The Pennsylvania State University
The Graduate School

ESSAYS IN BARGAINING WITH OUTSIDE OPTIONS

A Dissertation in
Economics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2024

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Abstract

This dissertation consists of three essays.

In the first chapter I study a bargaining game with outside options in an interdependent values setting. A seller makes sequential offers to a buyer who has private information about the value of the object. The seller has an exercisable outside option that is valued at $\alpha < 1$ proportion of the buyer's value. In the frequent offer limit of any equilibrium, the seller is able to extract full rents from a subset of buyer types. However, Coasian forces are present on two levels; the seller is not only tempted to lower prices to conclude trade, but is less inclined to exercise her outside option as she becomes more pessimistic about the value of the object (and therefore, the value of her outside option). In equilibrium, an improvement in the outside option (through an increase in α) could make the seller less willing to lower prices and helps her extract more rents from the middle type. As she is able to extract more rents, she is less willing to exit, which undermines her ability to extract rents from the high type. When the middle type occurs with sufficiently high probability, a marginal increase in α may make the seller worse off for some values of α.

In the second chapter I study a bargaining setting in which players are asymmetrically informed about the value of an outside option. A seller makes sequential offers to a buyer who has private information about the value of the good. The seller has an outside option whose value evolves over time. However, the buyer does not observe the outcome of this process. Further, if the seller exercises her outside option, the probability that the buyer trades depends on the value of the outside option. I construct a pooling equilibrium that exhibits Coasian dynamics and yields the seller a payoff equal to her payoff from an optimally timed take-it-or-leave-it offer. Trade dynamics follow a simple structure: the probability of trade initially increases at a continuous rate until some deterministic time at which point there is a discontinuous jump in trade probability.

In the third chapter I study a bargaining game with interdependent values in which the seller must invest in her outside option to maintain it. Although investment is costless, there is an equilibrium in which the seller does not maintain her outside option with positive probability on the equilibrium path. Under some conditions, this equilibrium may yield the seller a higher payoff than any equilibrium in which the seller maintains her outside option. When the seller gives up her outside option with positive probability, the buyer's belief about the seller's outside option becomes more adverse over time which in turn incentivizes the seller to exit early in the game. Because the seller underinvests with a very small probability, she is able to extract virtually all rents from high value buyers.

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Acknowledgements

I'm deeply indebted to my advisor Rohit Lamba and Miaomiao Dong for their continued academic, as well as emotional, support and guidance. I would like to thank Vijay Krishna and Kalyan Chatterjee for providing crucial feedback and guidance in the final stages of this project. I would like to thank all theory faculty members, especially Yuhta Ishii and Nageeb Ali, for their helpful and insightful feedback.

I am in perpetual debt to my Guru, Amma, my family and my Amma family without whom this would not have been possible.

I would like to express my heartfelt gratitude to my study group friends, Caio Figueiredo, Han Ng, Stephan Sagl, Wenhao Li and Faqiang Li, my childhood friend, Kalyani Padmakumar and my Penn State friends, Manasi D. Nair, Jibin Joy Kolliyil, Vivek Bhasi, Amritanand Sebastian, Krishnan M A and Kiran Bhaskar for their care, compassion and support.

I would like to thank Krista Winkelblech for all her help and support.

Chapter 1

Why Better Outside Options May Erode Bargaining Power

1.1 Introduction

The role of outside options in bargaining is a topic that has received a lot of attention in the literature. This question has been studied in a variety of settings, including the standard dynamic bargaining setting with one sided private information. In the standard one-sided incomplete information model sans outside options, the seller's temptation to lower prices in the future prevents her from securing a high price in the present. In essence, the seller's present self, who wishes to service high valuations of the buyer, is engaged in a price competition with her future self, who services lower valuations that rejected high offers in the past. This dynamic, however, unravels with the introduction of outside options. If the buyer has an outside option (see Board and Pycia (2014)), lower valuations of the buyer opt out in equilibrium early in the game, eliminating the seller's temptation to lower prices in the future. If the seller has an outside option in a private values setting (see Fudenberg et al. (1987)), the seller's future self would rather exercise her outside option than service buyer types with low valuations. The seller's problem (which is reminiscent of standard commitment problems) is resolved either because the seller's future self has no control over terminating negotiations or when they do have control, the future and present selves agree on the value of terminating negotiations, thus aligning future and present incentives in that regard. In light of this argument, if the value of the outside option depends on the buyer's value, the seller's present and future incentives to exercise the outside option may possibly be misaligned. In this paper, I study the robustness of Coasian dynamics to the introduction of outside options in an interdependent values setting.

In particular, I study the following model: A seller makes sequential offers to a buyer who has private information about the value of the object. The seller has an outside option that she may exercise at any point during the game. The seller receives $\alpha < 1$ proportion (therefore, there are always gains from trade) of the value of the object if she exercises her outside option, while the buyer receives nothing. Following the literature, we study Weak Markov Equilibria of this game, where the seller's belief acts like a state variable.

There are many bargaining settings in which the buyer of the object knows more about the (common) value of the object that the seller. For instance, art dealers are likely to be more informed about the value of the artwork than the seller of the artwork. When the seller has an outside option in such settings, she learns about the value of the object *and* the value of her outside option through negotiations; an obstinate buyer makes the seller more pessimistic about the value of the object and consequently, the value of her outside option. In this paper, I study bargaining dynamics and outcomes that emerge in such settings in the frequent offer limit[1]. I further study how changes in the cost of exercising the outside option affects the seller's payoff.

I first show existence for large discount factors(Proposition 1). I then (uniquely) pin down equilibrium dynamics and the seller's payoff in the frequent offer limit as a function of α (Proposition 2). Finally, I characterize necessary and sufficient conditions on the prior that ensure non-monotonicity of the seller's payoff function when the buyer's valuation can take three values(Theorem 1).

The limit structure of equilibria pinned down in Proposition 2 have the same pattern for all values of α that put the ex-ante value of the outside option above the lowest valuation in the support of the value distribution. In equilibrium, an initial burst of trade is followed by immediate exit. The price in the initial burst of trade is (approximately) equal to the valuation of the marginal type that is excluded from trade when the seller subsequently exits. This equilibrium structure resembles the equilibrium dynamic in Fudenberg et al. (1987). However, in Fudenberg et al. (1987), the seller exercises her outside option when there are no gains from trade. When there are gains from trade in a private values setting, the seller never exercises her outside option. In an interdependent values setting, on the other hand, the seller may exercise her outside option even though there are gains from trade.

The seller exits in equilibrium with positive probability. Moreover, the seller's (limit)

[1] Essentially, as players become arbitrarily patient.

payoff from exercising her outside option is always weakly lower than her equilibrium payoff from negotiating with the buyer. Therefore, the seller exercises her outside option when she is exactly indifferent between exercising her outside option and continuing negotiations. We show, in Proposition 2 that such indifference points exist.

As a consequence of Coasian forces, the seller's payoff may not be monotonically increasing in α. Theorem 1 lays down a sufficient and necessary condition for three value distributions that obtains a non monotonic payoff function for the seller. We provide a brief intuition for why this may happen. The intuition is discussed in detail in Section 1.5.

The seller may get a higher payoff for lower values of α because she may be able to extract a higher price from the buyer when the value of α is low. For the purpose of providing a coarse intuition, consider a seller who is optimistic on day one, agnostic on day two and pessimistic on day three. The seller's ability to extract a high price from the buyer today depends on her willingness to exit tomorrow. Conversely, the seller's willingness to exit today depends on the price she is able to extract today, should she choose to trade. The key point of difference between trade dynamics for low and high values of α is that the seller may prefer to exercise her outside option on day three for high values of α, while she prefers to trade at a low price for low values of α. As a seller with a better outside option will exercise her outside option on day three when she is pessimistic, the buyer is willing to pay a high price on day two. Consequently, the seller is unwilling to exercise her outside option on day two. On the other hand, when the value of α is low, the buyer will not accept a high price on day two, anticipating the price to fall on day three. This could make it worthwhile for the seller to opt out of negotiations on day two. The same argument implies that in period one, the buyer may accept a high price on day one when the value of α is low but may not accept a high price when α is high.

At the heart of the seller's dilemma are familiar Coasian forces, that now act on two fronts. The seller simultaneously faces a commitment problem on two fronts. When the buyer rejects a price offer targeted at a particular type, the seller becomes more pessimistic about the value of the object. This tempts her to not only lower her price offers, but also weakens her incentive to exercise her outside option. An increase in α affects the seller's commitment problem in two ways. On the one hand, an increase in α alleviates the seller's commitment problem with respect to her pricing decision. On the other hand, if the seller has more control over her pricing decision, the commitment problem with respect to her exit decision is exacerbated, i.e., she is less willing to opt out. As a consequence, the seller's bargaining position may weaken with an increase in α.

The rest of the paper is organized as follows: Section 1.2 provides a literature review, Section 1.3 describes the model, Section 1.4 describes strategies and lays down the equilibrium concept, Section 1.5 provides an illustration, Section 1.6 and Section 1.7 explain the main results the final section concludes.

1.2 Related Literature

This paper is related to the literature on outside options in bargaining with one sided asymmetric information. The classic Coasian force, examined in Gul et al. (1986) (henceforth,GSW) and Stokey (1981) also manifests in our setting, albeit in two ways. In context of bargaining with one sided asymmetric information, the question of outside options has been studied extensively. Fudenberg et al. (1987) study a setting where the seller has an outside option that she may exercise at any time. In their analysis, they discuss how the self fulfilling nature of equilibria means that the seller could end up playing either soft or tough in equilibrium. Board and Pycia (2014) study a game where the buyer has the outside option as well as private information about his type. The Coase conjecture breaks down in this setting and the seller gets her commitment payoff. In contrast, our setting examines the commitment problem the seller faces with respect to exercising her outside option. Nava and Schiraldi (2019) study a setting where the seller sells two products.

The role of interdependent values with and without outside options has been studied in a variety of contexts. Deneckere and Liang (2006) (henceforth, DL) study a dynamic bargaining model in a lemons market and find that trade happens in 'bursts' punctuated by periods of delay. In our setting, the seller quits negotiations instead of delaying negotiations. Fuchs and Skrzypacz (2010) study a bilateral trading game where the seller has an outside option that arrives at some exogenous rate and ends the game. In their setting, the value of the outside option is correlated with the buyer's value for the object. They find that in an atomless stationary equilibrium, the seller's payoff is reduced to the payoff she would get from simply waiting for the outside option to arrive. As a consequence, the seller is indifferent between different rates of trade in equilibrium. A similar logic is applied to our setting. The option value in our setting lies in quitting negotiations altogether. While she is able to extract some rents in the first few periods, the seller's payoff is eventually reduced to the payoff from her outside option. However, the fact that the outside option can be exercised at will by the seller plays a crucial role in the non monotonicity of the seller's payoff in α. In Fuchs and Skrzypacz (2010), an increase in the exogenous arrival rate is unambiguously better for the seller as her payoff is exactly

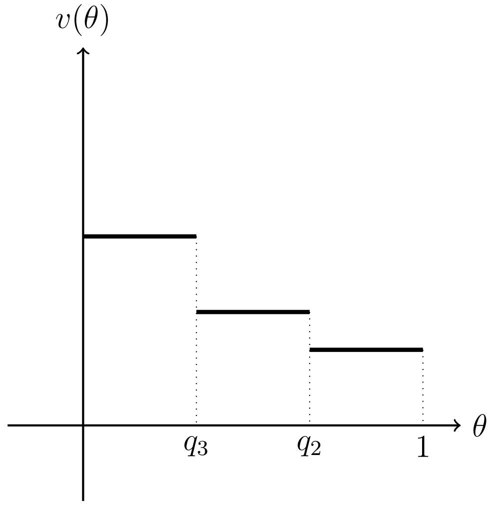

Figure 1.1: Example of value distribution

equal to the payoff she would get from waiting for the outside option to arrive. In Chaves (2019), negotiations between the seller and the buyer is observed by a third party that can endogenously disrupt negotiations. The paper examines the role of offer transparency in this setting. Daley and Green (2020) study trade dynamics in a lemons market when news arrives over time about the object's value. Similar to Fuchs and Skrzypacz (2010), they find that the buyer's payoff is reduced to the payoff from waiting to become sufficiently optimistic about the object's value. An endogenous interdependence arises in Ortner (2017) where the seller has stochastic cost. In this setting, the seller is tempted to lower her price and cater to low valued buyers as costs fall.

1.3 Model

Time is discrete and each period is of length Δ. We are interested in outcomes as Δ goes to zero. A seller negotiates with a buyer who is privately informed of her valuation. The buyer's valuation takes value in the set $\{v_1, v_2, v_3, ..v_N\}$ where $v_1 < v_2 < .. < v_N$. Following DL, we let the value of the buyer depend on the realization of a random variable $q \sim U[0, 1]$, i.e.,

$$v(q) = v_i \quad q \in (q_{i+1}, q_i] \tag{1.1}$$

for $i = 1, 2, ..N$, where $q_{N+1} = 0$ and $q_1 = 1$. We refer to the realization of q as the buyer's 'type'. Figure 1.1 illustrates the value distribution.

The seller has an outside option whose value depends on the buyer's valuation. In particular, if the buyer's type is q, the seller's payoff from opting out is $\alpha v(q)$. If the seller opts out, the buyer's payoff is zero.

The timeline of the game is as follows

- Seller decides whether or not to exit
- If seller doesn't exit, she makes an offer
- If an offer is made, buyer decides whether to reject or accept the offer

1.4 Strategies and Equilibrium

A public history consists of all past offers, i.e., a time t public history, denoted by h^t is the sequence of all offers made till period $t-1$. The set of all time t public histories is denoted by $\mathcal{H}^t$. Let p^t denote the price offer made at time t. The buyer's acceptance strategy at time t is a mapping from their type (i.e., the realization of q), time t public history and the current offer to a probability of acceptance. We denote the buyer's strategy by the function $\sigma^t : [0,1] \times \mathcal{H}^t \times \mathbf{R}_+ \to [0,1]$.

The seller makes two decisions-one with respect to her exit decision and the other with respect to the price offers. The seller's (pure) exit strategy is a mapping from a history to $\{0,1\}$ where 1 indicates exit. A pure offer strategy is a function $\sigma_p : \mathcal{H}^t \to \mathcal{R}_{++}$.

The skimming property is a standard result in the classic Coasian framework that shows that if lower valuation buyers accept a price, it must be acceptable to high valuation buyers. An implication of the skimming property is that in any PBE, the belief over the set of types following rejection of an offer is a right truncation of the prior belief. Thus, beliefs over types in each period can be summarized by a cutoff type at which truncation occurs. We show that the skimming property holds in any PBE.

Lemma 1. *Suppose $v_{\theta'} > v_\theta$ and suppose a positive mass of both types remain. If a type with valuation v_θ accepts an offer p with positive probability, then all types with valuation $v_{\theta'}$ accept p with probability 1*

Proof. Let σ_b' be any other arbitrary strategy for the buyer and let σ_s be the seller's strategy in equilibrium. By optimality, we have

$$v_\theta - p \geq E_{(\sigma_b', \sigma_s)}[\delta^t (v_\theta - p)] \tag{1.2}$$

Next, consider

$$\begin{aligned} E_{(\sigma_b', \sigma_s)}[\delta^t (v_{\theta'} - p)] - E_{(\sigma_b', \sigma_s)}[\delta^t (v_\theta - p)] \\ = E_{(\sigma_b', \sigma_s)}[\delta^t (v_{\theta'} - v_\theta)] < v_{\theta'} - v_\theta \\ \implies v_{\theta'} - E_{(\sigma_b', \sigma_s)}[\delta^t (v_{\theta'} - p)] > v_\theta - E_{(\sigma_b', \sigma_s)}[\delta^t (v_\theta - p)] \end{aligned} \tag{1.3}$$

Subtracting p on both sides, we get

$$v_{\theta'} - p - E_{(\sigma_b', \sigma_s)}[\delta^t (v_{\theta'} - p)] > v_\theta - p - E_{(\sigma_b', \sigma_s)}[\delta^t (v_\theta - p)] \geq 0 \tag{1.4}$$

where the last inequality follows from Equation (1.2). This implies that θ' accepts p with probability 1.

□

We consider a subclass of PBE called Weak Markov Equilibria (see Ausubel and Deneckere (1989),Fudenberg et al. (1985)). A PBE is a Weak Markov Equilibrium if the buyer's acceptance strategy depends only on the current price and the buyer's type (i.e., q) and the seller's offer and exit strategy depend only on the current belief and the seller's offer in the previous period. This class of equilibria has been studied extensively in the literature, is tractable and provide a natural point of comparison to standard Coasian results.

1.5 Illustration

In this section, we provide a heuristic argument for the non monotonicity of the seller's ex ante payoff. Suppose the buyer's valuation takes values in the set $\{v_1, v_2, v_3\}$ and suppose $\alpha < v_2/v_3$. We first examine when the seller offers v_1 and when she can get a higher price accepted with positive probability.

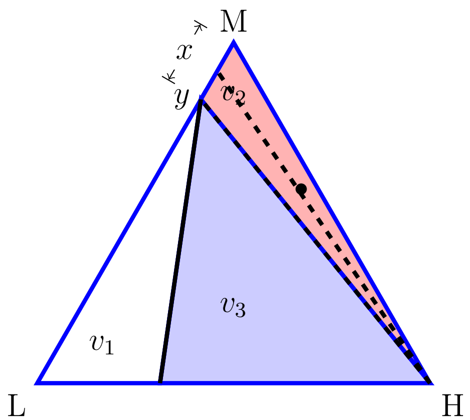

Figure 1.2: Illustration Figure 1

(a) The black dot represents the prior

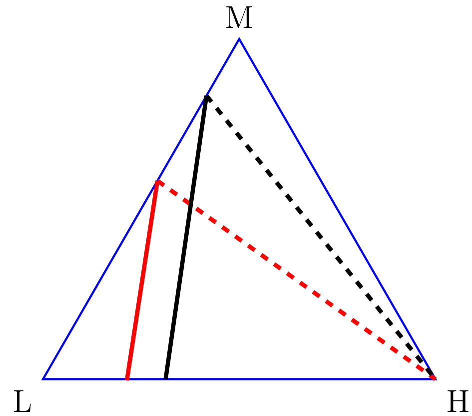

Figure 1.3: Illustration Figure 2

(a) Black lines correspnd to α_1 and red lines to α_2 where $\alpha_1 < \alpha_2$

Intuitively, the seller's ability to extract a high price depends on the credibility of her threat to exit if her offer is rejected. Figures 1.3a and 1.2a denote the belief simplex. The black dot denotes the prior belief. In both figures, v_1 is greater than the payoff from the outside option in the white region. Coasian forces operate in this region, which means that the seller offers v_1 almost immediately in the frequent offer limit. In the blue and pink regions, the seller would rather opt out than trade at v_1 with probability one. Along the solid black line, the seller is indifferent between trading at v_1 and exiting. We argue that the seller is able to extract a higher price with positive probability in the blue and pink regions. Further, the seller is able to trade at v_3 with positive probability in the blue region, but not the pink region.

