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Abstract

This dissertation consists of three essays in economics.

The first essay studies information trading in fixed networks of economic agents who can only observe and trade with other agents with whom they are directly connected. We study the nature of price competition for information in this environment. The linear network, when the agents are located at the integer points of the real line, is a specific example I completely characterize. For the linear network there always exists a stationary equilibrium, where the strategies do not depend on time. I show that there is an equilibrium where any agent has a nonzero probability of staying uninformed forever. Under certain initial conditions this equilibrium is a limit of equilibria of finite-horizon games. The role of a transversality condition is emphasized, namely that the price in the transaction should not exceed the expected utility of all the agents who get the information due to the transaction. I show that the price offered does not converge to zero with time.

The second essay focuses on the problem of using idiosyncratic shocks and random matchings. Many economic models use a continuum of negligible agents to avoid considering one person's effect on aggregate characteristics of the economy. Along with a continuum of agents, these models often incorporate a sequence of independent shocks and random matchings. Despite frequent use of such models, there are still unsolved questions about their mathematical justification. In this paper we construct a discrete time framework, in which major desirable properties of idiosyncratic shocks and random matchings hold. In this framework the agent space constitutes a probability space, and the probability distribution for each agent is replaced by the population distribution. Unlike previous authors, we question the assumption of known *identity* — the location on the agent space. We assume that the agents only know their *previous history* — what had happened to them before, — but not their identity. The construction justifies the use of numerous dynamic models of idiosyncratic shocks and random matchings.

The third essay examines inefficient equilibria in Games Played Through Agents. Games Played through Agents is a class of complete information games with multiple principles and multiple agents. Previous studies (Bernheim and Whinston (11), Prat and Rustichini (40)) found conditions for the existence of the efficient equilibria, in which the sum of all the payoffs is maximized. In this paper, we prove that for a game with two principals and one agent only, any inefficient equilibrium is Pareto-dominated by an efficient one. A counterexample shows that this result does not hold for more than two principals. We also demonstrate that any efficient equilibrium is weakly dominated by the Truthful Nash Equilibrium introduced in Bernheim and Whinston (11).

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Chapter 1

INFORMATION TRADING IN SOCIAL NETWORKS

1.1 Introduction

In this paper, we consider a network of agents, with each agent only able to observe and communicate with his direct neighbors. The social network is fixed. Initially, each agent becomes informed with a probability p, independently of other agents. The informed agents then offer to sell the information to their uninformed neighbors who decide to accept the offer or wait. The uninformed agents who buy the information can in turn sell it to their neighbors, if these neighbors are uninformed. We analyze the equilibria of this game.

"Neighbors" and "networks" need not be interpreted spatially. One can think of firms in similar markets as "neighbors" and the discovery of how to solve the problem of miniaturizing electronics, as in the 1970s, as the "information". Firms in similar industries become aware that their neighbors have solved a problem and might want to buy the solution. Similarly, prices need not be in terms of money but could be reciprocal exchange. Eric von Hippel (33) discusses a network of steel mini-mills, whose managers exchanged information on how to solve common problems, with the implicit contract being that each member would tell the others of relevant information. Exchange of gossip also falls into the category of such reciprocal exchange.

There have been many recent studies of learning through observing the actions or strategies of the neighbors. Boyd and Richerson (13) consider this learning a fundamental way of behavior pattern diffusion and call it cultural evolution. Empirical studies such as those of Banerjee and Munshi (8) show that the structure of the social network is especially important when the markets function imperfectly. These authors consider the effect of the social network on lending. In particular, they demonstrate that migrants prefer to be in places close to their community's lending resources. This serves as evidence that there are benefits to being in proximity to the social network. The authors show that those who migrate to places with no access to the lending network are characterized by higher production ability. The relative independence of these migrants emphasizes the importance of the network for all the others who are less productive and therefore rely more on the lending network's benefits.

Foster and Rosenzweig (26) show that the structure of the social network plays an important role in spreading information about new technologies. They demonstrate that for farmers in India, imperfect knowledge about the management of high-yielding seed varieties is a significant barrier to their adoption, and neighbors' familiarity with these seed varieties significantly increases profitability. This neighbor effect indicates that farmers rely not only on official directions provided by the producers of the seed varieties but also on the experience of the people they know, which reveals the importance of the social network structure in information diffusion. Conley and Udry (17) also argue that the learning process about new technology in agriculture (they consider pineapple growing in Ghana) is rather social, and depends on one's neighbors' experience. The social nature of adopting new technology is explained by different conditions (soil, temperature, and so on) for different regions. The people paying someone they know to do research on financial markets (which stock to invest in) is one more example of information diffusion in the social environment.

This "network effect" — the people learning from their direct neighbors in accordance with the established connections — is an evidence of market failure because the agents do not communicate with the rest of the group, and therefore information diffusion among the population is not socially optimal. This failure can be corrected only through the involvement of the government or related organizations; as Belli (9) observes, the agents themselves can not achieve a good level of communication.

In this paper we combine three main theoretical strains in the literature: information diffusion, exogenously given network structure, and rational agents who trade the information. Muto (38) discusses the sale of information but does not model a network structure. He addresses the question of diffusion of an information good from a monopolistic owner to a finite number of demanders, and a seller being allowed to charge a price for the information. Although this problem was considered within some community, the structure of the connections was not taken into account. The assumption is that everyone is connected to everyone. Muto stresses the role of information resale, and analyzes the monopolist and resellers behavior. If the resales are prohibited, then the outcome is always Pareto optimal (and therefore the society reaches maximum welfare), but if resales are allowed, then the outcome is not Pareto optimal. The author finds the number of final possessors of the information good.

Irrational agents whose response to the neighbors' actions is predetermined are studied in numerous papers. Chatterjee and Xu (16) consider myopic agents and place them at the integer points of the real line, i.e. everyone has exactly two neighbors. There are two types of technology, R(ed) and B(lue). Technology R is better than B because it provides a higher probability of success. Every period the agents decide on which technology to use. If there was a success in the technology the agent used during the last period, then he continues to use it. If there was a failure, then the agent chooses better technology based on his own and his neighbors' experience during the current period. The important finding of the paper is that sooner or later all the agents switch to the best technology.

Bala and Goyal (7) advance by taking into account an arbitrary structure. There is a finite connected social network of myopic agents who try to find the best action based on their own and their neighbors' experience. The agents do not have any beliefs about their neighbors. The information about the right action is not traded: for every agent the result of his action immediately becomes known to the neighbors. The authors show that an agent's belief converges to some limit with probability one; consequently, the utilities of all the agents are the same at infinity. As the network is finite, there is a chance that all the agents would not choose the right action. In an infinite network choosing the right action depends on the structure of the network. In particular, in the presence of a royal family — the agents who are connected to everyone else — there still exists some positive probability that the agents will not choose the best action.

Polanski (39) considers information good pricing in a network for the bargaining process at which only one pair of agents can trade at each period of time. The seller makes an offer, and the buyer either accepts it or rejects. In the case of rejection, the pair may be allowed to trade next period of time. There is no discounting. The author studies the role of cycles in the trading process. The infirmation always diffuses completely, and the price does not exceed the utility of those who can get the information good only due to this transaction: the price is zero if the buyer and seller are connected in more than one way. This result is explained by the absence of discounting which increases the patience of the agents, and the special trading structure which decreases the competition in the case of several connections between the seller and the buyer.

This paper investigates information trading and information diffusion in the social network. The focus is on how the people trade, the equilibrium strategies and prices, and the final information distribution across the agents.

By "*information*" we mean a good that has the following properties (see Muto (38)):

- It delivers some level of utility to a person who has it (commodity);
- It is possible to duplicate it without any loss in the utility (free replication);
- Once a person knows the information, it is impossible to prohibit him from knowing it (irreversibility); and
- It is impossible to get utility from a fraction of the information (indivisibility).

For example, some financial information, technology, political news, or even gossip might be considered as the information.

The important property of information is everyone's ability to trade it. It can be paid for by barter or money — we do not distinguish between the two. Again, one may argue that it is difficult to trade gossip for money. In this case by price here we mean an obligation to provide another gossip next time — we can hardly imagine a person with whom other people want to share gossip and who never gives anything back.

"*Social network*" ("*social environment*"), in which the information diffusion is considered, is a set of agents with the following properties:

- Some agents are connected to each other (these agents are called "*neighbors*");
- The agents are able to trade only with their neighbors.

This social network is conveniently represented by a graph, where the agents are located at the nodes, and the connections of the agents are represented by the edges.

There is only one sort of information in the model. At the beginning, every agent independently with the same probability learns this information. At every consequent period (time is discrete) the informed agents make offers to their uninformed agents by setting the prices in exchange for providing the information. If the buyer accepts an offer, he becomes informed and can resell the information in the following periods of time.

We make the following assumptions:

1. **Everlasting offers.** Once made, the offer stays forever and the seller can not change it later. This assumption is made for the sake of simplicity of proofs to avoid dealing with evolving prices.

2. **Limited observability.** Any agent knows if his neighbors have the information or not, and all the offers made to him during previous periods of time. The agents, however, have only general knowledge about the rest of the network and the game — who is whose neighbor, and what are the strategies, probabilities, distributions, and so on. No agent knows who, besides his neighbors, has the information, and the offers made to other agents.

The linear network is considered in the paper. We show that for any initial parameters there is a stationary equilibrium where the strategies do not depend on time, although the fraction of informed agents increases every period. This equilibrium is possible because the agents' beliefs about the distribution of the uninformed agents prior to the next informed agent do not change with time. The price in the stationary equilibria does not converge to zero as it does in the random network.

The research demonstrates that for a small probability of learning the information at the beginning, the sellers' strategy always includes a mass point above the value of the information. The existence of this mass point above the personal valuation of the information leads to the possibility of a "low probability trap," when some agents never get the information because both their neighbors make high enough offers at the same time, and each of these offers requires reselling in order to get a non-negative payoff. Moreover, the probability for the agent with two uninformed neighbors to stay uninformed forever does not change over time. For some initial parameters, this equilibrium is a limit of equilibria of finite-horizon games. For a high probability of learning the information at the beginning, the strategy of the stationary equilibrium has continuous distribution below the personal valuation of the information, which means that every agent gets the information.

This result confirms Foster and Rosenzweig's empirical finding that good technologies might not diffuse to the whole population even with rational agents. Our result can can be considered as a general explanation for empirical findings of this nature, though we do not specifically address Foster and Rosenzweig's setup.

The rest of the paper is organized as follows: in Section 2 the model is described and the equilibrium concept is defined. The importance of *the transversality condition* in an equilibrium is emphasized: the price an agent pays for the information does not exceed the expected discounted utility of all the agents who get the information due to the transaction. The linear model, where the agents are placed in the integer points of the real line, is considered in Section 4. The random network is considered in Section 5, where at every period the agents randomly meet each other.

1.2 The Model

In this section we define the game, describe the strategies, and establish the existence of the symmetric equilibrium.

1.2.1 The Game

Consider a network of agents without cycles, where every agent has exactly M neighbors. Because of the same number of neighbors for each agent, the network looks the same way no matter which agent we place at the center. An example of such a network for $M = 4$ is given in Figure 1.1.

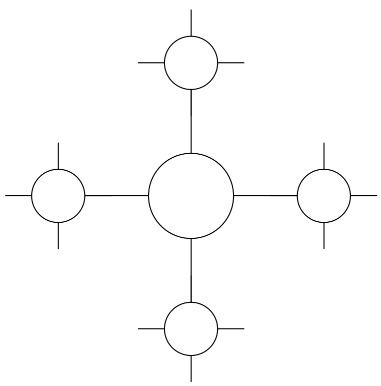

Fig. 1.1 An example of a network for $M = 4$.

There is one kind of information (for example, some particular technology) every agent can use to extract a one-time utility u. Time is discrete, $t \in \mathbb{N}$. At $t = 0$ the agents obtain independent realizations of a $\{0, 1\}$ random variable. If an agent gets a realization of 1 (this happens with exogenous probability p), he becomes "informed," otherwise "uninformed." Once an agent has the information, he remembers it forever. Every agent always knows who of his neighbors is informed; however, no one knows anything about his neighbors' neighbors.

At each period starting from $t = 1$, the informed agents (sellers) decide on making offers to their uninformed neighbors (buyers). If made, the offer is a price at which the seller agrees to share the information with a buyer. The sellers who decide to wait with an offer can make it next period of time, if the neighbor is still uninformed. The sellers make the decision about the offers and set the prices separately for each of their uninformed neighbors. At the end of the period, the uninformed agents who have at least one offer can accept one of them, or wait.

The discount factor equals $\delta \in (0,1)$. All the agents are risk neutral. The agent's utility at $t = 0$ equals

$$U = \begin{cases} 0, & \text{the agent is never informed;} \\ \delta^t(u - v) + W, & \text{the agent gets the information at period } t, \end{cases}$$

where v is the price the agent pays for the information, and W is the total discounted revenue from selling the information to the neighbors. The agents maximize their expected utility.

1.2.2 The Strategies

At every period t agent α has history

$$H_t^\alpha = (\{s_{tn}^\alpha\}_{n=1}^M, \{(s_{tn}^{\alpha B}, v_{tn}^{\alpha B})\}_{n=1}^M, \{(s_{tn}^{\alpha S}, v_{tn}^{\alpha S}, \tilde{s}_{tn}^{\alpha S})\}_{n=1}^M, (s_t^\alpha, m_t^\alpha)),$$

where

s_{tn}^α — the time when neighbor n got informed;

$s_{tn}^{\alpha B}, v_{tn}^{\alpha B}$ — the time when neighbor n made an offer, and the price offered;

$s_{tn}^{\alpha S}, v_{tn}^{\alpha S}, \tilde{s}_{tn}^{\alpha S}$ — the time of the offer to neighbor n, the price and the time of acceptance, if any;

s_t^α, m_t^α — the time when the agent got the information, and the neighbor from whom he got it.

All the histories are consistent across the agents and across time.[1] Denote $\mathbf{H}_t$ — the set of all possible histories at time t. The state of the world at time t is the set of all histories for all agents $\{H_t^\alpha\}_\alpha$.

The buyer pure strategy is the decision to buy the information from one of the neighbors, or to wait (0 corresponds to waiting):

$$R_t^{\alpha B} : \mathbf{H}_t \to \{0, 1, \ldots, M\}.$$

The sellers pure strategy for each of the uninformed neighbors is the decision to wait (represented by $\emptyset$, which is also played for the informed neighbors) or a price:

$$R_t^{\alpha S} : \mathbf{H}_t \to \{\mathbb{R}_+, \emptyset\}^M.$$

Denote the sets of pure strategies by $\mathbf{R}_t^B$ and $\mathbf{R}_t^S$ respectively. We allow mixed strategies, i.e. some probability measures $\mu_t^B(\cdot) \in \Delta(\mathbf{R}_t^B)$ and $\mu_t^S(\cdot) \in \Delta(\mathbf{R}_t^S)$.

1.2.3 Equilibrium Definition

To find the equilibrium strategies we use Perfect Bayesian Equilibrium concept. This means that the agents play the best response to their histories in accordance with the beliefs, even if the histories are not achievable under the given equilibrium strategies.

[1] By "consistent" we mean that the agents can not have contradictory histories. For example, if agent α got the information at some time t', then at all the consequent periods of time $s_t^\alpha = t'$.

This equilibrium concept shares with the Perfect Bayesian Equilibrium the idea that every agent maximizes the expected utility in every state, given the system of beliefs consistent with all the other players' strategies. Since there are infinitely many agents in the game, we can not directly apply the PBE concept, but need to generalize it in order to use it in our context. This generalization is similar in spirit to the local perfect equilibrium in Fudenberg, Levine, and Maskin (27) and is possible because at every period of time only a finite number of agents can influence the agent's history.

Definition. A symmetric equilibrium of the game is a set of strategies

$$\{\mu_t^B(\cdot), \mu_t^S(\cdot)\}_{t\geq 0}$$

such that no agent with any history (on or off the equilibrium path) can get extra payoff by deviating from his strategy given that all the other agents play the equilibrium strategies. The seller strategy $\mu_t^S(\cdot)\}_{t\geq 0}$ is a product of identical distribution functions towards each of the neighbors.

We consider symmetric equilibria, i.e. equilibria in which all the agents use the same strategies, and the strategies are symmetric with respect to different neighbors. The network does not contain cycles; therefore, the agent's action towards one neighbor can not influence the decision of another neighbor, and based on this we assume that the agents act independently towards their different neighbors.

In the definition of an equilibrium the strategies depend on the history. The knowledge of the whole history is excessive and the decision might depend on a smaller number of parameters than the history contains. Also, we want to reduce the number of equilibria in the game by introducing the concept of equivalence between the equilibria.

The equivalence of two equilibria is understood in the following way. For the sellers, their expected revenue from selling the information to a particular uninformed neighbor does not change. What we change is when the offer is made. However, the offer itself (or a distribution of the offers) is the same and, although it is made at different period of time, the time of the acceptance does not change. For the buyers, if an informed neighbor does not make an offer, it means that the future offer is such that, made at the current period of time, it would not change the buyer decision on buying the information: the earlier offer does not change the buyer behavior. Consequently, the buyers in the equivalent equilibrium face the same distribution of the offers and the sellers make the offers with the same distribution as before. The expected utility of the agents is the same, although in the original equilibrium we need to take expectation with respect to the offers of the informed neighbors who wait with the decision to make their offers. The information diffuses in the same manner, and the fraction of the informed agents as well as the spacial structure of the informed/uninformed agents also stays the same.

The following proposition describes the necessary parameters and reduces the number of equilibria by introducing an equivalent equilibrium in which the sellers make their offers immediately after acquiring the information.

Proposition 1.1. *For any equilibrium there exists an equivalent one, in which all the informed agents make their offers immediately, the seller strategy at time t is a distribution* *function of offer prices $F_t(v)$, and the buyer strategy is a function*

$$K_t : \{1, 2, \ldots, M\} \to \mathbb{R}_+,$$

which determines the reservation price for a given number of informed neighbors.

We described here the strategies on the equilibrium path. The only deviation these strategies do not take into account is the one when a neighbor gets the information and then does not make an offer. In this case, we assume that the agent believes that the neighbor will make an offer next period of time, and uses corresponding best response. This happens with probability zero, therefore we should not worry about the effect of such a deviation except for that this kind of a deviation should not be profitable.

We concentrate on the equilibria from Proposition 1.1 at which the offers are made immediately, and the offers to different neighbors are independently drawn from distribution $F_t(\cdot)$. Denote

$$V_t = \sup(\operatorname{supp} F_t(v))$$

— the highest possible price offered at time period t. The buyers with l informed neighbors accept the lowest offer if this offer does not exceed $K_t(l)$.

The game has infinitely many agents and infinite horizon. Therefore, the equilibria may have the property of the Ponzi game, where the prices are not consistent with the utility the agents get from knowing the information. To exclude such equilibria from consideration, we use a *transversality condition*. We require that for any period of time t and for any l

$$K_t(l) \leq u * \mathbf{E}\, A_{tl}, \tag{1.1}$$

where A_{tl} stands for the random variable representing the discounted number of the uninformed agents who will get the information due to the transaction between the agent and the seller, if the buyer has exactly l informed neighbors. This condition requires that the price does not exceed the expected discounted utility of all the agents who will get the information due to the transaction.

In equilibrium a buyer with all informed neighbors prefers to buy the information for any price not exceeding u. At the same time, buying the information for a price above u results in a negative payoff, therefore

$$K_t(M) = u. \tag{1.2}$$

A buyer immediately agrees on price $v \leq (1-\delta)u$ because the loss in the expected utility from waiting is $u - \delta u = u(1-\delta)$. Therefore,

$$K_t(l) \geq (1-\delta)u \qquad \forall l \in \{1, 2, \ldots, M-1\}.$$

1.3 Linear Network

This section considers a special case of an infinite symmetric network without cycles — the infinite linear network (see Figure 1.2), where every agent has exactly two neighbors.

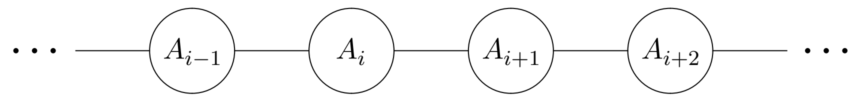

Fig. 1.2 Infinite linear network.

1.3.1 General Results

The linear network, along with its plain structure, has the advantage of simple beliefs of the agents, which we can formulate using the following notation. Denote event "agent i is informed at time t" by $\overline{A_i^t}$, and event "agent i is uninformed at time t" by A_i^t.

The following proposition describes the agent belief about the distance till the next informed agent. Although the fraction of the informed agents increases over time, this belief does not change as long as the agent himself and his neighbor stay uninformed.

Proposition 1.2. *Suppose that all the agents in the linear network act independently and use the same (even non-equilibrium) strategies. Then for any uninformed agent with an uninformed neighbor his belief that there are exactly k other uninformed agents beyond the uninformed neighbor has a geometric distribution with parameter p:*

$$\mathbf{P}(A_{i-1}^t A_i^t \dots A_{i+k}^t \overline{A_{i+k+1}^t} | A_{i-1}^t A_i^t) = p(1-p)^k. \quad (1.3)$$

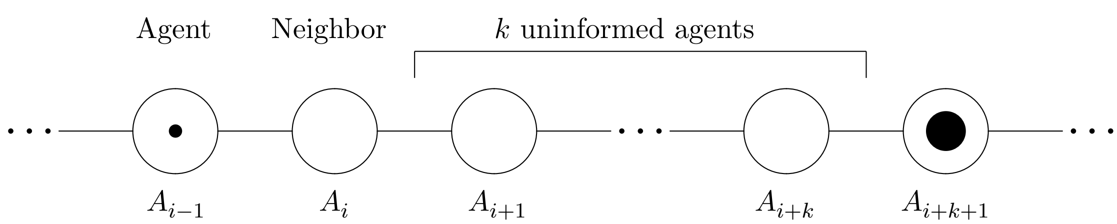

Fig. 1.3 Stationary beliefs in the infinite linear network.

Consider uninformed Agent A_{i-1}, whose neighbor A_i is uninformed (Figure 1.3). We do not need to assume that the agents use equilibrium strategies; the only assumption necessary is that the strategies are the same (mixed or pure) for all the agents. Then Agent A_{i-1} believes that the probability of k agents $A_{i+1}, A_{i+2}, \dots, A_{i+k}$ to be uninformed and agents A_{i+k+1} to be informed (this agent is marked with a black circle) equals $p(1-p)^k$. Probability that the next k agents are uninformed

$$\begin{aligned}\mathbf{P}(A^t_{i-1}A^t_i\dots A^t_{i+k}|A^t_{i-1}A^t_i) &= \sum_{l=0}^{\infty}\mathbf{P}(A^t_{i-1}A^t_i\dots A^t_{i+k}\overline{A^t_{i+k+l}}|A^t_{i-1}A^t_i)\\ &= \sum_{l=0}^{\infty}p(1-p)^{k+l} = (1-p)^k.\end{aligned}$$

This result of Proposition 1.2 holds because of the following reasoning. First, the belief is calculated conditionally on the fact that the agent himself and his neighbor are uninformed. In particular, Agent A_{i-1} does not know anything about agents A_l for $l \geq 1$. Second, the initial distribution of the number of uninformed agents preceding the first informed one is geometric with parameter p because at the beginning everyone learns the information independently. And finally, the geometric distribution has the property similar to the constant hazard rate of the exponential distribution: the distribution of the difference of a geometrically distributed random variable and a constant (which models the diffusion of the information towards the agent) is the same as the distribution of the random variable if the difference is non-negative.

Suppose that the strategies are such that an uninformed agent with one offer always buys the information, i.e. $K_t(1) \geq V_t$. Then the probability of acquiring the information by an uninformed neighbor of an uninformed agent equals p. (The product of $\mathbf{P}(A^t_{i-1}A^t_i\overline{A^t_{i+1}}|A^t_{i-1}A^t_i) = p$ and the probability that the information will be transferred, which equals one.) In other words,

$$\mathbf{P}\{A^t_{i-1}, A^t_i, \overline{A^{t+1}_i}|A^t_{i-1}, A^t_i\} = \mathbf{P}\{A^t_{i-1}, A^t_i, \overline{A^t_{i+1}}|A^t_{i-1}, A^t_i\} = p. \tag{1.4}$$

Consider a pure strategy equilibrium. In this equilibrium all the informed agents offer the information for the same price V_t at time t. The following proposition characterizes all such equilibria that satisfy the transversality condition.

Proposition 1.3. *For any $p \in (0,1)$ there exists at most one pure strategy equilibrium satisfying the transversality condition; for this equilibrium*

$$K_t(1) = V_t = \frac{u}{1-\delta(1-p)^2}.$$

As the buyers and sellers use the same strategy every period of time in this pure strategy equilibrium, the information always diffuses from an informed agent to his uninformed neighbor if this neighbor has only one offer. The range for the initial parameter p when pure equilibria exists will be found in the next subsection.

1.3.2 Stationary Equilibria

Proposition 1.3 showed that in all pure strategy equilibria the strategies do not depend on time. Such equilibria, in which the strategies do not depend on time, $F_t(\cdot) = F(\cdot)$, $K_t(1) = K$, we will call *stationary equilibria*. Equation 1.4 shows that if an agent with one only offer always buys the information, then the probability of an uninformed agent's uninformed neighbor becoming informed equals p. This argument allows us to guess that there might be other stationary equilibria except for pure strategy equilibria. In this subsection we characterize all such equilibria.

All the possible strategies of stationary equilibria can be characterized using the following proposition.

Proposition 1.4. *In any stationary equilibrium $K(1) = V$. For any $p \in (0, 1)$ there exists exactly one stationary equilibrium. All stationary equilibria satisfy the transversality condition. The type of the equilibrium depends on p:*

1. For $p \in \left(0, \frac{2\delta+1-\sqrt{4\delta+1}}{2\delta}\right]$

$$F(v) \equiv F^p(v) = \begin{cases} 0, & v < V^p; \\ 1, & v \geq V^p, \end{cases}$$

and $V^p = \frac{u}{1-\delta(1-p)^2}$.

