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STUDENT TEACHERS’ CONCEPTIONS OF PROOF AND FACILITATION OF ARGUMENTATION IN SECONDARY MATHEMATICS CLASSROOMS

A Thesis in
Curriculum and Instruction

by

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This multi-case study investigates the relationship between a student teacher’s conception of proof and how he or she facilitates collective argumentation in a secondary mathematics classroom. In addition to classroom artifacts, interviews and observations in prospective secondary mathematics teachers’ algebra, geometry, and calculus classes provide the data. Each participant’s support for argumentation was analyzed using Krummheuer’s adaptation of Toulmin’s model for argumentation. Using interviews and analysis methods based on current research on proof and proving, conceptions of proof were analyzed for each participant. Patterns in each participant’s support for argumentation were found to align with his or her conception of proof. In a cross-case analysis, participants’ support for argumentation was found to align most strongly with one aspect of their conceptions of proof: their perceptions of the purpose and role of proof in mathematics. While similarities in their support for argumentation were noted, their perceptions of the purpose and need for proof differed from each other: Karis saw proving as important for explaining the mathematics, Lynn saw proving as important for her to know the reason why things worked, and Jared saw proving as important to know how to do things. These perceptions were found to be aligned with the patterns of argumentation observed in participants’ classrooms. Karis provided most of the warrants for her students as she explained the mathematics to her students. Lynn’s view of proving as personally important to her and her students is evident in her asking students to be involved in contributing many claims, data, and warrants. Jared’s use and acceptance of
primarily rules and procedures as warrants in his classroom is consistent with his view of proof as important for knowing how to do things.
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Chapter 1

Rationale

Mathematical proof is one of the major themes of university mathematics courses, yet it is not prevalent in many secondary mathematics courses. Many advanced university mathematics classes are taught in the definition–theorem–proof tradition (Thurston, 1995). Even in courses not taught strictly in this tradition, mathematical proof plays an important role, reflective of the importance of proof in the discipline of mathematics. Proof is the way in which the validity of mathematical claims is established; as such, it is the cornerstone of mathematical activity. Many university programs now employ a “transition to proof” course to ease students’ transition from the calculus sequence to more advanced courses in which proof plays a pivotal role.

Proving and justifying has been identified as an important mathematical process by the National Council of Teachers of Mathematics (NCTM, 2000). However, it is questionable how often formal proof may be seen in secondary mathematics classrooms outside geometry or courses for “more advanced” students. For example, Knuth (2002a) found that the teachers in his study thought proof was only appropriate in upper-level classes for advanced students and conceived of proof as a topic in mathematics rather than a tool for doing mathematics. The relative difference in emphasis on proof and proving at the secondary and college levels despite its recognized importance raises a question regarding how a secondary mathematics teacher’s conception of proof and proving influences his or her teaching practice.
NCTM’s Principles and Standards for School Mathematics (2000), in addition to a standard for reasoning and proof, contains a standard entitled “Communication.” Reading the short summaries of these standards draws attention to the fact that reasoning and proving and communicating mathematically are intertwined in a classroom environment. Many of the examples in the communication standard involve students reasoning and proving or justifying answers and solution methods. Defending one’s answer and articulating one’s reasoning to others form a large part of the explanation offered in the “Reasoning and Proof” standard (NCTM, 2000). The concept of collective argumentation, as described by Krummheuer (1995), seems to enfold much of what the authors intended with these two standards. Krummheuer emphasizes the interactive nature of classroom argumentation, calling it collective argumentation to distinguish it from the more traditional Aristotelian argumentation that focuses on an individual convincing a group of listeners.

**Definitions**

Although collective argumentation captures the essence of NCTM’s (2000) Reasoning and Proof and Communication standards, argumentation, justification, and proof are not synonymous. For the purposes of this study, a mathematical proof (or a proof) will be defined as a logically correct deductive argument built up from given conditions, definitions, and theorems within an axiom system. The axiom system may or may not be apparent to the person who is building the argument. Proving, then, is the process of constructing or attempting to construct a deductive argument. The purpose of
proving could be convincing, explaining, or ensuring certainty (Hanna, 1990; Hersh, 1993). A person’s conception of proof includes his or her ability to prove and analyze proofs and his or her beliefs about proof. This is elaborated in Chapter 2. Justifying is giving reasons for a mathematical action or statement “in an attempt to communicate the legitimacy of one’s mathematical activity” (Yackel & Hanna, 2003, p. 229). Justifying includes proving and is not limited to proving. A person may be said to engage in the process of justification when he or she is justifying. Krummheuer defines collective argumentation as “a social phenomenon, when cooperating individuals tried to adjust their intentions and interpretations by verbally presenting the rationale of their actions” (Krummheuer, 1995, p. 229). Within argumentation, as conceptualized by Toulmin (1964), claims, data, warrants, and backings are provided in an attempt to convince an audience of the validity of a particular claim. Justifying often occurs within argumentation as participants reveal the data and warrants that support their claims. Argumentation includes justifying, and justifying includes proving, thus a proof is a particular kind of argument, as indicated by the definition above.

**Impact of Argumentation on Student Learning**

Collective argumentation occurs when participants in a discussion act in such a way as to work together in the construction of an argument. The actions involved in collective argumentation occur within the larger phenomenon of classroom discourse. Researchers in mathematics and science education (e.g., Erduran, Simon, & Osborne, 2004; Hiebert & Wearne, 1993; McNeal & Simon, 2000; Yackel & Cobb, 1996) have
found that classroom discourse practices impact students’ learning. In addition, Mewborn’s (2003) review of literature on teaching gives evidence that teachers’ actions influence students’ learning. One way, then, that teachers may influence students’ learning is through their actions with regard to classroom discourse, in particular, through initiating the negotiation of norms with regard to collective argumentation. Cobb (2002) calls this aspect of classroom mathematical practice “normative standards of argumentation” (p. 189).

A perspective on learning that views interaction as important within the learning process underlies this study. It is in this context that the importance of studying argumentation, and particularly the teacher’s role in facilitating collective argumentation becomes apparent. My perspective on learning aligns with Cobb and Yackel’s (1996) description of the “emergent perspective.” The basic premise of this perspective is that learning occurs in individuals through reflection, but an individual’s learning may be affected as he or she reflects on his or her understanding of what is taken-as-shared within the classroom or on his or her interpretation of interactions in the classroom. Some of what is taken-as-shared in a mathematics classroom may be the claims, data, and warrants that are contributed by students or the teacher. In addition, warrants or data that are taken-as-shared may be left implicit (not specified) by members of the classroom community.

The emergent perspective is a theory of learning that includes both social and individual dimensions. According to the emergent perspective, individuals construct their own understandings, but these understandings are not directly observable. Individuals contributing to a classroom community help to create understandings that are “taken-as-
shared” by the community (Cobb, Yackel, & Wood, 1993). The discourse by which this occurs is observable. The process by which understandings come to be taken-as-shared is negotiation, which is explicit only when a participant in the discussion realizes that there is not a fit between an explanation and his or her understanding of the idea being discussed (Cobb, Yackel, & Wood, 1993). An important component of this theory is reflexivity, in that individual understandings and beliefs constrain taken-as-shared understandings and acceptable or normative mathematical behaviors, and these classroom social and sociomathematical norms constrain individual understandings and beliefs (Yackel & Cobb, 1996), that is, “individual students’ mathematical activity and the classroom microculture are reflexively related” (Cobb & Bauersfeld, 1995). In the current study, the individual understandings and beliefs of most interest are those of the teacher as a member of the classroom community. The teacher’s conception of proof is part of his or her beliefs about mathematics that may constrain norms related to argumentation in the classroom, while the taken-as-shared understandings of his or her students may constrain his or her conception of proof or other beliefs about mathematics, particularly school mathematics.

**Connecting Proof and Argumentation**

Investigating classroom argumentation involves investigating what normative standards of argumentation have been and continue to be negotiated in a classroom (Cobb, 2002). A normative standard of argumentation is a norm observed within one part of the mathematical activity in a classroom: the collective argumentation surrounding
mathematical ideas. According to Cobb, a norm “refers to patterns in collective activity within a classroom” (2002, p. 190). Norms are not established but are “interactively constituted by the teacher and students” (Cobb, 2002, p. 190), making the teacher’s role crucial in the negotiation of norms (McClain, 2002). Of particular interest when considering normative standards of argumentation are sociomathematical norms. Sociomathematical norms include, among other things, what has been established in a classroom as an acceptable explanation or justification (McClain, 2002; Yackel & Cobb, 1996).

When negotiating sociomathematical norms, including normative standards of argumentation, the teacher may act as a representative of the external mathematical community within the classroom mathematical community (Yackel & Cobb, 1996). In an ideal setting, when the teacher takes on the role of a representative of the mathematical community in negotiating normative standards of argumentation in a mathematics classroom, he or she would model and support ways of arguing that are consistent with how knowledge is constructed in the larger mathematical community. In other words, since knowledge construction in the field of mathematics is dependent on proof, a teacher who is acting as a representative of the mathematical community would attempt to, in a way that is approachable by his or her students, facilitate argumentation that is consistent with acceptable mathematical justification that could eventually lead to proof.

It is probably not the case that every teacher consciously acts as a representative of a larger mathematical community in his or her classroom practice. Some teachers may not think of themselves in this way, and others, while they may consider themselves to be part of the larger mathematical community, may be influenced by a host of other factors,
such as organizational and assessment issues and conceptions of teaching and learning (see, e.g., Ball, Lubienski, & Mewborn, 2001).

In addition, the teacher who acts as a representative of the larger mathematical community would not necessarily require every argument be a deductive proof. Instead, he or she would support and model data, warrants, and backings for claims that are acceptable in the field of mathematics (Wood, 1999). However, all of this is mediated by the teacher’s understanding of justification and proof, as this is part of his or her conception of proof, which is part of his or her conception of mathematics. That is, the teacher represents only a mathematical community that is consistent with his or her conceptions of and experience with mathematics, because he or she is not able to represent a community of which he or she is not aware or with which he or she has no experience. Thus, the data, warrants, and backings for claims that he or she supports or models within the classroom may tend to align with the teacher’s conception of proof and justification, moderated by other personal, classroom, and institutional factors.

**Research Questions**

Acknowledging that there is much to be studied about the interaction of teachers’ conceptions of mathematics, conceptions of proof, classroom practice, and collective classroom argumentation, this study will address one small part of the larger picture by answering the following questions.
• How do prospective secondary mathematics teachers support claims, data, warrants, and backings as elements of argumentation in secondary mathematics classrooms?

• What characterizes the relationship between the argumentation observed in a particular classroom and the prospective secondary mathematics teacher’s conception of proof and justification?

These questions require the investigation of individual prospective mathematics teachers’ conceptions of proof and support for argumentation and then the coordination of observations across the conceptions and classroom practice of the teachers.

Overview of Study

This study is designed as a collective case study in which three individual prospective mathematics teachers’ conceptions of proof and support for argumentation are reported. Through the analysis of data sources, including interviews, classroom observations, and collected artifacts, the collective argumentation in each classroom and each student teacher’s conception of proof is described. Krummheuer’s (1995) adaptation of Toulmin’s (1964) conceptualization of argumentation to mathematics education is used as a way to analyze the structure of the observed collective classroom argumentation. The degree to which each teacher’s conception of proof is aligned with his or her support for argumentation is explored, and the similarities and differences between the conclusions for each teacher are described in a cross-case synthesis of the form described by Yin (2003).
The analysis of the individual and collective cases provides evidence for a relationship between these prospective mathematics teachers’ conceptions of proof, particularly their views of the purpose and need for proof, and the support for argumentation that was observed in their classrooms. A person’s view of the purpose and need for proof in mathematics had been described as a theoretically important part of his or her conception of proof (Hanna, 1995, 2000; Harel & Sowder, 1998; Hersh, 1993; Thurston, 1995); this study provides empirical evidence for the usefulness of this construct in the practice of these student teachers. In addition, in this study, the role of the teacher in orchestrating argumentation and the use of explicit or implicit warrants were found to be pivotal in understanding each student teacher’s support for argumentation, even in a situation where the student teachers were not explicitly attempting to facilitate particular argumentation in their classrooms, although the teacher’s role in orchestrating argumentation and his or her provision of warrants differed from what has been reported by Cobb (1999), Yackel (2002), and others (e.g., Forman & Ansell, 2002; Whitenack & Knipping, 2002). While there are clearly other factors that affect a teacher’s classroom practice and the argumentation that occurs within that practice, it seems clear from the evidence presented in this study that a student teacher’s conception of proof plays a role in his or her support for argumentation.

The next chapter provides a theoretical and empirical grounding for this study, expanding on ideas introduced in this chapter. After the methodology for the multi-case study is explained in Chapter 3, each of the next three chapters presents the case of one of
the participants, Karis, Lynn, and Jared\textsuperscript{1}. The cross-case synthesis with answers to the research questions is presented in Chapter 7, with implications in Chapter 8.

\textsuperscript{1} All names are pseudonyms to protect the identity of participants.
Chapter 2

Conceptual and Theoretical Grounding

This chapter situates the current study using previous empirical and theoretical work in mathematics education. In this chapter I describe three aspects of a conception of proof and how those aspects of a conception of proof are derived from previous research on proof and justification. I then describe what is meant by argumentation, how Toulmin (1964) conceptualized it, and how Krummheuer (1995) adapted Toulmin’s model of argumentation for use in mathematics education research. Finally, I describe other factors that might influence argumentation in a student teacher’s classroom, drawing from research on novice teachers and classroom practice.

Conception of Proof

I think of a conception of proof as a multi-faceted construct that includes a person’s ability to prove and analyze proofs and his or her beliefs about proof and justification. A person’s beliefs about proof and justification include his or her beliefs about the necessity of proof in mathematics and his or her beliefs about his or her ability to prove. The vocabulary currently in use by researchers in this area includes several terms with overlapping meanings. Using that vocabulary, a learner’s conception of proof and justification is made up of his or her proof schemes; his or her ability to construct a deductive proof; his or her implicit or explicit beliefs about the necessity, role and function of proof in mathematics; and his or her perceptions of proof. To avoid
overlapping ideas and to provide focus, the three aspects of a conception of proof that I investigated within this study are ability to prove and analyze arguments, perception of the purpose and need for proof in mathematics, and, to a lesser extent, affective perception of proof. At the beginning of this study, I conceptualized these three aspects of a conception of proof as the axes in a three-dimensional coordinate system as shown in Figure 2-1. I see the three aspects as interrelated, although they can be investigated independently to some extent. In the next three sections, I will describe how each aspect of a conception of proof arises from empirical and theoretical work on proof and justification.

Figure 2-1: Three Aspects of a Conception of Proof
Ability to Prove and Analyze Proofs

Much of the research on proof and proving has focused on learners’ abilities to prove and analyze proofs. These studies have focused on what convinces students of the truth of a mathematical statement (e.g., Harel & Sowder, 1998), what makes proving difficult for students (e.g., Moore, 1994), and how students validate texts as proofs (e.g., Selden & Selden, 2003). The research questions in Chapter 1 as well as the interview tasks and their analyses as described in Chapter 3 for the current study were based on a synthesis of findings from studies in these categories.

Proof: What Is Convincing?

Rodd (2000) reminds us that a person may be convinced by something other than a proof, and that a formal, deductive proof does not serve to convince everyone. One of the main studies in this area (Harel & Sowder, 1998) categorizes students’ proof attempts according to what evidence convinced them of the truth of mathematical statements. Harel and Sowder, and others following them (e.g., Housman & Porter, 2003; Recio & Godino, 2001), call a student’s source of conviction his or her “proof scheme” (Harel & Sowder, 1998, p. 244). In their landmark study, Harel and Sowder classified number theory, college geometry, linear algebra, and advanced linear algebra students’ sources of conviction into one or more sub-categories of empirical, external, or analytical proof schemes. They hypothesize that an empirical proof scheme is a natural extension of everyday reasoning and that the inductive reasoning that characterizes students with this proof scheme is a necessary part of learning mathematics. However, they state that
students need to come to understand that an inductive argument (such as a perceived pattern) does not prove a mathematical statement conclusively.

Some of the secondary teachers in Knuth’s (2002a, 2002b) study, while claiming that a proof was a deductive argument, were convinced by inductive arguments. He found that their decisions about what convinced them related “more to form than to substance” (Knuth, 2002a, p. 402). Knuth’s study of secondary mathematics teachers’ conceptions of proof, along with the work of Harel and Sowder and others (e.g., Housman & Porter, 2003) following them, formed the basis for part of the design of the first interview schedule for the current study. For instance, Knuth (2002b) found that questions like “What purpose does proof serve in mathematics?” (p. 83) and “What constitutes proof?” (p. 83) contributed to interesting discussions with teachers, especially when the teachers in his study were also asked to evaluate the validity of arguments from various areas of mathematics as proofs. The three participants in the current study were asked questions similar to Knuth’s (2002b) suggestions and were asked to evaluate three different arguments for the same mathematical claim in a way that was similar to one of the tasks in Knuth’s (2002b) study. Knuth found that teachers were often most convinced by arguments that were not proofs, and Harel and Sowder (1998) focused on differentiating students’ work with proof based on what kind of argument convinced them. In response to these findings, participants in the current study were also asked how convinced they were by several arguments. In this way each participant’s view of proof as a convincing argument could be hypothesized.
Proving: Difficult for Many

It has long been established that proving is difficult for students (see, e.g., Almeida, 1995; Bell, 1976; Moore, 1994; Recio & Godino, 2001). A variety of reasons for this difficulty have been investigated, including: their understanding (or lack thereof) of relevant mathematical concepts (Almeida, 1995; Chazan, 1993; Moore, 1994; Weber, 2001), a lack of knowledge of the principles of logic (Hazzan & Leron, 1996; Knuth, 2002a; Selden & Selden, 2003; Stylianides, Stylianides, & Philippou, 2004), an understanding of proof based on different institutional experiences with proof (Recio & Godino, 2001), and having a view of proof that differs from an ideal or practicing mathematician’s perspective (Almeida, 1995; Chazan, 1993). The three most pertinent to the difficulty in constructing and analyzing proofs for the purposes of the current study are knowledge of relevant mathematical concepts, knowledge of the principles of logic, and different institutional meanings of proof. A person’s view of proof and how that relates to an ideal or practicing mathematician’s view is discussed in the section that deals with the perception of the purpose and need for proof.

Moore (1994) and Weber (2001) each observed undergraduate students as they attempted to construct proofs. Moore’s students were in an introduction to proof class that involved topics such as mathematical logic, methods of proof, mathematical induction, elementary set theory, relations and functions, and the real number system. Weber’s participants attempted proofs in the context of abstract algebra, although they were not simultaneously enrolled in a formal abstract algebra class. Both Moore and Weber concluded that one reason students were unsuccessful in generating correct proofs
was that they did not possess the mathematical knowledge necessary for the proof or they
did not understand how to use the mathematical knowledge that they did have. For
instance, some students in Moore’s study were unable to prove mathematical statements
because they did not know relevant definitions or did not understand the concepts or
theorems needed to prove the statement. Some participants in Weber’s study could recall
the mathematical ideas necessary to prove a statement but did not recognize which ideas
were relevant or were unable to use these ideas in constructing a proof.

Part of the design of the analysis of argument tasks in this study was to have
available for students statements of relevant theorems with which they may not have been
familiar. This was accomplished by preparing a written statement of one theorem
beforehand and being prepared to offer other relevant definitions and theorems verbally if
asked by a participant. In this way, a participant’s lack of familiarity with relevant
mathematical concepts might not hinder a participant’s analysis of an argument.

Recio and Godino (2001) examined university students’ proof schemes and
difficulties with proof in arithmetic and geometry contexts and related these to what they
call different institutional meanings of proof. That is, students encounter different forms
of arguments in settings such as “daily life, experimental science classes, mathematics
classes, philosophy and logic classes” (p. 96), and Recio and Godino conclude that
students may have difficulty in differentiating when different types of argumentation, that
may be appropriate in different settings, should be used. In particular, students may
employ arguments appropriate to other institutional settings when asked to construct a
mathematical proof.
One of the things I looked for in participants’ statements was whether they talked about proof and proving in the discipline of mathematics and whether they referenced other institutional settings such as daily life or, in particular, school mathematics, when talking about proof. Recio and Godino’s (2001) finding concerning the different types of argumentation is related to Toulmin’s (1964) construct of a field of argument, which will be discussed later.

Analyzing Arguments: Validating Texts as Proofs

Selden and Selden (2003), in a number theory context, and Stylianides, Stylianides, and Philippou (2004), in an algebraic context and a non-mathematical context, found that students’ difficulties with or lack of knowledge of logic contributed to their incorrect validations of texts as proofs. In addition, several studies have shown that students have difficulty with understanding such principles of logic as logical implication (Hoyles & Kuchemann, 2002, in contexts of number, algebra, and geometry) and the appropriate formulation and use of a theorem (Lagrange’s Theorem in abstract algebra) and its converse (Hazzan & Leron, 1996). Selden and Selden found that students validated a text that proved the converse of a theorem as a proof of the theorem because each algebraic step of the “proof” was correct. Stylianides, Stylianides, and Philippou found that many students rejected a proof that used the law of contraposition and, when asked, stated that the correct equivalence to $p \rightarrow q$ is $\sim p \rightarrow \sim q$. From these studies, it

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2 The phrases “validate a text” and “evaluate (or analyze) an argument” are used synonymously in this study to mean the process of reading a given set of prose whose purported intention is to show that some statement is true and determining if indeed that set of prose is a proof or if it convinces the reader.
seems that distinguishing between the converse and contrapositive of a statement is
important for students to be able to analyze arguments effectively. Galbraith (1981)
points out another facet of knowledge of logic that is important for proving: the strength
of a counterexample. He suggests that some students do not believe that one example can
refute a general statement or they do not know how to construct or recognize a
counterexample. In addition, Knuth (2002a) found that secondary mathematics teachers
in his study identified non-proofs as proofs based on the form of the argument or whether
it promoted understanding.

One proof that students were asked to analyze (validate) in the current study
required them to recognize a proof by contradiction that included a correctly negated
statement. Another argument conformed to what some might think was the form of a
proof by referencing two cases, but did not, in fact, prove the statement. These choices
were made based on the findings of the researchers above that knowledge of the
principles of logic is necessary to be able to validate a proof, and that the form of an
argument might also be influential in the analysis of an argument.

Perception of the Purpose and Need for Proof

Some authors suggest that proof may have several different roles or functions
within mathematics and mathematics education. Hanna (1995, 2000) suggests that, in the
classroom, the role of proof is primarily to explain and to promote understanding. Hersh
(1993) claims that the primary role of proof in mathematics is convincing, while the
primary role of proof in education is explaining. Thurston (1995) suggests that one of the
roles of proof in mathematics is ensuring certainty. Various empirical studies have found that learners’ views of proof include these different roles. In a study of pre-service teachers’ backgrounds, beliefs, and attitudes about proof, Mingus and Grassl (1999) found that many of their participants believed the role of proof was to explain how and why concepts work, quoting one participant as saying, “It is easier to remember and understand how to do math if you know why it works and explore why it works for yourself instead of having someone just tell you how to do it” (p. 441). Coe and Ruthven (1994) found that their students, who were “towards the end of their first year in a sixth-form college” (p. 41), saw proof as a way to ensure certainty after completing a unit involving problem solving and proof.

Several authors theorize that students do not perceive a need for proof in mathematics. Dreyfus and Hadas (1996) discuss students’ lack of a perceived need for proof in the introduction of their study. Hanna (1995) suggests that one role of a mathematics teacher should be to emphasize the need for proof to his or her students. Interviews with participants in two studies confirm that some teachers and students do not believe proof is integral to mathematics: in Knuth’s (2000a, 2000b) study, the secondary mathematics teachers viewed proof as a topic in the curriculum rather than “an essential tool for studying and communicating mathematics” (Knuth, 2002a, p. 84), and some of Almeida’s (2000) participants, in his study of proof perceptions, viewed proof as important only for passing mathematics examinations.

Participants in the current study were asked questions designed to elicit their view of the role and function of proof in mathematics. In particular, participants were asked to
give important aspects of mathematics and then to talk about how important proof is in mathematics.

**Affective Perception of Proof**

Several researchers have investigated participants’ perceptions of proof (Almeida, 1995; Coe & Ruthven, 1994; Knuth, 2002b). These beliefs about proof and proving included whether they liked or disliked proof (Almeida, 1995), a belief that they should engage in proving only to pass examinations (Almeida, 1995), a belief that proof in secondary school was only appropriate for a limited number of advanced students (Knuth, 2002b), and a preference for empirical verification and seeing whether methods worked rather than why they worked (Coe & Ruthven, 1994).

Participants in the current study were asked to describe successful and unsuccessful experiences with proving for several reasons, one of which was to investigate their perceptions of proof. In addition, participants’ affective reaction when presented with proof tasks was observed, and their affect was taken into consideration as they proved and analyzed arguments.

**Perspective on Argumentation**

While a proof is an argument, argumentation has a much broader focus, particularly outside the discipline of mathematics. Argumentation is a major focus of study in rhetoric and communication theory. There are many definitions of and
perspectives on argumentation within communication theory. However, in mathematics education, one particular model of argumentation has been used to examine the argumentation that occurs within mathematics classrooms. This model is based on the work of Stephen Toulmin, a philosopher with an undergraduate degree in mathematics. In his 1964 book, *The Uses of Argument*, Toulmin introduced constructs of claim, data, warrant, and backing, which became popular and have been adopted into common usage in texts on communication (Casagrande, 1977). Toulmin’s model will be briefly described below, and then its application to research in mathematics education and the current study will be explained.

**Toulmin’s Model of Argumentation**

Toulmin describes a claim as a “conclusion whose merits we are seeking to establish” (p. 97). A simple example of a claim in a mathematics classroom is a student’s statement that the solution set to the quadratic equation $x^2 - 5x + 6 = 0$ is $\{2,3\}$. In general, an argument can be identified by the presence of a claim; one must have something for which to argue.

Toulmin defines data as “the facts we appeal to as a foundation for the claim” (p. 97). In the case of our quadratic equation, the student might cite the following data: $x^2 - 5x + 6 = (x-3)(x-2)$, if $x-3 = 0$ then $x = 3$, and if $x-2 = 0$ then $x = 2$. These data might be accepted by the audience to which the argument is made, or the audience may question the validity of the data. If questioned, the person making the argument would need to provide an argument for the validity of the data itself, thus engaging in a
sub-argument, or what Toulmin calls a “preliminary argument” or “lemma” (p. 97). A sub-argument, then, is an argument that supports the validity of some part of the data or warrant of an argument. I chose to use “sub-argument” rather than “preliminary argument” or “lemma” to emphasize that, in school mathematics, these arguments for the validity of the data or warrant are buried within the episode of argumentation. “Preliminary argument” gives the impression that the argument occurred before the main argument, when, in fact, sub-arguments often occur within other arguments.

According to Toulmin, warrants are “general, hypothetical statements, which can act as bridges, and authorise the sort of step to which our particular argument commits us” (p. 98). Often, data are stated explicitly, while warrants are left implicit unless specifically asked for or challenged (Toulmin, 1964). There are several warrants that the student in our hypothetical classroom might state if challenged to provide reasons that his or her data relates to the claim. A student might state the zero product property and the knowledge of how to solve a linear equation as warrants connecting the data to the claim. These warrants might be challenged in one of two ways. The validity of the warrant might be challenged, in which case a sub-argument (preliminary argument) would need to be made to support its validity. The relevance of the warrant in the field in which the argument was being made could also be challenged. So, for instance, if a student claimed that the data related to the claim because “that’s the way I learned to do it,” the warrant could be challenged on the grounds that appropriate warrants in mathematics classrooms should have a mathematical basis. This type of challenge to a warrant is really an appeal to the backing of the argument.
The backing of an argument, according to Toulmin, is usually unspecified. Backing is “other assurances without which the warrants themselves would possess neither authority nor currency” (p. 103). In the case of the quadratic equation, there are probably many unspecified backings, but one in particular is that we are working in the real number field and thus the zero product property holds. According to Toulmin, the kinds of warrants and backings that are acceptable for a particular argumentation are dependent upon the field (such as law, mathematics, sports, or art) in which the argumentation is occurring. That is, according to Toulmin, the criteria by which we judge the validity of an argument varies from one field (or discipline) to another. Thus an argument in law is not judged in the same way as an argument in a scientific journal, and the appropriate backings for arguments in these fields are understood but not (usually) stated.

Toulmin also included two other components of argumentation in his model: qualifier and rebuttal. A qualifier “indicates the degree of strength that is conferred by the warrant,” (p. 105). An example of a qualifier is “probably” or “presumably” (p. 106). A rebuttal indicates “circumstances in which the general authority of the warrant would have to be set aside” (p. 105). According to Toulmin, a rebuttal is necessary “in all cases where the application of a law may be subject to exceptions, or where a warrant can be supported by pointing to a general correlation only, and not to an absolutely invariable one” (p. 106). The student teachers and their students in this study acted as if their statements and arguments applied without qualification. Thus, while an awareness of these two components of Toulmin’s model was maintained, qualifiers and rebuttals were not found to be useful in characterizing argumentation in these classrooms. This is
consistent with Krummheuer’s (1995) adaptation of Toulmin’s model to mathematics education, as well as how this model is commonly used within mathematics education (see, e.g., Cobb, Stephan, McClain, & Gravemeijer, 2001; Forman, Larreamendy-Joerns, Stein, & Brown, 1998; Whitenack & Knipping, 2002; Yackel, 2002). However, it is reasonable that within collective argumentation in mathematics classes, these two components could be found to be useful (see Inglis, Mejia-Ramos, & Simpson, in press; Lakatos, 1976).

**Krummheuer’s Translation of Toulmin’s Constructs**

In his “Ethnography of Argumentation,” Krummheuer (1995) translated (relevant parts of) Toulmin’s (1964) model of argumentation into terms that he used to analyze argumentation in an elementary mathematics classroom. In collective argumentation such as that described by Krummheuer, the participants and the setting determine what are acceptable data, warrants, and backing. Krummheuer seems to have taken Toulmin’s conception of field and, using a social lens, in a sense both narrowed and expanded it. In one sense, since Krummheuer discusses argumentation in mathematics classrooms only, he is narrowly defining the field in which he studies argumentation as mathematics classrooms. This implies that what is accepted as backing for an argument need only be valid in the particular mathematics classroom in question. This narrow definition of field actually provides for a possible broadening of which warrants and backings are acceptable in that, depending on the classrooms, these may not be limited to what is accepted in the external mathematics community. Instead, these acceptable warrants and
backings may be negotiated as part of the sociomathematical norms of a classroom community.

Researchers have used argumentation, and particularly Krummheuer’s (1995) adaptation of Toulmin’s (1964) model of argumentation, to examine several aspects of teaching and learning in mathematics classrooms: the role of the teacher (Cobb, 1999; McCrone, 2005; Wood, 1999), the importance of warrants as components of argumentation (Yackel, 2002), and the interactive nature of learning (Krummheuer, 2000; Yackel, Cobb, & Wood, 1999; Zack & Graves, 2001). The role the teacher plays in argumentation and negotiation of classroom norms has emerged as an important aspect on which to focus when examining teaching and learning in mathematics classrooms.

Yackel and Cobb (1996) examined the teacher’s role in argumentation and propose that part of that role in argumentation is to serve as a representative of the mathematical community within the classroom. In addition, they suggest that both social and sociomathematical norms regulate argumentation in mathematics classrooms, and that students may become more autonomous learners of mathematics as they participate in the negotiation of sociomathematical norms. Cobb (1999) describes the key role of the teacher in structuring discussions in ways that capitalize on “students’ diverse ways of participating” (p. 35). Yackel (2002) suggests that teachers, in addition to fostering the negotiation of appropriate norms, may act to support students’ interactions within argumentation or to provide support for various implicit parts of arguments. These studies served to inform my observations and analysis of classroom argumentation. In particular, Yackel’s suggestion that teachers support argumentation in various ways, including providing support for implicit parts of arguments provided guidance for
thinking about what happens in arguments in which components are not (verbally) specified.

While the research thus far on collective argumentation has not specifically addressed the role of the teacher’s conception of proof, two studies have specifically mentioned the teacher’s need for detailed knowledge of the mathematical subject matter they are teaching (O'Connor, 2001; Yackel, 2002). One reason this knowledge is needed is the role of the teacher as a representative of the mathematical community in collective argumentation (Yackel & Cobb, 1996). The teacher is responsible for negotiating with his or her students the kinds of warrants and backings that are acceptable in the classroom, possibly through the negotiation of sociomathematical norms (Yackel & Cobb, 1996), as well as providing examples and counter-examples when necessary (O'Connor, 2001). This suggests that a teacher’s conception of counter-example (part of his or her conception of proving) and a teacher’s understanding of the kinds of warrants and backings that are acceptable for proofs in mathematics are important aspects of his or her successful facilitation of collective argumentation in a mathematics classroom.

Possible Intervening Factors on Classroom Practice

It is important to consider what factors beyond his or her conception of proof might influence a teacher’s actions with regard to collective classroom argumentation. The act of teaching is a complex endeavor, perhaps even more complex when the teacher is a student teacher, because many of the details involved in classroom practice have not become routinized and are matters of conscious concern for student teachers. There are
many possible factors that might influence a teacher’s practice (and thus his or her support for argumentation). Thus investigating the relationship between a student teacher’s conception of proof and any one aspect of his or her practice is not straightforward.

Several studies suggest that teachers’ knowledge of mathematics and students influences their practice (Borko, Eisenhart, Brown, Jones, & Agard, 1992; Cavey, Whitenack, & Lovin, 2007; Chinnappan & Lawson, 2005; Even, 1993; Even & Tirosh, 1995; Fernandez, 1997; Hill, Rowan, & Ball, 2005; Kahan, Cooper, & Bethea, 2003; Leinhardt, 1986; Stein, Baxter, & Leinhardt, 1990; Swafford, Jones, & Thornton, 1997). For instance, Stein, Baxter, and Leinhardt suggest that the depth of a teacher’s knowledge may influence his or her presentation of the subject matter, including whether or not he or she lays the groundwork appropriate for future learning and whether he or she makes appropriate connections with other mathematical ideas or representations. Swafford, Jones, and Thornton observed changes in teachers’ practice after professional development and attributed these changes to increased subject matter knowledge and knowledge of student cognition. Hill, Rowan, and Ball measured elementary teachers’ mathematical knowledge (using a direct, researcher-constructed measure of mathematical knowledge for teaching) and found that it correlated with student achievement. This finding is different from several earlier studies that suggested that teacher knowledge and student performance was not correlated (e.g., Eisenberg, 1977). Cavey, Whitenack, and Lovin suggest that the knowledge a teacher needs for practice extends beyond the content students are learning, providing an example from a teacher’s practice where she was teaching the concept of slope and drew on her knowledge of congruency and equivalence
classes of ratios. Taken together, these studies suggest that a teacher’s subject matter knowledge influences his or her practice, including how content is presented, what questions are asked, what activities are designed and chosen, what student suggestions to follow, what connections are made between ideas and representations, how lesson plans are designed, and, ultimately, what students learn.

In addition to subject matter knowledge, several researchers have investigated the influence of beliefs about mathematics and mathematics teaching on classroom practice. Thompson (1984) suggests that the relationship between teachers’ conceptions of mathematics and mathematics teaching and their classroom practice is complex, but that their conceptions of mathematics are related to their classroom practice. Borko and colleagues (1992) found that a student teacher’s beliefs about mathematics learning, based primarily on her own mathematics learning, influenced her classroom practice. Simmons and colleagues (1999) found that beginning teachers exhibited classroom practices that conflicted with their professed beliefs in their first year, but that their professed beliefs changed by their third year to be in line with their actual classroom practice.

Artzt and Armour-Thomas (1998) focused on teachers’ metacognition, positing a complex relationship between classroom practice and eight components of metacognition, including knowledge, beliefs, goals, planning, monitoring and regulating, and assessing and revising. For some teachers in their study, the components of knowledge, monitoring, and regulating seemed to influence their practice, while for others, knowledge, goals, and beliefs were the primary influences on classroom practice.
Thompson (1992) and Fennema and Franke (1992) argue that a teacher’s knowledge of mathematics and conceptions of mathematics are intertwined and difficult to separate. Literature in both of these areas suggests that teachers’ classroom practice is influenced by their knowledge of and conceptions of mathematics as well as by other environmental factors and their beliefs about mathematics teaching and learning. Especially for student teachers, knowledge of mathematical content and beliefs about mathematics, teaching, and learning are important mediators of classroom practice (Artzt & Armour-Thomas, 1998; Borko, Eisenhart, Brown, Jones, & Agard, 1992). Borko and colleagues (2000) agree, and suggest that even when a student teacher has strong content knowledge, he or she may not be able to enact the “discourse patterns from the university setting” (p. 204) in his or her own classroom.


Teachers’ conceptions of teaching and learning mathematics are not related in a simple cause-and-effect way to their instructional practices. Instead, they suggest a complex relationship, with many sources of influence at work; one such source is the social context in which mathematics teaching takes place with all the constraints it imposes and the opportunities it offers. Embedded in this context are the values, beliefs, and expectations of students, parents, fellow teachers, and administrators; the adopted curriculum; the assessment practices; and the values and philosophical leanings of the educational system at large. (Thompson, 1992, p. 138)
Important influences on student teachers’ classroom practice include elements of their teacher preparation programs, including mathematics methods courses and the influence of mentor teachers. Borko and colleagues (2000), in their study of influences on a student teacher’s practice in an algebra class, found that the student teacher’s mathematics methods course was an important influence on her classroom practice, including her selection of tasks and the structure of tasks and discourse that she wanted to implement. In addition, they suggest that her perspective on proof from her mathematics methods course influenced her practice more than the perspective she learned in her mathematics content courses. However, despite these influences, they found that she was unable to implement the discourse practices that she intended within her classroom.

Borko and colleagues (1992) also suggest that a mathematics methods course influenced how the student teacher in their study taught mathematics, while Frykholm (1999) found that “beliefs and teaching strategies of the cooperating teachers impacted the thinking of the student teachers” (p. 92) in his study. Earlier quantitative studies suggest that education coursework is a better predictor of teaching performance than other measures, including measures of content expertise (Ferguson & Womack, 1993; Guyton & Farokhi, 1987).

From the research reported above, we can see that there are many competing influences on a student teacher’s classroom practice. Some of the strongest influences may be his or her conceptions of mathematics, conceptions of teaching and learning, and knowledge of mathematics. A teacher’s conception of proof, as defined in this study, is part of his or her conception of and knowledge of mathematics. However, there are other factors mentioned in this section whose influences on classroom argumentation are yet to
be considered. The conceptual and theoretical considerations described in this chapter informed the methodology of this study. Details of the methods by which this study was conducted are found in Chapter 3.
Chapter 3
Methodology

This study was designed as a multicase study, as described by Merriam (1998). This is a qualitative design, appropriate for a study whose goals include contributing to an understanding of how teachers’ conceptions of mathematics relate to their classroom practice. In particular, this study uses rich descriptions based on their mathematical work and classroom practice to relate student teachers’ conceptions of proof to their support of classroom argumentation. Each case consists of the relationship between one student teacher’s conception of proof and his or her facilitation of argumentation within his or her classroom. This chapter describes the participants, data collection, and methods of analysis used to build each case as well as for the cross-case synthesis.

Research on teacher knowledge suggests that a qualitative design is appropriate when comparing a characteristic of teachers with an element of their practice. Fennema and Franke (1992) describe several early studies that tried to correlate knowledge of mathematics with student learning and determined that there was a low correlation between them. As Fennema and Franke point out, these studies used number of mathematics courses taken or score on a standardized test as a proxy measure of teacher knowledge. These measures do not account for some of the complexity involved in the knowledge necessary for teaching, nor do they describe the nature of a relationship that might exist between teacher knowledge and student learning. Thus, Fennema and Franke suggest that these measures may not be the best way to study this relationship and
describe several studies that use a qualitative methodology to look at the complex relationship between teacher knowledge and student learning. The examination of student teachers’ proving ability described below does not depend on the number of proof-oriented courses they have taken, but seeks to characterize their proficiency with proof by asking them to prove mathematical statements and to examine arguments and proofs of mathematical statements.

**Selection of Participants**

Participants in this study were three prospective secondary mathematics teachers engaged in their culminating student teaching experience. The choice to study how prospective secondary mathematics teachers rather than inservice teachers facilitate argumentation in classrooms was based on a belief about how teachers learn to facilitate classroom discourse. I contend that facilitation of classroom discourse is something that is learned through experience, and that the relationship between a teacher’s conception of proof and facilitation of argumentation is likely to be more apparent in a prospective secondary mathematics teacher who has recently been involved in college mathematics classes that include an emphasis on proof and who has not had the time to develop habits of discourse that might mask the relationship I intended to investigate in this particular study.

