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PARITY CONSIDERATIONS IN ANDREWS-GORDON IDENTITIES, AND THE k-MARKED DURFEE SYMBOLS

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Abstract

In 1974, Andrews discovered the generating function for the partitions of n considered in a theorem due to Gordon. In a more recent paper, he reconsidered this generating function and gave refinements where additional restrictions involving parities are imposed. A combinatorial construction for the partitions enumerated by the mentioned generating function is given, and some of the Andrews' refinements are proven combinatorially. Some open problems posed by Andrews are solved, and a conjecture is settled. In another recent paper, Andrews defined k-marked Durfee symbols, and gave a substantial analytic account. He left the combinatorial explanations as open problems. Some of those open problems are also solved.

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for İsmail Varsoy and İsmail Kurşungöz

1. Introduction

A partition λ of a positive integer n is a nonincreasing sequence of positive integers $\lambda_1 \geq \ldots \geq \lambda_k > 0$ such that $n = \lambda_1 + \ldots + \lambda_k$ [3, Ch.1]. One may impose some constraints such as requiring distinct parts, parts that belong to certain residue classes modulo some positive integer, and so on. A large class of theorems in partition theory is equinumerity between partitions of a given integer satisfying a condition and the partitions of that integer satisfying some other condition. Some theorems can be proven using generating functions. Some are proven using purely combinatorial methods and a following problem in such cases is to provide generating functions for partitions described in those results.

In 1961, Gordon proved that the number of partitions of n into parts that are not congruent to $0, \pm a$ modulo $2k+1$ equals the number of partitions of n in which pairs of consecutive parts appear at most $k-1$ times and 1 appears at most $a-1$ times [12]. It is easy to write the generating function for the former sort of partitions, as they are given by

$$\prod_{\substack{n \geq 1 \\ n \not\equiv 0, \pm a (\text{mod } 2k+1)}} \frac{1}{(1-q^n)} \tag{1.1}$$

However, a generating function was not given for partitions subject to the latter constraint unless $k = 2$. Gordon used purely combinatorial methods, and his proof is a generalization of Schur's [17] combinatorial proof of the Rogers-Ramanujan Identities. In case $k = 2$, Gordon's result is a combinatorial interpretation of the famous Rogers-Ramanujan identities.

In 1966, Andrews [1] found that the following series is a solution to functional equations which derive from recurrences satisfied by $b_{k,a}(m,n)$, the number of partitions of n into m parts such that pair of consecutive integers occur at most $k-1$ times together and 1 occurs at most $a-1$ times.

$$Q_{k,i}(x;q) = \sum_{n \geq 0} \frac{(-1)^n x^{kn} q^{(2k+1)n(n+1)/2 - in}(1 - x^i q^{(2n+1)i})}{(q)_n (xq^{n+1})_\infty} \tag{1.2}$$

This series previously appeared in [15] and [18]. Here,

$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

and

$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty} (a;q)_n$$

In this sense,

$$\sum_{m,n\geq 0} b_{k,a}(m,n)x^m q^n = Q_{k,i}(x;q) \tag{1.3}$$

Upon substituting $x = 1$ and using Jacobi's Triple Product identity [3, eq.(2.2.10)], Gordon's theorem follows.

In 1974, Andrews, in his Proceedings of the National Academy of Sciences paper [2, eq.(2.5)], discovered the generating function for $b_{k,a}(m,n)$ as

$$\sum_{m,n\geq 0} b_{k,a}(m,n)x^m q^n = \sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\ldots+N_{k-1}^2+N_a+\ldots+N_{k-1}} x^{N_1+\ldots+N_{k-1}}}{(q)_{n_1}\cdots(q)_{n_{k-1}}} \tag{1.4}$$

where

$$N_r = n_r + \ldots + n_{k-1}$$

He used the same functional equations for which $Q_{k,i}(x;q)$ is a solution, and established the right hand side of (1.4) as another. Here, the exponent of q is the number being partitioned (n), and the exponent of x is the number of parts (m).

For $x = 1$, together with (1.3), and Jacobi's Triple Product Identity, it follows that

$$\sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\ldots+N_{k-1}^2+N_a+\ldots+N_{k-1}}}{(q)_{n_1}\cdots(q)_{n_{k-1}}} = \prod_{\substack{n\geq 1 \\ n\not\equiv 0,\pm a (\text{mod } 2k+1)}} \frac{1}{(1-q^n)} \tag{1.5}$$

which has since been called the Andrews-Gordon Identities. Note that the proof does not use Gordon's original result.

The next problem was to provide a constructive explanation to explain Andrews' series (1.4) since it was not at all clear how the mentioned partitions are generated. In 1980, Bressoud gave a combinatorial argument using ordinary partitions [8, §5] which allowed an extension to (1.4) with one more parameter. Bressoud's definition is inductive. There are other approaches which explain (1.4) such as multipartitions [5] or overpartitions [9].

In **§2**, the *Gordon marking* of a partition is defined, and a set of attributes to the partition is described. Backward and forward moves are defined which are restrictions of adding one or subtracting one from some part in the partition. This way, it is possible to keep some of those attributes invariant. Those invariants are then related to the indices in the denominators of the generating function (1.4) thus arriving at a new combinatorial interpretation. The difference from Bressoud's approach is that in **§2**, the construction is direct instead of inductive. For $k = 2$, both constructions are clearly the same. For larger k, it is yet to be shown that the methods are exactly the same. Bressoud's method also seems to keep the mentioned attributes invariant, for the case of Andrews-Gordon identities. On the other hand, it is not possible to reach Bressoud's all generalizations while keeping the invariants.

In a recent paper [6], Andrews revisited his generating function (1.4) and extended his results by considering some additional restrictions involving parities. He achieved those generalizations by using double recursions satisfied by $b_{k,a}(m,n)$ where additional constraints are imposed. This in turn gave larger sets of functional equations the solutions of which are variants of (1.4). Andrews then left the combinatorial explanations of the resulting generating functions as open problems. He made a conjecture, and gave a list of open problems. The method employed in **§2** very naturally generalizes to explain most generating functions in [6] and proves Andrews' conjecture as well.

In the same paper [6] Andrews defined parity indices, and derived the corresponding generating functions, again by solving functional equations that follow from recurrences. In **§3** constructive methods are described to obtain the generating functions, or variants thereof, which reduce to the series given by Andrews upon standard algebraic manipulations. Several sieves are also constructed in connection to those generating functions.

One of the open problems Andrews listed was to investigate the function

$$\sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\ldots+N_{k-1}^2} \quad x^{N_1+\ldots+N_{k-1}} \quad (-yq)_{n_1}\cdots(-yq)_{n_{k-1}}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{k-1}}} \tag{1.6}$$

which seems to have properties related to parity indices, and which satisfy the conditions for $b_{k,k}(m,n)$ at the same time. In **§3.4** cluster parity indices are defined, hence the definition of parity indices is extended in a way compatible with Gordon marking, and an interpretation using ordinary partitions is given to (1.6).

In another recent paper [4], Andrews defined k-marked Durfee symbols, provided and proved a number of identities and congruences. He mainly used analytical methods, and left the combinatorial explanations to his results as open problems. In **§4**, an alternative definition of k-marked Durfee symbols is given. Using this alternative characterization, some combinatorial open problems listed at the end of [4] are solved. The method employed is to define an equivalence class of Durfee symbols of a given number, and to consider the possible ways to make the symbols in an equivalence class into k-marked Durfee symbols. Part of the results presented in **§4** will appear in [7], where Boulet and Kursungoz use the same alternative characterization of k-marked Durfee symbols, but more direct combinatorial methods. Also, Ji [14] proved the results in **§4** along with many more open problems posed by Andrews in [4]. Ji's approach is essentially different from **§4** and [7].

2. Parity Considerations in Partitions Enumerated by Andrews-Gordon Identities

2.1 Background

We begin with a few definitions from [6].

Definition 2.1. Let $k \geq 2$, $k \geq a \geq 1$. $b_{k,a}(m,n)$ denotes the number of partitions of n into m parts such that 1s appear at most $a-1$ times, and any pair of consecutive integers together appear at most $k-1$ times.

Definition 2.2. Let $k \geq 2$, $k \geq a \geq 1$. $w_{k,a}(m,n)$ denotes the number of partitions of n enumerated by $b_{k,a}(m,n)$ such that even parts appear an even number of times.

Definition 2.3. Let $k \geq 2$, $k \geq a \geq 1$. $\overline{w}_{k,a}(m,n)$ denotes the number of partitions of n enumerated by $b_{k,a}(m,n)$ such that odd parts appear an even number of times.

Let $\lambda = \lambda_1 + \ldots + \lambda_m$ be a partition of n.

Definition 2.4. The *Gordon marking* of a partition λ is an assignment of positive integers (marks) to λ such that parts equal to any given integer a are assigned distinct marks from the set $\mathbb{Z}_{>0} \backslash \{r \mid \exists\, r\text{-marked}\, \lambda_j = a-1\}$ such that the smallest possible marks are used first. Let $\lambda^{(r)}$ denote the sub-partition of λ that consists of all r-marked parts. Let N_r be the number of r-marked parts (i.e. the number of parts in $\lambda^{(r)}$), and let $n_r = N_r - N_{r+1}$ for any positive integer r.

For instance, if $\lambda = 18+17+16+15+15+13+13+11+9+7+6+6+5+4+3+2+2$, ($|\lambda| = 162$), then its Gordon marking would be

$$\lambda = 18_2+17_1+16_3+15_2+15_1+13_2+13_1+11_1+9_1+7_2+6_3+6_1+5_2+4_1+3_3+2_2+2_1$$

In fact, we can represent the Gordon marking by an array where the column indicates the value of a part, and the row (counted from bottom to top) indicates the mark, so the Gordon marking of λ above would be:

$$\lambda = \left\{ \begin{array}{ccccccccccccccccc} & 3 & & & 6 & & & & & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & & & & 13 & & 15 & & & 18 \\ 2 & & 4 & & 6 & & & 9 & & 11 & & 13 & & 15 & & 17 & \end{array} \right\}$$

we will use this representation throughout this section.

There are several things to note here. First of all, Gordon marking is unique. For if a is the smallest part appearing in the partition, then there is a unique way to mark parts that are equal to a, and then there is a unique way to mark parts that are equal to $(a+1)$ (if any) and so on.

$\lambda^{(r)}$s are sub-partitions with distinct non-consecutive parts, because no consecutive parts are assigned the same mark by definition. Also, for any r-marked λ_j, $r > 1$, there is a unique $(r-1)$-marked $\lambda_{j_0} = \lambda_j$ or $\lambda_{j_0} = \lambda_j - 1$. This implies $N_1 \geq N_2 \geq \ldots$, and hence $n_1, n_2, \ldots \geq 0$.

Lastly, if λ is enumerated by $b_{k,a}(m,n)$, then there are no k or greater marked parts, since each consecutive pair of integers together occur at most $(k-1)$ times. In this case, we can restrict our attention on $N_1, \ldots, N_{k-1}$, and $n_1, \ldots, n_{k-1}$.

Definition 2.5. Let $\lambda = \lambda_1 + \ldots + \lambda_m$ be a Gordon marked partition. Let $\lambda_j \neq 1$ be an r-marked part such that

(a) There are no $(r+1)$ or greater marked parts that are equal to λ_j or $\lambda_j + 1$.

(b) There is an $r_0 \leq r$ such that there is an r_0-marked $\lambda_{j_0} = \lambda_j$, but no r_0-marked parts that are equal to $\lambda_j - 2$.

Choose the smallest r_0 described in (b), and a *backward move of rth kind* on λ_j is replacing r_0-marked λ_{j_0} with an r_0-marked $\lambda_{j_0} - 1$, and hence $|\lambda| \leftarrow |\lambda| - 1$.

For instance,

$$\lambda = \left\{ \begin{array}{ccccccccccccc} & 3 & & & 6 & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array} \right\}$$

$$\downarrow \quad \text{a backward move of 3rd kind on 3-marked 6}$$

$$\lambda' = \left\{ \begin{array}{ccccccccccccc} & 3 & & \mathbf{5} & & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array} \right\}$$

Note that a backward move of 2nd kind is not possible for 2-marked 2, since (a) does not hold. Similarly, a backward move of 1st kind on 1-marked 11 is not possible, since (b) fails.

We claim that a backward move of rth kind preserves the marking of other unchanged parts. By (b) and the original marking, there are no r_0-marked parts that are equal to $\lambda_{j_0} - 1$ or $\lambda_{j_0} - 2$. It follows that there are no $r_0, r_0+1, \ldots$-marked parts that are equal to $\lambda_{j_0} - 1$, since by the marking, any part equal to $\lambda_{j_0} - 1$ that requires a higher mark than r_0 would be assigned r_0. Then, again by the marking, there are $r_0, r_0+1, \ldots, r$-marked parts equal to λ_j. By a similar reasoning, there are no

$r_0, r_0+1, \ldots$-marked parts equal to $\lambda_j + 1$, since any part that requires a higher mark than r_0 would be assigned $(r+1)$. That is ruled out by (a). This justifies the claim. Therefore, $N_1, N_2, \ldots$ are invariant under backward moves of any kind when conditions exist.

Definition 2.6. Let $\lambda = \lambda_1 + \ldots + \lambda_m$ be a Gordon marked partition. Let λ_j be an r-marked part such that

(c) There are no $(r+1)$ or higher marked parts equal to λ_j or $\lambda_j + 1$

and either

(d1) There is an r_0 marked part $\lambda_{j_0} = \lambda_j - 1$, $r_0 < r$ such that there are no r_0-marked parts equal to $\lambda_j + 1$, and $r_0 + 1$ or higher marked parts equal to $\lambda_j - 1$

or

(d2) there are $1, \ldots, r-1$ marked parts equal to λ_j or $\lambda_j + 1$, and no r-marked parts equal to $\lambda_j + 2$.

A *forward move of rth kind* is replacing r_0-marked λ_{j_0} with an r_0-marked $\lambda_{j_0} + 1$ if (c) and (d1) holds; and replacing r-marked λ_j with an r-marked $\lambda_j + 1$ if (c) and (d2) holds, and (d1) fails; hence $|\lambda| \leftarrow |\lambda| + 1$.

For example,

$$\lambda = \left\{ \begin{array}{ccccccccccccccc} & 3 & & & 6 & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array} \right\}$$

$$\downarrow \quad \text{a forward move of 3rd kind on 3-marked 16}$$

$$\lambda'' = \left\{ \begin{array}{ccccccccccccccc} & 3 & & & 6 & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & & \mathbf{16} & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array} \right\}$$

Observe that a forward move of 2nd kind is not possible on 2-marked 13 in λ, since neither (d1) nor (d2) holds. Similarly, a forward move of 2nd kind on 2-marked 2 is not possible, since (c) fails. To be more precise, we can replace that 2-marked 2 with a 2-marked 3 as some forward move, but that would be a forward move of 3rd kind for 3-marked 3.

We claim here also that a forward move of rth kind preserves the Gordon marking of unchanged parts. For when (c) and (d1) holds, there are no $r_0 + 1$ or higher marked parts that are equal to $\lambda_j - 1$ so that the deletion of r_0-marked $\lambda_{j_0} (= \lambda_j - 1)$ would spare the mark r_0 for them. Also, there are no r_0 marked part equal to λ_j, but there are

$r_0, r_0+1, \ldots, r$-marked ones by the Gordon marking. Once r_0-marked λ_{j_0} is deleted, however, we need to alter the marking of parts equal to λ_j. That is avoided by the introduced r_0-marked $\lambda_{j_0}+1 = \lambda_j$. By (c) and (d1), there are no r_0 or greater marked parts equal to $\lambda_j + 1$, therefore the marking of the other parts is not affected.

When (c) and (d2) holds, but (d1) does not, then there are $1, \ldots, (r-1)$-marked parts equal to either λ_j or $\lambda_j + 1$, so an extra $\lambda_j + 1$ would be assigned mark r after the deletion of r-marked λ_j. Also, similar to the reasoning following the definition of a backward move of rth kind, there are no $r+1$ or higher marked parts equal to $\lambda_j + 2$. So in this case as well, the Gordon marking of the other parts are unaltered, and hence $N_1, N_2, \ldots$ are invariant.

Proposition 2.7. Let $\lambda = \lambda_1 + \ldots + \lambda_m$ be a Gordon marked partition. Let $\lambda_j \neq 1$ be an r-marked part. If conditions exist for a backward move of rth kind on λ_j, then conditions will exist for a forward move of rth kind on the same part after the backward move is performed. Conversely, if conditions exist for a forward move of rth kind on λ_j, then conditions will exist for a backward move of rth kind on the same part after the forward move is performed. Moreover, the moves made in given orders will fix λ.

In other words, so many forward and that many backward moves, or vice versa, on the same part are inverse transformations on λ when conditions exist for the first sequence of moves.

Remark: 'Same part' refers to λ_j if another strictly smaller marked part is altered, and to $\lambda_j \pm 1$ if λ_j itself is altered.

Proof. Assume that a backward move of rth kind is performed on λ_j, and an r_0-marked λ_{j_0} is replaced by an r_0-marked $\lambda_{j_0} - 1$.

If $r_0 < r$, then the forward move is to be performed on λ_j. In this case, (a) implies (c), and by the arguments following **Definition 2.5**, (d1) holds for the above $r_0 < r$, so the r_0-marked $\lambda_{j_0} - 1$ is replaced back by r_0-marked $\lambda_{j_0} - 1 + 1 = \lambda_{j_0}$.

Else if $r_0 = r$, then the forward move is to be performed on r-marked $\lambda_j - 1$. (a) and the arguments following **Definition 2.5** imply (c) under the substitution $\lambda_j \leftarrow \lambda_j - 1$. By (b), $r_0 = r$ and (d1) fails. By the Gordon marking of the original λ (d2) holds, and r-marked $\lambda_j - 1$ is replaced back by r-marked $\lambda_j - 1 + 1 = \lambda_j$. In either case, λ is fixed, so the first claim is proven.

For the second claim, if (c) and (d1) held, then (a) holds. There cannot be any r_0 or higher marked part equal to $\lambda_j + 1$, because there were no r_0 marked parts equal to λ_j before the forward move. Also, for some $r_0 < r$, (b) holds by the Gordon marking of original λ. r_0 in (d1) will be the smallest such for (b), since for any smaller $r_1 < r_0$, there are r_1-marked parts that are equal to either $\lambda_j - 1$ or $\lambda_j - 2$, by the marking. In this case, r_0-marked λ_{j_0} will first be replaced by r_0-marked $\lambda_{j_0} + 1$, and then replaced back again by r_0-marked $\lambda_{j_0} + 1 - 1 = \lambda_{j_0}$.

Else if (c) and (d2) held, but (d1) failed, then by the Gordon marking and (c), there are no r or higher marked parts equal to $\lambda_j + 1$ or $\lambda_j + 2$, so (a) holds upon substituting $\lambda_j \leftarrow \lambda_j + 1$. Also, when $r = r_0$, then by the Gordon marking, there are r_1-marked parts ($r_1 < r$) equal to either λ_j or $\lambda_j - 1$ and by (d2), for all r_1 for which there is an r_1-marked part equal to $\lambda_j - 1$, there is also an r_1-marked part equal to $\lambda_j + 1$. Thus, (b) holds only for $r_0 = r$ and r-marked $\lambda_j + 1$ is replaced back by r-marked $\lambda_j + 1 - 1 = \lambda_j$. In either case, λ is fixed. This justifies the second claim and concludes the proof. □

As an example, please note that a forward move of 3rd kind on 3-marked 5 in λ' above gives us λ back. So does a backward move of 3rd kind on 3-marked 16 in λ''.

Proposition 2.8. Let $\lambda = \lambda_1 + \ldots + \lambda_m$ be a Gordon marked partition. Let $\lambda_{j_1} < \lambda_{j_2}$ be two r-marked parts such that there are no r or higher marked λ_{j_3} for which $\lambda_{j_1} < \lambda_{j_3} < \lambda_{j_2}$.

(i) If conditions exist for a backward move of rth kind on λ_{j_1}, and (a) is satisfied for λ_{j_2}, then the move made on λ_{j_1} will enable a backward move of rth kind on λ_{j_2}.

(ii) If conditions exist for a forward move of rth kind on λ_{j_2}, and either there are no $r+1$ or higher marked parts equal to $\lambda_{j_1} - 1$, or for all $r_0 < r$ there are r_0 marked parts equal to λ_{j_1}, then the move made on λ_{j_2} will enable a forward move of rth kind on λ_{j_1}.

Proof. **(i)** When (b) fails for λ_{j_2}, then for all $r_0 \leq r$, there is an r_0 marked part equal to $\lambda_{j_2} - 2$ whenever there is an r_0 marked part equal to λ_{j_2}. In particular, $\lambda_{j_1} = \lambda_{j_2} - 2$. Once a backward move of rth kind is made on λ_{j_1}, an $r_0 \leq r$ will be spared to satisfy (b) for λ_{j_2}.

(ii) (c) is satisfied both for λ_{j_1} and for λ_{j_2}.

For the first possibility, when both (d1) and (d2) fails for λ_{j_1}, then by (d2) there is an r-marked part equal to $\lambda_{j_1} + 2$. That is, $\lambda_{j_2} = \lambda_{j_1} + 2$. Once the move is performed on λ_{j_2}, if r-marked λ_{j_2} is replaced by r-marked $\lambda_{j_2} + 1$, (d2) will be satisfied for λ_{j_1} thanks to (d2) for λ_{j_2}. Or, some $r_0 < r$ will be spared after replacing an r_0 marked part equal to $\lambda_{j_2} - 1 = \lambda_{j_1} + 1$. By the remarks following the definitions of backward and forward moves, the same r_0 will satisfy (d1) for λ_{j_2}, since there are no $r_0 + 1$ or higher marked parts equal to $\lambda_{j_2} - 1 = \lambda_{j_1} + 1$.

For the second possibility, i.e. λ having $1, \ldots, r$-marked parts equal to λ_{j_1}, the only hindrance is that there are $1, \ldots, r$-marked parts equal to $\lambda_{j_2} + 2$, again, $\lambda_{j_1} = \lambda_{j_2} + 2$. In this case, the only possible forward move of rth kind for λ_{j_1} is replacing r-marked λ_{j_1} with an r-marked $\lambda_{j_1} + 1$, and hence enabling (d2) for λ_{j_2}. Observe that there are no parts equal to $\lambda_{j_2} + 1 = \lambda_{j_1} - 1$ at all in this case.

□

As an example, a backward move of 1st kind on 1-marked 9 in λ enables a backward move of 1st kind for 1-marked 11. Also, the hypotheses of case **(ii)** hold for the 3-marked 3 and 3-marked 6 in λ, but a forward move of 3rd kind is possible on 3-marked 3 regardless of whether a forward move is performed on 3-marked 6 or not. Therefore the result cannot claim necessity. This observation will be used below.

We provide another example and a non-example for part (**ii**) of **Proposition 2.8**. Let

$$\eta = \left\{ \begin{array}{ccccc} & 2 & & & \\ & 2 & & & \\ 1 & & 3 & & 5 \\ 1 & & 3 & & 5 \\ 1 & & 3 & & 5 \end{array} \right\}, \text{ and } \quad \eta' = \left\{ \begin{array}{cccccccc} & & & & 5 & & & \\ & & & 4 & & 6 & & 8 \\ & & 3 & & 5 & & 7 & \\ & 2 & & 4 & & 6 & & 8 \\ 1 & & 3 & & 5 & & 7 & \end{array} \right\}$$

For η, let η_{j_1} be the 3-marked 3, and η_{j_2} be the 3-marked 5, and for η', let η'_{j_1} be the 3-marked 6, and η'_{j_2} be the 3-marked 8. A forward move of 3rd kind on η_{j_2} enables a forward move of 3rd kind on η_{j_1}, in spite of the fact that there are 4- and 5-marked 2s. However, a forward move of 4th kind on η'_{j_2} does not enable a forward move of 4th kind on η'_{j_1}, since any such move will alter the invariants (n_is) of Gordon marking. These last examples indicate that we do need a longer hypothesis for the second part in **Proposition 2.8**.

Proposition 2.9. **(i)** When conditions exist, a single forward or backward move of rth kind switches the parity of number of occurrences of two consecutive parts.

(ii) When conditions exist, two successive backward or forward moves on the same part (hereafter a *double* backward or forward move) either preserve the parities of number of occurrences of each part, or else swap the parities of number of occurrences of a and $a+2$, which have opposite parities, for some positive integer a.

Proof. **(i)** This is obvious, since a part λ_{j_0} is replaced by $\lambda_{j_0} \pm 1$.

(ii) We will prove this for backward moves only, the proof will be complete by **Proposition 2.7**. There are two possibilities. Either there are $r_0 < r_1 \leq r$ such that r_0-marked λ_{j_0} will be replaced by r_0-marked $\lambda_{j_0} - 1$, and r_0-marked $\lambda_{j_1} = \lambda_{j_0}$ will be replaced by r_1-marked $\lambda_{j_1} - 1 = \lambda_{j_0} - 1$. This gives us the first option.

Else if an r-marked λ_j is first replaced by r-marked $\lambda_j - 1$, and then an $r_0 \leq r$-marked part equal to $\lambda_j - 1$ is replaced by an r_0-marked part equal to $\lambda_j - 2$; by (b) this means for any $r_0 < r$ for which there is an r_0 marked part equal to λ_j, there is an r_0-marked part equal to $\lambda_j - 2$. Moreover, there is no r-marked parts equal to $\lambda_j - 2$ while λ_j itself is r-marked. Thus the number of occurrences of λ_j is exactly one more than the number of occurrences of $\lambda_j - 2$, i.e. they have opposite parities. Those parities are swapped by the two successive moves described. With $a = \lambda_j - 2$, this gives us the second option, and proves the assertion.

□

For example,

$$\lambda = \left\{ \begin{array}{ccccccccccccc} & 3 & & & 6 & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array} \right\}$$

$$\downarrow \quad \text{a double backward move of 3rd kind on 3-marked 6}$$

$$\lambda''' = \left\{ \begin{array}{ccccccccccccc} & 3 & & \mathbf{5} & & & & & & & 16 & & \\ 2 & & \mathbf{4} & & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array} \right\}$$

the double backward move here swaps the opposite parities of number of occurrences of 4 and 6.

2.2 Main Results

Theorem 2.10. Let $k \geq 2$, $k \geq a \geq 1$. Then,

$$\sum_{m,n\geq 0} b_{k,a}(m,n)x^m q^n = \sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\ldots+N_{k-1}^2+N_a+\ldots+N_{k-1}} x^{N_1+\ldots+N_{k-1}}}{(q)_{n_1}\cdots(q)_{n_{k-1}}} \tag{2.1}$$

where $N_r = n_r + \ldots + n_{k-1}$.

This generating function is given by Andrews in [2, p.4082, eq.(1.7)].