2. For $p \in \left(\frac{2\delta+1-\sqrt{4\delta+1}}{2\delta}, p^*\right)$

$$F(v) \equiv F_1^m(v) = \begin{cases} 0, & v < (1-p)V_1^m; \\ \frac{1}{p} - \frac{(1-p)V_1^m}{pv}, & (1-p)V_1^m \leq v \leq u; \\ \frac{1}{p} - \frac{(1-p)V_1^m}{pu}, & u < v < V_1^m; \\ 1, & v \geq V_1^m, \end{cases}$$

and $V_1^m > u$ is uniquely determined by equation

$$\frac{u}{(1-p)V_1^m} + \delta(1-p) - \frac{1}{1-p} = \frac{\delta}{1-\delta(1-p)} \left(\frac{u}{(1-p)V_1^m} - 1 - \ln\left(\frac{u}{(1-p)V_1^m}\right)\right) \quad (1.5)$$

and decreases with p.

3. For $p \in [p^*, 1)$

$$F(v) \equiv F_1^m(v) = \begin{cases} 0, & v < (1-p)V_1^m; \\ \frac{1}{p} - \frac{V_1^m(1-p)}{pv}, & (1-p)V_1^m \leq v \leq V_1^m; \\ 1, & v > V_1^m, \end{cases}$$

and

$$V_1^m = \frac{u(1-\delta)}{(1-\delta(1-p))(1-\delta(1-p)^2) + \delta(1-p)\ln(1-p)} \in (0, 1) \quad (1.6)$$

is a decreasing function of p for $p \geq p^*$.

Constant p^* is the unique solution of equation

$$p - (1-\delta(1-p))(1-p)^2 + (1-p)\ln(1-p) = 0 \quad (1.7)$$

from interval $\left(\frac{2\delta+1-\sqrt{4\delta+1}}{2\delta}, 1\right)$.

Different strategies $F(v)$ for all three types of stationary equilibria are depicted at Figure 1.4.

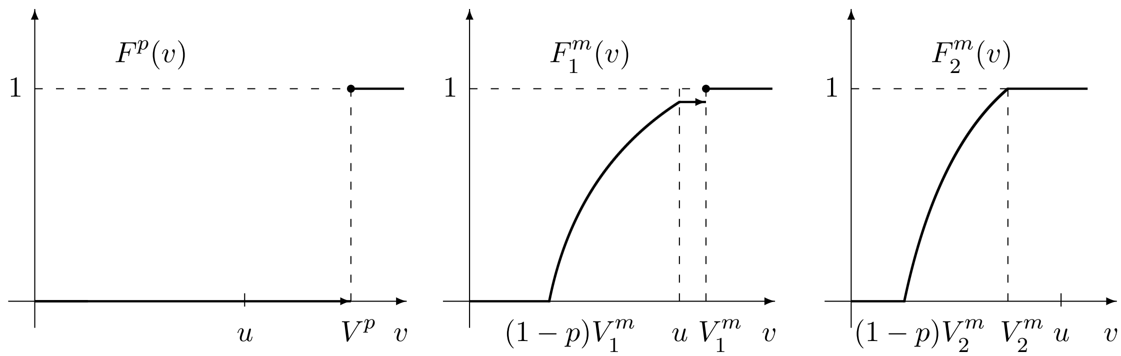

Fig. 1.4 Stationary equilibria strategies in the infinite linear network. Left graph: pure strategy $F^p(v)$; Center graph: strategy $F_1^m(v)$; Right graph: strategy $F_2^m(v)$.

For a small p strategy $F(v)$ is a degenerate distribution with the mass point at $V^p > u$, for a medium p strategy $F(v)$ has both continuous part on $[(1-p)V_1^m, u]$ and mass point at $V_1^m > u$, and for a high p strategy $F(v)$ is an absolutely continuous distribution on $[(1-p)V_2^m, V_2^m]$, where $V_2^m \leq u$.

Strategy $F(v)$ for stationary equilibria evolves in the following way as p increases (see Figure 1.5). For small p strategy $F(v) = F^p(v)$ is a degenerate distribution with a mass point at $V^p > u$, and this mass point decreases with p. After $p = \frac{2\delta+1-\sqrt{4\delta+1}}{2\delta}$, an absolutely continuous segment on $[(1-p)V_1^m, u]$ appears in $F(v) = F_1^m(v)$; this segment grows ($(1-p)V_1^m$ decreases) and the mass point V_1^m decreases to u with the mass at V_1^m decreasing to zero. At $p = p^*$, the mass point disappears, and the absolutely continuous segment starts moving towards zero. The lower bound decreases to 0, and the upper bound decreases to $u(1-\delta)$. Distribution $F(v)$ weakly converges to the degenerate distribution with mass point at 0.

For $p \in (0, p^*)$ there exists a mass point at $V > u$. Because this mass point is above the agent's personal valuation of the information, there is a non-zero probability that the agent will get two offers V at the same time, and therefore will stay uninformed forever.

Proposition 1.5. *In the stationary equilibrium with $p \in (0, p^*)$ probability that an uninformed agent with two uninformed neighbors will stay uninformed forever equals $\frac{p(1-F(u))^2}{2-p} > 0$ and does not depend on time.*

Probability that a randomly chosen agent will stay uninformed forever can be calculated as the sum of two probabilities: (1) the probability that the agent and his

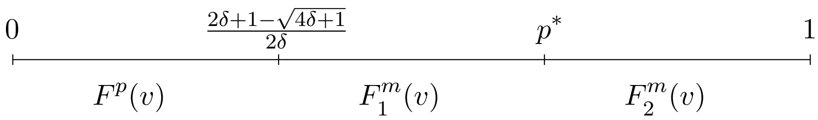

Fig. 1.5 Stationary equilibria regions.

neighbors were initially uninformed multiplied by the probability that the agent will stay uninformed forever, and (2) the probability that the agent has two informed neighbors each of which offers price above u:

$$(1-p)^3 \frac{p(1-F(u))^2}{2-p} + (1-p)p^2(1-F(u))^2 = (1-p)\frac{p(1-F(u))^2}{2-p}.$$

For $p \geq p^*$ every agent in the network will get informed. This threshold p^* divides interval (0,1) into the areas of efficient and non-efficient equilibria. In order to achieve efficiency, the central planner does not need to give the information to everyone; it is enough to give the information randomly to a sufficient fraction of the population.

The equilibria of the game, in particular the stationary equilibria, might not be robust with respect to some modifications of the game. The question is what happens with the strategies if we consider the same game with a finite horizon instead of the infinite one. Take a sequence of equilibria in the games with the time limited by T. We want to investigate how close are the equilibria in such finite horizon games to the infinite horizon game equilibria, i.e. the limit of the equilibria of the games with finite horizons.

Proposition 1.6. *For small enough p the equilibria for the finite horizon games converge to the pure strategy stationary equilibrium for the infinite horizon game.*

1.3.3 Equilibria with Unbounded Price

The transversality condition restricts the prices. In this subsection we construct an example with a family of strategies in which this condition is not satisfied. The prices offered exceed some level and increase to infinity with time. What the agents pay for the information is not justified by the utility of the agents who get the information due to the transaction; the current price is supported by the expectations of the higher prices in the future.

Consider the linear network. For simplicity, we restrict our attention to the equilibria with pure strategies only, in which a buyer with two offers will not buy the information because these offers exceed his personal valuation of the information. By V_t we denote the offer/acceptance price at period t. A buyer with one informed neighbor only should be indifferent between buying the information and waiting, therefore the following equation holds for any t:

$$u - V_t + \delta(1-p)^2 V_{t+1} = 0,$$

i.e. buying the information and offering it to the uninformed agent gives zero expected utility. After rearranging the terms, one can get

$$V_{t+1} - \frac{u}{1-\delta(1-p)^2} = \frac{V_t - \frac{u}{1-\delta(1-p)^2}}{\delta(1-p)^2}.$$

Taking into account that in the stationary pure strategy equilibrium price always equals $\bar{V} = \frac{u}{1-\delta(1-p)^2}$, we get

$$V_{t+1} - \bar{V} = \frac{V_t - \bar{V}}{\delta(1-p)^2}. \tag{1.8}$$

As $\delta(1-p)^2 < 1$, difference $V - \bar{V}$ grows exponentially if initial V_0 exceeds $\bar{V}$:

$$V_t = \bar{V} + (V_0 - \bar{V}) \left(\frac{1}{\delta(1-p)^2} \right)^t.$$

The only additional requirement for V_0 is that a seller does not deviate to offering u at $t = 0$, i.e. $V_0(1-p) > u$ (if the prices increase, it will also be true for arbitrary t). Therefore, for any

$$V_0 > \max \left(\frac{u}{1-p}, \frac{u}{1-\delta(1-p)^2} \right)$$

the equilibrium we get is a pure strategy equilibrium for which the transversality condition fails, and the prices increases to infinity with time.

1.4 Random Networks

The analysis of the fixed networks showed that some equilibria in such networks possess some properties, like the price does not converge to zero. In this section we want to consider random networks, and find the properties of equilibria in these random networks to compare them with the properties of equilibria of the fixed networks.

Suppose that every period of time the agents are randomly matched with exactly M other agents[2], and the network formed does not contain cycles. It means that at every period of time a new M-network or a set of them is formed, and no past history can influences the agents' current decisions. Therefore, the agents' actions are independent across the time and neighbors.

As before, all the informed agents can simultaneously make their offers to their uninformed neighbors, and the uninformed agents decide to accept one of the available offers or to wait. As we deal with the random network, the informed agents make their

[2] Random network is a controversial issue, although it is used in many models. In this paper we do not discuss the question of existence of such networks (although we believe that it is possible to construct a formal justification). We rather use some assumptions about such networks, namely that no two current neighbors can have any influence on each other in the future. In particular, the probability of being matched with the same partner twice is assumed to be zero.

offers to uninformed neighbors every period of time, and the offers made expire at the end of each period with the abortion of the connections.

At the beginning, every agent independently with probability p learns the information. The seller strategy is a distribution function of offers $F_t(v)$. The buyer strategy is a threshold K_t — the maximal price at which he is ready to buy the information. As new network is randomly formed each period of time, K_t does not depend on the number of informed neighbors. We consider only symmetric equilibria, i.e. the agents use the same strategies.

As before, denote $V_t = \sup \operatorname{supp} F_t(\cdot)$. Threshold $K_t \geq V_t$ because otherwise offer $V_t > K_t$ will never be accepted. Distribution function $F_t(\cdot)$ is absolutely continuous because $K_t \geq V_t$ and the agents will try to avoid the competition from other agents at the mass points. Also, $K_t \leq V_t$ because otherwise the agents selling the information for price $V_t < K_t$ will be able to increase their offer to K_t without decreasing the probability of the deal. Therefore, K_t coincides with V_t, and later in this section V_t will represent both constants.

From $K_t = V_t$ follows that an agent becomes informed once he has at least one informed neighbor. Denote the probability of being informed at the beginning of period t by p_t, with $p_1 = p$. Then

$$p_{t+1} = p_t + (1 - p_t)(1 - (1 - p_t)^M) = 1 - (1 - p_t)^{M+1};$$
$$1 - p_{t+1} = (1 - p_t)^{M+1},$$

and p_t monotonically approaches 1.

Denote

$$g_t = (1 - p_t)^{M-1}(M + (1 - p_t)). \tag{1.9}$$

As p_t monotonically approaches 1, g_t monotonically approaches 0.

As before, by the transversality condition we understand that the price in the transactions does not exceed the discounted expected utility of all the agents who get the information due to this transaction. The following proposition completely characterizes equilibria satisfying the transversality condition.

Proposition 1.7. *For any initial probability $p \in (0, 1)$ there is only one equilibrium that satisfies the transversality condition. In this equilibrium the seller strategy*

$$F_t(v) = \frac{1}{p_t} - \frac{1 - p_t}{p_t}\left(\frac{V_t}{v}\right)^{\frac{1}{M-1}}; \tag{1.10}$$

$$\operatorname{supp} F_t(\cdot) = [V_t(1 - p_t)^{M-1}, V_t]. \tag{1.11}$$

The highest price possible at period t

$$V_t = V_1 \prod_{i=2}^{t} \frac{1}{\delta g_i} - u(1-\delta) \sum_{i=2}^{t} \prod_{j=i}^{t} \frac{1}{\delta g_j}; \tag{1.12}$$

$$V_1 = u(1-\delta) \left(1 + \sum_{i=3}^{\infty} \prod_{j=2}^{i-1} \delta g_i \right) < \infty. \tag{1.13}$$

The highest possible price V_t monotonically decreases to $u(1-\delta)$, and the expected price $\mathbf{E}_{F_t} v$ converges to 0.

As we see, V_t is uniquely determined by constants M, p, u, and δ. $F_t(\cdot)$ weakly converges to the degenerate distribution with the mass point at zero.

1.5 Conclusion

In this paper we show that the structure of the social connections plays an important role in information diffusion. It determines the price pattern the sellers charge for the information and the buyers strategy. In particular, the price asked does not always converge to zero. The agents making an offer might believe that the probability of an uninformed neighbor getting another acceptable offer is small enough, therefore they do not decrease the price. In the case of many uninformed agents at the beginning, this belief leads to the price exceeding the personal valuation of the information.

Not all the agents might learn the information at the end if the price exceeds the personal valuation; it happens if the information is a scarce resource. The information diffuses to all the agents if the fraction of the initially informed agents is large enough. Therefore, if the government wants everyone to have the information, it does not need to give it to all the agents; it is enough to exceed some threshold, and after this the agents will successfully trade the information with each other.

The linear network considered in many papers does not constitute a representative example. It has the property which is particular only for such a network: the belief about the number of uninformed agents till the first informed one, conditional on the fact that the agent himself and his neighbor are uninformed, does not depend on time. Due to this there exists the stationary equilibrium where the strategies the agents use do not depend on time.

The equilibrium for the random network differs from the fixed network in the following aspects. The uninformed agents buy the information as soon as they get at least one offer. The average price offered at period t converges to 0; however, the upper bound of the price converges to $u(1-\delta)$. In the random network, every agent becomes informed with probability 1.

1.6 Proofs

Lemma 1.1. *Differential equation*

$$af'(x)x = 1 - bf(x)$$

for $b \neq 0$, $a \neq 0$ has solution

$$f(x) = \frac{1}{b} - Cx^{-b/a}. \tag{1.14}$$

Proof of lemma 1.1.

The solution is verified by substituting formula (1.14) for $f(x)$ into the original equation and the fact that the first-order differential equation has only one undetermined constant.

□

Proof of proposition 1.1.

The neighbors are connected only through the agent, therefore the seller strategy can be independent for each of his uninformed neighbors; if the offer is made, it follows some distribution function $F_t(v)$, which depends only on time.

Suppose that a buyer with exactly l informed neighbors accepts offer v. Then accepting offer $v' < v$ increases the buyer's expected payoff by $v - v'$ without changing his expectations of the future resales. The expected utility of waiting with the lowest offer $v' < v$ increases by less than $v - v'$ because the best difference is $v - v'$ and the discount factor decreases it. Therefore, the strategy of a buyer with l informed neighbors is to accept an offer either from interval $[0, K_t(l))$ or $[0, K_t(l)]$ for some $K_t(l) \geq 0$. The buyer is indifferent to accept offer $K_t(l)$ or to wait.

If for a buyer there is no mass of offers at $K_t(l)$, then these two options (to buy immediately and to wait) do not differ, and we can choose the closed interval. If there is a mass point, then the sellers who create the mass point ($F_t(\cdot)$ has a mass point) would prefer to deviate to $K_t(l) - \epsilon$, which means that this is not an equilibrium and $K_t(l)$ can not be a mass point of offers. Therefore, we can always assume that a buyer with l neighbors accepts any offer not exceeding $K_t(l)$.

To prove the existence of an equivalent equilibrium in which all the sellers make their offers immediately, consider one informed agent A and his uninformed neighbor B. By waiting agent A can observe only the fact that B gets the information from his other neighbor (what makes impossible selling the information to B). Agent A makes such offer v that maximizes his expected payoff.

There are 2 options:

Option 1. Agent B with non-zero probability may accept offer v earlier than agent A normally makes it. Then agent A is strictly better off by making the offer earlier, and therefore this is not an equilibrium to delay with making this offer.

Option 2. Agent B would not accept offer v earlier than agent A normally makes it. Then by making offer v earlier Agent A does not change his own payoff and the rest of his strategy. Suppose that Agent B has other lowest offer and making offer v earlier changes B's behavior. As B does not accept offer v (we excluded option 1) then the other offer he has is better, and B knows it because A does not make an offer. Consequently, revealing v does not change B's decision to accept other offers. Therefore, making offer v earlier does not change anything and making the offers as soon as possible is a new equivalent equilibrium.

□

Proof of proposition 1.2.

The proof has the following structure. First, we consider the following modification of the game: agents A_i, A_{i-1},... are always uninformed at the beginning (see Figure 1.3). Second, we demonstrate that random variables ξ_1 and $\xi_1 - \xi_t$ are independent for any t, where ξ_t is the number of uninformed agents A_{i+1}, A_{i+2},..., A_{i+k} till the first informed agent A_{i+k+1} at time t. Third, we show that ξ_t conditional on $\xi_t \geq 0$ has the same geometric distribution as ξ_1. And last, we return to the original game, and prove formula 1.3 from the Proposition.

Step 1. Defining the game and random variables.

Suppose that A_i, A_{i-1},... are always uninformed at the beginning. Define random variable $\xi_t \in \mathbb{Z}$ in the following way:

$$\xi_t = \min\{k : A_{i+k+1} \text{ is informed at the beginning of period } t\}.$$

Random variable $\xi_t \in \mathbb{Z}$ stands for the first informed agent in the network.

Step 2. Independence of ξ_1 and $\xi_1 - \xi_t$.

Consider agent l who acquires the information at period t. Let η_{lt} be the number of periods it takes for agent l to transfer the information to his left neighbor, if this neighbor has only one offer. The agents act independently, therefore all random variables $\{\eta_{lt}\}$ are independent of each other and ξ_1. The agents use the same strategies, therefore $\{\eta_{lt}\}_l$ are identically distributed for each t.

Let η_l stands for the number of agents the information diffused to the left by time t if initially agent l is the first informed agent. Variables η_l are determined by $\{\eta_{l't'}\}_{l't'}$ and therefore independent of ξ_1 for any l (but not from each other). As $\{\eta_{lt}\}_l$ are identically distributed for each t, η_l are identically distribute for every l. Denote this distribution by η.

Note that

$$\begin{aligned} \mathbf{P}(\xi_1 = m, \xi_1 - \xi_t = l) &= \mathbf{P}(\xi_1 = m, \eta_{\xi_1} = l) = \mathbf{P}(\xi_1 = m, \eta_m = l) \\ &= \mathbf{P}(\xi_1 = m)\,\mathbf{P}(\eta = l); \end{aligned}$$

$$\mathbf{P}(\xi_1 - \xi_t = l) = \sum_m \mathbf{P}(\xi_1 = m, \xi_1 - \xi_t = l) = \sum_m \mathbf{P}(\xi_1 = m)\,\mathbf{P}(\eta = l) = \mathbf{P}(\eta = l),$$

i.e. random variables ξ_1 and $\xi_1 - \xi_t$ are independent.

Step 3. Geometric distribution of ξ_t conditional on $\xi_t \geq 0$.

We want to prove

$$\mathbf{P}\{\xi_t = k | \xi_t \geq 0\} = p(1-p)^k, \qquad \forall t, k \in \mathbb{N}. \tag{1.15}$$

Note that this formula holds for $t = 1$ because the agents independently with probability p get the information at the beginning.

$$\mathbf{P}\{\xi_t = k|\xi_t \geq 0\} = \frac{\sum_{l\geq 0} \mathbf{P}(\xi_1 = k+l, \xi_1 - \xi_t = l)}{\sum_{l\geq 0, k'\geq 0} \mathbf{P}(\xi_1 = k'+l, \xi_1 - \xi_t = l)} = \frac{\sum_{l\geq 0} p(1-p)^{k+l}\,\mathbf{P}(\xi_1 - \xi_t = l)}{\sum_{l\geq 0, k'\geq 0} p(1-p)^{k'+l}\,\mathbf{P}(\xi_1 - \xi_t = l)}$$
$$= \frac{(1-p)^k \sum_{l\geq 0} p(1-p)^l\,\mathbf{P}(\xi_1 - \xi_t = l)}{\frac{1}{p}\sum_{l\geq 0} p(1-p)^l\,\mathbf{P}(\xi_1 - \xi_t = l)} = p(1-p)^k,$$

i.e. formula 1.15 holds for any t.

Step 4. Proof of formula 1.3 from the Proposition.

Consider the original game. In this game, agents A_{i-1}, A_i, can get the information by time t either at the beginning, from A_{i-2}, or from A_{i+1}. We considered the process from the right. We can make the same analysis from the left, and consider corresponding random variable ζ_t — the distance from the right informed agent to A_{i-1} in the hypothetical network where all the agents $A_{i-1}, A_i, \ldots$ are uninformed at the beginning, Then ξ_t and ζ_t are independent, and for any $k \geq 0$

$$\mathbf{P}(A^t_{i-1}A^t_i \ldots A^t_{i+k}\bar{A}^t_{i+k+1}|A^t_{i-1}A^t_i) = \mathbf{P}(\zeta_t \geq 0, \xi_t = k|\zeta_t \geq 0, \xi_t \geq 0) = \frac{\mathbf{P}(\zeta_t \geq 0, \xi_t = k)}{\mathbf{P}(\zeta_t \geq 0, \xi_t \geq 0)}$$
$$= \;= \frac{\mathbf{P}(\xi_t = k)}{\mathbf{P}(\xi_t \geq 0)} = p(1-p)^k.$$

□

Proof of proposition 1.3.

Consider first t such that $V_t \leq u$. Consider an agent who makes an offer at time t to his uninformed neighbor. There is a non-zero probability that the neighbor has the same offer V_t from his another neighbor, and will be choosing the best one. Then the agent will benefit by decreasing his offer to $V_t - \epsilon$: the probability of selling the information increases, and the payment stays almost the same. Therefore, $V_t > u$ for all t.

Suppose that there exists t such that $V_t < K_t(1)$. The agent accepts offer V_t only if this is the only offer, and another neighbor is uninformed. By increasing the offer to $K_t(1)$ the seller does not decrease the chance of the deal, but increases the payment. Therefore, V_t can not be less than $K_t(1)$.

At every period of time there is either no trade or all the agents with one informed neighbor only buy the information.

Consider first t such that $V_t = K_t(1)$, $V_{t+1} > K_{t+1}(1)$. Suppose that $t > 1$ (the proof with slight modification works for $t = 1$, too.) The agents with one offer V_{t+1} only at time period $t+1$ wait with the purchase until some period $\tau > t+1$ with $V_{t+1} \leq K_\tau(1)$, and there is no trade in periods $t+1, t+2, \ldots, \tau - 1$.

Any agent who buys the information at period t has utility zero because the offer from other neighbors will exceed u, and the seller has all the power. Therefore,

$$-V_t + u + \delta^{\tau-t}(1-p)^2 V_{t+1} = 0. \tag{1.16}$$

Suppose that some agent with one the only offer V_t at period t does not buy the information immediately, but waits till period $t+1$. If his neighbor still stays uninformed, then he pays V_t at period $t+1$, and offers it to his uninformed neighbor at time $t+2 \leq \tau$ for V_{t+1}. Then, using equation 1.16, his utility

$$\begin{aligned}\delta(1-p)(-V_t + u + \delta^{\tau-t-1}(1-p)V_{t+1}) &= \delta(1-p)(-V_t + u) + V_t - u \\ &= (V_t - u)(1 - \delta(1-p)) > 0\end{aligned}$$

because $V_t > 0$, which means that this is not an equilibrium. The intuition behind the fact the the utility increases if the agent waits is the following: by waiting the agent decreases the uncertainty about the possibility of reselling the information.

We have proved that for any period t holds $V_t = K_t(1)$. Equation 1.16 for $\tau = t+1$ gives us the the law of motion for V_t:

$$-V_t + u + \delta(1-p)^2 V_{t+1} = 0.$$

Fixed point

$$V = \frac{u}{1 - \delta(1-p)^2}.$$

Therefore,

$$V_t - V = \delta(1-p)^2(V_{t+1} - V),$$

which means that this fixed point is unstable: if $V_1 \neq V$ then V_t converges either to $-\infty$ or to $+\infty$. The first option contradicts $V_t \geq 0$, and the second one contradicts the transversality condition (the price is limited by some constant).

□

Proof of proposition 1.4.

First, we want to show that $V = K(1)$. We already know that $K(2) = u$. Distribution function $F(v)$ is continuous for $v \leq u$ because $K_t(2) = u$ and a seller offering the mass point price would better off by decreasing his offer by small ϵ to avoid the tie.

Suppose that $V \neq K(1)$. As $V \leq \max(K(1), u)$, the following four options are possible: $V < K(1) \leq u$, $K(1) < V \leq u$, $V < u \leq K(1)$, and $u \leq V < K(1)$.

Options $V < K(1) \leq u$ and $V < u \leq K(1)$ can not be an equilibrium because $F(v)$ is continuous below u, and a seller offering the information for price V is better off by asking $K(1)$ and u correspondingly.

Consider option $K(1) < V \leq u$. There is a non-zero probability of offers $v \in [0, K(1)]$ and $v \in (K(1), u]$ because otherwise an agent with offer $K(1)$ will not get be able to get a better offer in the future. An offer from $[0, K(1)]$ is always accepted, and an offer from $(K(1), u]$ is accepted if and only if there are two offers; in the case of one offer the agent always waits for the second one. Because of the waiting the agents change their belief about event "the first informed agent behind the uninformed neighbor got an offer above $K(1)$," which is impossible in stationary equilibrium. Therefore, $K(1) < V \leq u$ is not an equilibrium.

Consider option $u \leq V < K(1)$. Offers V and $K(1)$ have the same chance to be accepted (the neighbor's neighbor is uninformed and stays uninformed till the next round), but $K(1)$ delivers a higher payoff. Therefore, this is also not an equilibrium.

