The Pennsylvania State University
The Graduate School
Department of Economics

ESSAYS ON COOPERATION, COORDINATION, AND CONFORMITY

A Thesis in
Economics
by
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Abstract

This thesis consists of three essays, discussing three related aspects of human behavior, namely, cooperation, coordination, and conformity.

The first essay studies the repeated Prisoner’s Dilemma in a local interaction setup. We construct a sequential equilibrium in pure strategies that sustains cooperation for sufficiently patient players. The notion of sequential equilibrium is extended to extensive form games with infinite time horizon and additive payoffs across time. The strategy is embedded in an explicitly defined expectation system, which is a more compact way to describe strategies than machines in the local interaction setup, although essentially the expectation system can also be viewed as a finite state automaton. The belief system is derived by perturbing the strategy appropriately and following the principle that parsimonious explanations receive all the weight. The equilibrium has the property that after any global history, full cooperation will be restored after a finite number of periods. Therefore, the explicitly defined expectation system serves as a social norm. What matters is not a common observation of a physical outcome, what matters is a common understanding of the social norm, the understanding that everybody knows the norm and is willing to follow it after any history.

The second essay deals with coordination games. By adding a small amount of noise to the information structure, the theory of global games is able to select a unique equilibrium in coordination games with a finite number of players and two actions, a safe action and a risky action. As the noise vanishes, however, it is often the case that positive amount of inefficiency remains in the selected equilibrium. This essay argues that this is partly due to the simultaneity of the moves. If the game is played sequentially with the order of moves determined endogenously, and if the risky action is irreversible and the safe action is reversible, then efficiency will be asymptotically restored as the noise vanishes. However, if the safe action is irreversible, then dynamics will not make much difference to the possible inefficiency of equilibria. Thus two coordination games may look very similar if they are treated as simultaneous move games, yet they can be very different if they are treated as sequential move games. For example, there has been much recent interest in the phenomenon of currency attacks and its similarity to the well-known model of bank runs. However, we show that these games are quite different in the dynamic setting and endogenous timing might help to resolve inefficiencies in the first but not the second.

In the third essay we propose to use Polya urn processes to model the emergence of order in an environment where people interact with each other sequentially and indirectly, through a common physical facility. Examples include rewinding video tapes, erasing blackboards, and flushing toilets, etc. We find that a minimum amount of imitation is able to generate a maximum level of conformity.
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Chapter 1

Sustaining Cooperation in the Repeated Prisoner’s Dilemma with Local Interaction

1.1 Introduction

This chapter considers a society where individuals interact with others locally, but a social norm of cooperative behavior is nonetheless sought to be sustained in society as a whole. The development and stability of social norms of cooperation is usually studied through an infinitely repeated prisoner’s dilemma, and we adopt the same approach in this chapter.

An example might clarify the nature of the problem. Consider a typical road in residential suburban America where each house has a lawn in front. Each houseowner derives a utility \( v \) from a well maintained lawn, but can only see her own lawn and those of her neighbors. The cost \( c \) of maintaining one’s lawn is strictly greater than \( v \). In the case each homeowner has two neighbors, her payoff, if both neighbors maintain their lawns and she doesn’t, is \( 2v \), while if she does is \( 3v - c \). If neither neighbor maintains his lawn, her payoff is \( v - c \) if she does and 0 otherwise.

Would we expect to see the lawns well maintained along the road in the absence of police enforcement of regulations? This chapter argues that this is possible in pure strategies. In general, we want to model a situation in which local interaction doesn’t create an intrinsic barrier to global coordination.

1.1.1 The model

The model has the following features:

1. A straight line with finitely many integer points.

2. On each integer point lives an agent. Each agent has two neighbors except the end agents, who have only one neighbor.\(^1\) Let \( N(j) \) denote \( j \)’s neighborhood.

3. Time is discrete, \( t = 1, 2, ..., \infty \).

4. The stage game is the prisoner’s dilemma.

\[
\begin{array}{cc|cc}
 & C & D \\
C & 1,1 & -l,1+g \\
D & 1+g,-l & 0,0 \\
\end{array}
\]

\(^1\)For ease of exposition, we assume that the reference player (player \( j \)) has two neighbors. None of the results in the paper depends on this assumption.
where $g > 0$, $l > 0$.

5. In each period, each player plays the stage game with his two neighbors. He plays the same action against the two neighbors, and his stage game payoff is the sum of his payoffs against both neighbors.

6. Everybody has the same discount factor $\delta$.

7. Each agent only observes what happens in his own neighborhood.

8. Let $h_t^j$ denote player $j$’s private history: $h_t^j = (a_{s,t-1}, a_{s,t}, a_{s,t+1})_{s=1}^{t-1}$, where $a_s \in \{C, D\}$, $k \in N(j)$. Let $H_j$ denote the set of $j$’s private histories. A pure strategy of player $j$ is a mapping from $H_j$ to $\{C, D\}$.

We are looking for a sequential equilibrium\(^2\) that supports cooperation on the line. The first natural candidate is, of course, the trigger strategy.\(^3\) A trigger strategy plays cooperation after any history in which no one in the neighborhood has ever defected, and defection otherwise. It turns out that when $l \geq 1$, the trigger strategy works if $g + l \leq \frac{g'}{1+g'}$. When $l < 1$, the trigger strategy also works if $\delta \in \left[ \frac{g}{1+g}, \frac{g+l}{1+g} \right]$. The trigger strategy fails when $l < 1$ and $\delta$ is large enough for a simple reason. When the cost of being defected upon is smaller than the payoff to mutual cooperation, and when people are sufficiently patient, they have an incentive to block the spread of defection. That is, people would rather live between a good neighbor and a bad one, than to punish the bad neighbor and then live between two bad neighbors forever. The problem with the trigger strategy is that punishment is both too severe and too lenient. It is too severe in that even the slightest mistake will never be forgiven; it is too lenient in that further deviation (like blocking) is not further punished.

There is a quick fix to the trigger strategy when $l < 1$ and $\delta$ is too large. We know that the trigger strategy still works for $\delta \in \left[ \frac{g}{1+g}, \frac{g+l}{1+g} \right]$. By Lemma 2 in Ellison(1994), for any sufficiently large $\delta$, we can always use trigger strategy to support cooperation by diluting the original game into a certain number of replica games. Players play the trigger strategy in each replica, and when they play in some replica, they ignore observations from other replicas. Effectively dilution reduces players’ discount factor so that it falls back into the interval $\left[ \frac{g}{1+g}, \frac{g+l}{1+g} \right]$. The problems with dilution are that (a) The diluted strategy is not uniform with respect to $\delta$: the larger $\delta$ is, the more replicas are needed to

---

\(^2\)The standard notion of sequential equilibrium is defined for finite extensive form games. We have an extension of it in the next section.

\(^3\)The discussion on trigger strategy and mixed strategy below is derived in a note of V. Bhaskar.
make people less patient; (b) The equilibrium is not "globally stable" in the sense that after some histories, full cooperation will never be restored.

An alternative to resolving the problem is using a mixed strategy. The idea is to create some uncertainty about whether one's neighbor is going to punish or block defection, in such a way that this player himself is indifferent between punishing and blocking. The nice feature of the mixed strategy equilibrium is that bad behavior can be localized so that most part of the society is left unaffected even if some player always plays defection. A mixed strategy equilibrium of this form exists when the line is infinite in both directions. With a finite number of agents, there is an "endpoint" problem, to be explained in the next subsection.

The main result of this chapter is the following. In the repeated prisoner’s dilemma on the finite line, for any \( l < 1, g > 0 \), there is a sequential equilibrium in pure strategies that supports cooperation for any sufficiently large \( \delta \).

As in standard folk theorem type of strategies (Fudenberg and Maskin (1986)), we need a book keeping device to keep track of punishments and rewards as the game goes on. In standard theory this is done by a machine, but here a machine is a very inconvenient way to describe strategies, due to the fact that different players observe different things in the game. Instead of writing down a machine explicitly, I define a pair of expectation operators recursively, \((E_j(\cdot|h_j^t), E_j(E_k(\cdot)|h_j^t))\). For any private history \( h_j^t \), \( E_j(\cdot|h_j^t) \) is the expectation that \( j \) forms on the future path of play in his neighborhood, \( E_j(E_k(\cdot)|h_j^t) \) is the expectation that \( j \) forms on his neighbor \( k \)'s expectation on \( j \)'s future actions. After \( h_j^t \), \( j \) simply does what \( E_j(\cdot|h_j^t) \) expects him to do. As we will see in the chapter, the device of expectations keep track of things more efficiently than a machine. It also has the added advantage that when we go to sequential equilibrium, we can use these expectations as an intermediate step to prove sequential rationality.

1.1.2 An example

Let's illustrate the strategy by the following three player example.

```
1 2 3
```

Initially everybody expects everybody to play cooperation forever. Then if say player 2 is surprised by a defection from player 1, 2 expects himself to begin a punishment of, say \( T \) periods, and then to go back to cooperation forever. 2 expects 1 to play anything (i.e. neither \( D \) nor \( C \) will surprise him) in the next \( T - 1 \) periods, then 2 expects 1 to go back to \( C \) (the ambiguity in 2's expectation is not necessary here, since 2 knows that if 1 follows the strategy, 1 will have \( T - 1 \) periods of \( D \) to play for sure. The ambiguity will be useful after more complicated histories, so that when someone finishes punishing one neighbor, he
can safely go back to $C$ without further surprising the other neighbor). 2 also expects that 1 expects 2 to punish for $T$ periods, and if 2 fails to punish, 2 will surprise 1 and trigger a more severe punishment. At the same time, 2 anticipates that the punishment will surprise 3 in turn, so he expects 3 to punish him for $T$ periods. If 3 fails to fully carry out the punishment, then 2 expects himself to punish 3 for a much longer period of time, say $bT$ periods, $b > 1$. Meanwhile, it is the mutual expectation of 2 and 3 that 3 should keep playing cooperation after 3 blocks. Finally 2 should also anticipate that his long punishment will keep surprising 1 from time to time later on, and 1 should react to it appropriately, and so on. A kind of social norm can thus be established by specifying people’s expectations after any history. People then use the expectations to judge other people’s behavior, and to guide their own.

Given any history of player 2, if he expects himself to play $C$ in the next period, the strategy is going to be defined such that he also expects that at least one of the neighbors also expects him to play $C$, and if he deviates, he will postpone the time when the entire neighborhood goes back to $C$; if he expects himself to play $D$ in the next period, he doesn’t want to play $C$ because it is either unnecessary to do so (when both 2 and 3 expect anything from him), or too costly to do so (when he expects at least one neighbor to expect him to play $D$). So far, sequential rationality is relative to the artificially defined expectation operators. Sequential equilibrium requires that the strategy be optimal with respect to real expectations formed under a consistent belief system. As we will see in later sections, essentially these real expectations are going to be duplicated by the artificial expectations, if we perturb the strategy appropriately.

We can also see from this example why the mixed strategy doesn’t work for finite number of players: 2 has incentive to mix between punishing and blocking only if 1 and 3 also have such incentive, but they don’t.

**1.1.3 Related Literature**

The literature that initially studies the repeated prisoner’s dilemma with local interaction takes an evolutionary approach. Bergstrom and Stark (1993) consider a group of farmers living along a road that loops around a lake. Each farmer plays the prisoner’s dilemma with his two immediate neighbors. In the next period, the farmers’ sons take over and decide whether to cooperate or defect by imitating the most successful player within his neighborhood in the last period. Eshel, Samuelson and Shaked (1998) study a similar setup but add noises to the model. They find that in the long run, the system is going to spend most of its time in states in which there is a significant proportion of cooperators, provided that the mutation rate is small. Nowak and May (1993) simulate a similar game on a two dimensional lattice. Their simulation generates chaotically changing spatial patterns, in which both the cooperators and the defectors persist indefinitely. The main idea of these papers is that local interaction, combined with simple imitation rules, helps maintain cooperation.
because if cooperators are grouped together, the benefits of cooperation are enjoyed mainly among cooperators, who then earn relatively high payoffs and are imitated. The reason that local interaction might help sustain cooperation no longer applies when the players are fully rational, which is the case in this chapter. Nonetheless, we show that global cooperation is still possible to achieve, provided that the players are sufficiently patient.

This chapter can also be viewed as a special variation under a general theme, which is to disperse information in a repeated game among the players and ask whether the efficient outcome can still be maintained or not. There are other ways to disperse information. In Kandori (1992), people are pairwise matched at random in each period and play the prisoner’s dilemma in each match. Each player only observes what goes on in his own matches, but not in other matches. Kandori constructs a contagion strategy that supports the cooperative outcome, provided that the cost of being defected upon is sufficiently large.4 Bhaskar (1998) studies a simple transfer game in an overlapping generation setup. He found that with some mild informational constraints, transfers (from the young to the old in each period) cannot be supported by pure strategy equilibria. For example, if each generation only observes the actions of the past generation, then the only pure strategy equilibrium is global autarchy.

Apart from the superficial differences between our model and the random matching model, there are similarities as well as differences between the two. In both models, if $l$ is large enough, the trigger strategy works for sufficiently large $\delta$;5 if $l$ is small, the trigger strategy works for moderate values of $\delta$. Moreover, Ellison’s dilution idea applies to both models. The differences are more subtle: in the random matching model with public randomization, the supporting strategy is uniform with respect to $\delta$, so long as $\delta$ is large enough. Without public randomization, however, the supporting strategy is not uniform with respect to $\delta$. In addition, the equilibrium with public randomization is globally stable,6 but without public randomization, it is unknown whether global stability is possible. In our model, the strategy, call it $E$ from now on, is both uniform with respect to $\delta$ and globally stable, for $l < 1$ and $g > 0$.

In the anonymous random matching model, a player’s information about history can be summarized by a one dimensional statistic. It is impossible to keep track of other players’ actions, and it is not necessary either. Although a player needs to guess how many players have been infected after any history, Kandori and Ellison simplified the analysis by constructing a strategy that is optimal against any reasonable guesses. Hence consistency is not an issue. In

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4Ellison (1994) embedded a public randomizing device into the contagion strategy and showed that cooperation can be supported for any payoff parameters.

5In the random matching model the cutoff value of $l$ depends on the population size, but it doesn’t here.

6In the sense of Kandori (1992), global stability requires that players be able to return to efficient outcome eventually after any history.
our model, however, the information is two dimensional, and a player has to
treat them separately. Instead of implementing a $T$-period punishment scheme
probabilistically, as in the strategy with public randomization in the random
matching model, we carry it out deterministically here. The tradeoff is we have
to specify a consistent belief system, because the strategy cannot be a best
response for all belief systems.

This chapter is organized as follows. Section 1.2 defines the pure strategy $E$,
by defining a pair of expectation operators inductively. Section 1.3 shows that
the strategy is sequentially rational with respect to the expectation operators.
Section 1.4 derives a consistent belief system, under which real expectations can
be formed after any private history. It is then shown that the real expectations
formed after a history can be mimiced by the expectation operators after the
same history. Section 1.5 discusses extensions to circles and trees. Section 1.6
concludes.

1.2 The strategy $E$

1.2.1 The solution concept

The solution concept we use is sequential equilibrium. Extending the notion
of sequential equilibrium from finite extensive form games to infinite extensive
form games requires no conceptual innovation, but it involves some technical
difficulty. However, when the only infinite object is the number of informa-
tion sets, and the number of information sets is countably many, then there
is a natural extension of sequential equilibrium, which I give in the following
definitions.

Definition Given an assessment $(\sigma, \mu)$, $\mu$ is consistent with respect to $\sigma$
if there exists a sequence of completely mixed behavior strategy profiles $\sigma(n)$,
such that

(i) $\sigma(n)$ converges to $\sigma$ in $l_\infty$,

(ii) $\mu(n)$ converges to $\mu$ pointwise.

where $l_\infty$ is the space of bounded infinite sequences of real numbers with the
supremum norm, and $\mu(n)$ is the belief system derived from $\sigma(n)$ using Bayes’
rule.

Remark The reason I impose uniform convergence on the part of the strate-
gies, but only pointwise convergence on the part of the beliefs, is discussed in
Appendix A.

Definition An assessment $(\sigma, \mu)$ is a sequential equilibrium if
(i) $\sigma$ is sequentially rational with respect to $\mu$ and
(ii) $\mu$ is consistent with respect to $\sigma$.

**Remark** This definition doesn’t cover the possibility that an information set may contain a continuum of nodes, which is my case when there are infinitely many players. So I focus on finite number of players in the model. Infinite number of players case will be discussed in the appendix.

In practice, given $\sigma$, it is convenient to construct a consistent belief system $\mu$ in the following way. First let’s define a mistake pattern $M$. Fix $\epsilon > 0$, perturb $\sigma$ such that $\forall$ player $j$, $\forall$ information set $h_j$ of $j$, $\forall$ action $a_j$ of $j$ at $h_j$, $M$ assigns a polynomial of $\epsilon$ (could be a constant) to $a_j$ as the probability that $j$ will play $a_j$ at $h_j$. The perturbations are independent across information sets, and $\forall a_j' \in h_j$, $M(a_j') > 0$, $\sum_{a_j' \in h_j} M(a_j') = 1$, $\forall h_j, \forall j$.

Under a mistake pattern $M$, there is a perturbed strategy profile $\sigma(\epsilon)$. We require that $\exists (\epsilon_n)_{n}$, such that $\epsilon_n \to 0$ as $n \to \infty$, and $\sigma(\epsilon_n) \to \sigma$ in $l_{\infty}$ as $n$ goes to infinity. Now fix an information set $h_j$. Since there is a one to one correspondence between the nodes in $h_j$ and the histories that lead to those nodes, it suffices to form a belief vector over the histories. A history will also be called an explanation since it explains why the information set is reached. Fix a history $h^t$ that leads to $h_j$, compute the probability of $h^t$ according to $\sigma(\epsilon)$. The significant part is the smallest power of $\epsilon$ and the corresponding coefficient. Denote that power by $\pi(h^t)$. Define $H(h^t_j) = \arg \min \{ \pi(h^t) \mid h^t \text{ explains } h_j \}$. Only members in $H(h^t_j)$ survive as $\epsilon \to 0$. We say $H(h^t_j)$ is the collection of parsimonious explanations. Now allocate the whole probability mass among members in $H(h^t_j)$ in proportion to their coefficients. This is the belief vector over $h_j$. Clearly a belief system formed in this way is consistent with respect to $\sigma$.

### 1.2.2 Notations and Definitions

Before we define $E$, we introduce the following notations and definitions:

**Notations**

1. $j$’s expectation on $N(j)$’s continuation actions after $h_j^t$:

$$\left( (E_j(a^*_k|h^t_j))_{k \in N(j)} \right)_{s=t}^\infty$$

where $E_j(a^*_k|h^t_j) \in \{C, D\}$, and $E_j(a^*_k|h^t_j) \in \{C, D, D/C\}$, for any $k \in N(j) \setminus \{j\}$. When $j$ expects $D/C$ from $k$, it means that $j$ expects anything from $k$, i.e. neither $D$ nor $C$ will contradict $j$’s expectation, hence surprise $j$. 

7
2. $j$’s expectation on $k$’s expectation on $j$’s continuation actions after $h^t_j$, $k \in N(j) \setminus \{j\}$:

$$\left( (E_j(E_k(a^s_j|h^t_j))_{k \in N(j) \setminus \{j\}})_{s=t} \right)^{\infty}$$

where $E_j(E_k(a^s_j|h^t_j)) \in \{C, \ D, \ D/C\}$.

3. $j$’s "debt" to $k$ upto $h^t_j$: $\lambda_{jk} (h^t_j)$. This is the number of $C$’s $j$ owes to $k$ upto $h^t_j$.

4. $k$’s "debt" to $j$ upto $h^t_j$: $\lambda_{kj} (h^t_j)$. This is the number of $C$’s $k$ owes to $j$ upto $h^t_j$.

5. Let $h^t_j$ be the null history of $j$.

**Definitions**

1. $\forall h^t_j$, an action profile in period $t (a^t_k)_{k \in N(j)}$ is a surprise to $j$, if $(a^t_k)_k \neq (E_j(a^s_j|h^t_j))_k$. To slightly abuse notation, $\forall a^t_k \in \{C, \ D\}$, if $E_j(a^s_j|h^t_j) = D/C$, then $a^t_k = E_j(a^s_j|h^t_j)$.

2. $\forall h^t_j, \forall (a^t_k)_k, j$ expects $a^t_j$ to surprise $k \in N(j) \setminus \{j\}$, if $a^t_j \neq E_j(E_k(a^s_j|h^t_j))$. In this case we just say that $j$ surprises $k$. Again, to slightly abuse notation, $\forall a^t_j \in \{C, \ D\}$, if $E_j(E_k(a^s_j|h^t_j)) = D/C$, then $a^t_j = E_j(E_k(a^s_j|h^t_j))$. Similarly, $\forall E_j(a^s_j|h^t_j) \in \{C, \ D\}$, if $E_j(E_k(a^s_j|h^t_j)) = D/C$, then $E_j(a^s_j|h^t_j) = E_j(E_k(a^s_j|h^t_j))$.

3. $a \left[ (E_j(E_k(a^s_j|h^t_j))_{s=t}) \right]$: number of periods in which ($j$ expects that) $k$ expects $j$ to play $D$ conditional on $h^t_j$.

4. $a \left[ (E_j(a^s_k|h^t_j))_{s=t} \right]$: number of periods in which $j$ expects $k$ to play $D$ conditional on $h^t_j$.

5. $r(x) = \begin{cases} 
0 & \text{if } x < 0 \\
\lfloor \frac{x}{\tau} \rfloor + 1 & \text{if } x \geq 0 \text{ and } \frac{x}{\tau} \text{ is integer} \\
\lfloor \frac{\lfloor x/\tau \rfloor}{\tau} \rfloor & \text{if } x \geq 0 \text{ and } \frac{x}{\tau} \text{ is not integer,} \\
\text{where } \lfloor \frac{x}{\tau} \rfloor & \text{is the largest integer to the left of } \frac{x}{\tau}.
\end{cases}$
1.2.3 A Road Map

The strategy $E$ is going to be defined in a non-standard way. In this subsection I want to motivate the way I define $E$, and sketch a road map that I will follow in the remaining sections.

At the beginning I have some principles and assumptions in mind. The principles are: 1. Unexpected defection must be punished. 2. Unexpected cooperation must be punished even harder. 3. A certain amount of tolerance in expectations (ambiguities in expectations) is needed to cushion the uncertainty in the environment.

To implement these principles, I define a pair of expectation operators and a pair of "debt" operators. Both operate on a player’s private histories. $E_j(\cdot|\cdot)$ is the first order operator, and $E_j(E_k(\cdot)|\cdot)$ is the second order operator, which is used to define the first order operator: only when $j$ knows how $k$ thinks about $j$ can $j$ predict future reactions of $k$ to $j$’s actions. (Below I’ll explain why I don’t need higher order operators.) The "debt" operators $\lambda_{jk}(\cdot)$ and $\lambda_{kj}(\cdot)$ are convenient to keep record of "rights" and "obligations" between two neighbors. They are also used in the definition of the first order operator. Here is a schematic illustration:

$$
\begin{array}{c}
\lambda_{jk}(\cdot) \\
\downarrow \\
E_j(\cdot|\cdot) \\
\downarrow \\
E_j(E_k(\cdot)|\cdot) \\
\downarrow \\
\text{pure strategy } E
\end{array}
$$

Now suppose that the above principles are common knowledge among the players. For player $j$ to be able to form expectations on his neighbors, $j$ has to make some assumptions on the behavior of those he can not see. The assumption $j$ has in mind is that $(N(j))^c$ never make mistakes, i.e., $j$ imagines that the background is always clean. If this assumption holds, (1) $E_j(\cdot|h_j^t)$ is indeed what will happen in $N(j)$ from $t$ on, (2) $E_j(E_k(\cdot)|h_j^t)$ is indeed what $k$ expects $j$ from $t$ on, and (3) when $j$ is surprised by $k$, $k$ knows that; when $j$ thinks that $k$ surprises $j$ or will surprise $k$, $k$ is indeed or will indeed be surprised. It turns out that even if $(N(j))^c$ is not clean, we still have (1), (2), and (3) above, as long as the background is as clean as possible (or, the explanation is a parsimonious one).

Now fix $h_j^t$. Let $h_j^t_{jk}$ be the restriction of $h_j^t$ to $\{j,k\}$. Let $h_k^t$ be the private history of $k$ in which the common history between $j$ and $k$ is $h_j^t_{jk}$, and the other neighbor of $k$ is "clean", i.e. has followed the strategy with no deviation. Now let’s go back to a question asked before: why do I not need higher
order expectation operators? For example, why do I not need something like $E_j (E_k (E_j (a_k^t)) | h_j^t)$? Because it’s redundant: if $j$ wants to guess what $k$ expects $j$ to expect $k$ to do, given $j$’s information $h_j^t$, then $j$ can simply use $k$’s second order operator $E_k (E_j (a_k^t) | h_j^t)$. What about $E_j (E_k (E_j (a_j^t)) | h_j^t)$? Again, it’s redundant, because given $h_j^t$, it is common knowledge between $j$ and $k$ what $k$ expects $j$ to do. In one word, the information $h_j^t$ doesn’t support any higher order expectations. Once $E_j (\cdot | h_j^t)$ and $E_j (E_k (\cdot) | h_j^t)$ are there, any higher order operators are automatically defined.

Once we have $E$, the goal is to find a consistent belief system $B$, such that $(E, B)$ is a sequential equilibrium. To this end, we need to show $\forall h_j^t$, $E$ is optimal with respect to the expectations formed under $B$. However, it is easier to show that $E$ is optimal with respect to the expectations built in the definition of $E$. Hence to achieve the goal, it suffices to show that the expectations formed under $B$ can be mimicked by the built in expectations. In the next subsection, we define $E$. In section 1.3, we show $E$ is optimal with respect to the built in expectation operators. We define $B$ in section 1.4 and show that the real expectations formed under $B$ essentially coincide with the auxiliary expectations in the definition of $E$.

1.2.4 The definition

Now we define the strategy $E$, by defining $E_j (\cdot | \cdot)$, $E_j (E_k (\cdot) | \cdot)$, $\lambda_{jk} (\cdot)$ and $\lambda_{kj} (\cdot)$ inductively. Once the expectation pair is well defined for any $h_j^t$, $j$ simply does what $E_j (\cdot | \cdot)$ expects him to do after $h_j^t$. In the following definition, the $d (\cdot)$ function and the $r (\cdot)$ function are as defined in Section 1.2.2 on notations and definitions. In the following definition, we include some intuitive explanations in brackets.

Initial configurations

$\lambda_{jk} (h_j^1) = 0$

$\lambda_{kj} (h_j^1) = 0$

$E_j (E_k (a_j^s) | h_j^t) = C \quad \forall k \in N (j) \setminus \{j\}, \; s \geq 1$

$E_j (a_k^s | h_j^t) = C \quad \forall k \in N (j), \; s \geq 1$

[Initially $j$ expects everybody in his neighborhood to play $C$ forever, $j$ also expects that his neighbors expect him to play $C$ forever.]
Fix $t$, $h^t_j$, $\lambda_{jk} (h^t_j)$, $\lambda_{kj} (h^t_j)$, $E_j (\cdot | h^t_j)$, $E_j (E_k (\cdot) | h^t_j)$, and $\left( a^t_k \right)_k$. Let $h^{t+1}_j = (h^t_j, \left( a^t_k \right)_k)$

0 First we define $\lambda_{jk} (h^{t+1}_j)$ and $\lambda_{kj} (h^{t+1}_j)$.

If $\lambda_{jk} (h^t_j) = 0$, then $\lambda_{jk} (h^{t+1}_j) = \begin{cases} \left( \begin{array}{c} bT \text{ if } a^t_j = C \neq E_j (E_k (a^t_j) | h^t_j) \text{ and } \sim \{ a^t_k = C \neq E_j (a^t_k | h^t_j) \} \\ 1 \text{ if } a^t_j = D \neq E_j (E_k (a^t_j) | h^t_j) \text{ and } a^t_k = E_j (a^t_k | h^t_j) \\ 0 \text{ otherwise} \end{array} \right) \\
\lambda_{jk} (h^t_j) - 1 \text{ if } a^t_j = C = E_j (E_k (a^t_j) | h^t_j) \neq D/C \\
\lambda_{jk} (h^t_j) \text{ otherwise} \end{cases}$

If $\lambda_{jk} (h^t_j) > 0$, then $\lambda_{jk} (h^{t+1}_j) = \begin{cases} \left( \begin{array}{c} 0 \text{ if } a^t_k \neq E_j (a^t_k | h^t_j) \\ \lambda_{jk} (h^t_j) - 1 \text{ if } a^t_j = C = E_j (E_k (a^t_j) | h^t_j) \neq D/C \\ \lambda_{jk} (h^t_j) \text{ otherwise} \end{array} \right) \\
\lambda_{jk} (h^t_j) - 1 \text{ if } a^t_k = C = E_j (a^t_k | h^t_j) \neq D/C \\
\lambda_{jk} (h^t_j) \text{ otherwise} \end{cases}$

1 Second we define $E_j (E_k (\cdot) | h^{t+1}_j)$

Fix $k \in N (j) \setminus \{ j \}$

1.1 If $a^t_j = E_j (E_k (a^t_j) | h^t_j)$

1.1.1 If $a^t_k = E_j (a^t_k | h^t_j)$, then $E_j (E_k (a^t_j | h^{t+1}_j)) = E_j (E_k (a^t_j | h^t_j)) \quad s \geq t + 1$

[If neither $j$ nor $k$ surprises the other, then the updated expectation is just the continuation of the old one.]