Because reviews of research on teacher education suggest that “cooperating teachers have a powerful influence on the nature of the student teaching experience” (Wilson, Floden, & Ferrini-Mundy, 2001, p. 19), and that there are many conflicting
influences on prospective teachers as they learn to teach (Brown & Borko, 1992), elements of this study design were constructed to account for the influence of the mentor teachers and other factors. For instance, mentor teachers were interviewed and participants were directly asked about the influence of the mentor teacher on task selection and classroom interaction. While student teachers may not be able to fully articulate the influence of their mentor teachers on their practice, the participants in this study were able to articulate the extent to which they chose tasks or activities based on suggestions from their mentor teachers and whether or not they talked with their mentor teacher about classroom interactions. By discussing these potential influences with both the student teacher and mentor teacher, a more complete picture of these influences was obtained. Classroom observations began several weeks after the student teachers started teaching and were conducted over a period of several weeks rather than at any one time during the student teaching experience. This was done so that the student teachers were more comfortable with the students in the classes they were teaching and had time to have already begun to negotiate norms within their classes, thus possibly reducing the impact on this study of a student teacher’s beginning focus on classroom management or other administrative details. As will be described in the next few paragraphs, participants were purposefully selected, with one of the criteria relating to the amount of freedom the student teachers would be given to plan and implement their lessons.

Participants were purposefully selected for this study based on recommendations of previous instructors, recommendations by their student teaching supervisor, and available information about their field placements. According to Merriam (1998, p. 61), “purposeful sampling is based on the assumption that the investigator wants to discover,
understand, and gain insight and therefore must select a sample from which the most can be learned.” A purposeful sample is selected by first outlining a set of criteria and then selecting participants based on those criteria (Merriam, 1998). Important criteria for selection of participants for this study included the following elements: grade level, mathematics courses taught, and timing of student teaching; particular school district in which placement occurred; and level of comfort of student teachers in expressing their thoughts in an interview context. Specific criteria and how the selected participants met those criteria are outlined in the following paragraphs.

The first criterion for participant selection was the grade level at which they were placed for student teaching. Specifically, since little had been written about argumentation in high school mathematics classrooms, this study was designed to add to the existing literature by describing collective argumentation in high school mathematics classrooms. Thus the participants were selected from amongst those who were student teaching at the high school level. All three participants taught high school mathematics for their student teaching placement. Jared taught a primarily ninth grade course, Karis taught a primarily twelfth grade course, and Lynn taught a primarily tenth grade course.

Having a range of areas of mathematics represented was important for the study. In particular, observing a student teacher in a geometry class was considered to be important since geometry is often a class in which students are expected to prove mathematical statements. However, it was important to not observe solely in geometry classes in order to see how a student teacher’s conception of proof might manifest itself outside of a course in which proof is seen as an important part. Thus one criterion for selection of participants was that some but not all should teach in geometry classes, and a
range of mathematics classes should be represented. A secondary criterion was that the classes include an upper-level and a lower-level class. Each of the three participants taught a different mathematics course in his or her student teaching placement. Jared taught a first-year algebra course, Karis taught calculus, and Lynn taught geometry. These placements include a geometry course as well as other courses different from geometry. In addition, the two non-geometry courses differed in their mathematical content as well as level of students, algebra having average to lower ability students and calculus relatively higher ability students (although this was not the highest level calculus class in the school). The algebra students were primarily ninth grade students, as was usual in this district; the highest level calculus class was AP Calculus AB (the class observed was not an advanced placement course).

As a logistical consideration, one criterion was that the participants needed to student teach during the time period in which the data could be collected. All participants were prospective secondary mathematics teachers who completed their student teaching experience during the spring semester of 2006. The participants were selected after student teaching placements had been completed for the semester; they were not given placements based on study criteria but were selected for the study based on the extent to which they met the criteria.

The school district was to be selected based on two criteria: administrators in the district needed to be willing for research to be conducted within the district, and the culture of the district, at least in the particular classrooms in which they were placed, needed to be such that student teachers were given the freedom to plan and enact lessons with minimal structure from their mentors. The experience of the student teaching
supervisor was invaluable in the process of selecting a school district. She recommended a district in which her observations in previous years suggested that the student teachers were free to plan and enact lessons, and this district was also willing for research to be conducted within the particular high school. All three participants had been placed in the same high school within the recommended district.

The participants needed to have high levels of comfort with expressing their thoughts in an interview setting in order for the data collection process by which their conceptions of proof were inferred to be effective. Final selection of participants was accomplished after conversations with a previous instructor and the student teaching supervisor. The three participants were recommended as ones who were likely to be both willing and capable of expressing their thoughts verbally in an interview setting. Each participant was told that the purpose of the study was “to explore the relationship between how prospective teachers think about mathematics and the conversations among teacher and students” (Participant Informed Consent Form).

All three student teachers were completing their final semester of the same teacher preparation program. They were enrolled in the teacher certification option of a program of study leading to a B.S. in Mathematics. In addition to a calculus sequence and courses in differential equations, matrices, and linear algebra, the required mathematics courses for this program included a course in discrete mathematics in which students were introduced to proofs, a concepts of real analysis course, a course in foundations of geometry, and a basic abstract algebra course. Three mathematics education courses were required (two methods of teaching mathematics and one technology-specific mathematics methods course), and one general methods course as well as two earlier practica were
required. All three student teachers maintained an overall grade point average higher than 3.85 on a 4-point scale.

Student teaching placements were made based on information gathered by the student teaching supervisor in an interview with each student teacher. Some factors influencing the placements were preferences of the student teachers with regard to geographic location, desired mentor characteristics (such as a desire to be able to try new things, or someone who was organized), and grade level or area of mathematics preferences. The student teaching supervisor balanced the students’ desires with the available placements. For instance, Karis requested to teach algebra or calculus, and she wanted a mentor who was organized and used technology. Karis did get to teach algebra and calculus, and her mentor was somewhat organized, but he did not often use technology. Lynn was concerned that her mentor be one that was open to letting her try new things and be involved in the day-to-day classroom procedures. She did not express a preference to teach a particular mathematical area, but she did express a geographic preference. Her placement in this school was based primarily on geographic location and the openness of her mentor. Jared preferred to teach algebra over geometry, and wanted to focus on the early years of high school (ninth and tenth grades). He did not express a geographic preference, so he was placed in this district based on his algebra and early high school preferences. Further details about Karis, Lynn, and Jared and each of their classes can be found in Chapters 4, 5, and 6, respectively.
Data Collection and Sources

To answer research questions using a case study methodology requires collection of multiple sources of data for triangulation (Creswell, 1998; Merriam, 1998). Data sources for this study included videotapes of and artifacts from semi-structured interviews of prospective secondary mathematics teachers including both proof tasks and questions about beliefs about mathematics and proof within mathematics; field notes from observations of classes; audio recordings of classroom discourse; notes from interviews of mentor teachers; and lesson plans, tests, quizzes, and worksheets created by the student teachers (see Table 3-1 below).
Table 3-1: Sources of Data and Intended Use in Study

<table>
<thead>
<tr>
<th>Source</th>
<th>Timing</th>
<th>Address</th>
<th>Data format/Notes</th>
</tr>
</thead>
</table>
| 1st Prospective Secondary Mathematics Teacher (PSMT) interviews | Before observations                              | PSMT conceptions of proof, specifically  
- PSMT definition of proof,  
- what arguments are accepted as proofs,  
- what arguments are offered as proofs,  
- what warrants and backings are made explicit or left implicit in proofs, and  
- PSMT beliefs about the role of proof in mathematics. | Video and audio recordings              |
| Classroom observations                      | Several 2-3-day blocks in alternating weeks, total of 8-10 class days | PSMT facilitation of argumentation, specifically  
- what the PSMT accepts as backings and warrants from students,  
- what the PSMT offers as backings and warrants to students,  
- how the PSMT supports student argumentation (verbally, nonverbally) or fails to support argumentation, and  
- what normative standards of argumentation are apparent in the classroom. | Field notes, audio recordings           |
| 2nd PSMT interviews                         | After all observations were completed            | PSMT conceptions of proof and beliefs about students, specifically  
- criteria for acceptable student explanation,  
- beliefs about students’ abilities to do proof,  
- beliefs about students’ abilities to provide written or oral justification, and  
- follow-up on 1st interview. | Video and audio recordings              |
| Mentor interviews                           | As convenient for mentor, after several observations | Mentor’s influence on observed classroom activities and norms, specifically  
- Mentor beliefs about students’ abilities to do proof,  
- Mentor beliefs about students’ abilities to provide written or oral justifications, and  
- Restrictions placed on PSMT regarding task selection or types of questions to be asked in class. | Audio recordings, field notes           |
| Teacher-created or selected documents such as tests, quizzes, handouts, lesson plans, answer keys | During the observation time; lesson plans and handouts for observed lessons; tests and quizzes with ideal solutions for the appropriate units | PSMT’s conceptions of proof and expectations for written arguments, specifically  
- Beliefs about students’ abilities,  
- What is an acceptable explanation or justification, and  
- What is valued by the PSMT. | Photocopies, scans, or photos of actual classroom materials |
Each of the data sources detailed in Table 3-1 was used to discern the student teacher’s conception of proof and justification or to describe the argumentation in his or her classroom. For instance, the two interviews were primarily used in the analysis of the student teachers’ conceptions of proof and justification, while the classroom observations, with support from the mentor interviews and collected documents, were primarily used in the analysis of argumentation. In the following sections, each data source and method of collection is described in detail.

The First Interview

Before observations commenced, participants were interviewed about their conceptions of proof. This interview consisted of questions about their beliefs about mathematics in general and then proof in particular, questions about their experiences with proof and proving, and tasks asking them to prove mathematical statements and evaluate various arguments as proofs or non-proofs (see “Views of Mathematics and Mathematical Arguments” in Appendix A). During this semi-structured interview, the interviewer prompted participants to explain their thinking as they wrote and analyzed proofs. In addition, the interviewer provided support for participants in the form of information such as definitions or statements of theorems or strategies such as example-generation for starting a proof but did not provide hints or other assistance in structuring or producing the proof of a statement.

The main purposes of the first interview were to find out how each participant defined proof, his or her views of himself or herself as successful or unsuccessful in
proving, and his or her prior experiences with proof; to determine what arguments each participant would accept as proofs and would offer as proofs; to clarify what warrants and backings each participant would tend to make explicit or leave implicit in proofs; and to ascertain each participant’s beliefs about the role of proof in mathematics. To these ends, the interview began with general questions about important aspects of mathematics and requests for opinions about proof. The next section of the interview involved questions designed to elicit participants’ successful and unsuccessful experiences with proof. The final part of the interview involved several proof tasks as described below.

During the first interview, there were three main mathematical tasks for participants. The first task involved constructing a proof for the statement: The sum of the first \( n \) natural numbers is \( \frac{n(n+1)}{2} \). The second task asked participants to respond to the argument shown in Figure 3-1 for the statement: The square root of a positive integer is either an integer or it is irrational. The third mathematical task in the first interview involved three arguments for the Triangle Inequality Theorem. Participants were given the arguments in Figure 3-2, Figure 3-3, and Figure 3-4, one at a time and asked whether each argument convinced them and which argument or arguments were proofs. For the argument in Figure 3-2, each participant was also shown a sketch in Geometer’s Sketchpad with which he or she could interact if he or she chose. Lynn, who was the first participant to be interviewed, was actually given Figure 3-5, which contains a typographical error in the labels of the vertices of the triangle, rather than Figure 3-4, so she responded to a somewhat different task from the other two participants.
The square root of a positive integer is either an integer or it is irrational.

Suppose \( n \) is a positive integer, \( \sqrt{n} \) is not an integer, and \( \sqrt{n} \) is rational. Then there exist relatively prime integers \( a \) and \( b > 1 \) with \( \sqrt{n} = \frac{a}{b} \).

Squaring both sides, \( n = \frac{a^2}{b^2} \), from which \( nb^2 = a^2 \). Now if we factor \( a \) and \( b \) into primes, there are no common factors. So the factorizations of \( a^2 \) and \( b^2 \) have no common factors. Consequently, the factorizations of the equal numbers \( nb^2 \) and \( a^2 \) are different. Since two different factorizations of \( a^2 \) are impossible by the Fundamental Theorem of Arithmetic, the supposition must be false. So if \( \sqrt{n} \) is not an integer, it must be irrational.

Figure 3-1: Square Root Argument to Critique (from Usiskin, Peressini, Marchisotto, & Stanley, 2003, pp. 23-24)

Drag any point. The sum of the measures of any two sides of the triangle is greater than the measure of the other sides.

Figure 3-2: Geometer’s Sketchpad-situated Argument for the Triangle Inequality Theorem
The Triangle Inequality Theorem
In any triangle, the sum of any two sides is greater than the remaining one.

Argument 2

Case 1:
Given three segments, if the sum of the lengths of any two of them is less than the length of the third segment, then no triangle can be formed by connecting their endpoints. The endpoints of two of the segments cannot be connected. (See figure below.)

Case 2:
Given three segments, if the sum of the lengths of any two of them is equal to the length of the third segment, then no triangle can be formed by connecting their endpoints. The endpoints, and hence the segments themselves, are collinear (justification: the Betweenness Postulate). (See figure below.)

Figure 3-3: Geometry Textbook Argument for the Triangle Inequality Theorem (Schultz, Hollowell, Ellis, & Kennedy, 2001, p. 274)
The Triangle Inequality Theorem
In any triangle, the sum of any two sides is greater than the remaining one.

Argument 3

Let $ABC$ be a triangle.

Draw $BA$ through to the point $D$, and make $DA$ equal to $CA$. Join $DC$. 
Since $DA$ equals $AC$, therefore the angle $ADC$ also equals the angle $ACD$. Therefore the angle $BCD$ is greater than the angle $ADC$. 
Since $DCB$ is a triangle having the angle $BCD$ greater than the angle $BDC$, and the side opposite the greater angle is greater, therefore $DB$ is greater than $BC$. 
But $DA$ equals $AC$, therefore the sum of $BA$ and $AC$ is greater than $BC$. 
Similarly we can prove that the sum of $AB$ and $BC$ is also greater than $CA$, and the sum of $BC$ and $CA$ is greater than $AB$. Therefore in any triangle the sum of any two sides is greater than the remaining one.

Figure 3-4: Euclid’s Argument for the Triangle Inequality Theorem (http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI20.html)
The Triangle Inequality Theorem
In any triangle, the sum of any two sides is greater than the remaining one.

Argument 3

Let $ABC$ be a triangle.

Draw $BA$ through to the point $D$, and make $DA$ equal to $CA$. Join $DC$.
Since $DA$ equals $AC$, therefore the angle $ADC$ also equals the angle $ACD$.
Therefore the angle $BCD$ is greater than the angle $ACD$.
Since $DCB$ is a triangle having the angle $BCD$ greater than the angle $BDC$, and the side opposite the greater angle is greater, therefore $DB$ is greater than $BC$.
But $DA$ equals $AC$, therefore the sum of $BA$ and $AC$ is greater than $BC$.
Similarly we can prove that the sum of $AB$ and $BC$ is also greater than $CA$, and the sum of $BC$ and $CA$ is greater than $AB$. Therefore in any triangle the sum of any two sides is greater than the remaining one.

Figure 3-5: Lynn’s Version of Euclid’s Argument for the Triangle Inequality Theorem (http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI20.html)

The Second Interview

After all classroom observations were completed, each participant was
interviewed a second time. The main purposes of the second interview were to clarify
participants’ conceptions of proof and to find out some of their beliefs about students. In
particular, the second interview was designed to elicit participants’ criteria for acceptable student explanations, beliefs about students’ abilities to do proof, and beliefs about students’ abilities to provide written or oral justification. The interview schedules for these semi-structured interviews were individualized in that several questions were asked that were specific to the classes observed for each participant. In addition, some questions were asked in the second interview in order to clarify questions from the first interview or events observed during the classroom observations. The actual interview schedule for each participant is in Appendix B.

During the second interviews, participants were asked to respond to situations constructed from my observations of their classroom practice. The student teachers responded to questions situated in the mathematical concepts they had taught, as well as responding to questions about ideal answers to problems from worksheets, quizzes, and tests they had constructed or selected, but they were not asked to respond to specifically recalled statements that were made during instruction. The questions asked of each participant were designed to address the same aspects of argumentation and justification, despite being contextualized in different mathematical ideas, and the form of the questions was the same across participants. For instance, each participant was asked to respond to a hypothetical situation in which two students disagreed about something that was discussed in class. For Karis, this disagreement was situated in the context of the similarities and differences between Rolle’s Theorem and the Mean Value Theorem (see Figure B-6). For Jared, the context involved whether a system of equations had many or no solutions (see Figure B-3), and for Lynn, the context concerned whether or not a set of three side lengths could form a triangle (see Figure B-9).
Mentor Interviews

Each mentor teacher was interviewed near the end of the classroom observations. The goal of interviewing each mentor was to investigate how the mentor may have influenced the observed sociomathematical norms of the classroom community, particularly those norms that involved argumentation. Mentor teachers were told that the purpose of the study was “to explore the relationship between how prospective teachers think about mathematics and the conversations among teacher and students” (Mentor Informed Consent Form). Each mentor teacher was asked about his or her interactions with the student teacher he or she mentored. He or she was asked to describe the kinds of questions the student teachers asked of him or her as well as the advice he or she gave to the student teacher, particularly in the areas of asking questions, responding to student questions, and selecting tasks. The mentors were asked about these specific areas because these areas would likely influence the argumentation in a classroom, the first two because of the (usually) verbal nature of the argumentation, and the last because much of the classroom argumentation was observed in the context of tasks presented to the students. In addition, each mentor was asked about his or her beliefs about students’ abilities to do proofs and his or her classroom practice regarding students justifying their answers. Each mentor was asked about the level of autonomy experienced by the student teacher in his or her classroom. In particular, some questions addressed how tasks were selected or constructed for students. The interview schedule for the interview with each mentor teacher can be found in Appendix C.
Classroom Observations

Participants were observed in two to three day segments for several non-consecutive weeks. Each participant was observed for a total of eight or nine class periods. Lynn was observed February 20, 21, 22; March 6, 7, 8, 29; April 3, 4. Karis was observed February 27, 28; March 1, 13, 15, 29; April 3, 4. Jared was observed February 20, 21; March 7, 8, 20, 21, 22, 28. Each class period was audio recorded and the teachers’ statements were transcribed from the recordings. In addition, I took detailed field notes, including students’ statements as well as any writing on the board or overhead. Each participant provided copies of his or her lesson plans, worksheets, and quizzes for each day he or she was observed. The transcribed recordings and field notes were combined to form as complete a record of classroom proceedings as possible. These records were supplemented by the lesson plans, worksheets, and quizzes as necessary.

Due to restrictions by the Office for Research Protections, I was only able to use and transcribe the parts of audio recorded conversations spoken by the prospective secondary mathematics teachers, not the parts of conversations spoken by the students. Any words spoken by students and written in field notes could be used. Thus the transcripts used for the analysis were incomplete, containing only the parts of conversations spoken by students that were captured in field notes augmented by the prospective secondary mathematics teachers’ words from the audio recordings.

The annotated transcripts derived from the recordings, field notes, and supplemental materials formed the basis for the analysis of the teacher’s role in facilitating classroom argumentation. In particular, the purpose of the classroom
observations was to determine what normative standards of argumentation were apparent in the classroom practices of each participant and how this argumentation was supported: what warrants each participant accepted from his or her students; what each participant offered as backings and warrants to students; and how each participant supported student argumentation, both verbally and nonverbally, or failed to support argumentation.

Additional Documents

On the days on which participants were observed, various documents such as lesson plans, tests, quizzes, handouts, and worksheets as well as corresponding answer keys were collected from each participant. Lesson plans, handouts, and worksheets were only collected on those days on which a participant was observed. Tests and quizzes with accompanying answer keys were collected for the units during which the observations occurred. Some tests and quizzes were created by the student teachers from potential problems available in the resource materials; others were created by a student teacher with input from his or her mentor. These additional documents provided evidence of the kinds of problems the prospective secondary mathematics teacher expected students to answer as well as evidence of the kinds of answers he or she expected students to give. Since teachers assess what they value (Stiggins & Conklin, 1992), the assessment documents collected helped to paint a picture of what the participants valued, in particular, what kinds of explanations and justifications they valued and to what warrants and backings they expected students to appeal in their arguments.
Analysis

The analysis was conducted using the transcripts from the various interviews, the documents obtained by combining the field notes and transcripts from classroom observations, notes from mentor interviews, and the additional documents collected from each participant. Analysis of each teacher’s support for argumentation and conception of proof proceeded individually. For each teacher, the analysis of classroom argumentation was completed first, followed by analysis of the interview transcripts for his or her conception of proof. These two analyses were then coordinated in order to compare each participant’s conception of proof with his or her support of classroom argumentation. In order to answer the research questions, similarities and differences among the three teachers were examined in a cross case synthesis. Details of the analysis process are described below.

Analysis of Classroom Argumentation

To describe and analyze classroom argumentation, the records of classroom observations (transcripts from audio recordings combined with field notes and supplemented with material from lesson plans and worksheets) were used. The analysis was done in several passes, each of which built on the previous one. The analysis process is described in general terms in the next few paragraphs and illustrated with the example in the next section.

The first pass of analysis consisted of identifying episodes of argumentation. An episode of argumentation consists of a claim, the data and warrants that support the claim
(as written or spoken in class), and any sub-arguments necessary to support the validity of the data and warrants for that claim. I searched each record of classroom observation for the claims that were made during the class. Each major claim, together with the talk that occurred around it, was identified as an argument; for many of these arguments, sub-arguments were identified as well. In identifying these episodes and focusing the analysis on them, I ignored classroom talk that did not contribute directly to argumentation in the classroom. Talk that was ignored includes non-mathematical questions and answers, the teacher’s directions to the class (such as assigning homework problems), and other classroom procedures not essential to the analysis.

The second pass of analysis involved writing a coherent description of each of the identified episodes of argumentation. These descriptions were written in paragraph form, contained as much detailed information as possible about the argument and sub-arguments, and contained the participants’ words (from the transcripts) and actions, but did not include interruptions and unrelated details. The intent of these descriptions was to provide a coherent picture of what happened in the classroom from which an analysis of the teacher’s and students’ roles in argumentation could be obtained. The initial coherent descriptions were examined by a second mathematics educator (a university faculty member) to ensure that an appropriate level of detail was maintained. Each coherent description was labeled with a statement of the major claim being made in that episode of argumentation. These coherent descriptions are used as transcript excerpts in Chapters 4, 5, and 6.

During the third pass of analysis, claims, data, and warrants from each argument and associated sub-argument were identified. In some arguments, warrants or data were
not explicit within the classroom data. These missing components were labeled simply “unspecified” in most cases. In some cases, however, it was possible to infer what the warrant or data might have been. These cases were labeled unspecified, and the possible warrant or data was included parenthetically. Each specified claim, datum, or warrant was attributed to a student, the teacher, or to a combination of students and teacher; unspecified components were not attributed to anyone.

This third pass of analysis involved the most inference, as it was sometimes difficult to ascertain precisely who was making the claim or providing the data or warrant. The situations in which it was difficult to determine who was providing the particular component of the argument were, in general, ones in which the teacher asked a question. In some cases the attribution was clear. For instance, if the teacher asked, “What is the solution to this problem?” and a student gave a solution, the claim would be attributed to the student. That the teacher stated the problem in his or her question does not mean he or she was making a claim about the problem. Or, if the teacher asked, “Why does A equal B?” and a student gave a reason, the claim A=B would be attributed to the teacher (assuming it had not been introduced earlier by someone else) and the datum (the reason) would be attributed to the student. However, in some cases the attribution was not as clear. For instance, if the teacher asked, “Can we say A and B are equal?” and a student said, “Yes,” the claim would be attributed to both the teacher and student. In this case, the teacher is not only suggesting that a claim should be made, but is suggesting what the claim should be. In this case, the decision to attribute the claim to both the teacher and student was easier than in the following situation. If the teacher asked, “What can we say about A and B?” and a student said, “They are equal,” the claim would be
attributed to the student unless the teacher had in a previous statement suggested that two things would be equal, in which case the claim would be attributed to both the teacher and student. In this case, while the teacher is suggesting that a claim should be made, and is even suggesting part of the content of the claim, the student could have made any number of observations about A and B, only one of which being that they are equal. When a component of an argument is attributed to “both,” it should be interpreted as containing possibly unequal contributions from the teacher and the student.

At various points in this pass of analysis, a second mathematics educator examined the classifications of some statements as claims, data, and warrants, and the attributions to teacher, students, or both. I outlined specific criteria for attributing statements to the teachers, students, or both, and the second mathematics educator challenged those criteria, offering alternative interpretations, until we were able to reach agreement about the attributions. Similarly, on several occasions the second mathematics educator challenged my classifications of statements as claims, data, and warrants, causing me to revisit parts of the data to ensure I was being consistent in my classifications.

After each argument was outlined, I diagrammed the argument according to the scheme developed by Toulmin (1964) and used by others in both mathematics education and science education (e.g., Forman, Larreamendy-Joerns, Stein, & Brown, 1998; Krummheuer, 1995). Toulmin’s diagram of the components of argumentation can be seen in Figure 3-6. As can be seen in this diagram, Toulmin’s description of argumentation includes not only claims, data, warrants, and backings, but also qualifiers and rebuttals or exceptions. However, Krummheuer’s application of Toulmin’s work to mathematics
education research only retained the four main parts of the argument: claim, data, warrant, and backing. As discussed in Chapter 2, I chose, while not discounting qualifiers and rebuttals, to focus the analysis of argumentation in these classrooms on these four main parts of argumentation.

![Diagram of Argumentation]

Figure 3-6: Example of Diagram of Argumentation (Toulmin, 1964, p. 104) where D=Data, W=Warrant, B=Backing, Q=Qualifier, C=Conclusion, R=Rebuttal or exception

**Example of Analysis of Argumentation**

An example of one small argument and its analysis appears in the following paragraphs. This example was chosen mainly for its brevity; most of the arguments analyzed involved much longer segments of transcript. The original transcript of the identified episode of argumentation appears in Figure 3-7. This segment of transcript was identified as containing an argument in the first pass of analysis. This episode occurred toward the end of a class in which Karis’ calculus class discussed the Mean Value Theorem. Notice that the transcript contains words that are not necessary for our analysis.

---

3 The word “conclusion” here is used for consistency with Toulmin’s diagram. He uses “claim” and “conclusion” interchangeably throughout his work; the word “claim” is used consistently throughout the current study.
understanding of the argument that is taking place, for instance, “Do you still have the handout that you did on Friday?” (Karis Calculus Class, February 27, line 286).

Figure 3-7: Transcript for Example of Argumentation (Karis Calculus Class, February 27, lines 286-296)

Figure 3-8 illustrates the product of the second pass of analysis, the coherent description of the episode of argumentation. The description begins with a statement identifying the main claim of this episode of argumentation. The shaded lines in Figure 3-8 indicate the presence of a sub-argument within the main argument. As this figure illustrates, in the second pass of analysis, nonessential details are omitted, while enough prose is added to give a narrative account of what happened in the classroom during this episode of argumentation.

Claim: We can apply the Mean Value Theorem to \( f(x) = x^2 \) on \([-2,1]\).

The teacher asks “Can we apply the mean value theorem?” (290). A student says “Yes” (291). The teacher asks “Why?” (292). The student says “Because it’s continuous and differentiable” (293). The teacher explains “you know x squared we just have the graph of a parabola, so we know it’s continuous everywhere, and every place you can take the derivative. So now, what’s my first step?” (294-295).

Figure 3-8: Coherent Description of ‘We can apply the Mean Value Theorem to \( f(x) = x^2 \) on \([-2,1]\)’ (from Karis Calculus Class, February 27, lines 286-296)
The result of the third pass of analysis is shown in Figure 3-9 and Figure 3-10. The main argument is identified in Figure 3-9, with the sub-argument in Figure 3-10. Each argument is color-coded according to the following scheme: parts of arguments attributed to students are colored blue, parts of arguments attributed to the teacher are colored red, and (although not illustrated in this example) parts of arguments attributed to both are colored violet.

<table>
<thead>
<tr>
<th>Claim: We can apply the Mean Value Theorem to $f(x) = x^2$ on $[-2, 1]$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data: $f(x) = x^2$ is continuous and differentiable.</td>
</tr>
<tr>
<td>Warrant: If a function is continuous on $[a, b]$ and differentiable on $(a, b)$, then it meets the conditions of the Mean Value Theorem [stated earlier in the class by the teacher].</td>
</tr>
</tbody>
</table>

Figure 3-9: Example of Components of Main Argument Attributed to Students and Teacher (from Karis Calculus Class, February 27, lines 286-296)

<table>
<thead>
<tr>
<th>293-295(sub)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Claim: $f(x) = x^2$ is continuous and differentiable.</td>
</tr>
<tr>
<td>Data: It is the graph of a parabola.</td>
</tr>
<tr>
<td>Warrant: Unspecified</td>
</tr>
</tbody>
</table>

Figure 3-10: Example of Components of Sub-argument Attributed to Students and Teacher (from Karis Calculus Class, February 27, lines 286-296)

Each argument and sub-argument was diagrammed separately, and the diagrams of individual arguments were combined into diagrams of each episode of argumentation. For this example, the main argument is diagramed in Figure 3-11, the sub-argument in Figure 3-12, and the episode of argumentation in Figure 3-13. Notice that the claim of the sub-argument is part of the data for the main argument. This is typical of sub-arguments. The purpose of a sub-argument is often to establish the validity of a statement that one wishes to use as part of the data or warrants in a larger argument (Toulmin, 1964).
If a function is continuous on [a, b] and differentiable on (a, b), then it meets the conditions of the Mean Value Theorem [stated earlier in the class by the teacher].

Figure 3-11: Example of Diagram of Main Argument (from Karis Calculus Class, February 27, lines 286-296)

It is the graph of a parabola

Since

Unspecified

Figure 3-12: Example of Diagram of Sub-argument (from Karis Calculus Class, February 27, lines 286-296)
We can apply the Mean Value Theorem to $f(x) = x^2$ on $[-2,1]$.

If a function is continuous on $[a, b]$ and differentiable on $(a, b)$, then it meets the conditions of the Mean Value Theorem [stated earlier in the class by the teacher].

So, $f(x) = x^2$ is continuous and differentiable.

Since

It is the graph of a parabola

So,

Unspecified
The diagrams of episodes of argumentation served to give an overall picture of the structure of argumentation within each class, whereas the diagrams of individual arguments were used to classify the arguments according to which parts were attributed to whom. After diagrams of all arguments were constructed, the diagrammed arguments were examined for patterns in terms of the teacher’s role in each argument. For example, arguments in which the teacher made the claim were examined to see if there were patterns in whether a warrant was given and by whom it was given. Arguments in which only students participated were examined to determine if particular kinds of claims were made or particular kinds of warrants were given. Arguments, like the two illustrated in Figure 3-11 and Figure 3-12 in which the claims were made by students, were examined to see if there were patterns in who contributed the data and warrants. These two arguments illustrate two different classes of arguments that were found in Karis’ teaching. In one (Figure 3-11), Karis provides the warrant, and in the other (in Figure 3-12) the warrant is left implicit. Many of the arguments in Karis’ class had either an implicit warrant or a warrant attributed to Karis. These will be discussed at length in Chapter 4, and similar patterns in Lynn’s and Jared’s classes will be discussed in Chapters 5 and 6, respectively.

As part of the analysis, the total number of arguments was counted, and the number attributed to only the teacher, only the students, and each possible combination of teacher and students were counted. These classifications and counts assisted in identifying patterns within the large number of arguments involved. Within this process, a second mathematics education researcher reviewed the classifications, and different interpretations were discussed. As will be seen in subsequent chapters, identifying the
patterns within the observed argumentation allowed various conclusions to be drawn. These conclusions were based primarily on the patterns of warrants and backings observed in the various types of arguments.

**Analysis of Conceptions of Proof**

Each participant’s conception of proof was characterized using primarily his or her statements in the first interview, with supporting data from the second interview. A participant’s statements about the definition and use of proof in his or her interviews provided insight into his or her conception of proof. In addition, each participant’s proof production and critiques of arguments were examined in terms of consistency with standards of formal mathematical proof. That is, each constructed argument was analyzed as to the extent to which it could be characterized as a logically correct deductive argument built up from given conditions, definitions, and theorems, and each critique of an argument was examined for the extent to which the participant analyzed the logical structure and mathematical content of the argument. Each participant’s statements about intended or ideal student explanations or justifications were also examined for insights into his or her conception of proof. The analysis process is described in general terms in the next few paragraphs and illustrated by an example in the following section.

Three aspects of each participant’s conception of proof were initially examined. These aspects of a conception of proof were conceptualized as in Figure 3-14 as lying along three continua: the participant’s ability to prove mathematical statements and analyze purported proofs of statements (from *less proficient* to *more proficient*), the
participant’s perception of the purpose and need for proof in mathematics (from *not necessary* to *absolutely necessary*), and the participant’s affective perceptions of proof (from *dislikes or feels incapable* to *likes and is confident*). While these three aspects appear as continua in Figure 3-14, there was no expectation of precisely locating each participant’s conception as a discrete point in this space. The model and its three aspects of a conception of proof guided but did not limit the analysis, and, as will be discussed later, this model was revised to better reflect observations from this study.

Figure 3-14: Aspects of a Conception of Proof

Within each continuum, distinctions were made based on participants’ statements within the interviews. For instance, a participant’s affective perception of proof was inferred primarily from his or her statements in answer to interviewer questions about his or her experiences with proving and when he or she felt successful or unsuccessful when proving. These questions were asked in the early part of the first interview. While
originally I thought that a participant’s affective perception of proof might be able to be situated globally along this continuum, during the analysis process it became clear that a participant’s affective perception of proof could be dependent on the context in which the proving was taking place. For instance, Lynn spoke about how much she liked proving in one course but how much she disliked and felt incapable of proving in a different course. Thus her affective perception of proof could best be described as context-dependent.

Situating participants’ perceptions of the purpose and need for proof in mathematics was similarly problematic. It was possible to characterize a participant’s perception of the purpose of proof as, for instance, absolutely necessary for personal understanding of mathematics. Thus it was only possible to characterize participants’ views of the need for proof with respect to particular purposes or roles of proof in mathematics about which they spoke. Other possible purposes for proof in mathematics, such as the need for proof to ensure certainty, were not mentioned by participants, so it was not possible to characterize participants’ views regarding this purpose of proof in mathematics. Participants’ perceptions of the purpose of and need for proof in mathematics were inferred primarily from their answers to questions about what is important in mathematics, how important proof is in mathematics, and how important proof is in teaching mathematics. Supporting evidence for these inferences was obtained by examining participants’ expectations for students’ justifications in the second interview.

Making distinctions along the “ability to prove and analyze proofs” continuum was slightly more straightforward. However, there were still some contextual issues in this part of the analysis. Participants’ proofs were analyzed as to the extent to which they
were logically correct deductive arguments. Participants’ critiques of arguments were examined as to the extent to which the critique involved logical structure and knowledge of mathematics as well as whether it was consistent with being a logically correct deductive argument. This aspect of a conception of proof was primarily inferred from students’ actual proofs and critiques of proofs in the first interview.

Analysis of conceptions of proof from the interviews proceeded in a nonlinear fashion. Multiple passes through the data were necessary to clarify the meanings of participants’ words as they responded to interview questions. After an initial reading of a transcript, I marked passages of the transcript that seemed to address one of the three continua described above. In addition, passages that informed a participant’s views of mathematics or in which he or she described some aspect of proving or justifying were also marked as possibly influencing his or her conception of proof. Each passage was examined in detail and summarized. From each marked passage, a conjecture was made about one of the three aspects of a participant’s conception of proof or his or her view of mathematics. If the passage did not relate to one of the three identified aspects of conception of proof, a conjecture was made about the statement itself as well as how it would inform a hypothesis of the participant’s conception of proof. The conjectures about a participant’s conception of proof and justification arising from one passage were compared to information and inferences made from other passages to develop as complete a picture as possible from the information available. Differences within the practice of individual student teachers were examined to determine if they could be explained by the mathematical context in which the student teacher was operating or if the conception seemed to change over time. At the beginning of the analysis,
hypothesized themes tended to center on the three aspects of conceptions of proof identified prior to the study, but not all themes were limited to these three aspects. For example, use of examples was a theme that arose from Lynn’s interviews that does not fit neatly into one of the three previously described aspects of a conception of proof. Themes arising from the various passages were noted and then compared to develop a complex picture of each participant’s conception of proof and justification. After these themes were identified and compared, a summary description of each participant’s conception of proof was written. A second mathematics educator read and responded to these summaries, pointing out alternative interpretations and places where additional support was necessary. These summaries and reactions were discussed, and differences in interpretation were resolved.

**Example of Analysis of Conceptions of Proof**

To illustrate the analysis of conceptions of proof, this section contains an excerpt from Lynn’s first interview that pertains to her perception of the purpose and need for proof. My notes, conjectures, and inferences about that excerpt and three others from the two interviews are also included in this section. This aspect of Lynn’s conception of proof was selected as an example simply because there were a small number of short excerpts pertaining to it. Because of the length of the transcripts, this illustration of the method of analysis begins after the first pass of analysis in which relevant passages from the interviews have been identified. One relevant passage is copied below, followed by the notes in which summaries, conjectures, and inferences were made from three other
excerpts. The final paragraph in this section details how all of this information was coordinated into a picture of her perception of the purpose and need for proof. As this analysis developed, it was discussed with a mathematics education researcher, and revisions were made based on alternative interpretations.

Early in the first interview, the interviewer asked Lynn “How important do you think proving is in mathematics?” (Lynn Interview 1, lines 103-104). After clarifying that she would answer as a student, Lynn gave the answer shown in Figure 3-15. My notes on this passage are shown in Figure 3-16. Notes for the subsequent three passages are shown in Figure 3-17, Figure 3-18, and Figure 3-19. “I1” and “I2” in the notes denote Interview 1 and Interview 2, respectively.