We have proved that either $V = K(1) < u$ or $u < V = K(1)$. Later in the proof we will always use V instead of $K(1)$, The support of $F(v)$ below u constitutes a connected set; if not, an agent can increase his expected payoff by increasing the offer in the gap as the probability of the deal does not change.

There are 3 cases: $F(u) = 0$, $F(u) \in (0, 1)$, and $F(u) = 1$. If $F(u) < 1$, then the distribution of prices $F(v)$ has mass $1 - F(u)$ at $V > u$. If $F(u) > 0$, then the expected payoff maximization problem for $v \in [0, \min(u, V)$ gives

$$(1-p)v + pv(1 - F(v)) \to \max; \tag{1.17}$$

$$1 - pF(v) - pvf(v) = 0.$$

Applying Lemma 1.1,

$$F(v) = \frac{1}{p} - \frac{C}{v} \qquad \text{for } v \in [pC, \min(u, V)].$$

Offer pC is always accepted, and offer V is accepted only of there is the neighbor does not have other offer. Both these prices are in the support of $F(\cdot)$ and deliver the same expected payoff, therefore $pC = (1-p)V$, and

$$F(v) = \frac{1}{p} - \frac{(1-p)V}{pv} \qquad \text{for } v \in [(1-p)V, \min(u, V)].$$

Now we want to find $F(v)$ for each of the three cases.

Case 1. $F(u) = 0$, pure strategy with mass 1 at $V > u$.

Denote this distribution function of offers by $F^p(v)$. In accordance with Proposition 1.3, $V = \frac{u}{1-\delta(1-p)^2}$. This equilibrium exists if and only if the agents do not want to offer price u which is always accepted, i.e.

$$V(1-p) \geq u;$$

$$p \leq \delta(1-p)^2; \tag{1.18}$$

Case 2. $F(u) \in (0, 1)$, some mass at $V > u$ and a continuous part on $[(1-p)V, u]$.

Denote this distribution function of offers by $F_1^m(v)$.

An agent with offer V and one uninformed neighbor is indifferent between accepting the offer and waiting for another one. The expected payoff from buying the information immediately equals

$$-V + u + \delta(1-p)^2 V. \tag{1.19}$$

If the agent waits for another offer, he gets the information if and only if his another neighbor offers the information for a price $v \le u$. The expected payoff from waiting is

$$\begin{aligned}
\mathbf{E}\left(\sum_{t\ge 1}\delta^t p(1-p)^{t-1}(u-v)I_{\{v\le u\}}\right) &= \frac{p\delta}{1-\delta(1-p)}\left(F(u)u-\int\limits_{(1-p)V}^{u} v\,dF(v)\right)\\
&= \frac{(1-p)V\delta}{1-\delta(1-p)}\left(\frac{u}{(1-p)V}-1-\int\limits_{(1-p)V}^{u}\frac{dv}{v}\right)\\
&= \frac{(1-p)V\delta}{1-\delta(1-p)}\left(\frac{u}{(1-p)V}-1-\ln\left(\frac{u}{(1-p)V}\right)\right).
\end{aligned}$$

Equating the expected payoff of from buying the information immediately (formula 1.19) and waiting, one gets equation 1.5:

$$\frac{u}{(1-p)V}+\delta(1-p)-\frac{1}{1-p}=\frac{\delta}{1-\delta(1-p)}\left(\frac{u}{(1-p)V}-1-\ln\left(\frac{u}{(1-p)V}\right)\right). \quad (1.20)$$

The condition for the existence of such equilibrium $F(u)\in(0,1)$ is equivalent to $x\equiv\frac{u}{(1-p)V}\in\left(1,\frac{1}{1-p}\right)$. Rewriting equation 1.20 using x gives

$$x-\frac{\delta}{1-\delta(1-p)}(x-1-\ln x)=\frac{1}{1-p}-\delta(1-p). \quad (1.21)$$

Denote the left-hand side of equation 1.21 by $h(x,\delta,p)$. For any $x\in\left[1,\frac{1}{1-p}\right]$ the derivative

$$\frac{\partial h(x,\delta,p)}{\partial x} = 1-\frac{\delta(1-1/x)}{1-\delta(1-p)}\ge 1-\frac{\delta(1-(1-p))}{1-\delta(1-p)}=\frac{1-\delta}{1-\delta(1-p)}>0;$$

Therefore, there exists $x\in\left(1,\frac{1}{1-p}\right)$ satisfying equation 1.21 if and only if

$$h(1,\delta,p)<\frac{1}{1-p}-\delta(1-p)<h\left(\frac{1}{1-p},\delta,p\right); \quad (1.22)$$

$$1<\frac{1}{1-p}-\delta(1-p)<\frac{1}{1-p}-\frac{\delta}{1-\delta(1-p)}\left(\frac{p}{1-p}-\ln\frac{1}{1-p}\right).$$

Therefore, this equilibrium exists if and only if the following two inequalities hold:

$$p>\delta(1-p)^2; \quad (1.23)$$

$$-(1-\delta(1-p))(1-p)^2+p+(1-p)\ln(1-p)<0. \quad (1.24)$$

Case 3. $F(u)=1$, $\mathrm{supp}(F(v))=[(1-p)V,V]$, $V\le u$.

Denote this distribution function of offers by $F_2^m(v)$. A buyer is indifferent between buying at the maximal price V (formula 1.19) and waiting:

$$\begin{aligned}
-V+u+\delta(1-p)^2V &= \sum_{t\geq 1}\delta^t p(1-p)^{t-1}\int (u-v)\,dF(v) = \frac{(1-p)V\delta}{1-\delta(1-p)}\int\limits_{(1-p)V}^{V}\frac{u-v}{v^2}\,dv \\
&= \frac{(1-p)V\delta}{1-\delta(1-p)}\left(\left(-\frac{u}{V}+\frac{u}{(1-p)V}\right)+\ln(1-p)\right) \\
&= \frac{p\delta u}{1-\delta(1-p)}+\frac{(1-p)V\delta\ln(1-p)}{1-\delta(1-p)};
\end{aligned}$$

$$V=\frac{u(1-\delta)}{(1-\delta(1-p))(1-\delta(1-p)^2)+\delta(1-p)\ln(1-p)}.$$

This equilibrium exists if and only if $V\leq u$, or

$$-(1-\delta(1-p))(1-p)^2+p+(1-p)\ln(1-p)\geq 0. \tag{1.25}$$

We want to show that for any $\delta\in(0,1)$ interval $(0,1)$ is divided into three parts by $p'\in(0,1)$ and $p''\in(p',1)$. On $(0,p']$ inequality 1.18 holds (Case 1), on (p',p'') inequalities 1.23 and 1.24 hold (Case 2), and on $[p'',1)$ inequality 1.25 holds (Case 3).

Inequality 1.18 holds on $(0,p']$ and inequality 1.23 holds on $(p',1)$, where

$$p'=\frac{2\delta+1-\sqrt{4\delta+1}}{2\delta}.$$

Denote left-hand side of inequalities 1.24 and 1.25 as $g(p,\delta)$. The second derivative

$$\begin{aligned}
\frac{\partial^2}{\partial p^2}\left(\frac{g(p,\delta)}{1-p}\right) &= \frac{\partial}{\partial p}\left((1-\delta(1-p))-\delta(1-p)+\frac{1}{1-p}+\frac{p}{(1-p)^2}-\frac{1}{1-p}\right) \\
&= 2\delta+\frac{1}{(1-p)^2}+2\frac{p}{(1-p)^3}>0,
\end{aligned}$$

therefore $\frac{g(p,\delta)}{1-p}$ either increases or first decreases and then increases. As $g(0,\delta)<0$ and $g(0,\delta)>0$, equation $g(p,\delta)=0$ has exactly one solution $p''\in(0,1)$, and on $(0,p'')$ inequality 1.24 holds, and on $[p'',1)$ inequality 1.25 holds. The only fact we have to prove is that $p''>p'$. To do this, it is enough to show that there exists p satisfying both inequalities in 1.22.

We know that $h(1,\delta,p')<h\left(\frac{1}{1-p'},\delta,p'\right)$, $h(1,\delta,p)=1$, $h(1,\delta,p')=\frac{1}{1-p'}-\delta(1-p')$, and function $h(x,\delta,\tilde{p})$ is continuous in all arguments. The middle part of 1.22 increases with p because

$$\frac{1}{1-p}-\delta(1-p)=(1-\delta)+p(1+\delta)+p^2+p^3+\ldots.$$

Therefore, in some neighborhood of p' for $p > p'$ both inequalities in 1.22 hold, and therefore $p'' > p'$.

Value V_2^m decreases with p because $p > \delta(1-p)^2$ and therefore

$$\begin{aligned}\frac{d}{dp}\left(\frac{u(1-\delta)}{\delta V_2^m}\right) &= 1-\delta(1-p)^2 + 2(1-p)(1-\delta(1-p)) - 1 - \ln(1-p)\\ &\geq 1-p-1-\ln(1-p) \geq 0.\end{aligned}$$

Value V_1^m decreases with p because $p > \delta(1-p)^2$ and therefore

$$\begin{aligned}\frac{d}{dp}\left(\frac{u(1-\delta)}{\delta V_2^m}\right) &= 1-\delta(1-p)^2 + 2(1-p)(1-\delta(1-p)) - 1 - \ln(1-p)\\ &\geq 1-p-1-\ln(1-p) \geq 0.\end{aligned}$$

□

Proof of proposition 1.5.

The agent will get two offers simultaneously only if the informed agents on the opposite sides are located on the same distance. Therefore, the probability of staying uninformed forever equals

$$\sum_{t\geq 0}((1-p)^t p)^2(1-F(u))^2 = p^2(1-F(u))^2\frac{1}{1-(1-p)^2} = \frac{p(1-F(u))^2}{2-p}.$$

□

Proof of proposition 1.6.

We will denote all the strategies in the game with horizon T by upper index T. We are looking for the finite horizon equilibria at which the agents with one informed neighbor only always buy the information, i.e. $V_t^T = K_t^T(1)$.

At the last period $K(1) = u$ and therefore $F_T^T(u) = 1$,. The seller's problem is the same as problem 1.17, which means that the solution is also the same:

$$F_T^T(v) = \frac{1}{p}\left(1 - \frac{u(1-p)}{v}\right), \qquad v \in [(1-p)u, u].$$

The expected payoff of the agent who gets only one offer is

$$\pi \equiv \int_{(1-p)u}^{u} (u-v)\,dF_T^T(v) \leq \int_{(1-p)u}^{u} (u-(1-p)u)\,dF_T^T(v) = pu.$$

Suppose that for any $t < T$ distribution function $F_t^T(v)$ has mass 1 at $V = K_t^T(1) > u$ and does not have the continuous part below u. Then to make the buyer indifferent between buying and waiting till the last period the following equation should hold:

$$-V_t^T + u + \delta(1-p)^2 V_{t+1}^T = \delta^{T-t}(1-p)^{T-t-1}p\pi \tag{1.26}$$

Note that

$$V_{T-1}^{T} = -\delta p\pi + u + \delta(1-p)(1-p)u > u(1 + \delta(1-p)^2 - p).$$

Note that $V_T^T < \frac{u}{1-\delta(1-p)^2}$. By induction,

$$V_t^T = -\delta^{T-t}(1-p)^{T-t-1}p\pi + u + \delta(1-p)^2 V_{t+1}^T < u + \frac{\delta(1-p)^2 u}{1-\delta(1-p)^2} = \frac{u}{1-\delta(1-p)^2}.$$

Note that $V_{T-1}^T > V_T^T$. By induction, $V_{t-1}^T > V_t^T$ because

$$\begin{aligned} V_{t-1}^T &= -\delta^{T-(t-1)}(1-p)^{T-(t-1)-1}p\pi + u + \delta(1-p)^2 V_t^T \\ &> -\delta^{T-t}(1-p)^{T-t-1}p\pi + u + \delta(1-p)^2 V_{t+1}^T = V_t^T. \end{aligned}$$

No seller will deviate from V_t^T because the expected payoff from V_t^T is greater than the expected payoff from u:

$$(1-p)V_t^T \geq (1-p)V_{T-1}^T > (1-p)(u(1+\delta(1-p)^2) - pu) > u,$$

where the last inequality holds for small enough p.

Finally, for any t values V_t^T converge as T increases because V_t^T are limited, increase, and $V_{t+1}^{T+1} = V_t^T$. Denote $V_t = \lim_{T\to\infty} V_t^T$. Then

$$V - V_t^T = \delta(1-p)^2(V - V_{t+1}^T) + \delta^{T-t}(1-p)^{T-t-1}p\pi;$$

$$V - V_t = \delta(1-p)^2(V - V_{t+1}).$$

V_t are limited, and have the same law of motion as V_t for the pure strategy equilibrium, therefore $V_t = \frac{u}{1-\delta(1-p)^2}$ for any t.

□

Proof of proposition 1.7.

The structure of the proof is the following. First, we show that $F_t(\cdot)$ does not have mass points and has a connected support for any t. Second, we prove that

$$F_t(v) = \frac{1}{p_t} - C_t v^{-\frac{1}{M-1}}. \tag{1.27}$$

and find formula for the support (formulas 1.10 and 1.11). Third, we prove the law of motion for V_t:

$$\frac{1}{\delta}(V_{t-1} - u) = V_t g_t - u. \tag{1.28}$$

Forth, based on the law of motion for V_t we establish formulas 1.12 and 1.13 for V_t. And last, we show monotonicity and convergence of V_t and convergence of $\mathbf{E}_{F_t} v$.

Step 1. Properties of $F_t(\cdot)$.

No offer above V_t will be accepted, therefore $F_t(V_t) = 1$. The distribution function $F_t(\cdot)$ does not have mass points because otherwise a seller would prefer to decrease his offer from these mass points by some small ϵ.

The support $\text{supp}(F_t(\cdot))$ is connected because by increasing the offer in the gap, a seller will increase his expected payoff as the acceptance probability of the offer stays the same, and the price increases.

Step 2. The proof of formulas 1.10 and 1.11 for $F_t(\cdot)$ and for its support.

The expected payoff from one uninformed neighbor is equal to

$$\begin{aligned}\pi_t(v) &= v\,\mathbf{P}\{v \leq \text{other offers }\} \\ &= v\prod_{i=1}^{M-1}\left(\mathbf{P}\{v \leq \text{offer from neighbor } i\} + \mathbf{P}\{\text{no offer from neighbor } i\}\right) \\ &= v\prod_{i=1}^{M-1}\left((1-F_t(v))p_t + (1-p_t)\right) = (1-p_tF_t(v))^{M-1}\,v.\end{aligned}$$

All the points in the support of $F_t(\cdot)$ should deliver the same utility π, we have

$$\pi_t'(v) \equiv (1-p_tF_t(v))^{M-1} - p_tf_t(v)v(1-p_tF_t(v))^{M-2}(M-1) = 0;$$

$$p_tf_t(v)v(M-1) = 1 - p_tF_t(v).$$

Applying Lemma 1.1,

$$F_t(v) = \frac{1}{p_t} - C_tv^{-\frac{1}{M-1}}$$

for some constant $C_t > 0$ (formula 1.27).

One can verify that

$$V_t \equiv \sup\operatorname{supp} F_t(\cdot) = \left(\frac{p_tC_t}{1-p_t}\right)^{M-1},$$

therefore $C_t = \frac{1-p_t}{p_t}V_t^{\frac{1}{M-1}}$ and we have proved formula 1.10 for $F_t(\cdot)$ and formula 1.11 for the support of $F_t(\cdot)$.

Step 3. Law of motion for V_t (formula 1.28).

Let U_t^i be the expected payoff of an informed agent at the beginning of period t, and let U_t^u be the expected payoff of an uninformed agent at the beginning of period t. Then

$$U_t^i = M(1-p_t)\pi_t + \delta U_{t+1}^i = \sum_{i=t}^{\infty}\delta^{i-t}M(1-p_i)\pi_i; \tag{1.29}$$

$$U_t^u = (u - \mathbf{E}\,v_t + \delta U_{t+1}^i)(1-(1-p_t)^M) + \delta U_{t+1}^u(1-p_t)^M, \tag{1.30}$$

where $\mathbf{E}\,v_t$ stands for the expected price an agent pays for acquiring the information at period t, conditional on the fact that there is at least one offer.

In an equilibrium the buyer with the highest possible offer V_t is indifferent between accepting the offer and waiting, therefore

$$u - V_t + \delta U^i_{t+1} = \delta U^u_{t+1}. \tag{1.31}$$

Substituting expressions for U^i_t (formula 1.29) and U^u_t (formula 1.30) into 1.31 one can get

$$\begin{aligned} \frac{1}{\delta}(V_{t-1} - u) &= M(1-p_t)\tilde{V}_t + \delta U^i_{t+1} \\ &\quad -(u - \mathbf{E}\, v_t + \delta U^i_{t+1})(1-(1-p_t)^M) - \delta U^u_{t+1}(1-p_t)^M \\ &= (V_t - u)(1-p_t)^M + M(1-p_t)^M V_t + (\mathbf{E}\, v_t - u)(1-(1-p_t)^M) \\ &= (M+1)V_t(1-p_t)^M + \mathbf{E}\, v_t(1-(1-p_t)^M) - u. \end{aligned} \tag{1.32}$$

In order to simplify expression 1.32, we need formula for $\mathbf{E}\, v_t$.

The average minimal price from l independent offers $v, v_2, \ldots, v_l$ is equal to

$$\begin{aligned} \mathbf{E}_{F_{lt}}\, v \equiv \int v\, d\, \mathbf{P}(\min(v_1, \ldots, v_l) \leq v) &= \int vd \left(1 - (1 - \mathbf{P}(v_1 \leq v))^l\right) \\ &= \int vd \left(1 - \left(h_t(v) - \left(\frac{1}{p_t} - 1\right)\right)^l\right) \\ &= \int \frac{h_t(v) l}{M-1} \left(h_t(v) - \frac{1}{p_t} + 1\right)^{l-1} dv, \end{aligned} \tag{1.33}$$

where $h_t(v) = \frac{1-p_t}{p_t}\left(\frac{V_t}{v}\right)^{\frac{1}{M-1}}$ for simplicity of notation. Note that

$$\begin{aligned} \sum_{l=1}^{M} l x^{l-1} C^l_M p^l_t (1-p_t)^{M-l} &= M p_t \sum_{l=0}^{M-1} C^l_{M-1} x^l p^l_t (1-p_t)^{(M-1)-l} \\ &= M p_t \,(px + 1 - p_t)^{M-1} \end{aligned} \tag{1.34}$$

Equation 1.34 for $x = h_t(v) - \frac{1}{p_t} + 1$ gives

$$\sum_{l=1}^{M} l \left(h_t(v) - \frac{1}{p_t} + 1\right)^{l-1} C^l_M p^l_t (1-p_t)^{M-l} = M p_t (h_t(v) p_t)^{M-1}. \tag{1.35}$$

Combining 1.33 and 1.35 and swapping the integral and the sum, one can get

$$\begin{aligned}
\sum_{l=1}^{M} C_M^l p_t^l (1-p_t)^{M-l}\,\mathbf{E}_{F_{lt}} v &= \int\limits_{V_t(1-p_t)^{M-1}}^{V_t} \frac{M}{M-1}(h_t(v)p_t)^M\,dv \\
&= \int\limits_{V_t(1-p_t)^{M-1}}^{V_t} \frac{M}{M-1}\left((1-p_t)\left(\frac{V_t}{v}\right)^{\frac{1}{M-1}}\right)^M dv \\
&= V_t \int\limits_{(1-p_t)^{M-1}}^{1} \frac{M}{M-1}(1-p_t)^M v^{-\frac{1}{M-1}-1}\,dv. \qquad (1.36)
\end{aligned}$$

Taking into account the fact that the probability of exactly l informed neighbors is equal to $C_M^l p_t^l (1-p_t)^{M-l}$ and applying equation 1.36, we have

$$\begin{aligned}
\mathbf{E}\, v_t(1-(1-p_t)^M) &= \sum_{l=1}^{M} C_M^l p_t^l (1-p_t)^{M-l}\,\mathbf{E}_{F_{lt}} v \\
&= V_t \int\limits_{(1-p_t)^{M-1}}^{1} \frac{M(1-p_t)^M}{M-1} v^{-\frac{1}{M-1}-1}\,dv \\
&= -V_t\, M(1-p_t)^M v^{-\frac{1}{M-1}}\Big|_{(1-p_t)^{M-1}}^{1} = V_t M p_t (1-p_t)^{M-1} \qquad (1.37)
\end{aligned}$$

Substituting $\mathbf{E}\, v_t(1-(1-p_t)^M)$ (formula 1.37) into formula 1.32 and taking definition for g_t (formula 1.9), we have the law of motion for V_t (formula 1.28).

Stage 4. Finding expression for V_t (formulas 1.12 and 1.13).

Rearranging terms in formula 1.28, one can get

$$V_t = \frac{V_{t-1}}{\delta g_t} - \frac{u(1-\delta)}{\delta g_t}. \qquad (1.38)$$

Formula 1.38 for $t = 2$ corresponds to the expression for V_t (formula 1.12). Using formula 1.38 again,

$$\begin{aligned}
V_{t+1} &= \frac{V_1 \prod_{i=2}^{t} \frac{1}{\delta g_i} - u(1-\delta)\sum_{i=2}^{t}\prod_{j=i}^{t}\frac{1}{\delta g_j}}{\delta g_{t+1}} - \frac{u(1-\delta)}{\delta g_{t+1}} \\
&= V_1 \prod_{i=2}^{t+1} \frac{1}{\delta g_i} - u(1-\delta)\sum_{i=2}^{t}\prod_{j=i}^{t+1}\frac{1}{\delta g_j} - \frac{u(1-\delta)}{\delta g_{t+1}} \\
&= V_1 \prod_{i=2}^{t+1} \frac{1}{\delta g_i} - u(1-\delta)\sum_{i=2}^{t+1}\prod_{j=i}^{t+1}\frac{1}{\delta g_j},
\end{aligned}$$

which by induction proves formula 1.12 for any $t > 2$. Now we want to prove formula 1.13 for V_1. Expressing V_1 through V_t using formula 1.12 , we get

$$\begin{aligned} V_1 &= \left(V_t + u(1-\delta) \sum_{i=2}^{t} \prod_{j=i}^{t} \frac{1}{\delta g_j} \right) \prod_{i=2}^{t} \delta g_i \\ &= \left(V_t \prod_{i=2}^{t} \delta g_i + u(1-\delta) \left(1 + \sum_{i=3}^{t} \prod_{j=2}^{i-1} \delta g_j \right) \right). \end{aligned} \tag{1.39}$$

Values V_t are limited by some constant because of the transversality condition, therefore

$$\lim_{t\to\infty} V_t \prod_{i=2}^{t} \delta g_i = 0$$

as $\lim_{t\to\infty} g_t \equiv \lim_{t\to\infty}(1-p_t)^{M-1}(M+1-p_t) = 0$. (Probabilities p_t converge to 1.) Therefore, taking limits both parts of 1.39 for $t \to \infty$, one gets formula 1.13 for V_1.

To prove $V_1 < \infty$ notice that as values $g_t = (1-p_t)^{M-1}(M+(1-p_t))$ converge to 0, for any $\epsilon \in (0,1)$ there exists t_0 such that $g_t < \epsilon$ for any $t > t_0$. Therefore,

$$\left| \frac{V_1}{u(1-\delta)} - \left(1 + \sum_{i=3}^{t_0} \prod_{j=2}^{i-1} \delta g_j \right) \right| = \sum_{i=t_0+1}^{\infty} \prod_{j=2}^{i-1} \delta g_j < \prod_{j=2}^{t_0} \delta g_j \sum_{i=t_0}^{\infty} \epsilon^{i-t_0} < \infty.$$

Step 5. Properties of V_t and $\mathbf{E}_{F_t} v$.
Find expression for V_t in terms of p_t and g_t:

$$\begin{aligned} V_t &= V_1 \prod_{i=2}^{t} \frac{1}{\delta g_i} - u(1-\delta) \sum_{i=2}^{t} \prod_{j=i}^{t} \frac{1}{\delta g_j} \\ &= u(1-\delta) \left(1 + \sum_{i=3}^{\infty} \prod_{j=2}^{i-1} \delta g_j \right) \prod_{i=2}^{t} \frac{1}{\delta g_i} - u(1-\delta) \sum_{i=2}^{t} \prod_{j=i}^{t} \frac{1}{\delta g_i} \\ &= u(1-\delta) \left(1 + \sum_{i=t+2}^{\infty} \prod_{j=t+1}^{i-1} \delta g_j \right). \end{aligned}$$

Values g_t decrease with time to zero. Therefore,

$$\sum_{i=t+2}^{\infty} \prod_{j=t+1}^{i-1} \delta g_j < \sum_{i=t+2}^{\infty} \prod_{j=t+1}^{i-1} \delta g_{j-1} = \sum_{i=(t-1)+2}^{\infty} \prod_{j=(t-1)+1}^{i-1} \delta g_j,$$

and V_t decreases with t. Also, if $g_j < g$ for any $j > t$, then

$$\sum_{i=t+2}^{\infty} \prod_{j=t+1}^{i-1} \delta g_j \le \sum_{i=t+2}^{\infty} (\delta g)^{i-(t+1)} = \frac{\delta g}{1-\delta g},$$

and $V_t \to u(1-\delta)$ as $g_t \to 0$.