1.1.2 If $a^t_k \neq E_j (a^t_k | h^t_j)$

1.1.2.1 $a^t_k = C \quad E_j (E_k (a^t_j) | h^{t+1}_j) = \begin{cases} D \text{ next } bT \text{ periods} \\ C \text{ thereafter} \end{cases}$

1.1.2.2 $a^t_k = D \quad E_j (E_k (a^t_j) | h^{t+1}_j) = \begin{cases} D \text{ next } T - 1 + \lambda_{kj} (h^{t+1}_j) \text{ periods} \\ C \text{ thereafter} \end{cases}$
If $k$ surprises $j$ by playing defection, then $k$ should not only expect $j$ to punish this defection, but also expect $j$ to collect whatever he owes to $j$ in the past.

1.2 If $a^t_j \neq E_j (E_k (a^t_j) | h^t_j)$

1.2.1 $a^t_j = C$ \hspace{1cm} $E_j (E_k (a^s_j) | h^{t+1}_j) = C$ \hspace{1cm} $s \geq t + 1$

1.2.2 $a^t_j = D$

1.2.2.1 $a^t_k = E_j (a^t_k | h^t_j)$ \hspace{1cm} $E_j (E_k (a^s_j) | h^{t+1}_j) = \begin{cases} D/C & \text{next } T-1 \text{ periods} \\ C & \text{thereafter} \end{cases}$

1.2.2.2 $a^t_k \neq E_j (a^t_k | h^t_j)$

1.2.2.2.1 $a^t_k = C$ \hspace{1cm} $E_j (E_k (a^s_j) | h^{t+1}_j) = \begin{cases} D/C & \text{next } bT \text{ periods} \\ C & \text{thereafter} \end{cases}$

1.2.2.2.2 $a^t_k = D$ \hspace{1cm} $E_j (E_k (a^s_j) | h^{t+1}_j) = \begin{cases} D/C & \text{next } T-1 \text{ periods} \\ \text{thereafter} \end{cases}$

[In case of mutual surprise of defection, $k$ should expect $j$ to play defection for $T-1$ periods, then go back to $C$.]

2 Third we define $E_j (\cdot | h^{t+1}_j)$.

2.1 If $(a^t_k)_k = (E_j (a^t_k | h^t_j))_k$, then $(E_j (a^s_j | h^{t+1}_j))_k = (E_j (a^s_j | h^t_j))_k$ \hspace{1cm} $s \geq t + 1$

[If $j$ is not surprised in period $t$, then the updated expectation is just the continuation of the old one.]

2.2 If $(a^t_k)_k \neq (E_j (a^t_k | h^t_j))_k$, let $S = \{k \in N(j) \mid a^t_k \neq E_j (a^t_k | h^t_j)\}$

2.2.1 If $S = \{j\}$, and $a^t_j = C$, and $E_j (E_k (a^s_j) | h^t_j) = D/C$ \hspace{1cm} $\forall k \in N(j) \setminus \{j\}$, then $(E_j (a^s_j | h^{t+1}_j))_k = (E_j (a^s_j | h^t_j))_k$ \hspace{1cm} $s \geq t + 1$
2.2.2 Otherwise

2.2.2.1 If \( \max_k d \left( \left(E_j \left( E_k \left( a_j^* \right) \vert h_j^{t+1} \right) \right)_{s \geq t+1} \right) > 0 \)

then \( E_j \left( a_j^* \vert h_j^{t+1} \right) = \begin{cases} D & \text{next } \max_k d \left( \left(E_j \left( E_k \left( a_j^* \right) \vert h_j^{t+1} \right) \right)_{s \geq t+1} \right) \text{ periods} \\ C & \text{thereafter} \end{cases} \)

[j will always fully carry out his punishment obligations.]

If \( \max_k d \left( \left(E_j \left( E_k \left( a_j^* \right) \vert h_j^{t+1} \right) \right)_{s \geq t+1} \right) = 0 \)

then \( E_j \left( a_j^* \vert h_j^{t+1} \right) = \begin{cases} D & \text{as long as } E_j \left( E_k \left( a_j^* \right) \vert h_j^{t+1} \right) = D/C \forall k \\ C & \text{otherwise} \end{cases} \)

2.2.2.2 (corresponds to 1.1.2.1) For \( k \in S, \ k \neq j, \text{ and } a_k^* = C \)

2.2.2.2.1 If \( a_j^* = C \neq E_j \left( E_k \left( a_j^* \right) \vert h_j^t \right) \), then

\[
E_j \left( a_k^* \vert h_j^{t+1} \right) = \begin{cases} C & \text{next period} \\ D & T/ \tau \left( d \left( \left(E_j \left( a_j^* \vert h_j^{t+1} \right) \right)_{s \geq t+1} \right) \right) \text{ periods} \\ C & \text{thereafter} \end{cases}
\]

2.2.2.2.2 Otherwise

\[
E_j \left( a_k^* \vert h_j^{t+1} \right) = \begin{cases} C & \text{next } bT + 1 \text{ periods} \\ D & \text{next } T/ \tau \left( d \left( \left(E_j \left( a_j^* \vert h_j^{t+1} \right) \right)_{s \geq t+1} \right) - bT \right) \text{ periods} \\ C & \text{thereafter} \end{cases}
\]

[When \( j \) is surprised by \( k \) playing \( C \), \( j \) expects \( k \) to pay him a long string of \( C \)'s, but it might be the case that \( j \) will surprise \( k \) when the punishment is over, so \( j \) should anticipate that down the road a punishment from \( k \) will be triggered.]

2.2.3 For \( k \in S, \ k \neq j, \text{ and } a_k^* = D \)

2.2.3.1 (corresponds to 1.1.2.2) If \( a_j^* = E_j \left( E_k \left( a_j^* \right) \vert h_j^t \right) \)
\begin{equation}
E_j(a^*_k|h^{t+1}_j) = \begin{cases} 
  D/C & \text{next } T - 1 \text{ periods} \\
  C & \text{next } \lambda_{kj} (h^{t+1}_j) + 1 \text{ periods} \\
  D & \text{next } T \cdot r \left( d \left[ (E_j(a^*_j|h^{t+1}_j))_{s \geq t+1} \right] - (T - 1 + \lambda_{kj} (h^{t+1}_j)) \right) \text{ periods} \\
  C & \text{thereafter} 
\end{cases}
\end{equation}

\[ j \text{ gives } k \text{ some room } (T - 1 \text{ periods}) \text{ to "breathe"}, \text{ just in case } k \text{ is punishing other people. Then } j \text{ continues to collect whatever } k \text{ owes to him after the surprise. } j \text{ also anticipates possible punishments from } k \text{ if he has to play defection for such a long time that he surprises } k \text{ later on.} \]

2.2.2.3.2 If \( a^*_j \neq E_j \left( E_k (a^*_j) | h^t_j \right) \)

If \( a^*_j = C \) (corresponds to 1.2.1)

\begin{equation}
E_j(a^*_k|h^{t+1}_j) = \begin{cases} 
  D & \text{next } T - 1 \text{ periods} \\
  C & \text{next } 1 \text{ period} \\
  D & \text{next } T \cdot r \left( d \left[ (E_j(a^*_j|h^{t+1}_j))_{s \geq t+1} \right] - (T - 1) \right) \text{ periods} \\
  C & \text{thereafter} 
\end{cases}
\end{equation}

2.2.2.4 For \( k \notin S, \ k \neq j \)

2.2.2.4.1

If \( a^*_j \neq E_j \left( E_k (a^*_j) | h^t_j \right) \)

If \( a^*_j = C \) (corresponds to 1.2.1)

\begin{equation}
E_j(a^*_k|h^{t+1}_j) = \begin{cases} 
  D & \text{next } T - 1 \text{ periods} \\
  C & \text{thereafter} 
\end{cases}
\end{equation}

If \( a^*_j = D \) (corresponds to 1.2.2.2)

\begin{equation}
E_j(a^*_k|h^{t+1}_j) = \begin{cases} 
  D & \text{next } T \cdot r \left( d \left[ (E_j(a^*_j|h^{t+1}_j))_{s \geq t+1} \right] + bT \right) \text{ periods} \\
  C & \text{thereafter} 
\end{cases}
\end{equation}

\begin{equation}
E_j(a^*_k|h^{t+1}_j) = \begin{cases} 
  D & \text{next } T - 1 \text{ periods} \\
  C & \text{next } 1 \text{ period} \\
  D & \text{next } T \cdot r \left( d \left[ (E_j(a^*_j|h^{t+1}_j))_{s \geq t+1} \right] - (T - 1) \right) \text{ periods} \\
  C & \text{thereafter} 
\end{cases}
\end{equation}

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After $j$ surprises $k$ by playing defection, $j$ should expect $k$ to punish this defection and possible surprises in the future, as well as to collect whatever debt ($\lambda_{jk}(h_j^{t+1})$ periods of $C$) $j$ owes to $k$ up to $h_j^{t+1}$.]

2.2.2.4.2 (corresponds to 1.1.1)

Otherwise, let $\tau = \min\{s \geq t + 1|E_j(a_j^n|h_j^{t+1}) \neq E_j(E_k(a_j^n)|h_j^{t+1})\}$

If $E_j(a_k^n|h_j^{t+1}) = C$

$$E_j(a_k^n|h_j^{t+1}) = \begin{cases} C \text{ upto } \tau \\ D \text{ } T \cdot r \left[d \left(E_j(a_j^n|h_j^{t+1})\right)_{s \geq t+1}\right] - (\tau - (t + 1)) \text{ periods} \\ C \text{ thereafter} \end{cases}$$

[In this case $j$ should anticipate all the possible punishment by $k$ from period $\tau$ on.]

If $E_j(a_k^n|h_j^{t+1}) \neq C$ and $E_j(a_j^{t+1}|h_j^{t+1}) = D$, then

If $\lambda_{jk}(h_j^{t+1}) > 0$

$$E_j(a_k^n|h_j^{t+1}) = \begin{cases} E_j(a_k^n|h_j^{t+1}) \text{ upto } \tau \\ D \text{ } T \cdot r \left[d \left(E_j(a_j^n|h_j^{t+1})\right)_{s \geq t+1}\right] - (\tau - (t + 1)) + \lambda_{jk}(h_j^{t+1}) - 1 \text{ periods} \\ C \text{ thereafter} \end{cases}$$

[In this case $j$ not only anticipates all the punishment by $k$ from period $\tau$ on, but also expects $k$ to collect whatever he owes to $k$ up to period $\tau$.]

If $\lambda_{jk}(h_j^{t+1}) = 0$

$$E_j(a_k^n|h_j^{t+1}) = \begin{cases} E_j(a_k^n|h_j^{t+1}) \text{ upto } \tau \\ D \text{ } T \cdot r \left[d \left(E_j(a_j^n|h_j^{t+1})\right)_{s \geq t+1}\right] - (\tau - (t + 1)) \text{ periods} \\ C \text{ thereafter} \end{cases}$$

If $E_j(a_k^n|h_j^{t+1}) \neq C$ and $E_j(a_j^{t+1}|h_j^{t+1}) = C$, then

$$E_j(a_k^n|h_j^{t+1}) = \begin{cases} D \lambda_{jk}(h_j^{t+1}) + \# \{s \geq t + 1|E_j(E_k(a_j^n)|h_j^{t+1}) = D/C\} \text{ periods} \\ C \text{ thereafter} \end{cases}$$

**Remark**

Complicated as it looks, the definition is trying to capture some simple principles in this society: 1. Unexpected defection must be punished, no matter what
causes the defection. 2. Unexpected cooperation must be punished much more severely so as to keep people from defecting in the first place. 3. Ambiguities in expectations are essential when observations are partial, they lubricates the society.

Before moving on to the next section, it is worth pointing out that even though defined in the same way, the strategy for the end players is dramatically simpler than the strategy for a middle player. We are able to explicitly write the end players’ strategy as a finite state automaton, which we present in Figure 1. For the ease of exposition, let $T = 2$, and $b = 2$. In this automaton, initially the end player is in state $C$, where she is supposed to play cooperation. If she observes $(C, C)$, she stays in the same state. If she observes $(C, D)$, she moves to state $D1$ and begins a two period punishment. If she executes the punishment as planned, and if her neighbor returns to cooperation on time, then she goes back to state $C$. If she fails to carry out the punishment, say, in $D1$, then she moves to state $I1$, and begins to accept a punishment of four periods, and so on. There is not enough space to write out fully all the incoming and outgoing arrows. In particular, if any dotted $D$ in the diagram becomes a $C$, then the corresponding outgoing arrow should go to either state $B1$, or state $B1$, depending on whether it is the end player’s duty to punish (in which case go to $B1$), or it is her neighbor’s duty to punish (in which case go to $B1$).

Notice that the expectation operators essentially defines a finite state automaton, where a state is a possible expectation generated by the operator; the initial state is the initial expectation; the action prescribed in a state is the action prescribed in an expectation; and the transition rules from one state to another are given by the transition rules from one expectation to another expectation.

### 1.3 $E$ is sequentially rational with respect to the built-in expectations

In the definition of $E$, we build players’ expectations on their neighborhood’s continuation actions into the strategy. Now we are in a position to check sequential rationality of $E$ with respect to such expectations. To this end, we ask the following two questions:

**Question 1:** $\forall t, \forall h^t_j$, if $E_j (a^t_j | h^t_j) = C$, is it profitable for $j$ to play $D$?

**Question 2:** $\forall t, \forall h^t_j$, if $E_j (a^t_j | h^t_j) = D$, is it profitable for $j$ to play $C$?

Let $h^{t+1}_j = (h^t_j; (E_j (a^t_{j-1} | h^t_{j-1}), E_j (a^t_j | h^t_j), E_j (a^t_{j+1} | h^t_{j+1})))$.

Let $\tilde{h}^{t+1}_j = (\tilde{h}^t_j; (E_j (a^t_{j-1} | h^t_{j-1}), \tilde{E}_j (a^t_j | h^t_j), E_j (a^t_{j+1} | h^t_{j+1})))$,
Figure 1: End player machine
where \( \tilde{E}_j (a^*_j | h^t_j) = \begin{cases} 
C & \text{if } E_j (a^*_j | h^t_j) = D \\
D & \text{if } E_j (a^*_j | h^t_j) = C
\end{cases} \).

In the spirit of the one step deviation property, \( \forall t, \forall h^t_j \), player \( j \) compares

\[
\left( (E_j (a^*_k | h^t_j) )_{k \in N(j)} \right)_{s=t}^{\infty} = \left( (E_j (a^*_j | h^t_j) , E_j (a^*_j | h^t_j) , E_j (a^*_j | h^t_j) ) , \left( (E_j (a^*_k | h^{t+1}_j) )_{k \in N(j)} \right)_{s=t+1}^{\infty} \right) \quad (i)
\]

with \( \left( (E_j (a^*_j | h^t_j) , E_j (a^*_j | h^t_j) , E_j (a^*_j | h^t_j) ) , \left( (E_j (a^*_k | h^{t+1}_j) )_{k \in N(j)} \right)_{s=t+1}^{\infty} \right) \quad (ii) \)

Notice that for some \( h^t_j \), it might be that \( E_j (a^*_j | h^t_j) = D/C \) and/or \( E_j (a^*_j | h^t_j) = D/C \), so that \( h^{t+1}_j \) and \( \tilde{h}^{t+1}_j \) do not take single values. In this case, just replace the \( D/C \)'s in the conditioning histories in \( \left( (E_j (a^*_k | h^{t+1}_j) )_{k \in N(j)} \right)_{s=t+1}^{\infty} \) and

\[
\left( (E_j (a^*_k | \tilde{h}^{t+1}_j) )_{k \in N(j)} \right)_{s=t+1}^{\infty}
\]

by \( D \) or \( C \), to obtain well defined continuation expectations. By construction of \( E \), it doesn’t matter how we replace \( D/C \): they all generate the same continuation expectations. Also notice that \( \forall h^t_j \),

\[
\left( (E_j (a^*_k | h^{t+1}_j) )_{k \in N(j)} \right)_{s=t+1}^{\infty}
\]

is always single valued, i.e. there is no \( D/C \) in it. But \( \left( (E_j (a^*_k | h^{t+1}_j) )_{k \in N(j)} \right)_{s=t+1}^{\infty} \) might contain some ambiguity. When we compare \((i)\) with \((ii)\), we always replace \( D/C \) in \( \left( (E_j (a^*_k | h^{t+1}_j) )_{k \in N(j)} \right)_{s=t+1}^{\infty} \) by \( D \), which is the worst case of \((i)\).

The two questions then ask whether \((i)\) is always preferred to \((ii)\) by \( j \). To answer these questions, we need the following claims, all of which follow from the definition of \( E \), and can be proved by induction and definition. We leave the proofs in Appendix B.

**Claim 1** \( \forall h^t_j, \forall k \in N(j) \setminus \{j\}, \text{if } E_j (a^*_j | h^t_j) = E_j (a^*_k | h^t_j) = C, \text{then } E_j (a^*_j | h^t_j) = E_j (a^*_k | h^t_j) = C, \forall s \geq t, \text{and } E_j (E_k (a^*_j | h^t_j) = C). \)

**Claim 2** \( \forall h^t_j, \forall s \geq t, \forall k \in N(j) \setminus \{j\}, \text{if } E_j (E_k (a^*_j | h^t_j) = D, \text{then } E_j (a^*_j | h^t_j) = D, \).

**Claim 3** \( \forall h^t_j, \forall s \geq t, \forall k \in N(j) \setminus \{j\}, \text{if } E_j (E_k (a^*_j | h^t_j) = D/C, \text{then } E_j (a^*_j | h^t_j) = D). \)

**Claim 4** \( \forall h^t_j, \forall k \in N(j) \setminus \{j\}, \text{if } E_j (E_k (a^*_j | h^t_j) = D/C, \text{then} \)
Claim 5 \( \forall h_j^t, \forall k \in N \setminus \{j\}, \) if \( E_j (a_j^t | h_j^t) = C \) and \( j \) doesn’t owe \( k \) any \( C \), then \( E_j (a_k^t | h_j^t) = C \).

Claim 6 \( \forall h_j^t, \) if \( E_j (a_j^t | h_j^t) = C \), and \( E_j (a_k^t | h_j^t) \neq C \), \( \forall k \in N \setminus \{j\} \), then \( \exists k' \in N \setminus \{j\} \) such that \( E_j (E_k (a_j^t | h_j^t) | h_j^t) = C \). Moreover, \( \forall k \in N \setminus \{j\} \), player \( j \) owes \( k \) \( m(k) \) \( C \)'s, \( 1 \leq m(k) \leq bT \).

Claim 7 \( \forall h_j^t, \) if \( E_j (a_j^t | h_j^t) = D \), and if \( E_j (E_k (a_j^t | h_j^t) | h_j^t) \neq D \), \( \forall k \in N \setminus \{j\} \), then \( E_j (E_k (a_j^t | h_j^t) | h_j^t) = D/C \), \( \forall k \in N \setminus \{j\} \).

Claim 8 \( \forall h_j^t, \) if \( E_j (a_j^t | h_j^t) = D \), and \( \exists k \) s.t. \( E_j (E_k (a_j^t | h_j^t) | h_j^t) = D \), then \( E_j (a_k^{t'} | h_j^t) = C \), \( \forall s \geq t + T - 1 \), where \( k' = \arg \max_{k \in N \setminus \{j\}} d \left( (E_j(E_k (a_j^s | h_j^t))_{s \geq t} \right) \).

Now we are ready to answer questions 1 and 2. In the following argument, (i) represents the future described by \( E_j (\cdot | h_j^t) \) if \( j \) does not deviate in period \( t \), (ii) represents the future described by \( E_j (\cdot | h_j^t) \) if \( j \) deviates in period \( t \).

Fix \( h_j^t \).

1 \( E_j (a_j^t | h_j^t) = C \)

1.1 \( \exists k \) s.t. \( E_j (a_k^t | h_j^t) = C \). In the analysis under 1.1, we need claims 1 through 5.

1.1.1 \( j \) owes \( m C \)'s to \( k' \), \( 1 \leq m \leq bT \)

1.1.1.1 \( E_j (E_k^{t'} (a_j^t) | h_j^t) = C \)

(i)

\( k : C(\infty) \)

\( j : C(\infty) \)

\( k' : D(m) + C(\infty) \)
That is, \( j \) expects \( k \) and himself to play \( C \) forever, \( j \) expects \( k' \) to play \( D \) for \( m \) periods, then play \( C \) forever. The notations that follow are interpreted similarly.

\[(ii)\]
\[k : \ C(1) + D(T) + C(\infty)\]
\[j : \ D(T) + C(\infty)\]
\[k' : \ D(T+m) + C(\infty)\]

\[(i) \succeq_j (ii)\]

1.1.1.2 \( k' \) expects \( j \) to play \( D/C \) in period \( t \).

\[(ii)\] is better than \((i)\) in period \( t \). \((i)\) is better than \((ii)\) in the next \( T \) periods, because in the next \( T \) periods, whenever it’s \((C \ C \ D)\) in \((i)\), it’s \((D \ C \ D)\) or \((D \ D \ D)\) in \((ii)\), and whenever it’s \((C \ C \ C)\) in \((i)\), it’s \((D \ C \ C)\) in \((ii)\). After the next \( T \) periods, \((i)\) and \((ii)\) coincide.

1.1.2 \( j \) doesn’t owe \( k' \) any \( C \).

In this case it must be that \( E_j (E_{k'}(a_j^t)|h_j^t) = C \).

\[(i)\]
\[k : \ C(\infty)\]
\[j : \ C(\infty)\]
\[k' : \ C(\infty)\]

\[(ii)\]
\[k : \ C(1) + D(T) + C(\infty)\]
\[j : \ D(T) + C(\infty)\]
\[k' : \ C(1) + D(T) + C(\infty)\]

\[(i) \succeq_j (ii)\]

1.2 No one is playing \( C \) with \( j \). In the analysis under 1.2, we need claims 2, 3, 4, and 6. We know \( \exists k' \) s.t. \( E_j (E_{k'}(a_j^t)|h_j^t) = C \), and \( j \) must owe \( k' \) \( m \ C's, \ 1 \leq m \leq bT \).

1.2.1 \( E_j (E_{k'}(a_j^t)|h_j^t) = \hat{C} \). Since \( j \) doesn’t expect \( k \) to play \( C \) with him, \( j \) must also owe \( k \) \( m' \ C's, \ 1 \leq m' \leq bT \).
\( k : D(m') + C(\infty) \)
\( j : C(\infty) \)
\( k' : D(m) + C(\infty) \)

\((ii)\)
\( k : D(T + m') + C(\infty) \)
\( j : D(T) + C(\infty) \)
\( k' : D(T + m) + C(\infty) \)

\((i) \succeq_j (ii)\)

1.2.2 \( E_j (E_k(a^j_t)|h^j_t) = D_j \hat{C} \). Let \( n \) be the number of periods in which \( k \) expects anything from \( j \), let \( m' \) be the number of \( C's \) that \( j \) owes \( k \). \( 1 \leq n \leq T - 1, \quad 1 \leq m' \leq bT \).

1.2.2.1 \( m' + n \leq m \)

We compare the undiscounted sum of payoffs of \((i)\) and \((ii)\) in the next \( m + T \) periods, since after the next \( m + T \) periods, \((i)\) and \((ii)\) coincide. Denote the payoffs by \( \pi_1 \) and \( \pi_2 \), respectively.

\[ \pi_1 = (m' + n)(-2l) + (m - (m' + n))(1-l) + T \cdot 2 \]
\[ \pi_2 = n \cdot 0 + m'(-2l) + (m + T - (m' + n))(1-l) \]

Since \( l < 1, T > n, \pi_1 > \pi_2 \), hence \((i) \succeq_j (ii)\) if \( \delta \) is large enough.

1.2.2.2 \( m' + n > m \)

1.2.2.2.1 \( m' + n \geq T \)
1.2.2.2.1.1 \( m' + n \geq m + T \)

Now we compare the undiscounted sum of payoffs of \((i)\) and \((ii)\) in the next \( m' + n \) periods, since after the next \( m' + n \) periods, \((i)\) and \((ii)\) coincide. Denote the payoffs by \( \pi_1 \) and \( \pi_2 \), respectively.

\[ \pi_1 = m(-2l) + (m' + n - m)(1-l) \]
\[ \pi_2 = n \cdot 0 + (m + T - n)(-2l) + (m' + n - T - m)(1-l) \]

Since \( l < 1, T > n, \pi_1 > \pi_2 \), hence \((i) \succeq_j (ii)\) if \( \delta \) is large enough.

1.2.2.2.1.2 \( m' + n < m + T \)
Now we compare the undiscounted sum of payoffs of \((i)\) and \((ii)\) in the next \(m + T\) periods, since after the next \(m + T\) periods, \((i)\) and \((ii)\) coincide. Denote the payoffs by \(\pi_1\) and \(\pi_2\), respectively.

\[
\pi_1 = m(-2l) + (m' + n - m)(1 - l) + (m + T - (m' + n))
\]

\[
\pi_2 = n \cdot 0 + m' (-2l) + (m + T - (m' + n))(1 - l)
\]

Since \(l < 1\), \(\pi_1 > \pi_2\), hence \((i) \succeq_j (ii)\) if \(\delta\) is large enough.

1.2.2.2.2 \(m' + n < T\)

This is covered by 1.2.2.2.1.2.

2 \(E_j (a'_j | h'_j) = D\)

2.1 If \(j\) deviates, no neighbor will be surprised. By claim 7, It must be that \(E_j (E_k (a'_j | h'_j) = D/C \quad \forall k \in N(j) \{j\})\). By definition of \(E(2.2.1)\), such deviation is not profitable.

2.2 If \(j\) deviates, exactly one neighbor, say \(k\), will be surprised. In the analysis under 2.2 we need claim 8.

2.2.1 \(j\) owes nothing to \(k'\).

(ii)

\[
\begin{align*}
k : & \quad D/C (1) + D (bT) + C (\infty) \\
j : & \quad C (\infty) \\
k' : & \quad C (\infty)
\end{align*}
\]

If \(j\) doesn’t deviate, then in the next \(bT + 1\) periods, deviation can be weakly better than no deviation in at most \(2T\) periods. In every other period, no deviation is better either because \(C \quad D \quad D \succeq_j D \quad C \quad C\), or because \(C \quad C \quad C \succeq_j D \quad C \quad C\). After the next \(bT + 1\) periods, it takes at most \(2T\) periods to make \((i)\) coincide with \((ii)\). Let no deviation take these \(2T\) periods. We can always choose \(b\) large enough so that \((i) \succeq_j (ii)\).

2.2.2 \(j\) owes one \(C\) to \(k'\).

Let \(n\) denote the uncertainty horizon of \(k'\) about \(j\), \(0 \leq n \leq T - 1\).
If $j$ doesn’t deviate, then in the next $bT + 1$ periods, deviation can be weakly better than no deviation in at most $2T$ periods. In every other period, no deviation is better either because $C D D \geq_j D C C$, or because $C C C \geq_j D C C$. After the next $bT + 1$ periods, it takes at most $2T$ periods to make $(i)$ coincide with $(ii)$. Let no deviation take these $2T$ periods. We can always choose $b$ large enough so that $(i) \succeq_j (ii)$.

2.2.3 $j$ owes $k'$ $m$ $C'$'s, $2 \leq m \leq bT$

Let $n$ denote the uncertainty horizon of $k'$ about $j$, $0 \leq n \leq T - 1$.

If $j$ doesn’t deviate, denote the undiscounted sum of $j$’s payoff in the next $(b + 2)T + m$ periods by $\pi_4$, denote the counterpart payoff in 2.2.2 by $\pi_2$, then $\pi_2 = \pi_4 + (m - 1)(1 + l)$. On the other hand, let $\pi_1$ be the corresponding payoff of $(ii)$ in 2.2.2, and $\pi_3$ be the counterpart payoff in 2.2.3, then $\pi_1 = \pi_3 + (m - 1)(1 + l)$. Since $\pi_2 > \pi_1$, it follows that $\pi_4 > \pi_3$.

2.3 $j$’s deviation will surprise both neighbors.

If $j$ can clear things up within the next $bT$ periods, then $(i) \succeq_j (ii)$. Otherwise it takes $j$ at most $2T$ periods to clear it up. Let $(ii)$ take these $2T$ periods, but $(i)$ will take all the $bT$ periods. If $b$ is large enough, no deviation is better.

In all the above analysis, we assume that $j$ has two neighbors. If $j$ is an end player, it’s straightforward to check that $E$ is optimal for him after any $h_j^t$ relative to $E_j \cdot \{h_j^t\}$ (notice that the expectation operators are well defined for end players too. In fact, they are well defined for any player in any graph).

By the definition of $E$, after any history, it takes at most $2(b + 1)T$ periods for $N(j)$ to return to full cooperation forever. Hence when we compare the
no deviation future with the one deviation future, it suffices to compare the undisccounted sum of future payoffs up to the next $2(b+1)T$ periods. Given $g > 0$, and $1 > l > 0$, we can always find $T > 0$, $b > 1$, such that no deviation is strictly better than one deviation in the sense of the undiscounted sum. By continuity of the payoffs with respect to $\delta$, in each case we can find $\delta$ large enough such that the comparison holds in terms of the discounted sum. There are finite number of cases we need to consider, hence there exists $0 < \delta < 1$, such that $\forall \delta > \delta$, the comparison holds for all cases.