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**Figure 3-15: Lynn Interview 1, lines 108-117**

```
L: When I was a student, um I wanted to know why. You know, why does this work? Why can I prove this? And it was very important for me to be able to prove. I mean on a scale from one to ten, that was probably a ten. If I got a problem right away my question if it wasn’t on the board was how do I prove that? You know, and that’s what you had to do, tests, exams, quizzes, show why, you know, why it happens, and then that was also very good if I knew that, um, then I thought okay, well, when I am teaching, something obviously simpler in high school, I’ll know why this works. So I came in as a student teacher with that idea. The first
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**Figure 3-16: Researcher Notes on Lynn Interview 1, lines 108-117**

I1 108-117 “As a student [seems to be university student b/c talks about preparing to teach] I wanted to know why. It was very important for me to be able to prove. I mean on a scale from one to ten, that was probably a ten…that’s what you had to do on tests, exams, quizzes, show why…it happens…when I am teaching, something obviously simpler in high school, I’ll know why this works.” Proving very important to her personally because she wants to know why. Proving important for future teaching because she would know why. Proving important because it’s what she had to do in her math classes.
As a teacher, … I would base it on what my classes are. … Normally if there’s a proof, I try to prove it for them, because I think it’s practice to go through the steps of the proof is good for them.” Proving is important for some classes; Lynn proves in her classes because it’s good practice for the students.

To convince a peer, prove it informally, like either the GSP argument or the textbook argument. To convince a student, do a hands-on activity, no matter if it’s an honors class or a regular class. To convince a mathematics professor, do a formal proof, not like the textbook argument, but like the Euclid argument.

Students should be able to prove mathematical statements in high school. It’s important because “I’m a reason person, I need a reason for anything. I want to know why it works and where it comes from and that also builds a stronger level of knowledge.” She did not prove a lot in class because she felt restricted, and this was hard for her.

I looked at the notes from the identified parts of interview transcripts, such as those in Figure 3-19, to see whether a general pattern existed or whether there were excerpts that differed greatly from one another. In this example, one very strong aspect of her view of the purpose and need for proof in mathematics is that proof is very important in mathematics because it is personally important to her. She states that proof is very important to her, giving it a ten on a scale of one to ten (Lynn Interview 1, lines 108-117) without being asked to quantify its importance. Both to convince a peer and to convince a
professor of the truth of a mathematical statement, she would do a “proof,” although the proof would be different depending on whether it was for a peer or a professor (Lynn Interview 1, lines 573-583; 592-606). She claims students should prove mathematical statements in high school, because “I’m a reason person” (Lynn Interview 2, lines 149-164). While she makes some statements that suggest that proof is not as important for some people (to convince a student she would do a hands-on activity, Lynn Interview 1, lines 584-591), in general Lynn is consistent in her claim that proving is important in mathematics because it is important to her personally. After this conclusion was hypothesized, the interviews were examined again, this time to look for evidence contrary to this conclusion. However, Lynn’s statements were very consistent in suggesting that proof is important to her to know why things work. Thus, Lynn’s perception of the purpose and need for proof would be placed relatively high on the continuum of perception of the purpose and need for proof in mathematics, noting that the reason for this perception is a personal need for proof, to know the reason why something works.

**Within-case Analysis**

Each participant’s support of classroom argumentation and his or her conception of proof were analyzed separately. After a participant’s support for argumentation and conception of proof were described, I examined the patterns of support for argumentation and the conjectured conception of proof and justification to see if common themes had arisen between the two analyses. For each aspect of a participant’s conception of proof, I considered the identified patterns of support for argumentation to see if there was
common ground between the aspect of conception and the pattern of support. Also, for each identified pattern of support, I examined the aspects of the participant’s conception of proof to see if the conception of proof could explain the support for argumentation. I also considered whether a combination of aspects of a participant’s conception of proof would explain observed patterns of support for argumentation. For instance, a participant’s view of justification as explanation might explain why he or she consistently provided warrants for arguments in his or her classroom. I also looked for inconsistencies between a participant’s conception of proof and his or her actions in support of classroom argumentation. For instance, a participant might claim that proof and justification were necessary and important in mathematics, but seldom provide or require justifications for solutions to problems. These three sets of observed commonalities and inconsistencies form the basis for the cross-case analysis and a hypothesis about a possible relationship between student teachers’ conceptions of proof and support of argumentation in secondary mathematics classrooms.

Cross-case Synthesis

The goal of a cross-case synthesis is to “build abstractions across cases” (Merriam, 1998, p. 195). To this end, I examined both the characterizations of the relationship between each participant’s conception of proof and his or her support for argumentation in his or her classroom and individual characterizations of support for argumentation from each case. “Word tables” (Yin, 2003, p. 134) were constructed to compare support for argumentation, conceptions of proof, and the observed relationship
between the two for each participant (for an example of an excerpt from a word table for support of argumentation—warrants, see Table 3-2). Similarities and differences between the three cases were recorded and analyzed until I obtained a unified characterization of a possible relationship between student teachers’ conceptions of proof and how they support argumentation in secondary mathematics classrooms.

Table 3-2: Example of a Word Table: Support for Argumentation—Warrants

<table>
<thead>
<tr>
<th>Karis (Calculus)</th>
<th>Lynn (Geometry)</th>
<th>Jared (Algebra)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Warrants (by Karis) were theorems, definitions, rules, procedures, explanations of theorems, explanations of relevant parts of definitions</td>
<td>Warrants (by Lynn) were statements of theorems, statements of properties, general references to properties</td>
<td>Warrants (by Jared) were most commonly rules and procedures, definitions, references to checking solutions</td>
</tr>
<tr>
<td>Implicit warrants: theorems, properties, procedures referenced earlier in class or algebraic properties (previous knowledge)</td>
<td>Few warrants (by Lynn) were statements of procedures, appeals to previous knowledge, or appeals to visual cues such as a figure or demonstration</td>
<td>Students’ warrants responses to Jared’s questions and were rules and procedures</td>
</tr>
<tr>
<td>One of Karis’ major roles in argumentation was providing warrants</td>
<td>All (but 1) students’ warrants were in response to a direct request for a warrant</td>
<td>Implicit warrants were rules and procedures that can be inferred from the data or had been previously discussed in class</td>
</tr>
<tr>
<td>Students’ warrants were theorems, rules, definitions in response to Karis’ questions</td>
<td>All (but 1) students’ warrants were names of theorems or properties of geometric objects</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Warrants (by both) appealed to theorems, definitions, and properties</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implicit warrants: definitions, properties, theorems that had been previously used as warrants or solving algebraic equations or inequalities (so previous knowledge)</td>
<td></td>
</tr>
</tbody>
</table>

For instance, within the word table for support for argumentation (part of which can be seen in Table 3-2), there was an indication that each participant asked questions to prompt claims, data, and warrants. This was noted as a common theme. Then I probed further, looking at the kinds of questions each participant asked to see if there were differences. I characterized the questions that each participant asked, and then looked for
similarities and differences in these characterizations. For example, I looked at the kinds of questions each participant asked, looking at the words used and the components of argumentation that seemed to be addressed. I looked at whether each participant gave information about the component of the argument for which he or she was looking. I also examined how students interpreted the questions as evidenced by their answers. I described the surface-level common elements, and then described the differences between the kinds of questions asked and the kinds of answers they elicited from the students. Within this part of the analysis, the second mathematics educator was instrumental in demanding support for hypothesized common elements and differences. In addition, she suggested some possible interpretations that led to connections between the argumentation in the three classrooms, such as looking for patterns in the tasks that were used in the three classrooms.

In the cross-case synthesis (Chapter 7), I use observations across cases to answer each of the research questions. The first question is about support for argumentation, so in the cross-case synthesis, I describe common themes across participants’ support for argumentation and suggest what is important in a student teacher’s support for argumentation. The second question relates to the relationship between a participant’s conception of proof and his or her support for argumentation, so the second part of the cross-case synthesis describes, not conceptions of proof alone, but common themes across the relationships between support for argumentation and conceptions of proof.

Within the analysis process, selected portions of every phase of analysis were submitted to a second mathematics educator. In each case, she offered alternative explanations or interpretations within the analysis. Especially when attributions required
more inference, we discussed explicit principles by which decisions (such as when to attribute a statement to the teacher, students, or both) should be made, and we often referred back to explicit definitions, such as the definition of claim, data, and warrant, when discussing the decisions that were made. In addition to these discussions, a group of four mathematics educators and mathematicians reacted to an earlier draft of this document, both verbally and in writing. Each member of this group suggested alternative explanations or interpretations of the data presented that were either incorporated as alternatives within the revised document or refuted from the data.

The next three chapters describe Karis’, Lynn’s, and Jared’s support for argumentation and how this relates to each student teacher’s conception of proof. Chapter 7 gives the cross-case synthesis and answers the research questions posed in Chapter 1. A particular aspect of his or her conception of proof is evident in all three participants’ support for argumentation: Each participant’s view of the role of proof in mathematics as an influence on support for argumentation will be important to the cross-case synthesis and conclusion.
Chapter 4

Karis: Proving as Explanation

Karis strongly believes the purpose of proof is to explain. At various times she seems to equate justification and explanation. Her focus on explanation is evident within her support for argumentation, particularly in how she provides warrants for arguments. In this chapter, I will describe Karis’ support for argumentation and then give evidence that this support for argumentation aligns with her conception of proof.

Argumentation in Karis’ Classroom

Classroom Setting

As part of her student teaching experience, Karis taught a calculus class every day for approximately fifty minutes. Her students were seniors who were typically college-bound and who seemed motivated to achieve. There were twenty-three students in this class, seated in five rows of desks facing the front of the room. In general, Karis taught from the front of the room, sometimes using an LCD projector or the classroom television to project GSP sketches to illustrate various ideas, but using the chalkboard for most of her notes and illustrations.

The fifty minutes of these class periods were usually divided into three parts: reviewing and answering questions about the homework assignment from the previous day, introducing concepts and skills that were considered to be new to the students
(hereafter called new material), and individual or group practice with the new material. The part of the class devoted to new material was usually longer than the other two. The last part of class, individual or group practice, usually consisted of students working on an assignment that would become the homework due the next day.

Most of the discourse in the class was whole group discussion, moderated by Karis. In general, students raised their hands to volunteer answers or to ask questions. Rarely did one student respond directly to another; most discourse was mediated by Karis. Karis occasionally called on a student who did not raise his or her hand to volunteer, however, she usually relied on volunteers for participation.

During the time that her classes were observed, Karis taught lessons involving Rolle’s Theorem\(^4\), the Mean Value Theorem\(^5\), integration by substitution, and simple first-order separable differential equations. In addition to introducing new material, Karis and her class spent several class periods during the time she was observed discussing every problem on a worksheet designed to give students practice with problems involving Rolle’s Theorem and the Mean Value Theorem prior to taking a test that contained similar problems.

\(^{4}\) The statement of Rolle’s Theorem used by Karis was the following: “If \(f\) is continuous on \([a, b]\) and differentiable on \((a, b)\) and \(f(a) = f(b)\), then there exists at least one point \(c\) in \((a, b)\) for which \(f'(c) = 0\)” (Karis’ answer key to the test “Limits to Infinity, Rolle’s Theorem, and the Mean Value Theorem”).

\(^{5}\) The statement of the Mean Value Theorem used by Karis was the following: “If \(f\) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\), then there exists a number \(c\) in \((a, b)\) such that \(f'(c) = \frac{f(b) - f(a)}{b - a}\)” (Lesson plan “Mean Value Theorem” for February 27).
Influence of Mentor

One of Karis’ mentor’s goals for her was to develop “her own [teaching] style” (Mr. A Interview, line 10). He purposefully discouraged her from trying to teach like he did. According to both Karis and Mr. A, they talked weekly (and sometimes more often) about what topics they would address in their calculus classes because they were each teaching different sections of the same course. They designed separate lessons with different activities and tasks for their students. However, they did collaborate about what homework problems to assign. Karis decided what problems should be on the tests and quizzes for both sections and designed the tests; She also graded most of the tests and quizzes from the beginning of her experience (Mr. A Interview, line 125). In their separate interviews, Karis and Mr. A each claimed not to have talked about classroom discourse other than Mr. A’s desire that Karis should talk to each student individually every day if possible (mentioned by both Mr. A in his interview and Karis in her second interview). Mr. A’s opinion seemed to be that Karis is a capable teacher who was able to ask good questions and whose classroom practice reflected her choices and behaviors.

Typical Argumentation

A typical episode of argumentation in Karis’ class involved Karis and one or more of her students. In the classes observed, more episodes of argumentation were observed during the parts of class dedicated to discussing problems already completed independently than occurred during the parts of class dedicated to introducing or discussing problems related to the new material. In fact, only fifteen of the forty-five
episodes of argumentation were observed during the introduction of new material. A possible reason for this discrepancy is the series of days spent discussing the problems involving Rolle’s Theorem and the Mean Value Theorem on a worksheet, during which time no new material was introduced. An episode of argumentation usually began with Karis writing a problem on the board. Sometimes the targeted problem was selected by Karis as an example of a new theorem or procedure that she would like the class to learn. Other times the problem was requested by a student as one that he or she did not complete successfully or did not understand how to complete. The main claim in an episode of argumentation was often the combination of the problem selected by Karis or a student and its answer.

An episode of argumentation often involved a main argument and one or more sub-arguments. Forty-five episodes of argumentation were identified within the classes observed, and twenty-three of these contained one or more sub-arguments. The other episodes were individual main arguments. The following example was selected as typical of an episode of argumentation in Karis’ classroom. The episode consists of a main argument and two sub-arguments, and it occurred during a discussion of problems that students had completed independently. The featured episode of argumentation occurred during class on March 13. The lesson began with students completing three integration problems as a review and discussing them as a class. The class then discussed questions from homework (dealing with integration by substitution), and then Karis introduced differential equations. Class ended with students working individually on solving problems involving differential equations. This particular episode of argumentation occurred after the discussion of the review problems and near the beginning of the
discussion of homework questions. In particular, a student had asked about problem five, which involved the expression in Equation 4-1, and Karis had written it on the board.

\[ \int \cos(3x + 2) \, dx \]  \hspace{1cm} \text{Eq. 4-1}

Karis begins by saying, “Now in this case it’s a little different, because we don’t have something raised to a power, but this is when Mr. A’s way helps you. If you were asked not to find the antiderivative, but to find the derivative, what rule would you have to use?” A student answers “Chain rule,” and Karis agrees, “Yes, you’d have to use the chain rule because you need to take the derivative of the cosine, but you can’t forget the derivative of the inside. So, if you think about it like that, we know, okay, we would need the chain rule, so that means we need substitution.” She then asks, “What should we substitute for?” A student answers, “3x+2.” Karis explains, “Yes. In the case when you have trig functions, it should always be the inside that you sub for.” She writes Equation 4-2 as she says, “So then, we have to find something for \( dx \), so \( du \) is three \( dx \). But I only have one \( dx \), so I need to multiply by the one third again, so that one third \( du \) is equal to \( dx \).”

\[
u = 3x + 2 \\
du = 3dx \\
\frac{1}{3}du = dx\]  \hspace{1cm} \text{Eq. 4-2}

As Karis says, “So I go back and sub in. I have the integral of the cosine, instead of putting this in, I use my substitution, so I say the cosine of \( u \) \( dx \) is one third \( du \), and then, I’m pretty much ready that I can find the antiderivative,” she writes Equation 4-3 on the board.

\[ \int \cos u \left(\frac{1}{3} \, du\right) \]  \hspace{1cm} \text{Eq. 4-3}

\[ \frac{1}{3} \int \cos u \, du \]

---

6 Mr. A’s way had been introduced earlier in this class. During the discussion of the three review problems at the beginning of class Karis said, “Mr. A likes to say, take a look at the function, if you would use the chain rule to take the derivative of the function, then you need substitution” (Karis Calculus Class, March 13, lines 27-28).
Karis then asks, “So using your chart\textsuperscript{7}, what’s the antiderivative of cosine?” A student answers, “Sine,” and Karis writes Equation 4-4 on the board.

\[
\frac{1}{3} \sin u \\
\text{Eq. 4-4}
\]

Karis then says, “So it’s one third sine of \(u\), have to put it in terms of \(x\)’s, so it’s one third times the sine of three \(x\) plus two plus our constant,” and writes Equation 4-5 on the board.

\[
\frac{1}{3} \sin(3x + 2) + C \\
\text{Eq. 4-5}
\]

Karis concludes by saying, “So the original problem was that we don’t have a rule for what the cosine of three \(x\) plus two is, we only know what the cosine of \(x\) is. So that’s why we need the substitution. Once we do the substitution, then we have the cosine of just the variable, turn back to your chart, see what the antiderivative is, and then your last step is just going back to \(x\)’s, putting your constant in.” (Karis Calculus Class, March 13, lines 82-109)

The claim, data, and warrants of the main argument and two sub-arguments were identified according to Toulmin’s (1964) description of argumentation. The general description of claims, data, and warrants as used in this analysis can be found in Chapter 2.

The main argument has the following components:

- **Claim by Karis:** \(\int \cos(3x + 2)\,dx = \frac{1}{3} \sin(3x + 2) + C\).\textsuperscript{8}

- **Data by students and Karis:** We need to use substitution; substitute \(u = 3x + 2\); antiderivative of cosine is sine; then go back to \(x\)’s; put constant in.

- **Warrant by Karis:** Substitution property, chart for antiderivatives.

\textsuperscript{7} Students have previously been given a chart of antiderivatives of some common functions. This chart of antiderivatives is commonly referred to as “your chart” by Karis and “the chart” by students.

\textsuperscript{8} The claim is given in two parts: the first part is the question; the second part is the answer.
In sub-argument 1, the following claims, data, and warrants have been identified:

- **Claim by Karis:** We have to use substitution to find this integral.
- **Data by student:** We would have to use the chain rule if we were finding the derivative instead of the antiderivative.
- **Warrant by Karis:** Mentor teacher’s rule is if you’d need to use the chain rule to find the derivative, you have to use substitution to find the antiderivative. Later, she also says, “we don’t have a rule for \( \cos (3x+2) \), just for \( \cos x \)” (Karis Calculus Class, March 13, lines 106-107).

Finally, sub-argument 2 has the following components:

- **Claim by student:** We should substitute for \( 3x+2 \).
- **Data:** unspecified (This is a trig function.)
- **Warrant by Karis:** “When you have trig functions, it should always be the inside that you sub for” (Karis Calculus Class, March 13, line 92).

The structure of this argument can be seen from the diagram in Figure 4-1. For the general structure of diagrams like this, see Figure 3-6. In Figure 4-1, the main argument can be seen on the right side, with the two sub-arguments on the left leading into the main argument. The colors in the diagram indicate to whom the statement is attributed: red for Karis, blue for students, and violet for collaborative effort between Karis and her students. Thus the bulleted items above are portrayed in the figure below for a visual depiction of the structure of the argument as a whole. Figure 4-1 illustrates how the two sub-arguments provide parts of the data for the main argument. Providing part of the data for the main argument seems to be one of the functions of sub-arguments. Figure 4-1 also illustrates how the argumentation in this episode is shared between the teacher and her
students. This sharing is one of the typical characteristics of argumentation in Karis’
classroom, and is discussed at length in later sections.
We would have to use the chain rule if we were finding the derivative instead of the antiderivative.

So, we have to use substitution to find this integral.

Mentor teacher’s rule is if you’d need to use the chain rule to find the derivative, you have to use substitution to find the antiderivative. Later, we don’t have a rule for cos (3x+2), just for cos x.

We need to use substitution; substitute $u = 3x + 2$; Antiderivative of cosine is sine; go back to the x’s; put constant in.

So, $\int \cos(3x + 2)\, dx = \frac{1}{3} \sin(3x + 2) + C$.

So, substitute for 3x+2

Since

When you have trig functions, it should always be the inside that you sub for.

Since

Substitution property, “chart”
Examining the episode description together with the structural diagram suggests a few important characteristics of argumentation in Karis’ classroom. First, many of the claims attributed to students were actually student answers to Karis’ questions. For example, Karis asked “what should we substitute for?” (Karis Calculus Class, March 13, line 90). A student answered “3x+2” (Karis Calculus Class, March 13, line 91). Karis is suggesting that a claim should be made and what that claim should include, but the student is providing the substance of the claim. Similarly, when students provided data for claims, they were often answering a question asked by Karis. For example, when Karis asked, “If you were asked not to find the antiderivative, but to find the derivative, what rule would you have to use?” (Karis Calculus Class, March 13, lines 82-85), a student answered, “Chain rule” (Karis Calculus Class, March 13, line 86). Thus the student identified the rule to be used for finding the derivative in answer to Karis’ question. In these types of cases, it is difficult to precisely categorize the data as coming from the student or from both the student and Karis. However, as can be seen in Figure 4-1, the contribution was attributed to the student since without the student’s contribution, no claim was made. The decision-making process for attributing components of arguments to the teacher, students, or both is described in more detail in Chapter 3.

Toulmin (1964) posited a fourth component of argumentation: backing. Although, according to Toulmin (1964), backings within arguments usually are not stated explicitly, the backings Karis and her students used in classroom argumentation can be inferred from the warrants identified above. Considering the possible backings for the warrants in this argument informs an understanding of Karis’ support of argumentation in her classroom. Backings provide the authority for the warrants in an argument (Toulmin,
1964), and thus examining Karis’ backings gives insight into who or what provides the authority for the arguments in her classroom.

The warrant in the first sub-argument is related to Karis’ mentor teacher’s rule for when to use substitution (See Figure 4-1). The authority in this case is a particular rule from a hypothetical list of rules the mentor possesses, presumably from his experience in teaching calculus. Later, as further warrant for the claim in sub-argument 1, Karis also said, “we don’t have a rule for cos (3x+2), just for cos x” (Karis Calculus Class, March 13, lines 106-107). This warrant seems to appeal to a list of rules to which the class has access, possibly recorded in the “chart” mentioned earlier. In sub-argument 2, the warrant is “When you have trig functions, it should always be the inside that you sub for” (Karis Calculus Class, March 13, line 92). In this case the warrant seems to be a ‘rule of thumb’ or general guideline for when the students see a problem of this type. The warrant for the main argument calls upon the substitution property and the students’ “chart” of antiderivatives. The students’ chart of antiderivatives seems to fall into the same category as the list of rules to which the class has access, and even the appeal to a ‘rule of thumb’ might fall into this category, even though the ‘rule of thumb’ is not a written rule. The authority for these rules could be considered to be the students themselves, since Karis spoke as if the rules were knowledge that was shared by all students. Alternatively, Karis could have been appealing to the source of the rules, which, for the students, might have been Karis herself, and for Karis, might have been generally accepted axioms, theorems, or properties of mathematics; the textbook; her mentor; or previous professors. Appealing to the substitution property of equality in the warrant for the main argument may suggest a different type of backing for this argument. On the other hand, her reference to the
substitution property may have been referring to the rules for integration by substitution
that she was introducing to the class and thus fall into the same category of backings as
the other rules. In general, Karis seemed to appeal to an authority consisting of rules
accepted by the class, suggested by her mentor, and commonly accepted in mathematics
as backings for her arguments.

In this particular episode, as can be seen in Figure 4-1, Karis provided all the
warrants as well as two of the claims and part of the data. This is typical of the
distribution of parts of arguments observed in Karis’ classroom, as is demonstrated in the
next section. By examining the patterns in who provides claims, data, and warrants,
Karis’ role in supporting classroom argumentation becomes clear.

**Parts of Arguments Attributed to Karis and Her Students**

In the previous section, an episode of argumentation thought to be typical of
argumentation in Karis’ classroom was examined in detail to provide a picture of Karis’
interactions with her students. This episode included one main argument and two sub-
arguments, for a total of three arguments. Altogether, seventy-five arguments, including
both main and sub-arguments, were observed in Karis’ class. These occurred in forty-five
episodes of argumentation identified over the course of eight class sessions.

Table 4-1 contains counts of the number of arguments in which claims, data, and
warrants were attributed to various combinations of Karis and her students. The three
arguments contained in the episode examined in the previous section are included in the
following cells of Table 4-1: the main argument and first sub-argument are two of the
seven arguments classified as claim by teacher, data by both, warrant by teacher; the second sub-argument is the one argument classified as claim by both, data by neither, and warrant by teacher. In this section, I focus on who provided warrants to link the data and claims within an argument and when these warrants were made explicit or left implicit. Several categories of arguments will be examined: arguments in which only students participated, arguments in which only Karis participated, and several kinds of arguments in which both Karis and her students participated. Within this section, the counts used to describe the frequency of occurrence of the classified arguments can be seen in summary form in Table 4-1.

Table 4-1: Parts of Arguments as Attributed to Karis and her Students

<table>
<thead>
<tr>
<th>Claim</th>
<th>Data</th>
<th>Warrant</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Student</td>
<td>Teacher</td>
</tr>
<tr>
<td>Student</td>
<td>Student</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Teacher</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Both</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Neither</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Teacher</td>
<td>Student</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Teacher</td>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>Both</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Neither</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Both</td>
<td>Student</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Teacher</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Both</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>Neither</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>3</td>
<td>42</td>
</tr>
</tbody>
</table>
Arguments by Students Alone

Karis’ role in supporting argumentation includes participating or contributing to arguments as well as refraining from overt participation in particular arguments. Five arguments were identified as ones in which no part of the claims, data, or warrants of the argument was attributed directly to Karis. In three of these arguments, represented by the 3 in the first line of Table 4-1, the claim and data of the argument were attributed to students and the warrant was left implicit. The two remaining arguments, identified by the 2 in the fourth line of data in Table 4-1, consisted only of claims attributed to students, with neither data nor warrants made explicit. Within these five arguments, no warrants were identified. Two of the five claims were mathematically incorrect, one claim was a suggestion for a procedure to follow (this procedure was unhelpful in solving the problem), and the remaining two claims were correct mathematical statements.

Even though Karis did not contribute claims, data, and warrants to the five arguments described above, examining Karis’ actions in response to each of the claims contributes to an understanding of her role in supporting argumentation in her classroom. In the two arguments in which students made mathematically incorrect claims, Karis replied with a counter-claim, as illustrated by the following example. On February 28, the class was discussing the solutions to problems on a worksheet where they had been asked whether or not the Mean Value Theorem applied to several different problems and to find the point described by the theorem if it applied. Up to this point in the class, two problems (numbers 26 and 34) had been discussed, involving functions to which the Mean Value Theorem applied on the given intervals, and the appropriate points had been
found. At the beginning of this episode, a student makes a claim about whether the Mean Value Theorem applies to two different functions on particular intervals (numbers 28 and 30 on the worksheet).

A student says “So for problems like 28 \[ f(x) = \frac{x+1}{x} \text{ on } \left[ \frac{1}{2}, 2 \right] \] and 30 \[ f(x) = \sqrt{x-2} \text{ on } \left[ 2, 6 \right] \], you just look at them and say they’re not continuous, so you just say they don’t exist.” Karis disagrees and says “Okay, well, yes, for problem 28, and 30 [the Mean Value Theorem] actually does apply, and we can look quickly at why it does. So 28 said f of x equals x plus one over x and your interval is one half, two [writes \[ f(x) = \frac{x+1}{x} \text{ on } \left[ \frac{1}{2}, 2 \right] \]. My guess is that many of you said that this function is not continuous because if x equals zero, we have a hole in the graph. We can’t divide by zero. The problem with that argument is your interval. Zero is not in the interval they ask you to check. So this function is continuous on the interval one half to two. So it’s not just a matter of if there are any issues of continuity, once you see, okay, here’s our problem, then you have to think zero, is it in my interval, no, so you would apply the theorem to this problem. And the same with 30, 30 just says f of x equals the square root of x minus two from two to six [writes \[ f(x) = \sqrt{x-2} \text{ on } \left[ 2, 6 \right] \].” (Karis Calculus Class, February 28, lines 164-175)

The student supplied both a claim (the theorem does not apply to these problems) and data to support the claim (the functions are not continuous). Karis replied with a counter-claim that contradicted the student’s claim and then provided new data, focusing on continuity on the appropriate intervals. Karis’ support for argumentation in the cases of mathematically incorrect claims by students consisted of an emphasis on mathematically correct claims. She redirected the argument, providing both the claim and data for the new direction.

In both cases of mathematically correct claims by students within the arguments in which only students participated, the student provided both the claim and supporting
data but no explicitly stated warrant. Since Karis provided warrants for more than half of all arguments observed, it is reasonable to ask why she might not have provided a warrant for these arguments, or at least why she did not prompt the students for a warrant in these cases. In one case, the claim and data addressed a topic quite familiar to students, that of a graph of a linear equation. Thus Karis might have assumed an implicit warrant would be understood by students since it involved prerequisite knowledge for this course. In the other case, the claim and warrant were given during the public solution of a problem similar to other problems discussed during the class session. Since the claim and data were given correctly, and other problems were discussed in which the warrant was made explicit, Karis may have assumed that the warrant was understood to be the same as for the arguments discussed in the public solutions to the other problems. Karis’ reasons for leaving these warrants implicit may be related to her view of proof as explanation, which will be elaborated in subsequent sections. These two arguments were the only two arguments observed in which the teacher did not play an active role in providing or prompting claims, data, or warrants.

**Arguments by Karis Alone**

From the arguments identified as by Karis alone one can derive some information about when Karis naturally provides a warrant and what she calls upon as a warrant for her own arguments. Within arguments in which students did not participate, the majority (11 out of 17) contained a complete set of claim, data and warrant, or what Erduran, Simon, and Osborne (2004) claim is a more sophisticated argument. One argument
included only claim and warrant with implicit data, and five contained claims and data with implicit warrants. The lone sub-argument with implicit data was somewhat of an anomaly, with the claim concerning the fact that \( f(x) = x^2 - 5x + 7 \) is differentiable, and the warrant stated as “polynomials are nice” (Karis Calculus Class, February 27, line 260). This warrant links the implicit data that \( f(x) = x^2 - 5x + 7 \) is a polynomial to the claim that it is differentiable, if the words “are nice” are taken to mean that polynomials are a class of mathematical objects about which the students have extensive knowledge, part of which is that the mathematical behavior of polynomials is predictable. Alternatively, Karis could simply mean that polynomials are differentiable by saying “polynomials are nice” (Karis Calculus Class, February 27, line 260). However, this interpretation would assume the same implicit data that \( f(x) = x^2 - 5x + 7 \) is a polynomial.

In four of the five arguments with implicit warrants, the implied warrant was a theorem, property, or procedure referenced earlier in class (these are similar to the mathematically correct student-only arguments discussed above), or the warrant was an algebraic property such as the transitive property of equality. For instance, in class on February 27, Karis claimed that Rolle’s Theorem did not apply to \( f(x) = x^2 - 5x + 7 \) on \([-1,3]\), and provided the data that \( f(-1) \neq f(3) \). The implicit warrant here, which was stated earlier in this class, was that one of the requirements of Rolle’s Theorem is \( f(b) = f(a) \). For these arguments with implicit warrants, Karis may have assumed that the students understood the warrant to be the same as in the other similar arguments in which the warrant was explicitly stated. This interpretation is consistent with the
previously described interpretation of the implied warrants for mathematically correct claims within arguments made by students alone.

The fifth argument with an implicit warrant was a counter-claim to a student’s claim. The student claimed that the formula for the slope of a line was $y = mx + b$, and the teacher claimed that it was not because $y = mx + b$ contains $m$, which is slope. This implicit warrant relied on the students’ knowledge of mathematics prerequisite to calculus; thus it may be that Karis assumed the warrant was understood even though it was not stated. From this examination of implicit warrants, it seems clear that there were particular situations in which Karis did not intervene by providing a warrant or prompting students to provide a warrant. These situations included when an argument involved a problem similar to ones discussed previously in class in which a warrant was provided as well as when the warrant would have involved mathematical considerations with which the students could safely be assumed to be familiar since the mathematical ideas were prerequisite to calculus.

The explicit warrants Karis provided in arguments in which she alone participated included theorems, definitions, rules, and procedures as well as explanations of the meaning of theorems or relevant parts of definitions. For instance, in several arguments Karis referred to a formula, such as using $\Delta x = \frac{b-a}{n}$ to find the length of the sub-intervals when using Riemann Sums to approximate the area under a curve. On several occasions, Karis gave a warrant by describing a procedure, either one that she had followed to solve a problem or one that was connected to the use of a formula. For
instance, when discussing the telescoping sum \( \sum_{k=1}^{40} \left( \frac{1}{k} - \frac{1}{k+1} \right) \), Karis described her general method for finding such sums: “I write out the terms, as many as I need to, to see the pattern, cross off what cancels, and then just simplify it” (Karis Calculus Class, March 29, lines 88-89). Her description seems to serve two purposes. It provides a warrant for her specific solution in the problem being discussed, and it also seems to describe for students a general method for solving such problems.

On March 29, Karis and her students discussed a problem in which students had been asked to find \( \sum_{i=1}^{10} a_i + b_i \) [sic], where \( \sum_{i=1}^{10} a_i = 40 \) and \( \sum_{i=1}^{10} b_i = 50 \). Karis described a procedure for evaluating this expression, and used “the properties on your card” to warrant writing \( \sum_{i=1}^{10} a_i + b_i = \sum_{i=1}^{10} a_i + \sum_{i=1}^{10} b_i = 40 + 50 = 90 \). The warrant, in this case, referred to the list of summation properties that had been discussed during previous classes.

On February 27, when stating that \( f(x) = \frac{x}{2} - \sin \frac{\pi x}{6} \) was differentiable, Karis seemed to use an operational definition of differentiability as her warrant. She stated, “And we talked about on Friday places to check for whether it’s differentiable. We need to check if there’s like a rational exponent, a fraction, if it has a denominator, or absolute value” (Karis Calculus Class, February 27, lines 24-26). Again, this statement seems to serve two purposes: one, to provide a warrant for the data that she provided for her claim that the function is differentiable, and two, as a general statement that students could use in the future when faced with the same type of problem.
In her counter-argument to the student’s claim regarding problems 28 and 30 on February 28 (described above), Karis used an explanation of part of the Mean Value Theorem as her warrant for the data that she provided for the claim that the Mean Value Theorem did apply to 28 \( \frac{x+1}{x} \) on \( \left[ \frac{1}{2}, 2 \right] \) and 30 \( \sqrt{x-2} \) on \( [2,6] \). In the part of the argument that applied to the function \( f(x) = \frac{x+1}{x} \) on \( \left[ \frac{1}{2}, 2 \right] \), she states, “So it’s not just a matter of if there are any issues of continuity, once you see, okay, here’s our problem, then you have to think zero, is it in my interval, no, so you would apply the theorem to this problem” (Karis Calculus Class, February 27, lines 171-173). Her warrant for the application of the Mean Value Theorem to this problem was basically a reminder that the Mean Value Theorem requires a function to be continuous only on the appropriate closed interval, not continuous everywhere. On another occasion, she used the Mean Value Theorem itself as a warrant within an argument.

Karis’ warrants for her arguments, whether stated explicitly or left implicit, seem to consist of similar mathematical objects. She refers to theorems, parts of theorems, rules, procedures, and formulas both implicitly and explicitly. Those warrants that she left implicit seem to be within arguments that occurred later in a class period when similar problems were discussed earlier or to refer to ideas with which she could assume students were familiar. Karis’ reasons for making some warrants explicit and leaving others implicit may be related to her view of proof as explanation, which will be elaborated in subsequent sections. She made more warrants explicit than she left implicit,
and these arguments seemed to occur, for the most part, earlier in the class periods than
the ones with implicit warrants.

Arguments With Contributions From Karis and Her Students

Most (53 out of 75) arguments in Karis’ calculus class contained contributions
from both Karis and her students. These arguments could take various forms, with
various people contributing claims, data, and warrants. Regardless of who made the claim
and provided the data, Karis usually provided warrants, and contributions by students
usually followed a question from Karis. As can be seen in the second line of Table 4-1,
rarely (3 out of 18 student claims) did Karis provide both data and warrant for a student’s
claim without input from students, although she often (23 out of 45 instances of data with
student input) provided the warrant for a claim after students (or a combination of
students and teacher) provided data for a claim. On 23 occasions, and especially when
mathematical ideas that might have been new to the students were being discussed, Karis
and her students both contributed to making a claim. Sometimes, Karis and her students
both contributed to both the claim and the data, and no warrant was made explicit. This
was observed in seven arguments, and seemed to happen at times when the problem
involved symbolic manipulation of a type with which the students might be familiar. The
following episode from the second class in which Karis and her class discussed
differential equations illustrates this type of argument. As the episode began, Karis had
written Equation 4-6 on the board, and the class was attempting to solve it together.
Karism begins by stating, “So we need to separate the x’s and y’s on
different sides of the equation.” She asks, “How should we start?” A
student answers, “Split up the square root.” Karis says, “Okay, we can
split up the square root and make it two times square root of x times
square root of y dy dx.” She writes Equation 4-7 on the board.

\[ 2\sqrt{xy} \left( \frac{dy}{dx} \right) = 1 \quad \text{Eq. 4-6} \]

A student says, “Divide by \( \frac{dy}{dx} \),” and Karis says, “Divide by this?” The
student answers, and the Karis asks, “So you get it to the other side?” The
student answers again [not recorded]. Karis says, “If we divide it’s the
same thing as multiplying by the reciprocal.” She writes Equation 4-8 on
the board.

\[ \frac{dx}{dy} \cdot 2\sqrt{x} \sqrt{y} \frac{dy}{dx} = 1 \cdot \frac{dx}{dy} \quad \text{Eq. 4-8} \]

Karism then says, “So then these would cancel on the left so we have two
times square root of x times square root of y is now equal to \( dx \ dy \)” and
writes Equation 4-9.

\[ 2\sqrt{x} \sqrt{y} = \frac{dx}{dy} \quad \text{Eq. 4-9} \]

Karism asks, “Where do you think we should go next?” A student answers.
Karism says, “Okay, we could do that,” and writes \( dy \) on both sides of the
equation so that Equation 4-10 is on the board.

\[ dy \cdot 2\sqrt{x} \sqrt{y} = \frac{dx}{dy} \cdot dy \quad \text{Eq. 4-10} \]

Karism continues to explain, “He said multiply both sides by \( dy \) in order so
that on this side we just have \( dx \), and that is still two times the square root
of x times square root of y now times \( dy \).” She writes Equation 4-11 on the
board.

\[ 2\sqrt{x} \sqrt{y} dy = dx \quad \text{Eq. 4-11} \]
A student makes a suggestion, and Karis says, “Okay. Divide by two square roots of $x$? So we’re left with $y$, square root of $y \, dy$, and divide this, we have one over two times square root of $x \, dx$.” She writes Equation 4-12 Equation 4-13 on the board.

$$\sqrt{y}dy = \frac{1}{2\sqrt{x}} \, dx \quad \text{Eq. 4-12}$$

Karis explains, “Now see we have all the $y$’s on the left and all the $x$’s on the right. Now our next thought is to find the antiderivative, but you can rewrite this if you want so that you can apply the power rule. So you change these to fractional exponents, $y$ to the one half $dy$ is equal to, I like to take out the one half out front and make this $x$ to the negative one half. She writes Equation 4-13 on the board.

$$y^{1/2}dy = \frac{1}{2} x^{-1/2} \, dx \quad \text{Eq. 4-13}$$

So we have $y$ to the one half $dy$ is equal to one half $x$ to the negative one half $dx$.” Karis continues, “So now we can find the antiderivative. Take the integral of both sides.” She inserts integral signs on both sides so that the board now reads what is shown in Equation 4-14.

$$\int y^{1/2}dy = \int \frac{1}{2} x^{-1/2} \, dx \quad \text{Eq. 4-14}$$

Karis continues, “Using the power rule on the left, what would be my antiderivative?” A student answers, “Two thirds $y$ to the three halves.” Karis repeats, “Two thirds $y$ to the three halves,” and then continues, “Increase the exponent by one, divide by the new exponents which is three halves, multiply by the reciprocal.” She asks “On the right side, if we have $x$ to the negative one half, using the power rule?” A student answers, and Karis says, “$x$ to the positive one half, and we would divide that by one half so we can multiply by two and we get,” and she writes Equation 4-15 on the board.

$$\frac{2}{3} y^{3/2} = \frac{1}{2} x^{1/2}(2) + C \quad \text{Eq. 4-15}$$

Karis continues, “Now simplify the right side, still two-thirds $y$ to the three-halves, one half and two cancel,” and she writes Equation 4-16 on the board.
\[ \frac{2}{3} y^{\frac{3}{2}} = x^{\frac{3}{2}} + C \]  \quad \text{Eq. 4-16}

Karis continues, “Now, we’ve got y’s, we took the antiderivative, but we don’t want to know what y to the three-halves is, we want y, so...” A student makes a suggestion, and Karis says, “Okay if we multiply by three-halves,” and she writes Equation 4-17 Equation 4-18 on the board.

\[ \frac{3}{2} \cdot \frac{2}{3} y^{\frac{3}{2}} = \left( x^{\frac{3}{2}} + C \right) \cdot \frac{3}{2} \]  \quad \text{Eq. 4-17}

Karis continues, “And then with the C, you don’t have to distribute to the C, you can, it will work out the same, but it’s not necessary since we’re just saying it’s any constant. So we just have three-halves x to the one half plus C,” and she writes Equation 4-18 on the board.

\[ y^{\frac{3}{2}} = \frac{3}{2} x^{\frac{3}{2}} + C \]  \quad \text{Eq. 4-18}

Karis continues, “Now we need to do one last thing.” A student says something. Karis says, “So we have three halves x to the one half, quantity squared, all under a cubed root.” She writes Equation 4-19 on the board.

\[ y = \sqrt[3]{\left( \frac{3}{2} x^{\frac{3}{2}} + C \right)^2} \]  \quad \text{Eq. 4-19}

(Karis Calculus Class, March 15, lines 27-78)

From this episode, I extracted the following argument summary and attributions.

- Claim by both Karis and her students: If \( 2\sqrt{xy} \frac{dy}{dx} = 1 \), then \( y = \sqrt[3]{\left( \frac{3}{2} x^{\frac{3}{2}} + C \right)^2} \).

- Data by both Karis and her students: \( 2\sqrt{xy} \frac{dy}{dx} = 1 \Rightarrow 2\sqrt{x} \sqrt{y} \frac{dy}{dx} = 1 \)

\[ \Rightarrow \frac{dx}{dy} \cdot 2\sqrt{x} \sqrt{y} \frac{dy}{dx} = 1 \Rightarrow \frac{dx}{dy} = \frac{1}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow 2\sqrt{x} \sqrt{y} \frac{dy}{dx} = 1 \]

\[ \Rightarrow 2\sqrt{x} \sqrt{y} \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow \frac{1}{2\sqrt{x}} \Rightarrow \int y^{\frac{3}{2}} dy = \frac{1}{2} x^{\frac{3}{2}} dx \]
\[ \frac{2}{3} y' = \frac{1}{2} x^{\frac{2}{3}}(2) + C \] \rightarrow \[ \frac{3}{2} \cdot \frac{2}{3} y' = \left( x^{\frac{2}{3}} + C \right) \cdot \frac{3}{2} \rightarrow y' = \frac{3}{2} x^{\frac{2}{3}} + C \]
\[ y = \sqrt[3]{\left( \frac{3}{2} x^{\frac{2}{3}} + C \right)^2} \]

- Warrant: unspecified (steps given previously for solving differential equations).