The average price at period t for $M > 2$

$$\begin{aligned}
\mathbf{E}_{F_t} v &= \int_{\tilde{V}_t}^{V_t} v \, d\left(\frac{1}{p_t} - C_t v^{-\frac{1}{M-1}}\right) = \int_{\tilde{V}_t}^{V_t} \frac{C_t}{M-1} v^{-\frac{1}{M-1}} \, dv = \frac{C_t}{M-2} v^{\frac{M-2}{M-1}} \Big|_{\tilde{V}_t}^{V_t} \\
&= \frac{V_t^{\frac{1}{M-1}}}{M-2} \frac{1-p_t}{p_t} \left(1 - (1-p_t)^{M-2}\right) V_t^{\frac{M-2}{M-1}} \overset{t\to\infty}{\longrightarrow} 0,
\end{aligned}$$

and the average price for $M = 2$

$$\begin{aligned}
\mathbf{E}_{F_t} v &= \int_{\tilde{V}_t}^{V_t} \frac{C_t}{M-1} v^{-\frac{1}{M-1}} \, dv = \frac{C_t}{M-1} \ln v \Big|_{\tilde{V}_t}^{V_t} = \\
&= \frac{V_t^{\frac{1}{M-1}}}{M-2} \frac{1-p_t}{p_t} \left(1 - (M-1)\ln(1-p_t)\right) \ln V_t \overset{t\to\infty}{\longrightarrow} 0.
\end{aligned}$$

□

Chapter 2

A THEORY OF CONTINUUM ECONOMIES WITH IDIOSYNCRATIC SHOCKS AND RANDOM MATCHINGS

2.1 Introduction

The Problem

A large number of economic models consider an uncountable number of negligible agents who experience idiosyncratic shocks and randomly meet each other. Some examples of such models are given by Aliprantis et al. (2), Alós-Ferrer (6), and Boylan (14). The assumptions in these models are made in the spirit of the Law of Large Numbers. For example, the mixing property assumption is often used, which states that the fraction of the agents from one set who are matched with the agents from another set equals the measure of the second set.[1] Until recently there was no formal mathematical evidence that such models exist. Moreover, some economists pointed out serious contradictions among the standard assumptions used.

The existence of many economic models with both idiosyncratic shocks and random matchings[2] makes it impossible to discuss all the discrepancies that arise. Therefore, we give here only two most famous examples of contradictions in the standard assumptions.

One of the most famous contradictions was described by Judd (34).[3] It has been assumed that the average of a continuum of independent and identically distributed random variables (idiosyncratic shocks) is nonrandom (no aggregate uncertainty property). Judd shows that the nonrandomness of the average neither contradicts nor follows from the independence and identical distribution of the shocks. Moreover, Judd proves that if the population is represented by the unit interval with the Borel σ-algebra, then most of the shock realizations on the agent space are not measurable and therefore the average shock cannot be calculated. Feldman and Gills (24) notice that measurable functions on the unit interval are in some sense "almost continuous." However, "almost continuity"

[1]Shi (41) describes the following matching process with the mixing assumption: "... the distribution of different types of matches for each household is almost surely nonrandom, although each member in the household is uncertain about the kind of agent he will meet."

[2]The word "meeting" will be used with respect to one agent. The one-time process of all the agents of the population being paired up with each other will be called "matching."

[3]This and the random matching example will be considered in details in Section 2.3.3 "Standard Model Inconsistencies."

of shock realizations contradicts the independence of shocks for different agents implied by the idiosyncratic nature of the shocks.[4]

A second contradiction is about random matching assumptions. It was described by McLennan and Sonnenschein (37) in footnote 4. Using Proposition 1 from Feldman and Gilles (24), the authors show that a measure preserving matching can not be mixing. The source of the contradiction is also measurability: the agents who are "close" to each other on the agent space should be paired up with the agents who are also "close" to each other,[5] which contradicts the randomness of the meetings.

There exist several solutions to the idiosyncratic shocks problem (see the literature review for the details.) There are also several papers dealing with the random matching problem. At the same time, to our knowledge, there is no paper dealing with both idiosyncratic shocks and random matching. Furthermore, no existing paper completely justifies the mixing property of a random matching simultaneously for all the measurable subsets of the agents, which is the cornerstone of many economic models. In this paper we build a mathematically valid discrete time model of idiosyncratic shocks and random matchings, which resolves the conflicting issues.

The Causes of the Problem

All the contradictions among the standard assumptions stem from an uncountable number of the agents. The very idea of considering a non-atomic agent space comes from the convenient properties of a finite but large population of agents. For example, because of the Law of Large Numbers, in a large but finite population the average shock is close to the average shock for each agent. Thus, by using a continuum of agents, one wants to achieve two important goals. The first one is to have negligible agents who have no influence on the aggregate characteristics of the economy. The second goal is to have an analogue of the Law of Large Numbers with respect to shocks and meetings.[6]

A replacement of a large but finite agent space with an uncountable space of negligible agents has several hidden problems arising from significant differences in the structures of these spaces. For a finite space, the measure comes naturally (the counting measure) and in the unique way. One-to-one matching automatically guarantees the measure preserving property. The natural discrete σ-algebra consists of all the subsets, making all the functions measurable. For an uncountable space, the choice of the measure is ambiguous. Additional assumptions should be made to guarantee the measure preserving property of the matching. Measurability seriously restricts the set of functions and matchings.[7]

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[4] Measurable functions on the unit interval have a good approximation by the continuous functions (Luzin's theorem); Al-Najjar (1) in Theorem 8 shows that "a typical realization of an i.i.d. process can not be approximated by a continuous function."

[5] As Alós-Ferrer (6) puts it, "the very concept of matching destroys the most basic independence aspirations."

[6] As Gilboa and Matsui (29) wrote, "...there are uncountably many agents of various types, each of which has no effect whatsoever on the aggregate behavior, thus eliminating strategic considerations which extend beyond a specific encounter." Alós-Ferrer (4) notices that "...oftentimes, agents have to be modeled as being negligible." Alós-Ferrer (6) also mentions convenience in using a continuum of agents because of analytical simplicity and anonymity properties.

[7] Al-Najjar (1) discusses the measurability problem in Section 4. As he notices, this problem emerges because the agents who are ex ante similar have to have independent shocks ex post.

The Main Idea

The existing models assume that the agents know their identity — the exact location on the agent space. For example, the requirement that the shocks are independent for different agents is only necessary if the agents know who they are. Because of the known identity, the standard setup requires the space of the states of the world, which allows us to model the uncertainty the agents face about their future shocks and meetings. Therefore, two different spaces are needed: the space of the agents and the space of the states of the world. The shocks and matchings are defined on their product.

However, in many economic models the knowledge of identity is excessive and has no influence on the results. What the models really require is that the agents have their own attributes, like type, initial endowment, etc.[8] Another requirement is that the agents perceive their future as random.[9] With these assumptions, the agents use strategies which depend only on the history and some initial attributes.[10] The idea of such history-dependent strategies implies that the agents do not know their identities; otherwise they would include it in their strategies.

Based on the idea of unknown identity, we suggest to simplify the setup and consider only one space — the agent space, which also serves as the probability space. With this approach the agents still face some randomness. It does not come from the realization of the state of the world, but from the unknown identity. Each agent believe that he was placed randomly (and uniformly) on the agent space. The shocks and meetings are predetermined.[11] However, the agents do not know their identities and therefore perceive thir shocks and meetings as random. The events help the agents to refine the knowledge of their identities, but do not resolve it completely. Based on the previous shocks and meetings, the agents update their beliefs about the future.[12]

The Results

The main result of the paper (Theorem 2.2) states that there exists a mathematically correct dynamic discrete time model of negligible agents with idiosyncratic shocks and random matchings. The shocks and matchings are measurable. The matchings are measure preserving. For any agent his shocks and meetings are independent of the past events. The equivalent of the Law of Large Numbers holds with respect to the σ-algebras generated by the histories.

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[8] Kandori (35) describes the following rules: "1. A label is attached to each agent. 2. Before executing trade each agent observes his and his partner's label. 3. A player and his partner's actions and labels today determine their labels tomorrow."

[9] Gale (28) requires the following form of randomness: "the probability of an active agent meeting an agent whose history <belongs to some set> is independent of the first agent's history."

[10] Green and Zhou (31) write: "an agent's strategy is a function of only his own trading history and initial money holdings..."

[11] This makes the shocks and meetings of different agents to be dependent. This idea is not new; it was explored in Feldman and Gilles (24) and Alós-Ferrer (4) and (6). Boylan (14) and Alós-Ferrer (6) showed that the matching scheme should be dependent on the assignment of types (shocks). However, in these papers the agents know their identities and therefore this dependence can influence the choice of the strategies.

[12] This approach is similar to the Kolmogorov definition of the probability space. Namely, in Kolmogorov's definition random variables are some measurable functions on the probability space Ω. The realization $\omega \in \Omega$ is not known to the observer(s); initially it is chosen at random. The observers(s) can update the belief about ω based on the events.

Many discrete time economic models with idiosyncratic shocks and random matchings can be reformulated in the new setup. The model is very flexible. It can incorporate some additional characteristics of the agents, such as type, product consumed, etc. It also permits shocks and matchings that depend on the agents' histories or other characteristics.

Plan of the Paper

The rest of the paper is organized as follows. Section 2 contains the literature review. Section 3 discusses the standard assumptions used in economic models and shows the problems arising from the use of these assumptions. In section 4, we give the formal definition and prove the existence of idiosyncratic shocks and random matchings. Section 5 concludes. All the proofs are given in the Appendix.

2.2 Literature Review

All the related literature naturally falls into two different categories: idiosyncratic shocks and random matchings.

2.2.1 Idiosyncratic Shocks

Judd (34) and Feldman and Gilles (24) were among the first economists to notice that for the unit interval of agents with the Borel σ-algebra and Lebesgue measure either the shocks are not measurable across the population, or the equivalent of the Law of Large Numbers does not hold.[13] Several remedies to the problem of idiosyncratic shocks were offered by Judd, Feldman and Gilles, and other economists.

Judd (34) solves the problem of idiosyncratic shocks on the whole agent space. The idea is to build an extension of the agent space so that the Law of Large Numbers is satisfied. The author notices that the extension is not unique, and for other extensions the Law of Large Numbers might fail.[14] Feldman and Gilles (24) suggest relaxing the assumption of shocks independence or considering finite or countably infinite approximations of the agent space. A countably infinite population provides an appropriate idealization of a large economy. The authors show that there exists a density charge on the agent space such that the Law of Large Numbers is satisfied. Al-Najjar (1) also considers a sequence of finite but increasingly large economies. The continuum-like laws, including the Law of Large Numbers for any subinterval of the $[0, 1]$ set of agents, hold in important aspects, although the measure is not countably additive and the integral might not coincide with the Lebesgue integral.

[13] Judd proves that the measure of the realizations for which the sample distribution function on the unit interval of agents does not exist has inner measure zero and outer measure one. He also shows that there is an extension of the whole space for which the Law of Large Numbers holds for the whole agent space. Feldman and Gilles show that there is no extension for which "a law of large numbers is valid simultaneously for all members of σ-algebra." As Sun (42) mentions, Doob (20) noticed long ago that "the sample functions of such a process are usually too irregular to be useful."

[14] Several other authors, including Green (30) and Alós-Ferrer (4), mention non-uniqueness of the extension. From our point of view, non-uniqueness does not really constitute a problem. The goal is to find a viable mathematical setup, and the question we want to answer is if we can justify using an uncountable number of agents."

Developing the idea of dependent shocks, Alós-Ferrer (4) suggests to consider a population extension. To illustrate the population extension, consider a randomly rotated circumference of agents which is naturally mapped onto the original circumference. The shock (chosen from a finite set) of an agent is the one that was originally at the point of the circumference before the rotation. The shocks in this model are not independent, but the Law of Large Numbers holds on the whole space.

Instead of the Lebesgue integral, Uhlig (43) uses the Pettis integral to calculate the average shock, which captures the idea of a countable normalized sum of shocks. On the unit interval of agents the Pettis integral is equivalent to the L_2-Riemann integral. If the shocks are pairwise uncorrelated and have the same mean and uniformly bounded variances, the Pettis integral exists. Uhlig shows that the Law of Large Numbers holds: the Pettis integral on any measurable subset of the agents is almost everywhere constant.

Green (30) changes the agent space by endowing the unit interval with a σ-algebra richer then the Borel σ-algebra. He constructs a family of iid variables on this nonatomic space so that for any subset of the population with a positive measure the Law of Large Numbers holds almost surely. Sun (42) uses hyperfinite Loeb spaces to demonstrate the same type of no aggregate uncertainty.[15] The results on the hyperfinite space can be routinely translated into large finite populations models.

2.2.2 Random Matching

The problems of random matching and idiosyncratic shocks have the same roots. Nevertheless, the random matching problem is more challenging because of the additional assumptions on the structure of the matching. Several authors suggested their remedies to the problem, which essentially use the same ideas as the solutions of the idiosyncratic shocks problem.

Gilboa and Matsui (29) were the first to approach the random matching problem. They consider a continuous time model with only a few individuals out of a countable population meeting at each period of time. Each individual meets someone only once. The probability measure in the model is finitely additive. The authors satisfy the properties of no aggregate uncertainty for any measurable set of the agents and randomness of the meetings. Boylan (14) also considers countably many agents. The Law of Large Numbers is formulated with respect to a finite set of agent types. The randomness implies that the type proportions evolve in accordance with the mixing property. The author develops a matchings scheme in which the mixing property holds, however the matching scheme can not be independent of the assignment of types. In (15) Boylan discusses the conditions under which a finite deterministic matching approximates a random matching process with continuous time as the population grows to infinity.

Alós-Ferrer (6) considers a random matching of a continuum of agents. He uses the tool of population extension developed in Alós-Ferrer (4) and proves the existence of a random matching satisfying the Law of Large Numbers properties with respect to a finite set of types. In (5), the author extends the results to several populations. Duffie

[15]Judd (34) suggested using hyperfinite discrete models from nonstandard analysis to solve the problem of idiosyncratic shocks.

and Sun (22) employ the framework of hyperfinite Loeb spaces (see Sun (42) above). The matching agents are assigned a finite number of types. The authors find an agent space and a random matching satisfying the following properties: measure preserving, uniform distribution, and mixing (for any two given sets of the agents the mixing property holds with probability one). For any two different agents their partners' types are independent. The matching scheme is independent in types for any assignment of types. In (21) the authors study the type distribution evolvement induced by the matchings and random mutations.

Although Aliprantis et al. (2) and (3) do not suggest any solution to the random matching problem, they use a set-theoretical approach to build a foundation for the random matching models. The basic object they consider is a *cluster* — several agents meeting at some period of time. The matching rule consists of non-intersecting k-element clusters. The stochastic matching rule is defined as a probability measure over all k-clustering matching rules. Although the authors mainly concentrate on finite-agent models, they define such general concepts as direct and indirect partners, along with anonymous and strongly anonymous sequences of k-clustering matching rules.

2.2.3 Why Further Solution?

Although the papers mentioned partially solve the problems of idiosyncratic shocks and random mmatchings, no solution guarantees no aggregate uncertainty/mixing property simultaneously for all the measurable subsets of the agent space (many of them also do not have other important characteristics, like countably additive measure, anonymous meetings, Lebesgue integral, independent shocks/meetings, etc.) No paper deals with both idiosyncratic shocks of a general distribution and random matchings.

Duffie and Sun (22) suggest a random matching in which the properties do not hold almost surely for all the subsets; they hold for a given set/sets of agents almost surely. For example, the mixing property is formulated in the following way: for any two sets of agents the mixing property (or an equivalent of it with respect to a finite set of types) holds with probability one. It means that with non-zero probability one may be able to find two sets for which the mixing property does not hold. McLennan and Sonnenschein in (37) emphasize that the properties should hold almost surely simultaneously on all the measurable subsets of the agents.[16]

2.3 Standard Assumptions

In this section we list the assumptions usually made about idiosyncratic shocks and random matchings. Particular assumptions may vary, therefore we give here the most general setup. For simplicity and without loss of generality, time is not taken into account in this section. After defining the assumptions, we discuss their possible alternative formulations. Several problems that arise under the standard assumptions conclude this section.

[16] For a discussion of different formulations of the properties, see subsection 2.3.2 "Different Formulations of the Properties." In this subsection we provide different formulations of the properties and discuss which formulation was used by which author.

2.3.1 The Model

Let A be the agent space with σ-algebra $\mathcal{A}$ and probability measure μ. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the state space. The assumption of an agent negligibility is usually made to ensure that no particular agent can influence the aggregate characteristics of the economy, i.e. measure μ is non-atomic. The unit interval $[0, 1]$ of the real line $\mathbb{R}$ is often taken as the agent space A.

Every agent $a \in A$ first experiences shock ξ and then meets with other agent in accordance with some rule $\mathbf{M}$. The rest of the assumptions describe those two objects and fall into three different categories: assumptions about the shocks (shocks are idiosyncratic), assumptions about the matching (meetings are random), and assumptions about the joint properties of the shocks and matching.

Idiosyncratic Shocks

An idiosyncratic shock[17] is a function $\xi : A \times \Omega \to \mathbb{R}$ such that the following properties hold:[18]

A1. Measurability: for any $a \in A$ function $\xi_a(\cdot)$ is $\mathcal{F}$-measurable; for any $\omega \in \Omega$ function $\xi_\omega(\cdot)$ is $\mathcal{A}$-measurable;

A2. Identical distribution: for any $a \in A$ shock $\xi_a(\cdot)$ has cdf $F(x)$;

A3. Independence: for any different $a_1, a_2, \ldots, a_l \in A$ corresponding shocks $\xi_{a_1}(\cdot)$, $\xi_{a_2}(\cdot), \ldots, \xi_{a_l}(\cdot)$ are independent;

A4. No aggregate uncertainty:[19] the sample distribution of the shock equals $F(x)$ for any positive-measured $B \in \mathcal{A}$, i.e.

$$\forall \omega \in \Omega,\ \forall x \in \mathbb{R} \quad \mu(\{a \in B : \xi_\omega(a) \leq x\}) = F(x)\mu(B).$$

We used the formulation of the no aggregate uncertainty property with respect to the distribution (Glivenko-Kantelly type). Judd (34), Green (30), Alós-Ferrer (4) use the same formulation. Some authors prefer to formulate it with respect to the averages, see Feldman and Gilles (24):[20]

$$\int\limits_B \xi_\omega(a)\, d\mu(a) = \mu(B) \int\limits_{\mathbb{R}} x\, dF(x).$$

[17] The singular form "shock" is used because this is just one function. Plural "shocks" will be used for many functions at different periods of time, or to refer to the shocks of different agents at one period of time. The exact meaning will be clear from the context.

[18] Some parts of this definition are given in many papers, including Alós-Ferrer (4), Feldman and Gilles (24), Green (30), Judd (34), and Sun (42).

[19] Sun (42) says that this property with respect to the average is often referred to as "aggregation removes uncertainty" and Feldman and Gilles say that "risks disappear in the aggregate." Alós-Ferrer (4) writes "individual uncertainty vanishes upon aggregation." Green (30) calls it the "idealized Glivenko-Cantelli property."

[20] Alós-Ferrer (4) calls this formulation "less demanding."

Random Matching

A random matching, which determines whom everyone meets for each $\omega \in \Omega$, is a mapping $\mathbf{M} : A \times \Omega \to A$ such that the following properties hold:[21]

B1. Everyone is met by someone:

$$\forall \omega \in \Omega \qquad \mathbf{M}_\omega(A) = A;$$

B2. No agent meets himself:

$$\forall \omega \in \Omega,\ \forall a \in A \qquad \mathbf{M}_\omega(a) \neq a;$$

B3. The partner's partner is the agent himself:

$$\forall \omega \in \Omega,\ \forall a \in A \qquad \mathbf{M}_\omega(\mathbf{M}_\omega(a)) = a;$$

B4. Measurability: for any $a \in A$ operator $\mathbf{M}_a(\cdot)$ is $\mathcal{F}$-measurable; for any $\omega \in \Omega$ operator $\mathbf{M}_\omega(\cdot)$ is $\mathcal{A}$-measurable;

B5. Measure preserving:

$$\forall \omega \in \Omega,\ \forall B \in \mathcal{A} \qquad \mu(\mathbf{M}_\omega(B)) = \mu(B);$$

B6. Uniform distribution: every agent has equal probability of meeting everyone else:[22]

$$\forall a \in A,\ \forall B \in \mathcal{A} \qquad \mathbf{P}(\{\omega : \mathbf{M}_a(\omega) \in B\}) = \mu(B);$$

B7. Independence: for any different $a_1, a_2, \ldots, a_l \in A$ random variables $\mathbf{M}_{a_1}(\cdot)$, $\mathbf{M}_{a_2}(\cdot), \ldots, \mathbf{M}_{a_l}(\cdot)$ are independent;[23]

B8. Mixing: the fraction of the agents from one set who meet the agents from another set equals the measure of the second set:

$$\forall \omega \in \Omega, \forall B_1, B_2 \in \mathcal{A} \qquad \mu(\mathbf{M}_\omega(B_1) \cap B_2) = \mu(B_1)\mu(B_2),$$

i.e. sets $\mathbf{M}_\omega(B_1)$ and B_2 are independent.[24]

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[21]Gale (28) and McLennan and Sonnenschein (37) define random matching in a similar way; Alós-Ferrer (6) and Boylan (14) give an analogous list of properties, including some properties with respect to types. Duffie and Sun (22) give the closest definition.

[22]Boylan (14) shows that in an economy with countably many agents it is impossible to satisfy this property directly; he therefore formulates it (Property II) with respect to types. Alós-Ferrer (6) requires that "given a fixed individual, any other agent were equiprobable as its partner." Alós-Ferrer (6) calls this property with respect to a finite set of types "Type's proportional law," and the general property "General proportional law." Duffie and Sun (22) give the closest formulation.

[23]Duffie and Sun (22) formulate this property with respect to a finite set of types.

[24]Boylan (14) formulates this property (Properties III or IV) with respect to a finite set of types. Alós-Ferrer (6) requires that "the proportion of matches between agents of two given types is equal to (twice) the product of the proportions of agents of those types." Alós-Ferrer (6) calls this property with respect to a finite set of types "Type's mixing", and the general property "Strong mixing." McLennan and Sonnenschein (37) and Duffie and Sun (22) give the closest formulation.

Joint Independence

Joint independence between shocks and meetings may be formulated in numerous ways.[25] Many papers have the following common element in the formulation: based on their past histories the agents can not infer any informative conclusion about current event. Different formulations of the property come from different kinds of information the agents remember or share during the meetings. For example, possible information available to the meeting parties can be the payment during the transaction, the identification of the partner, or the payments in all the partner's previous transactions, and so on.[26]

Assuming that the agents first learn their shock and then meet and learn their partner's shock, joint independence means that the partner's shock is independent of the agent's own shock. Suppose that the agents have types as in Boylan (14), Duffie and Sun (22), and Alós-Ferrer (6). We see the types as shocks. Boylan (14) considers the following form of independence: (1) for any agent his probability of being matched with an agent of some type equals the fraction of the agents of this type; (2) the fraction of the agent of one type who meet the agents of another type equals the product of the fractions of the agents for both types.[27] Boylan shows that a random matching scheme can not be independent of the assignment of types.[28] Duffie and Sun (22) require that (1) for almost every agent the probability of meeting a partner with some type equals the fraction of the agents of this type, and (2) for almost every agent his partner's shock is pairwise independent of other agent partner's shocks for almost all other agents. Alós-Ferrer (6) considers the following forms of independence: (types' proportional law) for all agents the probability of meeting a partner with some type equals the fraction of the agents of this type; (types' mixing) the fraction of the agent of one type who meet the agents of another type equals the product of the fractions of the agents for both types.

2.3.2 Different Formulations of the Properties

We formulated the properties for the idiosyncratic shocks and random meetings for all $\omega \in \Omega$, although we also could use "for almost all ω" formulations. Significant differences in the formulations may arise for the properties dealing with the subset of the agents. For example, consider no aggregate uncertainty property. The following different formulations, in addition to A4, are possible:

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[25] Green and Zhou (31) give the following kind of joint independence: "the probability distribution of the trading partners bid and offer should be identical to the sample distribution."

[26] Kandori (35) describes the following information rules: "1. A label is attached to each agent. 2. Before executing trade each agent observes his and his partner's label. 3. A player and his partner's actions and labels today determine their labels tomorrow."

[27] Boylan (14) describes it as a "subpopulation facing the distribution of types equal to the population distribution."

[28] Boylan (14) shows that in his setup of countably many agents the independence of the random matching scheme from the assignment of types contradicts the condition that for any agent his probability of being matched with an agent of some type equals the fraction of the agents of this type.

A4'. For any measurable subset of the agent space with probability one there is no aggregate uncertainty, i.e.

$$\forall B \in \mathcal{A} \qquad \mathbf{P}\left(\{\omega : \ \forall x \in \mathbb{R} \quad \mu(\{a \in B : \xi_\omega(a) \leq x\}) = \mu(B)F(x)\}\right) = 1.$$

A4". With probability one there is no aggregate uncertainty for any measurable subset of the agent space, i.e.

$$\mathbf{P}\left(\{\omega : \ \forall B \in \mathcal{A}, \forall x \in \mathbb{R} \quad \mu(\{a \in B : \xi_\omega(a) \leq x\}) = \mu(B)F(x)\}\right) = 1.$$

The difference between A4' and A4" is in the location of the clause "for almost all ω." In definition A4' we first fix the subset of the agents, and then say that for this subset for almost all ω there is no aggregate uncertainty on this subset. For A4", for almost all ω, for any measurable subset of the agents there is no aggregate uncertainty. Obviously, A4" follows from A4, and A4' follows from A4", but not a vice versa:

$$\text{A4} \Rightarrow \text{A4"} \Rightarrow \text{A4'}.$$

The following example shows the difference between A4' and A4".[29] Consider a set of independent random variables $\{\tau_i\}_{i \in \mathbb{N}}$, uniformly distributed on the unit interval $[0, 1]$. For a given $B \subset [0, 1]$ define $\nu_\omega(B) = \lim_{n \to \infty} \frac{\#\{i<n : \tau_i \in B\}}{n}$, if it exists. From the Law of Large Numbers, for any Borel-measurable B with Lebesgue measure $\mu(B)$ we have $\nu_\omega(B) = \mu(B)$ with probability 1. However, for any particular ω sequence τ_i consists of a countable number of elements (with Lebesgue measure 0), therefore there are plenty sets B with non-zero measure such that $\nu_\omega(B) = 0$ (and many measurable sets B for which $\nu_\omega(B)$ is not defined.)

