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OPTIMAL INTERMEDIATED INVESTMENT IN A LIQUIDITY-DRIVEN BUSINESS CYCLE

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by

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Abstract

The first essay of this dissertation presents a general equilibrium model of a financial intermediary that extends the model first introduced by D. Diamond and P. Dybvig (*JPE,* 1983) to an infinite-horizon environment. This extension enables the relationship between the real business cycle and the composition of assets held in the banking sector to be studied. As in the D-D model, the bank is modeled as an optimal financial intermediary coalition here. Moreover, the bank's optimal policy involves decisions about liquidity that vary systematically over the business cycle. The second essay studies a version of the model in the first essay with a broader set of contracts–the direct mechanisms. We design an efficient allocation of the direct mechanism such that truth telling reporting strategy is an Perfect Bayesian Nash equilibrium of the mechanism. Further analysis show that there may still exist other equilibrium. The third essay describes the numerical algorithm we use to solve the model and provides an estimate for the approximation error.

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Acknowledgments

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Chapter 1

Introduction

The research reported in this thesis is inspired by the paper "Bank Runs, Deposit Insurance, and Liquidity" by Douglas Diamond and Philips Dybvig(1983) , which is at the core of most models in the literature on bank-centred financial crisis. The significant difference between the Diamond Dybvig model and previous models about a "bank run" is that the previous models assume that a "bank run" is caused by some unexpected objective events. In the Diamond and Dybvig model, even without real changes in the economy, a "bank run" will happen when the depositors arbitrarily lose their confidence in the bank. In that event, all agents, no matter whether or not they need to consume, will rush to the bank and try to make a withdrawal before resources of the bank run out.

A typical Diamond-Dybvig style model has 3 periods, $t = 0, 1, 2$ and a large number of identical agents. Each agent is endowed with one unit of a homogeneous consumption good and has an opportunity to invest into 2 technologies in period 0. One is a long-term technology (illiquid), which will yield higher returns, the other one is a short term (liquid) technology, which will yield lower returns. The gross return of the liquid technology is one unit of good after the first period. The long term technology is illiquid in the sense that it will yield higher gross return than the liquid technology only after its maturity, and it is costly to liquidate it in the middle period. If it is interrupted in the first period, the gross return from the illiquid technology is even lower than the liquid one.

There are 2 consumption preference types—patient and impatient. Impatient agents will only value the consumption in period 1, while consumption in period 1 and period 2 are perfect substitutes for patient agents. Agents will not know their own preference types until period 1. Preference types are private information to each agent. Thus in the initial period $t = 0$, each individual agent faces the choice between return and liquidity. If an agent chooses to invest in the illiquid technology, he/she will lack liquidity if he/she turns out to be an impatient agent in the first period. While if he/she chooses to invest in the liquid technology, then his/her consumption is lower if he/she turns out to be a patient agent. The uncertainty of agents' preference types creates preference for liquidity, and what an individual agent wants is an insurance against the uncertainty of preference types.

In the model the bank is mutually owned by all agents and compared to individual agents, it has an advantage of providing liquidity insurance by offering a deposit contract to individual agents. The deposit contract offered by the bank promises agents a combination of liquidity and higher returns that an individual can not match using the investment. The simplest case is when there is no uncertainty about how many agents will turn out to be impatient, only each single agent will face an idiosyncratic risk about his/her own preference type. In period $t = 0$, each agent can deposit his/her endowment into the bank and the bank will do the investment for all agents who deposit. In exchange, agents are promised contractually specified amounts of consumption in each following period depending on which preference type they report to the bank. The deposit contract offered by the bank is designed to maximize the ex ante expected discounted utility of agents. And only the consumption in the second period is discounted.

Since the preference type is the private information of each agent, the bank can not identify a "fake" report. There are 2 possible outcomes of this economy.[1] One result is that all agents will make the truthful report, in which case that the expected utility is higher than in autarky. The other one is that all agents will claim being impatient and withdraw in the first period regardless of their genuine preference types.

The following literatures have made important contributions based on Diamond Dybvig model. Chari and Jagannathan (1988) rigorously model "bank runs" with or without adverse information. And they show that "bank runs" can be avoided by "suspension of payment". Jacklin and Bhattacharya (1988) contrast panics and information based "bank runs" in an effort to provide a robust and empirically plausible model of how "bank runs" are plausibly triggered. Postlewaite and Vives (1987) consider a standard deposit contract with no exogenous event that agents can condition on, they show that there exists an equilibrium with a positive probability of "bank run". Jacklin (1987) first

[1] In the Diamond-Dybvig model, they use the Constant Relative Risk Averse (CRRA) function as the utility function $u(c) = \dfrac{c^\gamma}{\gamma}$ for $\gamma < 1$, and they assume that the relative risk averse constant $\gamma < -1$. As a matter of fact, if $-1 < \gamma < 0$, there is only one possible outcome of this economy.

studies the incentive problems related to the nonintermediated finance in the Diamond-Dybvig model. His key observation is that if trade is allowed in later period(s), then the agents have incentive to invest directly. Wallace (1988; 1990) responds the critique of Jacklin by arguing that deposit arrangements should be viewed as an alternatives rather than as complements to trading mechanisms. Green and Lin (2000, 2003) study the Diamond-Dybvig model with a broader set of contracts than the one in Diamond and Dybvig (1983), and they notice that with a broader set of contracts, the “bank run” disappears. Peck and Shell(2004) extend the Diamond-Dybvig model to capture the payment and transaction roles of checking accounts and allow for sunspots triggered bank runs. They compare the bank’s portfolio with/without restriction and conclude that the bank is immune to “bank run” with no restriction on their portfolio, while it will ration depositors with positive probability.

The model will be presented in this thesis is also based on the Diamond-Dybvig model, but different from all previous literatures, our model is an infinite horizon model, the periods from $t = 1, 2, \ldots, \infty$. Inspired by the paper “Demand Deposits, Trade Restriction and Risk Sharing” [2], we assume that agents will face a preference shock in each period. With the extended model, we focus on 2 topics. First, if there is an uncertainty about the number of agents who will turn out to be impatient in each period, what should be the optimal investment decision of the bank? Second, we study a version of

[2] Charles Jacklin mentioned that an utility function in the form of $U(c_1, c_2) = u(c_1) + \rho^i u(c_2)$, where $u(\cdot)$ in the utility function in each period, and different ρ^i represents the different preference type of agents

the model with the direct mechanism. Direct mechanism contains a broader set of contracts than those studied in Diamond and Dybvig (1983). We use a simplified version of the extended model by assuming that the liquid technology is the only investment choice of the bank and there is no uncertainty about the number of impatient agents in each period. According to the revelation principle we design an efficient allocation of the direct mechanism by extending the efficient allocation of an incentive compatible deposit contract. Further analysis shows that other equilibrium might exist with certain parameter values. The last chapter of the thesis describes the numerical algorithm we use to solve the model in the first chapter. The numerical algorithm we use is based on time iteration. For comparison, we also use value iteration to solve our model. Computation results show that time iteration is more efficient than value iteration. While different from value iteration, time iteration is not based on the standard dynamical programming procedure which guarantees the convergence of the method, time iteration is based on the Euler's equation of the model. With the presence of inequality constraint, additional conditions are needed to guarantee the convergence of the method. We use the result in Santos (2000) to provide an estimate of the approximation error, the computation results show that a sharper error estimation is needed for using time iteration.

The main contributions of this thesis can be stated as follows: first, we extend the 2-period Diamond Dybvig model to infinite-horizon and study the optimal investment of an financial intermediary with the existence of a liquidity driven business cycle; second we use a version of the extended model to study the optimal contract between a financial intermediary and it's depositors, and we find a sufficient condition of the existence of multi-equilibria; lastly, to our knowledge, this study is the first application of time iteration to a stochastic economic model with heterogeneous agents. The result is that, for this model, time iteration is more efficient than value iteration.

Chapter 2

Optimal Intermediated Investment in a Liquidity-Driven Business Cycle

2.1 Introduction

The asset portfolio of the U.S. banking sector seems to be strongly correlated with the real business cycle. Motivated by this phenomenon, which is clearly reflected in figure 1, we construct here a general equilibrium model of the portfolio decisions of the banking sector. This model provides an explanation of why the liquidity of banks' portfolios varies systematically with the business cycle, and suggests that such variation is economically efficient.

Figure 1[1] shows 2 things about the portfolio of the banking sector. First, the portfolio becomes more liquid throughout the contraction period of the business cycle. Second, to close approximation, the part of the sample period during which banks are increasing the liquidity of their portfolio strictly includes the NBER contraction episodes. In fact, between January 1973 and November 2001, banks increased the fraction of their portfolio held in liquid form during a total of 10 years and 1 month, while the NBER

[1]Figure 1 shows the evolution of the illiquid asset portion of US banking sector from Jan.1973—Aug.2007. Federal Reserve Board H.8. (510) data (commercial banks in the United States, not seasonally adjusted) has been smoothed by an HP filter to produce the figure. We categorize treasury and agency securities, securities in loans and leases and cash assets as liquid assets. We omit the interbank loans, which are not an asset of the consolidated banking sector. The shaded area shows the period from peak to trough in business cycles determined by NBER.

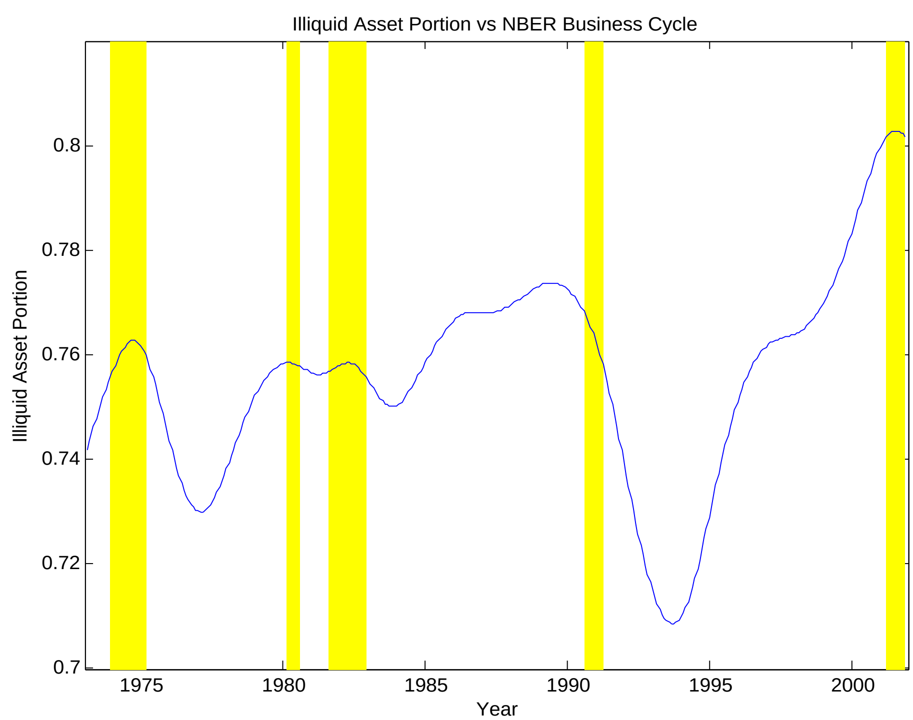

Fig. 2.1. HP-filtered liquidity ratio of bank's portfolio from Jan.1973-Aug.2007.

contractions span only 4 years and 6 months in total.

Our model provides a rigorous justification for this observation. It is an infinite-horizon, dynamic, stochastic model with two production technologies. One of these two technologies has a higher gross rate of return than the other does. However, adjusting investment in the higher return technology incurs a convex cost. In contrast, investment in the other technology is freely adjustable. We refer to these two technologies as the illiquid technology and the liquid technology respectively.

Section 2 presents a baseline model of an autarkic agent. The agent adjusts his/her portfolio in response to fluctuations in his/her rate of time preference. We show analytically (for a deterministic cycle) and by numerical simulation (for a random preference-shock process) that the agent may transfer some investment to the liquid technology in anticipation of choosing high consumption in the near future.

Section 3 presents the main model of this chapter. In this model, a financial intermediary maximizes the aggregate expected discounted utility of a continuum of agents who are identical ex ante. These agents will face a random shock to time preference each period. These shocks are independent across the agents, so time preference risk can be fully diversified in principle. However, consumption preferences of each individual are private information. Patient agents have the incentive to misrepresent themselves to be impatient, even when they actually are patient, in order to collect an insurance indemnity. Since there is no way to detect this misrepresentation, no one would be willing to be a counterparty in such a direct market for insurance.

However, a financial intermediary that offers a deposit contract can provide a partial substitute for explicit insurance. As in Diamond and Dybvig's model, it is economically efficient for a financial intermediary to play this role. Instead of a 2-period model, our model has an infinite horizon. Thus we can use it to study the relationship between the real business cycle and the composition of assets held in the banking sector. We show banks' optimal portfolio evolves a in similar fashion to the autarkic agent's portfolio in the baseline model.

2.2 Baseline Model

The model presented in this section is an infinite-horizon model with a single agent. There are two production technologies which differ in 2 respects. First, one technology has a higher gross rate of return than the other technology does. Second, the higher-gross-return technology is subject to an investment adjustment cost, from which the lower-gross-return technology is exempt. Thus we call the higher-gross-return technology the "illiquid" technology and the lower-gross-return technology the "liquid" technology. We would like to study the agent's optimal investment portfolio in three cases: the degenerate case in which there is no change in his /her consumption preference; the case in which the consumption preference is a deterministic 2-period cycle; and the case in which the preference shock sequence is a sequence of i.i.d random variables subject to some Bernoulli distribution. In the degenerate case we introduce, the agent will hold either purely liquid or purely illiquid investment at all times. Otherwise, if preferences fluctuate either deterministically or randomly, there will be some time at which the agent will hold both assets.

2.2.1 Shared Features of Different Versions of the Model

Environment There exists one composite good in the economy which is both the input and output of both technologies. In each period, the goods can be either reinvested in the current technology, or transfered to the other technology, or liquidated for consumption. Time is discrete $t = 1, 2, \cdots$.

Investment Choices There are two production technologies, illiquid and liquid[2], with one period gross return R_I and R_L, where $1 < R_L < R_I$. We use I and L to denote the level of investment in two technologies.

The crucial feature of the production technology is the following. In each period investment return from these two production technologies is divided into 2 parts: the agent's consumption of current period c_t and investment in 2 technologies in the next period I_{t+1} and L_{t+1}. In a typical model which does not have an adjustment cost, the feasibility constraint is given by $R_I I_t + R_L L_t - (c_t + I_{t+1} + L_{t+1}) \geq 0$.

[2]To understand this, imagine that the composite goods is corn. It can be grown or eaten by the agent. Two production technologies are like 2 farms in this case, one is more productive but far away from where the agent lives, the other one is less productive but very close to where the agent lives. Thus in order to consume the corn from the more productive farm the agent will have to pay transportation fee, while getting corn from the less productive but on site farm, no transportation fee is needed.

While in this model we assume the agent is not free to transfer the goods in or out of illiquid technology, i.e., the rate of transfer is not 1-1. Transfer is subject to a convex adjustment cost[3] deduction. The further I_{t+1} deviates from $R_I I_t$, the fewer composite goods there are to be allocated to $c_t + I_{t+1} + L_{t+1}$. In contrast, adjusting the investment level of the liquid technology will have no effect on $c_t + I_{t+1} + L_{t+1}$. Thus the feasible constraint for investment and consumption[4]is given by:

$$\begin{aligned} R_I I_t + R_L L_t - (c_t + I_{t+1} + L_{t+1}) &= \theta(1 - \frac{I_{t+1}}{R_I I_t})^2 I_t, \\ I_{t+1} &\geq 0, \\ L_{t+1} &\geq 0, \end{aligned}$$

where $\theta(1 - \frac{I_{t+1}}{R_I I_t})^2 I_t$ is the adjustment cost function and θ is a positive constant. The adjustment cost function is a convex function of $1 - \frac{I_{t+1}}{R_I I_t}$ and it is homogeneous of degree 1 of the investment leve I_t. To rule out trivial solutions of the investment problem, we assume $R_I - \theta < R_L$. Thus the return from liquid technology is not dominated all the time.

Agent There is a single agent in the economy. In the initial period, the agent is endowed with a portfolio (I_1, L_1) which is exogenously invested in 2 technologies, where $I_1 + L_1 = 1$. We use $\log(\cdot)$ as the utility function of the agent. Since consumption

[3]The adjustment cost here is a physical feature of the production technology, not merely a financial term.

[4]Since the goods liquidated from investment technologies are perishable, the agent will consume all liquidated goods from both technologies.

is uniquely determined by the sequence of investment portfolio, the agent will choose the optimal portfolio sequence to maximize his /her discounted utility.

2.2.2 Constant Discount Factor

To provide an intuition before we expand our model, we assume the agent's discount factor is a constant, $\delta_t \equiv \delta \in [0,1)$. This is a deterministic dynamic programming model. And the optimization problem of the agent is given by the following:

$$\max_{\{I_t, L_t\}_{t=1}^{\infty}} \sum_{t=2}^{\infty} \delta^{t-1} \log(c_t) \tag{2.1}$$

subject to

$$\begin{aligned} c_t &= R_I I_t + R_L L_t - \theta(1 - \frac{I_{t+1}}{R_I I_t})^2 I_t - I_{t+1} - L_{t+1}; \\ 0 &\leq I_{t+1}; \\ 0 &\leq L_{t+1}; \end{aligned}$$

for all $t > 1$.

In order to study the composition of the agent's optimal investment portfolio, it is helpful to introduce another set of coordinates.

Definition 2.1. *Define* B_t *to be the portfolio size of the agent* $B_t = I_t + L_t$ *and* q_t *to be the liquidity ratio*[5] *of the portfolio* $q_t = \dfrac{B_t - L_t}{B_t}$.

Thus we can rewrite the constraints of the agent's problem:

$$
\begin{aligned}
c_t &= R_I B_t q_t - \theta(1 - \frac{B_{t+1} q_{t+1}}{R_I B_t q_t})^2 B_t q_t + R_L B_t (1 - q_t) - B_{t+1}; \\
0 &\leq B_{t+1}; \\
0 &\leq q_{t+1} \leq 1;
\end{aligned}
\tag{2.2}
$$

for all $t \geq 1$.

Definition 2.2. *The agent's investment portfolio path is in a steady state* (b, q) *in period* t*, if for all* $\tau \geq t$,

- $\dfrac{B_{\tau+1}}{B_\tau} \equiv b > 0$ *and*
- $q_\tau \equiv q \in [0, 1]$.

The following proposition states that when the agent has constant consumption preference, he /she will hold purely liquid investment or purely illiquid investment in the optimal steady state.

Proposition 2.3. *q is 0 or 1 in the optimal steady state of the agent's investment portfolio.*

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[5] *The higher liquidity ratio is, the more illiquid the asset portfolio is.*

Proof The Proposition is proved by contradiction. If in the optimal steady state $q_t \equiv q \in (0,1)$, then there exists a perturbation from the optimal steady state, which improves the agent's consumption.

Suppose the optimal steady state is given by $q \in (0,1)$ and $b > 0$.

Since the agent's portfolio sequence $(B_\tau, q_\tau)_{\tau=t}^{\infty}$ is in the optimal steady state, it must satisfy Euler's equation:

$$
\begin{aligned}
\frac{1-\frac{2\theta}{R_I}(1-\frac{B_{\tau+1}q_{\tau+1}}{R_I B_\tau q_\tau})q_{\tau+1}}{c_\tau} &- \delta\frac{(R_I-\theta+\theta(\frac{B_{\tau+2}q_{\tau+2}}{R_I B_{\tau+1}q_{\tau+1}})^2-R_L)q_{\tau+1}+R_L}{c_{\tau+1}} \\
&= 0 \\
\frac{\frac{2\theta}{R_I}(1-\frac{B_{\tau+1}q_{\tau+1}}{RB_\tau q_\tau})}{c_\tau} &+ \delta\frac{R_I-\theta+\theta(\frac{B_{\tau+2}q_{\tau+2}}{RB_{\tau+1}q_{\tau+1}})^2-R_L}{c_{\tau+1}} \\
&= 0
\end{aligned}
\tag{2.3}
$$

Using the condition of a steady state, $\frac{B_{\tau+1}}{B_\tau} \equiv b$, and $q_\tau \equiv q$, we rewrite the 2 equations in (2.3):

$$
b(1-\frac{2\theta}{R_I}(1-\frac{b}{R_I})q) - \delta((R_I-\theta+\theta(\frac{b}{R_I})^2-R_L)q+R_L) = 0 \tag{2.4}
$$

$$
b\frac{2\theta}{R_I}(1-\frac{b}{R_I}) + \delta(R_I-\theta+\theta(\frac{b}{R_I})^2-R_L) = 0 \tag{2.5}
$$

Since b denotes the growth rate of B_τ, $b < R_I$. Thus we have $\frac{2\theta}{R_I}(1-\frac{b}{R_I}) > 0$. Since b is positive and $q \in (0,1)$, b and q are the solutions to (2.4) and (2.5). According to equation (2.5), $R_I - \theta + \theta(\frac{B}{R_I})^2 - R_L < 0$. Thus if $b > \delta$,

$$\begin{aligned} R_I - \theta(1 - \frac{b}{R_I})^2 - R_L &= R_I - \theta + \theta(\frac{b}{R_I})^2 - R_L + 2\theta\frac{b}{R_I}(1 - \frac{b}{R_I}) \\ &= (1 - \frac{\delta}{b})(R_I - \theta + \theta(\frac{b}{R_I})^2 - R_L) \\ &< 0. \end{aligned}$$

Consider a slight perturbation from steady state (b, q) with $\widetilde{q} = (1 - \varepsilon)q$, where $\varepsilon > 0$. Now compare $\widetilde{c}_t$ when $q_t \equiv \widetilde{q}$ with c_t when $q_t \equiv q$. Since gross return from both production technologies are homogeneous of degree 1, without loss of generality we may assume $B_t = 1$. Then

$$\widetilde{c}_t - c_t = (-\varepsilon)(R_I - \theta(1 - \frac{b}{R_I})^2 - R_L) > 0.$$

This perturbation can be applied again: let $\widetilde{\widetilde{q}} = (1 - 2\varepsilon)q$ and so on until $q \equiv 0$. Thus in the optimal steady state $q = 0$.

If $b < \delta$, we can consider a deviation with $\widetilde{q} = (1 + \varepsilon)q$. Follow the same logic as above, we can conclude $q = 1$ in the optimal steady state.

If b is the solution to equation (2.5), where $b = \delta$, the agent is indifferent between $q = 0$ or 1.

If the solution to equation (2.5), where $b \leq 0$, then the constraint on q must be binding, $q = 0$ or 1.

So we can conclude that in the optimal steady state $q = 0$ or 1.

2.2.3 2-Period Cycle Discount Factor

In this section, we assume that the agent's discount factor follows a 2-period cycle, which reflects a cyclical consumption time preference. Let $0 \leq \delta^l < \delta^h < 1$. Assume the discount factor δ_t discounts the utility after period t, in which case given the same portfolio, the agent is more willing to consume that period when $\delta_t = \delta^l$ than when $\delta_t = \delta^h$. Thus we call a period in which the agent has the discount factor δ^l an impatient period, and we call a period in which the agent has the higher discount factor δ^h a patient period.

Without loss of generality, assume period 1 is an impatient period, then

$$\delta_t = \begin{cases} \delta^l & : \text{ t is odd} \\ \delta^h & : \text{ t is even.} \end{cases}$$

where $0 < \delta^l < \delta^h < 1$.

The maximization problem of the agent is:

$$\max_{(I_t, L_t)_{t=2}^{\infty}} \sum_{t=1}^{\infty} (\delta^l)^{\lfloor \frac{t}{2} \rfloor} (\delta^h)^{\lfloor \frac{t-1}{2} \rfloor} \log(c_t) \tag{2.6}$$

subject to feasibility constraints given by (2.2) where $\lfloor \cdot \rfloor$ denotes the floor function, i.e. $\lfloor n + \frac{1}{2} \rfloor = n, \forall n \in N$.

Since the agent's consumption preference is cyclical, there may exist a cycle in a steady state of agent's investment portfolio.

DEFINITION 2.4. *The agent's investment portfolio path is in a steady state*$((b^l, q^l), (b^h, q^h))$ *in period* t, *if* $\forall \tau \geq \frac{t}{2}$

- $\frac{B_{2\tau+1}}{B_{2\tau}} \equiv b^l > 0, \frac{B_{2\tau+2}}{B_{2\tau+1}} \equiv b^h > 0;$ *and*

- $q_{2\tau+1} \equiv q^l \in [0,1], q_{2\tau+2} \equiv q^h \in [0,1].$

ASSUMPTION 2.5. *Assume* $R_I - R_L < \theta < \frac{R_I^2}{R_I - 1}$.

6

PROPOSITION 2.6. *If assumption (2.5) is satisfied, then the liquidity ratio is identically* 0 *or* $0 < q^l \leq q^h = 1$ *in the optimal steady state.*

LEMMA 2.7. *For any* $t > 0$, *if* $q_t = 0$, *then* $q_\tau = 0$ *for all* $\tau \geq t$.

Proof According to the adjustment function, if $q_t = 0$ and $q_{t+1} > 0$, then the adjustment cost in period $t+1$ is $+\infty$, which is a contradiction. Therefore, $q_{t+1} = 0$, the conclusion follows directly.

LEMMA 2.8. *In the optimal steady state, if the liquidity ratio* q_t *is not identically* 0, *then either* $q^l = 1$ *or* $q^h = 1$.

[6] This assumption is quite general, for example if $R_I = 1.2$ we only require $0.02 < \theta < 7.2$.

Proof We prove this lemma by contradiction.

Suppose $0 < q^l < 1$ and $0 < q^h < 1$.

Here we only consider when the portfolio path is in the optimal steady state. Thus $q_t = q^l$ if t is odd, or $q_t = q^h$ if t is even.

