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TWO APPROACHES TO MEASURES FOR ALGEBRAIC INDEPENDENCE

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Abstract

Patrice Philippon and Yuri V. Nesterenko have proven results for measuring the algebraic independence of certain numbers, which they achieved by using properties of resultants of Chow forms. In this thesis we generalize the two results, and use the research on Chow forms in positive characteristic undertaken by John Zuehlke to extend the work to that setting.

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Chapter 1

Introduction

In 1949, A.O. Gelfond proved that for algebraic numbers α and β with $\alpha \neq 0$, $\log \alpha \neq 0$, and β a cubic irrational over $\mathbb{Q}$, α^{β} and α^{β^2} are algebraically independent. Soon after Gelfond and his student N.I. Feldman found a measure of algebraic independence for these numbers. They showed that for all $\epsilon > 0$, there exists a constant $t = t(\epsilon) > 0$ such that for all polynomials $P \in \mathbb{Z}[X_1, X_2]$ satisfying $\deg(P) + h(P) > t$ (where $h(P)$ is the logarithm of the maximum of the absolute values of the coefficients of P) we have

$$\log |P(\alpha^{\beta}, \alpha^{\beta^2})| > -\exp((\deg(P) + h(P))^{4+\epsilon}). \tag{1.1}$$

This is considered the initial application of elimination theory to transcendence theory. These results have been refined over the years by A.A. Smelev, G.V. Chudnovsky, Michel Waldschmidt, W. Dale Brownawell, and Guy Diaz.

More recently, Yuri V. Nesterenko and Patrice Philippon have independently developed more general (and more flexible) tools to find measures of algebraic independence. Both men's results take the numbers as coordinates of a given point and then work with families of polynomials with bounds on their degrees, logarithmic heights, and values when evaluated at that given point. The procedure to obtain these families is sometimes called the "Transcendence Machine" in the literature.

Once this family of auxiliary polynomials is in hand, the two procedures assume that certain technical constraints are satisfied and construct polynomials in new variables known as Chow forms (see Chapter 2) to obtain a lower bound on the evaluation of an algebraic cycle at the given point.

There are some significant differences between the two men's work. Nesterenko originally began this line of research in the realm of function fields, looking at orders of zeros at a point when substituting a solution of a system of linear differential equations. Philippon started with number fields instead of function fields, though both men have since worked in both settings. Nesterenko uses the standard valuations exclusively in the calculation of his heights; Philippon sometimes substitutes Mahler measures which require more delicate usage. Nesterenko uses hyperplanes defined by linear forms, whereas Philippon uses the more general hypersurfaces defined by more general forms. To account for this generality, Philippon uses a multidimensional degree in his results, whereas Nesterenko does not. Nesterenko explicitly introduces the concept of a resultant (see Section 2.3), whereas in Philippon it is "hidden" as a specialization of a map obtained by substituting the coefficients of the auxiliary polynomials in for the coefficients of the hypersurfaces.

Philippon's student El Mostafa Jabbouri constructed a measure of algebraic independence based on his adviser's that slightly improves the technical hypothesis and that assumes the auxiliary polynomials are without any common zero in a closed ball around the given point. Jabbouri requires his degrees and heights to have a fixed constant upper bound, whereas Nesterenko allows the bound to be in terms of a chosen parameter. Brownawell and Rémond have further developed these tools.

These results have been used to prove a variety of algebraic independence measures, involving notably

- periods and quasi-periods of elliptic functions by Jabbouri in [5]
- values of Ramanujan functions by Nesterenko in [7], and
- improvements of the Gelfond-Feldman inequality in Equation (1.1) by Nesterenko in [6] and by Diaz in [4].

The purpose of this thesis is to provide generalizations of the Jabbouri-Philippon and Nesterenko results in the number field case, and create analogues of their results in a form suitable for function field applications. In particular with the results of John Zuehlke in [9], the results in this thesis can be used for t-modules, the higher dimensional generalizations of Drinfeld modules. We present two applications in the number field setting and indicate further possibilities.

Chapter 2

Chow forms and auxiliary results

In this chapter we present the main tool used in our results: the Chow form (also known as the Cayley-Chow form, the eliminant form, and the associated form). For a more detailed look at Chow forms, [2] summarizes the Nesterenko point of view. We provide an even more abbreviated introduction below.

2.1 Introduction

Let R be a unique factorization domain with fraction field $\mathbb{K}$, and let V be an irreducible variety in $\mathbb{P}^n(\mathbb{K})$ of dimension d corresponding to a homogeneous prime ideal $\mathfrak{P}$ in $\mathbb{K}[x_0, \dots, x_n]$. Define $\underline{u}_j = (u_{j0}, \dots, u_{jn})$ to be an $(n+1)$-tuple of new variables for $j = 0, \dots, d$. Denote by H_j the hyperplane

$$H_j : \; u_{j0}x_0 + \cdots + u_{jn}x_n = 0.$$

Then the hyperplanes $H_1, \dots, H_{d-1}$ intersect V in $\deg \mathfrak{P}$ points in $\mathbb{P}^n(\overline{\mathbb{K}})$; call them $\underline{\alpha}_k = (\alpha_{k0} : \cdots : \alpha_{kn})$ for $k = 1, \dots, \deg \mathfrak{P}$. These points are considered "generic zeros" of $\mathfrak{P}$, and their coordinates are algebraic over $R[\underline{u}_0, \dots, \underline{u}_{d-1}]$.

It can be shown (see [3]) that we can choose $a \in R[\underline{u}_0, \ldots, \underline{u}_{d-1}]$ so that the polynomial
$$F(\underline{u}_0, \ldots, \underline{u}_d) = a \prod_{k=1}^{\deg \mathfrak{P}} (\alpha_{k0} u_{d0} + \cdots + \alpha_{kn} u_{dn})$$
has coefficients in R, but no (non-unit) factors in R. We call this a *Chow form* of $\mathfrak{P}$. Any other choice of a Chow form of $\mathfrak{P}$ will be a non-zero scalar multiple of this one. Note that $F(\underline{u}_0, \ldots, \underline{u}_d) = 0$ if and only if H_d passes through a point of $V \cap H_0 \cap \cdots \cap H_{d-1}$, i.e. that the H_i intersect in at least one point of V.

Let $\mathcal{Z}$ be an unmixed cycle. That is,
$$\mathcal{Z} = \sum_{i=1}^{r} e_i \mathfrak{P}_i$$
is a formal linear combination of distinct prime ideals $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ of $\mathbb{K}[x_0, \ldots, x_n]$ with assigned multiplicities $e_1, \ldots, e_r$, respectively. Then the Chow form of $\mathcal{Z}$ is defined to be
$$F = F_{\mathcal{Z}} = \prod_{i=1}^{r} F_i^{e_i}, \tag{2.1}$$
where F_i is a Chow form of the corresponding prime ideal $\mathfrak{P}_i$. Each F_i can be written as
$$F_i = a_i \prod_{k=1}^{\deg \mathfrak{P}_i} (\alpha_{k0,i}\, u_{d0} + \cdots + \alpha_{km,i}\, u_{dm}).$$

For everything that follows in this thesis, let $\mathcal{M}$ be a set, and let an absolute value $|\cdot|_v$ be given on the field $\mathbb{K}$ for each $v \in \mathcal{M}$. Moreover, assume that the set of these absolute values is such that for each $a \in \mathbb{K}^{\times}$, there exists only a finite number of elements $v \in \mathcal{M}$ with $|a|_v \neq 1$, and $\prod_{v \in \mathcal{M}} |a|_v = 1$. In addition, assume that the subset $\mathcal{M}_\infty \subset \mathcal{M}$ consisting of all elements $v \in \mathcal{M}$ for which the corresponding absolute value $|\cdot|_v$ is archimedean and finite. Let ν denote the number of elements in the set $\mathcal{M}_\infty$. Unless otherwise indicated, we will assume we have chosen $w \in \mathcal{M}$ and we let $|\cdot| = |\cdot|_w$. We will also let $\mathcal{K}$ be the completion of the algebraic closure of the completion of $\mathbb{K}$ with respect to $|\cdot|$. Let $|P|$ be the maximum of the absolute values of the coefficients of the polynomial P.

2.2 Heights and absolute values

Let $\mathcal{Z}$ again be an unmixed cycle, and let F be its Chow form. We define the *degree* of $\mathcal{Z}$ to be $\deg_{\underline{u}_0} F = \sum e_i \deg \mathfrak{P}_i$.

For any nonzero polynomial P, we define the (logarithmic) *height* of P,

$$h(P) := \sum_{v \in \mathcal{M}} \log |P|_v.$$

Then we let the height of $\mathcal{Z}$ be defined as the height of F. (Note that in [8], Philippon allows a Mahler measure to replace the maximum of the coefficients for his height in certain situations.)

