SINGLE-POINT NONLINEARITY INDICATORS FOR THE
PROPAGATION OF HIGH-AMPLITUDE ACOUSTIC SIGNALS

A Thesis in
Acoustics
by
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Abstract

In the study of jet noise, prediction schemes and impact assessment models based on linear acoustic theory are not always sufficient to describe the character of the radiated noise. Typically, a spectral comparison method is employed to determine whether nonlinear effects are important. A power spectral density recorded at one propagation distance is extrapolated to a different distance using linear theory and compared with a measurement at the second distance. Discrepancies between the measured and extrapolated spectra are often attributed to nonlinearity. There are many other factors that can influence the outcome of this operation, though, including meteorological factors such as wind and temperature gradients, ground reflections, and uncertainty in the source location. Therefore, an improved method for assessing the importance of nonlinearity that requires only a single measurement is desirable.

This work examines four candidate single-point nonlinearity indicators derived from the quantity $Q_{p^2p}$ found in the work of Morfey and Howell. These include: $Q_{neg}/Q_{pos}$, a ratio designed to test for conservation of energy; $Q_{pos}/p_{rms}^3$, a band-limited quantity that describes energy lost from a certain part of the spectrum due to nonlinearity; the spectral Gol’dberg number $\Gamma_s$, a dimensionless quantity whose sign indicates the direction of nonlinear energy transfer and whose magnitude can be used to compare the relative importance of linear and nonlinear effects; and the coherence indicator $\gamma_Q$, which also denotes the direction of nonlinear energy transfer and which is bounded between -1 and 1.

Two sets of experimental data are presented. The first was recorded in a plane wave tube built of 2" inner-diameter PVC pipe with four evenly-spaced microphones flush-mounted with the inside wall of the tube. One or two compression drivers were used as the sound source, and an anechoic termination made of fiberglass served to minimize reflections from the far end of the tube. Both single-frequency signals and band-limited noise were used as sources, and waveforms were
recorded at all four propagation distances. The second set of data was obtained at the model-scale jet facility at the University of Mississippi’s National Center for Physical Acoustics. A computer controlled microphone boom was constructed to hold an array of six microphones. The array was rotated about the presumed location of the acoustic source center (4 jet diameters downstream of the nozzle exit), and two stationary microphones were mounted on the walls. Measurements were made for several jet conditions; data presented here represent Mach 0.85 and Mach 2 conditions.

Application of the four candidate nonlinearity indicators to the experimental data reveals that each indicator has advantages and disadvantages. $Q_{\text{neg}}/Q_{\text{pos}}$ does not detect the presence of shocks as postulated, but it does conform to expectations in the shock-free region and support the use of $Q_{\text{pos}}$ as an indicator. The main advantage of $Q_{\text{pos}}/p_{\text{rms}}^2$ is that it can be used for band-limited measurements. Increased indicator values are seen for signals with higher source frequencies and amplitudes that are expected to undergo stronger nonlinear evolution. However, no physical meaning can yet be derived from the numerical value of the indicator. The spectral Gol’dberg number $\Gamma_s$ is the most promising of the candidate quantities. It has the ability to indicate the direction of nonlinear energy transfer as well as provide a comparison between the strengths of linear and nonlinear effects. These attributes allow it to be used to qualitatively predict the evolution of a spectrum. The coherence indicator $\gamma_Q$ also specifies the direction of nonlinear energy transfer, but its numerical value holds less meaning. However, it is bounded between -1 and 1, so values near zero denote very weak or no nonlinearity, and values near -1 or 1 denote strong nonlinearity. Further, because it is bounded, it does not become unstable for spectral components beneath the system noise floor.
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\( a \)  Tube radius (m)

\( A \)  Acoustic pressure amplitude (Pa) or coefficient of first term in state equation \( \rho_0 \left( \frac{\partial P}{\partial \rho} \right)_{s,0} \) (Pa)

\( B \)  Coefficient of second term in state equation \( \rho_0^2 \left( \frac{\partial^2 P}{\partial \rho^2} \right)_{s,0} \) (Pa)

\( B \)  Adiabatic bulk modulus (Pa)

\( B_n \)  Normalized amplitude of \( n^{th} \) harmonic component

\( c \)  Sound speed (m/s)

\( c_0 \)  Equilibrium or small-signal sound speed (m/s)

\( C_{p^2} \)  Real part of the cross spectrum of the pressure and the square of the pressure; also called cospectral density (Pa^3/Hz)

\( d \)  Experimental jet nozzle diameter (m)

\( d_0 \)  Full-scale jet nozzle diameter (m)

\( E \)  Energy density (J/m^2) or expected value operator

\( f \)  Frequency (Hz)

\( F \)  Scaled frequency \( f/p_s \) (Hz/atm)

\( F_{r,N} \)  Scaled nitrogen relaxation frequency \( f_{r,N}/p_s \) (Hz/atm)
Scaled oxygen relaxation frequency $f_{r,O}/p_s$ (Hz/atm)

Time-averaged acoustic intensity (W/m$^2$)

Source angular wave number $\omega_0/c_0$ (1/m)

Absorption length $1/\alpha$ (m)

Second-order Lagrangian density (Pa)

Harmonic number

Time domain acoustic pressure (Pa)

Frequency domain acoustic pressure, or Fourier transform of $p$ (Pa·s)

Imaginary part of frequency domain acoustic pressure (Pa·s)

Real part of frequency domain acoustic pressure (Pa·s)

Root-mean-square pressure $\sqrt{\text{mean}(p^2)}$ (Pa)

Atmospheric pressure (atm)

Acoustic pressure amplitude at the source (Pa)

Total pressure (Pa)

Equilibrium ambient pressure (Pa)

Prandtl number

Frequency domain squared acoustic pressure, or Fourier transform of $p^2$ (Pa$^2$·s)

Imaginary part of the cross spectrum of the pressure and the square of the pressure; also called quadspectral density (Pa$^3$/Hz)

The integration over all frequencies of $\omega QSD_{neg}$ (Pa$^3$/Hz)

The integration over all frequencies of $\omega QSD_{pos}$ (Pa$^3$/Hz)

A function whose value is that of the QSD where it is negative and zero where it is not (Pa$^3$/Hz)
$QSD_{pos}$ A function whose value is that of the QSD where it is positive and zero where it is not (Pa$^3$/Hz)

$r$ Propagation distance of a spherical wave (m)

$S_p$ or PSD Single-sided power (or auto-) spectral density of a pressure signal (Pa$^2$/Hz)

$S_{p^2}$ Single-sided autospectral density of the square of a pressure signal (Pa$^4$/Hz)

$S_{p^2p}$ Single-sided cross-spectral density of the square of the pressure and the pressure $C_{p^2} + jQ_{p^2}$ (Pa$^3$/Hz)

$t$ Time (s)

$T$ Temperature (K) or record length of a time series (s)

$T_0$ Reference atmospheric temperature (K)

$u$ Acoustic particle velocity vector (m/s)

$V$ Volume (m$^3$)

$x$ Propagation distance of a plane wave (m)

$\bar{x}$ Lossless plane wave shock formation distance (m)

$y$ Dimensionless retarded time $\omega \tau$

$\alpha$ Classical absorption coefficient (Np/m)

$\alpha'$ Complex absorption and dispersion coefficient, $\alpha + j\beta_d$ (Np/m + j(1/m))

$\beta$ Coefficient of nonlinearity $((\gamma + 1)/2$ in air)

$\beta_d$ Dispersion coefficient (j(1/m))

$\gamma$ Ratio of specific heats $C_P/C_V$

$\gamma_{xy}^2(f)$ Coherence function of spectra $S_x$ and $S_y$

$\gamma_Q(f)$ Coherence indicator $\frac{Q_{p^2}}{\sqrt{S_{p^2}S_p}}$

$\Gamma$ Gol’dberg number $1/\alpha \bar{x}$
\( \Gamma_s \)  Spectral Gol’dberg number \( \frac{\omega \beta}{2\alpha_0 \rho_0 c_0} \frac{Q_p^2}{s_p} \)

\( \delta \)  Diffusivity (Np\cdot s^2/m)

\( \epsilon \)  Acoustic Mach number \( u/c_0 \)

\( \epsilon \)  Nonlinear coefficient \( \beta/(\rho_0 c_0^3) \) (s^3/kg)

\( \mu \)  Shear viscosity (Pa\cdot s)

\( \mu_B \)  Bulk viscosity (Pa\cdot s)

\( \nu \)  Kinematic viscosity \( \mu/\rho_0 \) (Np\cdot s^2/m)

\( \rho \)  Total density (kg/m^3)

\( \rho_0 \)  Equilibrium ambient density (kg/m^3)

\( \rho' \)  Acoustic density \( \rho - \rho_0 \) (kg/m^3)

\( \sigma \)  Normalized propagation distance \( x/\bar{x} \)

\( \tau \)  Retarded time \( t - x/c \) (s)

\( \phi \)  Phase of complex frequency-domain pressure

\( \Phi \)  Phase of any point on time-domain pressure waveform

\( \Phi_s \)  Phase of shock front

\( \psi \)  Phase of complex frequency-domain squared pressure

\( \omega \)  Angular frequency (rad/s)

\( \omega_0 \)  Source angular frequency (rad/s)

\( \Box^2 \)  d’Alembertian operator \( \nabla^2 - \frac{1}{c_0^2} \left( \frac{\partial^2}{\partial t^2} \right) \)
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Dedication

This thesis is dedicated to my husband, Matt. His patience and understanding throughout my time at Penn State were more than I could have asked for. His gentle encouragement, always considerate and never unwelcome, helped me to complete this work and is greatly appreciated.
Chapter 1

Introduction

1.1 Jet Noise

1.1.1 Motivation for the Study of Jet Noise

Jet noise has been a problem for communities and the aerospace industry alike since the introduction of the jet engine in the 1950’s. The noise radiated from aircraft is nearly always undesirable: for those who live and work near airports, it can range from a nuisance to an annoyance; in the case of military aircraft, it can be detrimental to health, stealth and safety.

Due to design improvements implemented over the last several decades, today’s commercial aircraft tend to radiate much less acoustic power than did early models. However, the problem of jet noise is still far from being solved. Typical sound pressure levels for military aircraft are well above 120 dB within a few hundred meters of the source [21], and increased air traffic, expanding airports and bases, and growing residential developments translate to higher levels and longer exposure times for many people. Coupled with ever stricter noise certification requirements for new aircraft, these trends mean that the demand for jet noise reduction is not likely to wane in the near future.

Another issue that must be addressed is the type of analysis used to assess noise impact. Prediction models used to assess the impact of jet noise on communities near airports and bases are generally based on linear acoustics. However, given the sound pressure levels generated by aircraft—especially military aircraft— it
is possible that linear acoustics is not sufficient to describe the propagation of jet noise. Thus, an important first step in an improved assessment of the impact of military jet noise is to determine if nonlinearity is important.

Finally, currently available noise metrics for the classification of a signal’s “loudness” do not sufficiently capture the character of nonlinearly propagated noise. Nonlinear propagation not only redistributes energy within the spectrum; it can fundamentally alter the subjective character of a signal as perceived by a listener. Simple A- and C-weighted overall sound pressure levels (OASPLs) are not well-suited for application to nonlinear propagation. These weightings de-emphasize spectral content at low and high frequencies where the largest nonlinear energy gains are seen. Weighted OASPLs are also calculated based on loudness levels much lower than those typically found in jet noise. A metric called the Mark-VII perceived loudness (Mark-VII PL) [44] assigns slightly higher levels to a nonlinearly propagated noise signal than to a linearly propagated one (see Chap. 5 in Gee [21]), but the difference is hardly significant. Further, certain subjective qualities that have their origins in nonlinear propagation, such as crackle, cannot be quantified with any known metric [21]. Even so, most listeners can easily identify the presence of such effects and tend to classify signals in which they are present as louder and more annoying.

A nonlinearity indicator would serve to alert researchers to the presence of important nonlinear effects and thus instruct them to use appropriate methods of analysis. For cases in which nonlinearity is important, the use of nonlinear propagation algorithms along with metrics and/or descriptions of the received signal that reflect the quality of the sound will result in more accurate objective and subjective assessments of the impact of jet noise.

1.1.2 Early History

The theoretical study of jet noise began in the early 1950’s with the publication of two papers by Sir James Lighthill [31, 32] which set forth an approach to the problem that has been widely used ever since [46, 34]. This approach, called the “Acoustic Analogy,” begins with the Navier-Stokes equations and collects the terms representing acoustic propagation on the left hand side. The remaining terms
are moved to the right hand side and are assumed to represent acoustic sources (monopoles, dipoles, and quadrupoles). Lighthill showed that the total acoustic power radiated by the turbulence term (quadrupoles) is proportional to the eighth power of the jet velocity. Results published by Ffowcs Williams in 1963 suggest that the peak radiation angle is somewhat downstream of the nozzle exit, but not along the axis of the jet flow. In the following decades, improved model scale test facilities enabled exploration of the differences between subsonic and supersonic jets and of specific types of noise sources associated with certain jet conditions [46].

1.1.3 Characteristics

The main component of jet noise in subsonic jets is generated by the turbulent mixing that occurs between the fast-moving jet exhaust and the ambient air [34]. Known as turbulent mixing noise, this component generally has a large frequency bandwidth. It is directional, with a peak radiation angle usually between 135° and 155° from the inlet axis (25° to 45° from the jet exhaust axis) [45].

Supersonic jets typically radiate more acoustic power than subsonic jets. While turbulent mixing noise is important in these jets, they are more likely than their subsonic counterparts to have large-scale structures in the jet exhaust, especially when operated off-design. These large-scale structures are responsible for other components of the radiated noise that tend to dominate turbulent mixing noise.

One such large-scale component is broadband shock-associated noise. This is generated by the interaction of turbulence with shocks in the jet plume, which only form in supersonic jets. While the shock-associated noise is broadband, it usually has a narrower spectrum than the turbulent mixing noise. It is also more directional, with a peak frequency that increases with angle from the jet inlet axis [45].

Not all components of jet noise are “noisy.” Screech, a phenomenon generated by a feedback mechanism in standing wave patterns in the jet plume, is tonal. The standing waves, which can be seen in Fig. 1.1, are called shock cells, and the source of the screech tone is usually 3–5 shock cells downstream from the nozzle exit. Screech tones are highly directional and are strongest in the upstream direction.
Although typical screech “tones” already contain harmonics at the source, given a large enough amplitude, nonlinear acoustic propagation can increase the energy contained in these harmonics [45].

Another large-scale structure known to exist in supersonic jets is an instability wave. This is a downstream-travelling disturbance which, when it attains supersonic speeds (with respect to the surrounding air), radiates sound in a very specific direction. The radiated sound is in the form of Mach waves, which can be seen in Fig. 1.2.

A characteristic common to all types of jet noise (subsonic and supersonic, full-scale and model-scale) is that the acoustic source is not compact and can extend several jet diameters beyond the nozzle exit. This fact can make centering a microphone array a complicated matter. It also means that spreading often does not become spherical until several tens of jet diameters from the nozzle exit.
1.2 Nonlinearity in the Propagation of Acoustic Signals in Air

The foundation upon which linear acoustics is built includes assumptions about two of the constitutive equations used to derive the linear wave equation. These assumptions are valid for a wide range of situations, as linear acoustics is sufficient to describe most acoustical phenomena. However, instances do arise in which these assumptions are no longer valid and the theoretical basis of a problem must be re-evaluated. In order to determine what circumstances require re-evaluation, it is helpful to examine the assumptions used in linear acoustics.

1.2.1 The Linear Wave Equation

The linear wave equation is derived from the equations of state, continuity, and momentum; these are known collectively as the constitutive equations. The following
derivation, which can be found in more detail in Ref. [28], outlines the operations and assumptions used to obtain a linear wave equation from the constitutive equations.

The adiabatic equation of state describes the relationship between pressure and density for a gas undergoing an adiabatic process—that is, a gas whose entropy is constant (meaning that no heat is exchanged with the surrounding medium). Most acoustic processes are well-approximated as being adiabatic. The state equation can be written

\[
P - P_0 = \rho_0 (\frac{\partial P}{\partial \rho})_{s,0} (\frac{\rho'}{\rho_0}) + \frac{1}{2!} \rho_0^2 (\frac{\partial^2 P}{\partial \rho^2})_{s,0} (\frac{\rho'}{\rho_0})^2 + \ldots ,
\]

(1.1)

where \( P \) is total pressure, \( P_0 \) ambient pressure, \( \rho \) total density, and \( \rho_0 \) ambient density. Here the subscript \((s,0)\) indicates that entropy remains constant and that the operation is evaluated at equilibrium pressure and entropy. The assumption commonly made in linear acoustics is that the acoustic density \( \rho' \) is much smaller than the ambient density \( \rho_0 \). This allows for the truncation of the equation of state so that only one term is retained in the series. The resulting linearized equation of state is then

\[
P - P_0 = (\frac{\partial P}{\partial \rho})_{s,0} \rho',
\]

(1.2)

or

\[
p = B \frac{\rho'}{\rho_0},
\]

(1.3)

where \( p = P - P_0 \) is acoustic pressure and \( B = \rho_0 (\partial P/\partial \rho)_{s,0} \) is the adiabatic bulk modulus. The assumption that \( \rho'/\rho_0 \) is very small is not made in the development of the nonlinear wave equation, leading to a state equation that contains both first-order and higher-order terms in \( \rho' \).

The equation of continuity provides a relationship between the particle velocity and the instantaneous density. The full equation,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,
\]

(1.4)

where \( \rho = \rho_0 + \rho' \) is total density and \( \mathbf{u} \) is the acoustic velocity vector, states that the rate of increase in density of a certain volume (first term on the left-hand side)
must equal the net influx of density (or mass) into the volume (second term on the left-hand side). Assuming that $\rho_0$ is independent of time and of space and that $\rho' \ll \rho_0$ allows the continuity equation to be rewritten

$$\frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} + \nabla \cdot \mathbf{u} = 0. \tag{1.5}$$

The third equation used in the derivation of the linear wave equation is the momentum equation, also known as the force equation or Euler’s equation. It results from the application of Newton’s second law ($F = ma$) to a fluid element through which an acoustic wave is propagating. Neglecting the effects of gravity and dissipation, the incremental force $\mathbf{d}f$ on a fluid element with volume $dV$ is

$$\mathbf{d}f = -\nabla P dV. \tag{1.6}$$

The acceleration of the fluid element, which has mass $\rho dV$, is approximated by

$$\mathbf{a} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}, \tag{1.7}$$

where

$$(\mathbf{u} \cdot \nabla) \equiv u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}. \tag{1.8}$$

Both terms on the right-hand side of Eq. (1.7) are necessary because the velocity of the fluid element is a function of both time and space. Substituting Eqs. (1.6) and (1.7) into the relation $\mathbf{d}f = \mathbf{a}dm$ gives

$$-\nabla P = \rho(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}). \tag{1.9}$$

Assuming that the effect of convection is small, or $|\mathbf{u} \cdot \nabla \mathbf{u}| \ll |\partial \mathbf{u}/\partial t|$, and that the acoustic density is small relative to the ambient density, or $\rho \approx \rho_0$, a linearized version of Eq. (1.9) can be written

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p. \tag{1.10}$$

These last two assumptions are not made in the derivation of the nonlinear wave equation.
To arrive at the linear wave equation, the divergence operator is applied to Eq. (1.10),
\[ \nabla \cdot (\rho_0 \frac{\partial \mathbf{u}}{\partial t}) = -\nabla^2 p. \] (1.11)

The time derivative of Eq. (1.5) is
\[ \frac{\partial^2 \rho'}{\partial t^2} + \nabla \cdot (\rho_0 \frac{\partial \mathbf{u}}{\partial t}) = 0, \] (1.12)
assuming that \( \rho_0 \) is not time-dependent. The second term in Eq. (1.12) is equivalent to the left-hand side of Eq. (1.11); equating the other terms,
\[ \nabla^2 p = \frac{\partial^2 \rho'}{\partial t^2}. \] (1.13)

From Eq. (1.3) we have \( \rho' = \rho_0 p / B \). Assuming that \( B \) is not time-dependent, we can write
\[ \nabla^2 p = \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2}, \] (1.14)
where the small-signal sound speed is defined by
\[ c_0^2 = B / \rho_0. \] (1.15)

Equation 1.14 is the linear wave equation and is valid (along with Eq. (1.15)) for the lossless propagation of waves for which acoustic variables are small compared to ambient variables, and where convection is negligible.

1.2.2 Nonlinear Propagation Speed and Coefficient of Non-linearity

When the acoustic density \( \rho' \) becomes large enough, the relation \( \rho' \ll \rho_0 \) no longer holds. This usually corresponds to a sound pressure level of about 140 dB re 20 \( \mu \)Pa in air. At this amplitude, the value of the ratio \( \rho' / \rho_0 \) is only 0.002, but the cumulative nature of nonlinear effects renders it significant. The small error incurred with each small propagation step accumulates into something which can no longer be ignored over larger propagation distances. Thus, the second term in the equation of state (1.1) can no longer be neglected. However, for virtually all
cases, there is no need to retain further terms; nonlinear effects are very accurately represented by the second-order term. The importance of nonlinearity in the equation of state can be gauged by examining the relative magnitudes of the first and second terms in the state equation expansion. Defining

$$A = \rho_0 \left( \frac{\partial P}{\partial \rho} \right)_{s,0}$$  \hspace{1cm} (1.16)

$$B = \rho_0^2 \left( \frac{\partial^2 P}{\partial \rho^2} \right)_{s,0},$$  \hspace{1cm} (1.17)

the quantity $B/A$ is known as the parameter of nonlinearity. For an ideal gas

$$\frac{B}{A} = \gamma - 1.$$  \hspace{1cm} (1.18)

For a diatomic gas such as air, $\gamma = 1.4$, giving $B/A = 0.4$.