For convenience, we introduce the following notation: define c^l to be the consumption in an impatient period with portfolio size $B = 1$.

$$c^l = R_I q^l + R_L(1 - q^l) - \theta(1 - \frac{b^h q^h}{R_I q^l})^2 q^l - b^h. \tag{2.7}$$

Correspondingly define c^h to be the consumption in a patient period with portfolio size $B = 1$.

$$c^h = R_I q^h + R_L(1 - q^h) - \theta(1 - \frac{b^l q^l}{R_I q^h})^2 q^h - b^l. \tag{2.8}$$

Since the portfolio path is in the optimal steady state with $q^l < 1$ and $q^h < 1$, consider Euler's equation of this problem:

$$\frac{1 - \frac{2\theta}{R_I}(1 - \frac{b^l q^l}{R_I q^h}) q^l}{c^h} = \delta^h \frac{(R_I - \theta + \theta(\frac{b^h q^h}{R_I q^l})^2 - R_L) q^l + R_L}{b^l c^l}; \tag{2.9}$$

$$\frac{1 - \frac{2\theta}{R_I}(1 - \frac{b^h q^h}{R_I q^l}) q^h}{c^l} = \delta^l \frac{(R_I - \theta + \theta(\frac{b^l q^l}{R_I q^h})^2 - R_L) q^h + R_L}{b^h c^h}; \tag{2.10}$$

and

$$\frac{\frac{2\theta}{R_I}(1-\frac{b^l q^l}{R_I q^h})}{c^h} + \delta^h \frac{(R_I - \theta + \theta(\frac{b^h q^h}{R_I q^l})^2 - R_L)}{b^l c^l} = 0; \tag{2.11}$$

$$\frac{\frac{2\theta}{R_I}(1-\frac{b^h q^h}{R_I q^l})}{c^l} + \delta^l \frac{(R_I - \theta + \theta(\frac{b^l q^l}{R_I q^h})^2 - R_L)}{b^h c^h} = 0. \tag{2.12}$$

Multiply equation (2.11) by q^l and equation (2.12) by q^h and add the resulting equations to equations (2.9) and (2.10) respectively:

$$\frac{1}{c^h} = \delta^h \frac{R_L}{b^l c^l}; \tag{2.13}$$

$$\frac{1}{c^l} = \delta^l \frac{R_L}{b^h c^h}. \tag{2.14}$$

Substituting equation (2.13) and (2.14) into equation (2.11) and (2.12) correspondingly:

$$\frac{2\theta}{R_I}(1-\frac{b^l q^l}{R_I q^h}) + \frac{1}{R_L}(R_I - \theta + \theta(\frac{b^h q^h}{R_I q^l})^2) - 1 = 0; \tag{2.15}$$

$$\frac{2\theta}{R_I}(1-\frac{b^h q^h}{R_I q^l}) + \frac{1}{R_L}(R_I - \theta + \theta(\frac{b^l q^l}{R_I q^h})^2) - 1 = 0. \tag{2.16}$$

By subtracting equation (2.15) from (2.16), we conclude

$$\frac{b^h q^h}{q^l} = \frac{b^l q^l}{q^h}.$$

Then according to the expression of c^l and c^h:

$$c^l = (R_I - R_L - \theta(1 - \frac{b^h q^h}{R_I q^l})^2) q^l + R_L - b^h \tag{2.17}$$

$$c^h = (R_I - R_L - \theta(1 - \frac{b^l q^l}{R_I q^h})^2) q^h + R_L - b^l. \tag{2.18}$$

Depending on whether $R_I - R_L - \theta(1 - \frac{b^l q^l}{R_I q^h})^2$ is positive or negative, simultaneously increasing and decreasing q^l or q^h will increase the consumption of the agent in both odd and even periods. Since $q_t \in [0, 1]$,if decreasing q improves the agent's consumption, then $q^l = q^h = 0$, while if increasing q improves the consumption, then either q^l or q^h is 1.

The next lemma states that if the agent holds purely illiquid investment in an impatient period, then he/she holds purely illiquid investment in a patient period. Intuitively, with a convex adjustment cost, the agent tries to liquidate assets in illiquid technology gradually over 2 periods rather than all at once. Specifically, the agent starts to decrease the illiquid investment level in patient period and consumes up all liquid assets in patient periods. Thus $q^h = 1$ and $q^l < 1$ would imply the agent liquidate excess the illiquid investment in an impatient period and consume all the liquid assets in a patient period which contradicts the intuition.

Lemma 2.9. *If assumption (2.5) is satisfied, then in the optimal steady state,* $q^l = 1$ *implies* $q^h = 1$.

Proof We will prove this lemma by contradiction. Suppose the optimal steady state of the liquidity ratio is given by $q^l = 1$ and $q^h < 1$. Define c^l and c^h as in (2.7) and (2.8). Again we consider the Euler equations of the problem. Since the constraint $q^l \leq 1$ is binding, we only have 3 equations:

$$\frac{1 - \frac{2\theta}{R_I}(1 - \frac{b^l}{R_I q^h})}{c^h} = \delta^h \frac{R_I - \theta + \theta(\frac{b^h q^h}{R_I})^2}{b^l c^l}; \tag{2.19}$$

$$\frac{1 - \frac{2\theta}{R_I}(1 - \frac{b^h q^h}{R_I})q^h}{c^l} = \delta^l \frac{(R_I - \theta + \theta(\frac{b^l}{R_I q^h})^2 - R_L)q^h + R_L}{b^h c^h}; \tag{2.20}$$

$$\frac{\frac{2\theta}{R_I}(1 - \frac{b^h q^h}{R_I})}{c^l} + \delta^l \frac{(R_I - \theta + \theta(\frac{b^l}{R_I q^h})^2 - R_L)}{b^h c^h} = 0. \tag{2.21}$$

As before we have

$$\frac{1}{c^l} = \delta^l \frac{R_L}{b^h c^h} \tag{2.22}$$

Since the constraint on q^l is binding

$$\frac{1}{c^h} > \delta^h \frac{R_L}{b^l c^l} \tag{2.23}$$

By (2.22) and (2.23) we can conclude:

$$(\frac{c^h}{c^l})^2 < \frac{\delta^l b^l}{\delta^h b^h} \tag{2.24}$$

Consider the following 2 cases:

1) Assume $b^h q^h < \frac{b^l}{q^h}$.

By equation (2.22), we can write equation (2.21) as follows:

$$\frac{2\theta}{R_I}(1 - \frac{b^h q^h}{R_I}) + \frac{R_I - \theta - R_L}{R_L} + \frac{\theta}{R_L}(\frac{b^l}{R_I q^h})^2 = 0 \tag{2.25}$$

By assumption,

$$\frac{2\theta}{R_I}(1 - \frac{b^l}{R_I q^h}) + \frac{\theta}{R_L}(\frac{b^l}{R_I q^h})^2 + \frac{R_I - \theta - R_L}{R_L} < 0 \tag{2.26}$$

Let $x = \frac{b^l}{q^h}$, and rewrite inequality (2.26) as follows:

$$\frac{\theta}{R_L}(\frac{x}{R_I})^2 + \frac{2\theta}{R_I}(1 - \frac{x}{R_I}) + \frac{R_I - \theta - R_L}{R_L} < 0 \tag{2.27}$$

According to assumption (2.5), $\theta < \frac{R_I^2}{R_I - 1} < \frac{R_I^2}{R_I - R_L}$. Thus the discriminant of inequality (2.27) is

$$\frac{4\theta}{R_I^4 R_L^2}(R_I - R_L)(\theta(R_I - R_L) - R_I^2) < 0,$$

and so there are no solutions to inequality (2.27), which is a contradiction.

2)Assume $b^h q^h \geq \frac{b^l}{q^h}$.

According to equation (2.19) and inequality (2.23),

$$1 - \frac{2\theta}{R_I}(1 - \frac{b^l}{R_I q^h}) > R_I - \theta + \theta(\frac{b^h q^h}{R_I})^2.$$

Let $x = b^h q^h$. The condition $b^h q^h \geq \frac{b^l}{q^h}$ implies:

$$\theta(\frac{x}{R_I})^2 - \frac{2\theta}{(R_I)^2}x + (R_I - \theta - 1 + \frac{2\theta}{R_I}) < 0. \tag{2.28}$$

By assumption (2.5), $\theta < \frac{R_I^2}{R_I - 1}$. Thus the discriminant of inequality (2.28) is

$$\frac{4\theta}{R_I^2}[\theta(\frac{1}{R_I} - 1)^2 - (R_I - 1)] < 0.$$

Again there are no solutions to this inequality (2.28), which is a contradiction. Thus we can conclude that $q^l = 1$ implies $q^h = 1$. This completes the proof.

To find out when the liquidity ratio has a nontrivial cycle in the optimal steady state, we need an auxiliary problem of the bank: in this problem, assume the illiquid technology is the only investment choice, thus B_t is composed of purely illiquid investment.

The auxiliary problem can be described as follows:

$$\max_{(B_t)_{t=2}^{\infty}} \sum_{t=1}^{\infty} (\delta^l)^{\lfloor \frac{t}{2} \rfloor} (\delta^h)^{\lfloor \frac{t-1}{2} \rfloor} \log(c_t) \tag{2.29}$$

subject to

$$\begin{aligned} c_t &= R_I B_t - \theta(1 - \frac{B_{t+1}}{R_I B_t})^2 B_t - B_{t+1} \\ 0 &\leq B_{t+1} \end{aligned}$$

for all $t \geq 1$. [7]

Since the gross returns from both technologies are homogeneous of degree 1, it is sufficient to consider the case with initial investment $B_1 = 1$.

Definition 2.10. *The agent's investment portfolio path of the auxiliary problem is in a steady state* (x^l, x^h) *in period* t*, if* $\forall \tau \geq \frac{t}{2}$

- $\frac{B_{2\tau+1}}{B_{2\tau}} \equiv x^l > 0,\ \frac{B_{2\tau+2}}{B_{2\tau+1}} \equiv x^h > 0$

The following theorem provides a sufficient condition for the existence of a non-trivial case with a cyclical liquidity ratio in the optimal steady state. It is proved that if a perturbation from the investment with purely illiquid asset is profitable, then the liquidity ratio in the optimal steady state is cyclical with $q^h = 1$ and $q^l < 1$.

[7] $\lfloor \cdot \rfloor$ is the floor function defined previously.

Theorem 2.11. *Let (x^l, x^h) be the optimal auxiliary steady state we defined in definition (2.10). If*

$$x^l \frac{-\frac{2\theta}{R_I}(1-\frac{x^l}{R_I})}{R_I - \theta(1-\frac{x^l}{R_I})^2 - x^l} - \delta^h \frac{(R_I - \theta + \theta(\frac{x^h}{R_I})^2 - R_L)}{R_I - \theta(1-\frac{x^h}{R_I})^2 - x^h} > 0, \qquad (2.30)$$

and assumption (2.5) are satisfied, then the liquidity ratio is cyclical with $q^l < 1$ and $q^h = 1$ for all $t \geq 1$ in the optimal steady state .

Proof Suppose $q^l = q^h = 1$, so the optimal steady state of the agent's original problem coincides with the optimal steady state of the auxiliary problem. According to definition (2.10), if the investment portfolio path is in the optimal auxiliary steady state in period t, then $x^l = \dfrac{B_{2\tau+1}}{B_{2\tau}}$ and $x^h = \dfrac{B_{2\tau+2}}{B_{2\tau+1}}$ for $2\tau > t$.
Since the gross returns from both production technologies are homogeneous of degree 1, without loss of generality, we may assume $B_{2\tau} = 1$. Consider a two-period perturbation from the optimal auxiliary steady state with $B_{2\tau+1} = x^l, \widetilde{q}_{2\tau+1} = 1-\varepsilon$, $B_{2\tau+2} = x^l x^h$, and $\widetilde{q}_{2\tau+2} = 1$, for a given $\varepsilon > 0$.
Then

$$c_{2\tau} = R_I - \theta(1 - \frac{x^l}{R_I})^2 - x^l$$

and the consumption in period 2τ after the perturbation is given by

$$\widetilde{c}_{2\tau} = R_I - \theta(1 - \frac{x^l(1-\varepsilon)}{R_I})^2 - x^l.$$

The difference in consumption due to the perturbation is

$$\widetilde{c}_{2\tau} - c_{2\tau} = -x^l \varepsilon \frac{2\theta}{R_I}(1 - \frac{x^l}{R_I}) + O(\varepsilon^2); \tag{2.31}$$

Similarly, we have $c_{2\tau+1}$ and $\widetilde{c}_{2\tau+1}$ as follows:

$$c_{2\tau+1} = x^l[R_I - \theta(1 - \frac{x^h}{R_I})^2 - x^h],$$

$$\widetilde{c}_{2\tau+1} = x^l[R_I(1-\varepsilon) + R_L\varepsilon - \theta(1 - \frac{x^h}{R_I(1-\varepsilon)})^2(1-\varepsilon)] - x^l x^h.$$

So

$$\widetilde{c}_{2\tau+1} - c_{2\tau+1} = x^l \varepsilon[-R_I - \theta + \theta(\frac{x^h}{R_I})^2 + R_L] + O(\varepsilon^2) \tag{2.32}$$

According to inequality (2.30),

$$\frac{\widetilde{c}_{2\tau} - c_{2\tau}}{c_{2\tau}} + \delta^h \frac{(\widetilde{c}_{2\tau+1} - c_{2\tau+1})}{c_{2\tau+1}} > 0,$$

In order to compare $[\log(\widetilde{c}_{2\tau}) - \log(c_{2\tau})] + \delta^h[\log(\widetilde{c}_{2\tau+1}) - \log(c_{2\tau+1})]$, we use the first order approximation:

$$[\log(\widetilde{c}_{2\tau}) - \log(c_{2\tau})] + \delta^h[\log(\widetilde{c}_{2\tau+1}) - \log(c_{2\tau+1})] = [\frac{\widetilde{c}_{2\tau} - c_{2\tau}}{c_{2\tau}} + \delta^h \frac{(\widetilde{c}_{2\tau+1} - c_{2\tau+1})}{c_{2\tau+1}}]\varepsilon + O(\varepsilon^2) > 0.$$

Since the amount of $B_{2\tau+2}$ is not affected, this perturbation can be applied to any 2-period cycle. Thus it generates an investment portfolio path with which the agent will have a higher consumption than in the investment portfolio path generated by the optimal steady state of the auxiliary problem. This is a contradiction. Thus we can conclude that the liquidity ratio of the optimal steady state is cyclical with $q^l < 1$ and $q^h = 1$.

2.2.4 Random Discount Factor

In this section we assume that the discount factor $\{\Delta_t\}_{t=1}^{\infty}$ is a sequence of independent identically distributed random variables. For each period t, Δ_t is subject to a Bernoulli distribution. In particular, for all $t > 1$, the support of $\Delta_t = \{\delta^l, \delta^h\}$ and $P(\Delta_t = \delta^l) = p$.[8]

The optimization problem of the agent is:

$$E[\max_{(I_t, L_t)_{t=2}^{\infty}} \sum_{t=1}^{\infty} (\prod_{\tau=0}^{t-1} \Delta_\tau) \log(c_t)] \tag{2.33}$$

subject to the constraints (2.2) again.

[8] Since we do not discount the consumption in period 1, without loss of generality, let $\Delta_0 \equiv 1$.

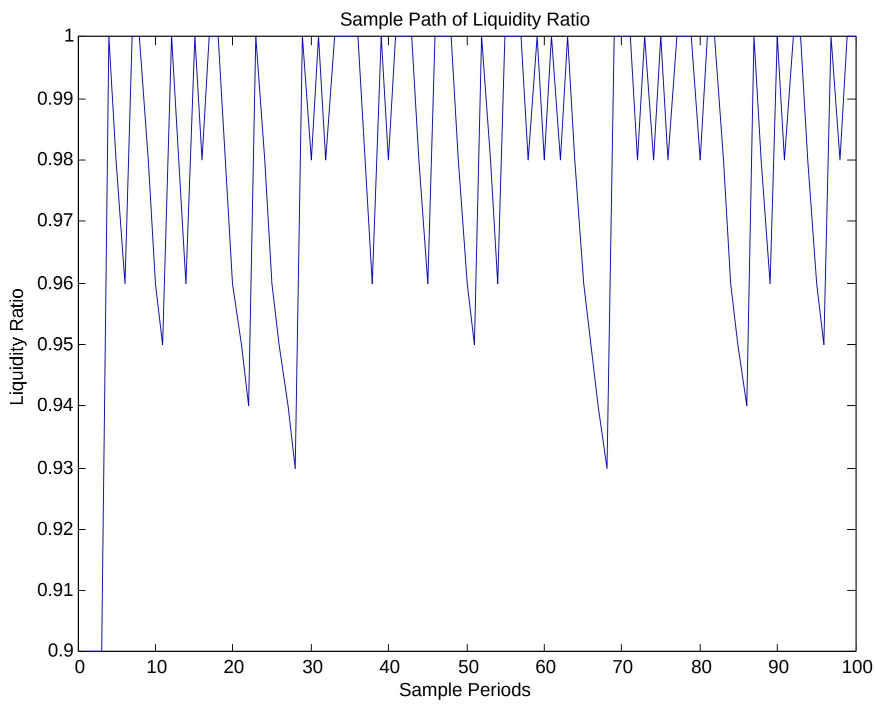

Fig. 2.2. A sample path

As the previous section, during a patient period, the agent will cautiously build up liquid investment stock to smooth out the liquidation process. To illustrate the statement, we will use a conceptual numerical example .(See figure 2)

Figure (2.2) shows a sample path of the liquidity ratio of the agent's investment portfolio for $R_I = 1.2$, $R_L = 1.18$, $\theta = 2$, with $\delta^l = 0.05$, $\delta^h = 0.95$ and $P(\delta_t = \delta^l) = 0.5$.

2.3 Main Model

In this section, we will present the main model of the paper which has the same production technologies as the previous baseline model. With this model, we study how the liquidity of asset portfolio in banking sector varies with the consumption preference of its depositors.

Agents and Bank There are a continuum of individual agents and a financial intermediary which we call a bank. Formally let the measure space $(A, \mathcal{A}, \mu)$ be the space of agents. Assume $\mu(\cdot)$ is a non-atomic measure with $\mu(A) = 1$.

Each agent in this model is specified in the same way as the agent in the single agent economy. In every period, each agent will face a random consumption-preference shock, which is independent of the other agents' consumption-preference shocks. For given $t \geq 1$, Δ_t^α has a Bernoulli distribution with support $\{\delta^l, \delta^h\}$ and

$P(\Delta_t^{\alpha} = \delta^l) = p_t$. [9] Assume[10] for every period t, the measure of agents with discount factor δ^l

$$\mu\{\alpha | \Delta_t^{\alpha} = \delta^l\} = p_t. \tag{2.34}$$

Each agent privately learns his/her own preference shock at the beginning of each period t. Agents who have $\Delta_t = \delta^l$ are called impatient agents in period t, since they value future consumption less than those agents with discount factor δ^h in period t. Correspondingly, agents with discount factor δ^h are called patient agents.

Agents are risk averse and would potentially gain from sharing risk with one another. From an ex post perspective, insurance against such shocks is a subsidy provided by patient agents to impatient agents. The opportunities for mutually beneficial trade are restricted, however by an incentive problem due to private information. Specifically, an individual preference shock is private information of that agent, thus agents can collect the subsidy by claiming to be impatient, and no one can detect whether this is a truthful report. Therefore, such an insurance market will not work since truthful reporting violates agents' incentive constraints. In contrast, the bank can provide at least partial insurance in an incentive compatible way. So agents will only make contracts with the bank.

---

[9] Later in this section, we will consider 2 different cases of p_t: first, we let p_t be a constant p for all periods; second, assume p_t is subject to a deterministic cycle with $p_t = p_1$ when t is an odd number, and $p_t = p_2$ when t is an even number.

[10] This in the same spirit of the law of the large number and its logical consistency with probability theory is shown by Robert M Anderson.[3]

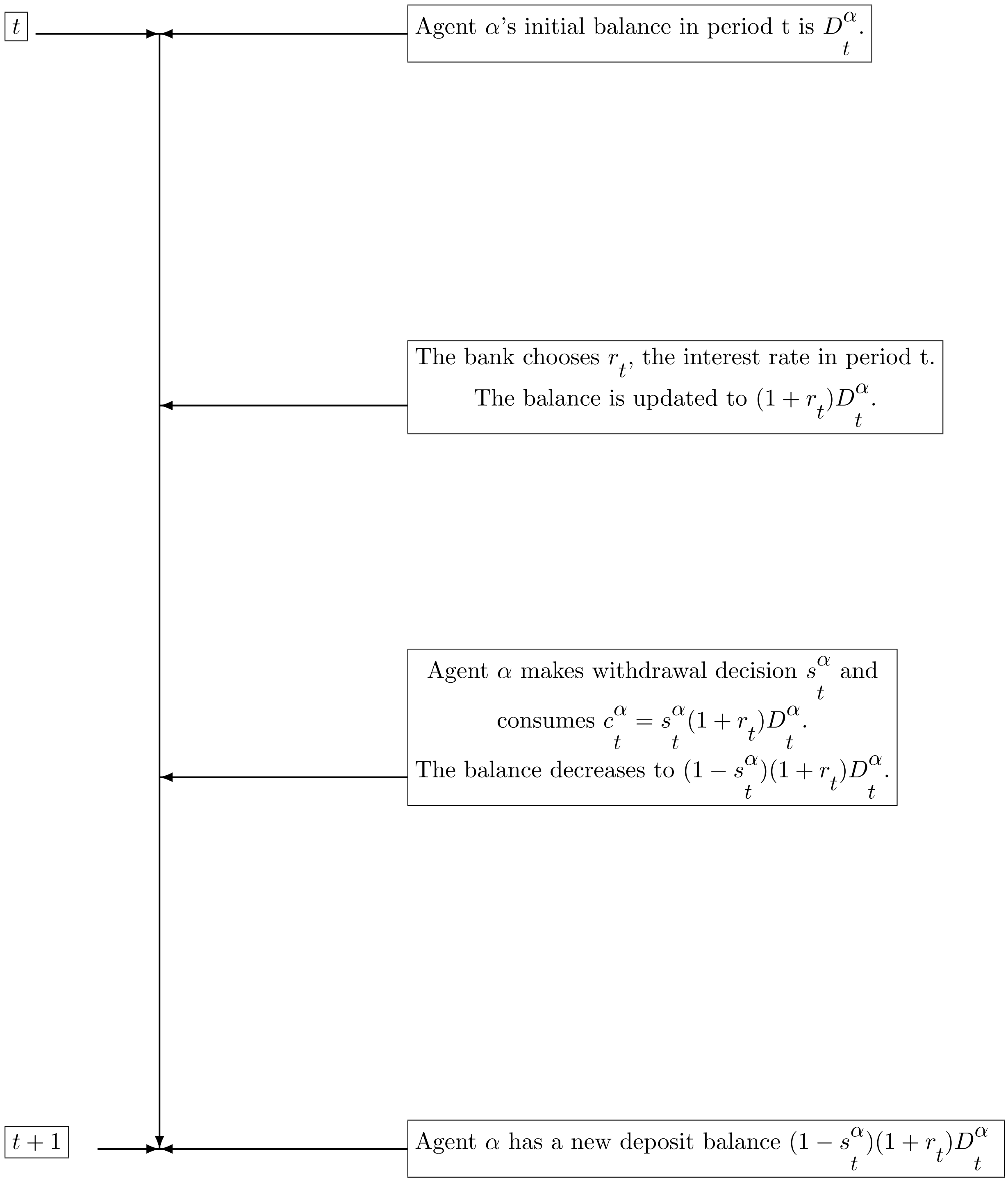

Fig. 2.3. Sequence of events in period t.

At the the beginning of the initial period, each agent is endowed with a portfolio (I_1, L_1) of assets which are exogenously invested in 2 production technologies. At the same time, the agent will make an irreversible choice whether to retain the endowment portfolio and manage it directly themselves or to transfer it to the bank in return for a deposit balance D_1.[11] This deposit will accumulate interest and agents can subsequently withdraw from it to satisfy their consumption needs.

Figure 3 shows the sequence of events that happen in period t. Let D_t^α be the deposit balance of agent α at the beginning of period t, and let s_t^α be the withdrawal decision of agent α, which is the fraction of the deposit balance that agent α wants to withdraw in period t, so $0 \le s_t^\alpha \le 1$. After each agent α learns his/her consumption preference of that period, he/she comes to the bank and withdraws a fraction s_t^α of his/her deposit balance to consume. The consumption constraint[12] is

$$c_t^\alpha = (1 + r_t) s_t^\alpha D_t^\alpha.$$

The law of motion of the deposit balance is

$$D_{t+1}^\alpha = (1 - s_t^\alpha)(1 + r_t) D_t^\alpha.$$

[11] Due to the parametric specification of the model, there is no loss of generality that an agent choose to transfer all endowment or none of it to the bank and assuming deposits are only accepted at the beginning of the initial period (before agents learn their own consumption type of period 1).

[12] As in the previous model, assume that goods withdrawn from the bank are perishable and since if agents choose to transfer their endowments to the bank, then they will not invest directly any more. Thus agents will consume all withdrawn goods.

Since the bank is mutually owned by its depositors, it makes investment decisions to maximize the aggregate expected discounted of their utilities. Let $(\bar{I}_t, \bar{L}_t)$ be the investment portfolio of the bank in period t. To meet the withdrawal demand, the bank will adjust its investment portfolio to $(\bar{I}_{t+1}, \bar{L}_{t+1})$.

Since all agents are identical ex ante, they will make the same decision whether or not they would transfer their endowment to the bank. If an agent chooses to retain his/her endowment portfolio rather than exchange it for a bank deposit, then his/her investment decision is the same as the single agent economy. If an agent chooses to transfer his/her endowment to the bank, then the relevant optimization problem is

$$\max_{\{s_t\}_{t=1}^{\infty}} E[\sum_{t=1}^{\infty} (\prod_{\tau=0}^{t-1} \Delta_\tau) \log \Big(s_t D_t (1 + r_t) \Big)], \tag{2.35}$$

subject to

$$0 \leq s_t \leq 1$$

for all $t \geq 1$.

