The Pennsylvania State University
The Graduate School

DYNAMIC PRICING IN AN URBAN FREIGHT ENVIRONMENT

A Thesis in
Industrial Engineering
by
Matthew Alden Rigdon

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Master of Science

May 2008

The thesis of Matthew Alden Rigdon was reviewed and approved* by the following:

Terry L. Friesz
Chaired Professor of Industrial Engineering
Thesis Advisor

Richard J. Koubek
Professor
Head of the Department of Industrial Engineering

Venky Shankar
Associate Professor of Civil Engineering

Tao Yao
Assistant Professor of Industrial Engineering

*Signatures are on file in the Graduate School.

Abstract

This thesis proposes a dynamic, game-theoretic model of dynamic pricing in an urban freight environment with three distinct agent types: sellers, transporters and receivers. The sellers and transporters are modelled as non-cooperative Nash agents. The sellers compete to capture receiver demands, while the transporters compete to capture the transportation demand generated by the seller-receiver transactions. Each competing agent's best response problem is formulated as an optimal control problem and the set of these coupled optimal control problems is transformed into a differential variational inequality representing the general Nash equilibrium problem. A time discretization approximation is utilized to recast the game as a finite dimensional nonlinear complementarity problem. The solution of a small numerical example gives insights into the equilibrium strategies of the different agents in an urban freight system.

Table of Contents

List of Figures

List of Tables

List of Symbols

The notation in this thesis is quite complex due to the multiple agent types being modeled. Therefore, a convention is used of concatenating vectors, as seen in the examples below, when possible in order to simplify the notation.

$$\begin{aligned}\pi^c &= \left\{\pi_{i,j}^{c,r,s} : r \in R, s \in S, i \in N_s, j \in N_r\right\} \\ \pi^{c,r,s} &= \left\{\pi_{i,j}^{c,r,s} : i \in N_s, j \in N_r\right\}\end{aligned}$$

If a subscript or superscript is missing, it indicates that the elements related to that script indice have been concatenated into the vector.

Sets

$\mathcal{S}$ set of sellers

$\mathcal{C}$ set of transporters

$\mathcal{R}$ set of receivers

$\mathcal{N}_s$ set of nodes where seller $s \in \mathcal{S}$ has locations

$\mathcal{N}_r$ set of nodes where receiver $r \in \mathcal{R}$ has locations

Variables

$p_i^{r,s}(t)$ delivered price charged by the seller $s \in \mathcal{S}$ charged to the receiver r located at node $i \in \mathcal{N}_r$

$q_i^s(t)$ production rate of seller $s \in \mathcal{S}$ at location $i \in \mathcal{N}_s$

$v_{i,j}^{r,s}(t)$ flow of goods sent by seller $s \in \mathcal{S}$ from its location $i \in \mathcal{N}_s$ for delivery at receiver $r \in \mathcal{R}$ at its location $j \in \mathcal{N}_r$

$I_i^s(t)$ inventory level of seller $s \in \mathcal{S}$ at location $i \in \mathcal{N}_s$ at time t

$d_j^{r,s}(p_j(t), t)$ demand of goods by receiver $r \in \mathcal{R}$ located at location $j \in \mathcal{N}_r$ captured by seller $s \in \mathcal{S}$ at time t

$\Psi_i^s(I_i^s(t), t)$ inventory holding cost of seller $s \in \mathcal{S}$ at location $i \in \mathcal{N}_s$ when inventory level is $I_i^s(t)$

$\theta_i^s(q_i^s(t), t)$ unit production cost of seller $s \in \mathcal{S}$ located at node $i \in \mathcal{N}_s$ when production level is $q_i^s(t)$

$\pi_{i,j}^{c,r,s}(t)$ price charged by transporter $c \in \mathcal{C}$ for delivering goods from supplier $s \in \mathcal{S}$ at location $i \in \mathcal{N}_s$ to receiver $r \in \mathcal{R}$ at location $j \in \mathcal{N}_r$ at time t

$\rho_{i,j}^{c,r,s}(t)$ flow of goods delivered by transporter $c \in \mathcal{C}$ at time t to the receiver r at its location $j \in \mathcal{N}_r$ shipped by the seller $s \in \mathcal{S}$ from location $i \in \mathcal{N}_s$

$x^{c,s}(t)$ total backlogged service of transporter $c \in \mathcal{C}$ for seller $s \in \mathcal{S}$ at time t

$u_{i,j}^{c,r,s}(\pi(t), x(t), t)$ amount of demand of service produced by transporter $c \in \mathcal{C}$ to deliver goods from location $i \in \mathcal{N}_s$ to the location $j \in \mathcal{N}_r$ at time t

$w^{c,s}(x^{c,s}(t), t)$ cost of lost goodwill from seller $s \in \mathcal{S}$ for transporter c due to the level of backlogged shipments at time t.

$k^c(\rho^c(t), t)$ unit transportation cost of transporter c shipping ρ units of goods at time t

Parameters

t time index of problem over the planning horizon $[t_0, t_f]$

$D_j^r(t)$ amount of goods desired by receiver $r \in \mathcal{R}$ at its facility $j \in \mathcal{N}_r$ at time t

$I_{i,0}^s$ inventory level of seller $s \in \mathcal{S}$ at its location $i \in \mathcal{N}_s$ at time t_0

$p_{\min}^s$ lower limit of price for firm $s \in \mathcal{S}$

$p_{\max}^s$ upper limit of price for firm $s \in \mathcal{S}$

$q_{i,\max}^s$ upper limit of production at node $i \in \mathcal{N}_s$ of seller $s \in \mathcal{S}$

$\pi_{\min}^c$ lower limit of price for transporter $c \in \mathcal{C}$

$\pi_{\max}^c$ upper limit of price for transporter $c \in \mathcal{C}$

Acknowledgments

I would like to acknowledge Professor Terry L. Friesz for his invaluable input, supervision and guidance throughout the process of writing this thesis. I would also like to acknowledge Reetabrata Mookherjee for inspiring my interest in dynamic pricing problems.

Dedication

This thesis is dedicated to my parents, Michael and Susan, my sister, Heather and my loving wife, Amanda. Your love, laughs and support has kept me sane.

Chapter 1

Introduction

This thesis studies a model of dynamic pricing of goods and services in an urban freight transportation system. There are three types of agents that are of specific interest in urban freight systems: sellers, transporters and receivers. This thesis proposes a differential Nash game in which the seller and transporter firms are in non-cooperative oligopolistic competition with firms of their own type; i.e. a seller competes with other seller firms, but not with transporter firms. The purpose of this research is to obtain equilibrium price, production and shipment trajectories of the firms in an effort to obtain insights of the system.

This research follows the paradigm set in the field of revenue management for nonlinear pricing in a dynamic, competitive environment. There are many applications of dynamic pricing in game theoretic settings. Bernstein and Federgruen (2003, 2004) and Cachon and Harker (2002) study dynamic pricing in supply chain networks, while Chen et al. (2004) studies dynamic pricing in the newsboy vendor problem. Altman and Wynter (2004) provide an overview of pricing in transportation and telecommunication networks. Lederer (2003) examines static price and production competition between spatially separated, profit maximizing firms. Zhang et al. (2005) study dynamic game theoretic models of infrastructure networks. However, the model presented in this thesis is believed to be the first application of dynamic pricing to urban freight transport systems that models the dynamic pricing competition among both the producers of goods in the network and the transportation firms that carry the goods.

What makes the model presented herein an urban freight model rather than simply a freight model is its purposeful and detailed representation of each of the key categories of decision makers who influence the demand and supply of freight services. In particular, the model proposed involves the three classes of spatially separated firms mentioned previously: sellers, transporters and receivers. This contrasts sharply with the two categories – shippers and carriers – included in the often-cited models of intercity freight transportation put forth by Friesz et al. (1986) and Harker and Friesz (1986a, 1986b).

The main focus of the formulation discussed in this thesis is the dynamic pricing behavior of the competing firms as opposed to other undoubtedly important topics such as the decisions made by receivers about delivery time windows. For that reason, the receiver demands and delivery schedules are assumed to be exogenous to the model. This is an assumption that is made in order to simplify the problem and maintain the focus on the dynamic pricing of goods and services. There are many complexities of the transporter-receiver interactions that may be studied by simple extensions of this model. Holguín-Veras et al. (2006a, 2006b, 2006c) provide a thorough overview of the importance of modeling receivers in urban freight transportation systems.

For the urban setting studied herein, the sellers are considered to be competing firms who produce goods that are sold to the receivers. The receivers are those firms that desire delivery of goods. The transporters are the competing firms that are contracted to deliver the goods from the sellers to the receivers. It is assumed that any transporter can service any seller-receiver origin-destination pair. The outcome of these interactions builds a relationship network connecting the different classes of firms. This scenario stands in sharp contrast with integrated production-transport-receiver operations that are conducted by vertically integrated conglomerates (e.g., Walmart, 7/11) that have central control over all aspects of the operation.

We assume that both the sellers and transporters are Nash agents in a network economy and they are profit optimizers with pricing power. Each seller of commodities competes with other sellers and each transporter competes with other transporters. However, the sellers and transporters do not compete with each other. The receivers are not modelled as competing agents, though such an extension poses no increased difficulties beyond notation. The decision to not explicitly model the receivers in this thesis was made to keep the notation from becoming yet more complex.

The presence of the receivers in this model is required as they generate demands that must be met by the sellers. We begin with the assumption that the receivers' demands are fixed for the time scale of one abstract "day." The sellers compete to capture these demands. Each seller has a demand function that depends upon the pricing decisions of all the sellers. Likewise, each transporter has a demand function which depends on its own price as well as its competitors' prices. The demand for the transporters is derived from the spatial separation of supply and consumption activities of the sellers and receivers. Similar to the sellers, the transporters must compete with each other to capture this demand for transportation services.

Because the sellers and transporters are assumed to be optimizers, there is an optimization problem associated with each competing firm. The optimization problem for each seller and transporter is formulated as a continuous time optimal control problem that depends on the strategies of the other firms. For this reason, these optimization problems are termed *best response problems.* This leads to a set of coupled optimal control problems that describe the game. The best response problems that are faced by the sellers and transporters are discussed in Chapters 2 and 3, respectively. The necessary conditions for each firm's optimization problem are then analyzed and used to create a differential variational inequality (DVI) representation of the differential Nash game. Many different solution schemes may be implemented from the DVI formulation including fixed point problems, nonlinear complementarity problems and descent in Hilbert space methods (Friesz et al., 2006). In this thesis, a time discretization approximation is employed to formulated a finite dimensional nonlinear complementarity problem (NCP) in Chapter 5. A numerical example is presented and solved in Chapter 6 for illustration.

Chapter 2

Dynamic Game Theoretic Model of the Sellers

The set of sellers $\mathcal{S}$ in this model are in dynamic oligopolistic competition. Each seller $s \in \mathcal{S}$ is assumed to be an optimizer with the goal of maximizing profit over the finite planning horizon $t \in [t_0, t_1]$. The sellers are assumed to produce a homogenous good that is consumed by the set of receivers $\mathcal{R}$. Each seller has cost functions associated with production and inventory holding. The sellers have the ability to dynamically set the price of the commodities they are selling to the receivers. The price that is charged to the receivers is assumed to be the delivered price and includes the price for the goods as well as the price for the transportation. The sellers then pay the transporters for freight service. The sellers must non-cooperatively choose the price of their goods, production rate and rate of outgoing shipments in order to maximize their profits.