The students had been solving differential equations for the previous two class days, and this was the second differential equation solved during class on March 15. The steps used to solve this differential equation were quite similar to those used to solve other differential equations during earlier classes, and, in fact, were made explicit during the solution of a differential equation earlier in this same class period. In this example, Karis could have assumed that students were familiar with the steps for solving differential equations, having discussed them on previous occasions, and thus not felt a need to give an explicit warrant for her and their symbolic work. This is consistent with the previously discussed occasions of implicit warrants since the possible rationale for those involved an assumption that either the students remembered prerequisite mathematical ideas or ideas discussed previously in the class session.

Eight instances of argumentation were observed in which a claim was made by both Karis and her students, both Karis and her students contributed data, and Karis provided a warrant. Three of these arguments involved the first three examples of differential equations discussed during the introductory lesson on differential equations on March 13, and two others were sub-arguments within discussions of finding the area under a curve using Riemann sums during the class period when this notion was introduced. Thus, five of the eight arguments occurred during the introduction of
mathematical ideas that were new to this class. The other three arguments occurred
during the discussion of the first problem on two different class days (two of the
arguments were sub-arguments in a larger episode that involved the solution to a problem
involving Riemann sums and one occurred during the discussion of a problem dealing
with the Mean Value Theorem). Five of the arguments occurred within larger episodes of
argumentation: four were sub-arguments and one the main argument. The other three
involved the solution of differential equations, problems that involved several steps.

The arguments in which Karis provided a warrant and both Karis and her students
created data and claim, then, involved solving problems requiring several steps or the
coordination of several sub-arguments within an episode of argumentation. In contrast, in
the two arguments in which students provided warrants where the claim and data were
contributed by both Karis and her students, the problems both involved simplifying sums
using previously learned formulas, and students stated the formulas as warrants. The
warrants Karis provided included explanations of the steps taken, statements of
definitions, statements of rules to be followed, and theorems such as the Mean Value
Theorem. In general, within the arguments in which both Karis and her students
contributed to both the claim and data, the more complex of a problem to be solved, or
the more complex the argumentation required to make a major claim, the more likely it
was that Karis contributed a warrant within the argument.

Within arguments in which both Karis and her students participated in some way,
Karis, her students, or both provided warrants for 39 out of 53 arguments and left the
warrants implicit for only 14 arguments. Of the warrants provided, students contributed
three warrants, Karis provided 30 warrants, and a combination of Karis and her students
contributed six warrants. The kinds of warrants that Karis provided mirror the types discussed in the previous paragraphs. Thus, Karis used theorems such as the Mean Value Theorem, rules such as the power rule for integrals, properties such as the distributive property, and general rules of thumb such as when finding the integral of a trigonometric function, always substitute for “the inside” (Karis Calculus Class, March 13, line 92) as warrants for arguments in her class.

Karis’ Support of Argumentation

Table 4-1 illustrates the fact that Karis participated in every main or sub-argument except five, which were attributed entirely to students. However, four of these arguments were sub-arguments within episodes of argumentation in which Karis participated. The one argument that might be seen as being attributed entirely to students was a student’s answer to a problem worked independently in class. However, after the student’s explanation, Karis provided an explanation for the student’s answer in response to another student’s question. Thus Karis participated in every major episode of argumentation in some way.

Karis made only seventeen arguments without input from students, and eight of these were sub-arguments in larger episodes of argumentation in which students did participate, so both Karis and her students participated in most of the argumentation in her classroom. In some ways, it could be said that Karis orchestrated the argumentation in her classroom. Karis often asked questions that suggested that a claim should be made and what it might address mathematically. She asked questions and made comments that
suggested to students that data or warrants were necessary at various points. If she was not satisfied with a student’s claim, data, or warrant, she added to it or prompted another student to expand upon it.

One of Karis’ major roles in argumentation was providing warrants, the links between data and claims. The nature of these warrants varied, from statements of procedure (such as “You only have to use substitution if you have a quantity raised to the fourth, or the fifth, or one third, something where you wouldn’t want to expand it out…if you would use the chain rule to take the derivative of the function, then you need substitution” [Karis Calculus Class, March 13, lines 25-30]) to statements of theorems (such as “So what the mean value theorem is saying is that if we have a continuous function that we can take the derivative of everywhere on the open interval, and we form the secant line, then there’s a point where the tangent is parallel to that secant line” [Karis Calculus Class, February 28, lines 22-25]) to properties or definitions that had been discussed earlier and recorded by students (such as “So using your chart, what’s the antiderivative of cosine?” [Karis Calculus Class, March 13, line 99]).

Karis supported argumentation in her classroom by providing and prompting claims, data, and warrants. She was involved in most arguments, and provided warrants to connect data with claims within many arguments. The warrants she provided included statements of theorems, properties, definitions, and procedures. She was consistent in her use of implicit warrants, most of which were used in arguments that addressed mathematics to which most of her students had been previously exposed. The particular warrants that she chose suggested that her backing for these arguments was based on her
mentor’s or other professors’ rules or guidelines as well as mathematical theorems, properties, definitions, and procedures.

In the next section, I focus on Karis’ conception of proof and justification and its relationship to her facilitation of argumentation. In particular, how the backings inferred from her implicit warrants relate to her conception of proof and justification will be discussed.

**Alignment of Conception of Proof and Support for Argumentation**

Karis’ conception of proof can be summarized in three words: explanation, audience, and understanding. In the rest of this chapter, these three interrelated aspects of Karis’ conception of proof and the relationship of these three to her support for argumentation are explored. Of particular interest are the circumstances under which Karis provides warrants or leaves them implicit. Another aspect of argumentation relating to her conception of proof that will be considered is the kinds of warrants she provides or leaves implicit.

**Explanation**

Karis primarily views justification to be synonymous with explanation. She views the role of proof in mathematics to be explanation, and she values explanation in her classroom. This leads to patterns within her classroom argumentation. Karis provided many of the warrants in her class and she expanded and rephrased warrants that her
students gave. In this section, I illustrate this aspect of Karis’ conception of proof and relate it to patterns in her support for classroom argumentation.

The idea that proof relates to explanation is not one that is new to mathematics education. In fact, de Villiers’ (1990) description of the functions of proof in mathematics, he mentions explanation along with verification, systematization, discovery, and communication. Nordström (2004) found that the mathematicians in her study cited explanation as a reason (among others) for learning proof. However, the secondary teachers in Knuth’s (2002b) study did not explicitly mention explanation as a role for proof in mathematics, although some of their actions and statements suggested an explanatory role for proof as a way to show how something came to be true rather than “promoting insight of the underlying relationships” (p. 80). Karis’ view of proof and explanation seems to align more closely with providing insight into underlying relationships or, as Hersh (1993) described, seeing the primary role of proof in a classroom as to “provide insight into why the theorem is true” (p. 396). Hanna (1990) also argued that one very important function of proof in mathematics is explanation, and further suggested that proofs in mathematics classes should be explanatory whenever possible. While Karis did not engage her students in constructing proofs in her calculus class, the argumentation observed in her classroom as well as her description of and work with proofs in her interviews suggest that she would agree with Hanna’s (1990) statement.

Dreyfus (1999) highlighted an inherent difficulty in differentiating between explanation and justification, suggesting that sometimes an explanation is a proof and that teachers and students may differently interpret an instruction to explain. Karis’ use of
“explanation” and “justification” synonymously illustrates this difficulty in her case. When Karis was asked about justification in her second interview, she often used the word explain or explanation in her response. For instance, when asked, “Could you estimate how frequently you ask students to justify their answers in class?” (Karis Interview 2, lines 208-209), Karis replied, “[Do] you mean, when I ask a question, saying things like why, or like explain?” (Karis Interview 2, lines 210-211). When she was asked, “Can you give me an example…of a time when you asked a student to justify something and he or she just gave a really good justification?” (Karis Interview 2, lines 274-277), Karis gave the following reply.

Well, I guess today in calculus they’re doing some more area under a curve stuff, and I gave them an example, uh, just the general parabola, like it was $x$ squared, but from negative two to positive two, so they had two sides, it was the first time they saw one like that, so they had to think of, whoa, what do I have to do differently kind of thing, because we’re doing inscribed rectangles, so if they used the left endpoints the whole way, then on one side they’re circumscribed instead of, so I asked one of the students to explain, you know, like what can we do? And they gave pretty good explanations of that, explaining like, oh, well, why don’t we just split it in half? And because if I use right endpoints on this side I’m fine, and if I use left on this side it’s fine, so they went through that whole thing, and then another student added on and said, well, look, it’s symmetrical, can’t we just do two times one, so I guess that was a good explanation, they just really saw and like were building off of each other’s suggestions, so I thought that was a good explanation. (Karis Interview 2, lines 278-293)

Karis’ description of a good justification ended with her statement that it “was a good explanation” (Karis Interview 2, line 293). In this description, she referred more to how one might do a problem of this sort than to a basis for why one might do a problem of this sort in this way. Thus she described an explanation of how one might approach a problem like this, calling it an explanation, when asked to describe a justification.
Similarly to her description of a good justification above, some of the warrants Karis gave explained how to do a particular thing (rather than why such a thing should be done), such as her warrant relating to the claim \( \lim_{x \to \infty} \frac{2x^3 + 7}{x^3 - x^2 + x + 7} = 2 \). She says, “So whenever we have \( x \)'s in the numerator we have to do something to that function before we can take the limit. And what we do is we determine what the highest power of \( x \) is in the denominator and even though we have three \( x \)'s in this one in the denominator, \( x \) cubed, \( x \) squared, and \( x \), it’s just whichever the highest power is” (Karis Calculus Class, March 1, lines 41-44).

In the statement below, Karis began to talk about providing an argument for a claim. In the end, she referred to a step by step explanation. I asked, “What do you count as evidence of a really good, strong justification?” (Karis Interview 2, lines 297-298); her reply included both referring to theorems previously discussed and “going step by step”:

> I guess putting thought into it and using, maybe, like the theorems or concepts or whatever it is we talked about in class, as backing up, instead of just well, this is what I did and that’s it kind of thing, just really showing, well, I remember we talked about this theorem, and this is, I don’t know, just going through and applying the stuff we’re learning and really going step by step, and things like that. (Karis Interview 2, lines 299-304)

Karis’ description of justification as including references to theorems as well as “going step by step” in the quote above suggests that she may view theorems as defining steps to complete a problem or as steps to be applied to a problem.

Karis only mentioned two theorems in her observations and interviews: Rolle’s Theorem and the Mean Value Theorem. In the second interview, I asked Karis to describe an “A” answer and a “C” answer to the question in Figure 4-2. Karis’ response
to this question reveals that she thinks about these theorems in terms of “these are the requirements, this is what the theorem would say if I could apply it” (Karis Interview 2, lines 34-35). Since these two theorems were the only two mentioned during the observations and interviews, it is difficult to determine whether this conception of theorem as steps to complete a problem is limited to these two particular theorems or is more general.

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<table>
<thead>
<tr>
<th>Can we apply Rolle’s Theorem to the function $f(x) = x + \frac{1}{x}$ on the interval $[\frac{1}{2}, 3]$? Why or why not?</th>
<th>Can we apply the Mean Value Theorem to the function $f(x) = x + \frac{1}{x}$ on the interval $[\frac{1}{2}, 3]$? Why or why not?</th>
</tr>
</thead>
</table>

Figure 4-2: Question about Rolle’s Theorem and the Mean Value Theorem (Karis Interview 2)

Karis’ references to explanation when talking about proving were not limited to her second interview. In her first interview, when asked what is involved in proving a mathematical statement, Karis replied, “I guess showing the steps to get to it, I guess starting with what they give you and using whether it’s algebra or reasoning to show the steps between what they give you and what you’re looking for and explaining each step” (Karis Interview 1, lines 134-138). Thus Karis seemed to define proving as showing steps with accompanying explanations, but did not provide a clear elaboration of what sufficed for an explanation. Karis considered the purpose of proof to be explanation, at least in secondary school mathematics. She also saw explanation as important in the construction of a proof since she claimed that each step in the proof should be explained. To Karis, it seems that showing the steps is part of giving an explanation, but it is not the totality of
giving an explanation. That is, an explanation should include something more than showing the steps; precisely what else should be included is not clear from her statements.

Karis’ view of proof as important for explanation, and her tendency to equate justification and explanation, may explain why she provided almost all of the warrants that are provided for arguments in her class. Even when she did not provide the warrant, Karis often repeated, rephrased, or expanded a student’s warrant into more of an explanation to the class. For instance, when Karis and her students were discussing whether a function was continuous or not, a student said, “As long as there are no fractions” (Karis Calculus Class, February 27, line 257), and Karis said, “Yep, if we’re just given a polynomial, no fractions at all, it’s continuous. That’s easy.” (Karis Calculus Class, February 27, line 258). Karis repeated the student’s phrase, and then continued to make the bridge between the function being a polynomial and the function being continuous clear to the students.

In addition, Karis often used explanations of procedures as warrants. For example, when discussing a collapsing sum, Karis’ warrant to connect data (we don’t have a formula for this) to claim (to find the sum, start by substituting numbers), was the following.

We have formulas for like the sum of $i$, $i$ squared, $i$ cubed, and $i$ to the fourth, that help prevent doing this, but in terms of collapsing sums, this is actually what you need to do. You don’t need to write out all these terms, but you have to write out enough until you start to see the pattern, what’s dropping out, what’s collapsing. So I generally write three or four, if it’s a smaller upper limit, sometimes I write them all, then I can definitely see what cancels out.
Karis explained to her students not only that they needed to write out several terms (a procedure), but also why they needed to write out several terms “to see the pattern…what’s collapsing” (Karis Calculus Class, March 29, lines 63-67). Karis was consistent in her view of the importance of explanation and in her actions involving explanations. One of the reasons explaining is very important to Karis is for the purposes of student understanding, as described in the next section.

**Understanding the Mathematics**

Karis intended to use proof with her students to help them to understand mathematics. In particular, she wanted them to “understand where it’s coming from” (Karis Interview 1, lines 63-64) and the “background behind the mathematics” (Karis Interview 1, line 68). She contrasted the way proof was used in her learning of school mathematics with how she intended to use proof in her classes by saying, “So it [proof in school mathematics] wasn’t used in the same way that I hope to use it, which is more of explanation and helping to understand” (Karis Interview 1, lines 82-83).

Karis’ understanding of mathematics allowed her to be able to prove mathematical statements. She credited understanding the mathematics required for the proof with allowing her to be successful in proving in her college mathematics courses. This is clear from her description of a successful proving experience from her abstract algebra course.

Okay, I guess that, by the end of the course, I really understood first of all, what rings were, so I understood the question, and so I could think about how to go about proving it, and use everything I had learned to prove it. So I guess that’s why I felt successful. I knew I wasn’t just, I hadn’t just
memorized it and written it down. I understood the question and everything that I was writing. So I felt like I understood what I put down. (Karis Interview 1, lines 102-108)

In this statement, Karis related her success in completing this proof on the final exam to her understanding of the mathematics involved in the proof. This is consistent with Moore’s (1994) findings that understanding the mathematical ideas involved is very important in being successful in proving.

Karis contrasted this successful experience with one that occurred earlier in her college mathematics experience. This earlier experience involved memorizing a proof, a practice in which she engaged when she was less successful in proving in an earlier course, which she described as her “first proofs course” (Karis Interview 1, line 112). According to the official course description, this course consists of topics such as introduction to proofs, elementary number theory, and group theory. In discussing this unsuccessful proving experience, Karis again talked about her understanding of the mathematics, this time negatively.

In my first proofs course…, my first test, I did not do well at all, and I felt very unsuccessful in it because I did, I just memorized stuff to write down, and so then when something wasn’t quite right, we got a lot of points off because you weren’t saying the right thing. And so I guess I really learned from that that it wasn’t just memorizing and writing it down, I needed to know what it was that I was doing in each step to make sure all the little things were right along the way. (Karis Interview 1, lines 112-118)

Karis’ description of this experience can be compared to Weber’s (2001) description of the participants in his study who were less successful in constructing proofs. In Weber’s (2001) study, participants who were less successful, while they may have had the knowledge necessary to complete the proofs, did not seem to know which theorems and proof techniques were important or applicable to the given situations. In other words,
they lacked what he called “strategic knowledge” (Weber, 2001). While it is not completely clear from Karis’ description whether or not she actually knew the appropriate theorems, her description of memorization implies that she had at least studied and memorized those things she thought would be important, but that was not adequate for constructing the proofs. She needed the additional knowledge of “what I was doing in each step” (Karis Interview 1, line 117) in order to be more successful. That is, Karis acted as if a construct similar to Weber’s strategic knowledge was important to her being able to prove mathematical statements. Her warrants were as much about how to do the problems, sometimes in general, as they were about knowing particular definitions and theorems. Weber includes “knowledge of the domain’s proof techniques” (p. 111) and “knowledge of which theorems are important and when they will be useful” (p. 112) within strategic knowledge. Similarly, Karis went beyond stating that and acting as if being able to state a theorem was the most important thing. She prioritized knowing when, how, and why to apply the theorem, which is very similar to the second aspect of strategic knowledge as described by Weber.

Karis described the differences between feeling successful and feeling unsuccessful as hinging on her understanding of the mathematical topics involved in the proof. When asked, “If you had to, to kind of summarize what differentiates for you between feeling successful when you’re proving and feeling unsuccessful when you’re proving, what would you say?” (Karis Interview 1, lines 119-122), Karis began by saying, “I guess, first of all, understanding exactly what it is I’m trying to prove, and then from there, knowing enough about that topic to think about what I can use to get to the proof” (Karis Interview 1, lines 123-125). Thus Karis linked her success with proving to
her understanding of the mathematical ideas required for the proof rather than to any skills specific to proving.

Karis was also able to describe her process of generating a proof. I first asked Karis, “What would you say is involved in proving a mathematical statement?” (Karis Interview 1, lines 132-133). Karis’ answer indicates that she saw a mathematical proof as a sequence of steps.

I guess showing the steps to get to it, I guess starting with what they give you and using whether it’s algebra or reasoning to show the steps between what they give you and what you’re looking for and explaining each step. (Karis Interview 1, lines 134-138)

When prompted for more details, Karis gave the following explanation.

I guess I start with like a brainstorm and organize kind of thing…I try and decide like which type of proof I would use I guess, like whether it’s induction or contradiction so choose a method like that and then, I don’t know, just try things, just start trying to prove it. (Karis Interview 1, lines 145-155)

When asked how she knew when she was finished, Karis referred to the logical structure of a proof.

I guess you know when you’re finished when you’ve gone through everything and you’ve written it all out, and if you read through it, it makes sense in that someone else, even if they maybe haven’t studied the math could sit down and read it and they should be able to understand where you’re going in the proof and end up where you want them to, so that they agree, yes, that’s true. (Karis Interview 1, lines 164-170)

Karis’ description of being able to understand and validate a proof “even if they maybe haven’t studied the math” is reminiscent of Selden and Selden’s (1995) description of a “proof framework” as “a representation of the ‘top-level’ logical structure of a proof, which does not depend on detailed knowledge of the relevant mathematical concepts, but which is rich enough to allow the reconstruction of the statement being proved or one
equivalent to it” (p. 129). According to Selden and Selden, proof frameworks may be useful for constructing proofs, validating proofs, and learning new mathematics. Karis seems to be referring to validating a proof in her description.

Karis’ description of her process when proving, especially her explanation of how she knew when she was finished, lends support to the claim that she was confident and able to prove mathematical statements in appropriate mathematical areas. However, her confidence about proving and her ability to prove are both linked to her understanding of the mathematics involved in the proof.

Karis believed that understanding mathematics was important for proving, and also that proving was important for understanding. Her view about understanding and its link to proving and explaining seems to be that it is important to understand the mathematics in order to be able to do the proof; it is important to prove in order to be able to explain to others so that they can understand the mathematics. That Karis wanted her students to understand the mathematical concepts that she was teaching and saw explaining as important in demonstrating that understanding can be inferred from several questions on her tests. For example, one of the questions on a test was “In your own words, explain Rolle’s Theorem” (Karis Calculus Test, Limits to Infinity, Rolle’s Theorem, and the Mean Value Theorem). Karis wanted her students to be able to explain mathematical ideas. That she considered knowing relevant theorems, definitions, procedures, and properties to be important can also be seen in the kinds of warrants she gave in support of arguments. Karis called on theorems, definitions, and properties for many of her warrants. At times, she would name the theorem, such as the Mean Value
Theorem. At other times, she would describe and explain procedures as warrants within arguments.

Understanding the mathematical concepts involved was important to Karis, both for herself when she was proving and for her students. The other important aspect of proving for her was audience.

**Audience Is Important**

Karis showed a tendency to consider her mathematical audience when constructing proofs and evaluating arguments. In the first interview, Karis was asked to prove a mathematical statement: The sum of the first $n$ natural numbers is $\frac{n(n+1)}{2}$. Karis’ first question was to clarify the audience for whom she was writing: “Well I guess the first thing that comes to mind is induction to show them, does it, am I supposed to like aim for a certain age level or anything?” (Karis Interview 1, lines 179-181). When asked to prove it for one of her peers, she proved it by induction as shown in Figure 4-3.
Figure 4-3: Karis’ Written Work for Her Induction Proof

I then asked, “Do you think this would convince just anyone you met on the street?”

(Karis Interview 1, lines 208-209). Karis replied by saying she would use a different approach and then wrote what is shown in Figure 4-4 as she said the following.

So if we just took like $n$ numbers, and if I line them up backwards, then $n$ plus $n$ minus one plus $n$ minus two [writes Equation 4-20]

\[
1 + 2 + 3 + \ldots + n \\
= n + (n - 1) + (n - 2) + \ldots + 1
\]

back down to one, and then I sum this way [indicates that she would add columns ($1+n$, etc.)], what happens is that I can notice a pattern, and here
I’ll get \( n \) plus one, and for these two I get \( n \) plus one also, because it’s \( n \) minus one plus two, which is \( n \) plus one, and here I have three plus \( n \) minus two, so this is also \( n \) plus one, and so I notice this pattern all the way down, so that they’re always \( n \) plus one [writes Equation 4-21].

\[
\begin{align*}
1 + 2 + 3 + \ldots + n \\
\frac{n + (n-1) + (n-2) + \ldots + 1}{n + 1} + \\
\frac{n-1 + 2 + 3 + n - 2}{n + 1} + \\
\frac{n}{n + 1}
\end{align*}
\]

Eq. 4-21

And so this would happen then \( n \) times, cause I have \( n \) different numbers, so we could think of it as \( n \) times \( n \) plus one, is what we have. But what we’re actually adding there then is two \( n \) numbers, so you divide by the two. And so that’s why the first sum, the sum of the first \( n \) numbers is \( n \) times \( n \) plus one over two [writes \( \frac{n(n+1)}{2} \)]. (Karis Interview 1, lines 218-230)

Karis chose two different methods to prove the statement, and these methods were dependent on her views of the mathematical sophistication of her audience. When asked to prove a statement to her peers, she chose the relatively mathematically sophisticated method of induction, while using a more visual argument based on patterns when describing how she would prove it to a less mathematically sophisticated audience.

---

Figure 4-4: Karis’ Written Work for Her Pictorial Proof
Karis also was concerned about the audience for the proof she was asked to analyze. When asked to critique an argument written by someone else that the square root of a positive integer is either an integer or it is irrational (see Figure 4-5), Karis asked from what point of view she was to critique it. She said, “Am I supposed to just analyze it like kind of as a mathematician and like the clarity of it? Or like from the point of view of a student?” (Karis Interview 1, lines 332-334). Thus Karis took audience into consideration when reading and critiquing an argument written by someone else. In particular, she considered a student’s viewpoint as different from that of a mathematician and possibly as different from her own. She concluded that the proof worked from the point of view of someone who was knowledgeable about mathematics, but said that it might not convince a high school student.

The square root of a positive integer is either an integer or it is irrational.

Suppose \( n \) is a positive integer, \( \sqrt{n} \) is not an integer, and \( \sqrt{n} \) is rational. Then there exist relatively prime integers \( a \) and \( b \) with \( \sqrt{n} = \frac{a}{b} \). Squaring both sides, \( n = \frac{a^2}{b^2} \), from which \( nb^2 = a^2 \). Now if we factor \( a \) and \( b \) into primes, there are no common factors. So the factorizations of \( a^2 \) and \( b^2 \) have no common factors. Consequently, the factorizations of the equal numbers \( nb^2 \) and \( a^2 \) are different. Since two different factorizations of \( a^2 \) are impossible by the Fundamental Theorem of Arithmetic, the supposition must be false. So if \( \sqrt{n} \) is not an integer, it must be irrational.

Figure 4-5: Square Root Argument to Critique (from Usiskin, Peressini, Marchisotto, & Stanley, 2003, pp. 23-24)
Later in the first interview, Karis was shown three arguments in support of the Triangle Inequality Theorem. The first argument used a researcher-created dynamic Geometer’s Sketchpad-situated sketch (see Figure 4-6). This dynamic sketch allowed Karis to drag any vertex of the triangle and see the measures of each side compared to the sum of the other two. The second argument was altered only slightly from a geometry textbook (Schultz, Hollowell, Ellis, & Kennedy, 2001, p. 274) and was presented as seen in Figure 4-7. The final argument was taken from Euclid’s proof of the Triangle Inequality Theorem and was presented as in Figure 4-8.

\[
\begin{align*}
\text{m}_{BC} + \text{m}_{CA} &= 11.63 \text{ cm} \\
\text{m}_{AB} + \text{m}_{CA} &= 11.02 \text{ cm} \\
\text{m}_{AB} + \text{m}_{BC} &= 8.29 \text{ cm} \\
\text{m}_{BC} &= 4.45 \text{ cm} \\
\text{m}_{CA} &= 7.18 \text{ cm}
\end{align*}
\]

Drag any point. The sum of the measures of any two sides of the triangle is greater than the measure of the other sides.

Figure 4-6: Geometer’s Sketchpad-situated Argument for the Triangle Inequality Theorem
The Triangle Inequality Theorem
In any triangle, the sum of any two sides is greater than the remaining one.

Argument 2

Case 1:
Given three segments, if the sum of the lengths of any two of them is less than the length of the third segment, then no triangle can be formed by connecting their endpoints. The endpoints of two of the segments cannot be connected. (See figure below.)

Case 2:
Given three segments, if the sum of the lengths of any two of them is equal to the length of the third segment, then no triangle can be formed by connecting their endpoints. The endpoints, and hence the segments themselves, are collinear (justification: the Betweenness Postulate). (See figure below.)

Figure 4-7: Geometry Textbook Argument for the Triangle Inequality Theorem (Schultz, Hollowell, Ellis, & Kennedy, 2001, p. 274)
Figure 4-8: Euclid’s Argument for the Triangle Inequality Theorem

When asked whether or not each argument would convince herself, a high school geometry student, a college geometry student, a college mathematics professor, and a high school geometry teacher, Karis claimed that all three arguments would convince herself, a high school geometry teacher, or a high school geometry student. She said that a college geometry student or college math professor would not be convinced by the Geometer’s Sketchpad-situated argument, but would be convinced by the other two arguments. When asked whether each argument was a proof, she replied that the Geometer’s Sketchpad-situated argument was not a proof, the geometry textbook
argument was a form of proof but not rigorous, and Euclid’s argument was a proof and would be most convincing to herself and a high school geometry teacher. When asked to elaborate on what she meant by a form of proof but not rigorous, Karis stated, “It’s just not um a proof you’d find in like a maths book in the math library. I don’t know. It’s not like, I guess it’s more of a hands-on type of explanation with the diagrams” (Karis Interview 1, lines 438-442).

After examining all three of the arguments above, Karis stated that a high school geometry teacher would be most convinced by the third argument, finding it most convincing because “It uses a lot of the theorems and concepts given in the class, so it really kind of pulls everything together and, I don’t know, states everything with these clear, set ideas in there” (Karis Interview 1, lines 535-537). In addition, she herself would be most convinced by the third argument because it was most like the proofs she had seen. However, she stated that a high school geometry student would probably find the first argument to be the most convincing “because it’s dynamic, they can check out all kind of cases” (Karis Interview 1, lines 520-521).

When asked to elaborate on how to convince various audiences of the truth of mathematical statements, Karis gave several general rules. To convince a peer, use a form of proof with visuals. To convince a mathematics professor, use precise theorems and official terms. To convince a student, use examples first, then show why the concepts and theorems apply to prove it. Karis seemed to see proof as fluid, not static, meaning different things to different people. She preferred to consider each argument in context, analyzing or constructing it with reference to its audience. She was able to construct two variations of a proof for different audiences, and her responses utilized precise
Karis’ concern for her audience can be seen most clearly in the warrants she left implicit in her classroom. Karis left the warrant implicit in 24 arguments. These arguments can be separated into two categories by their mathematical content: arguments that address new material or ideas that are new to the students in a calculus class and arguments that address mathematical ideas that students could reasonably be assumed to know from classes previous to calculus. For arguments in the first category (new material), Karis left the warrant implicit only if a warrant was provided for a similar argument within that class period. Karis’ rationale for withholding the warrants for arguments in both of these categories could be related to her perception of her audience (her students). That is, she may have thought that her students did not need to hear an explanation for ideas that they already knew, and she may have defined what they already knew as things they had talked about in that class and things they should have know from previous classes.

Karis’ concern for the audience of a proof is consistent with the way proofs have been accepted by the mathematical community throughout history. Dreyfus (1999) illustrates this fact by two examples of proofs that were convincing to earlier mathematicians but would now not be recognized as proofs. In addition, he calls attention to historical disagreements between mathematicians who were contemporaries about whether particular arguments were legitimate proofs. Thurston (1995), in describing his work as a mathematician, describes a process of careful scrutiny by which a proof is accepted; this careful scrutiny is a social process and thus related to audience. Hersh
(1993) goes farther, as he defines proof as “convincing argument, as judged by qualified judges” (p. 389), and reminds readers that “formal proof can exist only within a formalized theory…has to be expressed in a formal vocabulary founded on a set of formal axioms, reasoned about by formal rules of inference” (p. 390). He goes on to say that most mathematical proofs are not formal, although some would say that they could “in principle…be turned into formal proofs” (p. 391). Hanna (1990) aptly sums up the importance of audience in proving by stating, “The acceptance of a theorem by practicing mathematicians is a social process which is more a function of understanding and significance than of rigorous proof” (p. 8). While Karis’ concern seems to be less for the formality of the proof and more for the ability of her audience to understand, her concern for audience reflects that of the mathematical community.

Conclusion

Karis’ views of justification as explanation may explain her choices to provide or withhold warrants during classroom argumentation. Her support for argumentation included providing warrants for many arguments, and she often provided expanded explanations for students’ data and warrants. Karis’ view of the importance of her audience, when considered in light of her view of the importance of explanation, may explain why she did not provide warrants when they involved mathematical ideas prerequisite to calculus or had been previously discussed in the class session. She did not seem to perceive a need for explanation in these cases. Karis’ view of the importance of understanding mathematical ideas may explain her use of theorems, definitions,
properties, and procedures, including explanations of these, as warrants. Her provision of warrants as a way of supporting classroom argumentation does seem to align with her view of the importance of proof and justification in explaining the background of the mathematics to students.
Chapter 5

Lynn: “I’m a Reason Person”

Lynn’s classroom was full of energy, and activity in her classroom progressed at high speed. Lynn expected and encouraged her students to participate in class, and infused the classroom with energy that overflowed from her outgoing personality and contagious laugh. Lynn seemed to take a personal interest in each of her students; this was evident in her individual interactions with them. Lynn saw the role of proof in mathematics as personally relevant for her and her students. She described the personal relevance of proof by saying, “I’m a reason person” (Lynn Interview 2, line 153). In this chapter, I describe the argumentation in Lynn’s classroom and how she supported that argumentation. I then outline how this support for argumentation aligns with her conception of proof.

Argumentation in Lynn’s Classroom

Classroom Setting

Lynn’s student teaching experience included teaching several sections of geometry. The geometry class on which this analysis focuses consisted of twenty-seven students who seemed to be representative of the racial and socio-economic characteristics of the general population of the school. Desks were arranged in six rows of five desks, all facing the front of the room. The classroom in which she taught was equipped with a
small chalkboard at the front of the room, an overhead projector at the front of the room, and a larger chalkboard to the side of the room. Lynn generally taught from either the front or side of the room, illustrating her points by writing on one of the chalkboards or on the overhead projector. Often Lynn had problems or figures relating to problems written on one of the boards for students to copy as they started class.

Lynn encouraged her students to participate verbally in class. She called upon her students to answer questions whether they volunteered or not. Students in this class rarely responded directly to one another’s statements or questions. Instead, students generally answered Lynn’s questions or asked questions of Lynn, and she answered their questions or redirected questions to other students. She expected students to take notes in class, and encouraged them to compile a list of theorems from each chapter.

Lynn structured her class time in several different ways during the classes that were observed. At various times, the class engaged in discussing problems completed as homework, completing and discussing proofs of mathematical statements, solving problems involving missing measures, and discussing properties and definitions relating to geometric figures. Some classes involved material that was new to the students, while other classes were dedicated to reviewing material that had been previously discussed with the students. Some class time was devoted to individual or small group work on assigned problems, but most class time was used for whole class discussion. The geometry classes observed focused on triangle inequality theorems; properties of parallelograms; and ratio, proportion, and similarity.
Influence of Mentor

Lynn’s mentor regarded her as a capable student teacher, who brought creativity in choices of tasks and activities to his classroom. He characterized her classroom practice as different from his own in that he tended to “lecture” (Mr. B Interview, line 134) and then give time for them to work on homework assignments, while she used a variety of other practices (e.g., involving students in completing problems on the board, asking students to complete problems individually on white boards at their desks, “sending them all around the room” [Mr. B. Interview, line 141]). Mr. B claimed that he stressed justification as important for his students, and suggested that he normally asked students to justify their answers rather than telling them whether their answers were right or wrong. However, when asked if he thought this influenced the questions Lynn asked, Mr. B seemed to imply that Lynn asked different questions, and that Lynn’s preference for justification (as discussed in the remainder of this chapter) was not something that she had learned from his classroom practice. However, it is possible that students were more receptive to whatever type of questions Lynn asked simply because they were accustomed to answering questions from Mr. B.

Typical Argumentation

A typical argument in Lynn’s class involved Lynn and one or more of her students. More episodes of argumentation were observed during the introduction and discussion of new material than during any other particular type of classroom activity. Of the fifty-five episodes observed in her classes, twenty-seven (or nearly one half) occurred
during the introduction of new material. The other episodes were approximately evenly divided between those that occurred when going over homework and those that occurred when discussing review problems. A typical episode of argumentation began with Lynn writing a problem or statement on the board or overhead and soliciting student claims about the problem or statement. For instance, Lynn started the second class with the following statements and questions.

I’d like you to draw this figure, triangle PDQ. We are going to do a proof today, so draw this figure. All right, let’s take a look at our figure up here, figure QPD, that triangle, okay? My given information says start with the first statement, QC, which is this segment right here, it bisects PD. Okay? Then I say that measure of angle PCQ, this angle right here, is greater than measure of angle DCQ, this angle right here. Okay. What I want to prove in the end is that angle D, the measure of angle D right here, is greater than the measure of angle P, that angle right there. Okay, now, to start our proof, let’s start with the given information. …QC bisects PD. What does it mean to bisect? What does it mean to bisect, Student? (Lynn Geometry Class, February 21, lines 12-23)

As seen above, Lynn often referred students to previously learned theorems, definitions, or properties of the objects to be discussed as she introduced the problem or statement in question.

Of the fifty-five episodes of argumentation identified in Lynn’s class, thirty-four involved only a main argument and twenty-one contained a main argument with one or more sub-arguments. (For a description of sub-arguments, see Toulmin’s Model of Argumentation in Chapter 2.) Altogether, eighty-seven sub-arguments were identified, yielding an average of approximately four sub-arguments per episode containing sub-arguments. These sub-arguments were often connected into a chain of argumentation involving sub-arguments, sub-sub-arguments, and sometimes even arguments in support of these sub-sub-arguments.
The following two episodes were selected as typical of the argumentation in Lynn’s classes. The episodes are both from the first class observed, and were selected to illustrate the two most prevalent kinds of episodes of argumentation in Lynn’s class: episodes containing only a main argument and episodes containing a main argument and several sub-arguments. In addition, the distribution of claims, data, and warrants attributed to the various involved parties in these examples is consistent with the overall distribution of these attributions throughout the data. That is, in general, students made more claims, alone and with Lynn’s assistance, than Lynn made alone. This is reflected in the claims in this example, which are made by the students and by both Lynn and her students, but not by Lynn alone. Warrants in Lynn’s class were given by either students alone, Lynn, both Lynn and students, or left unspecified. There are examples of each of these in this episode.

The first episode contains a main argument and several sub-arguments and occurred in the middle of class on February 20. Lynn began this class by explaining the SAS (Side–Angle–Side\textsuperscript{9}) and SSS (Side–Side–Side\textsuperscript{10}) triangle inequality theorems. She then drew a figure on the board and asked questions about the relationships of sides and angles within the figure. The first episode falls within this part of class. The second

\textsuperscript{9} Lynn used the following statement of the SAS Inequality Theorem in her class: “SAS Inequality/Hinge Theorem – if two sides on one triangle are congruent to two sides of another triangle, and the included angle in one triangle is greater than the included angle in the other, then the third side of the first triangle is longer than the third side in the second triangle” (Lesson Plan “Inequalities involving two triangles” for February 20-21).

\textsuperscript{10} Lynn used the following statement of the SSS Inequality Theorem in her class: “SSS Inequality – if two sides of one triangle are congruent to two sides of another triangle and the third side in one triangle is longer than the third side in the other, then the angle between the pair of congruent sides in the first triangle is greater than the corresponding angle in the second triangle” (Lesson Plan “Inequalities involving two triangles” for February 20-21).
episode occurred toward the end of class on February 20 and contains only a main
argument. Lynn drew a different figure on the board, asked students to quantify
relationships between sides and angles independently, and then discussed their answers
with them.

Typical Episode 1: Main Argument With Sub-arguments

As this episode began, Lynn had drawn Figure 5-1 on the board and had written

\[ m\angle AFB \quad m\angle BFD \]

on the board.

![Figure 5-1: Quadrilateral ABDC from Typical Episode 1](image)

Claim: \( m\angle AFB > m\angle BFD \)

Lynn says, “All right, the second pair of angles that I’d like you to look at,…I’d like you to look at measure of angle AFB and measure of angle BFD, so AFB, this angle right here, I want you to compare that to measure of angle BFD.” Lynn then asks, “Now, looking at this figure, do we have two triangles that we could say have two congruent sides?” A student answers, “Yeah, we do because um if F bisects AD, then AF and FD are congruent.” Lynn says, “Okay, AF and FD are congruent and they’re already marked congruent because I said F was our midpoint.” A student says, “And then BF is reflexive to itself so it’s the other congruent side.” Lynn says, “Right, so these two triangles here we can compare. We can compare AF to FD, those are congruent, and then FB is congruent to itself because of the reflexive property, very good. Now, do we have an angle measure?” A student answers, and Lynn says, “No, so which inequality do you think we are going to use?” A student says, “SSS.” Lynn replies, “Right, side-side-side because what do we know about these sides? What do we know? What is AB?” A student says, “AB is greater than BD.”
Lynn asks, “You said AB is greater than what?” The student says, “BD.” Lynn asks, “BD, what is AB?” A student says, “5.” Lynn asks, “And BD is?” A student says, “3.” Lynn says, “Okay, so if AB is bigger than BD, Student, what do you think we can say about angle AFB and BFD? Which one’s larger?” A student says, “AFB.” Lynn replies, “AFB is larger. So AFB is greater than BFD. Good. Any questions on that one?” (Lynn Geometry Class, February 20, lines 161-192)

The claim, data, and warrants of the main argument, three sub-arguments, and two sub-sub-arguments were identified according to Toulmin’s (1964) description of argumentation. The general description of claim, data, and warrant, as well as a definition of sub-argument as used in this analysis can be found in Chapter 2 in the section titled “Toulmin’s Model of Argumentation.”

The main argument has the following components:

- **Claim by students:** $m\angle AFB > m\angle BFD$.
- **Data by both Lynn and her students:** We have two triangles with two sides congruent; we can use the SSS Inequality Theorem; $AB > BD$.
- **Warrant by both Lynn and her students:** SSS Inequality Theorem.

The first sub-argument has the following components:

- **Claim by both Lynn and her students (asked by Lynn, student answers):** We have two triangles that we could say have two congruent sides.
- **Data by students:** Segment AF and segment FD are congruent and segment BF is the other congruent side.
- **Warrant:** Unspecified (possibly segments AF and BF are in triangle AFB and segments FD and BF are in triangle DFB).

This first sub-argument has two sub-arguments of its own. These are called sub-sub-arguments hereafter. The first sub-sub-argument has the following components:
• Claim by students: $\overline{AF} \cong \overline{FD}$.

• Data by students: F bisects $\overline{AD}$.

• Warrant by Lynn: “I said F was our midpoint.”

The second sub-sub-argument has the following components:

• Claim by students: $\overline{BF} \cong \overline{BF}$.

• Data by students: It ($\overline{BF}$) is itself.

• Warrant by students: Reflexive [property].

The second sub-argument has the following components:

• Claim by students: We are going to use the SSS Inequality Theorem.

• Data by both Lynn and students: We do not have an angle measure; we know something about the sides.

• Warrant: Unspecified (possibly statement of the SSS Inequality Theorem).

The third sub-argument has the following components:

• Claim by students: $AB > BD$.

• Data by both Lynn and students: AB = 5; BD = 3.

• Warrant: Unspecified ($5 > 3$).

This episode of argumentation, considered as individual parts above, can also be visually depicted by a diagram as seen in Figure 5-2. For the general structure of diagrams like this, see Figure 3-6 and the accompanying explanation in Chapter 3. The colors in the diagram indicate to whom each component of each argument was attributed: red for Lynn, blue for students, and violet for collaborative effort between Lynn and her students. Examining the colors in this diagram gives insight into patterns of
