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THERMODYNAMICAL FORMALISM FOR MAPS WITH INDUCING SCHEMES

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Abstract

We study the thermodynamical formalism for a class of systems admitting inducing schemes. Following [PS06],[PS05], we discuss the general procedure for the existence and uniqueness of equilibrium measure among the liftable class. This will be covered in Chapter 1.

To obtain equilibrium measure in the usual sense we need to study the liftable class, and prove that the equilibrium measure is in this class. In Chapter 3 we study this problem. We demonstrate two approaches to liftability: first, we use techniques of [Kel89] and [Zwe05] to study the structure of the inducing scheme directly and provide a sufficient condition and some examples. Second, following [Bru95], we use liftabilty result to the Markov extension of a piecewise invertible map to show the liftability of the inducing schemes that can be "embedded" to the Markov extension. These results appeared in [PZ07], [PSZ07].

We will also use this theory to study certain examples. In Chapter 2 we study uniformly expanding system with a class of non-Hölder potential, prove existence and uniqueness of equilibrium measure, and an example of a phase transition. In Chapter 4, we show that for systems admitting Young's tower with potentials of the type $\varphi_t(x) = -t \log|\det Df^u(x)|$, there is an equilibrium measure among the liftable class.

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List of Symbols

Card E	Cardinality of the finite set E. p. 34
$\bar{\varphi}$	Induced potential function for φ, p. 10
φ^+	Induced potential function for $\varphi - s_\varphi$, p. 12
φ_ϵ^+	Induced potential function for $\varphi - s_\varphi + \epsilon$, p. 12
$\gamma^u(\gamma^s)$	The unstable (stable) disk for Young's tower. p. 53
$h_\mu(f)$	Metric entropy of μ, p. 1
$h_{top}(f)$	Topological entropy of f, p. 47
$(\tilde{I}, \tilde{f})$	The abstract tower system constructed from an inducing scheme. p. 32
$\check{I}, \check{f}$	The Markov extension of a piece-wise invertible system (I, f). p. 45
$\mathcal{I}, f_e$	The natural extension of the system (I, f). p. 46
$i(\mu)$	Induced measure for μ, p. 11
inc	The inclusion map from W to the first level of the Markov extension $(\check{I}, \check{f})$. p. 45
μ_φ	The equilibrium measure in the class $\mathcal{M}_L(X, f)$ for the potential φ. p. 2, p. 11
$\mathcal{M}(\cdot)$	Set of all Borel invariant probability measure for a certain dynamical system, p 10
$\mathcal{M}_L(X, f)$	Set of all liftable measures in (X, f), p. 11
∂P	The boundary set for a partition P. p. 44

$P_G(\Phi)$	Gurevich pressure of Φ, p. 5
$\pi(\nu)$	Lifted measure for ν, p. 10
Q_ν	The integral of the inducing time with respect to ν. p. 10
Σ_A	The countable Markov shift with transition matrix A. p. 16
$\tau(J)$	The inducing time of the inducing interval J. p. 2, 3
(W, F)	The induced system (with respect to the original system (X, f)), p. 2, 3
$V_n(\Phi)$	$n-$variation of Φ, p. 5

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Chapter 1

Inducing schemes and equilibrium measures

1.1 Introduction

Thermodynamical formalism is a collection of methods aimed to produce special invariant measures for dynamical systems. More precisely, let f be a continuous map of a compact topological space I and $\mathcal{M}(f, I)$ the class of all invariant Borel probability measures on I. Given a continuous real valued *potential* function φ on I, one considers the *equilibrium* measures for φ, i.e., invariant Borel probability measures μ on I for which the supremum

$$\sup_{\mu \in \mathcal{M}(f,I)} \left\{ h_\mu(f) + \int_I \varphi \, d\mu \right\} \tag{1.1}$$

is attained, where $h_\mu(f)$ denotes the metric entropy of the map f. In the classical theory if the system is a subshift of finite type and the potential is Hölder continuous, the equilibrium measure exists, is unique, and has good mixing properties. Systems that admits a Markov partition (systems with uniformly hyperbolic structure, e.g. Anosov maps and axiom A maps) can be represented by subshifts of finite type, and thermodynamical formalism can be effected for those systems.

It is hard to build Markov partitions for general systems that is not uniformly hyperbolic. However, of some systems with nonzero Lyapunov exponents one can build *towers* – in this case, the partition is not Markov, but it will be under a certain iterate of f depending on the partition element. Further more, the partition is countable instead of finite. This allows the study of potentials that is not Hölder continuous, for example the potential $\varphi = -t \log |df|$, which goes to infinity near a critical point. We will need to use the theory for countable Markov shifts, see section 1.3.

We will first describe intuitively the strategy to effect thermodynamical formalism for maps with towers, rigorous definitions will be given in the next section.

For a map f a tower is determined by a subset $W \subset I$, the *base* of the tower, a positive integer-valued function τ on W, the *inducing time*, the map $F : W \to W$, the *induced map*, and a countable partition $\mathcal{R}$ of W. The function τ is constant on each partition element J and is a return time of J to W but it is not necessarily the first return time to the base. The relation between the original map f and the induced map F is given by $F(x) = f^{\tau(J)}(x)$ for each $x \in J$. A crucial feature of the tower is that $\mathcal{R}$ is a generating Bernoulli partition for the induced map F so that it is equivalent to the full (one- or two-sided) shift on a countable set of states.

In [PS06, PS05], Pesin and Senti developed thermodynamical formalisms for general systems admitting tower constructions. In particular, they described a class of potential functions for which equilibrium measures exist and are unique; see Section 1.5 for detailed definitions and results. The crucial elements in constructing equilibrium measures are the following:

1) Starting with a continuous potential function φ on I one obtains the *induced potential function* $\bar{\varphi}$ on the base W by the formula $\bar{\varphi}(x) = \sum_{k=0}^{\tau(x)-1} \varphi(f^k(x))$.

2) One finds equilibrium measures for the induced map F with respect to the induced potential function $\bar{\varphi}$. Since F is conjugate to the full shift on a countable set of states one can apply the known results for the symbolic spaces (for 2-sided shifts some additional discussion is needed), see section 1.3. This leads to certain requirements on potential function φ and thus determines the desired class of potentials.

3) One then lifts equilibrium measures to the tower. This produces an f-invariant Borel probability measure μ_φ which is a unique equilibrium measure in somewhat restricted sense: it minimizes the free energy $E = -(h_\mu(f) + \int_I \varphi \, d\mu)$ among not all but only *liftable measures*, i.e., the measures that can be obtained

by lifting to the tower F-invariant measures on the base. The exact form of this result is presented in Proposition 1.5.3.

One immediate question is, then, is the measure we found the equilibrium measure among all invariant probability measures, not just liftable ones. This raises the liftability problem: *Given a map f admitting an inducing scheme $\{S, \tau\}$, describe the class of all liftable measures.* The discussion of this question will be postponed to Chapter 3, in the remaining part of this chapter we will first describe the methods of Pesin and Senti.

1.2 Inducing Schemes

We describe the class of systems admitting inducing schemes which were introduced in [PS06, PS05]. Let $f : I \to I$ be a continuous map of a compact topological space I, S a countable collection of disjoint Borel sets, and $\tau : S \to \mathbb{N}$ a positive integer-valued function. Let also $W = \bigcup_{J \in S} J$ be the *inducing domain* and $\tau : I \to \mathbb{N}$,

$$\tau(x) = \begin{cases} \tau(J), & x \in J \in S, \\ 0, & x \neq W \end{cases}$$

the *inducing time*.

Let $\overline{J}$ denote the closure of the set J. We assume that the following conditions hold:

(H1) For each $J \in S$ there exists a connected neighborhood U_J of J such that $f^{\tau(J)}|U_J$ is a homeomorphism onto its image and $f^{\tau(J)}(J) \subseteq W$;

(H2) the partition $\mathcal{R}$ of W induced by the sets $J \in S$ is "one-sided" generating: for any countable collection of elements $\{J_k\}_{k \in \mathbb{N}}$, the intersection

$$\overline{J_1} \cap \Big(\bigcap_{k \geq 2} f^{-T_k}(\overline{J_k}) \Big)$$

is nonempty and consists of a single point, where $T_k = \sum_{m=1}^{k-1} \tau(J_m)$.

Define the *induced map* $F : W \to W$ by $F(x) = f^{\tau(x)}(x)$ and set

$$X = \bigcup_{J \in S} \bigcup_{k=0}^{\tau(J)-1} f^k(W \cap J). \tag{1.2}$$

The set X is forward invariant under f.

In view of (H2), the induced map $F : W \to W$ is conjugate to the one-sided Bernoulli shift σ on a countable set of states S. More precisely, this means the following (see [PS06, PS05]). Define the coding map $h : S^{\mathbb{N}} \to \overline{W}$ by

$$h : \omega = (a_0, a_1, \cdots) \mapsto x_\omega, \tag{1.3}$$

where $x_\omega = \bigcap_{n \geq 0} F^n(J_{a_n})$.

Proposition 1.2.1. *The following statements hold:*

1. *the map h is well-defined, continuous and $W \subset h(S^{\mathbb{N}})$;*
2. *h is one-to-one on $h^{-1}(W)$;*
3. *the induced map $F : W \to W$ is topologically conjugate to the one-sided Bernoulli shift $\sigma : S^{\mathbb{N}} \to S^{\mathbb{N}}$ via h, i.e., $h \circ \sigma | h^{-1}(W) = F \circ h | h^{-1}(W)$.*

In what follows we assume that the following condition holds

(H3) the set $S^{\mathbb{N}} \setminus h^{-1}(W)$ is negligible with respect to any ergodic measures which are positive on open sets.

If ν is a Gibbs measure for the shift σ (see the next section) then Condition (H3) allows one to transfer it via the conjugacy map h to a measure which gives full measure to the base W and is invariant under the induced map F.

Inducing schemes satisfying Conditions (H1)–(H3) can be constructed for many one-dimensional maps (including some unimodal and multimodal maps) and some expanding maps. In these cases one has $f^{\tau(J)}(J) = W$. However, in the case of (nonuniformly) hyperbolic dynamical systems admitting tower construction Condition (H1) holds with $f^{\tau(J)}(J)$ strictly inside of W and Condition (H2) should be replaced with the following one:

(H2') the partition $\mathcal{R}$ of W induced by the sets $J \in S$ is "two-sided" generating: for any countable collection of elements $\{J_k\}_{k\in\mathbb{Z}}$, the intersection

$$\overline{J_0} \cap \Big(\bigcap_{k\geq 1} f^{-T_k}(\overline{J_k})\Big) \cap \Big(\bigcap_{k\leq -1} f^{T'_k}(\overline{J_k})\Big) \tag{1.4}$$

consists of a single point, where

$$T_k = \sum_{m=0}^{k-1} \tau(J_m), \quad T'_k = \sum_{k=m+1}^{0} \tau(J_m).$$

In this case the induced map $F : W \to W$ is conjugate to the two-sided Bernoulli shift σ on a countable set of states S.

1.3 Thermodynamics for countable Bernoulli shifts

Here we briefly describe the theory needed to treat the countable Bernoulli shifts (both one-sided and two-sided). For more details see Mauldin and Urbański [MU01] and Sarig [Sar99, Sar03] (see also Aaronson, Denker and Urbanski [ADU93], Yuri [Yur99], and Buzzi and Sarig [BS03]).

1.3.1 One-sided Bernoulli shifts

Given a function $\Phi : S^{\mathbb{N}} \to \mathbb{R}$, the $n-$*variation* $V_n(\Phi)$ is defined by

$$V_n(\Phi) = \sup_{[b_0,\cdots,b_{n-1}]} \sup_{a,a'\in[b_0,\cdots,b_{n-1}]} \{|\Phi(a) - \Phi(a')|\}$$

where the cylinder set $[b_0, \cdots, b_{n-1}]$ consists of all sequences $a = (a_k)_{k\in\mathbb{N}}$ such that $a_k = b_k$, $k = 0, \cdots, n-1$.

The *Gurevich pressure* of Φ is defined by:

$$P_G(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n(a)=a} \exp \Phi_n(a) 1_{[b]}(a) \tag{1.5}$$

where $b \in S$ and $\Phi_n(a) = \sum_{k=0}^{n-1} \Phi(\sigma^k(a))$. One can show (see [Sar99]) that if

$\sum_{n\geq 1} V_n(\Phi) < \infty$ the limit in (1.5) exists, does not depend on b and

$$P_G(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n(a)=a} \exp \Phi_n(a).$$

We call a measure $\nu = \nu_\Phi$ a *Gibbs measure* for Φ if there exist constants C_1 and C_2 such that for any cylinder set $[b_0, \cdots, b_{n-1}]$ and any $a \in [b_0, \cdots, b_{n-1}]$ we have

$$C_1 \leq \frac{\nu([b_0, \cdots, b_{n-1}])}{\exp(-nP_G(\Phi) + \Phi_n(a))} \leq C_2.$$

Denote by

$$\mathcal{M}_\Phi(\sigma) = \{\nu : \sigma - \text{invariant Borel probability measure on } S^{\mathbb{Z}} \\ \text{with } -\int_{S^{\mathbb{N}}} \Phi d\nu < \infty.\}$$

We call a $\sigma-$invariant measure $\nu = \nu_\Phi$ an equilibrium measure for Φ if

$$\sup_{\nu\in\mathcal{M}_\Phi(\sigma)} \{h_\nu(\sigma) + \int \Phi d\nu\} = h_{\nu_\Phi}(\sigma) + \int \Phi d\nu_\Phi. \tag{1.6}$$

Proposition 1.3.1. *The following statements hold:*

1. *Assume that* $\sup_{a\in S^{\mathbb{N}}} \Phi < \infty$ *and* $\sum_{n\geq 2} V_n(\Phi) < \infty$. *Then the variational principle for* Φ *holds:*

$$P_G(\Phi) = \sup_{\nu\in\mathcal{M}_\Phi(\sigma)} \{h_\nu(\sigma) + \int \Phi d\nu\};$$

2. *Assume that* $\sup_{a\in S^{\mathbb{N}}}$, $P_G(\Phi) < \infty$ *and*

$$\sum_{n\geq 1} V_n(\Phi) < \infty.$$

Then there exists an ergodic $\sigma-$*invariant Gibbs measure* ν_Φ *for* Φ. *If in addition, the entropy* $h_{\nu_\Phi}(\sigma) < \infty$ *then* $\nu_\Phi \in \mathcal{M}_\Phi$ *and is a unique Gibbs and equilibrium measure.*

1.3.2 two-sided Bernoulli shifts

We will need to deal with two-sided shift space in the case of a hyperbolic inducing scheme, i.e. an inducing scheme that satisfies condition (H1) and (H2'). In this case, let us consider a potential function $\Phi : S^{\mathbb{Z}} \to \mathbb{R}$, the n-variation is defined by:

$$V_n(\Phi) = \sup_{[b_{-n+1},\cdots,b_{n-1}]} \sup_{a,a' \in [b_{-n+1},\cdots,b_{n-1}]} \{|\Phi(a) - \Phi(a')|\},$$

note that the cylinder $[b_{-n+1}, \cdots, b_{n-1}]$ refers to the 2-sided cylinder from position $-n+1$ to $n-1$. The Gurevich pressure of Φ is defined as:

$$P_G(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n(a)=a} \exp \Phi_n(a) 1_{[b]}(a),$$

where $b \in S$ and

$$\Phi_n(a) = \sum_{k=0}^{n-1} \Phi(\sigma^k(a)).$$

Note that the definition only depend on the positive side of the sequences, it is no longer obvious that the limit must exist, the fact that it does is covered Proposition 1.3.4. We call a measure $\nu = \nu_\Phi$ a *Gibbs measure* for Φ if there exist constants C_1 and C_2 such that for any *one-sided* cylinder set $[b_0, \cdots, b_{n-1}]$ and any $a \in [b_0, \cdots, b_{n-1}]$ we have

$$C_1 \leq \frac{\nu([b_0, \cdots, b_{n-1}])}{\exp(-nP_G(\Phi) + \Phi_n(a))} \leq C_2. \tag{1.7}$$

Notice that in this definition we still only take the *positive cylinders* although we are considering the 2-sided shift space.

We have the following:

Proposition 1.3.2. *1. Assume that* $\sup_{a\in S^{\mathbb{Z}}} \Phi < \infty$ *and*

$$\sum_{n\geq 1} nV_n(\Phi) < \infty. \tag{1.8}$$

then $P_G(\Phi)$ exists and the variational principle for Φ holds:

$$P_G(\Phi) = \sup_{\nu \in \mathcal{M}_\Phi(\sigma)} \{h_\nu(\sigma) + \int \Phi d\nu\};$$

2. *Assume that $\sup_{a \in S^{\mathbb{Z}}} \Phi < \infty$, $P_G(\Phi) < \infty$ and (1.8) hold. Then there exists an ergodic $\sigma-$invariant Gibbs measure ν_Φ for Φ. If in addition, the entropy $h_{\nu_\Phi}(\sigma) < \infty$ then $\nu_\Phi \in \mathcal{M}_\Phi$ and is a unique Gibbs and equilibrium measure.*

We remark that for the 2-sided version we require a stronger summability condition in (1.8).

To extend the 1-sided results to 2-sided shifts, a technique in [Bow75] can be applied with minor modification. We have the following:

Proposition 1.3.3 (See [Bow75])**.** *Suppose that $\sum_{n \geq 1} nV_n(\Phi) < \infty$, then $\exists$ a bounded function $u(a)$ such that*

$$\Psi(a) = \Phi(a) + u(\sigma(a)) - u(a)$$

satisfies: $\Psi(a) = \Psi(a')$ whenever $a_i = a'_i$ for all $i \geq 0$. Also:

$$\sum_{n \geq 1} V_n(\Psi) < +\infty.$$

Further more, if $\exists A > 0$, $0 < \gamma < 1$, such that $V_n(\Phi) < A\gamma^n$, for $n \geq 1$ then $\exists B > 0$, $0 < \eta < 1$ such that

$$V_n(\Psi) < B\eta^n; \quad n \geq 1.$$

Proof. For $s \in S$, pick a sequence $\{x_{k,s} \in S\}_{k=-\infty}^{0}$ such that $x_{0,s} = s$. Define $r : S^{\mathbb{Z}} \to \mathbb{Z}$ by $r(a)_k = a_k$ for $k \geq 0$ and $r(a)_k = x_{k,a_0}$ for $k < 0$. Define

$$u(a) = \sum_{j=0}^{\infty} (\Phi(\sigma^j(a)) - \Phi(\sigma^j r(a))). \tag{1.9}$$

We have $|\Phi(\sigma^j(a)) - \Phi(\sigma^j(r(a)))| \leq V_j(\Phi)$, hence (1.9) is well defined. We have the following calculations:

For $a, a' \in [b_{-n+1}, \cdots, b_{n-1}]$,

$$|u(a) - u(a')| \leq 4 \sum_{j \geq [\frac{n-1}{2}]} V_j(\Phi),$$

from which we obtain $\sum_{n\geq 1} V_n(u) \leq \infty$. We also have

$$\Psi(a) = \sum_{j=0}^{\infty} (\Phi(\sigma^j r(a)) - \Phi(\sigma^j r(\sigma a))),$$

which only depend on a_i with $i \geq 0$. Finally we have

$$|\Psi(a) - \Psi(a')| \leq V_n(\Phi) + 8 \sum_{j \geq [\frac{n-2}{2}]} V_n(\Phi),$$

hence $\sum_{n\geq 1} V_n(\Psi) < \infty$. When $V_n(\Phi) < A\gamma^n$, it follows that $V_n(\Psi) < B\eta^n$ for some $0 < \eta < 1$ and $B > 0$. □

Let $\tilde{\Psi} : S^{\mathbb{N}} \to \mathbb{R}$ be such that $\tilde{\Psi}(x) = \Psi(\pi^{-1}(x))$, where $\pi : S^{\mathbb{Z}} \to S^{\mathbb{N}}$ is the canonical projection. $\tilde{\Psi}$ is well defined since Ψ only depends on the positive side.