The seller can extract v_3 from type H with positive probability subject to two restrictions– (1) the posterior upon rejection of v_3[2] satisfies Bayes' plausibility and (2) Type v_3 has incentive to accept the offer v_3, i.e., the seller does not strictly prefer to wait for lower future prices. The first point implies that in Figure 1.2a, the seller's posterior belief upon rejection must lie along the dashed black line. The second point implies that the seller must not lower prices in the future, which in turn implies that the seller exits with probability one in the future. In Figure 1.2a, the seller's posterior upon rejection must lie on the solid black line.[3]

[2]Note that v_3 is rejected with positive probability by the buyer. If the buyer is of types v_2 or v_1, she rejects the offer.

[3]There is one further step involved here, which is to show that the seller's posterior cannot lie in the interior of the blue or pink region upon rejection. Suppose the seller exits at some point in the interior

When the seller's prior lies in the blue region, there is exactly one posterior belief that satisfies the above conditions, denoted by the point of intersection between the solid black line and the dashed black line. It is, therefore, possible for the seller to extract price v_3 from type v_3 with positive probability in equilibrium.

We now turn our attention to the pink region. Since $\alpha < v_2/v_3$, the seller never opts out if she can trade at a price of v_2 with type v_3. Indeed, if the belief lies on the line segment labeled x, the seller can get v_2 to accept v_2 in equilibrium with positive probability since the point y satisfies Bayes' plausibility (which implies that the posterior must lie along the blue line joining the v_2 and L vertices) and v_2's IC constraint (which implies that the posterior must lie on the black solid line). By the skimming property, it must be the case that type v_3 accepts v_2 with probability one.

Since the seller never allows the price to fall to v_1 in the pink region, there are two possibilities: either the seller can get v_3 to accept v_3 with positive probability or the seller trades with type v_3 at price v_2.

Towards a contradiction, suppose there is a way to get type v_3 to accept v_3 with positive probability. Once again, the seller's posterior must lie at the intersection of the dashed black line and the solid black line. However, note that the dashed black line has no intersection with the solid black line in the pink region. Since, the seller can always charge v_2 in the pink region, there is no incentive to opt out at any point in this region. The seller cannot credibly follow through on her threat to opt out of negotiations after an offer of v_3. Therefore, the seller's initial price offer is v_2 in equilibrium.

Why is the seller unable to extract v_3 from type v_3 in the pink region? The reason is that the presence of type v_2 acts as a sort of buffer against lowering prices to v_1. When the belief assigned to type v_2 isn't sufficiently high, the seller knows that if she stays in negotiations for too long, she is likely to lower prices quickly. Given her (future) temptation to lower prices, she preemptively opts out while she is still optimistic. This enables her to extract v_3 (with some probability) from type v_3. In the pink region, the probability assigned to type v_2 is sufficiently high that the seller is not tempted to lower prices to v_1,

of the blue region, where the seller strictly prefers to opt out. If, at this point the seller makes an offer that concedes some small rents to type v_3, it has to be accepted with positive probability, which in turn implies that the seller does not strictly prefer to opt out.

regardless of type v_3's acceptance probability. In fact, even if v_3 accepts v_3 with probability one, the posterior upon rejection assigns a very high probability to type v_2, which prevents the seller from lowering prices. Consequently, the seller is able to extract price v_2 from type v_2 before exiting. The seller's temptation to trade with type v_2 adversely affects her ability to trade with v_3 at price v_3.

We shall now see what may happen when α increases. Suppose the initial value of α is α_1 and we increase α to α_2. This makes the price v_1 unacceptable to the seller for a larger set of priors. The contraction of the white region is illustrated in Figure 1.3a. The solid black line is the indifference set associated with α_1 and the red line is the indifference set associated with type α_2. Another consequence of an increase in α is the expansion of the pink region at the expense of the blue region. Therefore some priors that lay in the blue region under α_1 lies in the pink region under α_2. Consequently, there is a fall in the initial price offer. However, since the initial price offer is lower, there is also a corresponding increase in the probability of trade. Since the dashed line shifts continuously with α, for small changes in α, the fall in price dominates the increase in trade probability, causing a fall in the seller's payoff. An increase in α improves the marginal value of the outside option given the expected value of the object. This means that for a larger region of priors that assign a sufficiently high probability to type v_2, v_1 is an unacceptable trade price. This consequently guarantees that the seller can get the buyer to accept v_2 with positive probability later in the game, which in turn adversely affects her ability to charge v_3 earlier in the game. The seller is able to extract v_3 from type v_3 because she can credibly exit in the future before lowering prices any further and she can credibly exit in the future because she lacks confidence in her future conduct, i.e., she may hastily lower prices. When α increases, the seller does not run the risk of lowering prices as long as the belief assigns a sufficiently high probability to type v_2. This means that the seller can always get the buyer to accept v_2 with positive probability. The seller, thus, settles for the lower, albeit, guaranteed price of v_2.

1.6 Existence and Uniqueness of Limit Outcomes

We first present the existence results for the general model for small values of Δ. Aside from some details, the proof proceeds as the existence proof in DL.

Proposition 1. ***[Existence for small*** Δ***]*** *There exists* $\Delta' > 0$ *s.t. for* $\Delta < \Delta'$*, there exists an equilibrium.*

The proof is constructive and as in DL, we 'backward induct' the seller's payoff function, the buyer's reservation price strategy and the induced state in the next period. The induction process stops at state zero or when the seller's payoff is equal to her payoff from the outside option, whichever happens earlier. Our assumption that Δ is small is key at this point. When Δ is small, the seller never makes a price offer that induces a state above the point of indifference. This allows us to repeat the induction process for states below the indifference point in the same manner.

Proposition 2. ***[Uniqueness of Limit Outcomes]*** *As Δ goes to zero, the seller's payoff is uniquely pinned down.*

We outline the proof for uniqueness of limit outcomes below. Suppose $W(q) = E[v(x)|x \geq q]$. Let $\overline{q}_1$ be such that

$$v_1 = \alpha W(\overline{q}_1) \tag{1.5}$$

and $\overline{q}_i$ is s.t.

$$\alpha W(\overline{q}_i) = v_i \frac{\overline{q}_{i-1} - \overline{q}_i}{1 - \overline{q}_i} + \frac{1 - \overline{q}_{i-1}}{1 - \overline{q}_i} \alpha W(\overline{q}_{i-1}) \tag{1.6}$$

for $i \leq N$, where $\overline{q}_i \in [0, 1]$. Intuitively, $\overline{q}_i$ represent points at which the seller's incentive to lower prices is matched by her incentive to opt out. If the seller initially prefers to opt out than lower prices, she will continue to stay in only if she anticipates that her future selves will not lower prices too quickly. However, upon rejection of her offers, the seller becomes pessimistic over time, which in turn makes her more willing to lower offers and less inclined to opt out. In equilibrium, the seller's future incentive to lower prices must match her willingness to opt out. In order to rationalize trade in the current period, the seller must opt out in the future with sufficiently high probability.

The proof involves three steps.

Step 1: Whenever $\overline{q}_i$ is between zero and one, for Δ small enough, there exists $\overline{q}_i(\Delta)$ where the seller is indifferent between continuing trade and opting out. As Δ goes to zero, $\overline{q}_i(\Delta)$ converges to $\overline{q}_i$.

Step 2: When Δ is small, if $\overline{q}_{i+1}(\Delta) < q < \overline{q}_i(\Delta)$, then the induced belief in the next period is atmost $\overline{q}_i(\Delta)$.

Step 3: When $\overline{q}_{i+1}(\Delta) < q < \overline{q}_i(\Delta)$, the price charged converges to $v(\overline{q}_i(\Delta))$ as Δ goes to zero.

Let $V(\alpha)$ denote the unique limit ex-ante payoff of the seller as a function of α. We first note that $V(\alpha)$ is continuous almost everywhere. If $V(.)$ is decreasing at α, there are two possibilities-(1) $V(.)$ is discontinuous at α or (2) $V(.)$ is continuous at α. Regardless of whether $V(.)$ is continuous or discontinuous at the point at which it decreases, we have that for an interval $[\underline{\alpha}, \overline{\alpha}]$, there exists $a < \underline{\alpha}$ s.t. $V(a) > V(x)$, for every $x \in [\underline{\alpha}, \overline{\alpha}]$, i.e., a lower α yields a higher payoff. In the next section, we examine sufficient and necessary conditions under which the seller's payoff may be non-monotonic in the three values case.

1.7 Necessary and Sufficient Condition: Three Values Case

Let $\beta_{ij} = v_i/v_j$. We assume that $v_1/W(q_3) \neq \beta_{23}$. We first describe a condition on the prior.

Definition 1. *The prior is said to exhibit* ***median prominence*** *if*

$$\frac{v_1}{W(q_3)} < \frac{1-\beta_{13}}{2-\beta_{23}-\beta_{12}} \tag{1.7}$$

We now state the main result:

Theorem 1. ***[Decreasing Payoffs]*** *For any prior distribution there exists $\alpha \in (0,1)$ s.t. $V(.)$ is decreasing at α iff the prior exhibits median prominence*

The fundamental tradeoff that the seller faces is encapsulated in the relation between $\overline{q}_1$ and $\overline{q}_2$. Ideally, the seller would prefer higher both values to be as high as possible, as an increase in these values increase the probability of trade. However, changes in α affect $\overline{q}_1$ and $\overline{q}_2$ differently: an increase in α increases the former quantity and but may decrease the latter in equilibrium.

The explanation for the increase in $\overline{q}_1$ is straightforward. An increase in α increases the value of the outside option at the original value of $\overline{q}_1$: the value of the outside option is now strictly greater than v_1 when the state is $\overline{q}_1$, i.e., the left hand side of Equation (1.5) is strictly lower than the right hand side. In order to offset the increase in α

in Equation (1.5), the *expected* value of the object must fall and so the new $\overline{q}_1$ must be larger. Therefore, an increase in α increases the probability of trade with the middle type.

Why may an increase in α cause $\overline{q}_2$ to fall? The value $\overline{q}_2$ represents the state at which the seller is indifferent between opting out and offering v_2, which in equilibrium is accepted by all types between $\overline{q}_2$ and $\overline{q}_1$. If $\overline{q}_1$ increases, the seller gets a higher payoff from trade and so is less willing to exit at $\overline{q}_2$. This inverse relation between $\overline{q}_1$ and $\overline{q}_2$ can be obtained from Equation (1.6). Upon manipulating Equation (1.6), we get

$$(\alpha v_3 - v_2)(q_3 - \overline{q}_2) = v_2(1-\alpha)(\overline{q}_1 - q_3) \tag{1.8}$$

Fixing α, we depict the inverse relationship between $\overline{q}_1$ and $\overline{q}_2$ in Figure 1.5a by the black solid line. We will refer to this line as the $\overline{q}_2,\overline{q}_1$ feasibility line, as it pins down feasible (i.e., the seller can credibly take the outside option at $\overline{q}_2$) $\overline{q}_2$ as a function of $\overline{q}_1$. An increase in α shifts the line up, depicted by the dashed black line. The exact value of $\overline{q}_1$ is determined independently of $\overline{q}_2$, depicted in Figure 1.5a by the vertical solid red line. An increase in α shifts the red line to the right (depicted by the dashed red line). The point of intersection of the solid(dashed) red line and the solid (dashed) black line pins down the value of $\overline{q}_2$ in equilibrium. If the upward shift in the black line is smaller than the shift in the red line, $\overline{q}_2$ drops with an increase in α[4].

We now trace the seller's indifference curves in the $\overline{q}_1$,$\overline{q}_2$ plane. The seller's ex-ante payoff is given by

$$v_3\overline{q}_2 + v_2(\overline{q}_1 - \overline{q}_2) + v_1(1-\overline{q}_1) \tag{1.9}$$

Therefore, the slope of the indifference curve is given by $-\dfrac{v_2 - v_1}{v_3 - v_2}$ while the slope of the $\overline{q}_2,\overline{q}_1$ feasibility line is $-\dfrac{v_2(1-\alpha)}{\alpha v_3 - v_2}$.

Now, we examine the median prominence condition. Consider an α between $v_1/W(q_3)$ and $\dfrac{1-\beta_{13}}{2-\beta_{23}-\beta_{12}}$[5]. Rearranging $\alpha < \dfrac{1-\beta_{13}}{2-\beta_{23}-\beta_{12}}$, we get

$$-\frac{v_2 - v_1}{v_3 - v_2} > -\frac{v_2(1-\alpha)}{\alpha v_3 - v_2} \tag{1.10}$$

[4]This may happen when α is close to $v_1/W(q_3)$

[5]Provided $v_1/W(q_3) > v_2/v_3$

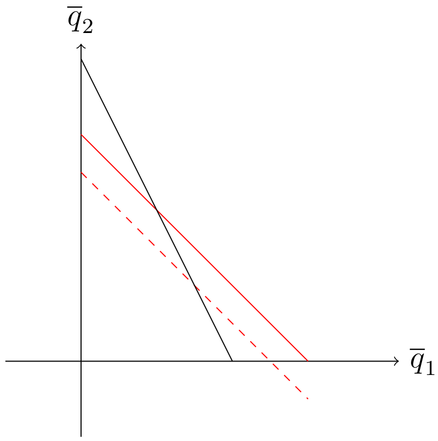

Figure 1.4: Theorem 1 Illustration 1

(a) The red lines are indifference curves and the black line is the $\overline{q}_2, \overline{q}_1$ feasibility line. The dashed line corresponds to a higher value of α

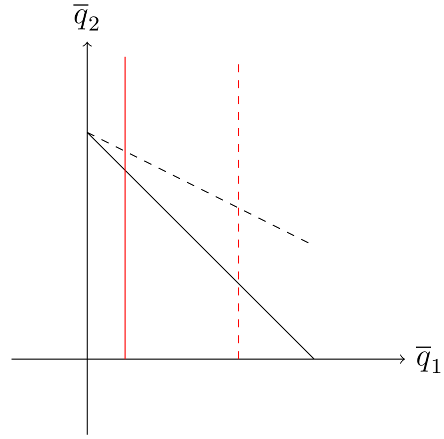

Figure 1.5: Theorem 1 Illustration 2

(a) The black lines are $\overline{q}_2, \overline{q}_1$ feasibility lines. The red lines mark the value of $\overline{q}_1$ associated with each α. The dashed lines correspond to a higher value of α

i.e., the condition implies the existence of an α that makes the slope of the $\overline{q}_2, \overline{q}_1$ feasibility line steeper than the slope of the indifference curve. This is depicted in Figure 1.4a.

When α is close to $v_1/W(q_3)$, both $\overline{q}_1$ and $\overline{q}_2$ are close to q_3, and so the benefit to be had from an increase in α is a relatively small quantity compared to the loss the seller bears from a decrease in $\overline{q}_2$ due to an increase in $\overline{q}_1$. The slope of the $\overline{q}_2, \overline{q}_1$ feasibility line therefore is a good approximation of the effect on $\overline{q}_2$ of a marginal change in α when α is close to $v_1/W(q_3)$. When the $\overline{q}_2, \overline{q}_1$ feasibility line is steeper than the indifference curve, the increase in payoff from an increase in $\overline{q}_1$ does not compensate for the loss in payoff from a fall in $\overline{q}_2$ and so, the seller's payoff falls.

Before we proceed to the next section, we state a sufficient condition for non-monotonicity of the seller's payoff in the general N value case:

Proposition 3. *$V(.)$ is non-monotonic in α if*

$$\frac{v_1}{W(q_N)} < \frac{v_{N-1}}{v_N}$$

1.8 Discussion

In this paper, I study a bilateral trade setting where the buyer has private information about the value of the object and the seller has an outside option whose value depends on the value of the object. In particular, the value of the outside option is $\alpha < 1$ times the value of the object. I find that under some conditions, an increase in the value of α may hurt the seller. In the three value case, the seller's limit payoff is non monotone in the scale parameter α if and only if the prior assigns a sufficiently high probability to the middle type. I call this condition median prominence. Intuitively, when the middle type occurs with a sufficiently high probability, the seller is able to extract surplus from the middle type as α increases. This makes her unwilling to exclude the middle type from trade, compromising her ability to extract surplus from the high type.

This paper highlights a potential risk involved in improving the value of outside options in interdependent value settings. As long as the seller is unable to resume negotiations with the buyer once she has walked away, the main result suggests the existence of a force that adversely affects the seller's ability to exercise her outside option when the value of her outside option improves.

This result also has potential policy implications in labor market settings. A recent report on the state of labor market competition (Department of Treasury (2022)) cites workers' "informational disadvantage relative to firms" as a source of market power for firms that hire them. They point out that workers often do "not [know] what other, similarly placed workers earn, the competitive wages for their labor, or the existence of workplace problems like discriminatory conduct or unsafe working conditions". Further, "workers also may have a limited or no ability to switch locations and occupations quickly and may lack the financial resources to support themselves while they search for jobs that pay more and better match their skills and abilities". To summarize the concerns raised by the report, not only do firms have more information than workers, but workers may also be constrained in their ability to switch jobs. Policy makers motivated to alleviate this problem may be inclined to implement policies that improve workers' outside option (for eg., policies that effectively reduce search costs)[6]. In this paper, I argue that certain interventions targeted at improving outside options may make the party with the exit option (workers, in this case) worse off.

The non monotonicty of the seller's payoff in α is a manifestation of familiar Coasian forces in a setting with outside options: the seller's temptation to extract surplus from the buyer prevents her from breaking off negotiations. This problem is, in fact, so severe that the seller's payoff falls in spite of a decrease in the cost of exercising the outside option.

[6] Indeed, the report proposes measures that promote competition among firms and improve job mobility

1.9 Proofs

1.9.1 Existence

Stationary Triplet Construction

Lemma 2. *If $v_1 > \alpha W(q)$, then the seller never opts out in any PBE*

Proof. Consider any history h at which $q(h) \geq q_2$, where $q(h)$ is the cutoff at history h. First we show that in any equilibrium, type L's payoff is 0. Suppose not. Let Π denote the set of equilibrium payoffs for L when the belief cutoff is q where $q \geq q_2$ and let π denote the supremum of Π(which exists because prices are bounded below by 0 and so payoff is bounded above by v_1). For any $x > 0$ s.t. $x \in \Pi$, it must be the case that the seller trades with positive probability in some equilibrium (σ, μ) (otherwise the buyer's payoff is 0). The seller's payoff associated with this equilibrium is bounded above by $v_1 - x$ (since the total surplus is v_1 and the buyer gets x). It must then be the case that $v_1 - x \geq \alpha v_1$ (since the seller trades with positive probability). Since this must hold for all $x \in \Pi$, $v_1 - \pi \geq \alpha v_1$. Suppose the seller offers $v_1 - \delta\pi - \varepsilon$ where $\varepsilon < (1-\delta)\pi$. Since $v_1 - (v_1 - \delta\pi - \varepsilon) > \delta\pi$, the buyer accepts the offer with probability 1 in any equilibrium. Also, $v_1 - \delta\pi - \varepsilon > v_1 - \pi \geq \alpha v_1$, so having this offer accepted is better than opting out. Since $v_1 - \delta\pi - \varepsilon > v_1 - x$, the seller has a profitable deviation as the seller would rather offer $v_1 - \delta\pi - \varepsilon$ when $x \in (\delta\pi + \varepsilon, \pi]$. But then this means that for all $x \in (\delta\pi + \varepsilon, \pi]$, $x \notin \Pi$. This implies that there exists $0 < \varepsilon' < (1-\delta)\pi - \varepsilon$ s.t. for all $x \in \Pi$, $x < \pi - \varepsilon'$. This in turn implies that π cannot be the supremum of Π, a contradiction. So, type L's payoff is always zero.