argumentation in Lynn’s class. In this episode, the arguments are constructed collaboratively by Lynn and her students. Only one of the sub-sub-arguments was attributed wholly to the students (and this is, in fact, the only argument observed in any of Lynn’s classes in which the students contributed all three components of an argument).
We have two triangles with two sides congruent; we can use the SSS Inequality Theorem.

Unspecified

Since

We have two triangles that we could say have two congruent sides. (Asked by Lynn; student answers yes)

We do not have an angle measure. (Asked by Lynn, student answers) We know something about the sides

Unspecified

Since

I said F was our midpoint.

AF and FD are congruent and BF is the other congruent side

So,

AF ≅ FD

F bisects AD

Since

It is itself

So,

BF ≅ BF

Reflexive [property]

Since

We are going to use the SSS inequality theorem

Since

Unspecified

AB = 5; BD = 3

So,

AB > BD

Unspecified

Since

Unspecified (5>3)

So,

m∠AFB > m∠BFD

SSS Inequality Theorem

Figure 5-2: Diagram of Example of Typical Longer Episode of Argumentation in Lynn’s Geometry Class (from Lynn Geometry Class, February 20, lines 161-192)
**Typical Episode 2: Main Argument**

Toward the end of class on February 20, Lynn assigned three problems (numbered 5, 6, 7) for students to work individually. All three problems dealt with Figure 5-3, which Lynn drew on the board while students were working.

![Figure 5-3: Triangle ACD](image)

The episode began after students had worked quietly for several minutes.

Claim: $\angle ABF > \angle DBF$

Lynn says, “We want to compare the measure of angle DBF and ABF.” She asks, “Which one do you think is going to be bigger?” A student says, “Angle ABF.” Lynn says, “Angle ABF is going to be greater, so, ABF is going to be greater, and how did you decide that?” A student says, “AF is bigger than FD.” Lynn says, “Good, AF is 10, and FD is only 8. They share a side of 6, and nine and nine are also congruent AFB and FBD.”

(Lynn Geometry Class, February 20, lines 341-350)

This short episode contains only a main argument, the components of which were attributed to Lynn and her students as follows:

- **Claim by students:** $\angle ABF > \angle DBF$.
- **Data by students:** “AF is bigger than FD.”

---

11 This figure is not constructible as labeled ($\Delta ABD$ has side lengths not meeting the requirements of the Triangle Inequality Theorem). However, this was not mentioned in class, apparently not being noticed by either Lynn or her students.
Warrant by Lynn: “AF is 10, and FD is only 8. They share a side of 6, and nine and nine are also congruent.” (Lynn seems to be referring to two triangles [AFB and FBD] that have two congruent sides; AF is opposite $\angle AFB$ and FD is opposite $\angle DBF$, so she is using the SSS Inequality Theorem without mentioning it by name.)

The structure of the argument outlined in the bulleted list above can be diagrammed as in Figure 5-4. The diagram in Figure 5-4 illustrates that both the claim and data were attributed to students, while the warrant was attributed to Lynn. Many, but not all, of the episodes containing only main arguments contain similar patterns, with some components of an argument attributed to the students and other parts attributed to Lynn. In addition, the warrant is specified in this main argument, as it is in all but nine of the 34 episodes containing only main arguments.

![Diagram]

Figure 5-4: Diagram of Example of Typical Episode Containing Only a Main Argument in Lynn’s Geometry Class (from Lynn Geometry Class, February 20, lines 341-350)
**Backings for the Typical Arguments**

Toulmin describes a fourth component of argumentation: backing. The backing for an argument is seldom stated, but it can sometimes be inferred from warrants that are stated. Backings provide the authority for the warrants in an argument (Toulmin, 1964); thus, examining Lynn’s backings gives insight into who or what provides the authority for the arguments in her classroom.

Three warrants were specified within the first example of argumentation. The warrant for the main argument was the SSS Inequality Theorem; the warrant for the first sub-sub-argument was “I said it was the midpoint,” and for the second sub-sub-argument was the reflexive property. Two of these warrants call upon previously stated theorems or properties, while the third involves what amounts to a restatement of a property of the mathematical object under consideration. When Lynn and her students used these warrants, they seemed to be relying on the mathematical authority of theorems and properties that had been accepted as true by the mathematical community of their classroom. It is not clear from this episode precisely why these had been accepted by this community, although from some later observations, hypotheses can be made. At one point Lynn asked students to look back in their books to remember a particular theorem or property. This suggests that theorems that were recorded in the book and had been previously discussed were accepted as true by their classroom community. At another point there was a conversation between Lynn and a student about whether or not one could use a theorem that had just been proved in the process of proving something else. Thus one authority for arguments was what had been proved by the class, at least from
Lynn’s point of view. It is not clear if this was an understanding that was shared by the students, or if they accepted the validity of previously proved theorems because of Lynn’s authority as the teacher.

The warrant for the argument described in the second example episode was “AF is 10, and FD is only 8. They share a side of 6, and nine and nine are also congruent” (Lynn Geometry Class, February 20, lines 101-102). As explained above, Lynn seemed to be referring to two triangles (AFB and FBD) that had two congruent sides; she also seemed to be using the fact that AF was opposite $\angle ABF$ and FD was opposite $\angle DBF$. Thus Lynn, in this warrant, was using the SSS Inequality Theorem, although the theorem itself was not mentioned. This warrant, even though it was not clearly specified, lends further credence to the claim that Lynn’s backings were the theorems and properties accepted as true in this class. In this case, she was calling on some elements of the theorem without referring to it by name.

**Parts of Arguments Attributed to Lynn and Her Students**

The previous section describes two different episodes that typify the argumentation in Lynn’s classroom. However, there are aspects of the argumentation in Lynn’s classroom that are apparent only after closely examining the patterns that occur within individual arguments, as opposed to larger episodes of argumentation, in her class. While 55 episodes of argumentation were observed, a total of 142 arguments were identified. Ten of these arguments occurred when a student made an incorrect claim, and so in addition to some combination of claim, data, and warrant, these arguments included
a counter-claim, the negation of a claim, or both. The other 132 arguments each included a claim and either data, warrants, or both.

In order to examine Lynn’s support of classroom argumentation, each component of each individual argument was attributed to Lynn, her students, both, or neither, as described in Chapter 3. Table 5-1 contains counts of arguments whose components were attributed to Lynn, her students, both, or neither. These counts, together with the organization of the table, form a context from which a particular pattern of Lynn’s support for classroom argumentation can be identified. In addition, these counts indicate which particular arguments might be more useful to examine in more detail in order to answer relevant questions. For instance, Table 5-1 indicates that Lynn provided warrants for 44 arguments (see last line of the column labeled Teacher under Warrant). For which of these arguments was Lynn solely responsible for the argument, and how are these arguments different from those in which others participated? What combination of student and teacher input characterized the claims and data for these arguments, and what was Lynn’s role in supporting these claims and data?
Table 5-1: Parts of Arguments as Attributed to Lynn and her Students

<table>
<thead>
<tr>
<th>Claim</th>
<th>Data</th>
<th>Warrant</th>
<th></th>
<th></th>
<th></th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Student</td>
<td>Teacher</td>
<td>Both</td>
<td>Neither (implied)</td>
</tr>
<tr>
<td>Student</td>
<td>Student</td>
<td>1</td>
<td>7 (1)</td>
<td>2</td>
<td>2</td>
<td>50 (5)</td>
</tr>
<tr>
<td></td>
<td>Teacher</td>
<td>4 (1)</td>
<td>2 (1)</td>
<td>2</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Both</td>
<td>2</td>
<td>6</td>
<td>4</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Neither</td>
<td>5</td>
<td>0 (1)</td>
<td>1</td>
<td>0 (1)</td>
<td></td>
</tr>
<tr>
<td>Teacher</td>
<td>Student</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>33 (3)</td>
</tr>
<tr>
<td></td>
<td>Teacher</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>5 (3)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Both</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Neither</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Both</td>
<td>Student</td>
<td>0</td>
<td>1 (1)</td>
<td>2</td>
<td>4</td>
<td>49 (2)</td>
</tr>
<tr>
<td></td>
<td>Teacher</td>
<td>1</td>
<td>9 (1)</td>
<td>0</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Both</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Neither</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>19 (1)</td>
<td>40 (5)</td>
<td>19</td>
<td>54 (4)</td>
<td>132 (10)</td>
</tr>
</tbody>
</table>

Numbers in parentheses indicate arguments containing an incorrect claim. Each argument is counted once, so the total number of arguments is 142.

The next several sub-sections within this section explicate patterns in arguments in which warrants were either made explicit or left implicit. Of particular consideration is who provided warrants to link the data and claims within arguments. Lynn’s role in providing or prompting claims, data, and warrants or in leaving data or warrants implicit is explored. Within the following sub-sections, the counts used to describe the frequency of occurrence of the classified arguments can be seen in summary form in Table 5-1.

Arguments With Warrants Attributed to Lynn

Lynn provided several kinds of warrants as bridges between data and claims within argumentation in her classroom. The warrants Lynn provided tended to be
statements of theorems, statements of properties, and general references to properties, with a few statements of procedures, appeals to previous knowledge, or appeals to visual cues such as a figure or a demonstration. (A demonstration was generally a physical example or model of the situation. It was usually dynamic, as when Lynn used the large chalkboard compass and a rubber band to illustrate the “Hinge Theorem” for her class.) As an example of using a theorem as warrant, Lynn’s warrant on February 22 for the connection between a student’s claim that \( m\angle D > m\angle P \) in a particular figure and the piece of data that “PQ is greater than PD and the side opposite” was “you could say [theorem] 5-9 or you could put a little note larger angle opposite larger side” (Lynn Geometry Class, February 22, line 136). On other occasions, Lynn simply stated all or part of a theorem as a warrant without naming it. For instance, Lynn used “two triangles that have two congruent sides” (Lynn Geometry Class, February 22, line 107) as a warrant when the claim involved writing an inequality relating sides of two triangles and the data involved statements of which sides and angles were opposite each other. The use of the SSS Inequality Theorem as a warrant was implied here, even though Lynn only stated part of the theorem.

Lynn also used general references to properties as well as statements of properties as warrants. For instance, on March 7, Lynn’s warrant for the main argument in which the values of \( x \) and \( y \) were found in the parallelogram shown in Figure 5-5 was a general reference to properties: “We used our properties” (Lynn Geometry Class, March 7, line 290). Within the same episode of argumentation on March 7, when giving a warrant for a sub-argument, Lynn stated a specific property of parallelograms: “Consecutive interior angles are supplementary” (Lynn Geometry Class, March 7, line 280).
In general, most of Lynn’s warrants could be classified as relating to properties or theorems. However, on six occasions Lynn used a statement like “Now this is a little bit more of prior knowledge” (Lynn Geometry Class, February 20, line 197), or a procedure such as “you set up a proportion…and then you just cross-multiply” (Lynn Geometry Class, April 4, lines 218-219) when relating data to a claim.

**Arguments With Warrants Attributed to Students or Both**

Nearly all of the warrants that were given by students were given in response to a direct request from Lynn for a warrant. All but one of the warrants were names of theorems or properties of geometric objects. For instance, the following episode of argumentation was observed on April 4. The students had completed some problems from their textbook for homework, and the episode occurred as Lynn and her students discussed the second of the completed problems. The triangles being referenced were not drawn on the board but were in the textbook.

Claim: $\angle ABF > \angle DBF$

Lynn says, “For number 14, similarly, now we use the congruent marks. Using the congruent marks, are those triangles similar? Are those triangles similar, Student?” The student says, “Yes.” Lynn says, “Yes, they are
similar, because we have one tic mark, the angle in between them, and two
tic marks, so we would say yes, by what theorem?” A student says, “Side-
angle-side.” Lynn says, “Side-angle-side, okay, and how would we name
them similar? Say my first one I named it STU. How would I name that
next one? Student?” The student says, “XVW.” Lynn says, “XVW, good,
just follow your tic marks there,” and writes \( \triangle STU \sim \triangle XVW \).

In this argument, a student gave “Side-angle-side” (Lynn Geometry Class, April 4, line
44) as the warrant for a student’s claim that the triangles were similar and Lynn’s data,
“because we have one tic mark, the angle in between them, and two tic marks” (Lynn
Geometry Class, April 4, line 42). Lynn’s statement of data was immediately followed by
a question, “by what theorem?” (Lynn Geometry Class, April 4, line 43). In each case
when a student provided a warrant in an argument, Lynn asked a similar question,
sometimes directly asking for a theorem, and other times asking questions such as “Why
can we say that?” (Lynn Geometry Class, March 3, line 299) or “What’s our reason?”
(Lynn Geometry Class, March 3, line 313).

Five arguments were observed in which students gave both a claim and a warrant,
but the data was not specified, as shown in the fourth line of Table 5-1. In each of these
cases, the unspecified data is something that might be read from the figure included with
the problem. In fact, the claims for three of these arguments were that a specific segment
is congruent to itself, with the warrant being the reflexive property. In each of these
cases, Lynn’s support for the argumentation consisted of asking what sides were
congruent and then asking why they are congruent. While a question of why something is
true could prompt a response that included either data or a warrant, the students could
have interpreted the question as requesting why the data that could be read from the
figure related to the statement that the sides are congruent. The other two arguments are
similar in that a claim was made, then Lynn asked why it was true, and in response the student provided a warrant by stating a property or theorem, leaving the data implicit. For example, the following excerpt is from an episode of argumentation that took place on March 6.

Claim: In the parallelogram, \( z = 120 \)
The student says, “And then \( z \) is equal to 120?” Lynn says, “Yes, and then Student said \( z \) will equal to one twenty. And how did you conclude that? How did you know that \( z \) is equal to one twenty? What did you use?” The student says, “Opposite angles are congruent.” Lynn says, “Right, Student used the fact that opposite angles are congruent to conclude that \( z \) is equal to one twenty. (Lynn Geometry Class, March 6, lines 243-248)

This excerpt was part of a larger episode of argumentation, during which Lynn had drawn Figure 5-6 on the board. The fact that this was a parallelogram was stated earlier in the episode; it seems that Lynn and her students could view the fact that the angles with measures 120 degrees and \( z \) degrees are opposite angles in a parallelogram as apparent from the figure. In cases like this, Lynn’s students often interpreted questions of how something was concluded or why something could be concluded as requests for warrants. They provided theorems or properties as warrants in answer to these requests even when the data were not provided.

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Figure 5-6: Figure for Example of Student Argument With Unspecified Data
Relatively few arguments were attributed solely to Lynn’s students. One of these arguments was the second sub-sub-argument described in Typical Episode 1, at the beginning of this chapter. In this argument, located in the first column, first row of Table 5-1, students provided the claim, data, and warrant. Lynn’s support of this argument was limited to asking if there were two triangles with two congruent sides and then restating the warrant for the claim ($\overline{BF} \cong \overline{BF}$) by saying, “And then FB is congruent to itself because of the reflexive property, very good” (Lynn Geometry Class, February 20, line 175). This was unusual for an argument in Lynn’s class in that she neither provided nor asked a student to provide a warrant, yet one was provided.

When warrants were given by a combination of both Lynn and her students, they again appealed to theorems, definitions, and properties. The difference between Lynn supporting a warrant that was attributed to students and a warrant being attributed to a combination of Lynn and her students depended upon the type of question that was asked to elicit the warrant. (For more information on how parts of arguments were attributed to Lynn, her students, or both, see page 55 in Chapter 3.) For instance, when solving for one of the missing angles in Figure 5-7, Lynn asked the following series of questions.

How do you think we might solve for this angle up here? How do you think we might do that? Student, how do you think we might have to do that? Remember, this is a straight line, how many degrees are in a straight line? (Lynn Geometry Class, March 7, lines 150-152).

When a student answered, “180,” (Lynn Geometry Class, March 7, line 153) he contributed to the warrant, but the warrant for the argument was, for the most part, introduced by Lynn. Thus the warrant was attributed to both Lynn and her students rather
than simply to the student. The argument within which this occurred had the following components:

- Claim by a student: The angle in question measures 118 degrees.
- Data by both Lynn and her students: This is a straight line; the angle is 180 minus 62.
- Warrant by both Lynn and her students: There are 180 degrees in a straight line. This warrant appeals to a property of lines, and the way in which it was contributed was similar to the other 18 warrants contributed by a combination of Lynn and her students.

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![Parallelogram With Missing Angles from March 7](image)

Figure 5-7: Parallelogram With Missing Angles from March 7

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**Arguments With Implicit Warrants**

In addition to examining the warrants Lynn and her students contributed to arguments, it is informative to examine the warrants Lynn allowed to be left implicit. In 21 cases, the warrants that were left implicit were definitions, properties, or theorems that had been used previously as warrants in similar situations. It is possible that Lynn felt it unnecessary to state such warrants verbally if she believed they were understood by her
students. This is consistent with Lynn’s conception of proof as it relates to her conception of teaching and learning mathematics, as will be discussed later in this chapter. In many of the remaining cases, the arguments involved solving algebraic equations or inequalities, skills that Lynn could safely assume had been taught in courses taken previous to geometry. The claims in these cases were generally the solution to the equation or inequality, and the data were a set of procedures needed to solve the equation or inequality.

Lynn’s support for these arguments regarding the solution of equations and inequalities may be illustrated by the following short excerpt from an episode of argumentation occurring on March 7.

Claim: If \(5y = 60\), then \(y = 12\).
Lynn says, “Five \(y\) equals sixty [writes \(5y = 60\)]. How do we get \(y\)? How do we get \(y\)? What do we get? Student, what do we do?” A student says, “Divide by 5.” Lynn says, “Divide by five, and what’s sixty divided by five?” A student says, “12.” (Lynn Geometry Class, March 7, lines 285-289)

In this and similar cases, Lynn asked questions about procedures (“What do we do?”) and results (“What’s sixty divided by five?” or “What do we get?”). She prompted students to explain how they got their answers or to provide data for their claims. There is no indication that a warrant was needed. In cases like this, the precise mathematical warrants for relating the procedures to the solutions may have been assumed as previous knowledge or left implicit as outside the scope of the geometry class.

Five arguments in which the warrants were left implicit were similar to an argument observed on March 6. Lynn and her students were discussing some properties of parallelograms and were given a diagram of a parallelogram (\(ABCD\)) in which the
diagonals ($\overline{AC}$ and $\overline{BD}$) had been drawn. $E$ was labeled as the point of intersection, and appropriate pairs of segments (such as $\overline{AE}$ and $\overline{EC}$) had been marked congruent. Lynn and her students claimed that $\overline{AE}$ was half of $\overline{AC}$. The data, given by Lynn, was “by the congruent marks you know that it’s cut in half” (Lynn Geometry Class, March 6, lines 111-112). An implicit warrant in this case could include some aspect of the statement that if a segment is composed of exactly two congruent segments with only one endpoint in common, then the length of each of the congruent segments is half the length of the original segment. However, such a statement might not have been necessary for Lynn’s students to connect the figure with its congruent marks to the statement that the length of one of the segments is half the length of the larger segment, since they regularly worked with relationships within similarly marked figures.

**Warrants Related to Incorrect Claims**

Only ten instances of students making incorrect claims were observed (these occurred in the arguments whose counts appear in parentheses in Table 5-1). Lynn’s reaction to incorrect claims made by students was consistent across observations. When an incorrect claim was made, Lynn either asked a question that made it clear that the claim was incorrect and asked that student or another student to make a different claim or she gave a reason the initial claim could not be correct and made a different claim herself. In general, the arguments that were made to counter students’ incorrect claims used the same kinds of warrants as other arguments in Lynn’s class. (The parenthetical counts that appear in Table 5-1 are categorized according to the attribution of the counter-claim,
data, and warrant; all incorrect claims were made by students and consisted only of the claim without data or warrant.) Lynn acted as if the incorrect claims themselves did not merit either data or warrants. Lynn’s support for students providing data and warrants, then, extended only to support of correct claims; incorrect claims were immediately quashed.

Lynn’s Support of Argumentation

Considering the various aspects of argumentation that have been described thus far evokes a picture of how Lynn supported argumentation in her classroom. One of the main ways in which Lynn supported collective argumentation in her classroom was in contributing or prompting the contribution of warrants. For each warrant that was provided, Lynn directly provided it, she provided part of it with students filling in details, or she asked a question that required a warrant. Because this was a geometry class, and some of the arguments were observed as the class constructed two-column proofs, it may not be surprising that some of the argumentation was more formal in nature than other argumentation that was observed in this class and others. It may have been more natural in this context than in other mathematics classes for Lynn to ask for warrants and more natural for her students to provide them in the form of theorems, properties, and definitions. Most of the claims made during the observations of Lynn’s class were correct, and Lynn and her students contributed either data or warrants or both for these claims. Lynn’s support for argumentation did not apply to incorrect claims made by students; instead, she immediately corrected a claim before asking for data or a warrant.
Lynn also supported argumentation in her classroom by providing, modifying, and directing student attention to diagrams and demonstrations illustrating the mathematical ideas being discussed. She consistently marked segments and angles congruent and then explicitly drew students’ attention to the marks during episodes of argumentation. Lynn’s verbal and nonverbal support for argumentation resulted in a classroom where she and her students collaboratively constructed arguments for the validity of mathematical claims by contributing data and warrants.

Various aspects of Lynn’s conception of proof suggest why Lynn might have supported argumentation in the ways that she did. In the next part of this chapter, I describe these aspects of Lynn’s conception of proof and relate them to her support for argumentation. I focus on three main aspects of her conception proof and how these relate to her provision or prompting of warrants, her use of implicit warrants, and her use of diagrams and demonstrations as examples.

**Aligning Lynn’s Support for Argumentation With Her Conception of Proof**

Lynn’s conception of proof can be seen as part of her view of mathematics, although when asked about the important aspects of mathematics, she did not mention proof. I briefly suggest some of her views of mathematics and how these views are consistent across her statements, support of argumentation, and assessment documents. I then describe several themes that arose as parts of her conception of proof, relating each of these to appropriate aspects of her support for argumentation.
When Lynn was asked, “What do you think are the important aspects of math?” (Lynn Interview 1, line 47), she gave the following answer.

Well, maybe I’m thinking of this because the subject I’m teaching right now is geometry, but definitions and theorems I think are huge to being able to solve any type of problem, and actually thinking back to my math classes, I had topology last semester and that’s the main thing I did, just studied theorems, studied definitions, because that was the basis of solving any type of problem. Um, another thing that’s important in math is um as a student seeing a lot of examples, being able to see them work through and think if there’s anything else I can think of back to my math classes that I liked when the teachers did. Ummm, that’s all I can think of, just seeing examples and really knowing definitions and theorems. (Lynn Interview 1, lines 50-61)

Lynn emphasized definitions, theorems, and examples as important aspects of mathematics, or as important aspects of learning mathematics. Within her teaching, Lynn often used statements of or references to theorems as warrants to connect data to the claim in an argument. Examining the four tests that she created and used during the time of the observations provides confirming evidence that she did see theorems as important. She included problems on every test that required students to either state a theorem as an answer or know what a particular theorem says and apply it to a given problem.

The Purpose of Proof is to Know the Reason Why

While Lynn did not mention proof as an important aspect of mathematics, she does view proof as personally important so that she can know the reason why. Proof is personally necessary for her, and it is seen by her as important for her students because she is “a reason person” (Lynn Interview 2, line 153). Her view of the importance of proof as a student is evident from the following statement.
When I was a student, um I wanted to know why. You know, why does this work? Why can I prove this? And it was very important for me to be able to prove. I mean on a scale from one to ten, that was probably a ten. If I got a problem right away my question if it wasn’t on the board was how do I prove that? You know, and that’s what you had to do, tests, exams, quizzes, show why, you know, why it happens, and then that was also very good if I knew that, um, then I thought okay, well, when I am teaching, something obviously simpler in high school, I’ll know why this works. (Lynn Interview 1, lines 108-117)

That Lynn sees proof as equally important for her students for the same basic reason can be inferred from a statement Lynn made during her second interview. She was asked, “Do you think students should prove mathematical statements in high school?” (Lynn Interview 2, lines 149-150). She replied, “I still think they should be able to do so, yes” (Lynn Interview 2, line 151), and when asked, “Why?” (Lynn Interview 2, line 152), her answer highlighted the importance of proving because knowing why something works is important.

Because I’m a reason person, I need a reason for anything. I want to know why it works and where it comes from and that also builds a stronger level of knowledge. It’s just not [inaudible] it’s not skill it’s more generalization, um, more concepts, you know, understanding why the Pythagorean theorem is the way it is, understanding, you know, what works and what’s going on behind it so you can apply it to different situations. (Lynn Interview 2, lines 153-159)

This statement seems to be consistent with the views of many teachers in Knuth’s (2002b) study who focused “not so much on an argument’s illumination of the underlying mathematical concepts which determine why a statement is true as much as …showing

12 It is curious that Lynn refers here to proving a “problem.” This may be indicative of her college mathematics experience, in which she completed many problem sets composed of statements to be proved. It also may be a simple mis-speak in which she said “If I got a problem” instead of “If I have a statement to be proved.” Alternatively, she may be referring to situations in which she was presented with a method or formula for solving particular problems and wanted to prove that the method or formula (such as the quadratic formula) was valid.
how a statement came to be true” (Knuth, 2002b, p. 80). Thus, while Lynn’s focus was on reasons, and as such could be called explanatory, she focused more on proof as a reason for why things are true and where they come from than on proof as a way to understand the fundamental mathematical concepts related to an idea.

In the first interview, Lynn said proving was important for high school students because “to go though the steps of the proof” (Lynn Interview 1, line 127) was good for them. Lynn seems to think proving is a process of constructing a particular correct argument, starting with given information and writing the steps necessary to obtain the conclusion being sought. In the second interview, Lynn’s reason for students being asked to prove in high school was highly personalized, “I’m a reason person, I need a reason for anything” (Lynn Interview 2, line 153). In addition, she claimed that proving “builds a stronger level of knowledge” (Lynn Interview 2, line 155), so it is important for students to prove. Lynn seemed to be saying that proving satisfies her need to know the reason why things work they way they do. In other words, proving provides Lynn with the reasons that she needs to know why things work and where they come from.

That Lynn holds proving as important is supported by an examination of four tests created by her that related to the lessons taught during the observations. The tests were created by her in the sense that she chose, wrote, or modified each problem that was on each test; her mentor approved the tests before they were administered. Each test contained at least one problem that asked students to justify their answers. In addition, two of the tests contained partially constructed two-column proofs in which students needed to complete the missing statements and reasons. However, the fact that there were no proofs for students to complete independently illustrates Lynn’s struggle to include
proving in a way that was approachable for her students. The continuation of Lynn’s earlier statement about the importance of proving in mathematics demonstrates this struggle.

The first lesson I taught I tried to prove all the theorems in there. And my [mentor] teacher goes, no, first of all, it will take them a lot longer, you know, to go through each section and um it is not as easy for first year geometry students to grasp how to go through a proof as it is as you know as I was, you know, four years down the road in math. …So, as a student I think it’s important. As a teacher, um, I think I would base it on what my classes are. I have lower to average classes right now. And so I normally if there’s a proof, I try to prove it for them, because I think it’s practice to go through the steps of the proof is good for them. (Lynn Interview 1, lines 117-128)

Lynn’s first inclination was to “prove all the theorems” (Lynn Interview 1, line 118). However, intervention from her mentor teacher and her own observation of the level of her classes led to a change in her practice. Her view of proof as the reason why is evident in her support for argumentation in her classroom, even through proving was not as prevalent in her classroom as she might have liked for it to be. As Lynn supported argumentation, she and her students provided warrants for many of the arguments in her classroom. The kinds of warrants that were provided and accepted in Lynn’s classroom generally related to theorems and properties. This is significant since these kinds of warrants can be seen as justifications that are appropriate for proving. This aspect of Lynn’s conception of proof can also be seen in her questioning of students. When asking for a warrant, Lynn’s questions tended to be similar to “What would our reason be?” (Lynn Geometry Class, March 8, line 315).

Lynn’s view of proving as important to know the reason why was clearly stated in her interviews. This aspect of her conception of proof is reflected in the warrants she
prompted and provided, in the questions she asked in class, and in the ways she assessed her students. However, her views of her students and her mentor teacher’s advice about proving are also reflected in these aspects of her classroom practice. Lynn’s support for argumentation with warrants that were appropriate in the discipline of mathematics seemed to provide an outlet for her belief that proving is important, while allowing her to scale back the amount of proving that she and her students did formally in her classroom.

**Role of Context**

While Lynn believed proving was important, as illustrated in the previous section, her ability to prove and analyze proofs as well as her expectations for proving seemed to differ depending on the mathematical context involved. This was reflected, perhaps most vividly, in how she left some warrants implicit within the argumentation in her classroom. Within her own proving, one aspect of the context that stands out as important is her level of familiarity with the mathematical content and common methods of proof in particular areas of mathematics.

As part of her first interview, Lynn described how she knew when she had finished a proof, differentiating between three contexts, which, in this case, were classes in somewhat different areas of mathematics.

How do I know that I’m finished? Um, well, in geometry it’s usually a given proof, you know, so if I get that prove line I’m good. In a case like this math analysis if I can do, you know, for \( n + 1 \), and I can get it to the form that I need it, then it’s done. With topology I never knew. (Lynn Interview 1, lines 219-223)
Lynn’s reference to a “given proof” (Lynn Interview 1, line 220) in geometry is probably a reference to requests to prove mathematical statements in the form: given [these facts or characteristics of a mathematical object], prove [this mathematical statement]. Thus if she could start her proof with the given facts or characteristics and end with the mathematical statement following “prove,” she was certain that she was finished. Lynn did not refer to what might come between the “given” statement and the “prove” statement; this lack of reference to making a deductive argument is, in fact, notable by its absence throughout Lynn’s talk about proof and justification. In geometry, then, Lynn was finished when she reached her conclusion. In talking about proving in her college analysis\textsuperscript{13} class, she referenced only proof by induction when she said, “if I can do…for \( n + 1 \), and I can get it to the form that I need it, then it’s done” (Lynn Interview 1, lines 221-222). In Lynn’s view, an analysis proof was finished when it was in the correct form. Her negative stance when it came to topology was evident in this quote, as she “never knew” (Lynn Interview 1, line 223) when she was finished, and, as she stated earlier in this interview, she rarely knew where to start when asked to prove something in her topology class.

In this study, the arguments Lynn was asked to construct and analyze were in the contexts of number theory and geometry. Lynn was somewhat more able to analyze the geometry arguments than to construct and analyze the number theory arguments. One reason for this might be her familiarity with geometry, because she was teaching geometry at the time of the interview. In particular, Lynn was familiar with the triangle

\textsuperscript{13} Topics in this introductory analysis course include a review of calculus concepts, properties of real numbers, sequences, infinite series, and limits and continuity. Many of the early proofs in this class are done by mathematical induction.
inequality theorem, because she was intending to introduce it to her class during the few
days following the interview, an unanticipated coincidence. In the next few sub-sections,
I briefly describe parts of Lynn’s process of constructing and analyzing arguments in the
first interview as they relate to her overall ability to construct and analyze arguments.
Other aspects of her processes in relation to these tasks will be discussed in later sections.

Proving in a Number Theory Context: The Sum of the First n Natural Numbers

Lynn was asked to prove that the sum of the first n natural numbers is \( \frac{n(n+1)}{2} \).
She first looked for a pattern by substituting numbers into \( \frac{n(n+1)}{2} \), but found that to be
unproductive. At one point in the process she pointed at the statement, “the sum of the
first n natural numbers” and said, “I don’t remember what this equals to. But I know if I
could prove that” (Lynn Interview 1, lines 243-244). At this point I decided to intervene,
and said, “This is saying the sum of the first n natural numbers, so you might think of one
plus two plus dot, dot, dot, up to n is how I might write that” (Lynn Interview 1, lines
247-249) as I wrote 1 + 2 + … + n. Lynn’s reaction indicated that this was the step with
which she was struggling: “Oh, that’s it. Equals n, n plus one over two. That’s what I was
thinking. Thank you” (Lynn Interview 1, lines 252-253). As she said this she wrote
\( \frac{n(n+1)}{2} \). While her statement could be interpreted to mean that she was already thinking
of representing “the sum of the first n natural numbers” as 1 + 2 + … + n, she had not made
progress toward doing so, and had explicitly stated that she did not remember how to
represent that phrase. Thus, “that’s what I was thinking” (Lynn Interview 1, line 253) can be interpreted to be an affirmation of her need for assistance with that particular part of the statement rather than an indication that she had thought of writing it that way.

When she finished writing $1 + 2 + \ldots + n = \frac{n(n + 1)}{2}$, Lynn immediately proceeded with a procedure that resembled a proof by mathematical induction, seeming to recognize and respond to the form of the statement as one for which this would be productive.

Figure 5-8 contains Lynn’s argument that $1 + 2 + \ldots + n = \frac{n(n + 1)}{2}$. Lynn seemed to know, in general, the form that a proof by mathematical induction might take. She knew she needed to establish a base case that was true, which she did, and then said she needed to assume it was true. She then seemed to think she had to assume that something else was true, and what she assumed to be true is not mathematically meaningful. That is, she said, “Suppose um, $n, n$ true for all natural numbers” (Lynn Interview 1, lines 258-259) as she wrote, “Suppose $n$ true for all $\mathbb{N}$” as in Figure 5-8. While she did not say this, she may have meant to suggest that the statement to be proved was true for all natural numbers up to $n$. In this case, her algebraic work would have allowed her to state her conclusion. However, saying that “$n$ is true for all natural numbers” has no mathematical meaning unless $n$ is interpreted as some quality or relationship. Thus it seems that she was focusing more on the form of the argument than its content. Her algebraic work was mathematically correct, and her conclusion, if her original assumption had been made concerning the actual statement, would be correct. Her focus was not on the words she had said or written within the argument; she focused on getting the argument into what
appears to be an appropriate form. Her argument, as written, does not meet the standards of proof as a logically correct deductive argument built up from given conditions, definitions, and theorems. It does correspond, to some degree, with the form of a proof by mathematical induction.

Figure 5-8: Lynn’s Argument that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

Lynn’s work in this context contained numerous starts and stops. She seemed committed to finishing her work, but, at least at the beginning, she did not seem to know how to start. Her process was not smooth; she was not confident in what she was doing,
as illustrated by her pauses and statements, “So, start with one plus, okay, so then we add two to get this number, three, um, so I’m trying to find [pause]. Start with n, and then this would be [pause]. I don’t remember what this equals to” (Lynn Interview 1, lines 241-244). When I assisted her with “what this equals to,” Lynn seemed to react to the form of $1 + 2 + ... + n = \frac{n(n+1)}{2}$ by immediately starting an argument involving induction. Her process seemed more motivated by form than by meaning; she relied on her memory of what an argument like this should look like rather than on whether what she was saying and writing made sense mathematically.

Analyzing in a Number Theory Context: The Square Root of a Positive Integer is Either an Integer or It Is Irrational

When asked to decide if the argument given in Figure 5-9 proved the statement, “the square root of a positive integer is either an integer or it is irrational,” Lynn began by reading the statement and the first two lines of the argument aloud. She then said, “So suppose $n$, say this is any positive integer. Say it’s three. Well the square root of a non-perfect square is irrational. Oh, we’re assuming it is rational” (Lynn Interview 1, lines 293-295). Lynn’s knowledge that the square root of three is irrational seemed to conflict with her reading of the argument, as at this point she paused, and then repeats, “Okay, so now we’re assuming it is rational” (Lynn Interview 1, line 297). She again paused, and I asked, “Why would you or they assume it’s rational?” (Lynn Interview 1, lines 298-299). Lynn responded with a question: “Oh, are they proving by contradiction?” (Lynn Interview 1, line 300). At my! noncommittal response, Lynn said, “Okay, so we’re first
proving that it’s not an integer. Okay? So if it’s not an integer, I’m assuming that this is not a perfect square” (Lynn Interview 1, lines 302-303). By “this,” Lynn seems to mean $n$. Lynn seems to want to characterize $n$ more specifically. She was not satisfied with knowing that $n$ is an integer as is required by the statement to be proved. Lynn, although with hesitation, identified the form of this argument as a proof by contradiction. As she read through the remainder of the argument, she made few observations. When agreeing with the conclusion that $a^2$ and $b^2$ have no common factors, she concluded this is true because $a$ and $b$ are prime numbers, when in fact they are only relatively prime. However, this did not interfere with a conclusion that $a^2$ and $b^2$ have no factors in common, and so she concluded that, “I would say that yes you could then state the square root of a positive integer is either an integer or it is irrational, but you’re missing this part, you have to prove when it would be an integer” (Lynn Interview 1, lines 324-327). To me, Lynn’s statement, “yes you could then state the square root of a positive integer is either an integer or it is irrational” (Lynn Interview 1, lines 324-325), sounded like a claim that the argument indeed proved the statement, so I asked, “Okay, so, um, you’re saying that this is a proof mostly but it’s incomplete?” (Lynn Interview 1, lines 330-331). Lynn’s answer concluded with, “you still need to prove that it’s an integer…could be an integer” (Lynn Interview 1, line 338-340).
The square root of a positive integer is either an integer or it is irrational.

Suppose \( n \) is a positive integer, \( \sqrt{n} \) is not an integer, and \( \sqrt{n} \) is rational. Then there exist relatively prime integers \( a \) and \( b > 1 \) with \( \sqrt{n} = \frac{a}{b} \).

Squaring both sides, \( n = \frac{a^2}{b^2} \), from which \( nb^2 = a^2 \). Now if we factor \( a \) and \( b \) into primes, there are no common factors. So the factorizations of \( a^2 \) and \( b^2 \) have no common factors. Consequently, the factorizations of the equal numbers \( nb^2 \) and \( a^2 \) are different. Since two different factorizations of \( a^2 \) are impossible by the Fundamental Theorem of Arithmetic, the supposition must be false. So if \( \sqrt{n} \) is not an integer, it must be irrational.

Figure 5-9: Square Root Argument (from Usiskin, Peressini, Marchisotto, & Stanley, 2003, pp. 23-24)

While Lynn correctly concluded that the proof-writer was using proof by contradiction, she did not correctly conclude that this was a proof. She did not recognize that the statement to be proved was correctly negated, and she was left with her conclusion that the argument was not complete. Thus her identification of this argument as a proof by contradiction was helpful only insofar as she understood the form. That is, if she had been able to identify the first sentence as a correct negation of the statement to be proved, Lynn might have concluded that the argument was a proof.

Lynn was not comfortable with analyzing this proof in a number theory context. Lynn’s work in this context was hesitant; she was unsure at first what to do with the assumption that the square root of \( n \) was irrational. Lynn identified the argument as a proof by contradiction, but then concluded that it proved only part of the statement, due,
at least in part, to not recognizing the first line of the argument as a correct negation of the statement to be proved. This could be due to Lynn being less familiar with proof by contradiction. On the other hand, Lynn’s difficulty seems to relate to her wanting to see a proof that the square root of $n$ could be an integer; she could be interpreting “either…or” in the statement to be proved as a need for a proof with two cases. If this is the case, Lynn’s difficulty in concluding that this was a proof might be related more to expecting a proof with two cases based on the “either…or” than only a difficulty with the form of proof by contradiction.

**Analyzing in a Geometric Context: The Triangle Inequality Theorem**

Lynn was presented with three arguments, one at a time, for the triangle inequality theorem and asked to state how convinced she and her students would be by the arguments. (For the arguments as they were presented, see Figure 3-2, Figure 3-3, and Figure 3-4 in Chapter 3.) Perhaps because this was a geometric context, or perhaps because she was given a figure for each, she did not generate any examples. Her conclusions about the three arguments can be summarized as follows. For the examples-based Geometer’s Sketchpad (GSP)-supported argument, she would not be convinced; she would need a proof. Her students would be convinced by this argument. For the textbook argument by cases, she would be convinced, and her students would be convinced by a combination of this argument and the GSP-supported argument through a hands-on activity. (Lynn’s description of a hands-on activity in this case was the way she presented the triangle inequality theorem to her students. The desks in her classroom
measured 25 inches in width, so she gave each student two 12-inch rulers and asked them to make a triangle with the desk as a base.) For Euclid’s argument (the diagram for which contained a typographical error in the labeling of the vertices), she assigned more weight to the diagram than to the verbal text surrounding the diagram, and at first concluded that it was not a proof. When the typographical error was pointed out, she was convinced by the argument, claimed it was a proof, and claimed her students would have a difficult time understanding the proof, but that it might be more helpful if it were in a list rather than a paragraph. This statement was Lynn’s only reference to the form of a proof in this section of the interview. Also, possibly because of the geometric context of these arguments and her familiarity with it, or because each argument contained at least one figure, Lynn did not use examples to instantiate any aspect of the statement, and did not seem to rely as much on the form of the arguments as she analyzed them.

**Summary of Lynn’s Context-Dependence**

Lynn had never taken a course in number theory, and she both had taken a college geometry course and was teaching a high school geometry course. Thus it is likely that she was more familiar with the ideas and methods of proof in geometry than she was in number theory. The context in which Lynn was asked to prove or analyze arguments, particularly whether or not she was comfortable with the specific mathematical ideas or methods of proof involved, affected her ability to construct and analyze arguments. Her description of how she knew she had finished a proof suggests that she saw proving as different in different mathematical contexts. When taken together, her description of
proving and my observations of her proving and analyzing proofs suggest that her ability to prove is dependent on the mathematical ideas involved in the proof and her perception of what a proof should look like in that mathematical area. That the form of a proof is important to Lynn is further supported by the fact that she exclusively used two-column proofs when constructing proofs in class. Although the focus here was not on proof schemes, the fact that Lynn proved and analyzed proofs differently in different contexts may be related to Harel and Sowder’s (1998) observations that a student may exhibit different proof schemes in different areas of mathematics, “reflecting his or her familiarity for, and relative expertise in, the contexts” (p. 277). These observations were subsequently supported by Housman and Porter (2003), who found that individual students generally exhibited multiple proof schemes.

Lynn’s support for argumentation in her classroom occurred in a geometric context. Thus the warrants she did provide, especially their mathematical relevance, may relate to her comfort and ability in geometric contexts. The pattern of warrants that Lynn left implicit also relates to her context-dependence, or perhaps a context-preference. In many cases, Lynn left warrants implicit in arguments that involved algebraic equations or inequalities. These arguments involved a context different from geometry. Her reasoning for not providing or prompting these warrants could relate to a perception that students ought to know how to solve these kinds of problems before entering this geometry class. However, it could also relate to viewing algebra as different from geometry insofar as when justifications are required and what needs to be justified. In geometry, Lynn seemed to justify or ask her students to justify every statement. Perhaps, from her perspective, arguments in the context of algebra did not require as much justification.