For each non-zero point $\underline{\omega} = (\omega_0, \ldots, \omega_n) \in \mathcal{K}^{n+1}$. we set $|\underline{\omega}| = \max_{0 \leq j \leq n} |\omega_j|$. Throughout this thesis we will use the notation

$$||P||_{\underline{\omega}} = \frac{|P(\underline{\omega})|}{|P|\,|\underline{\omega}|^{\deg P}} \tag{2.2}$$

for a homogeneous polynomial $P \in \mathbb{K}[x_0, \ldots, x_n]$.

Let $S^{(i)}$, $0 \leq i \leq d$ be generic $n \times n$ skew-symmetric matrices so $s^{(i)}_{jk} + s^{(i)}_{kj} = 0$ but no other algebraic relations exist among the entries. Then we define the *absolute value* of a Chow form at $\underline{\omega}$ to be

$$|F|_{\underline{\omega}} := \frac{|F(S^{(0)}\underline{\omega}, \ldots, S^{(d)}\underline{\omega})|}{|F| \, |\underline{\omega}|^{(d+1)\deg F}}.$$

Here we consider $F(S^{(0)}\underline{\omega}, \ldots, S^{(d)}\underline{\omega})$ to be the polynomial in the variables $s^{(i)}_{jk}$, $j < k$, obtained as a result of the substitution of the vector $S^{(i)}\overline{\omega}$ for the variable $\underline{u}_i$.

2.3 Resultants

The foundations for the resultant go back to Zariski. We will be using Nesterenko's formation of the resultant of a Chow form and a polynomial.

If we have a Chow form as in Equation (2.1) and take $Q \in R[x_0, \cdots, x_n]$ an ordinary form, we can define the *resultant* of a Chow form F and Q to be

$$\operatorname{Res}(F, Q) := \prod_{i=1}^{r} \left(a_i^{\deg Q} \prod_{k=1}^{\deg \mathfrak{P}_i} Q(\alpha_{k0,i}, \cdots, \alpha_{km,i}) \right)^{e_i}.$$

This resultant is a product of the resultants of Q with each F_i. Note that the resultant is zero if and only if $Q \in \cup \mathfrak{P}_i$ since Q is being evaluated at the generic zeros of the $\mathfrak{P}_i$. When $\dim F = 0$, then $\operatorname{Res}(F, Q) \in \mathbb{K}$ is zero exactly when Q vanishes at one of the zeros of some $\mathfrak{P}_i$. In particular, Nesterenko showed the important fact that if $\operatorname{Res}(F, Q) \neq 0$ and the dimension of F is greater than 0, then this resultant is itself a Chow form. Moreover, Nesterenko has shown that $\mathrm{Res}(F, Q)$ has dimension one lower than F.

We can describe the degree, height, and absolute value of a resultant in terms of the ideal and polynomial used to construct it. Nesterenko proved the following inequalities in the number field case in [7]; John Zuehlke has shown the function field version in [9].

PROPOSITION 2.1. *Let $R[X]$ be a ring of polynomials in the variables $x_0, \ldots, x_n$ over R. Let $\mathfrak{P}$ be a homogeneous prime ideal of $R[X]$, $\dim \mathfrak{P} \geq 0$, with associated Chow form F, and let Q be a homogeneous polynomial from $R[X]$ with $Q \notin \mathfrak{P}$. Assume, in addition, that $|\cdot| = |\cdot|_w$, $w \in \mathcal{M}$, is an absolute value on $\mathbb{K}$.*

If $r = 1 + \dim \mathfrak{P}$, then

1. $\deg \mathrm{Res}(F, Q) = \deg F \ \deg Q$,
2. $h(\mathrm{Res}(F, Q)) \leq h(F) \ \deg Q + h(Q) \ \deg F + \nu n(r+1) \deg F \ \deg Q$,
3. *for any point $\underline{\omega} \in \mathcal{K}^{n+1}$ and for $\rho = \min ||\underline{\omega} - \underline{\beta}||$, where the minimum is taken over all nontrivial zeros $\underline{\beta}$ of F in $\mathcal{K}^{n+1}$, the following inequality holds:*

$$\log |\mathrm{Res}(F, Q)|_{\underline{\omega}} \leq \log \delta + h(F) \ \deg Q + h(Q) \ \deg F + 11\nu n^2 \deg F \ \deg Q,$$

where

$$\delta = \begin{cases} ||Q||_{\underline{\omega}}, & \text{if } \rho < ||Q||_{\underline{\omega}} \\ |F|_{\underline{\omega}}, & \text{if } \rho \geq ||Q||_{\underline{\omega}}. \end{cases}$$

Here we are using the notation

$$||\underline{\varphi} - \underline{\psi}|| = \left(\max_{0 \le i < j \le n} |\varphi_i \psi_j - \varphi_j \psi_i| \right) |\underline{\varphi}|^{-1} |\underline{\psi}|^{-1}$$

for the projective distance between two non-zero points $\underline{\varphi} = (\varphi_0, \ldots, \varphi_n)$, $\underline{\psi} = (\psi_0, \ldots, \psi_n)$, both in $\mathcal{K}^{n+1}$. Note that in the function field setting, $\nu = 0$ so the terms which are factors of $(\deg F \deg Q)$ are absent in inequalities 2 and 3. This proposition also holds even when $\dim F = 0$; then $|\mathrm{Res}(F, Q)|_{\underline{\omega}} = 1$.

In some applications we will apply the following alternate version of Proposition 2.1:

Corollary 2.2. *Under the same conditions as Proposition 2.1, assume that F and Q satisfy*

$$|F|_{\underline{\omega}} \le \exp(-S), \quad S > 0, \quad ||Q||_{\underline{\omega}} \le \exp(-2n\nu \deg Q).$$

If the integer $\eta > 0$ satisfies the inequality

$$-\eta \log ||Q||_{\underline{\omega}} \ge 2 \min \left\{ S, \log \frac{1}{\rho} \right\}, \tag{2.3}$$

then

1. $\deg \mathrm{Res}(F, Q) \le \eta \deg F \deg Q$,
2. $h(\mathrm{Res}(F, Q)) \le \eta \left(h(Q) \deg F + h(F) \deg Q + \nu n(r+2) \deg F \deg Q \right)$,
3. $\log |\mathrm{Res}(F, Q)|_{\underline{\omega}} \le -S + \eta \left(h(F) \deg Q + h(Q) \deg F + 12 \nu n^2 \deg F \deg Q \right)$.

Just as in the previous proposition, the terms which are factors of $(\deg F \deg Q)$ are not present in inequalities 2 and 3 in the function field case, and the corollary still holds when $\dim F = 0$.

2.4 Additional inequalities

We will need the following results to proceed with our two main theorems. All of the following except Proposition 2.4 were proven by Nesterenko in the number field case in [7], and by John Zuehlke in the function field setting in [9]. Proposition 2.4 was proven for archimedean absolute values by Nesterenko in [6] and is proven for non-archimedean situations in the appendix.

Proposition 2.3. *Let $\mathfrak{I} \subset \mathbb{K}[X]$ be a homogeneous unmixed ideal, F its associated Chow form, $r = 1 + \dim \mathfrak{I} \geq 1$, and let $|\cdot|$ be an absolute value. For each nonzero point $\underline{\omega} \in \mathcal{K}^{n+1}$, there exists a zero $\underline{\beta} \in \mathcal{K}^{n+1}$ of F such that*

$$\deg F \, \log ||\underline{\omega} - \underline{\beta}|| \leq \frac{1}{r} \log |F|_{\underline{\omega}} + \frac{1}{r} h(F) + (\nu + 3) n^3 \deg F.$$

If the absolute value $|\cdot| = |\cdot|_w$, $w \in \mathcal{M}$, is non-archimedean, then the factor $\nu + 3$ on the right-hand side of this inequality can be replaced by ν. (In the function field setting, $\nu = 0$ so the final term disappears entirely.)