Retaining second-order terms in both the momentum and state equations results in a particle-velocity-dependent sound speed. In terms of the particle velocity $u$ and the small-signal (or equilibrium) sound speed $c_0$, the propagation speed becomes

$$c = c_0 + u + \frac{B}{2A}u.$$  \hspace{1cm} (1.19)

Here the contributions of the nonlinear terms in the two constitutive equations are apparent: the first $u$ on the right-hand side is a result of convection, which is accounted for in the momentum equation; the $B/2A$ term comes from the state equation [10]. Substituting Eq. (1.18) into Eq. (1.19) yields

$$c = c_0 + \frac{1}{2}(\gamma + 1)u = c_0 + \beta u,$$  \hspace{1cm} (1.20)

where

$$\beta = 1 + \frac{B}{2A} = \frac{1}{2}(\gamma + 1)$$  \hspace{1cm} (1.21)

is the coefficient of nonlinearity. For air, $\beta = 1.2$.

### 1.2.3 Wave Steepeening and Shock Formation

The amplitude-dependent sound speed results in wave steepening in the time domain. Where $u = 0$, $c = c_0$, but portions of the wave with higher particle velocities
Nonlinear Steepening of a 2000 Hz Sinusoid

Figure 1.3. Nonlinear wave steepening can be seen in these numerically propagated waveforms. The source is a 2000 Hz sinusoid and is propagated according to the Fubini solution. Waveforms are shown before shock formation at normalized distances of 0.01, 0.5, 0.8, and 0.99.

travel faster than $c_0$, and portions with lower particle velocities travel slower than $c_0$. Thus, the portion of the wave that is between a low velocity point and a high velocity point steadily increases in slope while the relative locations of the zero-particle-velocity points (often called zero crossings) remain constant. Wave steepening for an initially sinusoidal wave can be observed in Fig. 1.3.

Once the wave has travelled a certain distance, the steepened slope becomes infinite and a shock is formed. For an initially sinusoidal plane wave in the absence of absorption, this propagation distance, called the shock formation distance, can be written \[12\]

$$\bar{x} = \frac{\rho_0 c_0^3}{\beta \rho_0 \omega} = \frac{1}{\beta \epsilon k},$$

where $p_0$ is the pressure amplitude at the source, $\omega$ the source angular frequency, $\epsilon$
Figure 1.4. When propagation using Eq. (1.19) results in a multi-valued wave, it is necessary to draw a vertical line to define the shock front and to make the wave single-valued. In this figure [30], the dashed line represents the multi-valued parts of the waveform that are being removed and the solid vertical line represents the shock. Reprinted with permission of John Wiley & Sons, Inc.

the acoustic Mach number (equal to the particle velocity amplitude at the source divided by the equilibrium sound speed), and 

\[ k = \frac{\omega_0}{c_0} \]

the angular wave number of the source. Often the propagation distance \( x \) is normalized by the shock formation distance, resulting in the variable

\[ \sigma = \frac{x}{\bar{x}}. \]  (1.23)

For \( \sigma < 1 \), no shocks have formed, and for \( \sigma > 1 \), shocks are present. In the absence of absorption, the wave attains a sawtooth shape at approximately \( \sigma = 3 \).

It should be noted that even if atmospheric absorption is assumed to be negligible, nonlinear propagation is not lossless. Once a shock has formed, simply using Eq. (1.19) to determine the time series would result in a multi-valued wave. Since this is not physically possible, a vertical line must be drawn to represent the shock front so that the area to the left of the line is equal to the area to the right of the line (see Fig. 1.4 [30]). This procedure is known as the **equal area rule** and suggests that energy is being lost at the shock front. In fact, it can be shown [12] that the rate of change of energy \( E \) with time for a shock whose pressure amplitude is \( \Delta P \) is given by

\[ \frac{dE}{dt} = -\frac{\beta}{6\rho_0^2c_0^3}(\Delta P)^3. \]  (1.24)
The energy lost by the wave is transferred to the medium as heat.

### 1.2.4 Spectral Implications

As the shape of the wave evolves nonlinearly, so too does its spectrum. Spectral components present in the source signal “add” and “subtract” to create sum- and difference-frequency components. Once formed, these new components participate in the creation of yet more frequency components, significantly increasing the spectral content of the signal. For an initially sinusoidal wave, harmonics are generated at integer multiples of the source frequency, and a zero-frequency or “DC” component is also created. (It should be noted that the presence of the “DC” component is not evident in Figs. 1.4 and 1.3; in fact, this component is often ignored altogether in the analysis of nonlinearly evolving signals.) Since energy must be conserved in the pre-shock region (assuming absorption is negligible), the energy gained at the harmonic frequencies is lost from the fundamental frequency. (The amount of energy gained at the zero-frequency component is small compared with the amount transferred upward in the spectrum; because this component is often of little interest acoustically, it is usually ignored.)

Several researchers have derived expressions for the amplitudes of the harmonic components as a function of propagation distance for an initially sinusoidal wave. One such researcher was Fubini, whose solution is valid for lossless plane waves for $\sigma < 1$ and is described in Ref. [12]. The Fubini solution is given by

$$p(\sigma, \tau) = p_0 \sum_{n=1}^{\infty} \frac{2}{n\sigma} J_n(n\sigma) \sin n\omega_0 \tau, \quad (1.25)$$

where $\sigma$ is the propagation distance normalized by the shock formation distance, $\tau = t - x/c_0$ the retarded time, $J_n$ the Bessel function of the first kind, and $\omega_0$ the angular frequency of the source. In terms of harmonic amplitudes, the solution is

$$B_n = \frac{2}{n\sigma} J_n(n\sigma), \quad (1.26)$$

where $B_n$ is the amplitude of the $n^{th}$ harmonic normalized to the amplitude of the fundamental at the source.

Fay [19] also derived a solution for the propagation of an initially sinusoidal
plane wave, but his expression is only valid for $\sigma > 3$. Harmonic amplitudes according to the Fay solution are

$$B_n = \frac{2/\Gamma}{\sinh n(1+\sigma)/\Gamma},$$  \hspace{1cm} (1.27)

where $\Gamma$ is the Gol’dberg number (see Sec. 1.2.5). In the limit of large $\Gamma$ (strongly nonlinear propagation), the Fay solution becomes

$$B_n = \frac{2}{n(1+\sigma)},$$  \hspace{1cm} (1.28)

which is a sawtooth wave. Thus, the Fay solution is often said to be valid in the “sawtooth region.”

Blackstock [8], formulated a “bridge” solution that connects the Fubini and Fay solutions and is valid for all $\sigma$; his solution is described in detail later in this section.

In order to observe both sum- and difference-frequency generation, it is easiest to consider a source having more than one spectral component. The simplest of such sources is one composed of two discrete frequencies, also called a bi-frequency source. Fenlon [20] derived a solution for the evolving spectrum of such a source and, through comparison with experiments, confirmed the presence of both sum- and difference-frequency generation. The same phenomenon can be observed for a noise source, as energy is lost from the peak frequency region (or middle) of the spectrum and gained at high and low frequencies. Because a noise signal already has some higher frequency components at the source, energy transferred upward in the spectrum during nonlinear propagation can give the appearance of a reduced attenuation coefficient at these frequencies. Conversely, spectral components in the peak frequency region (or, for a propagating sinusoid, the source frequency) may appear to have a larger than expected attenuation coefficient [7].

1.2.4.1 The Blackstock Bridging Function

Figure 1.5 shows the normalized harmonic amplitudes of an initially sinusoidal wave calculated with Blackstock’s “bridging function.”

This solution [8] for a nonlinearly propagating sinusoid is used in this research.
Figure 1.5. Normalized harmonic amplitudes of an initially sinusoidal wave as a function of normalized distance. This plot was generated using the Blackstock bridging function, which assumes plane wave propagation with no atmospheric or boundary layer absorption.

as a tool for investigating the physical meaning of the quantities described in Chap. 2 and as a theoretical reference point against which to compare results for measured data. The function expresses the harmonic amplitudes of an initially sinusoidal signal as a function of normalized distance for plane wave propagation in the absence of absorption. It effectively “bridges” the gap between the Fubini and Fay solutions and is valid for all values of normalized distance $\sigma$. In formulating his solution, Blackstock begins with the Earnshaw solution and uses weak shock theory (described in Ref. [12]), which assumes that shocks in the waveform are weak and have a rise time of zero (that is, they are true discontinuities), and that dissipation is concentrated at the shocks and negligible elsewhere in the waveform.

The harmonic amplitudes of an initially sinusoidal wave according to the Blackstock bridging function can be written as follows,

\[
B_n = \frac{2}{n\pi} \sin \Phi_s + \frac{2}{n\pi\sigma} \int_{\Phi_s}^{\pi} \cos nyd\Phi, \tag{1.29}
\]

where $B_n$ is the amplitude of the $n^{th}$ harmonic, $\Phi_s$ the phase of the shock front, $\Phi$
the phase of any point on the waveform, $\sigma$ the normalized distance, and $y = \omega \tau$ the dimensionless retarded time.

For this work, harmonic amplitudes were calculated using the Blackstock bridging function and saved at very small intervals of $\sigma$. These amplitudes were then used to reconstruct time domain waveforms for source parameters matching those used in the measurements. Thus, the measured waveforms (and any spectral quantities or indicators derived from them) could be compared directly with theory at virtually any propagation distance.

### 1.2.5 Nonlinearity and Absorption

Because nonlinearity tends to increase the energy content at the upper end of the spectrum and absorption (atmospheric and/or boundary layer) acts to preferentially reduce the energy content at these frequencies, these two phenomena can be viewed as working against one another. It is therefore helpful to be able to gauge the relative importance of nonlinearity and absorption so that accurate predictions and assessments can be made. For single-frequency sources propagating as plane waves, this can be achieved by calculating a quantity known as the Gol’dberg number [7], which is given by

$$\Gamma = \frac{l_a}{\bar{x}},$$

where $\bar{x}$ is the lossless plane wave shock formation distance and $l_a$, called the absorption length, is equal to the inverse of the absorption coefficient $\alpha$ for the source frequency.

Gol’dberg was the first to show that $\Gamma = 1$ is the threshold for the importance of nonlinearity [7]. For $\Gamma \ll 1$, absorption dominates and nonlinear effects are negligible, and for $\Gamma \gg 1$, nonlinearity dominates absorption and shocks are likely to form. In light of this interpretation, the Gol’dberg number can function as a kind of nonlinearity indicator. As such, it will be used in this work as a quantity against which to compare the results obtained with the nonlinearity indicators derived in Chap. 2. The most significant limitation of the Gol’dberg number is that it exists only for a sinusoidal source; however, one of the nonlinearity indicators derived in this work will function much like a “spectral” Gol’dberg number.
1.2.6 The Burgers Equation

The Burgers Equation, perhaps the most widely used model equation in nonlinear acoustics, is a second-order parabolic wave equation that accounts for nonlinearity as well as absorption and dispersion. It was initially derived by Burgers [13] for use in turbulence theory and was modified for use in nonlinear acoustics by Mendousse [35]. Although the form presented by Mendousse is the one most commonly used in nonlinear acoustics, it is widely known as the Burgers Equation, perhaps because of a reference to it by that name by Lighthill [33]. The equation has since been generalized for arbitrary geometrical spreading [27] and for arbitrary absorption and dispersion [9]. A more complete history appears in Blackstock et al. [12].

To derive the Burgers equation, begin with the lossy second-order wave equation

\[ \Box^2 p + \frac{\delta}{c_0^4} \frac{\partial^3 p}{\partial t^3} = -\frac{\beta}{\rho_0 c_0^4} \frac{\partial^2 p}{\partial t^2} - (\nabla^2 + \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2}) L. \]  

(1.31)

Here \( \delta \) is the diffusivity of sound given by

\[ \delta = \nu \left( \frac{4}{3} + \frac{\mu_B}{\mu} + \frac{\gamma - 1}{Pr} \right), \]

(1.32)

where \( \nu = \mu/\rho_0 \) is the kinematic viscosity, \( \mu_B \) the bulk viscosity, \( \mu \) the shear viscosity, \( \gamma \) the ratio of specific heats, and \( Pr \) the Prandtl number. The operator \( \Box^2 \) is the d’Alembertian operator

\[ \Box^2 = \nabla^2 - \frac{1}{c_0^2} \left( \frac{\partial^2}{\partial t^2} \right). \]

(1.33)

The linear wave equation operator from Eq. (1.14)), and \( L \) is the second-order Lagrangian density

\[ L = \frac{1}{2} \rho_0 u^2 - \frac{p^2}{2\rho_0 c_0^2}. \]

(1.34)

Equation (1.31) can be derived from the constitutive equations by retaining first- and second-order terms, as suggested in Sec. 1.2.1, and by including dissipative terms. This wave equation is valid outside the boundary layer for weakly thermoviscous fluids; linear theory has been used to approximate vorticity. Equation (1.31) accounts for the interaction of nonlinear and dissipative processes, may
be used for three-dimensional propagation, and is not restricted to progressive waves.

Assuming that cumulative nonlinear effects dominate local nonlinear effects allows the Lagrangian density operator $\mathcal{L}$ to be neglected ($\mathcal{L}$ is equal to zero for plane progressive waves anyway). The result is the Westervelt equation,

$$\Box^2 p + \frac{\delta}{c_0^4} \frac{\partial^3 p}{\partial t^3} = -\frac{\beta}{\rho_0 c_0^4} \frac{\partial^2 p^2}{\partial t^2}. \quad (1.35)$$

Simplifying the one-dimensional form of the Westervelt equation,

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2}\right)p + \frac{\delta}{c_0^4} \frac{\partial^3 p}{\partial t^3} = -\frac{\beta}{\rho_0 c_0^4} \frac{\partial^2 p^2}{\partial t^2}, \quad (1.36)$$

and using the retarded time $\tau = t - x/c_0$ results in the Burgers Equation

$$\frac{\partial p}{\partial x} - \frac{\beta p}{\rho_0 c_0^3} \frac{\partial p}{\partial \tau} = \frac{\delta}{2c_0^3} \frac{\partial^2 p}{\partial \tau^2}. \quad (1.37)$$

Here $p$ is the time domain acoustic pressure, $x$ the propagation distance, $\beta$ the coefficient of nonlinearity, $\rho_0$ the ambient density, $c_0$ the equilibrium sound speed, $\tau$ the retarded time, and $\delta$ the diffusivity of sound. A detailed derivation of the Burgers equation can be found in Ref. [25].

Aside from its inclusion of both nonlinear and thermoviscous effects, the Burgers Equation is attractive for use in nonlinear acoustics because it has an exact analytical solution. Forms of this solution have been developed for sound propagating from a monofrequency source [35] and for strongly nonlinear waves ($\Gamma \gg 1$) at normalized distances $\sigma > 3$ from a monofrequency source [19], among others. Many numerical propagation schemes [40, 16] make use of the solution, as well.

### 1.2.6.1 The Mendousse Solution to the Burgers Equation

The Mendousse solution to the Burgers equation can be viewed as the “lossy” analog of the Blackstock bridging function. It assumes an initially sinusoidal signal propagating as a plane wave, but accounts for the effects of both nonlinearity and
absorption. It is given in series form in Ref. [12] as

\[ p = p_0 \frac{4\Gamma^{-1} \sum_{n=1}^{\infty} (-1)^{n+1} n I_n \left(\frac{1}{2} \Gamma\right) e^{-n^2\alpha x} \sin n\omega_0 \tau}{I_0 \left(\frac{1}{2} \Gamma\right) + 2 \sum_{n=1}^{\infty} (-1)^{n+1} I_n \left(\frac{1}{2} \Gamma\right) e^{-n^2\alpha x} \cos n\omega_0 \tau}, \]  

(1.38)

where \( p_0 \) is the acoustic pressure amplitude at the source, \( I_n \) a modified Bessel function of the first kind of order \( n \), \( \alpha \) the absorption coefficient at the source frequency, \( \Gamma \) the Gol’dberg number, \( \omega_0 \) the source frequency, and \( \tau \) the retarded time.

The Mendousse solution is used in this research in much the same way as the Blackstock bridging function; harmonic amplitudes were calculated using Eq. (1.38) and saved at small intervals of \( \sigma \) so that waveforms could be reconstructed for conditions matching those of measured data. With this method, though, more care must be taken to match the theoretical source conditions to the experimental ones. Assigning values to the absorption coefficient and the Gol’dberg number in Eq. (1.38) amounts to specifying the source frequency and amplitude.

1.2.6.2 Anderson-Type Algorithm

The Anderson-type algorithm used here was written by Gee and is described in detail in Ref. [21]. The code essentially solves the Burgers equation as a means of numerically propagating an acoustic signal. It is a split-step propagation algorithm that treats nonlinear effects in the time domain and dissipation effects in the frequency domain. The code accepts any waveform at any initial propagation distance as an input and produces an output waveform at any desired final propagation distance. It can be used with any type of geometrical spreading and with any type of absorption (i.e. atmospheric and/or boundary layer). For propagation within the plane wave tube, both atmospheric and boundary layer absorption were included (see Sec. 1.2.6.3). The Anderson-type algorithm has been shown to be accurate in the propagation of full-scale and model-scale jet noise as well as in plane wave tube propagation. It has been called an “arbitrary wave Anderson algorithm” and will be referred to in this document as the “AWAA.”
1.2.6.3 Absorption and Dispersion

For both the Anderson-type algorithm and the Mendousse solution to the Burgers equation, atmospheric absorption was calculated according to Bass et. al [2, 3]. The formula combines the effects described in the classical absorption coefficient (thermal and viscous effects) with those of oxygen and nitrogen relaxation; it can be written as follows:

\[
\frac{\alpha}{p_s} = F^2 (1.84 \times 10^{-11} \left( \frac{T}{T_0} \right)^{1/2} + \left( \frac{T}{T_0} \right)^{-5/2} [0.01275 \left( \frac{e^{-2239.1/T}}{F_{r,O} + F^2 / F_{r,O}} \right) + 0.1068 \left( \frac{e^{-3352/T}}{F_{r,N} + F^2 / F_{r,N}} \right)]. \tag{1.39}
\]

Here \(\alpha\) is the atmospheric absorption coefficient [Np/m], \(p_s\) the atmospheric pressure [atm], \(F = f/p_s\) the scaled frequency [Hz/atm], \(T\) the atmospheric temperature [K], \(T_0 = 273.16\text{K}\) the reference atmospheric temperature, \(F_{r,O} = f_{r,O}/p_s\) the scaled oxygen relaxation frequency [Hz/atm], \(F_{r,N} = f_{r,N}/p_s\) the scaled nitrogen relaxation frequency [Hz/atm], and \(f\) the frequency in Hz.

Atmospheric dispersion resulting from relaxation processes was calculated according to ANSI standard S1.26-1978; it can also be found using relations from Ref. [9] and updated formulae from the Bass et. al method as follows:

\[
\frac{\beta_d}{p_s} = -fF^2 \left( \frac{T}{T_0} \right)^{-5/2} [0.01275 \left( \frac{e^{-2239.1/T}}{F_{r,O} + F^2 / F_{r,O}} \right) + 0.1068 \left( \frac{e^{-3352/T}}{F_{r,N} + F^2 / F_{r,N}} \right)]. \tag{1.40}
\]

Here \(\beta_d\) is the atmospheric dispersion coefficient [Np/m] and all other variables are as described above.

The algorithm for boundary layer absorption and dispersion in the Anderson-type algorithm as well as in the Mendousse solution accounted for both thermal and viscous processes and accepted any value of (circular cross-section) tube radius \(a\) as an input. The formula used,

\[
\alpha = \frac{1}{a} \sqrt{\frac{\omega \nu}{2 c_0^2} \left( 1 + \frac{\gamma - 1}{\sqrt{Pr}} \right)}, \tag{1.41}
\]

is Eq. (D-9b) from Ref. [11], where \(\alpha\) is the boundary-layer absorption coefficient (and \(j\alpha\) is the boundary-layer dispersion coefficient), \(a\) the tube radius, \(\omega\) the
frequency or array of frequencies, $\nu$ the kinematic viscosity coefficient, $\gamma$ the ratio of specific heats, and $Pr$ the Prandtl number. Equation (1.41) is valid for

$$\delta_v \ll a, \text{ where } \delta_v = \sqrt{\frac{2\nu}{\omega}}. \quad (1.42)$$

Here $\delta_v$ is the viscous boundary layer thickness. This condition was satisfied in all the measurements made for this study, with $a = 2.6 \text{ cm}$ and $\delta_v \leq 0.14 \text{ mm}$ for all frequencies of interest.

1.3 Prediction and Assessment of Jet Noise in Light of Nonlinearity

1.3.1 Evidence of Nonlinearity in Jet Noise Propagation

Many researchers [48, 14, 39, 40] have studied the nonlinear propagation of plane waves in a tube. Phenomena such as harmonic generation, energy transfer to higher frequencies, and apparent reduced attenuation at higher frequencies are well documented in these works. Several of these researchers postulate the applicability of their work to jet noise. Webster and Blackstock [49] carried out a series of experiments with both tones and noise emitted from an array of conventional loudspeakers. They also found substantial evidence of nonlinearity and made comparisons between their experiments and jet noise. Comparatively few researchers have found evidence of nonlinear propagation in actual jet noise measurements. Morfey and Howell [37] present flyover data that shows evidence of nonlinear energy transfer to higher frequencies, and they refer to a 1978 report by Gallard and Gower as another example of the same phenomenon. More recently, Petitjean et al. [42] have offered evidence for the presence of nonlinearity in model scale jet noise, and Gee et al. [23, 22, 21] have shown nonlinearity to be of importance in full scale jet noise.
1.3.2 Traditional Methods for the Assessment of Nonlinearly Propagated Signals

The power spectral density (PSD) of a signal is the quantity typically used to assess that signal’s impact upon the propagation environment. While the PSD is not sufficient to reconstruct a time waveform (it is a magnitude-only quantity and therefore contains no phase information), it is a good measure of how acoustic energy is distributed across the frequency spectrum. Assuming linear propagation in the far field and ignoring atmospheric effects such as temperature gradients, wind shear, and turbulence, it is straightforward to predict the PSD of a signal at a propagation distance $r$ given its PSD at a distance $r_0$. In such a case, geometrical spreading and atmospheric absorption, both of which are well known, are the only factors that cause the PSD to change.