Next, exercising the outside option cannot be optimal since an offer of $v_1 - \varepsilon > 0$ is always accepted with probability 1 by the buyer. For ε small enough, the seller would prefer to make this offer than exercise her outside option.

Therefore, there exists an equilibrium in which v_1 is offered and accepted by the buyer with probability 1 and this is the unique equilibrium when $q \geq q_2$.

Now suppose $q < q_2$ and $v_1(1-q) > \alpha W(q)$. If an offer is accepted by L with positive probability (not equal to 1) in equilibrium, this offer must be v_1. Owing to the skimming property, if type L accepts an offer with positive probability, type M accepts it with probability 1. This means that the next period's cutoff is greater than q_2 and the offer is v_1. This means that the offer that is accepted with positive probability is also v_1. Therefore, if an offer less than v_1 is made in equilibrium, it must be accepted by all types with probability 1. Suppose there is an equilibrium in which the seller offers $p < v_1$ and L accepts this offer with probability 1. Let p^* be the infimum of such offers. Now suppose the seller offers $\delta p^* + (1-\delta)v_1 - \varepsilon$ where $\varepsilon < (1-\delta)(v_1 - p^*)$. The buyer would accept this price with probability 1 since $v_1 - (\delta p^* + (1-\delta)v_1 - \varepsilon) = \delta(v_1 - p^*) + \varepsilon > \delta(v_1 - p^*)$[7] and the seller gets a payoff strictly higher than p^*. So any price offer between $[p^*, \delta p^* + (1-\delta)v_1 - \varepsilon)$ is not optimal. This contradicts the fact that $p^* < v_1$ is the infimum of all price offers that are accepted with probability one. Therefore, type L's payoff in equilibrium is always 0 and an offer of v_1 is always accepted in any equilibrium(the same argument above implies that the seller does not opt out). And since v_1 is always accepted by all types in equilibrium, opting out is suboptimal in any equilibrium. □

Lemma 3. *There exists $q_1 < q_2$ s.t. for all $q > q_1$, it is optimal to offer v_1 in any PBE*

Proof. First we show that the optimal commitment payoff in a setting without outside options forms an upper bound to equilibrium payoffs for the seller in this setting when q is close enough to q_2. For q close enough to q_2, $v_1 > \alpha W(q)$ and so the seller never opts out (by the previous Lemma, the seller could get a higher payoff from offering v_1). Consider an arbitrary equilibrium (σ, μ). Since the seller never opts out following any history at which the state is q, the equilibrium outcome is feasible in the commitment setting without outside options. Therefore, for q close enough to q_2, the commitment payoff is an upper bound. Since the optimal mechanism is a posted price mechanism and since the optimal price is v_1, the commitment payoff can be obtained by offering v_1 (which, as we noted in the previous Lemma, is accepted with probability 1 by the buyer). In particular, the optimal offer with commitment is v_1 iff $q > \hat{q}_1$ for some $\hat{q}_1 > 0$. Let $q_1' := \max\{\hat{q}_1, \hat{q}_2\}$ where $v_1(1 - \hat{q}_2) = \alpha W(\hat{q}_2)$. For $q > q_1'$, the outside option is never invoked and the optimal offer price with commitment is v_1 and so v_1 is offered in any equilibrium. □

Lemma 4. *[ADAPTED FROM DL] $P(q) = v(q)(1-\delta) + \delta v_1$ is a reservation price strategy*[8] *on $(q_1', q_2]$*

Proof. Suppose $p > P(q)$ is offered and accepted by all types $q' > q$. In the next period, the price is v_1 (since the state is atleast $q > q_1'$). So, $v(q') - p = v(q') - P(q) + [P(q) - p] =$

[7]Although p^* is the infimum of offers at any history where the belief cutoff is q, the buyer cannot (a) accept the proposed alternative with probability 1(only the offer v_1 is accepted with positive probability) or (b) reject it with probability one (the payoff in the next period is atmost $v - p^*$)

[8]see Gul et al. (1986)

$\delta(v(q')-v_1)-(1-\delta)(v(q)-v(q'))+(P(q)-p) < \delta(v(q')-v_1)$, which gives us a contradiction.

Suppose $p < P(q)$ is offered and rejected by some $r < q$. Next period price offer is atleast v_1 and so $v(q) - p > v(q) - P(q) = \delta[v(q) - v_1]$, a contradiction. $\square$

Following DL, given a left continuous, weakly decreasing function $P(.)$, we define

$$V(q) = \max_{z \geq q} P(z)\frac{z-q}{1-q} + \delta V(z)\frac{1-z}{1-q} \tag{1.11}$$

$$t(q) = \min \arg\max_{z \geq q} P(z)\frac{z-q}{1-q} + \delta V(z)\frac{1-z}{1-q} \tag{1.12}$$

We say that $(V(.), P(.), t(.))$ is a *consistent triplet* if $P(.)$ is non increasing and left continuous and given $P(.)$, $V(.)$ and $t(.)$ satisfy Equation (1.11) and Equation (1.12) respectively. We say that a stationary triplet $(V(.), P(.), t(.))$ is consistent with $P(.)$ if $(V(.), P(.), t(.))$ is a consistent stationary triplet.

We say that a consistent triplet is *generated* by a stationary equilibrium if there exists a stationary equilibrium s.t. (1) the buyer follows a reservation price strategy $P(.)$ (2) the seller's equilibrium payoff is given by $V(.)$ and (3) the induced state in the next period given today's state q is $t(q)$.

We show that given the reservation price strategy $P(.)$ on $[q_1', 1]$ there exists a stationary triplet on $[q_1', 1]$.

Lemma 5. *Given the reservation price strategy* $P(q) = (1-\delta)v(q) + \delta v_1$ *on* $(q_1', 1]$*, there exists a stationary triplet on* $(q_1', 1]$ *that is consistent with* $P(.)$ *and generated by a stationary equilibrium.*

Proof. Define

$$V_1(q) = \max_{z \in [q, q_2]} P(z)\frac{z-q}{1-q} + \delta\frac{1-z}{1-q}v_1 \tag{1.13}$$

and

$$t_1(q) = \arg\max_{z \in [q, q_2]} P(z)\frac{z-q}{1-q} + \delta\frac{1-z}{1-q}v_1 \tag{1.14}$$

and let $V(q) = v_1$ and $t(q) = 1$. We show that $(V(.), P(.), t(.))$ constitute a consistent triplet.

We first note that $V_1(q_1') < v_1$. This follows directly from the definition of q_1'. By Lemma 2, v_1 is accepted with probability one and consequently $t(q) = 1$. This means that the triplet is consistent with $P(.)$.

Next, we show that $(V(.)P(.), t(.))$ is generated by a stationary equilibrium.

Consider the following strategies for players:

- Seller:The seller offers v_1 when $q \in (q_1, 1]$
- Buyer: Buyer follows the reservation price strategy $P(.)$

We note that the above strategies constitute an equilibrium. Suppose $q > q_1$. From the fact that $V_1(q) < v_1$, it is optimal for the seller to offer v_1, if v_1 is accepted with probability one. Since $P(q) > v_1$ for $q \in [q_1, q_2]$ and $P(q) = v_1$ for $q \in [q_2, 1]$, it follows that v_1 is indeed accepted with probability one. From Lemma 4, it follows that the buyer's strategy is also optimal. Furthermore, the seller's payoff in this equilibrium is v_1 and $t(q) = 1$. Thus, the triplet is generated by a stationary equilibrium.

□

Suppose there exists a consistent stationary triplet $(V(.), P(.), t(.))$ on $(q, 1]$. Define[9]

$$G(x; q) \equiv \max_{y \geq q} P(y)\frac{y-x}{1-x} + \delta V(y)\frac{1-y}{1-x} - \alpha W(x) \tag{1.15}$$

and

$$x(q) := \max\{0, \max\{x \leq q | G(x; q) = 0\}\} \tag{1.16}$$

We next show that the stationary triplet on $(q, 1]$ can be extended either (1) over the entire unit interval or (2) until some point q at which $x(q) = q$. But first, we prove a result which shall help us establish the continuity of the value function. We first state a preliminary result from Ausubel and Deneckere (1993) that shall help us prove this result.

Lemma 6. *[Theorem 2,Ausubel and Deneckere(1993)] Let X be a regular topological space and let Λ be a topological space, and $\gamma : \Lambda \to X$ be a u.h.c correspondence that is non-empty and compact valued. Suppose $f : X \times \Lambda \to \mathbf{R}$ is a u.s.c function and $\Pi : \Lambda \to \mathbf{R}$ a*

[9] If the set $\{x \leq q | G(x; q) = 0\}$ is empty, set $x(q) = 0$

l.h.c correspondence. Then (a) $M(\lambda) = \max_{x\in\gamma(\lambda)} f(x;\lambda)$ *is a continuous function and (b)* $m(\lambda) = \arg\max_{x\in\gamma(\lambda)} f(x;\lambda)$ *is a non-empty and compact valued, u.h.c correspondence.*

Where $\Pi(\lambda) := \{y | y \leq f(x;\lambda), x \in \gamma(\lambda)\}$.

Define

$$V_1(q) = \max_{x\geq q} P(x)\frac{x-q}{1-q} + \delta V(x)\frac{1-x}{1-q} \tag{1.17}$$

$$\mathcal{T}_1(q) = \arg\max_{x\geq q} P(x)\frac{x-q}{1-q} + \delta V(x)\frac{1-x}{1-q} \tag{1.18}$$

Lemma 7. *Suppose* $V(.)$ *is a continuous function and* $P(.)$ *is left continuous and weakly decreasing. Then,* $V_1(.)$ *is continuous and* $\mathcal{T}_1(.)$ *is non empty, compact valued and a u.h.c correspondence*

Proof. We show that the conditions of Lemma 6 are satisfied. Let $f(x;q) := P(x)\frac{x-q}{1-q} + \delta V(x)\frac{1-x}{1-q}$ Define $\gamma(q) \equiv [q, 1]$.

That $\gamma(q)$ is u.h.c is a standard result (see for eg., pg 59, Stokey et al. (1989)). We note that $P(.)$ is u.s.c. Let $x_n \to x$ be an increasing sequence, then $\lim_{n\to\infty} P(x_n)$ is a decreasing sequence and by left continuity of $P(.)$, we have that $\lim_{n\to\infty} P(x_n) = P(x)$. Suppose $x_n \to x$ is a decreasing sequence, then since $P(x_n)$ is weakly increasing, we have that $\lim_{n\to\infty} P(x_n) \leq P(x)$. Thus $P(.)$ is u.s.c. Since $V(.)$ is continuous and $P(.)$ is u.s.c, and since $f(.,.)$ is the sum of two upper continuous functions, it is u.s.c.

Next, we show that $\Pi(.)$ is l.h.c. Suppose $y \in \Pi(x)$ and $x_n \to x$. By definition, there exists $z \in [x, 1]$ s.t. $y \leq f(z;x)$. Suppose $z > x$. Then there exists N s.t. $z \in [x_n, 1]$ for all $n > N$. Since $f(z;.)$ is continuous, we have that $y_n \equiv f(z;x_n) \in \Pi(x_n)$[10] converges to $f(z;x) \geq y$. Similarly, if $z = x$, $f(z;x) = \delta V(x)$. Since $V(.)$ is continuous, taking the sequence $x_n \in [x_n, 1]$ gives us our result.

Applying Lemma 6 gives us our result. □

We show that $\mathcal{T}_1(.)$ is an increasing correspondence, i.e., if $x > y$ and $a \in \mathcal{T}_1(y)$, then $a \leq b$ for all $b \in \mathcal{T}_1(x)$.

[10] For n large enough

Lemma 8. *If $x > y$ and $a \in \mathcal{T}_1(y)$, then $a \geq b$ for all $b \in \mathcal{T}_1(x)$.*

Proof. Let $x > y$. First note that for any $a > x$,

$$f(a;y) = P(a)\frac{x-y}{1-y} + \frac{1-x}{1-y}f(a;x)$$
$$\implies f(a;y)(1-y) - f(a;x)(1-x) = P(a)(x-y)$$

And so if $x < a < b$,

$$f(a;y)(1-y) - f(a;x)(1-x) = P(a)(x-y) \geq P(b)(x-y) \tag{1.19}$$
$$= f(b;y)(1-y) - f(b;x)(1-x) \tag{1.20}$$

So, if $(f(a;x) - f(b;x))(1-x) \geq 0$, then $(f(a;y) - f(b;y))(1-y) \geq 0$, which gives us our result.

□

Given some stationary triplet $(V(.), P(.), \mathcal{T})$, if $V(.)$ is continuous and $\mathcal{T}$ is uhc, compact valued and increasing, we say that the stationary triplet is *continuous*. In the next lemma, we show that a continuous stationary triplet defined on an interval can be extended to a larger interval.

Lemma 9. *Suppose there exists a continuous stationary triplet on $[q_n, 1]$ and suppose $V(q) > \alpha W(q)$ for $q \in [q_n, 1]$. Then there exists $q_{n+1} < q_n$ s.t. a continuous stationary triplet exists on $[q_{n+1}, 1]$ with the property that $V(q) \geq \alpha W(q)$ for all $q \in [q_{n+1}, 1]$ and $V(q) > \alpha W(q)$ for $q \in (q_{n+1}, 1]$.*

Proof. Note that $G(x; q_n)$ is continuous in x and if $G(q_n; q_n) > 0$, and if for some $x < q_n$, $G(x; q_n) < 0$, then there exists $x' \in (x, q_1)$ s.t. $G(x'; q_1) = 0$. Let $\{P(.), V(.), t(.)\}$ denote a stationary triplet on $[q_n, 1]$, where $q_n > x(q_n)$. Let

$$V_1(q) = \max_{x \geq q_n} P(x)\frac{x-q}{1-q} + \delta V(x)\frac{1-x}{1-q} \tag{1.21}$$

$$V_2(q) = \max_{x \in (q, q_n]} P_1(x)\frac{x-q}{1-q} + \delta V_1(q)\frac{1-x}{1-q} \tag{1.22}$$

Where $P_1(x) = \delta P(t_1(x))+(1-\delta)v(x)$ and $t_1(x) = \min \mathcal{T}_1(x) \equiv \arg\max_{x \geq q_n} P(x)\dfrac{x-q}{1-q}+ \delta V(x)\dfrac{1-x}{1-q}$.

Next, let $q_{n+1} := \max\{q \in [x(q_n), q_n] | V_1(q) \leq V_2(q)\}$[11]. So for all $q > q_{n+1}$, $V_1(q) > V_2(q)$. It is easy to see that $q_{n+1} < q_n$. Let $(V_1(.), P_1(.), t_1(.))$ be the candidate stationary triplet on $[q_{n+1}, q_n]$. We consider two cases:

Case I** ($q_{n+1} > x(q_n)$)**: Let $V(.) = V_1(.)$,$P(.) = P_1(.)$ and $t(.) = t_1(.)$ on $[q_{n+1}, q_n]$. We show that $(V(.), P(.), t(.))$ is a continuous and consistent triplet.

That $(V(.), P(.), t(.))$ is continuous follows from Lemma 7. Given $P(.)$, for all $q \in (q_{n+1}, q_n]$, $V_1(q) > V_2(q)$. This follows from the definition of q_{n+1}. Consequently, $t_1(q)$ as defined above is an optimizer.

Further, note that for $q < q_{n+1}$, we must have that $V_2(q) \geq V_1(q)$. We omit the proof of this fact as it is identical to the proof in DL.

Note that since $q_{n+1} > x(q_n)$, we have that $G(q_{n+1}; q_{n+1}) > 0$, i.e., $V(q_{n+1}) > \alpha W(q_{n+1})$. By definition of $x(q)$, $G(q_{n+1}; q_n) > 0$ which implies that $G(q_{n+1}; q_{n+1}) > 0$. Consequently, $q_{n+1} > x(q_{n+1})$.

Case II**($q_{n+1} = x(q_n)$)**: Note that when the buyer follows the reservation price strategy given by $P(.)$, we once again have that $V_1(q) > V_2(q)$ for $q \geq q_{n+1}$. Consequently, $V(q) = V_1(q)$ for $q > q_{n+1}$ and so, if $G(q_{n+1}; q_n) = 0$, this implies that $G(q_{n+1}; q_{n+1}) = 0$.

We now show that either (1) for some finite k, $q_k = x(q_k)$ or (2) there exists a finite k s.t. $q_k \leq 0$. Suppose (1) doesn't hold and suppose there exists a sequence $\{q_k\}_{k=1}^{\infty}$ with $q_k > x(q_k)$ and $q_k \to q^*$.

Note that

[11] If the set is empty, then set $q_{n+1} = 0$

$$
\begin{aligned}
V(q_{k+2}) &= P(t(q_{k+2}))\frac{t(q_{k+2}) - q_{k+2}}{1 - q_{k+2}} + \delta V(t(q_{k+2}))\frac{1 - t(q_{k+2})}{1 - q_{k+2}} \\
&< v_3 \frac{q_k - q_{k+2}}{1 - q_{k+2}} + \delta V(q_{k+2}) \\
&\implies V(q_{k+2})(1-\delta) < v_3 \frac{q_k - q_{k+2}}{1 - q_{k+2}}
\end{aligned}
$$

Where the first inequality follows from the fact that $V()$ is decreasing, $P(t(q_{k+2})) < v_3$(since it is accepted with positive probability) and $q_k \geq t(q_{k+2})$. Since $V(q) > 0$ for any $q \in [0, 1]$ (since the seller could simply offer v_1), the left hand side of the inequality is strictly bounded away from zero, even as $k \to \infty$. But this contradicts the fact that q_k is convergent. Thus, we have that iterations end after $k < \infty$ rounds. Therefore, after $k < \infty$ rounds, $q_k = 0$ or $q_k = x(q_k)$.

□

Let $q_1(\Delta)$ be such that $x(q_1(\Delta)) = q_1(\Delta)$. Note that the construction above is valid for any $\Delta > 0$. We now show that for Δ small enough, (1) $q_k = x(q_k)$ for $k < \infty$ and (2) it is possible to construct a stationary equilibrium when the state is less than $q_1(\Delta)$.