Proposition 2.4. *Suppose that $\underline{\omega} \in \mathbb{K}^{n+1} \setminus \{0\}$, $\mathcal{Z}$ is an unmixed cycle with associated Chow form F, Q is a homogeneous polynomial with Q not in any of the component prime ideals of $\mathcal{Z}$. Suppose further that there exists μ such that $0 < \mu \leq 1$*

and for all $\underline{\beta}$ *a zero of* $\mathcal{Z}$*,*

$$||\underline{\omega} - \underline{\beta}||^{\mu} \geq |Q(\underline{\omega})| \, |\underline{\omega}|^{-\deg Q}.$$

Then $\mathrm{Res}(F,Q)$ *satisfies the inequality:*

$$\log|(\mathrm{Res}(F,Q))(\underline{\omega})| < \mu \log|F(\underline{\omega})| + \deg Q \, h(F) + \deg F \, h(Q) + 8n^2 \deg F \, \deg Q.$$

If $|\cdot|$ *is non-archimedean, then the factor* $8n^2 \deg F \, \deg Q$ *can be omitted.*

Proposition 2.5. *Assume* $F = \prod_{i=1}^{r} F_i^{e_i}$ *as in Equation 2.1, and let* $\underline{\omega} \in \mathcal{K}^{n+1}$*,* $\underline{\omega} \neq 0$*. Then*

1. $\displaystyle\sum_{i=1}^{r} e_i \deg F_i = \deg F,$
2. $\displaystyle\sum_{i=1}^{r} e_i h(F_i) \leq h(F) + \nu n^2 \deg F,$
3. $\displaystyle\sum_{i=1}^{r} e_i \log|F_i|_{\underline{\omega}} \leq \log|F|_{\underline{\omega}} + n^3 \deg F.$

If $\mathcal{M}_\infty = \emptyset$*, then the equality holds in (2). If* $|\cdot|$ *is a non-archimedean absolute value, then the equality also holds in (3); moreover, the term* $n^3 \deg F$ *on the right-hand side must be omitted.*

Proposition 2.6. *If* V *is a homogeneous polynomial of the ring* $\mathbb{K}[X]$ *and* $\underline{\omega}, \underline{\xi}$ *are nonzero points, and moreover if* $V(\underline{\xi}) = 0$*, then the following inequality holds:*

$$||V||_{\underline{\omega}} \leq ||\underline{\omega} - \underline{\xi}|| \, \exp((2n+1)\deg V).$$

If the absolute value $|\cdot|$ *is non-archimedean, then the factor* $\exp((2n+1)\deg V)$ *can be omitted.*

Chapter 3

Lower bounds based on Jabbouri-Philippon

In this chapter we will prove the first of our two results. This theorem generalizes the Jabbouri-Philippon measure for algebraic independence in number fields and creates a new analogue in function fields.

3.1 Hypotheses

Let $\underline{\omega} \in \mathbb{K}^{m+1} \setminus \{0\}$ for a field $\mathbb{K}$. Suppose that $\mathcal{Z}$ is an unmixed cycle of dimension $d \geq 0$. Let C and c_1 be constants greater than 1 not depending on $\underline{\omega}$ or $\mathcal{Z}$. Let $c_2 > 3c_1$ and let c_3 be any constant greater than or equal to $11(\nu+3)m^3$. Let $D_Q \in \mathbb{N}$, $h_Q > 1$,

$$N_0 = \left\lfloor \frac{c_3}{c_1(3C)^{d+1}} \left(\frac{D_Q}{(d+1)\deg \mathcal{Z}} + h_Q \right) \right\rfloor, \text{ and}$$

$$N_1 = \left\lceil \frac{c_3}{c_2} {D_Q}^d \left(D_Q h(\mathcal{Z}) + (d+1) h_Q \deg \mathcal{Z} \right) \right\rceil.$$

Assume that for every N in the interval $[N_0, N_1]$ there exists a finite family of polynomials $Q_j \in \mathbb{K}[X_1, \ldots, X_m]$ such that

- $\max_j |Q(\underline{\omega})| \leq \exp(-c_2(3C)^{d+1}N)$
- $\deg Q_j \leq D_Q$
- $h(Q_j) \leq h_Q$

- The Q_j lack a common zero in $B(\underline{\omega}, \exp(-c_1(3C)^{d+2}N))$.

3.2 Lower bound

THEOREM 1. *Under the above conditions,*

$$\log|\mathcal{Z}|_{\underline{\omega}} \geq -c_3(3C)^{d+1}{D_Q}^d(D_Q\, h(\mathcal{Z}) + (d+1)h_Q \deg \mathcal{Z}).$$

3.3 Proof of result

We define F_d to be the Chow form of $\mathcal{Z}$, and then define F_i inductively by letting $F_i = \mathrm{Res}(F_{i+1}, Q_j)$ for a sequence of Q_j's defined later.

The proof involves linking a sequence of assertions $\mathcal{A}_i$, one for each codimension $d-i$, $i = 0, \ldots, d$. Assertion $\mathcal{A}_d.1$ is equivalent to the negation of Theorem 1, and the major part of the proof consists of establishing that $\mathcal{A}_i \Rightarrow \mathcal{A}_{i-1}$ for $i = d, d-1, \ldots, 1$.

Assertion $\mathcal{A}_i$:

1. $\log|F_i|_\omega \leq -c_3(3C)^{i+1}{D_Q}^i\left(D_Q\, h(F_i) + (i+1)h_Q \deg F_i\right)$.
2. $\deg F_i \leq (\deg \mathcal{Z}){D_Q}^{d-i}$
3. $h(F_i) \leq h(\mathcal{Z})\, {D_Q}^{d-i} + (d-i)h_Q(\deg \mathcal{Z}){D_Q}^{d-i-1}$

We will prove the induction step for all three parts. Examining the base case, we will see that $\mathcal{A}_0.1$ fails as we interpret $|F_0|_{\underline{\omega}}$ to be 1 and $\deg F_0$ to be 0.

3.3.1 Inductive step for $\mathcal{A}_i.1$

Denote by ρ_i the minimum projective distance from ω to a zero of F_i. Consider the interval $[N_0, N_1]$. Choose M_i maximal in this interval such that $\rho_i \leq \exp(-c_1(3C)^{d+2} M_i)$.

First, we show that M_i actually lies in this interval; that is, N_0 satisfies

$$\rho_i \leq \exp(-c_1(3C)^{d+2} N_0).$$

By Proposition 2.3, we have

$$\log \rho_i \leq \frac{\log |F_i| + h(F_i) + (\nu+3)(i+1)m^3 \deg F_i}{\deg F_i (i+1)},$$

where the factor $\nu + 3$ can be replaced by ν in a non-archimedean case.

So it is sufficient to show that

$$\frac{\log |F_i| + h(F_i) + (\nu+3)(i+1)m^3 \deg F_i}{\deg F_i (i+1)} \leq -c_1(3C)^{d+2} N_0,$$

which is equivalent to

$$N_0 \leq -\frac{\log |F_i| + h(F_i) + (\nu+3)(i+1)m^3 \deg F_i}{c_1(3C)^{d+2}(i+1)(\deg F_i)}.$$

By our definition of N_0, we have:

$$
\begin{aligned}
N_0 &\leq \lfloor \frac{c_3}{c_1(3C)^{d+2}} \left(\frac{D_Q}{(d+1)\deg \mathcal{Z}} + h_Q \right) \rfloor \\
&\leq \frac{c_3 {D_Q}^i}{c_1(3C)^{d-i+2}} \left(\frac{{D_Q}^{1-d+i}}{(i+1)\deg \mathcal{Z}} + h_Q \right) \text{ for } 0 \leq i \leq d \\
&\leq \frac{c_3 {D_Q}^i}{c_1(3C)^{d-i+2}} \left(\frac{D_Q h(F_i)}{(i+1){D_Q}^{d-i}\deg \mathcal{Z}} + h_Q \right) \text{ since } h(F_i) \geq 1 \\
&\leq \frac{c_3 {D_Q}^i}{c_1(3C)^{d-i+2}} \left(\frac{D_Q h(F_i)}{(i+1)\deg F_i} + h_Q \right) \\
&= \frac{c_3(3C)^i {D_Q}^i (D_Q h(F_i) + (i+1) h_Q \deg F_i)}{c_1(3C)^{d+2}(i+1)\deg F_i} \text{ by } \mathcal{A}_i.2\text{, justified shortly} \\
&\leq \frac{c_3(3C)^{i+1} {D_Q}^i (D_Q h(F_i) + (i+1) h_Q \deg F_i)}{c_1(3C)^{d+2}(i+1)\deg F_i} \\
&\quad + \frac{-h(F_i) - (\nu+3)(i+1)m^3 \deg F_i}{c_1(3C)^{d+2}(i+1)\deg F_i} \text{ by our bounds on } c_3 \\
&\leq -\left(\frac{\log|F_i| + h(F_i) + (\nu+3)(i+1)m^3 \deg F_i}{c_1(3C)^{d+2}(i+1)\deg F_i} \right).
\end{aligned}
$$

So $N_0 \leq M_i$, as desired.