We can use similar different formulations of other properties, like measure preserving (B5), uniform distribution (B6), etc. Some properties allow additional interpretations. For example, independence (A3 or B7) allows the following formulation: for almost every agent his shock (partner) is independent of the shock (partner) of almost every other agent.[30]

Different authors use different formulations of no aggregate uncertainty property in their papers. Feldman and Gilles (24) use several of them: in equation 1 they use A4'-type property; in Proposition 1 they switch to A4"-type; in Proposition 2 (the existence of idiosyncratic shocks) they use no aggregate uncertainty on the whole space; in Proposition 3 they again adopt A4'-type. Judd (34) and Alós-Ferrer (4) require no aggregate uncertainty the whole space only. Al-Najjar (1) provides a model in which the Law of Large Numbers holds only on all subintervals.[31] Green (30) satisfies the no aggregate uncertainty A4' with respect to any measurable subset of the population. Uhlig (43) and Sun (42) also use property A4'.

[29] This example has the same idea as the one given by Al-Najjar (1) in footnote 17.

[30] Duffie and Sun (22) use this formulation.

[31] Al-Najjar provides an example in Footnote 17, showing that although the Law of Large Numbers holds for all the subintervals, for any ω there exists a subset of agents on which the Law of Large Numbers fails. The reason for this is the example considered above.

No paper resolving the random matching problem deals with A4"-type of mixing property. Gilboa and Matsui use A4'-type property (they fix the sets, and then define the probabilities.) Duffie and Sun (22) also deal with A4'-type properties. At the same time, some authors need for their models A4"-type and not A4'. In particular, McLennan and Sonnenschein (37) (footnote 4) require (using our notation) that "with probability one, $\mu(\mathbf{M}(B_1, \omega) \cap B_2) = \mu(B_1)\mu(B_2)$ for all Borel sets B_1, B_2."

The question is, which formulation suits better a discrete time economic model with idiosyncratic shocks and random matchings? Consider an arbitrary agent. This agent should not be able to infer any informative conclusion about his future from the past. The agent has history — what had happened to him before. Therefore, he can associate himself with many subsets of the agents (those who had some shock at the previous period of time, those whose partner had specific shock, etc.) Formulation A4' guarantees that for a particular subset there is no aggregate uncertainty with probability one. But it does not guarantee that no agent belongs to a set for which aggregate uncertainty exists. For formulation A4', with probability one there is no aggregate uncertainty for any of the sets to which the agent belongs. And exactly this formulation will be used in our main result.

2.3.3 Standard Model Inconsistencies

Although the standard assumptions given above look intuitively natural, some of them contradict each other. Here we consider several possible problems which one can face while using them.

There Always Exists a Subset with Aggregate Uncertainty

It is often assumed that the no aggregate uncertainty property holds simultaneously on all the measurable subsets of the agent space (property A4"). The following proposition shows that this is impossible for any non-degenerate $F(\cdot)$.[32]

Proposition 2.1. *Suppose that $\xi_\omega(a)$ is $\mathcal{A}$-measurable and is not constant $\mu(\cdot)$-almost surely. Then there exists a positive-measured $B \in \mathcal{A}$ such that*

$$\int_B \xi_\omega(a)\, d\mu_B(a) \neq \int_A \xi_\omega(a)\, d\mu(a), \tag{2.1}$$

where $\mu_B(\cdot)$ stands for the measure on B induced by $\mu(\cdot)$: for any $X \subset B$, $X \in \mathcal{A}$

$$\mu_B(X) = \mu(X)/\mu(B).$$

The result is a generalization of Proposition 1 from Feldman and Gilles (24). (We do not specify the distribution function $F(\cdot)$ and consider a general agent space A.) Although Feldman and Gilles claim that the problem is in "almost" continuity of measurable functions, the proposition holds for an arbitrary space. This result demonstrates that the standard model does not allow no aggregate uncertainty simultaneously on all

[32] Alós-Ferrer (4) refers to this problem as "Absence of homogeneity."

the measurable subsets of the agent space. The other authors bypass this difficulty by considering other types of no aggregate uncertainty: A4'-type no aggregate uncertainty, no aggregate uncertainty on the whole space, or no aggregate uncertainty on a countable set of agent subsets.

Measure Preserving Matching is not Mixing

It is assumed (assumption B8) that an agent is matched with the agents from some set B_2 with probability $\mu(B_2)$, or exactly $\mu(B_2)$ fraction of the agents from set B_1 are matched with the agents from set B_2. Formally, for any $\omega \in \Omega$ and for any $B_1, B_2 \in \mathcal{A}$ holds

$$\mu(\mathbf{M}_\omega(B_1) \cap B_2) = \mu(B_1)\mu(B_2). \tag{2.2}$$

Take any B_1 such that $\mu(B_1) \in (0,1)$ and $B_2 = \mathbf{M}_\omega(B_1)$. Then

$$\mu(\mathbf{M}_\omega(B_1) \cap B_2) = \mu(B_1) > \mu(B_1)^2 = \mu(B_1)\mu(B_2),$$

which says that the mixing property does not hold simultaneously on all the measurable subsets of the agent space.[33]

This result shows that we can not provide a random matching satisfying the measure preserving and mixing properties. To prove it, we found two such sets of agents on which the mixing property does not hold. Therefore, for any measure preserving random matching for any realization there always exist two subsets on which the mixing property does not hold. However, we did not rule out the existence of a measure preserving random matching such that for any two sets of agents the mixing property holds almost surely for these two sets.[34]

No Aggregate Uncertainty is not the Only Option

Consider no aggregate uncertainty in the form of equality of the sample distribution on the whole space and the theoretical distribution. The probability space $(\Omega, \mathcal{F}, \mathbf{P})$ does not impose any restrictions on measure μ. Independence of the shocks does not restrict measure μ as well. Therefore, if no aggregate uncertainty follows from other properties, it should not matter which measure on $\mathcal{A}$ we consider. For a given $\omega \in \Omega$ consider $\mathcal{A}_\omega$ — the minimal σ-algebra in which $\xi_\omega(\cdot)$ is measurable as a function of $a \in A$.

Proposition 2.2. *Suppose that there is no aggregate uncertainty for measure μ. Then there exists aggregate uncertainty for measure $\overline{\mu}$ for the same ω if and only if there exists $B \in \mathcal{A}_\omega$ such that $\mu(B) \neq \overline{\mu}(B)$.*

The proposition states that if we change measure μ so that at least one of the sets from $\mathcal{A}_\omega$ changes its measure, then there exists aggregate uncertainty for the given ω. If measure μ is changed in such a way that with non-zero probability there exists aggregate uncertainty for some $B \in \mathcal{A}$, $\mu(B) > 0$, then there exists aggregate uncertainty in terms of definition A4'. Therefore, each set of idiosyncratic shocks requires its own measure μ (if it exists). Also, arbitrary measure μ might require construction of its own shocks.

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[33] As Alós-Ferrer (6) proves (Corollary 3.2.), even weaker mixing contradicts measure preserving.

[34] The random matching which was found by Duffie and Sun (22).

The fact that idiosyncratic shocks with no aggregate uncertainty is just one of many options was mentioned in several papers, including Judd (34) and Green (30).

Randomness and σ-Algebra

Suppose that there exist two agents such that for any $B \in \mathcal{A}$ if one of them belongs to B, then another one also belongs to B. Obviously, the shocks for these two agents have to be the same, which follows from the measurability of the shocks across the agents. This leads us to the conclusion that independent nondegenerate shocks do not exist for all σ-algebras, if measurability of these shocks over the agent space is required for any $\omega \in \Omega$. Therefore, the choice of the agent space can not arbitrary.

2.3.4 Intuitiveness of the Random Matching Properties

Green and Zhou (31) describe the following model. "Agents are nonatomic. Each agent has a type in $(0, 1]$. The mapping from the agents to their types is a uniformly distributed random variable, independent of all other random variables in the model. Similarly, there is a continuum of differentiated goods, each indexed by a number $j \in (0, 1]$.<...> Each agent of type i receives an endowment of one unit of i good in each period. An agent can consume his own endowment and half of the other brands in the economy; agent i consumes goods $j \in [i, i+\frac{1}{2}]$ (mod 1)<...> He prefers other goods in his consumption range to his endowment good <...> Agents randomly meet pairwise each period. By the assumed pattern of endowments and consumption sets, there is no double coincidence of wants in any pairwise meeting." The last statement means that agents whose indexes differ by .5 (for example, .3 and .8) never meet. The authors explain it by the following argument: "Strictly speaking, there is a double coincidence of wants only when types i and j are matched, with $i \equiv j + 1/2$ (mod 1). Such a match occurs with probability zero. Hence, we ignore this possibility."

Indeed, for a particular agent i the probability to be matched with agent $j = i + 1/2$ (mod 1) equals zero. However, this does not guarantee that event "there is no agent i who is matched with agent $j = i + 1/2$ (mod 1)" has probability zero (if measurable), because this event is a union of uncountable number of probability zero events (for each i). Simulations show that for a large finite population model, probability of the event "there exists agent i who is matched with agent $i + n/2$ (mod n)" is about .4 (100000 simulations with 100 agents.) Therefore, in finite models the probability of the event in the exact Green and Zhou's setup does not converge to zero. There are many ways to bypass this difficulty. Therefore, all the results of the the Green and Zhou's paper hold. Nevertheless, it is obvious that we should be very careful with "intuitive" random matching properties in nonatomic models.

In the conclusion of the section, we want to emphasize that the inconsistencies considered here and nonintuitive properties of idiosyncratic shocks and random matchings demonstrate that the standard model needs to be reconsidered. Different authors tried to solve the problem by considering different deviations from the standard setup, but no solution satisfies what we believe numerous economic models require.

2.4 Examples

The standard setup needs to be reconsidered in order to satisfy the main requirement — the formulation of the properties in terms "for almost all realizations for all the subsets of the agents space." Before proceeding to our solution of the problem, we provide here two simple examples showing the main idea.

One-Period Model with Discrete Shocks

Agent space A consists of eight agents a_1,a_2,a_3,a_4,a_5,a_6,a_7, and a_8. At the beginning every agent learns his shock. After that, the agents meet and learn their partner's shock. Suppose that the shocks and partners are determined by Table 2.1.

Agent a	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
Shock $\xi(a)$	1	-1	1	-1	1	1	-1	-1
Partner $\mathbf{M}(a)$	a_5	a_6	a_7	a_8	a_1	a_2	a_3	a_4
Partner's shock $\xi(\mathbf{M}(a))$	1	1	-1	-1	1	-1	1	-1

Table 2.1 One-period model with discrete shocks.

The a priori probability of being one of the agents equals 1/8, and based on the shock and the partner's shock the agent modifies his belief about who he actually is. Suppose that the shock equals 1. Only agents a_1, a_3, a_5, and a_6 have shock 1, therefore his new belief of being each of these agents equals 1/4, and 0 for all the other agents. Half of these four agents have a partner with shock 1 (a_1 and a_5), and half of them have a partner with shock -1 (a_3 and a_6). Therefore, for the agent with shock 1 the probability of having a partner with shock 1 equals 1/2. Suppose that the agent learns that his partner's shock is 1, too. Only two agents have both their own and their partner's shock 1 — agents a_1 and a_5. Therefore, the agent concludes that with equal probability 1/2 he is one of these two agents.

In general, for any $x_1, x_2 \in \{-1, 1\}$ we have

$$\mathbf{P}\left(\xi(\mathbf{M}(a)) = x_1 \middle| \xi(a) = x_2\right) = 1/2,$$

i.e. the probability of an agent with shock x_1 having a partner with shock x_2 equals 1/2 (a priory probability of having shock x_2), and depends neither on x_1 nor on x_2. For example,

$$\begin{aligned}\mathbf{P}\left(\xi(\mathbf{M}(a)) = 1 \middle| \xi(a) = 1\right) &= \frac{\mathbf{P}\left(\xi(\mathbf{M}(a)) = 1, \xi(a) = 1\right)}{\mathbf{P}\left(\xi(a) = 1\right)} \\ &= \frac{\mathbf{P}\left(\{a_1, a_5\}\right)}{\mathbf{P}\left(\{a_1, a_3, a_5, a_6\}\right)} = 1/2.\end{aligned}$$

σ-algebra $\sigma(\xi)$ generated by shock ξ consists of four elements:

$$\sigma(\xi) = \{\emptyset, \{a_1, a_3, a_5, a_6\}, \{a_2, a_4, a_7, a_8\}, A\}.$$

Matching operator $\mathbf{M}$ maps σ-algebra $\sigma(\xi)$ into σ-algebra

$$\mathbf{M}(\sigma(\xi)) = \{\emptyset, \{a_5, a_7, a_1, a_2\}, \{a_6, a_8, a_3, a_4\}, A\}.$$

It is easy to show that any two sets from $\sigma(\xi)$ and $\mathbf{M}(\sigma(\xi))$ are independent, which means that the agent's shock is independent of the partner's shock, and the value of the agent's shock does not change his belief about the shock of his partner.

One-Period Model with Continuous Shocks

For any distribution function $F(\cdot)$ there always exists a probability space $\tilde{A}$ with two independent random variables ξ_1 and ξ_2 distributed in accordance with cdf $F(\cdot)$. Take A_1 and A_2 — two copies of the probability space $\tilde{A}$ with random variables ξ_1 and ξ_2. Let A be a disjoint union of A_1 and A_2 with ξ defined as ξ_1 on A_1 and ξ_2 on A_2 (see Figure 2.1.) Let $\mathbf{M} : A \to A$ be the natural mapping from A_1 onto A_2 and vice versa, and let σ-algebra $\mathcal{A} = \sigma(\xi, \xi \circ \mathbf{M})$.

For any $B \in \mathcal{A}$ define measure $\mathbf{P}$ as

$$\mathbf{P}(B) = \frac{\mathbf{P}_{A_1}(B \cap A_1) + \mathbf{P}_{A_2}(B \cap A_2)}{2}.$$

Shock ξ has a cdf $F(\cdot)$.

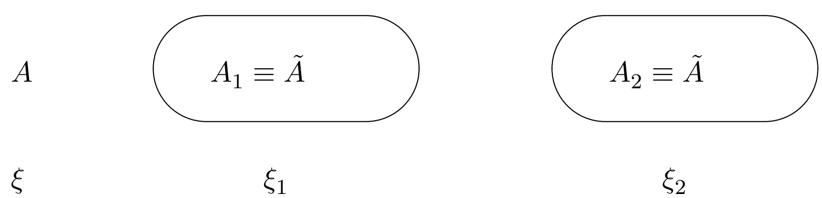

Fig. 2.1 One-period model with continuous shocks.

Take any two $\mathcal{A}$-measurable sets C_1 and C_2. Then

$$\{a \in A_1 : \xi(\mathbf{M}(a)) \in C_1, \xi(a) \in C_2\} = \{a \in A_1 : \xi_2(\mathbf{M}(a)) \in C_1, \xi_1(a) \in C_2\},$$

$$\{a \in A_2 : \xi(\mathbf{M}(a)) \in C_1, \xi(a) \in C_2\} = \{a \in A_2 : \xi_1(\mathbf{M}(a)) \in C_1, \xi_2(a) \in C_2\},$$

and therefore ξ and $\xi \circ \mathbf{M}$ are independent because

$$\begin{aligned}\mathbf{P}(\xi(\mathbf{M}(a)) \in C_1, \xi(a) \in C_2) &= \sum_{i=1,2} \mathbf{P}(a \in A_i : \xi(\mathbf{M}(a)) \in C_1, \xi(a) \in C_2) \\ &= \frac{\mathbf{P}_{\tilde{A}}(\xi_1 \in C_1, \xi_2 \in C_2) + \mathbf{P}_{\tilde{A}}(\xi_2 \in C_1, \xi_1 \in C_2)}{2} \\ &= \mathbf{P}(\xi_1 \in C_1)\, \mathbf{P}(\xi_2 \in C_2).\end{aligned}$$

The examples considered in this section demonstrate the main idea of the construction of idiosyncratic shocks and random matchings: the agents do not know who they are, and based on the previous event (their own shocks) the agents are not able to infer any additional information about their future (partners' shocks).

2.5 Idiosyncratic Shocks and Random Matchings

This section defines idiosyncratic shocks and random matchings using independence of the history. We prove that there exists a space with a sequence of idiosyncratic shocks and random matchings (the random matchings are anonymous and commutative). We show how the old assumptions relate to the new ones.

2.5.1 Basic Definitions

Consider an agent space $(A, \mathcal{A}, \mu)$. The same space serves as the probability space: $\Omega = A$, $\mathcal{F} = \mathcal{A}$, and $\mathbf{P} = \mu$. The uncertainty an agent faces comes from his unknown identity $a \in A$.

Definition. A *shock* is a random variable on $(A, \mathcal{A}, \mu)$.

The definition of a random matching consists of three parts: matching operator, measure preservation, and independence.

Definition. A *matching operator* is an operator $\mathbf{M} : A \to A$ with the following properties:

C1. $\mathbf{M}$ is a bijection:

$$\forall\ a \in A\ \exists!\ a' \in A : \mathbf{M}(a') = a;$$

C2. No agent meets himself:

$$\forall\ a \in A \quad \mathbf{M}(a) \neq a;$$

C3. The partner's partner is the agent himself:

$$\mathbf{M}^{-1} = \mathbf{M}.$$

Definition. A matching operator $\mathbf{M}$ is *measurable* if

C4. It maps measurable sets into measurable sets:

$$\forall B \in \mathcal{A} \quad \mathbf{M}^{-1}(B) \in \mathcal{A}.$$

Definition. A measurable matching operator $\mathbf{M}$ is *measure-preserving* if

C5. It does not change the measure of any measurable subset:

$$\forall B \in \mathcal{A} \quad \mu(\mathbf{M}^{-1}(B)) = \mu(B).$$

Before proceeding with defining the concepts of idiosyncratic shocks and random matchings, we need the concept of history in order to capture the idea of independence of the current events from the past.

2.5.2 History

Time is discrete, $t \in \mathbb{T} \equiv \mathbb{Z}$. A sequence of shocks $\{\xi_t\}_{t\in\mathbb{T}}$ and measure-preserving matching operators $\{\mathbf{M}_t\}_{t\in\mathbb{T}}$ is given. At the beginning of each period, the agents experience shocks, and at the end of each period they meet.

Wither the agents can infer any additional information about the future depends on how much they remember from the past. The information available to an agent at each period of time is called *history*. We assume that the agent's own shocks and the information sharing during the meetings is the only possible source for the history. We denote the agent a's history measured right before the meeting by $H_t(a)$, and right before the shock by $H_t'(a)$.

Let $\mathcal{A}_t$ be the minimal σ-algebra in which history $H_t(a)$ is a measurable function: $\mathcal{A}_t = \sigma(H_t(\cdot))$, and $\mathcal{A}_t'$ be the minimal σ-algebra in which history $H_t'(a)$ is a measurable function: $\mathcal{A}_t' = \sigma(H_t'(\cdot))$. We say that sigma-algebras $\mathcal{A}_t$ and A_t' are generated by the history.[35] The shocks and matchings are $\mathcal{A}$-measurable, therefore $\mathcal{A}_t, \mathcal{A}_t' \subset \mathcal{A}$.

If the agents remember everything and during the meetings they share their full histories, then history $H_{Mt}(a)$ is called *maximal history* and includes:

1. Current shock $\xi_t(a)$;
2. History at the previous period of time $H_{Mt-1}(a)$;
3. Previous period partner's history $H_{Mt-1}(\mathbf{M}_{t-1}(a))$.

Therefore,

$$H_{Mt}(a) = (\xi_t(a), H_{Mt-1}(a), H_{Mt-1}(\mathbf{M}_{t-1}(a)). \tag{2.3}$$

The maximal history before the shock equals

$$H_{Mt}'(a) = (H_{Mt-1}(a), H_{Mt-1}(\mathbf{M}_{t-1}(a)). \tag{2.4}$$

Since the shocks and meetings is the only source for the history, from equation 2.3 follows

$$\begin{aligned}\mathcal{A}_{Mt} \equiv \sigma(H_{Mt}(\cdot)) &= \sigma(\xi_t, \mathcal{A}_{Mt-1}, \mathbf{M}_{t-1}(\mathcal{A}_{Mt-1})) \\ &= \sigma\left(\xi_t, \{\xi_{t_0} \circ \mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}\}_{t_0 \le t_1 < t_2 < \ldots < t_l < t}\right).\end{aligned}$$

Example 1. Suppose that Agent 3 meets with Agent 4 at period 1 and with Agent 2 at period 3 (see Figure 2.2.) Suppose also that Agent 2 meets with Agent 1 at Period 2, and Agent 4 meets with Agent 5 at period 2.

[35] This definition helps us to avoid the problem of dynamic coalition formation mentioned by Alós-Ferrer (4) "it would be expected that any large coalition of traders could be able to form a risk-pooling coalition." The same problem is stated in footnote 4 "it is important to keep track of the sets of agents that have experienced a specific realization." The problem is avoided by defining a sequence of σ-algebras.

Then for the maximal history at the end of period 3 (after the matching) Agent 3 knows Agent 4's history up to period 1 (when they met), Agent 2's history up to period 3, and Agent 1's history up to period 2 (when Agent 2 met Agent 1). At the same time, Agent 3 at the end of period 3 does not know anything about Agent 5 because their common partner — Agent 4 — met with Agent 3 before he met Agent 5.

Denoting the agents as a_1, a_2, a_3, a_4, and a_5 correspondingly, we can write:

$$\mathbf{M}_1(a_3) = a_4;\ \mathbf{M}_2(a_1) = a_2;\ \mathbf{M}_2(a_4) = a_5;\ \mathbf{M}_3(a_2) = a_3,$$

and maximal histories

$$H_{M3}(a_2) = (\xi_3(a_2), \xi_2(a_2), \xi_1(a_2), \xi_2(a_1), \xi_1(a_1));$$

$$H_{M3}(a_3) = (\xi_3(a_3), \xi_2(a_3), \xi_1(a_3), \xi_1(a_4)).$$

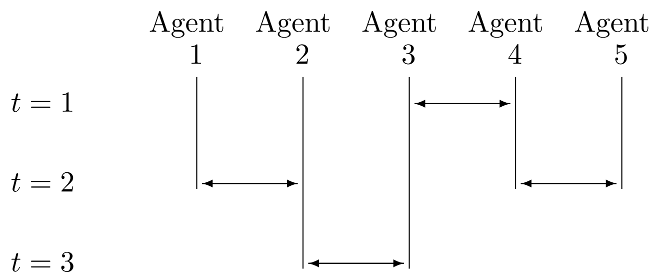

Fig. 2.2 Scheme of matchings $\mathbf{M}_t$ for example 1.

2.5.3 Independence

Mixing property B8 was not defined correctly because no measure-preserving matching operator can be mixing on the whole non-trivial σ-algebra $\mathcal{A}$.[36] At the same time, if we consider a family of σ-algebras $\{\mathcal{A}_t\}$ generated by the history, then the sequence of matching operators $\{\mathbf{M}_t\}$ can be mixing in the sense of $\mathbf{M}_t$ being mixing on $\mathcal{A}_t$ for each t. To define mixing correctly, we need the concept of independence of a matching operator from a σ-algebra.

Definition. A matching operator $\mathbf{M}$ is *independent of a σ-algebra $\mathcal{A}' \subset \mathcal{A}$* if σ-algebras $\mathcal{A}'$ and $\mathbf{M}\mathcal{A}'$ are independent.

The definition is a reformulation of the mixing property and says that for any $B, C \in \mathcal{A}'$ events B and $\mathbf{M}C$ are independent, or

$$\mu(B \cap \mathbf{M}C) = \mu(B)\mu(\mathbf{M}C) = \mu(B)\mu(C).$$

[36] See "A Measure Preserving Matching can not be Mixing" inconsistency, page 42.

The definition is similar to the definition of independence of a random variable from a σ-algebra.

Definition. A measure-preserving matching operator $\mathbf{M}_t$ is called a *random matchings* if

C6. Matching operator $\mathbf{M}_t$ is independent of $\mathcal{A}_t$.

Definition. A shock ξ_t is called *idiosyncratic* if

C7. Shock ξ_t is independent of σ-algebra $\mathcal{A}'_t$.

Both definitions require current event (partner's history or own shock) to be independent of the past. For the matching, the past is $H_t(\cdot)$ and the event is $H_t(\mathbf{M}(\cdot))$ — the partner's history. For the shock, the past is $H'_t(\cdot)$ and the event is $\xi_t(\cdot)$ — the shock.

2.5.4 Old and New Assumptions

Now we want to demonstrate (see Table 2.2) how the old assumptions A1-A4 and B1-B7 relate to new assumptions C1-C7.

Old Assumption	**Explanation of the Assumption**	**Replaced by**
	Assumptions about the Shocks	
A1	Measurability	Definition
A2	Identical distribution	Unknown identity
A3	Independence	Obsolete, C7
A4, A4'	No aggregate uncertainty	Definition, C7
A4"	No aggregate uncertainty	Obsolete
	Assumptions about the Matchings	
B1-B5	Everyone meets someone, no agent meets himself, partner's partner is the agent himself, measurability, measure preserving	C1-C5
B6	Uniform distribution	Obsolete
B7	Independence	Obsolete
B8	Mixing	C6
	Assumption of Joint Independence	
Independence of the shocks and matchings		C6, C7

Table 2.2 Equivalence of the old and new assumptions.

Measurability of the shocks (A1) follows from the definition of a shock. Identical distribution of the shocks (A2) comes from the fact that an agent does not know who he is, and that he sees the future as random. Independence of the shocks becomes obsolete. No aggregate uncertainty comes from the definition of the shocks and the independence of the current and future shocks from the past. Everyone meets someone, no agent meets himself, the partner's partner is the agent himself, measurability of the matchings, and measure preserving (B1-B5) come from C1-C5 respectively. Uniform distribution of the partners (C6) and independence of the partners (C7) become obsolete because of the assumption of unknown identity. Mixing (B8) follows from the measure preserving and independence of σ-algebras $\mathcal{A}_t$ and $\mathbf{M}_t(\mathcal{A}_t)$.