Let’s summarize this section by the following lemma:

**Lemma 1**: In the repeated prisoner’s dilemma on the line (finite or infinite), $\forall 0 < l < 1$, $\forall g > 0$, $\exists 0 < \delta < 1$, $\exists T > 0$, $b > 0$, such that $\forall \delta > \delta$, $E$ is sequentially rational with respect to $E_j(\cdot|\cdot)$.

1.4 The belief system $B$

First we classify three types of mistakes that a player could possibly make.

**Definition** $a^t_j$ is a defection mistake by player $j$ after $h^t_j$ if $a^t_j = D \neq E_j(a^t_j|h^t_j)$

**Definition** $a^t_j$ is a naive mistake by player $j$ after $h^t_j$ if $a^t_j = C \neq E_j(a^t_j|h^t_j)$ and $E_j(E_k(a^t_j)|h^t_j) = D/C, \forall k \in N(j) \{j\}$.

**Definition** $a^t_j$ is a blocking mistake by player $j$ after $h^t_j$ if $a^t_j = C \neq E_j(a^t_j|h^t_j)$ and $E_j(E_k(a^t_j)|h^t_j) = D$, $\exists k \in N(j) \{j\}$.

**Remark** A surprise is not necessarily a mistake, for example, $j-1$ surprises $j$ by playing $D$, but it might be that $j-1$ is punishing $j-2$, so it is not a defection mistake by $j-1$. A mistake is not necessarily a surprise, for example, naive mistakes never surprise any neighbor. However, a surprise of unexpected cooperation is always a blocking mistake; conversely, a blocking mistake is always a surprise of unexpected cooperation, because of the following claim.

**Claim 9** For any global history $h^t$, for any $j$ and $k$ who are neighbors,

$$E_j(E_k(a^t_j)|h^t_j) = E_k(a^t_j|h^t_k)$$

where $h^t_j$ and $h^t_k$ are the restrictions of $h^t$ to $N(j)$ and $N(k)$, respectively.

**Proof**: See Appendix B. ■

Let the mistake pattern $M$ be defined in the following way. $M$ assigns probability $\epsilon^{bT+1}$ to blocking mistakes, $\epsilon$ to naive mistakes, and $\epsilon^{\frac{1}{2}+\frac{1}{2}T}$ to defection mistakes made in period $t$. Define $H(h^t_j) = \{h^t| h^t$ explains $h^t_j$ most parsimoniously $\}$.
∀h^t ∈ H (h^t_j), if j uses h^t as the explanation, then he knows everything in the past, and he can fully predict everything in the future. The future path within N (j) is deterministic, denote it by E_j (⋅|h^t), and call it the real expectation that j forms after h^t_j, in the explanation h^t. Formally, fix h^t_j, fix h^t ∈ H (h^t_j), E_j (⋅|h^t) is formed by the following steps.

1. Given h^t, j calculates the actions each player is going to take in period t, denote it by (a^t_k)_k.

2. Let h^{t+1} = (h^t, (a^t_k)_k), j then calculates the actions each player is going to take in period t + 1, denote it by (a^{t+1}_k)_k.

3. Let h^{t+2} = (h^{t+1}, (a^{t+1}_k)_k), and so on.

The future actions of everybody can be derived following these steps. The restriction of these actions to N (j) is E_j (⋅|h^t). Our goal in this section is to show that the mistake pattern is such that the real expectations formed after any parsimonious explanation can be essentially duplicated by the auxiliary expectations we defined in Section 1.2.

Any explanation consists of three parts:

h^t = (h^t_{j−}, h^t_j, h^t_{j+})

We say h^t_{j−} explains h^t_j parsimoniously if π (h^t_{j−}), the power of ε associated with h^t_{j−} attains the minimum among all h^t_{j−}’s that are compatible with h^t_j. Notice that ∀h^t = (h^t_{j−}, h^t_j, h^t_{j+}) ∈ H (h^t_j), h^t_{j−} and h^t_{j+} also explain h^t_j parsimoniously. On the other hand, if h^t_{j−} and h^t_{j+} both explain h^t_j parsimoniously, then h^t = (h^t_{j−}, h^t_j, h^t_{j+}) ∈ H (h^t_j). Hence without loss of generality, in the following lemma we can assume that j is the right end player. Let k denote j’s left neighbor.

**Lemma 2** Given the mistake pattern M, ∀h^t_j, ∀h^t ∈ H (h^t_j), E_j (⋅|h^t) = E_j (⋅|h^t_j), the equality is upto the difference between D/C and D or C.

**Proof:** Since j is the right end player and k is j’s left neighbor, it suffices to show that

E_j (a^t_k|h^t) = E_j (a^t_k|h^t_j) ∀s ≥ t

To this end, we need the following claim, the proof of which is in Appendix B.

**Claim 10** If ∀h^t_j, ∀h^t ∈ H (h^t_j), E_j (a^t_k|h^t) = E_j (a^t_k|h^t_j), then ∀h^t_j, ∀h^t ∈ H (h^t_j), E_j (a^t_k|h^t) = E_j (a^t_k|h^t_j) ∀s ≥ t.
By Claim 10, it suffices to show \( \forall h^t_j, \forall h^t \in H(h^t_j), E_j(a^t_k|h^t_j) = E_j(a^t_k|H(h^t_j)) \).

If \( E_j(a^t_k|h^t_j) = D \), then by Claim 9, \( E_k(E_j(a^t_k)|h^t_k) = D \), where \( h^t_k \) is the restriction of \( h^t \) to \( N(k) \). Hence by the definition of \( E \), \( E_k(a^t_k|h^t_k) = D \), hence \( E_j(a^t_k|h^t_j) = D \). Need to show that if \( E_j(a^t_k|h^t_j) = C \), then \( E_k(a^t_k|h^t_k) = C \).

For simplicity we assume that \( j \) has only two people to his left, \( k \) and \( k' \). The case of more players can be proved analogously.

Suppose by way of contradiction that \( \exists h^t_j, \exists h^t \in H(h^t_j) \), such that \( E_j(a^t_k|h^t_j) = C \), but \( E_k(a^t_k|h^t_k) = D \). Then it must be that \( k \) need to punish the last mistake by \( k' \) in \( h^t \). Suppose this last mistake occurs in period \( s \) (see Figure 2), then it must be that in \( h^t \) \( k \) plays \( D \) in period \( s + 1, \ldots, t - 1 \).

![Figure 2: Lemma 2](image)

Suppose that the period \( s \) mistake is a blocking mistake, then since blocking is so unlikely to happen according to the mistake pattern \( M \), by the time the blocking mistake of \( k' \) is realized by \( j \), \( k \) should have already finished punishing it. Since \( k' \) makes no further mistake, there is no punishment obligation of \( k \) coming from \( k' \)’s side. Hence it is not possible that \( E_j(a^t_k|h^t_j) = C \), while \( E_k(a^t_k|h^t_k) = D \).
If the period \( s \) mistake is a defection mistake, then there are two cases to consider.

Case 1. \( j \) is not surprised by \( k \)'s defection after period \( s \) till period \( t - 1 \). Three possibilities. 1. \( j \) expects(auxiliary expectation) \( k \) to play \( D \) from \( s + 1 \) to \( t - 1 \). In this case \( k \) makes no mistake from \( s + 1 \) to \( t - 1 \), hence the period \( s \) defection mistake by \( k' \) is redundant; 2. \( j \) expects \( k \) to play \( D/C \) from period \( s + 1 \) to \( t - 1 \). Since uncertainty horizon lasts at most for \( T - 1 \) periods, there are at most \( T - 1 \) periods from \( s + 1 \) to \( t - 1 \). Consider the alternative explanation in which \( k' \) does not make the mistake in period \( s \), and from \( s \) on till \( t - 1 \), \( k' \) keeps following her strategy. Call the alternative explanation \( \tilde{h}^t \). \( \tilde{h}^t \) releases a period \( s \) defection mistake, at a cost of at most one defection mistake by \( k \) from \( s + 1 \) to \( t - 1 \). Since the probability of defection mistakes is ascending in time, \( \tilde{h}^t \) is more efficient at explaining \( h_s^t \) than \( h_s^t \), a contradiction. 3. From \( s + 1 \) to \( t - 1 \), \( j \) first expects \( k \) \( D/C \), then expects \( k \) \( D \). Combining the arguments in the first two possibilities, we can also find a better explanation of \( h_s^t \) than \( h_s^t \), a contradiction.

Case 2. \( j \) is surprised by \( k \)'s defection after period \( s \). If \( j \) is surprised by \( k \)'s defection after period \( s \), then from \( s + 1 \) to \( t - 1 \) there must be more than \( T - 1 \) periods, and \( j \) must be surprised prior to the last \( T - 1 \) periods upto \( t - 1 \)(see Figure 3). During the "surprise" interval \( k \) is supposed to punish \( k' \) even in the absence of the period \( s \) mistake. Hence this interval of \( D \)'s cannot be mistakes. Therefore we essentially return to case 1, and if we release the period \( s \) mistake by \( k' \), we creat at most one defection mistake by \( k \) during the last \( T - 1 \) periods upto \( t - 1 \). Hence there is a better explanation than \( h_s^t \), a contradiction.

Combining Lemma 2 and lemma 1, we are now ready to prove the following proposition:

**Proposition 1** In the repeated prisoner’s dilemma on the line (finite), \( \forall 0 < l < 1, \ \forall g > 0, \ \exists 0 < \delta < 1, \ \exists T > 0, \ \exists b > 0 \), such that \( \forall l > \delta > \delta \), \((E, B)\) is a sequential equilibrium that supports global cooperation.

**Proof:** Fix \( h_s^t \), fix \( h_s^t \in H \left( h_s^t \right) \).

Let \( (i') = \left( (E_j \left( a_k^s | h_s^t \right))_{k \in N(j)} \right)_{s=t}^\infty \)

\[ = \left( (E_j \left( a_{j-1}^t | h_s^t \right), E_j \left( a_j^t | h_s^t \right), E_j \left( a_{j+1}^t | h_s^t \right)); \left( (E_j \left( a_k^s | h_s^t+1 \right))_{k \in N(j)} \right)_{s=t+1}^\infty \right) \),

\( (ii') = \left( (E_j \left( a_{j-1}^t | h_s^t \right), \hat{E}_j \left( a_j^t | h_s^t \right), E_j \left( a_{j+1}^t | h_s^t \right)); \left( (E_j \left( a_k^s | \tilde{h}_s^t+1 \right))_{k \in N(j)} \right)_{s=t+1}^\infty \right) \).
where $h^{t+1}$ is $h^t$ augmented by period $t$ in which everybody in the world followed the strategy, and $\tilde{h}^{t+1}$ is $h^t$ augmented by period $t$ in which everybody in the world except $j$ followed the strategy, and

$E_j (a_j^t | h^t) = \begin{cases} C & \text{if } E_j (a_j^t | h^t) = D \\ D & \text{if } E_j (a_j^t | h^t) = C \end{cases}$

To establish the link between $(i)$, $(ii)$ and $(i')$, $(ii')$, we consider two cases:

1. $E_j (a_j^t | h_j^t) = C$. In this case there is no ambiguity in $(i)$ and $(ii)$. Moreover, $\tilde{h}^{t+1} \in H (\tilde{h}^{t+1})$, where $\tilde{h}^{t+1} = \left( h_j^t; (E_j (a_j^t | h_j^t), E_j (a_j^t | h_j^t), E_j (a_j^t | h_j^t)) \right)$. Therefore $(i) = (i')$, and $(ii) = (ii')$.

2. $E_j (a_j^t | h_j^t) = D$. In this case there might be some ambiguities in $(i)$ and $(ii)$, but in $(ii)$, there is no ambiguity in $\left( E_j (a_k^t | h_j^t) \right)_{k \in N(j)}$, even if $\tilde{h}^{t+1}$ itself may not be single-valued.

Let $\tilde{h}_j^{t+1} = (h_j^t, (E_j (a_j^t | h^t), E_j (a_j^t | h_j^t), E_j (a_j^t | h_j^t)))$. 

Figure 3: Lemma 2
Let $\hat{h}^{t+1}_j = \left( h^t_j, (E_j (a^t_{j-1}|h^t_j), \hat{E}_j (a^t_j|h^t_j), E_j (a^t_{j+1}|h^t_j)) \right)$.

Let $(i'') = \left( (E_j (a^t_{j-1}|h^t_j), E_j (a^t_j|h^t_j), E_j (a^t_{j+1}|h^t_j)); \left( (E_j (a^t_k|h^t^{t+1}_j))_{k\in N(j)} \right)_{s=t+1}^\infty \right)$.

Let $(ii'') = \left( (E_j (a^t_{j-1}|h^t_j), \hat{E}_j (a^t_j|h^t_j), E_j (a^t_{j+1}|h^t_j)); \left( (E_j (a^t_k|h^t^{t+1}_j))_{k\in N(j)} \right)_{s=t+1}^\infty \right)$.

Since $\hat{h}^{t+1}_j \in H \left( \hat{h}^{t+1}_j \right)$, $(ii'') = (i''')$, by Lemma 2. Notice that there might still be some ambiguities in $\left( (E_j (a^t_k|h^t^{t+1}_j))_{k\in N(j)} \right)_{s=t+1}^\infty$, but Lemma 1 implies that the worst case of $(i'')$ is preferred by $j$ to $(ii'')$. Since $h^{t+1}_j \in H \left( h^{t+1}_j \right)$, $(i')$ is one case of $(i'')$, by Lemma 2. Hence $(i') \succeq_j (ii'') = (i''')$, as was to be shown.

**Proposition 2** Under the same conditions in proposition 1, $\forall h^t, \exists 0 < T (h^t) < \infty$, such that global cooperation is restored after $T (h^t)$ periods.

**Proof:** By induction on the number of players.

### 1.5 Circles and Trees

The analysis so far seems to rely on the fact that each agent is the only "bridge" between his two neighbors, i.e. there are no cycles in the graph. This is why we have very little to say about circles. We summarize what we can say about circles in the following proposition.

**Proposition 3** Consider the same model as in section 1.1, but replace the line by a circle. Trigger strategy supports cooperation for the circle with four players if $\delta \geq \frac{g}{1+g}$; it supports cooperation for the circle with more than four players if $l \geq 1$ and $\delta \geq \frac{l}{1+g}$, or $l < 1$ and $\delta \in \left[ \frac{g}{1+g}, \frac{g+1}{1+g} \right]$.

**Proof:** Just carry out the one step deviation calculation.

For trees (connected graphs with no cycles), we can use the expectation-surprise type of strategy to achieve the following.

**Proposition 4** Consider the same model as in section 1.1, but replace the line by a tree. Let $n$ be the maximum number of neighbors one can have on the tree. If $g < 1 + l$ and $(n - 1) g < 1 + nl$, then there exists a sequential equilibrium in pure strategies that supports cooperation if $\delta$ is large enough.
**Proof:** The type of strategy and argument is similar to that in sections 1.2 to 1.4, but much simpler. In particular, we make the following modifications to $E$. Let $T = 1$, $b = 1$. Moreover, if $j$ blocks $k$ in some period, instead of playing $C$ in the next period as prescribed by $E$, $j$ is supposed to play $D$. In this case, we can perturb the strategy uniformly, and it is easy to show that the real expectations formed under the corresponding consistent belief system can be exactly duplicated by the auxiliary expectations. Notice that once the payoff parameters belong to the range specified in the proposition, the cutoff value of $\delta$ doesn’t depend on the size of the neighborhood. ■

1.6 Conclusion

I construct a sequential equilibrium in pure strategies to support cooperation on the line when $l < 1$ and $\delta$ is large. I first define a pair of expectation operators inductively to keep track of "rights" and "obligations" during the play. Then I show that the pure strategy thus obtained is sequentially rational with respect to the built-in expectations. The real expectations formed under the consistent belief system, moreover, can be mimiced by the built-in expectations, if we perturb the strategy appropriately. Here is a schematic illustration of the approach.

```
Expectation $\rightarrow$ Strategy
            ↑   ↓
Expectation $\leftarrow$ belief
```

The main message of this chapter is this. The ultimate source of stability in this simple society is shared belief, or mutually compatible expectations on each other. An explicitly defined expectation system can be used as a social norm. What is important is not a common observation of a physical outcome, what is important is a common understanding of the social norm, the understanding that everybody knows the norm and is willing to follow it after any history.

1.7 Appendices

1.7.1 Appendix A

In this appendix I will explain why I impose uniform convergence on the part of the strategies, but only pointwise convergence on the part of the beliefs. I will also discuss the infinitely many players case.

A.1 I require that the perturbed strategies converge to the equilibrium strategy in supremum norm in order to rule out the formation of unreasonable beliefs, as illustrated by the example in Figure 4.

This is a signaling game with infinite time horizon where player 1 has two types, $t_1$ and $t_2$. Player 2 has only one type. The number after each terminal
Figure 4: Signaling game
node is player 1’s payoff at that node. For simplicity, assume that player 2 is indifferent at all terminal nodes. Consider the strategy profile represented by the arrowed actions, and the belief system in which player 2 always assign probability one to player 1 being type $t_1$. I want to rule this out as a sequential equilibrium. The reason is it takes more and more mistakes on the part of $t_1$ to reach 2’s information set down the road, but it only takes one mistake on the part of $t_2$ to do the same. Eventually 2 should flip his belief and begin to assign probability one to 1 being type $t_2$. If we agree that player 2 will question his initial belief after he is surprised for sufficiently many times, then we should not merely require that the perturbed strategies converge pointwise to the original strategy, because it’s easy to construct pointwise convergent perturbations such that the belief system in which 2 always believes 1 is $t_1$ is the pointwise limit of the corresponding belief systems, derived from the perturbations using Bayes’ rule. Simply assign probability $\epsilon^2$ to $t_2$’s first mistake, and $\epsilon^{\frac{1}{n}}$ to $t_1$’s $n$th mistake.

A.2 While sup norm convergence seems to be reasonable for the strategy part, it is too strong for the belief part. Consider the following example.

\[\begin{array}{c|c|c|c|c}
\mu & h^1 & h_3^1 & h^t & \mu_{\epsilon} \\
1 & CCC & CC & CCC & 1 - \epsilon \\
1 & CCC & CC & CCC & (1 - \epsilon)^2 \\
\end{array}\]

In this example players 1, 2 and 3 play the repeated prisoner’s dilemma. Let the strategy they play simply be "cooperate after any history". Let the perturbation be such that after any history, cooperation is played with probability $1 - \epsilon$, and defection is played with probability $\epsilon$. Obviously such perturbation converges to the strategy in $l_\infty$. However, the sequence of player 3’s belief systems doesn’t converge in $l_\infty$. To see this, we first enumerate player 3’s information sets (private histories), then within each information set, we enumerate all the nodes (global histories). Hence a belief system of player 3 is just a sequence of real numbers between 0 and 1, one to one corresponding to the enumeration. Consider the following subsequence.
\begin{center}
\begin{tabular}{ccc}
CCC & CC & CCC \\
1 & CCC & CC & CCC & (1-\epsilon)^3 \\
CCC & CC & CCC \\
CCC & CC & CCC \\
\end{tabular}
\end{center}

where the $h^t_3$ column is a subsequence of player 3’s information sets, and the $h^t$ column picks a node out of each of the information sets. The $\mu_\epsilon$ column is the probability that player 3 assigns to the nodes given the perturbation. The $\mu$ column is the pointwise limit of $\mu_\epsilon$ as $\epsilon$ goes to zero. If \{\mu_\epsilon\}_\epsilon converges in $L^1$, then it must converge to $\mu$. But $\forall \epsilon > 0$, $\rho(\mu, \mu_\epsilon) = 1$, where $\rho(\cdot, \cdot)$ is the sup norm distance between two sequences.

A.3 When there are infinitely many players, each information set contains a continuum of nodes. It is not clear how to use Bayes’ rule to update beliefs. Kolmogorov Extension Theorem might be an approach to deal with this but it’s not clear how to assign probabilities to the cylinder sets. Alternatively, given the perturbations to the strategy, we are able to compare the relative likelihood of any two nodes in an information set. As the perturbation vanishes, if there are only a finite number of nodes that are infinitely more likely than other nodes, then we can assign positive probabilities only to those nodes (parsimonious explanations). In general this may not be true, but this is the case in our model. With infinitely many players, each information set contains a continuum of nodes, but only a finite number of them can be parsimonious: mistakes from remote players are redundant. If our players only assign positive probabilities to parsimonious explanations, then the strategy $E$ is still sequentially rational with respect to their beliefs.
1.7.2 Appendix B

Proof of claim 1: The claim is obviously true if $h_j^t = h_j^1$, the null history. Now fix $h_j^t$, fix an action profile in period $t$, $(a_k^t)_k$. Suppose the claim holds for $h_j^t$, we need to show that it also holds for $h_j^{t+1} = (h_j^t, (a_k^t)_k)$. That is, we need to show that if $E_j (a_j^{t+1} | h_j^{t+1}) = E_j (a_k^{t+1} | h_j^{t+1}) = C$, then (a) $E_j (a_j^t | h_j^{t+1}) = E_j (a_k^t | h_j^{t+1}) = C$, $\forall s \geq t + 1$, and (b) $E_j (E_k (a_j^{t+1}) | h_j^{t+1}) = C$.

Part (b) follows from claims 2 and 3, we only need to prove part (a).

First, $E_j (a_j^{t+1} | h_j^{t+1}) = C \implies E_j (a_j^s | h_j^{t+1}) = C$, $\forall s \geq t + 1$, by the definition of the first order operator.

Second, we already have $E_j (a_k^{t+1} | h_j^{t+1}) = C$, which, by the definition of the second order operator, implies that $E_j (E_k (a_j^t) | h_j^{t+1}) = C$, $\forall s \geq t + 1$, which in turn, together with $E_j (a_j^t | h_j^{t+1}) = C$, $\forall s \geq t + 1$, implies that $j$ does not expect himself to surprise $k$ in period $t + 1, ..., \infty$. By the definition of $E_j$, the only time that $j$ expects $k$ to play $C$ in the current period, but $D$ in some future period is if $j$ expects himself to surprise $k$ in the future. Therefore, $E_j (a_j^t | h_j^{t+1}) = C$, $\forall s \geq t + 1$.

Proof of claim 2: The claim is obviously true if $h_j^t = h_j^1$, the null history. Now fix $h_j^t$, fix an action profile in period $t$, $(a_k^t)_k$. Suppose the claim holds for $h_j^t$, we need to show it also holds for $h_j^{t+1} = (h_j^t, (a_k^t)_k)$. That is, we need to show that if $E_j (E_k (a_j^t) | h_j^{t+1}) = D$, then $E_j (a_j^t | h_j^{t+1}) = D$, $\forall s \geq t + 1$.

In period $t$, if $k$ surprises $j$, then by 2.2.2.1,

$$E_j (E_k (a_j^t) | h_j^{t+1}) = D \implies E_j (a_j^t | h_j^{t+1}) = D, \forall s \geq t + 1.$$ 

If $k$ does not surprise $j$, then by the definition of the second order operator, $E_j (E_k (a_j^t) | h_j^{t+1}) = D$ implies that $j$ does not surprise $k$ either in period $t$, which implies that $E_j (E_k (a_j^t) | h_j^t) = D$, by 1.1.1 (which implies $E_j (a_j^t | h_j^t) = D$ by the definition of the second order operator). This in turn, implies that $E_j (a_j^t | h_j^t) = D$, by the induction hypothesis, which implies that $E_j (a_j^t | h_j^{t+1}) = D$ (since $E_j (E_k (a_j^t) | h_j^t) = D$, and $j$ does not surprise $k$ in period $t$, $j$ does not block in period $t$, hence if $j$ expects himself to play $D$ in period $s$ after $h_j^t$, $j$ does not change this expectation after $h_j^{t+1}$).

Proof of claim 3: The claim is obviously true if $h_j^t = h_j^1$, the null history. Now fix $h_j^t$, fix an action profile in period $t$, $(a_k^t)_k$. Suppose the claim holds for $h_j^t$, we need to show it also holds for $h_j^{t+1} = (h_j^t, (a_k^t)_k)$. That is, we need to show that if $E_j (E_k (a_j^t) | h_j^{t+1}) = D/C$, then $E_j (a_j^t | h_j^{t+1}) = D$.
If $j$ surprises $k$ in period $t$, then $E_j (a^*_k | h^t_j) = D$ by 1.2.2.1 and 2.2.2.4.1.

If $j$ does not surprise $k$ either in period $t$, then $E_j (E_k (a^*_j) | h^t_j) = D/C$, by 1.1.1. Let us denote this fact by (1).

$$(1) \implies E_j (E_k (a^*_j) | h^t_j) = D/C, \text{ by the definition of the second order operator.}$$

$$(1) \implies E_j (a^*_k | h^t_j) = D, \text{ by the induction hypothesis.}$$

If $E_j (a^*_k | h^{t+1}_j)$ is just the continuation of $E_j (a^*_k | h^t_j)$, then we are done. Otherwise, since $E_j (a^*_k | h^t_j) = D$, the second case of 2.2.2.4.2 applies. In this case, if $E_j (a^*_{j+1} | h^{t+1}_j) = C$, then $E_j (E_k (a^*_j) | h^{t+1}_j) = D/C \implies E_j (a^*_k | h^{t+1}_j) = D$; if $E_j (a^*_{j+1} | h^{t+1}_j) = D$, then it must be that $\tau > s$, hence $E_j (a^*_k | h^{t+1}_j) = E_j (a^*_k | h^t_j) = D$. ■

**Proof of claim 4:** The claim is obviously true if $h^t_j = h^1_j$, the null history. Now fix $h^t_j$, fix an action profile in period $t$, $(a^+_k)_k$. Suppose the claim holds for $h^t_j$, we need to show it also holds for $h^{t+1}_j = (h^t_j, (a^+_k)_k)$. That is, we need to show that if $E_j (E_k (a^*_{j+1}) | h^{t+1}_j) = D/C$, then

(a) $E_j \left(a^+_k | h^{t+1}_j\right) = D$

(b) $E_j \left(E_k \left(a^*_j \right) | h^{t+1}_j\right) = C \quad t' = \min \{ s \geq t + 1 | E_j \left(E_k \left(a^*_j \right) | h^{t+1}_j\right) \neq D/C \}$

$E_j (E_k (a^{t+1}_j) | h^{t+1}_j) = D/C \implies k$ does not surprise $j$ in period $t$.

If $j$ surprises $k$ in period $t$, then (a) and (b) follows from 2.2.2.4.1 and 1.2.2.1.

If $j$ doesn’t surprise $k$ in period $t$, then $E_j (E_k (a^*_j) | h^{t+1}_j) = E_j (E_k (a^*_j) | h^t_j)$ by 1.1.1. Hence $E_j (E_k (a^{t+1}_j) | h^{t+1}_j) = D/C$ implies that $E_j (E_k (a^{t+1}_j) | h^t_j) = D/C$, which by the definition of the second order operator, implies that $E_j (E_k (a^*_j) | h^t_j) = D/C$. Let $t'' := \min \{ s \geq t | E_j (E_k (a^*_j) | h^t_j) \neq D/C \}$, then by the induction hypothesis, $E_j \left(a^{t''}_k | h^t_j\right) = D$ and $E_j \left(E_k \left(a^{t''}_j \right) | h^t_j\right) = C$. Since $t' = \min \{ s \geq t | E_j (E_k (a^*_j) | h^{t+1}_j) \neq D/C \}$, and $E_j (E_k (a^*_j) | h^{t+1}_j) = E_j (E_k (a^*_j) | h^t_j)$, we have $E_j \left(E_k \left(a^*_j \right) | h^{t+1}_j\right) = C$.

If $E_j (a^*_k | h^{t+1}_j)$ is just the continuation of $E_j (a^*_k | h^t_j)$, then we are done because $E_j \left(a^{t''}_k | h^t_j\right) = D$. Otherwise, since $E_j (a^*_k | h^t_j) = D$ (this is because of
$E_j (E_k (a_j^t | h_j^t)) = D/C$ and claim 3), the second case of 2.2.2.4.2 applies. In this case, if $E_j (a_j^{t+1} | h_j^{t+1}) = C$, then $E_j (a_k^t | h_k^{t+1}) = D$; if $E_j (a_j^{t+1} | h_j^{t+1}) = D$, then it must be that $t \geq t'$, hence $E_j (a_k^t | h_k^{t+1}) = E_j (a_k^t | h_j^t) = D$.

**Proof of claim 5:** If $E_j (a_k^t | h_j^t) = D$, then at least $j$ owes $k$ one period of $C$; if $E_j (a_k^t | h_j^t) = D/C$, then $j$ should expect himself to play $D$, either way, we have a contradiction.

**Proof of claim 6:** If $\forall k' \in N (j) \setminus \{j\}$, $E_j (E_{k'} (a_j^t | h_j^t)) = D/C$, then $E_j (a_j^t | h_j^t) = D$, by 2.2.2.1. Hence $\exists k' \in N (j) \setminus \{j\}$, such that $E_j (E_{k'} (a_j^t | h_j^t)) = C$. Since $E_j (a_k^t | h_j^t) \neq C$, $\forall k$, it must be that $E_j (a_k^t | h_j^t) = D$, $\forall k$. Hence $j$ must owe each neighbor some debt.

**Proof of claim 7:** If $\exists k$ such that $E_j (E_k (a_j^t) | h_j^t) = C$, then $E_j (a_j^t | h_j^t) = C$, by 2.2.2.1.

