$. Or

$$\sum_{j=1}^n e^{2t_j} \begin{pmatrix} \mathbf{b_{1j}^2} \\ \vdots \\ \mathbf{b_{nj}^2} \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

which is equivalent to having an n-tuple of positive coefficients C_j, such that

$$\sum_{j=1}^n C_j \begin{pmatrix} \mathbf{b_{1j}^2} \\ \vdots \\ \mathbf{b_{nj}^2} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

which is equivalent to requirement 2. $\qquad\square$

Remark. If we only require that $(\mathbf{p_i})_{i=1}^n$ be reduced, then we only can show that the orbit of $(|\mathbf{b_{ij}}|)_{i,j=1}^n$ contains a Gauss reduced element. B itself could differ from $(|\mathbf{b_{ij}}|)_{i,j=1}^n$ by multiplication by a matrix from $\mathbf{M}$.

Our current goal is to describe a subset $\mathcal{B}_0$ of elements of $SL_n\mathbb{R}$, which we will call "reduced", and the set $\mathcal{B} = \bigsqcup_{g \in SL_n\mathbb{Z}} g(\mathcal{B}_0)$, which should be a cross section of

the Weyl chamber flow. Our conjecture is that $\mathcal{B}_0$ should be the set of the Gauss reduced elements or the reduced n-tuples in Θ^n. Also, since the first return map for such a cross section is not well defined, the coding will not be a linear structure such as a sequence, but will rather be a graph with the elements of the code on the edges of the graph.

For every orbit we would then be able to find elements in $\mathcal{B}$ and an $SL_n\mathbb{Z}$-equivalent "reduced" orbit, whose code we will apply to the original one.

5.5   Calculations for X_3

Lastly, we would like to display the results of the computations made for the case $n = 3$. Let (x, y, z, w, u) be the coordinates on X_3 with positive y and u. For an arbitrary $B \in SL_3\mathbb{R}$ let $\mathbf{b_i}$, $i = 1, 2, 3$, be the i-th row of B. First we give the formulas for the canonical representative of B:

$$x = \frac{\mathbf{b_2} \cdot \mathbf{b_3}}{\mathbf{b_3}^2}, \quad y = \frac{\|\mathbf{b_2} \times \mathbf{b_3}\|}{\mathbf{b_1}^2},$$

$$z = \frac{\mathbf{b_1} \cdot \mathbf{b_3}}{\mathbf{b_3}^2}, \quad w = \frac{(\mathbf{b_1} \times \mathbf{b_3}) \cdot (\mathbf{b_2} \times \mathbf{b_3})}{\mathbf{b_3}^2 \cdot \|\mathbf{b_2} \times \mathbf{b_3}\|}, \quad u = \frac{1}{\|\mathbf{b_3}\| \cdot \|\mathbf{b_3} \times \mathbf{b_2}\|}.$$

The u coordinate's numerator is actually $\mathbf{b_1} \cdot (\mathbf{b_3} \times \mathbf{b_2})$, but we know it is equal to 1.

Next we give the transformations of coordinates, corresponding to the following elements of $SL_3\mathbb{Z}$:

$$S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_{13} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

and

$$T_{abc} = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

These matrices generate $SL_3\mathbb{Z}$ and act on bases in $\mathbb{R}^3$ in the following ways: S_{12} interchanges the first and second vectors, preserving the orientation; S_{13} and S_{23}

do the same with the first and third and the second and third vectors of the basis, respectively. $T_{a,b,c}$ adds $b\mathbf{v_3}$ to $\mathbf{v_2}$ and $a\mathbf{v_2} + c\mathbf{v_3}$ to $\mathbf{v_1}$.

To avoid clutter we will just give the results.

$$T_{a,b,c} :(x, y, z, w, u) \mapsto (x + a, y, z + bx + c, w + by, u),$$

$$S_{23} :(x, y, z, w, u) \mapsto (-\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}, \frac{xz + yw}{x^2 + y^2}, \frac{xw - yz}{x^2 + y^2}, \frac{u}{\sqrt{x^2 + y^2}}),$$

$$S_{13} :(x, y, z, w, u) \mapsto$$
$$(\frac{xz + yw}{z^2 + w^2 + u^2}, \frac{\sqrt{(x^2 + y^2)u^2 + (xw - zy)^2}}{z^2 + w^2 + u^2}, -\frac{z}{z^2 + w^2 + u^2},$$
$$\frac{yzw - x(w^2 + u^2)}{(z^2 + w^2 + u^2)\sqrt{(x^2 + y^2)u^2 + (xw - zy)^2}},$$
$$\frac{yu}{\sqrt{z^2 + w^2 + u^2}\sqrt{(x^2 + y^2)u^2 + (xw - zy)^2}}),$$

$$S_{12} :(x, y, z, w, u) \mapsto (z, \sqrt{w^2 + u^2}, -x, \frac{-w}{\sqrt{w^2 + u^2}}y, \frac{u}{\sqrt{w^2 + u^2}}y).$$

Since the coordinates (x, y) are basically the coordinates of $\mathbf{v_2}$ relative to $\mathbf{v_3}$ and (z, w, u) are the coordinates of $\mathbf{v_1}$ relative to $\mathbf{v_2}$ and $\mathbf{v_3}$, the calculations for S_{12} are easy, because the ratios of the vectors relative to $\mathbf{v_3}$ do no change. To briefly check the entries of S_{23} one needs to realize that, for example, the norm of the vector (x, y) before the transformation should be the inverse of that after the transformation.

We also give the sides of the fundamental domain $\mathcal{F}_3$ for $GL_3\mathbb{Z}$ in X_3, which is the same as that for $SL_3\mathbb{Z}$, found in [Gre88] for arbitrary n and shown in [GGT87] specifically for $n = 3$ in a different parameterization, but rewritten in our coordinates for the reader's convenience:

$$\{x^2 + y^2 = 1\}, \quad \{z^2 + w^2 + u^2 = 1\}, \quad \{x^2 + y^2 = z^2 + w^2 + u^2\},$$
$$\{w^2 + u^2 = y^2\},$$
$$\{x = \frac{1}{2}, 0\}, \quad \{z = \pm\frac{1}{2}\}, \quad \{w = \frac{1}{2}y, 0\},$$
$$\{(z - x)^2 + (w - y)^2 + u^2 = z^2 + w^2 + u^2\},$$
$$\{(z - x + 1)^2 + (w - y)^2 + u^2 = z^2 + w^2 + u^2\},$$

which come from the identifications by the following elements of $SL_3\mathbb{Z}$ (not respectively, also adapted to our coordinates, i.e. transposed and rotated 180 degrees):

$$T_{1,0,0}, \quad T_{0,1,0}, \quad T_{0,0,1}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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