2.5.5 Anonymity in Matchings

The "anonymity" concept, when any two meeting agents do not have any chance of meeting later, directly or through their future partners, plays an important role in many economic models. [37] The anonymity implies that the current action of an agent can not influence his future. We want to establish how idiosyncratic shocks and random matchings are related to the concept of anonymity.

Following Aliprantis et al. (2), we define the concepts of indirect partners and strongly anonymous matchings.[38]

Definition. We define the *set of indirect partners* of agent a at time t (before the matching) as follows:

$$\tilde{H}_t(a) = \{a\} \cup \left\{\mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}(a)\right\}_{t_1 < t_2 < \ldots < t_l < t}.$$

The agent knows some of the shocks of his indirect partners. For the matching scheme from example 1, one may find that

$$\tilde{H}_3(a_5) = \{a_3, a_4, a_5\}; \quad \tilde{H}_3(a_3) = \{a_3, a_4\}.$$

Definition. A sequence of matchings $\mathbf{M}_t$ is called *strongly anonymous*, if for any time t no meeting agents have common indirect partners:

$$\forall t \in \mathbb{T}, \forall a \in A \qquad \tilde{H}_t(a) \cap \tilde{H}_t(\mathbf{M}_t(a)) = \emptyset.$$

The concept of strongly anonymous matchings is very strict; we can allow for a possibility of the future influence if it has zero probability. To do this, we introduce the concept of μ-strongly anonymous matchings.

[37] For example, Kocherlakota (36) requires that "there is no possibility of any direct or indirect contact between the two agents before the current match." Green and Zhou (31) define anonymity as "no pair meets more than once and also that each agent knows the variety of good which his partner is endowed but nothing else."

[38] By indirect partners, we understand both common and indirect partners from Aliprantis et al. (2).

Definition. A sequence of matchings $\{\mathbf{M}_t\}_{t\in\mathbb{T}}$ is called *μ-strongly anonymous*, if for any $t \in \mathbb{T}$ with probability one the meeting agents do not have common indirect partners:[39]

$$\forall t \in \mathbb{T} \;\; \mu\left(\{a : \tilde{H}_t(a) \cap \tilde{H}_t(\mathbf{M}_t(a)) = \emptyset\}\right) = 1.$$

The next theorem demonstrates that the random matchings for the maximal history should be μ-strongly anonymous.[40] If the history differs from the maximal history, the requirement of μ-strong anonymity can be weakened.

THEOREM 2.1. *Let $F(\cdot)$ be a continuous distribution function. Then the matchings are random for the maximal history only if they are μ-strongly anonymous.*

2.5.6 Existence

The following theorem constitutes the main result of the paper. It establishes the existence of a sequence of idiosyncratic shocks and random matchings for the maximal history as we defined them before.

THEOREM 2.2. *For any $F(\cdot)$ there exists a probability space $(A, \mathcal{A}, \mu)$ with a continuum of elements. On this probability space there exists a sequence of idiosyncratic shocks ξ_t and random matchings $\mathbf{M}_t$ for the maximal history. The matchings are strongly anonymous and commutative: for any t_1, t_2 holds*

$$\mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} = \mathbf{M}_{t_2} \circ \mathbf{M}_{t_1}.$$

The idea of the proof is the following. We construct a probability space with a sufficient number of random independent variables. Then, we create countably many copies of this space and allocate the random variables to these spaces. The agent space is the union of all the copies, and the shock at each period of time is a particular random variable on the corresponding copy of the original probability space. At each period of time, the agents from one copy of the probability space meet with the corresponding agents from some other copy. The copies whose agents meet are chosen in such a way that the matchings are strongly anonymous; to achieve this, we use Aliprantis et al. (3) mechanism of recursive block-partition on the copies of the probability space. The σ-algebra is generated by the shocks and matchings; the σ-algebras at each period of time are generated by the histories. It turns out that for any measurable set on the agent space its original measure on any of the copies does not depend on the copy. Therefore, we define the measure of a measurable set as the measure of its part on any of the copies of the original probability space. Then we prove that the shocks constructed are idiosyncratic and the matchings are random.

[39] Obviously, all strongly anonymous matchings are μ-strongly anonymous.

[40] In Theorem 2.2 we show that spaces with idiosyncratic shocks and strongly anonymous random matchings exist. Hence, we do not need to consider any other types of anonymity because, as was shown in Aliprantis et al. (3), all of them follow from strong anonymity.

If the history is not maximal, then the results of theorem 2.2 still holds because of the following proposition.

Proposition 2.3. *Suppose that the shocks $\{\xi_t\}_t$ are idiosyncratic and matchings $\{\mathbf{M}_t\}_t$ are random for the maximal history. Then they constitute a system of idiosyncratic shocks and random matchings for any function of the maximal history.*

2.6 Proofs

Proof of proposition 2.1.

As $\xi_\omega(\cdot)$ is measurable, then for any $x \in \mathbb{R}$ set $\{a : \xi_\omega(a) \leq x\}$ is measurable. Take $m = \int\limits_A \xi_\omega(a)\, d\mu(a)$ — the sample average of the shock, and

$$B = \{a \in A : \xi_\omega(a) < m\}$$

— all the agents who have the shock not greater than the average. The shock is not constant, therefore B has a positive measure, and

$$\int\limits_B \xi_\omega(a)\, d\mu_B(a) < \int\limits_B m\, d\mu_B(a) = m,$$

which proves inequality 2.1.

□

Proof of proposition 2.2.

Notice that the Borel σ-algebra on $\mathbb{R}$ is the minimal σ-algebra containing all the intervals $(-\infty, x]$. σ-algebra $\mathcal{A}_\omega$ is generated by the shocks, therefore there exists $B \in \mathcal{A}_\omega$ such that $\mu(B) \neq \overline{\mu}(B)$ if and only if there exists $x \in \mathbb{R}$ such that

$$\mu(\{a : \xi_\omega(a) \in (-\infty, x]\}) \neq \overline{\mu}(\{a : \xi_\omega(a) \in (-\infty, x]\}). \tag{2.5}$$

There is no aggregate uncertainty for measure μ, therefore

$$\mu(\{a : \xi_\omega(a) \leq x\}) = F(x) \qquad \forall x \in \mathbb{R}.$$

Then, using inequality 2.5,

$$\exists\ B \in \mathcal{A}_\omega : \mu(B) \neq \overline{\mu}(B) \Leftrightarrow$$

$$\exists\ x : \mu(\{a : \xi_\omega(a) \leq x\}) \neq \overline{\mu}(\{a : \xi_\omega(a) \leq x\}) \Leftrightarrow$$

$$\exists\ x : \overline{\mu}(\{a : \xi_\omega(a) \leq x\}) \neq F(x).$$

The last conditions means the existence of aggregate uncertainty for measure $\bar{\mu}$.

□

Proof of theorem 2.1.

Suppose that there exists a positive-measured subset of agents $B \in \mathcal{A}$ such that the matching at time period t for the agents from this set is not strongly anonymous.

Consequently, for any $a \in B$ there exist $t_1 < t_2 < \ldots < t_l < t$ and $t'_1 < t'_2 < \ldots < t'_m < t$ such that

$$\mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}(a) = \mathbf{M}_{t'_1} \circ \mathbf{M}_{t'_2} \circ \ldots \circ \mathbf{M}_{t'_m}(\mathbf{M}_t(a)).$$

Denote

$$B^{t'_1 t'_2 \ldots t'_m}_{t_1 t_2 \ldots t_l} = \{a \in B : \mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}(a) = \mathbf{M}_{t'_1} \circ \mathbf{M}_{t'_2} \circ \ldots \circ \mathbf{M}_{t'_m}(\mathbf{M}_t(a))\}.$$

Note that

$$B = \bigcup_{\substack{t_1, t_2, \ldots, t_l, \\ t'_1, t'_2, \ldots, t'_m}} B^{t'_1 t'_2 \ldots t'_m}_{t_1 t_2 \ldots t_l}.$$

Set B can be represented as a countable union of sets $B^{t'_1 t'_2 \ldots t'_m}_{t_1 t_2 \ldots t_l}$. If B has a positive measure, then for some indexes $t_1 < t_2 < \ldots < t_l < t$ and $t'_1 < t'_2 < \ldots < t'_m < t$ set $B^{t'_1 t'_2 \ldots t'_m}_{t_1 t_2 \ldots t_l}$ also has a positive measure.

Denote $\eta_{t_1 t_2 \ldots t_l} = \xi_{t_1}(\mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}(a))$ and $\eta_{t'_1 t'_2 \ldots t'_m} = \xi_{t'_1}(\mathbf{M}_{t'_1} \circ \mathbf{M}_{t'_2} \circ \ldots \circ \mathbf{M}_{t'_m}(a))$. To complete the proof, we need to use probability theory lemma 2.1 (the lemma is given after this proof). The matchings are independent, therefore $\mathbf{M}_t$ is independent of $\mathcal{A}_t$ and random variables $\eta_{t_1 t_2 \ldots t_l}$ and $\eta_{t'_1 t'_2 \ldots t'_m} \circ \mathbf{M}_t$ are independent. From Lemma 2.1 follows that $\mu\Big(\{a : \eta_{t_1 t_2 \ldots t_l}(a) = \eta_{t'_1 t'_2 \ldots t'_m} \circ \mathbf{M}_t(a)\}\Big) = 0$. However, these random variables coincide on $B^{t'_1 t'_2 \ldots t'_m}_{t_1 t_2 \ldots t_l}$, and therefore the measure of this set is not equal to zero. Contradiction. Thus, set B has measure zero for any time period t.

□

Lemma 2.1. *Suppose that random variables ζ_1 and ζ_2 are independent and that one of them has a continuous distribution. Then*

$$\mu(\{a : \zeta_1(a) = \zeta_2(a)\}) = 0.$$

Proof of lemma 2.1.

Suppose that ζ_2 has a continuous distribution. Then for any $\epsilon > 0$ there exists l and a set of real numbers $\{h_i\}_{i=0}^{l}$, such that $h_0 \equiv -\infty < h_1 < h_2 < \ldots < h_{l-1} < h_l \equiv +\infty$ and

$$\mu(\{a : \zeta_2(a) \in [h_i, h_{i+1})\}) < \epsilon \qquad \forall i = 0, \ldots, l-1.$$

Note that

$$\{a : \zeta_1 - \zeta_2 = 0\} \subset \bigcup_{i=0}^{l-1} \{a : \zeta_1 \in [h_i, h_{i+1}), \zeta_2 \in [h_i, h_{i+1})\}.$$

Therefore,

$$\begin{aligned}\mu(\zeta_1 - \zeta_2 = 0) &\leq \sum_{i=0}^{l-1} \mu(\zeta_1 \in [h_i, h_{i+1}), \zeta_2 \in [h_i, h_{i+1})) \\ &= \sum_{i=0}^{l-1} \mu(\zeta_1 \in [h_i, h_{i+1}))\mu(\zeta_2 \in [h_i, h_{i+1})) \\ &< \epsilon \sum_{i=0}^{l-1} \mu(\zeta_1 \in [h_i, h_{i+1})) = \epsilon.\end{aligned}$$

As ϵ is arbitrary, then

$$\mu(\zeta_1 - \zeta_2 = 0) = 0.$$

□

Proof of theorem 2.2.

Agent Space

By Kolmogorov theorem, for any distribution function $F(\cdot)$ there exists a probability space $(\Theta, \mathcal{Q}, \nu)$ with a countable number of independent random variables $\{\xi_t^i\}_{i \in \mathbb{N}, t \in \mathbb{T}}$, each distributed in accordance with $F(\cdot)$. Consider spaces A_i, $i \in \mathbb{N}$, of which every space is an exact copy of $(\Theta, \mathcal{Q}, \nu)$. Assume that functions S_i naturally map set Θ onto A_i (see Figure 2.3).

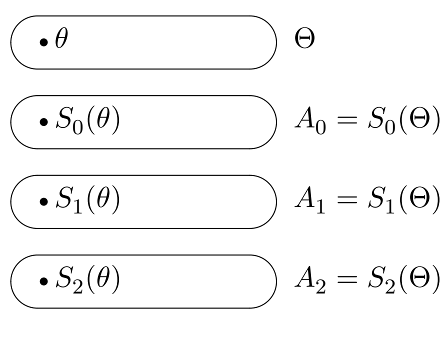

Fig. 2.3 Different instances of probability space $(\Theta, \mathcal{Q}, \nu)$.

Define agent space $A = A_0 \sqcup A_1 \sqcup A_2 \sqcup \ldots$ and function $s(a)$ — the space number to which $a \in A$ belongs: $a \in A_{s(a)}$. By $S^{-1}(a)$ we understand the corresponding $S_i^{-1}(a)$, where $i = s(a)$.

Random variables ξ_t^i on probability space $(\Theta, \mathcal{Q}, \nu)$ have only two indexes, $i \in \mathbb{N}$ and $t \in \mathbb{T}$. Therefore, by Kolmogorov theorem we can take $[0,1]^{\mathbb{T} * \mathbb{N}}$ as space Θ (see Wentzell (44)), and $\bigsqcup_{i=0}^{\infty} [0,1]^{\mathbb{T} * \mathbb{N}}$ as the agent space A.

By using cardinal arithmetic (see Halmos (32)), one can show that $\left|\bigsqcup_{i=0}^{\infty}[0,1]^{\mathbb{T}*\mathbb{N}}\right| = |[0,1]|$, i.e. the agent space A can have a continuum of elements.[41]

Shocks

Define shocks $\xi_t : A \to \mathbb{R}$ in the following way: $\xi_t|A_i = \xi_t^i$. Random variables $\xi_t(a)$ consist of components $\xi_t^i(S^{-1}(a))$, depending on the space A_i to which a belongs. We also can write $\xi_t(a) = \xi_t^{s(a)}(S^{-1}(a))$. Figure 2.4 illustrates the construction.

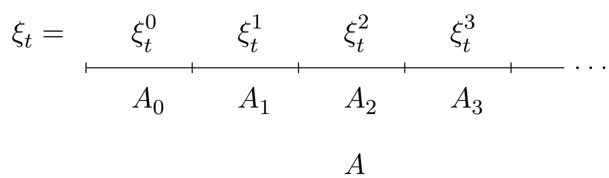

Fig. 2.4 Construction of random variables ξ_t.

Matchings

To define the matchings, consider any bijective index $k(\cdot) : \mathbb{T} \to \mathbb{N}$ (for example, $k(0) = 0$, $k(1) = 1$, $k(-1) = 2, \ldots$). All the results can be equivalently formulated in term of time t or index k. From the context it will be clear which concept, time or index, is used.

Take any A_i. For any k there exists the unique representation $i = m2^{k+1} + \delta 2^k + n$, where $m \in \mathbb{N}$, $\delta \in \{0,1\}$, and $n \in \{0, 1, \ldots, 2^k - 1\}$. We define the matching rule $\mathbf{M}_k$ at the time period with index k so that agent $a \in S_i$ meets with agent

$$\mathbf{M}_k(a) = S_{m2^{k+1}+(1-\delta)2^k+n}(S_i^{-1}(a)). \tag{2.6}$$

The scheme of the matchings is represented at Figure 2.5.[42] At $k = 0$, agents from A_0 meet with the corresponding agents from A_1, agents from A_2 meet with A_3, agents from A_4 meet with A_5, and so on. At $k = 1$, agents from $\{A_0, A_1\}$ meet with the corresponding agents from $\{A_2, A_3\}$, agents from $\{A_4, A_5\}$ meet with $\{A_6, A_7\}$, and so on. At $k = 2$, agents from $\{A_0, A_1, A_2, A_3\}$ meet with the corresponding agents from $\{A_4, A_5, A_6, A_7\}$, and so on. In other words, if we denote the binary expansion of i by $X(i) = \overline{\ldots x_2 x_1 x_0}$, then at a period with index k agents from $A_{\overline{\ldots x_k \ldots x_1 x_0}}$ meet with the corresponding agents from $A_{\overline{\ldots \overline{x}_k \ldots x_1 x_0}}$, where $\overline{x}_k = 1 - x_k$.

If we consider

$$a' = \mathbf{M}_{k_1} \circ \mathbf{M}_{k_2} \circ \ldots \circ \mathbf{M}_{k_l}(a),$$

[41] We showed that there exists an agent space A with a continuum of elements. However, there also might exist spaces with different cardinality.

[42] This scheme was introduced by Aliprantis et al. in (3). The authors call it *recursive block-partition*.

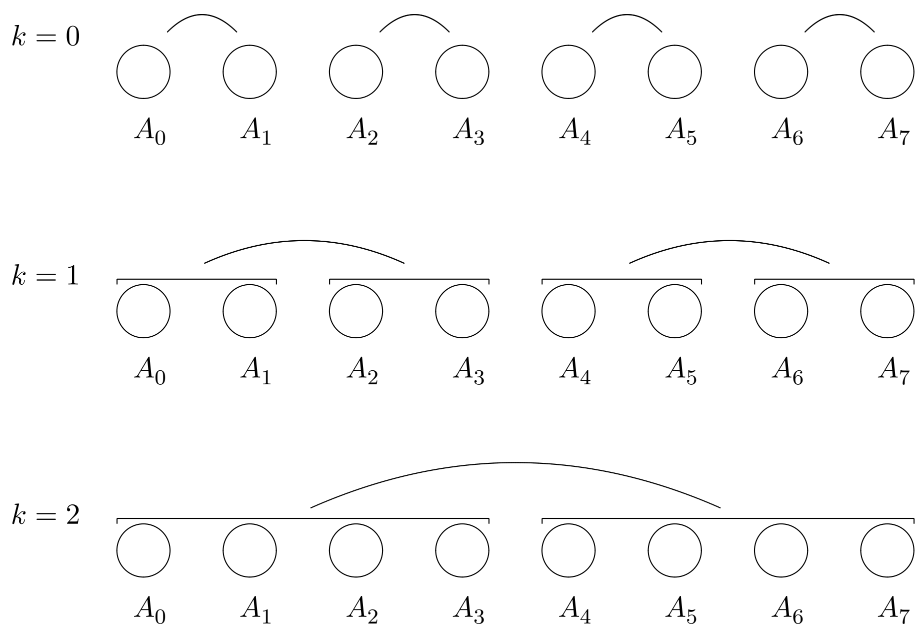

Fig. 2.5 Structure of the matchings.

then the binary expansion of number $s(a')$ of the copy of the space to which a' belongs differs from the binary expansion of $s(a)$ in digits $k_1, k_2, \ldots, k_l$. In other words,

$$a' = S_{G_{k_1 k_2 \ldots k_l}(s(a))}(S^{-1}(a)), \tag{2.7}$$

where $G_{k_1 k_2 \ldots k_l}(\cdot)$ denotes the function of changing $k_1, k_2, \ldots, k_l$-th digits of the binary expansion of the argument.

The matchings, as we defined them, satisfy the commutativity property. For example, the agent's t_1-partner is meeting at any other time the agent partner's t_1-partner. The situation is depicted at Figure 2.6. At time $t = 1$, agent 0 meets agent 2, and agent 1 meets agent 3. At time $t = 2$, agent 0 meets agent 1, and agent 2 meets agent 3.

Whom agent $a \in A$ meets at time with index k_i is determined by changing the k_i-th digit in the binary expansion of number $s(a)$. The result of multiple matchings at times $k_1, k_2, \ldots, k_l$ is changing digits $k_1, k_2, \ldots, k_l$ in the binary expansion of $s(a)$. For any permutation of the times the result is the same because it does not matter in which order to change the digits. Therefore, for any set of time indexes $k_1, k_2, \ldots, k_l$ and for any permutation τ of numbers $1, 2, \ldots, l$, the following equation (commutativity of matchings) holds:

$$\mathbf{M}_{k_1} \circ \mathbf{M}_{k_2} \circ \ldots \circ \mathbf{M}_{k_l} \equiv \mathbf{M}_{k_{\tau(1)}} \circ \mathbf{M}_{k_{\tau(2)}} \circ \ldots \circ \mathbf{M}_{k_{\tau(l)}}.$$

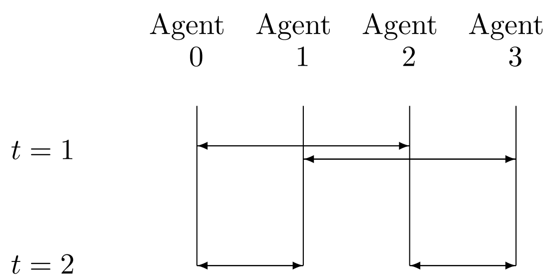

Fig. 2.6 Commutativity of the matchings.

The matchings are strongly anonymous because of the commutativity: if agent a meets agent $a' = M_{k_0}(a)$ at time with index k_0 and they have a common previous indirect partner, then

$$\mathbf{M}_{k_1} \circ \mathbf{M}_{k_2} \circ \ldots \circ \mathbf{M}_{k_l}(a) = \mathbf{M}_{k'_1} \circ \mathbf{M}_{k'_2} \circ \ldots \circ \mathbf{M}_{k'_m}(a');$$

$$\mathbf{M}_{k'_1} \circ \mathbf{M}_{k'_2} \circ \ldots \circ \mathbf{M}_{k'_m} \circ \mathbf{M}_{k_1} \circ \mathbf{M}_{k_2} \circ \ldots \circ \mathbf{M}_{k_l}(a) = a' \equiv M_{k_0}(a) \tag{2.8}$$

for some sets $\{k_1, k_2, \ldots, k_l\}$ and $\{k'_1, k'_2, \ldots, k'_m\}$. Matching at time k_i corresponds to changing k_i-digit in the binary expansion of the space number to which a belongs. Therefore equation 2.8 holds only on such sets $\{k_1, k_2, \ldots, k_l\}$ and $\{k'_1, k'_2, \ldots, k'_m\}$ that differ in one element k_0, which is impossible because the previous partners could not meet at time k_0.

σ-Algebra

Following the definition, σ-algebras $\mathcal{A}_t$ and $\mathcal{A}'_t$ are generated by the maximal history:

$$\mathcal{A}_t = \sigma\left(\xi_t, \{\xi_{t_0} \circ \mathbf{M}_{t_1} \circ \mathbf{M}_{t_2} \circ \ldots \circ \mathbf{M}_{t_l}\}_{t_0 \le t_1 < t_2 < \ldots < t_l < t}\right);$$

$$\mathcal{A}'_t = \sigma\left(\mathcal{A}_{t-1}, \mathbf{M}_{t-a}(\mathcal{A}'_{t-1})\right).$$

Note that $\mathcal{A}_t \subseteq \mathcal{A}'_{t+1} \subseteq \mathcal{A}_{t+1}$. Define σ-algebra $\mathcal{A}$ as the minimal σ-algebras in which all the shocks and matchings are measurable:

$$\mathcal{A} = \sigma\left(\{\xi_{t_0} \circ \mathbf{M}_{t_1} \circ \ldots \circ \mathbf{M}_{t_l}\}_{t_0, t_1, \ldots, t_l}\right).$$

Obviously,

$$\mathcal{A}_t, \mathcal{A}'_t \subseteq \mathcal{A}.$$

Probability

In order to define probability measure on $(A, \mathcal{A})$, we will use the probability measure ν on $(\Theta, \mathcal{Q})$. Namely, for any $B \in \mathcal{A}$ define

$$\mu(B) = \lim_{j \to \infty} \frac{1}{j+1} \sum_{i=0}^{j} \nu(S^{-1}(B \cap A_i)). \tag{2.9}$$

Measure $\mu(B)$ is defined correctly. Indeed, consider any finite set of indexes $W = \{k_0, k_1, \ldots, k_l\}$ and any measurable sets $B_{k_0,k_1,\ldots,k_l}$. From equation 2.7, for any $a \in A_i$

$$\xi_{t_0} \circ \mathbf{M}_{t_1} \circ \ldots \circ \mathbf{M}_{t_l}(a) = \xi_{t_0}^{G_{k_1k_2\ldots k_l}(i)}(S^{-1}(a)).$$

Hence, values

$$\begin{aligned} \nu^i &= \nu\left(\bigcap_{(k_0,k_1,\ldots,k_l)\in W} \{\theta \in \Theta : \xi_{k_0} \circ \mathbf{M}_{k_1} \circ \ldots \circ \mathbf{M}_{k_l}(S_i(\theta)) \in B_{k_0,k_1,\ldots,k_l}\}\right) \\ &= \nu\left(\bigcap_{(k_0,k_1,\ldots,k_l)\in W} \left\{\theta \in \Theta : \xi_{t_0}^{G_{k_1k_2\ldots k_l}(i)}(\theta) \in B_{k_0,k_1,\ldots,k_l}\right\}\right) \end{aligned}$$

do not depend on i because of the independence of the random variables $\left\{\xi_{t_0}^{G_{k_1k_2\ldots k_l}(i)}(\cdot)\right\}$. Therefore, for any i and j and for any $B \in \mathcal{A}$ holds

$$\nu(S^{-1}(B \cap A_j)) = \nu(S^{-1}(B \cap A_i)), \tag{2.10}$$

which means that measure $\mu(\cdot)$ was defined correctly, and for any i

$$\mu(B) = \nu(S^{-1}(B \cap A_i)).$$

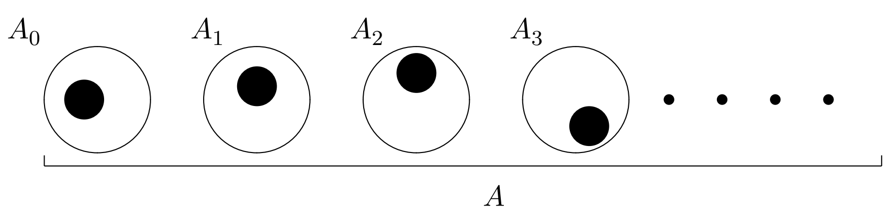

Fig. 2.7 A typical measurable set.