Predicting the evolution of a signal experiencing nonlinear propagation requires information about the phase of that signal. Thus, because the PSD contains no phase information, it is not possible to predict the nonlinear evolution of a signal given only its PSD. It is possible, however, to detect the presence of nonlinear effects using the PSD. This is done by propagating a measured PSD using the linear method described above to a larger propagation distance where it can be compared with another measured PSD. An underprediction of the amplitudes of the high and low frequency components and an overprediction of center frequency components indicate the presence of nonlinear effects. While this process may appear to be straightforward, there are many factors that can introduce complications. These will be discussed in Sec. 1.4.

1.4 Motivation

1.4.1 Pitfalls of Traditional (PSD) Jet Noise Assessment Methods

In general, large propagation distances and a large measurement bandwidth are necessary in order to see evidence of nonlinearity in sound propagation using the PSD. Propagation distances of hundreds of jet diameters (meaning hundreds of
meters for full-scale jets) are not unusual, and standard 1/4” or 1/8” microphones with frequency responses up to 70 or 100 kHz are often insufficient when used with the PSD method of detecting nonlinear effects.

1.4.1.1 Full-Scale Jets

Several problems can arise when using the PSD to assess the nonlinear propagation of sound radiated from a full-scale jet. Wind and temperature gradients as well as turbulence are often present along the propagation path, causing received levels to vary. Meteorological data taken alongside an acoustic measurement is usually spatially limited and therefore is not adequate to correct for the effects of these atmospheric phenomena. For ground-based measurements, microphones are located either in a plate that rests on the ground or on a stand up to a few meters above the ground surface. Because of the difficulty of placing a microphone in a ground plate and because microphones placed this way can end up in acoustic “shadow zones” if the terrain is not level or if the meteorological conditions cause upward refraction, most experiments are carried out with the microphones on stands. In this configuration, reflections result in multiple propagation paths that interfere at the microphone. Removal of these interference effects depends upon knowledge of the ground impedance, a quantity that can be difficult to measure.

Another difficulty of full-scale jet measurements arises from the diameter \( d_0 \) of the jet nozzle itself. Because nonlinear effects are cumulative, the propagation distances required to observe nonlinear effects in the PSD are typically large, on the order of tens or even hundreds of wavelengths. In jet noise, frequency scales as the inverse of the jet diameter, meaning that wavelength scales roughly with \( d_0 \). A typical full-scale jet with \( d_0 \) on the order of 1 m usually has peak radiation frequencies around a few hundred hertz. This means that capturing nonlinear effects in the PSD depends upon placing microphones hundreds of meters from the source.

All the factors described above contribute to make full-scale jet measurements costly and time-consuming. Microphones and meteorological data collection stations in a measurement array encompassing hundreds of meters require laying out hundreds of meters of cables and long setup and calibration times. Limited ability to compensate for atmospheric and ground effects leads to reduced confidence in
the acoustic data and difficulty in identifying nonlinear effects in the PSD.

1.4.1.2 Model-Scale Jets

While the controlled environment of a model-scale jet facility eliminates many of the problems with full-scale measurements, it also imposes its own unique restrictions. The scaling-up of frequency and the constraints of working in an anechoic chamber of finite dimension often make it difficult to detect nonlinear effects by examining the PSD. As mentioned above, nonlinear effects are cumulative, so maximizing the value of \( r/d \) of a measurement (where \( r \) is radial distance from the source and \( d \) is model jet diameter) is important to their detection. Because model-scale measurements are usually conducted in an enclosed space, \( r \) is limited.

Furthermore, decreasing \( d \) increases the frequency bandwidth needed for the measurements. Frequency is scaled as \( d_0/d \) (where \( d_0 \) is the full-scale jet diameter), meaning that a larger bandwidth is required for a smaller jet. Frequency bandwidth well above the peak frequency (one to two decades) is generally necessary to see significant differences between measurements and linear predictions and therefore to detect the presence of nonlinearity. Peak frequencies in model-scale measurements are usually on the order of a few kilohertz [42]. While it is possible to record data at a sample rate of roughly 200 kHz (thereby obtaining a Nyquist frequency of 100 kHz), amplitude and phase calibrations are not readily available for most microphones at these high frequencies. A typical 1/8” microphone (Brüel & Kjær type 4138) has a flat amplitude response to about 140 kHz; a similar 1/4” microphone (Brüel & Kjær type 4938) is reliable only to 70 kHz. Specifications on the phase response of these microphones as a function of frequency are not available from the manufacturer. This restriction, combined with the limitation on \( r \) posed by the measurement space, makes it very difficult to see nonlinearity in model-scale measurements of the PSD.

1.5 Goals

In light of the considerations described above, new methods for evaluating the effects of nonlinearity on the propagation of acoustic signals is desirable. It would be beneficial to be able to obtain information about the nonlinearity of a signal
using a measurement at a single location. It would also be preferable if this in-
formation could be gathered without the requirement that the measurement have
a large bandwidth. This work describes the development and application of such
methods. The result will be an improved understanding of the importance of non-
linearity in the propagation of jet noise which, in turn, will assist in the selection
and application of appropriate propagation models and assessment methods.

1.6 Overview of Thesis Contents

The main body of this work is concerned with the development and testing of
single-point indicators of nonlinearity. Chapter 2 summarizes and expands upon
the work of previous researchers as it pertains to the development of nonlinearity
indicators and presents derivations of several such indicators. Chapter 3 provides
detailed descriptions of the experiments that provided data for the analysis con-
tained in this work. In Chap. 4, the results of the application of the nonlinear-
ity indicators from Chap. 2 to the data described in Chap. 3 are presented and
discussed, and Chap. 5 contains assessments of the nonlinearity indicators and
suggestions for future work.
Chapter 2

Development of Nonlinearity Indicators

2.1 Introduction

This chapter provides the theoretical basis for the nonlinearity indicators investigated in this research. It begins with discussing the physical meaning of the quantity $Q_{\rho^2}$ that appears in the analysis of Morfey and Howell [37] with the goal of developing other physically meaningful nonlinearity indicators. Several such indicators are derived in this chapter. They are applied to experimental and numerical data in Chapter 4.

2.2 Analysis of Morfey and Howell

The theory upon which this work is based is laid out in a paper by Morfey and Howell [37]. Starting from the Burgers Equation, they derive an equation containing a quantity $Q_{\rho^2}$ that they call the “spectral transfer term” that “accounts for the nonlinear distortion of the spectrum which occurs during finite-amplitude propagation.” The current work builds upon their analysis by investigating the physical significance of $Q_{\rho^2}$ and developing methods of using it in the analysis of the nonlinear propagation of acoustic signals.
2.2.1 Derivation of Spectral Burgers’ Equation

Beginning with the Burgers Equation in a form similar to that of Eq. (1.37), Morfey and Howell apply the temporal Fourier transform to take the equation from the time domain to the frequency domain. Here we will define the Fourier transform \( \tilde{g}(\omega) \) of a function \( g(t) \) as follows:

\[
\tilde{g}(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} d\omega.
\] (2.1)

Morfey and Howell’s resulting frequency-domain Burgers equation is

\[
\frac{\partial \tilde{p}}{\partial x} - \frac{\beta}{2\rho_0 c_0^3} j\omega \tilde{q} = -\frac{\delta}{2c_0^2} \omega^2 \tilde{p},
\] (2.2)

where \( \tilde{p} \) is the Fourier transform of pressure and \( \tilde{q} \) is the Fourier transform of the square of the pressure. As mentioned in Sec. 1.2.6, the Burgers Equation is valid for arbitrary absorption and dispersion [9], so Morfey and Howell now take advantage of this by replacing \( \omega^2 \delta/2 \) with \( \alpha' = \alpha + j\beta_d \), where \( \alpha \) is a generalized absorption coefficient and \( \beta_d \) is a generalized dispersion coefficient. They also employ the fact that the equation can be used for problems with arbitrary geometrical spreading [27] and generalize their version for spherical spreading. The result is

\[
(r \frac{\partial}{\partial r} + \alpha') r \tilde{p} = \frac{\beta}{2\rho_0 c_0^3} j\omega r \tilde{q}.
\] (2.3)

Next, both sides of Eq. (2.3) are multiplied by \( r \tilde{p}^* \), where \( \tilde{p}^* \) is the complex conjugate of the frequency domain pressure, giving

\[
r \tilde{p}^* \frac{\partial}{\partial r} + (r + \alpha' r^2) \tilde{p}^* \tilde{q} = \frac{\beta}{2\rho_0 c_0^3} j\omega r^2 \tilde{p}^* \tilde{q},
\] (2.4)

or

\[
r^2 \tilde{p}^* \frac{\partial}{\partial r} + (r + \alpha' r^2) \tilde{p}^* \tilde{p} = \frac{\beta}{2\rho_0 c_0^3} j\omega r^2 \tilde{p}^* \tilde{q}.
\] (2.5)

Defining \( \tilde{p} = \tilde{p}_R + j\tilde{p}_I \) and \( \tilde{p}^* = \tilde{p}_R - j\tilde{p}_I \) leads to
\[ r^2 \left( \frac{\partial \tilde{p}_R}{\partial r} + j \tilde{p}_R \frac{\partial \tilde{p}_I}{\partial r} - j \tilde{p}_I \frac{\partial \tilde{p}_R}{\partial r} + \tilde{p}_I \frac{\partial \tilde{p}_I}{\partial r} \right) + (r + \alpha r^2)(\tilde{p}^* \tilde{p}) + j \beta r^2(\tilde{p}^* \tilde{p}) = \frac{\beta}{2 \rho_0 c_0^3} j \omega r^2 \tilde{p}^* \tilde{q}. \]  

(2.6)

Taking the real part of Eq. (2.6) gives

\[ r^2 \left( \frac{\partial \tilde{p}_R}{\partial r} + \tilde{p}_I \frac{\partial \tilde{p}_I}{\partial r} \right) + (r + \alpha r^2)(\tilde{p}^* \tilde{p}) = -\frac{\beta}{2 \rho_0 c_0^3} \omega r^2 \text{Im}[\tilde{p}^* \tilde{q}]. \]  

(2.7)

Recognizing that

\[ \tilde{p}_R \frac{\partial \tilde{p}_R}{\partial r} = \frac{1}{2} \frac{\partial \tilde{p}_R^2}{\partial r} \quad \text{and} \quad \tilde{p}_I \frac{\partial \tilde{p}_I}{\partial r} = \frac{1}{2} \frac{\partial \tilde{p}_I^2}{\partial r} \]  

(2.8)

and that \( \tilde{p}_R^2 + \tilde{p}_I^2 = \tilde{p}^* \tilde{p} \), and multiplying both sides by \( 2 e^{2 \alpha r} \), Eq. (2.7) becomes

\[ r^2 e^{2 \alpha r} \frac{\partial}{\partial r}(\tilde{p}^* \tilde{p}) + (2 re^{2 \alpha r} + 2 \alpha e^{2 \alpha r} r^2)(\tilde{p}^* \tilde{p}) = -\frac{\beta}{\rho_0 c_0^3} \omega r^2 e^{2 \alpha r} \text{Im}[\tilde{p}^* \tilde{q}]. \]  

(2.9)

The left-hand side of this result can be rewritten such that

\[ \frac{\partial}{\partial r}[r^2 e^{2 \alpha r} (\tilde{p}^* \tilde{p})] = -\frac{\beta}{\rho_0 c_0^3} \omega r^2 e^{2 \alpha r} \text{Im}[\tilde{p}^* \tilde{q}]. \]  

(2.10)

According to Bendat and Piersol [4], the single-sided cross-spectral density function \( S_{xy} \) between two random processes is given by

\[ S_{xy}(f) = \lim_{T \to \infty} \frac{2}{T} E[X_k^*(f, T)Y_k(f, T)], \]  

(2.11)

where \( T \) is the record length of the \( k^{th} \) record, \( E \) the expected value operator that denotes an averaging operation over the index \( k \), and \( X_k \) and \( Y_k \) the finite Fourier transforms of random processes \( x_k(t) \) and \( y_k(t) \) over the record length \( T \), respectively. A similar definition exists for the single-sided autospectral density \( S_{xx} \) (or simply \( S_x \)) of a single random process:

\[ S_{xx}(f) = \lim_{T \to \infty} \frac{2}{T} E[X_k^*(f, T)X_k(f, T)]. \]  

(2.12)
Applying this limit and expected value operation to both sides of Eq. (2.10) allows \( \tilde{p}^* \tilde{p} \) to be rewritten as the power spectral density, or \( S_p \), and \( \tilde{p}^* \tilde{q} \) to be rewritten as \( S_{p^2} \), which is the cross-spectral density of the square of the pressure and the pressure. Thus, the power spectral density (PSD) is defined as

\[
S_p(f) = \lim_{T \to \infty} \frac{2}{T} E[\tilde{p}_k^*(f, T)\tilde{p}_k(f, T)],
\]

and the cross-spectral density of the square of the pressure and the pressure is defined as

\[
S_{p^2}(f) = C_{p^2}(f) + jQ_{p^2}(f) = \lim_{T \to \infty} \frac{2}{T} E[\tilde{q}_k^*(f, T)\tilde{p}_k(f, T)].
\]

It should be noted that we define \( S_{p^2} = C_{p^2} + jQ_{p^2} \), which, although different from the definition given in Ref. [4], is correct given the fact that we started with \( \tilde{p}^* \tilde{q} \) and not \( \tilde{q}^* \tilde{p} \). Morfey and Howell use \( S_{pp^2} = C_{pp^2} + jQ_{pp^2} \), which is not consistent with the definition in Ref. [4]. Throughout this work we will use the former convention, i.e. \( S_{p^2} \), and assume (contrary to Ref. [4]) that this and \( S_p \) are single-sided quantities.

Applying the limit and expected value operation to Eq. (2.10), Morfey and Howell arrive at

\[
\frac{\partial}{\partial r} (r^2 e^{2ar} S_p) = -\frac{\beta}{\rho_0 c_0^3} \omega r e^{2ar} Q_{p^2}.
\]

Equation (2.15), a “spectral Burgers equation,” and the variant

\[
\frac{\partial}{\partial x} (e^{2ax} S_p) = -\frac{\beta}{\rho_0 c_0^3} \omega e^{2ax} Q_{p^2},
\]

which is valid for plane waves and where \( x \) is propagation distance, form the basis of the current work. Here \( Q_{p^2} \) is the imaginary part of the cross-spectral density of the square of the pressure and the pressure. The imaginary part of any cross-spectral density is also known as the quadranspectral density; thus \( Q_{p^2} \) will be referred to in this work as the QSD. Morfey and Howell call it the “spectral transfer term” for reasons discussed below.
2.2.2 Physical Meaning of the QSD

Because nonlinearity in acoustic propagation is predominantly quadratic, the idea that a term derived from the pressure and its square should represent nonlinearity is a reasonable one. In the case of a single-frequency source, energy is transferred from the fundamental frequency to the zero-frequency component and to the second harmonic component, which has twice the frequency of the fundamental. Since squaring a sine wave results in another sine wave with twice the frequency of the original wave and whose average value is no longer zero, it might be expected that evidence of this energy transfer could be found in the cross-spectral density of the signal and its square (of which the QSD is the imaginary part).

The argument that the QSD is related to the nonlinearity of a signal can also be made by examining Eq. (2.15). The argument of the derivative on the left-hand side is the PSD with the effects of geometrical spreading (spherical in this case) and absorption removed. The only remaining effect that could cause the spatial derivative of this term to be nonzero is nonlinearity. Because the spatial derivative of this “corrected” PSD is proportional to $Q_{pp}$, the QSD must have a direct relationship to the nonlinear evolution of the signal.

In light of the discussion of the evolution of a single-frequency source in Sec. 1.2, the left-hand side of Eq. (2.15) should be negative at the source frequency (where energy is being lost nonlinearly) and positive at the higher harmonic frequencies (where energy is being gained nonlinearly). (It should also be positive at the zero-frequency component, but, as discussed above, this component is usually ignored.) Thus, we expect the QSD to be positive at the source frequency (note the negative sign on the right hand side) and negative at the harmonic frequencies. As shown in Fig. 2.1, this is in fact the case. This figure shows the QSD at the fundamental and harmonic frequencies for a 2000 Hz source propagated numerically to a distance $\sigma = 0.8$ according to the Fubini solution. For noise, we expect the QSD to be negative at low and high frequencies and positive at frequencies in between. This will be discussed later.
Figure 2.1. The quantity $Q_{p^2}$, also known as the quadspectral density, is shown here for a sinusoid propagated numerically to a distance $\sigma = 0.8$. The value of $Q_{p^2}$ is positive at the source frequency, negative at the harmonic frequencies, and zero elsewhere.

2.3 Integration of the QSD and Conservation of Energy

A useful starting point in exploring the utility of the QSD as an indicator of nonlinearity is to examine the case of lossless propagation. For such a case, conservation of energy can serve as a helpful analysis tool. Here we consider just such a case and apply the analysis of Morfey and Howell in order to develop a single-point indicator of nonlinearity.

For a lossless plane travelling wave, the time-averaged intensity at any point in the fluid is given by $I(x) = \frac{p_{\text{rms}}^2(x)}{\rho_0 c_0}$, where $p_{\text{rms}}^2(x)$ is the mean-square acoustic pressure at location $x$. For a signal having a zero mean, the integral of the PSD over frequency yields the mean-square value. Therefore,

$$\int_{0}^{\infty} S_p(x) d\omega = p_{\text{rms}}^2(x),$$

(2.17)
and so

$$I(x) = \int_{0}^{\infty} \frac{S_p(x)}{\rho_0 c_0} d\omega.$$ \hspace{1cm} (2.18)

The intensity is constant for lossless propagation. Thus, its spatial derivative should be equal to zero, i.e.,

$$\frac{\partial I(x)}{\partial x} = \frac{1}{\rho_0 c_0} \frac{\partial}{\partial x} \int_{0}^{\infty} S_p(x) d\omega = 0,$$ \hspace{1cm} (2.19)

and so

$$\frac{\partial}{\partial x} \int_{0}^{\infty} S_p d\omega = 0.$$ \hspace{1cm} (2.20)

This result is, strictly speaking, valid only in lossless, pre-shock propagation. However, we would expect Eqs. (2.19) and (2.20) to hold to a reasonable degree of approximation in cases where losses due to absorption and shocks are small over the propagation distance of interest.

If we remove the restriction on absorption, we have $p(x) = p(0)e^{-\alpha x}$, or $p_{rms}^2(x) = p_{rms}^2(0)e^{-2\alpha x}$. Rearranging gives $p_{rms}^2(0) = p_{rms}^2(x)e^{2\alpha x}$, so we can write

$$\int_{0}^{\infty} S_p(0) d\omega = \int_{0}^{\infty} S_p(x)e^{2\alpha x} d\omega = \text{constant}.$$ \hspace{1cm} (2.21)

Thus,

$$\frac{\partial}{\partial x} \int_{0}^{\infty} e^{2\alpha x} S_p(x) d\omega = 0,$$ \hspace{1cm} (2.22)

which is the integral over frequency of the left-hand side of Eq. 2.16. Integrating the right-hand side in the same manner gives

$$\int_{0}^{\infty} \omega e^{2\alpha x} Qp d\omega = 0.$$ \hspace{1cm} (2.23)
In the lossless case,
\[ \int_{0}^{\infty} \omega Q_{p^2_p} d\omega = 0. \] (2.24)

This last equation yields much simpler and more intuitive results, so it will be used in the development of nonlinearity indicators instead of the lossy version.

As discussed in Sec. 2.2.2, the quantity \( Q_{p^2_p} \) represents the rate of energy transfer between spectral components as a result of nonlinear propagation. Because the primary mechanisms of nonlinear energy transfer are sum-frequency (or harmonic) and difference-frequency generation, we expect to see mid-frequency components lose energy nonlinearly and high and low frequencies gain energy nonlinearly. Thus, \( Q_{p^2_p} \) is positive at frequencies where energy is being lost and negative at frequencies where energy is being gained (note the negative sign on the right-hand side of Eq. (2.15)). Figures 2.2 and 2.3 illustrate this behavior as seen in the PSD and in the QSD.

For lossless, pre-shock propagation, conservation of energy holds, and the net amount of energy lost to nonlinearity must equal the net amount gained through nonlinearity. So, if we define

\[
Q_{SD_{pos}} = \begin{cases} 
Q_{p^2_p} & \text{if } Q_{p^2_p} > 0 \\
0 & \text{if } Q_{p^2_p} \leq 0
\end{cases}
\] (2.25)

and

\[
Q_{SD_{neg}} = \begin{cases} 
Q_{p^2_p} & \text{if } Q_{p^2_p} < 0 \\
0 & \text{if } Q_{p^2_p} \geq 0
\end{cases}
\] (2.26)

we should expect

\[ \int_{0}^{\infty} \omega Q_{p^2_p} d\omega = \int_{0}^{\infty} \omega Q_{SD_{pos}} d\omega + \int_{0}^{\infty} \omega Q_{SD_{neg}} d\omega = 0. \] (2.27)

Or, defining

\[ Q_{pos} = \int_{0}^{\infty} \omega Q_{SD_{pos}} d\omega \] (2.28)
Figure 2.2. A sample PSD for a noise signal propagating in one dimension along with a linear prediction for the same propagation distance. There is a net energy gain due to nonlinearity at low and at high frequencies and a net energy loss due to nonlinearity at mid-frequencies.

Figure 2.3. A sample QSD for a the noise signal from Fig. 2.2 is shown. The QSD is negative at low and at high frequencies and positive at mid-frequencies.
and

\[ Q_{\text{neg}} = - \int_{0}^{\infty} \omega Q S D_{\text{neg}} d\omega, \quad (2.29) \]

\[ Q_{\text{pos}} = Q_{\text{neg}}, \quad (2.30) \]

As discussed above, positive values of \( Q_{p^2} \) typically occur at mid-frequencies (where energy is being lost), and negative values of \( Q_{p^2} \) occur at lower and at higher frequencies (where energy is being gained). Because \( Q_{\text{pos}} \) and \( Q_{\text{neg}} \) are essentially equivalent in the pre-shock region, the same information is contained in both. However, \( Q_{\text{neg}} \) requires a large bandwidth measurement because it typically represents energy gains at higher frequencies. \( Q_{\text{pos}} \) represents energy losses from a limited range of frequencies and can therefore be calculated using a smaller bandwidth measurement. Thus, using \( Q_{\text{pos}} \) in a single-point nonlinearity indicator meets the goal of eliminating the need for large bandwidth measurements.