To see that Proposition 1.3.2 holds, we first observe the following facts:

Proposition 1.3.4. *1. If $\sup \Phi < \infty$, then $\sup \Psi < \infty$. Also $\sup_{a\in S^{\mathbb{Z}}} \Psi(a) = \sup_{a\in S^{\mathbb{N}}} \tilde{\Psi}(a)$.*

2. $P_G(\Phi)$ exists and $P_G(\Phi) = P_G(\Psi) = P_G(\tilde{\Psi})$. Note that by $P_G(\tilde{\Psi})$ we mean the Gurevich pressure defined in the space $S^{\mathbb{N}}$.

3. For any ν invariant Borel probability measure on $S^{\mathbb{Z}}$, we have: $\int \Phi d\nu = \int \Psi d\nu$. In particular, $\mathcal{M}_\Phi(\sigma) = \mathcal{M}_\Psi(\sigma)$.

4. For any ν invariant Borel probability measure on $S^{\mathbb{Z}}$, Let $\bar{\nu} = \nu \circ \pi^{-1}$ on $S^{\mathbb{N}}$, then $\int_{S^{\mathbb{Z}}} \Psi d\nu = \int_{S^{\mathbb{N}}} \tilde{\Psi} d\bar{\nu}$, $h_\nu(\sigma) = h_{\bar{\nu}}(\sigma)$.

On the other hand, for any invariant Borel probability measure $\bar{\nu}$ on $S^{\mathbb{N}}$, there exists a unique invariant Borel probability measure ν on $S^{\mathbb{Z}}$, such that $\bar{\nu} = \nu \circ \pi^{-1}$.

When Φ satisfies the conditions of Proposition 1.3.2, with the above construction we can see that $\tilde{\Psi}$ satisfies the conditions for Proposition 1.3.1. The variational principle holds because $P_G(\Phi) = P_G(\tilde{\Psi})$ and

$$\sup_{\nu \in \mathcal{M}_\Phi(\sigma)} \{h_\nu(\sigma) + \int \Phi d\nu\} = \sup_{\bar{\nu} \in \mathcal{M}_{\tilde{\Psi}}(\sigma)} \{h_{\bar{\nu}}(\sigma) + \int \tilde{\Psi} d\bar{\nu}\}.$$

Let $\bar{\nu}_{\tilde{\Psi}}$ be the unique Gibbs measure for $\tilde{\Psi}$, let ν_Φ be its natural extension to the 2-sided shift. There is a one-to-one correspondence between invariant measures of the original space and its natural extension which preserves entropy, it follows that n_Φ is the unique Gibbs and equilibrium measure for Φ.

1.4 Induced measure

Our next step is to describe some relations between invariant measures for f and those for F. For a Borel probability measure ν on W set

$$Q_\nu = \sum_{J \in S} \tau(J)\, \nu(J) = \int_W \tau(x)\, d\nu(x).$$

Define the measure $\pi(\nu)$ on the tower X (see (1.2)) as follows: for any Borel subset $E \subset X$,

$$\pi(\nu)(E) = \frac{1}{Q_\nu} \sum_{J \in S} \sum_{k=0}^{\tau(J)-1} \nu(f^{-k}(E) \cap J).$$

For the following result see for example, [dMvS93].

Proposition 1.4.1. *Let $\nu \in \mathcal{M}(F, W)$, and $Q_\nu < \infty$. Then $\pi(\nu) \in \mathcal{M}(f, I)$, $\pi(\nu)(X) = 1$, and $\pi(\nu)|_W \ll \nu$. If ν is ergodic, so is $\pi(\nu)$.*

Given a potential function $\varphi : I \to \mathbb{R}$, define the *induced potential function* $\bar{\varphi} : W \to \mathbb{R}$ by

$$\bar{\varphi}(x) = \sum_{k=0}^{\tau(J)-1} \varphi(f^k(x)), \quad x \in J. \tag{1.10}$$

Although the induced map F may *not* be the first return map, Abramov's formula, connecting the entropies of F, and f and Kac's formula, connecting the integrals of φ and $\overline{\varphi}$, still hold (see [Zwe05], for related results see also Keller [Kel89]).

Theorem 1.4.2. *Let ν be an F-invariant Borel probability ergodic measure on W with $Q_\nu < \infty$. Then*

$$h_\nu(F) = Q_\nu h_{\pi(\nu)}(f) < \infty.$$

If $\int_W \bar{\varphi} d\nu$ is finite then

$$-\infty < \int_W \bar{\varphi}\, d\nu = Q_\nu \int_X \varphi\, d\pi(\nu) < \infty.$$

We call a measure $\mu \in \mathcal{M}(f, I)$ *liftable* if $\mu(W) > 0$ and there exists a measure $i(\mu) \in \mathcal{M}(F, W)$ such that $i(\mu) \ll \mu$ and $\mu = \pi(i(\mu))$. We call $i(\mu)$ the *induced measure* for μ. The following result is proved in [Zwe05].

Proposition 1.4.3. *For any liftable ergodic measure $\mu \in \mathcal{M}(f, I)$, the measure $i(\mu)$ is unique, ergodic, and $Q_{i(\mu)} < \infty$.*

1.5 Equilibrium measure among the liftable class

For the simplicity of arguments, we will use the same terminology for (W, F) and its counter part in the symbolic space. For example, we will refer to a function ϕ on W as having summable variation if it does after conjugating to the symbolic space, and $P_G(\phi)$ its Gurevich pressure on the symbolic space, etc.

We establish the equilibrium measure among liftable class for the map (X, f).

Denote by $\mathcal{M}_L(f, X)$ the class of all liftable measures. We call a measure μ_φ an *equilibrium measure* (with respect to the class of measures $\mathcal{M}_L(f, X)$) if

$$s_\varphi := \sup_{\mathcal{M}_L(f,X)} \{h_\mu(f) + \int_X \varphi\, d\mu\} = h_{\mu_\varphi}(f) + \int_X \varphi d\, \mu_\varphi.$$

We stress that the definition of equilibrium measure introduced here differs from the classical one as only measures in $\mathcal{M}_L(f, X)$ are allowed.

Let $\bar{\varphi}$ be the induced potential function (see (1.10)).

Proposition 1.5.1 (see [PS06, PS05])**.** *Assume that $\bar{\varphi}$ has summable variations and finite Gurevich pressure. Then*

$$-\infty < s_\varphi < \max\{0, P_G(\Phi)\} < \infty.$$

Consider the induced potential function for the *normalized* potential $\varphi - s_\varphi$, i.e., $\varphi^+ = \overline{\varphi - s_\varphi} = \bar{\varphi} - s_\varphi \tau$.

As an immediate corollary of Proposition 1.3.2 we obtain the following result.

Proposition 1.5.2 (see [PS06, PS05]). *Assume that the induced potential function $\bar{\varphi}$ has summable variations and finite Gurevich pressure. Assume also that the function φ^+ has finite Gurevich pressure, and* $\sup_{x \in W} \varphi^+ < \infty$. *Then*

1. *there exists an F-invariant ergodic Gibbs measure ν_{φ^+} for φ^+;*
2. *if $Q_{\nu_{\varphi^+}} < \infty$, then $\nu_{\varphi^+} \in \mathcal{M}_{\varphi^+}(F, W)$ and*

$$\sup_{\mathcal{M}_{\varphi^+}(F,W)} \{h_\nu(F + \int_W \varphi^+ d\nu)\} = h_{\nu_{\varphi^+}}(F) + \int_W \varphi^+ d\nu_{\varphi^+}.$$

3. *if $Q_{\nu_{\varphi^+}} < \infty$, then $\mu_\varphi = \pi(\nu_{\varphi^+}) \in \mathcal{M}_L(f, X)$.*

The measure μ_φ is a *natural candidate* for the equilibrium measure for φ. It is indeed the equilibrium measure for φ provided the following *positive recurrent* condition holds: there exist $\varepsilon_0 > 0$ such that for any $0 \leq \varepsilon < \varepsilon_0$ the function

$$\varphi_\varepsilon^+ := \overline{\varphi - s_\varphi + \varepsilon} = \varphi^+ + \varepsilon\tau$$

has finite Gurevich pressure.

Proposition 1.5.3 (see [PS06, PS05]). *Assume that $\bar{\varphi}$ has summable variations. finite Gurevich pressure, and is positively recurrent. Assume also that* $\sup_{x \in W} \varphi^+ < \infty$ *and $Q_{\nu_{\varphi^+}} < \infty$. Then $\mu_\varphi = \pi(\nu_{\varphi^+})$ is the unique equilibrium measure for φ, i.e.,*

$$s_\varphi = h_{\mu_\varphi}(f) + \int_X \varphi \, d\mu_\varphi = \sup_{\mathcal{M}_L(f,X)} \{h_\mu(f) + \int_X \varphi \, d\mu\}. \tag{1.11}$$

For proof of the propositions we refer to [PS06],[PS05]. We will, however, prove a simple version of Proposition 1.5.3 which will nonetheless be useful:

Proposition 1.5.4. *Assume that $\bar{\varphi}$ has summable variations, finite Gurevich pressure and $-\infty < s_\phi < +\infty$. Assume also that $Q_{\nu_{\varphi^+}} < \infty$ and $P_G(\varphi^+) = 0$. Then $\mu_\varphi = \pi(\nu_{\varphi^+})$ is the unique equilibrium measure for φ in the class $\mathcal{M}_L(X, f)$.*

Proof. Theorem 1.5.2 implies that μ_φ is well defined and is in $\mathcal{M}_L(X,f)$.Since $Q_{\nu_{\varphi^+}}$ is finite, we have the following calculation:

$$\begin{aligned} 0 &= P_G(\varphi^+) = h_{\nu_{\varphi^+}}(F) + \int_W \varphi^+ d\nu_{\varphi^+} && (1.12) \\ &= Q_{\nu_{\varphi^+}} \cdot \left(h_{\mu_\varphi}(f) + \int_X (\varphi - s_\varphi) d\mu_\varphi \right) && (1.13) \\ &= Q_{\nu_{\varphi^+}} \cdot \left(h_{\mu_\varphi}(f) + \int_X \varphi d\mu_\varphi - s_\varphi \right) \le 0. && (1.14) \end{aligned}$$

The last inequality follows from the definition of s_φ ((1.11)) and that $\mu_\varphi \in \mathcal{M}_L(X,f)$. We then have

$$0 = h_{\mu_\varphi}(f) + \int_X \varphi d\mu_\varphi - s_\varphi,$$

which implies that μ_φ is the equilibrium measure for φ in $\mathcal{M}_L(X,f)$. The uniqueness follows from the uniqueness of ν_{φ^+} and Proposition 1.4.3. □

Verifying conditions of this theorem may not be an easy task and in [PS05] some stronger requirements on the potential function φ are given that ensure existence and uniqueness of equilibrium measures. For the sake of completeness we shall briefly describe these requirements.

We say that the induced potential function $\bar{\varphi}$ is *locally Hölder continuous* if there exists $A > 0$ and $0 < \gamma < 1$ such that

$$V_n(\Phi) \le A\gamma^n, \quad n \ge 1.$$

If $\bar{\varphi}$ is locally Hölder continuous then it has summable variations.

We say that the potential function φ satisfies:

1. the (FGP)-*condition* if

$$\sum_{J \in S} \sup_{x \in J} \exp \bar{\varphi}(x) < \infty;$$

2. the (INT)-*condition* if

$$\sum_{J\in S}\tau(J)\sup_{x\in J}\exp(\varphi^+(x))<\infty.$$

Proposition 1.5.5. *Assume that the potential function φ satisfies:*

1. *the (FGP)-condition, then $\bar\varphi$ has finite Gurevich pressure;*

2. *the (INT)-condition, then the function φ^+ satisfies the (FGP)-condition, and $\sup_{x\in W}\varphi^+<\infty$;*

3. *the (INT)-conditions, $\bar\varphi$ is locally Hölder continuous and has finite Gurevich pressure, then $Q_{\nu_{\varphi^+}}<\infty$.*

Inducing scheme of hyperbolic type

In the case of a hyperbolic inducing scheme (see Section 1.2), we use Proposition 1.3.2 instead of Proposition 1.3.1 to get equilibrium measure for the induced map. We also need to require condition (1.8) instead of summable variations. The proofs in the standard case will work in the hyperbolic case with the above modifications. We have:

Proposition 1.5.6. *Assume that $\bar\varphi$ satisfies* (1.8)*, has finite Gurevich pressure, and is positively recurrent. Assume also that* $\sup_{x\in W}\varphi^+<\infty$ *and* $Q_{\nu_{\varphi^+}}<\infty$*. Then $\mu_\varphi=\pi(\nu_{\varphi^+})$ is the unique equilibrium measure for φ:*

$$s_\varphi=h_{\mu_\varphi}(f)+\int_X\varphi d\mu_\varphi=\sup_{\mathcal{M}_L(f)}\{h_\mu(f)+\int_X\varphi d\mu\}. \tag{1.15}$$

Proposition 1.5.7. *If φ satisfies*

1. *the (FGP)-condition then $\bar\varphi$ has finite Gurevich pressure.*

2. *the (INT)-condition then the function φ^+ satisfies the (FGP)-condition, and* $\sup_{x\in W}\varphi^+<\infty$.

3. (1.8) *and (INT)-conditions, and $\bar\varphi$ has finite Gurevich pressure, then $Q_{\nu_{\varphi^+}}<\infty$.*

Chapter 2

An application to uniformly expanding maps

2.1 Introduction

We consider a uniformly piecewise expanding map f of the interval $I = [0,1]$, i.e., we assume that $I = I^{(1)} \cup I^{(2)} \cup \cdots \cup I^{(p)}$ and $intI^{(k)} \cap intI^{(j)} = \emptyset$ for $1 \leq k, j \leq p$, $j \neq k$ and that $f : I^{(j)} \to I$, $j = 1, \ldots, p$ is a C^1 diffeomorphism with $|f'| > 1$. The classical thermodynamical formalism states that if φ is (piecewise) Hölder continuous, then f admits a unique equilibrium measure.

By representing this map as by a special countable Markov shift (*the renewal shift*), we describe a class of potentials which admit equilibrium measures. We will build an inducing scheme and use Proposition 1.5.4 to prove the existence and uniqueness of equilibrium measure. This class of potentials includes the Hölder continuous ones, we will also show that this class is strictly larger. In particular, building on results of Sarig [Sar01a], we will construct a family of continuous (but not Hölder continuous) potentials φ_c exhibiting phase transitions: there exists a critical value $c_0 > 0$ such that for every $0 < c < c_0$ there is a unique equilibrium measure for φ_c which is supported on $(0,1]$ and for $c < c_0$ the equilibrium measure is the Dirac measure at 0. This result appeared in [PZ06].

In Section 2.2 we lay out some of the techniques needed to study the symbolic space, in Section 2.3 these techniques are applied to our map f. In the last section we describe the phase transition phenomenon.

2.2 More on the thermodynamics of countable Markov shifts

Let $S = \{1, 2, \cdots, \}$ be an infinite alphabet and Σ_A the *Markov shift* with the set of states S and a transition matrix $A = (a_{ij}), i, j \in \mathbb{N}$. This means that Σ_A is the set of all one-sided infinite sequences $\omega \in S^{\mathbb{N}}$ which are admissible by A, i.e., $a_{\omega_i \omega_{i+1}} = 1$ for all $i \in \mathbb{Z}$. Assume that (Σ_A, σ) is topologically mixing. We write $[a_1, \cdots, a_n]$ for the cylinder set $\{(x_k) : x_i = a_i, 1 \leq i \leq n\}$.

We will first state come general results on the countable Markov shifts. Recall that if a potential $\phi : \Sigma_A \to \mathbb{R}$ has summable variation then the Gurevich pressure $P_G(\phi)$ is well defined. Also we have that ϕ is *weakly Hölder continuous* if there exists $A > 0$ and $0 < \gamma < 1$ such that for all $n \geq 1$,

$$V_n(\phi) \leq A\gamma^n. \tag{2.1}$$

If ϕ is weakly Hölder continuous then it has summable variations.

Write

$$Z_n(\phi, a) = \sum_{\sigma^n(x)=x} \exp \phi_n(x) 1_{[a]}(x),$$

where $\phi_n(x) = \sum_{i=0}^{n-1} \phi(\sigma^i(x))$. Note that the Gurevich pressure is given by:

$$P_G(\phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi, a).$$

A Borel σ-invariant probability measure μ_ϕ on Σ_A is called an *equilibrium measure* for ϕ if

$$h_{\mu_\phi}(\sigma) + \int_{\Sigma_A} \phi \, d\mu_\phi = \sup\{h_\mu(\sigma) + \int_{\Sigma_A} \phi \, d\mu\},$$

where the supremum is taken over all Borel $\sigma-$invariant probability measures on Σ_A for which $\int_{\Sigma_A} \phi \, d\mu_\phi > -\infty$.

We consider an inducing scheme obtained by the first return map to the cylinder

$[a]$. For $x \in [a]$, define

$$r_a(x) = \inf\{n \geq 1; \sigma^n(x) \in [a]\}$$

the first return time of x to $[a]$. We have

$$[a] = \bigcup_{n \geq 0,\, \xi_i \neq a} [a, \underline{\xi}, a] \cup N,$$

where $\underline{\xi} = \xi_1, \cdots, \xi_{n-1}$, $r_a(x) = n$ for $x \in [a, \underline{\xi}, a]$, and N is a null set for any $\sigma-$invariant probability measure. Let $J_{\underline{\xi}} = [a, \underline{\xi}, a]$ we have that $(\{J_{\underline{\xi}}\}, r_a)$ defines an inducing scheme.

Given a potential ϕ, recall that the *induced potential* is defined by $\overline{\phi}(x) = \phi_{r_a(x)}(x)$. In order to describe thermodynamics of σ with respect to the potential ϕ we introduce the following crucial property.