Proof. Let $\lambda = \lambda_1 + \ldots + \lambda_m$ be a partition of n enumerated by $b_{k,a}(m,n)$. We will find integers $N_1 \geq \ldots \geq N_{k-1}$ such that $m = N_1 + \ldots + N_{k-1}$; a *base partition* $\widetilde{\lambda}$ which is enumerated by $b_{k,a}(m, \widetilde{n})$ where

$$\widetilde{n} = |\widetilde{\lambda}| = N_1^2 + \ldots + N_{k-1}^2 + N_a + \ldots + N_{k-1}$$

and for which no further backward moves are possible without altering N_r for some $r = 1, \ldots, k-1$ or without violating the conditions for $b_{k,a}(m,n)$; and $k-1$ partitions $\mu^{(r)}$ such that $\mu^{(r)}$ has at most $n_r = N_r - N_{r+1}$ parts, $r = 1, \ldots, k-1$; all unique. This will give us an injective mapping from the partitions enumerated on the left hand side of identity (2.1) to the partitions enumerated on the right hand side, where the exponent of q in the numerator will account for $|\widetilde{\lambda}|$, the factors on the denominator will account for $\mu^{(r)}$s, $r = 1, \ldots, k-1$, and the exponent of x for the number of parts.

Conversely, let $N_1 \geq \ldots \geq N_{k-1}$, $n_r = N_r - N_{r+1}$, $r = 1, \ldots, k-1$; $k-1$ partitions $\mu^{(r)}$ such that $\mu^{(r)}$ has at most $n_r (= N_r - N_{r+1})$ parts, $r = 1, \ldots, k-1$ be given. Let

$m = N_1 + \ldots + N_{k-1}$. We will first construct a *base partition* $\widetilde{\lambda}$ which is enumerated by $b_{k,a}(m, \widetilde{n})$ where

$$\widetilde{n} = |\widetilde{\lambda}| = N_1^2 + \ldots + N_{k-1}^2 + N_a + \ldots + N_{k-1}$$

and for which no further backward moves are possible without altering N_r for some $r = 1, \ldots, k-1$ or without violating the conditions for $b_{k,a}(m, n)$. Then we will produce a partition λ of

$$n = |\lambda| = N_1^2 + \ldots + N_{k-1}^2 + N_a + \ldots + N_{k-1} + \sum_{r=1}^{k-1} |\mu^{(r)}|$$

enumerated by $b_{k,a}(m, n)$, again uniquely. This in turn will give us an injective mapping from the partitions listed on the right hand side of (2.1) to the partitions listed on the left hand side.

Then we will argue that the two constructions are inverses to each other, proving the theorem bijectively.

Given $\lambda = \lambda_1 + \ldots + \lambda_m$, a partition of n enumerated by $b_{k,a}(m, n)$, let N_r be the number of r-marked parts in the Gordon marking of λ. Let $n_r = N_r - N_{r+1}$, $r = 1, \ldots, k-1$. Observe that $0 = N_k = N_{k+1} = \ldots$, since there are no more than $k-1$ occurrences of any two consecutive parts. Let $\lambda_{j_{n_{k-1}}} < \ldots < \lambda_{j_1}$ be the $k-1$ marked parts. Then both (a) and (b) are satisfied for $\lambda_{j_{n_{k-1}}}$ unless the partition ends with

$$\overbrace{2, \ldots, 2}^{k-a}, \overbrace{1, \ldots, 1}^{a-1}$$

Note that with this configuration, one more backward move of $(k-1)$th kind on $(k-1)$-marked 2 will bring exactly a 1s in the partition, violating a condition for $b_{k,a}(m, n)$. We make a sequence of backward moves of $(k-1)$th kind on $\lambda_{j_{n_{k-1}}}$, until the partition ends as above. We call the number of required moves $\mu_{n_{k-1}}^{(k-1)}$. By **Proposition 2.8** and the following remark, we can perform at least $\mu_{n_{k-1}}^{(k-1)}$ backward moves on $\lambda_{j_{n_{k-1}-1}}$, possibly more. So we call the number of backward moves of $(k-1)$th kind on $\lambda_{j_{n_{k-1}-1}}$ required to make the partition end with

$$\overbrace{4, \ldots, 4}^{k-a}, \overbrace{3, \ldots, 3}^{a-1}, \overbrace{2, \ldots, 2}^{k-a}, \overbrace{1, \ldots, 1}^{a-1}$$

$\mu_{n_{k-1}-1}^{(k-1)}$. One more backward move of $(k-1)$th kind on $(k-1)$-marked 4 will bring a k-marked 3, will hence violate a condition for $b_{k,a}(m, n)$ and will alter the Gordon marking.

We repeat this process for the remaining $(k-1)$-marked parts in their increasing order, forming the partition $\mu^{(k-1)}$ with at most n_{k-1} parts $\mu_{n_{k-1}}^{(k-1)} \leq \ldots \leq \mu_1^{(k-1)}$. So far the transformed λ looks like

parts $\geq (2N_{k-1} + 1)$,

$$\underbrace{\overbrace{2N_{k-1},\dots,2N_{k-1}}^{k-a},\overbrace{(2N_{k-1}-1),\dots,(2N_{k-1}-1)}^{a-1},\dots,\overbrace{2,\dots,2}^{k-a},\overbrace{1,\dots,1}^{a-1}}_{(k-1)n_{k-1}\text{ parts}}$$

and we have made a total of $|\mu^{(k-1)}|$ backward moves of $(k-1)$th kind.

As an example on the fly, we take λ as given far above, noting that it is a partition listed by $b_{4,3}(17, 162)$. Namely, $k-1 = 3$, $a-1 = 2$, $m = 17$ and $n = 162$. By the Gordon marking of λ, $N_1 = 8$, $N_2 = 6$, and $N_3 = 3$. Thus, $n_1 = 2$, $n_2 = 3$, and $n_3 = 3$. Following the above notation, $\lambda_{j_3} =$ 3-marked 3, $\lambda_{j_2} =$ 3-marked 6, and $\lambda_{j_1} =$ 3-marked 16. It is easy to see that after $\mu_3^{(3)} = 3$ backward moves of 3rd kind on 3-marked 3, and $\mu_2^{(3)} = 5$ moves of 3rd kind on 3-marked 6, λ will be made into

$$\left\{\begin{array}{ccccccccccccc} & 2 & & 4 & & & & & & & 16 & & \\ 1 & & 3 & & & 7 & & & 13 & 15 & & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\}$$

It would be instructive to make the backward moves of 3rd kind on 3-marked 16 one by one. The boldfaced part at each step is the decreased one.

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccc} & 2 & & 4 & & & & & & \mathbf{15} & & \\ 1 & & 3 & & & 7 & & & 13 & 15 & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & 13 & 15 & 17 & \end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{ccccccccccccc} & 2 & & 4 & & & & & & \mathbf{14} & & & \\ 1 & & 3 & & & 7 & & & 13 & & 15 & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & 13 & & 15 & 17 & \end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccc} & 2 & & 4 & & & & & \mathbf{13} & & & \\ 1 & & 3 & & & 7 & & & 13 & 15 & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & 13 & 15 & 17 & \end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{ccccccccccccc} & 2 & & 4 & & & & & & 13 & & & \\ 1 & & 3 & & & 7 & & & \mathbf{12} & & 15 & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & & 13 & 15 & 17 & \end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{ccccccccccccc} & 2 & & 4 & & & & & \mathbf{12} & & & & \\ 1 & & 3 & & & 7 & & & 12 & & 15 & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & & 13 & 15 & 17 & \end{array}\right\}$$

$\downarrow$ after 10 more analogous moves

$$\left\{\begin{array}{cccccccccccc} & 2 & & 4 & \mathbf{6} & & & & & & & \\ 1 & & 3 & & 6 & & 9 & & & 15 & & 18 \\ 1 & & 3 & & 6 & & 9 & 11 & 13 & 15 & 17 & \end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccc} & 2 & & 4 & & 6 & & & & & & \\ 1 & & 3 & & & 6 & 9 & & & 15 & & 18 \\ 1 & & 3 & & \mathbf{5} & & 9 & 11 & 13 & 15 & 17 & \end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccc} & 2 & & 4 & & 6 & & & & & & \\ 1 & & 3 & & \mathbf{5} & & 9 & & & 15 & & 18 \\ 1 & & 3 & & 5 & & 9 & 11 & 13 & 15 & 17 & \end{array}\right\}$$

we have made $\mu_1^{(3)} = 17$ backward moves of 3rd kind on 3-marked 16, and have reached a configuration in which no more backward moves of 3rd kind are possible without violating the conditions for $b_{4,3}(m,n)$ or without altering at least two of N_1, N_2, N_3. We have made a total of $25 = |\mu^{(3)}|$ backward moves of 3rd kind.

We repeat the above process for $r = k-2, \ldots, 1$ in decreasing order as follows: For each r, we choose the n_r largest r-marked parts $\lambda_{j_{n_r}} < \ldots < \lambda_{j_1}$ beginning with the smallest of these, and pick the next smallest after we are done with the one at hand. We perform $\mu_i^{(r)}$ backward moves of rth kind, $i = n_r, \ldots, 1$, so that no more moves are possible without violating the conditions for $b_{k,a}(m,n)$ or without altering some N_ls, $l = 1, \ldots, k-1$. When all $\mu^{(r)}$s are constructed, λ will be transformed into

$$\widetilde{\lambda} = \underbrace{(2N_1-1), (2N_1-3), \ldots, (2N_2+1)}_{n_1 \text{ parts}},$$

$$\vdots$$

$$\underbrace{\overbrace{(2N_r-1), \ldots, (2N_r-1)}^{\max\{r-(a-1),0\}}, \overbrace{(2N_r-2), \ldots, (2N_r-2)}^{\min\{r,a-1\}}, \ldots \overbrace{(2N_{r+1}+1), \ldots, (2N_{r+1}+1)}^{\max\{r-(a-1),0\}}, \overbrace{2N_{r+1}, \ldots, 2N_{r+1}}^{\min\{r,a-1\}}}_{rn_r \text{ parts}},$$

$$\vdots$$

$$\underbrace{\overbrace{2N_{k-1}, \ldots, 2N_{k-1}}^{k-a}, \overbrace{(2N_{k-1}-1), \ldots, (2N_{k-1}-1)}^{a-1}, \ldots, \overbrace{2, \ldots, 2}^{k-a}, \overbrace{1, \ldots, 1}^{a-1}}_{(k-1)n_{k-1} \text{ parts}}$$

It is straightforward to check that (b) is not satisfied for any part in $\widetilde{\lambda}$ for a backward move of some kind, save for the $(k-1)$-marked 2. Yet, the impossibility of the backward move of $(k-1)$th kind on $(k-1)$-marked 2 was discussed above. Also, it is readily seen that $\widetilde{\lambda}$ is enumerated by $b_{k,a}(m, \widetilde{n})$. Therefore, $\widetilde{\lambda}$ is the *base partition* we are looking for. It remains to show that

$$\widetilde{n} = |\widetilde{\lambda}| = N_1^2 + \ldots + N_{k-1}^2 + N_a + \ldots + N_{k-1}$$

This follows by the Gordon marking of $\widetilde{\lambda}$, for

$$\widetilde{\lambda}^{(r)} = \begin{cases} 1, 3, \ldots, 2N_r - 1 & \text{if} \quad r < a \\ 2, 4, \ldots, 2N_r & \text{if} \quad r \geq a \end{cases}$$

Continuing with our example, for $r = 2$, the $n_2 = 3$ largest 2-marked parts are $\lambda_{j_3} =$ 2-marked 9, $\lambda_{j_2} =$ 2-marked 15, and $\lambda_{j_1} =$ 2-marked 18. After $\mu_3^{(2)} = 4$ backward moves of 2nd kind on 2-marked 9, the so far transformed λ looks like

$$\left\{ \begin{array}{ccccccccccccc} & 2 & & 4 & & 6 & & & & & & & \\ 1 & & 3 & & 5 & & 7 & & & & 15 & & 18 \\ 1 & & 3 & & 5 & & 7 & & 11 & 13 & 15 & 17 & \end{array} \right\}$$

after $\mu_2^{(2)} = 8$ backward moves of 2nd kind on 2-marked 15 like

$$\left\{ \begin{array}{ccccccccccccc} & 2 & & 4 & & 6 & & & & & & & \\ 1 & & 3 & & 5 & & 7 & 9 & & & & & 18 \\ 1 & & 3 & & 5 & & 7 & 9 & & 13 & 15 & 17 & \end{array} \right\}$$

and after $\mu_1^{(2)} = 9$ backward moves of 2nd kind on 2-marked 18 like

$$\left\{ \begin{array}{cccccccccccc} & 2 & & 4 & & 6 & & & & & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & & 15 & 17 \end{array} \right\}$$

Lastly, for $r = 1$, the largest $n_1 = 2$ of the 1-marked symbols are $\lambda_{j_2} =$ 1-marked 15, and $\lambda_{j_1} =$ 1-marked 17. After the obvious $\mu_2^{(1)} = 2$ and $\mu_1^{(1)} = 2$ backward moves of 1st kind, which happen to be mere subtractions, we obtain

$$\widetilde{\lambda} = \left\{ \begin{array}{ccccccccccc} & 2 & & 4 & & 6 & & & & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & 13 & 15 \end{array} \right\}$$

At this point,

$$|\widetilde{\lambda}| = 8^2 + 6^2 + 3^2 + 3 = 112$$

and

$$n = |\lambda| = 162 = |\widetilde{\lambda}|$$

$$= N_1^2 + \ldots + N_{k-1}^2 + N_a + \ldots + N_{k-1} + \sum_{r=1}^{k-1} |\mu^{(r)}| = 112 + |\mu^{(3)}| + |\mu^{(2)}| + |\mu^{(1)}|$$

$$= 112 + (17 + 5 + 3) + (9 + 8 + 4) + (2 + 2)$$

as desired.

Conversely, let $n_r \geq 0$, partitions $\mu^{(r)}$ with at most n_r parts, $r = 1, \ldots, k-1$ be given. Let $N_r = n_r + \ldots + n_{k-1}$, and let $\widetilde{\lambda}$ be a base partition such that

$$\widetilde{\lambda}^{(r)} = \begin{cases} 1, 3, \ldots, 2N_r - 1 & \text{if} \quad r < a \\ 2, 4, \ldots, 2N_r & \text{if} \quad r \geq a \end{cases}$$

Then, as above,

$$\widetilde{n} = |\widetilde{\lambda}| = N_1^2 + \ldots + N_{k-1}^2 + N_a + \ldots + N_{k-1}$$

$\widetilde{\lambda}$ has $m = N_1 + \ldots + N_{k-1}$ parts. It is enumerated by $b_{k,a}(m, \widetilde{n})$, and any backward move of any kind, if at all possible, either violate the conditions for $b_{k,a}(m, \widetilde{n})$ or alter some N_r, $r = 1, \ldots, k-1$.

We take $\mu^{(1)} = \mu_1^{(1)} \geq \ldots \geq \mu_{n_1}^{(1)}$, and the largest n_1 of the 1-marked parts. (c) and (d2) (possibly (d1)) are always satisfied for the largest 1-marked part. Also, for the remaining largest $n_1 - 1$ 1-marked parts, the hypotheses of **Proposition 2.8 (ii)** are satisfied. Backed by **Proposition 2.8**, we perform $\mu_i^{(1)}$ forward moves of 1st kind on the ith largest 1-marked part for $j = 1, \ldots, n_1$ beginning with the largest and then picking the next largest once we are done with the one at hand.

Then, for $r = 2, \ldots, k-1$ in increasing order, we observe that (c) and (d2) are always satisfied for the largest of r-marked parts provided that $n_r \geq 1$, and the hypotheses of **Proposition 2.8 (ii)** hold for the next largest $n_r - 1$ r-marked parts when $n_r > 1$. So we take $\mu^{(r)} = \mu_1^{(r)} \geq \ldots \geq \mu_{n_r}^{(r)}$ and perform $\mu_i^{(r)}$ forward moves of rth kind on the ith largest r-marked part for $j = 1, \ldots, n_r$ beginning with the largest and then picking the next largest once we are done with the one at hand. We call the final partition λ, after all moves are made.

By virtue of forward moves, $N_1, \ldots, N_{k-1}$ remain invariant, and hence the conditions for $b_{k,a}(m, n)$ are satisfied for λ, and by construction,

$$n = |\lambda| = N_1^2 + \ldots + N_{k-1}^2 + N_a + \ldots + N_{k-1} + \sum_{r=1}^{k-1} |\mu^{(r)}|$$

as claimed.

Lastly, we note that in the two constructions above the steps are performed in exact reverse order. By the observation following **Proposition 2.7**, the two transformations are inverses to each other. □

The example given for the first part of the proof can be worked backwards.

The following result is a refinement to **Theorem 2.10** [6, Thm.3, eq.(1.8)].

Theorem 2.11. (Andrews) Let $k \geq 2$, $k \geq a \geq 1$.

$$\sum_{m,n \geq 0} w_{k,a}(m,n) x^m q^n$$

$$= \sum_{n_1,\ldots,n_{k-1} \geq 0} \frac{q^{N_1^2+\ldots+N_{k-1}^2+2N_a+2N_{a+2}+\ldots+2N_{k-2}} x^{N_1+\ldots+N_{k-1}}}{(q^2;q^2)_{n_1} \cdots (q^2;q^2)_{n_{k-1}}} \quad \text{if} \quad k \equiv a (\text{mod } 2) \quad (2.2)$$

$$= \sum_{n_1,\ldots,n_{k-1} \geq 0} \frac{q^{N_1^2+\ldots+N_{k-1}^2+2N_a+2N_{a+2}+\ldots+2N_{k-1}} x^{N_1+\ldots+N_{k-1}}}{(q^2;q^2)_{n_1} \cdots (q^2;q^2)_{n_{k-1}}} \quad \text{if} \quad k \not\equiv a (\text{mod } 2) \quad (2.3)$$

Proof. Observe that by **Proposition 2.9 (ii)**, if in both λ and λ' the number of occurrences of even numbers are all even (or just as well all odd), and if $\lambda \rightsquigarrow \lambda'$ by a sequence of forward or backward moves of any kind, then the number of moves made must be even.

Imitating the proof of **Theorem 2.10**, we begin with a given λ enumerated by $w_{k,a}(m,n)$. we perform double backward moves of some kind at each step, instead of single ones. This will ensure that we obtain $\mu^{(r)}$s with all even parts, $r = 1, \ldots, k-1$. Now, if $r \geq a$, and r-(a-1) is odd, we miss $\widetilde{\lambda}$ by exactly n_r single moves to be performed on the n_r largest r-marked parts by the remark at the beginning of the proof. In other words, for $r = a, a+2, \ldots$, the largest n_r r-marked parts are even numbers that occur an odd number of times. This will account for $n_a + n_{a+2} + \ldots$ backward moves on the total. Therefore, $\mu^{(r)}$s must be accompanied by a separate n_r for $r = a, a+2, \ldots$.

Conversely, beginning with $\widetilde{\lambda}$ as in the proof of **Theorem 2.10**, for the moves on n_r largest r-marked parts $r = a, a+2, \ldots$, we first need to perform n_r single moves on each, so as to ensure that all even parts occur an even number of times. Then we continue with making half as many double moves as parts of $\mu^{(r)}$s, $r = 1, \ldots, k-1$. Again, by the remark at the beginning of the proof, the constructed λ will satisfy the conditions for $w_{k,a}(m,n)$.

The fact that the two constructions above are inverses to each other follows by the argument employed in the last part of the proof of **Theorem 2.10**.

The contribution of the extra moves should be added to the exponent of q on the numerator on the right hand side

$$N_1^2 + \ldots + N_{k-1}^2 + N_a + \ldots + N_{k-1} + n_a + n_{a+2} + \ldots = N_1^2 + \ldots + N_{k-1}^2 + 2N_a + 2N_{a+2} + \ldots$$

and the result follows. □

We give a simpler example here, and work backwards. Let $k-1 = 3$, $a-1 = 1$, $N_1 = 4$, $N_2 = 3$, and $N_1 = 1$. Let $\mu^{(1)} = 2$, $\mu^{(2)} = 4, 2$, and $\mu^{(3)} = 8$. Then the base partition described as in the proof of **Theorem 2.10** will be

$$\widetilde{\lambda} = \left\{ \begin{array}{ccccccc} & 2 & & & & & \\ & 2 & & 4 & & 6 & \\ 1 & & 3 & & 5 & & 7 \end{array} \right\}$$

We begin with $\mu^{(1)}$, and simply make a $2/2 = 1$ double move on the largest 1-marked part 7

$$\left\{ \begin{array}{cccccccc} & 2 & & & & & & \\ & 2 & & 4 & & 6 & & \\ 1 & & 3 & & 5 & & & 9 \end{array} \right\}$$

For $r = 2$, $r - (a-1) = 2 - 1 = 1$ is odd. Thus we need to make $n_2 = 2$ single forward moves of 2nd kind on the 2 largest 2-marked parts 4 and 6.

$$\left\{ \begin{array}{cccccc} & 2 & & & & \\ & 2 & 4 & 6 & & \\ 1 & & 4 & 6 & & 9 \end{array} \right\}$$

Then we can realize $\mu^{(2)}$ as $4/2 = 2$, and $2/2 = 1$ double forward moves of 2nd kind on 2-marked 6 and 2-marked 4, respectively. 2-marked 6 goes first, as mentioned in the above proofs, and here we see why: both (d1) and (d2) fails for 2-marked 4.

$$\left\{ \begin{array}{ccccc} & 2 & & & \\ & 2 & 5 & & 9 \\ 1 & & 5 & 7 & 9 \end{array} \right\}$$

Lastly, we perform $8/2 = 4$ double moves on 3-marked 2

$$\left\{ \begin{array}{ccccc} & & 6 & & \\ 3 & & 6 & & 9 \\ 3 & 5 & & 7 & 9 \end{array} \right\}$$

which is a partition enumerated by $w_{4,2}(8, 36)$.

We also see that the order the moves are made is very important. If we introduce the $n_2 = 2$ single forward moves of 2nd kind in advance of introducing any $\mu^{(r)}$s,

$$\left\{ \begin{array}{ccccc} & 2 & & & \\ & 2 & & 5 & 7 \\ 1 & & 3 & 5 & 7 \end{array} \right\}$$

then no forward moves of 1st kind is possible anymore, and $\mu^{(1)}$ is rendered useless. Although a specific order of forward moves may exist so that all $\mu^{(r)}$s can be realized as forward moves, this violates the uniqueness of the constructions above. Furthermore, this specific order does not exist in general.

Theorem 2.12. (Andrews) Let $k \geq a \geq 2$, and a be even. Then,

$$\sum_{m,n\geq 0} \overline{w}_{k,a}(m,n)x^m q^n = \sum_{m,n\geq 0} \overline{w}_{k,a-1}(m,n)x^m q^n \tag{2.4}$$

$$= \sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\ldots+N_{k-1}^2+N_{a-1}+\ldots N_{k-1}+n_1+n_3+\ldots+n_{a-3}} x^{N_1+\ldots+N_{k-1}}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{k-1}}} \tag{2.5}$$

This is Theorem 3 in [6, eq.(1.9)]

Proof. (2.4) is obvious, since 1s cannot appear exactly $(a-1)$ (an odd number) of times. For (2.5), we argue as in the proof of **Theorem 2.11**. The only difference is that the extra single moves are required for n_r largest r-marked odd parts, $r = 1, 3, \ldots, a-3$. Because in the base partition $\widetilde{\lambda}$ described in the proof of **Theorem 2.10**, these are precisely the odd parts that appear an odd number of times, hence the extra $n_1 + n_3 + \ldots + n_{a-3}$ in the exponent of q in the numerator on the rightmost infinite sum. The rest of proof of **Theorem 2.11** applies word by word. □

Example: For $k = 6$ and $a - 1 = 5$, given

$$n_1 = 2, \quad n_2 = 3, \quad n_3 = 2, \quad n_4 = 0, \quad n_5 = 2$$

$$\mu^{(1)} = 2, \qquad \mu^{(2)} = 6, 4, 2, \qquad \mu^{(3)} = 6, 6,$$

$$\mu^{(4)} = \text{the empty partition} \qquad \mu^{(5)} = 28, 24$$

We will construct a partition λ enumerated by $b_{6,5}(23, 238)$ in which odd parts appear an even number of times,

$$238 = N_1^2 + \ldots + N_5^2 + N_5 + n_1 + n_3 + |\mu^{(1)}| + \ldots + |\mu^{(5)}|, \text{ and}$$

$$23 = N_1 + \ldots + N_5$$

We begin by constructing the base partition defined in the proof of **Theorem 2.10**

$$\widetilde{\lambda} = \left\{ \begin{array}{ccccccccc} & 2 & & 4 & & & & & \\ 1 & & 3 & & & & & & \\ 1 & & 3 & & 5 & 7 & & & \\ 1 & & 3 & & 5 & 7 & 9 & 11 & 13 & & \\ 1 & & 3 & & 5 & 7 & 9 & 11 & 13 & 15 & 17 \end{array} \right\}$$

We observe that 5, 7, 15, and 17 are odd parts that appear an odd number of times. We need single forward moves performed on the 1-marked 17, 1-marked 15, 3-marked 7, and 3-marked 5. If the forward moves of the appropriate kind are performed in decreasing order of the mentioned parts, hypotheses of **Proposition 2.8 (ii)** will be satisfied so that the moves are possible. This is the point where the $n_1 + n_3$ is realized as part of the exponent of q in the numerator in the generating function (2.5). The base partition is transformed into

$$\left\{\begin{array}{cccccccccccccccccc}
 & 2 & & 4 \\
1 & & 3 \\
1 & & 3 & & & 6 & & 8 \\
1 & & 3 & & 5 & & 7 & & 9 & & 11 & & 13 \\
1 & & 3 & & 5 & & 7 & & 9 & & 11 & & 13 & & & 16 & & 18
\end{array}\right\}$$

Then we take into account the partitions $\mu^{(r)}$s as described in the proof of **Theorem 2.10**. After realizing $\mu^{(1)}$, we have

$$\left\{\begin{array}{cccccccccccccccccccc}
 & 2 & & 4 \\
1 & & 3 \\
1 & & 3 & & & 6 & & 8 \\
1 & & 3 & & 5 & & 7 & & 9 & & 11 & & 13 \\
1 & & 3 & & 5 & & 7 & & 9 & & 11 & & 13 & & & 16 & & & & 20
\end{array}\right\}$$

$\downarrow$ after applying $\mu^{(2)}$

$$\left\{\begin{array}{cccccccccccccccccccc}
 & 2 & & 4 \\
1 & & 3 \\
1 & & 3 & & & 6 & & 8 \\
1 & & 3 & & 5 & & 7 & & & 10 & & & & 14 & & & 17 \\
1 & & 3 & & 5 & & 7 & & & 10 & & 12 & & 14 & & & 17 & & & 20
\end{array}\right\}$$

$\downarrow$ after applying $\mu^{(3)}$

$$\left\{\begin{array}{cccccccccccccccccccc}
 & 2 & & 4 \\
1 & & 3 \\
1 & & 3 & & & & & & 9 & & 11 \\
1 & & 3 & & & 6 & & & 9 & & 11 & & & 14 & & & 17 \\
1 & & 3 & & & 6 & & 8 & & 10 & & 12 & & 14 & & & 17 & & & 20
\end{array}\right\}$$

We keep in mind that we interpret the parts of $\mu^{(r)}$s as half as many double moves instead of as many single moves unlike in the example in the proof of **Theorem 2.10**. It is instructive to go step by step for performing 14 double forward moves of 5th kind (28 moves) on the 5-marked 4.