2.1 Constraints and Equations

A seller s is assumed to have a linear demand function for each location $j \in \mathcal{N}_r$ of receiver $r \in \mathcal{R}$

$$d_j^{s,r}\left(p_j^r(t)\right) = a_{1,j}^{s,r} - a_{2,j}^{s,r} p_j^{s,r}(t) + \sum_{g \in \mathcal{S} \setminus s} a_{3,j}^{s,r,g} p_j^{r,g}(t) \tag{2.1}$$

with coefficients $a_{1,j}^{s,r},\ a_{2,j}^{s,r},\ a_{3,j}^{s,r,g} \in \Re_+^1$. Therefore, a seller's demand increases as its own price decreases and its competitors' prices increase.

Note that the demand function (2.1) does not depend upon the location $i \in \mathcal{N}_s$ of the seller $s \in \mathcal{S}$. Let $v_{i,j}^{r,s}$ denote the rate at which seller $s \in \mathcal{S}$ releases product from location $i \in \mathcal{N}_s$ for transport to receiver $r \in \mathcal{R}$ at location $j \in \mathcal{N}_r$. It is then apparent that the following conservation equation is necessary

$$d_i^{r,s}\left(p^r\left(t\right)\right) = \sum_{i \in \mathcal{N}_s} v_{i,j}^{r,s}(t) \quad \forall\ r \in \mathcal{R},\ j \in \mathcal{N}_r \tag{2.2}$$

with $v_{i,j}^{r,s}$ non-negative.

$$v_{i,j}^{r,s}\left(t\right) \geq 0 \text{ for all } i \in \mathcal{N}_s,\ r \in \mathcal{R},\ j \in \mathcal{N}_r,\ t \in [t_0, t_f] \tag{2.3}$$

Let $D_j^r(t)$ be the total demand of receiver $r \in \mathcal{R}$ at location $j \in \mathcal{N}_r$ at time t for the homogenous good produced by the sellers. The demand functions of all the sellers must then satisfy

$$\sum_{s \in \mathcal{S}} d_j^{r,s}\left(p_j^r\left(t\right)\right) = D_j^r(t) \text{ for all } r \in \mathcal{R},\ j \in \mathcal{N}_r \tag{2.4}$$

so that the total demands of all receivers is satisfied at each time instant. It is important to note that this is a *joint constraint*. A joint constraint is a constraint which is faced by multiple competing agents in which the actions of one agent affects the feasible strategy space of the competitors. This is more apparent when the constraint is rewritten as

$$d_j^{r,s}\left(p^r\left(t\right)\right) = D_j^r(t) - \sum_{g \in \mathcal{S} \backslash s} d_j^{r,g}\left(p^r\left(t\right)\right) \text{ for all } r \in \mathcal{R},\ j \in \mathcal{N}_r$$

Constraints of this form cause the game to take the form of a generalized Nash equilibrium problem (GNEP). Such problems have been researched by Harker (1991), Pang and Hobbs (2004), and Pang (2007). These joint constraints will require special attention when the complementarity problem is formulated in Section 5.

Each seller may keep some inventory on hand at each of its locations. Let the inventory level of seller $s \in \mathcal{S}$ at location $i \in \mathcal{N}_s$ at time $t \in [t_0, t_f]$ be denoted by

$I_i^s(t)$. The inventory levels follow the dynamics

$$\frac{dI_i^s(t)}{dt} = q_i^s(t) - \sum_{r\in\mathcal{R}} \sum_{j\in\mathcal{N}_r} v_{i,j}^{r,s}(t) \text{ for all } i \in \mathcal{N}_s \tag{2.5}$$

where the right hand side is a flow balance between the production rate of the firm $q_i^s(t)$ at location $i \in \mathcal{N}_s$ and the rate of outgoing shipments $v_{i,j}^{r,s}(t)$. The initial inventory at time $t = t_0$ of firm $s \in \mathcal{S}$ at node $i \in \mathcal{N}_s$ is

$$I_i^s(t_0) = I_{i,0}^s \text{ for all } i \in \mathcal{N}_s \tag{2.6}$$

There are naturally constraints on the minimum and maximum inventory level allowed at each location following

$$I_{i,\min}^s \leq I_i^s(t) \leq I_{i,\max}^s \text{ for all } i \in \mathcal{N}_s \tag{2.7}$$

Note that a negative lower bound $I_{\min}^s$ on inventory indicates that the firm may take back-orders. If a firm is allowed to take back-orders, i.e. $I_{\min}^s < 0$, a terminal time constraint should be considered so that the inventory level at terminal time is non-negative.

$$I_{\min}^s(t_f) \geq 0$$

Otherwise, a firm will take a strategy of taking back-orders and never fulfilling them within the time horizon. It is assumed from here on that the selling firms do not take back-orders. This assumption is made to simplify the model of the transporters which will be presented in Section 3.

2.2 Revenue and Costs

The *instantaneous revenue* of a seller $s \in \mathcal{S}$ is represented as the product of the demand capture and the corresponding price for all receivers $r \in \mathcal{R}$ at all locations $j \in \mathcal{N}_r$.

$$\sum_{r\in\mathcal{R}} \sum_{j\in\mathcal{N}_r} \left[d_j^{r,s}(p_j^r(t)) \cdot p_j^{r,s}(t)\right]$$

The *cumulative revenue* of seller $s \in \mathcal{S}$ over the planning horizon may then be represented by

$$\int_{t_0}^{t_f} \sum_{r \in \mathcal{R}} \sum_{j \in \mathcal{N}_r} \left[d_j^{r,s}(p_j^r(t)) \cdot p_j^{r,s}(t) \right] dt$$

The sellers are assumed to have a *production cost* function of the quadratic form

$$\theta_i^s(q_i^s(t)) = f_{1,i}^s + f_{2,i}^s q_i^s(t) + \frac{1}{2} f_{3,i}^s (q_i^s(t))^2$$

with $f_{1,i}^s,\ f_{2,i}^s,\ f_{3,i}^s \in \Re_+^1$. Note that the modeling framework used herein is general enough to allow different forms of functions; the quadratic form is simply assumed to move forward with the analysis of the problem.

The sellers are also assumed to have an *inventory holding cost* function

$$\Psi_i^s(I_i^s(t)) = e_{1,i}^s I_i^s(t) + \frac{1}{2} e_{2,i}^s [I_i^s(t)]^2$$

where $e_{1,i}^s,\ e_{2,i}^s \in \mathbb{R}_+^1$ for all $s \in \mathcal{S}$ and $j \in \mathcal{N}_s$.

Finally, each seller has a *cost for transportation services.* Recall that the prices $p_j^{r,s}(t)$ are delivered prices. Therefore, the sellers must pay the transporters for the cost of transportation. The price of transportation $\pi_{i,j}^{c,r,s}(t)$ and the rate of shipments $u_{i,j}^{c,r,s}\left(\pi_{i,j}^{c,r,s}(t)\right)$ will be determined by the transporters as they compete for the transportation service demands of the sellers. Therefore, the cost of transportation is exogenous to the seller firms and takes the form

$$\sum_{c \in \mathcal{C}} \sum_{j \in \mathcal{N}_s} \sum_{r \in \mathcal{R}} \sum_{i \in \mathcal{N}_r} u_{i,j}^{c,r,s}(\pi(t)) \cdot \pi_{i,j}^{c,r,s}(t)$$

where $u_{i,j}^{c,r,s}(\pi(t))$ is the transportation service demand function for transporter $c \in \mathcal{C}$ to carry goods from supplier $s \in \mathcal{S}$ at location $i \in \mathcal{N}_s$ to receiver $r \in \mathcal{R}$ at location $j \in \mathcal{N}_r$. The associated price for this transportation, as determined in the transporter's model, is $\pi_{i,j}^{c,r,s}(t)$. As will be seen in Section 3, the cost of transportation for the sellers will be the revenues of the transporters.

Therefore, the *cumulative profit* function of a seller $s \in \mathcal{S}$ depends on both the actions of its competitors as well as the actions of the transporters and takes the form

$$J_s\left(p^s, q^s, v^s; I^s; p^{-s}, \pi^s, u^s; t\right) = \int_{t_0}^{t_1} \left\{ \left[d^s(p(t))\right]^T p^s(t) - \sum_{i\in N_s} \left[\theta_i^s\left(q_i^s(t)\right)\right] - \sum_{i\in N_s} \left[\Psi_i^s\left(I_i^s(t)\right)\right] - \left[u^s(\pi(t))\right]^T \pi^s(t) \right\} dt \quad (2.8)$$

where p^{-s} is defined by

$$p^{-s} \equiv \{p^g : g \in \mathcal{S} \backslash s\}$$

and represents information, taken as exogenous by seller s, regarding the actions of its competitors. Likewise, π^s, u^s are also taken as exogenous, though these values come from the model of the transporters.

2.3 A Seller's Best Response Problem

Each seller seeks to maximize their profit in response to the actions taken by their competitors. In this case, each seller $s \in \mathcal{S}$ seeks to solve the following optimal control problem, taking the prices of its competitors p^{-s} and transportation quantities and costs u^s and π^s as exogenous:

$$\max_{p,q,v} J_s\left(p^s, q^s, v^s; I^s; p^{-s}, \pi^s, u^s; t\right) \quad (2.9)$$

subject to

$$\frac{dI^s(t)}{dt} = q^s(t) - \sum_{r\in\mathcal{R}}\sum_{j\in\mathcal{N}_r} v_j^{r,s}(t) \tag{2.10}$$

$$I^s(t_0) = I_0^s \tag{2.11}$$

$$d^s(p(t)) = \sum_{i\in\mathcal{N}_s} v_i^s(t) \tag{2.12}$$

$$\sum_{s\in\mathcal{S}} d^s(p(t)) = D(t) \tag{2.13}$$

$$v^s \geq 0 \tag{2.14}$$

$$I_{\min}^s \leq I^s(t) \leq I_{\min}^s \tag{2.15}$$

$$p_{\min}^s \leq p^s(t) \leq p_{\max}^s \tag{2.16}$$

$$0 \leq q^s(t) \leq q_{\max}^s \tag{2.17}$$

Note that this problem takes the form of a continuous time optimal control problem with state variables I^s and control variables p^s, q^s, and v^s. Constraints (2.10) - (2.15) arise from the previous discussion of (2.4) - (2.7). Constraints (2.16) - (2.17) are upper and lower bounds on the price and production rate control variables.

The set of optimal control problems (2.9) - (2.17) for all sellers $s \in \mathcal{S}$ are coupled due to the demand function (2.1) and the joint constraints (2.13). This set of coupled optimal control problems defines the differential Nash game played by the sellers.

The equilibrium solution of the differential Nash game is a solution in which no seller can increase its profit without another seller decreasing its profit. This equilibrium must hold at each instant of the planning horizon. Therefore, the equilibrium is termed a *dynamic* or *moving equilibrium*. In the next section, the necessary conditions for the set of coupled optimal control problems are analyzed and used to form a differential variational inequality (DVI) representation of the game.

2.4 Differential Variational Inequality Formulation

The set of coupled optimal control problems (2.9) - (2.17) represents only part of the game of interest, namely the competition among the sellers. The other part of the game will consist of the model of the transporters. It is useful to put this set of optimal control problems into a more compact formulation that represents this portion of the game. The differential variational inequality (DVI) is a natural construct for accomplishing this and may be easily manipulated to obtain forms that are useful for the computation of equilibria (Friesz et al., 2006).