**Proof of claim 8:** There are two kinds of punishment in the strategy. Punishment of unexpected defection and punishment of unexpected cooperation. The punishee is expected to play $C$ in the second punishment, and $D$ for at most $T - 1$ periods in the first punishment, provided that the punishee is not further surprised by the punisher in the future. But the punishee who expects the longest punishment from the punisher will not be further surprised by the punisher in the future, hence the punishee is expected to play $C$ forever after at most $T - 1$ periods.

**Proof of claim 9:** Let $h^1$ denote the global null history. By definition,

$E_j (E_k (a_j^t | h_j^t)) = E_k (a_j^t | h_k^t) = C$

Induction hypothesis: Fix $h^t$, fix each player’s period $t$ action $(a_j^t)$. Let $h^{t+1} = (h^t, (a_j^t))$. Suppose the claim is true for any subhistory $h^s$ of $h^t$, including $h^t$ itself. We need to show that the claim also holds for $h^{t+1}$.

First consider the case where

$a_j^t = E_j (E_k (a_j^t | h_j^t))$ and $a_k^t = E_j (a_k^t | h_j^t)$

In this case, $E_j (E_k (a_j^{t+1} | h_j^{t+1}) = E_j (E_k (a_j^{t+1} | h_j^t))$, by 1.1.1 in the definition of the strategy.

If $E_j (E_k (a_j^{t+1} | h_j^t) = D/C$, then it must be that after some subhistory $h^s$ of $h^t$, $D = a_j^s \neq E_j (E_k (a_j^s | h_j^s))$. Let $h^s$ be the longest such subhistory. By the induction hypothesis, $a_j^s \neq E_k (a_j^s | h_k^s)$. From period $s + 1$ to period $t - 1$, it
must be that neither \( j \) nor \( k \) further surprises the other, otherwise either \( h^s \) is not the longest history, or \( E_j (E_k (a_j^{t+1}) | h_j^t) \neq D/C \).

Since \( D = a_j^s \neq E_j (E_k (a_j^s) | h_j^0) = E_k (a_j^s | h_k^0) \), and \( E_j (E_k (a_j^{t+1}) | h_j^t) = D/C \), it must be that \( E_j (E_k (a_j^{t+1}) | h_j^{t+1}) = D/C \), by repeated application of 1.1.1. Hence \( E_k (a_j^{t+1} | h_k^{t+1}) = D/C \), since the first order operator and the second order operator are always matched to each other. Then since no surprise occurs between \( j \) and \( k \) in period \( s + 1 \ldots t \), we have \( E_k (a_j^{t+1} | h_k^{t+1}) = D/C \), as was to be shown.

If \( E_j (E_k (a_j^{t+1}) | h_j^t) = D \), then it must be that after some subhistory \( h^s \), \( a_k^s \neq E_j (a_k^s | h_j^0) = E_k (E_j (a_k^s) | h_k^0) \), where the equality is by the induction hypothesis. Again, let \( h^s \) be the longest such subhistory. From period \( s + 1 \) to \( t - 1 \), it must be that neither \( j \) nor \( k \) further surprises the other.

Since \( a_k^s \neq E_j (a_k^s | h_j^0) = E_k (E_j (a_k^s) | h_k^0) \), and \( E_j (E_k (a_j^{t+1}) | h_j^t) = D \), it must be that \( E_j (E_k (a_j^{t+1}) | h_j^{t+1}) = D \), by repeated application of 1.1.1. Since the first order operator always matches with the second order operator, it must be that \( E_k (a_j^{t+1} | h_k^{t+1}) = D \). Since no surprises occur between \( j \) and \( k \) in period \( s + 1 \ldots t \), \( E_k (a_j^{t+1} | h_k^{t+1}) = D \), as was to be shown.

If \( E_j (E_k (a_j^{t+1}) | h_j^t) = C \), then need to show \( E_k (a_j^{t+1} | h_k^{t+1}) = C \). Suppose otherwise that \( E_k (a_j^{t+1} | h_k^{t+1}) = D \). Then there must exist a longest subhistory \( h^s \) of \( h^t \), such that \( a_k^s \neq E_k (E_j (a_k^s) | h_k^0) = E_j (a_k^s | h_j^t) \), where the equality is by the induction hypothesis. From period \( s + 1 \) to period \( t - 1 \), it must be that \( j \) does not further surprise \( k \), and \( k \) does not further surprise \( j \), either.

Since \( a_k^s \neq E_j (a_k^s | h_j^0) \), and the first order operator always matches the second order operator, it must be that \( E_j (E_k (a_j^{t+1}) | h_j^{t+1}) = D \), which implies \( E_j (E_k (a_j^{t+1}) | h_j^t) = D \), by repeated application of 1.1.1. But we begin with \( E_j (E_k (a_j^{t+1}) | h_j^t) = C \), a contradiction.

Similarly, when \( E_j (E_k (a_j^{t+1}) | h_j^t) = C \), it cannot be that \( E_k (a_j^{t+1} | h_k^{t+1}) = D/C \).

Second consider the case where \( a_j^t \neq E_j (E_k (a_j^t) | h_j^t) \). Notice by the induction hypothesis, \( E_j (E_k (a_j^t) | h_j^t) = E_k (a_j^t | h_j^t) \).

1. \( a_j^t = C \). In this case \( E_j (E_k (a_j^{t+1}) | h_j^{t+1}) = C = E_k (a_j^{t+1} | h_k^{t+1}) \).

2. \( a_j^t = D \).

2.1 If \( a_k = E_j (a_k^t | h_j^t) \), then \( a_k = E_k (a_k^t | h_k^0) \), by the induction hypothesis. In this case \( E_j (E_k (a_j^{t+1}) | h_j^{t+1}) = D/C = E_k (a_j^{t+1} | h_k^{t+1}) \).
2.2 If \( a^t_k \neq E_j (a^t_k | h^t_j) \), then \( E_j (E_k (a^{t+1}_j | h^{t+1}_j)) = D = E_k (a^{t+1}_j | h^{t+1}_j) \).

Third consider the case where \( a^t_j = E_j (E_k (a^t_k | h^t_j)) \), but \( a^t_k \neq E_j (a^t_k | h^t_j) \). In this case \( E_j (E_k (a^{t+1}_j | h^{t+1}_j)) = D = E_k (a^{t+1}_j | h^{t+1}_j) \).

Proof of Claim 10: The claim is trivially true when \( s = t \). Now we show that the claim is true for \( s = t + 1 \).

Let \( a^i_t = E_j (a^i_t | h^t_j) \), \( \forall \) player \( i \). Let \( h^{t+1} = (h^t, (a^i_t)_i) \), and \( h^{t+1}_j = (h^t_j, (a^i_t)_i \in N(j)) \).

Then it must be that \( h^{t+1} \in H (h^{t+1}_j) \), since otherwise \( h^t \notin H (h^{t+1}_j) \). Then we have

\[
E_j (a^{t+1}_k | h^t) = E_j (a^{t+1}_k | h^{t+1}) = E_j (a^{t+1}_k | h^{t+1}_j) = E_j (a^{t+1}_k | h^t_j)
\]

where the first equality is by the definition of real expectations, the second equality is by the condition in the claim, and the third equality is because \( j \) is not surprised by \( (a^i_t)_{i \in N(j)} \), hence the new expectation is just the continuation of the old.

The proof is analogous for \( s > t + 1 \).

References


2.1 Introduction

Many economic and political situations can be modelled as a coordination game with a finite number of players and two actions, a risky action and a safe action. One example is industrialization, where simultaneous industrialization of many sectors can be profitable even if no single sector can make money by industrializing alone. In this example the risky action is to commit to industrializing a sector, and the safe action is not to commit to industrializing. Such coordination games usually have multiple equilibria, but the theory of "global games" provides methods for selecting a unique equilibrium. This equilibrium selection theory is developed by modelling the coordination situation as a simultaneous move game with incomplete information. In this chapter, we argue that if it is more appropriate to model the situation as a sequential move game, then the prediction of the outcome may be very different from the prediction made by the theory of global games. If the risky action is irreversible, and the safe action is reversible, then the static model will exaggerate the likelihood of a coordination failure, compared to the dynamic model. However, if the risky action is reversible, and the safe action is irreversible, then dynamics will not make much difference to the possible inefficiency of equilibria. Hence two coordination games may look very similar if they are treated as simultaneous move games, yet they can be very different if they are treated as sequential move games.

For ease of exposition, we model the coordination situation as a stag hunt game. In the stag hunt game, a group of hunters need to decide whether to hunt for a stag or not. The risky action is to hunt for the stag, and the safe action is not to hunt for the stag. The stag will be caught if and only if all of them choose to hunt. Each hunter has private information about his cost of participating in the hunt. Each hunter must decide whether and when to commit to hunting. Commitment is a one time decision and irreversible in that once a hunter makes a commitment, his cost is sunk even if the stag is not caught. Each hunter in each period observes accurately how many hunters have committed to hunting so far. If there is no discounting, and if there are no dominant strategy types who strictly prefer not to hunt, then it is not hard to see that in any equilibrium the stag is caught for sure. I show that this result is robust to the introduction of dominant strategy types and discounting. It also holds if there is a common shock to the payoffs. This is in contrast to the static game in which there is always a "no hunting" equilibrium, and this equilibrium is unique in the presence of dominant strategy types under certain conditions on the distribution function.
2.2 A Simple Example

We illustrate the ideas using the following simple example.

\[
\begin{array}{c|cc}
   & H & NH \\
\hline
H & 1-c_1,1-c_2 & -c_1,0 \\
NH & 0,-c_2 & 0,0 \\
\end{array}
\]

Two hunters must decide whether to hunt for a stag or not. The risky action is to hunt for the stag, and the safe action is not to hunt for the stag. The stag will be caught if and only if both players choose to hunt. Once caught, the stag is worth 1 to each player. Let \( c_i \) denote player \( i \)'s cost of hunting for the stag. If player \( i \) chooses to hunt, she loses \( c_i \) no matter what the other player plays. Let us first assume that \( c_i \in [0,1], \ i = 1,2 \), and that the values of \( c_1 \) and \( c_2 \) are common knowledge between the two players. The complete information game has two pure strategy Nash equilibria, \((H,H)\) and \((NH,NH)\). Now we assume that it takes two periods for the players to play this game, and there is no discounting. The payoff of each player is read off from the simultaneous move game, according to the final actions of the two players. In the two period game, players choose whether and when to commit to hunting. Commitment is a one time decision and irreversible in that once a player makes a commitment, her cost is sunk even if the stag is not caught. We also assume that at the beginning of the second period, each hunter is able to observe whether the other hunter has committed to hunting or not in the first period. To be sure, this two period game also has multiple subgame perfect equilibria, but it is easy to see that the stag will be caught for sure in any subgame perfect equilibrium.

Next we come back to the static game, but let us assume that \( c_1 \) and \( c_2 \) are independently and uniformly distributed over \([0,1]\). Now we have an incomplete information game, but it is still common knowledge between the two players that it is in the interest of both to hunt for the stag. There are multiple Bayesian Nash equilibria in this game. In particular, "always hunt no matter what the type is" is one of them, and "never hunt no matter what the type is" is another. Again, if we add one more period and allow the players to choose the timing of moves, the stag will be caught in any perfect Bayesian equilibrium.

Now let us assume that \( c_1 \) and \( c_2 \) are independently and uniformly distributed over \([0,1+\epsilon]\), where \( \epsilon > 0 \). The introduction of \( \epsilon \) represents a small amount of doubt that the players have about each other’s willingness to hunt. Now for some type \( c \) less than 1 but very close to 1, it is not profitable to hunt because the potential profit is so small that it is not worth taking the risk that the other player's type is above 1. Anticipating this, it is not profitable for some type \( c' \) less than \( c \) but very close to \( c \) to hunt either, and so on. With uniform distribution, the argument keeps going to the left until it reaches 0. Hence the unique Bayesian Nash equilibrium in the static game is for nobody
to hunt no matter what the type is. What happens to the two period game with endogenous timing? Suppose that whenever a player is indifferent between hunting and no hunting, she always chooses hunting. Then it is easy to show that there is a unique perfect Bayesian equilibrium in the two period game, in which player \( i \) hunts in period 1 if and only if

\[
C_i \leq \frac{2(1-\epsilon) + \sqrt{4(1-\epsilon)^2 + 4 \epsilon^2}}{2}.
\]

Hence for small \( \epsilon \), the probability that the stag is caught in the dynamic game is still near 1, even if the probability that the stag is caught in the static game is 0.

These simple examples suggest that if players have no doubt about other players’ willingness to play the risky action, then the static game still has multiple equilibria, but the dynamic game refines away the inefficient ones. On the other hand, if players slightly doubt about other players’ willingness to play the risky action, then the static game has a unique equilibrium, which is the most inefficient equilibrium. Dynamics gives a qualitatively different set of equilibria, in which the probability of the inefficient outcome is small in every equilibrium. Dynamics will not make such a difference, however, unless the risky action is irreversible, and the safe action is reversible. We will illustrate this by comparing currency attacks and bank runs in Section 2.6, and by studying an arms race game in Section 2.7. The main message of the chapter is thus the following. Endogenous timing, combined with an irreversible risky action and a reversible safe action, will overcome coordination failure even in the presence of incomplete information. We will confirm this intuition in a generalized version of the above two player game.

The rest of the chapter is organized as follows. Section 2.3 discusses some related literature. Section 2.4 presents the model and the asymptotic result. Section 2.5 establishes the existence of PBE in both the no discounting case and the discounting case. A common shock model is studied in Section 2.6, and the issue of irreversibility and the arms race game will be discussed in Section 2.7. Section 2.8 concludes.

### 2.3 Related Literature

To resolve the multiplicity problem in games described in the introduction, Carlsson and van Damme (1993) introduce a small amount of uncertainty about the payoff from the safe action. They create dominant strategy types of players who always choose the risky action or the safe action in the Bayesian game. These extreme (remote) types exert an influence on other types’ reasoning, so that a process of iterative elimination of strictly dominated strategies will generate a unique Bayesian Nash Equilibrium. This idea is further developed by Morris and Shin in a series of papers. Morris and Shin find that introducing small heterogeneity in the information structure of games with strategic complementarity is likely to generate a unique equilibrium. Again, the games they consider are simultaneous move games. In a simultaneous move game, players eliminate strictly dominated strategies iteratively by introspection, which selects
a unique strategy. An important difference between our model and the global games model is that we have independent types, while the global games model usually considers correlated types. The correlation is created by the players’ noisy observations of a common fundamental variable. We will investigate a common shock model in Section 2.6.

While the coordination games literature is largely concerned with the static setting, a different set of papers has focused on how rational behavior could lead to individuals’ rationally following the actions of others who moved earlier. This is the so-called informational cascade literature, initiated by Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992). In these papers players might ignore their own information about the environment and blindly yet rationally follow their predecessors’ choices. One player’s action has no direct impact on other players’ payoff, and the order of moves is exogenously given. Chamley and Gale (1994) endogenize the order of actions. In their model, a random fraction of people have an investment opportunity. The payoff to investment depends on the number of people who have the opportunity, but not on the number of people who actually execute it. Another endogenous timing model with pure informational externalities is studied in Zhang (1997), where different players receive signals about the fundamentals with different precision. In equilibrium, the player with the highest precision waits the shortest, and her choice will be imitated immediately by everyone else. Like the informational cascade literature, herding might occur in this chapter, too, but for a different reason. People follow other people because they rationally expect themselves to be followed, and this is due to the strategic complementarity in our model.

Our model also has some similarity with the war of attrition game. In fact, if only one hunter is required to catch the stag, and if the stag is a public good so that once it is caught everybody has a share, then it is exactly a war of attrition game (Bliss and Nalebuff (1984)). In the war of attrition game, it can be strictly optimal for a player to wait for a while and contribute to the public good only if there is no contribution during the waiting period. In our model this could also be the case if a player finds it worthwhile to postpone her action in order to synchronize her action with the “bottleneck” players. As a consequence, it might be that in equilibrium a player is not willing to move when she is more optimistic about other players’ types, but she is willing to move then she becomes more pessimistic about their types. We will discuss this in Section 2.5.

Dynamic games with strategic complementarity and incomplete information are also studied in Choi (1997) and Dasgupta (2001). Choi (1997) studies a sequential technology adoption game in which agents refrain from experimenting with a new technology unless the expected value of it is sufficiently higher than the realized value of the established technology. The agents tend to herd on an established technology for fear of being stranded. Choi concludes that sequential adoption may be worse than simultaneous adoption in terms of ex-ante welfare.
The major difference between Choi’s model and our model is that in his model (i) both actions are risky and irreversible, (ii) players are symmetric and (iii) uncertainty about a technology is resolved immediately after the technology is tried by someone. All the three factors discourage the ex-ante efficient outcome from being realized.

Contrary to Choi’s conclusion, Dasgupta (2001) argues that dynamics may be good when network effects and uncertainty are both present. Dasgupta considers a continuum of agents, who make private observations of the underlying state of the world. Agents choose whether and when to switch from a safe project to a risky one. Network externalities are present in the risky project. In particular, the fraction of investors it takes to succeed in the risky project is inversely related to the state of the world. Dasgupta shows that when heterogeneity is small enough, i.e. when agents’ observations are close enough, there is a unique monotone equilibrium in which agents switch early if and only if their observations exceed some threshold value. Moreover, endogenous timing produces higher welfare than exogenous timing and the static counterpart of the switching game. Dasgupta (2001) is in spirit the closest paper to ours, yet there remain several major differences: (a) He has a continuum of agents, so that an individual player need not worry about her impact on the rest of the players, while we have an arbitrary but finite number of players, and an individual player’s impact is not negligible; (b) There are only two periods in his model, while there is a finite yet arbitrary number of periods in our model; (c) In his model agents have private observations of a common fundamental of the economy, and their observations are close to each other, in our model agents have private information about their own types, and they don’t have to be close to each other and (d) Agents’ observations are assumed to follow a normal distribution in his model, but we put no specific restrictions on the distribution of types except some regularity conditions.

The common shock section is based on an investment game in Morris and Shin (2000). Morris and Shin (2000) find that in the simultaneous move game, inefficiency does not disappear even if players observe the fundamentals of the economy with more and more accuracy. We show that inefficiency will disappear as the amount of noise vanishes, if the game is played out sequentially. The arms race game is studied in Baliga and Sjöström (2002), in which they use a cheap talk approach to restore efficiency in the "security dilemma". Dynamics does not change much in their model because the risky action (not to build weapons) is a reversible decision. It is hard to destroy weapons that have already been built, so the safe action (to build weapons) is irreversible.

If it is hard to renege on the risky choice and easy to switch from the safe one, then there is a role for leaders to play, and coordination with incomplete information may be achieved by sequential moving. Thus a crucial aspect of our model, missing in the static model, is whether or not the risky action is reversible. One example is political revolution. A successful revolution requires a
critical mass of participants. However, people are uncertain about other people's willingness to participate. Usually it is hard for someone who already takes part to renege, but it's easier for someone who is silent earlier to change position and follow suit. Kuran (1989) gives a theory of revolution that relies on "preference falsification". Formally Kuran's model is static, but it implies a dynamic process in which people who are privately more pro-revolution revolt early, followed by people who are slightly less pro-revolution, and so on. Eventually a bandwagon effect takes on and a majority of people switch position publicly. We formalize this reasoning explicitly in a dynamic model. A significant difference is that we assume people differ in their cost, not their "political ideal point". If we take people's "political ideal point" seriously, then after a revolution only those who join the revolution late should be repressed, since they are most sympathetic for the old regime. What we observe, however, is that those who join the revolution at the very beginning sometimes get wiped out first. Our model implies that people with smaller cost to participate a revolution will take part earlier, and it is these people who are most capable of launching another revolution, hence they are likely to be repressed immediately after the success of the revolution.

Murphy, Shleifer and Vishny (1989) consider a model where industrialization of many sectors can be profitable even if no single sector can break even by industrializing alone. They propose several scenarios in which a good equilibrium where all sectors industrialize and a bad equilibrium where no sector does can coexist. Thus industrialization requires a certain level of coordination, which in turn, seems to require the intervention of some conscious planning. In fact, Murphy, Shleifer and Vishny (1989) point out the role of government intervention: "..., the government can use investment subsidies as long as they are widely enough spread to bring about a critical mass of investment needed to sustain a big push." This chapter suggests that even without conscious planning, coordination among many sectors might still be achieved spontaneously through a herding mechanism. Indeed, in the industrialization scenario the risky action is likely to be irreversible.

2.4 The Model and The Asymptotic Result

The simultaneous move stag hunt game is as follows. There are \( n \) players who simultaneously choose whether to hunt or not. The stag is caught if and only if all players hunt. Each player gets a benefit of \( 1 \) if the stag is caught. Players' costs of hunting are \( i.i.d. \) with distribution function \( F \) over support \([0, 1 + \epsilon]\), where \( \epsilon \geq 0 \) and \( f (\cdot) \) is bounded. A player loses her cost if she chooses to hunt, no matter what other players do. A player receives \( 0 \) if she chooses not to hunt. The only private information is cost.

The following proposition is essentially Theorem 1 in Baliga and Sjöström (2002).
Proposition 1 In the static game, there always exists a Bayesian Nash Equilibrium (BNE) in which there is no hunting with probability 1. If \( \epsilon > 0 \), and \( F(c) < c, \forall c \in (0, 1 + \epsilon) \), then the no hunting equilibrium is unique, for any \( n \).

Proof: It is clear that nobody hunting whatever her type may be is always a Nash equilibrium of the Bayesian game. It is also clear that any BNE has the cutoff property: if no hunting is a best response for type \( c_i \) of player \( i \), then it is also a best response for type \( c'_i > c_i \).

Now fix any BNE. Let \( \hat{c}_i \) denote player \( i \)'s cutoff type. If no hunting is not the unique equilibrium, then there must exist another equilibrium in which \( 1 > \hat{c}_k > 0, \forall k \). Without loss of generality assume \( \hat{c}_i = \max_k \{ \hat{c}_k \} \), then we have

\[
\hat{c}_i = \prod_{k \neq i} F(\hat{c}_k) < \prod_{k \neq i} \hat{c}_k \leq \hat{c}_i^{n-1},
\]

which is a contradiction. \( \blacksquare \)

Notice that Proposition 1 holds for any \( \epsilon > 0 \), so long as the condition on the distribution function holds. Hence in the limit when \( \epsilon = 0 \), as long as the condition on the distribution function still holds, the equilibrium in which nobody hunts no matter what the type is can be selected as the unique prediction of the noise free (\( \epsilon = 0 \)) game. As we will see next, this strong prediction is completely a consequence of the simultaneity of the moves.

From now on, we will consider a dynamic version of the stag hunt game. Assume that it takes the \( n \) players \( T \) periods to finish the game, where \( n \leq T < \infty \). Players choose whether and when to hunt. Hunting is a one time, irreversible decision, i.e. once a player chooses to hunt, her cost is not refundable, no matter whether the stag is caught or not. No hunting is a reversible decision. In each period, each player observes accurately how many players have committed to hunting so far. Players discount both cost expenditures and potential rewards by the same discount factor \( \delta, 0 < \delta \leq 1 \). At the end of the \( T \)th period, a player’s payoff is determined by the final decisions of everybody, i.e. the payoff is read off from the simultaneous move game, according to the final decisions of each player, up to certain adjustments of discounting. Formally, let \( h^T \) denote an arbitrary terminal history, let \( \pi_j (h^T, c_j) \) denote type \( c_j \) of player \( j \)'s discounted payoff attached to \( h^T \). If \( j \) never hunts, then \( \pi_j (h^T, c_j) = 0 \); if \( j \) hunts in period \( t \), but at least one player never hunts, then \( \pi_j (h^T, c_j) = \delta^{t-1} (-c_j) \); if \( j \) hunts in period \( t \), and there is no player who never hunts, and the last hunt occurs in period \( t' \), then \( \pi_j (h^T, c_j) = \delta^{t-1} (-c_j) + \delta^{t'-1} \cdot 1 \).

Our solution concept is perfect Bayesian equilibrium (PBE). For the case where \( \delta = 1 \), it is convenient to restrict attention to the set of PBE that satisfies the following assumption.
(A) After any history, for any player \( j \), for any two types \( c_j \) and \( c'_j \) of player \( j \), if both types are indifferent between hunting and waiting, then they choose the same pure action.

As we will see in the following lemma, assumption (A) is not needed if \( \delta < 1 \). If \( \delta = 1 \), then there might exist a PBE in which a player plays a strategy that is not monotonic in her type. This is ruled out by assumption (A) and the following lemma.

**Lemma 1** In any PBE, if either \( \delta < 1 \), or \( \delta = 1 \) and (A) holds, then the following is true. After any history, if a player has not hunted so far, and if he is willing to hunt when his cost is \( c \), then he is also willing to hunt when his cost is below \( c \), i.e. any PBE has the cutoff property.\(^1\)

**Proof:** Fix a PBE, a player \( i \), and a history \( h \). Let \( c_1 \) and \( c_2 \) be two types of player \( i \). Let \( c_1 < c_2 \). Let \( H(c_j|h) \) denote the expected equilibrium payoff of type \( c_j \), \( j = 1, 2 \), if type \( c_j \) chooses to hunt, conditional on history \( h \); let \( W(c_j|h) \) denote the expected equilibrium payoff of type \( c_j \), \( j = 1, 2 \), if type \( c_j \) chooses to wait, conditional on \( h \).

Case 1. After choosing to wait at history \( h \), type \( c_1 \) and type \( c_2 \)'s equilibrium decisions in the continuation game are identical. In this case, we can write

\[
W(c_1|h) = \delta (\alpha(h)(-c_1) + D(h)),
\]

and

\[
W(c_2|h) = \delta (\alpha(h)(-c_2) + D(h)),
\]

where \( \alpha(h) \) and \( D(h) \) only depend on other players’ equilibrium strategies, and \( \alpha(h) \leq 1 \). Since \( \delta < 1 \), if \( H(c_2|h) \geq W(c_2|h) \), then \( H(c_1|h) > W(c_1|h) \).

Case 2. After choosing to wait at history \( h \), type \( c_1 \) and type \( c_2 \)'s equilibrium decisions in the continuation game are different. In this case, let \( c_2 \) mimic \( c_1 \)'s decision in each and every contingency in the continuation game. Let \( \hat{W}(c_2|h) \) denote the resulting expected payoff of \( c_2 \). Then it must be that \( \hat{W}(c_2|h) \leq W(c_2|h) \), by the incentive compatibility of perfect Bayesian equilibrium. Hence if \( H(c_2|h) \geq W(c_2|h) \), then \( H(c_2|h) \geq \hat{W}(c_2|h) \), which in turn, implies that \( H(c_1|h) > W(c_1|h) \), by the argument in case 1.

Since every PBE has the cutoff property, the belief system of the equilibrium can be easily determined from the equilibrium strategy, using Bayes’ rule. In

\(^1\)Usually in the literature, equilibrium with the cutoff property is called "monotone equilibrium", in that a player’s action is monotonic in her own types. In this paper we use the term "monotone equilibrium" in a different sense. An equilibrium is monotone if a player’s action is monotone in her belief about other players’ types. That is, if she moves when she is more pessimistic about other players’ types, then she also moves when she is more optimistic. We will refer to equilibrium with the cutoff property as "cutoff equilibrium".
particular, after any history, the belief about any remaining player’s types must be some truncated distribution above some cutoff value. The belief about the players who have already moved is irrelevant. Having said this, from now on we identify a PBE with a PBE strategy, omitting the supporting belief system, which can be derived from the strategy straightforwardly. Our proofs will use the following notation. Let \( \bar{x} \) denote an \( n \) dimensional vector of lower bounds on the \( n \) players’ types. The upper bound on each player’s cost is \( 1 + \epsilon \). Let \( (n, \epsilon, \delta, \bar{x}, T) \) denote the \( n \) player, \( T \) period game in which the discount factor is \( \delta \), the lower bounds on the players’ cost types are \( \bar{x} \), and the distribution over \( j \)’s costs is given by \( F \), truncated to the interval \([x_j, 1 + \epsilon]\), where \( x_j \) is the \( j \)th coordinate of \( \bar{x} \).

If there are no dominant strategy types and no discounting, then the results of Proposition 1 are completely reversed in the dynamic game.

**Proposition 2** If \( \epsilon = 0 \) and \( \delta = 1 \), then in any PBE that satisfies (A) the stag is caught with probability 1.

**Proof:** We show this by induction on the number of players. This is obvious if \( n = 1 \). Now assume that this is true for \( \Gamma(k, 0, 1, \bar{x}, T) \), \( \forall k \leq n \), \( \forall \bar{x} \prec (1, 1, ..., 1) \), \( \forall T \geq k \). We need to show that this is true for \( \Gamma(n + 1, 0, 1, \bar{x}, T) \), \( \forall \bar{x} \prec (1, 1, ..., 1) \), \( \forall T \geq n + 1 \). Suppose otherwise that there exists a PBE of \( \Gamma(n + 1, 0, 1, \bar{x}, T) \), such that the stag is caught with probability less than 1. Then there exists a player \( j \), a type \( c_j < 1 \), such that the equilibrium payoﬀ of \( c_j \) is less than \( 1 - c_j \). However, if \( c_j \) hunts in period 1, then by the induction hypothesis, \( c_j \) obtains a payoﬀ of \( 1 - c_j \), a contradiction.

Next we show that Proposition 2 is robust to introducing a small amount of dominant strategy types and discounting. That is, for small \( \epsilon > 0 \) and large \( \delta < 1 \), the probability that the stag is caught in any PBE is close to 1.