First we note that if there exists $q_k(\Delta) = 0$ and $x(q_k(\Delta)) < 0$, even as Δ goes to zero, the Coase Conjecture(see Lemma 10, GSW or DL), implies that the initial price offer (and the seller's payoff) converges to v_1[12]. However, this contradicts the fact that $\overline{q}_1 \in (0, q_2)$. And so, as Δ goes to zero, there exists $k(\Delta) < \infty$ s.t. $q_{k(\Delta)} = x(q_{k(\Delta)})$.

We have constructed a stationary equilibrium for states $x \geq q_{k(\Delta)}$(for Δ small). We now proceed to construct an equilibrium when the belief is less than $q_{k(\Delta)}$ for Δ small.

For $q \geq q_{k(\Delta)}$ and $x < q$, let

$$
F_1(x; q) := P(q)\frac{q - x}{1 - x} + \delta V(q)\frac{1 - q}{1 - x} - \alpha W(x)
$$

and $\overline{F}_1(x) = \max_{q \geq q_{k(\Delta)}} F_1(x; q)$. Let $S_1^F(\Delta) := \{x < q_{k(\Delta)} | \overline{F}_1(x) \geq 0\}$. Claim 1 shows that for Δ small $S_1^F(\Delta)$ is empty. We set $P(q_1(\Delta)) = v(q_1(\Delta))$ and iteratively extend the triplet by applying the same arguments. We now proceed to construct the equilibrium.

[12]Note that by definition, under our hypothesis, $q_l(\Delta) > x(q_l(\Delta))$ and so for any $x \geq q_k(\Delta)$, $G(x; x) > 0$

Equilibrium Strategies

- Buyer's Strategies
  - Type q buyer accepts a price iff $p \leq P(q)$
- Seller's Strategies
  - If the state is anything but $\overline{q}_i(\Delta) \in (0,1)$, the seller never exits[13]
  - If the state is $\overline{q}_i(\Delta) \in (0,1)$ and the previous period offer is p the seller exits with probability x, where[14]

$$v(\overline{q}_i(\Delta)) - p = \delta(1-x)(v(\overline{q}_i(\Delta)) - P(t'(\overline{q}_i(\Delta)))$$

    where $t'(\overline{q}_i(\Delta))$ is a member of $\arg\max \overline{F}_i(\overline{q}_i(\Delta))$.
  - If the state is q and the previous period's offer is p, the seller makes randomizes suitably between offers that induce states in $\mathcal{T}(q)$.

1.9.2 Limit Outcomes

We first prove that trade probability in each period is uniformly bounded below even as Δ goes to zero. The proof is similar to DL Lemma C-1.

Lemma 10. *There exists $\eta > 0$ s.t. if trade occurs with positive probability at any $q \in [0, 1-\eta)$, the probability of trade is atleast η in any Weak Markov Equilibrium*

Proof. Let $\overline{V}(q) \equiv (1-q)V(q)$. Let q be such that $t(q) < q_{N-1}$ as $\Delta \to 0$[15]. At any such on path q, we have

$$\begin{aligned} \overline{V}(q) &= P(t(q))(t(q)-q) + \delta\overline{V}(t(q)) \\ \implies P(t(q))(t(q)-q) &= \overline{V}(q) - \delta\overline{V}(t(q)) \end{aligned}$$

By optimality,

[13] For states reached on path.

[14] In the generic case that $\overline{q}_i(\Delta)$ is not exactly equal to q_3

[15] If $t(q) > q_2$, then trade ends in one period.

$$P(t(q))(t(q)-q) = \overline{V}(q) - \delta\overline{V}(t(q))$$
$$\geq P(t^2(q))(t(q)-q) + \overline{V}(t(q))(1-\delta)$$

where $t^2(q) = t(t(q))$. And so,

$$[P(t(q)) - P(t^2(q))](t(q)-q) \geq \overline{V}(t(q))(1-\delta)$$
$$(1-\delta)[v(t(q)) - P(t^2(q))](t(q)-q) \geq \overline{V}(t(q))(1-\delta)$$
$$(t(q)-q) \geq \frac{\overline{V}(t(q))}{v(t(q)) - P(t^2(q))}$$
$$\geq \frac{v_1(1-t(q))}{v_N - v_1}$$
$$\geq \frac{v_1(1-q_{N-1})}{v_N - v_1} \equiv \eta$$

where the second inequality comes from the fact that $P(t(q)) = \delta P(t^2(q)) + (1-\delta)v(t(q))$, the third inequality comes from the fact that $\overline{V}(q) \geq v_1(1-q)$ for any $q \in [0,1]$ and that $v(q) \leq v_N$ and $P(q) \geq v_1$ for any $q \in [0,1]$. The final inequality comes from the fact that $t(q) \leq q_2$.

□

Note that this implies that there exists $N < \infty$ s.t. trade ends in N periods even as Δ goes to zero.

As earlier, we define $G(x;q)$ and $x(q)$ for each $x \leq q$.

$$G(x;q) = \max_{y \geq q} P(y)\frac{y-x}{1-x} + \delta V(y)\frac{1-y}{1-x} - \alpha W(x)$$

$$x(q) := \max\{0, \max\{x \leq q | G(x,q) = 0\}\}$$

If $G(x,q) > 0$ for some $x \leq q$, but $G(x,q) \neq 0$ for all $x \leq q$, then set $x(q) = 0$.[16] Let $\hat{S}_\Delta(q) := [x(q), q]$ for all $q > 0$ and $\hat{S}_\Delta(0) = \{0\}$. (If $G(x,q) < 0$ for all $x \leq q$, then set $\hat{S}_\Delta(q) = \phi$).

[16] Note that by continuity of $G(x;q)$ in x, if $G(x,q) > 0$ for some $x \leq q$ and $G(x';q) < 0$ for some $x' \leq q$, then there must exist some x'' between x and x' s.t. $G(x'';q) = 0$

Let $q_{k+1} = x(q_k)$ and $q_1 = \overline{q}_1$. First, note that for $k < \infty$, $\hat{S}_\Delta(q_k)$ is non empty. We know that $\hat{S}_\Delta(q_1) \neq \phi$(since $G(q_1, q_1) \geq 0$). Suppose $\hat{S}_{k-1}(\Delta)$ is non empty and $q_k = 0$. Then $q_{k'} = 0$ for all $k' \geq k$ and $\hat{S}_{k'} = \{0\}$ and so is non empty. Suppose $q_k > 0$. Then since $G(q_k, q_{k-1}) = 0$, it must be that $G(q_k, q_k) \geq 0$ and so $\hat{S}_\Delta(q_k)$ is non empty. Next, we show that if $x \in \hat{S}_\Delta(q_k)$, then $G(x, q_k) \geq 0$.

Lemma 11. *For any $k \in N$, $G(x, q_k) \geq 0$ for all $x \in \hat{S}_\Delta(q_k)$. Moreover, if $x > x(q_k)$, then $G(x, q_k) > 0$.*

Proof. If $G(q_k, q_k) = 0$ then $q_{k'} = q_k$ for all $k' \geq k$, and so we have our result. Suppose for some $k < \infty$ and $x \in \hat{S}_\Delta(q_k)$, $x < q_k$, it is the case that $G(x, q_k) < 0$. Since $x < q_k$ belongs in $\hat{S}_\Delta(q_k)$, $G(q_k, q_k) > 0$(Note that $G(q_k, q_{k-1}) = 0$, so $G(q_k, q_k) \geq 0$). Further $x \in [q_{k+1}, q_k]$, so $x > q_{k+1}$. By continuity of $G(x, q)$ in x, there must be $x' \in (x, q_k)$ s.t. $G(x', q_k) = 0$, which contradicts the definition of $x(q)$. Further, if $x > x(q_k)$, then it must be the case that, by definition of $x(.)$, $G(x, q_k) > 0$. □

Note that $q_{k+1} \leq q_k$ and the sequence is bounded below by 0. Suppose $q_k \to q$. We show that $x(q) = q$. First, note that for $x \in (q_1, q)$ that is not part of the sequence, $G(x, x) > 0$. Since $q_1 > x$ and by convergence of q_k, there exists n s.t. $q_n < x$. Let q_n be the smallest member of the sequence below x. Then $q_{n-1} > x > q_n$, i.e., $x \in \hat{S}_\Delta(q_{n-1})$. By the previous lemma, we have that $G(x, q_{n-1}) > 0$ and so $G(x, x) > 0$. We also have that $G(q, q) \geq 0$. Suppose $G(q, q) < 0$. Then for $x > q$ close enough to q, $G(x, x) < 0$, which is a contradiction. This implies that $\hat{S}_\Delta(q) \neq \phi$ as q belongs in the set. We now prove a preliminary lemma that helps us solve the fixed point result. Let $t_q(y) \equiv \min \arg\max G(y, q)$.

Lemma 12. *Suppose $q_n \downarrow q$, where $x(q_n) < q_n$ and suppose there exists $y \in \hat{S}_\Delta(q)$, $y < q$. Then there exists a subsequence q_m s.t. $y' \in \hat{S}_\Delta(q_m)$, where $y' < q$ and $y' \in \hat{S}_\Delta(q)$.*

Proof. Let $q_n \downarrow q$. Suppose there exists $y < q$ s.t. $t_q(y) > q$ and $y \in \hat{S}_\Delta(q)$. Since $y < q$ belongs in $\hat{S}_\Delta(q)$ and since the set is convex, there exists $y' < q$ close enough to q s.t. $y' \in \hat{S}_\Delta(q)$ and $t(y') > q$. Note that $G(q, q) > 0$ and $x(q) < y$. Consider $z \in (y', q)$. Since $z > y'$, it must be that $t_q(z) \geq t_q(y')$. So, for n large enough s.t. $q_n < t(y')$, $G(z, q_n) = G(z, q) > 0$ for all $z \in [y', q]$. Let M be s.t. $q_m < t(y')$ for all $m \geq M$. Let $\varepsilon > 0$ be s.t. $G(x, x) > 0$ for all $x \in (q, q + \varepsilon)$ (It is possible to find such a ε since $G(x, x)(1-x) \geq (t(q)-x)P(t(q)) + \delta(1-t(q))V(t(q)) - \alpha W(x)(1-x)$ and the term on the right hand side of the inequality is continuous in x. So, for $\varepsilon > 0$ small enough, the term on the right hand side is strictly positive as $G(q,q) > 0$). Let M' be s.t. $q_m < q+\varepsilon$ for all $m > M'$. So, for $m > M'$ and any $z \in [q, q_m]$, we have that $G(z,z) = G(z, q_m) > 0$ (since $t(q) \leq t(z)$, we have that $G(z,z) = G(z, q_2)$ when $q_m < t(q)$). Let $K = \max\{M, M'\}$. For all $m > K$, we have that $G(z, q_m) > 0$ for all $z \in [y', q_m]$. This in turn means that $y' \in \hat{S}_\Delta(q_m)$. □

Since $q_k \to q$ and since $x(q_k)$ is a decreasing sequence, for any k, $y < q$ does not belong to $\hat{S}_\Delta(q_k)$ and so cannot belong to $\hat{S}_\Delta(q)$[If it did then by the previous Lemma, it is possible to construct a subsequence of q'_k s.t. $y' \in \hat{S}_\Delta(q'_k)$ for some $y' < q$, which contradicts the fact that $x(q'_k) = q'_{k+1} \geq q$]. This implies that $x(q) \geq q$ which implies (by definition) that $x(q) = q$.

We show that $q_1(\Delta)$ where $x_\Delta(q_1(\Delta)) = q_1(\Delta)$ (constructed in the preceding paragraphs) converges to $\overline{q}_1$ as Δ goes to zero. Let $\lim_{\Delta\to 0} q_1(\Delta) = q^*$[17]. First note that $G_\Delta(x,x) > 0$ for any $x > q_1(\Delta)$[18]. And so the the seller never opts out when the state is greater than $q_1(\Delta)$. So for any $x > q^*$, there exists Δ small enough such that the seller never opts out if the state is atleast x and continues to trade with all types. By Lemma 10 , the number of periods until trade ends is bounded above even as Δ goes to zero. Hence, q^* cannot be strictly less than $\overline{q}_1$. By construction, $q_1(\Delta) \leq \overline{q}_1$ and so q^* cannot exceed $\overline{q}_1$. Hence $q^* = \overline{q}_1$.

As in the previous section, for $q \geq q_1(\Delta)$ and $x < q$, let

$$F_1(x;q) := P(q)\frac{q-x}{1-x} + \delta V(q)\frac{1-q}{1-x} - \alpha W(x)$$

and $\overline{F}_1(x) = \max_{q\geq q_1(\Delta)} F_1(x;q)$. Let $S_1^F(\Delta) := \{x < q_1(\Delta) | \overline{F}_1(x) \geq 0\}$.

We show that $S_1^F(\Delta)$ is empty for Δ small enough.

Claim 1. *For Δ small enough, $S_1^F(\Delta)$ is empty.*

Proof. Since $q_1(\Delta)$ converges to $\overline{q}_1$, for Δ small enough, $|\overline{q}_1 - q_1(\Delta)| < \eta/2$, where η is as defined in Lemma 10. Since $t_{q_1(\Delta)}(q_1(\Delta)) > q_1(\Delta) + \eta$, it must be that for Δ small enough $t_{q_1(\Delta)}(q_1(\Delta)) > \overline{q}_1 + \eta/2 \equiv q'$. Note that by Lemma 10, as Δ goes to zero, $P(q')$ approaches v_1 and so does $\delta P(q') + (1-\delta)v(\overline{q}_1)$. Note that given $P(.)$ in any WME,

[17] Take any convergent subsequence if the limit does not exist

[18] We noted earlier that $G(x,x) > 0$ if x does not belong to the sequence. If x is a member of the sequence, either $G(x,x) = 0$ in which case, $x = q_1(\Delta)$, or $x(x) < x$, in which case $G(x,x) > 0$

$v(x) - P(x) \geq \delta[v(x) - P(t(x)))]$[19] which implies that $P(x) \leq \delta P(t(x)) + (1-\delta)v(x)$. Therefore, we have for Δ small enough s.t. $v(q_1(\Delta)) = v(q_1)$ that[20]

$$\begin{aligned} P_{q_1(\Delta)}(q_1(\Delta)) &\leq \delta P(t_{q(\Delta}(q_1(\Delta))) + (1-\delta)v(q_1(\Delta)) \\ &< \delta P(q') + (1-\delta)v(\overline{q}_1) \end{aligned}$$

where the inequality follows from the fact that $t_{q_1(\Delta)}(q_1(\Delta)) > q'$ and $P(.)$ is weakly decreasing. Therefore, for Δ small enough, $P_{q_1(\Delta)}(q_1(\Delta))$ is close to v_1 and so, $\alpha v(\overline{q}_1) > P_{q_1(\Delta)}(q_1(\Delta))$. So, for $x < q_1(\Delta)$[21]

$$\begin{aligned} \overline{F}_1(x) &= P(t_{q_1(\Delta)}(x))\frac{t_{q_1(\Delta)}(x) - x}{1-x} + \delta\frac{1 - t_{q_1(\Delta)}(x)}{1-x}V(t_{q_1(\Delta)}(x)) \\ &= P(t_{q_1(\Delta)}(x))\frac{q_1(\Delta) - x}{1-x} + F_1(q_1(\Delta), t_{q_1(\Delta)}(x))\frac{1 - q_1(\Delta)}{1-x} \\ &\leq P_{q_1(\Delta)}(q_1(\Delta))(\frac{q_1(\Delta) - x}{1-x}) + \alpha W(q_1(\Delta))\frac{1-q_1(\Delta)}{1-x} \\ &< \alpha v(\overline{q}_1)\frac{q_1(\Delta) - x}{1-x} + \alpha W(q_1(\Delta))\frac{1-q_1(\Delta)}{1-x} \\ &\leq \alpha W(x) \end{aligned}$$

where the first equality comes from rearranging terms, the second inequality comes from the fact that $F_1(q_1(\Delta), t_{q_1(\Delta)}(x)) \leq \overline{F}(q_1(\Delta)) = \alpha W(q_1(\Delta))$ and that $t_{q_1(\Delta)}(x) \geq q_1(\Delta)$ and $P(.)$ is weakly decreasing. The final inequality follows from the fact that $P_{q_1(\Delta)}(q_1(\Delta)) > \alpha v(q_1)$ for Δ small enough.

□

Next, we show that for Δ small enough, in any equilibrium, $P(q_1(\Delta)) = v(q_1(\Delta))$.

Lemma 13. *For Δ small enough, $P(q_1(\Delta)) = v(q_1(\Delta))$ in any equilibrium.*

Proof. Suppose $P(q_1(\Delta)) < v(q_1(\Delta))$. Suppose the offer $p = P(q_1(\Delta))\delta + (1-\delta)v(q_1(\Delta)) - \varepsilon$ is made for $\varepsilon > 0$ small s.t. $p > P(q_1(\Delta))$. We show that the induced state in the next

[19] Suppose x prefers to strictly reject the offer $\delta P(t(x)) + (1-\delta)v(x)$. Then, the next period state is strictly less than x, which means that the offer in the next period is atleast $P(t(x))$ and therefore type x's payoff from rejection is atmost $\delta P(t(x)) + (1-\delta)v(x)$, a contradiction.

[20] Where $P_{q_1(\Delta)}(q_1(\Delta)) = \lim_{q \downarrow q_1(\Delta)} P(q)$

[21] If $t_{q_1(\Delta)}(x) = q_1(\Delta)$, then $P(t_{q_1(\Delta)}(x)) = P_{q_1(\Delta)}(q_1(\Delta))$

period is $q_1(\Delta)$, which contradicts the fact that $P(q_1(\Delta))$ is type $q_1(\Delta)$'s reservation price. Since $P(q)$ is non increasing, the induced belief q' cannot be greater than $q_1(\Delta)$. So $q' \leq q_1(\Delta)$.

Suppose $q_1(\Delta) > q'$ and suppose $v(q') = v(q_1(\Delta))$. Rearranging $p = P(q_1(\Delta))\delta + (1 - \delta)v(q_1(\Delta)) - \varepsilon$, we get $v(q_1(\Delta)) - p = \delta(v(q_1(\Delta)) - P(q_1(\Delta))) + \varepsilon$. Since $v(q_1(\Delta)) = v(q')$, we have that $v(q') - p = \delta(v(q') - P(q_1(\Delta))) + \varepsilon > \delta(v(q') - P(q_1(\Delta)))$. This implies that in the next period, an offer $p'(\varepsilon)$ which is strictly less than $P(q_1(\Delta))$ is made with positive probability. Combined with the fact that $P(q)$ is non increasing, it must be that the induced belief is weakly greater than $q_1(\Delta)$. If the induced belief upon offering $p'(\varepsilon)$ is $q_1(\Delta)$, then it is not optimal for the seller to offer $p'(\varepsilon)$ when the state is q since offering $P(q_1(\Delta)) > p'(\varepsilon)$ gives a higher payoff since the probability of trade is the same, but the price is higher. So, the induced belief upon offering $p'(\varepsilon)$ for any $\varepsilon > 0$ is strictly greater than $q_1(\Delta)$. But since $S_1^F(\Delta)$ is empty, the seller strictly prefers to opt out than offer $P(x)$ for some $x > q_1(\Delta)$, we get a contradiction.