Define the *discriminant* of ϕ by

$$\Delta[\phi] = \sup_{p \in \mathbb{R}} \{P_G(\overline{\phi + p}); P_G(\overline{\phi + p}) < +\infty\}.$$

Proposition 2.2.1 (see [Sar01a])**.** *$\Delta[\phi] \geq 0$ if and only if there exists p such that $P_G(\overline{\phi + p}) = 0$.*

Set

$$Z_n^*(\phi, a) = \sum_{\sigma^n(x) = x,\, r_a(x) = n} e^{\phi_n(x)} 1_{[a]}(x).$$

Proposition 2.2.2 (see [Sar01a])**.** *We have*

$$\left| \Delta[\phi] - \log \sum_{n=1}^{\infty} \xi^n Z_n^*(\phi, a)|_{\xi = rad.} \right| \leq \sum_{n \geq 2} V_n(\phi),$$

where $\xi = rad.$ is the radius of convergence for the power series

$$\sum_{n=1}^{\infty} z^n Z_n^*(\phi, a).$$

Furthermore, if we define $p^*(\phi) = \sup\{p : P_G(\overline{\phi + p}) < \infty\}$, then $\log \xi = -p^*(\phi)$.

Set $\lambda = \exp P_G(\phi)$. Following [Sar01a], we call the potential ϕ *recurrent* if the sum

$$\sum_{n\geq 1} \lambda^{-n} Z_n(\phi, a)$$

diverges. Otherwise we call ϕ *transient.* We call ϕ *positive recurrent* if it is recurrent and

$$\sum_{n\geq 1} n\lambda^{-n} Z_n^*(\phi, a) < +\infty$$

and *null recurrent* if

$$\sum_{n\geq 1} n\lambda^{-n} Z_n^*(\phi, a) = \infty.$$

We note that the definition of positive recurrence here is strictly in the setting of countable Markov shifts, while definition given in Section 1.5 is in view of general inducing schemes. By [Sar01a], we have that ϕ is recurrent if and only if $\Delta[\phi] \geq 0$. ϕ is positive recurrent when $\Delta[\phi] > 0$ and is either positive recurrent or null recurrent if $\Delta[\phi] = 0$.

Proposition 2.2.3 (see [Sar01b], [Sar01a])**.** *Assume that the potential ϕ has summable variations, $P_G(\phi) < +\infty$ and $\sup \phi < \infty$. Then the variational principle holds:*

$$P_G(\phi) = \sup\Big\{h_\mu(\sigma) + \int_{\Sigma_A} \phi\, d\mu : \mu \circ \sigma^{-1} = \mu,\ \int_{\Sigma_A} \phi d\mu > -\infty\Big\}.$$

If ϕ is positive recurrent then $P_G(\phi) = -p(\phi)$, where $p(\phi)$ is the solution of the equation $P_G(\overline{\phi + p}) = 0$; if ϕ is transient then $P_G(\phi) = -p^*(\phi)$, where

$$p^*(\phi) = \sup\{p :\ P_G(\overline{\phi + p}) < +\infty\}.$$

The following proposition illustrates that if ϕ is transient, there is no equilibrium measure for ϕ.

Proposition 2.2.4 (see [BS03])**.** *Assume that ϕ has summable variations, finite Gurevich pressure and $\sup \phi < \infty$. If ϕ admits an equilibrium measure then it is positive recurrent.*

In general, positive recurrence does not imply existence of equilibrium measure.

We therefore, consider a particular case of the *renewal shift* (Σ_A, σ) where the entries of the transition matrix A are 1 at $(0, n)$ and $(n, n-1); n \geq 1$ and 0 otherwise.

To study existence and uniqueness of equilibrium measures for the renewal shift one can use results on thermodynamical formalism for general subshifts of countable type (see for example, [ADU93, BS03, MU01, Sar99, Wal78, Yur99]) and reexamine the conditions in this particular case. We however, will use results of Chapter 1 by considering the inducing scheme build on the symbol 0. The induced partition is formed by the cylinders $J_n = [0, n-1, \cdots, 1, 0]$ for $n \geq 1$. The collection of intervals $\{J_n\}_{n\geq 1}$ and the inducing time $\tau(J_n) = r_{[0]}(x) = n$, $x \in J_n$ form an inducing scheme with the inducing domain [0] and the induced map $\overline{\sigma}(x) = \sigma(x)^n$, $x \in J_n$. We then have the original system (X, f) and the induced system (W, F), also given a potential ϕ, we can consider its induced potential $\overline{\phi}$, see section 1.2, 1.4.

Recall that $\mathcal{M}_L(\sigma)$ is the set of all liftable σ-invariant Borel ergodic probability measures μ on X and

$$s_\phi = \sup_{\mathcal{M}_L(\sigma)} \left(h_\mu(\sigma) + \int_X \phi \, d\mu \right),$$

where the supremum is taken over all liftable measures.

We restate Proposition 1.5.4 here:

Proposition 2.2.5. *Assume that the induced potential function $\overline{\phi}$ has summable variations, $-\infty < s_\phi < +\infty$, and $P_G(\overline{\phi - s_\phi}) = 0$. Assume also that* $\sup(\overline{\phi - s_\phi}) <$ ∞ *and that*

$$\sum_{n=1}^{\infty} n \sup_{x\in J_n} \exp(\overline{\phi - s_\phi}) < \infty. \tag{2.2}$$

Then there is a unique equilibrium measure $\mu_\phi \in \mathcal{M}_L(\sigma)$, i.e., a measure for which

$$s_\phi = h_{\mu_\phi}(\sigma) + \int_I \phi \, d\mu_\phi = \sup_{\mathcal{M}_L(\sigma)} \{h_\mu(\sigma) + \int_I \phi \, d\mu\}.$$

Note that condition (2.2) is equivalent as $Q_{\nu_{\varphi^+}} < \infty$.

Combining Propositions 2.2.3 and 2.2.5, we establish a condition, which guarantees existence and uniqueness of equilibrium measures for the renewal shift.

Theorem 2.2.6. *Let* (Σ_A, σ) *be the renewal shift. Assume that:*

1. ϕ *has summable variations;*
2. $\sup \phi < \infty$*;*
3. ϕ *is positive recurrent.*

Then there exists a unique equilibrium measure.

Proof. We show that the conditions of Proposition 2.2.5 are satisfied. Since ϕ has summable variations, so does $\overline{\phi}$. Due to the special structure of the renewal shift, the sum in

$$Z_n(\phi, 0) = \sum_{\sigma^n(x)=x} e^{\phi_n(x)} 1_{[0]}(x)$$

has at most 2^n terms. Hence,

$$Z_n(\phi, 0) \leq 2^n e^{(\sup \phi)n}.$$

This implies that

$$P_G(\phi) \leq \log 2 + \sup \phi < \infty.$$

By Proposition 2.2.3,

$$P_G(\phi) = \Big\{ h_\mu(\sigma) + \int_{\Sigma_A} \phi \, d\mu : \mu \circ \sigma^{-1} = \mu, \ \int_{\Sigma_A} \phi \, d\mu > -\infty \Big\}.$$

Since the inducing time in our case is the first return time, we conclude that the right-hand side of the last equation is s_ϕ and hence, $s_\phi = P_G(\phi) < \infty$. Taking the Dirac measure over a periodic orbit, we know that there is at least one invariant measure with $\int \phi \, d\mu > -\infty$. Hence, $s_\phi > -\infty$.

Using the fact that ϕ is positive recurrent we have

$$\sum_{n \geq 1} n \lambda^{-n} Z_n^*(\phi, 0) < +\infty.$$

Since σ is the renewal shift we obtain that $Z_n^*(\phi, 0) = \exp(\overline{\phi}(x))$, where x is the unique periodic orbit of period n in the cylinder $[0, n-1, \cdots, 0]$. For $\lambda =$

$\exp P_G(\phi) = \exp s_\phi$, we have that

$$\Big|\sum_{n\geq 1} n\lambda^{-n} Z_n^*(\phi, 0) - \sum_{n=1}^{\infty} n \sup_{x\in J_n} \exp(\overline{\phi - s_\phi})\Big| < \sum_{n=2}^{\infty} V_n(\phi).$$

We know that positive recurrence implies Condition (2.2). Note that Condition (2.2) in turns, implies that $\sup(\overline{\phi - s_\phi}) < \infty$.

Furthermore, Proposition 2.2.3 implies that $P_G(\overline{\phi - s_\phi}) = 0$ and the desired result follows. □

2.3 Representation of f by the renewal shift

Recall that f is a uniformly expanding map of the unit interval $I = [0, 1]$. More precisely, we assume that $I = I^{(1)} \cup I^{(2)}$, $intI^{(1)} \cap intI^{(2)} = \emptyset$ and that the map $f : I^{(j)} \to I$, $j = 1, 2$ can be extended to a C^1 diffeomorphism of a small neighborhood of $I^{(j)}$; finally, we assume that $|f'(x)| > \lambda > 1$ for all $x \in intI^{(1)} \cup intI^{(2)}$. Consider the closed intervals $I_0 = I^{(2)}$, $I_n = f^{-1}(I_{n-1}) \cap I^{(1)}$, $n \geq 1$. They cover the semi-open interval $(0, 1]$ and form a countable Markov partition for f.

Define the *coding map* $h : \Sigma_A \to I \setminus \{0\}$ by

$$h(i_1, \cdots, i_n, \cdots) = \bigcap_{j=1}^{\infty} f^{-j}(I_{i_j}).$$

Here (Σ_A, σ) is the renewal shift.

Denote by Q the set of all end points of the partition intervals and all their preimages. Observe that the only invariant measure μ with $\mu(Q) = 1$ is the Dirac measure at 0, δ_0.

Proposition 2.3.1. *The map h is onto and is one-to-one and a topological conjugacy between $(\Sigma_A \setminus h^{-1}(Q), \sigma)$ and $(I \setminus Q, f)$.*

Given a potential function φ on I, we define $\phi = \varphi \circ h$ to be a potential function on Σ_A. We denote by $\mathcal{H}$ the class of potential functions φ for which the corresponding functions ϕ satisfy all the conditions of Theorem 2.2.6, i.e., ϕ has summable variations, is positive recurrent and $\sup \phi < \infty$.

The following result is an immediate corollary of Theorem 2.2.6 and Proposition 2.3.1.

Proposition 2.3.2. *If $\varphi \in \mathcal{H}$ then*

1. *there exists a unique equilibrium measure ν_ϕ for ϕ;*
2. *there exists an equilibrium measure μ_φ for φ that is either δ_0 or $\nu_\phi \circ h^{-1}$.*

The following example illustrates that for some potentials φ the measure μ_φ may not be an equilibrium measure for φ while the Dirac measure δ_0 at 0 may be such a measure.

Example 2.3.1. *Assume that the map f is $C^{1+\epsilon}$, and consider the potential*

$$\varphi(x) = \begin{cases} -t \log |f'(x)|, & x \neq 0 \\ f_0, & x = 0 \end{cases}.$$

Observe that φ is Hölder continuous on $(0, 1]$. By Theorem 2.3.3 below, if $f_0 < P_G(\varphi \circ h)$, the measure μ_φ is a unique equilibrium measure for φ (it is the absolutely continuous invariant measure) and if $f_0 > P_G(\varphi \circ h)$, the equilibrium measure is δ_0. Furthermore, if $f_0 = P_G(\varphi \circ h)$ there are two equilibrium measures for φ.

We show that the class of functions $\mathcal{H}$ is sufficiently large and in particular, includes all the Hölder continuous functions.

Theorem 2.3.3. *If φ is Hölder continuous on I then $\varphi \in \mathcal{H}$.*

Proof. Since φ is bounded and continuous, so is ϕ. Moreover, as the map h is Hölder continuous (while h^{-1} is not), we have that ϕ is Hölder continuous on Σ_A. This implies that ϕ has summable variations and is bounded from above.

To establish the positive recurrence condition we consider inducing on the cylinder [0] as described in Section 2.2. Note that the only induced cylinder with return time n is $[0, n-1, \cdots, 1, 0]$. Fix $x_i \in [0, i-1, \cdots, 1, 0]$ and consider the function ψ such that $\psi(x) = \phi(x_i)$ whenever $x \in [0, i-1, \cdots, 1, 0]$. For simplicity we write $a_i = \phi(x_i)$. The function ψ has the following properties:

1. $V_n(\psi) = 0$ (this is true since ψ is constant on every cylinder);

2. there exist a, $A > 0$ and $0 < \gamma < 1$ such that $|a_i - a| < A\gamma^i$ (this is true since $h(x_i) \to 0$ in I and ϕ is Hölder continuous);

3. $|\phi(x) - \psi(x)| \le V_n(\phi)$ for any $x \in [0, n-1, \cdots, 1, 0]$.

We claim that

$$\log \sum_{n=1}^{\infty} \xi^n Z_n^*(\psi, 0)|_{\xi = rad.} = +\infty.$$

To show this observe that for $x \in [0, n-1, \cdots, 1, 0]$,

$$Z_n^*(\psi, 0) = \exp\Big\{\sum_{i=0}^{n-1} \phi(\sigma^i x)\Big\} = \exp\Big\{\sum_{i=0}^{n-1} a_i\Big\}.$$

As $n \to +\infty$, we have that

$$Z_n^*(\psi, 0) \asymp e^{na}.$$

Hence, for $\xi = e^{-a}$,

$$\begin{aligned}
\log \sum_{n=1}^{\infty} \xi^n Z_n^*(\psi, 0) &= \log \sum_{n=1}^{\infty} e^{-an} Z_n^*(\psi, 0) \\
&= \log \sum_{n=1}^{\infty} \exp\left\{\sum_{i=0}^{n-1} (a_i - a)\right\} \\
&\ge \log \sum_{n=1}^{\infty} \exp\left\{-\sum_{i=0}^{n-1} A\gamma^n\right\} \\
&\ge \log \sum_{n=1}^{\infty} \exp\left\{-A\frac{1}{1-\gamma}\right\} = +\infty.
\end{aligned}$$

Using Property 3 of the function ψ, we obtain that

$$|Z_n^*(\phi, 0)) - Z_n^*(\psi, 0)| \le \sum_{k=2}^{n+1} V_n(\phi).$$

Since $\sum_{n \ge 2} V_n(\phi) < +\infty$, we conclude that the power series

$$\sum_{n=1}^{\infty} z^n Z_n^*(\phi, a), \quad \sum_{n=1}^{\infty} z^n Z_n^*(\psi, a)$$

have the same radius of convergence. It follows that

$$\begin{aligned}\Delta[\phi] &\geq \log \sum_{n=1}^{\infty} \xi^n Z_n^*(\phi, 0) - \sum_{n\geq 2} V_n(\phi) \\ &\geq \log \sum_{n=1}^{\infty} \xi^n Z_n^*(\psi, 0) - 2\sum_{n\geq 2} V_n(\phi) = +\infty.\end{aligned}$$

The desired result follows. □

We now show that the class of function $\mathcal{H}$ admits some non-Hölder continuous potentials which thus have equilibrium measures. Consider the one-parameter family of functions

$$\varphi_c(x) = \begin{cases} -c(1-\log x)^{-\alpha}, & x \in (0,1] \\ 0, & x = 0, \end{cases} \tag{2.3}$$

where $c \in \mathbb{R}$. Observe that φ_c is continuous on $[0,1]$ but is not Hölder continuous at zero.

Theorem 2.3.4. *For any $\alpha > 1$ the potential $\phi_c = \varphi_c \circ h \in \mathcal{H}$, i.e., it satisfies all the conditions of Theorem 2.2.6.*

Proof. Set $\beta = \min |f'|^{-1}$ and $\gamma = \max |f'|^{-1}$. Since f is uniformly expanding $\beta \leq \gamma < 1$. We write $I_0 = [d_0, 1]$, $I_1 = [d_1, d_0]$, $\cdots$, $I_n = [d_n, d_{n-1}]$, and $I_{i_1,\cdots,i_n} = [i_1, \cdots, i_n]$ as long as $i_1, \cdots, i_n$ is admissible. Note that $f(d_n) = d_{n-1}$ and that

1. $I_n \subset [d_n, 1]$ and $\frac{\beta}{\gamma} \leq \frac{|[0,d_n]|}{|I_n|} \leq \frac{\gamma}{\beta}$;
2. $|I_{i_1,\cdots,i_n}| \leq \gamma^n$;
3. $d_n \geq \beta^n$.

We first verify the summable variation condition. Choose $x, y \in I_{i_1,\cdots,i_n}$. Let a be an integer such that $\gamma < \beta^{\frac{1}{a}}$. Set $i_1 = m$. If $m \geq \frac{n}{a}$, we have that $I_{i_1,\cdots,i_n} \subset I_m \subset [d_m, 1]$ and hence, $\beta^m \leq x, y \leq \gamma^m$. It follows that

$$|\varphi_c(x) - \varphi_c(y)| = |\varphi_c{}'(\xi)||x-y|$$

$$\begin{aligned}
&= |c|\frac{\alpha(1-\log\xi)^{-\alpha-1}}{|\xi|}|x-y| \\
&\leq |c|\frac{\alpha(1-\log\xi)^{-\alpha-1}}{|[0,d_m]|}|I_m| \\
&\leq |c|\frac{|I_m|}{|[0,d_m]|}\alpha(1-m\log\gamma)^{-\alpha-1} \\
&\leq C|c||m|^{-\alpha-1} \leq C|c|\left(\frac{n}{a}\right)^{-\alpha-1}
\end{aligned}$$

If $m < \frac{n}{a}$, we have that

$$\begin{aligned}
|\varphi_c(x)-\varphi_c(y)| &= |\varphi_c{}'(\xi)||x-y| \\
&= |c|\frac{\alpha(1-\log\xi)^{-\alpha-1}}{|\xi|}|x-y| \\
&\leq |c|\frac{\alpha(1-\log\xi)^{-\alpha-1}}{|[0,d_m]|}|I_{i_1,\cdots,i_n}| \\
&\leq |c|\frac{|I_m|}{|[0,d_m]|}\frac{|I_{i_1,\cdots,i_n}|}{|I_m|}\alpha(1-m\log\gamma)^{-\alpha-1} \\
&\leq C'|c|\frac{\gamma^n}{\beta^{\frac{n}{a}}}
\end{aligned}$$

With n sufficient large, we can guarantee that

$$\left(\frac{\gamma}{\beta^{\frac{1}{a}}}\right)^n \leq C''\left(\frac{n}{a}\right)^{-\alpha-1}.$$

This yields that

$$V_n(\phi_c) \asymp n^{-1-\alpha},$$

and hence, ϕ_c has summable variations. Clearly, this function is bounded from above. We shall now show that

$$\log\sum_{n=1}^{\infty}\xi^n Z_n^*(\phi_c,0)|_{\xi=rad.} = +\infty.$$

In fact,

$$Z_n^*(\phi_c,0) = \exp\sum_{k=0}^{n-1}\varphi_c(x_k)$$

with $x_k \in I_k$. Taking into account that $\alpha > 1$ we find that

$$\begin{aligned} \exp\Big\{\sum_{k=0}^{n-1} \varphi_c(x_k)\Big\} &\geq \exp\Big\{-\sum_{k=0}^{n-1} |\varphi_c(d_k)|\Big\} \\ &\geq \exp\{-|c|\sum_{k=0}^{n-1}(1+k|\log\gamma|)^{-\alpha}\} \\ &\geq \exp\{-|c|(|\log\gamma|)^{-\alpha}\sum_{k=0}^{n-1}(k+1)^{-\alpha}\} > e^{-C}, \end{aligned}$$

where $C > 0$ is a constant. It follows that for $\xi = 1$,

$$\log\sum_{n=1}^{\infty}\xi^n Z_n^*(\phi, 0) = \log\sum_{n=1}^{\infty} e^{-C} = +\infty.$$

We have $\Delta[\phi_c] = +\infty$ and the desired result follows. □

Remark. *The above argument shows indeed, that the function ϕ_c has summable variations and is bounded from above for all $\alpha > 0$.*

2.4 Phase Transitions

Phase transitions for the renewal shift were studied by [Hof77], [Lop93], [Sar01a]. Let (Σ_A, σ) be the renewal shift, $\phi : \Sigma_A \to \mathbb{R}$ a potential function and $\overline{\phi}(x) = \phi_{r_a(x)}(x)$ the corresponding induced potential function.