We begin the formulation of the DVI through the inspection of the necessary conditions for each seller's optimal control problem (2.9) - (2.17). The Hamiltonian H_s for each seller's problem may be stated as the sum of the integrand of the performance function and the product of the adjoint variables and the dynamics. Temporarily ignoring the time argument to simplify the notation, the Hamiltonian takes the form

$$\begin{aligned} H_s\left(y^s; I^s; p^{-s}, \pi^s, u^s; \kappa^s; t\right) &= \left[d^s(p)\right]^T \left(p^s\right) - \sum_{i \in N_s} \left[\theta^s\left(q^s\right) + \Psi^s\left(I^s\right)\right] \\ &\quad - \left[u^s\left(\pi\right)\right]^T \pi^s + \left[\kappa^s\right]^T \left(q^s - \sum_{r \in \mathcal{R}} \sum_{j \in \mathcal{N}_r} v_j^{r,s}\right) \end{aligned}$$

where κ^s are the adjoint variables (dynamic dual variables) associated with the dynamics (2.5)

$$\kappa^s(t) \equiv \left\{\kappa_i^s(t) : \ i \in \mathcal{N}_s\right\}$$

and y^s is the vector of controls for seller $s \in S$

$$y^s \equiv \begin{bmatrix} p^s \\ q^s \\ v^s \end{bmatrix}$$

which is used to simplify the notation.

The Hamiltonian may be augmented by dualizing the state constraints. The state constraints (2.15) are priced out to allow easier analysis of the necessary conditions. The augmented Hamiltonian takes the form

$$\tilde{H}_s\left(y^s; I^s; p^{-s}, \pi^s, u^s; \kappa^s, \varpi_L^s, \varpi_U^s; t\right) = H_s + \varpi_{\min}^s\left(I_{\min}^s - I^s\right) + \varpi_{\max}^s\left(I^s - I_{\max}^s\right)$$

where ϖ^s are the dual variables

$$\begin{aligned} \varpi_{\min}^s(t) &\equiv \left\{\varpi_{\min,i}^s(t) : i \in \mathcal{N}_s\right\} \\ \varpi_{\max}^s(t) &\equiv \left\{\varpi_{\max,i}^s(t) : i \in \mathcal{N}_s\right\} \end{aligned}$$

associated with the state constraints (2.15).

The necessary conditions for this optimal control problem require that the partial derivative of the augmented Hamiltonian with respect to the control variables be equal to zero.

$$\frac{\partial \tilde{H}_s}{\partial y_s} = 0 \Rightarrow \begin{cases} \frac{\partial \hat{H}_s}{\partial p^s} = 0 \\ \frac{\partial \hat{H}_s}{\partial q^s} = 0 & \text{for all } s \in \mathcal{S} \\ \frac{\partial \hat{H}_s}{\partial v^s} = 0 \end{cases}$$

$$\begin{aligned} \frac{\partial \tilde{H}_s}{\partial p_j^{r,s}} &= \left(-a_{2,j}^{r,s}\right) p_j^{r,s} + d_j^{r,s}\left(p_j^r\right) \text{ for all } s \in \mathcal{S},\ r \in \mathcal{R},\ j \in \mathcal{N}_r \\ \frac{\partial \tilde{H}_s}{\partial q_i^s} &= -\left(f_{2,i}^s + f_{3,i}^s q_i^s\right) + \kappa_i^s \text{ for all } s \in \mathcal{S},\ i \in \mathcal{N}_s \\ \frac{\partial \tilde{H}_s}{\partial v_{i,j}^{r,s}} &= -\kappa_i^s \text{ for all } s \in \mathcal{S},\ i \in \mathcal{N}_s,\ r \in \mathcal{R},\ j \in \mathcal{N}_r \end{aligned}$$

In addition, the adjoint variables must satisfy the adjoint dynamics which are of the form

$$\frac{d\kappa_i^s}{dt} = -\frac{\partial \tilde{H}_s}{\partial I_i^s} = e_{1,i}^s + e_{2,i}^s I_i^s + \varpi_{\max}^s - \varpi_{\min}^s \text{ for all } s \in \mathcal{S},\ i \in \mathcal{N}_s \tag{2.18}$$

with terminal time conditions

$$\kappa_i^s(t_f) = 0 \text{ for all } s \in \mathcal{S},\ i \in \mathcal{N}_s \tag{2.19}$$

as there is no salvage value for inventory left over at the end of the time horizon.

Using the information from the necessary conditions, the differential variational inequality (DVI) may immediately be stated as

$$\text{find } y_s^* \in \Omega_s \text{ such that}$$

$$\sum_{s \in S} \int_{t_0}^{t_1} \left[\frac{\partial \hat{H}_s \left(y^{s*} \right)}{\partial y^s} \right]^T \left(y^s - y^{s*} \right) dt \leq 0 \tag{2.20}$$

$$\text{for all } y^s \in \Omega_s$$

where Ω_s is defined by the set of constraints

$$\Omega_s = \{ y^s : (2.10) \text{ - } (2.14),\ (2.16), (2.17),\ (2.18) \text{ and } (2.19) \text{ hold} \}$$

Chapter 3

Dynamic Game Theoretic Model of the Transporters

Similar to the sellers, the set of transporters $\mathcal{C}$ are Nash agents in dynamic oligopolistic competition. Each transporter is in competition with the other transporters to capture demand for transportation services and the associated revenues over the finite planning horizon. The demands for transportation services are exactly the origin-destination demands that arise from the model of the sellers. The transporters are allowed to dynamically set the price of transportation services in order to maximize their profits. The price that is charged by transporter $c \in \mathcal{C}$ to seller $s \in \mathcal{S}$ for transportation services between an origin-destination pair (i, j), $i \in \mathcal{N}_s$, $j \in \mathcal{N}_r$, $r \in \mathcal{R}$ at time t is $\pi_{i,j}^{c,r,s}(t)$; as was seen in Section 2, this term arises as the per unit transportation cost to the seller s. It is important to note that a transporter may also charge different prices to different sellers for the same destination transportation service at the same time instant.

3.1 Constraints and Equations

The demand function for services provided by transporter $c \in \mathcal{C}$ for transportation services between an origin-destination pair (i, j), $i \in \mathcal{N}_s$, $j \in \mathcal{N}_r$, $r \in \mathcal{R}$ at time t

is assumed to be of the linear form

$$u_{i,j}^{c,r,s}\left(\pi_{i,j}^{r,s}(t)\right) = h_{1,i,j}^{c,r,s} - h_{2,i,j}^{c,r,s}\pi_{i,j}^{c,r,s}(t) + \sum_{g\in C\backslash c} h_{3,i,j}^{c,g,r,s}\pi_{i,j}^{g,r,s}(t) \tag{3.1}$$

where $\omega_1^{c,r,s},\ \omega_2^{c,r,s},\ \omega_3^{c,g,r,s} \in \mathbb{R}_+^1$ for all $c \in \mathcal{C},\ r \in \mathcal{R},\ s \in \mathcal{S}$ and $g \in C\backslash c$. From this demand function, it is apparent that the demand of transporter c decreases as its own price increases and its competitors' prices decrease.

Let the flow of goods delivered by transporter $c \in \mathcal{C}$ from seller $s \in \mathcal{S}$ at location $i \in \mathcal{N}_s$ to receiver $r \in \mathcal{R}$ at location $j \in \mathcal{N}_s$ at time t be denoted by $\rho_{i,j}^{c,r,s}(t)$. The total shipments made by all carriers must equal the origin-destination demands $v_{i,j}^{r,s}$

$$\sum_{c\in\mathcal{C}} \rho_{i,j}^{c,r,s}(t) = v_{i,j}^{r,s}(t) \quad \forall\ r \in \mathcal{R},\ i \in \mathcal{N}_r,\ s \in \mathcal{S},\ j \in \mathcal{N}_s \tag{3.2}$$

where the origin-destination demands $v_{i,j}^{r,s}$ are taken as exogenous from the sellers' problems. Note that this equation is also a joint constraint, similar to that in (2.4).

It is assumed that the transporters are allowed to backlog some of the shipments. Let $x^{c,s}$ be the quantity of goods that are backlogged by transporter $c \in \mathcal{C}$ for seller $s \in \mathcal{S}$ at time t. The change in backlogged shipments may be represented by a set of system dynamics obeying

$$\frac{dx^{c,s}(t)}{dt} = \sum_{r\in\mathcal{R}} \sum_{i\in\mathcal{N}_r} \sum_{j\in\mathcal{N}_s} \left(u_{i,j}^{c,r,s}(\pi_{i,j}^{r,s}(t)) - \rho_{i,j}^{c,r,s}(t)\right) \tag{3.3}$$

in which the right hand side is a flow balance statement between the rate of demand for services and the rate of fulfillment of services.

There are associated end-point conditions for the dynamics in (3.3) such that

$$x^{c,s}(t_0) = 0 \tag{3.4}$$

$$x^{c,s}(t_f) = 0 \tag{3.5}$$

which indicates that there is no backlogged service at the start or end of the time horizon. In addition, there must be a constraint

$$x^{c,s}(t) \geq 0 \ \ \forall\ s \in \mathcal{S},\ t \in (t_0, t_1) \tag{3.6}$$

which ensures that deliveries are not made before the sellers have demanded them.

3.2 Revenue and Costs

The *instantaneous revenue* of a transporter is represented by the product of demand and price

$$\sum_{s\in\mathcal{S}}\sum_{j\in\mathcal{N}_s}\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}_r} u_{i,j}^{c,r,s}(\pi_{i,j}^{r,s}(t))\cdot\pi_{i,j}^{c,r,s}(t) \tag{3.7}$$

and is summed over all possible origin-destination pairs. Recall that these revenues are equal to the cost of transportation services for the sellers as seen in Section 2. Therefore, the models of the transporters and the sellers are coupled.

The costs for the transporters consists of transportation costs and loss of goodwill costs. The *unit transportation cost* for transporter $c \in \mathcal{C}$ shipping goods from seller $s \in \mathcal{S}$ at location $i \in \mathcal{N}_s$ to receiver $r \in \mathcal{R}$ at location $j \in \mathcal{N}_r$ at time $t \in [t_0, t_1]$ is assumed to be of the quadratic form

$$k_{i,j}^{c,r,s}\left(\rho_{i,j}^{c,r,s}(t)\right) = l_{1,i,j}^{c,r,s}\rho_{i,j}^{c,r,s}(t) + \frac{1}{2}l_{2,i,j}^{c,r,s}\left[\rho_{i,j}^{c,r,s}(t)\right]^2$$

where $l_{1,i,j}^{c,r,s},\ l_{2,i,j}^{c,r,s} \in \mathbb{R}_+^1$. Therefore, the *total instantaneous transportation cost* is

$$\sum_{s\in\mathcal{S}}\sum_{j\in\mathcal{N}_s}\sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}_r} k_{i,j}^{c,r,s}\left(\rho_{i,j}^{c,r,s}(t)\right)\rho_{i,j}^{c,r,s}(t) \tag{3.8}$$

The *loss of goodwill cost* is associated with a backlogging of service. If the carrier allows some of the goods for seller s to become backlogged, then seller s may be less likely to use that carrier in the future. Therefore, there is an indirect cost associated with the level of backlogged service to ensure that the transporters provide timely service to their customers. The instantaneous cost of lost goodwill due to the backlog of service $x^{c,s}$ by transporter $c \in \mathcal{C}$ for seller $s \in \mathcal{S}$ is assumed to be of the quadratic form

$$w^{c,s}\left(x^{c,s}(t)\right) = b_1^{c,s}x^{c,s}(t) + \frac{1}{2}b_2^{c,s}\left[x^{c,s}(t)\right]^2 \tag{3.9}$$

where $b_1^{c,s},\ b_2^{c,s} \in \mathbb{R}_+^1$.