Using Examples

Related to Lynn’s context-dependence and her work with the three tasks described above is her emphasis on using examples. Lynn saw examples as an important part of mathematics. When describing the important aspects of mathematics as both a student and a teacher, Lynn mentioned “seeing a variety of different examples” (Lynn Interview 1, lines 65-66). In addition, Lynn stressed doing examples in her classroom, both to introduce ideas to her students and to help them prepare for tests. When helping her students prepare for a test, Lynn told them which problems to do that were like ones that would be on the test.

In the second interview, Lynn was asked several questions relating to her expectations for students’ justifications and interaction. In her reply to each of these, Lynn employed the use of examples. Lynn was asked what an “A” answer would be to the question: Is a parallelogram a trapezoid? Why or why not? In her response, she drew a parallelogram and a trapezoid (see Figure 5-10). These can be interpreted as prototypical examples of the classes of parallelograms and trapezoids. She used these examples in her answer as she characterized parallelograms and trapezoids by the number of “parallel lines” each contains. This is the characteristic that was highlighted in her sketch of each. While her stated definition of a trapezoid did not contain the word “exactly,” it is clear from her conclusion (“In a trapezoid, you have the bases and the legs, bases are parallel, well, you wouldn’t have any legs. You’d have two sets of bases” [Lynn Interview 2, lines 38-40]) that she intended her definition to include that
restriction. What is not clear is whether her definition of trapezoid would be the same without the example sketch.

Figure 5-10: Lynn’s Sketch of a Parallelogram and a Trapezoid

Lynn was also asked to specify how many points out of three she would assign to a student who wrote “No solution” (as shown) as his or her answer to the prompt in Figure 5-11. In order to answer the question, she solved the system by substitution herself, possibly as an example of what a solution should look like. She began by saying, “Well, I’d have to solve this” (Lynn Interview 2, line 50). She wrote down her solution (see Figure 5-12), and then gave the following explanation, referring to the parts of her example solution that are each marked with a circled “1.”

Okay, to get the full three points, I’d want to see them set up some work and show me because five does not equal seven there’s no solution to the problem or vice versa, if they’d solve for y, you know, show me that something’s not equal. No solution I’d probably just give that one point. If I was grading this out of three points I’d want to see are you solving for an x or y, one point, are you getting some reason, one point, have you concluded, one point, so do it that way. (Lynn Interview 2, lines 62-69)

Lynn used the example of a solution she generated to extract categories of work for which she would assign points in grading a student’s solution. This is evident in that she
points to (and labels) the different parts that she would expect to see, but refers to them in
general terms rather than by the specific work she did when generating the solution.

Solve the following system by substitution:

\[
\begin{align*}
3x + 2y &= 5 \\
3x + 2y &= 7
\end{align*}
\]

\text{No solution}

Figure 5-11: Hypothetical Student Solution to System of Equations Problem

In her answers to both the parallelogram and trapezoid prompt and the system of
equations prompt, Lynn generated and used examples to guide her explanation of a good
justification. In her response to my next prompt (I handed Lynn a sheet of paper
containing Figure 5-13 and said “Imagine the following happening in your classroom.
Describe for me what you would do and say in this situation”), Lynn used an example
apparently to situate the general situation that was described. She said, “Let’s say, for instance, we had a scenario where, let’s say it was the lengths were 24, um 11 and 12” (Lynn Interview 2, lines 83-84). She did not state why she chose to instantiate the situation as she did, and she alternately referred to the situation using the lengths 24, 11, and 12 and using general terms (such as least, greatest, first, second, third) for the three side lengths.

Figure 5-13: Description of Possible Student Discussion

Within Lynn’s description of what she would say to the two students, she referred not only to the particular numerical example she had created, but also to the hands on demonstration (with desks and rulers) that she used to introduce the idea of the triangle inequality theorem in her class. She stated that she would remind her students of this demonstration as a way to help them understand what the correct answer is.
Lynn consistently generated examples within her descriptions of good justifications as well as when she was asked to generate a proof or analyze an argument. The only time she did not do this was when she was analyzing the three arguments concerning the triangle inequality theorem. Each of those arguments was accompanied by a drawing (or, in the case of the GSP-supported argument, many drawings). Thus she may not have generated an example because she used the researcher-provided drawings as examples from which she could reason. Alternatively, her familiarity with the particular content of the triangle inequality theorem may have rendered her generation of examples unnecessary. Moore (1994) suggests that generating and using examples is an important aspect of proving, and one with which the participants in his study struggled. Housman and Porter (2003) related their participants’ example use and generation strategies to their proof schemes, finding that the one participant who tended to use examples more than any other exhibited an inductive proof scheme, while the students who had a transformational proof scheme used examples when prompted or when necessary to produce a counter-example. Lynn used examples more productively in geometric contexts than in number theory contexts, suggesting that productive example generation and use may also be somewhat context-dependent.

Lynn consistently provided her students with examples from which they could reason. Part of her support for argumentation involved providing figures, marking figures, and drawing students’ attention to the relevant parts of these figures. In addition, when asking her students for claims, data, and warrants, Lynn sometimes used references to examples they had previously discussed or demonstrations they had seen (such as the desk and ruler demonstration of the triangle inequality theorem).
Conclusion

Lynn’s support for argumentation aligns well with her multi-faceted conception of proof. The number of warrants that she prompted or provided is related to her view of proof as personally important to know the reason why. Lynn’s reliance on form in her own proving is apparent in her preference for two-column proofs when producing written proofs in class, but her more informal approach during other episodes of argumentation. Lynn’s use of examples and demonstrations in prompting warrants from her students relates to her use of examples in her own proving and justifying work. The fact that she often left warrants implicit in non-geometric contexts may relate to her thinking about proving differently in different mathematical contexts, or simply because they would relate to a mathematical topic different from what she was teaching. Leaving warrants implicit may also relate to her view that proving is important in order to know the reason why. Any warrant that was left implicit was in an argument in which the mathematical ideas had either been previously discussed in that class or could be assumed to be known from previous classes. Thus Lynn may not have seen a need to reiterate a warrant for something for which her students already know the reason why.
Chapter 6

Jared: “There’s not really much to prove in algebra”

Jared’s support of argumentation aligns with his conception of proof. As illustrated by the title of this chapter, Jared does not believe proving is very important in algebra, especially for his particular group of students. Yet Jared provided warrants or prompted his student to provide warrants for most of the arguments in his class. In this chapter, I will describe the argumentation I observed in his classroom and then discuss the relationship of his support for argumentation to his conception of proof, particularly proof in school mathematics.

Argumentation in Jared’s Classroom

Jared taught several sections of a first-year regular-level algebra class for his student teaching experience. The section that was observed met for approximately fifty minutes every day. During the time of the observations, Jared and his class discussed solving linear equations and inequalities; solving systems of linear equations; and adding, subtracting, and multiplying polynomials. A typical class began with a discussion of homework questions or questions from a recent test or quiz. This was followed by the introduction of the new material for the day’s intended lesson. Class often ended with students completing problems independently. These problems were sometimes discussed at the end of class and sometimes assigned to be completed for homework. Class
discussions were typically orchestrated by Jared. In general, he asked questions and his students answered them, or his students asked questions and he answered them. Rarely did students address or respond to each other in a discussion.

**Influence of Mentor**

Jared’s mentor claimed that his classroom practice was different from her own in that she stressed the “connectedness” (Ms. C Interview, line 35) of mathematical ideas. However, she did share with him her planning and disciplinary strategies, including the importance of making time for students to work on practice problems during each class period. According to both Jared and Ms. C, they did not talk about discourse practices or questioning within their discussions about teaching. In general, Jared seemed to feel constrained by his mentor’s expectations, while his mentor claimed to not discuss how a lesson should be taught until after the lesson had been taught. It seems that there were some expectations for how Jared would teach that were perceived by him, whether intentionally or not, and these expectations apparently shaped his practice. For instance, Ms. C’s description of her own practice contained elements (such as making time for students to work on practice problems during class) that were consistent with Jared’s practice.
Typical Argumentation

Eight observations of Jared’s class were analyzed to characterize his support for argumentation. The first two classes that were observed, along with the first part of the third class, dealt with solving linear inequalities and solving systems of equations by graphing. The typical episode of argumentation in these first few classes contained a main argument and between one and seven sub-arguments. Starting partway through the third class, episodes of argumentation generally consisted of a main argument with occasionally one or two sub-arguments. This coincides with the change in focus of the class discussions from solving equations, inequalities, or systems of equations to adding, subtracting, and multiplying polynomials. This change in focus may explain the noticeable shift from longer episodes of argumentation in the earlier classes to shorter episodes in the later ones. In the following section, the typical argumentation in Jared’s classroom will be illustrated with episodes from two classes, one early in the sequence of observations and one later.

Typical Episode 1: Main Argument With Sub-arguments

The first episode, typical of argumentation in the earlier classes observed, occurs during the first class that was observed, on February 20. Jared had handed back a quiz, and was discussing some problems from the quiz with his students. Later in this class, Jared introduced systems of linear equations and solving systems by graphing. This particular episode of argumentation involved solving a linear inequality from the third quiz problem they discussed.
Claim: If \(-4x + 9 \leq x - 21\), then \(x \geq 6\)

Jared writes \(-4x + 9 \leq x - 21\) on the board. He says, “We have negative four \(x\) plus five is less than and equal to \(x\) minus 21. If we want to solve this inequality for \(x\), what should we do first?” A student answers, “Subtract, no divide by negative 4.” Jared says, “We don’t want to divide by negative four yet. We want to combine all the like terms.” A student says, “Subtract nine.” Jared repeats, “Subtract 9 from both sides.” He writes \(-4x \leq x - 30\) and calls on another student who says, “Subtract \(x\).” Jared repeats, “Subtract \(x\)” and writes \(-5x \leq -30\). He says, “So negative 5 \(x\) is less than or equal to negative thirty. Let’s solve for \(x\), Student.” The student says, “Divide by negative 5.” Jared says, “What do we have to keep in mind, when we’re dividing by a negative number and we have an inequality?” The student says, “Flip the inequality sign.” Jared says, “When do we flip the sign?” A student answers. Jared clarifies, “Dividing or.” A student answers. Jared says, “Not by an integer but by a?” A student answers. Jared repeats, “By a negative number. So \(x\) is greater than or equal to six.” He writes \(x \geq 6\). Jared says, “There’s another way we could have looked at this. Instead of taking the \(x\) to the negative four \(x\), we could try to keep our variable positive, right? So we have nine less than or equal to five \(x\) minus 21.” He writes \(9 \leq 5x - 21\). He says, “What should we do to solve for \(x\)?” A student says, “Add 21.” Jared says, “Add 21 to both sides. 30 is less than or equal to 5 \(x\).” He writes \(30 \leq 5x\). He says, “Divide by five. Get six is less than or equal to \(x\).” He writes \(6 \leq x\). He asks, “What does this tell us about \(x\)? So six is less than or equal to \(x\), what does that tell us about \(x\)?” A student says, “It’s a solid line and \(x\) can be less than 6.” Jared says, “No,” and another student says, “\(x\) is greater than or equal to 6.” Jared says, “\(x\) is greater than or equal to 6.” We just have one variable here, we’re going to use a number line, we’re not going to do coordinate planes. We’ll do coordinate planes with two variables, here we have \(x\) is greater than or equal to 6. So six is less than or equal to \(x\), we know \(x\) is greater than or equal to 6. Which is the same answer. We did it two different ways, same answer.” (Jared Algebra Class, February 20, lines 56-93)

This episode contains one main argument, two sub-arguments, and two sub-sub-arguments, each of which is identified and components of which are attributed to either Jared, his students, or both below. A description of claims, data, and warrants and how they are used in this analysis may be found in Chapters 2 and 3.

The main argument has the following components:

- Claim by Jared and his students: If \(-4x + 9 \leq x - 21\), then \(x \geq 6\).
• Data by Jared and his students: We can subtract nine first or we can add four \( x \) first (and then perform appropriate steps).

• Warrant by Jared: We solved this two different ways and got the same answer.

The first sub-argument has the following components:

• Claim by students: We should subtract nine first.

• Data by Jared and his students: 
  \[-4x + 9 \leq x - 21 \Rightarrow -4x \leq x - 30 \Rightarrow -5x \leq -30 \Rightarrow x \geq 6.\]

• Warrant by students: We have to flip the inequality sign.

The first sub-sub-argument, which supports the warrant of the first sub-argument, has the following components.

• Claim by students: We have to flip the inequality sign.

• Data by Jared: We’re dividing by a negative number.

• Warrant by Jared and his students: When we have an inequality and are [multiplying or] dividing by a negative number, we have to flip the sign.

The second sub-argument has the following components:

• Claim by Jared: We could add four \( x \) first.

• Data by Jared and his students: 
  \[-4x + 9 \leq x - 21 \Rightarrow 9 \leq 5x - 21 \Rightarrow 30 \leq 5x \Rightarrow 6 \leq x \Rightarrow x \text{ is greater than or equal to six.}\]

• Warrant by Jared: This is the same answer as we got when we subtracted nine first.

The second sub-sub-argument, that supports the warrant of the second sub-argument, has the following components.
• Claim by students: We have a solid line; $x$ is less than or equal to 6.
• Data by Jared: $6 \leq x$.
• Warrant: Unspecified.
• Counter-Claim by Jared: This is not a solid line; we’re going to use number lines.
• Data by Jared: We have one variable.
• Warrant: We do coordinate planes with two variables.

This episode of argumentation, considered as individual parts above, can also be visually depicted in a diagram as in Figure 6-1. For the general structure of diagrams like this, see Figure 3-6 and the accompanying description in Chapter 3. The colors in the diagram indicate to whom each component of each argument was attributed: red for Jared, blue for students, and violet for collaborative effort between Jared and his students. Examining the colors in this diagram gives insight into patterns of argumentation in Jared’s class. In this episode, the arguments were constructed collaboratively by Jared and his students. Students made three of the six claims; Jared offered three of the five warrants. In the diagram, blue arrows can be seen within two of the boxes containing data. These blue arrows between red inequalities indicate that students gave the operation by which the next equivalent inequality was reached, while Jared performed the operations and wrote the equivalent inequality on the board. Arguments in Jared’s class were typically constructed by a combination of Jared and his students, and this collaborative construction of data was typical of their argumentation in the first few classes. Patterns in the contributors of each component of each argument observed in Jared’s class will be discussed in later sections.
Figure 6-1: Diagram of Longer Episode of Argumentation in Jared’s Algebra Class (from Jared Algebra Class, February 20, lines 56-93)
Typical Episode 2: Main Argument Only

The second sample episode occurred during the fifth observation on March 20. At the beginning of class, Jared and his class had discussed some questions from a test. Then Jared began to introduce new material, part of which concerned the degree of a polynomial. This episode occurred soon after Jared defined the degree of a monomial, saying, “The degree of a monomial is the sum of the powers of the variables” (Jared Algebra Class, March 20, lines 152-153).

Claim: The degree of \(6a^2b^2c\) is 5.

Jared says, “Now in the case where there’s multiple variables in the monomial, such as six a squared b squared c, the degree represents the sum of the powers.” He writes \(6a^2b^2c\) on the board. A student says, “Five.” Jared says, “Right, two plus two plus one. Five.” He writes 5 on the board. (Jared Algebra Class, March 20, lines 162-166)

This (main) argument has the following components:

- Claim by students: The degree of \(6a^2b^2c\) is 5.
- Data by Jared: \(2 + 2 + 1\)
- Warrant by Jared: The degree represents the sum of the powers.

This episode can also be represented by use of a diagram (see Figure 6-2).

Examining the colors in this diagram suggests what is a common pattern within the episodes of argumentation that contain only main arguments: a student made a claim (usually in answer to a question from the teacher), and then Jared provided the data and warrant for the claim.
Backings for the Typical Arguments

Examining the warrants that Jared used in these episodes leads to some inferences regarding what backings might be acceptable in this classroom. The warrants used in the first episode basically fall into two categories: warrants that address when to use particular procedures and warrants that concern checking answers. The warrant that concerns checking answers suggests that, in this class, an appropriate backing was arriving at the same solution using different mathematical steps. Jared typically gave warrants of this type in the first three classes observed, but not in later classes.

The warrants that address using particular procedures include Jared’s and his students’ statements about reversing the direction of an inequality sign and Jared’s statements about using number lines and coordinate planes. Within and beyond the first three days observed, Jared and his students frequently used warrants composed of rules or procedures during episodes of argumentation in their classroom. These frequently-observed warrants suggest that a backing for argumentation in Jared’s classroom was a rule or set of procedures that had been established as valid in the class. While the
observed warrants do not give insight into how a rule or set of procedures might have been established as valid, observations of the class indicate that rules, procedures, or sets of procedures gained validity by being discussed in class or written on the board. They were often explicitly referred to as “rules” or “steps” to be followed.

The warrant in the second typical episode was basically a definition. Jared said, “the degree represents the sum of the powers” (Jared Algebra Class, March 20, line 164), but he seemed to use “represents” to mean “is” or “is defined as.” Jared and his students occasionally appealed to a classroom definition in their arguments. These definitions seem to have been established as valid in this class in much the same way that rules or sets of procedures became valid in this class: Jared wrote them on the board or discussed them in class, and then they could be called upon to support data and claims.

The way that definitions, rules, and sets of procedures achieved validity through Jared’s actions suggests that Jared was the mathematical authority in the classroom. This suggestion is somewhat challenged by Jared’s appeals to check solutions, but Jared was the only person who initiated the checking. Thus the backing for warrants in Jared’s classroom seems to be Jared’s mathematical authority. Jared might not have considered the backing for these arguments to be his own mathematical authority; he might have considered these rules and procedures to be mathematics. As will be discussed later, his view of mathematics seems to hold knowing how to do things as highly important. Thus by appealing to these rules and procedures, Jared may have felt that he was appealing to the discipline of mathematics in these backings.
Patterns in Jared’s and His Students’ Contributions to Arguments

While an outline of Jared’s support for argumentation can be seen from the structure and details of the typical arguments described above, examining how Jared and his students contributed various components of individual arguments fills in some details about how he supported argumentation in his classroom. Each of the 112 arguments observed within 61 episodes of argumentation is represented in Table 6-1; several interesting observations can be made simply by examining who contributed which components of arguments. In this section, several observations about the patterns of argumentation will be elaborated. Following this, the rest of this section is organized around the kinds of warrants Jared provided or left implicit and the situations in which he provided prompted, and withheld warrants.

Table 6-1: Parts of Arguments as Attributed to Jared and his Students

<table>
<thead>
<tr>
<th>Claim</th>
<th>Data</th>
<th>Warrant</th>
<th></th>
<th></th>
<th></th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Student</td>
<td>Teacher</td>
<td>Both</td>
<td>Neither</td>
<td>(implied)</td>
</tr>
<tr>
<td>Student</td>
<td></td>
<td>2</td>
<td>15</td>
<td>0</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>Teacher</td>
<td>0</td>
<td>5</td>
<td>1</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Both</td>
<td>2</td>
<td>10</td>
<td>4</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Neither</td>
<td>0</td>
<td>2</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Teacher</td>
<td></td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Teacher</td>
<td>0</td>
<td>6</td>
<td>3</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Both</td>
<td>0</td>
<td>9</td>
<td>5</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Neither</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Both</td>
<td></td>
<td>2</td>
<td>2</td>
<td>3</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Teacher</td>
<td>0</td>
<td>2</td>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Both</td>
<td>4</td>
<td>9</td>
<td>4</td>
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<td></td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>11</td>
<td>62</td>
<td>21</td>
<td>18</td>
<td>112</td>
</tr>
</tbody>
</table>
The most striking observations from Table 6-1 concern the number of claims attributed to students and the number of warrants attributed to Jared. Students made nearly half (53 out of 112) of the claims in Jared’s class. Most of these claims were made in response to a question asked by Jared, so, as will be discussed later, he played a role within each argument, even the ones to which he did not contribute a component directly. However, students participated in 81 of the 112 claims made in this class, between the ones that were attributed to students only and the ones that were attributed to both Jared and his students. If a claim was attributed to both Jared and his students, then the claim was made in a way that both contributed a part of the claim without which part that particular claim would not have been the same. The number of claims attributed to Jared alone was only 31, so students participated in generating a majority of claims in this class.

In contrast, Jared contributed warrants for more than half (62 out of 112) of the arguments observed. Students contributed warrants for only eleven arguments, and including warrants to which both Jared and his students contributed raises that number to only 32. Thus in Jared’s class, students seemed to be expected to make claims, usually expressed as answers to Jared’s questions, while Jared provided more of the warrants.

A natural question, then, is who was responsible for providing data? The answer to this question is not immediately apparent from Table 6-1, but adding across the rows for students, Jared, and both produces an interesting observation. Students provided data for 35 arguments, Jared provided data for 23 arguments, and data for 52 arguments was provided by a combination of Jared and his students in such a way that each provided a crucial part. Thus the norms of argumentation in Jared’s class were such that arguments
were constructed collaboratively, with students responsible for more of the claims, Jared and his students together providing more of the data, and Jared responsible for more of the warrants. Warrants, as the bridges between data and claims, play an especially crucial role in argumentation, and the remainder of this section is organized by how Jared’s support for argumentation is revealed by the warrants he provided, the warrants he asked his students to provide, and the warrants he chose not to provide.

**Warrants Provided by Jared**

Jared provided more than half of the warrants in his class and contributed significantly to warrants in 83 out of 112 arguments. The kinds of warrants that Jared provided and to which he contributed included stating rules and procedures, stating definitions, checking solutions by substituting into equations or solving equations in different ways, and occasionally observing a pattern or a feature of a graph.

By far the most common kinds of warrants that Jared provided or to which he contributed were rules and procedures. This was true across the various classes observed, true across different mathematical topics, true whether the argument took place within discussion of review problems or new material, and true whether the problems were first completed individually or as part of a whole class discussion. For instance, in the sub-sub-argument within the episode of argumentation described in the first example of a typical episode of argumentation, Jared provided the data that they were dividing by a negative number, a student claimed that they should “Flip the inequality sign” (Jared Algebra Class, February 20, line 69), and then together Jared and a student provided the
rule for when to reverse an inequality sign, with Jared clarifying and supplementing the student’s answers to his prompts. In the fifth class observed (March 20), Jared directly referenced “our rules” in his warrant as the class discussed a question that dealt with the new material (degrees of polynomials) for that class.

Claim: The degree of $4x^5 y^2 - 3x^3 y^2 + 8xy$ is 7.
Jared has written three polynomials on the board, and students are finding the degree of each. The first one is $4x^5 y^2 - 3x^3 y^2 + 8xy$. Jared says, “All right, number one, the polynomial is four $x$ to the fifth $y$ squared minus three $x$ cubed $y$ squared plus eight $x$ $y$. Anybody tell me the degree of this polynomial?” A student says, “Three.” Jared asks, “How did you get three?” The student says, “I mean, seven.” Jared says, “Seven, that’s better. What did you do to find the degree of seven?” The student says, “Add them.” Jared says, “Okay, so you got seven for this term; what did you get for this term?” The student says, “Five.” Jared points to the next term, and the student says, “Two.” Jared says, “Two. So, based on our rules, we know that seven is greater than five or two so seven is the degree of this polynomial.” (Jared Algebra Class, March 20, lines 220-234)

In this argument, the student’s claim was that the degree of the polynomial is seven (the statement that the degree is three likely was a mis-speak on the part of the student, referencing the next polynomial, since the degree of the next polynomial was three and he changed his answer immediately upon looking up at the board, where the polynomials, not their degrees, were written, to answer Jared’s question as to his procedure). The student and Jared each contributed part of the data, with the student supplying the degree of each term and Jared reminding students that seven was the greatest of the three numbers. As he did this, Jared gave the warrant “Based on our rules” (Jared Algebra Class, March 20, line 234) that provided the connection between the data and the claim. Other rules and procedures Jared cited as warrants included “we add the powers with the same base” (Jared Algebra Class, March 8, lines 32-33), “solve for one
of the variables in one of the equations and substitute it into the other equation” (Jared Algebra Class, March 7, lines 68-69), and “Found our y-intercept, down two over one, down two over one, connect our points and we have our line” (Jared Algebra Class, February 21, lines 202-203).

While Jared’s warrants were dominated by rules and procedures, he also used definitions as warrants. For example, on March 20, while discussing the degree of monomials, Jared asked what the degree of the number 12 would be. A student claimed it would be 0, indicating that there were no variables. Jared connected this data to the claim by saying, “Degree is the sum of the powers of the variables. There are no variables present, so we have nothing to add, so it’s zero” (Jared Algebra Class, March 20, lines 174-175).

Finally, as mentioned in the explanation of the backings of typical arguments, in the first few observed classes, Jared used references to checking solutions as backings for arguments. For instance, on February 21, Jared and his students were solving the system of equations in Equation Error! Reference source not found.. Jared stated that the solution to the system was (4, 2) because the lines intersect at (4, 2), and then said, “Let’s check this to make sure” (Jared Algebra Class, February 21, lines 217-218) and substituted the values into the equations to check.

\[
\begin{align*}
  y & = 10 - 2x \\ 
  y & = \frac{1}{2} x 
\end{align*}
\]  

(6.1)

In the main argument of the first example episode, when solving the inequality \(-4x + 9 \leq x - 21\), Jared’s warrant for the validity of the solution was, “We did it two
different ways, same answer” (Jared Algebra Class, February 20, lines 92-93). These warrants referencing a way to check a solution appeared only in the first few classes. One possible explanation for this is that Jared viewed checking solutions to be appropriate for the mathematical content of these classes (solving equations, inequalities, and systems of equations), while he did not view checking solutions as relating to working with polynomial expressions, the topic of the later classes observed. This is not to imply that Jared did not think his students could verify their answers when working with polynomial expressions. On one occasion (March 22), Jared demonstrated that $x^2$ and $x^3$ were not equivalent by substituting three for $x$ and showing that the resulting values were not the same.

**Warrants Provided by Jared’s Students**

Warrants that Jared’s students provided were generally answers to questions that Jared asked. For instance, he asked questions such as “What do we need to do?” (Jared Algebra Class, February 21, lines 68-69) to which a student replied, “Subtract $x$ from both sides” and “Divide by…negative one” (Jared Algebra Class, February 21, lines 70, 72) in the context of claiming that the slope-intercept form of the line $x - y = 6$ is $y = x - 6$. Similarly, on March 28, when asking his students to multiply a binomial and a trinomial, he asked, “What do we need to do?” (Jared Algebra Class, March 28, line 288), and a student said, “Distribute” (Jared Algebra Class, March 28, line 289). While Jared’s students did not tend to provide many warrants, the warrants that they did provide tended to be of the same kind that Jared himself provided: rules and procedures, and the warrants
were given in answer to Jared’s questions regarding how students had proceeded or would proceed to the next step.

**Implicit Warrants**

Jared left relatively few warrants implicit in the argumentation in his classroom (in 112 arguments, a total of 18 warrants were left implicit). The warrants that were left implicit often can be inferred from the data that are given. For instance, when a student incorrectly claimed that $b^3(b)(b^5) = b^8$, the given data was “Added three plus five in the exponents” (Jared Algebra Class, March 8, line 21). While no explicit warrant was given, the implicit warrant would seem to be that one should add the exponents. This can be seen within the given data, as the student reported her process of simplifying the expression to be adding the numbers that can be seen as exponents in the expression. Similar arguments were observed with correct claims and implicit warrants, such as when a student said to “distribute the negative two $x$ squared $y$ to the parentheses” (Jared Algebra Class, March 28, line 54) when asked to multiply $-2x^2y(x^3 - 3xy + y^2)$.

Other warrants that were left implicit were observed in arguments such as one that occurred on February 21. As the class was discussing questions from the previous night’s homework assignment, a student claimed that a particular system of equations had no solution. He gave the reason for this as “Parallel lines” (Jared Algebra Class, February 21, line 247). This was accepted, and the class went on to discuss the next question. One hypothesis as to the reason for a lack of warrant here could be that, earlier in this same
class period, the class was engaged in discussing the solutions to several systems of
equations, the graphs of most of which intersected, and they had discussed classifications
of solutions to systems of equations. This discussion included the fact that parallel lines
do not intersect and thus have no solution, so when the student stated that there was no
solution because the lines were parallel, Jared did not see a reason to restate a warrant
that connected the fact that the lines were parallel to the claim that there was no solution.
Other arguments in which warrants were left implicit took place under similar
circumstances. That is, they occurred within arguments in a discussion of a series of
similar problems during which warrants were given within some of the arguments.
Several of the arguments about solutions to systems of equations in the class on February
21 fell into this category as did several arguments regarding whether two monomials
were like terms on March 20.

Jared’s Support for Argumentation

One of Jared’s main contributions to argumentation in his classroom was in
providing warrants within arguments. He individually provided many of the warrants in
the arguments in his classroom, and provided many more in conjunction with his
students. Most of the warrants in Jared’s classroom were rules or statements of
procedure. This may have been due to an emphasis in this class on knowing these rules
and knowing how to do the problems that were discussed in class. Jared left relatively
few warrants implicit. The warrants that were left implicit can generally be inferred from
the given data (so the warrant was just not separate from the data) or were given earlier
within a different argument or as a definition, fact, or procedure explicitly discussed. Jared’s questions that elicited warrants from his students generally focused on what they should do and resulted in students providing warrants that dealt with rules or procedures. Students’ claims, data, and warrants typically were provided in response to questions from Jared; rarely did a student spontaneously provide data or a warrant for a claim.

Jared’s support for argumentation can be seen to align with some aspects of his conception of proof. In the next part of this chapter, I describe Jared’s conception of proof as it relates to the argumentation he supported in his classroom. Several aspects of Jared’s conception of proof will be related to the kinds of warrants he and his students provided and when those warrants are provided.

**Jared’s Support for Argumentation Aligns With His Conception of Proof**

Since Jared’s conception of proof is related to his views of school mathematics, I first situate his conception of proof within his view of school mathematics. I briefly show how Jared’s overall view of school mathematics and proof’s place within it relates to the argumentation that occurs in his classroom. I then describe several specific aspects of Jared’s conception of proof and discuss how these aspects contribute to explaining some parts of the argumentation that occurred in his classroom.
**Proof’s Place Within School Mathematics**

When asked, “What are the important aspects of math?” (Jared Interview 1, line 7), Jared immediately replied, “Understanding” (Jared Interview 1, line 8). He gave the following elaboration in which, significantly, proof was not mentioned.

> Understanding. Knowing math is a building process, like, once you learn something you really can’t forget it. You have to practice, practice, practice so that you know how to do it, because one thing builds upon another. And then, um, understanding is a big thing, especially when you’re doing word problems. You need to understand what you’re doing, or else you’re not going to be able to do it well if you don’t understand what the problem is asking you. (Jared Interview 1, lines 8-14)

In Jared’s answer, there is a sense that he was referring not to mathematics as a discipline but specifically to school mathematics or how one might learn mathematics. This reference to school mathematics or learning mathematics rather than mathematics as a discipline continued throughout the interviews. Jared’s references to mathematics did not seem to reference any mathematics outside of what one might learn in secondary school, except when talking about his college mathematics experiences, but even these were mentioned in the context of learning mathematics. The idea of mathematics as a field or a discipline to which one could contribute was not mentioned. Jared mentioned knowing or understanding how to do problems, understanding what one is doing, and understanding what the problem is asking. These three aspects of understanding also appeared later when Jared talked about proving. Other important aspects of mathematics for Jared include knowing it is a building process, so you have to remember what you have learned, and practicing, so that you know how to do it.
As Jared explained his view of the importance of proving in mathematics, he made the statement, “I feel we need to spend more time getting the students to be able to do the math, do the skills” (Jared Interview 1, lines 40-41). In this statement, Jared seems to be equating math with skills. This is consistent with one of the three aspects of understanding that Jared talked about as important in mathematics: understanding how to do problems. This view of skills as important in or equivalent to mathematics was also evident in Jared’s report of his classroom practice (and in an examination of the tests and quizzes he used in his class). In reference to asking his students to solve systems of equations, Jared specified which method they were to use, both in group work and on tests. As he reflected on his practice, he said that one thing he would change was that he would have asked his students to choose which method to use.

I think with systems I might have done more you choose, more practical problems where the kids really had to think which method would be best. Um, maybe because on all the group work and the tests I said use, do that, do this, use this method. I think that might have been more beneficial. (Jared Interview 2, lines 290-296)

Examining Jared’s tests and quizzes reveals that he usually specified how students were to do each problem. Following directions in this regard was very important to Jared. This is apparent from an exchange in the second interview in which, as shown in the next few paragraphs, Jared would have assigned two out of three points to an answer because the student did not use the indicated method, even though he or she gave a complete solution.

The original prompt asked how many points out of three Jared would give a student who answered the question as shown in Figure Error! Reference source not found.. Jared first requested clarification that these were the only directions and then said he would give it full credit if the directions did not say “show your work.”
Solve the following system by substitution:

\[3x + 2y = 5\]
\[3x + 2y = 7\]

No solution

Figure 6-3: Hypothetical student solution to system of equations problem

When asked how he would assign credit if the directions instructed student to show their work, he said he would give the answer only one point out of three. Jared was then asked how many points would be given for the following answer.

No solution because if three \(x\) plus two \(y\) equals five and three \(x\) plus two \(y\) equals seven, that doesn’t make sense. You can’t have the same \(x\) and \(y\) substituted in that will give you two different answers. (Jared Interview 2, lines 60-63)

He replied, “Two out of three points” (Jared Interview 2, lines 74-75) and gave the following rationale.

I’d probably give it two out of three points, because they did validate this point. In this case [the answer of simply “no solution”] they didn’t validate their point. But, they kind of, they didn’t do any work but they kind of showed the process that they used to think of their work. (Jared Interview 2, lines 74-79)

Thus, because the question asked the student to solve by substitution and he or she did not show the steps that Jared thought should accompany the process of solving by substitution, he or she could not receive full credit for their answer. To Jared, an explanation was not as good if it did not contain the algebraic work required to solve it by a specific method.
In addition to his direction on how problems should be solved on tests and quizzes, Jared’s questions to his students within argumentation often seemed to signal that he wanted particular answers. For instance, as described in Typical Episode 1, when he and his students were discussing how to solve the inequality and a student suggested dividing by a negative number, Jared and his student had the following exchange about reversing the inequality sign.

Jared says, “What do we have to keep in mind, when we’re dividing by a negative number and we have an inequality?” The student says, “Flip the inequality sign.” Jared says, “When do we flip the sign?” A student answers. Jared clarifies, “Dividing or.” A student answers. Jared says, “Not by an integer but by a.” A student answers. Jared repeats, “By a negative number. (Jared Algebra Class, lines 67-76)

Jared seemed to want his student to give the same reason for the procedure of reversing the inequality sign as he had given in the first sentence, and he continued to ask questions until that procedure had been reiterated.

Jared’s focus on directions may explain his and his mentor’s differing perceptions of her influence on his classroom practice. Jared said that his mentor told him what to do and how to do it. However, in the mentor interview, his mentor said she primarily gave advice on how the lesson went after it was taught. It seems Jared might see school mathematics as about following rules and procedures, beyond any influence of his mentor. It is possible that he sees teaching school mathematics in a similar way. A view of teaching mathematics as primarily a set of rules and procedures would further explain Jared’s emphasis on rules and procedures within his support for classroom argumentation.
In the first interview, Jared revealed that he viewed proof as appropriate for students in higher level classes but not for students in lower level classes. This is consistent with Knuth’s (2002b) report of the views of most teachers in his study.

Depends on the type of students. I think that in lower level classes, no, it’s not very important. You just need to be able to get them to use the generalizations, theorems, whatever. But, in an honors class, in a calculus class, even a trig class, because trig seems to almost be an elective here, … in cases like that, proving is good. But, in classes where kids still can’t take a negative times a negative and get a positive number, what’s the point? To me, that’s what I feel we need to spend more time getting the students to be able to do the math, do the skills. (Jared Interview 1, lines 32-41)

In separate conversations with Jared, he would characterize many of his students as ones who “can’t take a negative times a negative and get a positive number” (Jared Interview 1, lines 39-40). Jared’s viewpoint on learning mathematics may have been influenced by his beliefs about the students he was currently teaching. He did not see his students, who were in a regular-level algebra class, as particularly mathematically able, and this may have been a factor in how he structured his classes. In particular, Jared provided or assisted his students in providing many more warrants than he asked his students to provide on their own. This provision of warrants is also consistent with the slightly revised view of the role of proof in teaching mathematics Jared demonstrated in the second interview. When asked, “So, in your opinion, now that you’ve taught for a couple of months, do you think students should prove mathematical statements in high school?” (Jared Interview 2, lines 124-126), Jared gave the following response.

Yes. Especially the geometry classes, and especially in the high level classes, where kids are more intrinsically motivated to learn, I think in my experiences, still kind of shaky. But there’s not really much to prove in algebra. So, I do still believe that it’s valuable to show proofs, just because, in a way, that’s also making students more critical thinkers and
more, you know, using their brain more, because a lot of these kids are not going to use the math, but they’re going to need to think logically, they’re going to need to think critically. And doing stuff like that teaches them more than just the content or the generalization involved. (Jared Interview 2, lines 127-136)

Even though he again mentioned proof as important for the high level classes, Jared’s view of proof seems to have changed in focus to one that sees proof as important for the development of critical thinking in students. Thus, although it does not seem that Jared viewed proving as important for all students, he did seem to view exposure to proofs as important for all students. His statement that “there’s not really much to prove in algebra” (Jared, Interview 2, lines 127-128) may have been his way of explaining why, even though he thought proof was important, he did not engage in it during his teaching of algebra. Alternatively, this statement may mean that he truly thought that algebra consists of the skills he was teaching his students, and thus he did not see opportunities to prove in algebra.

Jared suggested that proof was not particularly important to him when he was learning mathematics, but as he looked back on the experiences he could see some value in proving. Jared first gave his opinion as a learner of mathematics that the important things to know were those that would be assessed on the test, so proving was not important: “Honestly I’m like why do I need to know this if I don’t need to know it for the test?” (Jared Interview 1, line 44). He then clarified that, looking back on the experiences he has had with proving, he saw some value in proving during his mathematics learning experiences. This value had to do with knowing how to do something in mathematics because he saw the proof of it.
But, as I look back, I think I appreciated a few ideas more because I knew where it came from. Like at the time it was like this is so stupid, but now, like looking back, it’s like oh, that does make sense now, and I do understand that more, and I do remember how to do this because we did the proof, you know. (Jared Interview 1, lines 47-51)

Jared seems to view school mathematics, and proving within school mathematics, as focused on understanding what to do and how to do it. This may be similar to the beliefs of Mingus and Grassl’s (1999) participants, who believed the role of proof was to explain how concepts work. Jared’s classroom practice and reflections on hypothetical student answers reflect this orientation. His conception of proof reflects this view, but maintains a separate identity. He talked about proof or proving as it related to learning mathematics, but talked about proving as a separate activity. The warrants Jared provided and accepted from his students align with this perspective. Many of the warrants observed in his class were rules and procedures. These rules and procedures told Jared and his students what they were to do and how they were to do it. In order to be successful, they needed to apply the appropriate rule, and often Jared specified which rule was to be applied to each problem. Jared did not see proving as important in algebra; this is consistent with his being satisfied with warrants that simply stated the rule or procedure. Finally, exposure to proof but not engaging in proving was important for lower level students, and since Jared thought of his students in this way, he provided many more warrants than they provided.
Jared’s Proving: Proof as Procedure and Missing Warrants

In response to a question about when he felt unsuccessful in proving, Jared described an experience with the midterm exam in his college geometry course. On this exam, he wrote a proof for a theorem, and the proof, while incorrect for the stated theorem, was appropriately written for a different theorem. He had been given a list of theorems to be proved on the test and had memorized the proofs and then recalled the wrong one for the particular theorem. He concluded, when discussing this experience, that the problem was not with his proving but with his memorization. However, the fact that he memorized the proofs and then wrote one down without noticing that it was for the wrong theorem is consistent with the possibility that he viewed proving as a procedural event and not one that made sense. It may be that his idea of a proof may have been able to be separated from the statement to be proved. In addition, his associating the proof with the wrong theorem suggests that he was not paying attention to the warrants for his claims. Warrants are the bridges between the data and claim of an argument, and his data did not relate to the claim being made.

Jared’s description of proving, however, suggests that he thought that proving should make sense, or at least that it was dependent on what he knew. He described the following as being involved in proving a mathematical statement: “Background knowledge, understanding of what you have to prove, and patience” (Jared Interview 1, lines 121-122). He described proving as involving multiple attempts (“you’re not going to always get the right answer right away”) as well as sudden insight (“you could be like in the shower and oh, that’s what I need to do”) (Jared Interview 1, lines 126-129). Jared