If the bandwidth of a measurement is not a concern, both \( Q_{\text{pos}} \) and \( Q_{\text{neg}} \) can be used. For such a case it is useful to rewrite Eq. (2.30) as the ratio

\[ \frac{Q_{\text{neg}}}{Q_{\text{pos}}} = 1. \quad (2.31) \]

As with Eqs. (2.27) and (2.30), Eq. (2.31) is only valid in the absence of absorption and shocks. However, if absorption is negligible, this ratio could be useful in determining if shocks are present. (The assumption that atmospheric and boundary layer absorption are negligible compared with losses at the shocks is made in weak shock theory [12], a widely used model in nonlinear acoustics.) The presence of shocks introduces a loss mechanism independent of atmospheric or boundary layer absorption. Blackstock et al. [12] quantify this loss by introducing a finite-amplitude absorption coefficient \( \alpha_f \) to account for the net loss of energy due to nonlinear effects. This could mean that less energy is gained nonlinearly than is lost, or that \( Q_{\text{neg}} < Q_{\text{pos}} \). Thus, we would expect the ratio in Eq. (2.31) to be less than one when shocks are present.
2.4 Nondimensional Forms of the QSD

In the study of jet noise, several different types of measurements are typically available. Most commonly, these are full-scale jet measurements and model-scale jet measurements. While every attempt is usually made to emulate the conditions of full-scale measurements when making model-scale measurements, it remains difficult to make direct comparisons between the two. This difficulty is mainly due to the fact that not all acoustic propagation effects (geometrical spreading, absorption, and nonlinearity) scale the same way. Frequency, and therefore nonlinearity, scales as the inverse of the jet nozzle diameter, while atmospheric absorption has a complicated frequency dependence due to molecular relaxation. Within the scope of this work, both model-scale jet measurements and plane wave tube measurements are discussed, adding to the difficulty of scaling. In order to minimize this difficulty in the current analysis, the nonlinearity indicators developed in this section are normalized so that they are dimensionless (or nearly so); normalization facilitates comparisons between measurement conditions.

2.4.1 Normalized $Q_{pos}$

As discussed above, using $Q_{pos}$ as a nonlinearity indicator eliminates the need for large bandwidth measurements. While this is a useful quality, it is difficult to extract much physical meaning from the value of the quantity itself (which has units of $[\text{Pa}^3/\text{Hz}]$). In order to simplify it, $Q_{pos}$ is divided by $p_{rms}^3$, the cube of the root-mean-square pressure of the signal. This removes the dependence of the value on signal amplitude and leaves an indicator $Q_{pos}/p_{rms}^3$ with units of $[\text{Hz}^{-1}]$. For periodic waveforms (or for narrow band noise), this quantity could be rendered dimensionless with a multiplication by the source frequency (or the source center frequency). However, this would merely scale the results by a constant value, adding little meaning. Further, there is no obvious value (or values) of frequency by which to multiply the indicator for broadband noise data. Thus, we retain the quantity $Q_{pos}/p_{rms}^3$ as a “nearly dimensionless” nonlinearity indicator.

While the above normalization of $Q_{pos}$ is dimensionally simple, its physical meaning still is not very intuitive. It is difficult to draw conclusions about the nonlinear evolution of a signal based solely on a calculated value of $Q_{pos}/p_{rms}^3$. 
Thus, when this indicator is applied to data in the following chapter, it is used to make comparisons between measurements and data sets rather than as an absolute indicator of the importance of nonlinearity.

2.4.2 Spectral Gol’dberg Number

Nonlinearity is but one effect that influences the propagation of an acoustic wave. Meteorological and ground effects, as described in Sec. 1.4, can significantly alter the received signal. Even in a controlled environment where these effects are absent, spreading and absorption can be at least as important as nonlinearity; in fact, both work against nonlinearity in a sense by reducing the amplitude of the propagating wave along with the degree of distortion it undergoes. For an initially sinusoidal signal, the relative importance of nonlinearity and absorption can be gauged with the Gol’dberg number. No parameter exists that performs this function for noise signals or that weighs the importance of geometrical spreading against that of nonlinearity and absorption. Here we derive an indicator that does all of these things.

From Eq. (2.15), repeated here for convenience,

\[
\frac{\partial}{\partial r} (r^2 e^{2\alpha r} S_p) = -\frac{\beta}{\rho_0 c_0^2} \omega r^2 e^{2\alpha r} Q_{p^{2p}}, \quad (2.32)
\]

expand the derivative on the left-hand side and divide both sides by \(e^{2\alpha r}\), giving

\[
2r S_p + r^2 (2\alpha S_p + \frac{\partial S_p}{\partial r}) = -\frac{\omega \beta}{\rho_0 c_0^2} r^2 Q_{p^{2p}}. \quad (2.33)
\]

Divide both sides by \(2\alpha r^2\) and regroup terms on the left-hand side:

\[
\left(\frac{1}{\alpha r} + 1\right)S_p + \frac{1}{2\alpha} \frac{\partial S_p}{\partial r} = -\frac{\omega \beta}{2\alpha \rho_0 c_0^3} Q_{p^{2p}}. \quad (2.34)
\]

Now divide both sides by \(-S_p\) and use the identity

\[
\frac{1}{g} \frac{\partial g}{\partial r} = \frac{\partial \ln g}{\partial r} \quad (2.35)
\]
to rewrite as
\[ -\frac{1}{2\alpha} \frac{\partial (\ln S_p)}{\partial r} = \frac{1}{\alpha r} + 1 + \frac{\omega \beta}{2\alpha \rho_0 c_0^3 \bar{Q}_p} \frac{Q_p}{S_p}. \] (2.36)

Equation (2.36) describes the evolution of the logarithm of the PSD (which is proportional to the decibel level of the spectral components) relative to absorption. In the absence of geometrical spreading and nonlinearity the left-hand side of Eq. (2.36) is equal to unity, or
\[ \frac{\partial (\ln S_p)}{\partial r} = -2\alpha, \] (2.37)
which describes the change in the PSD due to linear absorption. The relative importance of geometrical spreading and nonlinearity compared to absorption are contained in the first and third terms on the right-hand side of Eq. (2.36), respectively. If $1/\alpha r$ is small (large) compared to one, spreading has a relatively small (large) influence on the rate of change of the PSD. Likewise for nonlinearity. If the third term is small (large) compared to one, nonlinearity has a relatively small (large) influence on the rate of change of the PSD.

As outlined in Sec. 1.2.5, a Gol’dberg number greater than one indicates that nonlinearity dominates absorption, and a Gol’dberg number less than one means that absorption dominates nonlinearity. The same logic can be applied to the nonlinear term, the last term on the right-hand side of Eq. (2.36). As a result, this term will be referred to as the *spectral Gol’dberg number*, $\Gamma_s$. Specifically,
\[ \Gamma_s = \frac{\omega \beta}{2\alpha \rho_0 c_0^3} \frac{Q_p}{S_p}. \] (2.38)

Recall that for initially sinusoidal plane waves of amplitude $P_0$
\[ \Gamma = \frac{1}{\alpha \bar{x}}, \] (2.39)
where
\[ \bar{x} = \frac{\rho_0 c_0^3}{\beta P_0 \omega}. \] (2.40)
Comparing the expressions for $\Gamma$ and $\Gamma_s$, it is seen that the quantity $Q_p/2S_p$ plays the role of $P_0$ for the individual spectral components of a noise signal.
Γ_s is useful not only because can be used to determine the relative importance of absorption and nonlinear evolution at any frequency and propagation distance, but because it is dimensionless and therefore can be used to compare different measurements with relative ease. One other significant advantage of Γ_s is its ability to indicate the direction of nonlinear energy transfer at any given frequency. On the right-hand side of Eq. (2.38), all the quantities that act as coefficients to $Q_{p^2_p}$, as well as the PSD $S_p$, are strictly positive. This leaves the QSD itself to determine the sign of the indicator. Equation (2.15) and discussion in Sec. 2.2.2 establish that positive values of the QSD indicate nonlinear energy loss and negative QSD values indicate nonlinear energy gain. Because the sign of the QSD is preserved in the spectral Gol’dberg number, we can assign the same meaning to the sign of Γ_s.

It should be noted that a result similar to Eq. (2.36) can be obtained for a plane wave:

$$-\frac{1}{2\alpha} \frac{\partial (\ln S_p)}{\partial x} = 1 + \frac{\omega \beta}{2\alpha \rho_0 c_0^3} Q_{p^2_p} S_p.$$

(2.41)

Because a plane wave does not experience geometrical spreading, the spreading term is absent from Eq. (2.41).

A final observation about the spectral Gol’dberg number indicator comes from examining Eqs. (2.41) and (2.36). The left-hand side of each of these equations contains the spatial derivative of the PSD. Thus, like $Q_{p^2_p}$ itself, we can expect Γ_s to be a measure of the rate of nonlinear evolution of a signal at the measurement location.

2.4.3 Coherence Indicator

We have already established that the QSD is descriptive of the nonlinear evolution of a signal (see Sec. 2.2.2), and we have argued in this section that a dimensionless nonlinearity indicator is desirable. Here we develop another dimensionless form of the QSD.

$Q_{p^2_p}$ can be normalized by calculating a kind of coherence function. According to Refs. [4] and [6], the coherence function between two signals is defined by

$$\gamma^2_{xy}(f) = \frac{|S_{xy}(f)|^2}{S_{xx}(f)S_{yy}(f)}, \quad 0 \leq \gamma^2_{xy}(f) \leq 1.$$

(2.42)
The quantity used in this work as a nonlinearity indicator is not exactly the same as the quantity defined in Eq. (2.42). We will begin by using \((Q_{p^2})^2\) in the numerator rather than \(|S_{p^2p}|^2\). First define

\[
\gamma_{Q}^2(f) = \frac{(Q_{p^2p}(f))^2}{S_{p^2p}(f)S_p(f)}, \quad 0 \leq \gamma_{Q}^2(f) \leq 1 (2.43)
\]

where \(S_{p^2}\) is the autospectral density of the square of the pressure. Like the general coherence function \(\gamma_{xy}^2(f)\), the quantity \(\gamma_{Q}^2(f)\) is bounded between zero and one and could itself be used as a nonlinearity indicator. However, this would eliminate one of the main advantages of the spectral Gol’dberg number indicator, which is the ability to specify the direction of nonlinear energy transfer through the preservation of the sign of \(Q_{p^2p}\). A simple way of incorporating this beneficial quality into the coherence indicator is simply to take the square root of Eq. (2.43). This can be done without ambiguity because the autospectra in the denominator are both positive quantities. Thus, the coherence indicator is

\[
\gamma_Q = \frac{Q_{p^2p}}{\sqrt{S_{p^2}S_p}}, \quad (2.44)
\]

where the frequency dependence of all the quantities involved has been omitted for simplicity of notation. Again, because the autospectra in the denominator are both positive quantities, the sign of \(\gamma_Q\) is determined by the sign of \(Q_{p^2p}\), and the reasoning from Sec. 5.2.3 can be applied here in asserting that \(\gamma_Q\) indicates the direction of nonlinear energy transfer. Because \(0 \leq \gamma_{Q}^2 \leq 1\),

\[
-1 \leq \gamma_Q \leq 1. \quad (2.45)
\]

Thus, the coherence indicator \(\gamma_Q\) is a bounded nonlinearity indicator whose sign specifies the direction of nonlinear energy transfer.

Inspection of Eq. (2.44) reveals a significant difference between the coherence indicator and the spectral Gol’dberg number. Whereas the spectral Gol’dberg number is related to the spatial evolution of the PSD of the signal, the coherence indicator is simply proportional to the QSD and thus is a measure of the phase coherence between the signal and its square at the measurement location. Any
and all phase coherence (or quadratic phase coupling) present in the signal at the measurement location is identified by the coherence indicator. Because quadratic phase coupling becomes more and more pronounced as a signal continues to undergo nonlinear distortion, $\gamma_Q$ is a measure of the cumulative effects of nonlinear evolution from the source to the measurement location rather than an instantaneous measure of nonlinear distortion.

2.5 Chapter Summary

This chapter has expanded upon the analysis of Morfey and Howell to develop several candidate nonlinearity indicators. Quantities involving $Q_{pos}$ offer the promise of a low-bandwidth indicator. The spectral Gol’dberg number $\Gamma_s$ is a broadly applicable quantity that will be used to compare the significance of linear and nonlinear processes at the measurement location and to indicate the direction of nonlinear energy transfer. The coherence indicator $\gamma_Q$ is a bounded quantity that also retains the ability to indicate the direction of nonlinear energy transfer.

In Chap. 3, two sets of experimental data are described. Chap. 4 presents the results of applying the candidate indicators to the experimental data.
Design and Setup of Experiments

3.1 Introduction

Two sets of experiments were performed to assess the utility of the indicators discussed in the previous chapter in predicting the importance of nonlinearity in the propagation of high-amplitude noise from jets. The first used a plane wave tube constructed at Penn State, and the second was done in the model-scale jet facility at the National Center for Physical Acoustics at the University of Mississippi. This chapter describes the setup of both of these experiments and presents some sample data from the plane wave tube. In order to verify that propagation in the plane wave tube is planar and that all loss mechanisms are accounted for, these sample data are compared with numerically-generated data from the arbitrary wave Anderson algorithm (AWAA).

3.2 Plane Wave Tube

3.2.1 Advantages of Using a Plane Wave Tube

As was discussed in Chapter 1, making acoustic measurements in an outdoor setting presents a host of challenges. The inhomogeneity of the atmosphere, the presence of a finite impedance ground surface, and the uncertainty of the source geometry contribute to the complexity of data acquisition and analysis. The use of a plane wave tube (also called a shock tube or waveguide) eliminates all of these
Figure 3.1. This schematic shows the setup of the plane wave tube along with dimensions, microphone locations, and the shape of the anechoic termination. Two drivers are pictured here; for measurements made with only one driver, the “tee” joint was removed and the driver installed directly onto the end of the tube.

...factors, allowing for much more predictable measurements. It also affords control of the source waveform. A plane wave tube offers an additional advantage when studying nonlinear propagation: spherical spreading “losses” are eliminated, allowing shock formation to happen more easily. Admittedly, propagation in a plane wave tube is simplistic when compared with jet noise, but its simplicity allows for better study of the phenomena of interest.

3.2.2 Specifications

The plane wave tube used in these experiments was constructed primarily of four 3.05 m (10 ft.) lengths of PVC pipe with an inner diameter of 5.21 cm (2.05 in.). The lengths were joined with straight couplers. One end contained an anechoic termination, described below, and the other was fitted with a “tee” joint that allowed two compression drivers to be installed as the sound source. A diagram of the plane wave tube can be found in Fig. 3.1.

All single-frequency source data were taken with a pair of JBL 2426H compression drivers, and all noise data were taken with a single JBL 2426H compression driver fitted directly to the end of the first length of pipe (without the “tee” joint).
Figure 3.2. This plot shows the frequency response of a Brüel & Kjær 1/4” type 4938 pressure-field microphone whose grid cap has been removed.

The frequency response of the drivers is nearly flat between about 300 Hz and 5000 Hz and does not drop off very sharply outside that range. See Appendix A for manufacturer specifications.

The last 2.52 m of the tube (nearest the end cap) contained a fiberglass anechoic termination consisting of a 1-m section in which the density of the fiberglass tapered linearly from zero to 28 kg/m$^3$ and a 1.52-m section of constant 28 kg/m$^3$ density.

Brüel & Kjær 1/4” type 4938 microphones were placed 0.10 m, 3.25 m, 6.40 m, and 9.55 m from the joint between the “tee” connector and the first section of pipe, respectively. A plot of the frequency response of one of these microphones can be found in Fig. 3.2. Because the source location was difficult to define accurately (especially given that the number and configuration of the compression drivers varied), the microphone at 0.10 m was used primarily as a reference microphone to monitor the source conditions; propagation effects were investigated using the other three microphones. The grid caps were removed from all four microphones, and they were fitted inside specially made PVC sleeves with an inside diameter to match the microphone and preamplifier and an outside diameter of 5/8 in. (16 mm). The sleeves, each with two o-rings around its exterior, were then inserted into ports with cylindrical holes having a diameter just large enough to accommodate the sleeve and ensure an airtight fit. The microphone-sleeve assembly was mounted
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Figure 3.3. This figure shows a photograph of a microphone sleeve from the plane wave tube. The diaphragm of the microphone was approximately flush with the curved surface at the bottom of the picture.

at a depth such that the end of the sleeve and the microphone diaphragm were as flush as possible with the interior tube wall. Photographs of the microphone sleeves appear in Figs. 3.3 and 3.4, and further description can be found in Sec. 3.2.3.2.

3.2.3 Validation

3.2.3.1 Dissipation

Comparisons of initial experiments with predictions from the AWAA (described in Sec. 1.2.6.2) revealed that propagation in the shock tube was largely as expected. The frequency content, phasing, and, to a large extent, amplitudes of the measured and predicted waveforms were very similar for both linear and nonlinear propagation. In all cases, however, there was a small discrepancy in the received amplitudes, with the measured amplitude slightly smaller than the predicted amplitude.

Because the algorithm included the effects of atmospheric and boundary layer absorption and dispersion as well as nonlinear effects, the predicted amplitude was assumed to be correct. The discrepancy between measurement and prediction was
attributed to “excess” losses in the tube which were not modeled in the algorithm. Such losses most likely occurred near the tube joints and microphone ports; thus, improvements were made to the design of these components. The joints between sections of tube were sealed with high vacuum grease and fastened securely so as to prevent slippage. The joints of the “tee” fitting (drivers to “tee,” “tee” to first section of tubing) were also sealed with vacuum grease. Finally, each microphone preamplifier was wound with a small amount of teflon plumbers’ tape just behind the diaphragm to ensure an airtight fit in the microphone sleeve. These alterations resulted in excellent agreement between theory and experiment, as can be seen in Figs. 3.5–3.9. Figures 3.5 and 3.6 show comparisons for a signal whose source had a relatively low sound pressure level (SPL) of 105 dB re 20 $\mu$Pa, meaning that the propagation could be described adequately with linear theory (throughout this work, quantities with units of dB are assumed to be referenced to 20 $\mu$Pa). Figures 3.8 and 3.9 show comparisons for a nonlinearly propagating signal whose source SPL was 145 dB.

Figure 3.7 compares the measured value of the PSD with the predicted value for the harmonic components that are above the noise floor. The error in the measurement of the first harmonic (source frequency) is very small, about 3%. The
Figure 3.5. This plot shows measured and predicted waveforms for a linearly propagated signal after improvements were made to the tube to reduce excess losses. The source signal for this case had a frequency of 2500 Hz and an SPL of 105 dB. While the measurement and prediction do not match exactly, the agreement is very good.

Figure 3.6. This plot shows the measured and predicted PSD’s for the waveforms in Fig. 3.5. Values for the first few harmonics (those above the noise floor) show excellent agreement.
second and third harmonics have larger percent errors, but these are not cause for concern, as the second harmonic is approximately 45 dB below the fundamental, and the third harmonic is about 40 dB below the second. For such small spectral levels, the degree of agreement seen here is acceptable.

Figure 3.10 shows the error in the measured values of the PSD relative to the AWAA prediction. Agreement is generally good, with errors of about 15% or less for the first nine harmonics. Larger error percentages occur for harmonics whose levels are more than 25 dB below the level of the fundamental.
Figure 3.8. This plot shows measured and predicted waveforms for a nonlinearly propagated signal after improvements were made to the tube to reduce excess losses. The source signal for this case had a frequency of 2000 Hz and an SPL of 145 dB. There is a higher frequency disturbance present in the measured waveform that does not appear in the prediction, but overall the agreement is very good.

Figure 3.9. This plot shows the measured and predicted PSD’s for the waveforms in Fig. 3.8. Values are in excellent agreement for the first several harmonics.
3.2.3.2 Scattering Effects

Initial measurements in the plane wave tube revealed significant deviations from expected levels for frequency components in the 15–18 kHz range. This was visible not only in the received spectra, an example of which can be seen in Fig. 3.11, but in the received waveforms, as shown in Fig. 3.12. It was postulated that these deviations were due to scattering from the microphone sleeves. Calculations revealed that the radius of the microphone sleeve was roughly equal to a half wavelength of an 18 kHz signal, indicating that scattering likely caused the discrepancies.

To reduce the effects of scattering, the ends of the microphone sleeves were milled to match the curvature of the inside of the tube, and their lengths were adjusted to match closely the depths of the microphone ports. Upon re-assembly, the microphone diaphragms were positioned flush with the ends of the sleeves (or as flush as possible given that the diaphragms are flat and the surface of the sleeves curved), and the ends of the sleeves were mounted flush with the interior wall of the tube. Figs. 3.3 and 3.4 contain photos of a microphone sleeve and port.

Subsequent measurements showed significant improvements in the smoothness of the waveforms and spectra in the 15–18 kHz range. An example measurement using source conditions similar to those from Figs. 3.11 and 3.12 is shown in Figs. 3.13 and 3.14.

**Figure 3.10.** This figure displays the error in the measured values from Fig. 3.9.
Figure 3.11. A spectrum of a signal propagated in the plane wave tube before improvements were made to reduce the effects of scattering. Source conditions for this signal are 3000 Hz and 148.6 dB. The progression of harmonic levels is relatively smooth until the 6th harmonic at 18 kHz.

Figure 3.12. A waveform of a signal propagated in the plane wave tube before improvements were made to reduce the effects of scattering. Source conditions for this signal are 3000 Hz and 148.6 dB. The 18 kHz disturbance is clearly visible in the waveform.
Figure 3.13. A spectrum of a signal propagated in the plane wave tube after improvements were made to reduce the effects of scattering. Source conditions for this signal are 3000 Hz and 134 dB. The progression of harmonic levels is very smooth, even through the 6th harmonic at 18 kHz.