Proposition 2.4.1. *Assume that ϕ satisfies Conditions 1 and 2 of Proposition 2.2.3 (i.e., φ has summable variations and is bounded from above) and that $\overline{\phi}$ is weakly Hölder continuous (see (2.1)). Then there exists a critical value $0 < c_0 \leq +\infty$ such that the potential $c\phi$ is positive recurrent for all $0 < c < c_0$ and is transient for $c > c_0$.*

Note that this result does not exclude the case when $c_0 = \infty$.

In fact, if φ is a Hölder continuous function on I then the function $c\varphi$ is Hölder continuous and hence, is positive recurrent for all c (see Theorem 2.3.4). We provide an example of a one-parameter family of functions for which the phase transition actually occurs.

Theorem 2.4.2. *For the function $\varphi_c(x)$, given by (2.3) with $0 < \alpha \le 1$, the potential function $\phi_c = \varphi_c \circ h$ has summable variations and $\overline{\phi}_c(x)$ is weakly Hölder continuous. There exists $c_0 > 0$ such that*

1. *ϕ_c is positive recurrent for $c_0 > c > 0$ and there is a unique equilibrium measure for φ_c among all invariant measures. This measure is supported on $(0, 1]$.*

2. *ϕ_c is transient for $c > c_0$ and there is no equilibrium measure for φ_c among the measures supported on $(0, 1]$, the Dirac measure at 0 is the equilibrium measure.*

Proof. The fact that ϕ_c has summable variations and is bounded from above for $0 < \alpha < 1$ (and actually for all $\alpha > 0$) was established in the proof for Theorem 2.3.4 (see Remark in the end of Section 3).

We shall show that the induced potential $\overline{\phi}_c(x)$ is weakly Hölder continuous. This will imply that it has summable variations. Choose two points x and y in the interval $[0, n_1, \cdots, 0, n_2, \cdots, 0, \cdots, n_k, \cdots, 0]$. We have

$$\begin{aligned}
|\overline{\phi}_c(x) - \overline{\phi}_c(y)| &= |\sum_{i=0}^{n_1}(\phi_c(f^i(x)) - \phi_c(f^i(y)))| \\
&\le \sum_{i=0}^{n_1} |c| \frac{\alpha(1-\log\xi_i)^{-1-\alpha}}{|\xi_i|} |\phi_c(f^i(x)) - \phi_c(f^i(y))| \\
&\le \sum_{i=0}^{n_1} |c| \frac{\alpha(1-(n_1-i)\log\gamma)^{-1-\alpha}}{|[0, d_{n_1-i}]|} \cdot \\
&\quad \big|[n_1 - i, \cdots, 0, n_2, \cdots, 0, \cdots, n_k, \cdots, 0]\big| \\
&\le \sum_{i=0}^{n_1} |c| \frac{|I_{n_1-i}|}{|[0, d_{n_1-i}]|} \gamma^{k-1} |\alpha(1-(n_1-i)\log\gamma)^{-1-\alpha}| \\
&\le |c| C \sum_{i=0}^{n_1} (n_1 - i)^{-\alpha-1}\gamma^{k-1} \le A\gamma^{k-1},
\end{aligned}$$

where $A = |c| C \sum_{n=0}^{\infty} n^{-\alpha-1}$. Hence, $\overline{\phi}_c(x)$ is weakly Hölder continuous.

We shall show that for c sufficiently close to zero, the potential ϕ_c is positive recurrent and for c sufficiently large, it is transient. Set

$$F_n(\phi_c) = \exp\Big\{\sum_{k=0}^{n-1} \sup_{x\in I_k} \varphi_c(x)\Big\}.$$

Observe that

$$\log \sum_{n\geq 1} \xi^n F_n(\phi_c)|_{\xi=rad.} - 2\sum_{n\geq 2} V_n(\phi_c) \leq \Delta[\phi_c] \leq \log \sum_{n\geq 1} \xi^n F_n(\phi_c)|_{\xi=rad.}$$

and that

$$F_n(\phi_c) = \exp\Big\{\sum_{k=0}^{n-1} \varphi_c(d_k)\Big\},$$

where d_k were introduced in the proof of Theorem 2.3.4 and are the endpoints of the basic intervals I_n (i.e., $I_n = [d_n, d_{n-1}]$).

We consider the two cases.

Case I: $\alpha = 1$**.** We have

$$\begin{aligned} F_n(\phi_c) &= \exp\Big\{\sum_{k=0}^{n-1} \varphi_c(d_k)\Big\} \\ &\geq \exp\Big\{\sum_{k=0}^{n-1} -c(1-\log\gamma^k)^{-\alpha}\Big\} \\ &\geq \exp\Big\{\sum_{k=0}^{n-1} -c|\log\gamma|(k+1)^{-\alpha}\Big\} \\ &\geq \exp\{-cM\log(n+1)\} \geq C(n+1)^{-cM}, \end{aligned}$$

where $C > 0$ and $M > 0$ are constants. Similarly, one can show that for some $C' > 0$ and $M' > 0$,

$$F_n(\phi_c) \leq C'(n+1)^{-cM'}.$$

Therefore, the radius of convergence $\xi = 1$ and

$$\sum_{n\geq 1} F_n(\phi_c) \geq \sum_{n\geq 1} C(n+1)^{-cM} = +\infty$$

as long as $-1 < -cM < 0$ (i.e., $0 < c < \frac{1}{M}$). For these values of c we have $\Delta[\phi_c] = +\infty$ and hence, ϕ_c is positive recurrent.

Observe that

$$\log \sum_{n\geq 1} F_n(\phi_c) \leq \log \sum_{n\geq 1} C'(n+1)^{(-cM')}.$$

The expression in the right-hand side of this inequality tends to $-\infty$ as $c \to \infty$. Hence, for some c we have

$$\Delta[\phi_c] \leq \log \sum_{n\geq 1} F_n(\phi_c) < 0.$$

This implies that for c sufficiently large, ϕ is transient.

Case II: $0 < \alpha < 1$. We find that

$$\exp\{-cM'(n+1)^{1-\alpha}\} \leq F_n(\phi_c) \leq \exp\{-cM(n+1)^{1-\alpha}\}.$$

The radius of convergence $\xi = 1$ and

$$\sum_{n\geq 1} F_n(\phi_c) \geq \sum_{n\geq 1} \exp\{-cM'(n+1)^{1-\alpha}\}.$$

Therefore,

$$\sum_{n\geq 1} F_n(\phi_c) \geq ke^{-1} \tag{2.4}$$

as long as $cM' < \frac{1}{(k+1)^{1-\alpha}}$. Observe that $\sum_{n\geq 2} V_n(\phi) \to 0$ as $c \to 0$. In view of (2.4), this implies that for c sufficiently close to zero,

$$\Delta[\phi_c] \geq \log \sum_{n\geq 1} F_n(\phi_c) - 2\sum_{n\geq 2} V_n(\phi_c) > 0$$

and hence, ϕ_c is positive recurrent.

We also have that

$$\log \sum_{n\geq 1} F_n(\phi_c) \leq \log \sum_{n\geq 1} \exp\{-cM(n+1)^{1-\alpha}\} \to -\infty$$

as $c \to +\infty$, and hence, ϕ_c is transient for sufficiently large c.

Since $\xi = 1$, by Proposition 2.2.2, $0 = \log \xi = p^*(\phi_c)$. When c is small, $\Delta[\phi_c] > 0$ implies that $p(\phi_c) < p^*(\phi_c) = 0$. Proposition 2.2.3 give that $P_G(\phi_c) = -p(\phi_c) > 0$. Because ϕ_c is positive recurrent, by Theorem 2.2.6, there is a unique equilibrium measure for the renewal shift, by topological conjugacy, there is a unique equilibrium measure μ_{φ_c} among the measures supported on $(0, 1]$. Since the only invariant measure supported outside of $(0, 1]$ is the Dirac measure δ_0, and from

$$h_{\delta_0}(f) + \int_I \varphi_c d\delta_0 = \varphi_c(0) < P_G(\phi_c)$$

we know that δ_0 is not an equilibrium measure, hence μ_{φ_c} is the equilibrium measure among all invariant measures.

In the case when c is sufficiently large, $\Delta[\phi_c] < 0$ and Proposition 1.5.3 implies that $P_G(\phi_c) = -p^*(\phi_c) = 0$. In this case ϕ_c is transient, by Proposition 2.2.4, there is no equilibrium measure supported on $(0, 1]$.

$$h_{\delta_0}(f) + \int_I \varphi_c d\delta_0 = 0 = P_G(\phi_c)$$

implies that the Dirac measure is the equilibrium measure. □

Chapter 3

Liftability question

3.1 Introduction

Recall that given a system (I, f) which admits an inducing scheme (S, τ), it is possible to obtain an invariant measure μ of (X, f) from an invariant measure ν of (W, f) by taking $\mu = \pi(\nu)$ (See Section 1.4). We have seen in Chapter 1 that it is possible to obtain equilibrium measure among the class $\mathcal{M}_L(X, f)$, which are the liftable measures (See Section 1.4).

We would like to study the class $\mathcal{M}_L(X, f)$. The size of this class depends on both the original system and the actual inducing scheme. We intend to prove that, after imposing some conditions on both the system and the inducing scheme, the class $\mathcal{M}_L(X, f)$ contains most of the meaningful measures.

Zweimüller proved that any measure μ such that $\int \tau d\mu < \infty$ is liftable (See [Zwe05]). His approach is to consider an abstract tower extension and try to obtain an invariant measure on this extension. Similar techniques were also used in Hoffbauer ([Hof79],[Hof81]) and Keller ([Kel89]) though in their setting the tower is a different construction known as the Hoffbauer-Keller tower. Hoffbauer and Keller proved that on this tower, for one-dimensional maps, any measure with positive entropy is liftable. In Section 3.2 we will use techniques similar to theirs and provide a sufficient condition for liftability. Some examples will be provided in Section 3.3. These results can be found in [PZ07].

In [PS06],[PS05], Pesin and Senti also proved that for the unimodal map and a suitable inducing scheme any measure with positive entropy is liftable. Their approach is based on a result of Bruin (See [Bru95]) which asserts that particular inducing schemes can be embedded in the Hoffbauer-Keller tower and hence, the results of Hoffbauer and Keller applies. In Section 3.4 we will introduce a version of the Hoffbauer-Keller tower studied by Buzzi([Buz99]) and in section 3.5 we will apply the approach by Bruin to this more general setting. These results can be found in [PSZ07].

3.2 Liftability property

3.2.1 Characterizations of liftability

Let's recall that for an inducing scheme (S,τ) and ν an invariant measure for (W,F), then $\pi(\nu)$ is defined by

$$\pi(\nu)(E) = \sum_{J\in S} \sum_{k=0}^{\tau(J)-1} \nu(f^{-k}(E)\cap J).$$

a measure μ is called liftable if there exists $\nu \ll \mu$ such that $\mu = \pi(\nu)$.

We describe the liftability property in terms of the abstract representation of f via the tower construction. We shall exploit the techniques from [Kel89] and [Zwe05]. Set

$$\tilde{I} = \{(x,k) :\, x \in W\cap J,\, k = 0,\cdots,\tau(J)-1,\, J\in S\},$$

and define the map $\tilde{f} : \tilde{I} \to \tilde{I}$ by

$$\tilde{f}(x,k) = \begin{cases} (x,k+1), & x\in J,\ 0\le k<\tau(J)-1 \\ (f^{\tau(J)}(x),0), & x\in J,\ k=\tau(J)-1 \end{cases}.$$

In what follows we shall denote by $\mathcal{B}$ the Borel σ-algebra in the corresponding measure space. We define the projection map $\tilde{p} : \tilde{I} \to X$ by $\tilde{p}(x,k) = f^k(x)$ whenever $x \in W\cap J$. It is easy to see that $f\circ\tilde{p} = \tilde{p}\circ\tilde{f}$. We also use the notations: $\tilde{x} = (x,k)$, $J(\tilde{x}) = J \in S$ if $x \in J$ and $\tau(\tilde{x}) = \tau(J(\tilde{x}))$.

Liftability of an f-invariant measure μ is equivalent to its liftability to the abstract tower $\tilde{I}$. More precisely, the following statement holds. Define the set $\tilde{W} \subset \tilde{I}$ by $\tilde{W} = \{(x, 0) : x \in W\}$.

Proposition 3.2.1 (see [Zwe05])**.** *For an f-invariant Borel probability measure μ on X there exists an induced measure ν on W such that $\pi(\nu) = \mu$ if and only if there exists a $\tilde{f}$-invariant finite Borel probability measure $\tilde{\mu}$ on $\tilde{I}$ for which $\mu = \tilde{\mu} \circ \tilde{p}^{-1}$. In this case, $\nu = \tilde{\mu}|_{\tilde{W}} \circ \tilde{p}^{-1}$.*

Let $\mu_0 = \mu|_W$. Define a measure $\tilde{\mu}_0$ on $\tilde{I}$ by setting $\tilde{\mu}_0(E, k) = \mu_0(E)$. Note that $\mu = \tilde{\mu} \circ \tilde{p}^{-1}$ implies that $\tilde{\mu} \ll \tilde{\mu}_0$ and hence, one can try to construct the lift of μ using its density with respect to $\tilde{\mu}_0$.

Define the operator $\tilde{\mathcal{L}} : L^1(\tilde{I}, \mathcal{B}, \tilde{\mu}_0) \to L^1(\tilde{I}, \mathcal{B}, \tilde{\mu}_0)$ by the following relation

$$\int (\tilde{\mathcal{L}} g_1) g_2 d\tilde{\mu}_0 = \int g_1 (g_2 \circ f) d\tilde{\mu}_0$$

for all $g_1 \in L^1(\tilde{I}, \mathcal{B}, \tilde{\mu}_0)$ and $g_2 \in L^\infty(\tilde{I}, \mathcal{B}, \tilde{\mu}_0)$.

We say that a subset $\tilde{E} \subset \tilde{I}$ is bounded if the inducing time of any point in $\tilde{E}$ is bounded, i.e. there exists $N \in \mathbb{N}$ such that $\tau(x) \leq N$ for all $(x, k) \in \tilde{E}$.

Consider the sequence of functions

$$\tilde{\mathcal{A}}_n = \frac{1}{n} \sum_{k=0}^{n-1} \tilde{\mathcal{L}}^k 1_{\tilde{W}}.$$

There exists an increasing sequence of numbers $\{n_k\}$ and a function $\tilde{h} : \tilde{I} \to [0, 1]$ such that for any bounded set $\tilde{E}$, we have

$$\tilde{\mathcal{A}}_{n_k}|_{\tilde{E}} \xrightarrow{w} \tilde{h}|_{\tilde{E}}. \tag{3.1}$$

Proposition 3.2.2 (see [Zwe05])**.** *There exist a lift $\tilde{\mu}$ for μ if there exists a limit point $\tilde{h}$ of $\mathcal{A}_n$ in the sense of* (3.1) *that is not identically* 0 *w.r.t. $\tilde{\mu}_0$. In this case, $\tilde{\mu} = \tilde{h} d\tilde{\mu}_0$ is the lift of μ to the tower.*

3.2.2 Criteria for liftability

Using the above result we shall obtain the following criterion for liftability. Given a Borel set $A \subset X$ and $J \in S$, define

$$\epsilon(J, A) = \frac{1}{\tau(J)} \text{Card}\{0 \leq k \leq \tau(J) - 1; \quad f^k(J) \cap A \neq \emptyset\},$$

where Card E denotes the cardinality of the set E.

Theorem 3.2.3. *For any f-invariant Borel ergodic probability measure μ such that $\mu(W) > 0$, if there exists a number $N \geq 0$ and a subset $A \subset I$ such that*

$$\mu(A) > \sup_{\tau(J) > N} \epsilon(J, A), \tag{3.2}$$

then there exists a lift $\tilde{\mu}$ for μ.

Proof. Assume that $\tilde{h} = 0$ for $\tilde{\mu}_0$-almost every point. Consider the sets $\tilde{E}_N \subset \tilde{I}$ given as

$$\tilde{E}_N = \{\tilde{x} \in \tilde{I} : \tau(\tilde{x}) \leq N\}.$$

Clearly, $\tilde{E}_N$ are bounded sets and we have

$$\begin{aligned} 0 &= \int_{\tilde{E}_N} \tilde{h}\, d\tilde{\mu}_0(\tilde{x}) = \lim_{k \to \infty} \int_{\tilde{E}_N} \tilde{\mathcal{A}}_{n_k} d\tilde{\mu}_0(\tilde{x}) \\ &= \lim_{k \to \infty} \int_{\tilde{I}} \frac{1}{n_k} \sum_{j=0}^{n_k - 1} \tilde{\mathcal{L}}^j 1_{\tilde{W}} 1_{\tilde{E}_N} d\tilde{\mu}_0(\tilde{x}) \\ &= \lim_{k \to \infty} \int_{\tilde{I}} \frac{1}{n_k} \sum_{j=0}^{n_k - 1} 1_{\tilde{W}} 1_{f^{-j}(\tilde{E}_N)} d\tilde{\mu}_0(\tilde{x}) \\ &= \lim_{k \to \infty} \int_{\tilde{W}} \delta_k^N(\tilde{x}) d\tilde{\mu}_0(\tilde{x}), \end{aligned} \tag{3.3}$$

where for $\tilde{x} \in \tilde{W}$,

$$\delta_k^N(\tilde{x}) = \frac{1}{n_k} \text{Card}\{0 \leq j \leq n_k - 1 : \tilde{f}^j(\tilde{x}) \in \tilde{E}_N\}.$$

It follows that the sequence δ_k^N converges to zero in measure (with respect to $\tilde{\mu}_0$) as $k \to \infty$ and hence, by passing to a proper sub-sequence if necessary, we may assume that it converges $\tilde{\mu}_0$-almost everywhere. Define:

$$A_n^N(x) = \{0 \leq i \leq n-1 : \tilde{f}^i((x,0)) \notin \tilde{E}_N\}$$
$$= \{0 \leq i \leq n-1 : \tau(\tilde{f}^i((x,0))) > N\}.$$

It follows from (3.3) that for μ-almost every x,

$$\lim_{k\to\infty} \frac{\mathrm{Card}A_{n_k}^N(x)}{n_k} = 1. \tag{3.4}$$

By Birkhoff's Ergodic Theorem, the following limit

$$r(x) := \lim_{n\to\infty} \frac{1}{n}\mathrm{Card}\{0 \leq i \leq n-1, \quad f^i(x) \in A\} \tag{3.5}$$

exists almost everywhere and is equal to $\mu(A)$.