Since the utility function is given by a logarithmic function, the withdrawal decisions of the agent are proportional to his/her wealth level. Thus s_t only depends on the discount factor in period t : $s_t \in \{s^l, s^h\}$, where s^l (s^h) is the optimal withdrawal decision when being impatient (patient.)

Given the withdrawal decisions of depositors, the bank will maximize the aggregate discounted utility values by choosing an optimal sequence of interest rates.[13] The interest rate[14] given by the bank is uniquely determined by the bank's investment portfolio,

$$(1+r_t)\int_A s_t^{\alpha} D_t^{\alpha} d\alpha = R_I \bar{I}_t - \theta(1-\frac{\bar{I}_{t+1}}{R_I \bar{I}_t})^2 \bar{I}_t + R_L \bar{L}_t - \bar{I}_{t+1} - \bar{L}_{t+1}.$$

Thus the maximization problem of the bank is given by:

$$\max_{\{\bar{I}_t, \bar{L}_t\}_{t=2}^{\infty}} \int_A \sum_{t=1}^{\infty} \left(\prod_{\tau=0}^{t-1} \Delta_{\tau}^{\alpha} \right) \log(c_t^{\alpha})\, d\alpha, \tag{2.36}$$

subject to

$$\begin{aligned}
c_t^{\alpha} &= s_t^{\alpha} D_t^{\alpha} (1+r_t); \\
(1+r_t)\int_A s_t^{\alpha} D_t^{\alpha} d\alpha &= R_I \bar{I}_t - \theta(1-\frac{\bar{I}_{t+1}}{R_I \bar{I}_t})^2 \bar{I}_t + R_L \bar{L}_t - \bar{I}_{t+1} - \bar{L}_{t+1}; \\
0 &\le \bar{I}_{t+1}; \\
0 &\le \bar{L}_{t+1}
\end{aligned}$$

for all $t \geq 1$.

[13] With this payment mechanism, just as [16], the run equilibrium still exists. In this paper we will focus on the welfare of the no run equilibrium, we do not discuss the bank run equilibrium here.

[14] As in the baseline model, goods adjusted from investment are perishable and this constraint is binding in the optimal case.

In the remainder of this section we first show that transferring the endowment to the bank is beneficial to an individual agent and then study the evolution of the bank's optimal investment portfolio of two variations of p_t.

2.3.1 Direct Investing vs Depositing In The Bank

PROPOSITION 2.12. *The expected discounted utility level that agents can achieve by depositing in the bank is at least as high as that which they would achieve by investing directly.*

Proof It is sufficient to show that a deviation from directly investing to depositing in the bank will not lower the agents' consumption .

Since all agents are identical ex ante, they will make the same decision at the initial period. Assume agents will only deposit in the bank for the first period and then stop depositing by taking back the same investment portfolio as they would have when investing directly. This deviation is feasible due to the following:

Since all agents transfer their endowments to the bank at the beginning of period 1, $\bar{I}_1 = \int_A I_1 d\alpha = I_1$, and $\bar{L}_1 = \int_A L_1 d\alpha = L_1$, which means that the bank will start with the same initial portfolio as any individual agent.

If agents choose to directly invest, then all those agents with the same discount factor in period 1 will make the same optimizing investment choice. Let $(I_2^l, L_2^l), (I_2^h, L_2^h)$ be the optimal investment choice of an individual agent with discount factor δ^l and δ^h respectively in the direct investing case.

Consider the following bank's suboptimal investment choice:

$$\bar{I}_2 = pI_2^l + (1-p)I_2^h,$$

and

$$\bar{L}_2 = pL_2^l + (1-p)L_2^h.$$

This is a feasible investment plan since

$$\begin{aligned}
\bar{I}_2 + \bar{L}_2 &= (pI_2^l + (1-p)I_2^h) + (pL_2^l + (1-p)L_2^h) \\
&= R_I I_1 + R_L L_1 - (p\theta(1 - \frac{I_2^l}{R_I I_1})^2 + (1-p)\theta(1 - \frac{I_2^h}{R_I I_1})^2)I_1 - pc_1^l - (1-p)c_1^h \\
&\leq R_I I_1 + R_L L_1 - \theta(1 - \frac{pI_1^l + (1-p)I_2^h}{R_I I_1})^2 I_1 - pc_1^l - (1-p)c_1^h
\end{aligned}$$

with equality only when $I_2^l = I_2^h$.

Also

$$\bar{c}_1 = R_I I_1 + R_L L_1 - \theta(1 - \frac{pI_1^l + (1-p)I_2^h}{R_I I_1})^2 I_1 - (\bar{I}_2 + \bar{L}_2),$$

and so we can conclude $\bar{c}_1 \geq pc_2^l + (1-p)c_2^h$.

The above deviation allows agents to take back the same investment portfolio as they directly invest, and to obtain at least the same level of consumption in period 1 as they would have in the first period by investing directly.

Depositing in the bank for a longer time is the same as applying a similar deviation in period 2, period 3 etc. Thus we proved that depositing in the bank will allow agents to have at least the same consumption as they would achieve by investing directly.

Example 2.13. *Let* $R_I = 1.2$, $R_L = 1.18$, $\theta = 2$, $\delta^l = 0.05$, $\delta^h = 1.0$ *and* $P(\delta_t = \delta^l) = 0.05$. *All agents are endowed with investment portfolio* $(I_1, L_1) = (1, 0)$. *By numerical simulation, the expected discounted utility value of the autarkic case is -6.8804, while the expected discounted utility value of depositing in the bank is -0.2551, which is higher than the autarkic case.*[15]

2.3.2 Bank's Optimal Portfolio

2.3.2.1 Constant p_t

In this section, we consider the simplest case of p_t, $P(\delta_t = \delta^l) = p_t \equiv p$. This implies $\mu\{\alpha \in A | \delta^\alpha_t = \delta^l\} \equiv p$ for all t.

For the convenience of studying the bank's optimal portfolio, as in the single agent economy, we will change the coordinates.

Definition 2.14. *Define* $\bar{B}_t$ *to be the portfolio size of the bank in period t,*

$$\bar{B}_t = \bar{I}_t + \bar{L}_t$$

[15] *All formulae for computing the expected discounted value of agents are provided in the appendix. The interest rate is given by* $\dfrac{\bar{B}_{t+1}}{\bar{B}_t} = \dfrac{(1+r_t)\int_A (1-s^\alpha_t) D^\alpha_t d\alpha}{\int_A D^\alpha_t d\alpha}$. *The initial deposit balance is given by* $\bar{c}_1 = (1+r_1)\int_A s^\alpha_1 D_1 d\alpha$. *For this example, the withdrawal choice is* $s^l = 0.4872$ *and* $s^h = 0.0453$, *the steady state interest rate is* $r_t = 0.2301$ *and the initial deposit of each agent is* $D_1 = 0.6018$.

and define $\bar{q}_t$ to be the liquidity ratio of the bank's portfolio,

$$\bar{q}_t = \frac{\bar{B}_t - \bar{L}_t}{\bar{B}_t}.$$

Thus $\bar{I}_t = \bar{B}_t\bar{q}_t$ and $\bar{L}_t = \bar{B}_t(1-\bar{q}_t)$. With the new set of coordinates $(\bar{B}_t, \bar{q}_t)$, we rewrite the aggregate consumption constraint as follows:

$$\bar{c}_t = (1+r_t)\int_\alpha s_t^\alpha D_t^\alpha d\alpha = R_I\overline{B}_t\bar{q}_t - \theta(1 - \frac{\overline{B}_{t+1}\bar{q}_{t+1}}{R_I\overline{B}_t\bar{q}_t})^2\overline{B}_t\bar{q}_t + R_L\overline{B}_t(1-\bar{q}_t) - \overline{B}_{t+1}$$

Since we use a logarithmic utility function in this model, it can be shown that the optimal investment decision of the bank is independent of the distribution of agents' wealth level and withdrawal choices.

Lemma 2.15. *The solution to the bank's problem is equivalent to the Simplified Bank's Problem [SBP] stated below:*

$$\max_{\{\overline{B}_t,\overline{q}_t\}_{t=2}^\infty}\sum_{t=1}^{\infty}\left(\int_A \prod_{\tau=0}^{t-1}\Delta_\tau^\alpha d\alpha\right)\log(\bar{c}_t) \tag{2.37}$$

subject to

$$\begin{aligned} \bar{c}_t &= R_I\overline{B}_t\overline{q}_t - \theta(1 - \frac{\overline{B}_{t+1}\overline{q}_{t+1}}{R_I\overline{B}_t\overline{q}_t})^2\overline{B}_t\overline{q}_t + R_L\overline{B}_t(1-\overline{q}_t) - \overline{B}_{t+1} \\ 0 &\le \bar{I}_{t+1} \\ 0 &\le \bar{L}_{t+1} \end{aligned} \tag{2.38}$$

for all $t \geq 1$.

Proof Consider an agent α, and let σ_t^{α} be the ratio of D_t^{α} to the mean balance of all agents $\overline{D}_t$.[16] By exchanging the order of integration and summation[17], we rewrite the bank's maximizing problem as follows:

$$\max_{\{\overline{B}_t,\overline{q}_t\}_{t=2}^{\infty}} \sum_{t=1}^{\infty} \int_A (\prod_{\tau=0}^{t-1} \Delta_{\tau}^{\alpha}) \log(\sigma_t^{\alpha}\overline{c}_t) d\alpha, \tag{2.39}$$

Since $\log(\sigma_t^{\alpha}\bar{c}_t) = \log(\sigma_t^{\alpha}) + \log(\bar{c}_t)$, problem (2.39) can be written as follows:

$$\max_{\{B_t,q_t\}_{t=2}^{\infty}} \{\sum_{t=1}^{\infty} \left(\int_A (\prod_{\tau=0}^{t-1} \Delta_{\tau}^{\alpha}) \log(\sigma_t^{\alpha}) d\alpha \right) + \sum_{t=1}^{\infty} \left(\int_A \prod_{\tau=0}^{t-1} \Delta_{\tau}^{\alpha} d\alpha \right) \log(\overline{c}_t)\}, \tag{2.40}$$

We notice the first term in (2.40) is not affected by the bank's investment choice. This completes the proof of lemma (2.15).

According to the lemma (2.15), the bank's investment choice only depends on the aggregate consumption preference.

Corollary 2.16. *The equivalent functional problem to SBP is*

$$V(\overline{B}_t, \overline{q}_t) = \max_{\{\overline{B}_{t+1},\overline{q}_{t+1}\}} \{\log(\overline{c}_t) + (p\delta^l + (1-p)\delta^h) V(\overline{B}_{t+1}, \overline{q}_{t+1})\} \tag{2.41}$$

subject to constraints (2.38).

---

[16] Since we normalize $\mu(A) = 1$, the mean balance coincides with the aggregate balance of the agents.

[17] This exchange is justified by the Dominated Convergence Theorem.

Proof Since $p_t \equiv p$ is a constant,

$$\int_A (\prod_{\tau=0}^{t} \Delta_\tau^\alpha) d\alpha = \sum_{k=0}^{t} ({}_t\mathbf{C}_k) p^k (1-p)^{t-k} (\delta^l)^k (\delta^h)^{t-k} = (p\delta^l + (1-p)\delta^h)^t.$$

The SBP can be written:

$$\max_{\{\overline{B}_t, \overline{q}_t\}_{t=2}^\infty} \sum_{t=1}^{\infty} (p\delta^l + (1-p)\delta^h)^{t-1} \log(\overline{c}_t)$$

By the principle of optimality, we can write the equivalent value functional problem of the SBP

$$V(\overline{B}_t, \overline{q}_t) = \max_{\{\overline{B}_{t+1}, \overline{q}_{t+1}\}} \{\log(\overline{c}_t) + (p\delta^l + (1-p)\delta^h) V(\overline{B}_{t+1}, \overline{q}_{t+1})\}$$

Definition 2.17. *The bank's investment portfolio path is in steady state $(\bar{b}, \bar{q})$ in period t, if for all $\tau \geq t$,*

- $\dfrac{\bar{B}_{\tau+1}}{\bar{B}_\tau} \equiv \bar{b} > 0$ *and*
- $\bar{q}_\tau \equiv \bar{q} \in [0, 1]$.

Proposition 2.18. *$\bar{q}$ is 0 or 1 in the optimal steady state of the bank's investment portfolio.*

Proof According to corollary (2.15), the proof is similar to the proof of proposition (2.6).

2.3.2.2 Cyclical p_t

In this section, we assume that the probability of an agent being impatient is cyclical over the period: $P(\delta_t = \delta^l) = p_i$, where $i = 1$ if t is odd and $i = 2$ if t is even. Without loss of generality, we assume $p_1 > p_2$, which implies that more agents are impatient in an odd period than in an even period.

Corollary 2.19. *The equivalent functional equation to SBP is*

$$V(\overline{B}_t, \overline{q}_t) = \max_{\{\overline{B}_{t+1}, \overline{q}_{t+1}\}} \{\log(\overline{c}_t) + \bar{\delta}_t V(\overline{B}_{t+1}, \overline{q}_{t+1})\}, \tag{2.42}$$

subject to constraints (2.38), and where $\bar{\delta}_t$ is given by

$$\bar{\delta}_t = \begin{cases} p_1\delta^l + (1-p_1)\delta^h & : \quad t \text{ is odd} \\ p_2\delta^l + (1-p_2)\delta^h & : \quad t \text{ is even.} \end{cases}$$

Proof For any $t > 0$,

$$\int_A \prod_{\tau=0}^{t} \Delta_\tau \, d\alpha = \sum_{k_1=0}^{t-[\frac{t}{2}]} \sum_{k_2=0}^{[\frac{t}{2}]} ({}_{t-[\frac{t}{2}]}\mathbf{C}_{k_1})({}_{[\frac{t}{2}]}\mathbf{C}_{k_2}) p_1^{k_1}(1-p_1)^{t-[\frac{t}{2}]-k_1} p_2^{k_2}(1-p_2)^{[\frac{t}{2}]-k_2}$$

$$\cdot \quad (\delta^l)^{k_1+k_2}(\delta^h)^{t-k_1-k_2}$$

$$= \left(\sum_{k_1=0}^{t-[\frac{t}{2}]} ({}_{t-[\frac{t}{2}]}\mathbf{C}_{k_1})(\delta^l)^{k_1}(\delta^h)^{t-[\frac{t}{2}]-k_1} p_1^{k_1}(1-p_1)^{t-[\frac{t}{2}]-k_1}\right) \cdot$$

$$\left(\sum_{k_2=0}^{[\frac{t}{2}]} ({}_{[\frac{t}{2}]}\mathbf{C}_{k_2}) p_2^{k_2}(1-p_2)^{[\frac{t}{2}]-k_2}(\delta^l)^{k_2}(\delta^h)^{[\frac{t}{2}]-k_2}\right)$$

$$= \left(p_1\delta^l + (1-p_1)\delta^h\right)^{t-[\frac{t}{2}]} \left(p_2\delta^l + (1-p_2)\delta^h\right)^{[\frac{t}{2}]}.$$

Thus we can write the SBP:

$$\max_{\{\overline{B}_t, \overline{q}_t\}_{t=2}^{\infty}} \sum_{t=1}^{\infty} (p_1\delta^l + (1-p_1)\delta^h)^{t-1-[\frac{t-1}{2}]}(p_2\delta^l + (1-p_2)\delta^h)^{[\frac{t-1}{2}]} \log(\bar{c}_t).$$

By the principle of optimality,

$$V(\overline{B}_t, \overline{q}_t) = \max_{\{\overline{B}_{t+1}, \overline{q}_{t+1}\}} \{\log(\bar{c}_t) + \bar{\delta}_t V(\overline{B}_{t+1}, \overline{q}_{t+1})\},$$

where $\bar{\delta}_t$ is the average discount factor of each period,

$$\bar{\delta}_t = \begin{cases} p_1\delta^l + (1-p_1)\delta^h & : \text{ if t is odd} \\ p_2\delta^l + (1-p_2)\delta^h & : \text{ if t is even.} \end{cases}$$

This completes the proof.

Definition 2.20. *The investment portfolio path of the bank is in steady state* $((\bar{b}_1, \bar{q}_1), (\bar{b}_2, \bar{q}_2))$ *in period* t*, if* $\forall \tau \geq \frac{t}{2}$,

- $\frac{\overline{B}_{2\tau+1}}{\overline{B}_{2\tau}} \equiv \bar{b}_1 > 0, \frac{\overline{B}_{2\tau+2}}{\overline{B}_{2\tau+1}} \equiv \bar{b}_2 > 0$
- $\bar{q}_{2\tau+1} \equiv \bar{q}_1 \in [0,1], \bar{q}_{2\tau+2} \equiv \bar{q}_2 \in [0,1]$.

As in the cyclical case of the baseline model, we study the auxiliary problem of the bank's investment to find out when the liquidity ratio of the bank's investment portfolio is cyclical in the optimal steady state.

The corresponding auxiliary problem for the bank is: assume the illiquid technology is the only one investment choice,

$$V(\overline{B}_t) = \max_{\overline{B}_{t+1}} \{\log(\bar{c}_t) + \bar{\delta}_t V(\overline{B}_{t+1})\} \tag{2.43}$$

subject to

$$\begin{aligned} \bar{c}_t &= R_I \overline{B}_t - \theta(1 - \frac{\overline{B}_{t+1}}{R_I \overline{B}_t})^2 \overline{B}_t - \overline{B}_{t+1}. \\ 0 &\leq B_{t+1} \end{aligned}$$

for all $t \geq 1$.

Since the return from illiquid technology is homogeneous of degree 1, without loss of generality we assume $\bar{B}_1 = \bar{I}_1 = 1$.

Definition 2.21. *The investment portfolio path is in the steady state* $(\bar{x}_1, \bar{x}_2)$ *of the auxiliary problem in period* τ*, if* $\forall t \geq \frac{\tau}{2}$*:*

- $\frac{\overline{B}_{2t+1}}{\overline{B}_{2t}} \equiv \bar{x}_1 > 0, \frac{\overline{B}_{2t+2}}{\overline{B}_{2t+1}} \equiv \bar{x}_2 > 0$ *for all* $t \geq \frac{\tau}{2}$.

  Denote

$$
\begin{aligned}
\bar{\delta}^l &= p_1\delta^l + (1-p_1)\delta^h \\
\bar{\delta}^h &= p_2\delta^l + (1-p_2)\delta^h
\end{aligned}
$$

As in the single agent economy we have the following theorem:

Theorem 2.22. *Let* $(\bar{x}_1, \bar{x}_2)$ *be the optimal steady state of auxiliary problem. If*

$$
\bar{x}_1 \frac{-\frac{2\theta}{R_I}(1-\frac{\bar{x}_1}{R_I})}{R_I - \theta(1-\frac{\bar{x}_1}{R_I})^2 - \bar{x}_1} - \bar{\delta}^h \frac{(R_I - \theta + \theta(\frac{\bar{x}_2}{R_I})^2 - R_L)}{R_I - \theta(1-\frac{\bar{x}_2}{R_I})^2 - \bar{x}_2} > 0 \tag{2.44}
$$

and assumption (2.5) are satisfied, then in the optimal steady state of the bank's investment portfolio, the liquidity ratio is cyclical with $0 < \overline{q}_1 < 1$ *and* $\overline{q}_2 = 1$.

Proof According to corollary (2.19), this proof is similar to the proof of theorem (2.11).

Example 2.23. *Let* $R_I = 1.2, R_L = 1.18, \theta = 0.5, p_1 = 0.9, p_2 = 0, \delta^l = 0$ *and* $\delta^h = 0.99$. *By computation, we have* $\bar{q}_1 = 0.936$ *and* $\bar{q}_2 = 1.000$*, and the average growth rate* *of portfolio size is given by* $\sqrt{\bar{b}_1 \bar{b}_2} = \sqrt{0.7143 * 0.3327} = 0.4876$. *The portfolio size is shrinking because* 90% *of the agents are impatient during odd periods.*

2.4 Conclusion

The model developed here makes a beginning towards explaining features of the asset portfolio of the U.S banking sector, which we described in the introduction. Since the aggregate preference shock is a deterministic 2-period cycle in the model, the priority for further research is to generalize it to a Markovian process and the ultimate goal is to calibrate or estimate such a generalized model.

Chapter 3

Direct Mechanism

3.1 Introduction

In the previous chapter, we study the optimal investment portfolio of the bank and solve for the efficient withdrawal behavior of the agents under the assumption that it is incentive compatible for agents to truthfully reveal their preference types. While the contract we study in the last chapter is ad ad hoc contract, we did not question whether or not the contract is optimal.In this chapter, we are going to study the direct mechanism,[1] which is a mechanism that is broad enough to contain all efficient deposit contacts according to the *revelation principle*.

THEOREM 3.1. *(revelation principle) To any equilibrium of a game of incomplete information, there corresponds an associated direct mechanism that has an equilibrium where the players truthfully report their types.*

The difference between an allocation of a mechanism and an allocation of a contract is that the allocation of a contract is a sequence of functions which specify the consumption of an agent under the assumption that there is no joint deviation. While

[1]Direct mechanism: A particular mechanism representing a game of incomplete information where the players act simultanously, and where each player's action only consists of a report about his type, i.e. private information. In a revealing equilibrium of a direct mechanism, for each player the incentive constraints for each type not to mimic another one are met, as well as the constraints of individual rationality that each type at least earning his reservation utility.

the allocation of a mechanism is a sequence of functions that specify the consumption of an agent with an arbitrary reporting profile.

According to the revelation principle, we can design an efficient allocation of the direct mechanism by extending the efficient allocation of a deposit contract. This allocation is efficient in the sense that truth-telling reporting strategy is an equilibrium of the direct mechanism, and the allocation sequence maximizes the ex ante utility of the agents without violating the incentive compatible condition. Inspired by Green and Lin (2003), we study whether or not other inefficient equilibrium still exists with a broader set of contracts than those studied by Diamond-Dybvig.

The model in this chapter is the same as in the previous chapter. The bank acts as a social planner of the economy, who will offer insurance to agents in the economy against their preference shocks. To keep the model simple, we assume there is no aggregate uncertainty about the preference type, that is, the probability of being impatient type is a constant, and we assume the bank only can invest in liquid technology.[2] In each period, each agent can get consumption by reporting to the bank his/her preference type of the current period.

[2]Since in the last chapter, we proved that if there is no aggregate uncertainty, the bank will only invest in one type of technology, without of generality we assume that bank will only invest in liquid technology here.

In this chapter, first we point out that since the preference types are private information of agents, the full insurance allocation (first best allocation) is also not implementable under the direct mechanism. [3] Thus the insurance offered by the bank should at least be incentive compatible.

To find an efficient allocation of the direct mechanism, we design an efficient incentive compatible deposit contract first. Use a similar framework as in Atkeson and Lucas (1992) , which was first introduced by Green (1987) , the maximization problem is solved by its dual problem. The incentive compatible problem is solved by announcing a promised utility for the future as well as giving the consumption goods: on one hand, it is incentive compatible for impatient agents to tell the truth by offering those "impatient" agents (lower discount factor for the next period) a higher current consumption level but a lower promised utility for the future; on the other hand, with a higher promised utility for the future, it is also optimal for "patient" agents to tell the truth even though they have a lower current consumption level.

The efficient allocation of the direct mechanism is derived by extending the efficient allocation rule of the deposit contract: the allocation of the agents is the same as the optimal deposit contract if the reporting strategy profile is on the truth-telling

[3] Since with the allocation which offers full insurance to all agents, the marginal value of having consumption in the future period should be equal to the marginal consumption value in the first period. According to our assumption, the preference shocks affect the discount factor in the next period, which means all impatient agents will have lower consumption in the future if they would tell the truth. Thus all impatient agents will lie being patient in each period, which means this insurance market won't work.

equilibrium path; while if there exists a joint deviation in some period, the bank will adjust agents' promised utility value to stay in solvent. Since the bank can not identify the lying agents in the economy, the adjustment will affect all agents. With the efficient allocation specified, we can show that although truth telling reporting strategy is an equilibrium of the direct mechanism, there might exist other equilibrium reporting strategy under certain parameters.

3.2 Definitions

3.2.1 Economic Environment

The environment of the model is the same as in the previous chapter. Time is discrete, $t = 1, 2, \ldots$. There are a continuum of agents in the economy $i \in I$. In each period t, all agents face a random consumption shock which is represented by the discount factor δ^l(impatient) or δ^h(patient).

Assume in the initial period, the bank has an investment portfolio L_1. Agents can get consumption from the bank in each period by reporting their current preference types to the bank.

In period $t+1$, denote agent i's history of preference types as $\delta^{i,t}$ which is given by $\delta^{i,t} = \{\delta^i_1, \ldots, \delta^i_t\}$, correspondingly the history of consumption level agent i is $c^{i,t} = \{c^i_1, \ldots, c^i_t\}$ and the history of the bank's investment portfolio $L^t = \{L_1, \ldots, L_t\}$.

Definition 3.2. *Let* $\Delta = \{\delta^l, \delta^h\} = \Delta^1$ *and* Δ^{t+1} *be the* $(t+1)$*-fold product space. Let* μ *be the measure on* Δ*, and* μ^{t+1}*, the product measure, be the distribution of the history of preference type shocks. Let* Δ^∞ *and* μ^∞ *be the corresponding infinite product space and probability measure.*

3.2.2 Reporting Strategy

Since preference types are private information, the reporting strategy of a single agent depends on the realized current preference type as well as his/her own history of preference types, consumption levels and the history of the bank's investment portfolio.