Formula 2.10 explains why we need the agent set A to consist of a countable number of sets A_i. In order to define a countable sequence of measure-preserving matchings, we need to find a countable number of equally probable sets A_i. Because of the equal probability of these sets and countability, the measure of each of these sets should equal zero (and the words "equally relatively probable" are not applicable to these sets). The measure of a union of a countable number of sets with measure zero itself has measure zero, which is impossible because it constitutes the whole space. We overcome this paradox by constructing a σ-algebra $\mathcal{A}$ which does not contain any of A_i by itself (see Figure 2.7): if $B \in \mathcal{A}$ contains some part of A_0, then it contains the equivalent parts of all the other A_i's, in accordance with the measure ν on $S_i^{-1}(A_i)$.

Random variables $\xi_t(\cdot)$ have distribution function $F(X)$ because

$$\mu(\xi_t \leq x) = \nu(S^{-1}(A_i \cap \{a : \xi_t \leq x\})) = \nu(\xi_t^i \leq x) = F(x).$$

By definition, a matching operator $\mathbf{M}_k$ naturally maps A_i onto A_j, where j's binary expansion differs from the i's binary expansion in the k-th digit. Therefore, for any $B \in \mathcal{A}$ measure $\nu(S^{-1}(\mathbf{M}_k(B) \cap A_i))$ also does not depend on i, which by formula 2.9 gives us the measure preserving of the matchings.

Independence

Now we can show that for the maximal history the shocks $\{\xi_t\}$ are idiosyncratic and the matchings $\{\mathbf{M}_t\}$ are random, i.e. assumptions C6 and C7 hold.

First we want to prove independence of ξ_t from $\mathcal{A}'_t$ (assumption A6). Consider an arbitrary A_i, for example A_0. Note that the history of an agent before the shock at time t is a function of the shocks at the previous periods of time. Therefore

$$\mathcal{A}'_t \cap A_0 \subset \sigma(\{\xi_{t'}^i\}_{t'<t, i\in\mathbb{N}});$$

$$\sigma(\xi_t) \cap A_0 = \sigma(\xi_t^0).$$

Since random variables $\{\xi_t^i\}$ are independent, we have

$$\sigma(\{\xi_t^i\}_{t<t,i\in\mathbb{N}}) \perp \sigma(\xi_t^0);$$

$$\mathcal{A}'_t \cap A_0 \perp \sigma(\xi_t) \cap A_0,$$

and therefore ξ_t is independent of $\mathcal{A}'_t$ on A_0, which in accordance with formula 2.9 means that ξ_t is independent of $\mathcal{A}'_t$, and therefore shocks $\{\xi_t\}$ are idiosyncratic.

To prove independence of $\mathbf{M}_t$ from $\mathcal{A}_t$ (assumption A7), note that the history of an agent from A_0 includes his current shock and the shocks of his previous direct or indirect partners. These previous direct or indirect partners belong to such A_i that $x_{k(t)}(i) = 0$. The history of an A_0 agent's partner at time t includes this partner's current shock and the shocks of this partner's indirect partners. These previous indirect partners belong to such A_i that $x_{k(t)}(i) = 1$. Therefore

$$\mathcal{A}_t \cap A_0 \subset \sigma(\{\xi_{t'}^i\}_{t'\leq t, x_{k(t)}(i)=0});$$

$$(\mathcal{A}_t \circ \mathbf{M}_t) \cap A_0 \subset \sigma(\{\xi_{t'}^i\}_{t'\leq t, x_{k(t)}(i)=1}).$$

Since random variables $\{\xi_t^i\}$ are independent, we have

$$(\mathcal{A}_t \cap A_0) \perp (\mathcal{A}_t \circ \mathbf{M}_t \cap A_0);$$

$$\mathcal{A}_t \perp (\mathcal{A}_t \circ \mathbf{M}_t);$$

$$\mathbf{M}_t \perp \mathcal{A}_t,$$

i.e. the matching operators are random.

Proof of proposition 2.3.

Any functions of two independent variables are independent, too. Therefore, the shocks are idiosyncratic and the matchings are random under any history which is a function of the maximal history.

□

Chapter 3

PARETO IMPROVEMENT IN GAMES PLAYED THROUGH AGENTS

3.1 Introduction

Common agency games (see Bernheim and Whinston (10)) are widespread in different areas of economics, including political economy, public economics, trade, etc. These games extend the standard principal-agent model to multiple principals. Dixit (18) gives the following example of a common agency: "... government agencies in the United States are common agencies with several principals who are engaged in a non-cooperative game to influence their actions."

A Game Played Through Agents is a perfect information game with multiple principals and multiple agents played in two stages. In the first stage, the principals simultaneously and independently set the transfer schedules (payment, or incentive schemes) for each of the agents. The transfer depends on the action (effort) the agent chooses. During the second stage, the agents simultaneously and independently decide which action to implement. The vector of actions chosen by the agents is the outcome of the game. An agent's profit (net payoff) is the sum of the transfers from all the principals depending on the action the agent chose minus the cost (disutility) of choosing the action. A principal's profit (net payoff) is a predetermined payoff based on the vector of actions chosen by the agents, minus all the transfers to the agents. The principals and agents maximize their expected net payoffs. The equilibrium concept used is similar to the Nash equilibrium, with the only exception that the agents always (even on the out of the equilibrium path) choose one of the actions delivering the highest net payoff, if possible.

In the simplest case of the game, with one principal only, one can easily show that the outcome is always efficient — the total payoff of the agents and the principal is maximized. The principal compensates the agents exactly for implementing the efficient vector of actions. As a result, the agents are indifferent between choosing the equilibrium action (some effort) and not participating in the game; the principal gets the maximal net payoff.

In a game with multiple principals, their behavior differs from the behavior of only one principal. If their interests differ significantly, each principal competes for the agents to implement the action vector which is more favorable for him. Consequently, the agents might get a positive net payoff because of the competition among the principals, and the action vector implemented might be inefficient, leading to the higher losses in the total principals' net payoff.

In Games Played Through Agents, the principals simultaneously attempt to influence the agents' choices of actions. Clearly, these principals would be better off if they could agree (in order to avoid a game among themselves) on a strategy that suits their joint interests better, and then split the gain. As long as the principals fail to cooperate in implementing an efficient outcome, their overall net payoff decreases. Bernheim and Whinston (10) showed that asymmetric information also might lead to the welfare losses.

One of the first steps in developing the theory of multiple principal games with complete information was taken by Bernheim and Whinston (11). The authors considered one agent and introduced the concept of Truthful Nash Equilibrium. In this Truthful Nash Equilibrium, the principals offer a simplified transfer schedules, in which each principal's net payoff is as close to a constant as possible, no matter what action the agent chooses. This means that a principal's transfer function equals the maximum of the principal's payoff minus this constant, and zero. A Truthful Nash Equilibrium always exists and is coalition-proof. The outcome of a Truthful Nash Equilibrium is always efficient.

Bernheim and Whinston (10) focused on the moral hazard problem in one agent multiple principles games. In this framework, the agent chooses not a particular action, but rather a distribution over a finite set of possible actions from a finite predetermined set of such distributions. The agent's effort is unobservable; however, the principals know the outcome of the effort. The authors find that the action implemented in an equilibrium is always implemented efficiently (the cost of the effort is minimized). For a risk averse agent and only two possible distributions over the set of actions, an equilibrium always exists.

Dixit (19) considered the problem of several risk averse principals and one agent. In this problem, the agent makes a k-dimensional unobservable effort and gets an observable and verifiable m-dimensional output. In this setup, providing a stronger incentive to one task draws away effort from the other tasks. The interaction among many principals results in a loss of the power of incentives. The incentives are weaker compared with the single principal problem. Moreover, the incentives turn out to be less for different interests of the principals in the implemented action and stronger for similar interests of the principals in the implemented action (conflicting interests as opposed to common interests).

Prat and Rustichini (40) considered the general case of complete information game with multiple principals and multiple agents. By analogy with the Truthful Nash Equilibrium in Bernheim and Whinston (11), the authors introduced the concept of Weakly Truthful Equilibrium and showed that any Weakly Truthful Equilibrium is efficient. It was shown that the existence of the efficient equilibria is related top the "balancedness" of the game.

This paper documents several important results about Games Played Through Agents. First, we show that for any equilibrium there exists an equivalent one in terms of the net payoffs and outcome, where the principals use only simple transfer functions, which equal zero everywhere except for a finite number of actions. This result allows us to reduce the set of strategies used by the principals in an equilibrium to the simple ones. In the previous studies the main point of interest was on some specific types of equilibria.

Second, we show that in a game with two principals and one agent, for any equilibrium and any efficient outcome, the equilibrium is dominated by some equilibrium with this predetermined efficient outcome. Therefore, for any inefficient equilibrium there always exists an efficient equilibrium which dominates it. This dominating equilibrium can be found for any efficient outcome. The strategies of the new equilibrium differ from the the strategies of the original equilibrium only at the new outcome. We provide an example of a three principal game with an equilibrium which is not dominated by an efficient one, which means that the dominance result is not extendable to more than two principals. The previous studies did not consider inefficient equilibria and other than Truthful Nash or Weakly Truthful equilibria and thus did not analyze the question of the different equilibria.

We also show that any efficient equilibrium is weakly dominated by the Truthful Nash Equilibrium with the same outcome; the agent's net payoff does not increase and the principals divide the additional amount with each other. Therefore, the Truthful Nash Equilibrium always dominates any other equilibrium.

The paper is organized as follows. Section 2 describes the game and defines equilibrium. Section 3 characterizes equilibria. In section 4, we consider an example of a game with two principals and one agent and find the Truthful Nash Equilibrium. Section 5 discusses efficient equilibria and provides the main results on equilibria dominance.

3.2 The Game

Consider a game with $M \geq 2$ principals and $N \geq 1$ agents. With some abuse of notation, we also denote $M = \{1, 2, \dots M\}$, and $N = \{1, 2, \dots N\}$. Let $m \in M$ denote a typical principal (usually used as the upper index), and $n \in N$ denote a typical agent (used as the lower index). There is a compact space of actions S_n for each agent $n \in N$. We assume that each S_n has metric $\rho_n(\cdot, \cdot)$. Denote $S = \prod_{n \in N} S_n$.

The game proceeds in the following two steps. First, the principals simultaneously and independently set transfer functions $t_n^m(s_n)$ for each agent, depending on the action $s_n \in S_n$ chosen by the agent. Then, each of the agents chooses actions $\hat{s}_n$ based on his transfer functions.

At the end of the game, the principals get net payoffs

$$G^m(\hat{s}) - \sum_{n \in N} t_n^m(\hat{s}_n),$$

and the agents get net payoffs

$$F_n(\hat{s}_n) + \sum_{m \in M} t_n^m(\hat{s}_n).$$

Each function $F_n(\hat{s}_n)$ captures the effort cost agent n incurs by choosing action $\hat{s}_n$, and each function $G^m(\hat{s})$ captures principal m's payoff from the agents' choice of action vector $\hat{s} = (\hat{s}_1, \hat{s}_2, \dots \hat{s}_N)$. The agents and principals maximize their expected net payoffs. The agents maximize their net payoffs by choosing which actions to implement; the principals maximize their net payoffs by choosing the transfer functions that influence the agents' choice of action vector.

We make the following assumptions:

A1. For every agent, there exists action 0 at which the agent does not incur any cost: $F_n(0) = 0$. If all agents choose action 0, the principals have zero payoffs: for any m holds $G^m(0) = 0$;

A2. Cost functions $F_n(s_n)$ are non-positive and continuous on S_n;

A3. Principals' payoffs $G^m(s)$ are non-negative and continuous on S.

Note that on a discrete space all the functions are continuous.

A principal's pure strategy consists of the transfer function for each of the agents $t_n^m(\cdot) : S_n \to \mathbb{R}$. An agent's pure strategy is a mapping $\sigma_n : 2^{S_n} \to S_n$, which defines one action based on the set of all the actions delivering the highest net payoff. Function σ_n returns one action from any set. Set $\hat{S}_n$ of the actions maximizing the agent's net payoff is determined by the transfer functions:

$$\hat{S}_n \left(\left\{ t_n^m(\cdot) \right\}_m \right) = \operatorname*{argmax}_{s_n \in S_n} \left(F_n(s_n) + \sum_{m \in M} t_n^m(s_n) \right).$$

The set of the best actions $\hat{S}_n \left(\left\{ t_n^m(\cdot) \right\}_m \right)$ might be empty if the transfer functions are discontinuous.

Definition. We call strategies $t_n^m(s_n)$ *allowable* if there is no "bluff," i.e., the principals set the transfers which they can afford for any vector of actions:

$$G^m(s) \geq \sum_{n \in N} t_n^m(s_n) \qquad \forall m \in M, \quad \forall s_n \in S_n.$$

For the rest of the paper we suppose that the principals use only allowable strategies $\{t_n^m(\cdot)\}$. With this restriction, the set of principals' possible strategies is limited so that the principals can promise not more than they have.

The other papers (Bernheim and Whinston (11), Prat and Rustichini (40)) concentrated on Truthful Nash Equilibria (Weakly Truthful Equilibria), in which the transfer functions are defined in such a way that they do not exceed the principals' gross payoffs. Therefore, "no bluff" condition was automatically satisfied. However, in general setup the set of equilibria in the game may be much reacher if we allow the principals to use all the possible strategies. In some equilibria the principals might not have enough gross payoff to compensate the agents under different deviations of other principals. Although these equilibria might be considered, we assume that the principals act responsibly and do not use this option.

To simplify the analysis of equilibria, we consider only upper semi-continuous strategies, i.e., such transfer functions $t_n^m(\cdot)$ that for any $s_n \in S_n$ holds

$$\forall \epsilon > 0 \exists \delta > 0 : \forall s_n', \rho_n(s_n, s_n') < \delta \Rightarrow t_n^m(s_n') \leq t_n^m(s_n) + \epsilon.$$

Upper semi-continuity means that the transfer functions are continuous from above, i.e. the limit when the argument converges to some action does not exceed the value of the transfer function at this action. This condition restricts the set of the strategies and ensures that the set of actions maximizing an agent's net payoff is not empty. On the restricted set of the strategies, we can further reduce the strategies to the simple ones, which equal zero everywhere except for a finite set of actions. Note that all the simple strategies are upper semi-continuous.

Definition. A pure strategy equilibrium with outcome $\hat{s}$ is a set of pure strategies $\{t_n^m(\cdot)\}$ and $\{\sigma_n(\cdot)\}$ such that:

1. Given the transfer functions $\{t_n^m(\cdot)\}$ chosen by the principals, the agents choose vector of actions $\hat{s}$:

$$\hat{s}_n = \sigma_n\left(\hat{S}_n\left(\left\{t_n^m(\cdot)\right\}_m\right)\right);$$

2. No principal m can benefit by deviating from his strategy: for any vector of actions $\tilde{s}$ (even $\tilde{s} = \hat{s}$) holds

$$G^m(\hat{s}) - \sum_{n \in N} t_n^m(\hat{s}_n) \geq G^m(\tilde{s}) - \sum_n \left(\sup_{s_n \in S_n} \left(\sum_{l \neq m} t_n^l(s_n) + F_n(s_n) \right) - \sum_{l \neq m} t_n^l(\tilde{s}_n) - F_n(\tilde{s}_n) \right) \quad (3.1)$$

The last condition (no principal wants to deviate) means that no principal can get increase his net payoff by deviating from his equilibrium strategy. To change the outcome, a principal needs to make the agents switch to action vector $\tilde{s}$. To achieve this switch, the principal can use a simple deviation strategy: transfer functions $\{t_n^m(\cdot)\}$ which have some non-zero value at $\tilde{s}_n$ and equal zero everywhere else.

In equation 3.1, the left-hand side represents the principal's net payoff in equilibrium. Non-negative value

$$\sup_{s_n \in S_n} \left(\sum_{l \neq m} t_n^l(s_n) + F_n(s_n) \right)$$

is agent n's best net payoff without principal m's participation. To compensate this net payoff, principal m needs to offer such transfer at $\tilde{s}_n$ that the agent gets at least the same net payoff. As the agent spends $F_n(\tilde{s}_n)$ and gets from other principals

$$\sum_{l \neq m} t_n^l(\tilde{s}_n),$$

principal m needs to compensate for the difference between the best net payoff the agent can achieve without this principal and what the agent gets at $\tilde{s}_n$. As a result, principal m's net payoff equals the difference from what he gets and what he spends on enforcing $\tilde{s}$.

The equilibrium definition is close to the subgame perfect equilibrium. The principals and the agents choose the best response, if it exists. The only problem is when the best response does not exist — it happens for the agents only. In this case, however, we assume that the agents choose action 0 because they can not decide which action is the best - whichever one they choose, there always exists a better one.

Note that equilibrium defines in fact coincides with the subgame perfect equilibrium on the equilibrium path; also, both concepts give the same result on a finite set of actions. If one of the agents chooses action 0 because empty set of best responses, one of the principals can always change his strategy in such a way that the agent chooses a non-zero action; therefore in an equilibrium it never happens that one of the agents has an ampty best response set. Therefore, our equilibrium concept naturally extends the perfect equilibrium concept to the considered multiple agents multiple principals problem.

Definition. We say that equilibrium $\{t_n^m(\cdot)\}$, $\{\sigma_n(\cdot)\}$ with outcome $\hat{s}$ (weakly) dominates equilibrium $\{\tilde{t}_n^m(\cdot)\}$, $\{\tilde{\sigma}_n(\cdot)\}$ with outcome $\tilde{s}$ if each principal in the first equilibrium is not worse than in the sedcond, and at least one principal is strictly better off:

$$G^m(\hat{s}) - \sum_{n\in N} t_n^m(\hat{s}_n) \geq \tilde{G}^m(\tilde{s}) - \sum_{n\in N} \tilde{t}_n^m(\tilde{s}_n) \qquad \forall m \in M;$$

$$\sum_{m\in M}\left(G^m(\hat{s}) - \sum_{n\in N} t_n^m(\hat{s}_n)\right) > \sum_{m\in M}\left(\tilde{G}^m(\tilde{s}) - \sum_{n\in N} \tilde{t}_n^m(\tilde{s}_n) \qquad \forall m \in M\right).$$

In other words, we define dominance with respect to principals only; all the principals should not be worse off, and at least one principal should be strictly better off.

Definition. We say that action $\hat{s}$ is *(Pareto) efficient*, if

$$\hat{s} \in \operatorname*{Argmax}_{s\in S}\left(\sum_{m\in M} G^m(s) + \sum_{n\in N} F_n(s_n)\right),$$

i.e., the sum of the payoffs (principals' and agents') is maximized. Correspondingly, we say that an equilibrium is efficient if its outcome is efficient.

Bernheim and Whinston (11) introduced the concept of the Truthful Nash Equilibrium for games with one agent. In particular, they say that strategies $\{t^m(\cdot)\}$ and $\{\sigma(\cdot)\}$ (we drop the agents' index because there is only one agent) constitute a Truthful Nash Equilibrium if they constitute a Nash equilibrium, and for any m there exists a constants c^m such that

$$t^m(s) = \begin{cases} c^m, & G^m(s) \geq c^m \\ G^m(s), & G^m(s) < c^m. \end{cases}$$

In a Truthful Nash Equilibrium the principals use the transfers that are shifted by a constant from the principal's payoff, where possible. Therefore, the transfers become "truthful," and the principals are almost indifferent to which action is implemented. In a Truthful Nash Equilibrium the agent has the highest net payoff at an efficient action. A Truthful Nash Equilibrium is coalition proof — no coalition can get more by deviating. The existence of Truthful Nash Equilibria was proved by Bernheim and Whinston (11); Prat and Rustichini (40) extended this concept to several agents by introducing the concept of the Weakly Truthful Equilibrium.

We say that an equilibrium is *cooperative* if all the agents are only compensated for their efforts: for any n holds

$$\sum_{m \in M} t_n^m(\hat{s}) + F(\hat{s}_n) = 0.$$

This equation means that no agent gets a positive net payoff. As the principals compensate the agents for their efforts only, the principals may choose transfer functions which equal zero everywhere except for the equilibrium outcome:

$$t_n^m(s_n) = 0 \quad \forall s_n \neq \hat{s}_n.$$

We say that an equilibrium is *competitive* if at least one agent gets a positive net payoff: there exists n such that

$$\sum_{m \in M} t_n^m(\hat{s}_n) + F_n(\hat{s}_n) > 0.$$

3.3 Equilibria Characterization

Equilibrium strategies might have a very complicated structure because of the general action space. Nevertheless, the main points of our interest are the outcome of an equilibrium and equilibrium transfers. First, we want to characterize equilibria with upper semi-continuous transfer functions.

LEMMA 3.1. *Let* $(\{t_n^m(\cdot)\}, \{\sigma_n(\cdot)\})$ *be an equilibrium with outcome* $\hat{s}$ *and upper semi-continuous transfer functions* $\{t_n^m(\cdot)\}$. *Then for any* m *and for any* n *there exists* s_n^m *(we can take* $s_n^m = \hat{s}_n$ *if* $t_n^m(\hat{s}_n) = 0$*) such that*

$$\sum_{p \in M} t_n^p(\hat{s}_n) + F_n(\hat{s}_n) = \sum_{p \neq m} t_n^p(s_n^m) + F_n(s_n^m). \tag{3.2}$$

The lemma says that for each principal and agent, the agent's transfers at the equilibrium outcome are balanced by the transfers at some other action at which the agent gets the same net payoff from all the other principals. We call this other action, along with the agent's net payoff, a *threat*. The existence of the threats means that in the equilibrium all the principals act together, and together compensate the agents for the cost of choosing the equilibrium outcome in addition to providing an additional net payoff. However, no principal can profitably deviate and decrease his transfer in the equilibrium, because there are threats at which the agents will get the same net payoffs as in the equilibrium, and the action of the threat will not deliver to the deviating principal a high enough payoff. In other words, the threats keep the principals from decreasing their incentives in the equilibrium and making the agent choose a different action. Such a threat exists for any principal-agent combination, although some threats (for different principals) may be posed at the same actions. As each agent has his own action space, different agents have different threat spaces.

If a principal wants to change the equilibrium outcome, he has to take into account all the threats. In the new outcome vector the principal, in addition to other principals' transfers, must compensate the agents for their costs and give an appropriate additional amount to compensate for the threats posed by the other principals. As a result, the agents choose the action vector which is favorable to him.

In the Lemma, upper semi-continuity is necessary for such actions $\{s_n^m\}$ to exist. Indeed, in the general case there would be a sequence of converging actions s_{nt}^m such that the agent would not be able to benefit much by switching to this sequence:

$$\left(\sum_{p\in N} t_p^m(\hat{s}_n) + F_n(\hat{s}_n)\right) - \left(\sum_{p\neq n} t_p^m(s_{nt}^m) + F_n(s_{nt}^m)\right) \to 0 \qquad (t\to\infty).$$

The difference is not negative and equals zero at the limit because of the upper semi-continuity of the transfer functions.

The key idea of the proof is the following. Suppose that for some principal m and agent n there is no such s_n^m. Then the principal can decrease his transfer at $\hat{s}$ by

$$\sum_{p\in M} t_n^p(\hat{s}_n) + F_n(\hat{s}_n) - \sup\left(\sum_{p\neq m} t_n^p(s_n^m) + F_n(s_n^m)\right).$$

The difference is non-negative because the agent in equilibrium chooses $\hat{s}_n$. The difference is positive because there is no such s_n^m satisfying equation 3.2 and the transfer functions are upper semi-continuous.

Based on Lemma 2.1, we are ready to characterize all equilibria. Namely, only equilibrium outcome and threats matter for the sustainability of an equilibrium. On the other hand, for any principal m it is enough to have one threat to prevent this principal from deviating from his transfer function for any particular agent. Therefore, a finite number of threats and the equilibrium outcome describe any equilibrium. We have informally proved the following theorem.

THEOREM 3.1. *Let $(\{t_n^m(\cdot)\},\{\sigma_n(\cdot)\})$ be an equilibrium with outcome $\hat{s}$ and upper semi-continuous transfer functions $\{t_n^m(\cdot)\}$. Then there exists an equivalent equilibrium $(\{\tilde{t}_n^m(\cdot)\},\{\tilde{\sigma}_n(\cdot)\})$ in terms of the equilibrium outcome and equilibrium transfers such that for any principal m and any agent n the corresponding transfer function $\tilde{t}_n^m(\cdot)$ equals zero everywhere except for a finite number of actions.*

Therefore, in equilibrium any principal can use only simple transfer functions. These transfer functions are equivalent to a menu: each principal sets his transfers for a finite set of actions. If the agent chooses one of these actions, the principal makes the corresponding transfer. The Theorem establishes a connection with the finite actions models considered in other papers. Namely, the Theorem says that any game with a general agent space is equivalent to a game with finite set of actions, and therefore the major results for finite action games are applicable for the general setup.

3.4 Example

To illustrate the definition of the game, equilibrium, Truthful Nash Equilibrium, efficient equilibrium, and inefficient equilibrium, consider the following two principal one agent game with three actions (see Table 3.1).

	s_1	s_2	s_3
$F(s)$	0	-3	-2
$G^1(s)$	3	4	0
$G^2(s)$	0	4	5

Table 3.1 Definition of the game.

The main characteristic of this game is that at action s_1 Principal 2 does not get any payoff, and Principal 1 does not get any payoff at action s_3. Action $s^* = s_2$ is the only efficient action, at which the total payoff

$$G^1(s^*) + G^2(s^*) + F(s^*) = 4 + 4 - 3 = 5$$

is maximized. There does not exist a cooperative equilibrium because Principal 1 on his own can get $3 - 0 = 3$ at action s_1, and Principal 2 on his own can get $5 - 2 = 3$ at action s_2, and the sum of the individual best net payoffs (6) exceeds the total payoff at s_2 (5).

Corollary 1 from Bernheim and Whinston gives direct formulas to calculate the only Truthful Nash Equilibrium (see Table 3.2). In this equilibrium Principal 1 poses

	s_1	s_2	s_3
$t^1(s)$	1	2	0
$t^2(s)$	0	2	3
$t^1(s) + t^2(s) + F(s)$	1	1	1
$G^1(s) - t^1(s)$	2	2	0
$G^2(s) - t^2(s)$	0	2	2

Table 3.2 Truthful Nash Equilibrium in the game.

a threat of value $d \equiv t^1(s^*) + t^2(s^*) + F(s^*) = 1$ against Principal 2 at action s_1, and Principal 2 poses a threat of value d against Principal 1 at action s_3. Consequently, the agent is indifferent to the choice between s_1, s_2, and s_3, and chooses s_2. For each principal, the equilibrium net payoff equals 2.