Fix some x such that both (3.4) and (3.5) hold. Let $x_0 = x$, $\tau_0 = \tau(x)$ and define $x_n = f^{\tau_{n-1}}(x_{n-1})$, $\tau_n = \tau(x_n)$ for $n \geq 0$. Note that we can rewrite the trajectory of x in the following form

$$x_0, \ldots, f^{\tau_0 - 1}(x_0), x_1, \ldots, f^{\tau_1}(x_1), \ldots$$

We also set $m_0 = 0$ and $m_j = \sum_{i=0}^{j-1} \tau_i$. For any k, there exists $l(k)$ such that

$$m_{l(k)-1} < n_k \leq m_{l(k)}.$$

Lemma 3.2.4. *We have that*

$$\lim_{k\to\infty} \frac{m_{l(k)} - CardA_{m_{l(k)}}^N(x)}{m_{l(k)}} = 0.$$

Proof. Note that the inducing time of the point $f^{m_{l(k)-1}}(x)$ is $\tau_{l(k)-1} = m_{l(k)} - m_{l(k)-1}$.

If $\tau_{l(k)-1} > N$ the point $\tilde{f}^j(x,0)$ for $j \in \{m_{l(k)-1}, \ldots, m_{l(k)}-1\}$ has inducing time larger than N and therefore $\{m_{l(k)-1}, \ldots, m_{l(k)}-1\} \subset A_{m_{l(k)}}^N$ and $\{m_{l(k)-1}, \ldots, n_k -$

$1\} \subset A^N_{n_k}$. It follows that

$$\{0, \dots, m_{l(k)} - 1\} \setminus A^N_{m_{l(k)}} = \{0, \dots, n_k - 1\} \setminus A^N_{n_k},$$

and hence,

$$\frac{m_{l(k)} - \mathrm{Card}A^N_{m_{l(k)}}}{m_{l(k)}} = \frac{n_k - \mathrm{Card}A^N_{n_k}}{m_{l(k)}} \leq \frac{n_k - \mathrm{Card}A^N_{n_k}}{n_k}.$$

If $\tau_{l(k)-1} \leq N$, we have

$$\frac{m_{l(k)} - \mathrm{Card}A^N_{m_{l(k)}}}{m_{l(k)}} \leq \frac{m_{l(k)} - \mathrm{Card}A^N_{m_{l(k)}}}{n_k} \leq \frac{n_k - \mathrm{Card}A^N_{n_k}}{n_k} + \frac{N}{n_k}.$$

In either case we obtain

$$0 \leq \lim_{k\to\infty} \frac{m_{l(k)} - \mathrm{Card}A^N_{m_{l(k)}}}{m_{l(k)}} \leq \lim_{k\to\infty} \left(\frac{n_k - \mathrm{Card}A^N_{n_k}}{n_k} + \frac{N}{n_k} \right) = 0,$$

where the last equality follows from (3.4). □

By Lemma 3.2.4 we have

$$\begin{aligned} r(x) &= \lim_{k\to\infty} \frac{1}{m_{l(k)}} \mathrm{Card}\{0 \leq i \leq m_{l(k)} - 1, \quad f^i(x) \in A\} \\ &= \lim_{k\to\infty} \frac{1}{\mathrm{Card}A^N_{m_{l(k)}}} \mathrm{Card}\{i \in A^N_{m_{l(k)}}, \quad f^i(x) \in A\}. \end{aligned}$$

Finally, we claim that

$$\frac{1}{\mathrm{Card}A^N_{m_{l(k)}}} \mathrm{Card}\{i \in A^N_{m_{l(k)}}, \quad f^i(x) \in A\} \leq \sup_{\tau(J)>N} \epsilon(J, A).$$

Observe that $A^N_{m_{l(k)}}$ is the set of those $0 \leq i \leq m_{l(k)} - 1$ for which $\tau(\tilde{f}^i(x, 0)) > N$. It follows that

$$A^N_{m_{l(k)}} = \bigcup_{j=0,\, \tau_j > N}^{l(k)-1} \{m_j, \cdots, m_{j+1} - 1\}.$$

This implies that

$$\begin{aligned}
&\frac{1}{\mathrm{Card}A^N_{m_{l(k)}}}\mathrm{Card}\{i \in A^N_{m_{l(k)}}, \quad f^i(x) \in A\} \\
&= \frac{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \mathrm{Card}\{m_j \le i \le m_{j+1}-1,\ f^i x \in A\}}{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \tau_j} \\
&= \frac{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \mathrm{Card}\{0 \le i \le \tau_j-1,\ f^i(x_j) \in A\}}{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \tau_j} \\
&\le \frac{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \mathrm{Card}\{0 \le i \le \tau_j-1,\ f^i(J(x_j)) \cap A \ne \emptyset\}}{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \tau_j} \\
&= \frac{1}{\sum_{j=0,\,\tau_j>N}^{l(k)-1} \tau_j} \sum_{j=0,\,\tau_j>N}^{l(k)-1} \tau_j \epsilon(J(x_j), A) \\
&\le \sup_{0\le j\le l(k)-1,\,\tau_j>N} \epsilon(J(x_j), N) \le \sup_{\tau(J)>N} \epsilon(J, A).
\end{aligned}$$

It follows that

$$\mu(A) = r(x) \le \sup_{\tau(J)>N} \epsilon(J, A)$$

and it contradicts to our assumption. □

As immediate corollaries of the above result we obtain the following statements.

Corollary 3.2.5. *Assume that*

$$\sup_{\tau(J)>N} \epsilon(J, W) \to 0$$

as $N \to \infty$. Then every measure μ with $\mu(W) > 0$ is liftable.

Corollary 3.2.6. *Assume that the inducing time τ is the nth return time to the base, i.e. for $x \in J$,*

$$\tau(x) = \sum_{j=1}^{n} \rho(f^j(x))$$

(where ρ is the first return time to the base). Then any invariant measure μ with $\mu(W) > 0$ is liftable.

Proof. It is easy to see that

$$\sup_{\tau(J)>N} \epsilon(J,W) = \frac{n}{N+1} \to 0 \quad \text{as } N \to \infty$$

and Corollary 3.2.5 applies. □

A substantial generalization of the last result is the criterion for liftability due to Zweimüller, see [Zwe05].

Proposition 3.2.7. *An f-invariant Borel probability measure μ is liftable provided that $\int_X \tau d\mu < \infty$.*

3.3 Examples, liftable and nonliftable

Example 3.3.1.

We shall construct an example of a one-dimensional map of a compact interval possessing inducing schemes which illustrate various phenomena associated with the liftability property. Namely,

1) for some potential function φ there exists a unique equilibrium measure μ_φ (with respect to the class of measures $\mathcal{M}(f,X)$) which has integrable inducing time;

2) for some potential function φ there exists a unique equilibrium measure μ_φ (with respect to the class of measures $\mathcal{M}(f,X)$) which is liftable but has non-integrable inducing time; in fact, all the invariant ergodic measures for the map in the example are liftable;

3) for some potential function φ there exists a unique equilibrium measure μ_φ (with respect to the class of measures $\mathcal{M}(f,I)$) which is supported outside the tower.

Our example is built upon a construction by Zweimüller [Zwe05] for abstract towers.

The map f is defined on the unit interval I. Set $I^{(1)} = [0, \frac{1}{2}]$, $I^{(2)} = (\frac{1}{2}, 1]$. We choose f such that: 1) it is continuous on I; 2) it maps $I^{(1)}$ diffeomorphically onto $[0,1]$ and it maps $I^{(2)}$ diffeomorphically onto $(0,1]$; 3) $|f'(x)| > a > 1$ for $x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$; 4) $f(0) = 0$, $f(\frac{1}{2}) = 1$.

Let S be the set of intervals I_n such that $I_0 = I^{(2)}$ and $I_n = f^{-1}(I_{n-1}) \cap I^{(1)}$ for $n \geq 1$. Let also $\tau(I_n) = n+1, n \geq 0$. Then $\{S, \tau\}$ is an inducing scheme for f. As a result we have the map $f : X \to X$ (see (1.2)) and the induced map $F = f^\tau : W \to W$. Note that both X and W differ from I by a countable set.

For $\alpha > 1$, define

$$p_0(\alpha) = \frac{1}{1 + \sum_{k \geq 1} k^{-(1+\alpha)}}$$

and

$$p_n(\alpha) = \frac{n^{-(1+\alpha)}}{1 + \sum_{k \geq 1} k^{-(1+\alpha)}}, \quad n \geq 1,$$

and the potential function $\varphi_\alpha : I \to \mathbb{R}$ given as follows:

$$\varphi_\alpha(x) = \begin{cases} \log p_0(\alpha), & x \in I_0 \\ \log \frac{p_n(\alpha)}{p_{n-1}(\alpha)}, & x \in I_n,\ n \geq 1. \end{cases}$$

Although the function $\varphi_\alpha(x)$ is not continuous on I the induced potential function $\bar{\varphi}_\alpha(x)$ is continuous on W. We claim that the following statements hold:

1. *For the above inducing scheme $\{S, \tau\}$ every measure in $\mathcal{M}(f, I)$ with $\mu(W) > 0$ is liftable, and there exists a unique equilibrium measure μ_α for φ_α with respect to $\mathcal{M}(f, X)$.*

2. *For the potential φ_α with $\alpha > 2$, the equilibrium measure μ_α has integrable inducing time.*

3. *For $1 < \alpha \leq 2$ the inducing time is not integrable with respect to the equilibrium measure μ_α.*

4. *Suppose f is $C^{1+\epsilon}$ on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, and set*

$$\psi(x) = \begin{cases} -\log |f'(x)|, & x \neq 0, \frac{1}{2} \\ a_0, & x = 0 \end{cases}.$$

Then for a_0 large enough, the Dirac measure at 0 is the equilibrium measure for ψ (note that this measure is supported outside the tower as $0 \in I \setminus X$).

To prove the first statement observe that for any interval I_n,

$$\epsilon(I_n, I_0) = \frac{1}{n+1}.$$

(See Theorem 3.2.3) Hence,

$$\sup_{\tau(I_n)\geq N} \{\epsilon(I_n, I_0)\} = \frac{1}{N} \to 0$$

as $N \to \infty$. It is easy to see that any invariant Borel probability measure μ on X must have $\mu(I_0) > 0$ and hence, by Theorem 3.2.3, every measure μ with $\mu(W) > 0$ is liftable.

To establish existence and uniqueness of equilibrium measures observe that the partition $\{I_n\}$ is a countable Markov partition for the map f on the tower, so that f is topologically conjugate to a subshift of countable type via the coding map $h_r : \Sigma_A \to X$. Here the transition matrix A is such that $a_{0,n} = 1$, $n \geq 0$ and $a_{n,n-1} = 1$, $n \geq 1$ while $a_{ij} = 0$ in all other cases. The subshift of countable type σ_r given by this matrix, is the renewal shift we considered in Section 2.2. We also have that the induced map F is topologically conjugate to the full countable Bernoulli shift σ via the coding map $h_i : S^{\mathbb{N}} \to W$.

We could, of course, use Theorem 2.2.6 to show that there is a unique equilibrium measure for this system. However, we need a little more than that, we will construct the equilibrium measure directly

We start by considering the Bernoulli measure κ on $S^{\mathbb{N}}$ given by $\kappa_\alpha([n]) = p_n(\alpha)$. Note that $\sum_{n\geq 0} p_n(\alpha) = 1$ and that κ is invariant under the shift map σ. The measure $\nu_\alpha = (h_i)_*\kappa_\alpha$ is invariant under the induced map F. Since

$$Q_{\nu_\alpha} = 1 + \sum_{k\geq 1} kp_k(\alpha) < \infty,$$

we can consider $\mu_\alpha = \pi(\nu_\alpha)$, which is a Markov measure on the shift space Σ_A.

We claim that $\int_X \tau d\mu_\alpha < \infty$ when $\alpha > 2$ and $\int_X \tau d\mu_\alpha = \infty$ when $1 < \alpha \leq 2$. In fact,

$$\mu_\alpha(I_i) = \frac{1}{Q_{\nu_\alpha}} \sum_{k\geq i} p_k(\alpha) = \frac{\sum_{k\geq i} p_k(\alpha)}{1 + \sum_{k\geq 1} kp_k(\alpha)}.$$

Furthermore,

$$\int_X \tau d\mu_\alpha = \sum_{k\geq 0}(1+k)\mu_\alpha(I_k) = 1 + \frac{1}{Q_{\nu_\alpha}}\sum_{k\geq 1}\frac{k(k-1)}{2}p_k(\alpha) \asymp \sum_{k\geq 1}k^2 p_k(\alpha)$$

and the desired claim follows.

We shall now show that the measure μ_α is the unique equilibrium measure for φ_α. Observe that the induced potential

$$\overline{\varphi}_\alpha(x) = \sum_{k=0}^{n-1}\varphi_\alpha(f^k(x)) = \log p_n(\alpha), \quad x \in I_n.$$

Set $\phi_\alpha = \varphi_\alpha \circ h_r$ and $\overline{\phi_\alpha} = \overline{\varphi}_\alpha \circ h_i$. It suffices to show that $\mu_\alpha \circ h_r$ is the unique equilibrium measure for ϕ_α. We fix α and in what follows we simplify our notations by writing $\phi = \phi_\alpha$, $\varphi = \varphi_\alpha$, $p_n = p_n(\alpha)$, etc.

To obtain our result we shall apply some methods in the theory of countable Markov shifts, see for example, [Sar99, BS03]. Observe that $V_n(\phi) = 0$ for any n. Define

$$Z_0(\phi, [0]) = 1, \quad Z_n(\phi, [0]) = \sum_{\sigma^n x = x}\exp \phi_n(x) 1_{[0]}(x)$$

and

$$Z_n^*(\phi, [0]) = \sum_{\sigma^n x = x, \rho(x) = n}\exp \phi_n(x) 1_{[0]}(x),$$

where $\phi_n(x) = \sum_{k=0}^{n-1}\phi(\sigma^k x)$ and $\rho(x)$ is the first return time to $[0]$. Note that

$$P_G(\phi) = P_G(\phi, [0]) = \lim_{n\to\infty}\frac{1}{n}\log Z_n(\phi, [0]).$$

We first show that $P_G(\phi) = 0$. Set

$$T(z) = \sum_{n\geq 0} z^n Z_n(\phi, [0]), \quad R(z) = \sum_{n\geq 1} z^n Z_n^*(\phi, [0]).$$

The fact that $V_n(\phi) = 0$ for all n implies that

$$Z_n(\phi, [0]) = Z_1^*(\phi, [0]) Z_{n-1}(\phi, [0]) + \cdots + Z_{n-1}^*(\phi, [0]) Z_1(\phi, [0]) + Z_n^*(\phi, [0]).$$

This in turn implies the following equation of formal power series known as the *renewal equation*:

$$T(z) = \frac{1}{1 - R(z)}.$$

Observe that $\exp P_G(\phi, [0])$ is exactly the radius of convergence for $T(z)$, and the latter is exactly the solution of the equation $R(z) = 1$. Notice that $Z_n^*(\phi, [0]) = p_{n-1}$, and the fact that $\sum_{k\geq 0} p_k = 1$ implies that $R(1) = 1$. We conclude that $P_G(\phi) = 0$.

By the variational principle (see Theorem 1.3.2), to show that $\mu \circ h_r$is a equilibrium measure, it suffices to show that

$$h_{\mu\circ h_r}(\sigma_r) + \int_{\Sigma_A} \phi \, d(\mu \circ h_r) = 0. \tag{3.6}$$

In fact, if ν is the induced measure for μ, by Abramov's and Kac's formulas (see Proposition 1.4.2),

$$h_\mu(f) + \int_X \varphi \, d\mu = \frac{1}{Q_\nu}\Big(h_\nu(F) + \int_W \overline{\varphi} \, d\nu\Big).$$

Since $\nu \circ h_i = \kappa$ is a Bernoulli measure, we find that

$$h_\nu(F) = -\sum_{k\geq 0} p_k \log p_k, \qquad \int_W \overline{\varphi} d\nu = \sum_{k\geq 0} p_k \log p_k.$$

It follows that

$$h_\mu(f) + \int_X \varphi \, d\mu = \frac{1}{Q_\nu}\Big(h_\nu(F) + \int_W \overline{\varphi} \, d\nu\Big) = 0.$$

This implies (3.6). We conclude that $\mu \circ h_r$ is an equilibrium measure for ϕ on Σ_A. As the potential ϕ has summable variations, is bounded from above, and has finite Gurevich pressure, Theorem 1.1 from [BS03] implies that the equilibrium measure is unique. By the topological conjugacy, μ is the unique equilibrium measure for φ on X.

To prove the last statement note that $0 \notin X$. Pick any

$$a_0 > \sup_{\mu \in \mathcal{M}(f,X)} \{h_\mu(f) + \int_X \psi d\mu\}$$

(observe that the function ψ is bounded from above and hence, the supremum is finite). Since $\mathcal{M}(f, X)$ only contains measures supported on X, it is clear that in this case the Dirac measure at 0 is the equilibrium measure among all the measures supported on I.

Example 3.3.2.

We describe an example due to Bruin (oral communication) that shows that there are inducing schemes which allow nonliftable measures. Indeed, we show that such a measure can be a unique equilibrium measures for an appropriately chosen potential function. A similar construction was used by Zweimüller [Zwe05].

Consider the map $f = 2x \pmod 1$ of the unit interval I and the countable partition of I by intervals I_n constructed in the previous example. This partition codes f into the renewal shift. Now subdivide any interval I_n into 2^{2^n} intervals of equal length and call them I_n^j, $j = 1, \ldots, 2^{2^n}$. Define the inducing time $\tau(I_n^j) = 2^n + n$. We claim that the Lebesgue measure λ, which is invariant under f, is not liftable to this inducing scheme. Observe that λ is an equilibrium measure (for the potential $\varphi = 0$). We have that

$$\int_I \tau \, d\lambda = \sum_{n,j} \tau(I_n^j)\lambda(I_n^j) = \sum_n (2^n + n)\lambda(I_n) = \infty.$$

To show the λ is not liftable let us note that any induced measure of λ must be F-invariant and absolute continuous with respect to λ, and that λ itself is the only candidate. We then have $\pi(\lambda) = \lambda$, however this is impossible as $\int_I \tau \, d\lambda = \infty$. Hence, λ is not liftable.