Case 1: $M_i < N_1$. Then by the maximality of M_i, we have

$$\exp(-c_1(3C)^{d+2}(M_i+1)) < \rho_i \leq \exp(-c_1(3C)^{d+2}M_i).$$

Then we choose Q_{j_i} and let $F_{i-1} := \mathrm{Res}(F_i, Q_{j_i})$. Then from the bounds on Q_{j_i} we have:

$$\begin{aligned}
\log|Q_{j_i}(\underline{\omega})| &\leq -c_2(3C)^{d+1}M_i \\
&< -c_1 3^{d+2}C^{d+1}M_i \\
&\leq \frac{M_i}{C(M_i+1)}(-c_1(3C)^{d+2}(M_i+1)) \\
&\leq \frac{1}{2C}(-c_1(3C)^{d+2}(M_i+1))
\end{aligned}$$

This yields $|Q_{j_i}(\underline{\omega})| < |\rho_i|^{\frac{1}{2C}}$.

Applying Proposition 2.4, we get

$$\begin{aligned}
\log|F_{i-1}|_\omega &\leq \frac{1}{2C}\log|F_i|_\omega + \deg F_i\, h_Q + h(F_i)\, D_Q + 8m^2 \deg F_i\, D_Q \\
&\leq \frac{1}{2C}\left(-c_3(3C)^{i+1}{D_Q}^i(D_Q\, h(F_i) + (i+1)h_Q \deg F_i)\right) \\
&\quad + \deg F_i\, h_Q + h(F_i)\, D_Q + 8m^2 \deg F_i\, D_Q \\
&\leq \frac{3}{2}\left(-c_3(3C)^{i}{D_Q}^i(D_Q\, h(F_i) + (i+1)h_Q \deg F_i)\right) \\
&\quad + \deg F_i\, h_Q + h(F_i)\, D_Q + 8m^2 \deg F_i\, D_Q \\
&\leq -c_3(3C)^{i}{D_Q}^i(D_Q\, h(F_i) + (i+1)h_Q \deg F_i) \\
&\quad + \frac{1}{2}(-c_3(3C)^{i}{D_Q}^i(D_Q\, h(F_i) + (i+1)h_Q \deg F_i)) \\
&\quad + \deg F_i\, h_Q + h(F_i)\, D_Q + 8m^2 \deg F_i\, D_Q \\
&\leq -c_3(3C)^{i}{D_Q}^i(D_Q\, h(F_i) + (i+1)h_Q \deg F_i) \text{ by our bounds on } c_3
\end{aligned}$$

Note that from Proposition 2.1(1), we know that

$$\deg F_{i-1} \leq \deg F_i \, D_Q \text{ or equivalently, } ih_Q \deg F_{i-1} \leq ih_Q \deg F_i D_Q$$

and, similarly, from Proposition 2.1(2) we have

$$h(F_{i-1}) \leq D_Q h(F_i) + h_Q \deg F_i, \text{ or } D_Q h(F_{i-1}) \leq D_Q(D_Q h(F_i) + h_Q \deg F_i).$$

Applying this to our above inequality for $\log |F_{i-1}|_\omega$, we get

$$\begin{aligned}
\log |F_{i-1}|_\omega &\leq -c_3 (3C)^i {D_Q}^i (D_Q h(F_i) + (i+1) h_Q \deg F_i) \\
&\leq -c_3 (3C)^i {D_Q}^{i-1} ({D_Q}^2 h(F_i) + D_Q (i+1) h_Q \deg F_i) \\
&\leq -c_3 (3C)^i {D_Q}^{i-1} ({D_Q}^2 h(F_i) + D_Q h_Q \deg F_i + i D_Q h_Q \deg F_i) \\
&\leq -c_3 (3C)^i {D_Q}^{i-1} (D_Q h(F_{i-1}) + i h_Q \deg F_{i-1}),
\end{aligned}$$

which proves $\mathcal{A}_i(1)$ in this case.

Case 2: $M_i = N_1$. Here, we can find a Q_{j_i} such that

$$F_{i-1} := \mathrm{Res}(F_i, Q_{j_i}) \neq 0.$$

Otherwise, we would have that all of the Q_j in our family have the zeroes of F_i in common. But F_i has zeroes within ρ_i of ω, whereas the Q_j do not have such zeros in common by definition.

Examining $|Q_{j_i}(\underline{\omega})|$, we find

$$
\begin{aligned}
\log |Q_{j_i}(\underline{\omega})| &\leq -c_2(3C)^{d+1} N_1 \\
&\leq -c_2(3C)^{d+1} \frac{c_3}{c_2} {D_Q}^d \left(D_Q h(\mathcal{Z}) + (d+1) h_Q \deg \mathcal{Z} \right) \\
&= -c_2(3C)^{d+1} \frac{c_3}{c_2} (3C)^{i-d} {D_Q}^i \\
&\quad \times \left(D_Q h(\mathcal{Z}) {D_Q}^{d-i} + (d+1) h_Q \deg \mathcal{Z} {D_Q}^{d-i} \right) \text{ for } 1 \leq i \leq d \\
&\leq -c_2(3C)^{d+1} \frac{c_3}{c_2} (3C)^{i-d} {D_Q}^i \\
&\quad \times \Bigg(D_Q \left(h(\mathcal{Z}) {D_Q}^{d-i} + (d-i) \deg \mathcal{Z} {D_Q}^{d-i-1} \right) \\
&\quad + (i+1) h_Q \deg \mathcal{Z} {D_Q}^{d-i} \Bigg) \\
&\leq -c_2(3C)^{d+1} \frac{c_3}{c_2} (3C)^{i-d} {D_Q}^i \left(D_Q h(F_i) + (i+1) h_Q \deg F_i \right) \\
&\quad \text{as we will show in 3.3.3} \\
&= -c_3(3C)^{i+1} {D_Q}^i (D_Q\, h(F_i) + (i+1) h_Q \, \deg F_i).
\end{aligned}
$$

Now

$$\begin{aligned}
\log|F_{i-1}|_{\underline{\omega}} &\le \max\{\log|F_i|_\omega, |Q_{j_i}(\underline{\omega})|\} + h(F_i)D_Q + h_Q \deg F_i \\
&\quad + 11\nu m^2 \deg F_i D_Q \\
&\le -c_3(3C)^{i+1} {D_Q}^i (D_Q\, h(F_i) + (i+1)h_Q\, \deg F_i) \\
&\quad + h(F_i)\, D_Q + h_Q\, \deg F_i + 11\nu m^2 \deg F_i D_Q \\
&\le -c_3(3C)^i {D_Q}^i (D_Q\, h(F_i) + (i+1)h_Q \deg F_i) \text{ by our bounds on } c_3.
\end{aligned}$$

In either case, we have constructed an F_{i-1}, and we proceed to prove the remainder of the inductive hypothesis.

3.3.2 Inductive step for $\mathcal{A}_i.2$

We start at $i = d$. By Proposition 2.1(1) we get

$$\begin{aligned}
\deg F_{d-1} &\le \deg F_d\, D_Q \\
&= \deg \mathcal{Z}\, D_Q.
\end{aligned}$$

Now assume $\deg F_i \leq \deg \mathcal{Z}\, D_Q^{d-i}$. Then as above, we get

$$\begin{aligned}\deg F_{i-1} &\leq \deg F_i\, D_Q \\ &\leq (\deg \mathcal{Z}\, {D_Q}^{d-i})\, D_Q \\ &\leq \deg \mathcal{Z}\, {D_Q}^{d-(i-1)}.\end{aligned}$$

3.3.3 Inductive step for $\mathcal{A}_i.3$

We again start at $i = d$. By Proposition 2.1(2), we have

$$\begin{aligned}h(F_{d-1}) &\leq h(F_d)D_Q + h_Q \deg F_d \\ &= h(\mathcal{Z})D_Q + h_Q \deg \mathcal{Z}.\end{aligned}$$

Now assume $h(F_i) \leq h(\mathcal{Z}){D_Q}^{d-i} + (d-i)h_Q \deg \mathcal{Z}\, {D_Q}^{d-i-1}$. Then by Proposition 2.1,

$$\begin{aligned}h(F_{i-1}) &\leq h(F_i)D_Q + h_Q \deg F_i \\ &\leq \left(h(\mathcal{Z}){D_Q}^{d-i} + (d-i)h_Q \deg \mathcal{Z}{D_Q}^{d-i-1}\right) D_Q + h_Q \deg \mathcal{Z}\, {D_Q}^{d-i} \\ &\leq h(\mathcal{Z})\, {D_Q}^{d-(i-1)} + (d-(i-1))h_Q \deg \mathcal{Z}{D_Q}^{d-(i-1)-1}.\end{aligned}$$

3.3.4 Base case

This completes the inductive step in the proof of our assertion. Now consider $\mathcal{A}_0.1$. Here, F_0 is the product of the polynomials Q_j evaluated at the generic zeroes of the ideal underlying F_1. In other words, F_0 is simply a constant, which by definition has degree 0. Thus F_0 must satisfy $\mathcal{A}_0.2$ and $\mathcal{A}_0.3$. However, if we try to apply $\mathcal{A}_0.1$, we get

$$\log|F_0|_{\underline{\omega}} \leq -c_3(3C)D_Q\, h(F_0).$$

But by definition,

$$\begin{aligned}\log|F_0|_{\underline{\omega}} &= \log|F_0(S^{(0)}\underline{\omega},\ldots,S^{(d)}\underline{\omega})| - (d+1)\deg F_0(\log|\underline{\omega}|)\\ &= h(F_0).\end{aligned}$$

This means $h(F_0) \leq -c_3(3C)D_Q\, h(F_0)$, which is false as $h(F_0)$, C, c_3, and D_Q are all greater than zero. Thus our inductive hypothesis cannot hold for any value of $i = 0,\ldots,d$. So we must have

$$\log|F_d|_{\underline{\omega}} = \log|\mathcal{Z}|_{\underline{\omega}} > -c_3(3C)^{d+1}{D_Q}^d(D_Q h(\mathcal{Z}) + (d+1)h_Q \deg \mathcal{Z}),$$

as desired.