The *cumulative profit* function of transporter $c \in \mathcal{C}$ over the time horizon

$t \in [t_0, t_f]$ thus takes the form

$$J_c\left(\pi^c, \rho^c; x^c; \pi^{-c}, \rho^{-c}, v^s; t\right) = \int_{t_0}^{t_1} \Big\{ \left[u^c(\pi(t))\right]^T \pi^c(t) \\ - \left[w^c\right]^T \left(x^c(t)\right) - \left[k^c\left(\rho^c(t), t\right)\right]^T \rho^c(t) \Big\}\, dt \tag{3.10}$$

where π^{-c} and ρ^{-c} defined by

$$\begin{aligned} \pi^{-c} &\equiv \{\pi^g : g \in \mathcal{C} \backslash c\} \\ \rho^{-c} &\equiv \{\rho^g : g \in \mathcal{C} \backslash c\} \end{aligned}$$

are taken as exogenous by transporter c from competing transporters and v^s is taken as exogenous from the problems of the sellers.

3.3 A Transporter's Best Response Problem

Each transporter is assumed to be an optimizer with the goal of maximizing their individual profit. The optimization problems of the transporters are assumed to take the form of optimal control problems which are solved simultaneously over the planning horizon. In particular, each transporter seeks to solve the following optimal control problem while taking the price π^{-c} and transportation flows ρ^{-c} of competing transporters and shipment flows v^s by the sellers as exogenous:

$$\max_{\pi,\rho}\ J_c\left(\pi^c, \rho^c; x^c; \pi^{-c}, \rho^{-c}, v^s; t\right) \tag{3.11}$$

subject to

$$\frac{dx^c(t)}{dt} = \sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}_r}\sum_{j\in\mathcal{N}_s}\left(u^{c,r}_{i,j}(\pi^r_{i,j}(t)) - \rho^{c,r}_{i,j}(t)\right) \tag{3.12}$$

$$x^c(t_0) = 0 \tag{3.13}$$

$$x^c(t) \geq 0 \tag{3.14}$$

$$x^c(t_f) = 0 \tag{3.15}$$

$$\sum_{c\in\mathcal{C}}\rho^c = v \tag{3.16}$$

$$\rho^c \geq 0 \tag{3.17}$$

$$\pi^c_{\min} \leq \pi^c \leq \pi^c_{\max} \tag{3.18}$$

In this problem, the states are the total backlogged service x^c and the controls are the prices of transportation service π^c and flow of commodities ρ^c. Constraints (3.12) - (3.16) are discussed in (3.2) - (3.6) and constraints (3.17) and (3.18) are simply bounds on the control variables. The set of optimal control problems (3.11) - (3.18) for all transporters $c \in C$ are coupled through the demand function and the joint constraints (3.16). This set of coupled optimal control problems describes the differential Nash game of the transporters. As with the model of the sellers, the equilibrium solution of the game is a moving equilibrium of the strategies of the transporters over the planning horizon.

3.4 Differential Variational Inequality Formulation

In this section, the DVI representation of the non-cooperative game among transporters is formulated through the inspection of the necessary conditions for each transporter's optimal control problem (3.11) - (3.18). The Hamiltonian for each transporter's optimal control problem may be stated as the sum of the integrand of the profit function and the product of the adjoint variables and the dynamics

$$H_c\left(y^c; x^c; \pi^{-c}, \rho^{-c}, \upsilon^c; \kappa^c; t\right) = [u^c]^T \pi^c - [w^c]^T (x^c) - [k^c]^T \rho^c$$
$$+ [\kappa^c]^T \left(\sum_{r\in\mathcal{R}} \sum_{i\in\mathcal{N}_r} \sum_{j\in\mathcal{N}_s} \left(u^{c,r}_{i,j} (\pi^r_{i,j}(t)) - \rho^{c,r}_{i,j} \right) \right)$$

where κ^c are the adjoint variables associated with the dynamics (3.12) and y^c is the vector of controls for transporter $c \in \mathcal{C}$ defined by

$$y^c \equiv \begin{bmatrix} \pi^c \\ \rho^c \end{bmatrix}$$

The definition of this vector of controls is made to simplify the notation.

The Hamiltonian may be augmented by dualizing the state constraints. The state constraints (5.31) are dualized resulting in the augmented Hamiltonian

$$\tilde{H}_c\left(y^c; x^c; \pi^{-c}, \rho^{-c}, \upsilon^c; \kappa^c, \eta^c; t\right) = H_c\left(y^c; x^c; \pi^{-c}, \rho^{-c}, \upsilon^c; \kappa^c; t\right) + [\varrho^c]^T (-x^c)$$

where $\varrho^{c,s}$ are the dual variables associated with the state constraint (3.14).

Because there is a terminal time constraint (2.13) on the states $x^{c,s}$, the terminal time conditions for the associated adjoint variables $\kappa^{c,s}$ are

$$\kappa^{c,s}(t_f) = C$$

where C is a constant.

The necessary conditions are given by

$$\frac{\partial \tilde{H}_c}{\partial y^c} = 0 \Rightarrow \begin{cases} \frac{\partial \hat{H}_c}{\partial \pi^c} = 0 \\ \frac{\partial \hat{H}_c}{\partial \rho^c} = 0 \end{cases}$$

$$\begin{aligned} \frac{\partial \tilde{H}_c}{\partial \pi^{c,r,s}_{i,j}} &= u^{c,r,s}_{i,j} - h^{c,r,s}_{2,i,j} + \kappa^{c,r,s}_{i,j} \\ \frac{\partial \tilde{H}_c}{\partial \rho^{c,r,s}_{i,j}} &= -\kappa^{c,r,s}_{i,j} - \left(l^{c,r,s}_{1,i,j} + l^{c,r,s}_{2,i,j} \rho^{c,r,s}_{i,j} \right) \rho^{c,r,s}_{i,j} \end{aligned}$$

In addition, the adjoint variables must satisfy the adjoint dynamics which are of the form

$$\frac{d\kappa^{c,s}}{dt} = -\frac{\partial \tilde{H}_c}{\partial x^{c,s}} = b_1^{c,s} + b_2^{c,s} x^{c,s} + \varrho^{c,s} \text{ for all } c \in \mathcal{C},\ s \in \mathcal{S} \tag{3.19}$$

Using the information from the necessary conditions, the differential variational inequality (DVI) may immediately be stated as

$$\text{find } y^{c*} \in \Omega_c \text{ such that}$$

$$\sum_{c\in\mathcal{C}} \int_{t_0}^{t_1} \left[\frac{\partial \tilde{H}_c\left(y^{c*}\right)}{\partial y^{c*}} \right]^T \left(y^c - y^{c*}\right) dt \leq 0 \tag{3.20}$$

$$\text{for all } y^c \in \Omega_c$$

where Ω_c is defined by the set of constraints

$$\Omega_c = \{y^c : (3.12), (3.13), (3.15) \text{ - } (3.18) \text{ and } (3.19) \text{ hold}\}$$

Chapter 4

Complete DVI Formulation

The DVI representing the entire game is formed by combining the DVI statements (2.20) and (3.20) for the models of the sellers and the transporters, respectively.

$$\text{find } \left(y^{s*}, y^{c*}\right)^T \in \Omega \text{ such that}$$

$$\sum_{s \in S} \int_{t_0}^{t_1} \left[\frac{\partial \hat{H}_s\left(y^{s*}\right)}{\partial y^s} \right]^T \left(y^s - y^{s*}\right) dt + \int_{t_0}^{t_1} \left[\frac{\partial \tilde{H}_c\left(y^{c*}\right)}{\partial y^{c*}} \right]^T \left(y^c - y^{c*}\right) dt \leq 0$$

$$\text{for all } y^s \in \Omega_s,\ y^c \in \Omega_c$$

The DVI is a mathematical construct that describes the equilibrium solution of the game and is not directly useful for computation as its statement relies on knowledge of the optimal control strategies y^{s*} and y^{c*}. However, the DVI may be shown to be equivalent to a fixed point problem or a functional complementarity problem for which many different solution algorithms exist (Friesz et al., 2006).

Chapter 5

Time Discretization Approximation

In this section, a time discretization approximation will be made to the optimal control problems and the Karush-Kuhn-Tucker (KKT) conditions will be applied to the resulting finite dimensional mathematical programs in order to formulate a finite dimensional complementarity problem formulation representing the solution of the game. When discretizing the optimal control problems, the following approximations will be used. There will be $N+1$ number of discrete time points indexed by $n = 0, 1, ..., N$. The time between each discrete time point will be represented by Δ where

$$\Delta = \frac{t_f - t_0}{N} \tag{5.1}$$

Differential equations stated as

$$\frac{dx}{dt} = f(x, u, t)$$

will be approximated as

$$\frac{x_{n+1} - x_n}{\Delta} = f_n(x_n, u_n, \Delta n)$$

or the equivalent form

$$x_{n+1} = x_n + \Delta f_n(x_n, u_n, \Delta n) \tag{5.2}$$

Integrals stated as

$$\int_{t_0}^{t_f} F\left(x,u,t\right)dt$$

will be approximated by the summation

$$\Delta \sum_{n=0}^{N} F_n\left(x_n,u_n,\Delta n\right) \tag{5.3}$$

5.1 Seller Complementarity Problem

In this section, the best response problems of the sellers (2.9) - (2.17) will be time discretized to obtain a finite dimensional nonlinear program approximation. Under a suitable constraint qualification, the KKT conditions will be employed to form a nonlinear complementarity problem (NCP).

Using the discretization approximations (5.1) - (5.3) and denoting the discretization index by t, the optimal control problem for each seller $s \in \mathcal{S}$ may be approximated by the following finite dimensional nonlinear program:

$$\max \; F_s = \Delta \sum_{t=0}^{N} \left[\left[d_t^s(p_t)\right]^T p_t^s - \sum_{i\in N_s} \left[\Psi_{i,t}^s\left(I_{i,t}^s\right) + \theta_{i,t}^s\left(q_{i,t}^s\right)\right] - \left[\pi_t^s\right]^T u_t^s \right] \tag{5.4}$$

subject to

$$I_{t+1}^s = I_t^s + \Delta \left[q_{j,t}^s - \sum_{r\in\mathcal{R}} \sum_{j\in\mathcal{N}_r} v_{j,t}^{r,s} \right] \;\; \forall\; t = 0...N-1 \tag{5.5}$$

$$I_0^s \text{ given} \tag{5.6}$$

$$\sum_{s\in\mathcal{S}} d_t^s = D_t \;\forall\; t = 0...N \tag{5.7}$$

$$d_t^s = \sum_{i\in\mathcal{N}_s} \sum_{j\in\mathcal{N}_r} v_{i,j,t}^s \;\; \forall\; t = 0...N \tag{5.8}$$

$$0 \leq v_t^s \;\; \forall\; t = 0...N \tag{5.9}$$

$$I_{\min}^s \leq I_t^s \leq I_{\max}^s \;\; \forall\; t = 0...N \tag{5.10}$$

$$p_{\min}^s \leq p_t^s \leq p_{\max}^s \;\; \forall\; t = 0...N \tag{5.11}$$

$$0 \leq q_t^s \leq q_{\max}^s \quad \forall\; t = 0...N \tag{5.12}$$

where the subscript t now denotes the discretization index.