3.4 Markov extensions

Many examples with inducing schemes falls into the category of *piecewise invertible maps*, which will be defined below. For these maps one can build a Hoffbauer-Keller tower (See [Hof81],[Kel89]) which is a Markov extension of the system. In the one dimensional case, any measure with positive entropy is liftable.

In [Bru95], Bruin established liftability of absolutely continuous invariant measures of positive entropy for piecewise continuous piecewise monotone interval maps with inducing schemes satisfying some additional assumptions. These assumptions allow one to "embed" the inducing scheme into the Hoffbauer-Keller tower and express the induced map as the first return time map to a certain subset (in the Hoffbauer-Keller tower). Using the techniques of Bruin, Pesin and Senti [PS06] showed that for any unimodal map satisfying the Collet-Eckmann condition every measure $\mu \in \mathcal{M}(f, X)$ of positive metric entropy is liftable.

In [Buz99], Buzzi constructed the connected Markov extension which is a version of the Hoffbauer-Keller tower for multidimensional maps. We will modify the approach by Bruin for the setting of connected Markov extension and establish liftability of measures with large entropy for a class of inducing schemes. To be more specific, we will prove, in Section 3.4:

Theorem 3.4.1. *Let f be a piecewise invertible continuous map of a compact metric space admitting inducing scheme $\{S, \tau\}$ satisfying Conditions (L1) (or respectively, (L1$^+$)) and (L2). Assume that the connected Markov extension of (I, f) satisfies Conditions (M1) and (M2). Then there exists $0 \le H < h(f)$ such that any measure $\mu \in \mathcal{M}(f, X)$ with $h_\mu(f) > H$ is liftable (here $h(f)$ is the topological entropy of f).*

In this section, we will introduce and discuss some of the properties of the connected Markov extension.

Let I be a compact metric space. A map $f : I \to I$ is said to be *piecewise invertible* if there exists a collection of open disjoint subsets $P = \{A_i \subset I\}_{i=1}^s$ satisfying:

1. $\bigcup_{i=1}^s \overline{A}_i = I$;

2. for each i there is a connected open set U_i and a homeomorphism $f_{U_i} : U_i \to I$ for which $\overline{A_i} \subset U_i$ and $f_{U_i}|A_i = f|A_i$;

3. the boundary $\partial P := \bigcup_{A_i \in P} \partial A_i$ is the critical set for f, i.e., for any open set U for which $U \cap \partial P \neq \emptyset$, $f|U$ is not a homeomorphism.

Set

$$\partial_0 P := \partial P, \quad \partial_n P := \bigcup_{k=0}^{n-1} f^{-k}(\partial P),\ n \geq 1$$

and for $x \notin \partial P$ denote by $P(x)$ the element of P containing x. Further, for $x \notin \partial_n P$ we denote by $P_n(x)$ the element of $P \vee f^{-1}P \cdots \vee f^{-n+1}P$ containing x.

Following [Buz99] we describe the connected Markov extension of the map f. This construction is slightly different from the construction of the Hoffbauer-Keller tower. Set $\mathcal{D}_1 = P$ and then

$$\mathcal{D}_{n+1} := \{f(A) \cap B \neq \emptyset :\ A \in \mathcal{D}_n,\ B \in P\} \text{ and } \mathcal{D} := \bigcup_{n \geq 0} \mathcal{D}_n.$$

The *connected Markov extension* of f is the pair $(\check{I}, \check{f})$ where

$$\check{I} := \{(x, D) \in I \times \mathcal{D} :\ x \in \bar{D}\}$$

is the tower and $\check{f} : \check{I} \setminus \pi^{-1}(\partial P) \to \check{I}$ is the map given by

$$\check{f}(x, D) = (f(x), E),$$

where E is the *connected* component of $f(D \cap P(x))$ containing $f(x)$, $\pi : \check{I} \to I$ is the canonical projection, i.e., $\pi(x, \mathcal{D}) = x$. We refer to subsets of the type

$$\check{D} := \{(x, D) :\ x \in \bar{D},\ D \in \mathcal{D}\}$$

as *elements* of the Markov extension and we set

$$\check{\mathcal{D}} := \bigcup_{D \in \mathcal{D}} \check{D}.$$

Let $\mathrm{inc} : I \setminus \partial D \to \check{I}$ be the inclusion into the first level of the Markov extension, i.e., $\mathrm{inc}(x) = (x, P(x))$. For any $D \in \mathcal{D}$ we define the *level of* D as $\ell(D) = \min\{n \in \mathbb{N} \colon D \in \mathcal{D}_n\}$ and, by extension, we define the *level of* $\check{D}$ as $\ell(\check{D}) = \ell(D)$ and write $\check{D} = \check{D}_\ell$.

Note that the projection $\pi : \check{I} \to I$ is countable to one on $\check{I}$, but it is injective on each $\check{D} \in \check{\mathcal{D}}$.

The Markov extension has the following properties (see [Buz99]):

1. it is an extension of the system (I, f), i.e.,

$$\pi \circ \check{f}|\check{I} \setminus \pi^{-1}(\partial P) = f \circ \pi|I \setminus \partial P;$$

2. $\check{\mathcal{D}}$ is a Markov partition for $(\check{I}, \check{f})$ in the sense that for any $i \in \mathbb{N}$ and any $\check{D}_a, \check{D}_b \in \check{\mathcal{D}}$ we have that $\check{f}^i(\check{D}_a) \cap \check{D}_b \neq \emptyset$ if and only if $\check{f}^i(\check{D}_a) \supseteq \check{D}_b$;

3. for any connected open set $\check{E} \subset \check{D} \in \check{\mathcal{D}}$ and any $i \in \mathbb{N}$ we have that $\check{f}^i|\check{E}$ is a homeomorphism if and only if so is $f^i|\pi(\check{E})$ (see Property (3) in the definition of the piecewise invertible map);

4. for any $\check{D} \in \check{\mathcal{D}}$ of level n there exists a unique subset $E \subset A_i$ for some $A_i \in P$ such that $\check{f}^n$ maps $\operatorname{inc}(E)$ homeomorphically onto $\check{D}$.

Let f be a piecewise invertible map of a compact metric space I and $(\check{I}, \check{f})$ its connected Markov extension. We define $(\mathcal{I}, f_e)$ and $(\check{\mathcal{I}}, \check{f}_e)$ to be the *natural extensions* of f and $\check{f}$ respectively. Recall that the natural extension $(\mathcal{I}, f_e)$ of a map $f : I \to I$ is the space of all sequences $\{x_n\}_{n\in\mathbb{Z}}$, satisfying $f(x_n) = x_{n+1}$ (i.e., orbits of f), along with the map f_e, which is the left shift. There is a natural projection $p(\{x_n\}) = x_0$ from the natural extension to the original space. If f preserves a measure μ, there is a unique f_e-invariant measure μ_e on the natural extension, which projects to μ. If μ is ergodic then so is μ_e and $h_{\mu_e}(f_e) = h_\mu(f)$.

We denote by $p : \mathcal{I} \to I$ and $\check{p} : \check{\mathcal{I}} \to \check{I}$ the natural projections and by $\pi_e : \check{\mathcal{I}} \to \mathcal{I}$ the extension of the projection π to the natural extensions. We have the following commutative diagram

$$\begin{array}{ccc} (\check{\mathcal{I}}, \check{f}_e) & \xrightarrow{\pi_e} & (\mathcal{I}, f_e) \\ \downarrow{\scriptstyle \check{p}} & & \downarrow{\scriptstyle p} \\ (\check{I}, \check{f}) & \xrightarrow{\pi} & (I, f) \end{array} \tag{3.7}$$

Define the $\check{f}_e$-invariant set $\check{\mathcal{I}}' \subseteq \check{\mathcal{I}}$ as

$$\check{\mathcal{I}}' := \{\{\check{x}_n\}_{n\in\mathbb{N}} \in \check{\mathcal{I}} : \text{there exists } N \geq 0 \text{ such that } \check{x}_0 = \check{f}^n(\operatorname{inc}(\pi(\check{x}_{-n}))) \text{ for all } n \geq N\}$$

and set $\mathcal{I}' = \pi_e(\check{\mathcal{I}}')$. It is shown in [Buz99] that $\pi_e : \check{\mathcal{I}}' \to \mathcal{I}'$ is one-to-one and bi-measurable. Let $\Delta P = f(\partial P)$.

Proposition 3.4.2 (see [Buz99], Theorem A, Proposition 2.2)**.** *Assume that the connected Markov extension satisfies the following conditions:*

(M1) $h_{top}(\Delta P, f) < h_{top}(f)$*;*

(M2) there exist a measurable subset $I_0 \subset I$ *and a number* $0 \leq H_0 < h_{top}(f)$ *such that for any measure* $\mu \in \mathcal{M}(f, I)$ *with* $h_\mu(f) \geq H_0$ *we have* $\mu(I \setminus I_0) = 0$ *and* $diam(P_n(x)) \to 0$ *as* $n \to \infty$ *for* μ*-almost every* $x \in I_0$*.*

Then for any $\mu_e \in \mathcal{M}(f_e, \mathcal{I})$ *with*

$$h_{\mu_e}(f_e) > \max\{H_0, h_{top}(\Delta P, f)\}$$

we have $\mu_e(\mathcal{I}') = 1$*. The same statement holds under the same conditions for any* $\check{\mu}_e \in \mathcal{M}(\check{f}_e, \check{\mathcal{I}})$*.*

One can therefore lift any measure with sufficiently large entropy to its Markov extension. More precisely, the following statement holds.

Proposition 3.4.3. *Assume that the connected Markov extension satisfies Conditions (M1) and (M2) of Proposition 3.4.2. Then for any ergodic invariant probability measure* μ *with* $h_\mu(f) > \max\{H_0, h_{top}(\Delta P, f)\}$*,*

1. *there exists an* $\check{f}$*-invariant ergodic Borel probability measure* $\check{\mu}$ *on the connected Markov extension* $\check{I}$ *with* $\pi_*\check{\mu} = \mu$*;*

2. *there exists an* $\check{f}_e$*-invariant ergodic Borel probability measure* $\check{\mu}_e$ *on* $\check{\mathcal{I}}$ *with* $\check{p}_*\check{\mu}_e = \check{\mu}$ *and* $\check{\mu}_e(\check{\mathcal{I}}') = 1$*.*

Proof. Let μ be an f-invariant ergodic Borel probability measure with $h_\mu(f) > \max\{H_0, h_{top}(\delta P, f)\}$ and μ_e the unique lift of μ to the natural extension $\mathcal{I}$. By Proposition 3.4.2, π_e is an measurable isomorphism between $\check{\mathcal{I}}'$ and $\mathcal{I}'$ and the measure $\check{\mu}_e := (\pi_e^{-1})_*\mu_e|\mathcal{I}'$ is well defined. Furthermore, $\mu_e(\mathcal{I}') = 1 = \check{\mu}_e(\check{\mathcal{I}}')$. Since the diagram (3.7) commutes we have $\mu = \pi_*\check{\mu}$ for $\check{\mu} = \check{p}_*\check{\mu}_e = (\check{p} \circ \pi_e^{-1} \circ p^{-1})_*\mu$. $\square$

3.5 Connection between the Markov extension and the inducing scheme

We will study the liftability problem using the connected Markov extension. To do that we need to impose some additional stronger restrictions on the inducing schemes:

(L1) *minimality*: there is an open connected set U_J of J such that $f^{\tau(J)}|U_J$ is a homeomorphism onto its image with $f^{\tau(J)}(J) = W$ (see Condition (H1)); in addition, the inducing time is *minimal* in the following sense: for any $L \subset I$ and $m \in \mathbb{N}$ such that f^m is a homeomorphism of L onto W and $L \cap J \neq \emptyset$ for some $J \in S$ we have $m \geq \tau(J)$;

In the case of one-dimensional maps one often needs bounds on the distortion of the derivative of the induced map F. Such bounds can be obtained using Koebe's lemma, which applies under a somewhat different assumption than (L1):

(L1$^+$) *minimal extendability*: there is a connected open neighborhood W^+ of W and for each $J \in S$ there exists a connected open neighborhood J^+ of J such that $f^{\tau(J)}|J^+$ is a homeomorphism onto its image, $f^{\tau(J)}(J^+) = W^+$ and $f^{\tau(J)}(J) = W$; in addition, the inducing time is *minimal extendable*, in the following sense: for any $L \subset I$, $m \in \mathbb{N}$ and any connected neighborhood $L^+ \supset L$ such that f^m is a homeomorphism of L onto W and of L^+ onto W^+ we have that if $L \cap J \neq \emptyset$ for some $J \in S$ then $m \geq \tau(J)$.

Along with (L1) (or (L1$^+$) we require the inducing scheme to satisfy

(L2) *compatibility* with the connected Markov extension, i.e., for any $J \in S$ there exists an element $A_i \in P$ such that $J \subset U_J \subset A_i$ (or $J^+ \subset U_J \subset A_i$ in the case of Condition (L1$^+$)).

For each $J \in S$ define the map

$$\check{F}|\pi^{-1}(J) = \check{f}^{\tau(J)}|\pi^{-1}(J), \check{F}|\pi^{-1}(J^+) = \check{f}^{\tau(J)}|\pi^{-1}(J^+)$$

and set

$$\check{W} := \bigcup_{k\geq 0} \check{F}^k(\text{inc}\,(W)), \quad \check{W}^+ := \bigcup_{k\geq 0} \check{F}^k(\text{inc}\,(W^+)).$$

Theorem 3.5.1. *Assume that the inducing satisfies either of Conditions (L1) or (L1$^+$) and Condition (L2). Then the map $\check{F} : \check{W} \to \check{W}$ is the first return map of $(\check{I}, \check{f})$ to $\check{W}$. More precisely, for any $\check{x} \in \check{W} \cap \pi^{-1}(J)$ with some $J \in S$ we have that $\check{f}^i(\check{x}) \notin \check{W}$ for $0 < i < \tau(J)$.*

Proof. We will only prove the statement under Condition (L1$^+$). If the inducing scheme satisfies Condition (L1) the proof goes by replacing J^+ with J, W^+ with W and L^+ with L.

So, let us assume that the inducing satisfies Conditions (L1$^+$) and (L2). Since $\check{F}$ is a homeomorphism on any inc$\,(J^+)$, for any $\check{D} \in \check{\mathcal{D}}$ with $\check{D} \cap \check{F}(\text{inc}\,(J^+)) \neq \emptyset$ we have that $\check{F}(\text{inc}\,(J^+)) \subset \check{D}$. Hence $\pi(\check{D}) \supset W^+$. Using induction one can easily check that this also holds for any element $\check{D} \cap \check{W} \neq \emptyset$ of the connected Markov extension.

Assume, by contradiction, that there exist $\check{x} \in \check{W} \cap \pi^{-1}(J) \cap \check{D}_a$ and $0 < i < \tau(J)$ such that $\check{f}^i(\check{x}) \in \check{W} \cap \check{D}_b$. It follows from the previous observation that both $\pi(\check{D}_a) \supset W^+$ and $\pi(\check{D}_b) \supset W^+$. As $i < \tau(J)$, the map $\check{f}^i|(\pi^{-1}(J^+) \cap \check{D}_a)$ is a homeomorphism and we have that $\check{f}^i(\pi^{-1}(J^+) \cap \check{D}_a) \subset \check{D}_b$. By the Markov property of the Markov extension, $\check{f}^i(\check{D}_a) \supset \check{D}_b$ and we can take $\check{L}$ to be the unique homeomorphic preimage of $\pi^{-1}(W^+) \cap \check{D}_b$ under $\check{f}^i$ that contains $\check{x}$.

Condition (L1$^+$) implies the following fact: let $L^+ \subset I$ and $m \in \mathbb{N}$ be such that $f^m(L^+) = W^+$ and $f^m|L^+$ is a homeomorphism; if $L^+ \cap J \neq \emptyset$ for some $J \in S$, then $m \geq \tau(J)$. Setting $L^+ = \pi(\check{L})$ and $m = i$ we come to a contradiction. □

Theorem 3.5.2. *Assume that the connected Markov extension satisfies Conditions (M1) and (M2) of Proposition 3.4.2. Let μ be an f-invariant Borel probability measure on I and $\check{\mu}$ its lift to the connected Markov extension. Let also $E \subset X$ be such that $\mu(E) > 0$ and $E \cap \partial P = \emptyset$. Then for $\check{\mu}$-almost every $\check{x} \in \check{I}$ there exists $k \in \mathbb{N}$ and $\check{y} \in inc\,(E)$ such that $\check{f}^k(\check{y}) = \check{x}$, i.e.,*

$$\check{\mu}\Big(\bigcup_{k \geq 0} \check{f}^k(inc\,(E))\Big) = 1.$$

Proof. First, define the set $\mathcal{R} \subset \check{\mathcal{X}}$ by

$$\mathcal{R} := \{\{\check{x}_n\}_{n\in\mathbb{N}} \in \check{\mathcal{I}}' : \text{ there exists } n_k \to \infty \text{ such that } \\ \pi(\check{x}_{-n_k}) \in E,\ \check{x}_0 = \check{f}^{n_k}(\text{inc}\,(\pi(\check{x}_{-n_k})))\}.$$

We claim that if $\check{\mu}_e(\mathcal{R}) = 1$ then our statement holds. Indeed, set

$$R := \{\check{x} \in \check{I} : \text{there exists } k \in \mathbb{N} \text{ and } \check{y} \in \text{inc}\,(E) \text{ such that } \check{f}^k(\check{y}) = \check{x}\}.$$

We have that $\check{p}(\mathcal{R}) \subset R$ and hence by Proposition 3.4.2,

$$1 \geq \check{\mu}(R) \geq \check{\mu}(\check{p}(\mathcal{R})) \geq \check{\mu}_e(\mathcal{R}) = 1.$$

It follows that $\check{\mu}(R) = 1$, which implies the desired result.