Figure 3.14. A waveform of a signal propagated in the plane wave tube after improvements were made to reduce the effects of scattering. Source conditions for this signal are 3000 Hz and 134 dB. This waveform is free of the 18 kHz disturbance seen in Fig. 3.12.
3.2.3.3 Plane Wave Cutoff Frequency

To ensure that wave propagation in a waveguide is purely planar, all source frequencies must be below the *plane wave cutoff frequency*. Also called the *cut-on frequency* for the first cross mode (the (1,1) mode for a tube of circular cross-section), this is the frequency above which nonplanar modes can propagate. Keeping source frequencies below the cutoff frequency causes these modes to be *evanescent*, meaning that they decay exponentially with distance and are of little importance for typical propagation distances of interest. It should be noted that we expect nonlinearly-generated harmonics and frequency components with frequencies above cutoff to be present in the shock tube; see the discussion at the end of this section on the planar nature of these components.

For a circular waveguide such as the plane wave tube used here, the plane wave cutoff frequency is given by [29]

\[ f_{11} = \frac{0.92c}{\pi a}, \]  

(3.1)

where \(c\) is the equilibrium sound speed and \(a\) the radius of the tube. For the plane wave tube used in these experiments, \(f_{11} = 3856\) Hz.

Experiments were carried out to verify this value in the plane wave tube. A single frequency source was supplied to the drivers, and the amplitude of the received signal at the first microphone was monitored using an oscilloscope. The power supplied to the drivers was also monitored. The power required to produce a signal with a root-mean-square pressure of 1 Pa (SPL = 94 dB) at the first microphone was nearly constant for frequencies from 300 Hz to 3400 Hz. Maintaining this amplitude at 3500 Hz required a 14-fold increase in power, and another 7-fold increase was needed at 3600 Hz (note that the response of the drivers is nearly flat over this frequency range). While these frequencies are somewhat below the calculated value of 3856 Hz, source frequencies for all measurements were kept to 3400 Hz and below to ensure that no cross-modes were present.

Measurements at the other three microphones reveal that the plane wave cutoff frequency was very near 3900 Hz. Figure 3.15 shows the ratio of the measured pressure amplitude at each of these microphones to the predicted pressure amplitude. The prediction was made by starting with the measured pressure at the
Figure 3.15. The ratio of measured pressure to linearly predicted pressure is shown for a frequency range encompassing all source frequencies used in subsequent experiments.

first microphone, assuming planar, linear propagation (a reasonable assumption for a source level of 94 dB), and applying the effects of atmospheric and boundary layer absorption. The ratio of the measured and predicted pressures is very close to unity for most of the frequency range shown. The largest deviation occurs at 3000 Hz, very near to the theoretically calculated plane wave cutoff frequency. Some deviation is seen at the lower frequencies, as well. While the cause of this discrepancy is not clear, all source frequencies used in subsequent experiments were greater than or equal to 200 Hz.

Even though source frequencies are kept within the realm of plane wave propagation, many of the nonlinearly generated frequency components seen in the subsequent experiments have frequencies well above the threshold discussed above. One might expect that the presence of these frequency components would excite cross-modes and destroy the planar nature of the propagation. However, theory [24] and practice [40, 39, 48] find that frequency components generated nonlinearly by a plane wave in a waveguide remain planar even if they exceed the plane wave cutoff frequency.
3.2.3.4 Anechoic Termination

To determine the reflection coefficient provided by the fiberglass anechoic termination, a short pulse was propagated down the tube and measured before and after it was reflected from the termination. The pulse had a frequency range of 200 Hz to 3500 Hz (source signals for all subsequent measurements fall within this frequency range), a sound pressure level of 95 dB re 20 $\mu$Pa to assure linear propagation, and a duration of 9 ms that allowed the reflected pulse to be observed without interference from the incident pulse at the first three microphones. Sample incident and reflected waveforms at the third microphone location (6.40 m from the source, 3.26 m from the termination) can be seen in Fig. 3.16, and sample spectra for the same conditions are in Fig. 3.17.

Once the pulse was recorded upon incidence and reflection at each of the first three microphones, its power spectral density was calculated and, in the case of the reflected pulse, corrected for the atmospheric and boundary layer absorption effects incurred over the round trip from the microphone to the termination. The resulting reflection coefficient was averaged over the three microphones and reported as a loss in dB. For all frequencies, the reflection loss was at least 28 dB, and for frequencies of 500 Hz and greater it was at least 35.5 dB. The average over the entire bandwidth of the pulse was 43.6 dB.

3.2.4 Data Acquisition System

For the single-frequency source data, an Agilent 33120A 15 MHz Function/Arbitrary Waveform Generator was set to the desired source frequency and amplified with a Techron 5507 Power Supply Amplifier. This signal was monitored using a Hewlett-Packard 3478A Multimeter and then sent directly to the compression drivers. For the noise data, a numerically generated and filtered noise signal was played using the sound card of a laptop computer. This signal was amplified in turn by a Stanford Research Systems Model SR560 Low-Noise Preamplifier and by the Techron amplifier before being passed through the multimeter and to the drivers.

Signals from the four microphones were passed through a 4-channel Brüel & Kjær Nexus Conditioning Amplifier which filtered the data from 0.1 Hz to 100 kHz. Gain settings ranging from 10 mV/Pa to 100 mV/Pa were used depending upon the
Figure 3.16. Incident and reflected waveforms are shown here as recorded at the third microphone location (6.40 m from the source, 3.26 m from the termination). The source was a 9 ms pulse whose amplitude was 95 dB and whose frequency range was 200 Hz to 3500 Hz. Note the difference in the scale of the vertical axes of the plots.

Figure 3.17. Incident and reflected power spectral densities are shown here as recorded at the third microphone location (6.40 m from the source, 3.26 m from the termination). The source was a 9 ms pulse whose amplitude was 95 dB and whose frequency range was 200 Hz to 3500 Hz.
Figure 3.18. This schematic represents the data acquisition system used in conjunction with the plane wave tube. Elements depicted above the tube were used in the generation and amplification of the source, and elements depicted below were used in the conditioning and capture of the received signal. On the source side, the laptop computer and Stanford preamplifier were used for noise sources; the Agilent waveform generator was used for sinusoidal sources.

Amplitude of the acoustic signal. The filtered, amplified data were then received by a TEAC GX-1 Integrated Recorder operating at a 200 kHz sample rate with a low-pass filter at 80 kHz. The final signal was recorded to the hard drive of a laptop computer connected to the recorder. Calibration signals were acquired using the same system along with a Brüel & Kjær type 4228 pistonphone. A schematic of the entire data acquisition system can be found in Fig. 3.18.

3.3 Model-Scale Jet Measurements

3.3.1 Model-Scale Jet Facility

Acoustic measurements were made in the model-scale jet facility in the National Center for Physical Acoustics at the University of Mississippi. The facility consists of a main anechoic chamber treated with fiberglass wedges, an upstream room plenum leading to the jet nozzle exit in the main room, and a 4 ft. by 4 ft. (1.22 m by 1.22 m) acoustically treated exhaust duct from the main room to a downstream room plenum with a powered exhaust fan. The main room measures 19 ft. (5.79 m)
Figure 3.19. A typical frequency response curve is shown here for a Bruel & Kjaer 1/4” type 4939 free-field microphone.

Long by 20 ft. (6.10 m) wide by 8 ft. (2.44 m) high from wedge tip to wedge tip and has an absorption coefficient of at least 0.99 above 200 Hz. More detailed information can be found in Ref. [43], and a photograph of the chamber can be seen in Fig. 3.21.

3.3.2 Measurement Apparatus

3.3.2.1 Microphone Boom

Eight Bruel & Kjaer microphones were used to record the scale-model data; all were mounted with their diaphragms at the same height as the jet nozzle. Two of these were 1/4” type 4939 free-field microphones with type 2670 pre-amplifiers and were mounted at a distance of 80 jet diameters from the center of the approximated source location and at angles of 90° and 150° from the jet inlet axis, respectively. These microphones were mounted at normal incidence and with their grid caps removed. A plot of the frequency response of these microphones can be found in Fig. 3.19.

The other six microphones were mounted on a rotatable microphone boom spanning a 75-jet-diameter radius. These pressure-field microphones were mounted at grazing incidence and with their grid caps in place. Bruel & Kjaer 1/4” type
Figure 3.20. A typical frequency response curve is shown here for a Brüel & Kjær 1/8” type 4138 pressure-field microphone.

4938 microphones were located at 10, 20, 40, and 60 jet diameters from the source, respectively, and 1/8” type 4138 microphones were mounted at 30 and 75 jet diameters from the source, respectively. Plots of the frequency responses for these microphones can be found in Figs. 3.2 and 3.20, respectively.

The microphone boom was affixed to a rotary motor and thus could be positioned at any azimuthal angle. Data were recorded in five-degree increments for angles from 80° to 150° relative to the inlet axis. The boom was constructed primarily of 25.4 mm (1 in.) diameter aluminum tubing. In order to minimize reflections, it was wrapped in fiberglass insulation during the measurements. Figure 3.21 shows the microphones mounted on the boom before it was wrapped in fiberglass.

3.3.2.2 Data Acquisition System

The signals acquired by the microphones were amplified using two 4-channel Brüel & Kjær Nexus conditioning amplifiers with a bandpass filter from 0.1 Hz to 100 kHz. Gain settings for the amplifiers were adjusted as needed for each run and ranged from 316µV/Pa to 100 mV/Pa. The amplified signals were then received by a Motu Model 896 recorder with 8 channel differential inputs and were simultaneously sampled with individual 24 bit, 192 kHz A/D converters. The re-
Figure 3.21. The rotatable microphone boom used in the model-scale jet measurements is seen here with all six microphones attached but without the fiberglass insulation that was used to minimize reflections. The rotary motor is visible in the lower left portion of the figure, and the jet nozzle can be seen near the left side of the figure.

Resulting sampled signals were saved to a hard drive for later analysis along with calibration data obtained using a Bruel & Kjaer type 4228 pistonphone.

3.3.3 Jet Conditions and Data Points

Two different 34.9 mm (1.375 in.) jet nozzles were used at several different jet Mach numbers. For all conditions, the jet was neither heated nor heat-simulated. A convergent nozzle was used at an ideally-expanded Mach 0.85 condition, and a convergent-divergent nozzle was operated at an ideally-expanded Mach 2 condition, an over-expanded Mach 1.8 condition, and an over-expanded Mach 1.69 condition. An ideally-expanded jet has equal pressures inside and outside the nozzle. For an over-expanded jet, the pressure inside the nozzle is less than the ambient pressure outside the nozzle.
For each of these conditions, measurements were made with two setups: one with the axis of the microphone boom centered on the jet nozzle exit and one with the boom centered 14 cm (5.5 in. or 4 jet diameters) downstream of the nozzle exit, a location thought to be a more accurate approximation of the location of the center of the acoustic source (as explained in Sec. 1.1.3, the acoustic source is extended and tends to be centered several jet diameters downstream of the nozzle exit). To verify this assumption, the Mach 0.85 data were chosen for examination because of their lower acoustic amplitudes which allow for the use of linear theory. Extrapolating data obtained at smaller propagation distances to larger distances and then comparing with measured data at these distances revealed that atmospheric absorption and spherical spreading were sufficient to describe the propagation, suggesting that the chosen location for the origin of the microphone array (the axis about which the microphone boom rotated) corresponded with the center of the acoustic source. Thus, only the set of measurements with the microphone boom axis centered downstream of the nozzle exit are used in the present work.

3.4 Data Processing

For all periodic data (experimental, numerical, and theoretical data with a single-frequency source), spectral quantities such as the PSD were calculated using a standard ensemble-averaging procedure [4]. Each time series (pressure or squared pressure) was first truncated so that its length was a power of two and then broken into smaller “blocks” of \( n_s = 2^{13} \) samples each. The blocks had 50% overlap, meaning that the first block contained samples \( (1 : n_s) \), the second contained samples \( (n_s/2 + 1 : 3n_s/2) \), and so on. A Hanning window of length \( n_s \) was applied to each block; each windowed block was then transformed to the frequency domain using the fast Fourier transform (\text{fft}) function in MATLAB. Each transformed block was then truncated to \( n_s/2 \) points and multiplied by two to make it single-sided. To compute \( S_p \) (the PSD) or \( S_{p^2} \), each single-sided transform was multiplied by its complex conjugate, and the blocks were then averaged; for \( S_{p^2} \), the spectrum of the squared pressure was multiplied by the complex conjugate of the spectrum of the pressure before averaging. The resulting averaged spectrum was then scaled (divided) by the transform length \( n_s \), the sample frequency \( f_s = 200 \) kHz, and
the equivalent noise bandwidth scale factor (or mean-square value) of the Hanning window.

Noise data were treated somewhat differently. Instead of ensemble averaging, these data were processed using a frequency averaging procedure (see p. 437–439 of Ref. [5]). Each signal was truncated to a power-of-two length $N_r$ and windowed with a Hanning window of length $N_r$. A Fourier transform (again the $\text{fft}$ function in MATLAB) was applied to the entire record, and a “raw” single-sided spectrum was found by multiplying the result by two and truncating to $N_r/2$ samples. The narrow-band autospectrum was found, as above, by multiplying a spectrum by its conjugate (or, for the cross-spectrum, by multiplying the spectrum of the squared pressure by the conjugate of the spectrum of the pressure) and scaling by the transform length $N_r$, the sample frequency $f_s = 192$ kHz or 200 kHz, and the equivalent noise bandwidth scale factor of the Hanning window. The result was broken into smaller consecutive frequency “bands,” each with $n_d = 2^9$ points, and the average value of each band found. The same spectral resolution can be achieved with this method as with ensemble averaging, but frequency averaging generally results in cleaner spectra and higher signal-to-noise ratios. However, the bin center frequencies are different: rather than $[0, \Delta f, 2\Delta f, 3\Delta f, \ldots]$, they are now $[\Delta f/2, 3\Delta f/2, 5\Delta f/2, 7\Delta f/2, \ldots]$. This makes the representation of harmonics difficult and is the reason frequency averaging was not used for the periodic data.

### 3.5 Chapter Summary

This chapter has described the two sets of experiments that were undertaken for this work. Single-frequency and noise data were taken in a plane wave tube for a range of source amplitudes and frequencies. Model-scale jet data were acquired in an anechoic facility for both low and high mach number jets; microphones located on a rotatable microphone boom were capable of measuring waveforms at a range of propagation distances and angles. All data were processed using either ensemble averaging or frequency averaging.

In Chap. 4, the candidate nonlinearity indicators developed in Chap. 2 are applied to the data described in this chapter.
4.1 Introduction

This chapter presents results of applying the nonlinearity indicators developed in Chap. 2 to the data obtained as described in Chap. 3. The results are organized by data type, with single-frequency and noise data from the plane wave tube followed by model-scale jet noise.

4.2 Plane Wave Tube Data

4.2.1 Single-Frequency Source Data

Table 4.1 shows the source conditions used to generate initially sinusoidal waves in the plane wave tube. The amplitudes used in the measurements were within ±1 dB of the nominal values listed in the table. Figures 4.1 through 4.4 show sample waveforms and PSD’s for a 105 dB source and a 145 dB source.

The range of source frequencies was chosen to ensure plane wave propagation and adequate response from the compression drivers (two JBL 2426H drivers were used for these data; see Sec. 3.2.3 for more information on source frequencies and App. A for driver specifications). The 105 dB data generally approximate linear propagation, and the 145 dB data represent the maximum amplitude that could be achieved without significant distortion at the source or damage to the drivers.
Figure 4.1. The waveform as recorded at the first microphone in the shock tube is shown for a source amplitude of 105 dB and a source frequency of 2000 Hz.

Figure 4.2. The power spectral density (at source and harmonic frequencies only) is shown for the waveform in Fig. 4.1. The black curve represents background and system noise.
Figure 4.3. The waveform as recorded at the first microphone in the shock tube is shown for a source amplitude of 145 dB and a source frequency of 2000 Hz.

Figure 4.4. The power spectral density (at source and harmonic frequencies only) is shown for the waveform in Fig. 4.3. The black curve represents background and system noise.
Table 4.1. Source conditions used to generate initially sinusoidal waves in the plane wave tube.

<table>
<thead>
<tr>
<th>Amplitude (dB re 20µ Pa)</th>
<th>Frequency (Hz)</th>
<th>500</th>
<th>1000</th>
<th>1500</th>
<th>2000</th>
<th>2500</th>
<th>3000</th>
</tr>
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<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
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<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>135</td>
<td></td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>145</td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>

4.2.1.1 Ratio of $Q_{neg}$ to $Q_{pos}$

As was stated in Chap. 2, $Q_{neg}$ and $Q_{pos}$ are calculated by integrating over the negative and positive values, respectively, of $\omega Q_{p^2}$. The quantities are defined as

$$Q_{pos} = \int_{0}^{\infty} \omega Q_{SD_{pos}} d\omega \quad \text{and} \quad Q_{neg} = -\int_{0}^{\infty} \omega Q_{SD_{neg}} d\omega,$$

(4.1)

where

$$Q_{SD_{pos}} = \begin{cases} Q_{p^2} & \text{if } Q_{p^2} > 0 \\ 0 & \text{if } Q_{p^2} \leq 0 \end{cases}$$

(4.2)

and

$$Q_{SD_{neg}} = \begin{cases} Q_{p^2} & \text{if } Q_{p^2} < 0 \\ 0 & \text{if } Q_{p^2} \geq 0 \end{cases}.$$  

(4.3)

An example of the quantity $\omega Q_{p^2}$ is shown in Fig. 4.5 for the initial waveform shown in Fig. 4.3 propagated to a distance of 9.55 m. As expected, $\omega Q_{p^2}$ is positive at the source frequency and negative at the harmonics, indicating that energy is being transferred nonlinearly from the source frequency to the harmonics.

The quantity $Q_{neg}/Q_{pos}$ was calculated for all the waveforms acquired using the source conditions in Table 4.1. Results are shown in Fig. 4.6. Each point on the figure represents a measurement at a single location and for a single set of source conditions. The points are plotted as a function of normalized distance; this allows all of the data to be included on the same axes. The value of the indicator is within 0.5% of unity in the pre-shock region with the exception of one data point, and most points deviate from one by 0.1% or less. However, no significant change in
Figure 4.5. The quantity $\omega Q_{p^2}^2$ is shown here for the source waveform in Fig. 4.3 at a propagation distance of 9.55 m. $Q_{neg}$ and $Q_{pos}$ are calculated by integrating over the negative and positive regions, respectively, of this function.

value is observed for $\sigma > 1$. This is contrary to our expectation that the value of $Q_{neg}/Q_{pos}$ should be less than one when shocks are present, suggesting either that such an assumption is incorrect or that the drop in the value of the indicator for these data is too small to be noticed.

The scatter in the data near $\sigma = 10^{-4}$ is seen only in the measured data and the AWAA predictions that use measured signals as inputs; plots of $Q_{neg}/Q_{pos}$ for Blackstock “bridging function” (BBF) or Mendousse predictions (which assume a perfectly sinusoidal initial waveform) have no scatter at all. The scatter occurs for data with a 105 dB source amplitude or which a linear propagation model is sufficient. Very little nonlinear energy transfer occurs in these waveforms, meaning that $Q_{neg}$ and $Q_{pos}$ are small, especially near the source (the largest deviations from one in Fig. 4.6 occur at the smallest values of $\sigma$, which corresponds to the first microphone at 0.10 m). In taking the ratio of two small numbers, the error due to noise is magnified, resulting in the deviations from unity seen in the figure.

Figure 4.7 contains the same analysis applied to numerical predictions made using the Arbitrary Wave Anderson Algorithm. The measured waveform at the first microphone was used as the input to the algorithm, and predicted waveforms
Figure 4.6. The nonlinearity indicator $Q_{\text{neg}}/Q_{\text{pos}}$ is plotted here for all the waveforms whose source conditions are listed in Table 4.1. Each point represents a measurement for a single microphone location and set of source conditions. The independent variable $\sigma$ is propagation distance normalized by shock formation distance.

were saved at each of the subsequent microphone locations. This plot displays the same behavior as Fig. 4.6 for the interval $0 < \sigma < 1$, but it also shows a decrease in the value of the indicator for $\sigma > 1$. Based on the analysis that resulted in the $Q_{\text{neg}}/Q_{\text{pos}}$ indicator, this is to be expected after shocks form. The disagreement between Figs. 4.6 and 4.7 in this region is surprising given the agreement between measured and predicted waveforms and spectra. It is possible that the discrepancy in the indicator comes from the way in which the AWAA treats shocks: shock fronts are re-sampled at each propagation step, but the ability of the algorithm to represent high frequencies remains limited.

It should be noted that for the same source conditions, the Blackstock bridging function does not predict a deviation from unity in the value of $Q_{\text{neg}}/Q_{\text{pos}}$; in fact, both it and the Mendousse solution to the Burgers equation predict that the value of the indicator is identically one for distances up to $\sigma = 6$. Since shocks are present in these solutions for $\sigma > 1$, this implies that $Q_{\text{neg}}/Q_{\text{pos}}$ does not detect the presence of shocks, or that the deviation from unity of $Q_{\text{neg}}/Q_{\text{pos}}$ is negligible even in the presence of shocks.

While $Q_{\text{neg}}/Q_{\text{pos}}$ does not appear to indicate the presence of shocks in a wave-
Figure 4.7. The nonlinearity indicator $Q_{\text{neg}}/Q_{\text{pos}}$ is plotted here for numerical predictions based on the data in Table 4.1. For each source condition, the measured waveform from the first microphone was used as the input to the AWAA, and predicted waveforms were generated for each of the subsequent microphone locations. Each point represents a prediction for a single microphone location and set of source conditions.

form (i.e., its value does not deviate from one for $\sigma > 1$ for the measured data or theoretical predictions), the fact that its value is very near unity for much of the data considered here means that $Q_{\text{neg}}$ and $Q_{\text{pos}}$ contain essentially the same information. Thus, $Q_{\text{pos}}$ can be used as a small bandwidth descriptor of nonlinearity.