described his process of proving as involving reading what he “has to prove” (Jared Interview 1, line 134), thinking about “what I know that could help me” (Jared Interview 1, line 136), and then “devise a plan, pretty much, go from there, start with what I know” (Jared Interview 1, lines 138-139). He knew he had finished “when you’ve proved the statement” (Jared Interview 1, line 145), but “sometimes when I prove things I just keep on going and going and going” (Jared Interview 1, lines 150-151). He knew he had been successful when he got “the right answer” (Jared Interview 1, line 79). When asked how he knew it was right he said, “Because it made sense” (Jared Interview 1, line 81).

Describing proving as something that makes sense contrasts both with his description of memorizing a proof for a test and with his actions when asked to prove that the sum of the first \( n \) natural numbers is \( \frac{n(n+1)}{2} \). However, his description of proving a mathematical statement did provide him with something of a procedure for proving. His procedure for proving is general in nature, yet does not address deductive reasoning.

Jared’s first inclination when asked to prove the statement, the sum of the first \( n \) natural numbers is \( \frac{n(n+1)}{2} \), was to use mathematical induction. He described induction (and how he was applying it in this situation) by saying, “Okay, so first it’s to prove the base case, which is \( n \) is equal to one, so one equals one plus one, so one is equal to one, so the base case is true” (Jared Interview 1, lines 161-163). He wrote what is shown in Figure Error! Reference source not found. and then said, “Then we have to assume it’s true for the base case and prove it for all \( n \) plus one” (Jared Interview 1, lines 166-167). While Jared seems to have the basic idea of proof by mathematical induction, he left out a crucial part: the induction hypothesis.
Perhaps because of his interpretation of proof by induction, Jared began to substitute $n + 1$ for $n$ in \( \frac{n(n+1)}{2} \). He wrote $n + 1 \ \frac{(n+1)[(n+1)+1]}{2} = \frac{(n+1)(n+2)}{2}$. He wrote and then crossed out $(n+1)$, $n+1=n$, and $\frac{n(n+1)}{2}$. He said, “Cause I say like, I don’t know what I’m doing. Well, um, so we show that this, I don’t know what I’m doing” (Jared Interview 1, lines 182-183). He paused and seemed to look at the paper. I asked, “How would you write out this part of the statement?” (Jared Interview 1, line 184) while pointing at “the sum of the first $n$ natural numbers.” Jared wrote $\sum_{i=1}^{n} i$, said, “From one to $n$, or $i$ is equal to” (Jared Interview 1, line 190), and wrote $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Jared said, “So then we do” (Jared Interview 1, line 190), and wrote $\sum_{i=1}^{n+1} i$. He said, “Now I’m good now, I think” (Jared Interview 1, line 197), and wrote $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+1)}{2}$. He said, “And that’s equal to, I don’t know what I’m doing. And that’s equal to this plus $n$ plus one” (Jared Interview 1, lines 197-198). He pointed at $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and wrote
+n+1 beside it so that it read \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} + n + 1 \). I asked how he knew

\[
\sum_{i=1}^{n+1} i = \frac{(n+1)(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} + n + 1
\]

were equal, and he said, “Um, let me try.

Yes, because we can multiply this out, we have \( n^2 + 3n + 2 \) over two” (Jared Interview 1, lines 207-208) and wrote \( \frac{n^2 + 3n + 2}{2} \). He said, “If we do this we need a common denominator, um, so two \( n + 2 \)” (Jared Interview 1, lines 207-208), and wrote \( \frac{2n + 2}{2} \) over \( n+1 \) so that \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} + n + 1 \) became \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} + \frac{2n + 2}{2} \). He said, “so, yeah, \( n^2 \) plus \( n + 2 \) plus two” (Jared Interview 1, lines 210-211), wrote \( n^2 + n + 2n + 2 \), and said, “So \( n^2 \) plus three \( n \) plus two” (Jared Interview 1, lines 211-212), while writing \( \frac{n^2 + 3n + 2}{2} \). After a long pause, when Jared seemed to have completed his argument, I asked how well the argument would convince a high school calculus student. Jared replied, “Well, if it was as jumbled as this, right now, it might be a little” (Jared Interview 1, lines 223-224). What Jared had written as his argument can be seen in Figure Error! Reference source not found.. I asked what if he rewrote it, and Jared gave the following reply.

I think that I’d be satisfied with it as a high school student, I think, because I’d be like, oh, that’s cool that this is a sum plus \( n \) plus one is the same as taking this sum. Adding one more thing to this is the same as this. That’s cool, that works, that works out like that. (Jared Interview 1, lines 229-233)
Jared interpreted what he had written as a valid proof, and this statement gives further insight into what he was doing. He interpreted \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} + n + 1 \) to mean:

\[
\sum_{i=1}^{n} i + n + 1 = \frac{n(n+1)}{2} + n + 1,
\]

and said that \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} + n + 1 \) (meaning

\[
\sum_{i=1}^{n} i + n + 1 = \frac{n(n+1)}{2} + n + 1 \]

was the same as \( \sum_{i=1}^{n+1} i \). In his view, \( \sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + n + 1 \) seemed to be the important part of the argument. It is possible that he saw “add \( n + 1 \)” as a step in mathematical induction, and so he was finished when he had done so.

Figure 6-5: Jared’s Written Argument that the Sum of the First \( n \) Natural Numbers is \( \frac{n(n+1)}{2} \)
Jared’s argument that the sum of the first \( n \) natural numbers is \( \frac{n(n+1)}{2} \), as interpreted above, included many pieces of a proof by induction of this statement. However, one key element, explicit mention of the induction hypothesis, was missing. This is consistent with Segal’s (1998) report of his participants’ difficulties with induction proofs in which they often either used informal methods or failed to prove the appropriate implication. Toward the beginning of his work, Jared did say, “Then we have to assume it’s true for the base case and prove it for all \( n \) plus one” (Jared Interview 1, lines 166-167). Thus he knew there was an assumption to be made within induction. However, he assumed what he had already verified. Without an induction hypothesis, Jared proved, in his words, that “adding one more thing to this is the same as this” (Jared Interview 1, lines 232-233); his two procedures produced the same result.

Jared’s process of proving this statement, when considered with his description of memorizing a proof and his statement of when he knew he had finished a proof, suggests that Jared views the goal of proving as coming up with an answer. Whether this answer was the proof or the last line in the proof is not clear from what he said. However, this emphasis on getting to the answer was also apparent in his classroom. Jared often prompted his students to make claims by asking them for answers. Because of this norm, Jared made only about one third of the claims in his class. The other two-thirds were made by students, with or without some assistance from Jared.

Within Jared’s work with induction, he made a claim that \( \sum_{i=1}^{n+1} i = \frac{(n+1)(n+1)}{2} \) and \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} + n + 1 \) were equivalent. However, he did not provide data for that claim.
until I asked him to do so, at which time he manipulated each expression to be
\[ \frac{n^2 + 3n + 2}{2} \]. After he did this, he stopped and appeared to be finished. It seems that completing the symbolic manipulation has, for him, completed the proof. In his classroom, many arguments contained symbolic elements. These often appeared as the data in an argument, and often, while there was a warrant in the argument, there was no warrant for individual parts of the manipulation. For instance, in the first sub-argument in Typical Episode 1 (see Figure 6-1), Jared and his students solved the inequality by a number of algebraic steps, including subtracting the same number from both sides, but the only step to which the warrant refers was the last, when the inequality sign was reversed due to dividing by a negative number.

Jared’s statements about proving and his work when proving suggest that he viewed proving as a procedure whose goal is a right answer. The pattern of claims in his classroom was consistent with his emphasis on right answers. In addition, Jared did not provide a warrant for his symbolic work in his argument, and the warrants given in his classroom were often not in support of the symbolic work within arguments.

**Analyzing Proofs: Multiple Methods**

When asked if the argument given in Figure 6-1 proved the statement given there, Jared talks through the argument and, after a long pause, concluded that it did, “by contradiction” (Jared, Interview 1, line 302). When I asked if he was convinced, Jared did not answer, so I ask, “Did they leave anything out
that you would have liked to be in there to be more convincing?” (Jared, Interview 1, lines 303-305). This question was followed by the following exchange.

Jared: I don’t know, maybe go the other way?
AC: Okay
Jared: Just say like suppose it is irrational, hmm, go the other way.
AC: So, you’d rather they have done
Jared: Done both ways to show
AC: So both directly and by contradiction?
Jared: Right, I mean I’m convinced by this, but I’m just saying that that you could do it both ways. (Jared Interview 1, lines 306-313)

Jared eventually said he was convinced by this argument; however, it seems that he is saying that he would actually prefer it to be proved by both methods.

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The square root of a positive integer is either an integer or it is irrational.

Suppose \( n \) is a positive integer, \( \sqrt{n} \) is not an integer, and \( \sqrt{n} \) is rational. Then there exist relatively prime integers \( a \) and \( b \) with \( \sqrt{n} = \frac{a}{b} \). Squaring both sides, \( n = \frac{a^2}{b^2} \), from which \( nb^2 = a^2 \). Now if we factor \( a \) and \( b \) into primes, there are no common factors. So the factorizations of \( a^2 \) and \( b^2 \) have no common factors. Consequently, the factorizations of the equal numbers \( nb^2 \) and \( a^2 \) are different. Since two different factorizations of \( a^2 \) are impossible by the Fundamental Theorem of Arithmetic, the supposition must be false. So if \( \sqrt{n} \) is not an integer, it must be irrational.

Figure 6-6: Square Root Argument to Critique (from Usiskin, Peressini, Marchisotto, & Stanley, 2003, pp. 23-24)

Jared’s emphasis on checking solutions, evidenced in particular by his demonstrating alternative methods to solve an equation or inequality, may be related to his apparent preference here to see both methods of proof. He stated he was convinced by
the argument, but perhaps he was not as well-convinced as he would have liked to have been. Proving the statement in a different way may have, to him, provided additional confirming evidence. Alternatively, he may have seen a second method of proof as a way to check the first proof. He provided this type of confirming evidence for his students when he demonstrated alternative ways to solve problems, and he used this confirming evidence as warrants within his arguments. Jared’s apparent desire for additional evidence after seeing the proof is reminiscent of students in Chazan’s (1993) study who saw proof as simply evidence of the veracity of a claim.

Conclusion

Jared viewed proof as important for understanding how to do things in school mathematics, which he presented as skill-based. He acted as if proving was a procedure, and his tendency was to work within proving without making his warrants explicit. In addition, he operated as if additional confirming evidence would be better than simply one proof. These aspects of Jared’s conception of proof were evident in the differences between who contributed the majority of claims and warrants in his class, and in the kinds of warrants that were provided. Jared supported argumentation both by providing warrants and asking questions that prompted his students to make a claim or give the rule or procedure they used in support of their claim. The kinds of warrants that were observed in Jared’s class align closely with his conception of how proving is useful in mathematics.
Chapter 7

Cross-Case Synthesis and Conclusions: Questions, Warrants, and Purposes

Jared, Karis, and Lynn each represent a case of a student teacher whose facilitation of argumentation aligned with his or her conception of proof. Each taught a different mathematical subject to a different group of students, and each worked with a different mentor teacher. In the preceding chapters, I described how the patterns of argumentation observed in each classroom were influenced by the individual student teacher’s conception of proof and justification. This chapter involves looking across the cases to answer the research questions presented in Chapter 1 (and repeated here).

- How do prospective secondary mathematics teachers support claims, data, warrants, and backings as elements of argumentation in secondary mathematics classrooms?
- What characterizes the relationship between the argumentation observed in a particular classroom and the prospective secondary mathematics teacher’s conception of proof and justification?

First, I describe how student teachers support claims, data, warrants, and backings as elements of argumentation in secondary mathematics classrooms. I then discuss commonalities and differences in how these student teachers’ conceptions of proof align with their support for argumentation in their classrooms. Finally, I suggest a particular aspect of a student teacher’s conception of proof that has potential to influence the argumentation in his or her classroom.
Argumentation Supported by Student Teachers

The overarching similarity among arguments in all three classrooms was the extent to which arguments were constructed at the request of and with the assistance of the student teacher. The importance of the teacher in facilitating argumentation has been suggested by several researchers (Cobb, 1999; Cobb, Boufi, McClain, & Whitenack, 1997; McCrone, 2005; Yackel, 2002), although some of the specific ways in which these student teachers supported argumentation differed from those reported by other researchers, as described in the following sections.

Several common themes arose from an examination of the teachers’ roles within argumentation in their classrooms. The student teachers participated in most if not all arguments in their classrooms. The student teachers asked questions that prompted students to make claims, provide data, and provide warrants; these kinds of questions and answers varied across their classrooms. The pattern of arguments in terms of numbers of main and sub-arguments varied across the practice of the three student teachers, but some similarities were observed across the argumentation in their classrooms. The main support for argumentation by each of the three student teachers was in the provision of warrants within arguments, and the kinds of warrants provided varied across the three student teachers. Each of these themes will be explored in the next few sub-sections, beginning with the kinds of questions the student teachers asked of their students.
Kinds of Questions and Norms Governing Answers

All three student teachers asked questions that prompted students to make claims and asked other questions that prompted students to give data. If a student provided a warrant, it was probably in response to a question from the student teacher. In fact, the interaction between the student teachers and their students in these classrooms could be described as following the IRE (initiation by teacher, response by one or more students, and evaluation from the teacher) model, described by Forman and Ansell (2002) as a contrast to the discussion orchestration model employed by the teacher in their study. The kinds of questions that were asked within argumentation in these classrooms and the norms that governed students’ answers differed across the three classrooms.

In order to understand the patterns within the questions and answers in the three classrooms, consider the following examples of questions and the components of argumentation that they evoked from the students. In Lynn’s class, many questions were similar to “What can we say about $\overline{BE}$ and $\overline{ED}$?” (Lynn Geometry Class, March 6, line 87) and “And how did you conclude that? How did you know that?” (Lynn Geometry Class, March 6, lines 244-245). Students generally responded to these questions by providing a claim, providing a warrant, or providing both data and a warrant. In Jared’s class, questions often were of the forms “What do we get?” (Jared Geometry Class, March 8, line 75), “What did you do for number 20?” (Jared Geometry Class, March 8, line 41), and “How did you get $y$ to the fourth?” (Jared Geometry Class, March 8, line 43). In general, his students provided claims or data as answers to these questions. Karis’ questions varied a bit more than the other two student teachers’ questions, but seemed to
be similar to “So how do we get started?” (Karis Calculus Class, March 13, line 260) and “So, using our power rule, what is the integral of that? (Karis Calculus Class, March 13, line 275). Karis’ students provided data and claims in answer to these questions.

While the questions in the preceding paragraph appear similar on the surface, there are some noteworthy differences in these questions and in the kinds of argumentation in which they occur. In addition, the ways in which students answered these questions point to possible norms of argumentation in the classrooms. At first glance, “What can we say about $\overline{BE}$ and $\overline{ED}$?” (Lynn Geometry Class, March 6, line 87), “What do we get?” (Jared Geometry Class, March 8, line 75), and “So, using our power rule, what is the integral of that? (Karis Calculus Class, March 13, line 275) seem to be similar requests for claims. In each case, the student teacher provided some information, whether verbal or written, and students were asked to make a mathematical claim based on that information. However, Lynn’s question asked students to make a claim about two geometric objects, specifying the objects. Thus, Lynn provided support for the claim by specifying which part of the figure she wanted students to consider. She did not, in this question, provide any other component of the argument, and she asked for a claim about a mathematical relationship. Jared’s question was a direct request for a claim that was an answer to a calculation. The task to which he was referring was written on the board, and a student had previously stated the procedure for simplifying the given expression at Jared’s request, so the claim was requested after the data and warrant for the argument have already been stated. Karis’ question provided a warrant and requested that her students provide the result of a calculation, specifying how that calculation should be done (“using our power rule”).
In Karis’ class, data and warrants often preceded claims. As Karis and her students worked problems, the warrants were often theorems or general ways to do particular kinds of problems, the data was often calculational in nature, and the claim was made when the solution was reached. These arguments tended to be longer than those observed in the other classrooms, perhaps explaining why fewer arguments were observed in her classroom. The norm that governed student answers seems to be that the answers should be a mathematical calculation or the result of a mathematical calculation. Often, students could expect to know what mathematical calculation would be expected by attending to the warrant given by Karis. In many cases, the patterns in discourse in Karis’ class could be described as calculational discourse as defined by Cobb (2002). Cobb describes calculational discourse as “discussions in which the primary topic of conversation is any type of calculational process that is enacted to produce a result or arrive at a conclusion” (2002, pp. 198-199). However, in Karis’ class some patterns of discourse were more conceptual in nature, resembling what Cobb calls conceptual discourse, since when she provided data and warrants sometimes “the reasons for calculating in particular ways also become an explicit topic of conversation” (Cobb, 2002, p. 199).

In Lynn’s class, arguments often began with a claim, prompted by a question such as “What can we say about \( \overline{BE} \) and \( \overline{ED} \)?” (Lynn Geometry Class, March 6, line 87), and data and warrant were provided after the claim. However, when there was an algebraic component of the claim, the order was often reversed so that the data preceded the claim. (Often algebraic arguments in Lynn’s classroom did not include warrants.) A norm that governed student answers in Lynn’s class was associated with the order of geometric
arguments: a geometric claim was to be followed by data and warrant. The patterns of argumentation observed in Lynn’s class could be categorized as primarily calculational in nature, although some conceptual discourse was observed. Even though some discourse in Lynn’s class surrounded constructing proofs, some of these were constructed procedurally, with only passing reference to the underlying theorems and logical structure.

In Jared’s class, the order in which components of arguments were contributed varied. Sometimes a claim was made and then supported by (usually calculational) data and warrant, and other times the data came first, followed by claim and warrant. Students generally provided mathematical calculations or the results of mathematical calculations as answers to Jared’s questions, although they sometimes provided the rule or procedure that served as the warrant within an argument. A norm for student answers in Jared’s class seemed to be that students should provide the next step in the solution procedure or a numerical answer to a previously given step. The argumentation in Jared’s class could be characterized as primarily calculational in nature, according to Cobb’s (2002) description of calculational discourse.

The evidence presented above suggests that some of the norms that govern student contributions to arguments in the three classes can be inferred from the questions asked and kinds of answers given. These norms relate to what kinds of answers were expected from students and thus what components of arguments students provided. One additional observed norm governing discourse (and thus argumentation) in the three classes was that students waited to be acknowledged by the student teacher (verbally or nonverbally) before speaking, and that the conversation progressed generally in a student
teacher – student – student teacher – student pattern. This was true across all three classrooms, even though the norms regarding the contributions of the students and student teacher to argumentation differed across the classrooms. This suggests that observing patterns in who is speaking is not nearly as informative as observing what they are contributing to the conversation. That is, the argumentation was observed to be different in the three classrooms, even though the same basic student teacher – student – student teacher – student pattern in verbal contributions to the classroom discourse was observed. The differences in argumentation are apparent by examining which components of arguments were contributed by the teacher and by the students in each classroom.

**Trends in Student Contributions to Arguments**

One way to characterize the student teachers’ support for argumentation is by examining how frequently they supported students’ contributions to arguments as opposed to contributing the components themselves. While both students and the student teachers participated in many arguments in each classroom, much of this participation was in collaborative contributions of data to arguments. The trend in student participation in arguments in general and in claims and warrants in particular was similar in Jared’s and Lynn’s classrooms, while the participation of students in arguments in Karis’ classroom differed from either of these.

Jared’s and Lynn’s students contributed to more arguments than Karis’ students did. In each of Jared’s and Lynn’s classes, students made more claims than the student
teacher, and when arguments to which both student teacher and student contributed are considered, each of Jared and Lynn contributed only about one third of the claims without contributions from a student. This pattern does not hold for Karis’ class, in which students contributed to only about one half of the claims. This difference may be due, in part, to patterns in the order in which components were contributed. That is, in Karis’ class, claims were often made (or completed) after the data and warrants are given. Karis presented a question to the class, she and her students contributed data and warrants as they solved the problem, and then either Karis or a student gave the final answer, completing the claim. In this pattern as observed in Karis’ class, the claim consists of the problem or question that was presented to the class along with the solution or answer to the question. Thus the claim was made by the person who gives the final answer. In the process of solving the problem or finding an answer to the question, mathematical ideas, processes, or procedures were presented by Karis and her students; these ideas, processes, or procedures made up the data and warrants that support the claim. Karis was perhaps more likely to present the answer in this situation than either Jared or Lynn, since Lynn usually asked for the claim in advance of the data and warrant and Jared tended to ask his students for the final solution rather than giving it himself.

Arguments in the three classes varied in both the number and kinds of warrants contributed by students. Students contributed fewer warrants than student teachers in all three classes. However, students in Karis’ class contributed fewer warrants than students in either Jared’s class or Lynn’s class. This may be explained by Karis’ focus on explanation as paramount; she provided what she thought were the appropriate explanations for her students, and thus there were fewer opportunities for her students to
provide warrants. Jared’s and Lynn’s students contributed warrants to many more arguments (32 and 39, respectively) than Karis’ students did (9). Their contributions, however, were different in the three classrooms. Jared’s students’ warrants were rules and procedures, while Lynn’s students’ warrants were theorems or properties of geometric objects. Karis’ students’ warrants were theorems, rules (formulas), and definitions. These warrants were of the same general types as the warrants provided by the student teacher in the respective classroom.

The kinds of warrants provided by students, and, as discussed in the next section, also provided by the student teachers, may be related to the kinds of tasks presented by each of the student teachers. For instance, the tasks in Lynn’s class were often proofs or applications of generalizations to geometric figures, so it seems reasonable that she and her students would offer primarily definitions, theorems, and properties as warrants within her class. The tasks in Jared’s class tended to be procedural in nature, asking students to apply skills that they had learned, and warrants in his class tended to be procedural in nature. The tasks in Karis’ class are more difficult to categorize, but they seemed to include a mixture of types, from more straightforward applications of procedures to ones that required more thoughtful applications of generalizations, and the warrants in her class included a mixture of theorems, rules, and definitions. The following section elaborates on how the student teachers’ conceptions of proof, in addition to the overarching nature of the tasks, contributed to differences between the kinds of warrants commonly found in their classrooms.
Warrants Provided by Student Teachers

One of the main ways the student teachers supported argumentation in their classrooms was in providing warrants for arguments. Paying attention to warrants within argumentation has been found to be important in several studies of argumentation (Forman & Larreamendy-Joerns, 1998; Yackel, 2002). In each classroom, the student teacher provided more warrants than the students, even when students are credited with warrants to which the students contributed but for which they were not solely responsible. In each case, a different hypothesis is put forth to explain this phenomenon. In Karis’ case, her provision of warrants was related to her emphasis on explanation. Karis believed that proofs explain, so she provided the warrants to make sure her students understood the mathematical ideas. In Lynn’s case, her provision of warrants was linked to her view of proof as important to know the reason why things are true. Jared’s provision of warrants seems to be linked to his view that exposure to proof but not engaging in proof was important for lower level students. Regardless of the possible explanations, the argumentation in all three of these student teachers’ classrooms was characterized by student teachers providing warrants as part of their support for argumentation.

The kinds of warrants that were provided in the three classes included theorems, definitions, rules, procedures, properties, appeals to visual cues such as a figure, and references to checking solutions. The warrants that were common to arguments in all three student teachers’ classrooms were definitions and procedures. Lynn used definitions
much more frequently than did Jared or Karis, and she used procedures somewhat less
frequently than the other two.

Jared and Karis both regularly used procedures as warrants. However, Karis’
procedures-as-warrants differed from Jared’s procedures-as-warrants. Many of Jared’s
procedures-as-warrants were simple phrases, such as “add like terms” or “subtract $x$ from
both sides and divide by negative one” that were used frequently by both Jared and his
students. Karis’ procedures-as-warrants tended to be longer and of greater complexity.
Her procedures often seemed to function as both a warrant for the specific argument and
a general guideline for solving problems similar to the one in the argument. For instance,
in an argument regarding a collapsing sum, Karis said, “I write out the terms, as many as
I need to, to see the pattern, cross off what cancels, and then just simplify it” (Karis
Calculus Class, March 29, lines 88-89).

Part of the reason for the difference in complexity between the procedures used as
warrants in the two classes may be the mathematical content of the classes (calculus
versus algebra) and the choice of tasks for students. In general, the tasks that students
were asked to complete in Karis’ calculus class required more steps and a coordination of
ideas from calculus and algebra, while the tasks students were asked to complete in
Jared’s class were presented as straightforward applications of the ideas discussed in the
past few class periods. Considering these task differences, along with the differences in
Karis’ and Jared’s conceptions of proof, may explain the differences in warrants observed
in the two classes. Karis focused on explanation and understanding the mathematics; she
presented tasks that required a coordination of ideas from calculus and algebra, and she
provided warrants that were often a general guideline for solving a class of problems.
Thus, her explanatory focus, when coupled with the types of tasks she chose, seems to suggest both the range of warrants that she offers as well as an explanation for the particular ones that are general guidelines for solving a class of problems. Jared focused on *how* to do something; he presented tasks that could be completed using procedures recently presented in class, and his warrants reflected a focus on the immediate way to solve each problem. Thus, his focus on proof as important for understanding how to do something, taken together with the kinds of tasks he presented to his students, are consistent with the warrants that arise within his classroom. The differences between the particular kinds of procedural warrants arising in Karis’ and Jared’s classrooms can be jointly explained by the kinds of tasks presented and the student teachers’ conceptions of proof. These differences, along with themes that arose from examining the alignment of all three student teachers’ conceptions of proof with the argumentation they supported in their classrooms, will be examined in the next section.

A brief examination of the tasks assigned in each of the classes suggests that the differences in argumentation observed across the classes may be more related to the tasks that were used as part of the curriculum than the differences between teaching calculus, geometry, or algebra. Lynn and Karis both used tasks that required students to coordinate knowledge from several concepts discussed in their classes as well as knowledge from previous classes (usually algebra in both cases). In the beginning, Jared also assigned tasks that required students to coordinate knowledge from an earlier part of the course, but after the third observed class, Jared tended to assign tasks that required students to use knowledge from the current class and one or two previous classes. Lynn’s tasks also included some proving, always completed in two-column form. As illustrated in the
previous paragraph, examining the types of tasks presented, in conjunction with the student teachers’ conceptions of proof, may explain more of the differences in their support for argumentation than an explanation that relies on differences in the subject matter (calculus, algebra, geometry) they were teaching.

**Aligning Conception of Proof With Support for Argumentation**

While each student teacher’s conception of proof can be seen differently in his or her classroom argumentation, one aspect of a conception of proof emerged as a powerful influence on each student teacher’s support for argumentation. For each student teacher, his or her perception of the purpose and role of proof in school mathematics was apparent in how he or she supported classroom argumentation. A student teacher’s view of how proof is useful in mathematics was visible in several aspects of the argumentation in his or her classroom: the kinds of warrants used or accepted, who contributed the warrants and how this contribution was initiated, and how incorrect claims were treated. Before elaborating on the observations that related proof and support for argumentation in the three classrooms, I describe a new visual representation of a conception of proof that arose from this study and use it to summarize elements of each student teacher’s conception of proof.
Re-constructing a Representation of a Conception of Proof

The original conceptualization of a conception of proof (as illustrated in Figure 2-1) as the intersection of three rays representing three aspects of a conception of proof was found to be inadequate in representing student teachers’ conceptions of proof. A revised model emerged after closely examining the data from this study in light of previous findings regarding conceptions of proof in the literature. The original conceptualization of three aspects of a conception of proof was consistent with the overall themes in the literature, but did not capture one aspect of a conception of proof: the mathematical context in which a person is proving. Harel and Sowder (1998) and Housman and Porter (2003) suggest that a person’s proof scheme may differ based on the mathematical context in which he or she is proving, and Moore (1994) and Weber (2001) found that a person’s ability to prove is based in part on his or her understanding of the mathematical content and methods of proof in a mathematical area. Thus it is reasonable to suggest that a person’s conception of proof may differ based on the mathematical context in which he or she is engaged in proving or the context in which he or she perceives himself or herself to be working, and this is what the data from this study suggests.

Within the original model, two of the three continua included more than one aspect of a conception of proof. In order to use that model effectively, the affective component of “likes and is confident” to “dislikes and feels incapable” needed to be changed, as a person might like proof but feel incapable in certain proving situations. In addition, the perception of the purpose and need for proof is not able to be strictly placed along a continuum. Instead, a person may feel that proof is “very important” or “not as
important,” so that part is captured in a continuum (labeled Importance) in the revised model (see Figure Error! Reference source not found.), but each person may also have a different perception of what the purpose of proof in mathematics is, which was not captured in the strict continuum of the original model. In the revised model, a summary of a person’s perception of the purpose of proof in mathematics is entered in the bottom block, representing the foundational nature of this perception in his or her conception of proof. The original model did not account for the context-dependence of a person’s conception of proof, and this aspect of a conception of proof, more than any other, precipitated a revision of the model. The basis for the new model, as seen in Figure Error! Reference source not found. without any details of a particular person’s conception of proof, clearly portrays the possibility of context-dependence. However, it also allows for the possibility that a person may have elements of a conception of proof that do not appear to be context-dependent or can be hypothesized to be the same across contexts. The larger arrows representing continua in the four corners of the diagram will contain markings that illustrate components of a person’s conception of proof that appear either to cut across contexts or to refer to mathematics in general. For instance, a person may view proof as important in mathematics, without differentiating its importance in separate mathematical contexts. Alternatively, a person may demonstrate similar proficiency with proving in several mathematical areas, allowing a hypothesis that his or her overall proficiency with proving may be at that level.
Figure 7-1: Revised Starting Point for Diagramming a Conception of Proof

One of the difficulties with conceptualizing such a diagram of a conception of proof is that a person’s conception of proof may change and develop over time and that it may not be something a person articulates easily to others or to him or herself. In addition, a person’s conception of proof may not have been an explicit object of development. While mathematics classes may focus on developing a person’s ability to construct proofs, and this is one aspect of his or her conception of proof, the purpose and role of proof in mathematics may not be (and likely is not) an explicit focus of collegiate mathematics courses. In other words, the ability to prove, or to learn the mathematical ideas necessary for proving (as both Moore, 1994, and Weber, 2001, describe), is the primary object of development in many college mathematics courses, while the development of an appropriate conception of proof is, at best, a secondary object of development. Some aspects of Figure Error! Reference source not found. were
deliberately designed to reflect the tentative and growing nature of an individual’s conception of proof as inferred by another person from this individual’s statements and observed practice. For instance, each contextualized aspect of a conception of proof is located on a ray. One possible configuration of this diagram contained segments with lengths related to the strength of each aspect of a conception of proof, but the rays that are currently in place better reflect the potential for growth in each aspect of a conception of proof.

The diagram in Figure Error! Reference source not found. is incomplete, not reflecting a conception of proof without the symbols and placements that demonstrate a particular person’s conception of proof. For each aspect of a person’s conception of proof, the relative magnitude of the aspect is estimated and placed on the ray labeled with that aspect. A circle with the interior shaded is used to denote aspects and magnitudes derived from substantial evidence presented in interviews and possibly also observed in practice; a circle with the interior unshaded is used to denote aspects and magnitudes for which there is only a passing reference in the interviews (for which there is less evidence in the study); and a square is used to denote aspects and magnitudes based on evidence derived from observations of classroom practice and not from statements made in the interviews. A diagram of each student teacher’s conception of proof with a summary description of that conception is presented in the next few paragraphs as background for considering the relationship between the student teachers’ conceptions of proof and their support for collective argumentation in their classrooms.
Karis’ Conception of Proof

The primary aspect of Karis’ conception of proof was her view of proof as useful for explanation. A tentative model of Karis’ conception of proof is in Figure Error! Reference source not found. Karis demonstrated proficiency and confidence in her proving and validating proofs in the interviews. She talked about proof as important for explanation, and her talk about proof was consistently positive. She demonstrated proficiency in both geometry and number theory in the interviews, and she used an example from her algebra class when asked to describe a proving experience in which she felt successful (the circle on the algebra proficiency continuum is unshaded since she was not observed proving or analyzing proofs in the context of abstract algebra, but she did report successfully proving in that context). Since Karis’ conception of proof seems to be quite consistent across contexts, her overall conception is marked on the four large arrows in the corner to illustrate this relatively stable conception.
Figure 7-2: Tentative Representation of Karis’ Conception of Proof

**Lynn’s Conception of Proof**

Lynn’s perception of proof seems to be contextualized, at least when considering her college mathematics experiences. Her overarching perception of the purpose for proof in mathematics is to know the reason why things are true. As illustrated by Figure Error! Reference source not found., Lynn’s conception of proof seems to be more context-dependent than Karis’ conception of proof (see Figure Error! Reference source not found.). In fact, from the evidence available, it is difficult to determine an overall conception of proof for Lynn since, for example, her conception of proof in geometry includes a high level of proficiency, confidence, affect, and perception of importance, but her conception of proof in topology, while still exhibiting a high level of perception of
importance, shows a low level of confidence and affect, as well as proficiency, based on her self-report of experiences in that class. The only aspect of her conception of proof that seems to be relatively consistent is her perception of the purpose and need for proof in mathematics which, except for the view inferred from her treatment of algebraic ideas in her classroom (marked by the square in Figure Error! Reference source not found.), seems to be that proof is important to know why things are true.

Figure 7-3: Tentative Representation of Lynn’s Conception of Proof

**Jared’s Conception of Proof**

Jared’s conception of proof includes an aspect that proof is important for knowing how to do things. A tentative model of Jared’s conception of proof is in Figure Error! Reference source not found.. As can be seen from this figure, Jared seems to be quite
ambivalent to proof. He expressed neither a strong like nor a strong dislike of proving; he was able to prove and analyze proofs to some extent, but had some difficulty with proving and analyzing in each setting within his interviews; he was not particularly confident when engaging in proving; and he views proving as important for some students, not for others, and somewhat important to his own learning of mathematics. He does not view proving as important in algebra, stating that “there’s not really much to prove in algebra” (Jared, Interview 2, lines 129-130). Since he was not asked to prove in a strictly algebraic context, his proficiency, confidence, and affect about proving cannot be assessed in this context.

![Figure 7-4: Tentative Representation of Jared’s Conception of Proof](image)

From the representations of the three student teachers’ conceptions of proof, it is clear that these three student teachers’ conceptions of proof have few common elements. The contextual nature of Lynn’s conception of proof is clear from Figure Error!
Reference source not found., while Karis’ and Jared’s conceptions of proof are less context-dependent (see Figure Error! Reference source not found. and Figure Error! Reference source not found.). Karis is, in general, more positive about proving, more confident, and more proficient than Jared, while these aspects of Lynn’s conception are tied to context. However, Lynn’s view of the importance of proof in mathematics seems to be less context-dependent than the other aspects of her conception of proof. She views proof as extremely important; proof is more important to her than it is to either Jared or Karis.

While these diagrams are helpful in picturing each student teacher’s overall conception of proof, and even in comparing conceptions of proof, they cannot capture every nuance of a person’s conception of proof, and they do not show how a person’s conception of proof aligns with his or her support for argumentation. In the next few sections, various aspects of the student teachers’ conceptions of proof will be elaborated and related to their support for argumentation in their classrooms.

Alignment of Perception of the Purpose and Role of Proof in Mathematics With Support for Argumentation

The most significant aspect of Karis’ conception of proof is that proof is useful for the purpose of explanation. She was the primary contributor of warrants in her classroom, and the warrants included explanations of theorems and definitions as well as statements of theorems, definitions, rules, and procedures. Karis saw herself as responsible for explaining the mathematical ideas in her class. This view led to her being the dominant contributor to argumentation in her classroom. However, when students
might already know the ideas involved in the argument, such as when an argument involved ideas prerequisite to calculus or discussed previously in the same class period, she did not find it necessary to provide a warrant (or explanation) or to ask her students to do so. Karis’ treatment of incorrect claims was, in general, to provide a correct counter-claim with accompanying explanation. In some instances, she also provided an explanation for why someone might think the incorrect claim was correct, or how the incorrect claim was indeed incorrect.

Lynn’s overriding conception of proof is that proof is useful in mathematics because “I’m a reason person” (Lynn Interview 2, line 153). She views proof as useful in mathematics because she wants to know the reason why things are true. Because Lynn herself wants to know the reason why things are true, she also thinks it is important that her students know why things are true. Lynn’s view of how proof is useful in mathematics is slightly different from Karis’ view. Karis wants her students to know how things fit together in mathematics, the “background behind the mathematics” (Karis Interview 1, line 68). Lynn’s view of proof is more focused on the individual. She is interested in proving that something is true, or knowing why something is true because it is important to her, and it is important to her that her individual students know why things are true. The difference between these two views of proof is similar to that described by Knuth (2002b). Some teachers in Knuth’s study described a possible explanatory role for proof in mathematics, mentioning the usefulness of proof in seeing why something is true, and giving the example of a proof of the quadratic formula in knowing how the formula is derived. Lynn’s view of proof as important to know why seems to be consistent with this description. The contrasting view of proof as explanatory, according
to Knuth, is “proof as a means of promoting insight of the underlying mathematical relationships” (p. 80). This view seems closer to Karis’ description of the importance of proof as explanation, although Karis’ view of proof does seem to include aspects of knowing why a statement, such as the quadratic formula, is true.

Lynn’s view of proof as important for individual students to know why something works is apparent in her support for argumentation in that it provides impetus for her to ask her students to provide warrants for arguments. This is different from Karis’ class, where Karis’ emphasis on explanation relates to her providing more warrants herself. The kinds of warrants that were acceptable in Lynn’s class were generally ones that would be acceptable warrants in the context of proving. This relates to her view that proving is important, and it may also relate to the fact that she was teaching geometry and saw geometry as a place where proving should occur. Lynn’s treatment of incorrect claims was to immediately offer or prompt a counter-claim. She did not, in general, spend much time suggesting why the incorrect claim was incorrect. Instead, she proceeded as if the correct counter-claim was the original claim, providing or prompting data and warrant for it. This may relate to her view of proof as important for knowing why something is true. That is, to Lynn, if it is not true, then it is not important to know why.

Jared views proof in mathematics as important for understanding how to do things. This emphasis on proof for understanding procedures is most evident in the kind of warrants he and his students provide in his class. The majority of warrants in Jared’s class were statements of or references to rules and procedures, which is consistent with wanting students to know how to do things. The small number of arguments with implicit warrants also relates to this aspect of his conception of proof. He consistently provided
warrants or asked his students to do so in order that they might understand how to do these problems. Jared’s treatment of incorrect claims involved asking for data to support them and then pointing out the problem with the claim if the student did not recognize it himself or herself. This is also consistent with his view that the students should know how to do things, since by asking for the data he may be able to ascertain whether the reason for the incorrect claim concerns the procedure or is linked to some other difficulty (such as reading the problem incorrectly). Alternatively, Jared may have asked for data for incorrect claims so that he was consistent in asking for data for any claim. That is, he may have asked for data in these cases in order not to signal whether a claim was correct or incorrect simply by his initial reaction to it.

Alignment of Other Aspects of Conceptions of Proof With Support for Argumentation

Other aspects of these student teachers’ conceptions of proof include: Karis’ emphasis on understanding the mathematics and her audience, Jared’s focus on proof as procedure and multiple methods, and Lynn’s context-dependence and use of examples. All of these aspects can basically be subsumed under their overarching beliefs about the role and purpose of proof in school mathematics. Each student teacher’s beliefs about the role and purpose of proof in school mathematics seem to either influence or be influenced by the other aspects of his or her conception of proof. Not only are the other aspects consistent with the belief about the role and purpose of proof in school mathematics, but the impact of the other aspects of their conception of proof on the argumentation in their classrooms is not nearly as noticeable. In some cases, the observed patterns and norms of
argumentation could be explained by either aspect of a student teacher’s conception of proof.

Karis’ conception of proof is very cohesive. Her view of understanding of mathematical concepts as important for proving relates to her view of proof as important for explanation. Karis views understanding mathematical concepts as necessary for her to be able to prove, and she views proofs as important since she can use them to explain mathematical ideas so her students will understand the mathematical ideas. Thus her provision of theorem, definitions, and explanations of procedures as warrants within arguments, while explained by her view of understanding of mathematical concepts as important for proving, is also aligned with her view of proof as important for explanation. In addition, Karis’ view of audience as important was used to partially explain the warrants she left implicit in her classroom. However, part of the reason for leaving the warrants implicit was because her view of audience made it unnecessary for her to explain those ideas that she could have assumed her class as audience to already have seen. Thus her view of proof as explanation provides the overarching explanation into which her view of audience fits.