We therefore are left to prove that $\check{\mu}_e(\mathcal{R}) = 1$. Note that the set $\check{\mathcal{I}}'$ has full $\check{\mu}_e$-measure and that $\check{\mu}_e(\check{p}^{-1}(\pi^{-1}(E))) = \mu(E) > 0$. Since μ is ergodic with respect to f, by Proposition 3.4.3 yields that $\check{\mu}$ is ergodic with respect to $\check{f}$. Note that the inverse map $\check{f}_e^{-1}$ is well defined on the natural extension and hence, it is ergodic with respect to $\check{\mu}_e$. By Birkhoff's ergodic theorem, for $\check{\mu}_e$-almost every $\{\check{x}_n\}_{n\in\mathbb{N}} \in \check{\mathcal{I}}'$ there exists $n_k \to \infty$ such that $\check{f}_e^{-n_k}(\{\check{x}_n\}) \in \check{p}^{-1}(\pi^{-1}(E))$. This implies that

$$\check{x}_{-n_k} = \check{p}(\check{f}_e^{-n_k}(\{\check{x}_n\})) \in \pi^{-1}(E),$$

i.e., $\pi(\check{x}_{-n_k}) \in E$. For any $\{\check{x}_n\} \in \check{\mathcal{I}}'$ we have that $\check{x}_0 = \check{f}^n(\text{inc}\,(\pi(\check{x}_{-n})))$ for sufficiently large n. It follows that $\check{\mathcal{I}}' \subseteq \mathcal{R} \pmod{\check{\mu}_e}$ and hence,

$$1 = \check{\mu}_e(\check{\mathcal{I}}') = \check{\mu}_e(\mathcal{R}),$$

which implies the desired result. □

Corollary 3.5.3. *Let μ and $\check{\mu}$ be as in Theorem 3.5.2. Assume that $\mu(W) > 0$ then $\check{\mu}(\check{W}) > 0$.*

Proof. It follows from Theorem 3.5.2 that

$$\check{\mu}(\bigcup_{k\geq 0} \check{f}^k(\text{inc}\,(W))) = 1.$$

Since

$$\bigcup_{k\geq 0} \check{f}^k(\mathrm{inc}\,(W)) \subset \bigcup_{j\geq 0} \check{f}^{-j}(\check{W}),$$

it follows that $\check{\mu}(\check{W}) > 0$. □

Finally, we prove the following:

Theorem 3.5.4. *Let (I, P, f) be a piecewise invertible system and (S, τ) an inducing scheme satisfying Condition (L1) or (L1$^+$) and Condition (L2). Let $\mu \in \mathcal{M}(f, I)$ be an ergodic measure, which is supported on X and is liftable to the connected Markov extension $(\check{I}, \check{f})$, i.e., there exists an ergodic measure $\check{\mu}$ on $(\check{I}, \check{f})$ with $\pi_*\check{\mu} = \mu$. Then μ is liftable to the inducing scheme (S, τ).*

Proof. Since μ is ergodic and $X = \bigcup_{k\geq 0} f^{-k}(W)$, the fact that $\mu(X) = 1$ implies that $\mu(W) > 0$. Then Corollary 3.5.3 yields that $\check{\mu}(\check{W}) > 0$.

On the other hand, it follows from Theorem 3.5.1 that $(\check{W}, \check{F})$ is the first return time map of $(\check{I}, \check{f})$ to $\check{W}$. Since $\check{\mu}(\check{W}) > 0$, we have that $\check{\nu} = \frac{1}{\check{\mu}(\check{W})}\check{\mu}|\check{W}$ is $\check{F}$-invariant. Furthermore, since $\tau(J)$ is the first return time of $\check{f}^i(J)$ to $\check{W}$,

$$\check{\mu} = \sum_{J\in S} \sum_{k=0}^{\tau(J)-1} \check{\mu}|\check{W} \circ \check{f}^{-k} = \frac{1}{Q_{\check{\nu}}} \sum_{J\in S} \sum_{k=0}^{\tau(J)-1} \check{\nu} \circ \check{f}^{-k}, \tag{3.8}$$

where

$$Q_{\check{\nu}} = \sum_{J\in S} \sum_{k=0}^{\tau(J)-1} \nu(\pi^{-1}(J) \cap \check{W}) = \frac{1}{\check{\mu}(\check{W})}.$$

Note that we have the following two conjugacies:

$$\pi \circ \check{f}|(\check{I} \setminus \pi^{-1}(\partial P) = f \circ \pi|(I \setminus \partial P), \quad \pi \circ \check{F} = F \circ \pi.$$

It follows that $\pi_*\check{\nu}$ is an F-invariant Borel probability measure. Applying the map π_* to both sides of (3.8), gives

$$\mu = \frac{1}{Q_\nu} \sum_{J\in S} \sum_{k=0}^{\tau(J)-1} \nu \circ f^{-k},$$

which is what we need. □

Chapter 4

Thermodynamics for Young's tower

4.1 Introduction

In this chapter we apply the results discussed in Chapter 1 to a particular inducing scheme of hyperbolic type (See Section 1.2). The inducing scheme is obtained from a construction by Young ([You98]). We show that the techniques used in [PS06],[PS05] applies to this case, and in particular, show the existence of equilibrium measure for the potential $\varphi_t = -t \log|\det Df^u|$ (See section 4.3). We point out however that unlike in [PS06], the liftability question in this case is unknown, we can only obtain equilibrium measure for the class $\mathcal{M}_L(X, f)$. To be more specific, we have the following (see [Zha07] for details):

For a system admitting a Young's tower (See Section 4.2 below), then the system admits an inducing scheme of hyperbolic type. Assuming also condition (4.1) (See Section 4.3), then for the system (X, f), there are numbers $t_0 < 1 < t_1$ such that when $t_0 < t < t_1$ the potential $\varphi_t = -t \log|df^u|$ admits an unique equilibrium measure μ_t in the class $\mathcal{M}_L(X, f)$, i.e.

$$h_{\mu_t}(f) - t\int_X \log|\det Df^u| d\mu_t = \sup_{\mathcal{M}_L}\{h_\mu(f) - t\int_X \log|\det Df^u| d\mu\}.$$

This is proved in Section 4.3.

4.2 Young's Tower

Theorem 1.5.6 provides a way to construct equilibrium measure for maps with inducing scheme of hyperbolic type. Here we show that the class of maps admitting Young's tower admits an inducing scheme satisfying our conditions.

For a $C^{1+\epsilon}$ diffeomorphism $f : M \to M$, and we describe the conditions considered in [You98].

An embedded disk $\gamma \subset M$ is called an *unstable disk* if for $\forall x, y \in \gamma$, $d(f^{-n}x, f^{-n}y) \to 0$ as $n \to +\infty$. It is called an *stable disk* if $d(f^n x, f^n y) \to 0$ as $n \to \infty$.

A family of embedded disks $\Gamma^u = \gamma^u$ is a *continuous family of* C^1 *unstable disk* if there exists a homeomorphism $\Phi : K^s \times D^u \to \cup\gamma^u$ with the following properties:

- K^s is a compact set and $D^u \subset \mathbb{R}^n$ the unit disk.
- $x \to \Phi|_{\{x\}\times D^u}$ is a continuous map from K^s to the space of C^1 embedded disks in M.
- $\gamma^u = \Phi(\{x\} \times D^u)$ is an unstable disk.

Continuous family of C^1 *stable disks* are defined similarly.

We say that $\Lambda \subset M$ has a *hyperbolic product structure* if there exists continuous family of unstable disks Γ^u and a continuous family of stable disks Γ^s such that

- $dim\gamma^s + dim\gamma^u = dimM$.
- the $\gamma^u - disks$ are transversal to $\gamma^s - disks$ by an angle uniformly bounded away from 0.
- each $\gamma^u - disks$ intersects each γ^s in exactly one point.
- $\Gamma = (\cup\gamma^u) \cap (\cup\gamma^s)$.

In particular, Λ is a compact set.

A subset Λ_0 of Λ is called an *s-subset* if it is defined by the same family Γ^u of unstable disks as Λ and a subfamily $\Gamma^s_0 \subset \Gamma^s$ of stable disks. A *u-subset* is defined analogously. For $x \in \Lambda$ let $\gamma^u(x)$ denote the unstable disk containing x and $\gamma^s(x)$ the stable disk containing x. s-subsets and u-subsets are compact subsets of Λ.

The map f satisfies the following conditions:

(P1) There exists $\Lambda \subset M$ with a hyperbolic product structure and with $\mu_\gamma\{\gamma \cap \Lambda\} > 0$ for $\forall \gamma^u \in \Gamma^u$, where$\mu_\gamma$ is the Lebesgue measure on γ^u.

(P2) There are pairwise disjoint s-subsets $\Lambda_1, \cdots, \Lambda_n, \cdots \subset \Lambda$ with:

- On each γ^μ disk, $\mu_{\gamma^u}\{(\Lambda - \cup\Lambda_i) \cap \gamma^u\} = 0$
- For each i, $\exists \tau_i \in \mathbb{N}$ such that $f^{\tau_i}(\Lambda_i)$ is a u-subset of Λ, we require in fact that $\forall x \in \Lambda_i$, $f^{\tau_i}(\gamma^s(x)) \subset \gamma^s(f^{\tau_i}(x)))$, $f^{\tau_i}(\gamma^u(x)) \supset \gamma^u(f^{\tau_i}(x))$.
- For each n, there are at most finite many i's with $\tau_i = n$.
- $\min \tau_i \geq \tau_0 > 1$ depending only on f.

(P3) Contraction along γ^s disks: $\exists c > 0$, $0 < \alpha < 1$ such that for $y \in \gamma^s(x)$, $d(f^n(x), f^n(y)) < c\alpha^n$, $\forall n \geq 0$.

Define a function $s_0(x, y)$ ("the separation time") for $x, y \in \Lambda$. We say that $s_0(x, y) = n$ if x and y are "together" for their first n iterates and "separate" at the $n + 1^{th}$. We assume the following for $s_0(x, y)$:

(i) $s(x, y) \geq 0$ and only depend on the γ^s disks containing the x and y.

(ii) The maximum number of orbits staring from Λ that are pairwise separated before time n is finite for each n.

(iii) For $x, y \in \Lambda$, $s_0(x, y) \geq \tau_i + s_0(f^{\tau_i}x, f_i^{\tau}y)$; in particular, $s_0(x, y) \geq \tau_i$.

(iv) For $x \in \Lambda_i$ and $y \in \Lambda_j$, $i \neq j$ but $\tau_i = \tau_j$, we have $s_0(x, y) < \tau_i - 1$.

We then introduce the next condition:

(P4) Backward contraction and distortion along γ^u. Assume that for $y \in \gamma^u(x)$ and $0 \leq k \leq n < s_0(x, y)$,

(a) $d(f^n(x), f^n(y)) \leq c\alpha^{s_0(x,y)-n}$.

(b)
$$\log \prod_{i=k}^{n} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq c\alpha^{s_0(x,y)-n}, \quad \forall n \geq 0.$$

(P5) Convergence of $D(f^i|\gamma^u)$ and absolute continuity of Γ^s.

(a) For $y \in \gamma^s(x)$,

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq c\alpha^n$$

(b) For $\gamma, \gamma' \in \Gamma^u$, if $\Theta : \gamma \cap \Lambda \to \gamma' \cap \Lambda$ defined by $\Theta(x) = \gamma^s(x) \cap \gamma'$, then Θ is absolutely continuous and

$$\frac{d(\Theta_*^{-1}\mu_{\gamma'})}{d\mu_\gamma}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i x)}{\det Df^u(f^i \Theta x)}.$$

The system that satisfies the conditions described admits an inducing scheme and satisfies the conditions (H1), (H2') and (H3) described in section 1.1.

We define the map $F : \bigcup \Lambda_i \to \Lambda$ as

$$F|\Lambda_i = f^{\tau_i}|\Lambda_i,$$

and $W = \bigcap_{n=-\infty}^{\infty} F^n(\bigcup \Lambda_i)$. Then the inducing scheme is defined by $S = \{\Lambda_i \cap W\}$ and $\tau(\Lambda_i \cap W) = \tau_i$. Since W is F invariant, it is easy to see that condition (H1) is satisfied. For condition (H2), we need to show that $\bigcap_{n=-\infty}^{\infty} F^n(\Lambda_{a_n})$ is a single point. We note that the set $\bigcap_{n=1}^{\infty} F^n(\Lambda_{a_n})$ is a s-subset and $\bigcap_{n=0}^{-\infty} F^n(\Lambda_{a_n})$ is a u-subset. By hyperbolic product structure, the intersection (1.4) is always nonempty. The fact that it consists of only one point follows from the following Lemmas.

Lemma 4.2.1. *For $x, y \in \Lambda$, we define $[x, y] = \gamma^s(x) \cap \gamma^u(y)$. Then, $\forall x, y \in \bigcap_{k=-n}^{n} F^k(\Lambda_{a_k})$ and $z = [x, y]$, we have $f^k(z) = [f^k(x), f^k(y)]$ for $-n \leq k \leq n$.*

Proof. First we consider $k \geq 0$.

By (P2) and $z \in \gamma^s(x)$, we have $F^k(z) \in \gamma^s(F^k(x))$ for $0 \leq k \leq n$.

On the other hand, by $x, y \in \Lambda_{a_0}$ and the hyperbolic product structure of Λ_{a_0} we know that $z \in [x, y]$. Then $F(z) \in \gamma^u(F(y))$ follows from $z \in \gamma^s(y)$ and that $F(\Lambda_{a_0})$ is a u-subset. This proves that

$$F(z) = [F(x), F(y)].$$

In addition, $x, y \in F^{-k}(\Lambda_{a_{-k}})$. We have $F^k(x), F^k(y) \in \Lambda_{a_{-k}}$. The same arguments can be then applied to $[F^k(x), F^k(y)]$ and by induction

$$F^k(z) = [F^k(x), F^k(y)] \quad for\, 0 \le k \le n.$$

The case for $k < 0$ is similar. □

Lemma 4.2.2. *Let $w = (a_{-n}, \cdots, a_0, \cdots, a_n)$ be a $2n+1$ word in S^{2n+1}, and let $M_w = \bigcap_{k=-n}^{n} F^k(\Lambda_{a_k} \cap W)$.*

Then $diam(M_w) \le 2c\alpha^n$

Proof. Take $x, y \in M_w$, let $z = [x, y]$. From Lemma 4.2.1 we have $F^k(z) = [F^k(x), F^k(y)]$ for $-n \le k \le n$. In particular, this implies that $s_0(z, y) \ge \sum_{k=0}^{n-1} \tau_{a_k} \ge n$. From (P4) and definition of $s_0(x, y)$ we have for any $z \in \gamma^u(y)$,

$$d(F(z), F(y)) \le c\alpha^{s_0(F(x), F(y))}.$$

Hence

$$d(z, y) = d(F^n F^{-n}(z), F^n F^{-n}(y)) \le c\alpha^{s_0(z,y)} \le c\alpha^n,$$

and by (P3)

$$d(x, z) = d(F^n F^{-n}(x), F^n F^{-n}(z)) \le c\alpha^n,$$

we have $d(x, y) \le d(x, z) + d(z, y) \le 2c\alpha^n$. □

Condition (H3) is indeed satisfied in a stronger way, since all the Λ_i's are disjoint compact sets, the set $h^{-1}(W) = S^{\mathbb{Z}}$.

4.2.1 Expressing as a one-sided shift

We have seen above that the system admits an inducing scheme of hyperbolic type and hence $F : W \to W$ is conjugate to a two-sided Bernoulli shift. The methods in Section 1.3.2 then can be applied to get an Gibbs measure for an suitable potential. We remark that if we consider only the unstable direction, the induced system can be described by a 1-sided shift. Then one can construct the Gibbs measure by first finding a Gibbs measure on the unstable direction, this measure then can be lifted to W using a method Analogous to the construction of SRB measure in

[You98]. We note that this approach via conjugacy, is equivalent to the procedure in Section 1.3.2.

To be more specific, we can pick an arbitrary unstable manifold γ_0, and consider the partition $\Lambda_i \cap \gamma_0$ of $\Lambda \cap \gamma_0$. We have:

$$f^{\tau_i}(\Lambda_i \cap \gamma_0) = \Lambda \cap \gamma^s(f^{\tau_i} x_i),$$

where $\gamma^s(x)$ denotes the stable manifold through x and x_i is any point on $\Lambda_i \cap \gamma_0$. For any point $x \in \Lambda$ define the holonomy map $\Theta : \Lambda \to \gamma_0$ be $\Theta(x) := \hat{x} := \gamma^s(x) \cap \gamma_0$, and define a map

$$F_0 : \bigcup \Lambda_i \cap \gamma_0 \to \Lambda \cap \gamma_0$$

as $F|_{\Lambda_i \cap \gamma_0} = \Theta \circ f^{\tau_i}|_{\Lambda_i \cap \gamma_0}$. Let

$$W_o = \bigcup_{k=0}^{\infty} F_0^{-k}(\bigcup \Lambda_i) \cap \gamma_0 = W \cap \gamma_0,$$

Then $F_0 : W_0 \to W_0$ is topologically conjugate to a 1-sided Bernoulli shift $\sigma : \Sigma_+ \to \Sigma_+$.

Let ϕ be a potential function on W such that by conjugacy, the corresponding potential function on the symbolic space satisfies condition (1.8). For $x \in W$ we define a function

$$u(x) = \sum_{k=0}^{\infty} (\phi(F^i x) - \phi(F^i \circ \Theta x)).$$

Then this function $u(x)$, after conjugacy, has summable variations, and is bounded. Let

$$\psi = \phi + u \circ F - u,$$

then ψ is constant on any stable manifold, and since it is cohomologous to ϕ, it has the same equilibrium measures as ϕ does. This function is analogous to the function Ψ in Proposition 1.3.3.

If the function ϕ has finite Gurevich pressure on the 2-sided shift space, so does ψ. Let $\psi_0 = \psi|_{\gamma_0}$. Let h_0 be the conjugacy to the 1-sided symbolic space $S^{\mathbb{N}}$, and let $\tilde{\Psi} = \psi_0 \circ h$, by Proposition 1.3.1, there exists equilibrium and Gibbs measure

$\nu_{\tilde{\Psi}}$ on $S^{\mathbb{N}}$, and by conjugacy, $\nu_0 = \nu_\Psi \circ h_0^{-1}$ is the unique equilibrium measure for ψ_0. Note that this measure is supported on $W_0 = W \cap \gamma_0$.

To lift this measure to the whole W, we consider the measures $\nu_n = F_*^n \nu_0$ for $n = 1, 2, \cdots$. Let ν be any weak-*-limit of the sequence ν_n, we claim that $\Theta_* \nu = \nu_0$, and indeed, after conjugate to the symbolic space, ν is the natural extension of ν_0. Note the fact that if we write the projection from $S^{\mathbb{Z}}$ to $S^{\mathbb{N}}$ as π, we have $h \circ \Theta = \pi \circ h_0$. If we write ν_Ψ as the natural extension of the measure $\nu_{\tilde{\Psi}}$ on $S^{\mathbb{N}}$, we claim that $\nu = \nu_\Psi \circ h^{-1}$. Notice that ν and $\nu_\Psi \circ h^{-1}$ coincide because they do on any cylinder set $F^n(\Lambda_{a_n}) \cap F^{n+1}(\Lambda_{a_{n+1}}) \cdots F^m(\Lambda_{a_m})$ for $n < m \in \mathbb{Z}$.

4.3 Thermodynamics for $\varphi_t = -t \log |\det Df^u|)$

For $x \in X$ we define $E^u(x) = T_x f^k(\gamma^u(y))$ if $y \in W$ and $f^y = x$. We consider the potential $-t \log |\det(Df|E^u(x))|$. For simplicity we will write $Df^u(x)$ instead of $Df|E^u(x)$. Also for $x \in \Lambda_i$ we write $DF^u(x) = \prod_{k=0}^{\tau_i - 1} Df^u(f^k x)$. We write $\lambda_2 = \alpha^{-1}$, where α is as in condition (P3).

To apply the above theory to the potential $\varphi_t = -t \log \det |Df^u|$, impose the following condition to the map $f : M \to M$.