4.2.1.2 Normalized $Q_{\text{pos}}$

As explained in Chap. 2, the quantity $Q_{\text{pos}}$ represents energy being lost to nonlinearity from a certain portion of the spectrum. $Q_{\text{pos}}$ has units of $[\text{Pa}^3/\text{s}]$; normalizing by the cube of the root-mean-square pressure results in a dimensionally simpler quantity that better lends itself to comparisons between measurements. Waveforms experiencing stronger nonlinear effects should have larger values of this normalized $Q_{\text{pos}}$. This hypothesis is tested here using single-frequency data for which shock formation distance and Gol’dberg number, two traditional measures of the importance of nonlinearity, are easily calculated.

Figure 4.8 depicts the evolution of $Q_{\text{pos}}/p_{\text{rms}}^3$ along the length of the plane wave tube for three signals having the same source frequency but different source ampli-
tudes. As amplitude increases, shock formation distance decreases and Gol’dberg number increases; therefore, we expect the indicator to have a larger value for the higher-amplitude signals. This is in fact the case in Fig. 4.8, as \( Q_{pos}/p_{rms}^3 \) for the higher-amplitude signals attains larger values and increases more rapidly with \( \sigma \) than do the smaller amplitude signals.

![Figure 4.8.](image)

**Figure 4.8.** The nonlinearity indicator \( Q_{pos}/p_{rms}^3 \) is plotted here as a function of normalized distance for plane wave tube data with a source frequency of 2000 Hz and source amplitudes of 125, 135, and 145 dB, respectively. Gol’dberg numbers for these data are 0.31, 0.79, and 2.91.

Figures 4.9 and 4.10 show normalized \( Q_{pos} \) as a function of normalized distance for signals with the same source amplitude but different source frequencies. Because shock formation distance decreases and Gol’dberg number increases with increasing frequency, we expect to see stronger nonlinear effects and thus larger values of normalized \( Q_{pos} \) for the higher-frequency signals. Again, this is indeed the case in Figs. 4.9 and 4.10; the value of \( Q_{pos}/p_{rms}^3 \) is larger and grows more quickly for the higher-frequency signals. It should be noted that the magnitude of the indicator is much greater for the 145 dB data than for the 105 dB data. Also, although the spatial resolution of the plots is coarse, extrapolation suggests that every curve originates at a value of zero for \( \sigma = 0 \).

The \( Q_{pos}/p_{rms}^3 \) indicator is not shown here for predictions made using the AWAA because the results are very similar to those obtained for the measured
Figure 4.9. The nonlinearity indicator \( Q_{pos}/p_{rms}^3 \) is plotted here as a function of normalized distance for plane wave tube data with a source amplitude of 105 dB and source frequencies of 500, 1000, 1500, 2000, 2500, and 3000 Hz, respectively. Gol’dberg numbers for the data range from 0.014 to 0.032.

Data. However, to validate the trends seen in Figs. 4.8–4.10, the indicator has been plotted in Fig. 4.11 for predictions obtained with the Mendousse solution to the Burgers equation and with the Blackstock bridging function. Each curve represents a data set with a different Gol’dberg number \( \Gamma \), achieved using a 2000 Hz source and different source amplitudes. While the Gol’dberg numbers represented in Fig. 4.11 do not match exactly those from the previous figures, some of the values are similar, allowing for qualitative comparisons.

The trends seen in Figs. 4.8 and 4.10 are evident in Fig. 4.11 as well, confirming the correlation between \( Q_{pos}/p_{rms}^3 \) and Gol’dberg number (note the differences in the range of \( \sigma \) among these plots). Figures 4.12 and 4.13 show values of \( Q_{pos}/p_{rms}^3 \) predicted by the Mendousse solution alongside those calculated for measured data with the same source conditions and Gol’dberg number. Agreement between measurement and prediction is good for both values of \( \Gamma \) at all propagation distances.
Figure 4.10. The nonlinearity indicator $Q_{pos}/p_{rms}^3$ is plotted here as a function of normalized distance for plane wave tube data with a source amplitude of 145 dB and source frequencies of 1000, 1500, 2000, and 2500 Hz, respectively. Gol’dberg numbers are 2.15, 2.72, 2.91, and 3.20.

Figure 4.11. The nonlinearity indicator $Q_{pos}/p_{rms}^3$ is plotted here as a function of normalized distance for the Blackstock bridging function and for data obtained with the Mendousse solution using three different values of the Gol’dberg number. Source amplitudes for the finite Gol’dberg numbers are 134.1 dB, 143.9 dB, and 161.7 dB, respectively.
Figure 4.12. The nonlinearity indicator $Q_{pos}/p_{rms}^3$ is plotted here as a function of normalized distance for measured data and Mendousse predictions having source frequencies of 2000 Hz and source amplitudes of 125 dB.

Figure 4.13. The nonlinearity indicator $Q_{pos}/p_{rms}^3$ is plotted here as a function of normalized distance for measured data and Mendousse predictions having source frequencies of 2000 Hz and source amplitudes of 145 dB.
4.2.1.3 Spectral Gol’dberg Number

The dimensionless indicator $\Gamma_s = (\omega/2\alpha \rho_0 c_0^3)(Q_{p^2 p}/S_p)$ was developed to gauge the relative importance of nonlinearity, geometrical spreading, and absorption. Because spreading is absent in the plane wave tube, the indicator will be used here to compare nonlinearity and absorption. As discussed in Chap. 2, indicator magnitudes greater than one suggest that nonlinearity is more important than absorption, and magnitudes less than one suggest that absorption dominates nonlinearity.

Figures 4.14–4.18 provide insight into the evolution of nonlinearity with distance. These plots depict the values of the spectral Gol’dberg number indicator for measured and numerical data with varying source frequencies. Because of this, the results are plotted as a function of harmonic number $n$ (with $n = 1$ the source frequency) rather than as a function of frequency. The number of harmonics shown in each plot is limited because progressively smaller values of $S_p$ occur at higher harmonic numbers; this causes the $Q_{p^2 p}/S_p$ factor in the indicator to become unstable and take on very large (negative) values at high harmonic numbers. Most of the harmonic components shown in any given plot have spectral amplitudes above the system noise floor; specifics about this are noted in the captions. The four signals used for each plot were captured at the same propagation distance, but because of their varying source frequencies, each had a different shock formation distance and therefore a distinct value of normalized distance $\sigma$.

All the quantities used in the calculation of the spectral Gol’dberg number indicator are positive with the exception of $Q_{p^2 p}$ itself; thus, the sign of the indicator at each frequency reveals whether energy is being lost or gained nonlinearly at that frequency. According to this indicator, then, energy is being lost at the fundamental frequency and gained at the first few harmonic frequencies for all the single-frequency source cases depicted.

In each case, the magnitude of the dimensionless indicator increases with increasing harmonic number; for larger harmonic numbers than those shown, it increases extremely rapidly and quickly becomes lost in numerical noise (this is why the fourth harmonic component for the 1500 Hz data is not visible in Fig. 4.14). This is likely because the magnitude of the PSD decreases with increasing harmonic number until it drops below the noise floor. When the PSD is small, the
Figure 4.14. The nonlinearity indicator $\Gamma_s$ is shown as a function of harmonic number for waveforms with four different values of source frequency. All four signals were captured at a propagation distance of 3.25 m, but because of their different source frequencies, each signal has a distinct value of normalized distance. All of the harmonic components shown have spectral amplitudes above the system noise floor. The right-hand plot shows a close-up of the values for the source frequency components. While the value of the indicator is small for these points, it is positive, unlike at the higher harmonics.

As normalized propagation distance increases, the indicator’s magnitude at the harmonics gradually decreases, though it remains larger than one for most $n \geq 2$ for the cases shown here. This suggests that the nonlinear evolution of a given harmonic component is most significant when its spectral density value emerges from the noise. As it continues to propagate, nonlinear evolution becomes weaker at those frequencies, but may remain dominant over absorption past $\sigma = 1$. Thus, while the nonlinear growth at these harmonics becomes less significant with propagation distance, it is often still large enough past $\sigma = 1$ that the spectral levels continue to increase rather than being attenuated by absorption.

The opposite is true of the indicator value at the source frequency. While it is universally positive at the fundamental for the data shown here, the magnitude of $\Gamma_s$ at this component becomes larger with increasing normalized propagation distance, not smaller as it does at the higher harmonics. This could be because as the wave propagates, more harmonics are generated nonlinearly; in order to balance the energy gained by these components, the fundamental component loses
Figure 4.15. The nonlinearity indicator $\Gamma_s$ is shown as a function of harmonic number for waveforms with four different values of source frequency and a source amplitude of 135 dB, as in Fig. 4.14. All four signals were numerically propagated to a distance of 3.25 m using the Arbitrary Wave Anderson Algorithm. All of the harmonic components shown have spectral amplitudes above the system noise floor. The right-hand plot shows a close-up of the values for the source frequency components. These plots are very similar to the Fig. 4.14 which uses measured data.

more energy nonlinearly.

Generally, the AWAA predictions (Figs. 4.15 and 4.17 compare favorably with the results for the measured data (Figs. 4.14 and 4.16. The same trends in magnitude, harmonic number, and normalized propagation distance are seen, and agreement between measurement and prediction on the actual value of $\Gamma_s$ is very good for most data points.
**Figure 4.16.** The nonlinearity indicator $\Gamma_s$ is shown as a function of harmonic number for waveforms with four different values of source frequency. All four signals were captured at a propagation distance of 3.25 m; normalized distances are indicated in the legend. All of the harmonic components shown have spectral amplitudes above the system noise floor.

**Figure 4.17.** The nonlinearity indicator $\Gamma_s$ is shown as a function of harmonic number for waveforms with four different values of source frequency and a source amplitude of 145 dB, as in Fig. 4.16. All four signals were propagated numerically to a distance of 3.25 m. All of the harmonic components shown have spectral amplitudes above the system noise floor.
Figure 4.18. The nonlinearity indicator $\Gamma_s$ is shown as a function of harmonic number for waveforms with four different values of source frequency. All four signals were captured at a propagation distance of 6.40 m; normalized distances are indicated in the legend. All of the harmonic components shown have spectral amplitudes above the system noise floor.
4.2.1.4 Coherence Indicator

As discussed in Sec. 2.44, the coherence indicator $\gamma_Q$ quantifies the cumulative nonlinear effects experienced by a propagating signal. Its magnitude is bounded between zero and one (its value is bounded between negative one and one), so as a signal propagates and distorts nonlinearly, we expect the magnitude of its coherence function to approach one. Because the sign of the coherence indicator dictates the direction of nonlinear energy transfer, for an initially sinusoidal signal we expect a positive value of $\gamma_Q$ at the fundamental and negative values of $\gamma_Q$ at the nonlinearly generated harmonics.

Figures 4.19–4.22 depict the coherence indicator as a function of harmonic number for progressively larger normalized propagation distances. Within a given plot, the source amplitude and physical propagation distance of the four signals is the same; however, each signal has a different source frequency and thus a different normalized propagation distance. From one figure to the next, either source amplitude or physical propagation distance is increased; thus, the data used to generate each figure have undergone more nonlinear distortion than the data from the previous figure, and we expect to see $\gamma_Q$ magnitudes progressively closer to unity in each figure.

Examination of Figs. 4.19 and 4.20 reveals that our expectations appear to be correct. In Fig. 4.19, the data have normalized propagation distances ranging from 0.03 to 0.07. At the fundamental frequency, $\gamma_Q = 1$, and $\gamma_Q \approx -1$ for the next two or three harmonics. For the data in Fig. 4.20 with normalized distances from 0.11 to 0.21, the indicator value is again 1 for the fundamental, but now $\gamma_Q \approx -1$ for the next four harmonics. This observation is consistent with nonlinear propagation theory for an initially sinusoidal signal, which states that energy “cascades” from the fundamental upwards in the spectrum. As normalized propagation distance continues to increase, so too does the number of harmonics containing significant spectral energy.

The trend seen in Figs. 4.19 and 4.20 continues in Fig. 4.21, where the indicator is again equal to one at the fundamental and approximately equal to negative one for the next seven harmonics. Figure 4.22, however, appears to show that while nonlinear phase coupling continues to become important at higher harmonics, the magnitude of this importance begins to decrease as $\sigma \to 1$, possibly indicating
Figure 4.19. The coherence indicator is shown as a function of harmonic number for waveforms with four different values of source frequency. All four signals were captured at a propagation distance of 3.25 m, but because of their different source frequencies, each signal has a distinct value of normalized distance. For these signals, the first four harmonics have spectral amplitudes above the system noise floor.

that the rate of nonlinear energy transfer decreases in this region. For normalized propagation distances of $\sigma = 1.03$ and $\sigma = 1.32$, most or all of harmonics 2–20 have $\gamma_Q \leq -0.5$, but only some have $\gamma_Q \approx -1$. This apparent decrease in magnitude could be the result of absorption and dispersion in the plane wave tube. A plot of the indicator for BBF predictions with the same conditions as those in Fig. 4.22 reveals that this is likely the case. Figure 4.23 shows that, in the absence of absorption and dispersion, all nonlinearly significant harmonics have $\gamma_Q = -1$. Results using AWAA-predicted waveforms, which do include the effects of absorption and dispersion, lie in between those using the measurements and those using the BBF predictions. Thus, absorption and dispersion may account for some of the deviation in $\gamma_Q$ from negative one at the harmonics, but imperfections in the plane wave tube are likely partially responsible, as well.
Figure 4.20. The coherence indicator is shown as a function of harmonic number for waveforms with four different values of source frequency. All four signals were captured at a propagation distance of 3.25 m. More harmonics show significant nonlinear evolution here than in Fig. 4.19; five harmonics have spectral amplitudes above the system noise floor.

Figure 4.21. The coherence indicator is shown as a function of harmonic number for waveforms with four different values of source frequency. All four signals were captured at a propagation distance of 3.25 m. There is significant nonlinear evolution at a larger number of harmonics than for the 135 dB source; for these data, ten harmonics have spectral amplitudes above the system noise floor.
Figure 4.22. The coherence indicator is shown as a function of harmonic number for waveforms with four different values of source frequency. All four signals were captured at a propagation distance of 6.40 m. Significant nonlinear evolution is present in a large number of harmonics here, with the largest numbers seen at the larger normalized propagation distances. Ten harmonics have spectral amplitudes above the system noise floor.

Figure 4.23. The coherence indicator is shown for predictions made using the AWAA and the Blackstock Bridging Function. Source and propagation conditions are identical to those in Fig. 4.22. For the AWAA predictions, the indicator value is 1 at the fundamental and nearly -1 at all harmonics; values are identically 1 and -1 for the BBF predictions. In both cases, all harmonic components shown have spectral amplitudes above the system noise floor.
4.2.2 Noise Source Data

All noise source data in the plane wave tube were generated using a single JBL 2426H driver. Source conditions are given in Table 4.2. The frequencies cited in the table are the cutoff frequencies of the 24 dB/octave butterworth filter used to generate the source waveform.

<table>
<thead>
<tr>
<th>Amplitude (dB re 20\mu Pa)</th>
<th>Frequency Range (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>200-3200</td>
</tr>
<tr>
<td>95</td>
<td>x</td>
</tr>
<tr>
<td>105</td>
<td>x</td>
</tr>
<tr>
<td>115</td>
<td>x</td>
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<td>125</td>
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<td>135</td>
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<td>140</td>
<td>x</td>
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<tr>
<td>145</td>
<td>x</td>
</tr>
<tr>
<td>150</td>
<td>x</td>
</tr>
</tbody>
</table>

Table 4.2. Source conditions used to generate noise data in the plane wave tube.

4.2.2.1 Ratio of $Q_{neg}$ to $Q_{pos}$

Again, $Q_{pos}$ and $Q_{neg}$ are calculated by integrating over the quantity $\omega Q_{p^2_p}$ as discussed in Sec. 2.3. Sample plots of the PSD and $\omega Q_{p^2_p}$ for a noise signal can be found in Figs. 4.24 and 4.25.

Figure 4.26 contains values of the indicator $Q_{neg}/Q_{pos}$ for all of the noise data taken in the plane wave tube. As for the single-frequency data, each point represents a measurement at a single microphone location for a single set of source conditions. Here the data are plotted as a function of $\sigma_{eff}$, or effective normalized distance. This was calculated by dividing the propagation distance by an effective shock formation distance, found using the center frequency of the source noise bandwidth and the effective (or rms) pressure at the source.

All values of $Q_{neg}/Q_{pos}$ shown in Fig. 4.26 are within 0.5% of unity, the expected value of the indicator for shock-free propagation. As with the single-frequency source data, some scatter is present at the smaller values of $\sigma$ in Fig. 4.26. Shocks are present in some of the higher amplitude waveforms at large values of $\sigma_{eff}$, as can
Figure 4.24. The power spectral density is shown here for a noise measurement taken in the plane wave tube with a 140 dB source amplitude and a source frequency bandwidth of 1200–3500 Hz.

Figure 4.25. The quantity $\omega Q_{p}^{2} p$ is shown here for a noise measurement taken in the plane wave tube with a 140 dB source amplitude and a source frequency bandwidth of 1200–3500 Hz. $Q_{neg}$ and $Q_{pos}$ are calculated by integrating over the negative and positive regions, respectively, of this function.
Figure 4.26. The nonlinearity indicator $Q_{neg}/Q_{pos}$ is plotted here for all the waveforms whose source conditions are listed in Table 4.2. Each point represents a measurement for a single microphone location and set of source conditions.

be seen in Fig. 4.27. (Despite the fact that none of the data points have normalized distances significantly greater than one, the presence of shocks is not surprising: for noise signals, shock formation can happen for $\sigma_{eff} < 1$.) It is not clear why the presence of these shocks is not detected by the $Q_{neg}/Q_{pos}$ indicator. Either the energy lost at the shocks represents a negligible fraction of the total energy of the wave, or $Q_{neg}/Q_{pos}$ does not “see” the losses at the shocks. Nevertheless, the fact that the value of the ratio is so close to one supports the use of $Q_{pos}$ as an indicator of nonlinear spectral evolution.
Figure 4.27. A waveform measured in the plane wave tube having a source bandwidth of 1200–3500 Hz and a source amplitude of 150 dB. The waveform was captured at a propagation distance of 9.55 m; $\sigma_{eff} = 0.69$ at this distance. Well-developed shocks can be seen in the waveform.

4.2.2.2 Normalized $Q_{pos}$

Here the indicator $Q_{pos}/P_{rms}^3$ is applied to the noise data whose source conditions are given in Table 4.2. Applying the same logic that was used with the single-frequency source data, we expect that the magnitude of the indicator should increase with amplitude for signals having the same source frequency content. Figure 4.28 affirms that this is the case. Using three sets of data with the same source frequency range but differing source amplitudes, the behavior of the normalized $Q_{pos}$ is the same as for the single-frequency source data in Fig. 4.8.

In Fig. 4.29, the indicator is plotted for data with the same source frequency range but lower source amplitudes than those shown in Fig. 4.28. Because the values of $\sigma_{eff}$ are so small for these data, a log scale has been used. These data exhibit much the same behavior as the higher amplitude data, and they suggest that the minimum value of $Q_{pos}/P_{rms}^3$ for this kind of noise may be $\sim 12$. While we would not expect much nonlinear evolution to occur with these data (especially for the 95 dB and 105 dB source amplitudes), it is not surprising that the apparent minimum value of $Q_{pos}/P_{rms}^3$ is larger for noise than for a sinusoid. A noise
source contains many spectral components that may begin interacting nonlinearly immediately as the leave the source location, while a sinusoid must propagate some distance before multiple spectral components are available for interaction. Thus, we might expect larger values of $Q_{pos}/p_{rms}^3$ at small $\sigma$ for noise than for single-frequency sources.

Figure 4.30 compares the value of $Q_{pos}/p_{rms}^3$ for two sets of data with the same source amplitude but different frequency ranges. As in the single-frequency source case, the data with the higher center frequency generally exhibit higher values of the indicator.

Inspection of Figs. 4.10 and 4.28 reveals that a noise source with roughly the same amplitude and center frequency as a single-frequency source evolves nonlinearly at about the same rate. The center frequency for the data in Fig. 4.28 is 1700 Hz; comparison of the 145 dB curve in this figure with the 1500 Hz curve in Fig. 4.10 shows that the value of the normalized $Q_{pos}$ indicator is close to 100 for both cases at a(n) (effective) normalized propagation distance of 0.5. Thus, while it has been asserted [39] that noise signals exhibit stronger nonlinear evolution than tone signals, such a result is not seen here. Again, a difference between the

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**Figure 4.28.** The nonlinearity indicator $Q_{pos}/p_{rms}^3$ is plotted here as a function of distance normalized by effective shock formation distance for plane wave tube data with a source frequency range of 200 Hz-3200 Hz and source amplitudes of 135, 140, and 145 dB, respectively.
Figure 4.29. The nonlinearity indicator $Q_{pos}/p_{rms}^3$ is plotted here as a function of distance normalized by effective shock formation distance for plane wave tube data with a source frequency range of 200 Hz-3200 Hz and source amplitudes of 95, 105, 115, and 125 dB, respectively. The indicator seems to have an asymptotic value of about 12 for $\sigma_{eff} \to 0$. Note that the horizontal axis uses a log scale for clarity.

noise data and the periodic data should be noted. While the value of the indicator appears to extrapolate to zero at $\sigma = 0$ for the single-frequency source data, for the noise data in Figs. 4.28 and 4.30 it appears to be $\sim 20$ at $\sigma_{eff} = 0$. Given that the data in Figs. 4.28 and 4.30 have higher source amplitudes than the data in Fig. 4.29, stronger nonlinear interaction can be expected at all propagation distances, so this higher value of $Q_{pos}/p_{rms}^3$ (20 rather than 12) near the source is unsurprising.
Figure 4.30. The nonlinearity indicator $Q_{pos}/p_{rms}^3$ is plotted here as a function of distance normalized by effective shock formation distance for plane wave tube data with a source amplitude of 135 dB and source frequency ranges of 200 Hz-3200 Hz and 1200 Hz-3500 Hz, respectively.

4.2.2.3 Spectral Gol’dberg Number

Figures 4.31, 4.33, and 4.35 depict the evolution with distance of the spectral Gol’dberg number indicator for noise with a 1200–3500 Hz source band and two different source amplitudes. Each figure was generated using the same signal captured at three different propagation distances. Corresponding spectra can be found in Figs. 4.32, 4.34, and 4.36.