Definition 3.3. *Define* Z_t *to be the set of reporting strategies that each agent can adopt in period* t, $Z_t = \{z_t : \ \Delta^t \times \mathcal{R}_+^{t-1} \times \mathcal{R}_+^t \mapsto \{\delta^l, \delta^h\}\}$*, and let* $Z = \prod_{t=1}^\infty Z_t$*. Let* $j_{z^i,t} = z^i(\delta^{t,i}, c^{t-1,i}, L^t)$ *be the image of reporting type of agent* i *in period* t *given reporting strategy* $z^i \in Z$ *and* $j_{z^i}^{i,t} \in \Delta^t$ *be the reporting history of agent* i *in period* t*.*

Definition 3.4. *The truth telling strategy is the strategy* $z^* \in Z$ *such that for all* $t \geq 1$ $j_{z^*,t}^i = \delta_t^i$ *for all* $\delta^{t,i} \in \Delta^t$, $c^{t-1,i} \in \mathcal{R}_+^{t-1}$*, and* $L^{t-1,i} \in \mathcal{R}_+^{t-1}$*.*

Since all agents are identical ex ante, it is enough to consider the set of symmetric reporting strategy profiles.

Definition 3.5. *Let* $\widetilde{Z} = \{\widetilde{z} : \ I \mapsto Z | \widetilde{z}$ *is a constant map*$\}$ *be the space of symmetric reporting strategy profiles, and let* $\widetilde{j}_{\widetilde{z},t} = \prod_{i \in I} j_{z^i,t}$*, where* $\widetilde{z} = \prod_{i \in I} z^i \in \widetilde{Z}$*. Let* $\widetilde{J}^t = \Delta^{I \times t}$*.*

3.2.3 Feasibility Constraint

Definition 3.6. *Define the bank's investment decision as* K*:* $\Delta^I \times \mathcal{R}_+ \mapsto \mathcal{R}_+$*, for* $\widetilde{z} \in \widetilde{Z}$*,* $\widetilde{j}_{\widetilde{z},t} \in \Delta^I$*, let* $L_{t+1} = K(\widetilde{j}_{\widetilde{z},t}, L_t)$*, where* L_t *be the bank's investment level in liquid technology in period* t*.*

The consumption amount of agent i getting from the bank is a function of the current report as well as his/her own reporting history and the history of reporting profiles of all agents and the bank's current investment portfolio.

Definition 3.7. *Define* $\widetilde{C}_t = \prod_{i \in I} \widetilde{C}^i_t$ *to be the allocation function set of agents in period* t*,* $\widetilde{C}^i_t(j^t_{z^i}, \widetilde{j}^{t-1}_{\widetilde{z}}, L_t) : \Delta^t \times J^{t-1} \times \mathcal{R}_+ \mapsto \mathcal{R}_+$*.*

Thus the feasibility constraint of the bank is given by the following:

for $t \geq 0$ and all $\widetilde{z} \in \widetilde{Z}$:

$$\begin{aligned} 0 &\leq L_{t+1} \\ \int_I \widetilde{C}^i_t(j^{t,i}_z, j^{t-1}_{\widetilde{z}}, L_t)\, di &\leq R_L L_t - L_{t+1} \end{aligned} \tag{3.1}$$

3.2.4 Equilibrium

3.2.4.1 Definition of Equilibrium

The equilibrium concept we will use in this paper is the Perfect Bayesian Nash equilibrium, which means that each agent chooses an reporting strategy to maximize his/her utility conditional on his/her information in each period.

Assume the period utility function is given by $\rho(c) = \dfrac{c^\gamma}{\gamma}$. Given the allocation function sequence $\{\widetilde{C}^i_t\}$, there exists a unique utility sequence $u_{\widetilde{C}} = \{\widetilde{u}_t\} = \{\rho(\widetilde{C}^i_t)\}$ corresponding to the given allocation.

DEFINITION 3.8. *Given the ex ante expected discounted utility value according to truthful reporting w_1, the allocation $\widetilde{C}$ and the reporting strategy profile $\widetilde{z}$, then the total expected discounted utility value function $\widetilde{U}$ is given by*

$$\widetilde{U}(w_1, u_{\widetilde{C}}, \widetilde{z}) = \sum_{t=1}^{\infty} \prod_{\tau=0}^{t-1} \delta^\tau \int_{\Delta^t} \rho(\widetilde{C}_t(j^t_z, \widetilde{j}^{t-1}_{\widetilde{z}}, L_t))\, d\mu^t.$$

Thus each agent $i \in I$, they will choose $z^i \in Z$ to maximize their total expected discounted utility value.

DEFINITION 3.9. *For any $\widetilde{z} \in \widetilde{Z}$ and $z^i \in Z$, denote $(\widetilde{z}\backslash_i z^i)(j)$ as*

$$(\widetilde{z}\backslash_i z^i)(j) = \begin{cases} z^j & \text{if } j \neq i \\ z^i & \text{if } j = i \end{cases} \tag{3.2}$$

Then given the ex ante expected discounted utility w_1, the allocation $\widetilde{C}$, then $\widetilde{z}$ is a Bayesian Nash Equilibrium reporting strategy profile if it satisfies

for all $i \in I$

$$\widetilde{U}(w_1, u_{\widetilde{C}}, \widetilde{z}) \geq \widetilde{U}(w_1, u_{\widetilde{C}}, \widetilde{z}\backslash_i z^i) \tag{3.3}$$

DEFINITION 3.10. *An allocation is incentive compatible if the truthful reporting strategy satisfies the condition (3.3).*

Definition 3.11. *For any $\hat{z}, z \in Z$, let $\hat{z}^r$ be a truncated reporting strategy from the initial period up to period r. Define $z' = (\hat{z}^r, z) \in Z$ to be a reporting strategy such that*

$$j_{z',t} = \begin{cases} j_{\hat{z},t} & \text{if } t \leq r \\ j_{z,t} & \text{if } t > r \end{cases}$$

Given the ex ante expected discounted utility value w_1 and the allocation $\widetilde{C}$, let $\widetilde{U}_{r+1}(w_1, u_{\widetilde{C}}, (\widetilde{\hat{z}}^r, \widetilde{z}))$ be the function of expected utility from period $r+1$ on discounted to period $r+1$,

$$\widetilde{U}_{r+1}(w_1, u_{\widetilde{C}}, (\widetilde{\hat{z}}^r, \widetilde{z})) = \sum_{\tau=1}^{\infty} \int_{\Delta^\tau} \left(\prod_{k=0}^{\tau-1} \delta_k\right) \rho(\widetilde{C}_{r+\tau}(w_1, (\hat{j}^r_{\hat{z}}, j^\tau_z), (\widetilde{\hat{j}}^r_{\widetilde{z}}, \widetilde{j}^{\tau-1}_{\widetilde{z}}), L_{r+\tau}))\, d\mu^\tau$$

Definition 3.12. *Given the ex ante expected discounted utility w_1 and the allocation $\widetilde{C}$, the truthful reporting strategy z^* is a Perfect Bayesian Nash Equilibrium if it satisfies for any $\hat{z}, z^i \in Z$, $r > 1$,*

$$\widetilde{U}_r(w_1, u_{\widetilde{C}}, (\widetilde{\hat{z}}^r, \widetilde{z}^*)) \geq \widetilde{U}_r(w_1, u_{\widetilde{C}}, (\widetilde{\hat{z}}^r, \widetilde{z}^* \backslash_i z^i)),$$

where $\widetilde{z}^* \in \widetilde{Z}$ is a truthful reporting strategy profile.

3.3 First Best Allocation

Now we consider the first best allocation of the economy. Since the bank is mutually owned by all agents, it maximizes the expected discounted utility of all agents according to their reporting consumption types:

$$\bar{\bar{U}} = \max_{\{(\widetilde{C}_t(j_z^{t,i},\widetilde{j}_{\widetilde{z}}^{t-1},L_t),L_{t+1}\}_{t=1}^{\infty}} \int_I \sum_{t=1}^{\infty} \int_{\Delta^t} \left(\prod_{\tau=0}^{t-1} j^i_{z^i,\tau} \right) \rho(\widetilde{C}_t(j^i_{z^i,t}, \widetilde{j}^{t-1}_{\widetilde{z}}, L_t))\, d\mu^t d\,i \tag{3.4}$$

subject to constraint (3.1) defined above.

THEOREM 3.13. *The first best allocation is not implementable.*

DEFINITION 3.14. *If $j_z^{i,t-1} \in J^{t-1}$ is any agent i's reporting history of preference types using $z \in Z$ as a reporting strategy, define $n(\cdot)$ to be a counting function which counters the number when $j^i_{z,\tau} = \delta^l$ is true for $\tau \leq t-1$.*

In order to prove the theorem, we need the following lemma.

LEMMA 3.15. *Assume that all agents are telling the truth, the first best allocation specifies that for any agent $i_1, i_2 \in I$, if $n(j_{z^*}^{i_1,t-1}) = n(j_{z^*}^{i_2,t-1})$, then $\widetilde{C}_t(j_{z^*}^{i_1,t}, \widetilde{j}_{z^*}^{t-1}, L_t) = \widetilde{C}_t(j_{z^*}^{i_2,t}, \widetilde{j}_{z^*}^{t-1}, L_t)$ for any $j^{i_s}_{z,t} \in \{\delta^l, \delta^h\}$ and $s = 1, 2$.*

Proof Assume all agents are using truth telling reporting strategy z^*. According to Slater's condition, we rewrite the feasibility condition as follows:

$$\begin{aligned} \eta_t(\textstyle\int_I \widetilde{C}_t(j_z^{i,t}, \widetilde{j}_{\widetilde{z}}^{t-1}, L_t)\, di - R_L L_t - L_{t+1}) &= 0 \\ -\zeta_t L_{t+1} &= 0 \\ \eta_t &\geq 0 \\ \zeta_t &\geq 0 \end{aligned} \tag{3.5}$$

Let $\Gamma_t = \{(\widetilde{C}_t, L_{t+1})\, satisfies\ (3.1)\}$. We need to find the sequence of $\{\widetilde{C}_t\}$, $\{L_t\}$, $\{\eta_t\}$ and $\{\zeta_t\}$ such that it is the solution to the problem:

$$\min_{\{\eta_t\}\geq 0,\{\zeta_t\}\geq 0} \max_{(\widetilde{C}_t, L_{t+1})\in\Gamma_t} \{\bar{\bar{U}} - \eta_t(\int_I \widetilde{C}_t(j^i_{z^*,t}, j^{t-1}, L_t)\, di - R_L L_t - L_{t+1}) + \zeta_t L_{t+1}\} \tag{3.6}$$

where $\bar{\bar{U}}$ is defined by (3.13).

Take the first order condition with respect to the consumption function $\widetilde{C}_t(j^{t,i}_{z^*}, \widetilde{j}^{t-1}_{z^*}, L_t)$, we have

$$(\delta^l)^k (\delta^l)^{t-k-1} P(j^{t,i}_{z^*}) \rho'(\widetilde{C}_t(j^i_{z^*,t}, j^{t-1}, L_t)) = \eta_t P(j^{t,i}_{z^*}) \tag{3.7}$$

where $k = n(j^{i,t-1}_{z^*})$, and $P(j^t_{z^*})$ is the probability of an agent has $j^t_{z^*}$ as his/her preference history in period t.

Thus according the probability of the permutations, for $j^{i_s,t-1}_{z^*} \in J^{t-1}$, $s \in \{1,2\}$, if $n(j^{i_1,t-1}_{z^*}) = n(j^{i_2,t-1}_{z^*})$, then according to (3.7), $\widetilde{C}^i_t(j^{i_1,t}_{z^*}, \widetilde{j}^{t-1}_{z^*}, L_t) = \widetilde{C}^j_t(j^{i_2,t}_{z^*}, \widetilde{j}^{t-1}_{z^*}, L_t)$. And this concludes the proof.

With the result of lemma we can prove the theorem (3.13).

Proof According to the revelation principle, we only need to show that truthful reporting violates the incentive compatible condition of agents given the first best allocation.

Assume all agents would give truthful reports about their consumption types in every period.

According to lemma (3.15), we can let (l^k, h^{t-k}) denote the reporting history of certain agent in period $t+1$ with k periods being impatient and $t-k$ periods being patient and $c_{t+1}^{l^k,h^{t-k}}$ be the consumption amount of this agent in period $t+1$.

Then we can rewrite the the bank's maximizing problem as follows:

$$\max_{\{c_t,L_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty}\sum_{k=0}^{t-1} \frac{k!(t-k-1)!}{(t-1)!} p^k (\delta^l)^k (1-p)^{t-k-1} (\delta^h)^{t-k-1} \frac{(c_t^{l^k,h^{t-k-1}})^{\gamma}}{\gamma} \tag{3.8}$$

subject to

$$\begin{aligned} 0 &\leq L_{t+1} \\ \textstyle\sum_{k=0}^{t-1} \frac{k!(t-k-1)!}{(t-1)!} p^k (1-p)^{t-k-1} c^{l^k,h^{t-k-1}} &\leq R_L L_t - L_{t+1} \end{aligned}$$

for $1 \leq t$.

According to the first order condition (**??**), the efficient allocation can be characterized by

$$\frac{c_t^{l^{t-k},h^k}}{c^{l^t}} = \left(\frac{\delta^h}{\delta^l}\right)^{\frac{k}{1-\gamma}}, \tag{3.9}$$

thus in period t, agent i is impatient and he (she) reports to be patient, start from period $t+1$ his (her) future consumption will be improved by factor $(\frac{\delta^h}{\delta^l})^{\frac{1}{1-\gamma}}$. The above logic shows that always claiming being patient is a dominant reporting strategy for agent i regardless of their true consumption type. Thus given the first best allocation, truthful reporting strategy violates the incentive compatible condition, which is a contradiction. This concludes the proof.

3.4 An Incentive Compatible Deposit Contract

As we have shown that the first best allocation (full insurance allocation) is not implementable, next we want to design an efficient allocation of the direct mechanism where truth telling is a Perfect Bayesian Nash equilibrium. According to revelation principle, we can start from an efficient allocation of an incentive compatible deposit contract. Recall the distinction between the allocation of a deposit contract and a mechanism is that an allocation of a deposit contract is a sequence of functions which specify the consumption of the agent under the assumption that all agents are telling the truth. While an allocation of a mechanism is a sequence of functions that specify the consumption of the agents with an arbitrary reporting strategy profile.

The contract we design in this section is optimal in the sense that it maximizes the agents' utility subject to the incentive compatible constraints that prevent full insurance being offered.

Since the allocation of the deposit contract is a sequence of functions which specify the consumption of each agent in each period under the assumption that there is no joint deviation, the consumption function of the contract $C_t(w_1, j_z^t)$ in each period only depends on each agent's own reporting preference type history j_z^t and the ex ante expected discounted utility value w_1 with truthful reporting strategy . Since all agents are identical ex ante, at the beginning of the initial period, they have the same expected discounted utility value w_1. Assume in period t, the bank will tell the agent the amount of consumption goods $c_t = C_t(w_1, j_z^t)$ as well as his/her promised utility value w_{t+1}, which is the expected discounted utility value of the agents at the beginning of period $t+1$.

$$w_{t+1} = \sum_{\tau=1}^{\infty} \int_{\Delta^\tau} \left(\prod_{s=t}^{t+\tau-1} \delta_s \right) \rho(C_{t+\tau}(w_1, \{\delta^t, \delta_{t+1}, \ldots, \delta_\tau\}))d\mu^\tau$$

.

Assume the allocation $\{C_t\}_{t=1}^{\infty}$ of the deposit contract satisfies the following conditions:

1. it delivers the ex ante expected discounted utility w_1 to all agents;

$$w_1 = \sum_{t=1}^{\infty} \int_{\Delta^t} \left(\prod_{s=0}^{t-1} \delta_s \right) \rho(C_t(w_1, \delta^t))d\mu^t$$

2. it is incentive compatible (truthful telling is the dominant strategy of the deposit contract).

If the allocation also satisfy the following condition, then we say the allocation attains w_1 with initial investment level L_1: given all agents having the identical ex ante expected discounted utility value w_1, the corresponding aggregate consumption is feasible with the initial endowment L_1.

$$\sum_{t=1}^{\infty}(\frac{1}{R_L})^t \int_I C_t^i(w_1, \delta^t)\, d\mu^t \leq L_1$$

The incentive problem here is solved by conditioning the current and future consumption not only on the history of preference types but also the current preference type. Thus by allowing current impatient agents having a higher current consumption and lower promised utility level in the future, it is incentive compatible for impatient agents to report the true type. While for patient agents, although they will have a lower current consumption, the promised future utility is higher, thus it is also incentive compatible for patient agents to report the true consumption type.

Let D be the space of present discounted utility value (like an interval on R). Assume $\rho(c) = \frac{c^\gamma}{\gamma}$ is the one period utility function and let $\rho^{-1}(v)$ be the inverse function of the utility value $D \mapsto \mathcal{R}_+$, which characterizes the consumption goods needed to have utility value v in one period. Thus ρ defines a one-to-one mapping between an allocation sequence $\{C_t(w_1, \delta^t)\}$ and a utility sequence $\{u_t(w_1, \delta^t)\}_{t=1}^{\infty} = \{\rho(c_t(w_1, \delta^t))\}_{t=1}^{\infty}$. Moreover, assume the allocation satisfies the transversality condition: let $u_C = \{u_t(w_1, \delta^t)\}$ be the corresponding utility sequence of a given allocation C, $u_t(w_1, \delta^t)$ satisfies:

$$\lim_{t\to\infty}(\prod_{s=1}^{t-1}\delta_s)\sum_{\tau=1}^{\infty}\int_{\Delta^\tau}\left(\prod_{r=0}^{\tau-1}\delta_{t+r}\right)u_{t+\tau}(w_1, \delta^{t+\tau})\, d\mu^\tau = 0. \tag{3.10}$$

Given a utility sequence u_C, ex ante expected discounted utility value w_1, and reporting strategy z, we can define the total expected discounted utility[4] of an agent:

$$U(w,u,z) = \sum_{t=1}^{\infty} \int_{\Delta^t} \left(\prod_{\tau=0}^{t-1} \right) u_t(w, j_z^t)\, d\mu^t$$

Corresponding to the requirements of the allocation, the utility sequence $u = \{u_t(w,\delta^t)\}_{t=1}^{\infty}$ should satisfy the following requirements[5]:

1. $\{u_t(w_1,\delta^t)\}_{t=1}^{\infty}$ delivers the initial expected discounted utility to each agent

$$w_1 = U(w_1, u, z^*) \tag{3.11}$$

2. $\{u_t(w_1,\delta^t)\}_{t=1}^{\infty}$ is incentive compatible

$$U(w_1, u, z^*) \geq U(w_1, u, z) \tag{3.12}$$

for all $z \in Z$

[4]Since in this section, the deposit contract is incentive compatible under the assumption that all other agents use truthful reporting strategy. Thus the total expected utility only depends on the reporting strategy the agent use by himself/herself, instead of the complete reporting strategy profile of all agents.

[5]All these given conditions are under the assumption that there does not exist joint deviation. In the next section we will remove this assumption.

And the feasibility constraint can be given by: if the allocation can attain w_1 with initial investment L_1, then the corresponding utility sequence $\{u_t(w_1, \delta^t)\}_{t=1}^{\infty}$ satisfies:

$$\sum_{t=1}^{\infty} \left(\frac{1}{R_L}\right)^t \int_I \rho^{-1}(u_t)\, di \leq L_1 \tag{3.13}$$

Let S be the space of u, which satisfies all the conditions (3.11), (3.12) and (3.13). Thus for any $u = \{u_t\} \in S$, $\{\rho^{-1}(u_t)\}$ defines a feasible allocation of the deposit contract.

Follow the same logic as in Atkeson and Lucas (1992), which was first introduced by Green (1987) , we solve the maximization problem of the bank by a dual problem: Let $\mathcal{D}$ be the Borel measurable subsets of D and let M be the space of all probability measure on $\mathcal{D}$. Define a function φ^* which maps distributions of utility $\psi \in M$ to the real line. Let $\varphi^*(\psi)$ be the greatest lower bound of initial investment level in liquid assets needed to attain expected discounted utility distribution ψ. Instead of solving the original maximization problem, we solve the following dual problem: given the ex ante utility distribution ψ_1 what is the allocation such that it requires the lowest initial investment level?

$$\varphi^*(\psi_1) = inf_{\{u_t\} \in S} \sum_{t=1}^{\infty} \int_{D \times \Delta^t} (\frac{1}{R_L})^t \rho^{-1}[u_t(w_1, \delta^t)]\, d\psi d\mu^t\},$$

where the expected discounted utility value distribution in the initial period is a degenerate distribution with $P(w = w_1) = 1$, since all agents are identical ex ante, they have an identical expected discounted utility value w_1.

In order to write the dual problem in a recursive form, we reformulate the problem as follows:

given the expected utility value w_1, the promised utility value $w_2 = g_1(w_1, j_{z,1})$, which is the expected discounted utility value in period 2 depends on the reporting preference types $j_{z,1}$ in period 1. Then in period 2, the bank gives the current period consumption $C_2(w_2, j_{z,2})$ and promised utility $w_3 = g_2(w_2, j_{z,2})$ according to expected discounted utility w_2 and preference report $j_{z,2}$. And in period 3, the bank chooses another pair of values depending on $w_3, j_{z,3}$ and so on. Thus we can identify each agent with their discounted expected utility w_t and all agents have the same w_t will have the same treatment in the current period. Given the period utility function $\rho(\cdot)$, it is equivalent to think the above problem as the bank chooses a sequence of pairs of functions (f_t, g_t), depending on $(w_t, j_{z,t})$ in each period, where $f_t(w_t, j_{z,t}) = \rho(C_t(w_t, j_{z,t}))$ is the utility value agent w_t with current reporting preference type $j_{z,t}$, and $w_{t+1} = g_t(w_t, j_{z,t})$ is the promised utility value from $t+1$ on of the same agent's. Notice that they are the functions of only the current preference type $j_{z,t}$ since the history of reporting preference types and the initial promised utility level w_1 can be identified by the current promised utility level w_t.

Given the distribution of expected discounted utility level ψ_1 in the initial period, which is a degenerate distribution with $P(w = w_1) = 1$, the promised utility function g_1 defines an operator $S_{g_1} : M \mapsto M$ for any $D_0 \in \mathcal{D}$

$$(S_{g_1}\psi)(D_0) = \int_{B_{g_1}(D_0)} d\mu d\psi$$

where $B_{g_1}(D_0) = \{(w_1, \delta_1) \in D \times \Delta : g_1(w_1, \delta) \in D_0\}$. Thus $S_{g_1}\psi_1$ determines the utility distribution in period 2. Let $\sigma = \{f_t, g_t\}$ $f_t, g_t : D \times \Delta :\mapsto D \times D$ be a sequence of pairs of Borel measurable functions. Thus given any initial utility distribution ψ_1, σ will generate a sequence of utility distribution $\{\psi_t\}_{t=1}^{\infty}$, where $\psi_{t+1} = S_{g_t}\psi_t$.

For all $t > 1$, the promised utility value $w_t = g_t(w_{t-1}, j_{z,t-1})$ given initial value w_1 induce a function $W_t(w_1, j_z^t) = w_t : D \times \Delta^{t-1} \mapsto D$ for $t > 1$. Let $u_t(w_1, j_z^t) = f_t[W_t(w_1, j_z^{t-1}), j_{z,t}]$ be the utility value of the agent with the reporting history j_z^{t-1}, current reporting preference type $j_{z,t}$ and ex ante expected discounted utility value in the initial period w_1.

σ is called an allocation rule if it satisfies the following conditions[6]:

1. if the utility sequence it generates u satisfies the equation (3.10);

2. if the promised utility sequence $\{w_t\}$ satisfies the transversality condition as follows: given $w_1 \in D$ and all $\{\delta^t\} \in \Delta^{\infty}$

$$\lim_{t \mapsto \infty} \left(\prod_{\tau=0}^{t-1} \delta_{\tau} \right) W_t(w_1, \delta^t) = 0; \tag{3.14}$$

[6]Since according to the allocation rule, the allocation is temporarily incentive compatible, from now on we will assume that agents will tell the truth in each period, later it will be verified by proposition (3.16) that temporarily incentive compatible is equivalent to total incentive compatible condition in this model.

3. if it satisfies that for all $t \geq 0$ all $w_t \in D$, and $\delta_t = \delta^i, \delta^j \in \Delta,\ i \neq j$

$$\begin{aligned} w_t &= \int_\Delta [f_t(w_t, \delta) + \delta_t g_t(w_t, \delta)\, d\mu]; && (3.15) \\ f_t(w_t, \delta^i) + \delta^i g_t(w_t, \delta^i) &\geq f_t(w_t, \delta^j) + \delta^i g_t(w_t, \delta^j). && (3.16) \end{aligned}$$

Let Σ be the space of all allocation rules σ that satisfies the condition above.

If the allocation rule σ satisfies the following feasibility constraint:

$$\sum_{t=1}^{\infty} (\frac{1}{R_L})^t \int_{D\times\Delta} \rho^{-1}(f_t(w_t, \delta_t))\, d\psi_t d\mu \leq L_1 \tag{3.17}$$

we say the allocation rule attains the utility distribution ψ_1 with initial investment level L_1.

The following proposition gives the equivalence between an allocation and an allocation rule. According to this proposition, we can find the optimal allocation of the deposit contract by characterizing the optimal allocation rule.

Proposition 3.16. *Let $w_1 \in D$ and $\psi_1 \in M$, where ψ_1 is the degenerate distribution given by $P(w = w_1) = 1$. If there exists an allocation $\{C_t(w_1, \delta^t)\}$ and the corresponding utility sequence $u \in S$ such that it attains ψ_1 with initial investment L_1, then there exists an allocation rule σ that attains ψ_1 with initial investment L_1. And if there exists an allocation rule σ attains ψ_1 with initial investment L_1 and $\{w_t\}$ is the expected discounted* *utility sequence generated by* σ*, then* $\{\rho^{-1}(u_t)\}$ *is an allocation sequence and attains* ψ_1 *with initial investment* L_1.

In order to prove the equivalence between these 2 problems, first we need the following lemma, which says that if an allocation is incentive compatible, then given arbitrary reporting history $\hat{j}_{\hat{z}}^{r-1}$, it is always optimal to adopt the truthful reporting strategy from period r.