Note that the states $I^s_{i,t}$ are now intermediate variables and can be replaced by the controls v^s_t and q^s_t using the following representation.

$$I^s_{i,t} = I^s_{i,0} + \Delta \sum_{n=0}^{t-1} \left[q^s_{i,n} - \sum_{r \in \mathcal{R}} \sum_{j \in \mathcal{N}_r} v^{r,s}_{i,j,n} \right] \tag{5.13}$$

Because all of the constraints (5.5) - (5.12) are linear, Abadie's constraint qualification holds (Mangasarian, 1969) and we may inspect the KKT conditions of the mathematical program (5.4) - (5.12). The KKT identity is:

$$\begin{aligned} 0 \;=\; \nabla_{y_{s,t}} \Bigg\{ & -F_s + \sum_{t=0}^{N} \Big[\left[\xi^{s,-}_t\right]^T (I^s_{\min} - I^s_t) + \left[\xi^{s,+}_t\right]^T (I^s_t - I^s_{\max}) \\ & \left[\beta^+_t\right]^T \left(\sum_{s \in \mathcal{S}} d^s_t - D_t \right) + \left[\beta^-_t\right]^T \left(- \sum_{s \in \mathcal{S}} d^s_t + D_t \right) \\ & + \left[\gamma^{s,+}_t\right]^T \left(d^s_t - \sum_{i \in \mathcal{N}_s} v^s_{i,t} \right) + \left[\gamma^{s,-}_t\right]^T \left(-d^s_t + \sum_{i \in \mathcal{N}_s} v^s_{i,t} \right) \\ & + \left[\delta^s_t\right]^T (-v^s_t) + \left[\zeta^{s,+}_t\right]^T (p^s_t - p^s_{\max}) + \left[\zeta^{s,-}_t\right]^T (p^s_{\min} - p^s_t) \\ & + \left[\eta^{s,+}_t\right]^T (q^s_t - q^s_{\max}) + \left[\eta^{s,-}_t\right]^T (-q^s_t) \Big] \Bigg\} \end{aligned} \tag{5.14}$$

Note that the equations (5.7) and (5.8) are each being represented by two inequalities. Also, note that the dual multipliers $\beta^{r,+}_{j,t}$ and $\beta^{r,-}_{j,t}$ do not depend on the specific seller. This is due to the fact that the constraints (5.7) associated with these dual multipliers are joint constraints. That is, the constraint for a specific seller involves the decision variables of that seller as well as the decision variables of competing sellers. This method of using a common dual variable is adapted from the work of Harker (1991).

The following change of variables is introduced to further simplify the notation.

$$\bar{p}^{r,s}_t \equiv p^{r,s}_t - p^{r,s}_{\min} \geq 0$$

Manipulating the KKT identity (5.14) results in the following expression

$$
\begin{aligned}
0 &= \bar{y}_{s,t} \cdot \nabla_{\bar{y}_{s,t}} \Bigg\{ -F_s + \sum_{t=0}^{N} \Big[\left[\xi_t^{s,-}\right]^T \left(I_{\min}^s - I_t^s\right) + \left[\xi_t^{s,+}\right]^T \left(I_t^s - I_{\max}^s\right) \\
& \left[\beta_t^{+}\right]^T \left(\sum_{s \in \mathcal{S}} d_t^s - D_t \right) + \left[\beta_t^{-}\right]^T \left(-\sum_{s \in \mathcal{S}} d_t^s + D_t \right) \\
& + \left[\gamma_t^{s,+}\right]^T \left(d_t^s - \sum_{i \in \mathcal{N}_s} v_{i,t}^s \right) + \left[\gamma_t^{s,-}\right]^T \left(-d_t^s + \sum_{i \in \mathcal{N}_s} v_{i,t}^s \right) \\
& + \left[\zeta_t^{s,+}\right]^T \left(\bar{p}_t^s + p_{\min}^s - p_{\max}^s\right) + \left[\eta_t^{s,+}\right]^T \left(q_t^s - q_{\max}^s\right) \Big] \Bigg\} \\
&\equiv \bar{y}_{s,t} \cdot \Theta_s \left(\bar{y}_{s,t}; \beta_t^{+}, \beta_t^{-}, \gamma_t^{s,+}, \gamma_t^{s,-}, \zeta_t^{s,+}, \eta_t^{s,+} \right) \qquad (5.15)
\end{aligned}
$$

where

$$\bar{y}_{s,t} \equiv \begin{pmatrix} v_t^s & \bar{p}_t^s & q_t^s \end{pmatrix}^T$$

The complementary slackness conditions accompanying the KKT Identity (5.14) give that

$$
\begin{aligned}
\left[\xi_t^{s,-}\right]^T \left(I_t^s - I_{\min}^s\right) &= 0 \;\forall\; t = 0...N & (5.16) \\
\left[\xi_t^{s,+}\right]^T \left(I_{\max}^s - I_t^s\right) &= 0 \;\forall\; i \in \mathcal{N}_s,\; t = 0...N & (5.17) \\
\left[\beta_t^{+}\right]^T \left(-\sum_{s \in \mathcal{S}} d_t^s\left(p_t\right) + D_t \right) &= 0 \;\forall\; t = 0...N & (5.18) \\
\left[\beta_t^{-}\right]^T \left(\sum_{s \in \mathcal{S}} d_t^s\left(p_t\right) - D_t \right) &= 0 \;\forall\; t = 0...N & (5.19) \\
\left[\gamma_t^{s,+}\right]^T \left(-d_t^s\left(p_t\right) + \sum_{i \in \mathcal{N}_s} v_{i,t}^s \right) &= 0 \;\forall\; t = 0...N & (5.20) \\
\left[\gamma_t^{s,-}\right]^T \left(d_t^s\left(p_t\right) - \sum_{i \in \mathcal{N}_s} v_{i,t}^s \right) &= 0 \;\forall\; t = 0...N & (5.21) \\
\left[\delta_t^{s}\right]^T \left(v_t^s\right) &= 0 \;\forall\; t = 0...N & (5.22) \\
\left[\zeta_t^{s,+}\right]^T \left(p_{\max}^s - \bar{p}_t^s - p_{\min}^s\right) &= 0 \;\forall\; t = 0...N & (5.23) \\
\left[\zeta_t^{s,-}\right]^T \left(\bar{p}_t^s\right) &= 0 \;\forall\; t = 0...N & (5.24) \\
\left[\eta_t^{s,+}\right]^T \left(q_{\max}^s - q_t^s\right) &= 0 \;\forall\; t = 0...N & (5.25) \\
\left[\eta_t^{s,-}\right]^T \left(q_t^s\right) &= 0 \;\forall\; t = 0...N & (5.26)
\end{aligned}
$$

The manipulated KKT identity (5.15) and complementary slackness conditions (5.16) - (5.26) can be combined into the following complementary vectors.

$$G_{s,t}\left(z_t^s\right) \equiv \begin{pmatrix} \Theta_s\left(\bar{y}_{s,t};\beta_t^+,\beta_t^-,\gamma_t^{s,+},\gamma_t^{s,-},\zeta_t^{s,+},\eta_t^{s,+}\right) \\ I_{i,\max}^s - I_{i,t}^s \\ I_t^s - I_{\min}^s \\ -\sum_{s\in\mathcal{S}} d_t^s + D_t \\ \sum_{s\in\mathcal{S}} d_t^s - D_t \\ -d_t^s + \sum_{i\in\mathcal{N}_s} v_{i,t}^s \\ d_t^s - \sum_{i\in\mathcal{N}_s} v_{i,t}^s \\ v_t^s \\ p_{\max}^s - \bar{p}_t^s - p_{\min}^s \\ \bar{p}_t^s \\ q_{\max}^s - q_t^s \\ q_t^s \end{pmatrix}, \quad z_t^s \equiv \begin{pmatrix} \bar{y}_{s,t} \\ \xi_t^{s,+} \\ \xi_t^{s,-} \\ \beta_t^+ \\ \beta_t^- \\ \gamma_{j,t}^{s,+} \\ \gamma_{j,t}^{s,-} \\ \delta_t^s \\ \zeta_t^{s,+} \\ \zeta_t^{s,-} \\ \eta_t^{s,+} \\ \eta_t^{s,-} \end{pmatrix}$$

with

$$0 \le G_{s,t}\left(z_t^s\right) \perp z_t^s \ge 0 \tag{5.27}$$

The form stated in (5.27) is known as a complementarity problem. Because Θ_s is nonlinear in some of the variables z^s, this problem is more specificially a nonlinear complementarity problem (NCP). The NCP (5.27) only represents the portion of the game relating to the sellers. A similar analysis will be conducted for the transporters to obtain another NCP. The conditions from both NCPs will then be concatenated into a single NCP representing the solution of the entire game.

5.2 Transporter Complementarity Problem

Again, using the discretization rules (5.1) - (5.3) the optimal control problem (3.11) - (3.18) can be time discretized to obtain the following finite dimension mathematical program.

$$\max \ F_c = \Delta \sum_{t=0}^{N} \left[\left[u_t^c(\pi_t)\right]^T \cdot \pi_t^c - \sum_{s\in\mathcal{S}} w_t^{c,s}\left(x_t^{c,s}\right) - \left[k_t^c\left(\rho_t^c\right)\right]^T \cdot \rho_t^c \right] \tag{5.28}$$

subject to

$$
\begin{aligned}
x_{t+1}^{c} &= x_{t}^{c} + \Delta \sum_{r\in\mathcal{R}} \sum_{j\in\mathcal{N}_r} \sum_{i\in\mathcal{N}_s} \left(u_{i,j,t}^{c,r} - \rho_{i,j,t}^{c,r}\right) \;\forall\; t = 0...N-1 && (5.29)\\
x_{0}^{c} &= 0 && (5.30)\\
x_{t}^{c,s} &\geq 0 \;\forall\; s\in\mathcal{S},\; t = 1...N-1 && (5.31)\\
x_{N}^{c,s} &= 0 \;\forall\; s\in\mathcal{S} && (5.32)\\
v_{t} &= \sum_{c\in\mathcal{C}} u_{t}^{c} \;\forall\; t = 0...N && (5.33)\\
\rho_{t}^{c} &\geq 0 \;\forall\; t = 0...N && (5.34)\\
\pi_{\min}^{c} &\leq \pi_{t}^{c} \leq \pi_{\max}^{c} \;\forall\; t = 0...N && (5.35)
\end{aligned}
$$

where t now denotes a specific time discretization point.