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccccccccccc}
 & 2 & & 4 \\
1 & & & 4 \\
1 & & & 4 & & & & & 9 & & 11 \\
1 & & 3 & & & 6 & & & 9 & & 11 & & & 14 & & & 17 \\
1 & & 3 & & & 6 & & 8 & & 10 & & 12 & & 14 & & & 17 & & & 20
\end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccccc}
 & 2 & 4 & & & & & & & & & & & \\
1 & & 4 & & & & & & & & & & & \\
1 & & 4 & & & & & 9 & & 11 & & & & \\
1 & & 4 & & 6 & & & 9 & & 11 & & 14 & 17 & \\
1 & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20
\end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccccc}
 & 2 & & 5 & & & & & & & & & & \\
1 & & & 5 & & & & & & & & & & \\
1 & & 4 & & & & & 9 & & 11 & & & & \\
1 & & 4 & & 6 & & & 9 & & 11 & & 14 & 17 & \\
1 & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20
\end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccccc}
 & 2 & & & 6 & & & & & & & & & \\
1 & & & 5 & & & & & & & & & & \\
1 & & & 5 & & & & 9 & & 11 & & & & \\
1 & & 4 & & 6 & & & 9 & & 11 & & 14 & 17 & \\
1 & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20
\end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccccc}
 & 2 & & & 6 & & & & & & & & & \\
1 & & & & 6 & & & & & & & & & \\
1 & & & & 6 & & & 9 & & 11 & & & & \\
1 & & 4 & & 6 & & & 9 & & 11 & & 14 & 17 & \\
1 & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20
\end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccccc}
 & 2 & & & & 7 & & & & & & & & \\
1 & & & & & 7 & & & & & & & & \\
1 & & & & 6 & & & 9 & & 11 & & & & \\
1 & & 4 & & 6 & & & 9 & & 11 & & 14 & 17 & \\
1 & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20
\end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccccc}
 & 2 & & & & 7 & & & & & & & & \\
1 & & & & & 7 & & & & & & & & \\
1 & & & & & 7 & & 9 & & 11 & & & & \\
1 & & 4 & & & 7 & & 9 & & 11 & & 14 & 17 & \\
1 & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20
\end{array}\right\}$$

$\downarrow$ after 5 more double forward moves of 5th kind

$$\left\{\begin{array}{cccccccccccccc}
 & 2 & & & & & & & & & 12 & & & \\
1 & & & & & & & & & & 12 & & & \\
1 & & & & & 7 & & 9 & & 11 & & & & \\
1 & & 4 & & & 7 & & 9 & & 11 & & 14 & 17 & \\
1 & & 4 & & 6 & & 8 & & 10 & & 12 & 14 & 17 & 20
\end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{cccccccccccc}
 & 2 & & & & & & & 12 & & & \\
1 & & & & & & & & 12 & & & \\
1 & & & & 7 & & 9 & & 12 & & & \\
1 & & 4 & & 7 & & 9 & & 12 & 14 & 17 & \\
1 & & 4 & 6 & & 8 & & 10 & 12 & 14 & 17 & 20
\end{array}\right\}$$

$$\downarrow$$

$$\left\{\begin{array}{ccccccccccccc}
 & 2 & & & & & & & & 13 & & & \\
1 & & & & & & & & & 13 & & & \\
1 & & & & 7 & & 9 & & 12 & & & & \\
1 & & 4 & & 7 & & 9 & & 12 & & 14 & 17 & \\
1 & & 4 & 6 & & 8 & & 10 & 12 & & 14 & 17 & 20
\end{array}\right\}$$

Lastly, we make 12 double forward moves of 5th kind (24 moves) on the 5-marked 2 to obtain

$$\lambda = \left\{\begin{array}{ccccccccccccc}
 & & & & & & & 10 & & 13 & & & \\
 & & & & & & & 10 & & 13 & & & \\
 & & 5 & & 7 & & 9 & & 12 & & & & \\
2 & & 5 & & 7 & & 9 & & 12 & & 14 & 17 & \\
2 & 4 & & 6 & & 8 & & 10 & 12 & & 14 & 17 & 20
\end{array}\right\}$$

indeed a partition enumerated by $b_{6,5}(23, 238)$ in which odd parts appear an even number of times.

Before the next important theorem, we need an auxiliary result. We also need to recall the definition of *Gaussian Polynomials* [3, §3.3] before we proceed.

$$\begin{bmatrix} A \\ B \end{bmatrix}_r = \frac{(q^r; q^r)_A}{(q^r; q^r)_B (q^r; q^r)_{A-B}}$$

for $0 \leq B \leq A$, and r a non-zero integer. If $r = 1$, then the base can be omitted, and we write

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}_1$$

We will be using the fact that $\begin{bmatrix} A \\ B \end{bmatrix}$ generates partitions into at most B parts, all $\leq A - B$ [3, **Theorem 3.1**]

Lemma 2.13. Lists of exactly l elements $0 \leq r_1 < \ldots < r_l < n$ weighted by the sum of the smallest numbers in each maximal sublist of consecutive integers is generated by $\begin{bmatrix} n \\ l \end{bmatrix}$.

By a maximal sublist of consecutive integers, we mean a sublist that is not properly contained in any other strictly larger such. For instance, let the list $\{1, 2, 4, 5, 6, 9\}$ where $n = 11$ and $l = 6$ be given. The maximal sublists of consecutive integers are $\{1, 2\}$, $\{4, 5, 6\}$, and $\{9\}$, but not, say $\{5, 6\}$. This list is weighted by $1 + 4 + 9 = 14$.

Proof. Given $0 \le r_1 < \ldots < r_l < n$, we will construct a unique partition enumerated by $\begin{bmatrix} n \\ l \end{bmatrix}$ of the weight of the given list. Conversely, given a partition enumerated by $\begin{bmatrix} n \\ l \end{bmatrix}$, we will uniquely construct a list $0 \le r_1 < \ldots < r_l < n$ the weight of which is the number being partitioned. Then we will argue that the transformations are inverses to each other.

Given $0 \le r_1 < \ldots < r_l < n$, let $\lambda_1 = r_1$. For $i = 2, \ldots, l$ in their increasing order; if $r_i = r_{i-1} + 1$ (i.e. r_i is not the smallest element in a maximal sublist of consecutive integers), then $\lambda_i = 0$ (since r_i does not contribute to the weight). Else if $r_i > r_{i-1} + 1$ (so that r_i is the smallest element in a maximal sublist of consecutive integers), then

$$\begin{aligned} \lambda'_i &= \lambda_{i-1} + 1 \\ \lambda'_{i-1} &= \lambda_{i-2} + 1 \\ &\vdots \\ \lambda'_2 &= \lambda_1 + 1 \\ \lambda'_1 &= r_i - (i - 1) \end{aligned}$$

That $\lambda'_i \le \ldots \le \lambda'_2$ is immediate. In addition, we need to show that $\lambda'_2 \le \lambda'_1$, i.e. $\lambda_1 \le r_i - i$. By construction $\lambda_1 = r_{i_0} - (i_0 - 1)$ for some $i_0 < i$, and by the hypothesis r_i is not in the consecutive list of integers beginning with r_{i_0}. So $r_{i_0} + (i - i_0) < r_i$, which implies $r_{i_0} - (i_0 - 1) \le r_i - i$, and hence $\lambda'_2 \le \lambda'_1$. The total contribution is r_i if r_i is the smallest element in a maximal sublist of consecutive integers, zero otherwise. Then we call λ'_is λ_is and start over for $i + 1$ unless $i = l$. At the end, we have a partition $\lambda_l \le \ldots \le \lambda_1$ possibly containing zeros, and the sum of all parts equals the weight of the given list. Also, $\lambda_1 = r_l - (l - 1)$ together with the fact that $r_l < n$ implies that $\lambda_1 \le n - l$. Therefore the constructed partition is one enumerated by $\begin{bmatrix} n \\ l \end{bmatrix}$.

Conversely, given $0 \le \lambda_l \le \ldots \le \lambda_1 \le n - l$, let $j = l$. Let i be such that $\lambda_i \ne 0$ but $\lambda_{i+1} = \ldots = \lambda_j = 0$. Set

$$\begin{aligned} r_i &= \lambda_1 + (i - 1) \\ r_{i+1} &= r_i + 1 \\ r_{i+2} &= r_{i+1} + 1 \\ &\vdots \\ r_j &= r_{j-1} + 1 \end{aligned}$$

and then substitute

$$\begin{array}{rcl} \lambda_1 & \leftarrow & \lambda_2 - 1 \\ \lambda_2 & \leftarrow & \lambda_3 - 1 \\ & \vdots & \\ \lambda_{i-1} & \leftarrow & \lambda_i - 1 \\ \lambda_i & \leftarrow & 0 \end{array}$$

Then, set $j = i - 1$ and start over until $i = 1$.

If $i \neq 1$ in the first run, then there is an $i_0 < i$, $r_{i_0} < r_i$ for which in the original partition $\lambda_1 = r_i-(i-1)$ and $\lambda_2 = r_{i_0}-(i_0-1)+1$. $\lambda_2 \leq \lambda_1$ implies $r_{i_0}-i_0 \leq r_i-i-1$, and hence $r_{i_0}+i-i_0 < r_i$. That is, r_i is not in the sublist of consecutive integers containing r_{i_0} for any $i_0 < i$, but $\{r_i, r_{i+1}, \ldots, r_j\}$ form a maximal sublist of consecutive integers. The repeated application of this argument shows that $\lambda_1+\ldots+\lambda_l$ would be the weight of the constructed sublist, since for each smallest element r_i in a maximal sublist of consecutive integers, r_i is extracted from the partition.

The above assignments or substitutions together with the order they are performed clearly show that the described operations are inverses to each other. $\square$

Remark: This can be equivalently done using the notion of a *hook*, and adjunction of hooks to partitions under suitable conditions.

As an example, we construct a partition enumerated by $\begin{bmatrix} 11 \\ 6 \end{bmatrix}$ using the list $\{1, 2, 4, 5, 6, 9\}$.

To begin, $\lambda_1 = r_1 = 1$.

$2 = 1{+}1$, so $\lambda_2 = 0$.

$4 \neq 2 + 1$, so

$$\begin{aligned} \lambda_3 &= \lambda_2 + 1 = 1 \\ \lambda_2 &= \lambda_1 + 1 = 2 \\ \lambda_1 &= 4 - 2 = 2 \end{aligned}$$

$5 = 4 + 1$, so $\lambda_4 = 0$.

$6 = 5 + 1$, so $\lambda_5 = 0$.

$9 \neq 6 + 1$, so

$$\begin{aligned} \lambda_6 &= \lambda_5 + 1 = 1 \\ \lambda_5 &= \lambda_4 + 1 = 1 \\ \lambda_4 &= \lambda_3 + 1 = 2 \\ \lambda_3 &= \lambda_2 + 1 = 3 \\ \lambda_2 &= \lambda_1 + 1 = 3 \\ \lambda_1 &= 9 - 5 = 4 \end{aligned}$$

Thus, $\lambda = 1 \leq 1 \leq 2 \leq 3 \leq 3 \leq 4 \leq 11 - 6 = 5$, and $|\lambda| = 1 + 1 + 2 + 3 + 3 + 4 = 14 = 1 + 4 + 9 =$ the weight of the given list, as claimed.

Theorem 2.14. (Andrews' Conjecture) Given $1 \leq a \leq k$, $2 \leq k$,

$$\sum_{n_1,\ldots,n_{k-1}\geq 0} q^{N_1^2+\ldots+N_{k-1}^2+N_a+\ldots+N_{k-1}} \quad x^{N_1+\ldots+N_{k-1}} \quad y^{N_a+N_{a+2}+\ldots}$$

$$\times \frac{H_{n_1}\cdots H_{n_{a-1}}\widetilde{H}_{n_a}H_{n_{a+1}}\widetilde{H}_{n_{a+2}}\cdots}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{k-1}}} \tag{2.6}$$

generates the partitions enumerated by $b_{k,a}(m,n)$, where the exponent of x accounts for the number of parts, the exponent of y accounts for the number of even parts that appear an odd number of times, and Hs are the Rogers-Szegö polynomials

$$H_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_2 (qy)^j$$

and

$$\widetilde{H}_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_2 (\frac{q}{y})^j$$

For $k = 2$ and 3, this is Theorem 4 in [6, eq.(5.1)], and Andrews conjectured that the identity holds for all $k \geq 2$ in the same paper.

Proof. Given a partition $\lambda = \lambda_1 + \ldots + \lambda_m$, a partition of n enumerated by $b_{k,a}(m,n)$; for $r = 1,\ldots,k-1$, we will produce N_r, $n_r = N_r - N_{r+1} \geq 0$;

a base partition $\widetilde{\lambda}$ as in the proof of **Theorem 2.10** (the factor $q^{N_1^2+\ldots+N_{k-1}^2+N_a+\ldots+N_{k-1}}x^{N_1+\ldots+N_{k-1}}$);

partitions $\mu^{(r)}$ having at most n_r all even parts (the factors $\frac{1}{(q^2;q^2)_{n_r}}$);

partitions $\nu^{(r)}$ having exactly j_r odd parts $\leq 2n_r - 2j_r + 1$, $0 \leq j_r \leq n_r$ accounting for y^{j_r} or $y^{n_r - j_r}$ as appropriate (the factor H_r or $y^{n_r}\widetilde{H_r}$).

Conversely, given $n_1,\ldots,n_{k-1} \geq 0$;

partitions $\mu^{(r)}$ having at most n_r all even parts;

partitions $\nu^{(r)}$ having exactly j_r odd parts $\leq 2n_r - 2j_r + 1$, $0 \leq j_r \leq n_r$;

we will first construct a base partition $\widetilde{\lambda}$ as in the proof of **Theorem 2.10**, and then recover a partition λ enumerated by $b_{k,a}(m,n)$, where there is a factor of y raised to a power equal to the number of even parts that appear an odd number of times.

Lastly, we will argue that the two transformations are inverses to each other.

Proposition 2.9 implies that double backward or forward moves keeps the number of even parts that appear an odd number of times invariant. This is not true for those that appear an even number of times, since zero is just as even.

For each $r = k-1, k-2, \ldots, 1$ in decreasing order, we consider the n_r largest r-marked parts $\lambda_1 < \ldots \lambda_{n_r}$ and perform so many double backward moves of rth kind on $\lambda_1, \lambda_2, \dots, \lambda_{n_r}$ in their increasing order so that no more double backward moves of rth kind is possible as in the proofs of **Theorem 2.11** and **Theorem 2.12**. We obtain $\mu^{(r)}$ here. The resulting partition does not have to be the base partition we are looking for, since single backward moves may still be possible which will give rise to more double moves may be possible. Then, we look at $\lambda_1, \lambda_2, \dots, \lambda_{n_r}$ one by one in increasing order. There are two cases.

(i) $r > a-1$ and $r \equiv a-1 (\text{mod } 2)$, or $r \leq a-1$

In this case, the base partition $\widetilde{\lambda}$ has its n_r largest r-marked parts even numbers that appear an even number of times ($r > a-1$ and $r \equiv a-1 (\text{mod } 2)$), or odd numbers ($r \leq a-1$).

If λ_i is an even number that appears an even number of times, or is an odd number, we do not do anything, and continue with λ_{i+1} unless $i = n_r$. Because, λ_i is already where it would be in $\widetilde{\lambda}$, the base partition, no further backward moves of rth kind are possible on it.

Else if λ_i is an even number that appears an odd number of times, we perform one single move on it. If λ_{i+1} also is an even number that appears an odd number of times, we restart the procedure for λ_{i+1}. Otherwise, if λ_{i+1} is an even number that appears a even number of times, or an odd number, then by (b) the single backward move of rth kind on λ_i will enable a double backward move of rth kind on λ_{i+1}, hence on $\lambda_{i+2}, \lambda_{i+3}, \dots, \lambda_{n_r}$ by **Proposition 2.8**. We first perform these double backward moves, then proceed with λ_{i+2}.

The single backward move yields the factor qy, since an even part that appears an odd number of times is accounted for.

(ii) $r > a-1$ and $r \not\equiv a-1 (\text{mod } 2)$

In this case, the base partition $\widetilde{\lambda}$ has its n_r largest r-marked parts even numbers that appear $r-(a-1)$ (an odd number of) times. This explains the separate factor y^{n_r}. We proceed exactly as in the preceding case except that we check if λ_i is an even number that appears an *even* number of times to perform a single backward moves of rth kind, and if so we check if λ_{i+1} is an even number that appears an *odd* number of times to perform a sequence of double backward moves of rth kind on r-marked parts strictly greater than λ_i. Single backward moves of rth kind bring $\frac{q}{y}$s. This is because each described single backward move of rth kind switches the number of occurrences of an even part from even into odd.

Now, if i_1th, i_2th, …, i_{j_r}th largest parts required single backward moves of rth kind among $\lambda_1, \lambda_2, \dots, \lambda_{n_r}$, then we have also made $i_s - 1$ double backward moves of rth kind if $i_{s+1} \neq i_s - 1$. In other words, for the list $\{i_{j_r} - 1, \dots, i_2 - 1, i_1 - 1\}$, we require $i_s - 1$ double backward moves of rth kind if $i_s - 1$ is the smallest element in a maximal sublist of consecutive integers in the list. **Lemma 2.13** applies to give a corresponding partition listed by $\begin{bmatrix} n_r \\ j_r \end{bmatrix}_2$. Subscript 2 is due to double moves instead of single ones. Along with the single moves and the y factors, we have $\begin{bmatrix} n_r \\ j_r \end{bmatrix}_2 (qy)^{j_r}$ (case **(i)**), or $y^{n_r} \begin{bmatrix} n_r \\ j_r \end{bmatrix}_2 (\frac{q}{y})^{j_r}$ (case **(ii)**). Either generates partitions with j_r odd parts $\leq 2n_r - 2j_r + 1$. This gives us $\nu^{(r)}$ as described above. Summing over all possible j_r yields H_{n_r} (case **(i)**), or $y^{n_r} \widetilde{H}_{n_r}$ (case **(ii)**). This gives us an injection from the partitions enumerated by $w_{k,a}(m, n)$ to partitions enumerated by (2.6), and concludes the first half of the construction.

As in the proof of **Theorem 2.10**, let us work on an example on the fly. Let

$$\lambda = \left\{ \begin{array}{cccccccccccccccccc} & & 3 & & & 6 & & & & & & & & & & 16 & & \\ & 2 & & & 5 & & 7 & & & & & & 13 & & 15 & & & 18 \\ & 2 & & 4 & & 6 & & & 9 & & 11 & & 13 & & 15 & & 17 & \end{array} \right\}$$

be a partition enumerated by $b_{4,3}(17, 162)$. In this context, it is accompanied by a factor y^3, since 4, 16 and 18 are precisely the even parts that appear an odd number of times. After making 2 backward moves of 3rd kind on the r =3-marked 3, 4 moves on 3-marked 6, and 16 moves on 3-marked 16, (1, 2, and 8 double backward moves of 3rd kind, respectively) we obtain

$$\left\{ \begin{array}{cccccccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & & & & & \\ & 2 & & 4 & & 6 & & & 9 & & & & & & 15 & & & 18 \\ 1 & & 3 & & & 6 & & 8 & & & 11 & & 13 & & 15 & & 17 & \end{array} \right\}$$

Hence $\mu^{(3)} = 16, 4, 2$. Note that no more double backward moves of 3rd kind are possible without violating the condition for $b_{4,3}(m, n)$. Since $r = 3 \not\equiv 3 - 1 = a - 1 ($ mod 2), we are in case **(ii)**. 2 and 4 are 3-marked parts which appear an even number of times, and these are the 3rd and 2nd largest 3-marked parts, respectively. This means $j_3 = 2$, $n_3 = 3$, and the list $0 \leq 2 - 1 < 3 - 1 < n_r = 3$ will produce us the partition $\lambda_1 = 2 \times 1, \lambda_2 = 0$ using **Lemma 2.13**, indeed a partition enumerated by $\begin{bmatrix} 3 \\ 2 \end{bmatrix}_2$, the factor and the subscript 2 because of double moves instead of single ones. Along with q^2, the two single moves, we have the partition $\nu^{(3)} = 3, 1$. After performing the single backward moves of 3-rd kind on 3-marked 2 and 3-marked 4, we perform the remaining double move on 3-marked 6, and transform λ further to

$$\left\{ \begin{array}{cccccccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & & & & & \\ 1 & & 3 & & 5 & & & & 9 & & & & & & 15 & & & 18 \\ 1 & & 3 & & 5 & & & 8 & & & 11 & & 13 & & 15 & & 17 & \end{array} \right\}$$

Observe that these last backward moves also accounted for $y^{n_r}(\frac{1}{y})^{j_r} = y^3(\frac{1}{y})^2 = y$. Indeed, for parts $\geq$ the largest of the 3-marked parts, we still need to account for y^2, in accordance with the arguments above.

We now look at the 3 largest r =2-marked parts 9, 15, and 18. 1 double backward move of 2nd kind is possible on the 2-marked 9, followed by a 3 double backward moves on 2-marked 15, which is in turn followed by 3 double backward moves of 2nd kind on the 2-marked 18. Thus, $\mu^{(2)} = 6, 6, 2$, and the current state is

$$\left\{ \begin{array}{cccccccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & & & & & \\ 1 & & 3 & & 5 & & & 8 & & 10 & & & 13 & & & & & \\ 1 & & 3 & & 5 & & 7 & & & 10 & & 12 & & & 15 & & 17 & \end{array} \right\}$$

the 1st and 3rd smallest 2-marked parts among the 3 largest correspond to even numbers that occur an even number of times. Since $r = 2 \leq 2 = a - 1$, we are in case **(i)**. We need two single backward moves of 2nd kind to convert the partition at hand to the base partition except for the largest 2 of the 1-marked ones, hence $j_2 = 2$. And single backward move of 2nd kind on 2-marked 8 will grant the 2-marked parts 10 and 13 one double move each. Those single moves are accounted for by $(qy)^2$, and **Lemma 2.13** from the list $0 \leq 1 - 1 < 3 - 1 < n_r = 3$ produces $\lambda_1 = 1, \lambda_2 = 1$, both referring to double moves. This is a partition generated by $\begin{bmatrix} 3 \\ 2 \end{bmatrix}_2$. Along with the single moves q^2, this yields $\nu^{(2)} = 3, 3$ $(3 \leq 2n_2 - 2j_2 + 1 = 6 - 4 + 1 = 3)$. Performing the described backward moves, we get

$$\left\{ \begin{array}{cccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & & & & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & & & 15 & & 17 \end{array} \right\}$$

Note that the remaining y^2 is already accounted for, so we do not expect any even parts larger than the largest of the 2-marked parts that appear an odd number of times.

Obviously, $j_1 = 0$ for the 2 largest 1-marked parts 15 and 17. So $\nu^{(1)}$ is the empty partition, realizing the assertion in the previous paragraph. a double backward move of 1st kind on both brings $\mu^{(1)} = 2, 2$ and λ is eventually made into

$$\widetilde{\lambda} = \left\{ \begin{array}{cccccccccc} & 2 & & 4 & & 6 & & & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & & \\ 1 & & 3 & & 5 & & 7 & 9 & 11 & 13 & 15 \end{array} \right\}$$

For the other direction, given integers $n_1, \ldots, n_{k-1}$, we construct $\widetilde{\lambda}$ as in the proof of **Theorem 2.10**. For each $r = 1, 2, \ldots, k-1$ in increasing order, we consider the n_r largest of the r-marked parts $\lambda_1 < \lambda_2 < \ldots < \lambda_{n_r}$. Here, $\nu^{(r)}$ is a partition listed by $q^{j_r} \begin{bmatrix} n_r \\ j_r \end{bmatrix}_2$. The contribution from $\begin{bmatrix} n_r \\ j_r \end{bmatrix}$ fed into **Lemma 2.13** will give us a list $0 \leq i_1 - 1 < i_2 - 1 < \ldots < i_{j_r} - 1 < n_r$ weighted by the sum of the smallest elements in maximal sublists of consecutive integers. The j_r r-marked parts that will be applied the single forward moves of rth kind are the i_1th, i_2th, ..., i_{j_r}th largest r-marked parts. To realize those single moves, we take $\lambda_1 < \lambda_2 < \ldots < \lambda_{n_r}$ one by one in decreasing order.

For λ_i, if $i = i_s$ for some i_s in the constructed list, we perform a single forward move of rth kind on λ_i. However, unless $i = 1$ or $i - 1 = i_{s-1}$ we cannot make a forward move of rth kind on λ_i by construction of $\widetilde{\lambda}$. To make this move possible by **Proposition 2.8 (ii)**, we perform double moves on $\lambda_1, \lambda_2, \ldots, \lambda_{i-1}$ first, then make the (single) forward move of rth kind on λ_i. The sequence of double moves ensure that the designated parts, and no other parts, become the even parts or succeed even parts the number of occurrences of which is of a fixed parity. This is the case if $i_s - 1$ is the smallest element of a maximal subsequence of consecutive integers in the above list, and so the required double moves are granted since the list is weighted by the sum of those smallest numbers.

For the assignment of ys, there are two cases.

(i) $r > a-1$ and $r \equiv a-1 (\text{mod } 2)$, or $r \leq a-1$

In this case, $\nu^{(r)}$ is enumerated by $\begin{bmatrix} n_r \\ j_r \end{bmatrix}_2 (qy)^{j_r}$. So we assign ys to r-marked λ_is which were applied the single forward moves of rth kind.

(ii) $r > a-1$ and $r \not\equiv a-1 (\text{mod } 2)$

In this case, $\widetilde{\lambda}$ had the n_r largest r-marked parts even numbers that appear an odd number of times. So, initially $\widetilde{\lambda}$ was accompanied by y^{n_r}. Also, $\nu^{(r)}$ is enumerated by $\begin{bmatrix} n_r \\ j_r \end{bmatrix}_2 (\frac{q}{y})^{j_r}$, and each single move rules out an even part that occur an odd number of times. Thus, λ_is that were applied single forward moves of rth kind are assigned $\frac{1}{y}$.

Lastly, we realize $\mu^{(r)}$s as in the proofs of **Theorem 2.11** and **Theorem 2.12**. We apply $\frac{\mu_i^{(r)}}{2}$ double forward moves of rth kind on the ith largest r-marked part for $i = 1, 2, \ldots, n_r$, the largest first.

By construction, the exponent of y is the number of even parts that appear an odd number of times at all times.