Suppose that in an equilibrium the threat $d = 3$ and the agent chooses $s = s_2$ (efficient equilibrium, see Table 3.3). The threat determines $t^1(s_1) = 3$, $t^2(s_1) = 0$,

$t^1(s_3) = 0$, and $t^2(s_3) = 5$. At the same time, there is some flexibility in choosing $t^1(s_2)$ and $t^2(s_2)$ based on parameter $a \in [0, 2]$. Indeed, Principal 1 should get not less than he can get on his own at action s_1, i.e., he should get at least 1. At the same time, Principal 2 should get not less than he can get on his own at action s_3, i.e., he also should get at least 1. The two principals have a total amount $G^1(s^*)+G^2(s^*)+F(s^*)-d = 4+4-3-3 = 2$ to divide among each other, therefore their net payoffs constitute vector $(a, 2-a)$ with the vector of transfers in equilibrium $(4-a, 2+a)$. Required conditions $t^m(s^*) \leq G^m(s^*)$ (the strategies are allowable) hold for these equilibrium transfers.

	s_1	s_2	s_3
$t^1(s)$	3	2+a	0
$t^2(s)$	0	4-a	5
$t^1(s) + t^2(s) + F(s)$	3	3	3
$G^1(s) - t^1(s)$	0	2-a	0
$G^2(s) - t^2(s)$	0	a	0

Table 3.3 Efficient equilibrium with threat $d = 3$.

Therefore, the vector of the principals' payoffs belongs to a line connecting points $(0, 2)$ and $(2, 0)$. The agent's payoff increases to 3 in comparison with the Truthful Nash Equilibrium, and each of the principal's payoffs (weakly) decreases.

It is easy to see that there is no equilibrium with outcome s^* where threat $d < 1$. Indeed, Principal 1 can get net payoff $3-d$ on his own at s_1, and Principal 2 can get net payoff $3-d$ on his own at s_3, and at $s_2 = s^*$ their total net payoff $5-d$ is less than the sum of the individual net payoffs $6-2d$. Therefore, there is no cooperative equilibrium with the efficient outcome in this game.

To illustrate an inefficient outcome influences the net payoffs, consider an equilibrium with threat $d = 3$ and outcome $s = s_3$ (see Table 3.4). At s_3 only Principal 2 can offer a transfer to the agent. At the same time, with this threat Principal 1 can pose a threat only at action s_1, therefore $t^1(s_1) = 3$. There is some flexibility in choosing transfers at action s_2: no principal should want to make the agent choose s_2. It is possible if and only if $t^1(s_2), t^2(s_2) \in [0, 2]$ $(t^m(s_s) + G^{-m}(s_2) + F(s_2) \leq d)$. For simplicity we assume $t^1(s_2) = t^1(s_2)$.

To conclude this section, we provide Table 3.5 with the principals and agent net payoffs in different equilibria. We see that in the Truthful Nash Equilibrium the principals have the highest and the agent has the lowest net payoff. As we move from the Truthful Nash Equilibrium to an efficient equilibrium, and from an efficient equilibrium to an inefficient equilibrium with the same threat, we see that the principals' net payoffs decrease and the agent's net payoff increases.

	s_0	s_1	s_2	s_3
$F(s)$	0	0	-3	-2
$G^1(s)$	0	3	4	0
$G^2(s)$	0	0	4	5
$t^1(s)$	0	3	0	0
$t^2(s)$	0	0	0	5
$t^1(s) + t^2(s) + F(s)$	0	3	0	3
$G^1(s) - t^1(s)$	0	0	4	0
$G^2(s) - t^2(s)$	0	0	4	0

Table 3.4 Inefficient equilibrium with threat $d = 3$.

	TNE	Efficient	Inefficient
$\hat{s}$	s_2	s_2	s_3
$t^1(\hat{s}) + t^2(\hat{s}) + F(\hat{s})$	1	2	3
$G^1(\hat{s}) - t^1(\hat{s})$	2	2	0
$G^2(\hat{s}) - t^2(\hat{s})$	2	1	0

Table 3.5 Comparison of different equilibria.

3.5 Equilibria Dominance

Bernheim and Whinston (11) showed the existence of a Truthful Nash Equilibrium in a game with one agent. For two principals, this equilibrium is unique. Prat and Rustichini (40) extended this result and considered multiple agents. They showed that the existence of a Weakly Truthful Equilibria is closely related to the balansedness of the game. The outcome of a Weakly Truthful Equilibrium is efficient, as in the game with one agent only. However, no paper investigates equilibria other than Truthful Nash Equilibria and Weakly Truthful Equilibria. Moreover, no attempt has been made to compare efficient and inefficient equilibria, Truthful Nash Equilibria and other efficient equilibria. In this section we do not discuss how the principals and agents choose the strategies leading to an equilibrium different from the Truthful Nash Equilibrium. Instead, our question of interest is how the principals and agents can change their strategies and switch to a better equilibrium, and how the payoffs in the new equilibrium correspond to the original one.

First of all, the nature of an inefficient equilibrium is that the principals cooperate in making the agents choose an inefficient action profile. At the same time, no principal has enough resources to profitably make the agents switch to other actions because the principals need to cooperate for this switch, too. Could the principals cooperate, they would prefer to change their strategies and switch to a better outcome equilibrium.

The following theorem states that in any two principal one agent game, the principals and the agent can always simultaneously change their strategies in such a way that an efficient action will be the new equilibrium outcome. The net payoffs will increase for the principals, and the agent will get the same net payoff.

THEOREM 3.2. *For any inefficient equilibrium* $(t^1(s), t^2(s), \sigma(\cdot))$ *with outcome*

$$\hat{s} \notin \operatorname*{Argmax}_{s \in S} \left(\sum_{m=1}^{2} G^m(s) + F(s) \right) \tag{3.3}$$

and upper semi-continuous transfer functions $t^1(s)$ *and* $t^2(s)$ *in a two principal one agent game, and for any efficient action*

$$\tilde{s} \in \operatorname*{Argmax}_{s \in S} \left(\sum_{m=1}^{2} G^m(s) + F(s) \right) \tag{3.4}$$

there exists an equilibrium $(\tilde{t}^1(s), \tilde{t}^2(s), \tilde{\sigma}\cdot)$ *with outcome* $\tilde{s}$ *which (weakly) dominates the original equilibrium; in this new equilibrium the transfer functions are the same everywhere except for* $\tilde{s}$*, and the agent's net payoff does not change.*

The theorem claims that any inefficient equilibrium is weakly dominated by some equilibrium for any efficient outcome. Moreover, in this efficient equilibrium the agent gets the same net payoff, whereas the extra gain is distributed among the principals. The new transfer functions differ from the old ones only at the new outcome. For the new equilibrium the agent's strategy needs to be changed in such a way that he favors the new outcome by choosing it if it belongs to the set of the actions delivering the highest net payoff.

The result of theorem 3.2 does not hold for more than two principals. Consider the following example.

Example 2. Consider a game with three principals and one agent. Space S (we drop the lower index as there is only one agent) consists of 3 actions: $S = \{s_0, s_1, s_2\}$. All functions are given in Table 3.6. Action s_2 is the only efficient outcome.

s	s_0	s_1	s_2
$F(s)$	0	-2	-4
$G^1(s)$	0	2	0
$G^2(s)$	0	2	4
$G^3(s)$	0	0	4

Table 3.6 Example of a game with an equilibrium which is not dominated by a Truthful Nash Equilibrium.

Consider the following inefficient cooperative equilibrium with outcome $\hat{s} = s_1$:

$$\begin{aligned} t^1(s) = t^2(s) &= \begin{cases} 1, & s = s_1; \\ 0, & s \neq s_1, \end{cases} \\ t^3(s) &= 0. \end{aligned}$$

In this equilibrium the agent chooses s_1 if it belongs to the set of actions delivering the highest net payoff.

Principal 1 does not want to deviate because lowering his transfer at s_1 will make the agent switch to s_0, and he can not offer the agent anything at s_2. Principal 2 does not want to deviate because lowering his transfer at s_1 will, again, make the agent switch to s_0, and to make the agent choose s_2, the principal has to compensate the cost at s_2, which would leave him (the principal) with no positive net payoff. Principal 3 does not want to deviate because at s_1 he gets nothing, and at s_2 he can at most compensate the agent's cost, which again will leave him with zero net payoff.

In this equilibrium the first two principals get net payoffs 1 each. This equilibrium is not dominated by any efficient equilibrium with outcome s_2 because the first principal gets nothing at s_2, and consequently his net payoff at s_2 can not be higher than zero.

□

Bernheim and Whinston (11) showed the existence of equilibria for any efficient outcome. In particular, they state for any efficient action there exists a Truthful Nash Equilibrium with this outcome. However, the set of all equilibria is much richer because not every equilibrium is equivalent in terms of transfers to a Truthful Nash Equilibrium. We want to describe how these equilibria correspond to each other.

THEOREM 3.3. *In a game with two principals and one agent, for any equilibrium* $(t^1(\cdot), t^2(\cdot), \sigma(\cdot))$ *with outcome* $\hat{s}$ *and upper semi-continuous transfer functions* $t^1(\cdot)$ *and* $t^2(\cdot)$*, the principals' equilibrium net payoffs do not exceed the principals' net payoffs in the Truthful Nash Equilibrium. Correspondingly, the agent's net payoff outmatches the agent's net payoff in the Truthful Nash Equilibrium.*

Example 3. To illustrate how the possibility of bluffs influences the set of equilibria, consider the following example of a game presented in Table 3.7.

	s_0	s_1	s_2
$F(s)$	0	0	0
$G^1(s)$	0	2	0
$G^2(s)$	0	2	0

Table 3.7 Game for a bluff example.

There exists only one no bluff equilibrium with outcome s_2 where the principals always set zero transfers and get net payoffs zero. With bluff, for any $a_1, a_2 \in [0, 2]$ there exists an equilibrium with the principals' net payoffs $2 - a_1$ and $2 - a_2$ and the agent's net payoff $a_1 + a_2$. Indeed, to implement this equilibrium the principals can use transfer functions from Table 3.8. In this equilibrium, we see that the agent's net payoff increased in comparison with the Truthful Nash Equilibrium, and the principals' net payoffs decreased.

	s_0	s_1	s_2
$t^1(s)$	$a_1 + a_2$	a_1	0
$t^2(s)$	0	a_2	$a_1 + a_2$

Table 3.8 Transfers for the bluff equilibrium.

□

Any equilibrium is the result of a composition of three possible deviations from a Truthful Nash Equilibrium. First, the outcome might be different from an efficient outcome. Second, the threat might be different from the threat in the Truthful Nash Equilibrium. And last, the strategies do not need to be the Truthful Nash Equilibrium strategies.

In the first case of deviation (inefficient outcome), Theorem 3.2 states that there exists an efficient equilibrium in which the agent gets the same net payoff, and the principals' net payoffs are weakly better (the sum of the net payoffs is strictly higher). Second, if the outcome is efficient, but the threat is different from the Truthful Nash Equilibrium threat, then the agent gets more and the principals are weakly worse than in the Truthful Nash Equilibrium. Finally, if the strategies are different from the Truthful Nash Equilibrium strategies, but the outcome and the threats are the same, then the payoffs are the same, too. As we see, any deviation from the Truthful Nash Equilibrium does not make the agent worse, but might decrease the principals' net payoffs.

3.6 Proofs

Proof of lemma 3.1.

Fix principal m and agent n. Denote $a = \sum_{p \in M} t_n^p(\hat{s}_n) + F_n(\hat{s}_n) \geq 0$. For $a = 0$ the statement of the lemma holds, therefore we can consider only $a > 0$. For each $\epsilon \geq 0$ define

$$K_\epsilon = \left\{ s_n \,\middle|\, \sum_{p \neq m} t_n^p(s_n) + F_n(s_n) - a \geq -\epsilon \right\}.$$

Sets K_ϵ are compact (functions $t_n^m(\cdot)$ are upper semi-continuous) and embedded. Also, equation $K_0 = \lim_{\epsilon \longrightarrow 0} K_\epsilon$ holds from the construction of sets K_ϵ.

Suppose the lemma does not hold, i.e. $K_0 = \emptyset$. From compactness of sets S_n, there exists $\epsilon > 0$ such that $K_\epsilon = \emptyset$, and therefore

$$\sum_{p \neq m} t_n^p(s_n) + F_n(s_n) - a < -\epsilon \qquad \forall s_n \in S_n.$$

As $a > 0$, we have

$$t_n^m(\hat{s}) \equiv a - \left(\sum_{p \neq m} t_n^p(\hat{s}_n) + F_n(\hat{s}_n) \right) > \epsilon,$$

and principal m has an incentive to decrease his transfer function to

$$\tilde{t}_n^m(s_n) = \begin{cases} t_n^m(\hat{s}_n) - \epsilon, & s_n = \hat{s}_n; \\ 0, & s_n \neq \hat{s}_n. \end{cases}$$

With this new transfer function, agent n's net payoff is

$$\tilde{t}_n^m(s_n) + \sum_{p \neq m} t_n^p(s_n) + F_n(s_n) = \begin{cases} a - \epsilon, & s_n = \hat{s}_n \\ \sum_{p \neq m} t_n^p(s_n) + F_n(s_n) < a - \epsilon, & s_n \neq \hat{s}_n. \end{cases}$$

Therefore, agent n still will choose action $\hat{s}_n$, and principal m's net payoff will increase by ϵ because he decreases his transfer at $\hat{s}_n$ by ϵ. Contradiction with our initial assumption that $\left(\{t_n^m(\cdot)\}, \sigma_n(\cdot)\right)$ constitute an equilibrium. Hence, there has to be action s_n^m such that

$$\sum_{p \in M} t_n^p(\hat{s}_n) + F_p(\hat{s}_n) = \sum_{p \neq m} t_n^p(s_n^m) + F_n(s_n^m).$$

Now let us prove that $t_n^m(s_n^m) = 0$. If not so, then

$$\begin{aligned} \sum_{p \in M} t_n^p(s_n^m) + F_p(s_n^m) &= t_n^m(s_n^m) + \sum_{p \neq m} t_n^p(s_n^m) + F_p(s_n^m) \\ &= t_n^m(s_n^m) + \sum_{p \in M} t_n^p(\hat{s}_n) + F_p(\hat{s}_n) = t_n^m(s_n^m) + a > a, \end{aligned}$$

therefore in the equilibrium agent n could not choose action $\hat{s}_n$. Contradiction with the definition of the equilibrium.

□

Proof of theorem 3.1.

It is enough to prove that any principal can switch to the described simple strategy. We know that

$$\hat{s}_n \in \operatorname*{Argmax}_{s_n \in S_n} \left(\sum_{p \in M} t_n^p(s_n) + F_p(s_n) \right),$$

therefore agent n's payoff

$$\sum_{p\in M} t_n^p(\hat{s}_n) + F_p(\hat{s}_n) \geq \sum_{p\in M} t_n^p(0) + F_p(0) = \sum_{p\in M} t_n^p(0) = 0$$

because $t_n^p(0) \in [0, G^p(0)] = \{0\}$.

For any combination of n and m find s_n^m satisfying equation 3.2 from Lemma 2.1. Define

$$\tilde{t}_n^m(s_n) = \begin{cases} t_n^m(s_n), & s_n = \hat{s}_n; \\ t_n^m(s_n), & s_n = s_n^{m'} \text{ for some } m'; \\ 0, & s_n \neq \hat{s}_n,\ s_n \neq s_n^{m'} \text{ for all } m'. \end{cases}$$

Also redefine $\sigma_n(\cdot)$ in the following way: for any $X_n \subset S_n$, let

$$\tilde{\sigma}_n(X_n) = \begin{cases} \hat{s}_n, & \hat{s}_n \in X_n \\ \sigma_n(X_n), & s_n \notin X_n. \end{cases}$$

Note that we did not change the transfer functions at $\hat{s}$ and did not increase the transfer functions at all the other action. Therefore, $\hat{s}$ still delivers the highest payoff to the agents. Because we redefined $\sigma_n(\cdot)$ in such a way that the agents favor $\hat{s}$ ($\hat{s}_n$ belongs to the set of the best actions, the agent will choose this action), all the agents together will choose the original equilibrium outcome $\hat{s}$. No principal will benefit from making the agents choose other actions by increasing his transfer functions because he could not benefit from it before and all the other agents did not increase their transfer functions. No principal will benefit from decreasing his transfer functions because for any agent all the other principals pose a threat which the principal can not overcome. Therefore, strategies $(\{\tilde{t}_n^m(\cdot)\}, \{\tilde{\sigma}_n(\cdot)\})$ constitute an equilibrium with outcome $\hat{s}$ and with the the same equilibrium transfers as in the original equilibrium.

Note that if

$$\sum_{p\in M} t_n^p(\hat{s}_n) + F_p(\hat{s}_n) = 0$$

for some agent n, then 0 can be taken as the threat point against every principal and the new transfer functions can be defined as

$$\tilde{t}_n^m(s_n) = \begin{cases} t_n^m(s_n), & s_n = \hat{s}_n; \\ 0, & s_n \neq \hat{s}_n. \end{cases}$$

□

Proof of theorem 3.2.

From equation 3.4 follows that for any action $s \in S$, including $\hat{s}$, holds

$$(G^1(\tilde{s}) + G^2(\tilde{s}) + F(\tilde{s})) - (G^1(s) + G^2(s) + F(s)) \geq 0.$$

Therefore, in comparison with $\hat{s}$ there is some additional payoff at $\tilde{s}$ which equals

$$a = (G^1(\tilde{s}) + G^2(\tilde{s}) + F(\tilde{s})) - (G^1(\hat{s}) + G^2(\hat{s}) + F(\hat{s})) > 0.$$

This additional payoff could be divided among the principals in an equilibrium with outcome $\tilde{s}$.

From the fact that the agent chooses $\hat{s}$ in the original equilibrium, it follows that no principal can benefit by making the agent choose $\tilde{s}$, i.e. for any $m \in M$ holds

$$\begin{aligned} G^m(\tilde{s}) + F(\tilde{s}) + t^{-m}(\tilde{s}) - (t^1(\hat{s}) + t^2(\hat{s}) + F(\hat{s})) &\leq G^m(\hat{s}) - t^m(\hat{s}); \\ G^m(\hat{s}) - G^m(\tilde{s}) + F(\hat{s}) - F(\tilde{s}) + t^{-m}(\hat{s}) - t^{-m}(\tilde{s}) &\geq 0. \end{aligned} \quad (3.5)$$

To prove the theorem it is sufficient to show that we can change $t^1(s)$ and $t^2(s)$ at $\tilde{s}$ so that the new transfer functions along a choice function $\tilde{\sigma}(s)$ favoring $\tilde{s}$ constitute an equilibrium such that the new equilibrium principals' net payoffs do not decrease and the agent's net payoff (threat) does not decrease.

Define new transfer functions in the following way:

$$\tilde{t}^1(s) = \begin{cases} G^1(\tilde{s}) - G^1(\hat{s}) + t^1(\hat{s}) - y, & s = \tilde{s}; \\ t^1(s), & s \neq \tilde{s}, \end{cases}$$
$$\tilde{t}^2(s) = \begin{cases} t^2(\hat{s}) + F(\hat{s}) - F(\tilde{s}) - G^1(\tilde{s}) + G^1(\hat{s}) + y, & s = \tilde{s}; \\ t^2(s), & s \neq \tilde{s}, \end{cases}$$

where $y \in [0, a]$. The second transfer function was chosen to provide at $\tilde{s}$ the same net payoff to the agent as at $\hat{s}$. Note that

$$\begin{aligned} G^1(\tilde{s}) - \tilde{t}^1(\tilde{s}) &= G^1(\hat{s}) - t^1(\hat{s}) + y \geq G^1(\hat{s}) - t^1(\hat{s}); \\ G^2(\tilde{s}) - \tilde{t}^2(\tilde{s}) &= G^2(\tilde{s}) - t^2(\hat{s}) - F(\hat{s}) + F(\tilde{s}) + G^1(\tilde{s}) - G^1(\hat{s}) - y \\ &\geq G^2(\tilde{s}) - t^2(\hat{s}) - F(\hat{s}) + F(\tilde{s}) + G^1(\tilde{s}) - G^1(\hat{s}) - a \\ &= G^2(\tilde{s}) - t^2(\hat{s}) - F(\hat{s}) + F(\tilde{s}) + G^1(\tilde{s}) - G^1(\hat{s}) \\ &\qquad -(G^1(\tilde{s}) + G^2(\tilde{s}) + F(\tilde{s})) - (G^1(\hat{s}) + G^2(\hat{s}) + F(\hat{s}))) \\ &= G^2(\hat{s}) - t^2(\hat{s}), \end{aligned}$$

i.e. for any m and for any y holds

$$G^m(\tilde{s}) - \tilde{t}^m(\tilde{s}) \geq G^m(\hat{s}) - t^m(\hat{s}). \quad (3.6)$$

Therefore, with the new transfer functions no principal wants to make the agent to switch back to $\hat{s}$.

Now we have to verify that the values of the new transfer functions $\tilde{t}^m(\cdot)$ at $\tilde{s}$ exceed zero. (Inequalities "no bluff" $\tilde{t}^m(\tilde{s}) \leq G^m(\tilde{s})$ follows from inequality 3.6.) The principals at $\tilde{s}$ have net payoffs not less than at $\hat{s}$. Let us find the transfer of the first principal at action $\tilde{s}$:

$$\begin{aligned} \tilde{t}^1(\tilde{s}) &= G^1(\tilde{s}) - G^1(\hat{s}) + t^1(\hat{s}) - y \\ &\geq G^1(\tilde{s}) - G^1(\hat{s}) + t^1(\hat{s}) \\ &\qquad -(G^1(\tilde{s}) + G^2(\tilde{s}) + F(\tilde{s})) + (G^1(\hat{s}) + G^2(\hat{s}) + F(\hat{s})) \\ &= t^1(\hat{s}) - G^2(\tilde{s}) - F(\tilde{s}) + G^2(\hat{s}) + F(\hat{s}) \geq t^1(\tilde{s}) \geq 0. \end{aligned}$$

In the last line we applied inequality 3.5. In the same way one can prove the similar inequality for the second principal:

$$\tilde{t}^2(\tilde{s}) \geq t^2(\tilde{s}) \geq 0.$$

Finally, no principal wants to make the agent to choose some action $s \neq \tilde{s}$ because this principal not want to make the agent to switch to s in the original equilibrium.

□

Proof of theorem 3.3.

Based on Theorem 3.2, it is enough to show the result if $\hat{s}$ is an an efficient outcome,

$$\hat{s} \in \operatorname*{Argmax}_{s \in S} \left(\sum_{m \in M} G^m(s) + F(s) \right).$$

The proof has two steps. First, we show that the Truthful Nash Equilibrium has the minimal threat (which equals the agent's net payoff). Second, we demonstrate that for any other threat no principal can get more than in the Truthful Nash Equilibrium.

Step 1. For threat d in the Truthful Nash Equilibrium, suppose that Principal 1 poses the threat against Principal 2 at s_1, and Principal 2 poses the threat against Principal 1 at s_2. Then from the definition of the Truthful Nash Equilibrium, the following equation holds:

$$\begin{aligned}(G^1(s_1) + F(s_1) - d) + (G^2(s_2) + F(s_2) - d) &= (G^1(s^*) - t^1(s^*)) + (G^2(s^*) - t^2(s^*)) \\ &= G^1(s^*) + G^2(s^*) + F(s^*) - d.\end{aligned}$$

Therefore,

$$d = (G^1(s_1) + F(s_1)) + (G^2(s_2) + F(s_2)) - (G^1(s^*) + G^2(s^*) + F(s^*)).$$

Condition $d > 0$ means that there does not exist a cooperative equilibrium.

Suppose that there exists an equilibrium with $d' \in [0, d)$ and with outcome $\hat{s}$. Then the total principals' net payoff

$$\begin{aligned}&(G^1(\hat{s}) - t^1(\hat{s})) + (G^2(\hat{s}) - t^2(\hat{s})) \\ &\qquad = G^1(\hat{s}) + G^2(\hat{s}) + F(\hat{s}) - d' \\ &\qquad \leq G^1(s^*) + G^2(s^*) + F(s^*) - d' \\ &\qquad = (G^1(s_1) + F(s_1) - d) + (G^2(s_2) + F(s_2) - d) + d - d' \\ &\qquad = (G^1(s_1) + F(s_1) - d') + (G^2(s_2) + F(s_2) - d') + d' - d \\ &\qquad < (G^1(s_1) + F(s_1) - d') + (G^2(s_2) + F(s_2) - d'),\end{aligned}$$

i.e. one of the principals will have a profitable deviation, which contradicts to the equilibrium definition.

Step 2. Now we want to show that in any equilibrium $t^1(\cdot), t^2(\cdot), \sigma(\cdot))$ with threat $d' \geq d$ the principals can not get more than in the Truthful Nash Equilibrium. Suppose not, i.e. Principal 1 gets more than in the Truthful Nash Equilibrium, i.e.

$$\begin{aligned} G^1(s^*) - t^1(s^*) &> G^1(s_1) + F(s_1) - d \\ &= (G^1(s^*) + G^2(s^*) + F(s^*)) - (G^2(s_2) + F(s_2)). \end{aligned}$$

Then

$$\begin{aligned} G^2(s^*) - t^2(s^*) &= (G^1(s^*) + G^2(s^*) + F(s^*)) - (G^1(s^*) - t^1(s^*)) - d' \\ &< (G^2(s_2) + F(s_2)) - d', \end{aligned}$$

and by making the agent choose s_2 the second principal will get at least

$$(G^2(s_2) + F(s_2)) - d' + \epsilon,$$

which for small enough ϵ is greater than the current net payoff $G^2(s^*) - t^2(s^*)$. Contradiction with the equilibrium definition.

□

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Vita

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