Chapter 4

Lower bounds based on Nesterenko

4.1 Hypotheses

Let $\underline{\omega} \in \mathbb{K}^{m+1} \setminus \{0\}$ for a field $\mathbb{K}$. Let a and b be two rational numbers with $a > 1$. Let c_1, c_2, c_3, and c_4 be positive real constants with $c_3 > c_4$. Let $d \in \mathbb{Z}_{\geq 0}$, $d \neq a$, and $s \in \mathbb{Q}$. Define $\alpha_d := \frac{a}{a-d}$ and $\beta_d := \frac{-bd}{a-d}$.

Assume that there exists an integer N_0 such that for all integers $N \geq N_0$ there is a (finite) family of polynomials $\mathcal{F}_N = \{Q\}$ such that

- $\deg Q \leq c_1 N$
- $h(Q) \leq c_2 N (\log N)^s$
- $-c_3 N^a (\log N)^b \leq \log |Q(\underline{\omega})| \leq -c_4 N^a (\log N)^b$

4.2 Lower bound

Theorem 2. *Under the above conditions, there exists a constant $\mu_d > 0$ depending on c_3, c_4, and N_0 such that for any unmixed cycle $\mathcal{Z}$ of dimension d, the following inequality holds:*

$$\log |\mathcal{Z}|_{\underline{\omega}} \geq -\mu_d D^{\alpha_d} (\log D)^{\beta_d},$$

where D is an arbitrarily large number satisfying

$$D \geq (\max\{1, \exp(bd/a)\}) \left(h(\mathcal{Z}) + \deg \mathcal{Z}(\log(h(\mathcal{Z}) + \deg \mathcal{Z}))^{s}\right).$$

4.3 Proof of result

We shall prove the theorem by descending induction in d, extending the proof given by Nesterenko in [7] in the archimedean case where $0 \leq d \leq 3$, $a = 4$, $b = -3$, and $s = 1$. Let d be the least number for which the assertion of Theorem 2 is no longer true. Since d is the dimension of a cycle, $d \geq 0$.

Note that we will use the notation $//_{\kappa}$ to indicate an inequality where we are assuming the constant κ is sufficiently large.

4.3.1 Existence of unbounded D

Let c_5 be an arbitrarily large constant. We claim that the set of numbers D for which there exists a Chow form F of dimension d,

$$(\max\{1, \exp(bd/a)\}) \left(h(F) + \deg F \left(\log(h(F) + \deg F)\right)^{s}\right) \leq D, \tag{4.1}$$

$$\log |F|_{\underline{\omega}} < -2c_5 D^{\alpha_d} (\log D)^{\beta_d}, \tag{4.2}$$

is unbounded. Assume not. Define

$$\varphi(F) := (\max\{1, \exp(bd/a)\}) \left(h(F) + \deg F \left(\log(h(F) + \deg F)\right)^{s}\right).$$

Then there must exist a positive constant C such that

$$\log |F|_\omega \geq -C\,\varphi(F)^{\alpha_d} \left(\log \varphi(F)\right)^{\beta_d} \tag{4.3}$$

would hold for *all* Chow forms F with dimension d. Let G be such a Chow form of dimension d. Write $G = \prod_{i=1}^{t} F_i^{e_i}$ as in Equation 2.1.

Then by Proposition 2.5(1) and (2),

$$\sum_{i=1}^{t} e_i \varphi(F_i) = \sum_{i=1}^{t} e_i \left(\max\{1, \exp(bd/a)\}\right) \left(h(F_i) + \deg F_i \left(\log(h(F_i) + \deg F_i)\right)^s\right)$$

$$\leq \gamma_1 \varphi(G) \text{ for some constant } \gamma_1.$$

Now consider the function $f(x) = x^\lambda (\log x)^\delta$ for some constants λ and δ. By examining its derivative, we see that it monotonically increases when

$$x \geq \max\{1, \exp\left(-\frac{\delta}{\lambda}\right)\}. \tag{4.4}$$

Letting $\lambda = \alpha_d$, $\delta = \beta_d$, we note that $\varphi(F_i)$ is greater than or equal to the quantity on the right hand side of Equation (4.4), so we get

$$\sum_{i=1}^{t} e_i \varphi(F_i)^{\alpha_d} \left(\log \varphi(F_i)\right)^{\beta_d} \leq \left(\gamma_1 \varphi(G)\right)^{\alpha_d} \left(\log\left(\gamma_1 \varphi(G)\right)\right)^{\beta_d}.$$

Now Proposition 2.5(3) gives us

$$\begin{aligned}
\log |G|_{\underline{\omega}} &\geq \sum_{i=1}^{t} e_i \log |F_i|_{\underline{\omega}} - \nu m^2 \deg G \\
&\geq -C \sum_{i=1}^{t} e_i \varphi(F_i)^{\alpha_d} \left(\log \varphi(F_i)\right)^{\beta_d} - \nu m^2 \deg G \text{ by Equation (4.3)} \\
&\geq -C \left((\gamma_1 \varphi(G))^{\alpha_d} \left(\log \left(\gamma_1 \varphi(G)\right)\right)^{\beta_d} \right) \\
&=: \gamma_2 \left(\varphi(G)\right)^{\alpha_d} \left(\log \left(\varphi(G)\right)\right)^{\beta_d} \text{ for some constant } \gamma_2.
\end{aligned}$$

But this contradicts our assumption that d is the smallest number not satisfying the assertion of Theorem 2. Thus we can choose an arbitrarily large D and a Chow form F that satisfies conditions (4.1) and (4.2).

4.3.2 Choice of and bounds on N

Choose N such that

$$2c_3 N^a (\log N)^b = \min\{2c_5 D^{\alpha_d} (\log D)^{\beta_d}, \log \frac{1}{\rho}\}. \tag{4.5}$$

Proposition 2.3 tells us that N increases to infinity as D does.

From Equation (4.4), we know that the function $f(x) = x^a (\log x)^b$ increases for x sufficiently large. Choose γ_3 such that

$$\gamma_3 \log D \geq \frac{1}{a} \left(\log \frac{c_3}{c_5 {\gamma_3}^b} + \alpha_d \log D + (\beta_d - b) \log \log D \right),$$

and notice that

$$f\left(\left(\frac{c_5}{c_3\gamma_3{}^b}\right)^{\frac{1}{a}} D^{\frac{\alpha_d}{a}} (\log D)^{\frac{\beta_d-b}{a}}\right) = \frac{c_5}{c_3\gamma_3{}^b} D^{\alpha_d} (\log D)^{\beta_d-b}$$

$$\times \left(\frac{1}{a}\log\frac{c_5}{c_3\gamma_3{}^b} + \frac{\alpha_d}{a}\log D + \frac{\beta_d-b}{a}\log\log D\right)^b$$

$$\geq \frac{c_5}{c_3\gamma_3{}^b} D^{\alpha_d} (\log D)^{\beta_d-b} (\gamma_3 \log D)^b$$

$$= \frac{c_5}{c_3} D^{\alpha_d} (\log D)^{\beta_d}.$$

Thus

$$f(N) = N^a (\log N)^b \leq \frac{c_5}{c_3} D^{\alpha_d} (\log D)^{\beta_d} \leq f\left(\left(\frac{c_5}{c_3\gamma_3{}^b}\right)^{\frac{1}{a}} D^{\frac{\alpha_d}{a}} (\log D)^{\frac{\beta_d-b}{a}}\right).$$

Since we can choose D large enough so that $N \geq \max\{1, \exp\left(-\frac{b}{a}\right)\}$, we get

$$N \leq \left(\frac{c_5}{c_3\gamma_3{}^b}\right)^{\frac{1}{a}} D^{\frac{\alpha_d}{a}} (\log D)^{\frac{\beta_d-b}{a}}. \qquad //_D \tag{4.6}$$

Now the above also gives us

$$\log N \leq \frac{1}{a}\log\left(\frac{c_5}{c_3\gamma_3{}^b}\right) + \frac{\alpha_d}{a}\log D + \frac{\beta_d-b}{a}\log\log D$$

$$\leq \gamma_3 \log D \tag{4.7}$$

4.3.3 Induction step

We wish to apply Corollary 2.2 to $\mathrm{Res}(F,Q)$. First, we claim that Q is not contained in the ideal associated with the Chow form F. For if it were, then by Proposition 2.6 and Definition 2.2,

$$\begin{aligned}
\log|Q(\underline{\omega})| &\leq \log\rho + h(Q) + N\log|\underline{\omega}| + \exp\left((2m+1)\deg Q\right)\\
&\leq -2c_3N^a(\log N)^b + N(\log N)^s + N\log|\underline{\omega}| + \exp\left((2m+1)\deg Q\right)\\
&\leq -c_3N^a(\log N)^b, \qquad //_N
\end{aligned}$$

which contradicts the lower bound on $|Q(\overline{\omega})|$.