**Proposition 3** For all sequences \((\epsilon_k, \delta_k)_k \) such that \((\epsilon_k, \delta_k)_k \rightarrow (0, 1)\), and \( \delta_k < 1 \), \( \forall k \), for all sequences \((E_k)_k \) such that \( E_k \) is a PBE of \( \Gamma(n, \epsilon_k, \delta_k, \bar{0}, T) \), \( p_k \rightarrow 1 \), where \( p_k \) is the probability that the stag is caught in \( E_k \).

**Proof:** Fix a sequence \((\epsilon_k, \delta_k)_k \rightarrow (0, 1)\). Let \( \bar{x}_k \) denote the \( n \) dimensional vector of lower bounds on the \( n \) players’ types. Let \( S \) be a subset of the \( n \) players, let \( \bar{x}^S_k \) be the restriction of \( \bar{x}_k \) to \( S \). Let \( \Gamma(S, \epsilon_k, \delta_k, \bar{x}^S_k, T') \) denote the game in which players in \( S \) play the corresponding game for \( T' \) periods, where \(|S| \leq T' \leq T\). Let \( E_k \) denote an arbitrary equilibrium of \( \Gamma(n, \epsilon_k, \delta_k, \bar{x}_k, T) \), let \( E^S_k \) denote an arbitrary equilibrium of \( \Gamma(S, \epsilon_k, \delta_k, \bar{x}^S_k, T') \).
We prove the proposition by induction on the number of players. First we state the induction hypothesis (IH).

\text{(IH): } \forall x < 1, \forall (\overline{x}_k)_k \text{ such that } x_{kj} \leq x, \forall k, \forall j = 1, ..., n, \forall S, \forall (E^S_k)_k, P\left( \text{everybody in } S \text{ hunts in } E^S_k \right) \rightarrow 1.

The IH obviously holds when \( n = 1 \). Suppose it holds for \( n \geq 1 \). Now suppose there are \( n + 1 \) players. Let \( E_k \) denote an equilibrium of \( \Gamma_k := \Gamma(n+1, \epsilon_k, \delta_k, \overline{x}_k, T) \), where \( T \geq n + 1 \). We need to show that \( \forall x < 1, \forall (\overline{x}_k)_k \text{ such that } x_{kj} \leq x, \forall k, \forall j = 1, ..., n+1, \forall (E_k)_k, P\left( \text{everybody hunts in } E_k \right) \rightarrow 1. \)

Proof by way of contradiction. The contradiction hypothesis is

\text{(CH): } \exists (\epsilon_k, \delta_k)_k \rightarrow (0, 1), \exists x < 1, \exists (\overline{x}_k)_k \text{ such that } x_{kj} \leq x, \forall k, \forall j = 1, ..., n+1, \exists (E_k)_k, p_k := P\left( \text{everybody hunts in } E_k \right) \rightarrow p < 1.

The first implication of CH: By Proposition 2(i), \( E_k \) is characterized by a collection of cutoff points. Let \( c^i_j \) denote the first period cutoff type of player \( j \) in \( E_k \), \( j = 1, ..., n+1 \). Let \( c_k := \max_j \left\{ c^i_j \right\} \), then \( c_k \rightarrow 1 \), since otherwise \( p_k \rightarrow 1 \) by IH. Taking a subsequence if necessary let \( c_k \rightarrow c < 1 \). The first implication of CH is, \( \forall \text{ type } c' \text{ of any player } j, \text{ if } j \text{ hunts in period } 1 \text{ in } \Gamma_k, \text{ then her expected payoff is } -c' + \alpha_k \), where \( \alpha_k \rightarrow 1 \), by IH.

The second implication of CH: Let \( A_k \) denote the event that the stag is not caught in \( E_k \). Then \( A_k \subseteq [0, 1 + \epsilon]^{n+1} \). Moreover, by the cutoff property of \( E_k \), \( A_k \) is a finite union of mutually disjoint product sets. That is \( A_k = \bigcup_{i=1}^{I(k)} A^i_k \), where \( I(k) \leq B < \infty \), and \( B \) only depends on the number of players and the number of cutoff points. \( A^i_k \) is the \( i \)th product set such that if the players’ types fall into this set, then the stag will not be caught in \( E_k \). \( A^i_k \) can be written as

\[ A^i_k = \Pi_{j=1}^{n+1} A^i_{kj}, \]

where \( A^i_{kj} \) is the \( j \)th component of \( A^i_k \), \( j = 1, ..., n+1 \). Let \( A^{i(k)}_k \) denote the event that receives the highest probability among all the \( A^i_k \)'s, \( i = 1, ..., I(k) \). By CH, \( P\left( A^{i(k)}_k \right) \rightarrow q > 0 \). But \( P\left( A^{i(k)}_k \right) = \Pi_{j=1}^{n+1} P(A^{i(k)}_{kj}) \), hence \( \forall j, \left( P\left( A^{i(k)}_{kj} \right) \right)_k \rightarrow q' > 0. \) Hence \( \forall j, \exists c < 1, \exists K_0 > 0, \) such that \( \forall k \geq K_0, \exists c(k) \leq c, \) and \( c(k) \in A^{i(k)}_{kj} \). Notice that once \( c(k) \in A^{i(k)}_{kj} \), the equilibrium payoff to type \( c(k) \) of player \( j \) is bounded away from below \( 1 - c(k) \), because so long as \( c_l \in A^{i(k)}_{kl}, \forall l \neq j \), which happens with non-negligible probability \( \Pi_{l \neq j} P(A^{i(k)}_{kl}) \), type \( c(k) \) of player \( j \) gets at most 0 in equilibrium. More precisely, \( \exists z > 0, \exists K_1 > 0, \) such that \( \forall k \geq K_1, \Pi_{l \neq j} P(A^{i(k)}_{kl}) \geq z \). Therefore, \( \forall k \geq K := \max\{K_0, K_1\} \), type \( c(k) \) of player \( j \) gets at most \( (1 - z)(1 - c(k)) \) in \( E_k \). The second implication of CH is, therefore, \( \forall j, \exists c < 1, \exists z > 0, \exists K > 0, \) such that \( \forall k \geq K, \exists c(k) \leq c, \) and type \( c(k) \) of player \( j \)'s equilibrium payoff is at most \( (1 - z)(1 - c(k)). \)
By the first implication of CH, for \( k \) sufficiently large, type \( c(k) \) of player \( j \) can guarantee herself an expected payoff arbitrarily close to \( 1 - c(k) \). Hence if \( k \) is large enough, the two implications contradict each other.

Even if there is a significant discounting and the proportion of dominant strategy types is high, complete coordination failure is impossible in the dynamic game, as shown in the following proposition.

**Proposition 4** If \( \epsilon > 0 \) and \( \delta < 1 \), or \( \delta = 1 \) and \( (A) \) holds, then in any PBE that satisfies \( (A) \) the stag is caught with positive probability.

**Proof:** We show this by induction on the number of players. This is obvious if \( n = 1 \). Now assume that this is true for \( \Gamma\left(k, \epsilon, \delta, \overrightarrow{0}, T\right) \), \( \forall T \geq k \), \( \forall k \leq n \), where \( \overrightarrow{0} \) is the \( k \) dimensional vector \( (0, 0, \ldots, 0) \). We need to show that this is true for \( \Gamma\left(n + 1, \epsilon, \delta, \overrightarrow{0}, T\right) \), \( \forall T \geq n + 1 \), where \( \overrightarrow{0} \) is the \( n + 1 \) dimensional vector \( (0, 0, \ldots, 0) \). Suppose otherwise that there exists a PBE of \( \Gamma\left(n + 1, \epsilon, \delta, \overrightarrow{0}, T\right) \) in which the stag is caught with probability 0. Then it must be that in the first period, at least one player’s cutoff type is 0, since otherwise there is a positive probability that everybody moves in the first period. Let \( S \) denote the set of players whose cutoff types in period 1 are 0. If \( |S| < n + 1 \), then there is a positive probability that everybody in \( S^c \) moves in period 1, but then by the induction hypothesis, there is a positive probability that everybody in \( S \) follows up in the continuation game, which is a contradiction. If \( |S| = n + 1 \), then for any player \( j \), if \( c_j \) is small enough, then by the induction hypothesis, \( c_j \) should deviate by moving in period 1 and obtain a positive expected payoff. On the other hand, the equilibrium payoff to type \( c_j \) is 0, contradiction.

**2.5 Existence of Equilibrium**

The robustness result in the last section only applies when the set of PBE is non-empty. In this section, we establish the existence of equilibrium. For \( \delta = 1 \), we show existence by constructing an equilibrium that satisfies assumption \( (A) \). For \( \delta < 1 \), we offer a general existence proof, and explain why direct construction is difficult.

**2.5.1 No Discounting Case**

In this case we construct an equilibrium that not only satisfies assumption \( (A) \), but also meets the following three requirements.

(a) Symmetry.
(b) Whenever a player is indifferent between hunting and waiting, she always chooses to hunt.

(c) If a player does not hunt when she is more optimistic about other players’ types, then she doesn’t hunt when she is less optimistic.\(^2\)

We do it in two steps. First we claim that any equilibrium in the infinite horizon game \((T = \infty)\) that satisfies (a), (b) and (c) corresponds to an equilibrium in the finite horizon game \((n \leq T < \infty)\) that also satisfies (a), (b) and (c). Then we show that there exists a unique equilibrium in the infinite horizon game that satisfies (a), (b) and (c).

Claim 1 Let \(\Gamma\) denote the infinite horizon game, and \(\Gamma^T\) the \(T\) period game, \(T \geq n\). Let \(E\) be an equilibrium of \(\Gamma\) that satisfies (a), (b) and (c). Then the restriction of the equilibrium path of \(E\) to the first \(T\) periods can also be supported as an equilibrium path in \(\Gamma^T\).

Proof: Let \(E_T\) be the strategy such that whenever the number of players left is less than or equal to the number of periods left, play according to \(E\), otherwise don’t hunt. Notice that \(E_T\) trivially satisfies (a), (b) and (c). We show that \(E_T\) is a PBE in \(\Gamma^T\). We prove this in steps.

Step 1. After each history, if the number of players left is less than or equal to the number of periods left, then the expected payoff a player gets by following \(E_T\) is the same as she could get in \(\Gamma\). Before we prove this, we state it formally as follows. Let \(\pi^T(h^t, c_j)\) be the expected payoff to type \(c_j\) of player \(j\) in \(\Gamma^T\) after history \(h^t\), if \(j\) follows \(E_T\). Let \(\pi(h^t, c_j)\) be the expected payoff to type \(c_j\) of player \(j\) in \(\Gamma\) after history \(h^t\), if \(j\) follows \(E\). Let \(n(h^t)\) be the number of players left after \(h^t\), and \(T(h^t)\) the number of periods left after \(h^t\). Step 1 claims that \(\forall h^t, \text{ if } n(h^t) \leq T(h^t), \text{ then } \pi^T(h^t, c_j) = \pi(h^t, c_j), \forall c_j, \forall j\).

Now we prove Step 1.

Suppose after \(h^t\), \(E_T\) prescribes "hunt" for type \(c_j\) of player \(j\). Need to show \(\pi^T(h^t, c_j) = \pi(h^t, c_j)\). Let \(v(n, x, T)\) denote the expected discounted value of the stag in \(\Gamma^T\) when everybody follows \(E_T\),\(^3\) where \(x\) is the lower bound on the \(n\) players’ cost. Let \(v(n, x)\) denote the expected discounted value of the stag in \(\Gamma\) when everybody follows \(E\). First we prove that \(\forall x < 1, \forall n, \forall T \geq n, v(n, x, T) = v(n, x)\).

---

\(^2\)Being "more optimistic" means putting a smaller lower bound on other players’ types.

\(^3\)Suppose there is an "outside" player whose cost is 0, whose participation is not essential, and whose payoff function is the same as the rest of the players. Then \(v(n, x, T)\) is this player’s payoff in \(\Gamma^T\), if everybody else follows \(E_T\).
Proof by induction on the number of players. First it is obvious that \( v(1, x, T) = v(1, x) \). The induction hypothesis is that \( \forall k \leq n, \forall x, \forall T \geq k, v(k, x, T) = v(k, x) \). We need to show \( \forall T \geq n + 1, v(n + 1, x, T) = v(n + 1, x) \).

For \( i = 0, \ldots, n + 1 \), let \( p_i \) denote the probability that in the first period \( i \) players hunt in \( \Gamma^T \) where everybody plays \( E_T \). Let \( x' \) denote the first period cutoff prescribed by \( E_T \). If everybody’s type is above \( x' \), then nobody hunts in period 1. By (c), nobody hunts thereafter, the expected discounted value of the stag must be 0. Therefore,

\[
v(n + 1, x, T) = p_0 \cdot 0 + p_1 \cdot \delta \cdot v(n, x', T - 1) + \ldots + p_{n + 1} \cdot 1,
\]

and

\[
v(n + 1, x) = p_0 \cdot 0 + p_1 \cdot \delta \cdot v(n, x') + \ldots + p_{n + 1} \cdot 1.
\]

Hence by the induction hypothesis, \( v(n + 1, x, T) = v(n + 1, x) \). Now if after \( h^t \), \( E_T \) prescribes "hunt" for type \( c_j \) of \( j \), then

\[
\pi^T(h^t, c_j)
= p_0 \cdot \delta \cdot v(n(h^t) - 1, x', T(h^t) - 1) + p_1 \cdot \delta \cdot v(n(h^t) - 2, x', T(h^t) - 1) + \ldots + p_{n(h^t) - 1} \cdot 1 - c_j
\]

\[
= p_0 \cdot \delta \cdot v(n(h^t) - 1, x') + p_1 \cdot \delta \cdot v(n(h^t) - 2, x') + \ldots + p_{n(h^t) - 1} \cdot 1 - c_j
\]

\[
= \pi(h^t, c_j),
\]

where \( x' \) is the current period cutoff prescribed by \( E_T \), and \( p_k, k = 0, \ldots, n(h^t) - 1 \), is the probability that \( k \) out of \( n(h^t) - 1 \) players will move in the current period.

Suppose after \( h^t \), \( E_T \) prescribes "wait" for type \( c_j \) of player \( j \). In this case proof by induction on \( n(h^t) \). The proof is trivial if \( n(h^t) = 1 \). Now suppose step 1 is true for \( n(h^t) \leq n \). We need to show that step 1 is also true for \( n(h^t) = n + 1 \). Let \( p_k \), \( k = 0, \ldots, n(h^t) - 1 \), be the probability that \( k \) out of \( n(h^t) - 1 \) players will move in the current period. Let \( h^{t+1}_i, i = 0, 1, \ldots, n(h^t) - 1 \), be the history in which right after \( h^t \), \( i \) players hunt in period \( t \). Then,

\[
\pi^T(h^t, c_j)
= p_0 \cdot \delta \cdot \pi^{T-1}(h_0^{t+1}, c_j) + p_1 \cdot \delta \cdot \pi^{T-1}(h_1^{t+1}, c_j) + \ldots + p_{n(h^t) - 1} \cdot \delta \cdot \pi^{T-1}(h_{n(h^t) - 1}^{t+1}, c_j)
\]

\[
= p_0 \cdot \delta \cdot \pi(h_0^{t+1}, c_j) + p_1 \cdot \delta \cdot \pi(h_1^{t+1}, c_j) + \ldots + p_{n(h^t) - 1} \cdot \delta \cdot \pi(h_{n(h^t) - 1}^{t+1}, c_j)
\]

\[
= \pi(h^t, c_j),
\]
where the second equality is because of the induction hypothesis and the fact
that \( \pi^{T-1}(h_{0}^{t+1}, c_{j}) = \pi(h_{0}^{t+1}, c_{j}) = 0 \), due to (c).

Step 2. If the number of players left is more than the number of periods left, it is obviously optimal not to hunt given that nobody else does.

Step 3. The payoff a player gets by deviating after a given history \( h^{t} \), however, can be no higher than the payoff she gets by deviating in the infinite horizon game. To see this, notice that there are two types of deviations, (1) A player should hunt but does not hunt. First of all, it must be the case that before the player makes the decision, \( n(h^{t}) < T(h^{t}) \). If \( n(h^{t}) < T(h^{t}) \), then the payoff from such deviation is the same as the payoff from the same deviation in the infinite horizon game, because by step 1, the payoff from such deviation is the same convex combination of the same expected continuation payoffs as the payoff from such deviation in the infinite horizon game. If \( n(h^{t}) = T(h^{t}) \), then the continuation payoff after \( h_{0}^{t+1} \) of such deviation is 0 in \( \Gamma^{T} \), and the continuation payoff after \( h_{0}^{t+1} \) of the same deviation is non-negative in \( \Gamma \). At the same time, by step 1, the continuation payoff after \( h_{i}^{t+1} \) of the deviation in \( \Gamma^{T} \) is the same as the continuation payoff after \( h_{i}^{t+1} \) of the deviation in \( \Gamma \), for any \( i = 1, \ldots, n(h^{t}) - 1 \). Therefore, the expected payoff of the deviation in \( \Gamma^{T} \) can be no higher than the expected payoff of the deviation in \( \Gamma \). (2) A player should not hunt but hunts. If right before such deviation the number of players left is more than the number of periods left, then the deviation is clearly not profitable, otherwise the payoff from such deviation is the same as the payoff from the same deviation in the infinite horizon game, because by step 1, the payoff from such deviation is the same convex combination of the same expected continuation payoffs as the payoff from such deviation in the infinite horizon game.

By Claim 1, in order to establish existence for the no discounting, finite horizon case, it suffices to show existence for the no discounting, infinite horizon case.

**Proposition 5** In the \( n \) player stag hunt game with infinite horizon and no discounting, there is a unique PBE that satisfies (a), (b) and (c). The equilibrium is characterized by a sequence of cutoff values

\[
1 = g(1) > g(2) > \ldots > g(n) > 0,
\]

where \( g(k) \) is such that in the continuation game with \( k \) players, any type \( c \leq g(k) \) is indifferent between hunting and waiting in the current period, and any type \( c > g(k) \) strictly prefers to wait in the current period, \( 1 \leq k \leq n \). Moreover, \( g(n) \rightarrow 0 \), as \( n \rightarrow \infty \).

**Proof:** See the Appendix.
We will sketch the intuition for the two player game here. First we introduce some notations for the \( n \) player game in general. Let \( p(k,x) \) denote the probability of a failure in a \( k \)-player game, where \( 1 \leq k \leq n \), and \( x \) is the lower bound on the players’ type. Let \( p(F,k,x) \) denote the probability of a failure (the stag is not caught) in the \( k \) player game when one player hunts for sure and the other player’s cost is at least \( x \), where \( 2 \leq k \leq n \). Let \( p^k_m (c|x) \) denote the probability that \( k \) players out of \( m \) players have types no higher than \( c \), conditional on that the lower bound on everybody’s cost is \( x \), where \( 0 \leq k \leq m \leq n \). Let \( v(k,c) \) denote the equilibrium continuation payoff to the cutoff type in the \( k \) player game where the opponents’ types are above \( c \), where \( 2 \leq k \leq n \). The two player indifference equation can thus be written as

\[-c + (1 - p(F,2,x)) \cdot 1 = p^1_1 (c|x) (1 - c) + p^{0,1}_1 (c|x) v(2,c).\]  

(1)

By Lemma 1, if \( \delta = 1 \), then any PBE that satisfies (A) has the cutoff property. Since condition (b) implies (A), any PBE that satisfies (a), (b) and (c) must have the cutoff property. This implies that we can write out \( v(2,c) \) as follows.

\[v(2,c) = \max \{0, -c + (1 - p(F,2,c))\}.\]

We rewrite equation (1) as

\[LHS_2(x,c,c) = RHS_2(x,c,c),\]

where the first argument is the lower bound on cost, the second argument is the reference player’s type, and the third argument is the cutoff that the opponent uses.

Now any PBE that satisfies (a) and (b) can be characterized by a cutoff type \( \hat{c}_2 \), such that \( LHS_2(x,c,\hat{c}_2) \geq RHS_2(x,c,\hat{c}_2), \forall c \leq \hat{c}_2 \), and \( LHS_2(x,c,\hat{c}_2) < RHS_2(x,c,\hat{c}_2), \forall c > \hat{c}_2 \).

Such \( \hat{c}_2 \) must solve (1), but not every solution to (1) can be \( \hat{c}_2 \). I show that such \( \hat{c}_2 \) exists, it is unique, and it doesn’t depend on \( x \). As proved in the Appendix, (1) has a continuum of solutions, and \( \hat{c}_2 \) is the largest one. In fact, a solution to (1) can become \( \hat{c}_2 \) if and only if it solves

\[-c + (1 - p(F,2,c)) = 0.\]

It is easy to show that \( p(F,2,c) = p(1,c) \). Hence \( \hat{c}_2 \) solves

\[-c + (1 - p(1,c)) = 0.\]
Figure 5 summarizes the discussion.

In Figure 5, the l.h.s. and r.h.s. of (1) coincide up to $g(2)$, where $g(2) := \tilde{c}_2$, then the r.h.s. kinks upward. As we change $x$ into $x' > x$, both sides will shift downward, but the kink point remains the same.

In general, the sequence of the cutoffs can be found inductively as follows.
\[ g(1) = 1, \]
\[ p(1, x) = p^{0 \to 1}(g(1) | x), \]
\[ 1 - g(2) = p(1, g(2)), \]
\[ p(2, x) = p^{0 \to 2}(g(2) | x) \cdot 1 + p^{1 \to 2}(g(2) | x) p(1, g(2)), \]
\[ 1 - g(3) = p(2, g(3)), \]
\[ \vdots \]
\[ p(n - 1, x) = p^{0 \to n-1}(g(n - 1) | x) \cdot 1 + \ldots + p^{n-2 \to n-1}(g(n - 1) | x) p(1, g(n - 1)), \]
\[ 1 - g(n) = p(n - 1, g(n)). \]

As we can see from these equations, the cutoff values and the conditional probabilities of failure are like two strands that feed on each other.

2.5.2 Discounting Case

Now we focus on the case where players discount both cost and reward at the same rate \( \delta < 1 \). In this case, each player prefers the stag to be caught as early as possible, but also prefers his cost to be incurred as late as possible. The tradeoff between hunting and waiting is the following. Hunting may accelerate the process to success, it may even make the difference between success and failure; waiting may avoid the cost when it turns out that too few people hunt, it may also synchronize the expenditure with the reward, i.e., without affecting the time at which the stag is caught, a player always wants to be the last one who hunts.

We keep the three restrictions (a), (b) and (c) on equilibrium as before. If we can construct an equilibrium in the infinite horizon game that satisfies (a), (b) and (c), then by Claim 1 (which does not depend on \( \delta \)), we are done. Hence in the following discussion, up to but not including the existence proposition, we consider the infinite horizon game. As we will see next, the three restrictions can be satisfied in the two player game. Once there are more than two players, there may not exist any equilibrium that satisfies these restrictions.

Let \( c = g(n, x) \) denote the first period cutoff in an \( n \)-player game with lower bound on cost being \( x \). Let \( w(k, c) \) denote the expected gross reward to the cutoff player when there are \( k \) players left in the game (the cutoff player has already moved) whose types are above \( c, k = 1, \ldots, n - 1 \). Let \( v(k, c) \) denote the expected net reward to the cutoff player when there are \( k \) players left, including the cutoff player himself, whose type is \( c \), and the other \( k - 1 \) players costs are
above c. Note that \( g(n, x) \) could be empty-valued or multi-valued, but it must satisfy the following indifference equation.

\[
-c + p^{n-1} n^{-1} c(x) \cdot 1 + p^{n-2} n^{-1} c(x) \delta w(1, c) + \ldots + p^{0} n^{-1} c(x) \delta w(n-1, c) = \delta[p^{n-1} n^{-1} c(x)(1-c) + p^{n-2} n^{-1} c(x) v(2, c) + \ldots + p^{0} n^{-1} c(x) v(n, c)].
\]  

The value functions in (2) need to be specified to make sure that it is a well defined equation. Let us first solve the two player game.

The two player equation can be written as

\[
-c + p^{1} c(x) \cdot 1 + p^{0} c(x) \delta w(1, c) = \delta[p^{1} c(x)(1-c) + p^{0} c(x) v(2, c)],
\]  

where \( w(1, c) = p^{1}(1|c) \), and \( v(2, c) = \max\{0, -c + \delta w(1, c)\} \).

I can write out \( v(2, c) \) in this way because (i) in this situation, type \( c \) is the smallest type, so if equilibrium doesn’t allow type \( c \) to move, then by the cutoff property and symmetry of PBE, nobody else is allowed to move; (ii) if the opponent doesn’t move when he is more optimistic, then he doesn’t move when he is less optimistic, by condition (c).

Let \( c_{2}^{*} \) solve

\[
-c + \delta w(1, c) = 0.
\]

Since \( w(1, c) \) is non-increasing in \( c \), \( c_{2}^{*} \) is unique. Moreover, \( c_{2}^{*}(\delta) \) is increasing in \( \delta \), and \( c_{2}^{*}(\delta) \rightarrow g(2) \) as \( \delta \rightarrow 1 \).

**Lemma 2** \( \forall x \in (0, c_{2}^{*}], \exists g(2, x) \in [x, 1] \), such that \( g(2, x) \) solves (3).

**Proof:** Rewrite (3) as \( LHS_{2}(x, c) = RHS_{2}(x, c) \).
Since $-c + \delta w(1, c)$ is decreasing in $c$, we have $\text{LHS}_2(x, x) > \text{RHS}_2(x, x)$ and $\text{LHS}_2(x, 1) < \text{RHS}_2(x, 1)$, $\forall x \in [0, c^*_2]$. By the Intermediate Value Theorem, $\exists c \in [x, 1]$ that solves (2).

**Remark** If $x > c^*_2$, define $g(2, x) = 0^4$, i.e., if both players are above $c^*_2$, then neither hunts in the current period. It is equilibrium behavior since $-c + \delta w(1, x) < 0$ for $c \geq x$ and $x > c^*_2$.

Let $g (2, x)$ denote the smallest solution to (3) for $x \in [0, c^*_2]$. By Lemma 1 and continuity of both sides of (3), $g(2, x)$ is well defined.

**Lemma 3** $g(2, x)$ is strictly decreasing on $[0, c^*_2]$, and $g(2, c^*_2) = c^*_2$.

**Proof:** We first prove monotonicity. Let $p^k_m ([x, x'] | x)$ denote the probability that $k$ players out of $m$ players have types falling into the interval $[x, x']$, conditional on that their types being no less than $x$, where $0 \leq k \leq m \leq 1$. Let $I^k_m$ denote the event that $k$ players out of $m$ players fall into the interval $[x, x']$.

Decompose the conditional probabilities in (3) in the following way:

$$p(\bullet | x) = p^1 m ([x, x'] | x)p(\bullet | I^1 1) + p^0 m ([x, x'] | x)p(\bullet | I^0 1)$$

$$= p^1 m ([x, x'] | x)p(\bullet | I^1 1) + p^0 m ([x, x'] | x)p(\bullet | I^0 1).$$

Decompose $c$ into $c[p^1 m ([x, x'] | x) + p^0 m ([x, x'] | x)]$.

Plug the decompositions into $\text{LHS}_2(x, c)$ and $\text{RHS}_2(x, c)$, rearrange, we have

$$\text{LHS}_2(x, c) - \text{RHS}_2(x, c) = p^1 m ([x, x'] | x) (1 - c - \delta(1 - c))$$

$$+ p^0 m ([x, x'] | x) (\text{LHS}_2(x', c) - \text{RHS}_2(x', c)).$$

Fix $x \in [0, c^*_2)$, let $c = g(2, x)$, fix $x' \in (x, c)$, then by (4)

$$\text{LHS}_2(x, c) - \text{RHS}_2(x, c) = p^1 m ([x, x'] | x) (1 - c - \delta(1 - c)) + p^0 m ([x, x'] | x) (\text{LHS}_2(x', c) - \text{RHS}_2(x', c))$$

$$= 0.$$

\[4\] In fact, for $x > c^*_2$, if $\delta$ is large, then no solution exists for equation (3), hence $g(2, x) = 0$ is not only an equilibrium, but also the unique equilibrium. I leave the proof of this to the Appendix.
This implies that $LHS_2(x', c) - RHS_2(x', c) < 0$. Hence at $x'$ there is a solution below $c$, hence $g(2, x') < c$.

Finally $g(2, c^*_2) = c^*_2$ because $LHS_2(c^*_2, c^*_2) = RHS_2(c^*_2, c^*_2)$. It’s important to notice that even if $g(2, x)$ is not left continuous at $c^*_2$, there can not be an upper jump at $c^*_2$, therefore, $g(2, x) \geq c^*_2$, $\forall x \in [0, c^*_2]$.

**Lemma 4** \( \exists \delta < 1 \), such that $\forall \delta > \delta$, (3) has a unique solution for each $x \in [0, c^*_2(\delta)]$.

**Proof:** We rewrite (3) as

$$LHS_2(x, c, \delta) = RHS_2(x, c, \delta).$$

Let $\Delta_2(x, c, \delta) := LHS_2(x, c, \delta) - RHS_2(x, c, \delta)$. Notice that $\Delta_2(x, c, \delta)$ kinks at $c^*_2(\delta)$.

Recall from the no discounting case that $p(1, x)$ is the probability of a failure in a one player game, $x$ being the lower bound on the player’s type, and $p(1, x)$ reaches its minimum at $x = 0$, and $p(1, 0) < 1$. Let $\bar{c} := 1 - p(1, 0)$.