Note that the states $x_t^{c,s}$ can be represented in terms of the controls ρ_t and π_t using the following representation.

$$
x_{t}^{c,s} = x_{0}^{c,s} + \Delta \sum_{n=0}^{t-1} \left[\sum_{r\in\mathcal{R}} \sum_{i\in\mathcal{N}_r} \sum_{j\in\mathcal{N}_s} \left(u_{i,j,t}^{c,r,s} - \rho_{i,j,t}^{c,r,s}\right) \right] \tag{5.36}
$$

The constraints (5.29) - (5.35) for the transporters' problem are all linear, so again Abadie's constraint qualification holds (Mangasarian, 1969). The KKT identity is

$$
\begin{aligned}
0 \;=\; & \nabla_{y_{c,t}} \Bigg\{ -F_c + \left[\vartheta^{c,+}\right]^T \left(x_N^c\right) + \left[\vartheta^{c,-}\right]^T \left(-x_N^c\right) + \sum_{t=1}^{N-1} \left[\theta_t^c\right]^T \left(-x_t^c\right) \\
& + \sum_{t=0}^{N} \Bigg[\left[\lambda_t^+\right]^T \left(\sum_{c\in\mathcal{C}} u_t^c - v_t\right) + \left[\lambda_t^-\right]^T \left(-\sum_{c\in\mathcal{C}} u_t^c + v_t\right) \\
& + \left[\nu_t^{c,+}\right]^T \left(-\bar{\pi}_t^c\right) + \left[\nu_t^{c,-}\right]^T \left(\pi_t^c - \pi_{\max}^c\right) + \left[\mu_t^c\right]^T \left(-\rho_t^c\right) \Bigg] \Bigg\}
\end{aligned} \tag{5.37}
$$

where the following change of variable is used to simplify the notation.

$$
\bar{\pi}_t^c \equiv \pi_t^c - \pi_{\min}^c \geq 0
$$

Manipulating the KKT identity in (5.37), we can obtain the following statement.

$$
\begin{aligned}
0 \quad = \quad & \bar{y}_{c,t} \cdot \nabla_{\bar{y}_{c,t}} \Bigg\{ -F_c + \left[\vartheta^{c,+}\right]^T (x_N^c) + \left[\vartheta^{c,-}\right]^T (-x_N^c) + \sum_{t=1}^{N-1} [\theta_t^c]^T (-x_t^c) \\
& + \sum_{t=0}^{N} \Bigg[\left[\lambda_t^+\right]^T \left(\sum_{c \in \mathcal{C}} u_t^c - v_t \right) + \left[\lambda_t^-\right]^T \left(-\sum_{c \in \mathcal{C}} u_t^c + v_t \right) \\
& + \left[\nu_t^{c,-}\right]^T (\bar{\pi}_t^c - \pi_{\min}^c - \pi_{\max}^c) \Bigg] \Bigg\} \\
\equiv \quad & \bar{y}_{c,t} \cdot \Theta_c \left(y_{c,t}; \phi_t^c, \vartheta^{c,+}, \vartheta^{c,-}, \lambda_t^+, \nu_t^{c,-} \right)
\end{aligned}
\tag{5.38}
$$

where

$$\bar{y}_{c,t} \equiv \begin{pmatrix} \rho_t^c & \bar{\pi}_t^c \end{pmatrix}^T$$

The complementary slackness conditions associated with the KKT identity (5.37) are

$$[\phi_t^c]^T (x_t^c) = 0 \; \forall \; t = 1 \ldots N-1 \tag{5.39}$$

$$\left[\vartheta^{c,+}\right]^T (-x_N^c) = 0 \tag{5.40}$$

$$\left[\vartheta^{c,-}\right]^T (x_N^c) = 0 \tag{5.41}$$

$$\left[\lambda_t^+\right]^T \left(-\sum_{c \in \mathcal{C}} u_t^c + v_t \right) = 0 \; \forall \; t = 1 \ldots N \tag{5.42}$$

$$\left[\lambda_t^-\right]^T \left(\sum_{c \in \mathcal{C}} u_t^c - v_t \right) = 0 \; \forall \; t = 1 \ldots N \tag{5.43}$$

$$\left[\nu_t^+\right]^T (\bar{\pi}_t^c) = 0 \; \forall \; t = 1 \ldots N \tag{5.44}$$

$$\left[\nu_t^{c,-}\right]^T (-\bar{\pi}_t^c + \pi_{\min}^c + \pi_{\max}^c) = 0 \; \forall \; t = 1 \ldots N \tag{5.45}$$

$$[\mu_t^c]^T (\rho_t^c) = 0 \; \forall \; t = 1 \ldots N \tag{5.46}$$

Similar to the sellers' model, the dual multipliers $\lambda_{i,j,t}^{s,+}$ and $\lambda_{i,j,t}^{s,-}$ are common among the transporters due to the joint constraints (3.2).

Combining the complementary slackness conditions (5.39) - (5.46) with the manipulation of the KKT identity (5.38), the following complementary vectors may be defined:

$$G_{c,t}\left(z_t^c\right) \equiv \begin{pmatrix} \Theta_c \bar{y}_{c,t} \cdot \Theta_c\left(y_{c,t}; \phi_t^c, \vartheta^{c,+}, \vartheta^{c,-}, \lambda_t^+, \nu_t^{c,-}\right) \\ x_t^c \\ -x_N^c \\ x_N^{c,s} \\ -\sum_{c\in\mathcal{C}} u_t^{c,} + v_t \\ \sum_{c\in\mathcal{C}} u_t^c - v_t \\ \bar{\pi}_t^c \\ -\bar{\pi}_t^c + \pi_{\min}^c + \pi_{\max}^c \\ \rho_t^c \end{pmatrix}, \quad z_t^c \equiv \begin{pmatrix} \bar{y}_{c,t} \\ \phi_t^c \\ \vartheta^{c,+} \\ \vartheta^{c,-} \\ \lambda_t^+ \\ \lambda_t^- \\ \nu_t^{c,+} \\ \nu_t^{c,-} \\ \mu_t^c \end{pmatrix}$$

with

$$0 \leq G_c\left(z_t^c\right) \perp z_t^c \geq 0 \tag{5.47}$$

The statement (5.47) is the NCP representing the game played by the transporters.

5.3 Complete NCP Formulation

The complete NCP representing the game involving both the sellers and the transporters is created by concatenating the complementarity conditions from (5.27) and (5.47) that were obtained through the analysis of the seller and transporter models.

$$0 \leq G(z) \equiv \begin{pmatrix} G_{s,t}\left(z_t^s\right) \\ G_{c,t}\left(z_t^c\right) \end{pmatrix} \perp z \equiv \begin{pmatrix} z_t^s \\ z_t^c \end{pmatrix} \geq 0 \tag{5.48}$$

Such a complementarity problem may be coded in a modeling language such as GAMS and solved using a commercial complementarity solver. The NCP can be solved directly using these solvers or they may be solved by a successive linearization scheme; such a scheme results in series a linear complementarity subproblems that may be quickly and efficiently solved using a Lemke's type algorithm (Pang and Facchinei, 2003).

Chapter 6

Illustrative Numerical Example

In this section an example problem with 3 sellers, 3 transporters, and 3 receivers is considered. Each seller and receiver is located at only one node. The 3 transporters can each deliver from any seller to any receiver. This leads to the network representation seen in Figure 6.1. For this example, only one mode of transportation is considered. The values and ranges of the parameters used in the numerical example are given in Table 6.1. This example was formulated as a nonlinear complementarity problem following the analysis from Section 5. The problem was coded in GAMS and solved using the PATH solver. PATH was run with the options set to sequentially linearize the problem and use a Lemke's type algorithm to solve the linearized problem at each iteration. The time for solution was less than 15 seconds on a Pentium 4 desktop with 1GB of RAM.

The receivers are assumed to have demands that vary with time as evidenced by the sinusoidal curves in Figure 6.2. In Figure 6.3, it may be observed that the sellers start with a positive inventory and then quickly sell it off to keep inventory as close to zero as possible, thereby keeping inventory holding costs low. Figure 6.4 shows that the sellers start off with a low rate of production since they want to sell off their existing inventory. The sellers then begin to increase the production rates and take on a just-in-time production scheme that keeps their inventory level near zero. Note that the sellers do not produce anything in the final time period; this is due to the finite planning horizon. The sellers do not produce any goods in the final time period because, from their point of view, there is no following time period to sell the goods in. In fact, if the sellers do produce goods in the last time

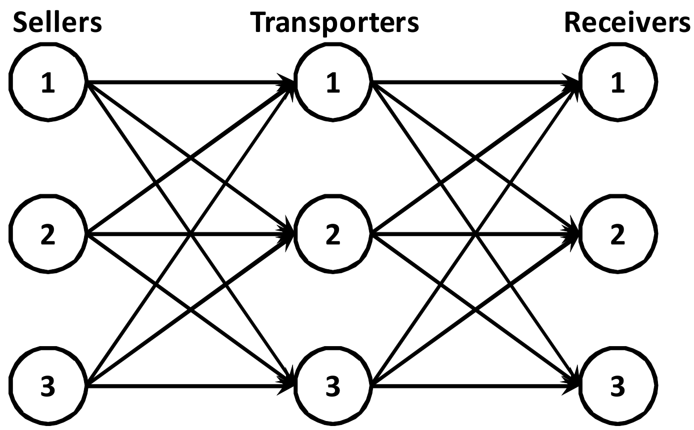

Figure 6.1. Network for numerical example

Parameter	**Range**	**Parameter**	**Range**
$a_1^{r,s}$	$47 - 57$	$b_1^{c,s}$	$0.05 - 0.15$
$a_2^{r,s}$	$0.45 - 0.525$	$b_2^{c,s}$	$0.02 - 0.12$
$a_3^{r,s,g}$	$0.025 - 0.075$	$\omega_1^{c,r,s}$	$9 - 10$
e_j^s	$0.45 - 0.55$	$\omega_2^{c,r,s}$	$0.45 - 0.525$
$f_{1,j}^s$	$0.25 - 0.35$	$\omega_3^{c,g,r,s}$	$0.1 - 0.15$
$f_{2,j}^s$	$0.05 - 0.15$	$I_{j,0}^s$	$0 - 10$
$f_{3,j}^s$	0	D_t^r	$48 - 52$
$l_{1,m}^{c,r,s}$	$15 - 15.5$	Δ	0.5
$l_{2,m}^{c,r,s}$	$0.3 - 0.4$	N	21
$p_{\min}$	0	$\pi_{\min}$	0
$p_{\max}$	100	$\pi_{\max}$	75
$q_{\max}$	100		

Table 6.1. Parameter values and ranges for the numerical experiment

period, they will have a positive inventory at the final time instant that will incur a holding cost.

The dynamic pricing strategies of the sellers are illustrated in Figure 6.5. The corresponding demands that are captured by each seller due to their choices of prices are shown in Figure 6.6. Note that seller 1 charges the maximum price of 100 to receiver 1 for most of the time horizon; this is the only case when a firm reaches its maximum price in this example. Also note that a seller's realized demand from a particular receiver decreases as the price charged to that receiver is increased. Also, recall that the prices charged by the receivers are the delivered

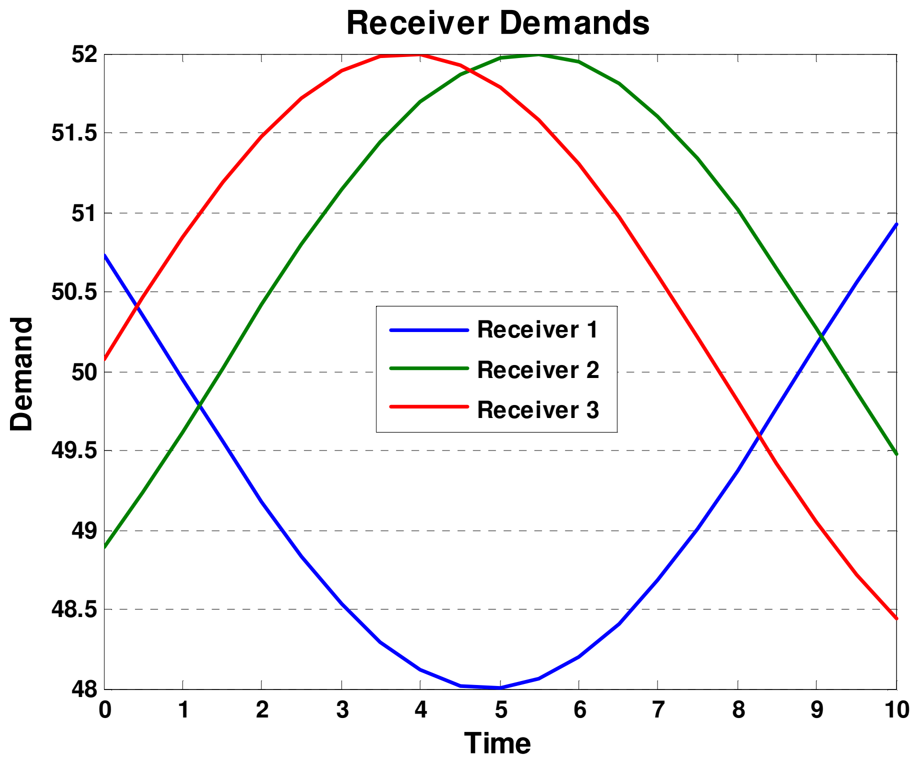

Figure 6.2. Receiver demands over the planning horizon

prices which include transportation charges.