Next, set $\eta = 1 + [4c_3/c_4]$. Since

$$\begin{aligned}
-\eta\log||Q||_{\overline{\omega}} &\geq -\eta\log|Q(\overline{\omega})|\\
&\geq \eta c_4N^a(\log N)^b\\
&> 4c_3N^a(\log N)^b\\
&= 2\min\{2c_5D^{\alpha d}(\log D)^{\beta d}, \log\frac{1}{\rho}\}
\end{aligned}$$

from (4.5), we get the inequality needed to apply Corollary 2.2 (with $S = 2c_5D^{\alpha d}(\log D)^{\beta d}$). Note that by our construction of D, $h(F) \leq D$ and

$\deg F \leq 2D(\log D)^{-s}$. Thus through Corollary 2.2, we see that

$$\begin{aligned}
\deg \operatorname{Res}(F,Q) &\leq \eta \deg F \deg Q \\
&\leq 2D(\log D)^{-s} c_1 N \\
&\leq 2D(\log D)^{-s} c_1 \left(\frac{c_5}{c_3 {\gamma_3}^b}\right)^{\frac{1}{a}} D^{\frac{\alpha_d}{a}} (\log D)^{\frac{\beta_d - b}{a}} \text{ by (4.6)} \\
&=: \gamma_4 D^{\frac{\alpha_d + a}{a}} (\log D)^{\frac{\beta_d - b - sa}{a}}; \qquad (4.8)
\end{aligned}$$

$$\begin{aligned}
h(\operatorname{Res}(F,Q)) &\leq \eta \left(h(F) \deg Q + h(Q) \deg F + \nu m(d+3) \deg F \deg Q\right) \\
&\leq D c_1 N + c_2 N (\log N)^s \, 2D (\log D)^{-s} \\
&\leq DN \left(c_1 + 2c_2 \left(\frac{\log N}{\log D}\right)^{-s}\right) + \nu m(d+3) \deg F \deg Q \\
&\leq DN \left(c_1 + 2c_2 {\gamma_3}^{-s}\right) + \nu m(d+3) \deg F \deg Q \text{ by (4.7)} \\
&\leq \left(c_1 + 2c_2 {\gamma_3}^{-s}\right) \left(\frac{c_5}{c_3 {\gamma_3}^b}\right) D^{\frac{\alpha_d + a}{a}} (\log D)^{\frac{\beta_d - b}{a}} \\
&\quad + \nu m(d+3) \deg F \deg Q \text{ by (4.6)} \\
&=: \gamma_5 D^{\frac{\alpha_d + a}{a}} (\log D)^{\frac{\beta_d - b}{a}}; \qquad //_D \qquad (4.9)
\end{aligned}$$

and

$$\begin{aligned}
\log|\mathrm{Res}(F,Q)|_{\underline{\omega}} &\leq -S + \eta\left(h(F)\deg Q + h(Q)\deg F\right) \\
&\leq -2c_5 D^{\alpha_d}(\log D)^{\beta_d} \\
&\quad + \gamma_5 D^{\frac{\alpha_d + a}{a}}(\log D)^{\frac{\beta_d - b}{a}} \text{ by (4.9)} \\
&\leq -c_5 D^{\alpha_d}(\log D)^{\beta_d}. \qquad //_{c_5} \qquad (4.10)
\end{aligned}$$

Put together, equations 4.8 and 4.9 mean that

$$
\begin{aligned}
\varphi(\operatorname{Res}(F,Q)) &= (\max\{1,\exp(bd/a)\}) \big(h(\operatorname{Res}(F,Q)) \\
&\quad + \deg\operatorname{Res}(F,Q)(\log(h(\operatorname{Res}(F,Q)) + \deg\operatorname{Res}(\mathrm{F},\mathrm{Q})))^{\mathrm{s}}\big) \\
&\le (\max\{1,\exp(bd/a)\}) \Bigg(\gamma_5 D^{\frac{\alpha_d+a}{a}} (\log D)^{\frac{\beta_d-b}{a}} \\
&\quad + \Bigg(\gamma_4 D^{\frac{\alpha_d+a}{a}} (\log D)^{\frac{\beta_d-b-sa}{a}}\Bigg) \\
&\quad \times \log\Bigg(\gamma_5 D^{\frac{\alpha_d+a}{a}} (\log D)^{\frac{\beta_d-b}{a}} + \gamma_4 D^{\frac{\alpha_d+a}{a}} (\log D)^{\frac{\beta_d-b-sa}{a}}\Bigg)^s\Bigg) \\
&\le (\max\{1,\exp(bd/a)\}) \Bigg(\gamma_5 D^{\frac{\alpha_d+a}{a}} (\log D)^{\frac{\beta_d-b}{a}} \\
&\quad + \Bigg(\gamma_4 D^{\frac{\alpha_d+a}{a}} (\log D)^{\frac{\beta_d-b-sa}{a}}\Bigg) \\
&\quad \times \Bigg(\log\gamma_5 + \frac{\alpha_d+a}{a}\log D + \frac{\beta_d-b}{a}\log\log D + \log\gamma_4 \\
&\quad + \frac{\alpha_d+a}{a}\log D + \frac{\beta_d-b-sa}{a}\log\log D\Bigg)^s\Bigg) \\
&< (\max\{1,\exp(bd/a)\}) \Bigg(\gamma_5 D^{\frac{\alpha_d+a}{a}} (\log D)^{\frac{\beta_d-b}{a}} \\
&\quad + \gamma_4 D^{\frac{\alpha_d+a}{a}} (\log D)^{\frac{\beta_d-b-sa}{a}} \left(2\,\frac{\alpha_d+a}{a}\log D\right)^s\Bigg) \qquad //_D \\
&=: \gamma_6 D^{\frac{\alpha_d+a}{a}} (\log D)^{\frac{\beta_d-b}{a}},
\end{aligned}
$$

which leads to

$$\log \varphi(\mathrm{Res}(F,Q)) = \log \gamma_6 + \frac{\alpha_d + a}{a} \log D + \frac{\beta_d - b}{a} \log\log D$$

$$=: \gamma_7 \log D. \qquad //_D$$

So now

$$\begin{aligned}(\varphi(\mathrm{Res}(F,Q)))^{\alpha_{d-1}} (\log \varphi(\mathrm{Res}(F,Q)))^{\beta_{d-1}} &\leq \gamma_6{}^{\alpha_{d-1}} \gamma_7{}^{\beta_{d-1}} D^{\left(\frac{\alpha_d+a}{a}\right)\alpha_{d-1}} \\ &\quad \times (\log D)^{\left(\left(\frac{\beta_d-b}{a}\right)\alpha_{d-1}\right)+\beta_{d-1}} \\ &=: \gamma_8 D^{\left(\frac{\alpha_d+a}{a}\right)\alpha_{d-1}} \\ &\quad \times (\log D)^{\left(\left(\frac{\beta_d-b}{a}\right)\alpha_{d-1}\right)+\beta_{d-1}}.\end{aligned}$$

Examining the coefficient of D, we see that

$$\begin{aligned}\left(\frac{\alpha_d + a}{a}\right)\alpha_{d-1} &= \left(\frac{\frac{a}{a-d} + a}{a}\right)\left(\frac{a}{a-d+1}\right) \\ &= \left(\frac{a}{a-d} + a\right)\left(\frac{1}{a-d+1}\right) \\ &= \left(\frac{a + a^2 - ad}{a-d}\right)\left(\frac{1}{a-d+1}\right) \\ &= \frac{a}{a-d} = \alpha_d.\end{aligned}$$