Jared’s conception of proof as a procedure to get an answer was used to explain his prompting students to make claims by asking them for answers. This conception of proof as procedure is very closely related to Jared’s view of the purpose and role of proof in mathematics as understanding how to do things. Jared’s procedural focus in mathematics also explains his missing warrants in his own work and in the symbolic work within his classroom as well as his preference for multiple methods of proof. Jared’s focus was on doing the procedure to obtain the right answer. Once the procedure
was completed, there was no longer a need for a warrant, since the warrant was for the purpose of understanding how to do things. He may have viewed multiple methods of proof as confirming evidence that he had done the right thing or that his conclusion was correct. His emphasis on checking solutions by using an alternative method, then, can be explained as providing evidence that the procedures used were correct. Thus Jared’s view of the purpose and role of proof in mathematics as understanding how to do things, when considered with his emphasis on procedures in mathematics, provides the overarching explanation for his view of proof as procedure, which explains the pattern of argumentation in his classroom.

The influence of context on Lynn’s personal work in proving was used to explain the fact that she left warrants implicit in an algebraic context and provided or prompted them in a geometric context. Combining the influence of context with her view of proof as important to know why something works provides a more complete explanation of her implicit warrants since the reason she left warrants implicit in algebraic contexts may be connected to her view that students already knew why things worked in algebra since they had learned that before coming to geometry. An alternative explanation for her implicit warrants in algebraic contexts could be that she agrees with Jared’s comment that there is not much to prove in algebra. Thus her view of the purpose and role of proof in mathematics as important to understanding why something works can explain her use of both explicit and implicit warrants.

Lynn’s use of examples, while not contradicting her view of proof as useful because she wants to know why things are true, does not seem to explain or be explained
by this view of proof. Her use of examples\(^{14}\) in proving does seem to relate to her provision of and attention to figures within her classroom argumentation. Neither Karis nor Jared used examples in proving or analyzing proofs, so a tendency to use examples was not part of their conceptions of proof, at least as revealed by this study. Neither Karis nor Jared was observed to use figures to support argumentation in their classrooms. However, Lynn was teaching in a geometry class, in which use of figures was necessary for explaining the geometric concepts. Both Karis and Jared tended to write the equations, expressions, and inequalities that were involved in an argument on the board, and both were observed to use graphs to further support argumentation on a few occasions. Thus it is not clear that use of examples in proving is an important aspect of a teacher’s conception of proof as it relates to support for argumentation, even though this aspect of Lynn’s conception of proof does seem to relate to her support for argumentation.

Looking across all three cases suggests that student teachers’ support for argumentation aligns most strongly with one aspect of their conceptions of proof: their beliefs about the purpose and role of proof in mathematics. While other aspects of their conceptions of proof do align with aspects of observed support for argumentation, as is illustrated by the results in Chapters 4, 5, and 6, it is the student teachers’ beliefs about the purpose and role of proof in mathematics that explain both individual ways of supporting argumentation and differences between the ways the student teachers supported argumentation in their classrooms. This implies that the mathematical

\(^{14}\) While Lynn used examples within proving and analyzing proofs, these were not generic examples as defined by Balacheff (1988).
preparation of teachers should include explicit attention to prospective mathematics teachers’ beliefs about the role and purpose of proof in mathematics, as this has the potential to influence their classroom practice as they support collective argumentation.

**Conclusion and Connections**

These three student teachers supported argumentation in their classrooms primarily by asking questions that prompted students to give claims, data, and warrants, and by providing warrants for arguments. While other actions in support of argumentation were observed in these classrooms, asking questions and providing warrants were observed most frequently and seemed to have the most impact on how arguments unfolded in their classrooms. The three student teachers in the current study structured the discussions in which argumentation occurred in slightly different ways, but all used questions as the primary mechanism to enact this structure. Other descriptions of argumentation in the literature have focused on the teacher’s role in argumentation in structuring discussions (Cobb, 1999; McCrone, 2005). The teacher’s role in these studies has been described as facilitating discussions by, for instance, making decisions about who should contribute solutions at what times based on observations of work in small groups. The three student teachers in this study were observed to structure discussions, and did so almost entirely by asking questions that prompted students to contribute claims, warrants, and backings, but did not tend to use small group work or to solicit solutions from students in a predetermined order. Instead, they structured arguments according to an order of claims, data, and warrants that had become normative in their
classrooms. For Karis, this order was warrants and data and then claim; for Lynn, claim, data, and then warrants. Jared used both of the previous orders of claims, data, and warrants.

Yackel (2002) described “recognizing the importance of warrants and backings” as part of the teacher’s role in argumentation (p. 439). Prompting and providing warrants were also shown to be important in the current study. However, I do not argue that the participants in this study necessarily recognized the importance of providing or prompting warrants. Instead, whether or not they prompted or provided warrants in various mathematical circumstances within particular classes reflected their conception of proof. This illustrates one of the differences between the context of this study and the context in which other research on classroom argumentation has been done. In the current study, the student teachers were not focused on supporting argumentation as an explicit goal of their instruction, nor did they have a goal of examining the argumentation of their students. In many previous studies involving argumentation, the teacher has been the researcher or the instruction was intentionally planned to foster collective argumentation within the classroom (e.g., Whitenack & Knipping, 2002; Yackel, 2002; see also Osborne, Erduran, & Simon, 2004). Yet, despite the differences in context between the studies focused on developing students’ collective argumentation and the current study focused on the nature of argument as it arises naturally in student teachers’ classrooms, very similar aspects of argumentation arose as critical aspects of the teacher’s role in the support of argumentation in the classroom. Implicit and explicit warrants within argumentation, and particularly the teacher’s use of warrants within argumentation, were described by Yackel and demonstrated by this study to be crucial in understanding the
collective argumentation in a classroom. In addition, the role of the teacher in structuring arguments was seen to be crucial in this study. Forman and Ansell (2002) and Yackel describe a similarly crucial role of the teacher in structuring the argumentation in their studies.

As described in the preceding paragraph, the classroom context for this study differed from previous studies in that the student teachers were not part of the research team or specifically attempting to facilitate collective argumentation within their classrooms. The fact that this study found similar aspects of argumentation to be important in a different classroom context is crucial not simply because the context in which the research was conducted is different. Argumentation was observed to occur in these three classrooms even though it was not a goal of these student teachers to specifically foster argumentation. While it is not possible to generalize from this observation of argumentation occurring “naturally” in three classrooms to the argumentation that might occur in all secondary mathematics classrooms, this study does provide evidence that argumentation may occur naturally in mathematics classrooms in a way that is reminiscent of communication in the discipline of mathematics or in scientific exchange as described by Forman and Ansell (2002). The argumentation observed in these classes is arguably not as rich as the argumentation in the classrooms described by other studies, in which argumentation was a specific focus of the teacher. However, this study suggests that there exists argumentation, even in beginning teachers’ classrooms, on which to build more robust practices of argumentation. Seeger (2001) calls for “more long-term studies on the formation of proficient discursive classrooms” (p. 296). While this study was not longitudinal, it does examine what argumentation looks like in the
classrooms of beginning teachers, and as such presents a picture from which future research can build.

The differences in how these student teachers support argumentation in their classrooms can be explained by differences in their conceptions of proof. In particular, one aspect of the student teachers’ conceptions of proof seems to align most strongly with their support for argumentation: their perception of the purpose and role of proof in mathematics. This aspect of a conception of proof is one about which several authors had theorized (Hanna, 1995, 2000; Hersh, 1993; Thurston, 1995). Harel and Sowder (1998) briefly mentioned having a sense of the necessity of proof as part of a conception of proof that they would like their students to develop, and a few studies have described their participants’ perceptions of the role of proof in mathematics or in teaching and learning mathematics (Almeida, 2000; Coe & Ruthven, 1994; Knuth, 2002a; Mingus & Grassl, 1999). The current study provides an empirically grounded description of the relevance of a perception of the purpose and role of proof to argumentation as a particular aspect of classroom practice of prospective secondary mathematics teachers. The extent to which these student teachers’ support for argumentation aligns with their conceptions of proof suggests that their conceptions of proof may influence the way in which they support collective argumentation in their classrooms.

In addition to providing evidence for the relevance of a student teacher’s perception of the purpose and role of proof in mathematics in his or her classroom practice, and in consideration of the current emphasis on instruments and methods, this study’s use of expanded versions of Toulmin’s (1964) diagrams provides a possibility for future research utilizing these diagrams to study related questions. Published research has
profitably used diagrams of single arguments, some diagrams more complex than others 
(see, e.g., Whitenack & Knipping, 2002 vs. Forman, Larreamendy-Joerns, Stein, & 
Brown, 1998) to examine the structure of arguments. Some diagrams have included 
information about the contributors of individual components of arguments. The expanded 
diagrams used in this study allow one to identify individual contributions both to 
individual arguments and to episodes of argumentation consisting of several levels of 
sub-arguments while maintaining a focus on the larger structure of an episode of 
argumentation. While complexity of argumentation was not the main focus of this study, 
these diagrams might be used profitably to study the complexity of argumentation within 
classrooms as well as to examine or classify the structure of these larger episodes of 
argumentation based on both the number and structure of sub-arguments and the 
connectedness between different levels of argument within an episode of argumentation. 
Forman, Larreamendy-Joerns, Stein, and Brown (1998) suggest that an argument’s 
complexity might be measured in terms of how many sub-arguments it has or the number 
of warrants and backings within it. Knipping (2003) used an expanded version of 
Toulmin’s (1964) diagrams, similar to the ones used in this study, in her investigation of 
the structure of argumentation in classrooms in two different countries. In Knipping’s 
diagrams, the overall structure of an episode of argumentation is readily apparent, more 
so than in the diagrams used in the current study, but the attribution of components to 
individuals, which is present in diagrams of individual arguments, is lost in the overall 
diagram of an episode of argumentation. It is possible that a combination of Knipping’s 
use of shapes to denote structural components with the current study’s use of color to
attribute components to the teacher or students would allow an even deeper investigation of the complexities of argumentation within classrooms.

This study provides empirical evidence for the relevance of a student teacher’s perception of the purpose and role of proof in mathematics to the facilitation of argumentation in his or her classroom, thus providing some empirical evidence for a construct long-theorized to be important and connecting one aspect of proving to one aspect of argumentation. Much future work remains to examine other aspects of a relationship between proof and argumentation, particularly with regard to student learning. In addition, the context of this study, and the fact that the argumentation occurred without explicit intentionality on the part of the student teachers, gives reason to believe that argumentation may often occur naturally in mathematics classrooms and may provide a base upon which to build more robust argumentation practices. Finally, the methodology of this study, in particular the use of expanded versions of Toulmin’s (1964) diagrams, provides another tool for the study of the complexities of teaching.
Chapter 8

Implications

This study has implications for teacher educators and for all those who work with mathematics teachers. In particular, this study should serve to sensitize those involved in the mathematical and pedagogical preparation of secondary mathematics teachers to some of the issues involved in relating what secondary mathematics teachers have learned in mathematics and mathematics education classes to how they facilitate argumentation in secondary classrooms. While proof is often central to undergraduate mathematics classes, the focus in these classes seems to be on learning to prove, and some students in these classes may focus on being able to prove or produce proofs for the test. This study suggests that an important additional focus for mathematics and mathematics education courses is how and why proving is important within mathematics. This study may also serve to inform those who work with teachers of a possible explanatory factor for observed patterns of argumentation in secondary mathematics classrooms and suggests that an appropriate focus of professional development may be the purpose and role of proof in school mathematics.

There is much more to be understood about the phenomenon of teachers’ conceptions of proof and their support for classroom argumentation. In the remaining sections, I describe some of the limitations of this study and then articulate some
questions and issues relating to how a teacher’s conception of proof might develop, how conceptions of proof relate to other influences on classroom argumentation, and the possible effects of some of these issues on student learning.

**Limitations of the Study**

As a qualitative examination of the conceptions and practice of three student teachers, this study is limited as to its generalizability. The evidence presented points to a possible relationship between the student teachers’ conceptions of proof and how they supported argumentation in their classrooms, but does not prove that such a relationship exists in all classrooms, even in classrooms of all student teachers. However, it does suggest a particular possible link between conceptions of proof and facilitation of argumentation that is worth pursuing in future research: a teacher’s perception of the purpose and need for proof in mathematics seems to relate to his or her support for argumentation. It also provides initial empirical evidence for the hypothesis that a person’s perception of the purpose and need for proof in mathematics is important. Thus, while not providing generalizability, this study does provide an initial description of a relationship and generates both hypotheses and further questions worth pursuing in this area.

This study focuses solely on conception of proof as one possible source of explanation for the differences in argumentation observed in the three classrooms. However, there are other possible explanations for differences that were observed. While it was not possible to account for all of these in this study, future work should include
investigating the influences of factors such as a teacher’s perception of teaching and learning on his or her support for argumentation, particularly in the provision of warrants within arguments. Other factors that merit additional investigation include the influences of teachers’ conceptions of mathematics in general (including content and pedagogical content knowledge), the curriculum used, the task choices of teachers, the particular courses taught (including the comfort of the teachers with the subject matter of the courses taught), and the practical knowledge of teaching obtained through years of experience of teaching mathematics. While these factors contribute to some possible alternative explanations for observations within this study, a task of future research is to tease out the connections and interactions among all of these factors.

A third limitation of the study, although it was a conscious choice, is that it examined the practice of student teachers rather than practicing teachers. The rationale for this choice, explained in Chapter 3, includes examining practice that has not yet been distanced from proving in university courses. This benefit must be weighed against the possibility that a student teacher’s classroom practice is influenced by his or her mentor’s practice in ways that may not be apparent to an observer. While this study attempted to document influences of the mentors on the actions of the student teachers, there may be other influences that have not been accounted for.

Another limitation of the study is the extent to which components of argumentation were labeled as data or warrants and attributed to the teachers or students by one or two researchers. While the intentions of the teachers and students were considered in the analysis, the analysis may have benefited from a larger team of researchers contributing and debating other alternative interpretations of the study data.
In addition, the limitations of the data collection itself, imposed by the external
regulators, limited the data available for analysis. Video recordings of classroom
observations, while admittedly still not objectively representing classroom events, would
provide a more complete record of the actions of teachers and students than audio
recordings and field notes.

Despite the limitations enumerated in this section, this study serves as a starting
point for examining an important aspect of classroom practice: teachers’ facilitation of
classroom argumentation, relating this support for argumentation to a particular aspect of
teachers’ knowledge and beliefs. Acknowledging these limitations adds to the questions
that are generated by this study.

Further questions

Developing a Conception of Proof

This study did not attempt to consider what conceptions of proof should be the
goal of teacher education, as the focus of the study was to consider the nature of a
relationship between conceptions of proof and support for argumentation if one existed.
The evidence presented in Chapter 7 suggests that a relationship does exist, that it
explains differences in support for argumentation in different classrooms, and that a
student teacher’s perception of the purpose and role of proof in mathematics is strongly
aligned with other aspects of his or her conception of proof as well as the support he or
she provides for argumentation in his or her classroom. Future research should consider
what perceptions of the purpose and role of proof in mathematics are more productive in facilitating classroom argumentation, perhaps by identifying and studying teachers who facilitate argumentation in ways that support student learning.

The suggestion that teachers’ conceptions of proof influence the argumentation in their classrooms implies that explicit attention should be paid to the conceptions of proof that they develop within teacher preparation programs. Karis described a change in her conception of proof from disliking it as a high school student and memorizing proofs in her first proofs course to feeling successful in proving in her abstract algebra course because she understood the mathematical ideas required for proving in that course. This suggests that it is possible for a person’s conception of proof to change during the time that he or she is enrolled in the mathematics and mathematics education courses of a teacher preparation program. These changes may be in the contextualized conception of proof related to the mathematical areas addressed by the courses in the program or they may be more global, as in a change in how a person views proof as useful in mathematics. However, while Karis’ conception of proof changed during her undergraduate program, it seems that Jared’s did not. In particular, he mentioned memorizing proofs for tests within one of his final two mathematics courses. What factors lead to one prospective teacher’s conception changing while conceptions of his or her peers stayed the same?
Other Influences Interacting With Conceptions of Proof in Argumentation

At several points in the study, reference was made to the possible impact of the subject matter on teachers’ support for argumentation. For instance, when comparing the nature of warrants in Jared’s and Karis’ classes, reference was made to possible differences due to Jared teaching algebra and Karis teaching calculus. In addition, some differences in Lynn’s support for argumentation might be explained by her beliefs about teaching geometry or by the nature of teaching geometry as opposed to algebra or calculus. Investigating differences in support for argumentation in different classes taught by the same teacher would give some insight into the impact of subject matter on support for argumentation and the relationship between conceptions of proof and the nature of the subject being taught.

In addition, teachers’ beliefs about students have been shown to influence their classroom practice. At several points, Jared and Lynn expressed beliefs about what was appropriate for different levels of students. If teachers’ conceptions of proof in school mathematics include that it is should be experienced differently for different levels of students, is their support for argumentation different when teaching different levels of students? What other beliefs about students influence either a teacher’s conception of proof or his or her support for argumentation?

The conceptions of proof of these student teachers have been shown to align with their support for argumentation in their classrooms. In particular, their explicit and implicit warrants seem to be consistent with their perceptions of the purpose and need for proof in mathematics. An open question that remains to be investigated is whether the
warrants given and the argumentation facilitated by practicing teachers would be similarly aligned with their conceptions of proof.

**Possible Impact on Student Learning**

The argumentation that occurred in the three classrooms can be characterized differently, and these differences were consistent with differences in the student teachers’ conceptions of proof. An open question is how student learning differed in the three classrooms, that is, what is the impact of these different ways of supporting argumentation on opportunities for students to learn mathematics? In addition, what are students learning about mathematics in contexts such as these?

One characteristic of collective argumentation that might affect student learning is the length or complexity of an episode of argumentation in terms of its number of sub-arguments and the connectedness of the arguments contained in a larger episode of argumentation. Does an argument with more sub-arguments simply reflect a problem with more steps? Is there a relationship between the length of an episode of argumentation and the time spent on a task? How might students’ learning differ between a classroom in which most episodes of argumentation contain only one argument and a classroom in which argumentation involves episodes with several embedded sub-arguments? These questions could begin to be examined using the extended versions of Toulmin’s (1964) diagram discussed previously and used in the analysis of this study or by a combination of color used in this study and shape used in Knipping’s (2003) study.
In addition to learning the mathematical concepts, skills, and generalizations that make up the content of mathematics courses, students also learn about the nature of mathematics as a discipline in their mathematics classrooms. One important role of the teacher in classroom argumentation is to represent the external mathematics community (Yackel & Cobb, 1996). As these student teachers took on that role, their conceptions of proof may have led to a portrayal of mathematics that is different from views held by the larger mathematical community. For instance, the view of mathematics that might be inferred by a student in Jared’s class probably has a much more procedural focus than a mathematician’s view of mathematics. It can be argued that Karis’ conception of proof was closer to that of the mathematical community. What view of mathematics might be inferred by a teacher’s students from his or her emphasis on a particular role of proof (such as explanation) and provision of warrants within arguments?

A possible hypothesis regarding learning mathematics can be made from considering the pattern of argumentation in Jared’s class. As described in Chapter 6, Jared’s classroom practice was focused on procedures to the extent that he specified what methods students were to use to solve problems on tests. Most of the warrants given in his class were rules and procedures, thus Jared justified mathematical results by focusing on what to do rather than why it is appropriate to do it. It has been observed that many students seem to learn skills, as evidenced by good assessment results in one particular class, but are unable to apply them in appropriate situations in a future mathematics course. Is the sort of understanding held by students who learn skills but not when to apply the skills fostered in environments characterized by particular patterns of warrants, such as those Jared provided for his students? That is, can an observed pattern in student
understanding where students are able to apply particular skills when asked specifically to do so but do not recognize when these skills might be helpful in solving problems be related to argumentation that is characterized by an emphasis on procedures and particularly on following directions that delineate which procedures are to be used? In addition, can an examination of collective argumentation in classrooms help to identify other patterns such as this emphasis on procedures and following directions in teaching and relate them to hypotheses about student understanding?

Other questions and issues in this section have focused on student learning as it occurs within argumentation in classrooms without explicit attempts to change the argumentation. Some research in science education has focused on the quality of students’ arguments and teaching them to argue effectively (e.g., Driver, Newton, & Osborne, 2000; Jiménez-Aleixandre, 2002; Osborne, Erduran, & Simon, 2004). What would be the effect of an explicit focus on effective collective argumentation in a course (such as a mathematics methods course) for prospective teachers? How would such a focus impact or conflict with prospective teachers’ conceptions of proof? What would be the impact of a professional development program such as what Osborne, Erduran, and Simon describe in science education for student learning in mathematics classrooms? How might teaching students to argue effectively impact their learning to prove deductively?
Concluding Thoughts

Investigating the relationship between a teacher’s conception of proof and how he or she supports argumentation in a secondary mathematics classroom is just one small part of a set of closely related questions. This study suggests that a relationship exists and shows that, at least for these three student teachers, their perception of the purpose and need for proof in school mathematics is consistent with how they support argumentation in their classrooms. Further research is needed to investigate how other characteristics of the teacher, the class, and the school might influence classroom argumentation. In addition, future research will be needed to describe conceptions of proof that lead to productive argumentation in classrooms and how that argumentation influences student learning.
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Appendix A

Interview 1 Schedule

Interview 1: Views of Mathematics and Mathematical Arguments

Materials:
- Microphones
- Audio recorder
- Video camera
- Computer
- Handouts (6)

Introduction

Thank you for agreeing to participate in this study, and for coming for this interview today. Is it okay if I go ahead and turn on the camera and recorder?

Upon participant's agreement, turn on camera and recorder.

The purpose of this study is to examine prospective teachers' conceptions of mathematics and the argumentation that occurs in their classrooms.

During this interview, I will be asking you to think about things both as a student and as a teacher. We can think about this as wearing two different hats, a "student hat" and a "teacher hat." At various times, I may ask you to put on a particular hat, or I may ask you which hat you are wearing when you say certain things. If you notice that you have "changed hats," that's okay, please just let me know.

I will be asking you a series of different questions, and different kinds of questions. If at any time you think of something that is relevant to something we talked about before, just let me know, and we can go back to it.

Part 1 - Background

Question 1: What drew you to choose to be a mathematics teacher? Probe as needed to elicit both why a teacher and why mathematics.

Question 2: What do you think are important aspects of mathematics? Probe for why should people study mathematics as well as why these aspects are important.

Question 3: How important do you think proving is in mathematics? Probe as needed for why proving is or is not important in mathematics and what aspects of mathematics proving is important in. This question may come up in the answer to question 2, if so, do not ask again.

Question 3b: (if not already addressed) What role do you think proof plays when you are learning mathematical ideas?
Question 4: Approximately how many courses have you taken in which you have had to prove mathematical statements?

Question 5: Describe a recent proving experience in which you felt successful.
   Question 5a: What parts of this contributed to your feeling of success?
   Question 5b: (If cannot describe a successful experience) What would it take for you to feel successful when you are proving something?

Question 6: Tell me about a proving experience in which you did not feel as successful.
   Question 6a: How was this experience different from your successful experience?
   Question 6b: What aspects of a situation contribute to your being successful or unsuccessful?

Question 7: What is involved in proving a mathematical statement?
   Question 7a: How do you approach a proof in general? Probe for steps, how to begin, how to progress.
   Question 7b: How do you know that you're finished a proof

Answers to the questions may bring up such things as being more or less successful based on the course, topic, instructor, proof method.

Part 2 - Proving

Question 1: Reveal handout 1. Depending on their conceptions of proof from above, ask one of the following questions:
   What could you do to convince someone that this is true? How would you explain to someone that this is true?
   Depending on their answer to the question asked, it may be good to then ask the other one.
   As they work, probe for thinking by asking Why? and reminding them to think out loud. Other questions might be What made you choose to do that? Why did you erase that? I saw that you were trying this. What made you change your mind?
   After they finish the proof, and express that their answer would convince someone or explain that it is true, ask Would your response be different if a professor asked you to prove this? If so, how would it be different?
   As a final question, ask Have you ever proved this statement before? Have you seen someone else prove this?

Question 2: Reveal handout 2. Here we have an argument that the square root of a positive integer is either an integer or irrational. Does this argument prove this statement?
   As they work through the proof, prompt them to talk out loud. Ask How convinced are you that this statement is true? Were you convinced by the argument? Are all the parts here necessary (would you leave any of them out)? Would you include any parts of the argument that are not here?
   If they mention the Fundamental Theorem of Arithmetic, ask What does the Fundamental Theorem of Arithmetic say? Would you like a copy of it? (If so, reveal handout 3).
   If they do not mention the Fundamental Theorem of Arithmetic, wait
until they are done and have decided that the argument proves or does not prove the statement, and then ask What does the Fundamental Theorem of Arithmetic say? If they are not sure, give them the statement. Then ask if this information changes their evaluation of the argument.

**Question 3: Reveal handout 3.** I am going to give you several arguments about the validity of this statement. For each one, please tell me how convinced you are by the argument that the statement is correct. Do not rely on your knowledge of whether the statement is correct or not, but on the given argument. For the first one, we will use a sketch created in Geometer’s Sketchpad. Open the file on the computer and allow participant to investigate using the mouse. How convinced are you by this argument? Do you think this argument proves the statement? Do you think this argument would convince high school geometry students? College geometry students? Why or why not?

**Reveal handout 4 and allow participant to read.** How convinced are you by this argument? Do you think this argument proves the statement? Do you think this argument would convince high school geometry students? College geometry students? Why or why not?

**Reveal handout 5 and allow participant to read.** How convinced are you by this argument? Do you think this argument proves the statement? Do you think this argument would convince high school geometry students? College geometry students? Why or why not?

**Part 3 - Audience**

Think now about others (implying previous part was thinking about self):

**Question 1:** As a student, if you wanted to convince or explain to (perhaps in keeping with their view of proving and then challenging it?) one of your peers that some mathematical statement was true, what would you do?

**Question 2:** As a teacher, if you wanted to convince one of your students that some mathematical statement was true, what would you do? Does it matter whether you are convincing a student in an honors class or a regular class?

**Question 3:** As a student, if you wanted to convince one of your math professors that some mathematical statement was true, what would you do?

**Conclusion**

Before we conclude, are there any questions that you would like to go back to and elaborate on? Thank you for your time today.
The sum of the first $n$ natural numbers is $\frac{n(n+1)}{2}$.
(Handout 2)

The square root of a positive integer is either an integer or it is irrational.

Suppose \( n \) is a positive integer, \( \sqrt{n} \) is not an integer, and \( \sqrt{n} \) is rational. Then there exist relatively prime integers \( a \) and \( b > 1 \) with \( \sqrt{n} = \frac{a}{b} \).

Squaring both sides, \( n = \frac{a^2}{b^2} \), from which \( nb^2 = a^2 \). Now if we factor \( a \) and \( b \) into primes, there are no common factors. So the factorizations of \( a^2 \) and \( b^2 \) have no common factors. Consequently, the factorizations of the equal numbers \( nb^2 \) and \( a^2 \) are different. Since two different factorizations of \( a^2 \) are impossible by the Fundamental Theorem of Arithmetic, the supposition must be false. So if \( \sqrt{n} \) is not an integer, it must be irrational.
The Fundamental Theorem of Arithmetic

Every integer \( n \geq 2 \) is either prime or can be represented as a product of prime numbers, and except for the order of the factors, there is only one such representation of \( n \).
The Triangle Inequality Theorem
In any triangle, the sum of any two sides is greater than the remaining one.

Argument 1

See GSP sketch "Triangle inequality.html"

Drag any point. The sum of the measures of any two sides of the triangle is greater than the measure of the other sides.
The Triangle Inequality Theorem
In any triangle, the sum of any two sides is greater than the remaining one.

Argument 2

Case 1:
Given three segments, if the sum of the lengths of any two of them is less than the length of the third segment, then no triangle can be formed by connecting their endpoints. The endpoints of two of the segments cannot be connected. (See figure below.)

Case 2:
Given three segments, if the sum of the lengths of any two of them is equal to the length of the third segment, then no triangle can be formed by connecting their endpoints. The endpoints, and hence the segments themselves, are collinear (justification: the Betweenness Postulate). (See figure below.)
The Triangle Inequality Theorem
In any triangle, the sum of any two sides is greater than the remaining one.

Argument 3

Let $ABC$ be a triangle.

Draw $BA$ through to the point $D$, and make $DA$ equal to $CA$. Join $DC$. Since $DA$ equals $AC$, therefore the angle $ADC$ also equals the angle $A CD$. Therefore the angle $BCD$ is greater than the angle $ADC$.

Since $DCB$ is a triangle having the angle $BCD$ greater than the angle $BDC$, and the side opposite the greater angle is greater, therefore $DB$ is greater than $BC$.

But $DA$ equals $AC$, therefore the sum of $BA$ and $AC$ is greater than $BC$. Similarly we can prove that the sum of $AB$ and $BC$ is also greater than $CA$, and the sum of $BC$ and $CA$ is greater than $AB$. Therefore in any triangle the sum of any two sides is greater than the remaining one.
Appendix B

Interview 2 Schedules

Jared’s Second Interview Schedule

Interview 2 – Beliefs About Student Justification/Clarification of Classroom Practice/Influence of Mentors

Classroom Scenarios:

1. Hand Jared Figure B-1. If you asked a student to answer the following question, what would you expect him or her to say? Does $3x^3y^4 \cdot 4x^4y = 12x^{12}y^4$? Why or why not? What would be an “A” answer? How about a “C” answer? What would be a good teacher explanation of this?

$$\text{Does } 3x^3y^4 \cdot 4x^4y = 12x^{12}y^4? \text{ Why or why not?}$$

Figure B-1: Handout 1 for Jared Interview

2. Hand Jared Figure B-2. If a student answered the question “Solve the following system by substitution: $3x + 2y = 5$,” by writing “No solution”, how good would you say his or her answer was? If the question was worth three points, how many points would you assign to his or her answer? Why?
Solve the following system by substitution:

\[3x + 2y = 5\]
\[3x + 2y = 7\]

No solution

Figure B-2: Handout 2 for Jared’s Interview

3. Hand Jared Figure B-3. Imagine the following situation happening in your classroom. Student A says (in reference to a system of equations written on the board) “There is no solution to this system of equations because the slopes are the same.” Student B says “But aren’t the y-intercepts the same, too?” Describe what you would do and say.

Probe with But what if he or she said ___?

Clarify to get at students’ interaction rather than mathematics.
There is no solution to this system of equations because the slopes are the same.

But aren’t the y-intercepts the same, too?

Opinions About the Teaching of Proof and Justification

1. In your opinion, should students prove mathematical statements in high school? Why or why not?

2. I wasn’t there for all of your classes, so I wonder if you could estimate how frequently you ask students to explain or justify their answers in class? On homework? On tests or quizzes?
3. If you taught a different class, to students of a different ability level, would you ask them to justify their answers differently? More or less frequently? Would your expectations for what their justifications would look like change?

4. When you see “justify your answer,” in a (textbook) task for your students, what do you think it is asking students to do?

5. Can you give me an example of a time when you asked a student to justify something and he or she gave a really good justification?

6. Did you find that some students were particularly strong in the area of justification? What do you count as evidence of strong justification?

**Influence of Curriculum Materials on Classroom Practice**

1. What materials did you draw upon for ideas of how to teach the concepts you taught?

2. What materials did you primarily use for sources of example problems? For sources of practice problems?

3. How did you decide what to teach and in what order to teach it?

**Opinions About the Influence of Mentor and Supervisor on Classroom Practice**

1. What kinds of things did you and Ms. B talk about with regards to teaching?

2. What advice did she give you with regards to asking questions and responding to student questions?

3. How did you act on her advice?

4. Are there things you changed about how you taught certain concepts based on advice from Ms. B? Can you think of an example?

5. Are there things you would have taught differently if you could have? Can you give me some examples?
Karis’ Second Interview Schedule

Interview 2 – Beliefs About Student Justification/Clarification of Classroom Practice/Influence of Mentors

Classroom Scenarios:

1. Hand Karis Figure B-4. If you asked a student to answer the following question, what would you expect him or her to say? Can we apply Rolle’s Theorem to the function \( f(x) = x + \frac{1}{x} \) on the interval \( \left[ \frac{1}{2}, 3 \right] \)? Why or why not? Can we apply the Mean Value Theorem to the function \( f(x) = x + \frac{1}{x} \) on the interval \( \left[ \frac{1}{2}, 3 \right] \)? Why or why not? What would be an “A” answer? How about a “C” answer? What would be a good teacher explanation of this?

Can we apply Rolle’s Theorem to the function \( f(x) = x + \frac{1}{x} \) on the interval \( \left[ \frac{1}{2}, 3 \right] \)? Why or why not? Can we apply the Mean Value Theorem to the function \( f(x) = x + \frac{1}{x} \) on the interval \( \left[ \frac{1}{2}, 3 \right] \)? Why or why not?

Figure B-4: Handout 1 for Karis’ Interview

2. Hand Karis Figure B-5. If a student answered the question “Solve the following system by substitution: \( 3x + 2y = 5 \) and \( 3x + 2y = 7 \)” by writing “No solution”, how good would you say his or her answer was? If the question was worth three points, how many points would you assign to his or her answer?
Solve the following system by substitution:

\[
\begin{align*}
3x + 2y &= 5 \\
3x + 2y &= 7
\end{align*}
\]

No solution

Figure B-5: Handout 2 for Karis’ Interview

3. Hand Karis Figure B-6. Imagine the following situation happening in your classroom. Student A says “Rolle’s Theorem and the Mean Value Theorem are basically the same theorem.” Student B says “But the conditions for Rolle’s theorem and the Mean Value Theorem are not the same.” How would you respond to this situation? Describe what you would do and say.

   Probe with But what if he or she said ___?

   Clarify to get at students’ interaction rather than mathematics.
Rolle’s Theorem and the Mean Value Theorem are basically the same theorem.

But the conditions for Rolle’s Theorem and the Mean Value Theorem are not the same.

Figure B-6: Handout 3 for Karis’ Interview

Opinions About the Teaching of Proof and Justification

1. In your opinion, should students prove mathematical statements in high school? Why or why not?

2. On one of your homework assignments, you asked students to “State whether the Mean Value Theorem applies. If it applies, find the points guaranteed to exist by the theorem. If not, explain why not.” What did you mean by explain why not?
What were some examples of acceptable answers? What were some examples of unacceptable answers?

3. I wasn’t there for all of your classes, so I wonder if you could estimate how frequently you ask students to justify their answers in class? On homework? On tests or quizzes?

4. If you taught a different class, to students of a different ability level, would you ask them to justify their answers differently? More or less frequently? Would your expectations for what their justifications would look like change?

5. Can you give me an example of a time when you asked a student to justify something and he or she gave a really good justification?

6. Did you find that some students were particularly strong in the area of justification? What do you count as evidence of strong justification?

Influence of Curriculum Materials on Classroom Practice

1. What materials did you draw upon for ideas of how to teach the concepts you taught?

2. What materials did you primarily use for sources of example problems? For sources of practice problems?

3. How did you decide what to teach and in what order to teach it?

Opinions About the Influence of Mentor and Supervisor on Classroom Practice

1. What kinds of things did you and Mr. A talk about with regards to teaching?

2. What advice did he give you with regards to asking questions and responding to student questions?

3. How did you act upon advice Mr. A gave you?

4. Are there things you changed about how you taught certain concepts based on advice from Mr. A?

5. Are there things you would have taught differently if you could have? Can you give me some examples?
Lynn’s Second Interview Schedule

Interview 2 – Beliefs About Student Justification/Clarification of Classroom Practice/Influence of Mentors

Classroom Scenarios:

1. Hand Lynn Figure B-7. If you asked a student to answer the following question, what would you expect him or her to say? **Is a parallelogram a trapezoid? Why or why not?** What would be an “A” answer? How about a “C” answer? What would be a good teacher explanation of this?

Is a parallelogram a trapezoid? Why or why not?

Figure B-7: Handout 1 for Lynn’s Interview

2. Hand Lynn Figure B-8. If a student answered the question “Solve the following system by substitution: \(3x + 2y = 5\), \(3x + 2y = 7\)” by writing “No solution”, how good would you say his or her answer was? If the question was worth three points, how many points would you assign to his or her answer? Why?

Solve the following system by substitution: \(3x + 2y = 5\), \(3x + 2y = 7\)

No solution

Figure B-8: Handout 2 for Lynn’s Interview

3. Hand Lynn Figure B-9. Imagine the following situation happening in your classroom. Student A says “These three lengths could be the lengths of the sides of a triangle since the first two add up to more than the third.” Student B says “But the second and third sides added together are less than the first side.” Describe what you would do and say.
Probe with But what if he or she said ___?

Clarify to get at students’ interaction rather than mathematics.

Figure B-9: Handout 3 for Lynn’s Interview
Opinions About the Teaching of Proof and Justification

1. In your opinion, should students prove mathematical statements in high school? Why or why not?

2. On one of your tests, you asked students to “state a reason that justifies your answer.” What did you mean by that? What were some examples of acceptable answers? What were some examples of unacceptable answers?

3. I wasn’t there for all of your classes, so I wonder if you could estimate how frequently you ask students to justify their answers in class? On homework? On tests or quizzes?

4. If you taught a different class, to students of a different ability level, would you ask them to justify their answers differently? More or less frequently? Would your expectations for what their justifications would look like change?

5. Can you give me an example of a time when you asked a student to justify something and he or she gave a really good justification?

6. Did you find that some students were particularly strong in the area of justification? What do you count as evidence of strong justification?

Influence of Curriculum Materials on Classroom Practice

1. What materials did you draw upon for ideas of how to teach the concepts you taught?

2. What materials did you primarily use for sources of example problems? For sources of practice problems?

3. How did you decide what to teach and in what order to teach it?

Opinions About the Influence of Mentor and Supervisor on Classroom Practice

1. What kinds of things did you and Mr. B talk about with regards to teaching?

2. What advice did he give you with regards to asking questions and responding to student questions?

3. How did you act upon advice Mr. B gave you?
4. Are there things you changed about how you taught certain concepts based on advice from Mr. B? Can you think of an example?

5. Are there things you would have taught differently if you could have? Can you give me some examples?
Appendix C

Mentor Interview Schedules

Karis’ Mentor’s Interview Schedule

**Mentor Interview Schedule - Mr. A**

1. What kinds of questions has Karis asked about teaching?
   a. Does she ever ask about finding tasks?
   b. Does she ever ask about getting students to participate?
2. What advice do you give her?
3. What advice have you given Karis about designing effective lessons?
4. What kinds of things have you shared with Karis to support her growth as a teacher?
5. What do you think student teachers need to learn most?
6. What advice do you give Karis about engaging students in talking about mathematics?
7. What kinds of mathematical questions do you encourage Karis to ask in class?
8. When you're teaching, how do you choose the tasks you ask your students to do in class and for homework?
9. How closely do you follow the textbook when designing your lessons?
10. When you're teaching, to what extent do you ask students to justify their answers?
Lynn’s Mentor’s Interview Schedule

**Mentor Interview Schedule - Mr. B**

1. What kinds of questions has Lynn asked about teaching?
   b. Does she ever ask about finding tasks?
   b. Does she ever ask about getting students to participate?

2. What advice do you give her?

3. What advice have you given Lynn about designing effective lessons?

4. What kinds of things have you shared with Lynn to support her growth as a teacher?

5. What do you think student teachers need to learn most?

6. What advice do you give Lynn about engaging students in talking about mathematics?

7. What kinds of mathematical questions do you encourage Lynn to ask in class?

8. When you're teaching, how do you choose the tasks you ask your students to do in class and for homework?

9. How closely do you follow the textbook when designing your lessons?

10. When you're teaching, to what extent do you ask students to justify their answers?
Jared’s Mentor’s Interview Schedule

**Mentor Interview Schedule - Ms. C**

1. What kinds of questions has Jared asked about teaching?
   c. Does he ever ask about finding tasks?
   b. Does he ever ask about getting students to participate?
2. What advice do you give him?
3. What advice have you given Jared about designing effective lessons?
4. What kinds of things have you shared with Jared to support him growth as a teacher?
5. What do you think student teachers need to learn most?
6. What advice do you give Jared about engaging students in talking about mathematics?
7. What kinds of mathematical questions do you encourage Jared to ask in class?
8. When you're teaching, how do you choose the tasks you ask your students to do in class and for homework?
9. How closely do you follow the textbook when designing your lessons?
10. When you're teaching, to what extent do you ask students to justify their answers?
VITA

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