Figure 4.31, calculated for a 115 dB source, reveals that nonlinear effects are most important in the frequency bands immediately above and below the source frequency range. The magnitude of the indicator (very near one) at these frequencies suggests that 115 dB is the source amplitude at which nonlinearity begins to dominate absorption in the plane wave tube (because boundary layer absorption depends on the radius of the tube, this conclusion does not necessarily apply to all waveguides). The prominence of the indicator in the higher frequency region, essentially a “second harmonic frequency band,” is consistent with the findings of Pernet and Payne [39] in their study of narrowband noise. Examination of the spectra in Fig. 4.32 confirms that nonlinear growth occurs in this region. Remem-
Figure 4.31. The nonlinearity indicator $\Gamma_s = (\omega/2\alpha\rho_0c_0^3)(Qp^2/S_p)$ is shown as a function of frequency for a waveform with a source frequency range of 1200 Hz–3500 Hz and a source amplitude of 115 dB. Each curve represents a different propagation distance, as given in the legend. The source frequency band is defined by the solid vertical lines. Nonlinearity is most important below the source frequency range and in the “second harmonic” frequency band.

Figure 4.32. Power spectral densities are shown here for the data used to calculate $\Gamma_s$ in Fig. 4.31.
bering Eq. 2.41,
\[- \frac{1}{2\alpha} \frac{\partial (\ln S_p)}{\partial x} = 1 + \frac{\omega \beta}{2 \alpha \rho c_0^2} \frac{Q_p^2}{S_p}, \tag{4.4}\]
the fact that spectral levels increase slightly with distance is consistent with the fact that the value of the indicator is close to negative one. The value of $\Gamma_s$ is negative, and its magnitude is generally greater than or equal to one, so slightly more energy is gained nonlinearly than is lost due to absorption.

For frequencies above this band (above $\sim 9$ kHz), the indicator suggests that nonlinearity is of little to no importance. Spectral levels at these frequencies decrease with propagation distance, as would be expected if absorption were dominant. The negative values of $\Gamma_s$ in the lower frequency band suggest that difference frequency components (at frequencies below the source band) are being generated for all propagation distances shown. Inspection of the PSD’s reveals that spectral levels in this frequency region remain constant for all propagation distances; this is consistent with the fact that $\Gamma_s \approx 1$, implying that nonlinear energy gains offset dissipative losses. Also, as with the single-frequency source data, nonlinearity seems to be of relatively small importance in the source frequency region.

Figures 4.33 and 4.35 show the indicator for a 140 dB source. Much stronger nonlinear effects are expected here than for the 115 dB source. Indeed, the magnitude of the indicator is larger for this data, and the frequency range for which nonlinearity is important is much broader. Sum and difference frequency generation are still evident for propagation distances larger than 3.25 m, both in the indicator and in the spectra (see Figs. 4.34 and 4.36). Generally, $\Gamma_s \leq -1$ immediately above and below the source band, and spectral levels in these bands increase with propagation distance. Nonlinear effects at the source frequency are still weak compared to those at the sum and difference frequencies. As it does for the single-frequency source data, nonlinearity becomes increasingly important at the highest frequencies as propagation distance increases. This can also be seen in the spectra, with significant nonlinear growth in progressively larger spectral bands as propagation distance increases.
Figure 4.33. The nonlinearity indicator $\Gamma_s = \left(\frac{\omega \beta}{2\alpha \rho_0 c_0^3}\right)\left(Q_p^2 p / S_p\right)$ is shown as a function of frequency for shock tube waveforms with a source frequency range of 1200 Hz–3500 Hz and a source amplitude of 140 dB. Each curve represents a different propagation distance, as given in the legend. Nonlinearity is most important for frequencies above and below the source frequency band.

Figure 4.34. Power spectral densities are shown here for the data used to calculate $\Gamma_s$ in Fig. 4.33.
Figure 4.35. The nonlinearity indicator \( \Gamma_s = (\omega\beta / 2c_\rho\rho c_0^3)(Q_{p^2}/S_p) \) is shown as a function of frequency for the same data used in Fig. 4.33. Here a larger frequency bandwidth is displayed to better illustrate the importance of nonlinearity at high frequencies.

Figure 4.36. Power spectral densities are shown here for the data used to calculate \( \Gamma_s \) in Fig. 4.35.
4.2.2.4 Coherence Indicator

Because the coherence indicator preserves the sign of the QSD, we expect plots of the coherence indicator to be at least qualitatively similar to plots of the spectral Gol’dberg number. Frequency regions where nonlinear energy loss occurs (i.e. the source frequency region) should have positive values of the indicator, and frequency regions where energy is gained nonlinearly should have negative values of the indicator. Additionally, in light of the coherence indicator results for the single-frequency source data, we expect the cumulative nature of nonlinearity to be evident in the results, as well.

![Coherence Indicator, 115 dB Source](image)

**Figure 4.37.** The coherence indicator is shown at three different microphone locations for the 115 dB source condition used to calculate $\Gamma_s$ in the previous section.

Figures 4.37–4.39 show the coherence indicator for the same data that were analyzed in the previous section using the spectral Gol’dberg number. Figure 4.37 represents a relatively low amplitude (115 dB) source condition, but nonlinear energy gain is apparent both below and immediately above (in the “second harmonic band”) the source frequency band for all three propagation distances.

Nonlinear energy gain is also seen in the low-frequency and second harmonic band regions for the larger amplitude (140 dB) data (see Fig. 4.38). Here, however, the growth at low frequencies is more significant, with the indicator attaining values of nearly -0.8. This is not surprising, as the spectral levels at these frequencies
actually increase with distance for the 140 dB case (see Fig. 4.34); for the 115 dB case, the nonlinear energy gain is only enough to offset absorption. Also, as expected, nonlinear energy loss is apparent in the source frequency region: $\gamma_Q$ takes on positive values in this region for the three largest propagation distances.

![Coherence Indicator, 140 dB Source](image)

**Figure 4.38.** The coherence indicator is shown at three different microphone locations for the 140 dB source condition used to calculate $\Gamma_s$ in the previous section.

The 140 dB case also differs from the lower amplitude case in that evidence of nonlinear phase coupling is seen at frequencies above the second harmonic band (i.e. above about 7 kHz). Figure 4.39 plots the data from Fig. 4.38 on a larger frequency scale so that this behavior can be more closely examined. As expected, evidence of nonlinear energy gain “cascades” up the spectrum as propagation distance increases. Because the coherence indicator reflects cumulative nonlinear effects, once the magnitude of $\gamma_Q$ becomes significant at a certain frequency, it tends to remain that way for all subsequent measurements.
Figure 4.39. The coherence indicator is shown at three different microphone locations for the 140 dB source condition used to calculate $\Gamma_s$ in the previous section. A large frequency bandwidth is used here to better depict the behavior of the indicator at high frequencies.

4.3 Model-Scale Jet Data

As was mentioned in Sec. 1.4, jet noise can be highly directional. In general, the peak radiation angle (the angle at which the most acoustic power radiates) is between 125° and 150° relative to the jet inlet axis (the front of the aircraft). Radiation at angles less than or equal to 90° is generally weakest.

The peak radiation angle for these model-scale measurements was found to be near 145°. Therefore, data recorded at this angle are included in many of the analyses below. Data from 90° are often included as a lower amplitude reference point.

4.3.1 Ratio of $Q_{neg}$ to $Q_{pos}$

Because the importance of nonlinear effects in model-scale jet noise depends on several different parameters, the indicator $Q_{neg}/Q_{pos}$ is presented here for a sampling of source conditions and measurement locations.

The first comparison, shown in Fig. 4.40, is of the $Q_{neg}/Q_{pos}$ indicator for several different propagation angles relative to the inlet axis of a low Mach number (0.85)
jet. At this Mach number, nonlinearity is not expected to play a significant role in the acoustic propagation of the noise. Typical sound pressure levels at a distance of 10 jet diameters from the source are between 100 and 110 dB. Thus, shocks should not form, and the $Q_{\text{neg}}/Q_{\text{pos}}$ indicator should have a value of one. While the data in Fig. 4.40 are somewhat noisy, the average value of the indicator is near one, and no trend away from that value is observed.

Figure 4.41 presents the $Q_{\text{neg}}/Q_{\text{pos}}$ indicator at the same propagation angles for a Mach 2 jet. For this source condition, nonlinear propagation effects are expected to be more important (amplitudes close to the source are as high as 135 dB), especially near the peak radiation angle of 145°. As a result, the development of shocks is likely for these angles at larger propagation distances, resulting in indicator values less than one. However, given the fact that no such trend has been seen in the plane wave tube data, we may not expect the indicator values to be measurably less than one. Indeed, no significant deviation from unity can be seen in the figure. Thus, either shocks have not formed, or $Q_{\text{neg}}/Q_{\text{pos}}$ is not a reliable indicator of the presence of shocks. However, the results indicate that the ratio is very near unity for several source conditions and two types of geometrical spreading, thus supporting the use of $Q_{\text{pos}}$ as a nonlinearity indicator.
4.3.2 Normalized $Q_{pos}$

Here the indicator $Q_{pos}/p_{rms}^3$ is applied to the same data used in the previous section. Figure 4.42 depicts the normalized $Q_{pos}$ indicator for one “off-peak” and three near-peak radiation angles. While the Mach 0.85 source condition is not expected to result in appreciable nonlinearity, any nonlinearity present would likely be stronger near the peak radiation angle. The plot, however, implies that the opposite is true: indicator values are largest for the 90° propagation angle and smallest for the 150° propagation angle. The likely answer lies in the fact that nonlinearity is unimportant at this source condition (recall the fact that numerical methods assuming linear propagation are adequate to describe these data). The value of the QSD, and therefore $Q_{pos}$, is relatively small throughout the propagation field. The amplitude of the radiated noise, however, is directional. Because the root-mean-square pressure occurs in the denominator of the indicator, angles near the peak radiation angle with larger acoustic pressures have smaller values of $Q_{pos}/p_{rms}^3$, while angles with smaller acoustic pressures have larger values of $Q_{pos}/p_{rms}^3$. While this explanation seems plausible for the present data, it does little to enhance the credibility of the $Q_{pos}/p_{rms}^3$ indicator.
Figure 4.42. The nonlinearity indicator $Q_{pos}/P_{rms}^3$ is shown at several different propagation angles for a Mach 0.85 jet.

Figure 4.43 uses data from a Mach 2 source condition at four different propagation angles. These results better align with expectations, showing 140° as the peak radiation angle and nonlinearity at 90° as relatively unimportant. Also significant is that the magnitude of the indicator is everywhere greater for the Mach 2 data than for the Mach 0.85 data.
Figure 4.43. The nonlinearity indicator $Q_{pos}/P_{rms}^3$ is shown at several different propagation angles for a Mach 2 jet.

4.3.3 Spectral Gol’dberg Number

The spectral Gol’dberg number indicator $\Gamma_s = (\omega/\beta)(\rho_0/c_0^3)(Q_p^2/S_p)$ was calculated for the peak radiation angle for Mach 0.85 and Mach 2 source conditions and for an off-peak radiation angle for the Mach 2 source condition. Power spectral densities for these conditions are shown in Figs. 4.44, 4.46 4.49, and 4.51, and the spectral Gol’dberg number results are shown in Figs. 4.45, 4.47 4.50, and 4.52.

Figure 4.45 presents the spectral Gol’dberg number indicator for data recorded at 145° to the inlet axis for a Mach 0.85 source condition. The traces represent propagation distances of 30, 40, 60, and 75 jet diameters, respectively. No trend in the data is distinguishable at any propagation distance, and the magnitude of the indicator is much less than one, suggesting that absorption dominates nonlinearity for these data. This is to be expected because of the low Mach number of the jet and because, according to Fig. 4.44, linear theory is sufficient to describe the propagation for these conditions.

Very similar results are seen for the 90°, Mach 0.85 condition. The collapsed spectra for this condition can be found in Fig. 4.44, and the spectral Gol’dberg number is plotted in Fig. 4.45. The similarity in the indicator for these two conditions affirms that nonlinearity is unimportant for both, and that our speculation
Figure 4.44. Power spectral densities are shown for several different propagation distances and a propagation angle of 90° for a Mach 0.85 jet. Linear theory has been used to extrapolate the spectra to a propagation distance of 75 jet diameters. Because the extrapolated spectra collapse well, it is likely that linear theory is sufficient to describe the propagation for these conditions.

Figure 4.45. The spectral Gol’dberg number is shown at several different propagation distances for a Mach 0.85 jet. All data were acquired at an angle of 90° relative to the jet inlet axis.
Figure 4.46. Power spectral densities are shown for several different propagation distances and a propagation angle of 145° for a Mach 0.85 jet. Linear theory has been used to extrapolate the spectra to a propagation distance of 75 jet diameters. Because the extrapolated spectra collapse well, it is likely that linear theory is sufficient to describe the propagation for these conditions.

Figure 4.47. The spectral Gol’dberg number is shown at several different propagation distances for a Mach 0.85 jet. All data were acquired at an angle of 145° relative to the jet inlet axis.
Figure 4.48. The spreading term $1/\alpha r$ is shown here for frequencies and propagation distances relevant to the model-scale jet data. Note the logarithmic scale on the vertical axis. The smallest value of $1/\alpha r$, which occurs at the largest frequency and largest propagation distance, is more than an order of magnitude larger than most values of $\Gamma_s$ for the model-scale jet data.

about the results for the 90° case in Sec. 4.3.2 are likely correct.

The spreading term $1/\alpha r$ has been calculated for the frequencies and propagation distances relevant to the model-scale jet data and is plotted in Fig. 4.48. This term comes from the equation used to derive $\Gamma_s$,

$$-\frac{1}{2\alpha} \frac{\partial (\ln S_p)}{\partial r} = \frac{1}{\alpha r} + 1 + \frac{\omega \beta}{2\alpha \rho_0 c_0^3} \frac{Q_p x_p}{S_p},$$

(4.5)

and its value represents the threshold of importance of nonlinearity relative to spreading. For the four propagation distances represented in Fig. 4.47, the value of the spreading term ranges from $\sim 10^3$ to $\sim 1$ over the frequency bandwidth of interest. These values are several orders of magnitude larger than the values of $\Gamma_s$ in Fig. 4.47, suggesting that geometrical spreading is responsible for the vast majority of the spectral evolution seen in this case.

In Figs. 4.49 and 4.50, the PSD’s and spectral Gol’dberg number are plotted as a function of frequency for the same four propagation distances, but for a 90° propagation angle and a Mach 2 jet. Thus, while nonlinear effects might be more
Figure 4.49. Power spectral densities are shown for several different propagation distances and a propagation angle of 90° for a Mach 2 jet. Linear theory has been used to extrapolate the spectra to a propagation distance of 75 jet diameters. While these spectra do not exactly collapse, they do not exhibit the characteristics commonly associated with nonlinear propagation.

Figure 4.50. The spectral Gol’dberg number is shown at several different propagation distances for a Mach 2 jet. All data were acquired at an angle of 90° relative to the jet inlet axis.
prominent overall for the Mach 2 case than for the Mach 0.85 case, these data were recorded at an off-peak radiation angle, likely minimizing the importance of nonlinearity. The dimensionless indicator suggests that nonlinearity is of very little importance for these data; as in Fig. 4.47, the magnitude of the indicator is much less than one at all distances and frequencies, and there is no observable trend. Comparison with Fig. 4.48 reveals that, once again, the evolution of the spectrum is mostly due to geometrical spreading.

Data recorded for a Mach 2 source condition at 145° reveal in Fig. 4.52 that nonlinearity is important for this set of conditions. The sign of the indicator makes it evident that energy is being lost nonlinearly in the 0–15 kHz region and gained nonlinearly above 15 kHz. The indicator has a magnitude near the critical value of one in the 2–10 kHz region at propagation distances of 30 and 40 jet diameters. Here, nonlinearity is at least as important as absorption (meaning that the spectral components should decay at a net rate about twice that due to absorption alone); elsewhere, it is less so, but still much more significant than for the data in Figs. 4.47 and 4.50. The spreading term $1/\alpha r$ (see Fig. 4.48) still has a larger magnitude than $\Gamma_s$ at all frequencies for these data, but it is now only about one or two orders of magnitude larger. Thus, while spreading continues to be the dominant influence on the spectrum for this jet condition, it is relatively less important compared to nonlinearity than for off-peak radiation angles and low Mach numbers.
Figure 4.51. Power spectral densities are shown for several different propagation distances and a propagation angle of 145° for a Mach 2 jet. Linear theory has been used to extrapolate the spectra to a propagation distance of 75 jet diameters. Although the spectra line up at high frequencies, the measured spectrum has more low-frequency content than the extrapolated spectra, and the peak frequency is shifted downward.

Figure 4.52. The spectral Gol’dberg number is shown at several different propagation distances for a Mach 2 jet. All data were acquired at an angle of 145° relative to the jet inlet axis.
4.3.4 Coherence Indicator

The coherence indicator is shown below for the same source conditions and microphone locations used in the spectral Gol’dberg number analysis. The indicator is bounded between negative one and one, with larger magnitudes denoting more significant nonlinear effects.

![Coherence Indicator, Mach 0.85, 90°](image)

**Figure 4.53.** The coherence indicator is shown at several different propagation distances for a Mach 0.85 jet. All data were acquired at an angle of 90° relative to the jet inlet axis.

Figures 4.53–4.55 show the coherence indicator for source conditions and/or microphone locations unlikely to exhibit significant nonlinear effects. Figure 4.53 represents a low (0.85) Mach number jet and an off-peak radiation angle (90°), Fig. 4.54 contains data taken at the peak radiation angle (145°) but for a low (0.85) Mach number jet, and Fig. 4.55 uses data taken at an off-peak radiation angle (90°) for a high (2) Mach number jet. The magnitude of the coherence indicator in all three plots is well below the significant value of one.

The coherence indicator suggests that nonlinearity is much more important for the conditions in Fig. 4.56. These data were taken at a propagation angle of 145° for a Mach 2 jet. The magnitude of the indicator is several times larger for these conditions than for the conditions in Figs. 4.54 and 4.55. It reveals that nonlinearity (specifically nonlinear energy loss) is most important in the 2–10 kHz
Figure 4.54. The coherence indicator is shown at several different propagation distances for a Mach 0.85 jet. All data were acquired at an angle of 145° relative to the jet inlet axis.

region, a result also obtained with the spectral Gol’dberg number. Above about 50 kHz, nonlinear energy gains become more significant.
Figure 4.55. The coherence indicator is shown at several different propagation distances for a Mach 2 jet. All data were acquired at an angle of 90° relative to the jet inlet axis.

Figure 4.56. The coherence indicator is shown at several different propagation distances for a Mach 2 jet. All data were acquired at an angle of 145° relative to the jet inlet axis.
Chapter Summary and Discussion

This chapter has presented the results of the application of the four candidate nonlinearity indicators developed in Chap. 2 to the experimental data described in Chap. 3. While each indicator has its advantages, some proved to be more useful than others.

The ratio of $Q_{neg}$ to $Q_{pos}$ does not seem to indicate the presence of shocks as postulated in Chap. 2; however, its value is very near unity for all the experimental data used. This result suggests that $Q_{neg}$ and $Q_{pos}$ contain essentially the same information and supports the idea of using $Q_{pos}$ or some variation as an indicator.

The normalization $Q_{pos}/p_{rms}^3$ correlates with the Gol’dberg number for single-frequency source data: signals with higher Gol’dberg numbers have higher values of $Q_{pos}/p_{rms}^3$. The same trends are observed in the noise data, both from the plane wave tube and the model-scale jet. Thus, $Q_{pos}/p_{rms}^3$ can be used to compare data with similar propagation conditions but different source conditions and to rank them according to the strength of nonlinear evolution. Care must be used in the application of the indicator when spectral levels are low. Division by the cube of the root-mean-square pressure can result in artificially high values of the indicator when the root-mean-square pressure is small.

The spectral Gol’dberg number $\Gamma_s$ proved to be a valuable tool in the evaluation of the importance of nonlinearity. Not only can it be used to determine the relative importance of linear and nonlinear effects (much like the original Gol’dberg number $\Gamma$) at any propagation distance and frequency, its sign indicates the direction of nonlinear energy transfer. Examination of power spectral densities alongside the spectral Gol’dberg number confirms that $\Gamma_s$ provides accurate information about the magnitude and direction of nonlinear energy transfer. Thus, given $\Gamma_s$ for a measurement, we know how (and how much) the spectrum should evolve with distance. The most significant disadvantage of the spectral Gol’dberg number is the fact that its denominator contains the PSD, meaning that it can become artificially large or unstable for signals with very low spectral levels.

Finally, the coherence indicator $\gamma_Q$ is also useful for its ability to specify the direction of nonlinear energy transfer. While its magnitude cannot readily be compared to a threshold of importance for linear effects (i.e. absorption and spread-
ing), the coherence indicator has the additional advantage of being bounded. It also serves to indicate how a given spectrum is evolving with distance. $\gamma_Q$ does not become unstable for small spectral levels and thus can be used with more confidence for data that contains very small levels of some frequency components.
Chapter 5

Conclusions and Discussion

5.1 Summary

This work was motivated by the need for more accurate modelling and assessment of the noise radiated from high performance jets and its impact on the environment. Past work suggests the amplitudes of the radiated noise may be high enough that linear acoustic models may not be adequate to predict the character of the noise. If this is true, then care must be taken to assess the importance of nonlinear effects and to include them in any models and prediction schemes when necessary. The first step in this process is to assess the importance of nonlinearity. The traditional method of conducting this assessment is to compare power spectral densities measured at different distances from the source. However, practical constraints such as measurement bandwidth, limited measurement ranges, and uncertainty in source location can make the results of this method difficult to interpret. A better assessment method would be one that relies upon measurements made at only a single location. This work has undertaken to develop single-point quantities which can serve as reliable indicators of the importance of nonlinear evolution. Four such candidate quantities, based on Morfey and Howell’s $Q_{p^2_p}$, have been derived and applied to plane-wave-tube and model-scale-jet data.
5.2 Nonlinearity Indicators

Four nonlinearity indicators have been developed based on the work of Morfey and Howell [37]. The results of the application of each to the experimental data described in Chap. 3 are discussed here.