Doing the same for the coefficient of $\log D$:

$$\left(\frac{\beta_d - b}{a}\right)\alpha_{d-1} + \beta_{d-1} = \left(\frac{\frac{-bd}{a-d} - b}{a}\right)\left(\frac{a}{a-d+1}\right) + \frac{-b(d-1)}{a-d+1}$$

$$= \left(\frac{-bd - ba + bd}{a-d}\right)\left(\frac{1}{a-d+1}\right) + \frac{-bd+b}{a-d+1}$$

$$= \frac{-ba}{(a-d)(a-d+1)} + \frac{-abd + bd^2 + ba - bd}{(a-d)(a-d+1)}$$

$$= \frac{-bd}{a-d} = \beta_d.$$

Thus

$$(\varphi(\mathrm{Res}(F,Q)))^{\alpha_{d-1}} (\log \varphi(\mathrm{Res}(F,Q)))^{\beta_{d-1}} \leq \gamma_8 D^{\alpha_d} (\log D)^{\beta_d}.$$

By our assumption on d, the inequality

$$\log |\mathrm{Res}(F,Q)|_{\underline{\omega}} \geq -\mu_{d-1}\varphi(\mathrm{Res}(F,Q))^{\alpha_{d-1}} (\log \varphi(\mathrm{Res}(F,Q)))^{\beta_{d-1}}$$

$$\geq -\mu_{d-1}\gamma_8 D^{\alpha_d} (\log D)^{\beta_d}$$

must hold for $\mathrm{Res}(F,Q)$. But if we choose $c_5 \geq \mu_{d-1}\gamma_8$, this contradicts Equation (4.10). This contradiction completes the proof of Theorem 2.

Chapter 5

Applications

In this chapter we present two applications of our results and suggest areas for future research.

5.1 Application of first result to Gelfond-Feldman situation

We apply our result from Chapter 3 to the Gelfond-Feldman measure, Equation (1.1), extending Diaz's result to the non-archimedean number field case.

Let $\mathbb{K}$ be a number field, $\alpha \in \mathbb{K} \setminus \{0,1\}$, and β be algebraic over $\mathbb{K}$ of degree 3. Fix an unmixed cycle $\mathcal{Z}$ of dimension 1 and let F be its chow form.

We use the same auxiliary polynomials as in [4]. Thus we obtain a family of polynomials $P(x, y)$ for every N in the interval $[N_0, N_1]$ with each polynomial being sufficiently small at the point $\underline{\omega} = (\alpha^{\beta}, \alpha^{\beta^2})$, lacking a common zero in a ball around $(\alpha^{\beta}, \alpha^{\beta^2})$, and satisfying

$$\deg P \leq N, \quad h(P) \leq N (\log N)^{s}.$$

We choose N such that $\log N \leq \gamma_1 \deg F\, h(F)$, and s such that $(\log N)^s \leq \gamma_2 h(F)/\deg F$. This puts us where we need to be to apply Theorem 1, obtaining

$$
\begin{aligned}
\log|F(\alpha^{\beta},\alpha^{\beta^2})| &\geq -c_3(3C)^2 D_Q(D_Q\,h(F)+2h_Q\deg F)\\
&\geq -c_3(3C)^2 N\left(Nh(F)+2N\,(\log N)^s\deg F\right)\\
&\geq -c_3(3C)^2\exp\left(\gamma_1\deg F\,h(F)\right)\Bigg(\exp\left(\gamma_1\deg F\,h(F)\right)h(F)\\
&\qquad +2\exp\left(\gamma_1\deg F\,h(F)\left(\gamma_2 h(F)/\deg F\right)\deg F\right)\Bigg)\\
&\geq -c_3\gamma_3(3C)^2 h(F)\exp\left(2\gamma_1\deg F\,h(F)\right).
\end{aligned}
$$

This produces a much sharper lower bound than in Equation (1.1), and it is comparable to Diaz's result.

5.2 Application of second result to Nesterenko situation

Let $\sigma_k(n)=\sum_{d|n}d^k$. Consider the Ramanujan functions

$$
\begin{aligned}
P(z) &= 1-24\sum_{n=1}^{\infty}\sigma_1(n)z^n,\\
Q(z) &= 1+240\sum_{n=1}^{\infty}\sigma_3(n)z^n,\\
R(z) &= 1-504\sum_{n=1}^{\infty}\sigma_5(n)z^n.
\end{aligned}
$$

Let $q\in\mathbb{C}$, $0<|q|<1$, and let $\omega_1,\omega_2,\omega_3\in\mathbb{C}$ be such that q, $P(q)$, $Q(q)$, and $R(q)$ are all algebraic over $\mathbb{Q}(\omega_1,\omega_2,\omega_3)$. Then there exists a constant

$\gamma_1 = \gamma_1(q, \omega_1, \omega_2, \omega_3)$ such that for any non-zero polynomial $A \in \mathbb{Z}[x_1, x_2, x_3]$,

$$|A(\omega_1, \omega_2, \omega_3)| \geq \exp\left(-\gamma_1 D^4 (\log D)^9\right),$$

where

$$D \geq \max\{h(A) + \deg A \times \log(\deg A + h(A)), e\}.$$

In other words, we have the Chow form of an unmixed ideal (of dimension 2) generated by the polynomials A and T, where T is a minimal polynomial vanishing at the point $(q, P(q), Q(q), R(q))$.

This implies, for example, that $\{\pi, e^{\pi}, \Gamma(1/4)\}$ are algebraically independent when $q = e^{-2\pi}$.

To apply the second result, let $d = 3$, $a = 4$, $b = -3$, and $s = 1$. Nesterenko constructs a family of polynomials $\{A\}$ such that

$$\deg A \leq N$$

$$h(A) \leq 2\gamma N \log N$$

$$\exp\left(-\varkappa_2 N^4 (\log N)^{-3}\right) \leq |A(\omega_1, \omega_2, \omega_3)| \leq \exp\left(-\varkappa_1 N^4 (\log N)^{-3}\right)$$

where $\gamma = 190(\log|\frac{r}{q}|)^{-1}$, $\varkappa_1 = \frac{1}{4}\log\frac{1}{r}$, $\varkappa_2 = 3c\log\frac{2}{|q|}$, $r = \min\{\frac{1+|q|}{2}, 2|q|\}$, and $c = 10^{47}$.

Thus by application of our result, we get that

$$|A(\omega_1, \omega_2, \omega_3)| > \exp\left(-\gamma_1 D^4 (\log D)^9\right),$$

as $\alpha_d = \frac{4}{4-3} = 4$ and $\beta_d = \frac{-(-3)(3)}{4-3} = 9$.

5.3 Possibilities for future applications

Our two theorems are suitable for application in the realm of number fields, just as the results they were based on are. However, combined with the power of John Zuehlke's work in [9], these results are among the first such measures of algebraic independence applicable to the function field case. For example in [1], Becker, Brownawell, and Tubbs proved a qualitative version of Gelfond's original theorem from Chapter 1 using the Drinfeld module equivalent of the exponential function. With the construction of appropriate polynomial families and zero estimates, these results should provide an analogue of the quantitative Gelfond-Feldman inequality.

Other results from what is known as the Gelfond-Schneider family of transcendence results can now be carried over to the function field setting using our methodology for measures of algebraic independence. It is also entirely possible that transcendence results unique to t-modules (i.e., with no known characteristic zero equivalent) may be uncovered as has happened in other branches of transcendence in positive characteristic.

Appendix

Proof of Proposition 2.4

This proof is almost identical to that for the Archimedean case in [6]; we include it here merely for completeness.

We cite two results needed for this proof. Again, both were proven in the number field case in [7], and in the function field case in [9]. We let $\varkappa(E)$ be the polynomial in the symmetric variables ${s_{jk}}^{(i)}$, $j < k$, from Chapter 2 with $i = 1, \ldots, r$, with coefficients in the field $\mathcal{K}$ which is obtained as a result of the substitution of the vector $S^{(i)}\underline{\omega}$ for the variables $\underline{u}_i$ in E.

Lemma A.1. *Let V and W be homogeneous polynomials from the ring $\mathbb{K}[X]$ of degree d. Then, for any nonzero vectors $\underline{\omega} = (\omega_0, \ldots, \omega_n) \in \mathcal{K}^{n+1}$, $\underline{\xi} = (\xi_0, \ldots, \xi_n) \in \mathcal{K}^{n+1}$, the following inequality holds:*

$$|V(\underline{\omega})W(\underline{\xi}) - V(\underline{\xi})W(\underline{\omega})| \leq ||\underline{\omega} - \underline{\xi}|| \times |V| \times |W| \times |\underline{\omega}|^d |\underline{\xi}|^d (d+1)^{2m+1}.$$

If the absolute value $|\cdot|$ is non-Archimedean, then the factor $(d+1)^{2m+1}$ on the right-hand side of the last inequality can be omitted.