To conclude the proof, we remind that forward and backward moves on the same part are inverses of each other. Also, everything in the second construction is done in exact reverse order of the first construction above, and vice versa. $\square$

Example: For $k = 4$, $a = 2$, given

$$n_1 = 3, \quad n_2 = 4, \quad n_3 = 3,$$

$$\mu^{(1)} = \text{the empty partition}, \qquad \mu^{(2)} = 4, 2, \qquad \mu^{(3)} = 14, 10, 8,$$
$$\nu^{(1)} = 5, \qquad \nu^{(2)} = 5, 5, \qquad \nu^{(3)} = 3, 3$$

we will construct a partition λ enumerated by $b_{4,2}(20, 225)$ accompanied by a power of y where the exponent counts the even parts that appear an odd number of times. We begin by constructing the base partition described in **Theorem 2.10**

$$\widetilde{\lambda} = \left\{ \begin{array}{cccccccccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & & & & & & & \\ & 2 & & 4 & & 6 & & 8 & & 10 & & 12 & & 14 & & & & & & \\ 1 & & 3 & & 5 & & 7 & & 9 & & 11 & & 13 & & 15 & & 17 & & 19 & \end{array} \right\}$$

note that by the generating function (2.6), we initially have an accompanying $y^{n_2} = y^4$, since 8, 10, 12, 14 are the even parts that appear an odd number of times. Looking at the number of parts in $\nu^{(r)}$s, we expect λ to be accompanied by

$$y^5 = y^4 y^1 y^{-2} y^2$$

where $4 = n_2$, and 1, 2, 2 are the number of parts in $\nu^{(r)}$s in their respective order. 2 has minus sign since $r = 2 \not\equiv 1 = a - 1 \pmod 2$.

We first apply $\nu^{(1)} = 5$. When we subtract 1 from all parts of $\nu^{(1)}$ (just one single move to be performed in this case), and divide by 2 (hence counting the double moves), we have the partition $2(\times 2)$ enumerated by $\begin{bmatrix} n_1 \\ j_1 \end{bmatrix}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_2$. **Lemma 2.13** produces the list $\{3 - 1\}$. This means, we need to apply a single move on the 3rd largest 1-marked part 15. And to enable that move without applying any other single moves, we need to perform double forward moves of 1st kind on the 1-marked 21 and 1-marked 19. These moves are granted by the partition $2(\times 2)$ mentioned above. At this point, we have

$$\left\{ \begin{array}{cccccccccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & & & & & & & \\ & 2 & & 4 & & 6 & & 8 & & 10 & & 12 & & 14 & & & & & & \\ 1 & & 3 & & 5 & & 7 & & 9 & & 11 & & 13 & & & 16 & & & 19 & 21 \end{array} \right\}$$

we have another y here thanks to the single move we made. With the initial y^4, that makes y^5 We now have another even part (16) that appears an odd number of times. $\mu^{(1)}$ does not change anything because it is the empty partition.

Then, for $r = 2$, we subtract 1 from all parts of $\nu^{(2)}$ (2 single moves to be made) and feed the remaining partition in **Lemma 2.13** after dividing the remaining even parts (4, 4) by 2 to get the list $\{2 - 1, 4 - 1\}$ from the partition 2, 2 enumerated by $\begin{bmatrix} n_2 \\ j_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Note that 2, 2 denotes double moves. Both elements in the list are smallest elements in maximal sublists, so the list is weighted by the sum of its elements in this case. Indeed, we need to apply single forward moves of 2nd kind on the 2nd and 4th largest 2-marked elements. Thus, we first need to perform a double move on the largest 2-marked part before applying the first single move, and a double move on each 1st, 2nd and 3rd largest 2-marked parts before applying the second single move.

We already see that 2-marked 12 cannot be moved forward since both (d1) and (d2) fail in **Definition 2.6**. We show the process step by step.

$\downarrow$ a double forward move performed on the largest 2-marked part

$$\left\{\begin{array}{ccccccccccccccccc}
 & 2 & & 4 & & 6 & & & & & & & & & & & \\
 & 2 & & 4 & & 6 & & 8 & & 10 & & 12 & & 15 & & & \\
1 & & 3 & & 5 & & 7 & & 9 & & 11 & & 14 & & 16 & 19 & 21
\end{array}\right\}$$

$\downarrow$ a single forward move performed on the 2nd largest 2-marked part

$$\left\{\begin{array}{cccccccccccccccc}
 & 2 & & 4 & & 6 & & & & & & & & & & \\
 & 2 & & 4 & & 6 & & 8 & & 10 & 12 & & 15 & & & \\
1 & & 3 & & 5 & & 7 & & 9 & & 12 & 14 & & 16 & 19 & 21
\end{array}\right\}$$

$\downarrow$ double forward moves performed on the 3 largest 2-marked parts

$$\left\{\begin{array}{cccccccccccccccc}
 & 2 & & 4 & & 6 & & & & & & & & & & \\
 & 2 & & 4 & & 6 & & 8 & & 11 & & 14 & & 17 & & \\
1 & & 3 & & 5 & & 7 & & 10 & & 12 & 14 & 16 & & 19 & 21
\end{array}\right\}$$

$\downarrow$ a single forward move performed on the 4th largest 2-marked part

$$\left\{\begin{array}{ccccccccccccccc}
 & 2 & & 4 & & 6 & & & & & & & & & \\
 & 2 & & 4 & & 6 & 8 & & 11 & & 14 & & 17 & & \\
1 & & 3 & & 5 & & 8 & 10 & & 12 & 14 & 16 & & 19 & 21
\end{array}\right\}$$

Note that single moves ruled out even parts that originally occurred an odd number of times, hence we have y^{-2} at this stage. With current y^5, this yields y^3. Indeed, there are 3 even parts (10, 12, and 16) that appear an odd number of times. Now we apply $\mu^{(2)}$ as described in the example following **Theorem 2.12** to obtain

$$\left\{\begin{array}{ccccccccccccccc}
 & 2 & & 4 & & 6 & & & & & & & & & \\
 & 2 & & 4 & & 6 & 8 & & 11 & & 15 & & & 20 & \\
1 & & 3 & & 5 & & 8 & 10 & & 12 & 15 & 17 & 19 & & 21
\end{array}\right\}$$

For $r = 3$, we proceed as in the previous paragraphs. We subtract 1 from parts of $\nu^{(3)}$ (which account for the single moves to be performed), and divide parts of the remaining partition by 2 (hence count the double moves) to get 1, 1. **Lemma 2.13** produces the list $\{2-1, 3-1\}$ with weight 1 ($3-1$ is not the smallest element in a maximal sublist). Therefore, we need to perform single forward moves of 3rd kind on the 2nd and 3rd largest 3-marked parts. In order to do that, one double forward move on the largest 3-marked part suffices, because once a single forward is performed on the 2nd largest 3-marked part, a single forward move is already possible on the 3rd largest 3-marked part by **Proposition 2.8 (ii)**.

$\downarrow$ a double move of 3rd kind performed on the largest 3-marked part 6

$$\left\{\begin{array}{ccccccccccccccc}
 & 2 & & 4 & & & 7 & & & & & & & & \\
 & 2 & & 4 & 6 & & 8 & & 11 & & 15 & & & 20 & \\
1 & & 3 & & 6 & & 8 & 10 & & 12 & 15 & 17 & 19 & & 21
\end{array}\right\}$$

$\downarrow$ single moves of 3rd kind performed on the 2nd and 3rd largest 3-marked parts 4 and 2

$$
\left\{ \begin{array}{ccccccccccccc}
2 & 4 & & 7 & & & & & & & & & \\
2 & 4 & 6 & & 8 & & 11 & & 15 & & & 20 & \\
2 & 4 & 6 & & 8 & 10 & & 12 & 15 & 17 & 19 & & 21
\end{array} \right\}
$$

We have y^2 here, since we introduced two even parts (2 and 4) that now occur an odd number of times. Together with y^3 from the previous run, we now have y^5 (2, 4, 10, 12, and 20 are the even parts that appear an odd number of times).

Lastly, we incorporate $\mu^{(3)}$ to get the final partition λ

$$
\lambda = \left\{ \begin{array}{cccccccccccccc}
& & & & 8 & & 11 & & & 16 & & & & \\
2 & & 5 & & 8 & & 11 & 13 & & 16 & & & 20 & \\
2 & 4 & & 6 & 8 & 10 & & 13 & 15 & & 17 & 19 & & 21
\end{array} \right\}
$$

along with y^5, as asserted.

3. Parity Indices of Partitions

3.1 Basic Constructions

Throughout the section, for a fixed positive integer n, let $\lambda = \lambda_1 + \ldots + \lambda_n$ be a partition such that $\lambda_1 \geq \ldots \geq \lambda_n$. Depending on context, we will impose certain restrictions on the parity of parts of λ. Also, we let $\mu = \mu_1 + \ldots + \mu_j$ be a partition consisting of distinct numbers such that $n \geq \mu_1 > \ldots > \mu_j$. Let $\mu' = \mu'_1 + \ldots + \mu'_r$ be the *conjugate* partition of μ (see **§4**). It follows that $j \leq n$, $\mu'_1 = j$, $\mu'_r = 1$, and that μ' contains instances of every integer from 1 to j.

3.1.1 Lower Parity Indices in Partitions with Distinct Parts

We begin with a definition from [6]

Definition 3.1. The *lower even (resp. odd) parity index* of a partition λ is the number of times the parity changes from the smallest part to the largest, beginning with an even (resp. odd) part.

Let λ have distinct even parts. Let $\widetilde{\lambda} = \widetilde{\lambda}_1 + \ldots + \widetilde{\lambda}_n$ such that

$$\begin{aligned} \widetilde{\lambda}_1 &= \lambda_1 - \mu'_1 = \lambda_1 - j \\ &\vdots \\ \widetilde{\lambda}_r &= \lambda_r - \mu'_r = \lambda_r - 1 \\ \widetilde{\lambda}_{r+1} &= \lambda_{r+1} \\ &\vdots \\ \widetilde{\lambda}_n &= \lambda_n \end{aligned} \tag{3.1}$$

Then the parities of parts of $\widetilde{\lambda}$ counted from the smallest look like

$$E, \ldots, E, \underbrace{O, \ldots, O}_{\text{1 subtracted}}, \underbrace{E, \ldots, E}_{\text{2 subtracted}}, \ldots, \underbrace{(O/E), \ldots, (O/E)}_{j\text{ subtracted}}$$

so that the lower odd parity index of $\widetilde{\lambda}$ is j, and the parts of $\widetilde{\lambda}$ are still distinct, since from two adjacent even parts, either the same numbers, or a consecutive pair of numbers, in their respective order, are subtracted. Moreover, either 1 or 0 is subtracted from the first part, so $\widetilde{\lambda}$, too, has n parts. Note that, in comparison to **§3.1.2**, adding parts of μ' to parts of λ does not serve the purpose. Doing that would introduce leaps of at least 3 when the parity is switched.

Conversely, given $\widetilde{\lambda}$, a partition into distinct parts with lower odd parity index j; we can add 1 to the first subsequence of adjacent odd parts, 2 to the following subsequence of adjacent even parts and so on to recover a partition λ with distinct even parts, and μ' whose conjugate is a partition into j distinct numbers $\leq n$.

The above argument proves us that

$$P_o(y,x;q) = \sum_{m,n,r\geq 0} p_o(r,m,n)x^m y^r q^n = 1 + \sum_{n=1}^{\infty} \frac{x^n q^{n(n+1)}(-y/q;1/q)_n}{(q^2;q^2)_n} \tag{3.2}$$

where $p_o(r,m,n)$ is the number of partitions of n into m distinct parts with lower odd parity index r. This is because $\frac{q^{n(n+1)}}{(q^2;q^2)_n}$ generates partitions into distinct even parts (λ), $(-y/q;1/q)_n$ generates the partitions into distinct parts, all $\leq n$ (μ), $1/q$ instead of q because we are subtracting. Each y indicates a parity change, the first to an odd number. Thus, the exponent of y keeps track of the lower odd parity index. If we switch the base to q in $(-y/q;1/q)_n$ [11, eq.(1.2.24)], then [6, eq.(7.4)] follows.

There is an alternative way to explain [6, eq.(7.4)], in its original form. $\frac{x^n y^n q^{n(n+1)/2}}{(q^2;q^2)_n}$ generates partitions with the largest possible lower odd index. Then for each factor in $(-q/y)_n$, say $(1+q^j/y)$ $(1 \leq j \leq n)$; 1 leaves the partition intact, and $(1+q^j/y)$ adds 1 to the first j parts, ruling out a parity change at the j the part.

How to obtain [6, eq.(7.3)] is already explained combinatorially in the mentioned paper.

Example: Let $\lambda = 18 + 16 + 14 + 10 + 6 + 4$, and $\mu = 5 + 4 + 2$. then, $\mu' = 3 + 3 + 2 + 2 + 1$, so that $\widetilde{\lambda} = 15 + 13 + 12 + 8 + 5 + 4$. The lower parity index of $\widetilde{\lambda}$ is 3. If we draw the Ferrers graphs of the partitions involved, we have the following picture:

λ :

λ, where μ' is indicated with $\times$s:

$$\begin{array}{llllllllllllllllll}
\times & \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
\times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & \\
\times & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & & & & & & & & & & & & & &
\end{array}$$

$$\widetilde{\lambda}: \begin{array}{lllllllllllllll}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & & & & & & & & & & &
\end{array}$$

3.1.2 Lower Parity Indices in Unrestricted Partitions

Suppose that λ has all odd parts. Let $\widetilde{\lambda} = \widetilde{\lambda}_1 + \ldots + \widetilde{\lambda}_n$ such that

$$\begin{array}{lcl}
\widetilde{\lambda}_1 & = & \lambda_1 + \mu'_1 = \lambda_1 + j \\
 & \vdots & \\
\widetilde{\lambda}_r & = & \lambda_r + \mu'_r = \lambda_r + 1 \\
\widetilde{\lambda}_{r+1} & = & \lambda_{r+1} \\
 & \vdots & \\
\widetilde{\lambda}_n & = & \lambda_n
\end{array} \tag{3.3}$$

Then the parities of parts of $\widetilde{\lambda}$ counted from the smallest look like

$$O, \ldots, O, \underbrace{E, \ldots, E}_{\text{parts added } 1}, \underbrace{O, \ldots, O}_{\text{parts added } 2}, \ldots, \underbrace{(E/O), \ldots, (E/O)}_{\text{parts added } j}$$

so that the lower even parity index of $\widetilde{\lambda}$ is j.

Note that, conversely, given a partition $\widetilde{\lambda}$ with lower even parity index j, we can subtract 1 from the first subsequence of adjacent even numbers, 2 from the following sequence of adjacent odd numbers, and so on, to obtain a partition λ into odd numbers, and a partition μ' in which all parts between 1 and j appear. Moreover, $\widetilde{\lambda}$ and λ are related as in (3.3).

This proves us that

$$U_e(y, x; q) = \sum_{m,n,r \geq 0} u_e(r, m, n) x^m y^r q^n = 1 + \sum_{n=1}^{\infty} \frac{x^n q^n (-yq; q)_n}{(q^2; q^2)_n} \tag{3.4}$$

where $u_e(r,m,n)$ is the number of partitions of n in to m parts, with lower even parity index r. This is because $\frac{q^n}{(q^2;q^2)_n}$ generates partitions into odd parts (λ). $(-yq;q)_n$ generates partitions into distinct parts, all $\leq n$ (μ), and the exponent of y is $j=r$ above, which is the lower even parity index as well as the number of parts in μ.

Example: We will work backwards in this example. That is, given $\widetilde{\lambda}$ with a certain lower even parity index, we will find a partition λ into odd parts, and a partition μ with distinct parts.

Let $\widetilde{\lambda} = 8+5+5+4+2+1+1$. The lower even parity index of $\widetilde{\lambda}$ is 3. By the above discussion, when we group the adjacent parts that have the same parity, we have:

$$1, 1, \underbrace{2, 4}_{\text{1 added}}, \underbrace{5, 5}_{\text{2 added}}, \underbrace{8}_{\text{3 added}}$$

So, $\mu' = 3+2+2+1+1$, $\mu = 5+3+1$, and $\lambda = 5+3+3+3+1+1+1$, as desired.

In the above construction, if we replace x by $\frac{x}{q}$, then we will have converted the lower even parity index into lower odd parity index. But then, partitions may include zeros counted by xs. If we multiply by $(1-x)$ in the front, then we will have eliminated the unwanted partitions. It follows that

$$(1-x)\left(1+\sum_{n=1}^{\infty}\frac{x^n(-yq;q)_n}{(q^2;q^2)_n}\right) = \sum_{m,n,r\geq 0} u_o(r,m,n)x^m y^r q^n = U_o(y,x;q) \tag{3.5}$$

This is [6, eq.(8.5)]. How to obtain [6, eq.(8.3)] is explained there combinatorially.

3.1.3 Upper Parity Indices in Unrestricted Partitions

We recall one more definition from [6].

Definition 3.2. The *upper even (resp. odd) parity index* of a partition λ is the number of times parts of λ changes parity from the largest to the smallest, beginning with an even (resp. odd) one, hence zero if all parts are odd (resp. even).

In [6, §10], the following generating functions are defined

$$F_e(N,y,x;q) := F_e(N) = \sum_{r,m,n\geq 0} \phi_e(N,r,m,n)y^r x^m q^n$$

$$F_o(N,y,x;q) := F_o(N) = \sum_{r,m,n\geq 0} \phi_o(N,r,m,n)y^r x^m q^n$$

where $\phi_e(N,r,m,n)$ (respectively, $\phi_o(N,r,m,n)$) is the number of partitions of n into m parts each at most N, with upper even (respectively, upper odd) parity index equal to r ([6, eq.s(10.1) and (10.2)]).

From this point onwards, we require that μ has distinct parts $< n$ (but not $\leq n$ as in **§3.1.1** and **§3.1.2**).

Given a partition λ with even parts, if $\widetilde{\lambda}$ is constructed as in (3.3), then for μ having $2j$ parts, the parities of parts of $\widetilde{\lambda}$ beginning with the smallest look like:

$$E,\ldots,E,\underbrace{O,\ldots,O}_{1\text{ added}},\ldots,\underbrace{O,\ldots,O}_{2j-1\text{ added}},\underbrace{E,\ldots,E}_{2j\text{ added}}$$

and for μ having $2j+1$ parts, the parities of parts of $\widetilde{\lambda}$ beginning with the smallest look like:

$$E,\ldots,E,\underbrace{O,\ldots,O}_{1\text{ added}},\ldots,\underbrace{E,\ldots,E}_{2j\text{ added}},\underbrace{O,\ldots,O}_{2j+1\text{ added}}$$

where both options have upper even parity index $2j$. Now, the partitions with exactly $2j+1$ (respectively $2j+1$) distinct parts, all $\leq n-1$ is generated by

$$q^{2j(2j+1)/2}\begin{bmatrix} n-1 \\ 2j \end{bmatrix}\left(\text{respectively, } q^{(2j+1)(2j+2)/2}\begin{bmatrix} n-1 \\ 2j+1 \end{bmatrix}\right)$$

On the other hand, given a partition λ with odd parts, if $\widetilde{\lambda}$ is constructed as in (3.3), then for μ having $2j$ parts, the parities of parts of $\widetilde{\lambda}$ beginning with the smallest look like:

$$O,\ldots,O,\underbrace{E,\ldots,E}_{1\text{ added}},\ldots,\underbrace{E,\ldots,E}_{2j-1\text{ added}},\underbrace{O,\ldots,O}_{2j\text{ added}}$$

and for μ having $2j-1$ parts, the parities of parts of $\widetilde{\lambda}$ beginning with the smallest look like:

$$O,\ldots,O,\underbrace{E,\ldots,E}_{1\text{ added}},\ldots,\underbrace{O,\ldots,O}_{2j-2\text{ added}},\underbrace{E,\ldots,E}_{2j-1\text{ added}}$$

where both options have upper even parity index $2j$. Now, the partitions with exactly $2j$ (respectively $2j-1$) distinct parts, all $\leq n-1$ is generated by

$$q^{2j(2j+1)/2}\begin{bmatrix} n-1 \\ 2j \end{bmatrix}\left(\text{respectively, } q^{(2j-1)2j/2}\begin{bmatrix} n-1 \\ 2j-1 \end{bmatrix}\right)$$

Observe that given $\widetilde{\lambda}$, we can form λ and μ, and the procedures are inverses of each other, as in (3.3).

Combining these computations in the light of the arguments employed above, namely adding parts of μ' to parts of λ, we get the generating function for the partitions with unrestricted parts where the exponent of x keeps track of the number of parts, and the exponent of y keeps track of the upper even parity index:

$$F_e(\infty) = \sum_{n\geq 0}\frac{x^n q^{2n}}{(q^2;q^2)_n}\left\{\sum_j\left(q^{2j(2j+1)/2}\begin{bmatrix} n-1 \\ 2j \end{bmatrix} + q^{(2j+1)(2j+2)/2}\begin{bmatrix} n-1 \\ 2j+1 \end{bmatrix}\right)y^{2j+1}\right\}$$

$$+\sum_{n\geq 0}\frac{x^n q^n}{(q^2;q^2)_n}\left\{\sum_j\left(q^{2j(2j+1)/2}\begin{bmatrix}n-1\\2j\end{bmatrix}+q^{(2j-1)2j/2}\begin{bmatrix}n-1\\2j-1\end{bmatrix}\right)y^{2j}\right\}$$

$$=\sum_{j,n\geq 0}\frac{x^n y^{2j+1}q^{2j(2j+1)/2+2n}}{(q^2;q^2)_n}\begin{bmatrix}n\\2j+1\end{bmatrix}+\sum_{j,n\geq 0}\frac{x^n y^{2j}q^{(2j-1)2j/2+n}}{(q^2;q^2)_n}\begin{bmatrix}n\\2j\end{bmatrix} \qquad (3.6)$$

by [3, p.35, eq.(3.3.4)]. This is [6, eq.(10.15)].

Here, we can impose an upper bound on the largest part of λ, and take $q^{2n}\begin{bmatrix}M+n-1\\n\end{bmatrix}_2$ for partitions into n even parts, all $\leq 2M$ in the first summand (instead of $\frac{q^{2n}}{(q^2;q^2)_n}$), and $q^n\begin{bmatrix}M+n-1\\n\end{bmatrix}_2$ for partitions into n odd parts, all $\leq 2M-1$ in the second sum (instead of $\frac{q^n}{(q^2;q^2)_n}$) in (3.6).

Note that, in the first sum, $\widetilde{\lambda}_1=\lambda_1+2j$, or $\widetilde{\lambda}_1=\lambda_1+2j+1$, and in the second sum, $\widetilde{\lambda}_1=\lambda_1+2j$, or $\widetilde{\lambda}_1=\lambda_1+2j-1$ in (3.6). Thus, taking $M\leftarrow M-j$ in the first sum and $M\leftarrow M-j+1$ in the second, we obtain the generating function for partitions with parts less than $2M+1$, where the exponent of x keeps track of the number of parts, and the exponent of y keeps track of the upper even parity index:

$$F_e(2M+1)=\sum_{j,n\geq 0}x^n y^{2j+1}q^{2j(2j+1)/2+2n}\begin{bmatrix}M-j+n-1\\n\end{bmatrix}_2\begin{bmatrix}n\\2j+1\end{bmatrix}$$
$$+\sum_{j,n\geq 0}x^n y^{2j}q^{(2j-1)2j/2+n}\begin{bmatrix}M-j+n\\n\end{bmatrix}_2\begin{bmatrix}n\\2j\end{bmatrix} \qquad (3.7)$$

This is [6, eq.(10.3)].

Making the obvious changes in the above lines, we can just as well obtain the generating function of partitions where the exponent of x keeps track of the number of parts, and the exponent of y keeps track of the upper odd parity index as

$$F_o(\infty)=\sum_{j,n\geq 0}\frac{x^n y^{2j+1}q^{2j(2j+1)/2+n}}{(q^2;q^2)_n}\begin{bmatrix}n\\2j+1\end{bmatrix}+\sum_{j,n\geq 0}\frac{x^n y^{2j}q^{(2j-1)2j/2+2n}}{(q^2;q^2)_n}\begin{bmatrix}n\\2j\end{bmatrix} \qquad (3.8)$$

which is [6, eq.(10.16)], and the same generating function where parts are $\leq 2M$ as

$$F_o(2M)=\sum_{j,n\geq 0}x^n y^{2j+1}q^{2j(2j+1)/2+n}\begin{bmatrix}M-j+n-1\\n\end{bmatrix}_2\begin{bmatrix}n\\2j+1\end{bmatrix}$$
$$+\sum_{j,n\geq 0}x^n y^{2j}q^{(2j-1)2j/2+2n}\begin{bmatrix}M-j+n-1\\n\end{bmatrix}_2\begin{bmatrix}n\\2j\end{bmatrix} \qquad (3.9)$$

which is [6, eq.(10.14)].

The example in **§3.1.2** can be reworked in this context with appropriate choices for M, and j.

3.1.4 Upper Parity Indices in Partitions with Distinct Parts

We recall two more generating functions from [6].

$$D_e(N, y, x; q) := D_e(N) = \sum_{r,m,n\geq 0} \delta_e(N, r, m, n) y^r x^m q^n$$

$$D_o(N, y, x; q) := D_o(N) = \sum_{r,m,n\geq 0} \delta_o(N, r, m, n) y^r x^m q^n$$

where $\delta_e(N, r, m, n)$ (respectively $\delta_o(N, r, m, n)$) is the number of partitions of n into m distinct parts each at most N and with upper even (respectively odd) parity index r ([6, eq.s(9.1) and (9.2)]).

If we require distinct parts, then we modify the construction in the preceding section as taking a partition λ into distinct parts, all odd or all even. Then as described in **§3.1.1** (3.1), we subtract the parts of μ', the conjugate of μ, a partition into distinct parts, all $< n$ (but not $\leq n$). Then we have the generating function of the partitions with distinct parts, where the exponent of x keeps track of the number of parts, and the exponent of y keeps track of the upper even parity index as

$$\begin{aligned} D_e(\infty) = &\sum_{j,n\geq 0} \frac{x^n y^{2j+1} q^{-2j(2j+1)/2+n(n+1)}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2j+1 \end{bmatrix}_{(-1)} \\ &+ \sum_{j,n\geq 0} \frac{x^n y^{2j} q^{-(2j-1)2j/2+n^2}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2j \end{bmatrix}_{(-1)} \end{aligned} \tag{3.10}$$

where $\begin{bmatrix} A \\ B \end{bmatrix}_{(-1)}$ is $\begin{bmatrix} A \\ B \end{bmatrix}$ with q replaced by $1/q$, since we are subtracting parts of μ'. This is [6, eq.(9.16)].