5.2.1 Ratio of $Q_{neg}$ to $Q_{pos}$

The PSD comparison method of assessing nonlinearity usually relies on the high frequency part of the spectrum to indicate the presence of nonlinearity. However, in model-scale facilities, this approach requires high fidelity, high bandwidth measurements. It would be beneficial to find an indicator that does not require high bandwidth. The indicators $Q_{pos}$ and $Q_{neg}$ were developed for this purpose.

$Q_{pos}$ and $Q_{neg}$ are determined by separating the quantity $\omega Q_{p}^2$ into negative and positive values, integrating each set of values over the corresponding frequencies, and taking the ratio of the results. The significance of $Q_{pos}$ is that it is a measure of the rate of energy transfer out of spectral components due to nonlinearity. Similarly, $Q_{neg}$ is a measure of the rate of nonlinear energy transfer into spectral components. Therefore, in the absence of losses and when no shocks are present, $Q_{neg}/Q_{pos}$ should equal unity. In jet noise $Q_{neg}$ usually involves the high frequency components of the spectrum, while $Q_{pos}$ involves the mid-frequencies. If the ratio is unity, $Q_{pos}$ can provide information about the importance of nonlinearity without requiring high bandwidth measurements. Of course, the propagation is not lossless and shocks may be present, so one would not expect that measured values of $Q_{neg}/Q_{pos}$ will equal unity. However, measurements of the ratio for initially single-frequency tones and band-limited noise in the plane wave tube and for broadband model-scale jet noise all show the ratio to be within a few percent (worst case) of unity for realistic propagation. Therefore, $Q_{pos}$ was investigated further as a candidate single-point indicator.

Another potential use of $Q_{neg}/Q_{pos}$ was also investigated. When shocks are present, there is a net loss of energy even in the absence of absorption. Thus, we postulated that the inequality $Q_{neg} < Q_{pos}$ should be true and that the ratio $Q_{neg}/Q_{pos}$ should be less than one if shocks are present.

Analysis of the single-frequency data taken in the plane wave tube result in
$Q_{neg}/Q_{pos} = 1$ to within 0.5% for $\sigma < 1$, but no decrease in the value of $Q_{neg}/Q_{pos}$ is seen for $\sigma > 1$. Likewise, the Blackstock bridging function and the Mendousse solution give indicator values of one over the range $0 \leq \sigma \leq 6$. Predictions made using the arbitrary wave Anderson algorithm do show $Q_{neg}/Q_{pos} = 1$ for $\sigma < 1$ $Q_{neg}/Q_{pos} < 1$ for $\sigma > 1$; the cause of this discrepancy is unknown.

Similar results were obtained for the noise data in the plane wave tube. Despite clear evidence of shocks in some of the waveforms, the value of the indicator does not deviate substantially from unity. The results for the model-scale jet data are very much like those for the noise data from the plane wave tube. These results have larger error in the shock-free region than do the single-frequency source data—on the order of 5%—but average values of the indicator are still close to one, and no trend away from one is seen even for higher source amplitudes, larger propagation distances, or conditions for which other indicators predict significant nonlinear effects.

Even though it is known [12] that shocks introduce energy losses in addition to atmospheric and boundary layer absorption, such losses are not detected by $Q_{neg}/Q_{pos}$, either because they represent too small a fraction of the total energy to be visible in $Q_{neg}/Q_{pos}$ or because the mechanism that causes losses at the shocks is separate from the nonlinear energy transfer mechanism seen by $Q_{neg}/Q_{pos}$. Regardless, the indicator $Q_{neg}/Q_{pos}$ cannot be used reliably to detect the presence of shocks.

Nevertheless, because the results do show that $Q_{pos} \approx Q_{neg}$ for many source and propagation conditions, $Q_{pos}$ alone is descriptive of the nonlinear energy transfer experienced by a waveform. Thus, $Q_{pos}$ can be used as a nonlinearity indicator.

### 5.2.2 Normalized $Q_{pos}$

Single-frequency source data in the plane wave tube confirm that the value of the indicator $Q_{pos}/P_{rms}^3$ is correlated with the Gol’dberg number, an assessment of the relative importance of nonlinearity and absorption. Increasing either the amplitude or the frequency of the source increases the value of the indicator at all propagation distances. When calculated for measured data, this normalized $Q_{pos}$ also indicates that the rate of nonlinear evolution grows initially as the signal propagates from
the source. Theory (the Blackstock bridging function and the Mendousse solution to the Burgers equation) concurs and suggests that nonlinearity’s importance, as indicated by the rate of change of harmonic amplitudes, decreases after a certain value of $\sigma$ has been reached. However, sufficient values of normalized distance were not attained in the laboratory to confirm this decrease experimentally.

Much the same conclusions can be drawn from the noise data in the plane wave tube. The importance of nonlinearity increases for higher source amplitudes, higher source frequency bands, and (within the propagation range available) for larger propagation distances.

The results for the model-scale jet data do not all follow the pattern seen in the plane wave tube data. For a low (0.85) Mach number jet, higher values of normalized $Q_{pos}$ are seen for off-peak radiation angles than for peak radiation angles. The explanation likely lies in the fact that noise levels for a Mach 0.85 jet are low enough that nonlinearity is unimportant, so $Q_{pos}$ is small and varies little with propagation angle. The root-mean-square pressure, however, is larger at the peak radiation angle than at off-peak angles. Thus, $Q_{pos}/P_{rms}^3$ takes on smaller values at the peak radiation angle than at off-peak angles for low Mach number jets.

Indicator values for a high (2) Mach number jet have the same amplitude dependence seen in the plane wave tube data. Higher values of the indicator occur near the peak radiation angle, where spectral levels are high, than at an off-peak angle, where spectral levels are lower.

A significant difference between the plane wave tube results and the model-scale jet results is the dependence of the indicator value on propagation distance. The plane wave data show that the value of the indicator increases with distance for small values of $\sigma$, but for the model-scale jet data, the indicator value decreases over the entire available propagation range. This is likely due to the fact that the model-scale jet data experience spherical spreading and the plane wave tube data do not. Unlike the dimensionless indicator discussed in Sec. 5.2.3, $Q_{pos}/P_{rms}^3$ does not provide information on the importance of nonlinearity relative to geometrical spreading.

In summary, the normalized $Q_{pos}$ indicator can be useful when comparing data with different source conditions but similar propagation conditions (i.e. geomet-
tical spreading). For both single-frequency and noise source data propagating as plane waves and for spherically propagating model-scale jet noise, the value of the indicator increases as source frequency and source amplitude increase. This is consistent with the assessment of the importance of nonlinearity provided by the Gol’dberg number. However, if the $Q_{\text{pos}}$ indicator is calculated for data whose spectral levels are too low and for which nonlinearity is unimportant, the value of the indicator tends to be larger for smaller-amplitude signals. Also, the dependence of the indicator value on propagation distance is not the same for all types of geometrical spreading, so the normalized $Q_{\text{pos}}$ cannot be used to compare measurements with different types of spreading.

5.2.3 Spectral Gol’dberg Number

The spectral Gol’dberg number $\Gamma_s$ proved very useful in assessing the potential of a signal to evolve nonlinearly. $\Gamma_s$ is a dimensionless quantity, so it can be used to compare signals with different source and propagation conditions. In keeping with the analogy between $\Gamma_s$ and the Gol’dberg number $\Gamma$, a spectral Gol’dberg number magnitude less than one indicates that absorption dominates nonlinearity in the evolution of a signal, and a magnitude greater than one indicates that nonlinearity dominates absorption. For nonplanar waves, a similar test can be done to weigh the effects of spreading against those of nonlinearity using the threshold value $1/\alpha r$, where $\alpha$ is the absorption coefficient and $r$ is the spherical propagation distance.

Another advantage of the spectral Gol’dberg number is its proportionality to $Q_{p^2p}$. In the definition

$$\Gamma_s = \frac{\omega \beta}{2\alpha \rho_0 c_0^3} \frac{Q_{p^2p}}{S_p},$$

(5.1)

all quantities on the right-hand side are strictly positive except for $Q_{p^2p}$. Thus, the sign of the quadspectral density (QSD) determines the sign of $\Gamma_s$. From the analysis in Chap. 2, we know that a positive value of the QSD indicates nonlinear energy loss and a negative value of the QSD indicates nonlinear energy gain; the same can be said for the spectral Gol’dberg number. Therefore, the direction of nonlinear energy transfer can be deduced from the sign of $\Gamma_s$. This fact, combined with the ability to compare linear and nonlinear effects discussed in the preceding paragraph, means that $\Gamma_s$ can be used to predict how a given spectrum will change,
both qualitatively and quantitatively, with propagation distance. For single-frequency source data, $\Gamma_s$ is only useful for frequency components whose spectral levels are above the system noise floor. For such components, the indicator can be used to assess the relative importance of nonlinear and absorption effects. Care must be taken, though, to determine whether a large value of the dimensionless indicator denotes strong nonlinear evolution or simply a small spectral density value. For the data analyzed in Chap. 4, the indicator is positive at the source frequency and negative at all subsequent harmonics. The rate of energy loss from the fundamental increases with propagation distance, while the rate of energy gain at the harmonics decreases with propagation distance. Nevertheless, the magnitude of the indicator at these harmonics remains larger than one regardless of propagation distance, indicating that the spectral amplitudes of the harmonics continue to grow for the normalized propagation distances achieved in the tube. Given that the largest value of normalized distance for the data shown is $\sigma = 1.32$, this is to be expected. The shape of the wave should continue to evolve until it becomes a sawtooth at $\sigma = 3$. The amplitudes of harmonics 2 and above are at their largest when the wave attains this most distorted sawtooth shape; the fact that $\Gamma_s$ indicates that these amplitudes are increasing for $\sigma < 3$ supports that conclusion.

For the noise data in the plane wave tube, the spectral Gol’dberg number indicates that nonlinearity is more important than absorption in the second harmonic band; the frequency range and magnitude of this dominance increase with amplitude. This is consistent with previous observations and with the fact that spectral amplitudes in this frequency region grow with distance. Significant nonlinear energy growth is also seen at frequencies below the source frequency. Where $\Gamma_s \approx -1$ in this region, nonlinear energy growth offsets losses due to absorption, and no change in level is seen in the PSD’s. Where $\Gamma_s < -1$ in this frequency region, nonlinearity dominates absorption, and the spectral levels increase with propagation distance.

The spectral Gol’dberg number serves to validate assumptions about the significance of nonlinearity in model-scale jet noise. Values of the indicator for low Mach number jets and off-peak radiation angles have magnitudes on the order of 0.01, well below the threshold of significance relative to absorption. For the high
Mach number (2) jet at an off-peak radiation angle ($90^\circ$), the linearly extrapolated spectra do not collapse well, possibly giving the impression that nonlinear energy transfer is occurring. However, the spectral Gol’dberg number does not predict any regions of nonlinear energy gain or loss for these data, and the magnitude of $\Gamma_s$ is less than 0.1 nearly everywhere.

Values of $\Gamma_s$ as high as 1.5 are seen for the peak radiation angle of a high Mach number jet. Evidence of nonlinear evolution in these data is seen for a large range of frequencies, with energy being lost from mid-frequencies and gained at higher and lower ones. The indicator places the most significance on nonlinearity in the peak frequency region of the model-scale jet data.

Comparisons of the spectral Gol’dberg number with the spreading term $1/\alpha r$ indicate that geometrical spreading is by far the dominant cause of spectral evolution for the model-scale jet noise. For the data examined, the spreading term was at least an order of magnitude greater than $\Gamma_s$ at most frequencies.

In summary, the spectral Gol’dberg number is most useful as a nonlinearity indicator for noise signals where noise floor issues are not as problematic as for signals with discrete frequency components. However, it can be very useful for noise signals for a number of reasons: its sign denotes the direction of nonlinear energy transfer; it can be used to assess the relative importance of nonlinearity and absorption; and it can provide a comparison of the importance of nonlinearity relative to geometrical spreading.

### 5.2.4 Coherence Indicator

The coherence indicator $\gamma_Q$ has many of the same advantages as the spectral Gol’dberg number $\Gamma_s$. It is dimensionless and thus facilitates comparisons between dissimilar experimental conditions. Like $\Gamma_s$, the sign of $\gamma_Q$ is completely determined by the sign of $Q_{p^2p}$; in the definition of the coherence indicator

$$\gamma_Q = \frac{Q_{p^2p}}{S_{p^2} S_p},$$

(5.2)

all quantities on the right-hand side are positive except $Q_{p^2p}$. As a result, a positive value of $\gamma_Q$ indicates nonlinear energy loss, and a negative value of $\gamma_Q$ indicates
nonlinear energy gain. These attributes make it a useful qualitative prediction tool.

While there is no “threshold of importance” value of the coherence indicator that can be used to compare the influence of linear and nonlinear effects, the value of $\gamma_Q$ is bounded between -1 and 1. An indicator value of zero suggests that no nonlinear distortion is occurring, while an indicator value of -1 or 1 reveals that nonlinear evolution is very important. Also, because $\gamma_Q$ is bounded, trends of very large magnitude are not seen for data with small spectral amplitudes near or beneath the noise floor. Thus, $\gamma_Q$ can be used for small-amplitude signals with more confidence than $\Gamma_s$.

Applied to the single-frequency source data from the plane wave tube, the coherence indicator clearly shows the nonlinear “cascading” of energy to progressively higher harmonics as propagation distance increases. Its value is almost universally one at the source frequency and negative one for all other harmonics with spectral levels above the noise floor, confirming our expectation that energy is transferred nonlinearly from the fundamental to the higher harmonic components.

For the noise data from the plane wave tube, $\gamma_Q$ predicts nonlinear energy gain at frequencies immediately above and below the source frequency band and nonlinear energy loss within the source band. Additionally, according to the indicator, nonlinear energy gain occurs at progressively higher frequencies as propagation distances increases. This behavior confirms our expectations for the evolution of the spectrum based on the theory of nonlinear acoustic propagation.

Results for the coherence indicator for model-scale jet noise are also promising. Unlike the spectral Gol’dberg number, it confirms common assumptions about the importance of nonlinearity for different Mach numbers and propagation angles. For low Mach numbers and off-peak radiation angles, the value of the indicator is very small, and no frequency regions can be identified in which either nonlinear growth or decay is predominant. For a high Mach number and a peak radiation angle, $\gamma_Q$ is positive in the peak frequency region, indicating that energy is being lost nonlinearly, and positive at higher frequencies, indicating that energy is being gained nonlinearly. While such behavior is not readily apparent in the spectra for this condition, it is what we would expect for a jet with high radiated acoustic power. Such a result indicates that the coherence indicator achieves the goal of
identifying nonlinear evolution even when such evolution cannot be seen in the spectra.

5.2.5 Summary of Nonlinearity Indicators

All the nonlinearity indicators discussed above have some advantages. $Q_{neg}/Q_{pos}$ confirms that energy is conserved for shock-free propagation and suggests that $Q_{pos}$ may be used alone as an indicator. The quantity $Q_{pos}/p_{rms}^3$ fulfills the goal of developing an indicator that does not require high bandwidth measurements. The coherence indicator $\gamma_Q$ denotes the direction of nonlinear energy transfer and does not become unstable when spectral components are in the noise floor.

The most useful indicator has proven to be the spectral Gol’dberg number $\Gamma_s$. Not only does this quantity indicate the direction of nonlinear energy transfer, its magnitude can be compared with one to ascertain the relative importance of nonlinearity and absorption, and with the quantity $1/\alpha r$ to determine the relative importance of nonlinearity and geometrical (spherical) spreading. With the results of these comparisons it is possible to predict the nature of the spectral evolution of a signal at any frequency component.

The only goal not met by the spectral Gol’dberg number is the small bandwidth requirement. Because $\Gamma_s$ is a spectral quantity, though, it is possible to learn something about the nonlinear evolution of a signal by examining only a limited bandwidth portion of $\Gamma_s$. As was seen in the plane noise data and model-scale jet data, $\Gamma_s$ is generally negative at low frequencies where energy is gained through difference-frequency generation and positive at mid-frequencies where energy is lost nonlinearly. Thus, knowledge of the spectral Gol’dberg number for only low and mid-frequencies is sufficient to convey some information about nonlinearity. What is not known is what the bandwidth of a measurement would have to be in order to ensure an accurate calculation of $\Gamma_s$ through the mid-frequency range. This could be done by filtering the existing data and comparing band-limited results with the full-bandwidth results presented in this work.
5.3 Future Work

It was postulated in this work that the ratio $Q_{neg}/Q_{pos}$ should be equal to one for shock-free propagation and should decrease after shocks have formed. This behavior was not seen for any of the experimental data or for analytic solutions, namely the Blackstock bridging function and the Mendousse solution to the Burgers equation. The results for these cases shown in Chap. 4 reveal that $Q_{neg}/Q_{pos}$ is essentially equal to one regardless of the presence of shocks. As was mentioned in Chap. 2, it is well-documented that the presence of shocks introduces a loss mechanism unrelated to atmospheric and boundary layer absorption; this mechanism can be characterized by defining a finite-amplitude absorption coefficient $\alpha_f$. However, although this loss mechanism is a direct consequence of nonlinear distortion, it does not appear to affect the value of $Q_{neg}/Q_{pos}$. Further investigation is warranted to determine whether losses at the shocks are truly not detected by the QSD and thus by $Q_{neg}/Q_{pos}$ or whether such losses simply represent a negligible fraction of the total energy in the wave and thus are not easily seen by any means.

While not completely dimensionless, the normalized $Q_{pos}$ indicator $Q_{pos}/p_{rms}^3$ proved useful for determining the relative importance of nonlinear evolution among signals with similar source and propagation conditions. Generally, periodic signals with larger Gol’dberg numbers exhibit higher values of the normalized $Q_{pos}$ indicator, as do noise signals with higher band-center frequencies and source amplitudes. However, no significance has been attached to the numerical value of the indicator. The usefulness of this indicator would be enhanced if a universal threshold of significance or benchmark value could be established. This would facilitate comparisons between dissimilar data sets and possibly eliminate the need for comparisons (as is true for the Gol’dberg number: the value of $\Gamma$ relative to unity provides information about the potential for nonlinear evolution of a signal).

While all of the indicators discussed in this work are “single-point” quantities—that is, they can be calculated from a measurement at a single point—it would be especially beneficial for jet noise if these spectral quantities could be reduced to a single value. This way, a two- or three-dimensional “map” of the importance of nonlinearity could be constructed, providing valuable information about the location and directivity of the source.
Appendix A

Compression Driver Specifications

A.1 JBL 2426H Compression Driver
Figure A.1. Specifications sheet for a JBL 2426H compression driver.
2426H/J Compression Driver

external attachment points are also provided. A machined ring of copper surrounds the pole piece to counteract the inductance of the voice coil at high frequencies. After manufacture, the frequency response of each driver is tested for conformity to rigid performance standards.

The 2420H is offered in an 8 ohm impedance, and the otherwise identical 2420J is offered in a 16 ohm impedance.

Architectural Specifications:

The compression driver shall consist of a ferrite magnetic structure. All magnetic assembly parts shall be machined from cast or extruded billet stock. The flange plug shall be assembled of concentric rings to minimize phase cancellations, and it shall be further coupled to a tapered throat. Driver mounting shall consist of a removable mount incorporating standard 25mm (1 in) three bolt pattern and fitted with three external attachment points. The mount shall be removable to allow the use of a 1.38” - 18 screw-thread attachment.

The diaphragm shall be 0.69 mm (0.027 in) polyimide nominally drawn to shape. High frequency response shall be controlled through the use of a three-dimensional suspension structure. The suspension may be suspended from titanium ribbon of not less than 48 mm (1 1/2 in) in diameter, operating in a magnetic field of not less than 1.8 Tesla (16,000 gauss). An impedance controlling ring shall be affixed to the pole piece in order to increase efficiency at high frequencies.

Performance specifications of a typical production unit shall be as follows: Measured sensitivity with a 1 mW input on a 25 mm (1 in) terminated tube, averaged from 800 Hz to 2.5 kHz, shall be at least 107 dB SPL. Measured sensitivity with a 1 W input in 1 m distance on-axis from the mouth of a horn with a Q of 6.3, averaged in the 2 kHz octave band, shall be at least 115 dB SPL. As an indication of electromechanical conversion efficiency, the B factor shall be at least 6.5 (0.5 newtons per ampere). Frequency response, measured on a terminated tube, shall be flat within ±1 dB from 800 Hz to 3.5 kHz, with a 6 dB/octave roll-off above that point. Nominal impedance shall be 8 [ohm] ohms and power capacity shall be at least 70 watts normal speech or music program material.

The compression driver shall be JBL Model 2426 H/J. Other drivers will be considered for equivalency provided that substantiated data from a recognized independent test laboratory verify that the above performance specifications are met.

Figure A.2. Specifications sheet for a JBL 2426H compression driver.
Bibliography


Lauren Elizabeth Mize was born on Dec. 20, 1978 in Columbus, OH, to Michael A. and Barbara B. Mize. She attended Woodrow Wilson Elementary School and Westminster School in Oklahoma City, OK, and graduated salutatorian from Bishop McGuinness Catholic High School in the same city on May 13, 1997. She attended Rhodes College in Memphis, TN under the auspices of a Morse scholarship. While at Rhodes, she participated in two summer research experiences. The first was at the High Altitude Observatory of the National Center for Atmospheric Research in Boulder, CO, and the second was at the Laboratoire Charles Fabry of the CNRS, in Orsay, France. Her degree, a B.S. in physics with a minor in music, was received in 2001 along with the Peyton Nalle Rhodes Phi Beta Kappa prize.

Lauren matriculated at The Pennsylvania State University in August 2001 as a Ph.D. student in The Graduate Program in Acoustics. While a student, she was active both on campus and off, assisting with hands-on acoustics demonstrations for middle- and high-school students, playing violin in the Nittany Valley Symphony, and participating as an active member of the State College Presbyterian Church. It was there that she met Matthew T. Falco, originally of Altoona, PA, and a 2001 graduate of Penn State in civil engineering. The two were married in Oklahoma City on May 17, 2003. They currently reside in State College and are expecting their first child in June of 2007.