First we show that

$$LHS_2(x, c, \delta) < 0, \forall x \in [0, c^*_2], \forall c > \bar{c}, \forall \delta \in (0, 1].$$

This is because

$$LHS_2(x, c, 1) = -c + 1 - p(1, x) \leq -c + 1 - p(1, 0)$$

$$\implies \forall c > \bar{c}, LHS_2(x, c, 1) < 0.$$

But $LHS_2(x, c, \delta) \leq LHS_2(x, c, 1)$, hence $LHS_2(x, c, \delta) < 0$.

By Lemma 3, there is no solution in $[0, c^*_2(\delta)]$ to (3) at any $x \in [0, c^*_2(\delta)]$. Moreover, there is no solution in $(\bar{c}, 1]$ either, because we show above that on that range $LHS_2(x, c, \delta) < 0$, and $RHS_2(x, c, \delta)$ is always nonnegative. Hence to prove uniqueness, it suffices to show that $\Delta_2(x, c, \delta)$ is strictly decreasing over $c \in [c^*_2(\delta), \bar{c}]$, for sufficiently large $\delta$, that is independent of $x$.

Notice that over this range of $c$, (3) becomes
\[-c + p^1(c|x) \cdot 1 + p^0(c|x) \delta w(1,c) = \delta p^1(c|x)(1 - c).\]

Pick \(c\) and \(c'\) in this range such that \(c < c'\). It suffices to show that \(\Delta_2(x, c, \delta) - \Delta_2(x, c', \delta) > 0\).

We can write \(w(1,c) = p^1([c,c']|c) \cdot 1 + p^0([c,c']|c) w(1,c')\). Substituting the decomposition into \(\Delta_2(x, c, \delta)\), we find that

\[
\Delta_2(x, c, \delta) - \Delta_2(x, c', \delta) = (c' - c) \left(1 - \delta p^1(c'|x)\right) + p^1([c,c']|x)(2\delta - \delta c - 1).
\]

Hence \(\exists \delta < 1\), such that \(\forall \delta > \delta\), \(\Delta_2(x, c, \delta) - \Delta_2(x, c', \delta) > 0\), regardless of \(x\), \(c\), and \(c'\).

**Lemma 5** If \(\delta\) is large enough, \(g(2,x)\) is continuous in \(x\) on \([0, c^*_2]\).

**Proof:** By Lemma 4, for sufficiently large \(\delta\), \(g(2,x) = \bar{g}(2,x) = g(2,x)\), where \(\bar{g}(2,x)\) is the largest solution to (3). We show that \(\bar{g}(2,x)\) is u.s.c. and \(\underline{g}(2,x)\) is l.s.c.

It suffices to prove the following general result.

Let \(F(x,y)\) be continuous in \((x,y)\). Suppose \(\forall x \in [a,b], \exists y \in [0,1]\), such that \(F(x,y) = 0\). Let \(\overline{F}(x) := \max \{ y \in [0,1] | F(x,y) = 0 \}\), then \(\overline{F}\) is u.s.c. in \([a,b]\).

Proof by way of contradiction. Suppose \(\exists x_0 \in [a,b]\), such that \(\overline{F}\) is not u.s.c. at \(x_0\). Then \(\exists \epsilon_0 > 0\), such that \(\forall \delta > 0\), \(\exists x \in B(x_0, \delta)\), such that \(\overline{F}(x) \geq \overline{F}(x_0) + \epsilon_0\). Then we can construct a sequence \(\{x_n\}\), such that \(x_n \rightarrow x_0\), and \(\overline{F}(x_n) \geq \overline{F}(x_0) + \epsilon_0\), \(\forall n\). Choosing a subsequence if necessary, let \(\overline{F}(x_{n_k}) \rightarrow y_0\) by continuity of \(F\), \(F(x_0,y_0) = 0\), contradicting to \(\overline{F}(x_0)\) being the largest solution.

That \(\underline{g}(2,x)\) is l.s.c is proved analogously.

We summarize the above results in the following proposition.

**Proposition 6** In the 2 player game with discounting, if \(\delta\) is sufficiently large, then there exists a unique PBE that satisfies (a), (b) and (c). Moreover, the equilibrium can be characterized by a continuous and strictly decreasing function \(g(2,x)\), such that \(g(2,x)\) is the cutoff type in the 2 player game with lower bound \(x\).
Now consider the 3-player equation. When I write down the three player equation below, I take \( g(2, x) \) to be the continuation policy function in the two player continuation game.

Let

\[
w(1, c) = p^1 1(1|c),
\]

\[
w(2, c) = \begin{cases} 
 p^2 2(g(2, c)|c) \cdot 1 + p^1 2(g(2, c)|c)\delta w(1, g(2, c)) & \text{if } c \leq c_2^* \\
 0 & \text{if } c > c_2^* \end{cases},
\]

\[
\delta(2, c) = \begin{cases} 
 -c + p^1 1(g(2, c)|c) \cdot 1 + p^0 1(g(2, c)|c)\delta w(1, g(2, c)) & \text{if } c \leq c_2^* \\
 0 & \text{if } c > c_2^* \end{cases},
\]

and

\[
v(3, c) = \max \{0, -c + \delta w(2, c)\}.
\]

The three player indifference equation can thus be written as

\[
-c + p^2 2(c|x) \cdot 1 + p^1 2(c|x)\delta w(1, c) + p^0 2(c|x)\delta w(2, c) \\
= \delta \left[ p^2 2(c|x)(1 - c) + p^1 2(c|x)\delta(2, c) + p^0 2(c|x)v(3, c) \right].
\]

(5)

Next we show that the solution to (5) may not satisfy condition (c). Hence an equilibrium that satisfies (a), (b) and (c) in the three player game may not exist. Why is it possible that some player is willing to hunt in period 2 when she is more pessimistic about other players, but she is not willing to hunt in period 1 when she is more optimistic? There are three effects going on here: Delay effect, if you anticipate that others are going to move early, you want to move early, too; Leading effect, if you anticipate that others are going to move late, you want to move early to encourage them to follow you; Synchronization effect, if you anticipate that someone is going to move early, and someone else is
going to move late, then you want to move late to synchronize your action with the "bottleneck" player. It is the third effect that might frustrate condition (c). Before we construct a counter-example, we make the following preparations.

**Lemma 6** \( w(2, c) < w(1, c) \), \( \forall c \in [0, c_2^*] \).

**Proof**: Let \( i \) be the player in a one player game. Imagine there is another player \( j \), who is a dummy player in the one player game, but is a normal player in a two player game. To slightly abuse notation, also let \( i \) and \( j \) denote \( i \)'s type and \( j \)'s type. Then the set of \( i \)'s types for which \( i \) will succeed in the one player game with lower bound \( c \) can be written as the union of \( \{(i, j) \geq (c, c) | i \in [c, g(2, c)], j \text{ is anywhere}\} \) and \( \{(i, j) \geq (c, c) | i \in [g(2, c), 1], j \text{ is anywhere}\} \). The union in turn, contains the union of \( \{(i, j) \geq (c, c) | i \in [c, g(2, c)], j \in [c, 1]\}\) and \( \{(i, j) \geq (c, c) | i \in [g(2, c), 1], j \in [c, g(2, c)]\}\), which is equal to the set of \( i \) and \( j \)'s types for which \( i \) and \( j \) will succeed in the two player game with lower bound \( c \). Now there are two reasons \( w(1, c) \) must be larger than \( w(2, c) \). One is that two players succeed only if one player succeed, the other is two players can never succeed earlier than one player does. ■

**Lemma 7** \( w(2, c) \) is strictly decreasing in \( c \) over \([0, c_2^*]\).

**Proof**: \( \forall c' \text{ s.t. } c < c' < c_2^* \). From Lemma 2 we know that \( c' < \overline{w}(2, c) \). We can decompose \( w(2, c) \) as follows.

\[
 w(2, c) = p^2 \cdot (|c, c'| | c) \cdot 1 + p^1 \cdot (|c, c'| | c) w(1, c') + p^0 \cdot (|c, c'| | c) w(2, c'). 
\]

Hence to show \( w(2, c) > w(2, c') \), it suffices to show that \( w(1, c') \geq w(2, c') \), which follows from Lemma 6. ■

Let \( c_3^* \) solve

\[
 -c + \delta w(2, c) = 0. 
\]

Then \( c_3^* \) is unique, and \( c_3^* < c_2^* \), by Lemma 6 and Lemma 7. Moreover, \( c_3^* (\delta) \rightarrow g(3) \) as \( \delta \rightarrow 1 \).

Let \( g(3, x) \) denote a solution to (5). If we can find \( \epsilon > 0, \delta < 1 \) and a distribution function \( F \), such that \( g(3, x) \) is the unique solution to (5) and \( g(3, x) < c_3^* \), then we have a counter-example. To see why, for type \( c \in (g(3, x), c_3^*) \), this type does not hunt in period 1 when she is more optimistic about other players' types. In any monotone equilibria, she should not hunt in the second period upon seeing inaction in the first period, and nobody should for the same reason. Hence her continuation payoff in equilibrium after seeing inaction in period 1 is 0. But if she deviates, her expected payoff is \( -c + \delta w(2, c) > -c_3^* + \delta w(2, c_3^*) = 0 \).
Synchronization effect is likely to make a difference if the distribution function $F$ is not skewed, so that the probability that the other two players are located on two sides of the cutoff is relatively high. At the same time, for large $\delta$, $c^*_3$ is close to $g(3)$, and for small $\epsilon > 0$, $g(3)$ is close to 1. So for small $\epsilon$ and large $\delta$, $c^*_3$ is large, hence makes it easier for $g(3, x)$ to fall below it. It is therefore no surprise that a counter-example occurs at a combination of small $\epsilon$, large $\delta$ and the least skewed distribution, uniform distribution. Numerical computation shows that at the combination where $\epsilon = 0.0001$, $\delta = 0.999$, $x = 0$, and $F =$ uniform distribution, $g(3, 0) \approx 0.52$, but $c^*_3 \approx 0.96$.

This example shows that in general, it is difficult to explicitly construct an equilibrium in the discounting case. Nevertheless, existence of equilibrium is still established by the following proposition.

**Proposition 7** $\forall \epsilon \geq 0, \forall \delta < 1, \forall 1 \leq n < \infty, \forall \vec{x} \in [0, 1 + \epsilon]^n, \forall 1 \leq T < \infty$, there exists a PBE in $\Gamma (n, \epsilon, \delta, \vec{x}, T)$.

**Proof:** For simplicity, we only prove the case where $\vec{x} = \vec{0}$. For other $\vec{x}$’s, the proof is essentially the same, except that more notations are needed. The basic idea of the proof is to approximate the original game by a sequence of games with finite type spaces. Existence of PBE in a game with finite type space is guaranteed, we then show that as the type space becomes arbitrarily finer, the limiting strategy profile exists, and it constitutes a PBE of the continuous type game. We establish this in steps.

Step 1 We first discretize the type space $\Theta := [0, 1 + \epsilon]$ in the following way. Let $\Theta_k := \{\theta_i = \frac{i(1 + \epsilon)}{2^k}, i = 0, \ldots, 2^k\}$. Let $P_k(\theta_i) := F(\theta_i) - F(\theta_{i-1})$ for $i \geq 1$ and $P_k(\theta_0) := 0$ be the probability distribution over $\Theta_k$. Let $F_k$ denote the c.d.f. induced by $P_k$. Since $F$ is continuous over the closed interval $[0, 1 + \epsilon]$, $F$ is uniformly continuous, which implies that $F_k$ converges to $F$ uniformly. Now let $\Gamma_k$ denote the game that is the same as the original game except we replace $\Theta$ and $F$ by $\Theta_k$ and $F_k$.

Step 2 By Theorem 4.6 in Myerson (1991), there exists a sequential equilibrium in $\Gamma_k$, hence there exists a PBE in $\Gamma_k$. We choose an arbitrary equilibrium of $\Gamma_k$, denote it by $E_k$.

Step 3 Suppose that there are $H$ non-terminal histories in the original game. Here by a history we mean public history that is observed by everybody in the game. Since $n < \infty$, $T < \infty$, it must be that $H < \infty$. Then $E_k$ is simply a collection of $n \times H$ functions, each mapping $\Theta_k$ to a probability distribution over $\{0, 1\}$, where 0 stands for no hunting, and 1 for hunting. $\forall k, \forall$ player $j$, $\forall$ history $h$, there is at most one type $\theta_i \in \Theta_k$ who is indifferent between 0 and
1, hence there is at most one type who mixes. To see this, notice that if the length of $h$ is $T - 1$, then if type $\theta$ is indifferent between 0 and 1, it must be that $\forall \theta' > \theta$, $\theta'$ strictly prefers 0, and $\forall \theta' < \theta$, $\theta'$ strictly prefers 1. If the length of $h$ is less than $T - 1$, the above claim also holds because $\delta < 1$(Refer to the proof of Proposition 2(i)). Therefore, $E_k$ is a collection of $n \times H$ nonincreasing functions.

Notice that $E_k$ is undefined over $\theta \notin \Theta_k$. Before we go to the next step, define $E_{kj}(h)(\theta) := E_{kj}(h)(\theta_{j+1})$, $\forall \theta \notin \Theta_k$, $\forall j$, $\forall h$, where $\theta_{j+1}$ is the closest point in $\Theta_k$ to the right of $\theta$, and $E_{kj}(h)(\cdot)$ is player $j$’s action in $E_k$ at $h$.

Step 4 By Helly’s selection theorem (Kolmogorov and Fomin 1970), there exists a monotone strategy profile $E$, such that $E_k \rightarrow E$ pointwise, meaning $E_{kj}(h)(\cdot) \rightarrow E_j(h)(\cdot)$ pointwise, $\forall j$, $\forall h$.

Step 5 Fix $j$ and $h$. The limiting function $E_j(h)$ has at most one point at which the value of the function is neither 0 nor 1. Consider Figure 6.

Suppose otherwise that there are two such points, $\theta_1$ and $\theta_2$. Let $p_1 = E_j(h)(\theta_1)$, $p_2 = E_j(h)(\theta_2)$, then by the monotonicity of $E_j(h)(\cdot)$ and the contradiction hypothesis, $0 < p_2 \leq p_1 < 1$. Since the grid can be made arbitrarily fine,
\[ \exists \theta_{-1}, \theta, \theta_{+1} \in \Theta_k \text{ for some } k \text{ such that } \theta_1 < \theta_{-1} < \theta < \theta_{+1} < \theta_2. \] By the monotonicity of \( E_j (h) (\cdot) \), \( E_j (h) (\theta) \in [p_2, p_1] \). By the convergence result, \( \exists K > 0 \), such that \( \forall k > K, E_{kj} (h) (\theta) \) is sufficiently close to \( E_j (h) (\theta) \). By Step 3, \( \forall k > K, E_{kj} (h) (\theta_{-1}) = 1 \) and \( E_{kj} (h) (\theta_{+1}) = 0 \). But \( E_j (h) (\theta_{-1}) \in [p_2, p_1] \) and \( E_j (h) (\theta_{+1}) \in [p_2, p_1] \), a contradiction.

**Step 6** Fix \( j \) and \( h \). Let \( \theta \) denote the cutoff point of the limiting function \( E_j (h) (\cdot) \). \( \forall k \), let \( \theta_{+1} \) denote the right closest grid point in \( \Theta_k \) to \( \theta \), similarly for \( \theta_{-1}, \theta_{+2}, \) and \( \theta_{-2} \). We claim that \( \exists K > 0 \), such that \( \forall k' > K, E_{k',j} (h) (\theta) = E_j (h) (\theta), \forall \theta \geq \theta_{+2}, \) and \( \forall \theta \leq \theta_{-2} \). Since \( k \) is chosen arbitrarily, this claim implies that the sequence of functions \( (E_{k',j} (h) (\cdot))_{k'} \) coincides with \( E_j (h) (\cdot) \) over an arbitrarily large set (relative to the type space). To prove the claim, consider Figure 7 (if \( \theta \) is an end point, the proof is analogous).

![Figure 7: Step 6 of Proposition 7](image_url)

By Step 5, \( E_j (h) (\theta_{-1}) = 1 \), \( E_j (h) (\theta_{+1}) = 0 \). By an argument similar to Step 5, \( \exists K > 0 \), such that \( \forall k > K, E_{kj} (h) (\theta_{+2}) = 0 \) and \( E_{kj} (h) (\theta_{-2}) = 1 \). The rest of the step is finished by the monotonicity of \( E_{kj} (h) (\cdot) \) and \( E_j (h) (\cdot) \).
Step 7  \( \forall h, \forall j, \forall \theta_j \in \Theta \) such that \( \theta_j \in \Theta_k \) for some \( k \), type \( \theta_j \) of player \( j \) does not want to deviate. To see this, let \( P (|h, \theta_j, \text{no dev}) \) denote the lottery over the terminal histories of the game induced by \( E \), conditional on history \( h \), player \( j \)’s type being \( \theta_j \), and player \( j \) following \( E_j \) throughout the continuation game. Let \( P (|h, \theta_j, \text{dev}) \) denote the lottery over the terminal histories of the game induced by \( E \), conditional on history \( h \), player \( j \)’s type being \( \theta_j \), and player \( j \) deviating right after \( h \) but following \( E_j \) for the rest of the continuation game. Let \( P^k (|h, \theta_j, \text{no dev}) \) and \( P^k (|h, \theta_j, \text{dev}) \) be defined similarly for \( \Gamma_{k'} \), induced by \( E_{k'} \). By Step 6 and the fact that \( F_{k'} \) converges to \( F \) uniformly, we have \( P^k (|h, \theta_j, \text{no dev}) \rightarrow P (|h, \theta_j, \text{no dev}) \), and \( P^k (|h, \theta_j, \text{dev}) \rightarrow P (|h, \theta_j, \text{dev}) \).

Since a player’s payoff is continuous in the lotteries over the terminal histories, if \( P (|h, \theta_j, \text{dev}) \succ \theta_j P (|h, \theta_j, \text{no dev}) \), then \( P^{k'} (|h, \theta_j, \text{no dev}) \succ \theta_j P^{k'} (|h, \theta_j, \text{no dev}) \) for sufficiently large \( k' \), contradiction.

Step 8  \( \forall h, \forall j, \forall \theta_j \in \Theta \) such that \( \theta_j \notin \Theta_k \) for all \( k \), if type \( \theta_j \) wants to deviate, then \( \exists \theta_{j}' \in \Theta_k \) for some \( k \), such that \( \theta_{j}' \) also wants to deviate.

There are two possibilities. 1. \( \theta_j \) is never a cutoff point in \( E_j \) at any history. In this case, \( \forall r > 0, \exists k, \exists \theta_{j}' \in \Theta_k \), such that (i) \( |\theta_{j}' - \theta_j| < r \) and (ii) \( E_j (h') (\theta_j) = E_j (h') (\theta_{j}') \), \( \forall h' \). Therefore, if we choose \( r \) sufficiently small, then a profit-deviation for type \( \theta_j \) implies a profit deviation for \( \theta_{j}' \), which is impossible by Step 7. 2. At some history \( h' \), \( \theta_j \) is a cutoff point in \( E_j \). First of all, it cannot be that \( \theta_j \) strictly prefers to hunt at \( h' \), since otherwise we can find a grid point \( \theta_{j}' \) close enough to the right of \( \theta_j \) who also strictly prefers to hunt at \( h' \), but \( E_j (h') (\theta_{j}') = 0 \), which implies that \( \theta_{j}' \) has a profit-deviation at \( h' \), impossible by Step 7. Hence in this case if necessary we can always redefine the value of \( E_j (h') (\cdot) \) at \( \theta_j \) to be equal to 0 without affecting any type of any player’s payoff. But then we go back to the first possibility, that is \( \forall r > 0, \exists k, \exists \theta_{j}' \in \Theta_k \), such that (i) \( |\theta_{j}' - \theta_j| < r \) and (ii) \( E_j (h') (\theta_j) = E_j (h') (\theta_{j}') \), \( \forall h' \).

If two types are arbitrarily close and they behave the same way at any history, then if one type has a profitable deviation at some history, so does the other.

2.6 Common Shocks

In this section we study the investment game in Morris and Shin (2000), and show that the asymptotic results in Section 2.2 applies to common shocks models to some extent. The model is as follows.

<table>
<thead>
<tr>
<th>( \text{Invest} )</th>
<th>( \text{Refrain} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta, \theta )</td>
<td>( \theta - z, 0 )</td>
</tr>
<tr>
<td>0, ( \theta - z )</td>
<td>0, 0</td>
</tr>
</tbody>
</table>
Two players must decide whether to invest or refrain from investing. If both invest, the payoff to each is $\theta$, which follows standard normal distribution $N(0, 1)$. If only one player invests, the investor receives $\theta - z$, where $z$ is a positive constant. Player $i$ observes $\theta$ with some noise $\epsilon_i$ that follows $N(0, 1/\beta)$. That is, player $i$'s signal $x_i = \theta + \epsilon_i$. Assume that $\epsilon_1$ and $\epsilon_2$ are independent, and they are independent from $\theta$. Morris and Shin (2000) show that if $\beta$ is large enough, namely if the players’ signals are precise enough, then there is a unique BNE of the game, which is characterized by a switching point $\bar{\epsilon}(\beta)$, such that player $i$ invests if and only if $x_i \geq \bar{\epsilon}(\beta)$. Interestingly enough, $\bar{\epsilon}(\beta) \to z/2 > 0$ as $\beta$ goes to infinity, hence positive amount of inefficiency remains as precision of observation goes to infinity.

What we are going to show next is that if we can think of the game as being played sequentially, with "invest" being an irreversible action, and "refrain" being a reversible action, then efficiency can be asymptotically restored. For the ease of exposition, assume that there are only two periods, the result still holds for more than two periods. We focus on cutoff equilibria$^5$ in which after any history, if a player is willing to invest at some signal, then she is also willing to invest at any higher signals.

**Proposition 8** $\forall (\beta_n)_n \to \infty$, for any sequence of cutoff equilibria $(E_n)_n$, $P(\text{both invest in } E_n | \theta > 0) \to 1$.

**Proof:** First we construct a symmetric equilibrium for fixed $\beta < \infty$, which can be characterized by a pair of numbers $(x^*(\beta), \bar{\epsilon}(\beta))$, such that a player invests in the first period if and only if $x \geq x^*(\beta)$; in the second period, if the opponent invests in the first period, follow him if and only if $x \geq \bar{\epsilon}(\beta)$, if the opponent refrains in the first period, refrain in the second period.

Given $x^*(\beta), \bar{\epsilon}(\beta)$ should satisfy

$$E(\theta|x_2 = \bar{\epsilon}(\beta), x_1 \geq x^*(\beta)) = 0.$$ (6)

On the other hand, $x^*(\beta)$ should make a player (say player 1) indifferent between investing and refraining in the first period. Notice that the payoff of investing in the first period is given by

$$P(x_2 \geq x^*(\beta) | x_1 = x^*(\beta)) \cdot E(\theta|x_1 = x^*(\beta), x_2 \geq x^*(\beta))$$

$$+ P(x_2 < x^*(\beta) | x_1 = x^*(\beta)) \cdot$$

$$(E(\theta|x_1 = x^*(\beta), x_2 < x^*(\beta)) - z \cdot P(x_2 < \bar{\epsilon}(\beta) | x_1 = x^*(\beta), x_2 < x^*(\beta))),$$

and the payoff of refraining in the first period is

$$P(x_2 \geq x^*(\beta) | x_1 = x^*(\beta)) \cdot E(\theta|x_1 = x^*(\beta), x_2 \geq x^*(\beta))$$

$$+ P(x_2 < x^*(\beta) | x_1 = x^*(\beta)) \cdot 0.$$  

$^5$Refer to footnote 1.
Hence \( x^* (\beta) \) must satisfy

\[
E[\theta|x_1 = x^* (\beta), x_2 < x^* (\beta)] - z \cdot P[x_2 < \bar{x} (\beta) | x_1 = x^* (\beta), x_2 < x^* (\beta)] = 0. 
\]

(7)

Next we show that (a) In the first period, \( \forall x > x^* (\beta) \), type \( x \) will invest, and \( \forall x < x^* (\beta) \), type \( x \) will refrain; (b) In the second period, upon seeing the opponent investing in the first period, type \( x \) will follow up if and only if \( x > \bar{x} (\beta) \); (c) In the second period, upon seeing the opponent not investing in the first period, it is optimal not to invest in the second period.

First of all, (b) and (c) immediately follow from equations (1) and (2), respectively. Next we prove that \( \forall x > x^* (\beta) \),

\[
E(\theta|x_1 = x, x_2 < x^* (\beta)) - z \cdot P(x_2 < \bar{x} (\beta) | x_1 = x, x_2 < x^* (\beta)) > 0.
\]

First we show that \( P(x_2 < \bar{x} (\beta) | x_1 = x, x_2 < x^* (\beta)) \) is decreasing in \( x \). Notice that \( (x_2|x_1 = x) \sim N \left( \frac{\beta_x}{1+\beta}, \frac{1}{1+\beta} + \frac{1}{2} \right) \), hence

\[
P(x_2 < \bar{x} (\beta) | x_1 = x, x_2 < x^* (\beta)) = \frac{P(x_2 < \bar{x} (\beta) | x_1 = x)}{P(x_2 < x^* (\beta) | x_1 = x)} = \frac{\Phi (a (\bar{x} - bx))}{\Phi (a (x^* - bx))},
\]

where \( a = \sqrt{\frac{3(1+\beta)}{1+2\beta}}, b = \frac{\beta}{1+\beta} \)

Differentiating w.r.t. \( x \), the numerator is

\[
ab (\Phi (a (\bar{x} - bx)) \Phi' (a (x^* - bx)) - \Phi (a (x^* - bx)) \Phi' (a (\bar{x} - bx))).
\]

Since \( \bar{x} < x^* \), it suffices to show that \( \frac{\Phi (z)}{\Phi' (z)} \) is increasing in \( z \) over \( \mathbb{R} \). Since \( \text{sign} \left( \frac{\partial}{\partial z} \left( \frac{\Phi (z)}{\Phi' (z)} \right) \right) = \text{sign} \left( \Phi' (z) + z \Phi (z) \right) \), and \( \frac{\partial}{\partial z} (\Phi' (z) + z \Phi (z)) > 0 \), and \( \lim_{z \to -\infty} (\Phi' (z) + z \Phi (z)) = 0 \), we have \( \Phi' (z) + z \Phi (z) > 0 \), as was to be shown.

Now let \( x < y \), let \( F \) denote the distribution of \( (x_2|x_1 = x) \) conditional on \( x_2 < x^* (\beta) \), let \( G \) denote the distribution of \( (x_2|x_1 = y) \) conditional on \( x_2 < x^* (\beta) \). Then by the above argument, \( \forall z < x^* (\beta) \),
\( P(x_2 < z|x_1 = x, x_2 < x^*(\beta)) > P(x_2 < z|x_1 = y, x_2 < x^*(\beta)). \)

Hence \( G \) first order stochastic dominates \( F \). Therefore,

\[
\int_{-\infty}^{x^*} x_2 dG \geq \int_{-\infty}^{x^*} x_2 dF.
\]

This implies that \( E(x_2|x_1 = x, x_2 < x^*(\beta)) \) is increasing in \( x \), which in turn, implies that \( E(\theta|x_1 = x, x_2 < x^*(\beta)) \) is increasing in \( x \).

Hence \( \forall x > x^*(\beta) \)

\[
E(\theta|x_1 = x, x_2 < x^*(\beta)) - z \cdot P(x_2 < \tilde{x}(\beta)|x_1 = x, x_2 < x^*(\beta)) > E(\theta|x_1 = x^*(\beta), x_2 < x^*(\beta)) - z \cdot P(x_2 < \tilde{x}(\beta)|x_1 = x^*(\beta), x_2 < x^*(\beta))
\]

\[
= 0.
\]

For fixed \( \beta < \infty \), it is easy to see that there is a unique pair \((x^*(\beta), \tilde{x}(\beta))\) that solves \( 6 \) and \( 7 \). Now fix any cutoff equilibrium of the two period game. Since it has the cutoff property, it can be characterized by six cutoff numbers, \( \tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{21}, \tilde{x}_{22}, \tilde{x}_{11}, \tilde{x}_{22} \), where \( \tilde{x}_{jt} \) is the cutoff type of player \( j \) in period \( t \) when nobody has invested yet, and \( \tilde{x}_{jt} \) is the cutoff type of player \( j \) in period \( t \) when the other player has already invested. We show that in this equilibrium,

\[
P(\text{both invest}|x_1 \geq x^*(\beta), x_2 \geq x^*(\beta)) = 1.
\]

Suppose not. Then it must be that \( \tilde{x}_{11} > x^*(\beta), \tilde{x}_{21} > x^*(\beta) \), and at least one of \( \tilde{x}_{12} \) and \( \tilde{x}_{22} \) is also greater than \( x^*(\beta) \), say it is \( \tilde{x}_{12} \). Consider type \( x_1 \) of player 1 such that

\[
x^*(\beta) < x_1 < \min\{\tilde{x}_{11}, \tilde{x}_{12}\}.
\]

If type \( x_1 \) follows her equilibrium strategy, her expected payoff is

\[
P(x_2 \geq \tilde{x}_{21}|x_1) \cdot E(\theta|x_1, x_2 \geq \tilde{x}_{21}).
\]

If she deviates by investing in the first period, her expected payoff is

\[
P(x_2 \geq \tilde{x}_{21}|x_1) \cdot E(\theta|x_1, x_2 \geq \tilde{x}_{21}) + P(x_2 < \tilde{x}_{21}|x_1) \cdot E(\theta|x_1, x_2 < \tilde{x}_{21}) - z \cdot P(x_2 < \tilde{x}_{22}|x_1, x_2 < \tilde{x}_{21})).
\]
Since \( x_1 > x^* (\beta) \), \( x_{21} > x^* (\beta) \), and \( x_{22} < \bar{x} (\beta) \), it must be that

\[
\begin{align*}
\left( E(\theta|x_1, x_2 < \bar{x}_{21}) - z \cdot P(x_2 < \bar{x}_{22}|x_1, x_2 < \bar{x}_{21}) \right) \\
> E(\theta|x_1 = x^* (\beta), x_2 < x^* (\beta)) - z \cdot P(x_2 < \bar{x} (\beta)|x_1 = x^* (\beta), x_2 < x^* (\beta)) \\
= 0.
\end{align*}
\]

Hence type \( x_1 \) has a profitable deviation, a contradiction.