Again, we can impose some bound on the parts, and we can replace $\frac{q^{n(n+1)}}{(q^2;q^2)_n}$ by $q^{n(n+1)} \begin{bmatrix} M \\ n \end{bmatrix}_2$, which gives us partitions into n distinct even parts $\leq 2M$ in the first sum, and $\frac{q^{n^2}}{(q^2;q^2)_n}$ by $q^{n^2} \begin{bmatrix} M \\ n \end{bmatrix}_2$, which is partitions into n distinct odd parts $\leq 2M - 1$ in the second sum in (3.10). Since we are subtracting $2j$ or $2j+1$ from λ_1 in the first sum, and $2j$ or $2j-1$ in the second, substituting $M \leftarrow M + j$ in both the first and the second, we obtain the generating function of partitions into distinct parts $\leq 2M$, where the exponent of x accounts for the number of parts, and the exponent of y accounts for the upper even parity index as

$$\begin{aligned} D_e(2M) = &\sum_{j,n\geq 0} x^n y^{2j+1} q^{-2j(2j+1)/2+n(n+1)} \begin{bmatrix} M+j \\ n \end{bmatrix}_2 \begin{bmatrix} n \\ 2j+1 \end{bmatrix}_{(-1)} \\ &+ \sum_{j,n\geq 0} x^n y^{2j} q^{-(2j-1)2j/2+n^2} \begin{bmatrix} M+j \\ n \end{bmatrix}_2 \begin{bmatrix} n \\ 2j \end{bmatrix}_{(-1)} \end{aligned} \tag{3.11}$$

Finally, we can repeat the arguments with the appropriate changes to obtain the generating function of partitions into distinct parts, where the exponent of x keeps track of the number of parts, and the exponent of y keeps track of the upper odd parity index as

$$D_o(\infty) = \sum_{j,n\geq 0} \frac{x^n y^{2j+1} q^{-2j(2j+1)/2+n^2}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2j+1 \end{bmatrix}_{(-1)} + \sum_{j,n\geq 0} \frac{x^n y^{2j} q^{-(2j-1)2j/2+n(n+1)}}{(q^2;q^2)_n} \begin{bmatrix} n \\ 2j \end{bmatrix}_{(-1)} \tag{3.12}$$

which is [6, eq.(9.15)], and the generating function of partitions into distinct parts $\leq 2M+1$, where the exponent of x keeps track of the number of parts, and the exponent of y keeps track of the upper odd parity index as

$$D_o(2M+1) = \sum_{j,n\geq 0} x^n y^{2j+1} q^{-2j(2j+1)/2+n^2} \begin{bmatrix} M+j \\ n \end{bmatrix}_2 \begin{bmatrix} n \\ 2j+1 \end{bmatrix}_{(-1)} + \sum_{j,n\geq 0} x^n y^{2j} q^{-(2j-1)2j/2+n(n+1)} \begin{bmatrix} M+j \\ n \end{bmatrix}_2 \begin{bmatrix} n \\ 2j \end{bmatrix}_{(-1)} \tag{3.13}$$

Noting that $\begin{bmatrix} A \\ B \end{bmatrix}_{(-1)} = q^{-B(A-B)} \begin{bmatrix} A \\ B \end{bmatrix}$, we get [6, eq.s (9.4) and (9.3)] from (3.11) and (3.13), respectively.

The example in **§3.1.1** can be reworked in this context with appropriate choices for M and j.

3.2 Sieves

3.2.1 Lower Odd Parity Index in Partitions With Distinct Parts

Here we explain [6, eq.(7.5)]

$$P_o(y,x;q) = (-xq)_\infty \sum_{n\geq 0} \frac{(y)_n(-xq)^n}{(q^2;q^2)_n} \tag{3.14}$$

The factor $\frac{(-xq)^n}{(q^2;q^2)_n}$ in the term under the sum generates exactly n odd numbers $\lambda = \lambda_1 + \ldots + \lambda_n$, not necessarily distinct, all having factor -1. $(yq)_{n-1}$ generates a partition μ having distinct parts $< n$. Each factor in $(yq)_{n-1}$, say $(1 - yq^j)$ for $0 < j < n$, either leaves the -1 weighted numbers intact (1), or adds 1 to the first j parts (another description of the method in **§3.1.2**), creating a parity change exactly at the jth part λ_j (that is, $\lambda_j \not\equiv \lambda_{j+1} (\text{mod } 2)$), and commutes the factor -1 of λ_j to y $(-yq^j)$. The remaining factor $(1-y)$ either leaves the smallest part λ_n (which is odd) as it is, or commutes its factor from -1 to y. This way, we form $\widetilde{\lambda}$ by adding all terms of μ'. If $-y$ is chosen in $(1-y)$, then the exponent y accounts for the lower odd parity index in $\widetilde{\lambda}$, otherwise it gives one less than the lower odd parity index of $\widetilde{\lambda}$. And remember that a part $\widetilde{\lambda}_j$ in $\widetilde{\lambda}$ has factor y if $\widetilde{\lambda}_j \not\equiv \widetilde{\lambda}_{j-1} (\text{mod } 2)$, -1 otherwise for $j = 2, \ldots, n$. Also, $\widetilde{\lambda}_1$ has factor either -1 or y. Finally, $(-xq)_\infty$ provides distinct terms, all having weight 1, and provides no ys. Note that for any partition generated by the right hand side, the exponent of y can never exceed the lower odd parity index.

Let $\beta = \beta_1 + \ldots + \beta_m$ be a partition generated by the right hand side, along with some factor of y to some power, which, as a matter of fact, is smaller than or equal to the lower odd parity index of β. It is clear that if a part is repeated in β, then one and only one of those parts could have come from $(-xq)_\infty$. We agree that it is the one with the largest index, namely the last of the occurrences.

When $\beta_i = \beta_{i-1} \neq \beta_{i+1}$ for some i between 2 and m ($\beta_{m+1} = 0$), then there seem to be two possibilities. If β_i does not have factor y, then β_i is either contributed by $(1 + xq^i)$ with weight 1, or it is contributed by a term under the sum, with weight -1. Thus, for any composition of β without β_i, β_i is introduced both with a plus and a minus sign. Else if β_i has a factor y, then β_{i-1} cannot assume a factor y. By the construction above, a factor y is not possible for adjacent parts without a parity change. Also, by our agreement, β_{i-1} could not have contributed by $(-xq)_\infty$, which forces a parity change $\beta_{i-1} \not\equiv \beta_i (\text{mod } 2)$ to justify the y factor for β_i, an impossibility. Therefore, βs with repetitions of parts are annihilated by the sieve on the right hand side.

Example: Let $\lambda = 7_{(-)} + 3_{(-)} + 3_{(-)}$ be given by $\frac{(-xq)^3}{(q^2;q^2)_3}$, where subscripts indicate the factors -1. Assume that $-y$ is chosen in $(1-y)$, and $\mu = 1$ is given. Then, $\widetilde{\lambda} = 8_y + 3_{(-)} + 3_y$. Moreover, assume that we take 1 from the whole factor $(-xq)_\infty$, that is, no contribution from the first factor in the generating function, so that $\beta = \widetilde{\lambda}$. On the other hand, given $\lambda = 7_{(-)} + 3_{(-)}$ by $\frac{(-xq)^2}{(q^2;q^2)_2}$, $-y$ from $(1-y)$, and $\mu = 1$, so that $\widetilde{\lambda} = 8_{(y)} + 3_{(y)}$ and 3 in $(1 + xq^3)$ from $(-xq)_\infty$, we obtain $\beta = 8_y + 3_{(+)} + 3_y$. Therefore, β along with y^2 is generated both by a plus and a minus sign. Although β has lower odd parity index 2, it is annihilated.

When β has distinct parts, along with some power of y, for a β_i having a factor of y, if there are parts $\beta_{i-1}, \beta_{i-2}, \ldots, \beta_{i-s}$ none having a factor of y, such that β_i and β_{i-s} have the same parity, then β_{i-s} could have come from $(-xq)_\infty$ with a plus sign, or from a term in the sum with minus sign. Such βs are annihilated by the sieve on the right hand side. This leaves us βs along with some power of y, whose single factors y are given to β_is for which β_{i-1} has opposite parity, and among the parts greater than the largest β_i to receive factor y, there are none with the same parity as β_i.

If the exponent of y is strictly less than the lower odd parity index of β, then there are exactly two distribution of ys. In one, β_1 receives y, and in the other β_i receives y, $\beta_1, \ldots, \beta_{i-1}$ receive none. In the latter case, $\beta_1, \ldots, \beta_{i-1}$ have opposite parity of β_i. Now, in exactly one of the cases, the smallest part to receive a y is even, and in the other, it is odd. If it is odd, then β has no parts with factor -1, hence it is counted as 1. Else if it is even, then by construction, then at least one odd part smaller than the smallest even part with a factor y has factor -1, forcing β to be counted -1. The mentioned odd part exists, as discussed above, to receive 1 from the factor $(1-y)$ in the sum. On the total, β is annihilated.

Therefore, only βs accompanied by a power of y equal to the lower odd parity index survive, and counted as 1. This establishes (3.14) ([6, eq.(7.5)]).

Example: Given $\beta = 11+10+8+7+5+4+3$ along with y^3, we need to give the factors y to parts that alternate in parity in increasing order. Suppose that we chose

$$\beta = 11 + \{10\} + 8_y + 7 + 5_y + 4_y + 3$$

but we did not indicate the source of the other parts, which can be either $(-xq)_\infty$, or a term inside the sum, without factors y. Note that the 10 in curly braces can be generated as $10_{(-)}$ or $10_{(+)}$. So that no matter what the signs of the other parts are, β is generated with plus sign as well as minus sign the same number of times. This is the case whenever the smaller one of a pair of parts of the same parity has factor y, and there are no other ys in between, hence no y for the larger part. Thus, there are only two distributions of ys for this β:

$$\beta = 11_y + 10_y + 8 + 7_y + 5 + 4 + 3$$

For this possibility, 8 cannot be generated by a minus factor, since in that case 10 could not have received a y by construction. $\ldots + 10_y + 8_{(-)} + \ldots$ in β means that 10 and 8 are adjacent parts without parity change in between, so that the larger part 10_y having factor y is not possible in the first place. Similar reasoning applies for 5 and 3, so that all 8, 5 and 3 have plus factors. For 4, it cannot have a minus factor because in that case it would be the smallest number in $\widetilde{\lambda}$, the part of β coming from a term inside the sum. However, that smallest term with a minus factor can only be an odd number by construction. Therefore, β as such is generated exactly once with plus sign.

The second possibility is

$$\beta = 11 + 10_y + 8 + 7_y + 5 + 4_y + 3$$

As discussed above, 8 and 5 have plus factors. 11 on the other hand has a plus sign for a different reason. Were 11 contributed by a term under the sum, i.e. if $\beta = 11_{(-)} + 10_y$, we should have a factor y for 11, thanks to the parity change in between. On the other hand, if the smallest part to receive a y is an even number, then there must be a smaller odd number, which is the smallest part of $\widetilde{\lambda}$, which evidently has a minus factor. So, 3 here has a minus factor. β here is generated once by a minus sign, therefore it is annihilated on the total.

However, if β as accompanied by y^5, then the only surviving distribution of ys would be

$$\beta = 11_y + 10_y + 8 + 7_y + 5 + 4_y + 3_y$$

which is generated exactly once with plus sign. Observe that the lower odd parity index of β is 5.

3.2.2 Lower Odd Parity Index in Unrestricted Partitions

Here we explain a variant of [6, eq.(8.4)].

$$U_o(y,x;q) = \frac{1}{(xq)_\infty} \sum_{n\geq 0} \frac{(-x)^n q^{n^2} (y;1/q)_n}{(q^2;q^2)_n} \tag{3.15}$$

which is converted back to [6, eq.(8.4)] by changing the base to q in $(y;1/q)_n$.

The factor $\frac{(-x)^n q^{n^2}}{(q^2;q^2)_n}$ under the sum generates partitions λ into distinct odd parts, each having factor -1. The factor $(y/q;1/q)$ generates partitions μ into distinct parts $< n$. We form $\widetilde{\lambda}$ by subtracting parts of μ' from parts of λ as described in **§3.2.1**with the following specifications. A factor in $(y/q;1/q)$, say $(1-y/q^i)$ for $0<i<n$ either leaves the partition intact (1), or subtracts 1 from the first j parts introducing a parity change at λ_j, namely $\lambda_j \not\equiv \lambda_{j+1} (\text{mod } 2)$, and commutes the factor of λ_j from -1 to y $(-y/q^i)$. $(1-y)$ either leaves the smallest part as it is, or changes its factor from -1 to y. So that the exponent of y accompanying $\widetilde{\lambda}$ gives us either the lower parity index of $\widetilde{\lambda}$, or one less than that, depending on the factor of the smallest part λ_n, which is odd. Finally, $\frac{1}{(xq)_\infty}$ brings unrestricted partitions, all parts having factor 1.

Let $\beta = \beta_1 + \ldots + \beta_m$ be a partition generated by the right hand side, along with a factor of y whose exponent is at most the lower odd parity index of β.

We can pick at least as many distinct parts of β as the exponent of y that alternate in parity, and argue that those parts come from the sum, each having factor -1 or y. This also introduces a distribution of ys among parts of β. We agree that the first part to appear among equal parts (the one with the smallest index) can possibly have come from the infinite sum, to fix an order of parts coming from the factor $\frac{1}{(xq)_\infty}$ or from the sum.

The argument from **§3.2.1** then applies almost word by word, except that βs are not required to have distinct parts. For a β_i having a factor of y, if there are parts $\beta_{i-1}, \beta_{i-2}, \ldots, \beta_{i-s}$ none having a factor of y, such that $\beta_i \neq \beta_{i-s}$ have the same parity, β_{i-s} could have come from $\frac{1}{(xq)_\infty}$ with a plus sign, or from a term in the sum with minus sign. Such βs are annihilated by the sieve on the right hand side. This leaves us βs along with some power of y, whose single factors y are given to β_is for which β_{i-1} has opposite parity, and among the parts greater than the largest β_i to receive factor y, there are none with the same parity as β_i.

If the exponent of y is strictly less than the lower odd parity index of β, then there are exactly two distribution of ys. In one, β_1 receives y, and in the other β_i receives y, $\beta_1, \ldots, \beta_{i-1}$ receive none. In the latter case, $\beta_1, \ldots, \beta_{i-1}$ have opposite parity of β_i. Now, in exactly one of the cases, the smallest part to receive a y is even, and in the other, it is odd. If it is odd, then β has no parts with factor -1, hence it is counted as 1. Else if it is even, then by construction at least one odd part smaller than the smallest even part with a factor y has factor -1, forcing β to be counted -1. On the total, β is annihilated.

Example: We rework $\beta = 11 + 10 + 8 + 7 + 5 + 4 + 3$ from the preceding section, this time accompanied by $y^1 = y$. Again, most of the alternatives to place y vanish in the sieve, and two remain to discuss. The first one is

$$\beta = 11_y + 10 + 8 + 7 + 5 + 4 + 3$$

Here, no part can assume minus factors, so β as such is generated once with plus sign. The second one is

$$\beta = 11 + 10_y + 8 + 7 + 5 + 4 + 3$$

Since the smallest part to receive factor y is even, there is at least one smaller odd part (there are three). At least one of them must receive a minus factor. The possibilities are

$$\begin{array}{ll} 7_{(-)}, 5_{(-)}, 3_{(-)} & \text{counted as } - \\ 7_{(-)}, 5_{(-)}, 3_{(+)} & \text{counted as } + \\ 7_{(-)}, 5_{(+)}, 3_{(-)} & \text{counted as } + \\ 7_{(-)}, 5_{(+)}, 3_{(+)} & \text{counted as } - \\ 7_{(+)}, 5_{(-)}, 3_{(-)} & \text{counted as } + \\ 7_{(+)}, 5_{(-)}, 3_{(+)} & \text{counted as } - \\ 7_{(+)}, 5_{(+)}, 3_{(-)} & \text{counted as } - \end{array}$$

Thus, β is generated exactly once with a minus sign when 10 receives a minus factor. Together with the former option, β is annihilated.

Note that exactly one among equal parts could receive factor y or $(-)$. All others come with $(+)$ sign. Therefore, the discussion would be just the same for $\beta_0 = 11 + 11 + 11 + 10 + 8 + 8 + 7 + 5 + 5 + 5 + 5 + 4 + 3 + 3$ along with $y^1 = y$

Therefore, only βs accompanied by a power of y equal to the lower odd parity index survive and counted as 1. This proves (3.15), hence [6, eq.(8.4)].

We remark that there is a similar way to prove [6, eq.(8.4)] in the form it is given:

$$U_o(y, x; q) = \frac{1}{(xq)_\infty} \sum_{n \geq 0} \frac{x^n y^n q^{n(n+1)/2} (1/y)_n}{(q^2; q^2)_n}$$

Here, $\frac{x^n y^n q^{n(n+1)/2}}{(q^2;q^2)_n}$ generates a partition with full lower parity index, and we interpret $(1/y)_n$ as ruling out parity changes, and replacing ys with -1s. This method is a bit more involved. The argument developed here seems to be more instructive.

3.2.3 An Unexpected Relation

Here we prove [6, eq.(8.13)]

$$U_o(y, x; q) = \frac{1}{(xq)_\infty} P_o(-yq, -x/q; q) \tag{3.16}$$

where $U_o(y, x; q)$ generates unrestricted partitions, and $P_o(y, x; q)$ generates partitions with distinct parts. In both functions, the exponent of x counts the number of parts, and the exponent of y keeps track of the lower odd parity index.

Given a partition $\lambda = \lambda_1 + \ldots + \lambda_n$ enumerated by $P_o(y, x; q)$, assume that the factors y are assigned to λ_is for which λ_{i+1} has opposite parity. Now, replacing y by $-yq$ and x by $-x/q$ subtracts 1 from the parts having no factor y and gives them factors -1. The parts having factor y are intact. Call the transformed partition $\widetilde{\lambda}$. If the smallest part is 1, it had the factor y, so that $\widetilde{\lambda}$ has n distinct parts.

The exponent of y is equal to the lower odd parity index of $\widetilde{\lambda}$, or one less than that. To see this fact, note that the parts without factor y that fall between two parts with factor y are all of the same parity as the smaller end before the decrease, and of the same parity as the larger end afterwards. The parts $\lambda_1, \ldots, \lambda_{s-1}$ without a factor y followed by λ_s with factor y must all be of the same parity as λ_s. Hence a parity change occurs after the decrease which is not accounted for by a y unless λ_1 and λ_2 are of opposite parities. This is the place where the lower parity index of $\widetilde{\lambda}$ increases by one. The parts $\lambda_n, \ldots, \lambda_{n-r+1}$ without factors y must all be even while λ_{n-r} with factor y must be odd. After the decrease, $\widetilde{\lambda}_n, \ldots, \widetilde{\lambda}_{n-r+1}$ become odd with $\widetilde{\lambda}_{n-r}$ having factor y also odd. If all parts are even, then the two previous cases coincide, and the lower odd parity index increases by one. Note that, unlike in **§3.2.1** and **§3.2.2**, $\widetilde{\lambda}_i$ assumes a factor y precisely when $\widetilde{\lambda}_{i-1}$ is of the opposite parity for i between 2 and n, and $\widetilde{\lambda}_1$ has a factor y when $\widetilde{\lambda}_2$ and $\widetilde{\lambda}_1$ are of opposite parities and $\widetilde{\lambda}_2$ has factor y, in which case λ_1 had a y to begin with. Otherwise $\widetilde{\lambda}_1$ may have either y or $(-)$. All other parts have minus factors.

Finally, $\widetilde{\lambda}$ is augmented by $\frac{1}{(xq)_\infty}$, which brings unrestricted partitions. Among the parts that are equal; we agree that the last occurrence (the one with the largest index) could have come from the infinite sum, and the following occurrences come from the infinite product; or else all occurrences come from the infinite product. Note that exactly one among the same parts is subject to having factor s y or $(-)$.

Let $\beta = \beta_1 + \ldots + \beta_m$ be a partition given by the right hand side along with some power of y. By the remark in the above paragraph, it suffices to pick one instance of repeated parts in β. So, without loss of generality we may assume that β has distinct parts. It is immediate that the lower odd parity index of β is at least as large as the exponent of y. Unlike in **§3.2.1** and **§3.2.2**, the smallest part to receive y must be odd by construction.

We employ a slightly different sieve here. Whenever β is given along with some power of y, there are two cases where β is annihilated. First, given a distribution of

ys, for a β_i having factor y, there is β_{i+s} of the same parity without factor y, and $\beta_{i+1}\ldots,\beta i+s-1$ have no ys either. In this case, β_{i+s}s could have come from the second factor in the right hand side of (3.16), hence with a minus sign, or from the first factor with a plus sign. Or else, for the smallest β_i to receive a factor y, there is a smaller β_{i-s} of the opposite parity. In this case, β_{i-s} could have been the largest part $\widetilde{\lambda}_1$ with the minus factor as discussed above, or $\beta i-s$ could have been contributed by $\frac{1}{(xq)_\infty}$, hence having plus sign. In either option, β is annihilated.

It is evident that when the accompanying power of y has exponent strictly less than the lower odd parity index of β, we fall in one of the cases above, thus β is annihilated. Also, for that exponent equal to the lower odd parity index of β, there is a unique distribution of ys so as to avoid the former case above, which assigns a plus sign to β. In other words, β survives along with a power of y the exponent of which equals the lower even parity index, and when the ys are assigned to β_is for which β_{i+1} is of the opposite parity ($\beta_{m+1}=0$).

Example: We continue with the same $\beta=11+10+8+7+5+4+3$ along with y^3. For the following distribution of ys,

$$\beta=11+10+8+7+5_y+4_y+3_y$$

both 8 and 10 may be contributed as parts of $\widetilde{\lambda}$, or parts from $\frac{1}{(xq)_\infty}$. Hence both may have plus or minus factors. β as such is annihilated. Other distributions may be considered, but all will be generated with minus sign as many times as they are generated with plus signs.

For $\beta=11+10+8+7+5+4+3$ along with y^5 (observe that β has lower odd parity index 5), we consider the surviving distribution of ys in **§3.2.1** and **§3.2.2**

$$\beta=11_y+10_y+8+7_y+5+4_y+3_y$$

Here, 8 has no ys, and it immediately follows 10, which does have factor y. By the argument above, 8 could have been contributed as a part of $\widetilde{\lambda}$, hence having a minus factor, or by $\frac{1}{(xq)_\infty}$ with plus sign, so β as such is annihilated by the sieve in this section. Same applies for 5 following 7. In case 8 and 5 are parts of $\widetilde{\lambda}$, then we have no contribution at all from $\frac{1}{(xq)_\infty}$, so that

$$\widetilde{\lambda}=11_y+10_y+8_{(-)}+7_y+5_{(-)}+4_y+3_y$$

To get λ, we just remove the minus factors from the parts that have it, and add 1 to those, as described. Thus,

$$\lambda=11_y+10_y+9+7_y+6+4_y+3_y$$

However, if we take

$$\beta=11_y+10+8_y+7+5_y+4_y+3_y$$

we fall in neither case discussed, so neither 10 nor 7 could have been generated with minus signs. In this case, the parts 10 and 7 come from $\frac{1}{(xq)_\infty}$.

3.3 Partitions with Ample Part Size

We recall another definition from [6].

Definition 3.3. A partition λ is said to have *even (resp. odd) ample part size* if each part λ_i is larger than the upper even (resp. odd) parity index of λ.

We will argue that

$$\varrho_e(y, x; q) = \sum_{n \geq 0} \frac{x^n q^{n^2} (-yq; q)_n}{(q^2; q^2)_n} \tag{3.17}$$

generates partitions into distinct parts with even ample part size, where the exponent of x keeps track of the number of parts, and the exponent of y the upper even parity index.

Now, $\frac{x^n q^{n^2}}{(q^2;q^2)_n}$ generates partitions $\lambda = \lambda_1 + \ldots + \lambda_n$ with distinct odd parts. Originally, the upper even index is zero, there is nothing to prove. For each factor $(1 + yq^i)$, $1 \leq i \leq n$, we add 1 to parts $\lambda_n, \lambda_{n-1}, \ldots, \lambda_{n-(i-1)}$, so that the parts of λ are still distinct, but there is a parity change $\lambda_i \not\equiv \lambda_{i-1} (\text{mod } 2)$. Observe that if λ_{i_0} is the largest part to be altered, then it becomes even. So that the exponent of y indeed counts the upper even parity index. Also, each parity change or each introduction of a factor y adds 1 to $\lambda_n \geq 1$. Therefore, $\widetilde{\lambda}_n$, the transformed λ_n is always greater than the upper even parity index, hence so are the other parts.

Conversely, given a partition into distinct parts with even ample part size, we can subtract 1s from so many smallest parts from the partition $\widetilde{\lambda}$ for each parity change, so as to get λ, a partition into distinct odd numbers. At each step, we recover a yq^i. The procedure is clearly reversible.

Example: In **§3.1.1**, $\widetilde{\lambda} = 15 + 13 + 12 + 8 + 5 + 4$ has upper even parity index 3, and all parts are greater than 3, so that $\widetilde{\lambda}$ has ample part size. The picture is

$$\widetilde{\lambda} = 15 + 13 + \underbrace{12 + 8}_{\text{1 added}} + \underbrace{5}_{\text{2 added}} + \underbrace{4}_{\text{3 added}}$$

so that

$$\lambda = 15 + 13 + 11 + 7 + 3 + 1$$

a partition into distinct odd parts, and $\mu' = 3 + 2 + 1 + 1$ (hence $\mu = 4 + 2 + 1$), as claimed.

Note that upper even parity index is even when the smallest part is odd, and it is odd when the smallest part is even. This shows that the inequality is strict in the definition. This fact is communicated by Andrews to the author, but it is not explicitly included in [6].

In the above argument, if we replace $\frac{x^n q^{n^2}}{(q^2;q^2)_n}$ by $\begin{bmatrix} N \\ n \end{bmatrix}_2 q^{n^2} x^n$, then we get $\lambda = \lambda_1 + \ldots + \lambda_n$ as a partition into distinct odd parts $\leq 2N-1$. So that $\widetilde{\lambda}_1 \leq 2N$, since $\lambda_1 \leq 2N-1$, and it may be added at most 1, coming from the factor $(1 + yq^n)$. This shows us that

$$\rho_e(N, y, x; q) = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_2 q^{n^2} x^n (-yq; q)_n \tag{3.18}$$

generates the partitions into distinct parts with even ample part size, each at most $2N$, where the exponent of x keeps track of the number of parts, and the exponent of y the upper even parity index. This is [6, eq.(11.2)].

If we begin with partitions into distinct even parts, and interpret the factor $(-yq; q)_n$ as above, we get the generating function for partitions into distinct parts with odd ample part size, and its variant where parts are bounded. Here there are instances where the equality can hold in the definition, so that we emphasize that the inequality is strict. For instance, the only partition of 1 is not counted as having odd ample part size. As argued in [6], this is equivalent to replacing x by xq in the above generating functions.