LEMMA 3.17. *A utility sequence satisfies (3.12) if and only if it satisfies :*

$$u_r(w_1, (\hat{j}_{\hat{z}}^{r-1}, \delta_r)) + \delta_r U_{r+1}(w_1, u, (\hat{z}^{r-1}, z^*)) \geq u_r(w_1, (\hat{j}_{\hat{z}}^{r-1}, j_{z,r})) + \delta_r U_{r+1}(w_1, u, (\hat{z}^{r-1}, z)) \tag{3.18}$$

for any $w_1 \in D$*,* $r \geq 0$*,* $\hat{j}_{\hat{z}}^{r-1} \in J^{r-1}$ *, reporting strategies* $z \in Z$ *and* $j_{z,r} \neq \delta_r \in \Delta$*.*

Proof Sufficiency is given by the fact that (3.12) is the special case of (3.18) when $r = 0$.

For necessity, we prove by contradiction. Assume that (3.12) holds, but for some period $r \geq 1$, $w_1 \in D$, some preference type history (δ^{r-1}, δ_r) and some $\hat{z}, z \in Z$, (3.18) fails to hold. Then there exists $\delta_r \in \Delta$

$$u_r(w_1, (\hat{j}_{\hat{z}}^{r-1}, \delta_r)) + \delta_r U_{r+1}(w_1, u, (\hat{z}^{r-1}, z^*) < u_r(w_1, (\hat{j}_{\hat{z}}^{r-1}, j_{z,r})) + \delta_r U_{r+1}(w_1, u, (\hat{z}^{r-1}, z))$$

Thus we can construct a reporting strategy z' such that if (1) $t < r$,let $j_{z',t} = j_{z^*,t}$ for all $\delta^t \in \Delta^t$; (2) for $t \geq r$, continue truth telling unless (δ^{r-1}, δ_r) is realized; (3) if (δ^{r-1}, δ_r) is realized, switch to reporting strategy z from period r.

Thus the new defined reporting strategy z' yields the same utility as z^* does in the first r periods, and will yield the same utility as z^* in the t periods for $t > r$ if (δ^{r-1}, δ_r) has not been realized. z' will yield strictly higher expected discounted utility if (δ^{r-1}, δ_r) is realized. Since the probability of having the preference type history as (δ^{r-1}, δ_r) is positive, thus $U(w_1, u, z') > U(w_1, u, z^*)$, which is a contradiction to (3.12). This ends the proof of the lemma.

Next, we prove the proposition (3.16). This proof follows the logic of the proof of Lemma (3.2) and Lemma (3.3) in the paper "On Efficient Distribution with Private Information" by Atkeson and Lucas . Given an allocation of the deposit contract, it can induce a sequence of pairs of Boral measurable functions $\{(f_t, g_t)\}_{t=1}^{\infty}$. Let $\sigma = \{(f_t, g_t)\}_{t=1}^{\infty}$, and we show that σ satisfies all the condition of being a feasible allocation rule. For the inverse direction, we show the utility sequence generated by the allocation rule satisfies all the condition of a utility sequence that corresponding to a feasible allocation. Since the utility sequence generated by an allocation rule only satisfies the temporary incentive compatible condition, we show the equivalence between the temporary incentive compatibility and the total incentive compatibility in the following manner: first we claim that if it is optimal to deviate from the truth telling reporting strategy in infinitely many periods, then according to the transversality condition, there must exist a reporting strategy which is different from reporting the truth in finite periods which will yield higher expected discounted utility value than truth telling reporting strategy. Then we prove by induction that if it is not optimal to deviate in any finite N periods, then it is not optimal to deviate in any $N + 1$ period for any $N > 0$. The conclusion that if the allocation satisfies the temporary incentive compatible condition, then the utility sequence it generates satisfies the inequality (3.12) follows. Thus according to the lemma (3.17) the corresponding allocation sequence is incentive compatible.

Here is the proof for proposition (3.16):

Proof

1. First we prove that if there exists an allocation sequence C such that the corresponding utility sequence $u_C \in S$ and it can attain expected discounted utility value w_1 with initial investment level L_1, then there exists an allocation rule $\sigma \in \Sigma$ such that the ex ante expected utility value is w_1 with initial investment L_1.

   Let $w_t = U_t(w_1, u, z^*)$ and define (f_t, g_t) as follows:

$$f_t(w_t, \delta_t) = u_t(w_1, (\delta^{t-1}, \delta_t))$$

$$g_t(w_t, \delta_t) = U_{t+1}(w_1, u, z^*)$$

   Thus according to its definition of U_{t+1} and the above lemma (3.17), the allocation rule satisfies (3.15) and (3.16).

   Next to prove the feasibility condition: according to the definition of ψ_t, we have

$$\int_{D\times\Delta} \rho^{-1}(f_t(w_t, \delta_t))d\psi_t d\mu \quad = \quad \int_\Delta \left(\int_{D\times\Delta} \rho^{-1}(f_t(g_{t-1}(w_{t-1}, \delta_{t-1}), \delta_t))d\psi_{t-1} d\mu \right) d\mu$$

By repeating the above step we have:

$$\begin{aligned}\int_{D\times\Delta}\rho^{-1}(f_t(w_t,\delta_t))d\psi_t d\mu &= \int_{D\times\Delta^{t-1}}\{\int_{\Delta}\rho^{-1}(f_t(g_{t-1}(U_{t-1}(w_1,u,z^*),\delta_t))d\mu\}d\mu^{t-1}d\psi_1\\ &= \int_{D\times\Delta^t}\rho^{-1}(f_t(U_t(w_1,u,z^*),\delta_t))d\psi_1 d\mu^t\\ &= \int_{D\times\Delta^t}\rho^{-1}(u_t(w_1,(\delta^{t-1},\delta_t)))d\psi_1 d\mu^t\end{aligned}$$

Thus if the allocation is feasible, the allocation rule defined above is feasible too.

2. Next we prove that if there exists a feasible allocation rule satisfies (3.14) (3.15), (3.16) and (3.17), then there exists an allocation sequence such that the corresponding utility sequence satiesfies (3.11), (3.12) and (3.13).

Given ex ante expected discounted utility value w_1, for any $t \geq 1$, $u_t, W_t : D\times\Delta^t \mapsto \mathcal{R}_+$ can be defined recursively by $u_t(w_1, j_z^t) = f_t(w_t, j_{z,t})$ and $W_{t+1}(w_1, j_z^t) = g_t(w_t, j_{z,t}) = w_{t+1}$.

In order to prove equation (3.11), which means the allocation rule delivers the ex ante expected discounted utility w_1, we show for any $t \geq 1$

$$W_t(w_1,\delta^{t-1}) = U_t(w_1,u,z^*) \tag{3.19}$$

Then (3.11) is the special case of (3.19) when $t=1$.

To prove (3.19), according to the definition of W_t and equation (3.15), for all t,

w_1, and δ^{t-1}

$$W_t(w_1, \delta^{t-1}) = \int_\Delta \{u_t(w_1, (\delta^{t-1}, \delta_t)) + \delta_t W_{t+1}(w_1, \delta^t) d\mu\}.$$

And

$$U_t(w_1, u, z^*) = \int_\Delta \{u_t(w_1, (\delta^{t-1}, \delta_t)) + \delta_t U_{t+1}(w_1, u, z^*) d\mu\}$$

Subtracting the above two equations, we have

$$|W_t(w_1, \delta^{t-1}) - U_t(w_1, u, z^*)| \leq (\delta^h)^s \sup_{w_1, \delta^{t+s-1}} |W_{t+s}(w_1, \delta^{t+s-1}) - U_{t+s}(w_1, u, z^*)|$$

for all s and t. Let $s \to \infty$, the right hand side of the above inequality goes to 0. Thus we prove equation (3.19), as well as equation (3.11).

Next we would like to verify the utility sequence the allocation rule generates satisfies the incentive compatible condition (3.12). According to the lemma, we only need to verify that the utility sequence generated by the allocation rule satisfies (3.18). Thus given the ex ante expected discounted utility value w_1 and an arbitrary reporting history $\hat{j}^r_{\hat{z}}$, if there exists a reporting strategy which differs from truth telling reporting strategy, while it will yield higher expected utility, since the allocation rule satisfies the trasversality condition (3.14), then there exists a sufficiently large N, such that there is a reporting strategy which differs from truth telling reporting strategy and yields higher expected discounted utility.

Now we prove by induction that given ex ante expected discounted utility value w_1 and arbitrary reporting preference type history $\hat{j}_{\hat{z}}^{r-1}$, there is no finite period deviation will yield higher expected discounted utility than truthful telling:

First assume that only in the r-th period, the reporting strategy z is different from the truth telling strategy , according to the definition of U_r and the temporary incentive compatible condition (3.16), we have

$$
\begin{aligned}
u_r(w_1,(\hat{j}_{\hat{z}}^{r-1},\delta_r)) &+ \delta_r U_{r+1}(w_1,u,(\hat{z}^{r-1},z^*)) \geq u_r(w_1,(\hat{j}_{\hat{z}}^{r-1},j_{z,r})) \\
&+ \delta_r U_{r+1}(w_1,u,(\hat{z}^{r},z^1,z^*))
\end{aligned}
$$

Then we assume that there is no reporting strategy that differs from truthful telling in N periods that will yield higher expected discounted utility value.

$$
\begin{aligned}
u_r(w_1,(\hat{j}_{\hat{z}}^{r-1},\delta_r)) &+ \delta_r U_{r+1}(w_1,u,(\hat{z}^{r-1},z^*)) \geq u_r(w_1,(\hat{j}_{\hat{z}}^{r-1},j_{z,r})) \\
&+ \delta_r U_{r+1}(w_1,u,(\hat{z}^{r-1},z^N,z^*))
\end{aligned}
\tag{3.20}
$$

We want to verify that (3.20) holds for reporting strategy that differs from truth telling in $N+1$ periods. According to the definition of U_{r+1}:

$$
\begin{aligned}
U_{r+1}(w_1,u,(\hat{z}^{r-1},z^1,z^N,z^*)) &= \int_\Delta \{u_{r+1}(w_1,j_{\hat{z}}^{r-1},j_{z,r},j_{z,r+1}) + \\
&\quad \delta_{r+1}U_{r+2}(w_1,u,(\hat{z}^{r-1},z^1,z^N,z^*)\}d\mu
\end{aligned}
$$

Since

$$\begin{aligned} U_{r+1}(w_1, u, (\hat{z}^{r-1}, z^1, z^N, z^*)) &\leq \int_\Delta \{u_{r+1}(w_1, (j_z^{r-1}, j_{z,r}, \delta_{r+1})) \\ &+ \delta_{r+1} U_{r+2}(w_1, u, (\hat{z}^{r-1}, z^1, z^*))\} d\mu \\ &= U_{r+1}(w_1, u, (\hat{z}^{r-1}, z^1, z^*)) \end{aligned}$$

for all $j_{z,t} \in \Delta$.

Thus

$$\begin{aligned} u_r(w_1, (\hat{j}_{\hat{z}}^{r-1}, j_{z,r})) &+ \delta_r U_{r+1}(w_1, u, (\hat{z}^{r-1}, z^1, z^N, z^*)) \\ &\leq u_r(w_1, (\hat{j}_{\hat{z}}^{r-1}, j_{z,r})) + \delta_r U_{r+1}(w_1, u, (\hat{z}^{r-1}, z^1, z^*)) \\ &\leq u_r(w_1, (\hat{j}_{\hat{z}}^{r-1}, \delta_r)) + \delta_r U_{r+1}(w_1, u, (\hat{z}^{r-1}, z^*)) \end{aligned}$$

Now we have shown that (3.20) holds for $N + 1$, which concludes the induction. Inequality (3.20) holds for any reporting strategy differs from truth telling in finite periods. Thus we can conclude that Inequality (3.20) holds. And the feasibility condition holds directly. This ends the proof of proposition

Recall the definition of φ^*: according to propostion (3.16), for any utility distribution $\psi \in M$, $\varphi^*(\psi)$ is the infinimum of the investment levels needed such that there exists an allocation rule σ that attains ψ. That is $\varphi^*(\psi)$ is the minimum investment level needed to attain the utility distribution ψ. Let $\varphi^* : M \mapsto R_+ \cup \{+\infty\}$ and $\varphi^*(\psi) = +\infty$ if the utility level ψ can not be attained by any finite investment level.

Thus we can write the dual problem in a recursive manner:

$$\varphi^*(\psi_t) = \inf_{\{f_t, g_t\}} \frac{1}{R_L} \int_{D\times\Delta} \rho^{-1}(f_t(w_t, \delta_t)) + \varphi^*(S_{g_t}\psi_t)\, d\mu d\psi_t$$

subject to equation (3.15) and the temporary incentive compatible condition (3.16).

Now we want to characterize the optimal allocation rule. Define an allocation rule σ (an allocation u) is efficient if it attains ψ with investment level $\varphi^*(\psi)$. Thus $\varphi^*(\cdot)$ can be found by solving the Bellman equation as follows:

$$\varphi(\psi) = \inf_{f,g\in B}\{\frac{1}{R_L} \int_{D\times\Delta} \rho^{-1}[f(w, \delta)] + \delta\, \varphi(S_g\psi) d\mu d\psi\} \tag{3.21}$$

where B is the set of Borel measurable functions $D \times \Delta \mapsto D \times D$ and f, g satisfies that for all $w \in D$,

$$w = \int_\Delta [f(w, \delta) + \delta g(w, \delta)] d\mu; \tag{3.22}$$

and incentive compatible condition that

$$f(w, \delta^i) + \delta^i g(w, \delta^i) \geq f(w, \delta^j) + \delta^i g(w, \delta^j) \tag{3.23}$$

for all $w \in D$ and $\delta^i, \delta^j \in \Delta$ and $i \neq j$.

Next we would show that φ^* is the solution to the Bellman equation (3.21) first, then we would like illustrate the steps to solve this Bellman equation.

Let X be the set of functions $\varphi : M \mapsto R_+ \cup \{+\infty\}$. Define the operator $T : X \mapsto X$ by

$$(T\varphi)(\psi) = \inf_{f,g\in B}\{\frac{1}{R_L}\int_{D\times\Delta}\rho^{-1}[f(w,\delta)] + \delta\,\varphi(S_g\psi)d\mu d\psi\} \tag{3.24}$$

Proposition 3.18. *φ^* is the fixed point of T.*

Proof First we prove $\varphi^* \leq T\varphi^*$, then $\varphi^* \geq T\varphi^*$ and we prove by contradiction. Assume there exists some promised utility level distribution ψ and $\varphi^*(\psi) > (T\varphi^*)(\psi)$, then there exists an $\varepsilon > 0$, such that there are f, $g \in B$ and

$$\varphi^*(\psi) - \frac{1}{R_L}\int_{D\times\Delta}[\rho^{-1}[f(w,\delta)] + \delta\,\varphi^*(S_g\psi)]d\mu d\psi > \varepsilon$$

Let $\psi' = S_g\psi$. According to the definition of φ^* and proposition 3.16, there exists an allocation rule σ' such that it attains ψ' with $\varphi^*(\psi') + \frac{1}{2}\varepsilon$. Define $\sigma_0 = \{(f,g),\sigma'\}$, where σ_0 denotes an allocation rule using (f,g) for the first period then switching to the allocation rule σ'. Thus we find an allocation rule such that attains ψ with resource $\varphi^*(\psi) - \frac{1}{2R_L}\varepsilon$, which is a contradiction.

Next we prove that $\varphi^* \geq T\varphi^*$.

Assume there exists some ψ and $\varphi^*(\psi) < (T\varphi^*)(\psi)$, then there exists an $\varepsilon > 0$, such that there are f, $g \in B$ and

$$\frac{1}{R_L}\int_{D\times\Delta}[\rho^{-1}[f(w,\delta)] + \delta\,\varphi^*(S_g\psi)]d\mu d\psi - \varphi^*(\psi) > \varepsilon$$

Let $\psi' = S_g\psi$. According to the definition of φ^* and proposition 3.16, there exists an allocation rule σ' such that it attains ψ' with $\varphi^*(\psi')+\frac{R_L}{2}\varepsilon$. Thus define $\sigma_0 = \{(f,g),\sigma'\}$, we find an allocation rule such that attains ψ with resource $(T\varphi^*)(\psi) - \frac{1}{2}\varepsilon$, a contradiction. This concludes the proof.

Since we assume the return from the liquid investment is homogeneous of degree 1 and we use the CRRA function as a utility function of one period $\rho(c) = \frac{c^\gamma}{\gamma}$, where $\gamma < 1$, the promised utility level distribution is multiplicate to the cost minimization problem.

Thus we can solve the Bellman equation by solving a static problem as follows:

$$\phi(\alpha) = \min_{r,y}\{\frac{1}{R_L}\int_\Delta \rho^{-1}(r(\delta)) + \alpha(\frac{\gamma}{|\gamma|}y(\delta))^{\frac{1}{\gamma}}\ d\mu\} \tag{3.25}$$

The interpretation of the static problem is to find the greatest lower bound of the initial investment level α such that $\frac{\gamma}{|\gamma|}$ is the initial expected discounted utility value. $r(\delta_t)$ is the utility value of current period with reporting preference type as δ_t, and $y(\delta_t)$ is the promised utility with current reporting preference type δ_t. Thus the solution to the original problem can be derived as follows

$$\begin{aligned} f(w,\delta) &= \frac{\gamma}{|\gamma|}wr(\delta) \\ g(w,\delta) &= \frac{\gamma}{|\gamma|}wy(\delta) \end{aligned} \tag{3.26}$$

and $\varphi^*(\psi) = \alpha\int_D(\frac{\gamma}{|\gamma|}w)^{\frac{1}{\gamma}}\ d\psi$.

Given $\mu(\delta = \delta^l) = p$,$\mu(\delta = \delta^h) = 1 - p$, and with a bit abuse of the notation, let $r^i = r(\delta^i)$, $y^i = y(\delta^i)$ for $i \in \{l, h\}$. The problem (3.25) can be written as follows:

$$\phi(\alpha) = \min_{(r^l,y^l),(r^h,y^h)} \{\frac{1}{R_L}\left\{p\left[\rho^{-1}(r^l) + \alpha(\frac{\gamma}{|\gamma|}y^l)^{\frac{1}{\gamma}}\right] + (1-p)\left[\rho^{-1}(r^h) + \alpha(\frac{\gamma}{|\gamma|}y^h)^{\frac{1}{\gamma}}\right]\right\}$$

subject to the following constraint:

$$p(r^l + \delta^l y^l) + (1-p)(r^h + \delta^h y^h) = \frac{\gamma}{|\gamma|}; \tag{3.27}$$

$$\begin{aligned} r^h + \delta^h y^h &\geq r^l + \delta^h y^l; && (3.28) \\ r^l + \delta^l y^l &\geq r^h + \delta^l y^h && (3.29) \end{aligned}$$

Let λ, ζ, η be the Lagrange coefficient to above 3 constraints correspondingly, thus we can write down the first order condition:

$$\begin{aligned} \frac{1}{R_L}p(\rho^{-1})'(r^l) &= \lambda p - \zeta + \eta; && (3.30) \\ \frac{1}{R_L}(1-p)(\rho^{-1})'(r^h) &= \lambda(1-p) + \zeta - \eta; && (3.31) \\ \frac{1}{R_L}\alpha\frac{1}{|\gamma|}p(\frac{\gamma}{|\gamma|}y^l)^{\frac{1}{\gamma}-1} &= \lambda p\delta^l - \zeta\delta^h + \eta\delta^l; && (3.32) \\ \frac{1}{R_L}\alpha\frac{1}{|\gamma|}(1-p)(\frac{\gamma}{|\gamma|}y^h)^{\frac{1}{\gamma}-1} &= \lambda(1-p)\delta^h + \zeta\delta^h - \eta\delta^l && (3.33) \end{aligned}$$

Remark 3.19. *Since* $\delta^h > \delta^l$ *, according to incentive compatible condition, we have* $r^l > r^h$*. And since the consumption function* ρ^{-1} *is increasing and convex in the utility* *level,* $(\rho^{-1})'(r^l) > (\rho^{-1})'(r^h)$. *While* $\lambda, \zeta, \eta \geq 0$ *and by equation (3.30) and (3.31), we can conclude that* $\eta > 0$ *and* $\zeta = 0$. *Thus the constraint (3.28) is not binding and the constraint (3.29) is binding. In other words, under the optimal allocation rule impatient agents in current period are indifferent claiming being impatient or patient, and patient agents are strictly better off by telling thte truth.*

Although the above minimization problem can not be solved explicitly, we can find the numerical solution to it.[7]

EXAMPLE 3.20. *Let* $R_L = 1.2$, $\delta^l = 0.1$, $\delta^h = 0.9$, $p = 0.5$ *and* $\gamma = -1$, *the numerical solution to (3.25) is given by* $r^l = -0.3581$, $y^l = -2.495$, $r^h = -0.5096$, $y^h = -0.9809$ *and* $\alpha = 4.8531$.

According to the computation result, if in the initial period, the investment of the bank's portfolio is $L_1 = 1$, *the expected utility of agents when having a bank is given by* $w_1 = -(\alpha^{-1})^{-1} = -4.8531$, *while the expected utility in the autarky case is* $\hat{w} = -(5.7687^{-1})^{-1} = -5.7687$. *Thus* $w_1 > \hat{w}_1$.

In the initial period, an agent with current preference type δ^l *will have the current utility value* $f(w_1, \delta^l) = -(-4.8531)(-0.3581) = -1.7379$, *that is, consumption goods* $c_1 = \dfrac{1}{1.7479} = 0.5754$, *and the promised utility is* $g(w_1, \delta^l) = -12.1065$. *Similarly*

[7]See the appendix for the numerical algorithm

an agent with current preference type δ^h *will have the current utility value* $f(w_1, \delta^l) = -(-4.8531)(-0.5096) = -2.4731$, *that is, consumption goods* $c_1 = \frac{1}{2.4731} = 0.4044$, *and the promised utility is* $g(w_1, \delta^l) = -4.7596$.

3.5 A Direct Mechanism

In the previous sections, we have shown that under the direct mechanism, the full insurance allocation is not implementable and then we design an efficient deposit contract which is incentive compatible and offer partial insurance to agents against their preference shocks. In this section, we want to find an efficient allocation rule of the direct mechanism by extending the optimal allocation rule of the incentive compatible deposit contract in the previous section[8], and show that given our way of extension, truth telling reporting strategy is a Perfect Bayesian Equilibrium of the direct mechanism. Further analysis shows that given certain parameter values, it is possible to have other equilibria.

Since the allocation rule of the incentive compatible deposit contract only characterizes what will happen when all the agents report their types truthfully, first we would like to extend the optimal allocation rule of the deposit contract to the case when there exists a joint deviation.[9] Then we will show that truth telling reporting strategy is a Perfect Bayesian Nash equilibrium of this mechanism. Under our way of extension,

[8]Since there exists a one to one mapping between the allocation and the utility sequence that the allocation rule generates, it is enough to study an allocation rule of the direct mechanism.

[9]In this section, "joint deviation" means a measurable set of agents with non zero measure will deviate. Since we only consider the symmetric reporting strategy profile, there are only 3 possible scenarios of the reporting profile: all agents are telling the truth, all agents claim being patient regardless of their true preference types or all agents claim being impatient regardless of their true preference types.

further analysis shows that truth telling may not be the unique equilibrium, we prove that if $\gamma < 0$, then all agents claim being patient regardless of their true preference types is also an equilibrium of the mechanism. At the last part of this section, we will show 2 numerical examples. The first one has a parameter $\gamma = -1 < 0$, always claim being impatient is also an equilibrium reporting strategy of the mechanism. While for the second example with $\gamma = 0.5$, we show that truth telling is the unique equilibrium of the direct mechanism.

In period t, an agent with expected discounted utility value w_t will make the preference type report to the bank after the current period preference shock is realized. Then in return the bank will notify the agent the current consumption value $\widetilde{C}_t(w_t, j^t, \widetilde{j}_{\widetilde{z}}^{t-1}, L_t)$ (or equivalently $u_t(w_1, j^t, \widetilde{j}_{\widetilde{z}}^{t-1}, L_t)) = \rho(C_t(w_1, j^t, \widetilde{j}_{\widetilde{z}}^{t-1}, L_t)))$ and the promised utility value w_{t+1}, as well as the updated investment portfolio L_{t+1} at the end of the period. Agents can tell whether or not there exists a joint deviation from the value of promised utility as well as the bank's updated investment portfolio.The promised utility should be feasible according to the investment portfolio.

Let $\widetilde{\sigma} = \{\widetilde{f}_t^p, \widetilde{g}_t^p\}$ denotes the allocation rule of the direct mechanism, with the underlying probability of being impatient agents in each period is p. Thus the utility sequence generated by the allocation rule is defined as $\widetilde{u}_t(w_1, j_z^t, \widetilde{j}_{\widetilde{z}}^{t-1}, L_{t-1}) = \widetilde{f}_t^p(w_t, j_{z,t})$ with $w_t = \widetilde{g}_{t-1}^p(w_{t-1}, j_{z,t-1}, L_t)$. Notice $\widetilde{f}_t^p$ is only a function of current expected discounted utility value w_t and current preference report $j_{z,t}$ since the information of the history of reporting profile, the specific reporting history of the agent and the history of the bank's investment portfolio can be determined from the expected discounted utility value w_t.

Denote $\{(f_t^p(w_t, \delta_t), g_t^p(w_t, \delta_t))\}$ to be the optimal allocation rule of the deposit contract generated by (r^p, h^p):

$$\begin{aligned} f_t^p(w_t, \delta_t) &= \frac{\gamma}{|\gamma|} w_t r^p(\delta_t) \\ g_t^p(w_t, \delta_t) &= \frac{\gamma}{|\gamma|} w_t y^p(\delta_t) \end{aligned}$$

where (r^p, y^p) is the solution of the static problem (3.25) with the underlying probability of being impatient agents p. Recall $(f_t^p(w_t, \delta_t), g_t^p(w_t, \delta_t))$ denotes the current utility value and promised utility value of an agent who has expected discounted utility level w_t and the current preference type δ_t. Let $\alpha(p)$ be the fix point of problem (3.25), which is the greatest lower bound of the initial investment portfolio to attain the degenerate expected discounted utility level distribution in the initial period ψ_1 with $P(w_1 = \frac{\gamma}{|\gamma|}) = 1$.