Lemma A.2. *Suppose that $\mathcal{Z} \subset \mathbb{K}[X]$ is an unmixed cycle, F its Chow form, $r = 1 + \dim \mathcal{Z} \geq 1$, x_0 is not in any prime component of $\mathcal{Z}$, $\underline{\omega} \in \mathcal{K}^{n+1}$, $|\underline{\omega}| = 1$, and let $\varkappa(a) \neq 0$. Suppose, in addition, that $|\cdot| = |\cdot|_w$, $w \in \mathcal{M}$, is an absolute value on* *the field $\mathcal{K}$. Then there exists a homomorphism*

$$\tau : \mathbb{K}[\underline{u}_1, \ldots, \underline{u}_{r-1}, {\alpha_1}^{(1)}, \ldots, {\alpha_m}^{(\deg F)}] \to \mathcal{K}$$

which is identical on $\mathbb{K}$ and such that for ${\beta_i}^{(j)} = \tau\left({\alpha_i}^{(j)}\right) \in \mathcal{K}$, the vectors $\underline{\beta}_j = \left(1, {\beta_1}^{(j)}, \ldots, {\beta_m}^{(j)}\right)$, $1 \le j \le \deg F$, are zeros of F, and, moreover,

- *(1)* $|\tau(a)| \prod_{j=1}^{\deg F} |\underline{\beta}_j| \le |F| \exp(2n^2 \deg F)$,
- *(2)* $|\tau(a)| \prod_{j=1}^{\deg F} \left(||\underline{\omega} - \underline{\beta}_j|| \times |\underline{\beta}_j|\right) \le |F|_{\underline{\omega}} \times |F| \exp(2n^2 \deg F)$.

If $|\cdot|$ is a non-Archimedean absolute value, then the factors $\exp(2n^2 \deg F)$ can be omitted in these inequalities.

In addition, for each polynomial $H \in \mathbb{K}[\underline{u}_1, \ldots, \underline{u}_{r-1}]$, the homomorphism τ can be chosen such that

- *(3a)* $|\varkappa(H)| = |\tau(H)|$ *if* $|\cdot|$ *is non-Archimedean, or*
- *(3b)* $|\varkappa(H)| \le (1+\epsilon)|\tau(H)|$ *for any given* $\epsilon > 0$ *if* $|\cdot|$ *is an Archimedean absolute value.*

We now prove Proposition 2.4:

Proof. We can suppose that $|\underline{\omega}| = 1$. By assumption, $||\underline{\omega} - \underline{\beta}_j||^{\mu} \ge |Q(\underline{\omega})|$ for $j = 1, \ldots \deg F$. Applying Lemma A.1 to the polynomials Q and $R = {x_k}^{\deg Q}$, where the index k is chosen so that $|\omega_k| = |\underline{\omega}| = 1$, we find that

$$|Q(\underline{\omega})\underline{\beta}_j{}^{\deg Q} - Q(\underline{\beta}_j)\underline{\omega}^{\deg Q}| \leq ||\underline{\omega} - \underline{\beta}_j||\,|Q|\,|x_k{}^{\deg Q}|\,|\underline{\omega}|^{\deg Q}\,|\beta|^{\deg Q}$$

$$|Q(\underline{\beta}_j)| - |Q(\underline{\omega})|\,|\underline{\beta}_j|^{\deg Q} \leq ||\underline{\omega} - \underline{\beta}_j||\,\exp(h(Q))\,1\,|\underline{\beta}_j|^{\deg Q}\,1$$

$$|Q(\underline{\beta}_j)| \leq |Q(\underline{\omega})|\,|\underline{\beta}_j|^{\deg Q} + ||\underline{\omega} - \underline{\beta}_j||\,|\underline{\beta}_j|^{\deg Q}\,\exp(h(Q)).$$

Hence,

$$\begin{aligned}\log|Q(\underline{\beta}_j)| &\leq \deg Q \log|\underline{\beta}_j| + \log\Big(|Q(\underline{\omega})| + ||\underline{\omega} - \underline{\beta}_j||\,\exp\left(h(Q)\right)\Big)\\ &\leq \deg Q \log|\underline{\beta}_j| + \log\Big(||\underline{\omega} - \underline{\beta}_j||^{\mu} + ||\underline{\omega} - \underline{\beta}_j||\,\exp\left(h(Q)\right)\Big)\\ &\leq \deg Q \log|\underline{\beta}_j| + h(Q) + \log\max\{||\underline{\omega} - \underline{\beta}_j||, ||\underline{\omega} - \underline{\beta}_j||^{\mu}\}\end{aligned}$$

Now from Lemma A.2, letting $\mathrm{Res}(F, Q)$ be written as

$$\mathrm{Res}(F, Q) = a^{\deg Q} \prod_{j=1}^{\deg F} Q(\underline{\beta}_j)$$

(where, due to possible inseparability, the $\underline{\beta}_j$ are not necessarily distinct) we use the homomorphism τ defined in Lemma A.2 to get

$$\tau(\mathrm{Res}(F,Q)) = \tau(a)^{\deg Q} \prod_{j=1}^{\deg F} Q(\underline{\beta}_j)$$

$$\log \tau(\mathrm{Res}(F,Q)) = \deg Q \log \tau(a) + \sum_{j=1}^{\deg F} Q(\underline{\beta}_j)$$

$$\leq \deg Q \log \tau(a)$$

$$+ \sum_{j=1}^{\deg F} \Bigg(\deg Q \log |\underline{\beta}_j| + h(Q)$$

$$+ \log \max\{||\underline{\omega} - \underline{\beta}_j||, ||\underline{\omega} - \underline{\beta}_j||^{\mu}\} \Bigg)$$

$$\leq \deg Q \left(\log \tau(a) + \sum_{j=1}^{\deg F} \log |\underline{\beta}_j| \right) + \deg F\, h(Q)$$

$$+ \mu \sum_{||\underline{\omega}-\underline{\beta}_j|| \leq 1} \log ||\underline{\omega} - \underline{\beta}_j|| + \sum_{||\underline{\omega}-\underline{\beta}_j|| \geq 1} \log ||\underline{\omega} - \underline{\beta}_j||$$

Since $||\underline{\omega} - \underline{\beta}_j|| < 1$,

$$\sum_{||\underline{\omega}-\underline{\beta}_j|| \geq 1} \log ||\underline{\omega} - \underline{\beta}_j|| \geq \mu \sum_{||\underline{\omega}-\underline{\beta}_j|| \geq 1} \log ||\underline{\omega} - \underline{\beta}_j||.$$

So our inequality becomes

$$
\begin{aligned}
\log \tau(\mathrm{Res}(F,Q)) &\le \deg Q \left(\log \tau(a) + \sum_{j=1}^{\deg F} \log |\underline{\beta}_j| \right) + \deg F\, h(Q) \\
&\quad + \mu \sum_{j=1}^{\deg F} \log ||\underline{\omega} - \underline{\beta}_j|| \\
&\le (\deg Q - \mu) \left(\log \tau(a) + \sum_{j=1}^{\deg F} \log |\underline{\beta}_j| \right) \\
&\quad + \mu \left(\log \tau(a) + \sum_{j=1}^{\deg F} \log |\underline{\beta}_j| \right) \\
&\quad + \deg F\, h(Q) + \mu \sum_{j=1}^{\deg F} ||\underline{\omega} - \underline{\beta}_j|| \\
&\le (\deg Q - \mu) \left(\log \tau(a) + \sum_{j=1}^{\deg F} \log |\underline{\beta}_j| \right) + \deg F\, h(Q) \\
&\quad + \mu \left(\log |\tau(a)| + \sum_{j=1}^{\deg F} (\log ||\underline{\omega} - \underline{\beta}_j|| + \log |\underline{\beta}_j|) \right) \\
&\le (\deg Q - \mu)(h(F)) + \deg F\, h(Q) + \mu \Big(\log |\tau(a)| \\
&\quad + \sum_{j=1}^{\deg F} (\log ||\underline{\omega} - \underline{\beta}_j|| + \log |\underline{\beta}_j|) \Big) \text{ by Lemma A.2} \\
&\le (\deg Q - \mu)(h(F)) + \deg F\, h(Q) + \mu(\log |F(\underline{\omega})| + h(F)) \\
&\quad \text{by Lemma A.2(2)} \\
&\le \deg Q\, h(F) + \deg F\, h(Q) + \mu \log |F(\underline{\omega})|.
\end{aligned}
$$

Finally, by Lemma A.2, $|\tau(\mathrm{Res}(F,Q))|_{\underline{\omega}} = |\mathrm{Res}(F,Q)|_{\underline{\omega}}$, which yields our desired result. $\square$

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Vita

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