Thus,

$$\varrho_o(y, x; q) = \sum_{n \geq 0} \frac{x^n q^{n(n+1)} (-yq; q)_n}{(q^2; q^2)_n} \tag{3.19}$$

generates partitions into distinct parts with odd ample part size, and

$$\rho_o(N, y, x; q) = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_2 q^{n(n+1)} x^n (-yq; q)_n \tag{3.20}$$

generates partitions into distinct parts $\leq 2N+1$ with odd ample part size. In both (3.19) and (3.20), the exponent of x keeps track of the number of parts, and the exponent of y the upper odd parity index.

3.4 Cluster Parity Indices

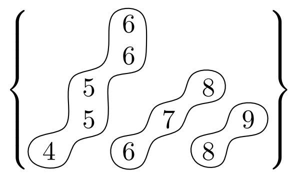
In this section, we reconcile ideas from **§2** and **§3**, and solve an open problem Andrews posed in [6]. We begin with a definition and related constructions.

Let a partition $\lambda = \lambda_1 + \ldots + \lambda_m$ along with its Gordon marking be given.

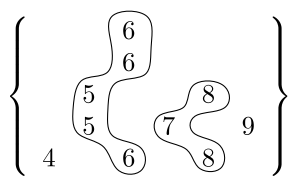
Definition 3.4. An *r-cluster* in $\lambda = \lambda_1 + \ldots + \lambda_m$ is a sub-partition $\lambda_{i_1} \leq \ldots \leq \lambda_{i_r}$ such that λ_{i_j} is j-marked for $j = 1, \ldots, r$, $\lambda_{i_{j+1}} - \lambda_{i_j} \leq 1$ for $j = 1, \ldots, r-1$, and there are no $(r+1)$-marked parts that are equal to λ_{i_r} or $\lambda_{i_r} + 1$.

Note that the absence of $(r+2)$-marked parts that are equal to λ_{i_r} or $\lambda_{i_r} + 1$ is not required.

Example: Below is an example where r-clusters of a Gordon marked partition is indicated. There is a 5-cluster that consists of parts $4, 5, 5, 6, 6$, a 3-cluster consisting of parts $6, 7, 8$, and a 2-cluster consisting of parts $8, 9$.



There is an obvious seemingly-alternative definition of an r-cluster. We would take an r-marked part λ_i where there are no $(r+1)$-marked parts that are equal to λ_i or $\lambda_i + 1$. Then, we associate to that r-marked part λ_i an $(r-1)$-marked part, an $(r-2)$-marked part, ..., a 1-marked part, all equal to λ_i or $\lambda_i - 1$. Observe that this definition is more intuitive as it reflects the idea of Gordon marking better. When we try to apply this definition to the above example, however, we would have



where the 2-marked 9 is dangling, or it defines a 2-cluster which overlaps with the 3-cluster that is defined by the 3-marked 8. This is not a desirable situation, and **Definition 3.4** leads to much nicer consequences, as the subsequent results show.

Proposition 3.5. Any partition $\lambda = \lambda_1 + \ldots + \lambda_m$ along with its Gordon marking has a unique decomposition into non-overlapping r-clusters.

Proof. It is evident that $\lambda = \lambda_1 + \ldots + \lambda_m$ is enumerated by $b_{k,k}(m, n)$ for some k. We will construct n_r r-clusters for $r = k-1, k-2, \ldots, 1$ in the given decreasing order. Here,

$$n_r = N_r - N_{r+1}$$

and N_rs are the number of r-marked parts in the Gordon marking of λ, as described in **Definition 2.4**.

Let

$$\lambda_1^{(r)} \leq \ldots \leq \lambda_{n_r}^{(r)}$$

be the r-marked parts that do not belong to any previously constructed r_1-cluster for $r_1 > r$. Then, there is no $(r+1)$-marked $\lambda_j^{(r+1)}$ that is equal to $\lambda_j^{(r)}$ or $\lambda_j^{(r)} + 1$ for $j = 1, \ldots, n_r$, by the construction of the previous clusters. Also,

$$\lambda_j^{(r)} - \lambda_i^{(r)} \geq 2$$

for $1 \leq i < j \leq n_r$, by the Gordon marking of λ.

Now, there are exactly n_r $(r-1)$-marked

$$\lambda_1^{(r-1)} \leq \ldots \leq \lambda_{n_r}^{(r-1)}$$

such that

$$\lambda_j^{(r-1)} = \lambda_j^{(r)} \quad \text{or} \quad \lambda_j^{(r-1)} = \lambda_j^{(r)} - 1$$

for $j = 1, \ldots, n_r$, again by the Gordon marking of λ. These $\lambda_j^{(r-1)}$s are the $(r-1)$-marked parts in the r-clusters that contain $\lambda_j^{(r)}$s for $j = 1, \ldots, n_r$. Then, given indices i, j such that for i, j, $1 \leq i < j \leq n_r$,

$$0 \leq \lambda_j^{(r)} - \lambda_j^{(r-1)} \leq 1 \quad \text{and} \quad 0 \leq \lambda_i^{(r)} - \lambda_i^{(r-1)} \leq 1$$

It follows that

$$-1 \leq (\lambda_j^{(r)} - \lambda_i^{(r)}) - (\lambda_j^{(r-1)} - \lambda_i^{(r-1)}) \leq 1$$

Since the difference in the first parenthesis is at least 2, the difference in the second parenthesis must be at least 1. So the $\lambda_j^{(r-1)}$s are distinct for $j = 1, \ldots, n_r$. Therefore,

$$\lambda_j^{(r-1)} - \lambda_i^{(r-1)} \geq 2$$

by the Gordon marking of λ. Moreover, $\lambda_j^{(r-1)}$s do not belong to any r_1 cluster for $r_1 > r$, $j = 1, \ldots, n_r$. Because if they did, this would force some $\lambda_j^{(r)}$s also to be contained in those r_1-clusters, contradicting the choice of $\lambda_j^{(r)}$s, $j = 1, \ldots, n_r$.

We repeat the procedure to find the s-marked parts in the n_r r-clusters that are being constructed for $s = r-2, r-3, \ldots, 1$. At each step, by the above arguments, we have $\lambda_1^{(s)} \leq \ldots \leq \lambda_{n_r}^{(s)}$, which are s-marked parts that are pairwise at least two apart, and which do not belong to any r_1 cluster for $r_1 > r$. This gives us n_r r-clusters which do not overlap with the already constructed r_1-clusters, $r_1 > r$.

For each $r = k-1, k-2, \ldots, 1$, the construction makes unique choices of parts, given the uniqueness of the previously constructed clusters. For $r = k-1$, there are only n_{k-1} $(k-1)$-marked parts in λ to begin with, so uniqueness of the whole decomposition follows.

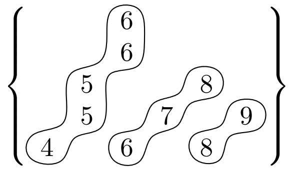
Finally, we have used rn_r parts for r-clusters, $r = k-1, k-2, \ldots, 1$, which are pairwise non-overlapping. Therefore, all $N_1 + \ldots + N_{k-1} = n_1 + 2n_2 + \ldots + (k-1)n_{k-1}$ parts eventually belong to an r-cluster for some $r = k-1, k-2, \ldots, 1$. □

The partition λ from the previous example has a unique such decomposition.

We need a few key definitions and a related auxiliary result before the main theorem of this section (**Theorem 3.10**).

Definition 3.6. *Parity of an r-cluster* is the opposite parity of the number of even parts in that r-cluster.

Example: We revisit



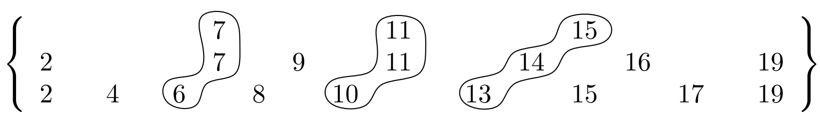
Here, the 5-cluster containing the 5-marked 6 is an even cluster, since there are an odd number of even parts in it, namely 4, 6, 6. The 3-cluster containing the 3-marked 8 is an odd cluster, and the 2-cluster containing the 2-marked 9 is an even cluster.

The reason for taking the opposite parity is that 1-clusters are just numbers, so we should keep the definition consistent with the parity of an integer. The reason why we look at the count of even parts instead of odd parts will become clear in the course of the proof of the main theorem of this section (**Theorem 3.10**).

Definition 3.7. *Lower even r-cluster parity index* of a partition λ is the number of times that the r-cluster parity changes from the r-cluster with the smallest r-marked part to the one with the largest r-marked part, beginning with an even r-cluster parity.

Definition 3.8. Given a partition $\lambda = \lambda_1 + \ldots + \lambda_m$ enumerated by $b_{k,k}(m, n)$, the *full lower even cluster parity index* of λ is the sum of all lower even 1-, 2-, …, $(k-1)$-cluster parity indices.

Example: Below, only the 3-clusters of a Gordon marked partition is shown.

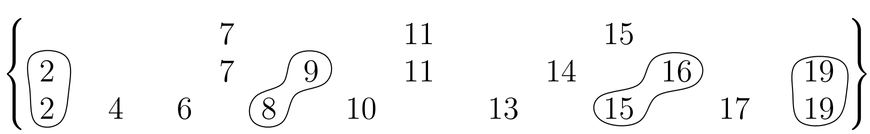


The 3-cluster containing the 3-marked 7 is an even cluster, the one with the 3-marked 11 is also an even cluster, and so is the one with the 3-marked 15. Therefore the 3-cluster parities written in increasing order of the 3-marked parts are

$$EEE$$

Beginning with an even 3-cluster parity, there is no parity change, so the lower even 3-cluster parity index of this partition is 1.

Now, the 2-clusters in the same Gordon marked partition is shown.



With similar reasoning as above, the 2-cluster parities written in increasing order of 2-marked parts are

$$OEEO$$

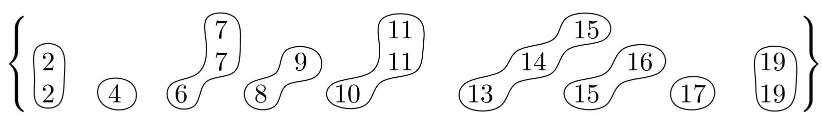
There is one parity change after an even 2-cluster parity to start with, so the lower even 2-cluster parity index of this partition is 2.

Next is the same Gordon marked partition where the 1-clusters are shown.

$$\left\{ \begin{array}{cccccccccccccc} & & & 7 & & & & 11 & & & 15 & & & \\ 2 & & & 7 & & 9 & & 11 & & 14 & & 16 & & 19 \\ 2 & \textcircled{4} & 6 & & 8 & & 10 & & 13 & & 15 & & \textcircled{17} & 19 \end{array} \right\}$$

It is easy to see that the lower even 1-cluster parity index is 2.

Finally, all clusters of the Gordon marked partition in this example is shown.



The full lower even cluster parity index of this partition is hence $5 = 2 + 2 + 1$.

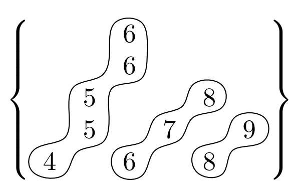
Proposition 3.9. Let $\lambda = \lambda_1 + \ldots + \lambda_m$ be a given partition along with its Gordon marking.

(i) If conditions exist for a single backward or forward move of rth kind on an r-marked λ_i, then λ_i is the r-marked part in an r-cluster. Moreover, that move made will commute the r-cluster parity of the r-cluster containing λ_i (possibly altering the r-cluster parity index), and will keep the other parity indices fixed.

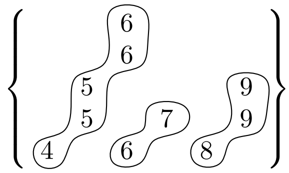
(ii) If conditions exist for a double backward or forward move of rth kind on an r-marked λ_i, then that double move made will fix all cluster parity indices in λ.

Before the proof is presented, we consider two examples.

Example: Note that in

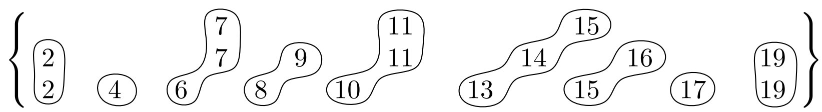


conditions exist for a forward move of 3rd kind on the 3-marked 8. After the move is performed, the partition is transformed into

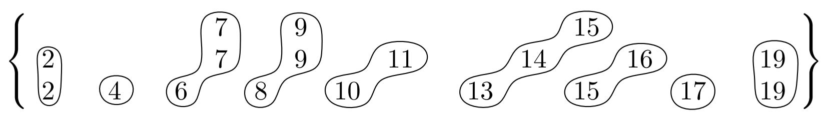


The 3-cluster is re-formed, and its parity changed from odd to even. The 5-cluster did not change at all; and although the 2-cluster is also re-formed, its parity is not altered.

Example: In



conditions exist for a double backward move of 3rd kind on the 3-marked 11. After the move is made, the partition becomes



Observe that all cluster parities are preserved, as well as the respective orders of 1-, 2-, and 3-cluster parities. That is, all lower even cluster parity indices are fixed, and so is the full lower even cluster parity index.

proof of **Proposition 3.9**. Throughout the proof, (c), (d1), and (d2) will refer to the conditions listed in **Definition 2.6**.

(i) We will prove this for a single forward move of rth kind. The full result will follow by **Proposition 2.7**, since backward and forward moves of the same kind on the same part are inverses to each other.

When (c) holds, then there are no $(r+1)$-marked parts that are equal to λ_i or $\lambda_i + 1$. Thus, λ_i does not belong to any r_1-cluster for $r_1 > r$, so it defines an r-cluster.

If (d1) held along with (c), then for some r_0, $r_0 < r$, there is an r_0-marked λ_{i_0} which is equal to $\lambda_i - 1$, and there are (r_0+1)-, (r_0+2)-, …, r-marked parts that are equal to λ_i. All of the mentioned (r_0+1)-, (r_0+2)-, …, r-marked parts together with r_0-marked λ_{i_0} belong to the r-cluster that contains the r-marked λ_i. The forward move of rth kind on the r-marked λ_i will replace the r_0-marked λ_{i_0} that equals $\lambda_i - 1$ by an r_0-marked $\lambda_{i_0}+1$ that then will equal λ_i, and will still belong to the r-cluster that contains the r-marked λ_i along with the previously mentioned (r_0+1)-, (r_0+2)-, …, r-marked parts that are equal to λ_i.

There are two cases to consider here:

case (i) There is an $(r_0 - 1)$-marked λ_j that is equal to λ_{i_0}

In this case, before the move, $\lambda_{i_0} - \lambda_j = 0$, and after the move $(\lambda_{i_0} + 1) - \lambda_j = 1$. Thus, the $(r_0 - 1)$-marked λ_j is in the r-cluster containing the r-marked λ_i either before or after the move. Since no $(r_0 - 1)$ or lower marked part is altered by the forward move, the r-cluster parity of the r-cluster containing the r-marked λ_i is altered while all other cluster parities remain unchanged. The proof follows for this case.

case (ii) There is an $(r_0 - 1)$-marked λ_j that is equal to $\lambda_{i_0} - 1$

In this case, there is also an $(r_0 - 1)$-marked λ_{j_1} that is equal to $\lambda_{i_0} + 1$ (and hence to $\lambda_j + 2$) by the Gordon marking of λ. Again, by the Gordon marking, (c), and (d1), there are no r_0 or higher marked parts that are equal to $\lambda_{i_0} + 1$ or $\lambda_{i_0} + 2$. In particular, there are no r_0-marked parts that are equal to λ_{j_1} or $\lambda_{j_1} + 1$. Thus, λ_{j_1} is the $(r_0 - 1)$-marked part in an $(r_0 - 1)$-cluster before the forward move is performed. Please note that the forward move replaces λ_{i_0} by $\lambda_{i_0} + 1$. Therefore, after the move is performed, the $(r_0 - 1)$-cluster that contains λ_{j_1} will become part of the r-cluster containing the r-marked λ_i, and another $(r_0 - 1)$-cluster will emerge, this time containing the $(r_0 - 1)$-marked λ_j.

To conclude the proof in this case, we need to verify that the $(r_0 - 1)$-clusters containing λ_j and λ_{j_1} possess the same number of even and odd parts, once forgetting about the r_0- and higher marked parts. Call λ_j and λ_{j_1} $\lambda_j^{(r_0-1)}$ and $\lambda_{j_1}^{(r_0-1)}$, respectively, where the superscript indicates the mark in the Gordon marking of λ. Call the s-marked parts in the $(r_0 - 1)$-cluster that contains the $(r_0 - 1)$-marked $\lambda_j^{(r_0-1)}$ $\lambda_j^{(s)}$, and the s-marked parts in the $(r_0 - 1)$-cluster that contains the $(r_0 - 1)$-marked $\lambda_{j_1}^{(r_0-1)}$ $\lambda_{j_1}^{(s)}$, $s < r_0 - 1$.

We inductively show that $\lambda_{j_1}^{(s)} = \lambda_j^{(s)} + 2$, $s \leq r_0 - 1$ We already know that $\lambda_{j_1} = \lambda_j + 2$, and that there is an r_0-marked part that is equal to $\lambda_j + 1$ in the original Gordon marking of λ, i.e. before the move is performed. This forms the base case of our induction on $s = r_0 - 1, r_0 - 2, \ldots, 1$.

For $s < r_0 - 1$, we assume that $\lambda_{j_1}^{(s+1)} = \lambda_j^{(s+1)} + 2$ and there is an $(s+1)$-marked part that is equal to $\lambda_j^{(s+1)} + 1$. By the construction of clusters, and by the Gordon marking of λ, $\lambda_{j_1}^{(s)} = \lambda_{j_1}^{(s+1)} - 1$, or $\lambda_{j_1}^{(s)} = \lambda_{j_1}^{(s+1)}$.

If $\lambda_{j_1}^{(s)} = \lambda_{j_1}^{(s+1)} - 1$, then it cannot be the case that $\lambda_j^{(s)} = \lambda_j^{(s+1)}$ since this would imply that $\lambda_{j_1}^{(s)} = \lambda_j^{(s)} + 1$, which violates the Gordon marking of λ. Thus, $\lambda_j^{(s)} = \lambda_j^{(s+1)} - 1$ also, and hence $\lambda_{j_1}^{(s)} = \lambda_j^{(s)} + 2$.

Else if $\lambda_{j_1}^{(s)} = \lambda_{j_1}^{(s+1)}$, then it cannot be the case that $\lambda_j^{(s)} = \lambda_j^{(s+1)} - 1$.

This is because there is an $(s+1)$-marked part that is equal to $\lambda_j^{(s+1)}+1$, and when $\lambda_j^{(s)} = \lambda_j^{(s+1)} - 1$, the mark s is spared for a part that is equal to $\lambda_j^{(s+1)} + 1$. This violates the Gordon marking of λ. Thus, $\lambda_j^{(s)} = \lambda_j^{(s+1)}$, and consequently $\lambda_{j_1}^{(s)} = \lambda_j^{(s)} + 2$. The proof is complete in this case.

So far, we have assumed (c) and (d1) holds. Else if (d1) fails, and (d2) holds along with (c), then the r-marked λ_i is replaced by r-marked $\lambda_i + 1$, and the above argument carries over word by word with the only changes $r_0 \leftarrow r$ and (d1) $\leftarrow$ (d2).

(ii) This is iterated application of the previous part.

□

Now we are ready to prove the main result.

Theorem 3.10. The function

$$\sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\ldots+N_{k-1}^2} \quad x^{N_1+\ldots+N_{k-1}} \quad (-yq)_{n_1}\cdots(-yq)_{n_{k-1}}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{k-1}}} \tag{3.21}$$

generates partitions enumerated by $b_{k,k}(m,n)$, where the exponent of q is the number being partitioned (n), the exponent of x is the number of parts (m), and the exponent of y is the full lower even cluster parity index.

Proof. We proceed as in the proof of **Theorem 2.10**. Let a partition $\lambda = \lambda_1 + \ldots + \lambda_m$ be given along with its Gordon marking enumerated by $b_{k,k}(m,n)$, and a non-negative integral power of y the exponent of which equals the full lower even cluster parity index. We produce

nonnegative integers $n_1, \ldots, n_{k-1}$,

a base partition $\widetilde{\lambda}$ as described in the proof of **Theorem 2.10** (contribution from the factor $q^{N_1^2+\ldots+N_{k-1}^2+N_a+\ldots+N_{k-1}} \quad x^{N_1+\ldots+N_{k-1}}$),

a partition $\mu^{(r)}$ into at most n_r even parts (contribution from the factor $\frac{1}{(q^2;q^2)_{n_r}}$),

and a partition $\nu^{(r)}$ into at most n_r distinct parts $\leq n_r$ (contribution from the factor $(-yq)_{n_r}$) for each $r = 1, \ldots, k-1$. This will constitute the backward phase of the proof.

Conversely, given nonnegative integers $n_1, \ldots, n_{k-1}$, and for each $r = 1, \ldots, k-1$ a partition $\mu^{(r)}$ into at most n_r even parts, and a partition $\nu^{(r)}$ into at most n_r distinct parts $\leq n_r$; we construct a partition $\lambda = \lambda_1 + \ldots + \lambda_m$ enumerated by $b_{k,k}(m,n)$ along with a nonnegative integral power of y the exponent of which equals the full lower even cluster parity index. This will be the forward phase of the proof.

Finally, we argue that the constructions are inverses to each other.

For the backward direction, let a partition $\lambda = \lambda_1 + \ldots + \lambda_m$ be given along with its Gordon marking enumerated by $b_{k,k}(m, n)$, and a non-negative integral power of y the exponent of which equals the full lower even cluster parity index.

Using **Definition 2.4**, we set $n_r = N_r - N_{r+1}$, where N_r is the number of r marked parts in λ. $\widetilde{\lambda}$ as described in the proof of **Theorem 2.10** is then constructed for $a = k$. That is, the r-marked parts in $\widetilde{\lambda}$ are

$$\widetilde{\lambda}^{(r)} = 1, 3, \ldots, 2N_r - 1$$

for $r = 1, \ldots, k-1$. Note that in $\widetilde{\lambda}$, all clusters are odd clusters (there are no even numbers at all), hence the full lower even cluster parity index is zero.

To transform λ into $\widetilde{\lambda}$, we consider each $r = k-1, k-2, \ldots, 1$ in the given decreasing order. At each step, we take the n_r-largest r-marked parts. By construction of $\widetilde{\lambda}$, all $(r+1)$-marked parts are strictly less than the n_r largest r-marked parts by the time we consider any particular r. Consequently, by **Definition 3.4**, those n_r r-marked parts are also the r-marked parts in the r-clusters.

We perform so many double backward moves of rth kind on the ith largest r-marked part for $i = n_r, n_r - 1, \ldots, 1$ in the given decreasing order until no more double backward moves are possible. This will give us $\mu^{(r)}$, partitions into even parts, as in proofs of **Theorems 2.11, 2.12**, and **2.14**. The lower even r-cluster parity index remain fixed by **Proposition 3.9 (ii)**.

Let λ_i be the smallest r-marked part that belongs to an even r-cluster. It is the jth largest r-marked part for some $j \leq n_r$. By the construction so far, the parts smaller than the r-marked λ_i are all odd, and λ_i is even. To justify this claim, please note that by the hypothesis, we have obtained the $(r+1), (r+2), \ldots, (k-1)$-clusters in $\widetilde{\lambda}$ in their respective places, which consist of all odd parts. Then, if there are r-clusters with smaller r-marked parts than λ_i, they are odd clusters. Hence, the number of even parts are even. Let λ_{i_0} be the smallest r-marked part $\lambda_{i_0} < \lambda_i$ such that there are at least two even parts in the r-cluster defined by λ_{i_0}. Then, a double backward move of rth kind is possible on λ_{i_0}, contradicting the construction of $\mu^{(r)}$.

Now, since λ_i is even, the smaller r-marked part (if any) λ_{i_0} is odd, then (a) and (b) in **Definition 2.5** are satisfied ((b) for $r_0 = r$), so a single backward move of rth kind is possible on r-marked λ_i so as to move λ_i into its respective place, as the jth largest r-marked part in $\widetilde{\lambda}$. By **Proposition 2.8 (i)**, then single backward moves of rth kind become possible for each of $(j-1)$th, $(j-2)$th, ..., 1st largest r-marked parts. This gives us exactly j moves.

Note that, λ_i is the smallest r-marked part to belong in an even r-cluster, and that even r-cluster becomes an odd one after the single backward move of rth kind is performed. Moreover, all r-cluster parities of r-clusters with larger r-marked parts than λ_i are commuted, which means that the lower even r-cluster parity index is decreased by one, hence so is the full lower even cluster parity index. Therefore, we let the j

single backward moves of rth kind be accompanied by a factor of y.

We repeat the procedure of finding the smallest r-marked part that belong to an even r-cluster until there are none, and keep track of the single backward moves of rth kind we perform to construct $\widetilde{\lambda}$. Eventually, the single moves that become possible make a partition into distinct parts $\leq n_r$ which is accompanied by a power of y whose exponent counts the lower even r-cluster parity index. That partition is $\nu^{(r)}$, as described above. This concludes the backward phase of the construction, and hence the first part of the proof.

Let us go over an example on the fly. Below is a Gordon marked partition from a preceding example in this section, where all 3-clusters are indicated. Note that the construction takes into account the r-clusters in decreasing order of $r = k-1, k-2, \ldots, 1$, and $k = a = 4$ here.

$$\left\{ \begin{array}{cccccccccccccc} & & & 7 & & & & 11 & & & 15 & & & \\ 2 & & & 7 & & 9 & & 11 & & 14 & & 16 & & 19 \\ 2 & 4 & 6 & & 8 & & 10 & & 13 & & 15 & & 17 & 19 \end{array} \right\}$$

We first perform 5 double backward moves on the 3-marked 7, 6 double backward moves on the 3-marked 11, and 8 double backward moves on the 3-marked 15. This will give us

$$\left\{ \begin{array}{cccccccccccccc} & 2 & & 4 & & 6 & & & & & & & & \\ 1 & & 3 & & 5 & & 8 & & & 13 & & 16 & & 19 \\ 1 & & 3 & & 5 & & 8 & 10 & 12 & & 15 & & 17 & 19 \end{array} \right\}$$

recovering $\mu^{(3)} = 16, 12, 10$, since the mentioned backward moves are double moves. Please note that no further double backward move of 3rd kind is possible on any of the 3-marked parts.