Next we would like to define the allocation rule of the direct mechanism which extends the optimal allocation rule of the deposit contract:

1. The allocation rule of the direct mechanism coincide with the optimal allocation rule of the deposit contract if there is no joint deviation.

2. If there exists a joint deviation:

   since the bank can not identify whether or not there exists a joint deviation at the beginning of the period, the current consumption $\widetilde{f}^p(w_t, j_{z,t})$ is given according to the contract regardless whether or not there is a joint deviation in the current period,

$$\widetilde{f}^p(w_t, j_{z,t}) = \begin{cases} f^p(w_t, \delta^l) & \text{if } j_{z,t} = \delta^l \\ f^p(w_t, \delta^h) & \text{if } j_{z,t} = \delta^h \end{cases}$$

Definition 3.21. *Denote $\hat{p}_{\widetilde{z},t}$ to be the measure of impatient agents given the reporting strategy profile $\widetilde{z} = \prod_{i \in I} z^i \in \widetilde{Z}$ in the period t*

$$\hat{p}_{\widetilde{z},t} = \mu(i \in I | j_{z^i,t} = \delta^l)$$

.

Given reporting strategy profile $\widetilde{z}$, the law of motion of the bank's investment portfolio $L_{t+1} = K(\widetilde{j}_{\widetilde{z},t}, L_t)$ is:

$$\begin{aligned} K(\widetilde{j}_{\widetilde{z},t}, L_t) &= R_L L_t - \int_{D \times \Delta} (\rho^{-1})(f(w_t, j_{z,t}, L^{t-1}))\, d\psi_t\, d\mu_{\widetilde{z},t} \\ &= R_L L_t - \int_{D \times \Delta} (\rho^{-1})(w_t r(j_{z,t}))\, d\psi_t\, d\mu_{\widetilde{z},t} \\ &= R_L L_t - \int_{D \times \Delta} (\frac{\gamma}{|\gamma|} w_t)^{\frac{1}{\gamma}} (\rho^{-1})(r^p(j_{z,t})\, d\psi_t\, d\mu_{\widetilde{z},t} \\ &= R_L L_t - (\hat{p}_{\widetilde{z},t}(\rho^{-1})(r^p(\delta^l)) + (1 - \hat{p}_{\widetilde{z},t})(\rho^{-1})(r^p(\delta^h)) \cdot \int_D (\frac{\gamma}{|\gamma|} w_t)^{\frac{1}{\gamma}}\, d\psi_t \end{aligned}$$

where $\mu_{\widetilde{z},t}$ is the measure associate with the reporting strategy profile $\widetilde{z}$ in period t.

The adjusted promised utility value should be feasible to the bank's portfolio. Again since we are using CRRA function as one period utility function and the gross rate of return from the investment is homogenous of degree 1, the adjusted promised utility value is feasible if and only if it satisfies the following:

$$\alpha(p) \cdot \left(\frac{\gamma}{|\gamma|} \int_D w_{t+1}\, d\psi_{t+1}\right)^{\frac{1}{\gamma}} \le L_{t+1} = K(\widetilde{j}_{\widetilde{z},t}, L_t)$$

and this constraint is binding in optimal.

For simplicity, assume $N(\cdot, \cdot)$ is linear in its first argument[10]:

$$\begin{aligned} w_{t+1} = \widetilde{g}^p_t(w_t, j_{z,t}, K(\widetilde{j}_{\widetilde{z},t}, L_t)) &= N(g^p_t(w_t, j_t), K(\widetilde{j}_{\widetilde{z}^*,t}, L_t)) \\ &= \frac{\gamma}{|\gamma|}\left(\frac{K(\widetilde{j}_{\widetilde{z},t}, L_t)}{\alpha(p)}\right)^{\gamma} \frac{g^p_t(w_t, j_{z,t})}{\int_{D\times\Delta} g^p_t(w_t, j_{z,t})\, d\mu_{\widetilde{z},t} d\psi_t} \end{aligned}$$

Thus in each period t, given the expected discounted utility value w_t, and current preference type shock δ_t, the agent choose the reporting type $j_{z,t} \in \Delta$ to maximize his/her expected discounted utility value in each period:

$$\max_{j_{z,t}\in\Delta} \{\widetilde{f}^p_t(w_t, j_t) + \delta_t \widetilde{g}^p_t(w_t, j_{z,t}, K(\widetilde{j}_{\widetilde{z},t}, L_t))\} \tag{3.34}$$

Theorem 3.22. *Given the allocation rule above, truth telling reporting strategy is an equilibrium reporting strategy of the direct mechanism.*

Proof According to the definition of the equilibrium reporting strategy, we only need to prove that for any $r > 1$ and $\hat{z}, z^i \in Z$, $\widetilde{U}_r(w_1, u, (\widetilde{\hat{z}}^r, \widetilde{z}^*)) \ge \widetilde{U}_r(w_1, u, (\widetilde{\hat{z}}^r, \widetilde{z}^* \backslash_i z^{i*})$, given the ex ante expected discounted utility value w_1, and utility sequence $\{u_t\}$ generated by

[10]Due to the homogeneity 1 of both investment and preference, this functional form works.

the allocation rule we defined above,

$$u_t(w_1, j_z^t, \widetilde{j}_{\widetilde{z}}^{t-1}, L_t) = f_t^p(w_t, j_{z,t}),$$

$$w_t = N(g_{t-1}^P(w_{t-1}, j_{z,t}), K(\widetilde{j}_{\widetilde{z},t-1}, L_{t-1})).$$

Now assume that all the other agents are using the truth telling reporting strategy in period t, then $\widetilde{g}^p(w_t, j_{t,z}, K(\widetilde{j}_{\widetilde{z}^*,t}, L_t)) = g_t^p(w_t, j_{z,t})$. Thus for any $w_t \in D$ and $\delta_t = \delta^i$,

$$\widetilde{f}_t^p(w_t, \delta^i) + \delta^i \widetilde{g}_t^p(w_t, \delta^i, K(\widetilde{j}_{\widetilde{z}^*,t}, L_t)) \geq \widetilde{f}_t^p(w_t, \delta^j) + \delta^i \widetilde{g}_t^p(w_t, \delta^j, K(\widetilde{j}_{\widetilde{z}^*,t}, L_t))$$

for $i \neq j$ and $\delta^i, \delta^j \in \Delta$.

Thus given that all the other agents are telling the truth, the allocation generated by the allocation rule is temporary incentive compatible. According to the proof the proposition (3.16) in the previous section, the inequality (3.18) holds. Thus according to lemma (3.17), the allocation sequence generated by the allocation rule we defined above is incentive compatible. Thus it is optimal for any agent i to tell the truth, which means that $\widetilde{U}(w_1, u, \widetilde{z}^*) \geq \widetilde{U}(w_1, u, \widetilde{z}^* \backslash_i z^{i*})$. So truth telling is an equilibrium reporting strategy of the direct mechanism.

Proposition 3.23. *If $\gamma < 0$, all agents claiming being patient regardless of their true preference types is an equilibrium reporting strategy of the direct mechanism.*

Before we prove the proposition, we need to show the following lemma:

LEMMA 3.24. *If $\gamma < 0$, $N(g^p(w_t, \delta^h), K(\widetilde{j}_{\widetilde{z},t}, L_t)) - N(g^p(w_t, \delta^l), K(\widetilde{j}_{\widetilde{z},t}, L_t))$ is a decreasing function of $\hat{p}_{z,t}$.*

Proof Since the expected discounted utility level is multiplicate to $\widetilde{f}^p_t$ and $\widetilde{g}^p_t$, it is enough to consider the following case: assume in the initial period, all agents have the identical expected utility value $w_1 = \frac{\gamma}{|\gamma|}$. Thus given an arbitrary reporting strategy profile $\widetilde{z}$, the adjusted promised utility value of impatient agents is given by:

$$
\begin{aligned}
N(g^p_1(w_1, \delta^l), K(\widetilde{j}_{\widetilde{z},1}, L_1)) &= \frac{\gamma}{|\gamma|}\Big(\frac{K(\widetilde{j}_{\widetilde{z},1}, L_1)}{\alpha(p)}\Big)^\gamma \frac{g^p_1(w_t, \delta^l)}{\int_\Delta g^p_1(w_1, j_{z,t})\, d\mu_{\widetilde{z},t}} \\
&= \frac{\gamma}{|\gamma|}\Big(\frac{R_L\alpha(p) - \hat{p}_{\widetilde{z},1}\rho^{-1}(r^p(\delta^l)) - (1-\hat{p}_{\widetilde{z},1})\rho^{-1}(r^p(\delta^h))}{\alpha(p)}\Big)^\gamma \\
&\cdot \frac{y^p(\delta^l)}{\hat{p}_{\widetilde{z},1}y^p(\delta^l) + (1-\hat{p}_{\widetilde{z},1})y^p(\delta^h)}
\end{aligned}
$$

and correspondingly the adjusted promised utility value of patient agents is

$$
\begin{aligned}
N(g^p_1(w_1, \delta^h), K(\widetilde{j}_{\widetilde{z},1}, L_1)) &= \frac{\gamma}{|\gamma|}\Big(\frac{K(\widetilde{j}_{\widetilde{z},1}, L_1)}{\alpha(p)}\Big)^\gamma \frac{g^p_1(w_t, \delta^h)}{\int_\Delta g^p_1(w_1, j_{z,t})\, d\mu_{\widetilde{z},t}} \\
&= \frac{\gamma}{|\gamma|}\Big(\frac{R_L\alpha(p) - \hat{p}_{\widetilde{z},1}\rho^{-1}(r^p(\delta^l)) - (1-\hat{p}_{\widetilde{z},1})\rho^{-1}(r^p(\delta^h))}{\alpha(p)}\Big)^\gamma \\
&\cdot \frac{y^p(\delta^h)}{\hat{p}_{\widetilde{z},1}y^p(\delta^l) + (1-\hat{p}_{\widetilde{z},1})y^p(\delta^h)},
\end{aligned}
$$

where $L_2 = K(\widetilde{j}_{\widetilde{z},1}, L_1)$ is the bank's investment portfolio at the beginning of period 2 with reporting strategy profile $\widetilde{z}$.

Denote $d(\hat{p}_{\widetilde{z},1}) = N(g^p(w_1, \delta^h), K(\widetilde{j}_{\widetilde{z},1}, L_1)) - N(g^p(w_1, \delta^l), K(\widetilde{j}_{\widetilde{z},1}, L_1))$

$$d(\hat{p}_{\widetilde{z},1}) = \frac{\gamma}{|\gamma|}\left(\frac{R_L\alpha(p) - \hat{p}_{\widetilde{z},1}\rho^{-1}(r^p(\delta^l)) - (1-\hat{p}_{\widetilde{z},1})\rho^{-1}(r^p(\delta^h))}{\alpha(p)}\right)^{\gamma}\frac{y^p(\delta^h) - y^p(\delta^l)}{\hat{p}_{\widetilde{z},1}y^p(\delta^l) + (1-\hat{p}_{\widetilde{z},1})y^p(\delta^h)}$$

Take the derivative of $d(\hat{p}_{\widetilde{z},1})$

$$\begin{aligned}\frac{d\, d(\hat{p}_{\widetilde{z},1})}{d\hat{p}_{\widetilde{z},1}} &= \frac{\gamma^2}{|\gamma|}\left(\frac{K(\widetilde{j}_{\widetilde{z},1}, L_1)}{\alpha(p)}\right)^{(\gamma-1)}\frac{-(\rho^{-1}(r^p(\delta^l)) - \rho^{-1}(r^p(\delta^h)))}{\alpha(p)}\frac{y^p(\delta^h) - y^p(\delta^l)}{\hat{p}_{\widetilde{z},1}y^p(\delta^l) + (1-\hat{p}_{\widetilde{z},1})y^p(\delta^h)} + \\ &\quad \frac{\gamma}{|\gamma|}\left(\frac{K(\widetilde{j}_{\widetilde{z},1}, L_1)}{\alpha(p)}\right)^{(\gamma)}\frac{(y^p(\delta^h) - y^p(\delta^l))^2}{(\hat{p}_{\widetilde{z},1}y^p(\delta^l) + (1-\hat{p}_{\widetilde{z},1})y^p(\delta^h))^2}\end{aligned}$$

Thus $\dfrac{d\, d(\hat{p}_{\widetilde{z},1})}{d\hat{p}_{\widetilde{z},1}} < 0$, if $\gamma < 0$, which concludes the proof.

Next we will use the conclusion of the lemma to prove the proposition (3.23).

Proof With the same reason as we stated in the proof of lemma (3.24), it is enough to consider the following case: assume in the initial period, all agents have the identical expected utility value $w_1 = \dfrac{\gamma}{|\gamma|}$.

Then we show that the impatient agent will choose to lie being patient given all the other impatient agents claim being patient.

Impatient agents will choose the report type to maximize the expected discounted utility

(3.34) by computing the deviation gain:

$$\begin{aligned}(f^p(w_1,\delta^h) &+ \delta^l N(g^p(w_1,\delta^h),K(\widetilde{j}_{\widetilde{z},1},L_1))) - (f^p(w_1,\delta^l)+\delta^l N(g^p(w_1,\delta^l),K(\widetilde{j}_{\widetilde{z},1},L_1))) \\ &= ((r^p(\delta^l)+\delta^l N(y^p(\delta^l),K(\widetilde{j}_{\widetilde{z},1},L_1))) - (r^p(\delta^h)+\delta^l N(y^p(\delta^h),K(\widetilde{j}_{\widetilde{z},1},L_1)))\end{aligned} \tag{3.35}$$

If the deviation gain is greater than 0, then impatient agents will choose to lie. And if it is less than 0, impatient agents will choose to tell the truth.

and similarly patient agents will make the report by computing their deviation gain:

$$\begin{aligned}(f^p(w_1,\delta^l) &+ \delta^h N(g^p(w_1,\delta^l),K(\widetilde{j}_{\widetilde{z},1},L_1))) - (f^p(w_1,\delta^h)+\delta^h N(g^p(w_1,\delta^h),K(\widetilde{j}_{\widetilde{z},t},L_t))) \\ &= ((r^p(\delta^h)+\delta^h N(y^p(\delta^h),K(\widetilde{j}_{\widetilde{z},1},L_1))) - (r^p(\delta^l)+\delta^h N(y^p(\delta^l),K(\widetilde{j}_{\widetilde{z},1},L_1)))\end{aligned} \tag{3.36}$$

The reporting type of patient agents also rely on the sign of their deviation gain.

If all other impatient agents deviate in the first period, that is $0 = \hat{p}_{\widetilde{z},1} < p$, then according to lemma (3.24), the value of (3.36) is still greater than 0, which means that all patient agents will still tell the truth, while the value of (3.35) is less than 0, which means the impatient agent will deviate if all other impatient agents lie being patient. Similar as the proof of proposition (3.16) in the previous section, if it is optimal to deviate in one period, then given an arbitrary reporting history, it is optimal to deviate in all future periods. Thus all impatient agents deviate is also an equilibrium of the direct mechanism if $\gamma < 0$.

Corollary 3.25. *If $\gamma < 0$, and*

$$\frac{\gamma}{|\gamma|}\frac{1}{y^p(\delta^l)}\left(\frac{R_L\alpha(p)-\rho^{-1}(r^p(\delta^l))}{\alpha(p)}\right)^{\gamma} \geq \frac{\delta^l}{\delta^h} \tag{3.37}$$

always claiming being patient is the only inefficient equilibrium reporting strategy of the direct mechanism.

Proof Here we show that always claiming being impatient is not an equilibrium reporting strategy.

For the same reason as above, we only need to consider the following case: assume in the initial period, all agents have the identical expected utility value $w_1 = \frac{\gamma}{|\gamma|}$. And according to lemma (3.24), the value of $N(g^p_t(w_t, \delta^h), K(\widetilde{j}_{\widetilde{z},t}, L_t)) - N(g^p(w_t, \delta^l), K(\widetilde{j}_{\widetilde{z},t}, L_t))$ is decreasing in $\hat{p}_{\widetilde{z},t}$, thus we only need to prove when $\hat{p}_{\widetilde{z},t} = 1$, that is almost all patient agents choose to claim being impatient, it is still optimal for an patient agent to tell the truth:

$$(f^p(w_1, \delta^h) + \delta^h N(g^p(w_1, \delta^h), K(\prod_{i\in I}\delta^h, L_1))) - (f^p(w_1, \delta^l) + \delta^h N(g^p(w_1, \delta^l), K(\prod_{i\in I}\delta^h, L_1))) > 0.$$

Since

$$K(\prod_{i\in I}\delta^h, L_1) = R_L\alpha(p) - \rho^{-1}(r^p(\delta^l)),$$

$$\begin{aligned} N(g^p(w_1, \delta^h), K(\prod_{i\in I}\delta^h, L_1)) \quad &- \quad N(g^p(w_1, \delta^l), K(\prod_{i\in I}\delta^h, L_1)) \\ &= \quad \frac{\gamma}{|\gamma|}\left(\frac{R_L\alpha(p)-\rho^{-1}(r^p(\delta^l))}{\alpha(p)}\right)^{\gamma}\frac{y^p(\delta^h)-y^p(\delta^l)}{y^p(\delta^l)} \end{aligned}$$

According to (3.37),

$$\begin{aligned}
\delta^h(N(g^p(w_1,\delta^h),K(\prod_{i\in I}\delta^h,L_1))-N(g^p(w_1,\delta^l),K(\prod_{i\in I}\delta^h,L_1))) &> \delta^l(y^p(\delta^h)-y^p(\delta^l)) \\
&= (r^p(\delta^l)-r^p(\delta^h)) \\
&= f^p(w_1,\delta^l)-f^p(w_1,\delta^h)
\end{aligned}$$

Thus it is optimal for the patient agent to tell the truth even if almost all the other patient agents deviate. While

$$\begin{aligned}
\delta^l(N(g^p(w_1,\delta^h),K(\prod_{i\in I}\delta^h,L_1))-N(g^p(w_1,\delta^l),K(\prod_{i\in I}\delta^h,L_1))) &< \delta^l(y^p(\delta^h)-y^p(\delta^l)) \\
&= (r^p(\delta^l)-r^p(\delta^h)) \\
&= f^p(w_1,\delta^l)-f^p(w_1,\delta^h)
\end{aligned}$$

Thus it is still for all impatient agents to tell the truth if almost all patient agents deviate.

Combine the argument above, always claim being patient is not dominant in one period, thus it is not an equilibrium reporting strategy of the direct mechanism. This ends the proof.

The following show that when $\gamma < 0$, there are 2 equilibrium reporting strategy of the direct mechanism:

Example 3.26. *Given* $R_L = 1.2$, $\delta^l = 0.1$, $\delta^h = 0.9$, $p = 0.5$ *and* $\gamma = -1$, *there are 2 equilibrium reporting strategy of the direct mechanism.*

The efficient allocation rule of the incentive compatible deposit contract can be found by solving the static problem (3.25), $\alpha(0.5) = 4.8531$ $r^{0.5}(\delta^l) = -0.3581$, $y^{0.5}(\delta^l) = -2.495$, $r^{0.5}(\delta^h) = -0.5096$ *and* $y^{0.5}(\delta^h) = -0.9809$.

Since the gross rate of return from the investment is homogeneous of degree 1 and we use CRRA function for the period utility, it is enough to consider the initial period and without loss of generality we can assume each agent is born with 4.8531 *units of liquid investment, thus the expected promised utility level of each agent is* -1 *at the beginning of the initial period.*

First we show that always claiming being impatient is not an equilibrium reporting strategy as follows:
assume $\hat{p}_{\widetilde{z},1} = 1$, *then* $f^{0.5}(-1, \delta^l) = -0.3581$, $f^{0.5}(-1, \delta^h) = -0.5096$, $L_2 = K(\prod_{i \in I} \delta^l, 4.8531) = 3.0312$ *and* $N(g^{0.5}(-1, \delta^l), 3.0312) = -1.601$ $N(g^{0.5}(-1, \delta^h), 3.0312) = -0.6294$, *first we verify that (3.37) is satisfied:*

$$\begin{aligned}
\frac{\gamma}{|\gamma|}\frac{1}{y^p(\delta^l)}(\frac{R_L\alpha(p)-\rho^{-1}(r^p(\delta^l))}{\alpha(p)})^\gamma &= -(\frac{3.0312}{4.8531})^{-1}\frac{-0.9809-(-2.495)}{-2.495} \\
&= 0.9716 \\
&> \frac{0.1}{0.9} = \frac{\delta^l}{\delta^h}
\end{aligned}$$

Thus according to the corollary, that all agents claim being impatient is not an equilibrium reporting strategy.

According to proposition (3.23), always claiming being patient is also an equilibrium of the mechanism, which also can be verified numerically as follows:
due to the fact that the promised utility level is multiplicate to f *and* g*, moreover according to the equivalence of the temporary incentive compatible and the total incentive compatibility and lemma (3.17), it is enough to consider the following case: in period* 1*, and all agents with the identical expected utility* -1*, if* $\hat{p}_1 = 0$*, that is almost all impatient agents lie being patient, then* $f^{0.5}(-1,\delta^l) = -0.3581$*,* $f^{0.5}(-1,\delta^h) = -0.5096$*,* $L_2 = K(\prod_{i\in I}\delta^h, 4.8531) = 3.8614$ *and* $N(g^{0.5}(-1,\delta^l), 3.8614) = -3.1968$ $N(g^{0.5}(-1,\delta^h), 3.8614) = -1.2568$*, for impatient agents*

$$\begin{aligned}
f^{0.5}(-1,\delta^l) + \delta^l N(g^{0.5}(-1,\delta^l),1) &= -0.3581 + 0.1*(-3.1968) \\
&= -0.6778 \\
f^{0.5}(-1,\delta^h) + \delta^l N(g^{0.5}(-1,\delta^h),1) &= -0.5096 + 0.1*(-1.2568) \\
&= -0.6353
\end{aligned}$$

$$f^{0.5}(-1,\delta^l)+\delta^l N(g^{0.5}(-1,\delta^l),3.8614) > f^{0.5}(-1,\delta^h)+\delta^l N(g^{0.5}(-1,\delta^h),3.8614) \tag{3.38}$$

which means that impatient agents claim being patient if almost all impatient agents lie in the first period.

Now we consider patient agents:

$$\begin{aligned} f^{0.5}(-1,\delta^h)+\delta^h N(g^{0.5}(-1,\delta^h),1) &= -0.5096+0.9*(-3.1968) \\ &= -1.6407 \\ f^{0.5}(-1,\delta^l)+\delta^h N(g^{0.5}(-1,\delta^l),1) &= -0.3581+0.9*(-1.2568) \\ &= -3.2352 \end{aligned}$$

$$f^{0.5}(-1,\delta^h)+\delta^h N(g^{0.5}(-1,\delta^h),1) > f^{0.5}(-1,\delta^l)+\delta^h N(g^{0.5}(-1,\delta^l),1) \tag{3.39}$$

which means patient agents will tell the truth even if almost all impatient agents lie.

Thus always claiming being impatient is temporarily a dominant reporting strategy, thus it is an equilibrium reporting strategy.

While when $0<\gamma<1$, truth telling might be the unique equilibrium reporting strategy.

EXAMPLE 3.27. *Let $R_L = 1.2$, $\delta^l = 0.1$, $\delta^h = 0.9$, $p = 0.5$ and $\gamma = 0.5$, again we find the optimal allocation rule for the incentive deposit contract first . We solve the static problem (3.25), $r^l = 0.5593$, $y^l = 0.1310$, $r^h = 0.4655$ and $y^h = 1.0690$.*

Now we verify numerically truth telling is the unique equilibrium in this case. The same as in the previous numerical example, it is enough to consider the first period and assume each agent is born with 0.1067 units of liquid investment, then the expected promised utility level of each agent is 1 at the beginning of the initial period.

- *First we verify that all agents claim being impatient is not an equilibrium. If $\hat{p}_1 = 1$, then $f^{0.5}(1, \delta^l) = 0.5563$, $f^{0.5}(1, \delta^h) = 0.4655$ $L_2 = K(\prod_{i \in I} \delta^l, 0.1087) = 0.0498$and $N(g^{0.5}(1, \delta^l), 0.0498) = 0.6832$ $N(g^{0.5}(1, \delta^h), 0.0498) = 5.5749$, for impatient agents*

$$
\begin{aligned}
f^{0.5}(1, \delta^l) + \delta^l N(g^{0.5}(1, \delta^l), 0.0498) &= 0.5563 + 0.1 * (0.6832) \\
&= 0.6246 \\
f^{0.5}(-1, \delta^h) + \delta^l N(g^{0.5}(-1, \delta^h), 0.0498) &= 0.4655 + 0.1 * (5.5749) \\
&= 1.023
\end{aligned}
$$

$$
f^{0.5}(1, \delta^l) + \delta^l N(g^{0.5}(1, \delta^l), 0.0498) < f^{0.5}(1, \delta^h) + \delta^l N(g^{0.5}(1, \delta^h), 0.0498) \quad (3.40)
$$

which means that impatient agents will tell the truth even if almost all patient agents will lie being impatient in the first period.

Now we consider patient agents:

$$
\begin{aligned}
f^{0.5}(1,\delta^h)+\delta^h N(g^{0.5}(1,\delta^h),0.0498) &= 0.4655+0.9*(5.5749)\\
&= 5.4829\\
f^{0.5}(1,\delta^l)+\delta^h N(g^{0.5}(1,\delta^l),0.0498) &= 0.5563+0.9*(0.6832)\\
&= 1.1712
\end{aligned}
$$

$$
f^{0.5}(-1,\delta^h)+\delta^h N(g^{0.5}(-1,\delta^h),0.0498) > f^{0.5}(-1,\delta^l)+\delta^h N(g^{0.5}(-1,\delta^l),0.0498) \tag{3.41}
$$

which means patient agents will tell the truth even almost all patient agents lie. Similar as before, we can conclude under this set of parameters, always claiming being impatient is not an equilibrium reporting strategy.