The dynamic pricing strategies of the transporters for transportation service between each seller-receiver pair are given in Figure 6.7 while Figure 6.8 shows the cumulative backlog of shipments as demanded by the three sellers in each time period. Figure 6.7 shows the pricing strategies followed by the transporters for each of the receiver-seller origin-destination pairs. It is important to note that the price trajectories of both the sellers and the transporters vary with time, suggesting that there is an incentive to the firms to change the price over time. In other words, the firms can gain more profit by charging a dynamic price than by charging a static price. It is observed in Figure 6.8 that the transporters start and end with no backlogged service as the model constrains. However, non-zero service backlogs are allowed during the interior of the planning horizon. The transporters allow some service to backlog because there is a trade-off between the loss-of-goodwill and transportation costs.

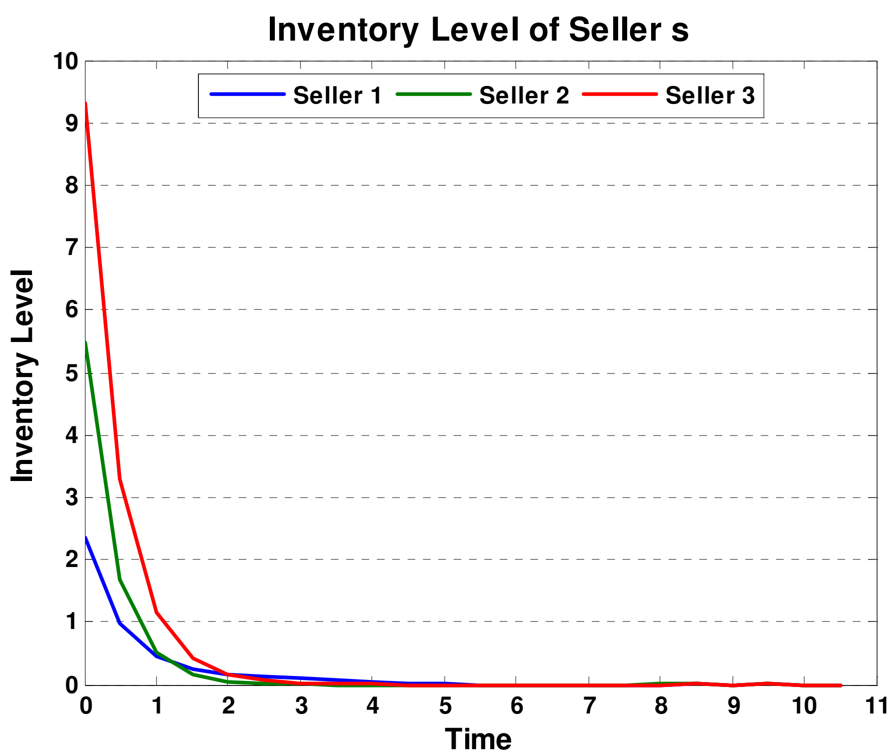

Figure 6.3. Inventory level of each seller over the planning horizon.

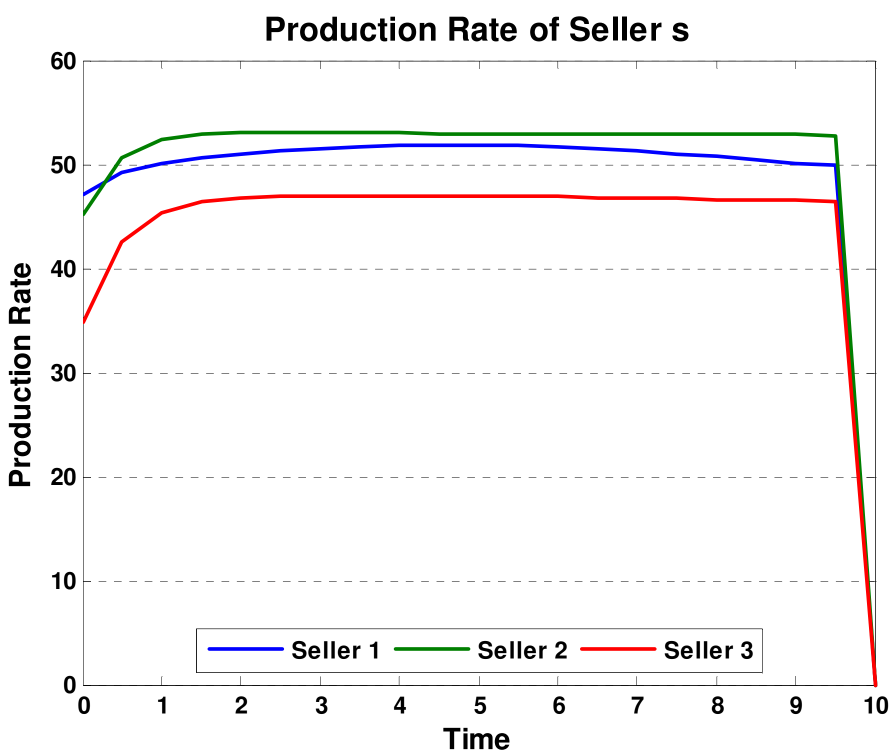

Figure 6.4. Production rate of each seller over the planning horizon.

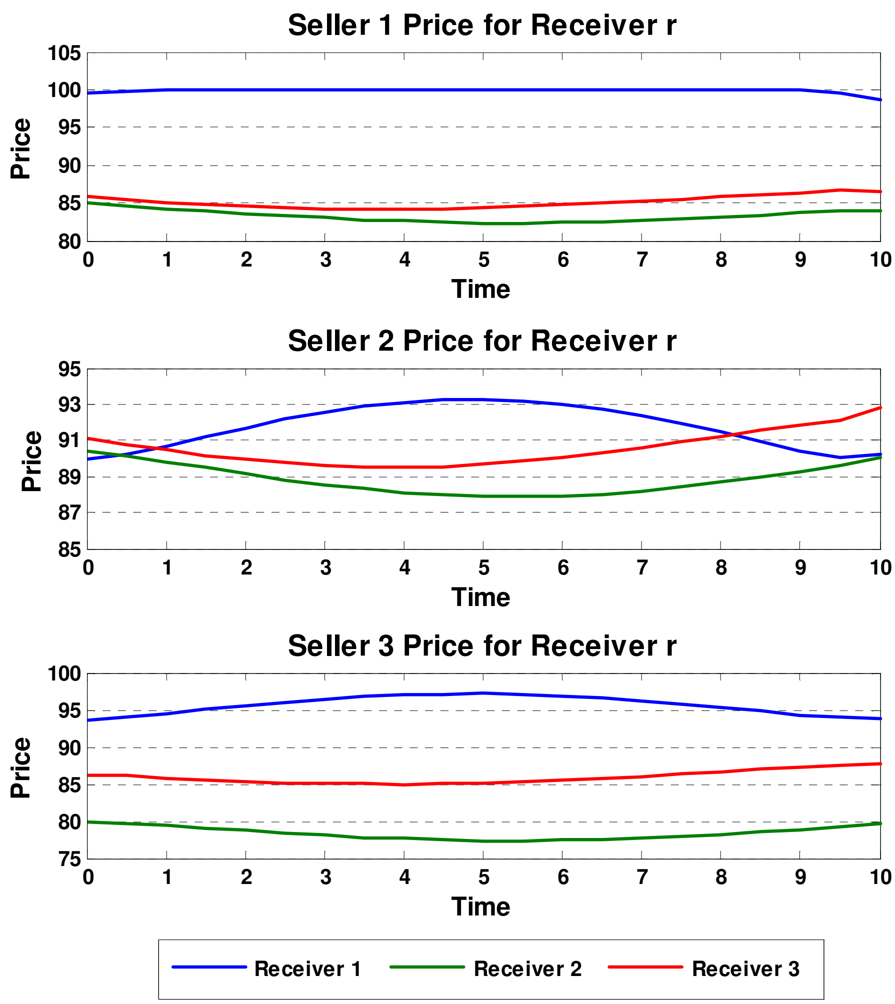

Figure 6.5. Dynamic price trajectories of the sellers

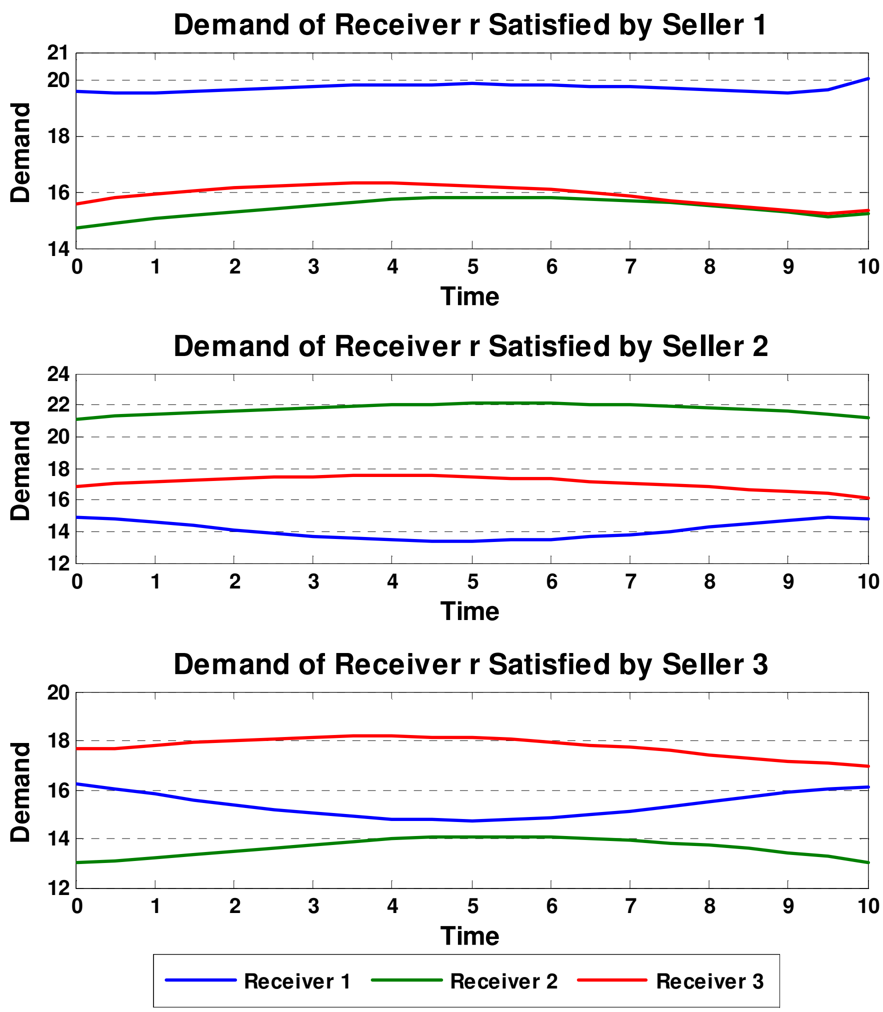

Figure 6.6. Demand that was captured by each seller over the planning horizon.