At this stage, we observe that a single backward move of 3rd kind is possible on the 3-marked 2, and hence two more single backward moves on the larger 3-marked parts. This will account for $\nu^{(3)} = 3$ along with a factor y, which contributed as the lower even 3-cluster parity index 1. We now have

$$\left\{ \begin{array}{ccccccccccc} 1 & 3 & 5 & & & & & & & & \\ 1 & 3 & 5 & 8 & & & 13 & & 16 & & 19 \\ 1 & 3 & 5 & 8 & 10 & 12 & & 15 & & 17 & 19 \end{array} \right\}$$

Next, we look at the 2-clusters

$$\left\{ \begin{array}{ccccccccccc} 1 & 3 & 5 & & & & & & & & \\ 1 & 3 & 5 & 8 & & & 13 & & 16 & & 19 \\ 1 & 3 & 5 & 8 & 10 & 12 & & 15 & & 17 & 19 \end{array} \right\}$$

1 double backward move of the 2nd kind on the 2-marked 8 is possible, after which 2 double backward moves on the 2-marked 13, then 3 double backward moves on the 2-marked 16, and lastly, 3 double backward moves on the 2-marked 19. This brings

$$\left\{\begin{array}{ccccccccccc} 1 & 3 & 5 & & & & & & & & \\ 1 & 3 & 5 & 7 & & 10 & & 12 & 14 & & \\ 1 & 3 & 5 & 7 & 9 & & 11 & & 14 & 16 & 19 \end{array}\right\}$$

No further double backward moves of 2nd kind are possible, so $\mu^{(2)} = 6, 6, 4, 2$. The lower even 2-cluster parity index is still 2.

A single backward move of 2nd kind is now possible on the 2-marked 10, enabling two more single backward moves on 2-marked 12, and 14. This accounts for 3 single moves, along with a factor y.

$$\left\{\begin{array}{cccccccccc} 1 & 3 & 5 & & & & & & & \\ 1 & 3 & 5 & 7 & 9 & 11 & & 14 & & \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & & 16 & 19 \end{array}\right\}$$

The $r = 2$ stage is not over, because we still need to take care of one more 2-cluster parity change. A single backward move of 2nd kind is now possible on the 2-marked 14 with no further single backward moves. This accounts for the second factor y for the lower even 2-cluster parity index 2, and yields $\nu^{(2)} = 3, 1$.

$$\left\{\begin{array}{ccccccccc} 1 & 3 & 5 & & & & & & \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & & \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 16 & 19 \end{array}\right\}$$

Lastly, we consider $r = 1$

$$\left\{\begin{array}{ccccccccc} 1 & 3 & 5 & & & & & & \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & & \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 16 & 19 \end{array}\right\}$$

No double backward moves of 1st kind is possible on any of the 2 largest 1-marked parts. In other words, we cannot subtract 2 from either. Therefore, $\mu^{(1)}$ is the empty partition.

It is easy to see that the lower even 1-cluster parity index is 2. A single backward move of 1st kind on the 1-marked 16 will enable a single backward move on the 1-marked 19. It accounts for a factor y as well.

$$\left\{\begin{array}{ccccccccc} 1 & 3 & 5 & & & & & & \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & & \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 18 \end{array}\right\}$$

Then, another single backward move of 1st kind is possible on the 1-marked 18, along with another factor y. This yields the base partition $\widetilde{\lambda}$, and $\nu^{(1)} = 2, 1$.

$$\widetilde{\lambda} = \left\{ \begin{array}{ccccccccc} 1 & 3 & 5 & & & & & & \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & & \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 \end{array} \right\}$$

We have accounted for 5 y factors, since the full lower even cluster parity index of the original partition is 5, and $\widetilde{\lambda}$ has all odd clusters, therefore its full lower even cluster parity index is zero.

At this point, it becomes clear why the definition of cluster parity counts the even parts in a cluster. We want the base partition $\widetilde{\lambda}$ to have full lower even cluster parity index zero. To ensure this, clusters consisting of a single file of odd numbers must have odd cluster parity.

For the forward phase of the proof, given integers $n_1, \ldots, n_{k-1}$, we first set $N_r = n_r + n_{r+1} + \ldots + r_{k-1}$ for $r = 1, \ldots, k-1$. Then, we construct $\widetilde{\lambda}$ as in the proof of **Theorem 2.10** for $a = k$. The r-marked parts in $\widetilde{\lambda}$ are

$$\widetilde{\lambda}^{(r)} = 1, 3, \ldots, 2N_r - 1$$

for $r = 1, \ldots, k-1$.

In the procedure that follows, we consider $r = 1, \ldots, k-1$ in the given increasing order.

For a particular r, we take the n_r largest r-marked parts, which are in their original places in $\widetilde{\lambda}$. Thus, they are the r-marked parts in the n_r odd r-clusters. The initial lower even r-cluster parity index is zero. For the largest r-marked part, (c) and (d1) in **Definition 2.6** are satisfied, so we can always perform a single forward move of rth kind on it.

Next, we realize $\nu^{(r)}$, accompanied with a power of y the exponent of which equals the number of parts in $\nu^{(r)}$. For each part j of $\nu^{(r)}$ from largest to smallest, we perform single forward moves of rth kind on the j largest r-marked parts. The forward move on the largest one is possible as argued in the preceding paragraph. The subsequent singe forward moves are possible by **Proposition 2.8 (ii)**.

After all parts of $\nu^{(r)}$ are realized as sequences of single forward moves of rth kind, note that we have exactly as many r-cluster parity changes counted from the r-cluster with the smallest r-marked part to the one with the largest as there are parts in $\nu^{(r)}$ by **Proposition 3.9 (i)**. The first parity change is to an even r-cluster, since all r-clusters were originally odd. Therefore, the exponent of the accompanying power of y now counts the lower even r-cluster parity index.

We realize $\mu^{(r)}$s as double forward moves of rth kind, as in proof of **Theorems 2.11, 2.12**, and **2.14**. The lower even r-cluster parity index remain fixed by **Proposition 3.9 (ii)**. This way, λ is constructed.

Finally, by **Definition 3.8**, the power of y accompanying the constructed λ has exponent equal to full lower even cluster parity index of λ. This establishes the forward phase of the construction, and hence the second part of the proof.

We see that the two constructions above consist of moves performed in the exact reverse order, thus they are inverses to each other, since backward and forward moves are, by **Proposition 2.7**. □

We give another example here, and work in the forward direction as described in the preceding proof. Let $n_1 = 0$, $n_2 = 4$, $n_3 = 0$, and $n_4 = 3$. Let

$$\mu^{(2)} = 2, 2 \quad \mu^{(4)} = 16, 10, 6$$

$$\nu^{(2)} = 4, 3, 1 \quad \nu^{(4)} = 2, 1$$

$\mu^{(1)}$, $\mu^{(3)}$, $\nu^{(1)}$, and $\nu^{(3)}$ are essentially empty partitions. Since $\nu^{(2)}$ and $\nu^{(4)}$ have three and two parts, respectively; the resulting partition λ will have lower even 2-cluster parity index 3, and lower even 4-cluster parity index 2. The full lower even parity index of λ will therefore be $5 = 3 + 0 + 2 + 0$.

We first construct the base partition $\widetilde{\lambda}$, as described in the proof

$$\widetilde{\lambda} = \left\{ \begin{array}{ccccccc} 1 & 3 & 5 & & & & \\ 1 & 3 & 5 & & & & \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 \end{array} \right\}$$

$\widetilde{\lambda}$ has all odd clusters, hence the full lower even cluster parity index is zero.

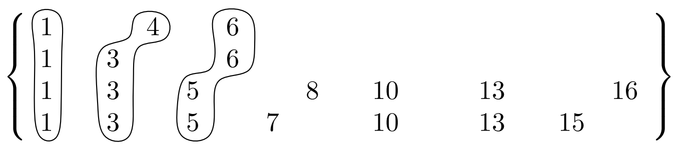
Since $n_1 = n_3 = 0$, we will skip the procedure for $r = 1$ and $r = 3$. We thus start with $r = 2$ and realize $\nu^{(2)} = 4, 3, 1$. This will introduce lower even 2-cluster parity index 3. We first apply a single forward move of 2nd kind on the largest 2-marked part 13. Then, single forward moves on the 3 largest 2-marked parts, and single forward moves on the 4 largest 2-marked parts follow.

$$\left\{ \begin{array}{ccccccccc} 1 & 3 & 5 & & & & & & \\ 1 & 3 & 5 & & & & & & \\ 1 & 3 & 5 & & 8 & 10 & 12 & & 15 \\ 1 & 3 & 5 & 7 & & 10 & 12 & 14 & \end{array} \right\}$$

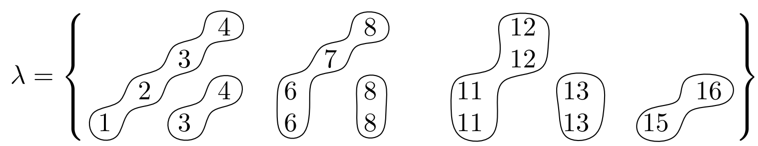
The 2-clusters are indicated, and their parities listed in increasing order of the 2-marked parts in them are $EOOE$. The lower even 2-cluster parity index is 3, as $\nu^{(2)}$ is accompanied with y^3. Then we realize $\mu^{(2)}$ as described in proof of **Theorem 2.12**

$$\left\{ \begin{array}{ccccccccc} 1 & 3 & 5 & & & & & & \\ 1 & 3 & 5 & & & & & & \\ 1 & 3 & 5 & & 8 & 10 & 13 & & 16 \\ 1 & 3 & 5 & 7 & & 10 & 13 & 15 & \end{array} \right\}$$

Next, for $r = 4$, we first take into account $\nu^{(4)} = 2, 1$ as a single forward move of 4th kind on the largest 4-marked part 5, followed by single forward moves on the 2 largest 4-marked parts. Indicating the 4-clusters, we have



Here, the lower even 4-cluster parity index is 2, since $\nu^{(4)}$ is accompanied by y^2. Then, we realize $\mu^{(4)}$, and indicate all clusters in the resulting partition λ



In this final picture, the full lower even cluster parity index of λ is 5, as expected.

It should be remarked that the proof of **Theorem 3.10** just with appropriate notational changes applies to prove a slightly more general result that

$$\sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\ldots+N_{k-1}^2} \quad x^{N_1+\ldots+N_{k-1}} \quad (-y_1q)_{n_1}\cdots(-y_{k-1}q)_{n_{k-1}}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{k-1}}} \tag{3.22}$$

generates partitions enumerated by $b_{k,k}(m,n)$, where the exponent of q is the number being partitioned (n), the exponent of x is the number of parts (m), and the exponent of y_j is the lower even j-cluster parity index for $j = 1, \ldots, k-1$. Upon setting $y_1 = \ldots = y_{k-1} = y$, **Theorem 3.10** follows.

For $k = 2$, **Theorem 3.10** reduces to the statement that

$$\sum_{n\geq 0} \frac{q^{n^2}x^n(-yq)_n}{(q^2;q^2)_n} \tag{3.23}$$

generates partitions with no same or consecutive parts, where the exponent of q is the number being partitioned, the exponent of x is the number of parts, and the exponent of y is the lower even parity index.

This interpretation is different from even ample part size ([6, §11]) as given in **§3.3**, and it will be convenient to rework the example given for even ample part size in this context. We have $k = 2$, $n = n_1 = 6$, $\mu = 4+4+4+2$, and $\nu = \nu^{(1)} = 4+2+1$. Then, the base partition will be

$$\widetilde{\lambda} = \{\ 1 \quad 3 \quad 5 \quad 7 \quad 9 \quad 11\ \}$$

which becomes

$$\{\ 1 \quad 3 \qquad 6 \quad 8 \qquad 11 \qquad 14\ \}$$

after the realization of ν, and

$$\lambda = \{\ 1 \quad 3 \qquad\qquad 8 \qquad\quad 12 \qquad 15 \qquad 18\ \}$$

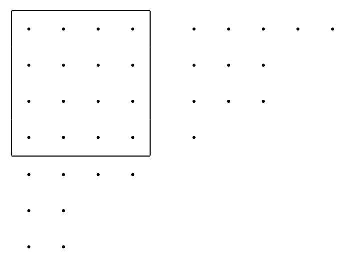
after the realization of μ as described in the proof of **Theorem 3.10**, with lower even parity index 3, but not having even ample part size; as opposed to the partition

$$15 + 13 + 12 + 8 + 5 + 4$$

which is not enumerated by $b_{2,2}(6, 57)$, but has even ample part size.

4. k-marked Durfee Symbols

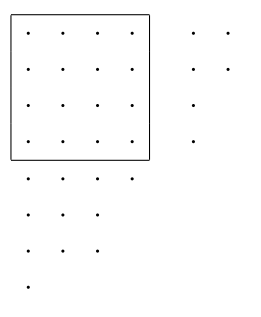
A pictorial representation for a partition is its Ferrers graph, a left indented table of dots such that the first row has λ_1 dots and so on. For instance, $36 = 9+7+7+5+4+2+2$ has Ferrers graph



The largest square that can be fit in the upper left corner of a Ferrers graph is called the Durfee square. So the partition above has a Durfee square of size 4 .

The *rank* of a partition is defined as the largest part minus the number of parts in a partition [10]. The rank of the above partition is $+2$.

The conjugate λ' of a partition λ is obtained by reflecting the Ferrers graph across the main diagonal. For the partition above, the conjugate is $7+7+5+5+4+3+3+1+1$ and has Ferrers graph



4.1 Definitions

Using the Ferrers graph of a particular partition, we follow Andrews in [4] and form another representation of the same partition, the Durfee symbol. It is a two-row array, with a subscript indicating the size of the Durfee square. The top row is obtained by reading the conjugate partition of the smaller partition to the right of the Durfee square (recording the columns instead of rows), and the bottom row by reading the smaller partition below the Durfee square. Thus, the partition $36 = 9+7+7+5+4+2+2$ has Durfee symbol

$$\begin{pmatrix} 4 & 3 & 3 & 1 & 1 \\ 4 & 2 & 2 & & \end{pmatrix}_4$$

Using the Durfee symbol, it is simple to find the rank; as it is the excess of the number of parts in the top row over the number of parts in the bottom row. Notice also that in terms of the Durfee symbol, taking conjugates corresponds to interchanging the top and bottom row.

Definition 4.1. Let $\mathcal{S}_d(m_1, \ldots, m_d)$ denote the collection of Durfee Symbols with Durfee square side d where the total number of appearances of j in the listed Durfee symbols is exactly m_j for $j = 1, \ldots, d$. This is a subset of partitions of $n = d^2 + \sum_{j=1}^{d} jm_j$.

example: $\mathcal{S}_2(2,3) = \left\{ \begin{pmatrix} 222 & 11 \\ & \end{pmatrix}_2, \begin{pmatrix} 222 & 1 \\ & 1 \end{pmatrix}_2, \begin{pmatrix} 222 & \\ & 11 \end{pmatrix}_2, \begin{pmatrix} 22 & 11 \\ 2 & \end{pmatrix}_2, \right.$
$\begin{pmatrix} 22 & 1 \\ 2 & 1 \end{pmatrix}_2, \begin{pmatrix} 22 & \\ 2 & 11 \end{pmatrix}_2, \begin{pmatrix} 2 & 11 \\ 22 & \end{pmatrix}_2, \begin{pmatrix} 2 & 1 \\ 22 & 1 \end{pmatrix}_2, \begin{pmatrix} 2 & \\ 22 & 11 \end{pmatrix}_2, \begin{pmatrix} & 11 \\ 222 & \end{pmatrix}_2,$
$\left. \begin{pmatrix} & 1 \\ 222 & 1 \end{pmatrix}_2, \begin{pmatrix} & \\ 222 & 11 \end{pmatrix}_2 \right\}.$

Definition 4.2. $\omega(\mathcal{F}, \mathcal{S}_d(m_1, \ldots, m_d))$ denotes the total weight given to $\mathcal{S}_d(m_1, \ldots, m_d)$, that is, the sum of all weights given to individual elements in $\mathcal{S}_d(m_1, \ldots, m_d)$ by $\mathcal{F}$, where $\mathcal{F}$ is a function generating any kind of partitions not necessarily with positive integer weights.

We will compute and compare the total weights given to $\mathcal{S}_d(m_1, \ldots, m_d)$ by various functions. It is clear that $\omega(\mathcal{F}, \mathcal{S}_d(m_1, \ldots, m_d))$ is linear in the first component for coefficients involving any variable but q, the exponent of which stands for the number being partitioned. It is also clear that $\omega(\mathcal{F}, \mathcal{S}_d(m_1, \ldots, m_d)) = \omega(\mathcal{G}, \mathcal{S}_d(m_1, \ldots, m_d))$ for all nonnegative $m_1, \ldots, m_d$ and positive d implies $\mathcal{F} = \mathcal{G}$.

Andrews extended the definition of Durfee symbols to *odd Durfee symbols*. In an odd Durfee symbol only odd numbers occur, and the Durfee square is reinterpreted. The detailed description can be found in [4].

Definition 4.3. Let $\mathcal{S}^o_d(m_1, \ldots, m_d)$ denote the collection of odd Durfee Symbols with Durfee square side d where the total number of appearances of $2j-1$ in the listed odd Durfee symbols is exactly m_j for $j = 1, \ldots, d$. This is a subset of partitions of $n = (2n^2 - 2n + 1) + \sum_{j=1}^{d}(2j-1)m_j$.

We recall one more definition from [4, §4], but rewrite it in an alternative form. As such, this definition provides the base in [7] for the combinatorial explorations of some results stated in [4]. The approach in [7] is a little different. The authors do not generate collections of k-marked Durfee symbols, but count each symbol in sieves.

Definition 4.4. A *k-Marked Durfee Symbol* τ is a concatenation of k two rowed arrays $\begin{pmatrix} a_{i,m_i} & \dots & a_{i,1} \\ b_{i,n_i} & \dots & b_{i,1} \end{pmatrix}$, $i = 1, \dots, k$, with $k-1$ posts $p_1, \dots, p_{k-1}$ in between. The first index indicates the mark. Either or both rows may be empty in an array, and the following monotonicity conditions hold:

$$a_{i,j} \leq a_{i,j+1}, \quad i = 1, \dots, k, \quad j = 1, \dots, m_i - 1 \quad \forall \text{ fixed } i,$$

$$b_{i,j} \leq b_{i,j+1}, \quad i = 1, \dots, k, \quad j = 1, \dots, n_i - 1 \quad \forall \text{ fixed } i,$$

$$\max\{a_{i,m_i}, b_{i,n_i}, p_{i-1}\} \leq p_i \leq \min\{a_{i+1,1}, b_{i+1,1}, p_{i+1}\} \quad i = 1, \dots, k$$

where $p_0 = 1$, $p_k = d$, d being the side of the Durfee Square.

$$\tau = \left(\begin{matrix} a_{k,m_k} & \dots & a_{k,1} \\ b_{k,n_k} & \dots & b_{k,1} \end{matrix} \quad p_{k-1} \quad \begin{matrix} a_{k-1,m_{k-1}} & \dots & a_{k-1,1} \\ b_{k-1,n_{k-1}} & \dots & b_{k-1,1} \end{matrix} \quad p_{k-2} \right.$$
$$\left. \begin{matrix} \dots \\ \dots \end{matrix} \quad p_1 \quad \begin{matrix} a_{1,m_1} & \dots & a_{1,1} \\ b_{1,n_1} & \dots & b_{1,1} \end{matrix} \right)_d$$

If we reinterpret the Durfee square ($2d^2 - 2d + 1$ instead of d^2), and allow odd numbers only, we get the definition of *k-marked odd Durfee symbols*.

Definition 4.5. The excess of entries in the top row of ith array over the entries in the bottom row in the same array is the *ith rank* of a k-marked Durfee symbol (or a k-marked odd Durfee symbol) τ.

Example:

$$\tau = \left(\begin{matrix} & 4 \\ 6 & 5 \end{matrix} \quad 3 \quad \begin{matrix} 2 & 2 \\ 1 & \end{matrix} \right)_8$$

is a 2-marked Durfee symbol with 1st rank 1 and 2nd rank -1.

For arbitrary but fixed k, k-marked Durfee Symbols are generated by

$$R_k(x_1, \dots, x_k; q) := \sum_{\substack{m_1 > 0, \\ m_2, \dots, m_k \geq 0}} q^{(m_1 + \dots + m_k)^2} \frac{q^{m_1}}{(qx_1; q)_{m_1} (q/x_1; q)_{m_1}}$$

$$\times \frac{q^{m_1 + m_2}}{(q^{m_1} x_2; q)_{m_2+1} (q^{m_1}/x_2; q)_{m_2+1}}$$

$$\vdots$$

$$\times \frac{q^{m_1 + \dots + m_{k-1}}}{(q^{m_1 + \dots m_{k-2}} x_{k-1}; q)_{m_{k-1}+1} (q^{m_1 + \dots m_{k-2}}/x_{k-1}; q)_{m_{k-1}+1}}$$

$$\times \frac{1}{(q^{m_1+\ldots m_{k-1}}x_{k-1};q)_{m_k+1}(q^{m_1+\ldots m_{k-1}}/x_{k-1};q)_{m_k+1}} \tag{4.1}$$

This is [4, eq.(2.3)]. k-marked odd Durfee symbols are generated by

$$R_k^o(x_1,\ldots,x_k;q) := \sum_{\substack{m_1>0,\\ m_2,\ldots,m_k\geq 0}} q^{(m_1+\ldots+m_k)^2-2(m_1+\ldots+m_k)+1} \frac{q^{2m_1-1}}{(qx_1;q^2)_{m_1}(q/x_1;q^2)_{m_1}}$$

$$\times \frac{q^{2m_1+2m_2+1}}{(q^{2m_1-1}x_2;q^2)_{m_2+1}(q^{2m_1-1}/x_2;q^2)_{m_2+1}}$$

$$\vdots$$

$$\times \frac{q^{2m_1+\ldots+2m_{k-1}+1}}{(q^{2m_1+\ldots 2m_{k-2}+1}x_{k-1};q^2)_{m_{k-1}+1}(q^{2m_1+\ldots 2m_{k-2}+1}/x_{k-1};q^2)_{m_{k-1}+1}}$$

$$\times \frac{1}{(q^{2m_1+\ldots 2m_{k-1}+1}x_{k-1};q^2)_{m_k+1}(q^{2m_1+\ldots 2m_{k-1}+1}/x_{k-1};q^2)_{m_k+1}} \tag{4.2}$$

which is [4, eq.(9.1)] where the exponent of x_i gives the ith rank in either generating function, $i = 1, \ldots, k$. Here,

$$(a;q)_0 = 1$$

$$(a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1})$$

The difference of the definition of k-marked Durfee symbols from that as given in [4, §4] is that the largest entries for each index in the top row are written as posts. It is obvious that there is a one to one correspondence between these modified Durfee Symbols and the original ones. To see the other direction of the correspondence, we write p_i as a_{i,m_i+1} in the top row. Therefore, $R_k(x_1,\ldots,x_k;q)$ enumerates these new symbols also, where the power of x_i indicates the ith rank $r_i(\tau) = m_i - n_i$.

4.2 Basic Constructions

Lemma 4.6.

$$\omega(R_1^o(z;q), \mathcal{S}_d^o(m_1,\ldots,m_d)) = \omega(R_1(z;q), \mathcal{S}_d(m_1,\ldots,m_d))$$

$$= \prod_{j=1}^{d} \left(\frac{z^{m_j+1} - z^{-m_j-1}}{z - z^{-1}} \right) \tag{4.3}$$

Proof. A Durfee symbol listed by $\mathcal{S}_d(m_1,\ldots,m_d)$ looks like

$$\begin{pmatrix} \overbrace{d\ldots d}^{j_d} & & \overbrace{1\ldots 1}^{j_1} \\ \underbrace{d\ldots d}_{m_d-j_d} & \cdots & \underbrace{1\ldots 1}_{m_1-j_1} \end{pmatrix}_d$$

It contributes to term q^n by weight $z^{j_1-(m_1-j_1)}\dots z^{j_d-(m_d-j_d)}$, where $n = d^2+\sum_{j=1}^{d} jm_j$. Therefore,

$$\omega(R_1(z,q),\mathcal{S}_d(m_1,\dots,m_d)) = \sum_{j_1=0}^{m_1}\dots\sum_{j_d=0}^{m_d} z^{j_1-(m_1-j_1)}\dots z^{j_d-(m_d-j_d)}$$

$$= \left(\sum_{j_1=0}^{m_1} z^{j_1-(m_1-j_1)}\right)\dots\left(\sum_{j_d=0}^{m_d} z^{j_d-(m_d-j_d)}\right) = z^{-m_1-\dots-m_d}\left(\sum_{j_1=0}^{m_1} z^{2j_1}\right)\dots\left(\sum_{j_d=0}^{m_1} z^{2j_d}\right)$$

$$= z^{-m_1-\dots-m_d}\left(\frac{1-z^{2m_1+2}}{1-z^2}\right)\dots\left(\frac{1-z^{2m_d+2}}{1-z^2}\right) = \prod_{j=1}^{d}\left(\frac{z^{m_j+1}-z^{-m_j-1}}{z-z^{-1}}\right)$$

The proof is identical for odd Durfee symbols, with obvious notational changes. □

Lemma 4.7. For $z \neq w$, $z \neq 1/w$,

$$\omega(R_2(z,w;q),\mathcal{S}_d(m_1,\dots,m_d)) = \omega(R_2^o(z,w;q),\mathcal{S}_d^o(m_1,\dots,m_d))$$

$$= \frac{1}{(z+1/z)-(w+1/w)}\left(\prod_{j=1}^{d}\frac{z^{m_j+1}-z^{-m_j-1}}{z-z^{-1}} - \prod_{j=1}^{d}\frac{w^{m_j+1}-w^{-m_j-1}}{w-w^{-1}}\right) \qquad (4.4)$$

Before we prove the Lemma, we give two corollaries:

Corollary 4.8.

$$R_2(z,w;q) = R_2(w,z;q)$$

$$R_2^o(z,w;q) = R_2^o(w,z;q)$$

This is [4, **Corollary 4**] and[4, **Corollary 23**] for $k = 2$.

Proof. Immediate by inspection of **Lemma 4.7**. □

Corollary 4.9.

$$R_2(z,w;q) = \frac{R_1(z;q)-R_1(w;q)}{(z+1/z)-(w+1/w)} \qquad (4.5)$$

$$R_2^o(z,w;q) = \frac{R_1^o(z;q)-R_1^o(w;q)}{(z+1/z)-(w+1/w)} \qquad (4.6)$$

This is [4, **Theorem 7**] and [4, **Theorem 25**] for $k = 2$.

Proof. Combine **Lemma 4.6** and **Lemma 4.7**. □

proof of **Lemma 4.7**. A 2-marking of a Durfee symbol listed by $\mathcal{S}_d(m_1,\dots,m_d)$ looks like

$$\left(\begin{matrix} \overbrace{d_2\dots d_2}^{j_d} \\ \underbrace{d_2\dots d_2}_{m_d-j_d} \end{matrix} \cdots \begin{matrix} \overbrace{r_2\dots r_2}^{j_r-k} \\ \underbrace{r_2\dots r_2}_{m_r-j_r-l} \end{matrix}\ r\ \begin{matrix} \overbrace{r_1\dots r_1}^{k-1} \\ \underbrace{r_1\dots r_1}_{l} \end{matrix} \cdots \begin{matrix} \overbrace{1_1\dots 1_1}^{j_1} \\ \underbrace{1_1\dots 1_1}_{m_1-j_1} \end{matrix}\right)_d$$

where the post is r, and subscripts indicate mark. We require that $m_r, j_r, k \geq 1$ by the definition of marked Durfee symbols. However, once we compute the factor of the weight due to rs, $m_r \geq 1$ will be implied. The factors of the weight due to other entries are found as in Lemma 1.