Suppose $q' < q_1(\Delta)$ and $v(q') = v_3$ and $v(q_1(\Delta)) = v_2$. Then, $v_3 - p > \delta(v_3 - P(q_1(\Delta)))$[22]. Suppose the offer made in the following period is $p'(\varepsilon)$. Then, $\delta(v_3 - p'(\varepsilon)) = v_3 - p > \delta(v_3 - P(q_1(\Delta)))$, which in turn implies that $p'(\varepsilon) < P(q_1(\Delta))$. We apply the same arguments as in the previous paragraph to obtain a contradiction. □

Finite iterations of the above arguments gives us our result.

We summarize the limit ex-ante payoffs below.

- If $v_1 \geq \alpha W(0)$, then the limit payoff is v_1.
- If $\overline{q}_1 \in (0, q_3]$, then the limit payoff is $\overline{q}_1 v_3 + \alpha(1 - \overline{q}_1)W(\overline{q}_1)$.
- If $\overline{q}_1 \in (q_3, q_2)$ and $v_2\overline{q}_1 + \alpha(1 - \overline{q}_1)W(\overline{q}_1) \geq \alpha W(0)$, then the payoff is $v_2\overline{q}_1 + \alpha W(\overline{q}_1)(1 - \overline{q}_1)$.
- Finally if $\overline{q}_1 \in (q_3, q_2)$ and $\overline{q}_2 \in (0, q_3)$, then the payoff is $v_3\overline{q}_2 + \alpha W(\overline{q}_2)(1 - \overline{q}_2)$.

In general, the seller's payoff is given by $v_j\overline{q}_{j-1} + \alpha W(\overline{q}_{j-1})(1 - \overline{q}_{j-1})$, if $v_j\overline{q}_{j-1} + \alpha W(\overline{q}_{j-1})(1 - \overline{q}_{j-1}) > \alpha W(0)$.

[22] Since $p < v_2(1 - \delta) + \delta P(q_1(\Delta)) < v_3(1 - \delta) + \delta P(q_1(\Delta))$

1.9.3 Theorem 1

We consider two cases. We first show that if $v_1/W(q_3) < v_2/v_3$ there exists α s.t. the seller's payoff is decreasing at α.

Let $H((q,\alpha);(q',\alpha')) := [v_2 q + (1-q)\alpha W(q)] - [q' v_3 + (1-q')\alpha' W(q')]$ where $q > q_3 > q'$. Simplifying $H((q,\alpha);(q',\alpha'))$, we get

$$\begin{aligned} H((q,\alpha);(q',\alpha')) &= [v_2(q-q'+q'-q_3+q_3) + \alpha v_2(q_2-q) + (1-q_2)\alpha v_1] - \\ &\quad [v_3 q' + (q_3-q')\alpha' v_3 + \alpha' v_2(q_2-q_3) + (1-q_2)\alpha' v_1] \\ &= (v_2-v_3)q' + (v_2-\alpha v_3)(q_3-q') + (1-\alpha)v_2(q-q_3) + (\alpha-\alpha')[v_2(q_2-q) + v_1(1-q_2)] \end{aligned}$$

Let $\underline{\varepsilon} = (v_3-v_2)/[(v_2 - v_3\alpha + (v_3 v_1/W(q_3)))]N$ and $\overline{\varepsilon} = (v_3-v_2)/[(v_2 - v_2 v_1 \alpha + (v_1 v_2/W(q_3)))]N$, where $\alpha = v_1/W(q_3)$ and N is large enough s.t. $q_3 - \underline{\varepsilon} > 0$ and $q_3 + \overline{\varepsilon} < q_2$. Let $\underline{\alpha} = v_1/W(\underline{q})$, where $\underline{q} = q_3 - \underline{\varepsilon}$. Let $\overline{\alpha} = v_1/W(\overline{q})$ where $\overline{q} = q_3 + \overline{\varepsilon}$.

Let $A = [\underline{\alpha}, \overline{\alpha}]$. Note that $\overline{q}_1(a) \in [\underline{q}, \overline{q}]$ for $a \in A$. Moreover, $\overline{q}_1(a)$ is increasing in a. If $a = v_1/W(x)$, then, $\overline{q}_1 = x$. Let $q_3 > q' > \max\{2/N, \underline{q}\}$, let $\overline{q} > q > q_3$ and let $a = v_1/W(q)$ and $a' = v_1/W(q')$. Thus,

$$\begin{aligned} &H((q,a);(q',a')) \\ &= (v_2-v_3)q' + (v_2 - a v_3)(q_3-q') + (1-a)v_2(q-q_3) + \\ &\quad (a-a')[v_2(q_2-q) + v_1(1-q_2)] \end{aligned}$$

Note that

$$
\begin{aligned}
&(a-a')[v_2(q_2-q)+v_1(1-q_2)] \\
&= (a-a')W(q)(1-q) \\
&= \frac{W(q')(1-q)-W(q)(1-q)}{W(q)W(q')}W(q)v_1 \\
&< \frac{W(q')(1-q')-W(q)(1-q)}{W(q')}v_1 \\
&= \frac{v_1v_3(q_3-q')+v_1v_2(q-q_3)}{W(q')}
\end{aligned}
$$

and so

$$
\begin{aligned}
&H((q,a);(q',a')) \\
&< (v_2-v_3)q' + (v_2-av_3)(q_3-q') + (1-a)v_2(q-q_3) + \\
&\quad \frac{v_1v_3(q_3-q')+v_1v_2(q-q_3)}{W(q')} \\
&= (v_2-v_3)q' + (v_2-av_3+(v_1v_3)/W(q'))(q_3-q') + ((1-a)v_2 + +(v_1v_2)/W(q'))(q-q_3) \\
&< (v_2-v_3)q' + (v_2-av_3+(v_1v_3)/W(q'))\underline{\varepsilon} + ((1-a)v_2+(v_1v_2)/W(q'))\overline{\varepsilon} \\
&< (v_2-v_3)[q'-2/N] < 0
\end{aligned}
$$

Where the final inequality comes from the fact that $\underline{\varepsilon} < \dfrac{v_3-v_2}{N(v_2-av_3+(v_3v_1/W(q'))}$ and $\overline{\varepsilon} < \dfrac{v_3-v_2}{N(v_2-av_2+(v_2v_1/W(q'))}$ as $a > \alpha$ and $W(q') > W(q_3)$. Since $H((q,a);(q',a')) < 0$, we have our result.

For the N value case, replacing v_2 and v_3 with v_{N-1} and v_N respectively will give us the result.

Next we consider the $v_2/v_3 < v_1/W(q_3)$ case. We show that $V(.)$ is decreasing at $v_1/W(q_3)$ iff the condition holds.

First note that, where both quantities are well defined, the following relation holds between $\overline{q}_2$ and $\overline{q_1}$

$$(q_3 - \overline{q}_2)(\alpha v_3 - v_2) = v_2(1-\alpha)(\overline{q}_1 - q_3) \tag{1.23}$$

By the implicit function theorem,

$$(q_3 - \overline{q}_2)v_3 + v_2(\overline{q}_1 - q_3) - \frac{d\overline{q}_2}{d\alpha}(\alpha v_3 - v_2) - \frac{d\overline{q}_1}{d\alpha}(v_2(1-\alpha)) = 0 \tag{1.24}$$

which implies that

$$\frac{d\overline{q}_2}{d\alpha} \geq -\frac{d\overline{q}_1}{d\alpha}\frac{v_2(1-\alpha)}{(\alpha v_3 - v_2)} \tag{1.25}$$

and so

$$(\frac{d\overline{q}_2}{d\alpha})/(\frac{d\overline{q}_1}{d\alpha}) \geq -\frac{v_2(1-\alpha)}{(\alpha v_3 - v_2)} \tag{1.26}$$

Next, consider the ex-ante payoff of the seller

$$v_3\overline{q}_2 + (\overline{q}_1 - \overline{q}_2)v_2 + v_1(1 - \overline{q}_1) \tag{1.27}$$

Differentiating Equation (1.27), we get

$$(v_3 - v_2)\frac{d\overline{q}_2}{d\alpha} + (v_2 - v_1)\frac{d\overline{q}_1}{d\alpha} \tag{1.28}$$

Equation (1.28) is less than zero if

$$(\frac{d\overline{q}_2}{d\alpha})/(\frac{d\overline{q}_1}{d\alpha}) < -\frac{v_2 - v_1}{v_3 - v_2} \tag{1.29}$$

A necessary condition for this to hold is

$$-\frac{v_2 - v_1}{v_3 - v_2} > -\frac{v_2(1-\alpha)}{(\alpha v_3 - v_2)} \tag{1.30}$$

which implies that

$$\alpha < \frac{1 - \beta_{13}}{2 - \beta_{12} - \beta_{23}}$$

Since $\alpha > v_1/W(q_3)$, we have our result.

Suppose $v_1/W(q_3) < \frac{1-\beta_{13}}{2-\beta_{12}-\beta_{23}}$. This implies that Equation (1.30) is satisfied. We show that it is possible to find an α s.t. $v_1/W(q_3) < \alpha < \frac{1-\beta_{13}}{2-\beta_{12}-\beta_{23}}$ at which the seller's payoff is decreasing. From Equation (1.23) and Equation (1.24), we have

$$v_2[1+\frac{v_3(1-\alpha)}{\alpha v_3 - v_2}](\overline{q}_1 - q_3) - \frac{d\overline{q}_2}{d\alpha}(\alpha v_3 - v_2) - \frac{d\overline{q}_1}{d\alpha}(v_2(1-\alpha)) = 0 \tag{1.31}$$

Let $\varepsilon' = 1/[N(v_2[1+\frac{v_3(1-\alpha)}{\alpha v_3 - v_2}])]$ where $\alpha = v_1/W(q_3)$. When α is $v_1/W(q')$ where $q' = q_3 + \varepsilon'$, and for N large enough, we have that Equation (1.29) holds.

Chapter 2

Bargaining With Unobservable Outside Options

2.1 Introduction

In this paper, I study a model of bargaining with asymmetric information about the outcome of exercising an outside option. In many bargaining settings, parties may have access to outside options. Further, the counter-party may be unsure of the possibility of trade if the option is exercised. For instance, consider an entrepreneur negotiating the sale of her firm with an acquirer. Two potential aspects of interest in this setting are: (1) the entrepreneur may be better informed than the acquirer about the interest among potential buyers in the market for acquiring her firm and (2) the entrepreneur always has the option of holding an auction by inviting bids. I study trade dynamics and potential causes of inefficiency in such settings when values are private and there are weak gains from trade. In particular, I consider how Coasian dynamics may operate in this setup.

I consider a model of two sided incomplete information where a seller(she) with an outside option that evolves stochastically over time, makes price offers to a buyer(he) who is privately informed of her valuation (which is either high or low). One way to interpret the increase in value is as an increase in demand for the seller's object. There are (weak) gains from trade in the sense that the value to the seller from exercising her outside option is equal to the low type buyer's valuation. The seller can choose to exercise her outside option in any period. If the seller exercises her outside option, the buyer receives the object if and only if the seller's outside option has not appreciated in value.

In contrast to standard Coasian settings with private values and weak gains from trade, there exists an equilibrium in stationary buyer strategies in this setting that involves inefficient delay. To better understand this phenomenon, I first consider an auxiliary benchmark setting in which the seller may only make a single offer to the buyer by way of negotiations. If the offer is rejected, the outside option is executed. Since the outside option evolves over time (or demand for the object accumulates over time), the seller has incentive to delay making her offer to the buyer. This game has a pooling equilibrium in which the seller delays trade until the buyer becomes sufficiently pessimistic about her chances of obtaining the object should the outside option be executed. The buyer's belief that the outside option has improved increases over time in equilibrium and the seller's choice of time for making her offer is uniquely pinned down in equilibrium as a function of her belief about the buyer's type.

In tandem with the observations in the benchmark setting, I construct a pooling equilibrium in which the seller delays trade in order to wear the buyer down. As opposed to classical incomplete information bargaining settings in which the buyer's trade probability jumps to one almost instantaneously, the trade probability in this equilibrium increases continuously over a period of time before jumping to one.

In this setting, the buyer's evolving uncertainty about the outside option is central to the existence of a stationary equilibrium with inefficient delay. The seller wishes to delay trade, not because it improves her outside option, but because it enhances her ability to extract a high price from the buyer by threatening him with the outside option. By virtue of construction, the seller is always guaranteed a payoff equal to her payoff from the benchmark setting in equilibrium. Since the gains made by the seller in the benchmark setting depend on the buyer's value as well as his belief about the outside option, the equilibrium in question inherits this dependence, resulting in inefficient delay.

Apart from the existence of a stationary equilibrium with delay, the probability of trade in equilibrium may also be of independent interest. The trade probability increases continuously over time and ends in an atom of trade. A point to note here is the absence of any 'gaps' (i.e., periods of time with zero probability of trade) in the probability distribution. This is distinct from equilibrium structures typically found in interdependent value settings (such as Deneckere and Liang (2006),Fuchs and Skrzypacz (2013)), where in the 'gap' case (i.e., when the buyer's valuation is bounded away from zero), the equilibrium features gaps in trade sandwiched between atoms. In these settings, the incentive to delay trade remains constant over time at the gaps. In the equilibrium in our setting, however, the seller's incentive to delay trade changes over time, given her belief about the buyer's type. In particular, given the seller's belief, as the buyer's belief about the outside option increases, the seller's incentive to delay trade reduces over time. This results in an equilibrium that exhibits a trade dynamic the literature calls 'atomless trade'(see Fuchs and Skrzypacz (2010)).

Finally, I pin down the seller's payoff in this equilibrium. Consistent with findings in other settings that feature atomless delay (such as Fuchs and Skrzypacz (2010)), the seller's payoff is reduced to the minimum payoff guaranteed to her in equilibrium, which in this case, is the payoff from the benchmark case. The seller fails to do any better than the benchmark case for the same reason as in other settings that feature atomless delay: the presence of Coasian forces imply that if the seller delays trade in equilibrium, her incentive to delay trade must exactly match her incentive to speed up trade.

The rest of the paper is organized as follows: Section 2.2 discusses related literature, Section 2.3 describes the model, Section 2.4 solves for the equilibrium in the benchmark case, Section 2.6 discusses the main result and the final section concludes.

2.2 Related Literature

This paper is related to the dynamic bargaining with incomplete information literature as well as the literature on bargaining with outside options.

In the complete information bargaining literature, the question of outside options has been addressed both in static and dynamic settings. While these threat points are crucial in determining bargaining outcomes in static settings, introducing exogenous outside options in dynamic bargaining settings with complete information impacts bargaining dynamics and outcomes only marginally, as shown in Binmore et al. (1989). In this paper, bargaining dynamics are driven by incomplete information on both sides-the seller is unsure of the buyer's valuation and the buyer cannot observe the seller's search outcomes.

There is extensive work on outside options in the dynamic bargaining literature with one sided incomplete information. In the classic, 'Coasian' bilateral bargaining problem with one sided incomplete information (see Fudenberg et al. (1985), Gul et al. (1986)), the seller's temptation to revise price offers result in efficient trade with all buyer types as discount factor goes to one. As opposed to complete information models, introducing outside options into the standard one sided incomplete information model almost always breaks the efficiency result. For instance, Board and Pycia (2014) show that when the buyer has an exogenous outside option, the seller obtains monopoly profits in equilibrium.

The dynamic bargaining literature with incomplete information has addressed the question of arriving outside options in a variety of settings. Among other papers, Ortner (2017) considers a setting where a monopolist's cost evolves stochastically, Daley and Green (2020) have exogenous arrival of information over time, Chaves (2019) studies a setting where entrants disrupt a negotiation endogenously. Similar to the equilibrium I construct, the equilibria in these settings exhibit Coasian dynamics in the sense that the seller's(buyer in the case of Daley and Green (2020)) payoff is reduced to her payoff from stalling negotiations (with the exception of Chaves (2019) who considers a private offers case where the payoff from stalling involves beliefs of the entrants). The main difference I would like to highlight between these papers and my setting is that the seller also possesses private information and the seller can credibly stall negotiations not necessarily because the outside option has some intrinsic value, but in order to influence the buyer's belief.

The work closest to this paper are Hwang and Li (2017) and Fuchs and Skrzypacz (2010) (henceforth, FS). Hwang and Li (2017) study a bargaining game between a seller and a privately informed buyer who has a stochastically arriving outside option. They compare bargaining outcomes when the arrival is public and when it is private. When the arrival is private, there are multiple equilibria including a deadlock equilibrium and a Coasian equilibrium. The forces that drive delay in their setting is distinct from those in my setting. In their setting, delay arises in the deadlock equilibrium because the buyer's opting out behavior neutralizes the change in belief from skimming, so that the seller can still extract rents from high valuation buyers. In my setting, delay is a result of growing buyer pessimism. Further, the equilibrium dynamics in their deadlock equilibrium and the equilibrium I consider are distinct. In the deadlock equilibrium, the seller's belief falls until it hits the point of deadlock where the seller randomizes between two offers. The equilibrium I characterize has no point of deadlock; the belief always falls and trade ends in finite time. Moreover, trade dynamics feature a smooth screening phase followed by an atom of trade.

Fuchs and Skrzypacz (2010) study a model in which an exogenous outside option arrives at a Poisson rate and ends the game. They characterize a stationary equilibrium in which the seller's payoff is reduced to her payoff from stalling trade until the outside option arrives and the equilibrium trade path is 'atomless',(i.e., screening is always smooth). The equilibrium I construct shares the feature that the seller's payoff is reduced to her payoff from stalling till the buyer becomes sufficiently pessimistic about the possibility of trade. Fuchs and Skrzypacz (2010) consider the equilibrium they construct in the 'no gap' case as a limit of 'DL equilibria'(Deneckere and Liang (2006)) in the gap case. In both papers, player valuations are correlated, which is not the case in my setting.

This paper is also closely related to Fudenberg et al. (1987). In their paper, the seller can end trade with the current buyer and start negotiations with a new buyer. They find that, under some conditions, in the unique stationary monotone equilibrium, the seller never haggles, i.e., the seller posts a price and switches buyers upon rejection. In contrast to their setting, in the setting I consider, the seller with an outside option has incentive to continue negotiations with the current buyer before exiting. Further, the seller has incentive to delay an agreement in order to extract some rents from the buyer. In the equilibrium I construct, the seller always haggles before exiting.