Now fix \( \beta > 0 \), fix a cutoff equilibrium \( E \). Since

\[ P(\text{both invest in } E | x_1 \geq x^* (\beta), x_2 \geq x^* (\beta)) = 1, \]

\[ P(\text{both invest in } E | \theta > 0) \geq P(x_1 \geq x^* (\beta), x_2 \geq x^* (\beta) | \theta > 0). \]

Therefore, it suffices to show that \( \lim_{\beta \to \infty} x^* (\beta) = 0 \). Suppose otherwise that \( \exists (\beta_k)_k \to \infty \), such that \( \lim_{k} x^* (\beta) = b > 0 \), then by (1) it must be that \( \lim_{k} \bar{x} (\beta_k) = -b < 0 \). But this implies that \( P(x_2 < \bar{x} (\beta)|x_1 = x^* (\beta), x_2 < x^* (\beta)) \to 0 \), which by (7) implies that \( x^* (\beta) \) converges to 0, a contradiction.

What if both "invest" and "refrain" are irreversible actions? Then the two period game with endogenous timing is the same as the simultaneous move game, hence Proposition 8 no longer holds, and we come back to the prediction of Morris and Shin (2000). The reversibility of the actions in a coordination game depends on the context of the game. In a bank run model, for example, if we replace "invest" by "withdraw late", and "refrain" by "withdraw early", then both actions are irreversible. In a currency attack model, however, if we replace "invest" by "attack", and "refrain" by "not attack", then "attack" is an irreversible action, and "not attack" is a reversible action. Bank runs and currency attacks are usually considered as very similar models in the literature, we argue that once we add endogenous timing so that reversibility becomes an issue, they fall into very different categories.

2.7 Irreversibility

In this section we follow up the discussion at the end of last section, by studying a dynamic version of the arms race game in Baliga and Sjöström (2002).

Two countries must decide whether and when to build new weapons. Building weapons is a one time and irreversible decision. Not building weapons is a reversible action. The cost to build weapons is a one time expense, and it is players’ private information. Without loss of generality, assume there are only two periods, and there is no discounting. Each country’s payoff is determined from the simultaneous move game according to the final decisions of the two
countries. The payoffs of the one period simultaneous move game are given in the following matrix.

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>(-c_1, -c_2)</td>
<td>(\mu - c_1, -d)</td>
</tr>
<tr>
<td>N</td>
<td>(-d, \mu - c_2)</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

where \(c_i\) is player \(i\)’s cost to build weapons. The \(c_i\)’s follow i.i.d. \(F\) over \([0, \pi]\). \(\mu > 0\) is the advantage of a better armed country over a less armed country; \(d > \pi\) is the disadvantage of a less armed country over a better armed country. Baliga and Sjöström show that if \(F(c) \cdot d \geq c, \forall c \in [0, \pi]\), then the only BNE is \((B, B)\) for all types. The question is, under the same conditions of the distribution function (the multiplier conditions in Baliga and Sjöström (2002)), is there an equilibrium where \((N, N)\) occurs with positive probability, if the two countries play the game sequentially with endogenous timing?

**Proposition 9** In the two-period arms race game with endogenous timing, if the multiplier condition holds, then in any PBE, there is probability 0 that \((N, N)\) is the final outcome.

**Proof:** Suppose by way of contradiction that there exists a PBE in which \((N, N)\) is the final outcome with positive probability. Let \(P_1\) denote the probability that country 1 will build in period 1. Let \(\pi_1\) denote the probability that country 1 will build in period 2, conditional on that \((N, N)\) is the outcome in period 1. By the contradiction hypothesis, \(P_1 < 1, \pi_1 < 1\). Now we show that for any country \(i\), for any type \(c_i\) of country \(i\), \(c_i\) does not build in period 1 in this equilibrium. Suppose otherwise that, say, type \(c_2\) of country 2 prefers to build in period 1. Then it must be that

\[-c_2 \geq P_1 (-c_2) + (1 - P_1) \max \{-c_2 + (1 - \pi_1) \mu, -\pi_1 d\}.\]

Hence \(-c_2 \geq \max \{-c_2 + (1 - \pi_1) \mu, -\pi_1 d\}\), which is impossible since \(\pi_1 < 1\) and \(\mu > 0\).

Now if \((N, N)\) is the outcome for sure in period 1, it does not reveal any information. We essentially go back to the one shot game in which, under the multiplier condition, \((B, B)\) is the unique outcome, which is a contradiction. ■

The arms race game is very similar to the stag hunt game in the chapter. It is also a coordination game that may have multiple equilibria if information is complete. When introducing some amount of dominant strategy types, it also generates a unique BNE in which the inefficient action is chosen by all, and so on. So what makes the two models behave so differently in a dynamic setup, when they behave so similarly in a static one? In the arms race game, the safe action \((B)\) is irreversible, while the risky action \((N)\) is reversible. In our
game, it is just the opposite. In both games, the risky action is the cooperative action. But if it is reversible, and if the safe (selfish) action is not, then leaders have no incentive to take the risk early. Only when the leaders anticipate that they will be followed by those who choose safe actions today are they willing to lead. In the arms race game, a leader will not be followed by countries which already build arms today, hence nobody wants to lead, and dynamics doesn’t make much difference.

2.8 Conclusion

We build a simple model that combines strategic complementarity and incomplete information in a sequential setup with endogenous timing. We use the model to argue that it makes a difference which way we model a coordination situation. If it is more appropriate to model the situation as a dynamic game, then the predictions in the static model might be over pessimistic, depending on the nature of the actions of the game. The equilibrium analysis reveals a recursive structure behind a herding mechanism. Initially there is a large scale coordination problem. Then players with extreme types sort out themselves, which reduces the coordination problem to a smaller scale, and the process repeats. Herding may be completed (everybody participates), it may also die out prematurely, depending on the realization of players’ types. People follow other people not because they are afraid of being left alone (Choi (1997)), or because they suppress their own information and free ride on predecessors’ information (Bikhchandani, Hirshleifer and Welch (1992)), or because doing so gives them a reputational advantage (Scharfstein and Stein (1990)), but simply because they rationally expect that they themselves will be followed. If the types are distributed "correctly", which will be judged according to the cutoff points in equilibrium in a recursive way, then we may expect to observe a domino effect, which unfolds itself quickly.

Several extensions remain to be investigated. First, we assume that it takes everybody to catch the stag. I don’t expect that the results will change qualitatively if it takes only a fraction of people to succeed, but it would still be interesting to check this out, especially when the rest of the players can free ride on the hunters’ catch. Second, we assume that once everybody hunts, the stag is caught for sure, what happens if there is some uncertainty about this and different players have different perceptions about such uncertainty? That is, what if we combine private shocks on players’ costs with common shocks on the fundamentals of the environment?

2.9 Appendix

Proof of Proposition 5: The idea of the proof is the following. Conditions (a), (b) and (c) allow us to write out the indifference equations that the cutoffs must satisfy. Once we have an explicit expression of the indifference equations,
we can prove the existence and uniqueness of the solutions. Then it is trivial to check that conditions (a), (b) and (c) are indeed satisfied.

The notations in this proof can be found in Section 2.5 right after Proposition 5.

We first consider the case where \( n = 2 \), and the lower bound on the players’ cost is \( x \in [0, 1] \). Let \( g(2, x) \) denote the first period cutoff type such that \( \forall c_i \leq g(2, x) \), player \( i \) hunts in period 1; \( \forall c_i > g(2, x) \), player \( i \) waits in period 1. We show that \( g(2, x) \) is unique, and doesn’t depend on \( x \).

We organize the argument into small steps.

Step 1  No discounting implies that in any equilibrium that satisfies (a), (b) and (c), the payoff of hunting today is equal to the payoff of waiting today but hunting tomorrow no matter what happens today.

Step 2  We write down the indifference condition that \( g(2, x) \) must satisfy:

\[
-g(2, x) + (1 - p(F, 2, x)) = p^1(1 - g(2, x)) (1 - g(2, x)) + p^0(1 - g(2, x)) v(1, g(2, x)).
\]

Step 3  We can decompose \( p(F, 2, x) \) as follows

\[
p(F, 2, x) = p^1(1 - g(2, x)) \cdot 0 + p^0(1 - g(2, x)) p(1, g(2, x)) = p(1, x).
\]

Step 4  Step 1, (a), (b) and (c) imply that

(i) On the equilibrium path, once there is inaction in some period, there will be inaction forever.

(ii) If type \( g(2, x) \) doesn’t hunt in period 1, then it hunts in period 2 no matter what.

(iii) \( v(2, g(2, x)) = -g(2, x) + 1 - p(1, g(2, x)) = 0 \).

It is easy to see that (i) directly follows from condition (c).

To see (ii), suppose otherwise that \( g(2, x) \) doesn’t hunt if he observes inaction in period 1, then the payoff to waiting in the first period is strictly higher than the
payoff to waiting in period 1 and hunting in period 2 no matter what. By step 1, the latter is equal to the payoff to hunting in period 1. Hence type \( g(2, x) \) is not indifferent between hunting and waiting in period 1, contradiction.

To see \((iii)\), note that \((ii)\) implies

\[
v(2, g(2, x)) = -g(2, x) + 1 - p(F, 2, g(2, x)) = -g(2, x) + 1 - p(1, g(2, x)),
\]

where the second equality comes from step 3.

The continuation value function \( v(2, g(2, x)) \) can not be less than 0, since otherwise \((ii)\) is violated; it can’t be greater than 0, since otherwise some type slightly above \( g(2, x) \) should also hunt in period 2, even if nobody hunts in the first period. But this violates \((i)\). Therefore \( v(2, g(2, x)) = 0 \).

Step 5 \( -g(2, x) + 1 - p(1, g(2, x)) = 0 \) has a unique solution in \((0, 1)\), which doesn’t depend on \( x \).

It suffices to show that \( -x + 1 - p(1, x) = 0 \) has a unique solution in \((0, 1)\). To that end, it suffices to show that \( p(1, x) \) is increasing in \( x \) over the interval \([0, 1]\). \( \forall x' s.t. x < x' < 1 \), We can decompose \( p(1, x) \) in the following way

\[
p(1, x) = p^1 1([x, x'] | x) \cdot 0 + p^0 1([x, x'] | x) \cdot p(1, x') < p(1, x'),
\]

as was to be shown.

Now denote the solution to the equation \( -x + 1 - p(1, x) = 0 \) by \( g(2) \). Note that if the lower bound \( x \) exceeds \( g(2) \), then no hunting will ever occur in equilibrium because \( \forall y \geq x \).

\[
\text{Payoff of hunting}
\begin{align*}
&= -y + 1 - p(1, x) \\
&\leq -x + 1 - p(1, x) \\
&< -g(2) + 1 - p(1, g(2)) \\
&= 0.
\end{align*}
\]

Now we consider the case where \( n = 3 \), and the lower bound on the players’ cost is \( x \in [0, 1] \). Let \( g(3, x) \) denote the first period cutoff type such that \( \forall c_i \leq g(3, x) \), player \( i \) hunts in period 1; \( \forall c_i > g(3, x) \), player \( i \) waits in period 1. We show that \( g(3, x) \) is unique, and doesn’t depend on \( x \).
We still have step 1 as in the two player case.

Step 1  No discounting implies that in any equilibrium that satisfies (a), (b) and (c), payoff of hunting today is equal to payoff of waiting today but hunting tomorrow no matter what.

Step 2  The indifference condition that \( g(3, x) \) must satisfy is

\[
-g(3, x) + (1 - p(F, 3, x)) = p^2 \cdot (g(3, x) | x) (1 - g(3, x)) + p^1 \cdot (g(3, x) | x) v(2, g(3, x)) + p^0 \cdot (g(3, x) | x) v(3, g(3, x)).
\]

Step 3  We can decompose \( p(F, 3, x) \) as follows.

\[
p(F, 3, x) = p^2 \cdot (g(3, x) | x) \cdot 0 + p^1 \cdot (g(3, x) | x) p(1, g(3, x)) + p^0 \cdot (g(3, x) | x) p(2, g(3, x)).
\]

Step 4  Step 1, (a), (b) and (c) imply that

(i) \( g(3, x) \) always hunts in period 2 no matter what happens in period 1;

(ii) \( g(3, x) \leq g(2), \forall x; \)

(iii) \( p(F, 3, x) = p(2, x). \)

To see (i), suppose otherwise that \( g(3, x) \) doesn’t hunt after some observation in period 1, then the payoff to waiting in the first period is strictly higher than the payoff of waiting in period 1 and hunting in period 2 no matter what. By step 1, the latter is equal to the payoff of hunting in period 1, hence type \( g(3, x) \) is not indifferent between hunting and waiting in period 1, a contradiction.

(ii) follows from (i) since if \( \exists x, s.t. \ g(3, x) > g(2) \), then by the two player argument type \( g(3, x) \) doesn’t hunt in period 2 if only one player hunts in period 1, contradicting (i).

(iii) follows from (ii) and step 3.

Step 5  (i) and (iii) implies that
\[ (iv) \; v(3, g(3, x)) = -g(3, x) + 1 - p(F, 3, g(3, x)) = -g(3, x) + 1 - p(2, g(3, x)). \]

Step 6 Condition (c) implies that

(v) On equilibrium path, if nobody hunts in period 1, then nobody hunts forever.

Step 7

\[
\begin{align*}
(i) \\
(iv) \\
(v)
\end{align*} \implies (vi) \; v(3, g(3, x)) = -g(3, x) + 1 - p(2, g(3, x)) = 0.
\]

It can’t be less than 0 since otherwise (i) is violated; it can’t be more than 0 since otherwise (v) is violated.

Step 8 \(-g(3, x) + 1 - p(2, g(3, x)) = 0\) has a unique solution in \((0, 1)\), that doesn’t depend on \(x\), and is smaller than \(g(2)\).

It suffices to show that \(-x + 1 - p(2, x) = 0\) has a unique solution in \((0, 1)\). To that end, it suffices to show that \(p(2, x)\) is increasing in \(x\) over the interval \([0, g(2)]\) \(p(2, x) = 1\) if \(x > g(2)\). \(\forall \; x' \; s.t. \; x < x' < g(2)\), we can decompose \(p(2, x)\) as

\[ p(2, x) = p(2/2 \in [x, x'] | x) \cdot 0 + p(1/2 \in [x, x'] | x) p(1, x') + p(0/2 \in [x, x'] | x) p(2, x'). \]

To show \(p(2, x) < p(2, x')\), it suffices to show that \(p(2, x) \geq p(1, x)\), \(\forall x\).

Let \(i\) be the player in a one player game. Imagine there is another player \(j\), who is a dummy player in the one player game, but is a normal player in a two player game. To slightly abuse notation, also let \(i\) and \(j\) denote \(i\)'s type and \(j\)'s type.

Now

\[ p(1, x) = p \{ (i, j) \geq (x, x) | i \in (1, 1 + \epsilon], \text{ } j \text{ is anywhere} \} \leq p \{ (i, j) \geq (x, x) | i \in (1, 1 + \epsilon], \text{ } j \text{ is anywhere} \} \cup \{ (i, j) \geq (x, x) | i \in [x, g(2)], \text{ } j \in (1, 1 + \epsilon) \} \leq p(2, x). \]
Finally, \( g(3) < g(2) \) since \(-g(2) + 1 - p(2, g(2)) = -g(2) < 0\). 

In general when there are \( n \) players, we can follow the same steps as above to show that

1. If type \( g(n, x) \) doesn’t hunt in period 1, then it will hunt in period 2 no matter what.

2. 
   \[
   \begin{align*}
   v(n, g(n, x)) &= -g(n, x) + 1 - p(F, n, g(n, x)) \\
   &= -g(n, x) + 1 - p(n - 1, g(n, x)) \\
   &= 0.
   \end{align*}
   \]

3. \( p(n, x) \) is increasing in \( n \) and \( x \).

The sequence of the cutoffs can be found inductively as follows.

\[
\begin{align*}
g(1) &= 1, \\
p(1, x) &= p^0(1 \mid |x|), \\
1 - g(2) &= p(1, g(2)), \\
p(2, x) &= p^0(2 \mid |x|) \cdot 1 + p^1(2 \mid |x|) p(1, g(2)), \\
1 - g(3) &= p(2, g(3)), \\
&\vdots \\
p(n - 1, x) &= p^{n-1}(n - 1 \mid |x|) \cdot 1 + \ldots + p^{n-2}(n - 1 \mid |x|) p(1, g(n - 1)), \\
1 - g(n) &= p(n - 1, g(n)).
\end{align*}
\]

Note that \( p(n, x) \geq p(n, 0), \forall x > 0 \), and \( p(n, 0) \rightarrow 1 \) as \( n \rightarrow \infty \), hence \( p(n, x) \rightarrow 1 \) as \( n \rightarrow \infty \), uniformly with respect to \( x \). Therefore, taking the limit of both sides of \( 1 - g(n) = p(n - 1, g(n)) \) as \( n \) goes to infinity, it must be that \( g(n) \) converges to 0.

**Proof of footnote 4:** Footnote 4 claims that if \( \delta \) is large enough, then \( \forall x > c^*_2 \), no solution \( g(2, x) \in [x, 1] \) exists for equation (3). Following the proof of Lemma 4, if \( x > \overline{c} \), then the claim is true; if \( x \in (c^*_2, \overline{c}] \), then \( \Delta_2(x, c, \delta) - \Delta_2(x, c', \delta) \) has the same expression as in the proof of Lemma 4. Moreover, \( LHS_2(x, x) < 0 = RHS_2(x, x) \), hence if \( \delta \) is large, then the r.h.s. is always larger than the l.h.s., for any \( x \in (c^*_2, \overline{c}] \), hence no solution.
References


Chapter 3

A Polya Urn Model of Conformity

3.1 Introduction

Everyday we give and receive numerous small courtesies. When we leave a building we hold the door for the person closely behind us. When we check out a video tape from a public library, we find that the tape is already rewound for us by the last borrower. When we arrive at a classroom to give a lecture, the blackboard is cleaned by the last instructor, and after we finish our lecture, we clean the board for whoever is the next instructor. In each of these examples, our everyday experience seems to suggest that there exists a prevalent pattern of behavior, or a norm, in which most of the people do the same thing most of the times. Such experience is so commonplace that we often take it for granted. On a second thought, it is not obvious how a norm in these contexts can ever be established, for the following reasons. 1. There is no policing from a central authority. 2. People are anonymous. We do not know who the last borrower of a video tape is, or who is going to be the next instructor. Often times we interact with strangers, whom we do not even see. Hence a punishment/reward scheme that targets specific individuals is not feasible. 3. People behave very naively. They simply respond to their past experience in some very mechanical way. For example, the more they observe other people do something in the past, the more likely they will do the same thing. 4. People are different in many ways. They come from different backgrounds, which endow them with different propensities to do one thing or another. They observe different things. They respond to their observations in different ways, some people may be more insulating from other people’s influences, while other people might be more conforming. In one word, we have a population of simple-minded, heterogeneous, and anonymous agents who live in the absence of a regulating authority. How could order emerge within such an environment?

In this chapter we propose to look at such problems through the lense of Polya urn processes. The simplest possible example of a Polya urn process is the following. Imagine an urn that contains a red ball and a black ball initially. Then randomly draw a ball from the urn, look at its color, put it back together with another ball of the same color. Then repeat this procedure. The question is, does the fraction of, say, red balls, converge in the long run? If so, where does it converge to? It turns out that the fraction of red balls converges almost surely, but it could converge to any point in the unit interval with equal probability (Johnson and Kotz (1977), Chung, Handjani, and Jungreis (2003)). Put it another way, if we run the process on 1,000 computers independently, then after a while we will find that all the processes will settle down, but the 1,000 processes will settle down around 1,000 different points in the unit interval.

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The above example, simple as it is, possesses two interesting properties. 1. A definite pattern is able to emerge spontaneously. 2. Ex ante it is hard to predict which pattern will emerge. We believe that these properties are also shared by the formation of a social norm. This is why we propose to use Polya urn processes to study the formation of norms.

In the economics literature, the Polya process has been applied to the problem of industry location (Arthur (1987)). The idea is that there is an initial distribution of firms across different regions. Each firm is then equally likely to "spin off" new firms. New firms stay in their parent region. The limiting regional pattern of an industry is thus determined by initial conditions and historical random events. Recently, Skyrms and Pemantle (2000) model the dynamics of network formation as a Polya urn process. In their benchmark model a finite number of agents decide with whom to interact in each period. The more agent A has visited agent B in the past, the more likely A will visit B in the future. Formally, such a reinforcement scheme is reduced to a Polya urn process.

The general theme of all these applications is about the spontaneous emergence of order. This chapter puts the theme in yet another concrete context. Consider a finite number of people interacting with each other sequentially and indirectly, through a common physical facility, a blackboard, for example. Each agent observes what the immediate predecessor does, and then chooses her own action according to her experience in the past, which consists of her observation of other people’s choices, and her own choices. We assume that agents behave very naively. They simply respond to their past experience in a monotonic way. Our concern is whether a pattern will emerge in such a system, and if so, what kind of pattern will emerge?

3.2 The Model

There are \( I \) players. Each player has an urn, with unlimited capacity. Initially player \( i \) has \( R_i > 0 \) red balls and \( B_i > 0 \) black balls. Let \( N_i = R_i + B_i \) be the total number of balls in player \( i \)'s urn initially. The \( I \) players make decisions sequentially and repeatedly. The order of moves is fixed to be \( 1, 2, \ldots, I, 1, 2, \ldots, I \), and so on. There are two actions to choose, the red action and the black action. At each point in time, each player chooses the red action with probability identical to the fraction of red balls in his urn, and the black action with the remaining probability. If player \( i \) chooses the red action, then he adds \( \gamma_i \geq 0 \) red balls to his own urn, and \( \gamma_{i+1} > 0 \) red balls to player \( i+1 \)'s urn. With slight abuse of notation, let \( I + 1 = 1 \), and \( 1 - 1 = I \). The rules are similar if player \( i \) chooses the black action.

The model has a simple interpretation. The initial configuration \((R_i, B_i)\) represents player \( i \)'s "prior", or background. Without any experience of other people’s choices or his own choices, player \( i \) will play the red action with probability \( \frac{R_i}{N_i} \). The parameters \( \gamma_i \) and \( \gamma'_{i} \) are trying to capture the idea of habit formation and
imitation, respectively. We allow $\gamma_i$ to be zero, but require $\gamma'_i$ to be strictly positive, so that the society is minimally "connected" in some sense, nobody is completely autistic. The players are heterogeneous in that they could come from different background, and the way they form habit and imitate others could be different. We show in the next section, nevertheless, that in the long run, their behavior in terms of the probability with which to choose a certain action, must be the same. That is, the minimal "connectivity" is able to generate the strongest "conformity".

3.3 Conformity

For ease of exposition, let one round be such that everybody has moved once, and the balls have been added to the urns according to the rules of the model. We illustrate this by an example in which $I = 3$. In the first round, player 1 moves first. Suppose player 1 chooses the red action, then he adds $\gamma_1$ red balls to his own urn, and $\gamma'_2$ red balls to player 2's urn. Now player 2 moves, suppose player 2 chooses the black action, then he adds $\gamma_2$ black balls to his own urn, and $\gamma'_3$ black balls to player 3's urn. Then player 3 moves and chooses, say, the black action, as a result player 3 adds $\gamma_3$ black balls to his own urn, and $\gamma'_1$ black balls to player 1's urn. This completes the first round. Hence after the first round, player $i$ has $N_i + \gamma_i + \gamma'_i$ balls in his urn. In general, after the nth round, player $i$ has $N_i + n (\gamma_i + \gamma'_i)$ balls in his urn.

For $i = 1, 2, \ldots, I$, let $x^i_n$ denote the fraction of red balls in $i$'s urn at the beginning of the nth round. Let $x_n = (x^i_n)_{i=1}^I$. We are interested in the limiting behavior of $x_n$. Does it converge? If so, where does it converge to? The following proposition answers these questions.

**Proposition 1** There exists a random vector $x$, such that (a) $x_n$ converges to $x$ almost surely, and (b) The support of $x$ is contained in the diagonal of $[0, 1]^I$.

The proposition says two things. First, the process is always going to settle down. The sample paths along which the process diverges can be neglected. Second, the probabilities with which people choose a certain action are going to be the same in the limit, even if different people have different background and different $\gamma$'s.

The proof is based on a theorem in Arthur, Ermoliev, and Kaniovski (1984), hereafter referred to as the AEK theorem. In the proof of Proposition 1, we first write the problem in the form as in the AEK theorem, then we prove that the conditions required by the theorem are met in our problem.

**Proof of Proposition 1:** Fix $n$ and $x_n$. Fix $i \in \{1, 2, \ldots, I\}$. Let $\Delta_i(n, x_n)$ denote the number of red balls added to $i$'s urn in round $n$, given $x_n$ being the fractions of red balls at the beginning of round $n$. Then,
\[ x_{n+1}^i = \left( \frac{1}{N_i + n (\gamma_i + \gamma_i')} \right) \left( x_n^i (N_i + (n - 1) (\gamma_i + \gamma_i')) + \Delta_i (n, x_n) \right) \]
\[ = x_n^i + \left( \frac{1}{N_i + n (\gamma_i + \gamma_i')} \right) (-x_n^i (\gamma_i + \gamma_i') + \Delta_i (n, x_n)). \]

By definition, 
\[ \Delta_i (n, x_n) = \begin{cases} 
0 & \text{with prob. } P ((i - 1) B \cap i B|n, x_n) \\
\gamma_i & \text{with prob. } P ((i - 1) B \cap i R|n, x_n) \\
\gamma_i' & \text{with prob. } P ((i - 1) R \cap i B|n, x_n) \\
\gamma_i + \gamma_i' & \text{with prob. } P ((i - 1) R \cap i R|n, x_n) 
\end{cases}. \]

where \( P ((i - 1) B \cap i B|n, x_n) \) is the probability that player \( i - 1 \) chooses the black action and player \( i \) also chooses the black action, conditional on the round being the \( n \)th round and the fractions of red balls at the beginning of round \( n \) being \( x_n \). The other probabilities have similar interpretations.

Let 
\[ \beta_i (x_n) := \frac{1}{\gamma_i + \gamma_i'} \Delta_i (n, x_n). \]

Then 
\[ x_{n+1}^i = x_n^i + \left( \frac{\gamma_i + \gamma_i'}{N_i + n (\gamma_i + \gamma_i')} \right) (\beta_i (x_n) - x_n^i). \]

Let 
\[ q_i (x_n) := \frac{1}{\gamma_i + \gamma_i'} E(\Delta_i (n, x_n)), \]

where \( E(\Delta_i (n, x_n)) \) denote the expectation of \( \Delta_i (n, x_n) \), conditional on fixed \( n \) and \( x_n \).

Then 
\[ x_{n+1}^i = x_n^i + \left( \frac{\gamma_i + \gamma_i'}{N_i + n (\gamma_i + \gamma_i')} \right) (q_i (x_n) - x_n^i) + \left( \frac{\gamma_i + \gamma_i'}{N_i + n (\gamma_i + \gamma_i')} \right) (\beta_i (x_n) - q_i (x_n)). \]

(1)

**Definition:** Let \( \{f_n\}_n \) be a sequence of functions, each mapping \([0, 1]^f \) into \([0, 1]^f \). Let \( \{f_n\}_n \) have a pointwise limit \( f \). We say that \( f_n \) converges to \( f \) **reasonably rapidly** if there exists a sequence of positive constants \( \{a_n\} \), such that \( \sup_{x \in [0, 1]^f} ||f_n (x) - f (x)|| \leq a_n \), and \( \sum_{n=1}^{\infty} a_n / n < \infty. \)
Lemma 1 (1) For any $i$, for any $x \in [0, 1]^I$,

$$q^n_i (x) \longrightarrow q^i (x) = \frac{\gamma_i}{\gamma_i + \gamma'_i} x_i + \frac{\gamma'_i}{\gamma_i + \gamma'_i} x_{i-1},$$

and (2) $q_n$ converges to $q$ reasonably rapidly, where $q_n (\cdot) := (q^n_i (\cdot))_i$, and $q (\cdot) := (q^i (\cdot))_i$.

Proof of Lemma 1:

(1) By definition of $q^n_i (x)$, it suffices to show that

$$E \triangle_i (n, x) \longrightarrow \gamma_i x_i + \gamma'_i x_{i-1}.$$

Consider the conditional probability $P ((i - 1) B \cap i B | n, x)$. If $i = 1$, then we show that

$$P (IB \cap 1B | n, x) \longrightarrow (1 - x_1)(1 - x_I).$$

Let $IB$ denote the event that "out of $N_I + (n - 1) (\gamma_I + \gamma'_I) + \gamma'_I$ balls, in which $x_I (N_I + (n - 1) (\gamma_I + \gamma'_I)) + \gamma'_I$ balls are red, a black ball is chosen in $I$'s urn".