The rth factor is given by:

$$\sum_{j_r=1}^{m_r}\sum_{k=1}^{j_r}\sum_{l=0}^{m_r-j_r} z^{(k-1)-l}w^{(j_r-k)-(m_r-j_r-l)} = \frac{w^{-m_r}}{z}\sum_{j_r=1}^{m_r} w^{2j_r}\sum_{k=1}^{j_r}\left(\frac{z}{w}\right)^k\sum_{l=0}^{m_r-j_r}\left(\frac{w}{z}\right)^l$$

$$= \frac{w^{-m_r}}{z}\sum_{j_r=1}^{m_r} w^{2j_r}\left(\frac{\frac{z}{w}-\left(\frac{z}{w}\right)^{j_r+1}}{1-\frac{z}{w}}\right)\left(\frac{1-\left(\frac{w}{z}\right)^{m_r-j_r+1}}{1-\frac{w}{z}}\right)$$

We then expand the product inside, compute the terminating geometric series and obtain a closed formula. We treat that as a Laurent expansion in w^{m_r+1}, and treat the coefficients of it as Laurent expansions in z^{m_r+1}. After obvious simplifications, the final expression is:

$$= \frac{1}{(z+1/z)-(w+1/w)}\left[\frac{z^{m_r+1}-z^{-m_r-1}}{z-z^{-1}} - \frac{w^{m_r+1}-w^{-m_r-1}}{w-w^{-1}}\right]$$

Now we take into account the other factors and sum over r:

$$\frac{1}{(z+1/z)-(w+1/w)}\sum_{r=1}^{d}\prod_{j=r+1}^{d}\left(\frac{w^{m_j+1}-w^{-m_j-1}}{w-w^{-1}}\right)$$

$$\times\left(\frac{z^{m_r+1}-z^{-m_r-1}}{z-z^{-1}} - \frac{w^{m_r+1}-w^{-m_r-1}}{w-w^{-1}}\right)\prod_{j=1}^{r-1}\left(\frac{z^{m_j+1}-z^{-m_j-1}}{z-z^{-1}}\right)$$

Upon expansion of the middle factor the sum telescopes and the result follows. □

Proposition 4.10.

$$\omega(R_{k+1}(1,\dots,1;q),\mathcal{S}_d(m_1,\dots,m_d)) = \sum_{k_1+\dots+k_d=k}\binom{m_1+k_1+1}{2k_1+1}\cdots\binom{m_d+k_d+1}{2k_d+1}$$

Proof. Assume that for some r between 1 and d, there are exactly m_r rs in a Durfee Symbol, j of which are in the top row, and $m_r - j$ are in the bottom row. To introduce k_r posts (so as to make rs (k_r+1)-marked), following the definitions in [4]; we need to choose k_r of the j elements in the top row, none repeated; and k_r possibly repeated posts from the $m_r - j + 1$ posts between the entries in the bottom row. There are

$$\binom{j}{k_r}\binom{m_r - j + 1 + k_r - 1}{k_r}$$

ways to do this [16, §1.1]. Upon summing over $j = 0, \ldots, m_r$ using [13, eq.(5.26)], we have

$$\binom{m_r + k_r + 1}{2k_r + 1}$$

Then, to make the whole symbol $k+1$-marked, we need to choose a total of k posts, k_r of them in rs in the Durfee symbol. This gives us the asserted sum. □

The *symmetrized kth moment* function is defined [4, eq.(1.13)] as

$$\eta_k(n) = \sum_{m=0}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m,n)$$

where $N(m,n)$ is the number of partitions of n with rank m.

Proposition 4.11.

$$\omega(\sum_{n \geq 0} \eta_{2k}(n) q^n, \mathcal{S}_d(m_1, \ldots, m_d))$$

$$= \sum_{\substack{0 \leq j_i \leq m_i \\ i = 1, \ldots, d}} \binom{2(j_1 + \ldots + j_d) - (m_1 + \ldots + m_d) + k - 1}{2k}$$

Proof. A Durfee symbol having j_r rs in the top row, $m_r - j_r$ rs in the bottom row for $r = 1, \ldots, d$ has rank $2(j_1 + \ldots + j_d) - (m_1 + \ldots + m_d)$. □

4.3 Applications

The line of reasoning in the proof of **Lemma 4.6** yields

$$\omega(R_1(-1;q), \mathcal{S}_d(m_1, \ldots, m_d)) = (-1)^{m_1 + \ldots + m_d}(m_1 + 1)\ldots(m_d + 1)$$

which is the number of partitions in $\mathcal{S}_d(m_1, \ldots, m_d)$ weighted by $(-1)^{\text{rank}}$. Note that the parity of rank is invariant in $\mathcal{S}_d(m_1, \ldots, m_d)$.

A more interesting computation is that

$$\omega(R_1(i;q), \mathcal{S}_d(m_1, \ldots, m_d)) = \Im(i^{m_1+1})\ldots\Im(i^{m_d+1})$$

This product vanishes unless all m_js are even, and it is $(-1)^{(m_1+\ldots+m_d)/2}$ then. We interpret that as follows. $\mathcal{S}_d(m_1,\ldots,m_d)$ is annihilated by $R_1(i,q)$ unless it contains a self conjugate partition, and self conjugate partitions are given the weight $(-1)^{\text{number of parts below the Durfee square}}$.

The formulae derived in **Lemma 4.6** prove [4, eq.s (13.1)-(13.4)], which follow. Of course, the equations are mere substitutions in the generating functions (4.1) and (4.2). The point is that the following identities are proven without having the exact formula of the generating function of the k-marked Durfee symbols (or k-marked odd Durfee symbols).

$$R_1(-1;q) = \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2} \tag{4.7}$$

$$R_1(i;q) = \sum_{n\geq 0} \frac{q^{n^2}}{(-q^2;q^2)_n} \tag{4.8}$$

$$R_1^o(-1;q) = \sum_{n\geq 0} \frac{q^{2n^2+2n+1}}{(q;q^2)_{n+1}^2} \tag{4.9}$$

$$R_1^o(i;q) = \sum_{n\geq 0} \frac{q^{2n^2+2n+1}}{(-q^2;q^4)_{n+1}} \tag{4.10}$$

It is also straightforward combine **Lemma 4.7**, the discussion following the proof of **Lemma 4.6**, and equations (4.7)-(4.10) to prove [4, eq.s (13.7) and (13.9)], which follow.

$$R_2(i,-1;q) = \frac{1}{2}\left(\sum_{n\geq 0} \frac{q^{n^2}}{(-q^2;q^2)_n} - \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2}\right) \tag{4.11}$$

$$R_2^o(i,-1;q) = \frac{1}{2}\left(\sum_{n\geq 0} \frac{q^{2n^2+2n+1}}{(-q^2;q^4)_{n+1}} + \sum_{n\geq 0} \frac{q^{2n^2+2n+1}}{(-q;q^2)_{n+1}^2}\right) \tag{4.12}$$

We do not actually need to deal with the factor 1/2 in front, since the idea of the proof was to show that $\mathcal{S}_d(m_1,\ldots,m_d)$ was assigned the same weights by both expressions. Yet, it is not that hard to explain the combinatorics behind that 1/2. Self conjugate partitions are annihilated if the number of parts below the Durfee square is even. Self conjugate partitions with an odd number of parts below the Durfee square gain weight 2, which is then halved. For non self conjugate pairs, exactly one partition is counted. We can even lexicographically order the rows in the Durfee symbol to make the choice.

It is possible to extend the results to obtain [4, **Theorem 7**], and hence or otherwise prove that $R_k(x_1,\ldots,x_k;q)$ is symmetric in $x_1,\ldots,x_k$. One needs to refine the definition of $\mathcal{S}_d(m_1,\ldots,m_d)$, and deal with two adjacent variables at once.

Definition 4.12. Given k and i such that $1 \le i \le k$, and a k-marked Durfee symbol τ (respectively, a k-marked odd Durfee symbol τ^o), Let $\mathcal{S}_{(k,i)}(\tau)$ (respectively, $\mathcal{S}^o_{(k,i)}(\tau^o)$) denote the collection of k-marked Durfee symbols (respectively, k-marked odd Durfee symbols) which are identical to τ except ith double arrays, and all have the same number of occurrences of each part in their ith double arrays.

Example:

$$\mathcal{S}_{(2,1)}\left(\begin{pmatrix} & 4 & & 2 & 2 & \\ 6 & 5 & 3 & 1 & & \end{pmatrix}_8\right) = \left\{\begin{pmatrix} & 4 & & 2 & 2 & 1 \\ 6 & 5 & 3 & & & \end{pmatrix}_8, \right.$$
$$\begin{pmatrix} & 4 & & 2 & 1 \\ 6 & 5 & 3 & 2 & \end{pmatrix}_8, \begin{pmatrix} & 4 & & 1 & \\ 6 & 5 & 3 & 2 & 2 \end{pmatrix}_8, \begin{pmatrix} & 4 & & 2 & 2 \\ 6 & 5 & 3 & 1 & \end{pmatrix}_8,$$
$$\left.\begin{pmatrix} & 4 & & 2 & \\ 6 & 5 & 3 & 2 & 1 \end{pmatrix}_8, \begin{pmatrix} & 4 & & & & \\ 6 & 5 & 3 & 2 & 2 & 1 \end{pmatrix}_8\right\}$$

Definition 4.13. Given a k-marked Durfee symbol τ (respectively, a k-marked odd Durfee symbol τ^o), and a function $\mathcal{F}$ generating partitions with various weights, let $\omega(\mathcal{F}, \mathcal{S}_{(k,i)}(\tau))$ (respectively, $\omega(\mathcal{F}, \mathcal{S}^o_{(k,i)}(\tau^o))$) denote the total weight assigned to $\mathcal{S}_{(k,i)}(\tau))$ (respectively, $\mathcal{S}^o_{(k,i)}(\tau^o)$) by $\mathcal{F}$.

Lemma 4.14. Let τ be a k-marked Durfee symbol with $(i-1)$th and ith posts p_{i-1} and p_i. Let r_j denote the jth rank of τ for $j = 1, \dots, k$, $j \ne i$. Let $m_{p_{i-1}}, m_{p_{i-1}} + 1, \dots, m_{p_i}$ be the number of occurrences of $p_{i-1}, p_{i-1} + 1, \dots, p_i$, respectively, in the ith double array of τ. Then,

$$\omega(R_k(x_1, \dots, x_k; q), \mathcal{S}_{(k,i)}(\tau)) = x_1^{r_1} \cdots x_{i-1}^{r_{i-1}} x_{i+1}^{r_{i+1}} \cdots x_k^{r_k} \prod_{j=p_{i-1}}^{p_i} \frac{x_i^{m_j+1} - x_i^{-m_j-1}}{x_i - x_i^{-1}} \tag{4.13}$$

Proof. A symbol listed by $\mathcal{S}_{(k,i)}(\tau)$ looks like

$$\left(\begin{matrix} \alpha_k \\ \beta_k \end{matrix}\; p_{k-1}\; \begin{matrix} \alpha_{k-1} \\ \beta_{k-1} \end{matrix}\; p_{k-2} \dots p_i\; \underbrace{\begin{matrix} \overbrace{p_i \dots p_i}^{j_{p_i}} \\ \underbrace{p_i \dots p_i}_{m_{p_i} - j_{p_i}} \end{matrix} \;\cdots\; \begin{matrix} \overbrace{p_{i-1} \dots p_{i-1}}^{j_{p_{i-1}}} \\ \underbrace{p_{i-1} \dots p_{i-1}}_{m_{p_{i-1}} - j_{p_{i-1}}} \end{matrix}}_{i\text{-marked double array}}\; p_{i-1} \dots p_2\; \begin{matrix} \alpha_2 \\ \beta_2 \end{matrix}\; p_1\; \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}\right)_d$$

and it contributes to the exponent of x_i in the term $q^n p$ (n is d^2+ sum of all entries in τ) with ith rank

$$x_i^{j_{p_i} - (m_{p_i} - j_{p_i})} \dots x_i^{j_{p_{i-1}} - (m_{p_{i-1}} - j_{p_{i-1}})}$$

then the computations in the proof of **Proposition 4.6** apply with the obvious notational changes. ∎

Lemma 4.15. Let τ be a k-marked Durfee symbol with $(i-1)$th and ith posts p_{i-1} and p_i. Let r_j be the jth rank of τ for $j = 1, \ldots, k$, $j \neq i-1$. Let $m_{p_{i-1}}, m_{p_{i-1}}+1, \ldots, m_{p_i}$ be the number of occurrences of $p_{i-1}, p_{i-1}+1, \ldots, p_i$, respectively, in the ith double array of τ. Then,

$$\omega(R_{k+1}(x_1, \ldots, x_{k+1}; q), \mathcal{S}_{(k,i)}(\tau)) = \frac{x_1^{r_1} \cdots x_{i-2}^{r_{i-2}} x_{i+1}^{r_i} \cdots x_{k+1}^{r_k}}{(x_i - 1/x_i) - (x_{i-1} - 1/x_{i-1})}$$

$$\times \left(\prod_{j=p_{i-1}}^{p_i} \frac{x_i^{m_j+1} - x_i^{-m_j-1}}{x_i - x_i^{-1}} - \prod_{j=p_{i-1}}^{p_i} \frac{x_{i-1}^{m_j+1} - x_{i-1}^{-m_j-1}}{x_{i-1} - x_{i-1}^{-1}} \right) \tag{4.14}$$

Proof. We will choose a post among the entries in the ith double array in τ, and make it into a $(k+1)$-marked Durfee symbol. We will then keep track of the ith rank as the exponent of x_i, and of the $(i+1)$th rank as the exponent of x_{i+1}. The other double arrays and posts are kept fixed by definition of $\mathcal{S}_{(k,i)}(\tau)$, the only portion of τ that can be played around with is the ith double array. τ after the choice of an extra post will look like

$$\left(\begin{matrix}\alpha_k\\ \beta_k\end{matrix} \; p_{k-1} \; \begin{matrix}\alpha_{k-1}\\ \beta_{k-1}\end{matrix} \; p_{k-2} \cdots p_i \right.$$

$$\underbrace{\overbrace{\begin{matrix}\overbrace{p_i\cdots p_i}^{j_{p_i}} \\ \underbrace{p_i\cdots p_i}_{m_{p_i}-j_{p_i}}\end{matrix} \cdots \begin{matrix}\overbrace{r\ldots r}^{j_r-k}\\ \underbrace{r\ldots r}_{m_r-j_r-l}\end{matrix}}^{\text{new }(i+1)\text{-marked double array}} \; r \; \overbrace{\begin{matrix}\overbrace{r\ldots r}^{k-1}\\ \underbrace{r\ldots r}_{l}\end{matrix} \cdots \begin{matrix}\overbrace{p_{i-1}\cdots p_{i-1}}^{j_{p_{i-1}}}\\ \underbrace{p_{i-1}\cdots p_{i-1}}_{m_{p_{i-1}}-j_{p_{i-1}}}\end{matrix}}^{\text{new } i\text{-marked double array}}}_{\text{initial } i\text{-marked double array}}$$

$$\left. p_{i-1} \cdots p_2 \; \begin{matrix}\alpha_2\\ \beta_2\end{matrix} \; p_1 \; \begin{matrix}\alpha_1\\ \beta_1\end{matrix} \right)_d$$

for some r between p_{i-1} and p_i. At this point, we make the substitutions

$$\begin{aligned} p_{k+1} &\leftarrow p_k \\ p_k &\leftarrow p_{k-1} \\ &\vdots \\ p_{i+1} &\leftarrow p_i \\ p_i &\leftarrow r \end{aligned}$$

along with the corresponding re-marking of αs and βs in between these new posts, so that the exponent of x_j now keeps track of jth rank for $j = 1, \ldots, (k+1)$.

The ith rank, as the exponent of x_i is

$$(j_{i-1} - (m_{i-1} - j_{i-1})) + \ldots + (j_{r-1} - (m_{r-1} - j_{r-1})) + (k - l)$$

and the $(i+1)$th rank as the exponent of x_{i+1} is

$$(j_r - k - (m_r - j_r - l))(j_{r+1} - (m_{r+1} - j_{r+1})) + \ldots + (j_i - (m_i - j_i))$$

Then, the computations in **Lemma 4.7** carry over with appropriate notational changes. $\square$

Corollary 4.16. $R_k(x_1, \ldots, x_k; q)$ is symmetric in $x_1, \ldots, x_k$.

This is [4, **Corollary 4**].

Proof. Upon inspection of **Lemma 4.15** under $k \leftarrow k-1$, we see that

$$R_k(x_1, \ldots, x_i, x_{i+1}, \ldots, x_k) = R_k(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_k)$$

Then, we use the well known fact that pairwise switching of adjacent symbols (in this case, x_js) generate the symmetric group on k letters. $\square$

Corollary 4.17. For $k \geq 2$, and $x_i \neq x_j$ for $i \neq j$, $i, j = 1, \ldots, k$,

$$R_k(x_1, \ldots, x_k; q) = \sum_{i=1}^{k} \frac{R_1(x_i; q)}{\prod_{j=1, j\neq i}^{k} \left[\left(x_i + \frac{1}{x_i}\right) - \left(x_j + \frac{1}{x_j}\right)\right]} \tag{4.15}$$

This is [4, **Theorem 7**].

Proof. We induct on k. For $k = 2$, this is **Corollary 4.9**. Then, we assume the claim up to some $k > 2$, and use **Lemma 4.14** and **Lemma 4.15** to write

$$\left[\left(x_1 + \frac{1}{x_1}\right) - \left(x_2 + \frac{1}{x_2}\right)\right] R_{k+1}(x_1, \ldots, x_{k+1}; q)$$

$$= R_k(x_1, x_3, x_4, \ldots, x_{k+1}; q) - R_k(x_2, x_3, x_4, \ldots, x_{k+1}; q)$$

by the inductive hypothesis,

$$= \sum_{i=1, i\neq 2}^{k+1} \frac{R_1(x_i; q)}{\prod_{j=1, j\neq 2, j\neq i}^{k+1} \left[\left(x_i + \frac{1}{x_i}\right) - \left(x_j + \frac{1}{x_j}\right)\right]} - \sum_{i=2}^{k+1} \frac{R_1(x_i; q)}{\prod_{j=2, j\neq i}^{k+1} \left[\left(x_i + \frac{1}{x_i}\right) - \left(x_j + \frac{1}{x_j}\right)\right]}$$

$$= \frac{R_1(x_1; q)}{\prod_{j=3}^{k+1} \left[\left(x_1 + \frac{1}{x_1}\right) - \left(x_j + \frac{1}{x_j}\right)\right]} - \frac{R_1(x_2; q)}{\prod_{j=3}^{k+1} \left[\left(x_2 + \frac{1}{x_2}\right) - \left(x_j + \frac{1}{x_j}\right)\right]}$$
$$+ \sum_{i=3}^{k+1} \frac{R_1(x_i; q)}{\prod_{j=3, j\neq i}^{k+1} \left[\left(x_i + \frac{1}{x_i}\right) - \left(x_j + \frac{1}{x_j}\right)\right]}$$

$$\times\left[\frac{1}{\left(x_i+\frac{1}{x_i}\right)-\left(x_1+\frac{1}{x_1}\right)}-\frac{1}{\left(x_i+\frac{1}{x_i}\right)-\left(x_2+\frac{1}{x_2}\right)}\right]$$

$$=\left[\left(x_1+\frac{1}{x_1}\right)-\left(x_2+\frac{1}{x_2}\right)\right]\sum_{i=1}^{k+1}\frac{R_1(x_i;q)}{\prod_{j=1,j\neq i}^{k+1}\left[\left(x_i+\frac{1}{x_i}\right)-\left(x_j+\frac{1}{x_j}\right)\right]}$$

Upon simplifying the factor $\left[\left(x_1+\frac{1}{x_1}\right)-\left(x_2+\frac{1}{x_2}\right)\right]$, the result follows. □

Theorem 4.18.

$$R_{k+1}(1,\ldots,1;q)=\sum_{n\geq 0}\eta_{2k}(n)q^n$$

This is [4, **Corollary 5**].

In other words, $\eta_{2k}(n)$ counts the number of $(k+1)$-marked Durfee symbols associated to n [4, **Corollary 13**].

Proof. It suffices to establish

$$\sum_{k_1+\ldots+k_d=k}\binom{m_1+k_1+1}{2k_1+1}\cdots\binom{m_d+k_d+1}{2k_d+1}$$

$$=\sum_{\substack{0\leq j_i\leq m_i\\ i=1,\ldots,d}}\binom{2(j_1+\ldots+j_d)-(m_1+\ldots+m_d)+k-1}{2k} \tag{4.16}$$

by **Proposition 4.10** and **Proposition 4.11**.

We induct on d. For $d=1$ we have:

$$\binom{m+k+1}{2k+1}=\sum_{j=0}^{m}\binom{2j-m+k-1}{2k}$$

Now, for $k=0$, we have $m+1=m+1$. For k positive,

$$\sum_{j=0}^{m}\binom{2j-m+k-1}{2k}=\sum_{0\leq j\leq\lfloor m/2\rfloor}\binom{2j-m+k-1}{2k}+\sum_{\lfloor m/2\rfloor<j\leq m}\binom{2j-m+k-1}{2k}$$

$$=\sum_{0\leq j\leq\lfloor m/2\rfloor}\binom{m-2j+k}{2k}+\sum_{\lfloor m/2\rfloor<j\leq m}\binom{2j-m+k-1}{2k}=\sum_{j=0\text{ or }1}^{m}\binom{j+k}{2k}$$

$$=\binom{m+k+1}{2k+1}$$

where we negated the upper index in the first half of the sum and used (5.10) in [13]. This provides the base case.

Assuming for some d, after applying the inductive hypothesis, and upon substituting

$$2(j_1 + \ldots + j_d) - (m_1 + \ldots + m_d) \leftarrow N,$$
$$m_{d+1} \leftarrow m,$$
$$j_{d+1} \leftarrow d,$$

we are down to proving

$$\sum_{i=0}^{k} \binom{m + (k-i) + 1}{2(k-i)+1} \binom{N+i-1}{2i} = \sum_{j=0}^{m} \binom{N + 2j - m + k - 1}{2k} \tag{4.17}$$

for $k, m \geq 0$, and $N \in \mathbb{Z}$.

In fact, it suffices to establish the result for non-negative N, since

$$\binom{N+i-1}{2i} = (-1)^{2i} \binom{(2i-1) - (N+i-1)}{2i} = \binom{(1-N)+i-1}{2i}$$

and

$$\binom{N+2j-m+k-1}{2k} = \binom{(2k-1) - (N+2j-m+k-1)}{2k}$$
$$= \binom{(1-N) + m - 2j + k - 1}{2k}$$

using [13, eq.(5.14)]. For the latter derivation, we are also allowed to substitute $j \leftarrow m - j$ under the sum over $0 \leq j \leq m$.

For integral $k, m, N \geq 0$, we define

$$L_e(N,k) = \sum_{i=0}^{k} \binom{m + (k-i) + 1}{2(k-i)+1} \binom{N+i-1}{2i}$$

$$R_e(N,k) = \sum_{j=0}^{m} \binom{N + 2j - m + k - 1}{2k}$$

$$L_o(N,k) = \sum_{i=0}^{k} \binom{m + (k-i) + 1}{2(k-i)+1} \binom{N+i-1}{2i-1}$$

$$R_o(N,k) = \sum_{j=0}^{m} \binom{N + 2j - m + k - 1}{2k-1}$$

Then, using the basic recurrence for binomial coefficients, we obtain

$$L_e(N,k) = \sum_{i=0}^{k} \binom{m + (k-i) + 1}{2(k-i)+1} \left(\binom{(N-1)+i-1}{2i} + \binom{(N-1)+i-1}{2i-1} \right)$$

$$= L_e(N-1,k) + L_o(N-1,k)$$

and

$$L_o(N,k) = \sum_{i=0}^{k} \binom{m+(k-i)+1}{2(k-i)+1} \left(\binom{N+i-2}{2i-1} + \binom{N+i-2}{2i-2} \right)$$

$$= L_o(N-1,k) + \sum_{i=0}^{k-1} \binom{m+((k-1)-i)+1}{2((k-1)-i)+1} \binom{N+i-1}{2i} = L_o(N-1,k) + L_e(N,k-1)$$

Similar reasoning shows that

$$R_e(N,k) = \sum_{j=0}^{m} \left(\binom{N-1+2j-m+k-1}{2k} + \binom{N-1+2j-m+k-1}{2k-1} \right)$$

$$= R_e(N-1,k) + R_o(N-1,k)$$

and that

$$R_o(N,k) = \sum_{j=0}^{m} \left(\binom{N-1+2j-m+k-1}{2k-1} + \binom{N+2j-m+(k-1)-1}{2k-2} \right)$$

$$= R_o(N-1,k) + R_e(N,k-1)$$

Thus, the Ls satisfy the same pair of recurrences as the Rs. The following base cases then establish

$$L_e(N,k) = R_e(N,k)$$

which is (4.17), and

$$L_o(N,k) = R_o(N,k)$$

which is a side result.

Please note that

$$L_e(0,k) = \binom{m+k+1}{2k+1} = \sum_{j=0}^{m} \binom{2j-m+k-1}{2k} = R_e(0,k)$$

as shown above in the case $d = 1$. Also,

$$L_e(N,0) = m+1 = R_e(N,0)$$

$$L_o(N,0) = m+1 = R_o(N,0)$$

is immediate.

$$L_o(0,k) = 0$$

and

$$R_o(0,k) = \sum_{j=0}^{m} \binom{2j-m+k-1}{2k-1}$$

$$= \sum_{0\le j\le \lfloor m/2\rfloor} \binom{2j-m+k-1}{2k-1} + \sum_{\lceil m/2\rceil \le j\le m} \binom{2j-m+k-1}{2k-1}$$

Now we substitute $j \leftarrow m-j$ in the second sum, and negate the upper index

$$= \sum_{0\le j\le \lfloor m/2\rfloor} \binom{2j-m+k-1}{2k-1}$$

$$+ \sum_{0\le j\le m-\lceil m/2\rceil} (-1)^{2k-1} \binom{((2k-1)-1)-(m-2j+k-1)}{2k-1}$$

$$= \sum_{0\le j\le \lfloor m/2\rfloor} \binom{2j-m+k-1}{2k-1} - \sum_{0\le j\le \lfloor m/2\rfloor} \binom{2j-m+k-1}{2k-1} = 0$$

as desired. □

The proof of (4.17) above is given by Andrews. The original proof is a double induction on m and k.

The side result of the double recursion in the preceding proof is

$$\sum_{i=1}^{k} \binom{m+(k-i)+1}{2(k-i)+1} \binom{N+i-1}{2i-1} = \sum_{j=0}^{m} \binom{N+2j-m+k-1}{2k-1}$$

In the context of k-marked Durfee symbols, this would be related to moments of odd ranks, which are zero, as shown in [4, **Theorem 1**].

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