Finally, this paper is related to the literature on bargaining with two sided incomplete information. Earlier papers in this literature include Cho (1990), Cramton (1984), Cramton (1992) and Chatterjee and Samuelson (1988). More recently, Ortner (2023) considers a setting where the seller's cost evolves stochastically and is unobserved by the buyer. The paper characterizes the class of separating equilibria and pins down the most efficient separating equilibrium. While inefficient delay arises even in the most efficient separating equilibrium in Ortner (2023), the cause of delay is quite different from that in my setting. In Ortner (2023), delay emerges as a result of the information revelation constraint, i.e., delay must arise in a separating equilibrium so that a low cost seller is not tempted to mimic the high cost seller. On the other hand, delay emerges in my setting because the buyer grows pessimistic over time. This kind of consideration does not arise in Ortner (2023) because cost drops over time, so even in a pooling equilibrium the buyer grows more optimistic over time about lower future prices.

Cho (1990) considers a setting where the buyer and seller have private information about their valuation and cost respectively and an efficient separating equilibrium is characterized. The equilibrium I construct, on the other hand, highlights inefficiencies that emerge when the seller can delay trade to influence the buyer's belief about an arriving outside option. Cramton (1984),Cramton (1992) and Chatterjee and Samuelson (1988) also study models of dynamic bargaining with two sided asymmetric information. In these papers, reputation building is the source of delay as more obstinate types are more likely to hold out for better offers. In the equilibrium I construct, there is no signaling through offers and both types of the seller are equally likely to hold out for better offers. The buyer's belief changes only owing to the arriving outside option.

2.3 Model

Time is discrete $t = 0, \Delta, 2\Delta ..., N(\Delta)\Delta$, where $N(\Delta)\Delta = T$. We focus on outcomes when T is large. The seller of a good bargains with a buyer who has private information about her private value for the good. The buyer's value for the good is either High (v_H) or Low(v_L). We denote the prior belief that the type is High by q_0. The seller makes a price offer to the buyer in each period, and the buyer decides whether to accept or reject the offer.

Outside Option: The seller has access to an outside option which yields her a payoff of v_L. If the seller exercises her outside option, the buyer's payoff depends on the status of the outside option, which is either 0 or 1. If the status of the outside option is zero, it increases to one at Poisson rate λ, where one is an absorbing state. The seller does not observe the outcome of the Poisson process and therefore, does not know the status of the outside option with certainty.

An interpretation: The above payoffs upon execution of the outside option are rationalized by the following reduced form game where the status of the outside option represents the arrival of a competitive buyer with valuation higher than v_H. In this scenario, the act of exercising outside option represents the seller's decision to hold a second price auction with reserve price v_L. If the status of the outside option is zero, the buyer has no competitors. If the status is one, as mentioned above, the seller has a competitor with a value higher than v_H. An equilibrium of this game yields the above payoffs to

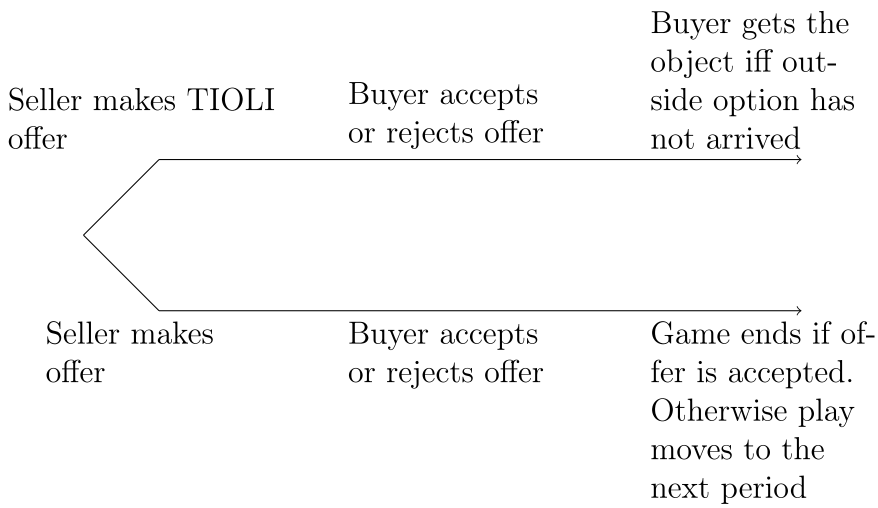

Figure 2.1: Timeline

players.

Timeline: The timeline of events in an arbitrary period t in illustrated in Figure 2.1. The seller initially decides whether to make a take-it-or-leave-it offer or a regular offer. The buyer observes the nature of the offer. If the seller makes a take-it-or-leave-it offer, the buyer decides whether or not to accept the offer. If the offer is rejected, the game ends with the execution of the outside option. If a regular offer is made, and if the offer is rejected, play moves to the next period. If any offer is accepted by the buyer, the game ends and payoffs from trading at the agreed terms are realized.

We make the following assumption about the parameters

Assumption 1. $\frac{\lambda}{\lambda + r}[v_H q_0 + (1 - q_0)v_L] > v_L$

We offer the following interpretation for Assumption 1– consider a game where the outcome of the Poisson random variable is observable to the primary buyer. If type H of the primary buyer agrees to trade at price v_H upon the change of status of the outside option, the seller prefers to wait for the status to change than trade with the buyer at price v_L.

2.4 TIOLI Benchmark

We first consider the seller's highest attainable payoff in the Take-It-Or-Leave-It benchmark in which the seller is only allowed to make a take-it-or-leave-it offer. If the offer is rejected, the outside option is evoked. In this setting, the seller chooses the optimal time to make the offer. Therefore, the seller's problem reduces to a stopping time problem. We consider the problem in continuous time for large T.

We first construct a pure strategy pooling equilibrium. For a pure stopping time strategy τ, the seller's payoff given μ, is given by

$$V_\tau(\mu) = \delta^\tau \left[(\mu_\tau v_H + (1-\mu_\tau)v_L)\frac{f_0}{1+f_0} + v_L \frac{1}{1+f_0} \right] \tag{2.1}$$

where $d\mu_t = (1-\mu_t)\lambda$. Therefore, the seller's problem reduces to choosing a threshold μ^* that maximizes

$$V_\tau(\mu) = \left(\frac{1-\mu^*}{1-\mu}\right)^{(r/\lambda)} \left[(\mu^* v_H + (1-\mu^*)v_L)\frac{f_0}{1+f_0} + v_L \frac{1}{1+f_0} \right] \tag{2.2}$$

Taking the first order condition and rearranging terms, we obtain a closed form solution for the threshold. The threshold is a function of f_0 and is given by

$$1 - \underline{\mu}(f_0) = \frac{r}{\lambda + r} \frac{v_H f_0 + v_L}{f_0(v_H - v_L)} \tag{2.3}$$

The seller therefore, follows the following strategy– the seller makes the offer iff $\mu_t \geq \underline{\mu}(f_0)$.

Note that $\underline{\mu}(.)$ is strictly increasing. In particular, this means that the more optimistic the seller is about the buyer's private value, the more she delays the offer. Intuitively, the more pessimistic type H buyer is about the outside option, the higher the offer she will accept. A higher likelihood of type H is associated with a higher marginal benefit from waiting, which in turn, rationalizes more delay. We denote the inverse of $\underline{\mu}(.)$ by $\underline{f}(.)$

Given (f, μ), we denote the seller's payoff from this TIOLI strategy by $W(f, \mu)$.

2.5 Strategies and Equilibrium

A public history h^t at period t is a sequence of prices $\{p_s\}_{s=0}^{t-1}$. A private history at time t, denoted by $\hat{h}^t$ consists of a sequence $\{a_s\}_{s=0}^{t}$ where a_s denotes the status of the outside option at time s. An acceptance rule for the buyer is a mapping from a public history h^t, the current offer and his type to a probability of acceptance. A (pure) offer strategy for the seller is a mapping from $h^t \sqcup \hat{h}^t$ to an offer. An (pure) exit strategy, denoted by e_t, for the seller with an outside option is a mapping from $h^t \sqcup \hat{h}^t$ to $\{1, 0\}$.

Let $W(N)$ denote the seller type with(without) an enhanced outside option. The respective equilibrium payoffs are denoted by $V_t^W(V_t^N)$. Note that the payoff to type L buyer is zero in any equilibrium. We denote type H buyer's payoff at time t by F_t. We let μ_t denote the buyer's belief that the seller has an outside option. The likelihood that the buyer's type is H is denoted by f.

We focus on Perfect Bayesian Equilibria in pure seller strategies with the property that the buyer's strategy is stationary and depends only on the belief pair (f, μ). As in the standard incomplete information setting, the skimming property also holds in this setting.

2.6 Coasian Equilibrium

In this section, we establish the existence of an equilibrium akin to the unique equilibrium in the standard Coasian framework. We construct a pooling equilibrium in which the buyer's strategies only depend on the 'state' variables (f, μ) and the seller's strategy may additionally depend on the offer in the previous period. We call such an equilibrium a *stationary pooling equilibrium*. Importantly, in this equilibrium the seller's payoff depends only on the belief pair (f, μ).

Proposition 1. *There exists a stationary pooling equilibrium*

The proof of the statement can be found in the Appendix. The statement is proved using construction techniques standard in the literature.

We consider outcomes in the frequent offer limit of this equilibrium. The state space can be divided into three regions. We state the proposition formally. Let $1 - \overline{\mu}(f) = \frac{r}{\lambda + r} \frac{v_L}{f(v_H - v_L)}$. We say that trade is smooth at (f, μ) if the trade probability is continuous in the frequent offer limit.

Proposition 2. *In the frequent offer limit:*

- *There is no limit delay at (f, μ) if $\overline{\mu}(f) \geq \mu \geq \underline{\mu}(f)$*
- *There is limit delay at (f, μ) if $\mu < \underline{\mu}(f)$. Further, trade is always smooth at (f, μ)*

- *The seller's payoff is $W(f,\mu)$ at state (f,μ)*

- *On the equilibrium path, seller makes a TIOLI offer at (f,μ) iff $\mu > \overline{\mu}(f)$*

We provide a rough sketch of the proof. First, we note that in an equilibrium in which the seller's payoff weakly exceeds the TIOLI payoff from the benchmark case considered in Section 2.4, there is no delay at any (f,μ) if $\mu > \underline{\mu}(f)$. We prove this in the Appendix.

Next, we provide a heuristic argument for why the payoff is equal to $W(f,\mu)$ when $\mu < \underline{\mu}(f)$.

First suppose that trade is always smooth when $\mu < \mu'$ for some μ'. The seller's equilibrium payoff function is given by[1]

$$V(f,\mu) = p(f,\mu)(f - f_+) + \delta V(f_+,\mu_+)$$

Where (f_+,μ_+) are the states induced in the next period in equilibrium. Dividing by Δ and taking limits, we get

$$rV(f,\mu) = -\frac{df}{dt}\left(p(f,\mu) - \frac{\partial V(f,\mu)}{\partial f}\right) + \frac{\partial V(f,\mu)}{\partial \mu}\frac{d\mu}{dt}$$

Due to the presence of Coasian forces, it has to be the case that $p(f,\mu) - \dfrac{\partial V(f,\mu)}{\partial f} = 0$. First notice that setting $p(f,\mu) - \dfrac{\partial V(f,\mu)}{\partial f} = 0$ yields an ODE. Substituting for $d\mu/dt$, we get

$$rV(f,\mu) = \lambda(1-\mu)\frac{\partial V(f,\mu)}{\partial \mu}\frac{d\mu}{dt}$$

Given the terminal condition at μ', the solution to the above ODE is the discounted payoff at (f,μ'). Since $df/dt > 0$, it cannot be the case that $p(f,\mu) - \dfrac{\partial V(f,\mu)}{\partial f} < 0$, as this would mean that the seller strictly prefers to stall trade. Next, we claim that it cannot be the case that $p(f,\mu) - \dfrac{\partial V(f,\mu)}{\partial f} > 0$. If it is so, then the seller gains from lowering prices slightly and this would lead to a jump in the trade probability, contradicting the fact that trade is smooth. Therefore, if trade is smooth, then the payoff equals to the payoff from stalling trade. Moreover, because $p(f,\mu) = \partial V(f,\mu)/\partial f$, the price offer at (f,μ) is equal to the discounted price offer at (f,μ'). In words, this means that when trade is smooth, the seller's payoff is equal to her payoff from delaying trade for a fixed period of time and

[1] We 'average' out the payoffs by multiplying by $(1+f)$.

the price offer she makes is equal to the discounted payoff she makes at the end of the period of the delay.

The above argument relies on the assumption that trade is smooth at (f, μ) i.e., there are no atoms or gaps in the trade probability. It is not hard to see that when the trade probability is zero over a time period, the seller's payoff equals the discounted payoff at the end of the gap. Next, we show that there cannot be any gaps or atoms in the region of delay. We proceed in the following manner.

First, we note that if the seller strictly prefers not to delay trade at (f, μ), there must be an atom of trade. Let the size of the atom be $f - f_+$. We show that the seller prefers not to delay trade iff the atom is followed by a gap.

1. If the seller strictly prefers not to delay trade $\iff$ atom of trade followed by a gap: If trade is smooth at (f_+, μ), the price offer is equal to the price offer made after a period of delay and the continuation payoff is equal to the payoff obtained at the end of the same period of delay. This would reduce the seller's payoff at (f, μ) to her payoff from delaying trade.

This implies that limit delay involves zero probability of trade at (f_+, μ), i.e., there is a gap in the trade probability.

Suppose an atom of trade is followed by a gap. Let μ_{+s} be the belief after a duration of length s. Then for $s > 0$ small enough

$$\begin{aligned}\delta^s V(f, \mu_{+s}) &= \delta^s p(f, \mu_{+s})(f - f_+) + \delta^s V(f_+, \mu_{+s}) \\ &= \delta^s p(f, \mu_{+s})(f - f_+) + V(f_+, \mu) \\ &< p(f, \mu)(f - f_+) + V(f_+, \mu)\end{aligned}$$

where the last line follows from the fact that $p(f, \mu) = p(f_+, \mu) = v_H(1 - \delta^s) + \delta^s p(f_+, \mu_{+s}) > p(f_+, \mu_{+s})$.

We next argue that if the seller is indifferent between trading and delaying trade at (f, μ) for f in some interval I, then $V(f, \mu) = W(f, \mu)$.

2. If the seller is indifferent between trading and delaying trade at (f, μ) for

f **in some interval** I**, then the payoff over the interval is equal to the benchmark payoff:** Suppose there is delay at (f, μ) and $V(f, \mu) > W(f, \mu)$ for $f \in I$. It must be the case that for some $m > \mu$ and $m < \underline{\mu}(f)$ there is an atom at (f, m).

Let $T > 0$ be the earliest time at which there is an atom of trade in I. Since an atom necessarily means that the probability of trade is non zero, we have that for beliefs in some interval I', the first atom occurs after a duration of length T. Moreover, if the length of the atom is k, for all beliefs $x \in (f, f - k)$, the size of the atom is $x - (f - k)$. This means that trade cannot be smooth over $(f, f - k)$. This is because $p(., \mu) = \partial V(., \mu)/\partial f$ is constant over this interval, contradicting the fact that there is delay at (f, μ).

Therefore if the seller is indifferent between trading and delaying trade at all (x, μ) for $x \in I$ for some interval I, then $V(x, \mu) = W(x, \mu)$.

This result has two implications. First, a region of smooth trade cannot be preceded by an atom of trade. This is because $W(f, \mu) < \partial W(g, \mu)/\partial g(f - g) + W(g, \mu)$ for all $g < f$. Second, if $V(f, \mu) > W(f, \mu)$, then there is necessarily at atom of trade at (f, μ). Conversely, if there is an atom of trade at (f, μ), then $V(f, \mu) > W(f, \mu)$.

Next, we argue that a gap cannot precede an atom.

3. A gap cannot precede an atom: Recall that an atom is followed by a gap. Suppose there is a gap before an atom. Suppose the initial gap occurs at f and the second gap at f_+ as illustrated in Figure 2.2a. If for some duration of time $s > 0$ the length of the atom stays constant at $f - f_+$, we have for f' close to f, $V(f', \mu) - \delta^s V(f', \mu_{+s}) = v_H(1 - \delta^s)(f' - f_+)$. By continuity of $V(., \mu)$, this means that $V(f, \mu) > \delta^s V(f, \mu_{+s})$, which contradicts the fact that the size of the atom remains constant. Since there is limit delay at (f_+, μ), if there is a gap at (f, μ), the size of the atom cannot increase through a fall in f_+. Suppose the atom shrinks through an increase in f_+. Suppose for some $\mu' > \mu$ close to μ, the size of the atom at μ' shrinks to $f - f'_+$. Suppose the total time taken for the state to evolve from μ to μ' is t. For t small enough, t is going to be less than the total delay at (f_+, μ). Let T denote the total delay at (f_+, μ). Since the atom shrinks between time 0 and time t, we have that $f - f_+ > f - f'_+$, i.e., $f'_+ > f_+$. However, because the seller can credibly delay trade at (f_+, μ'), the seller prefers to make an offer of $p(f_+, \mu')$ and immediately trade with probability $f'_+ - f_+$ when the

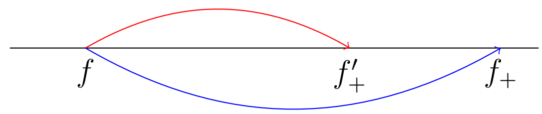

Figure 2.2: Illustration: Atom doesn't change in size

(a) The red path denotes the atom at (f, μ_+) and the blue path denotes the atom at (f, μ)

state is (f'_+, μ'), contradicting the fact that there is a gap at (f'_+, μ'). This is depicted in Figure 2.2a.

Therefore an atom cannot occur between two gaps. Consequently, a gap in trade has to be followed by smooth trade. This in turn implies that if there is a gap in trade at (f, μ), the payoff at (f, μ) is equal to $W(f, \mu)$. Next, we argue that an atom cannot occur between a region of smooth trade and a gap.