Let $\overline{IB}$ denote the event that "out of $N_I + (n - 1) (\gamma_I + \gamma'_I) + \gamma'_I$ balls, in which $x_I (N_I + (n - 1) (\gamma_I + \gamma'_I))$ balls are red, a black ball is chosen in $I$'s urn". Then,

$$P (IB) P (1B | n, x) \leq P (IB \cap 1B | n, x) \leq P (\overline{IB}) P (1B | n, x).$$

But

$$P (1B | n, x) = 1 - x_1,$$

$$P (IB) = 1 - \frac{x_I (N_I + (n - 1) (\gamma_I + \gamma'_I)) + \gamma'_I}{N_I + (n - 1) (\gamma_I + \gamma'_I) + \gamma'_I},$$

and

$$P (\overline{IB}) = 1 - \frac{x_I (N_I + (n - 1) (\gamma_I + \gamma'_I))}{N_I + (n - 1) (\gamma_I + \gamma'_I) + \gamma'_I}.$$

Hence

$$P (IB) \longrightarrow 1 - x_I,$$

and

$$P (\overline{IB}) \longrightarrow 1 - x_I.$$

By the sandwich theorem,
\[ P(IB \cap 1B \vert n, x) \longrightarrow (1 - x_1)(1 - x_I). \]

Similarly,
\[ P(IB \cap 1R \vert n, x) \longrightarrow x_1(1 - x_I), \]
\[ P(IR \cap 1B \vert n, x) \longrightarrow (1 - x_1)x_I, \]
and
\[ P(IR \cap 1R \vert n, x) \longrightarrow x_Ix_I. \]

Therefore,
\[ E\Delta_1(n, x) \longrightarrow \gamma_1 x_1 + \gamma_1' x_I. \]

If \( i > 1 \), then we show that
\[ P((i - 1)B \cap iB \vert n, x) \longrightarrow (1 - x_{i-1})(1 - x_i). \]

Let \((i - 1)B\) denote the event that "out of \( N_{i-1} + (n - 1) \left( \gamma_{i-1} + \gamma_{i-1}' \right) \) balls, in which \( x_{i-1} \left( N_{i-1} + (n - 1) \left( \gamma_{i-1} + \gamma_{i-1}' \right) \right) \) balls are red, a black ball is chosen in \( i - 1 \)'s urn".

Let \((i - 1)\overline{B}\) denote the event that "out of \( N_{i-1} + (n - 1) \left( \gamma_{i-1} + \gamma_{i-1}' \right) \) balls, in which \( x_{i-1} \left( N_{i-1} + (n - 1) \left( \gamma_{i-1} + \gamma_{i-1}' \right) \right) \) balls are red, a black ball is chosen in \( i - 1 \)'s urn".

Similarly, let \(iB\) denote the event that "out of \( N_i + (n - 1) \left( \gamma_i + \gamma_i' \right) \) balls, in which \( x_i \left( N_i + (n - 1) \left( \gamma_i + \gamma_i' \right) \right) \) balls are red, a black ball is chosen in \( i \)'s urn".

Let \(i\overline{B}\) denote the event that "out of \( N_i + (n - 1) \left( \gamma_i + \gamma_i' \right) \) balls, in which \( x_i \left( N_i + (n - 1) \left( \gamma_i + \gamma_i' \right) \right) \) balls are red, a black ball is chosen in \( i \)'s urn".

Then
\[ P\left((i - 1)B \cap iB \vert n, x\right) \leq P\left((i - 1)B \cap iB \vert n, x\right) \leq P\left(i - 1)\overline{B} \cap iB \vert n, x\right). \]

But
\[ P\left((i - 1)B \right) = 1 - \frac{x_{i-1} \left( N_{i-1} + (n - 1) \left( \gamma_{i-1} + \gamma_{i-1}' \right) \right) + \gamma'_{i-1}}{N_{i-1} + (n - 1) \left( \gamma_{i-1} + \gamma_{i-1}' \right) + \gamma'_{i-1}}, \]
\[ P\left(i - 1)\overline{B} \right) = 1 - \frac{x_{i-1} \left( N_{i-1} + (n - 1) \left( \gamma_{i-1} + \gamma_{i-1}' \right) \right) + \gamma'_{i-1}}{N_{i-1} + (n - 1) \left( \gamma_{i-1} + \gamma_{i-1}' \right) + \gamma'_{i-1}}. \]
\[ P(iB) = 1 - \frac{x_i (N_i + (n-1) (\gamma_i + \gamma'_i)) + \gamma'_i}{N_i + (n-1) (\gamma_i + \gamma'_i) + \gamma'_i}, \]

and
\[ P(iB) = 1 - \frac{x_i (N_i + (n-1) (\gamma_i + \gamma'_i))}{N_i + (n-1) (\gamma_i + \gamma'_i) + \gamma'_i}. \]

Hence
\[ P((i-1)B) \rightarrow 1 - x_{i-1}, \]
\[ P((i-1)B) \rightarrow 1 - x_{i-1}, \]
\[ P(iB) \rightarrow 1 - x_i, \]
\[ P(iB) \rightarrow 1 - x_i. \]

Again, by the sandwich theorem,
\[ P((i-1)B \cap iB | n, x) \rightarrow (1 - x_{i-1}) (1 - x_i). \]

Similarly,
\[ P((i-1)B \cap iR | n, x) \rightarrow (1 - x_{i-1}) x_i, \]
\[ P((i-1)R \cap iB | n, x) \rightarrow x_{i-1} (1 - x_i), \]
and
\[ P((i-1)R \cap iR | n, x) \rightarrow x_{i-1} x_i. \]

Therefore
\[ E\Delta_i (n, x) \rightarrow \gamma_i x_i + \gamma'_i x_{i-1}. \]

(2) In order to show that the conditional expectations converge reasonably rapidly, it suffices to show that the conditional probabilities converge reasonably rapidly. But by the proof of part (1), it is easy to see that, for example, \( P(iB) \) converges to \( 1 - x_i \) reasonably rapidly, hence it is easy to show, again by a sandwich argument, that \( P(iB \cap 1B | n, x) \) converges to \( (1 - x_1) (1 - x_i) \) reasonably rapidly. This completes the proof of Lemma 1.

At this point we digress a little bit, and consider a somewhat related problem studied by Arthur, Ermoliev, and Kaniovski (1984).

Consider an urn containing balls of \( I \) colors. Initially there are \( \gamma \) balls in the urn. At the beginning of round \( n \), a ball of color \( i \) is added to the urn with probability \( q^i_n (x_n) \), where \( x_n = (x_n^i)_{i=1}^I \) is the vector that summarizes the fraction of each color.
color at the beginning of round \( n \). After some manipulation, the law of motion of the process \((x_n)_n\) can be written as follows.

\[
x_{n+1}^i = x_n^i + \frac{1}{n+\gamma} (q_n^i(x_n) - x_n^i) + \frac{1}{n+\gamma} (\beta_n^i(x_n) - q_n^i(x_n)),
\]

where

\[
\beta_n^i(x_n) = \begin{cases} 1 & \text{with prob. } q_n^i(x_n) \\ 0 & \text{with prob. } 1 - q_n^i(x_n) \end{cases}.
\]

Let \( S \) denote the simplex contained in \([0,1]^I\), Arthur, Ermoliev, and Kaniovski (1984) proved the following theorem.

**Theorem** (Arthur, Ermoliev, and Kaniovski (1984)) Let \( \{q_n\} \) be continuous functions. If there exists a continuous function \( q : S \rightarrow S \), a sequence of constants \( \{a_n\} \), and a function \( v : S \rightarrow \mathbb{R} \), such that

(a) \( \sup_{x \in S} \|q_n(x) - q(x)\| \leq a_n \), and \( \sum_{n=1}^{\infty} a_n/n < \infty \).

(b) \( B = \{x \in S | q(x) = x\} \) contains a finite number of components.

(c) (i) \( v \) is twice differentiable.

(ii) \( v(x) \geq 0, \forall x \in S \).

(iii) \( < q(x) - x, v_x(x) > > 0, \forall x \in S \setminus B \).

Then \( x_n \) converges to a point in \( B \) or to the border of a connected component.

The proof of the AEK theorem is based on Theorem 7.3 in Nevelson and Hasminskii (1976). Nowhere in the proof of the AEK theorem requires that the \( q_n \) functions be probability functions. Hence the AEK theorem still holds if we replace \( S \) by \([0,1]^I\) everywhere in the statement of the theorem. Hence the problem of two colors with \( I \) urns is essentially the same as the problem of \( I \) colors with one urn. By Lemma 1, condition (a) of the AEK theorem is satisfied. Also by Lemma 1,

\[
q(x) - x = \begin{pmatrix} l_1 (x_1 - x_1) \\ \vdots \\ l_I (x_{I-1} - x_I) \end{pmatrix},
\]

where \( l_i = \frac{\gamma_i}{\gamma_i + \gamma_i} \).

Hence \( B = \text{diagonal of } [0,1]^I \), and \( B \) contains only one component, which is \( B \) itself. Moreover, the boundary of \( B \) is also \( B \) itself, since every neighborhood of every point along the diagonal contains both points in \( B \) and points not in
Therefore, to establish the proposition it suffices to construct a Lyapunov function $v$.

Before we give the general construction, we illustrate the idea in a simple example, where $I = 3$, and $\gamma_i = \gamma'_i = 1$, $i = 1, 2, 3$.

In this case

$$q(x) - x = \left( \begin{array}{c}
\frac{1}{4}(x_3 - x_1) \\
\frac{1}{4}(x_1 - x_2) \\
\frac{1}{4}(x_2 - x_3)
\end{array} \right).$$

We need to find a function $v(x)$ which is twice differentiable and non-negative, and the inner product between $v_x(x)$ and $(q(x) - x)$ is strictly negative at every non fixed point of $q$, and zero at every fixed point of $q$.

Such $v$ is easy to find. Let

$$v(x) = \frac{1}{4}(x_1^2 + x_2^2 + x_3^2) - \frac{1}{2}(x_1 x_2 + x_1 x_3 + x_2 x_3) + A,$$

where $A$ is some positive number to make sure that $v$ is non-negative. Then

$$v_x(x) = \left( \begin{array}{c}
\frac{1}{4}x_1 - \frac{1}{4}x_3 - \frac{1}{4}x_2 \\
\frac{1}{4}x_2 - \frac{1}{4}x_1 - \frac{1}{4}x_3 \\
\frac{1}{4}x_3 - \frac{1}{4}x_2 - \frac{1}{4}x_1
\end{array} \right) = (a) - (b)$$

This $v$ function works because $< q(x) - x, (a) > \leq 0$, with equality only at fixed points of $q$, and $< q(x) - x, (b) > = 0$ for all $x$. This is going to be the general approach as well: we construct $v$ as a quadratic function so that $v_x(x)$ can be written in two parts, $(a)$ and $(b)$, such that $< q(x) - x, (a) > \leq 0$, with equality only at fixed points of $q$, and $< q(x) - x, (b) > = 0$ for all $x$.

In general, let

$$v(x) = \sum_{i=1}^{I} a_i \frac{x_i^2}{2} - \sum_{1 \leq i < j \leq I} b_{ij} x_i x_j + A.$$ 

Since $x \in [0, 1]^I$, we can always choose $A$ large enough to make sure that $v$ is nonnegative. Hence conditions $c(i)$ and $c(ii)$ are trivially satisfied. It remains to be shown that there exist $a_i$ and $b_{ij}$ such that condition $c(iii)$ is also satisfied.

Let $b_{12} = a_2$, $b_{23} = a_3$, $b_{34} = a_4$, ..., $b_{I-1 I} = a_I$, and $b_{1 I} = a_1$, then
\[
\begin{align*}
\mathbf{v}_x (x) &= \begin{pmatrix}
  a_1 (x_1 - x_6) \\
  a_2 (x_2 - x_1) \\
  a_3 (x_3 - x_2) \\
  \vdots \\
  a_I (x_I - x_{I-1})
\end{pmatrix} = \begin{pmatrix}
  a_2 x_2 + b_{13} x_3 + b_{14} x_4 + b_{15} x_5 + \ldots + b_{1I-3} x_{I-3} + b_{1I-2} x_{I-2} + b_{1I-1} x_{I-1} \\
  a_3 x_3 + b_{23} x_4 + b_{25} x_5 + b_{26} x_6 + \ldots + b_{2I-2} x_{I-2} + b_{2I-1} x_{I-1} + b_{2I} x_I \\
  a_4 x_4 + b_{35} x_5 + b_{36} x_6 + b_{13} x_1 + \ldots + b_{3I-1} x_{I-1} + b_{3I} x_1 + b_{13} x_1 \\
  a_1 x_1 + b_{21} x_2 + b_{31} x_3 + b_{41} x_4 + \ldots + b_{I-4} x_{I-4} + b_{I-3} x_{I-3} + b_{I-2} x_{I-2}
\end{pmatrix}
\end{align*}
\]

On the other hand,
\[
q(x) - x = \begin{pmatrix}
  l_1 (x_I - x_1) \\
  l_2 (x_1 - x_2) \\
  l_3 (x_2 - x_3) \\
  \vdots \\
  l_I (x_{I-1} - x_I)
\end{pmatrix} = (c),
\]

where \( l_i = \frac{\gamma'}{\gamma_i} \).

We will show that there exist \((a_i)_{i=1}^I\) and \((b_{ij})_{i<j}\), all strictly positive, such that \((b) \cdot (c) = 0\), for all \( x \in [0,1]^I \).

When we take the inner product of \((b)\) and \((c)\), we obtain \( C_I^2 = \frac{I(I-1)}{2!} \) terms of cross products between \( x_i \) and \( x_j \). Letting the coefficient of each term be zero, we obtain \( C_I^2 \) equations. Notice that we also have \( C_I^2 \) unknowns, \( I \) a’s and \((C_I^2 - I)\) b’s. We write these equations out as follows.

\[
\begin{align*}
x_I x_I : & \quad l_I a_I = l_2 b_{2I} \\
x_1 x_2 : & \quad l_1 a_2 = l_3 b_{13} \\
x_2 x_3 : & \quad l_2 a_3 = l_4 b_{24} \\
x_3 x_4 : & \quad l_3 a_4 = l_5 b_{35} \\
x_4 x_5 : & \quad l_4 a_5 = l_6 b_{46} \\
\vdots
\end{align*}
\]

\[
\begin{align*}
x_{I-1} x_I : & \quad l_{I-1} a_I = l_1 b_{1I-1} \\
x_1 x_3 : & \quad (l_1 + l_3) b_{13} = l_2 a_3 + l_4 b_{14} \\
x_1 x_4 : & \quad l_1 + l_4 b_{14} = l_2 b_{24} + l_5 b_{15}
\end{align*}
\]

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\begin{align*}
x_1x_5 &: (l_1 + l_5) b_{15} = l_2 b_{25} + l_6 b_{16} \\
& \vdots \\
x_1 x_{i-1} &: (l_1 + l_{i-1}) b_{1i-1} = l_2 b_{2i-1} + l_i a_1 \\
x_2 x_4 &: (l_2 + l_4) b_{24} = l_3 a_4 + l_5 b_{25} \\
x_2 x_5 &: (l_2 + l_5) b_{25} = l_3 b_{35} + l_6 b_{26} \\
& \vdots \\
x_2 x_I &: (l_2 + l_I) b_{2I} = l_3 b_{3I} + l_1 a_2 \\
& \vdots \\
x_{I-2} x_I &: (l_{I-2} + l_I) b_{I-2I} = l_{I-1} a_I + l_1 b_{1I-2}
\end{align*}

System (3) is a system of $C_I^2$ equations with $C_I^2$ unknowns. Notice that all the $C_I^2$ unknowns are displayed on the l.h.s. of the system. We claim that every term on the l.h.s. of system (3) shows up exactly once on the r.h.s., and since the two sides have the same number of terms, this implies that if we add up all the $C_I^2$ equations, we obtain an identity, which in turn, implies that the coefficient matrix of system (3) is singular.

To see that every term on the l.h.s. of system (3) shows up exactly once on the r.h.s., consider $a_i$ first. When we take the inner product between $(b)$ and $(c)$, $a_i$ shows up twice, once in the coefficient of $x_i x_{i-1}$, and once in the coefficient of $x_i x_{i-2}$. The coefficient of both terms is $l_{i-1} a_i$, but the signs are opposite. Hence $l_{i-1} a_i$ appears once on both sides of system (3). Now consider $b_{ij}$. When we take the inner product between $(b)$ and $(c)$, $b_{ij}$ shows up four times, twice in the coefficients of $x_i x_j$, once in the coefficient of $x_i x_{j-1}$, and once in the coefficient of $x_{i-1} x_j$. The coefficient of $x_i x_j$ is $(l_i + l_j) b_{ij}$, the coefficient of $x_i x_{j-1}$ is $l_j b_{ij}$, and the coefficient of $x_{i-1} x_j$ is $l_i b_{ij}$. Again, $(l_i + l_j) b_{ij} x_i x_j$ shows up negative in the inner product, while $l_j b_{ij} x_i x_{j-1}$ and $l_i b_{ij} x_{i-1} x_j$ show up positive in the inner product. Hence $l_i b_{ij}$ shows up once on both sides of (3), and so does $l_j b_{ij}$.

Since system (3) is a homogenous system with singular coefficient matrix, it has a non-zero solution. Next we show that the system has a non-zero and non-negative solution. We prove a more general result in Lemma 2. The a’s and b’s in Lemma 2 are just for notational convenience. They are different from the a’s and b’s outside Lemma 2.

**Lemma 2** Consider the $n$–variate linear homogeneous system of equations,
\[ a_1x_1 = b_{12}x_2 + \ldots + b_{1n}x_n \]
\[ a_2x_2 = b_{21}x_1 + \ldots + b_{2n}x_n \]
\[ \vdots \]
\[ a_n x_n = b_{n1}x_1 + \ldots + b_{nn-1}x_{n-1} \]

If \( \forall j, a_j \geq 0 \); \( \forall i, j, b_{ij} \geq 0 \); and \( \forall j, \sum_{i \neq j} b_{ij} = a_j \), then \( \exists x \neq 0 \), and \( \forall i, x_i \geq 0 \), such that \( x \) solves the system.

**Proof of Lemma 2**: Proof by induction. Lemma 2 holds trivially for \( n = 2 \). Suppose it is true for \( n > 2 \), we need to show that it is true for \( n + 1 \). With \( n + 1 \) variables, the system becomes

\[ a_1x_1 = b_{12}x_2 + \ldots + b_{1n+1}x_{n+1} \]
\[ a_2x_2 = b_{21}x_1 + \ldots + b_{2n+1}x_{n+1} \]
\[ \vdots \]
\[ a_{n+1} x_{n+1} = b_{n+1,1}x_1 + \ldots + b_{n+1,n}x_n \]

If \( a_j = 0 \), \( \forall j \), then all the coefficients are zero, Lemma 2 is trivially true. Without loss of generality, assume \( a_1 \neq 0 \). Then \( x_1 = \frac{1}{a_1} (b_{12}x_2 + \ldots + b_{1n+1}x_{n+1}) \). Substituting it into the rest of the system, then multiplying both sides of the rest of the system by \( a_1 \), and rearranging, the rest of the system becomes,

\[ (a_1a_2 - b_{21}b_{12})x_2 = (b_{21}b_{13} + a_1b_{23})x_3 + \ldots + (b_{21}b_{1n+1} + a_1b_{2n+1})x_{n+1} \]
\[ (a_1a_3 - b_{31}b_{13})x_3 = (b_{31}b_{12} + a_1b_{32})x_2 + \ldots + (b_{31}b_{1n+1} + a_1b_{3n+1})x_{n+1} \]
\[ \vdots \]
\[ (a_1a_{n+1} - b_{n+1,1}b_{1n+1})x_{n+1} = (b_{n+1,1}b_{12} + a_1b_{n+1,2})x_2 + \ldots + (b_{n+1,1}b_{1n+1} + a_1b_{n+1,n})x_n. \]

By the assumptions on the a’s and the b’s, we are allowed to use the induction hypothesis. Therefore, there exists a non-zero and non-negative vector \( (x_2, x_3, \ldots, x_{n+1}) \) that solves the above system, hence there exists a non-zero and non-negative \( x \) that solves the original system. This completes the proof of Lemma 2.

Now we go back to system (3). Notice that each unknown is a positive linear combination (i.e., linear combination with positive coefficients) of one or two
other unknowns. If we think of each unknown as a node, and think of the unknowns in the linear combination as each receiving a link from the original unknown, we obtain the following directed graph from system (3).

\[ a_2 \rightarrow b_{13} \rightarrow b_{14} \rightarrow b_{15} \rightarrow \ldots \rightarrow b_{1I-3} \rightarrow b_{1I-2} \rightarrow b_{1I-1} \]
\[ a_3 \rightarrow b_{24} \rightarrow b_{25} \rightarrow b_{26} \rightarrow \ldots \rightarrow b_{2I-2} \rightarrow b_{2I-1} \rightarrow b_{2I} \]
\[ a_4 \rightarrow b_{35} \rightarrow b_{36} \rightarrow b_{37} \rightarrow \ldots \rightarrow b_{3I-1} \rightarrow b_{3I} \rightarrow b_{13} \]
\[ a_5 \rightarrow b_{46} \rightarrow b_{47} \rightarrow b_{48} \rightarrow \ldots \rightarrow b_{4I} \rightarrow b_{14} \rightarrow b_{24} \]

The pattern of the graph is, except for the leftmost and the rightmost columns, each node is pointing to the right next node, and the node in the next row left to it. For example, $b_{14}$ is pointing to $b_{15}$ and $b_{24}$, because when we take the inner product between $(b)$ and $(c)$, $b_{14}$, $b_{15}$, and $b_{24}$ are the only unknowns associated with the term $x_1x_4$. In general, for every $b$ not in the first, second and last column, $b_{ij}$ is pointing to $b_{ij+1}$ and $b_{i+1j}$ (remember that $I + 1 = 1$ and $b_{ij} = b_{ji} = \partial^2 v / \partial x_i \partial x_j$), because when we compute the inner product between $(b)$ and $(c)$, the only unknowns associated with the term $x_ix_j$ are $b_{ij}$, $b_{ij+1}$ and $b_{i+1j}$. For every $b$ in the second column, $b_{ij}$ is pointing to $b_{ij+1}$ and $a_j$, for the same reason. For the leftmost column, each $a$ points to the right next node, following directly from system (3). Each node in the rightmost column should also point to two nodes, but we consider only one of them, which suffices for our purposes. Finally, we copy the first row after the last row to indicate the cyclic property of the graph. We can imagine that it is drawn on a cylinder.

By Lemma 2, there exists a non-negative, non-zero solution to system (3). We fix this solution. It is easy to see from the graph that if any unknown is 0, then to keep non-negativity, all the other unknowns must also be 0, which is impossible since we begin with a non-zero solution. Therefore all the unknowns in this non-zero and non-negative solution must be strictly positive, which is what we need for the Lyapunov function. This completes the proof of Proposition 1.

3.4 A Related Model

In this chapter we are interested in a situation where people interact with each other anonymously, sequentially, and indirectly through a common physical facility, a blackboard, for example. Each person observes the current state of the
facility (whether the board is clean or not), uses the facility, then decides which state of the facility to leave to the next person (whether to clean the board or not). We assume that the agents behave very naively, they respond to their past experience in some monotonic and probabilistic way. We see in Proposition 1 that people are going to behave more and more like each other in the long run, and in the limit everybody behaves the same way, even if people differ from each other in terms of backgrounds and behavior rules. There is no particular reason that the agents can settle down at a "cooperative norm". In fact, we do not even specify payoffs in the model, in order to highlight the force of imitation to bring about conformity. Now if we add payoffs to the model, and endow the agents with full rationality, are they able to deliberately sustain a cooperative norm? This section examines this question. For concreteness, we use the blackboard example.

Let there be \( N \) players. Let \( N_1 \) players be patient players, and \( N_2 \) players be impatient players, \( N_1 + N_2 = N \). Patient players discount future payoffs by \( \delta \in (0, 1) \), impatient players are completely myopic. In each period, a player is chosen with probability \( \frac{1}{N} \) to arrive at the blackboard. If he finds the board clean, he receives a utility of \( b \), otherwise his utility is \( 0 \). After the player uses the board, he chooses whether to clean it for the next person. If he cleans it, he incurs a disutility of \( c \), otherwise his disutility is \( 0 \). We assume that \( b > c \).

The players are anonymous. They arrive at random. The impatient players never clean the board. The question is, can the patient players still maintain a norm to clean the board?

Grim trigger strategy does not support the good norm because with Grim trigger strategy, sooner or later the system is stuck with permanent defection (not cleaning the board). However, the next proposition shows that the patient players can still maintain the good norm probabilistically, under some conditions on the parameters of the model.

**Proposition 2** If \( \frac{(N_1+N_2)c-b\delta}{(N_1-1)\delta c} < 1 \), then there exists \( p \in (0, 1) \), and there exists a symmetric and stationary sequential equilibrium, in which the patient players clean the board with probability no less than \( p \) in each period.

Proof: We construct a mixed strategy for the patient players of the following form. Upon seeing a clean board, erase the board with probability \( p_1 \), upon seeing a dirty board, erase the board with probability \( p_2 \). Let \( v(1) \) denote the lifetime value of a clean board at the beginning of a period (before a player is chosen) to a patient agent, let \( v(0) \) denote the lifetime value of a dirty board at the beginning of a period to a patient agent. If every patient player plays \( (p_1, p_2) \), then
\[ v(1) = \frac{1}{N_1 + N_2} (b + p_1 (-c + \delta v(1)) + (1 - p_1) \delta v(0)) \]
\[ + \frac{N_2}{N_1 + N_2} \delta v(0) + \frac{N_1 - 1}{N_1 + N_2} (p_1 \delta v(1) + (1 - p_1) \delta v(0)), \]

and

\[ v(0) = \frac{1}{N_1 + N_2} (p_2 (-c + \delta v(1)) + (1 - p_2) \delta v(0)) \]
\[ + \frac{N_2}{N_1 + N_2} \delta v(0) + \frac{N_1 - 1}{N_1 + N_2} (p_2 \delta v(1) + (1 - p_2) \delta v(0)). \]

In a symmetric and stationary mixed strategy equilibrium, it must be that the continuation value of erasing the board is equal to the continuation value of not erasing the board. That is,

\[-c + \delta v(1) = \delta v(0) \]

Combining (4), (5), and (6), solving for \( p_1 \) and \( p_2 \), we have

\[ p_1 - p_2 = \frac{(N_1 + N_2) c - b \delta}{(N_1 - 1) \delta c}. \]

Let \( p := \min\{p_1, p_2\} \), this completes the proof of Proposition 2. ■

Consider another equilibrium in which nobody does anything good, patient and impatient players alike (call this the autarky equilibrium). Are the players necessarily better off in the \((p_1, p_2)\) equilibrium (call this probabilistic tit-for-tat equilibrium)? In the autarky equilibrium, everybody receives 0. In the probabilistic tit-for-tat equilibrium, all the impatient players are strictly better off. The patient players are also strictly better off, because \( v(0) > 0 \).

The above stationary, symmetric equilibrium is very restrictive. If, for example, \( N_2 > \frac{b}{c} \), then there no longer exists an equilibrium of the form \((p_1, p_2)\). In this case, if the players insist playing a stationary, symmetric equilibrium, it seems that the only thing they can do is the autarky equilibrium.

### 3.5 Conclusion

This chapter studies an environment in which people interact with each other anonymously, sequentially, and indirectly through a common physical facility. Examples include rewinding video tapes, erasing blackboards, and flushing toilets, etc. Can order emerge spontaneously in such an environment? By order we mean some recognizable pattern of behavior. We assume that people respond

\[ \text{From (4), (5), and (6), } v(0) = \frac{(N_1 - 1)p_2 c}{(N_1 + N_2)(1 - \delta)} > 0. \]
to their past experience in a monotonic and probablistic way. In particular, the more they observe other people do something, the more likely they will do it (imitation), and the more they observe themselves do something, the more likely they will do it (habit formation). We propose to use Polya urn process to model such behavior. We find that eventually people are going to behave in the same way as each other, even if they come from different backgrounds, and they use different parameters in the imitation and habit formation process. An arbitrarily small amount of imitation in the system will produce an arbitrarily strong level of conformity. We then consider a related model in which agents differ from each other in terms of patience, but they are all fully rational. We find that under some conditions on the parameters of the model, the patient people are able to sustain the good norm among themselves probablistically, even in the presence of impatient players who have no incentive to sustain the good norm.

In both models, many questions remain to be answered. In the Polya urn model, it would be interesting to know what is the limiting distribution of the process. We know that the support of the limiting distribution is contained in the diagonal of the $I$-dimensional cube, but is it properly contained? Computer simulations suggest that any point along the diagonal is a possible limit point, but we do not have a proof. A realistic extension of the Polya urn model is to introduce forgetting. In our model, things happening in the distant past have the same influence over decision making as things happening recently. A more plausible assumption is that people discount their past experience by a discount factor $\delta$. For example, a ball added ten periods ago is counted as $\delta^{10}$ balls, compared to a ball added today. If people forget, and if they forget at different rates, do we still have the conformity result? If so, what does the support of the limiting distribution look like? In the one person case, once discounting is added, the person will always settle down with either action deterministically. Only extreme points remain in the support (Skyrms and Pemantle (2000)). Is this also true in our model? In the repeated game model, the supporting equilibrium strategy does not work if there are too many impatient players, can the good norm still be sustained in this case?

References


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