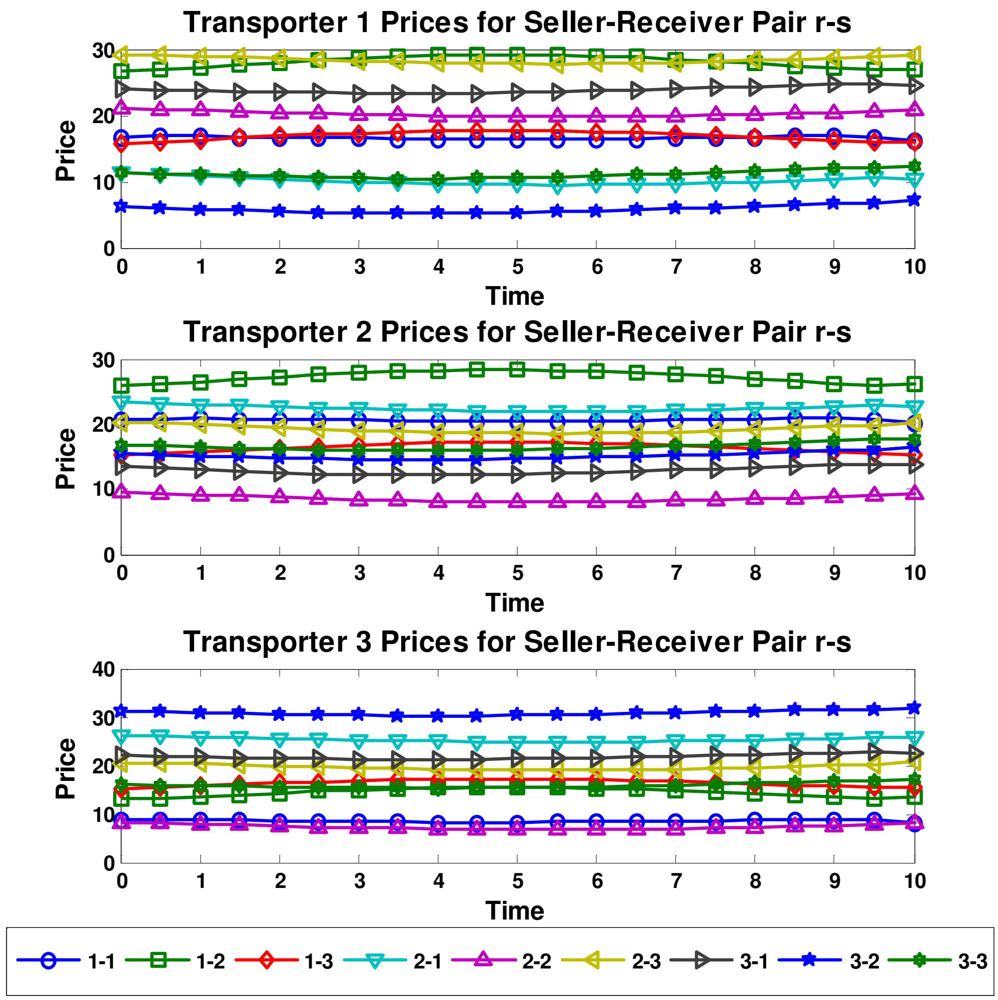

Figure 6.7. Dynamic price trajectories of the transporters for each seller-receiver origin destination pair

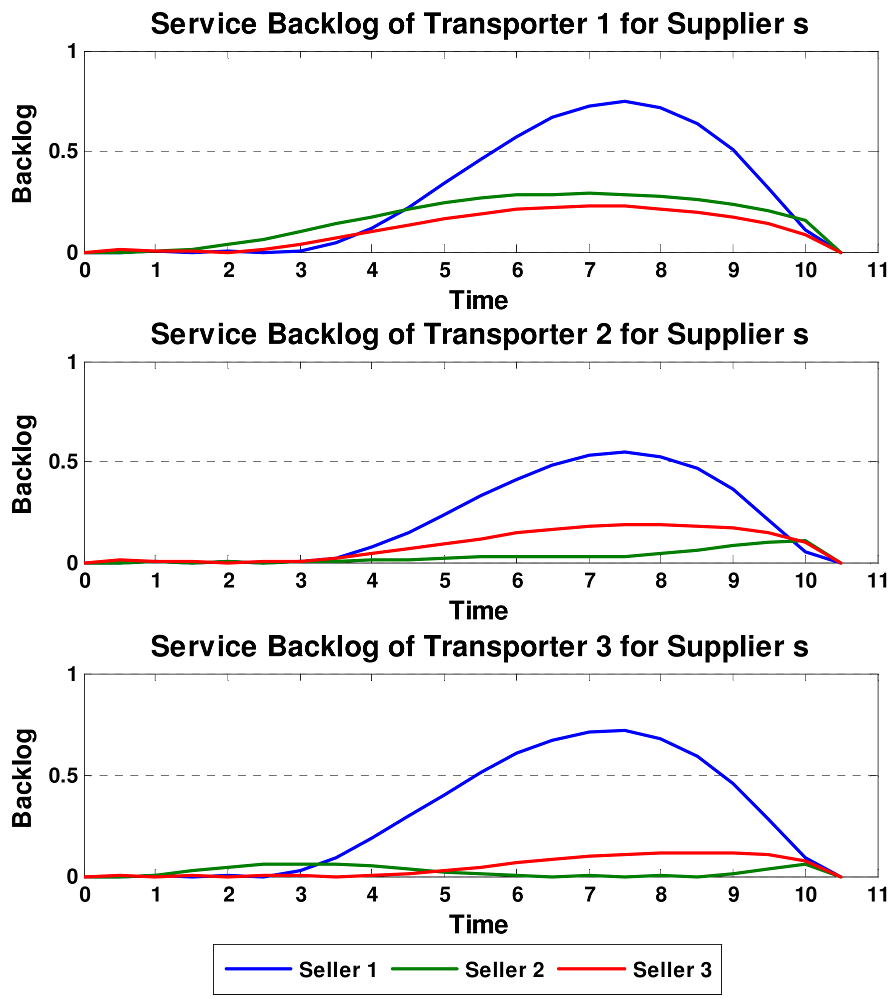

Figure 6.8. Service backlog strategy of each transporter for each seller

Chapter 7

Conclusions and Future Work

This thesis has put forth a differential Nash game of dynamic pricing in an urban freight environment. Each agent's best response problem was represented by a continuous time optimal control problem, and the analysis of the set of coupled optimal control problems representing the game was used to represent the solution of the game as a differential variational inequality. A time discretization method was applied to approximate the continuous time optimal control problems as nonlinear programs. The Karush-Kuhn-Tucker conditions were used under a suitable constraint qualification in order to examine the necessary conditions for each agent's optimization problem. These conditions were then used to formulate a finite dimensional nonlinear complementarity problem formulation of the solution of the game. This formulation was used to solve a numerical example and the properties of the equilibrium solution were discussed.

The numerical example showed that the firms should vary their prices with time in order to obtain the best possible revenues at equilibrium. In addition, there are instances in which it is advantageous for the transporters to allow some of their service to backlog. Finally, it is apparent from the wide range of pricing strategies that the equilibrium strategy is highly dependent upon the parameters of the cost and demand functions. These conclusions have been made from a relatively simple numerical model of the agents in urban freight environments. Extensions to this model may unveil other insights into the equilibrium strategies of these agents.

There are many possible extensions that could be made to the model presented here. The receivers could be explicitly modeled as competing agents with decisions about the timing and quantities of demands. Another clear extension is the introduction of stochasticity to the receiver demands which would result in a stochastic DVI representation of the game. In addition, other types of stochasticity may be introduced such as uncertain transportation costs for the transportation firms. Yet another extension would be to consider multiple modes of transportation for the transporters with considerations for full and partial loads. Clearly, there is much more research that could be accomplished by extending the basic model presented in this thesis.

Bibliography

[1] Altman, E., Wynter, L., 2004. Equilibrium, games, and pricing in transportation and telecommunication networks. *Networks and Spatial Economics* 4(1), 7-21.

[2] Bernstein, F., Federgruen, A., 2004. Dynamic inventory and pricing models for competing retailers. *Naval Research Logistics* 51(2), 713-746.

[3] Bernstein, F., Federgruen, A., 2003. Pricing and replenishment strategies in a distribution system with competing retailers. *Operations Research* 51(3), 409-426.

[4] Cachon, G., Harker, P., 2002. Competition and outsourcing with scale economies. *Management Science* 48(10), 1314-1333.

[5] Chen, Y.F., Yan, H., Yao, L., 2004. Newsvendor pricing game. *IEEE Transactions on Systems, Man and Cybernetics - Part A* 34(4), 450-456.

[6] Friesz, T.L., Rigdon, M.A., Mookherjee, R., 2006. Differential variational inequalities and shipper dynamic oligopolistic network competition. *Transportation Research Part B* 40(6), 480-503.

[7] Friesz, T., Gottfried, J., Morlok, E., 1986. A sequential shipper-carrier network model for predicting freight flows. *Transportation Science* 20(2), 80-91.

[8] Harker, P., 1991. Generalized nash games and quasivariational inequalities. *European Journal of Operations Research* 54, 81-94.

[9] Harker, P., Friesz, T., 1986a. Prediction of intercity freight flows, I: Theory. *Transportation Research Part B* 20(2), 139-153.

[10] Harker, P., Friesz, T., 1986b.Prediction of intercity freight flows, II: Mathematical formulations. *Transportation Research Part B* 20(2), 155-174.

[11] Holguín-Veras, J., Silas, M., Polimeni, J., Cruz, B., 2006a. An investigation on the effectiveness of joint receiver-carrier policies to increase truck traffic in the off-peak hours. Part I: The behavior of receivers. *Networks and Spatial Economics*, in press.

[12] Holguín-Veras, J., Silas, M., Polimeni, J., Cruz, B., 2006b. An investigation on the effectiveness of joint receiver-carrier policies to increase truck traffic in the off-peak hours. Part II: The behavior of carriers. *Networks and Spatial Economics*, in press.

[13] Holguín-Veras, J., Wang, Q., Xu, N., Ozbay, K., Cetin, M., Polimeni, J., 2006c. The Impacts of Time of Day Pricing on the Behavior of Freight Carriers in a Congested Urban Area: Implications to Road Pricing. *Transportation Research Part A* 40(9), 744-766.

[14] Lederer, P.J., 2003. Competitive delivered spatial pricing. *Networks and Spatial Economics* 3(4), 421-439.

[15] Mangasarian, O.L., 1969. *Nonlinear Programming*, McGraw-Hill.

[16] Pang, J.-S., 2007. Computing generalized nash equilibria. *Mathematical Programming Series A*, in revision.

[17] Pang, J.-S., Hobbs, B.F., 2004. Spatial oligopolistic equilibria with arbitrage, shared resources, and price function conjectures. *Mathematical Programming Series B* 101, 57-94.

[18] Pang, J.-S., Facchinei, F., 2003. *Finite-Dimensional Variational Inequalities and complementarity Problems*, Springer-Verlag.

[19] Zhang, P., Peeta, S., Friesz, T.L., 2005. Dynamic game theoretic model of multi-layer infrastructure networks. *Networks and Spatial Economics* 5(2), 147-178.