4.An atom of trade cannot be preceded by smooth trade: Suppose we have a region of smooth trade followed by an atom of trade. We argue that the seller has a profitable deviation in this case. Since the region of smooth trade is followed by an atom of trade, by continuity of the payoff function, for every μ there exists a decreasing function $r(.)$ where $r(\mu)$ which marks the boundary between these two regions. By continuity of the payoff function, the seller is indifferent between delaying trade and trading with positive probability at $(r(\mu), \mu)$. Suppose the size of the atom when the belief is μ is $r(\mu) - g$. Consequently there is a gap at g when the belief is μ. Recall that in a smooth trade region the seller must be indifferent between speeding up or slowing down trade, i.e., the seller must be indifferent between different rates of trade. Consequently, the seller obtains the same payoff from trading at different rates. In particular, the seller obtains her optimal payoff from trading at rate $\frac{dr(\mu)}{d\mu}\frac{d\mu}{dt}$. We argue that the seller has a profitable deviation. First, the seller's payoff at $(r(m), m)$ from trading at the above rate for small $t > 0$ is (approximately) equal to

$$
\begin{aligned}
V(r(m), m) &= p(r(m_t), m_t)(r(m) - r(m_t)) + e^{-rt}V(r(m_t), m_t) \\
\implies e^{-rt}(V(r(m), m_t) - V(r(m_t), m_t)) &= p(r(m_t), m_t)(r(m) - r(m_t)) \qquad (2.4)
\end{aligned}
$$

where the second line follows from the fact that delaying trade is optimal at $(r(m), m)$. However, we also have

$$V(r(m),m) = p(g,m)(r(m)-g) + W(g,m)$$
$$\implies e^{-rt}(V(r(m),m_t) - V(r(m_t),m_t)) = (r(m)-r(m_t))p(g,m) + (r(m_t)-g)(1-\delta^t)v_H \tag{2.5}$$

where the second line follows from the fact that $\delta^t V(r(m_t),m_t) = \delta^t[p(g,m_t)(r(m) - g) + W(g,m_t)]$, $W(g,m) = e^{-rt}W(g,m_t)$ and by the buyer's incentive compatibility, $p(g,m) = v_H(1-\delta^t) + \delta^t p(g,m_t)$. Combining Equation (2.4) and Equation (2.5) and taking limits, we have

$$-[p(r(m),m) - p(g,m)]\frac{dr(m)}{dm}\frac{dm}{dt} = (r(m)-g)rv_H$$

This implies that $p(r(m),m) > p(g,m)$. But since the seller is indifferent between speeding up trade and trading along $(r(.),.)$, we also have $p(g,m) = p(r(m),m)$, which yields a contradiction.

We conclude the argument by noting that a gap cannot be preceded by an atom of trade if there is no delay preceding the atom of trade. Once again, this is because as μ approaches $\underline{\mu}(.)$, the payoff at the atom falls below $W(.,\mu)$. By right continuity of the reservation price function, a gap can also not be preceded by a region of smooth trade. We have argued that a gap cannot be preceded by an atom or by smooth trade. Therefore, there cannot be any gaps in the region of delay. Since gaps are necessary for the occurrence of atoms, we also rule out atoms in the region of delay. Therefore, trade is always smooth in the region of delay and the payoff is equal to the payoff in the benchmark case.

2.7 Discussion

In this paper I study how Coasian forces may operate in a two sided incomplete information setting where the seller of a good does not know the buyer's value for the object and the buyer is unsure of his possibility of being excluded from trade in the future. I construct an equilibrium in which the buyer's acceptance strategy depends only on her own type and the belief each player holds about the other's type. In contrast to stationary equilibria in typical private values settings, this equilibrium features limit delay. The presence of limit delay can be ascribed to two features of the equilibrium: (1) the threat of being excluded by the outside option introduces artificial interdependence in values and (2) the payoff from exercising the outside option evolves over time. The presence of these two features not only result in delay, but also yields distinctive, yet familiar, trade dynamics; the equilibrium trade probability increases continuously over time and terminates in an atom. The continuity of trade probabilities further pins down the seller's payoff. In equilibrium, the seller's payoff is reduced to her payoff from an optimally timed take-it-or-leave-it offer.

2.8 Proofs

2.8.1 Proposition 1

We solve for the equilibrium by backward induction. Let $f_1^*(\mu;\Delta)$ be such that

$$f_1^*(\mu;\Delta)[v_H-\tilde{\delta}(1-\mu)(v_H-v_L)]+\delta v_L=f_1^*(\mu;\Delta)[v_H-(1-\mu)(v_H-v_L)]+v_L \quad (2.6)$$

Suppose $t=N(\Delta)-1$. Since the outside option is invoked in the next period, a type H buyer will accept an offer if it is weakly lesser than $v_H-\delta(v_H-(\mu_{N(\Delta)}v_H+(1-\mu_{N(\Delta)}v_L))=v_H-\delta(1-\mu_{N(\Delta)})(v_H-v_L)$. Then, for $0<f<f_1^*(\mu_{N(\Delta)};\Delta)$, it is optimal for the seller to exercise the outside option while for $f>f_1^*(\mu_{N(\Delta)-1};\Delta)$ it is optimal for the seller to offer a price of $v_H-\delta(1-\mu_{N(\Delta)})(v_H-v_L)$, which is accepted by the type H buyer with probability one. Note that $f_1^*(.)$ is strictly increasing.

We now show by induction that for any $t<N(\Delta)-1$, (1) the seller exercises the outside option with probability one if $0<f<f_1^*(\mu_t)$ and (2) type H buyer accepts the offer $v_H-\delta(1-\mu_{t+1})(v_H-v_L)$ with probability one. Suppose the statements above hold true for $t=N(\Delta)-k$. We show that it holds true for $t-1=N(\Delta)-(k+1)$. We first show that $v_H-\delta(1-\mu_t)(v_H-v_L)$ is accepted with probability one at $t-1$. Since $f_1^*(.)$ is strictly increasing, if $f<f_1^*(\mu_{t-1})$, then $f<f_1^*(\mu_t)$. Therefore, if type H buyer accepts an offer with probability less than one, the offer must be weakly greater than $v_H-\delta(1-\mu_t)(v_H-v_L)$. By the same logic as earlier, this implies that $v_H-\delta(1-\mu_t)(v_H-v_L)$ is accepted with probability one. By definition of $f_1^*(.)$ it follows that the seller prefers to exercise the outside option.

Next, let $f_2^*(\mu;\Delta)$ be such that

$$(f_2^*(\mu;\Delta) - f_1^*(\mu_+;\Delta))[v_H - \tilde{\delta}^2(1-\mu)(v_H - v_L)] + \delta V_1(f_1^*(\mu_+;\Delta), \mu_+;\Delta)$$
$$= f_2^*(\mu;\Delta)[v_H - \tilde{\delta}(1-\mu)(v_H - v_L)] + \delta v_L \tag{2.7}$$

where $V_1(f,\mu;\Delta) = f[v_H - \tilde{\delta}(1-\mu)(v_H - v_L)] + \delta v_L$.

Next, let $f_k^*(\mu;\Delta)$ be such that

$$(f_k^*(\mu;\Delta) - f_{k-1}^*(\mu_+;\Delta))[v_H - \tilde{\delta}^k(1-\mu)(v_H - v_L)] + \delta V_{k-1}(f_{k-1}^*(\mu_+;\Delta), \mu_+;\Delta)$$
$$= (f_k^*(\mu;\Delta) - f_{k-2}^*(\mu_+;\Delta))[v_H - \tilde{\delta}^{k-1}(1-\mu)(v_H - v_L)] + \delta V_{k-2}(f_{k-2}^*(\mu_+;\Delta), \mu_+;\Delta) \tag{2.8}$$

where $V_j(f,\mu;\Delta) = (f - f_{j-1}^*(\mu_+;\Delta))[v_H - \tilde{\delta}^j(1-\mu)(v_H - v_L)] +$ $\delta V_{j-1}(f_{j-1}^*(\mu_+;\Delta), \mu_+;\Delta)$, for $j = 3, 4..k-1$.

We now describe players' strategies:

- Buyer:
  - Any buyer with value L accepts an offer iff the price offer is weakly less than v_L
  - If the outside option is exercised in period t, all buyers with value H accept an offer p iff $p \leq v_H - (1-\mu_t)(v_H - v_L)$
  - In period $t = N(\Delta) - k$ and $f > f_k(\mu_t;\Delta)$, type H buyer accepts an offer with probability $(f - f_k(\mu_t;\Delta))/(1+f)$ accepts an offer iff the offer is atmost $v_H - \tilde{\delta}^k(1-\mu)(v_H - v_L)$. For $f \in (f_j(\mu_{t+1};\Delta), f_{j-1}(\mu_{t+1};\Delta)]$ for $j \leq k$, the buyer accepts an offer with probability $(f - f_{j-1}(\mu_{t+1};\Delta))/(1+f)$ if it is atmost $v_H - \tilde{\delta}^j(1-\mu_t)(v_H - v_L)$.
- Seller:
  - In period $t = N(\Delta) - k$, if $f_t > f_k(\mu_{N(\Delta)-k};\Delta)$, the seller offers $v_H - \tilde{\delta}^k(1 - \mu_{N(\Delta)-k})(v_H - v_L)$

- In period $t = N(\Delta) - k$, if $f_j(\mu_{N(\Delta)-k}; \Delta) > f_t > f_{j-1}(\mu_{N(\Delta)-k}; \Delta)$ for $j < k$, the seller offers $v_H - \tilde{\delta}^{j-1}(1 - \mu_{N(\Delta)-k})(v_H - v_L)$. The seller takes the outside option iff $j = 1$
- In period $t = N(\Delta) - k$, if $f_t = f_j(\mu_{N(\Delta)-k}; \Delta)$ for $j \leq k$, the seller's offer depends on the offer made in the previous period p. The seller offers $p_j(\mu_{N(\Delta)-k}) = v_H - \tilde{\delta}^j(1 - \mu_{N(\Delta)-k})(v_H - v_L)$ with probability q and offers $p_{j-1}(\mu_{N(\Delta)-k}) = v_H - \tilde{\delta}^{j-1}(1 - \mu_{N(\Delta)-k})(v_H - v_L)$ with probability $1 - q$ where q is such that

$$v_H - p = \delta(v_H - (qp_j(\mu_{N(\Delta)-k}) + (1 - q)p_{j-1}(\mu_{N(\Delta)-k})))$$

  Additionally if $j = 1$, the seller exercises the outside option with probability $1 - q$.

2.8.2 Proposition 2

Definitions

We define $\mu_{+t(\Delta)}(\mu; \Delta)$ as the belief t periods hence when current belief is μ,i.e., $(1 - \mu_{+t(\Delta)}(\mu; \Delta)) = e^{-t\lambda\Delta}(1-\mu)$ and $f_{+t}(f, \mu; \Delta)$ is the belief t period hence when the current state is (f, μ). Let $f_T(f, \mu) = \lim_{\Delta\to 0} f_{+t(\Delta)}(f, \mu; \Delta)$ and $(1 - \mu_T) = \lim_{\Delta\to 0}(1 - \mu_{+t(\Delta)})$ where $\lim_{\Delta\to 0} t(\Delta) = T$.

$$T(\mu, \mu') : \mu_{+T(\mu,\mu')}(\mu) = \mu'$$

$$\underline{T}(f, \mu) = T(\mu, \underline{\mu}(f))$$

$$T_f(f, g, \mu; \Delta) = \min\{t \geq 0 | f_t(f, \mu; \Delta) \leq g\}$$

$$f_+(f, \mu) = \inf\{g \leq f | \lim_{\Delta\to 0} T(f, g, \mu; \Delta)\Delta = 0\}$$

$$\hat{T}(f, \mu) = \inf\{t \geq 0 | f - f_t(f, \mu; \Delta) > \varepsilon, \varepsilon > 0\}$$

$$\tilde{\mu}(f, \mu) = \inf\{\mu' \geq \mu | f_+(f, \mu') - f > \varepsilon, \varepsilon > 0\}$$

and $\tilde{T}(f,\mu)$ is such that

$$(1-\tilde{\mu}(f,\mu)) = e^{-\lambda\tilde{T}(f,\mu)}(1-\mu)$$

$$\tilde{f}(f,\mu) = \lim_{\Delta\to 0} f_{+t(\Delta)}(f,\mu)$$

where $\lim_{\Delta\to 0} t(\Delta) = \tilde{T}(f,\mu)$.

We say that there is an **atom of trade** at (f,μ) if $f > f_+(\mu)$. We say that **limit trade is smooth** at (f,μ) if $f_+(\mu) = f$ and $\hat{T}(f,\mu) = 0$. We say that **limit trade is silent** at (f,μ) if $f_+(f,\mu) = f$ and $\hat{T}(f,\mu) > 0$. There is said to be **limit delay** at (f,μ) if limit trade is either smooth or silent at (f,μ).

Step 0: Ultimate Atom of Trade

Lemma 14. *There is no further limit delay at* (f,μ) *if* $\mu > \underline{\mu}(f)$

Proof. WLOG suppose $f_+(f,\mu) = 0$ and for all $\varepsilon > 0$, there is $g \in (f+\varepsilon, f]$ s.t. $g - f_+(g,\mu) = 0$. Suppose $f_n \to f$.

$$(v_H - \tilde{\delta}^{\Delta}(v_H - v_L)(1-\mu))(f_n - f_{+1}(f_n,\mu)) + \delta V(f_{+1}(f_n,\mu), \mu_{+1}(\mu)) \geq W(f_n,\mu)$$

Therefore,

$$\begin{aligned}
V(f,\mu) &= ((1-\mu)(v_H - v_L) - F(f_n,\mu))(f_n - f_{+1}(f_n,\mu)) \\
&\geq W(f_{+1}(f_n,\mu),\mu) - \delta V(f_{+1}(f_n,\mu), \mu_{+1}(\mu)) \\
\implies & \lim_{\Delta\to 0}\lim_{n\to\infty}((1-\mu)(v_H - v_L) - F(f_n,\mu))(f_n - f_{+1}(f_n,\mu)) \\
&\geq \lim_{\Delta\to 0}\lim_{n\to\infty} W(f_{+1}(f_n,\mu),\mu) - \delta V(f_{+1}(f_n,\mu), \mu_{+1}(\mu)) \\
\implies & 0 \geq r[v_H f + v_L] - (1-\mu_{T_\sigma})(v_H - v_L)(\lambda + r)f
\end{aligned}$$

But since $r[v_h f + v_L] = (1-\underline{\mu}(f))(v_H - v_L)f > (1-\mu)(v_H - v_L)(\lambda + r)f > (1-\mu_{T_\sigma})(v_H - v_L)(\lambda + r)f$, we have a contradiction. □

Step 1: Properties of a smooth trade region

We prove two main properties of a smooth trade region.

Lemma 15. *Suppose there is limit delay at (f,μ). Then, the seller's limit payoff at (f,μ) is $\delta^{\tilde{T}(f,\mu)}V(f,\tilde{\mu}(f,\mu))$.*

Proof. Let $\lim_{\Delta\to 0} t(T;\Delta) = T$, $p(f_{+1}(f_-(\mu;\Delta),\mu),\mu_{+1}) = p(f,\mu_{+1})$, let $T'(f,\mu) = \inf\{t \geq 0 | \lim_{\Delta\to 0} f_-(\mu_{t(t;\Delta)};\Delta) - f > 0\}\}$ and let $T''(f,\mu) = \min\{T'(f,\mu), \tilde{T}(f,\mu)\}$.

We first show that $V(f,\mu) = \delta^{T''(f,\mu)}V(f,\mu_{T''(f,\mu)})$.

Let $T < T''(f,\mu)$.[2]. Given Δ, consider the following function.

$$f'_-(\mu;\Delta) = \max\{f_-(\mu_{+t(\Delta)}(\mu;\Delta);\Delta), f'_-(\mu_{+(t+1)(\Delta)}(\mu;\Delta);\Delta)\} \quad \mu \in [\mu_{+t(\Delta)}(\mu;\Delta), \mu_{+(t+1)(\Delta)}(\mu;\Delta)]$$

Where $f'_-(\mu_{T-\Delta};\Delta) = f_-(\mu_{T-\Delta};\Delta)$. Note that $f'_-(\mu;\Delta)$ converges pointwise to f. Moreover, $f'_-(\mu;\Delta)$ is monotonic. Therefore, $f'_-(\mu;\Delta)$ converges uniformly (see Chapter 0, Resnick (2008)).

For any $\Delta > 0$, it must be the case that

$$\begin{aligned} &p(f_-(\mu;\Delta),\mu)(f_-(\mu;\Delta) - f) + \delta V(f,\mu_{+1}) \geq p(f,\mu)(f_-(\mu;\Delta) - f) + V(f,\mu) \\ &\implies (f_-(\mu;\Delta) - f)(p(f_-(\mu;\Delta),\mu) - p(f,\mu)) + \delta V(f,\mu_+) \geq V(f,\mu) \end{aligned} \tag{2.9}$$

Let $T(\Delta) \to T$ as $\Delta \to 0$. Iterating on Equation (2.9), we get

$$\sum_{j=0}^{T(\Delta)-1} \delta^j (f_-(\mu_{+j};\Delta) - f)(p(f_-(\mu_{+j};\Delta),\mu_{+j}) - p(f,\mu_{+j})) + \delta^{T(\Delta)}V(f,\mu_{T(\Delta)}) \geq V(f,\mu)$$

Note that for any $\varepsilon > 0$, there exists $\Delta' > 0$ s.t. $\sup_{t\leq T} |f'_-(\mu_t;\Delta) - f| < \varepsilon$ for all $\Delta < \Delta'$. This in conjunction with the fact that $p(f_-(\mu_{+j};\Delta),\mu_{+j}) - p(f,\mu_{+j}) \leq (1-\tilde{\delta})v_H$ and $f'_-(\mu;\Delta) \geq f_-(\mu;\Delta)$, implies that for all $\Delta < \Delta'$

[2] We let $T = \tilde{T}(f,\mu)$ if $\lim_{\Delta\to 0} f_-(\mu_{t(\tilde{T}(f,\mu);\Delta)}(\mu;\Delta);\Delta) - f = 0$

$$(1-\tilde{\delta})v_H\varepsilon \sum_{j=0}^{T(\Delta)-1} \delta^j + \delta^{T(\Delta)} V(f, \mu_{T(\Delta)})$$
$$\geq V(f,\mu) \tag{2.10}$$

Therefore,

$$\delta^T V(f, \mu_T) \geq V(f, \mu)$$

This follows from the fact the first term goes to zero as $(1-\tilde{\delta}^2)\sum_{j=0}^{T(\Delta)-1}\delta^j$ is bounded and ε can be made arbitrarily small.

Therefore $V(f,\mu) \leq \delta^T V(f,\mu_T)$ for all $T < T''(f,\mu)$ and by continuity of $V(f,.)$(see Lemma 17), we have that $V(f,\mu) \leq \delta^{T''(f,\mu)} V(f, \mu_{T''(f,\mu)})$.

Next, we show that there exists a sequence $t_n \to 0$, $f_{t_n}(f,\mu)$,$\mu_{t_n} = \mu_n$ s.t.

$$V(f,\mu) = \lim_{n\to 0} \delta^{\tilde{T}(f_n,\mu_n)} V(f_n, \tilde{\mu}(f_n, \mu_n))$$

Suppose limit delay is smooth at (f,μ). Then there exists $\varepsilon > 0, \nu > 0$ s.t. for all $s \in (o,\nu)$, $f - f_t(f,\mu_s) < \varepsilon$ implies $\tilde{T}(f,\mu_s(\mu)) \geq \tilde{T}(f_t(f,\mu_s),\mu_t(\mu_s)) \geq T''(f_t(f,\mu_s),\mu_t(\mu_s)) > 0$. Therefore $T''(f_t(f,\mu),\mu_t) > 0$ is bounded away from zero. If there exists a sequence $t_n \to 0$, such that $T''(f_n, \mu_n = \tilde{T}(f_n, \mu_n(\mu))$, then, $V(f_n,\mu_n) = \delta^{\tilde{T}(f_n,\mu_n)} V(f_n, \tilde{\mu}(f_n,\mu_n))$. Taking limits, by continuity, we have our result.