- *Next we show that always claim being patient is also not an equilibrium. If* $\hat{p}_1 = 0$, *then* $f^{0.5}(1,\delta^l) = 0.5563$, $f^{0.5}(1,\delta^h) = 0.4655$, $L_2 = K(\prod_{i\in I}\delta^h, 0.1087) = 0.0739$ *and* $N(g^{0.5}(1,\delta^l),0.0739) = 0.102$ $N(g^{0.5}(-1,\delta^h),0.0739) = 0.8322$, *for impatient agents*

$$
\begin{aligned}
f^{0.5}(1,\delta^l)+\delta^l N(g^{0.5}(1,\delta^l),0.0739) &= 0.5563+0.1*(0.102)\\
&= 0.5665\\
f^{0.5}(-1,\delta^h)+\delta^l N(g^{0.5}(-1,\delta^h),0.0739) &= 0.4655+0.1*(0.8322)\\
&= 0.5487
\end{aligned}
$$

$$f^{0.5}(-1,\delta^{l})+\delta^{l}N(g^{0.5}(-1,\delta^{l}),0.0739) > f^{0.5}(-1,\delta^{h})+\delta^{l}N(g^{0.5}(-1,\delta^{h}),0.0739) \tag{3.42}$$

which means it is not optimal for an impatient agent to lie being patient even if almost all impatient agents lie.

Now we consider patient agents:

$$\begin{aligned} f^{0.5}(1,\delta^{h})+\delta^{h}N(g^{0.5}(1,\delta^{h}),0.0739) &= 0.4665+0.9*(0.8322) \\ &= 1.2415 \\ f^{0.5}(1,\delta^{l})+\delta^{h}N(g^{0.5}(1,\delta^{l}),0.0739) &= 0.5563+0.9*(0.102) \\ &= 0.6481 \end{aligned}$$

$$f^{0.5}(-1,\delta^{h})+\delta^{h}N(g^{0.5}(-1,\delta^{h}),0.0739) > f^{0.5}(-1,\delta^{l})+\delta^{h}N(g^{0.5}(-1,\delta^{l}),0.0739) \tag{3.43}$$

which means patient agents will tell the truth even if almost all impatient agents lie.

Thus we can conclude, always claiming being patient is also not an equilibrium reporting strategy.

3.6 Conclusion

In this chapter, we study the optimal contract between the bank and its depositors with the direct mechanism. Here we use a simplified version of the model we introduce in the previous chapter. According to the revelation principle, the direct mechanism contains a broad enough set of deposit contracts. Assume the reporting strategy profile is symmetric, we study the Perfect Bayesian equilibrium of the direct mechanism. Again, under the direct mechanism, the full insurance allocation is not implementable. Thus we design an efficient allocation of the direct mechanism by extending the efficient allocation of an incentive compatible deposit contract. This allocation is optimal in the sense that it maximizes the expected discounted utility of agents without violating the incentive compatible constraints. Under the way of our extension, truth telling reporting strategy is an equilibrium reporting strategy of the mechanism. While depending on the risk aversion coefficient, further analysis shows that there might exist other inefficient equilibrium as well.

Chapter 4

Numerical Algorithm

As is typical of dynamic programming models, our model does not have an explicit solution. In this chapter we would like to describe the numerical method we use to solve the single agent model when the preference shock is a deterministic cycle. The case when the preference shock is subject to a Markovian Process is almost the same except for a different transition matrix. For the simplified problem of the bank, according to lemma (2.15), it is identical to the problem of the single agent.

4.1 Simplified Problem

Due to the homogeneity degree 1 of the gross returns of investing technologies and the adjustment cost, in this section, we will introduce a simplified Bellman equation, which reduces the state space of the Bellman equation of the original problem from 2 dimension to one. And the solution to the original equation can be derived from the simplified one.

Consider the case when the preference shock of the single agent is a deterministic 2-period cycle, which can be expressed by the following functional problem:

$$v(B, q, \delta^i) = \max_{\{B' \in \mathcal{R}_+, q' \in [0,1]\}} [\log(c(B, q, B', q', \delta^i)) + \delta^i v(B', q', \delta^j)],$$

subject to

$$
\begin{aligned}
c(B, q, B', q', \delta^i) &= R_I Bq - \theta(1 - \frac{B'q'}{R_I Bq})^2 Bq + R_L B(1-q) - B'; \\
0 &\leq B'; \\
0 &\leq q' \leq 1;
\end{aligned}
\tag{4.1}
$$

where $\delta^i \neq \delta^j \in \Delta$.

Since gross returns from both technologies, the adjustment cost of illiquid investment are homogeneous of degree 1 in the investment level, moreover we use the logarithm function as the one period utility function, the portfolio size is additive separable to the original problem. So we solve for v by solving a simplified maximization problem as follows, where the solution $\hat{v}$ is equal to $v(1, q, \delta^i)$.

$$
\hat{v}(q, \delta^i) = \max_{q' \in [0,1], b' \in (0, R_I)} [\log(\hat{c}(q, b', q', \delta^i)) + \delta^i (\hat{v}(q', \delta^j) + \frac{\delta^j}{1 - \delta^l \delta^h} \log(b'))]
\tag{4.2}
$$

where[1]

$$
\begin{aligned}
0 < \hat{c}(q, b', q', \delta^i) &= (R_I - \theta(1 - \frac{b'q'}{R_I q})^2 - R_L)q + R_L - b' \\
0 &< b' < R_I \\
0 &\leq q' \leq 1
\end{aligned}
\tag{4.3}
$$

[1]According to the feasibility constraint, if $\hat{c} > 0$, then $b' < R_I$.

And the solution of the original problem can be derived by:

$$
\begin{aligned}
v(B, q, \delta^{i}) &= \frac{\delta^{i} \log(B)}{1 - \delta^{l} \delta^{h}} + \hat{v}(q, \delta^{i}) \\
B' &= B \cdot b' \\
c'(B, q, B', q', \delta^{i}) &= B \cdot \hat{c}(q, b', q', \delta^{i})
\end{aligned}
\tag{4.4}
$$

and q' remains the same as the solution of in problem (4.2).

Thus next instead of solving for v, we will solve the simplified Bellman equation to find $\hat{v}(q, \delta^{i})$, and the value of $v(B, q, \delta^{i})$ can be computed by equation (4.4).

4.2 Overview

Given a Bellmen equation, generally speaking there are 2 families of numerical methods to solve it. One family of the methods seeks to approximate value and/or policy functions by iteration. The other one focuses on the system of Euler equations.

The first family of methods are widely known as a group of reliable methods, since the value iteration/policy iteration method is based on the method of dynamical programming, and if the problem satisfies some standard assumptions, the convergence to the fixed point of the functional mapping is guaranteed by the successive approximations. Moreover, arbitrary levels of accuracy can be achieved by using sufficiently fine grids on the state space. Although it is easy for this type of methods to deal with inequality constraints, they will face the problem of the curse of dimensionality.[2] To circumvent this difficulty, researchers have relied on continuous state approximation methods (or "parameterized dynamic programming"). These methods work well for interior problems where the value function is differentiable. However, given the interiority requirement of Benveniste and Scheinkman's (1979) envelope theorem, this result does not apply to the problem where inequality constraints may occasionally bind.

An alternative way is to use the other family of methods, which are based on Euler equation(s), such as time iteration, with a "penalty" item. Although it is well known that using computation methods based on Euler equation(s) requires "reasonable guess" of initial value, Rendahl (2006) shows that if certain condition is satisfied, time iteration is equivalent to value function iteration. Thus it is also a globally convergent method of finding equilibrium functions. And our computation result shows that time iteration is much more efficient than value iteration.

In order to provide a comparison of efficiency, we would implement both value iteration and time iteration to our model. According to the result of our numerical experiment, time iteration method significantly improves the efficiency of the algorithm of value iteration. Moreover, as the grid size decreases, the relative efficiency of using time iteration grows.

The following literatures have tried to provide the accuracy properties of numerical solutions of using time iteration. An early procedure of checking accuracy of

[2] If the policy space or/and state space have high dimensions, it may be not computable to use value iteration.

numerical solutions based on the Euler equation was proposed by den Haan and Marcet (1994). They conduct a test for the orthogonality of the Euler equation residuals over current and past information, and consider that such a statistic provides an accuracy measure for a given numerical solution. The criticisms of this approach are that Euler equation residuals maybe compatible with large deviations from the optimal policy and for a given numerical solution these residuals can be computed numerically with a very low cost without using formal statistical techniques. To avoid these shortcomings, Judd (1992) suggest an alternative test that need numerical computation of the Euler equation residuals over the whole state space. While both of them have not made an effort to infer the approximation error of the computed value and the policy function without an specific knowledge of the true functions. Santos (2000) offers a global estimate of the approximation error of using time iteration without the true solution to the problem, but the numerical experiments in this section show that further practical ways are needed to sharpen these error estimates are needed.

The rest part of this chapter will be organized as follows: first we verify that our problem satisfies all standard assumptions of a stochastic dynamical programming problem, and then we would introduce value iteration method as well as its implementation. Before we implement time iteration method, we would like to verify that our problem satisfies the addition conditions as listed in Rendahl(2006), thus time iteration is actually a global convergent method as value iteration. Then we describe the numerical algorithm using time iteration. And we show that with same arbitrary initial value and stopping tolerance of iteration, time iteration is much more efficient than value iteration.

The last but not the least, we will provide an estimate of approximation error of using time iteration.

4.3 Standard Assumption

Generally a Bellman equation can be expressed as follows:

$$v(x,z) = \max_{y \in \Gamma(x,z)} \{F(x,y,z) + \beta \int_Z v(y,z^{'})Q(z,dz^{'})\} \tag{4.5}$$

where $x \in X$ is the endogenous state, $z \in Z$ is the exogenous state with a law of motion determined by the stationary transition function Q. According to Section 9.2 in Stokey, Lucas and Prescott (1989), given the following standard assumptions:

1. X is a Borel set in $\mathcal{R}^l$ with Borel subsets $\mathcal{X}$, and Z is a compact Borel set in $\mathcal{R}^k$ with Borel subsets $\mathcal{Z}$. Denote the (measurable) product space of $(X,\mathcal{X})$ and $(Z,\mathcal{Z})$ as $(S,\mathcal{S})$.

2. The transition function, Q has the Feller property.

3. The feasibility correspondence $\Gamma(x,z)$ is nonempty, compact-valued, and continuous. Moreover, the set $A = \{(y,x) \in X \times X : \ y \in \Gamma(x,z)\}$ is convex in x, for all $z \in Z$.

4. The return function $F(\cdot,\cdot,z) : \ A \mapsto \mathcal{R}$ is , once continuously differentiable, strictly concave and bounded on A for all $z \in Z$.

5. The discount factor, β, is in the interval $(0,1)$.

if v_0 is weakly concave and the above assumptions hold, the following statements are true for any $n \in \mathcal{N}$

1. the sequences of functions defined by

$$\begin{aligned} v_{n+1}(x,z) &= \max_{y \in \Gamma(x,z)} \{F(x,y,z) + \beta \int_Z v_n(y,z')Q(z,z')\} \\ g_{n+1}(x,z) &= arg \max_{y \in \Gamma(x,z)} \{F(x,y,z) + \beta \int_Z v_n(y,z')Q(z,z')\} \end{aligned}$$

   converge pointwise (in the $sup - norm$ to the unique fixed points v and g).

2. v and v_n are strictly concave.

3. g and g_n are continuous functions.

To justify convergence to the right solution by iteration methods, we validate our model satisfies the standard assumptions first: recall our model:

$$\hat{v}(q,\delta^i) = \max_{q' \in [0,1], b' \in (0,\infty), c(q,b',q',\delta^i) > 0} \{\log(\hat{c}(q,b',q',\delta^i)) + \delta^i(a^j \log(b') + \hat{v}(q',\delta^j))\}, \tag{4.6}$$

$$\hat{c}(q,b',q',\delta^i) = R_I q - \theta(1 - \frac{b'q'}{R_I q})^2 q + R_L(1-q) - b', \tag{4.7}$$

where $i, j \in \{l, h\}$, $i \neq j$ and $a^i = \dfrac{1+\delta^i}{1-\delta^l \delta^h}$ for $i \in \{l, h\}$. Notice that in our model, the value of b' is determined as long as q' is fixed[3].

We verify that our problem satisfies the standard assumptions as follows:

1. In our model X is a finite interval $(0, 1.5)$[4]. $Z = \Delta = \{\delta^l, \delta^h\}$ is compact with 2 states $\{\delta^l, \delta^h\}$

2. The degenerate transition matrix satisfies the Feller property.

3. The feasibility correspondence[5]

$$\Gamma(q, \delta^i) = \{q' \in (0, 1.5) : \; q' \leq 1, -\hat{c}(q, b', q', \delta^i) + 10^{-6} \leq 0\},$$

   which satisfies the condition nonempty, compact valued, and continuous. Also the set $A = \{(q', q) \in X \times X : q' \in \Gamma(q, \delta^i)\}$ is convex in q for all $\delta^i \in \Delta$.

[3] Fix q', the value of b' can be found by solving the first order condition,

$$\frac{1 - \frac{2\theta q'}{R_I}(1 - \frac{b' q'}{R_I q})}{\hat{c}(q, b', q', \delta^i)} = \delta^i \frac{a^j}{b'}$$

[4] When $q = 0$, the optimal solution of q' has to be 0. Thus we consider only $q > 0$. The upper bound 1.5 is arbitrarily given, any finite number greater than 1 will do the job.

[5] The lower bound of $\hat{c}(q, b', q', \delta^i)$ is arbitrarily given, and it can be any small positive number. The constraint $b' > 0$ is not binding as long as $\theta < R_I$.

4. The return function $\log(\hat{c}(q, b', q', \delta^i)) : A \mapsto \mathcal{R}$ is once continuously differentiable, strictly concave[6] and bounded[7] on A for all $\delta^i \in \Delta$

5. The contraction mapping condition is guaranteed by the fact that $0 < \delta^l < \delta^h < 1$.

4.4 Value Iteration Algorithm

In this section we implement value iteration with our problem. According to equation (4.2), the state space of our problem have 2 dimensions. One is the endogenous state $q \in [0, 1]$, and the other one is the exogenous state $\delta^i \in \{\delta^l, \delta^h\}$. In the deterministic 2-period cycle case, the transition matrix is given by $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus the algorithm for value iteration can be described as follows:

1. Discretize $[0, 1]$, $0 = q_0 < q_1 < q_2 < \ldots < q_N = 1$. Here N denotes the discretization size.

2. Initialize the auxiliary value function $\hat{v}^0(q_n, \delta^i)$.

3. (a) given state (q_n, δ^i) search the optimal policy $(q')_n^{k+1}$ by testing all possible state of q_m.

[6] For $0 < q_1 < q_2 \leq 1$, let $r \in [0, 1]$ and $b'_{q_1,\delta^i} \in \Gamma(q_1, \delta^i)$, $b'_{q_2,\delta^i} \in \Gamma(q_2, \delta^i)$ $r\log(\hat{c}(q_1, b'_{q_1,\delta^i}, q'_1, \delta^i)) + (1 - r)\log(\hat{c}(q_2, b'_{q_2,\delta^i}, q'_2, \delta^i)) \leq \log(r\hat{c}(q_1, b'_{q_1,\delta^i}, q'_1, \delta^i) + (1 - r)\hat{c}(q_2, b'_{q_2,\delta^i}, q'_2, \delta^i)) \leq \log(\hat{c}(rq_1 + (1 - r)q_2, rb'_{q_1,\delta^i} + (1 - r)b'_{q_2,\delta^i}, rq'_1 + (1 - r)q'_2, \delta^i))$, for all $\delta^i \in \Delta$.

[7] Although log function is unbounded on $(0, \infty)$, without loss of generality, we can change the feasibility constraint to $\varepsilon_0 \leq \hat{c}(q, b', q', \delta) < \infty$ for any arbitrarily given small $\varepsilon_0 > 0$. From now on, assume $\varepsilon_0 = 10^{-6}$. Thus according to (4.5.2), $\hat{c}(q, b', q', \delta^i) < R_I$. Thus $\log(\hat{c}(q, b', q', \delta^i)$ is bounded on A.

(b) for each candidate $q' = q_m$, we can compute b' by solving the Euler's equation:the first order derivative of (4.2) taken with respect to b' should be 0.

$$\frac{1 - \frac{2\theta q_m}{R_I}(1 - \frac{b' q_m}{R_I q_n})}{\hat{c}(q_n, b', q_m, \delta^i)} = \delta^i \frac{a^j}{b'} \tag{4.8}$$

(c) compare the value of w^m

$$w^m(q_n, \delta^i) = \log(\hat{c}(q_n, b'_{q_n,q_m,\delta^i} q_m, \delta^i)) + \frac{\delta^i \delta^j}{1 - \delta^l - \delta^h} \log(b'_{q_n,q_m,\delta^i}) + \delta^i \hat{v}^k(q_m, \delta^j)$$

for each possible state q_m, where b'_{q_n,q_m,δ^i} is the solution to equation (4.8), and choose the maximum value of $w^m(q_n, \delta^i)$ as $\hat{v}^{k+1}(q_n, \delta^i)$.

4. Compute the difference between two adjacent iteration results, if the difference is greater that the tolerance, go back to step 3, otherwise the loop ends.

4.5 Time Iteration

According to the paper "Inequality Constraints in Recursive Economies" by Pontus Rendahl (2006) , if the occasionally binding inequality constraints of the maximization problem satisfies the following additional assumptions, time iteration method is equivalent to value iteration/ policy iteration.

Assumption 4.1. *The feasibility correspondence can be formulated as*

$$\Gamma(x, z) = \{y \in X : m_j(x, y, z) \leq 0, \, j = 1, \ldots, r\}$$

and the functions $m_j(x, y, z)$, $j = 1, \ldots, r$ *are, once continuously differentiable in* x *and* y*, and convex in* y*.*

Assumption 4.2. *Linear Independence Constraint Qualification (LICQ) : The Jacobian of the* d *binding constants has full (row) rank; i.e.* $rank((J)_m) = d$*.*

Assumption 4.3. *The following hold*

1. $\Gamma(x, z) \in int(X)$ *or*

2. X *is compact and* $g_n(x, z) \in int(X)$*.*

Define the operator T on $C^1(S)$, the space of bounded, strictly concave once continuously differentiable functions, as

$$(Tf)(x, z) = \max_{y \in \Gamma(x,z)} \{F(x, y, z) + \beta \int_Z f(y, z')Q(z, z')\} \tag{4.9}$$

Under above assumptions, the problem (4.9) can be expressed by

$$(Tf)(x, z) = \min_{\rho \geq 0} \max_{y \in X} L(x, y, z, \rho) = \max_{y \in X} \min_{\rho \geq 0} L(x, y, z, \rho) \tag{4.10}$$

$$L(x, y, z, \rho) = F(x, y, z) + \beta \int_Z f(y, z')Q(z, dz') - \sum_{j=1}^{r} \rho_j m_j(x, y, z)$$

where $L(x, y, z, \rho)$ is a saddle function.

Define time iteration as follows:

DEFINITION 4.4. *Denote the partial derivatives of F and m with respect to the ith element of y as $F_i(x,y,z)$ and $m_{j,i}(x,y,z)$, respectively. Then,* **time iteration** *is the iterative procedure that finds the sequence $\{g_n(x,z)\}_{n=0}^{\infty}$ as $y = g_{n+1}(x,z)$ such that*

$$\begin{aligned} 0 &= F_i(x,y,z) + \beta \int_Z [F_i(y, g_n(y,z'), z') \\ &\quad - \sum_{j=1}^{r} \rho_{j,n}(y,z') m_{j,i}(y, g_n(y,z'), z')] Q(z, dz') - \sum_{j=1}^{r} \rho_{j,n+1}(x,z) m_{j,i}(x,y,z) \end{aligned}$$

Rendahl proves that if the occasionally binding inequality constraints satisfy the additional assumptions listed above, then time iteration is equivalent to value iteration.

4.5.1 Validation

Next we need to verify that our problem satisfies these 3 additional assumptions:

1. The feasibility correspondence of our model can be formulated as

   $$\Gamma(q,\delta^i) = \{q' \in (0,1.5) : m_1 = q' - 1 \leq 0,\ m_2(q,q',\delta^i) = -\hat{c}(q,b',q',\delta^i) + 10^{-6} \leq 0\},$$

   where m_1 and m_2 are continuously differentiable in q and q', and convex in q'.

2. There is only one constraint that may be binding $q' \leq 1$, and it has full rank 1.[8]

3. For our model $\Gamma(q,\delta^i) \in int(X)$.

[8]The reason that the other 2 constraints will never bind can be given as follows: for the constraint $q' > 0$, since we are interested in the case when the investment portfolio have both assets, since $q'(0) \equiv 0$, and we assume $\frac{dq'}{dq}|_{q=0} > 0$, otherwise the investment portfolio will only contain the liquid asset. For the constraint $\hat{c} > 0$. Due to the same reason as above, we assume

According to Rendahl, time iteration method would be more efficient when nested with endogenous gridpoints method . While the application of endogenous grid points method is limited in models with such law of motion that state variable(s) of current period and that (those) of the next period are separable, i.e. growth models. The idea of endogenous grid points method is that by endogenously generating an intermediate variable, we can save the computation cost of solving Euler's equations numerically, which is typically a nonlinear system. While in our model, state variable of current period q and next period q' are not separable because of the functional form of the adjustment cost $\theta(1-\frac{b'q'}{R_I q})^2 q$, thus endogenous gridpoints method is not applicable for our model. While for our problem, the computation cost of solving the Euler equation(s) is similar to that of using the endogenous gridpoints method since in our model the Euler equation is a quadratic equation of the updated optimal policy given the state (q, δ^i), we can find the value of q' by quadratic formula.

4.5.2 Time Iteration Algorithm

Since we have verified that our model satisfy all the conditions of the equivalence of time iteration and value iteration, according to Rendahl, time iteration converges to

---

that the adjustment factor is not too large, $\theta < \frac{R_I}{2}$, thus according to the first order condition taken with respect to b',

$$\frac{1-\frac{2\theta}{R_I}(1-\frac{b'q'}{R_I q})}{\hat{c}(q, b', q', \delta^i)} = \delta^i \frac{a^j}{b'},$$

where the right hand side of the equation is greater than 0, and according to the assumption $\theta < \frac{R_I}{2}$, the numerator of left hand side is greater than 0, thus given the solution $b' > 0$, $\hat{c}$ must be greater than 0. As a matter of fact, according to the author, this constraint is dispensable.

the solution of the Bellman equation with any initial value.

Next we would like to briefly introduce the algorithm of using time iteration. Recall the functional problem we need to solve:

$$\hat{v}(q,\delta^i) = \max_{q'\in[0,1],b'\in(0,\infty),c(q,q',\delta^i)>0} \{\log(\hat{c}(q,b',q',\delta^i)) + \delta^i(a^i\log(b') + \hat{v}(q',\delta^j))\},$$

$$c(q,b',q',\delta^i) = R_I q - \theta(1-\frac{b'q'}{R_I q})^2 q + R_L(1-q) - b'.$$

According to the definition of time iteration, we use backward iteration to solve the optimal policy function q'_{k,δ^i} and the corresponding investment portfolio size ratio between 2 consecutive periods b'_{k,δ^i}. That is, if in today's portfolio the illiquid asset portion is $(1, q_n)$, assume we know the optimal choice of tomorrow's investment $(bp^k_{n,\delta^j}, qp^k_{n,\delta^j})$, then we can compute what is the illiquid asset portion in yesterday's investment portfolio $\widetilde{q}_n$ and the decision of investment portfolio size ratio $\widetilde{b}^{k+1}_{n,\delta^i}$ given the corresponding illiquid asset portion choice is q_{n,δ^i}. Then by linear interpolation between the relation $\widetilde{q}^{k+1}_{n,\delta^i}$ and q_n with q_n, we can update tomorrow's investment choice qp^{k+1}_{n,δ^i} and then bp^{k+1}_{n,δ^i}. Algorithm details are described as follows:

1. Discretize $(0,1]$, which is the feasible space, $0 < q_1 < q_2 < \ldots < q_N = 1$. Here N denotes the discretization size. Set the Lagrange coefficient $\mu^0_{n,\delta^i} = 0$.[9]

2. Updating optimal policy:

[9]As we show before, there is only 1 constraint that might be occasionally binding.

(a) Given the optimal policy $qp^k_{n,\delta^i}, bp^k_{n,\delta^i}$ of each state q_n and $i \in \{l, h\}$[10];

(b) Compute $\widetilde{b}^k_{n,\delta^i}$ by linear interpolation between qp^k_{n,δ^i} and bp^k_{n,δ^i} with q_n, which is the optimal investment portfolio size ratio decided on yesterday corresponding to the illiquid asset portion choice q_n.

(c) Update the state yesterday $\widetilde{q}^{k+1}_{n,\delta^i}$ when choosing q_n as an optimal policy:

$$\frac{\partial \log(\hat{c}(\widetilde{q}^{k+1}_{n,\delta^i}, \widetilde{b}^k_{n,\delta^i}, q_n, \delta^i))}{\partial q_n} + \delta^i \frac{\partial \log(\hat{c}(q_n, bp^k_{n,\delta^j}, qp^k_{n,\delta^j}, \delta^j))}{\partial q_n} = 0,$$

where $i \neq j \in \{l, h\}$.

(d) Interpolate the relation between $\widetilde{q}^{k+1}_{n,\delta^i}$ and q_n with q_n, we can update the value $\hat{qp}^{k+1}_{n,\delta^i}$, let $qp^{k+1}_{n,\delta^i} = \max(\hat{qp}^{k+1}_{n,\delta^i}, 1)$ and the value of Lagrange coefficient $\mu^{k+1}_{n,\delta^i} = \max\{0, \hat{\mu}^{k+1}_{n,\delta^i}\}$ by

$$(\partial_2 \log(\hat{c}(q_n, b'_{q_n,\delta^i}, 1, \delta^i)) - \hat{\mu}^{k+1}_{n,\delta^i}) + \delta^i \partial_1 \log(\hat{c}(1, bp^{k+1}_{N,\delta^j}, qp^{k+1}_{N,\delta^j}, \delta^j)) = 0,$$

where ∂_r means taking the partial derivative with respect to r-th argument.