(H4) $\exists c_1 > 0$, $\lambda_1 > 1$, $\gamma \in \Gamma^u$, such that

$$\mu_\gamma(\{x \in \gamma \cap \Lambda, \quad \tau(x) \geq n\}) \leq c_1 \lambda_1^{-n}, \quad n \geq 1, \tag{4.1}$$

Write $\lambda_2 = \alpha^{-1}$, where α is as in condition (P3). We could choose λ_1 to be smaller than λ_2 without changing the estimates.

With condition (H4) satisfied, we have that the following:

Corollary 4.3.1. *There exists positive constants c_3, c_4 and $\lambda_3 \geq \lambda_1$ such that for any $x \in \Lambda_i$, $\gamma \in \Gamma^u$,*

$$c_1^{-1} \lambda_1^{\tau_i} \leq c_3 \mu(\gamma \cap \Lambda_i)^{-1} \leq c^4 |\det DF^u(x)| \leq \lambda_3^{-n}. \tag{4.2}$$

Proof. Fix an arbitrary $\gamma \in \Gamma^u$. Note that (P4) and (P5) gives distortion estimates and absolute continuity of the Γ^u family along Γ^s, which implies that if we can prove the statements for one γ we have the same for any γ with uniform constants.

We note that for any i, the map F maps the set $\gamma \cap \Lambda_i$ diffeomorphically onto an entire unstable disk γ'. From which we obtain:

$$\mu_{\gamma'}(\gamma' \cap \Lambda) = \int_{\gamma \cap \Lambda_i} \det |DF^u(x)| d\mu_\gamma(x).$$

By (P1) and (P5)(b), There exists a constant $A > 0$ (which does not depend on the choice of γ') such that

$$A^{-1} \leq \mu_{\gamma'}(\gamma' \cap \Lambda) \leq A,$$

and (P4)(b) gives that

$$e^{-c} \leq \frac{\det DF^u(x)}{\det DF^u(y)} \leq e^c, \quad x, y \in \gamma.$$

Let $\lambda_3 = \sup_{x \in M} \det |Df^u(x)|$, and constants c_3 and c_4 can be chosen depending on A and c. □

Furthermore, let S_n be the number of Λ_i's with $\tau_i = n$, we have $S_n \leq c_1 c_3^{-1} c_5 \beta^n$, where $\beta = \frac{\lambda_3}{\lambda_1}$. Better estimates on S_n will lead to better results which will be seen in some of the calculations later.

The following statement hold:

Theorem 4.3.2. *Let $\varphi_t := -t \log \det |Df^u|$, then $\bar{\varphi}_t$ is locally Hölder continuous for $t > 0$.*

Proof. Consider $w = (a_{-n+1}, \cdots, a_{n-1})$ a $2n-1$ word, let $x, y \in M_w$ as defined in Lemma 4.2.2. It suffits to prove that there exists $A > 0$, $0 < \gamma < 1$ such that

$$|\bar{\varphi}(x) - \bar{\varphi}(y)| \leq A\gamma^n, \quad n \geq 1.$$

Let $z = [x, y]$ as defined in Lemma 4.2.1. We have $z \in \gamma^s(x)$ and $F^{-n}z \in \gamma^s(F^{-n}x)$. Applying Property (P4)(a) gives that

$$|\bar{\varphi}(x) - \bar{\varphi}(z)| \quad = \quad \left| -t \log \prod_{i=0}^{\tau_0 - 1} \frac{\det Df^u(f^i x)}{\det Df^u(f^i z)} \right|$$

$$\begin{aligned} &= \left| -t \log \prod_{i=\sum_{k=-n+1}^{0} \tau_{a_k}}^{\sum_{k=-n+1}^{1} \tau_{a_k}} \frac{\det Df^u(f^i F^{-n} x)}{\det Df^u(f^i F^{-n} z)} \right| \\ &\leq 2c\alpha^{\sum_{k=-n+1}^{0} \tau_{a_k}} \leq 2c\alpha^n = 2c\lambda_2^{-n}. \end{aligned}$$

On the other hand, it was proved in Lemma 4.2.2 that $s_0(y, z) \geq \sum_{k=0}^{n-1} \tau_{a_k}$ and $z \in \gamma^u(y)$, we have the following inequalities, applying Property (P5)(a):

$$\begin{aligned} |\bar{\varphi}(y) - \bar{\varphi}(z)| &= \left| -t \log \prod_{i=0}^{\tau_0 - 1} \frac{\det Df^u(f^i y)}{\det Df^u(f^i z)} \right| \\ &\leq c\alpha^{s_0(y,z) - \tau_0} \\ &\leq c\alpha^{\sum_{k=1}^{n-1} \tau_{a_k}} \leq c\alpha^{n-1} = c\lambda_2^{-n+1}. \end{aligned}$$

The statement follows. □

We have the following useful estimates for $\det |Df^u|$:

Lemma 4.3.3. *Let $\mu \in \mathcal{M}_L(f)$ be liftable, then the following inequalities hold:*

$$\log \lambda_1 \leq \int_X \log |\det df^u(x)| d\mu(x) \leq \log \lambda_3,$$

where the constants λ_1, λ_3 are as defined before.

Proof. The second part of the inequality is easy to show. Indeed, we know from previous estimates that:

$$|\det DF^u(x)| \leq \lambda_3^{\tau_i}, \quad x \in \Lambda_i.$$

It follows that:

$$\log |\det DF^u(x)| \leq \tau_i \log \lambda_3.$$

Integrate over each Λ_i with respect to $i(\mu)$ then sum up all i's:

$$\int_W |\det DF^u(x)| di(\mu) \leq Q_{i(\mu)} \log \lambda_3.$$

using

$$\int_W |\det DF^u(x)| di(\mu) = Q_{i(\mu)} \left(\int_X |\det Df^u(x)| d\mu(x) \right)$$

we get the desired estimate.

To prove the other part of the inequality, we first show the following properties of $\det DF^u(x)$:

1. (Summable variations) There exists $C > 0$ such that

$$\sum_{n\geq 1} V_n(\log|\det DF^u(x)|) \leq C.$$

2. (Uniformly expansion at return times) Denote by

$$\Lambda_{i_0,\cdots,i_{n-1}} = \Lambda_{i_0} \cap F^{-1}(\Lambda_{i_1}) \cdots \cap F^{-n+1}(\Lambda_{i_{n-1}}). \tag{4.3}$$

Then

$$|\prod_{k=0}^{n-1} \det DF^u(F^k x)| \geq c^{-2}\lambda_2^{(\tau_{i_1}+\cdots+\tau_{i_n})},$$

for $x \in \Lambda_{i_1,\cdots,i_n}$.

The first property follows from Theorem 4.3.2 while the second follows from conditions (P4) and (P5).

For any $1 \leq k \leq n$, let $x_{j_k,\cdots,j_1}$ be any arbitrarily chosen point in the cylinder set $\Lambda_{j_k,\cdots,j_1}$.

$$\begin{aligned}
& \int_W \log|\det DF^u(x)|d\nu \\
= & \sum_{j_1,\cdots,j_k} \int_{\Lambda_{j_k,\cdots,j_1}} \log|\det DF^u(x)|d\nu \\
\geq & \sum_{j_1,\cdots,j_k} \log|\det DF^u(x_{j_k,\cdots,j_1})|\nu(\Lambda_{j_k,\cdots,j_1}) - C \\
= & \sum_{j_1,\cdots,j_n} \log|\det DF^u(x_{j_k,\cdots,j_1})|\nu(\Lambda_{j_n,\cdots,j_1}) - C,
\end{aligned}$$

for the last inequality, note that ν is invariant, hence

$$\nu(\Lambda_{j_k,\cdots,j_1}) = \sum_{j_{k+1},\cdots,j_n} \nu(\Lambda_{j_n,\cdots,j_1}).$$

We have

$$\begin{aligned} & n\int_W \log|\det DF^u(x)|d\nu \\ \geq & \sum_{k=1}^{n-1}\sum_{j_1,\cdots,j_n} \log|\det DF^u(x_{j_k,\cdots,j_1})|\nu(\Lambda_{j_n,\cdot,j_1}) - C \\ \geq & \sum_{j_1,\cdots,j_n}\sum_{k=0}^{n-1} \log|\det DF^u(F^k x_{j_n,\cdots,j_1})|\nu(\Lambda_{j_n,\cdots,j_1}) - 2C \\ = & \sum_{j_1,\cdots,j_n} \log|\prod_{k=0}^{n-1}\det DF^u(F^k x_{j_n,\cdots,j_1})|\nu(\Lambda_{j_n,\cdots,j_1}) - 2C \\ \geq & \sum_{j_1,\cdots,j_n} (\tau_{j_1}+\cdots+\tau_{j_n})\log\lambda_2 - 2\log(A') - \log c \\ = & nQ_\nu\log\lambda_2 - 2C - \log(2c). \end{aligned}$$

Divide both sides by n and let $n\to\infty$ we obtain

$$\int_W \log|\det DF^u(x)|d\nu \geq Q_\nu\log\lambda_2 \geq Q_\nu\log\lambda_1, \tag{4.4}$$

and the statement follows. $\square$

Theorem 4.3.4. *For any t there exists a constant c_t such that $\varphi_t + c_t$ satisfies the (FGP)-condition.*

Proof. The (FGP)-condition for $\varphi_t + c_t$ can be explicitly written down as the following:

$$\sum_{i=i}^{\infty}\sup_{x\in\Lambda_i}|\det DF^u(x)|^{-t}e^{c_t\tau_i} < \infty.$$

We first assume $t\geq 1$. In this case, we only need to choose $c_t = 0$. We have

$$\begin{aligned} & \sum_{i=1}^{\infty}\sup_{x\in\Lambda_i}|\det DF^u(x)|^{-t} \leq c_4c_3^{-1}\sum_{i=1}^{\infty}\mu_\gamma(\gamma\cap\Lambda_i)^t \\ = & c_4c_3^{-1}\sum_{i=1}^{\infty}\mu_\gamma(\gamma\cap\Lambda_i)\mu_\gamma(\gamma\cap\Lambda_i)^{t-1} \leq c_4c_3^{-1}c_1^{t-1}\lambda_1^{1-t}\sum_{i=1}^{\infty}\mu_\gamma(\gamma\cap\Lambda_i) \\ \leq & c_4c_3^{-1}c_1^{t-1}\lambda_1^{1-t}\mu_\gamma(\gamma\cap\Lambda) < \infty. \end{aligned}$$

In the case when $t < 1$, we have

$$\begin{aligned}\sum_{i=1}^{\infty} \sup_{x\in\Lambda_i} |\det DF^u(x)|^{-t} e^{c_t \tau_i} &\leq B \sum_{i=1}^{\infty} \mu_\gamma(\gamma \cap \Lambda_i)^t e^{c_t \tau_i} \\ &= B \sum_{n=1}^{\infty} \sum_{\tau_i = n} \mu_\gamma(\gamma \cap \Lambda_i)^t e^{c_t n} \\ &= B \sum_{n=1}^{\infty} \sum_{\tau_i = n} \mu_\gamma(\gamma \cap \Lambda_i) \mu_\gamma(\gamma \cap \Lambda_i)^{t-1} e^{c_t n} \\ &\leq B \sum_{n=1}^{\infty} \sum_{\tau_i = n} \mu_\gamma(\gamma \cap \Lambda_i) c_1 \lambda_1^{n(1-t)} e^{c_t n} < \infty\end{aligned}$$

provided that $c_t < (t-1)\log\lambda_1$. □

To verify the condition (INT) we need the following lemmas:

Lemma 4.3.5. *Let s_t denote s_{φ_t}. Then*

1. *$s_1 = 0$, and there exists an equilibrium measure μ_1 for φ_1, i.e.*

$$h_{\mu_1}(f) + \int_X \varphi_1 d\mu_1 = s_1 = 0.$$

2.

$$s_t \geq \begin{cases} (1-t)\log\lambda_3, & t > 1 \\ (1-t)\log\lambda_1, & t < 1 \end{cases}.$$

Proof. For statement 1, we write $\Phi_1 := \bar{\varphi}_1 \circ h$. We have

$$\begin{aligned}P_G(\Phi_1) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n a = a} \exp \sum_{i=0}^{n-1} \Phi_1^i a 1_{[b]} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{F^n x = x} \exp |\det DF^u(x)| 1_{[\Lambda_b]} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{a_0,\cdots,a_n \in S \\ a_0 = a_n = b}} \mu(\gamma \cap (\bigcap_{i=0}^{n-1} F^{-i}\Lambda_{a_k})) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \mu(\gamma \cap \Lambda_b) = 0.\end{aligned}$$

By Proposition 1.5.1 we have $s_1 \le \max\{0, P_G(\Phi_1)\} = 0$.

For the other direction, we note that for $\epsilon > 0$, there exists $\nu \in \mathcal{M}_{\bar{\varphi}_1}(F)$ such that

$$h_\nu(F) + \int_W \bar{\varphi}_1 d\nu \ge P_G(\Phi_1) - \epsilon = -\epsilon.$$

We have $-\infty < \int_W \bar{\varphi}_1 d\nu < \infty$, since $\bar{\varphi}_1 = -\log|\det DF^u|$, apply (4.4) for ν we have

$$\log \lambda_1 \, Q_\nu \le -\int_W \bar{\varphi}_1 d\nu < \infty,$$

hence $Q_\nu < \infty$ and $\pi(\nu) \in \mathcal{M}_{\varphi_1}(f)$.

We have

$$s_1 \ge h_{\pi(\nu)}(f) + \int_X \varphi_1 d\pi(\nu) = \frac{h_\nu(F) + \int_W \bar{\varphi}_1 d\nu}{Q_\nu} \ge -\frac{\epsilon}{Q_\nu} \ge -\epsilon.$$

Since ϵ is arbitrary, $s_1 = 0$.

We already know that φ_1 satisfies the (SVN) and (FGP)-conditions. For (INT)-condition, since $s_1 = 0$, $\varphi_1^+ = \bar{\varphi}_1 = -\log|\det DF^u(x)|$. We have:

$$\sum_{i=1}^{\infty} \tau_i \sup_{x \in \Lambda_i} |\det DF^u(x)|^{-1} \le c_4 c_3^{-1} \sum_{n=1}^{\infty} n \sum_{\tau_i = n} \mu_\gamma(\gamma \cap \Lambda_i)$$
$$\le c_4 c_3^{-1} \sum_{n=1}^{\infty} c_1 n \lambda_1^{-n} < \infty.$$

The above estimate is still valid when $\bar{\varphi}_1$ is replaced with $\overline{\varphi_1 + \epsilon}$, from which we have the positive recurrence. By Theorem 1.5.5, the potential function φ_1 satisfies all the conditions for Theorem 1.5.3, and hence admits an unique equilibrium measure μ_1 in the class $\mathcal{M}_{\varphi_1}(f)$. This proves Statement 1.

For the second statement, we note that

$$\begin{aligned} s_t &= \sup_{\mu \in \mathcal{M}_{\varphi_t}(f)} \{h_\mu(f) - t \int_X \log|\det Df^u| d\mu\} \\ &\ge h_{\mu_1}(f) - t \int_X \log|\det Df^u| d\mu_1 \\ &= (1-t) \int_X \log|\det Df^u| d\mu_1. \end{aligned}$$

And the statements follows from Lemma 4.3.3. □

We write $\nu_1 := i(\mu_1)$. We have the following:

Theorem 4.3.6. *1. When $1 < t < t_1 := \frac{\log \lambda_3}{\log \lambda_3 - \log \lambda_1}$, the potential φ_t satisfies the (INT)-condition and is positive recurrent.*

2. For $t > t_0 := \frac{\log \beta - \log \lambda_1}{\log \beta}$ the potential φ_t satisfies the (INT)-condition and is positive recurrent;

Proof. We have

$$\sum_{i=1}^{\infty} \tau_i \sup_{x \in \Lambda_i} |\det DF^u(x)|^{-t} e^{-s_t \tau_i}$$
$$\leq c_4 c_3^{-1} \sum_{n=1}^{\infty} n e^{-s_t n} \sum_{\tau_i = n} \mu_\gamma(\gamma \cap \Lambda_i)^t := T$$

If $t > 1$, we have $s_t \geq (1-t)\log \lambda_3$ by Lemma 4.3.5:

$$\begin{aligned} T &\leq c_4 c_3^{-1} \sum_{n=1}^{\infty} n e^{-s_t n} \sum_{\tau_i = n} \mu_\gamma(\gamma \cap \Lambda_i) \mu_\gamma(\gamma \cap \Lambda_i)^{t-1} \\ &\leq c_4 c_3^{-1} \sum_{n=1}^{\infty} n e^{-s_t n} c_1^{t-1} \lambda_3^{(1-t)n} n c_1 \lambda_3^{-n} < \infty \end{aligned}$$

if $t \log \lambda_3 - s_t < 0$. This is true when $t < \frac{\log \lambda_3}{\log \lambda_3 + \log \lambda_1}$.

If $t < 1$, by Jensen's Inequality,

$$\begin{aligned} T &\leq c_4 c_3^{-1} \sum_{n=1}^{\infty} n e^{-s_t n} S_n^{1-t} \left(\sum_{\tau_i = n} \mu_\gamma(\gamma \cap \Lambda_i) \right)^t \\ &\leq c_4 c_3^{-1} \sum_{n=1}^{\infty} n e^{-s_t n} c_1^{1-t} c_3^{t-1} \beta^{(1-t)n} c_1^t \lambda_1^{-nt} \\ &= c_4 c_3^{-t} c_1 \sum_{n=1}^{\infty} n e^{-s_t n} \beta^{(1-t)n} \lambda_1^{-nt} < \infty \end{aligned}$$

provided $-s_t + (1-t)\lambda\beta - t \log \lambda_1 < 0$. By Lemma 4.3.5, this is true when

$$(t-1)\log \lambda_1 - t \log \lambda_1 + (1-t)\log \beta < 0,$$

or

$$t > \frac{\log\beta - \log\lambda_1}{\log\beta}.$$

Note that the estimates above will all hold when we replace $\overline{\varphi_t - s_t}$ with $\overline{\varphi_t - s_t + \epsilon}$, which proves the positive recurrence property for those maps. $\square$

This allows us to prove the main statement:

Corollary 4.3.7. *For a map (M, f) satisfying Young's conditions (P1)-(P5), and in addition, condition (4.1) is satisfied. Then for the system (X, f), there are numbers $t_0 < 1 < t_1$ such that when $t_0 < t < t_1$ the potential $\varphi_t = -t\log|Df^u|$ admits an unique equilibrium measure μ_t in the class $\mathcal{M}_L(X, f)$, i.e.*

$$h_{\mu_t}(f) - t\int_X \log|df^u|d\mu_t = \sup_{\mathcal{M}_L}\{h_\mu(f) - t\int_X \log|df^u|d\mu\}.$$

Remark. *If in addition, the condition $\beta < \lambda_1$ is satisfied for the system, then $t_0 < 0$ and the map admits measure of maximal entropy (for the class $\mathcal{M}_L(X, f)$).*

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