(e) with the value of qp^{k+1}_{n,δ^i}, we can update bp^{k+1}_{n,δ^i} by

$$\frac{\partial \log(\hat{c}(q_n, bp^{k+1}_{n,\delta^i}, qp^{k+1}_{n,\delta^i}, \delta^i))}{\partial bp^{k+1}_n} = \delta^i \frac{a^j}{bp^{k+1}_{n,\delta^i}}.$$

[10] The only reason that we assign the initial value to each $q_n \in (0, 1]$, instead of $(0, 1.5)$ is that today's investment portfolio should also be feasible in the view of yesterday.

(f) Compute the updated today's consumption $\hat{c}(q_n, bp^{k+1}_{n,\delta^i}, qp^{k+1}_{n,\delta^i}, \delta^i)$

(g) Update the $\tilde{b}^{k+1}_{n,\delta^i}$ by linear interpolation between qp^{k+1}_{n,δ^i} and bp^{k+1}_{n,δ^i} with q_i

3. Evaluate the difference between $\hat{c}(q_n, bp^{k+1}_{n,\delta^i}, qp^{k+1}_{n,\delta^i}, \delta^i)$ and $\hat{c}(q_n, bp^{k}_{n,\delta^i}, qp^{k}_{n,\delta^i}\delta^i)$. If the difference is below the tolerance, iteration ends, if not, then go back to step 3.

4.6 Error Estimate

In the previous section, we show that time iteration is equivalent to value iteration for our model. This guarantees the convergence of time iteration and the solution of time iteration converges to the unique solution of the Bellman equation. While as we state before, by decreasing the grid size, value iteration can reach arbitrarily precision as we want. While for time iteration, simply decreasing the grid size can not guarantee improving the precision of the numerical solution.

According to the paper "Accuracy of Numerical Solutions using the Euler Equation Residuals" by Manuel Santos, the computation error of time iteration can be bounded by the product of a constant and the residual of the Euler equation.

Recall the standard expression of a Bellman equation: assume $(X, \mathcal{X})$ and $(Z, \mathcal{Z})$ are measurable spaces, and let $(X \times Z, \mathcal{X} \times \mathcal{Z})$ be the product space, where X is the space of states and Z is the space of exogenous shocks. Let $A \subset X \times X \times Z$ be the feasible set, which is the graph of a continuous correspondence $\Gamma : X \times Z \mapsto X$. Let u be the return function on A. Assume the exogenous random shock z is subject to a Markov process with a weakly continuous transition function Q on $(Z, \mathcal{Z})$. Let

$(Z^t, \mathcal{Z}^t) = \{Z \times Z \cdots \times Z, \mathcal{Z} \times \mathcal{Z} \cdots \times \mathcal{Z}\}$ be the t-fold product space. Given any $z_0 \in Z$, Q can define a probability measure $\mu^t(z_0, \cdot)$ on $(Z^t, \mathcal{Z}^t)$.

The optimization problem is to find a sequence of measurable functions $\{x_t\}_{t=0}^{\infty}$, $x_t : Z^{t-1} \mapsto X$, as a solution to

$$v(x_0, z_0) = \max_{\{x_t\}} \sum_{t=0}^{\infty} \beta^t \int_{Z^t} u(x_t, x_{t+1}, z_t) \mu^t(z_0, dz^t) \tag{4.11}$$

subject to $(x_t, x_{t+1}, z^t) \in A$, (x_0, z_0) fixed, $0 < \beta < 1$ for $t \geq 0$.

Assumption 4.5. *Compactness and Convexity: The set $X \times Z \subset R^l \times R^m$ is compact, and for each fixed z the set $A_z = \{(x, x') | (x, x', z) \in A\}$ is convex.*

Assumption 4.6. *Smooth and Strong Concavity: The mapping $u : A \mapsto R$ is differentiable of class C^2. Also, there exists $\eta > 0$ such that for every z the function $u(x, x', z) + \frac{\eta}{2} ||x'^2||$ is concave in (x, x').*

Assumption 4.7. *Interiority: For each initial condition (x_0, z_0) in $X \times Z$ with $x_0 \in int(X)$, every optimal realization $\{x_t, z_t\}_{t \geq 0}$ has the property that $(x_t, x_{t+1}) \in int(A_{z_t})$ for each $t \geq 0$.*

Let $\hat{g}$ be the computed policy function and $v_{\hat{g}}$ be the resulting value function under the computed policy function $\hat{g}$. Thus

$$v_{\hat{g}}(x_0, z_0) = \sum_{t=0}^{\infty} \beta^t \int_{Z^t} u(\hat{g}^t(x_0, z^{t-1}), \hat{g}^{t+1}(x_0, z^t), z_t) \mu^t(z_0, dz^t)$$

where $\hat{g}^t(x_0, z^{t-1})$ denotes the composite mapping $\hat{g}(\hat{g}(\ldots,(\hat{g}(x_0, z_0),\ldots,), z_{t-2}), z_{t-1})$ for each possible realization $z^{t-1} = (z_1, z_2, \ldots, z_{t-1})$. Let

$$E_{\hat{g}}(x,z) = ||D_2 u(x, \hat{g}(x,z), z) + \beta \int_Z D_1 u(\hat{g}(x,z), \hat{g}^2(x,z'), z')Q(z, dz')||.$$

Then we define ε as the Euler residual

$$\varepsilon = \max_{(x,z)\in X\times Z} E_{\hat{g}}(x,z) \tag{4.12}$$

Santos claim that if the above 3 assumptions are satisfied, then we have the following results: for $\varepsilon > 0$ small enough:

$$||v - v_{\hat{g}}|| \leq \frac{2}{\eta(\frac{1}{\sqrt{\beta}} - 1)^2(1-\sqrt{\beta})^2}(\frac{L}{\eta})\varepsilon^2 \tag{4.13}$$

and

$$||g - \hat{g}|| \leq \frac{2}{\eta(\frac{1}{\sqrt{\beta}} - 1)(1-\sqrt{\beta})}(\frac{L}{\eta})^{\frac{1}{2}}\varepsilon \tag{4.14}$$

where $L = \max_{(x_0,z_0)\in X\times Z} ||D_{11}u(x_0, z_0, z_0)||$ and η is the same as in assumption (4.6).

Define the coefficient on the right hand side of the inequality (4.14) be the worst case coefficient

$$M^{WC} = \frac{2}{\eta(\frac{1}{\sqrt{\beta}} - 1)(1-\sqrt{\beta})}(\frac{L}{\eta})^{\frac{1}{2}},$$

and compute the corresponding "numerical" coefficient:

$$\hat{M}_{\hat{g}}^{NUM} = \sup_{(x_0,z_0)\in K\times Z} \frac{||\hat{g}(x_0, z_0) - \hat{g}_f(x_0, z_0)||}{E_{\hat{g}(x_0,z_0)}}$$

where $E_{\hat{g}}$ is the residual function under $\hat{g}$, and $\hat{g}_f$ is the estimate of the true policy function. In Santos(2000), all conducted numerical experiments show that M^{WC} is a good upper bound of M^{NUM}.

Before we use the result by Santos (2000) to estimate the approximation error, we need to show that our model satisfies the 3 assumptions above.

1. For the first assumption—Compactness and Convexity and the third assumption–Interiority: theses two are standard assumptions which we have verified before.

2. For the second assumption—Smooth and Strong Concavity: Since the second assumption require $u(k, k', z)$ to be concave and the concaveness should be bounded away from 0, we verify it through the Hessian Matrix of $\log(\hat{c}(q, b', q', \delta^i))$, with $\hat{c}(q, b', q', \delta^i) = (R_I - \theta(1 - \frac{b'q'}{R_I q})^2)q + R_L(1 - q) - b'$

$$\begin{pmatrix} \frac{\partial^2 \log(\hat{c}(q, b', q', \delta^i))}{\partial q^2} & \frac{\partial^2 \log(\hat{c}(q, q', \delta_t))}{\partial q\, \partial q'} \\ \frac{\partial^2 \log(\hat{c}(q, b', q', \delta^i))}{\partial q\, \partial q'} & \frac{\partial^2 \log(\hat{c}(q, b', q', \delta^i))}{\partial (q')^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \log(\hat{c}(q,b',q',\delta^i))}{\partial q^2} = -\frac{\frac{2\theta}{q}(\frac{b'q'}{R_I q})^2\hat{c}(q,b',q',\delta^i) + (R_I - \theta + \theta(\frac{b'q'}{R_I q})^2 - R_L)^2}{(\hat{c}(q,b',q',\delta^i))^2}$$

$$\frac{\partial^2 \log(\hat{c}(q,b',q',\delta^i))}{\partial (q')^2} = -\frac{\frac{2\theta(b')^2}{R_I^2 q}\hat{c}(q,b',q',\delta^i) + \frac{4\theta^2(b')^2}{R_I^2}(1-\frac{b'q'}{R_I q})^2}{(\hat{c}(q,b',q',\delta^i))^2}$$

$$\frac{\partial^2 \log(\hat{c}(q,b',q',\delta^i))}{\partial q\,\partial q'} = \frac{\frac{2\theta(b')^2 q'}{R_I^2 q^2}\hat{c}(q,b',q',\delta^i) - \frac{2\theta b'}{R_I}(1-\frac{b'q'}{R_I q})(R_I - \theta + \theta(\frac{b'q'}{R_I q})^2 - R_L)}{(\hat{c}(q,b',q',\delta^i))^2}$$

In the appendix, we show that the Hessian Matrix of $\log(c(q,q',\delta^i))$ is a negative definite matrix and the minimum eigenvalue (in the sense of absolute value) is bounded away from 0 for all $(q,\delta^i) \in [h,1] \times \{\delta^l, \delta^h\}$.[11]

4.7 Computation Result

We implement both value iteration and time iteration method to solve the Bellman equation (4.2). The parameter we use for the computation is:

$R_I = 1.2$, $R_L = 1.18$, $\theta = 0.5$ $\delta^l = 0.099$, and $\delta^h = 0.99$.[12]

[11]See appendix for proof. The reason we only consider $q \in [h,1]$ is that with $q = 0$, the optimal policy is determined with $q' = 0$. Thus the computation domain for q is $q \in [h,1]$.

[12]We use the same convergence criteria $||\frac{|c^i - c^{i+1}|}{1+|c^i - c^{i+1}|}|| < 10^{-8}$ for both algorithms (value iteration and time iteration) and the time listed here is the average computation time for running the program 1000 times. The technology spec of the machine we use has 2.8G Hz dual Xeon CPU and 2 GB RAM. The algorithm is written in C language and I use a gcc complier to compile it.

In order to compare the efficiency of using these 2 methods, we compute the problem on different grid sizes.

Grid Size	Time Iteration	Value Iteration
$h = 0.01$	5.5×10^{-3}	4.5×10^{-2}
$h = 0.005$	1.60×10^{-2}	1.85×10^{-1}
$h = 0.0025$	5.5×10^{-2}	7.43×10^{-1}
$h = 0.001$	3.7×10^{-1}	5.871
$h = 0.0005$	1.415	2.3055×10^{1}

The computation result shows that generally time iteration method is more efficient than value iteration method. Moreover, as the grid size decreases, the relative computation time of the time iteration method does not increase as much as value iteration.

Since the preference shock is a 2-period deterministic cycle, we also compute our problem with differenct average discount factors $\bar{\delta} = \sqrt{\delta^l \delta^h}$.

The grid size is 0.001.

$\bar{\delta}$	Time Iteration	Value Iteration
0.5422	5.31×10^{-1}	8.965
0.3131	3.7×10^{-1}	5.871
0.2214	3.33×10^{-1}	5.828

The computation result also shows that time iteration method dominates the value iteration method for different average discount factors.

Next, we would like to estimate the approximation error for using time iteration. For the purpose of comparison, here we will also compute the value of M^{NUM} for our model.

In order to compute the "worst case" coefficient $M^{WC} = \dfrac{2}{\eta(\frac{1}{\sqrt{\beta}} - 1)(1 - \sqrt{\beta})}(\dfrac{L}{\eta})^{\frac{1}{2}}$, we need the estimated value of $\dfrac{L}{\eta}$. According to Santos, the value L can be estimated by the maximum value (in absolute value) of $\dfrac{\partial^2 \log(\hat{c}(q, q', \delta^i))}{\partial q^2}$ evaluated at points $(q, \hat{g}(q, \delta^i), \delta^i)$ and η by the smallest eigenvalue (in absolute value) of the Hessian matrix of $\log(\hat{c}(q, b', q', \delta^i))$. And the best estimate for the value of the other η should be the minimum value of $\dfrac{\partial^2 log(\hat{c}(q, b', q', \delta^i))}{\partial (q')^2}$. As the discount factor is considered, in our case we would like to use average discount factor $\bar{\delta} = \sqrt{\delta^h \delta^l}$.

In order to compute M^{NUM}, we need a good estimate of $\hat{g}_f$. Here we use the computational result from value iteration with a smaller size grid:[13] for time iteration the grid size we use is 0.001 and in order to compute the M^{NUM}, the grid size we use for value iteration is 0.0001.

Our computation result shows that $||\hat{g} - \hat{g}_f||_\infty = 5.32 \times 10^{-5}$.[14]

[13] The grid size we use for value iteration is one tenth of the grid size we use for doing time iteration. For the problem where value iteration is not applicable, a candidate for $\hat{g}_f$ can be the computation result from the time iteration where the Euler residual is at least one tenth of the Euler residual of $\hat{g}$. According to the Santos, there is no significant difference from using the computation result of value iteration.

[14] According to the paper "Analysis of a Numerical Dynamic Programming Algorithm Applied to Economic Models" by Santos and Vigo-Aguiar, the computation error of $\hat{g}_f$ computed by

The computation result[15] can be shown by the following table:

h	M^{NUM}	M^{WC}	$\hat{\varepsilon}_{\hat{g}}$	$\|\|\hat{g} - \hat{g}_f\|\|$
0.01	17.298	2.2950×10^4	4.838488×10^{-5}	8.3697×10^{-4}
0.005	2.845	2.5226×10^4	3.838102×10^{-5}	1.092×10^{-4}
0.001	12.1687	2.7407×10^4	4.378175×10^{-6}	5.32×10^{-5}

As the grid size decreases, the computation error will decrease, while the value of M^{WC} remains quite stable. [16] Our computation result also shows that M^{WC} is an upper bound of M^{NUM}. While since close to 0.1, the smallest eigenvalue (in absolute value) of the Hessian Matrix of $\log(\hat{c}(q, b', q', \delta^i))$ is close to 0, thus the estimate of M^{WC} is significantly greater than M^{NUM}.

4.8 Conclusion

In this chapter, we implement 2 numerical algorithms to solve our model- value iteration, which is stable and easy to deal with inequality constraints and time iteration, which is based on Euler equation. Although, generally speaking, time iteration is not

value iteration is given by

$$||g - \hat{g}_h|| \leq (\frac{2\gamma}{\eta(1-\beta)})^{0.5} h,$$

where h is the grid size, and $\gamma = \max ||D^2 \hat{v}||$. Thus the estimated value of $(\frac{2\gamma}{\eta(1-\beta)})^{0.5} h$ of our model is given by 8.8912×10^{-4}.

[15] Since $q = 0$ is a singularity of this problem, all the computation result we show is given on the interval $[0.1, 1]$. For $q \in [h, 0.1]$, the difference between $\hat{g}$ $\hat{g}_f$ is small, while the value of M^{WC} is unreasonably large. We consider the reason of this is due to the singularity at 0.

[16] M^{WC} slightly increases, this is due to the factor that close to 0.1 which is the lower bound of our computation domain, the smallest eigenvalue (in absolute value) of the Hessian Matrix of $\log(\hat{c}(q, b', q', \delta^i))$ decreases as the grid size decreases (precision increases as the grid size decreases.)

as stable as value iteration, it is much more efficient than value iteration, where is applicable. Moreover the relative efficiency grows as the grid size decreases. According to Rendahl, as long as the occasionally binding inequality constraints satisfy certain additional assumptions, time iteration is equivalent to value iteration. Thus the convergence of the time iteration with arbitrary initial value is guaranteed. Compared with value iteration and policy iteration, it is harder to estimate the computation error for time iteration. Although the paper "Accuracy of Numerical Solutions using the Euler Equation Residuals" by Santos offers an estimate of an upper bound of the approximation error of using time iteration to solve general stochastic dynamical model. In practice the value of the "worst case" coefficient according to Santos is always much greater than the underlying computation error, especially for those models like our case, when near the singularity point, the curvature of the utility function with respect to the policy function is small, the "worst case" coefficient will magnify the computation error.

Appendix

A.1 Chapter 1

Given the deposit contract, we study the withdrawal behavior of a single agent. Since all agents are identical ex ante, we only need to consider a generic agent. Let $v(D_t)$ be the expected discounted value function of the generic agent with current deposit D_t. According to the principle of optimality

$$v(D_t) = \max_{s_t} E[u(s_t D_t(1 + r_t)) + \Delta_t v(D_{t+1})] \tag{A.1}$$

subject to $0 \leq s_t \leq 1$.

Since we use a logarithmic utility function, the withdrawal choice is independent of the agent's deposit balance. In other words, the withdrawal choice s_t depends only on the utility discount factor Δ_t, that is $s_t \in \{s^l, s^h\}$.

The law of motion is

$$D_{t+1} = (1 - s_t)(1 + r_t)D^t.$$

In order to find the optimizing value of s^l and s^h, we solve equation (A.1) by making a conjecture and verifying it. Assume the value function has the form $v(D_t) = a\log(D_t) + b$. Equation (A.1) induces a contraction mapping for a, b with the following solution:

$$
\begin{aligned}
a &= \frac{1}{1-p\delta^l-(1-p)\delta^h} \\
b &= \frac{\log(1-p\delta^l-(1-p)\delta^h)}{1-p\delta^l-(1-p)\delta^h} + \frac{\log(1+r_t)}{(1-p\delta^l-(1-p)\delta^h)^2} \\
&\quad + \frac{p\delta^l\log(\delta^l)+(1-p)\delta^h\log(\delta^h)}{(1-p\delta^l-(1-p)\delta^h)^2} \\
&\quad - \frac{p(1-(1-p)(\delta^h-\delta^l))\log(1-(1-p)(\delta^h-\delta^l))}{(1-p\delta^l-(1-p)\delta^h)^2} \\
&\quad - \frac{(1-p)(1+p(\delta^h-\delta^l))\log(1+p(\delta^h-\delta^l))}{(1-p\delta^l-(1-p)\delta^h)^2}
\end{aligned}
$$

This solution implies the optimal withdrawal choice s^l, s^h:

$$
s^l = \frac{1-p\delta^l-(1-p)\delta^h}{1-(1-p)(\delta^h-\delta^l)},
$$

$$
s^h = \frac{1-p\delta^l-(1-p)\delta^h}{1+p(\delta^h-\delta^l)}.
$$

A.2 Chapter 2

Although we can not solve the minimization (3.25) analytically, we can find the numerical solution as follows:

1) turn inequalities (3.28) (3.29) into equality constraints

$$
\zeta(r^h+\delta^h y^h - r^l + \delta^h y^l) = 0; \tag{A.2}
$$

$$
\eta(r^l+\delta^l y^l - r^h + \delta^l y^h) = 0 \tag{A.3}
$$

2) take the first order condition with respect to r^l, r^h, y^l and y^h and combine with the equation (3.27), we can deduce a linear system with 7 equations and 7 unknown variables. Thus, we have a well defined linear system which can be solved numerically.

The difficulty here is that α is also an unknown variable in the minimization problem and it is defined by the fixed point of a contraction mapping. We deal with this problem by taking α as an additional unknown variable and adding one more equal constraint given by the definition of α:

$$\alpha = \frac{1}{R_L}\{p(\rho^{-1}(r^l) + \alpha(\frac{\gamma}{|\gamma|}y^l)^{\frac{1}{\gamma}}) + (1-p)(\rho^{-1}(r^h) + \alpha(\frac{\gamma}{|\gamma|}y^h)^{\frac{1}{\gamma}})\} \tag{A.4}$$

Thus we have a linear system with 8 equations and 8 unknown variables, and the solution to this system is the fixed point of the contraction mapping defined by (3.25), the minimizer $(r^l, y^l), (r^h, y^h)$ and the corresponding lagrange coefficients.

A.3 Chapter 3

Here is the proof of that the Hessian Matrix of $\log(\hat{c}(q, b', q', \delta^i))$ is negative definite.

Proof The Hessian Matrix of $\log(\hat{c}(q,b',q',\delta^i))$ is given by

$$H^{q,q',\delta} = \begin{pmatrix} \dfrac{\partial^2 \log(\hat{c}(q,b',q',\delta^i))}{\partial q^2} & \dfrac{\partial^2 \log(\hat{c}(q,b',q',\delta^i))}{\partial q\,\partial q'} \\ \dfrac{\partial^2 \log(\hat{c}(q,b',q',\delta^i))}{\partial q\,\partial q'} & \dfrac{\partial^2 \log(\hat{c}(q,b',q',\delta^i))}{\partial (q')^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \log(\hat{c}(q,b',q',\delta^i))}{\partial q^2} = -\frac{\frac{2\theta}{q}(\frac{b'q'}{R_I q})^2 \hat{c}(q,b',q',\delta^i) + (R_I - \theta + \theta(\frac{b'q'}{R_I q})^2 - R_L)^2}{(\hat{c}(q,b',q',\delta^i))^2}$$

$$\frac{\partial^2 \log(\hat{c}(q,b',q',\delta^i))}{\partial (q')^2} = -\frac{\frac{2\theta(b')^2}{R_I^2 q}\hat{c}(q,b',q',\delta^i) + \frac{4\theta^2(b')^2}{R_I^2}(1-\frac{b'q'}{R_I q})^2}{(\hat{c}(q,b',q',\delta^i))^2}$$

$$\frac{\partial^2 \log(\hat{c}(q,b',q',\delta^i))}{\partial q\,\partial q'} = \frac{\frac{2\theta(b')^2 q'}{R_I^2 q^2}\hat{c}(q,b',q',\delta^i) - \frac{2\theta b'}{R_I}(1-\frac{b'q'}{R_I q})(R_I - \theta + \theta(\frac{b'q'}{R_I q})^2 - R_L)}{(\hat{c}(q,b',q',\delta^i))^2}$$

Thus it is clear that for all $(q,\delta_t) \in [h,1] \times \{\delta^l,\delta^h\}$ and $q' \in \Gamma(q,\delta_t)$, $H^{q,q',\delta}(1,1)$ and $H^{q,q',\delta}(2,2)$ is negative. We need to prove that

$$H^{q,q',\delta}(1,1)H^{q,q',\delta}(2,2) - H^{q,q',\delta}(1,2)H^{q,q',\delta}(2,1) > 0$$

and the minimum (absolute value) eigenvalue of H is bounded away from 0.As a matter of fact, the first claim can be verified as a corollary of the second one. Thus we prove the second claim first:

let λ_1 λ_2 be the unknown eigenvalue, λ can be computed by solving the eigenfunction of $H^{q,q',\delta}$

$$\lambda_1 = \frac{H^{q,q',\delta}(1,1) + H^{q,q',\delta}(2,2) + \sqrt{(H^{q,q',\delta}(1,1) - H^{q,q',\delta}(2,2))^2 + 4H^{q,q',\delta}(1,2)H^{q,q',\delta}(2,1)}}{2}$$

$$\lambda_2 = \frac{H^{q,q',\delta}(1,1) + H^{q,q',\delta}(2,2) - \sqrt{(H^{q,q',\delta}(1,1) - H^{q,q',\delta}(2,2))^2 + 4H^{q,q',\delta}(1,2)H^{q,q',\delta}(2,1)}}{2}$$

Since $H^{q,q',\delta}(1,1), H^{q,q',\delta}(2,2) < 0$, then we can conclude $0 \geq \lambda_1 \geq \lambda_2$.

We only need to show that

$$(H^{q,q',\delta}(1,1)+H^{q,q',\delta}(2,2))^2 > (H^{q,q',\delta}(1,1)-H^{q,q',\delta}(2,2))^2+4H^{q,q',\delta}(1,2)H^{q,q',\delta}(2,1)$$

By simple algebra,

$$\begin{aligned}(H^{q,q',\delta}(1,1) + H^{q,q',\delta}(2,2))^2 \quad &- \quad ((H^{q,q',\delta}(1,1) - H^{q,q',\delta}(2,2))^2 + 4H^{q,q',\delta}(1,2)H^{q,q',\delta}(2,1)) \\ &= \quad \frac{\frac{2\theta b'^2}{R_I^2 q}(R_I - \theta(1 - \frac{b'q'}{R_I q})^2 - R_L)^2}{(\hat{c}(q,q',\delta))}\end{aligned}$$

Notice $(R_I - \theta(1 - \frac{b'q'}{R_I q})^2 - R_L)q = \hat{c}(q, b', q', \delta^i) - R_L + b'$

Assume that $R_I - \theta(1 - \frac{b'q'}{R_I q})^2 - R_L = 0$, then $\hat{c}(q, b', q', \delta^i) = R_L - b'$, which means that the bank will only invest in liquid assets. This is not an interesting case in computation. Given the parameter set we choose, the bank will actively invest in both technologies, which is a contradiction.

Thus we prove that λ_1 is bounded away from 0.

Since both eigenvalues of $H^{q,q',\delta}$ is strictly negative, thus we can conclude

$$H^{q,q',\delta}(1,1)H^{q,q',\delta}(2,2) - H^{q,q',\delta}(1,2)H^{q,q',\delta}(2,1) > 0$$

The above conclusion holds for all $(q, \delta_t) \in [h, 1] \times \{\delta^l, \delta^h\}$ and $q' \in \Gamma(q, \delta_t)$.

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