Suppose for any sequence $t_n \to 0$, there exists N s.t. for all $m > N$, $T''(f_{t_m}(f,\mu),\mu_{t_m}(f,\mu)) = T'(f_{t_m}(f,\mu),\mu_{t_m}(f,\mu)) < \tilde{T}(f_{t_m}(f,\mu),\mu_{t_m}(f,\mu))$. If for some $t \in (0,\varepsilon)$, $f_+(f', \mu_{T'(f_t(f,\mu),\mu_t(\mu))}(\mu)) = f_t$ for some $f' \geq f$, then $\tilde{T}(f,\mu) \leq T'(f_t(f,\mu),\mu_t(\mu))$. However, since $T'(f_t(f,\mu),\mu_t(\mu)) < \tilde{T}(f_{t_m}(f,\mu),\mu_{t_m}(f,\mu)) \leq \tilde{T}(f,\mu)$, we have a contradiction. Therefore, for all $t \in (0,\varepsilon)$ with $f_+(f', \mu_{T'(f_t(f,\mu),\mu_t(\mu))}(\mu)) = f_t(f,\mu)$ $f' < f$.

Let $t_o \in (0,\varepsilon)$ and let $f_+(f_{n+1},\mu_{n+1}) = f_n$. Therefore $T'(f_{n+1},\mu_{n+1}) <$
$\tilde{T}(f_{n+1},\mu_{n+1}) \leq T'(f_{t_n}(f,\mu),\mu_{t_n}(\mu))$. Therefore,

$$
\begin{aligned}
V(f_n, \mu_n(\mu)) &= \delta^{T'(f_n(f,\mu),\mu_n)} V(f_n(f,\mu), \mu_{T'(f_n(f,\mu),\mu_n)}) \\
&< \delta^{T'(f_n,\mu_n)}[(f_{n+1} - f_n)p(f_n, \mu_{T'(f_{t_n}(f,\mu),\mu_{t_n}(\mu))}) + \\
&\qquad V(f_n, \mu_{T'(f_{t_n}(f,\mu),\mu_{t_n}(\mu))})] \\
&= \delta^{T'(f_{n+1}(f,\mu),\mu_{n+1})} V(f_{n+1}, \mu_{T'(f_{n+1},\mu_{n+1})}) \\
&\le \delta^{\tilde{T}(f_{n+1},\mu_{n+1})} V(f_{n+1}, \mu_{\tilde{T}(f_{n+1},\mu_{n+1})})
\end{aligned}
$$

Taking limits on both sides, we have $V(f,\mu) \le \lim_{n\to\infty} \delta^{\tilde{T}(f_n,\mu_n)} V(f_n, \tilde{\mu}(f_n,\mu_n))$. Further, note that $V(f,\mu) \ge \delta^{\tilde{T}(f_n,\mu_n)} V(f, \tilde{\mu}(f_n,\mu_n)) \ge \delta^{\tilde{T}(f_n,\mu_n)} V(f_n, \tilde{\mu}(f_n,\mu_n))$. Therefore, we have our result.

Let T_n be the sup of times such that $V(f_n,\mu_n) = \delta^{T_n} V(f_n, \mu_{T_n}(\mu_n))$. We show that $T \equiv \limsup_{n\to\infty} T_n = \tilde{T}(f,\mu)$. If $T < \tilde{T}(f,\mu)$, then trade is smooth at $(f, \mu_T(\mu))$. Therefore it possible to find a sequence (f_m, μ_m) such that $V(f,\mu) = \delta^T V(f, \mu_T(\mu)) = \lim_{m\to\infty} \delta^{\tilde{T}(f_m,\mu_m)+T} V(f_m,\mu_m)$. Since $\tilde{T}(f_m,\mu_m) > 0$ is bounded away from zero, we have a contradiction.

Since, $V(f,\mu) \ge \delta^{\tilde{T}(f,\mu)} V(f, \mu_{\tilde{T}(f,\mu)})$ (since the seller can always stall trade), we have our result. □

Next, we show that for (f,μ) in a smooth trade region, $p(f,\mu) \le \delta^{\tilde{T}(f,\mu)} p(f, \mu_{\tilde{T}(f,\mu)})$. The proof is similar to Lemma 5 in FS. However, for the sake of completeness, we provide the proof here.

Lemma 16. *Suppose trade is smooth at (f,μ). Then $p(f,\mu) \le \delta^{\tilde{T}(f,\mu)} p(f, \tilde{\mu}(f,\mu))$. Moreover if $\lim_{g\uparrow f} \delta^{\tilde{T}(g,\mu)} p(g, \tilde{\mu}(g,\mu)) = \delta^{\tilde{T}(f,\mu)} p(f, \tilde{\mu}(f,\mu))$, then $p(f,\mu) = \delta^{\tilde{T}(f,\mu)} p(f, \tilde{\mu}(f,\mu))$*

Proof. Suppose $p(f,\mu) > \delta^{\tilde{T}(f,\mu)} p(f, \tilde{\mu}(f,\mu))$. By continuity of the reservation price function, there must exist a $T > 0$ s.t. for all $t < T$ and a sequence $t(t,\Delta)$ with $\lim_{\Delta\to 0} t(t,\Delta) = t$, $\lim_{\Delta\to 0} p(f_{+t(\Delta)}(f,\mu), \mu_{+t(t,\Delta)}) > \delta^{\tilde{T}(f,\mu)-t} p(f, \tilde{\mu}(f,\mu))$ and $\lim_{\Delta\to 0} f_{+t(T,\Delta)}(f,\mu) < \tilde{f}(f,\mu)$. Because trade is smooth, we are guaranteed that $f_T < f$. Therefore,

$$
\sum_{j=0}^{t(T,\Delta)-1} \delta^j p(f_{+j}(f,\mu), \mu_{+j})(f_{+j}(f,\mu) - f_{+(j+1)}(f,\mu)) + \delta^{t(T,\Delta)} V(f_{+t(T,\Delta)}(f,\mu), \mu_{+t(T,\Delta)})
$$

We note that $\delta^j p(f_{+j}(f,\mu),\mu_{+j}) > \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))$ for Δ small and $\lim_{\Delta\to 0} V(f_{+t(T,\Delta)}(f,\mu),\mu_{+t(T,\Delta)}) \geq \delta^{\tilde{T}(f,\mu)-T}\hat{V}(f_T,\tilde{\mu}(f,\mu))$ (where $\hat{V}(f_T,\tilde{\mu}(f,\mu))$ is the seller's limit payoff at $(f_T,\tilde{\mu}(f,\mu))$ from offering $p(f,\tilde{\mu}(f,\mu))$ at (f_T,μ_T)). By taking limits, we have

$$V(f,\mu) > \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))(f - f_{t(T,\Delta)}(f,\mu)) + \delta^{\tilde{T}(f,\mu)}\hat{V}(f_T,\tilde{\mu}(f,\mu)) > \delta^{\tilde{T}(f,\mu)} V(f,\tilde{\mu}(f,\mu))$$

which contradicts Lemma 15. Next, suppose $p(f,\mu) < \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))$ and $\lim_{g\uparrow f} \delta^{\tilde{T}(g,\mu)} p(g,\tilde{\mu}(g,\mu)) = \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))$. Since $\lim_{g\uparrow f} \delta^{\tilde{T}(g,\mu)} p(g,\tilde{\mu}(g,\mu)) = \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))$, there exists $T > 0$ and a sequence $t(t,\Delta)$ with $\lim_{\Delta\to 0} t(t,\Delta) = t$ s.t. for $p(f,\mu) < \delta^{\tilde{T}(f_T,\mu_T)} p(f_T,\tilde{\mu}(f_T,\mu_T))$. Therefore by taking limits, we have

$$V(f,\mu) < \delta^{\tilde{T}(f_T,\mu_T)} p(f_T,\tilde{\mu}(f,\mu_T))(f - f_T) + \delta^{\tilde{T}(f_T,\mu_T)}\hat{V}(f_T,\tilde{\mu}(f_T,\mu_T)) \leq \delta^{\tilde{T}(f_T,\mu_T)}\hat{V}(f,\tilde{\mu}(f_T,\mu_T))$$

where the second inequality follows from the upper bound on $p(f,\mu)$ and the fact that $V(f_T,\mu_T) = \delta^{\tilde{T}(f_T,\mu_T)-T} V(f_T,\tilde{\mu}(f_T,\mu_T))$. Therefore, we get a contradiction.

□

Step 2: Seller's payoff in region of delay

We first show that $V(f,\mu)$ is continuous and weakly increasing in both its arguments.

Lemma 17. $V(f,.)$ *is weakly increasing and continuous*

Proof. Suppose $V(f,.)$ is not increasing. Then for some μ and $\mu' < \mu$, there exists a $\Delta' > 0$ s.t. for all $\Delta < \Delta'$, $V(f,\mu';\Delta) > V(f,\mu(\Delta);\Delta)$, where $\lim_{\Delta\to 0}\mu(\Delta) = \mu$. We show that this inequality cannot hold for any $\Delta > 0$. We first note that $V(f^*_{T-1},\mu_{T-1}(\Delta);\Delta) \leq V(f^*_T,\mu_T(\Delta);\Delta)$ at the points of indifference. Next, we show that if $V(f,\mu_{+1}(\mu;\Delta);\Delta) \geq V(f,\mu;\Delta)$ when the seller is indifferent between ending trade in T and $T-1$ periods at $(f,\mu_{+1}(\mu;\Delta))$, then if the seller is indifferent between ending trade in T and $T+1$ periods at (f',μ), we have $V(f',\mu_{+1}(\mu;\Delta);\Delta) \geq V(f',\mu;\Delta)$. Suppose not. Then,

$$\begin{aligned}(f'-f)(v_H - \delta(v_H - p(f,\mu_{+1}(\mu;\Delta)))) + \delta V(f,\mu_{+1}) > (f'-f)p(f,\mu_{+1}(\mu;\Delta)) + V(f,\mu_{+1}(\mu;\Delta))\\ \implies (f'-f)(v_H - p(f,\mu_{+1}(\mu;\Delta))) > V(f,\mu_{+1}(\mu;\Delta))\end{aligned}$$

By optimality, we also have

$$
\begin{aligned}
(f'-f)p(f,\mu)+V(f,\mu) &\geq (f'-f)(v_H-\delta(v_H-p(f,\mu_{+1}(\mu;\Delta))))+\delta V(f,\mu_{+1}(\mu;\Delta)) \\
\implies V(f,\mu)-\delta V(f,\mu_{+1}(\mu;\Delta)) &= (f'-f)[(1-\delta)v_H+\delta p(f,\mu_{+1}(\mu;\Delta))-p(f,\mu)]
\end{aligned}
\tag{2.11}
$$

Evaluating $(1-\delta)(v_H-p(f,\mu_{+1}(\mu;\Delta)))$, we get $(1-\delta)\tilde{\delta}\tilde{\delta}^{T-1}(v_H-v_L)(1-\mu_{+1}(\mu;\Delta))$. Evaluating the second term on the right hand side of Equation (2.11), we get $(f'-f)\delta(1-\tilde{\delta})\tilde{\delta}^{T-1}(v_H-v_L)(1-\mu_{+1}(\mu;\Delta))$. Therefore,

$$
\begin{aligned}
V(f,\mu)-\delta V(f,\mu_{+1}(\mu;\Delta)) &= (f'-f)[(1-\delta)v_H+\delta p(f,\mu_{+1}(\mu;\Delta))-p(f,\mu)] \\
\implies V(f,\mu_{+1}(\mu;\Delta))(1-\delta) &\geq (f'-f)(1-\delta)\tilde{\delta}\tilde{\delta}^{T-1}(v_H-v_L)(1-\mu_{+1}(\mu;\Delta)) \\
\implies V(f,\mu_{+1}(\mu;\Delta))(1-\delta) &\geq (f'-f)(1-\delta)(v_H-p(f,\mu_{+1}(\mu;\Delta)))
\end{aligned}
$$

which yields a contradiction. Therefore at any belief f that is a cutoff point at (f,μ), it must be the case that $V(f,\mu_{+1}(\mu;\Delta))\geq V(f,\mu)$. We now show that the result can be extended to points that are not cutoff points. Suppose (f',μ) is not a cutoff point. Let $f>f'$ be such that $p(f,\mu)=p(f',\mu)$ and $p(f,\mu_{+1}(\mu;\Delta))=p(f',\mu_{+1}(\mu;\Delta))$ and (f,μ) is a cutoff point. If $p(f',\mu)>p(f',\mu_{+1}(\mu;\Delta))$, we have

$$
\begin{aligned}
(f-f')p(f,\mu) &> (f-f')p(f,\mu_{+1}(\mu;\Delta)) \\
\implies V(f,\mu)-V(f'.\mu) &> V(f,\mu_{+1}(\mu;\Delta))-V(f',\mu_{+1}(\mu;\Delta)) \\
\implies V(f,\mu)-V(f,\mu_{+1}(\mu;\Delta)) &> V(f',\mu)-V(f',\mu_{+1}(\mu;\Delta))
\end{aligned}
$$

since $V(f,\mu)-V(f,\mu_{+1}(\mu;\Delta))<0$, we have that $V(f',\mu)<V(f',\mu_{+1}(\mu;\Delta))$.

Suppose $p(f',\mu)\leq p(f',\mu_{+1}(\mu;\Delta))$. Let $f''<f'$ be such that $p(f',\mu)$ is optimal at (f'',μ) and $p(f',\mu_{+1}(\mu;\Delta))$ is optimal at $(f'',\mu_{+1}(\mu;\Delta))$ and either (f'',μ) or $(f'',\mu_{+1}(\mu;\Delta))$ is a cutoff point. Then, we have

$$
\begin{aligned}
V(f',\mu) &= p(f',\mu)(f'-f'') + V(f'',\mu) \\
&\leq p(f',\mu_{+1}(\mu;\Delta))(f'-f'') + V(f'',\mu_{+1}(\mu;\Delta)) \\
&= V(f',\mu_{+1})(\mu;\Delta))
\end{aligned}
$$

Therefore, $V(f,.)$ is weakly increasing. Continuity follows from the fact that $V(f,\mu) \geq \delta^T V(f,\mu_T)$ for any μ and T. Taking T to zero, we have $V(f,\mu) \geq \lim_{T\to 0} V(f,\mu_T)$. Since $\lim_{T\to 0} V(f,\mu_T) \geq V(f,\mu)$ by monotonicity, we have our result.

□

Suppose $\tilde{T}(f,\mu) > 0$ and for some $f' < f$, $V(f',\mu) > \delta^T V(f',\mu_T(\mu))$ for any $T > 0$. Moreover, by Lemma 15, there must be an atom at (f',μ). Note that there is limit delay at $f_+(f',\mu)$.

Claim 2. *There is limit delay at* $(f_+(f',\mu),\mu)$

Proof. Suppose there is no limit delay at $(f_+(f',\mu),\mu)$. This implies that there exists $f'' < f_+(f',\mu)$ s.t. $\lim_{\Delta\to 0} T(f_+(f',\mu), f'', \mu;\Delta)\Delta = 0$. Moreover we have, $\lim_{\Delta\to 0} T(f', f_+(f',\mu),\mu;\Delta)\Delta = 0$ by definition. Since $T(f',f'',\mu;\Delta) = T(f',f_+(f',\mu),\mu;\Delta) + T(f_+(f',\mu),f'',\mu;\Delta)$, we have that $\lim_{\Delta\to 0} T(f',f'',\mu;\Delta) = 0$, contradicting the definition of $f_+(f',\mu)$. □

Next, we show that trade cannot be smooth at $f_+(f',\mu)$.

Claim 3. *Trade cannot be smooth at* $(f_+(f',\mu),\mu)$

Proof. Suppose it is smooth. Then the limit payoff at (f',μ) is

$$
\begin{aligned}
V(f',\mu) &= p(f_+(f',\mu),\mu)(f' - f_+(f'(\mu))) + V(f_+(f',\mu),\mu) \\
&\leq \delta^{\tilde{T}(f_+(f',\mu),\mu)} p(f_+(f',\mu),\mu(\tilde{T}(f_+(f',\mu),\mu))) + \delta^{\tilde{T}(f_+(f',\mu),\mu)} V(f_+(f',\mu),\mu(f_+(f',\mu),\mu)) \\
&\leq \delta^{\tilde{T}(f_+(f',\mu),\mu)} V(f',\tilde{\mu}(f_+(f',\mu),\mu))
\end{aligned}
$$

where the first inequality follows from Lemma 16 and Lemma 15. This inequality, however, contradicts the fact that $V(f',\mu) > \delta^T V(f,\mu_T(\mu))$ for all $T > 0$. □

Therefore, limit delay is silent at $(f_+(f',\mu),\mu)$.

Suppose there is limit delay on some interval I. Let $T_L(x,y) = \sup\{t \geq 0 | V(x,y) = \delta^t V(x, \mu_t(y))\}$, i.e., $T_L(x,y)$ is the maximum length of delay at (x,y) and there is an atom of trade at time T_L. Let I be such that $\min_{x\in I} T_L(x,\mu) > 0$. Note that if there is an atom of size $x - f_+(x,m)$ at (x,m), then there is an atom at (y,m) for $y \in (x, f_+(x,m))$. Therefore there is an interval $I' \subset I$ s.t. for all $x, x' \in I'$, $T_L(x,\mu) = T_L(x',\mu)$. This in turn implies that $\dfrac{\partial V(x,\mu)}{\partial f} = \dfrac{\partial V(x',\mu)}{\partial f} = \delta^{T_L(x,\mu)-T(\mu,m)} p(x, \mu_{+T_L(x,m)}(\mu))$ for all $m \in [\mu, \mu_{+T_L(x,\mu)}(\mu)]$,i.e., delay is not smooth over the interval I' when the belief is μ.

Therefore, it must be the case that if there is an interval I s.t. there is limit delay at (x,μ) for all $x \in I$, then $V(x,\mu) = W(x,\mu)$. We first note that smooth trade cannot be preceded by an atom of trade. So if there is an atom of trade, it must be followed by silent delay. Second, we argue that if there is silent delay at (x,μ), then for every $m > \mu$ there is an interval (x,x') s.t. for all $y \in (x,x')$, $V(y,m) = W(y,m)$. If not, then there is an $m > \mu$ s.t. $V(y,m) > W(y,m)$ for all y close to x. This in turn implies that there is an atom of trade at (y,m) followed by silent trade at $(f_+(y,m),m)$. We also have that $V(y,b) > \delta^t V(y, \mu_{+t}(b))$ for all $t > 0, b > \mu$. Therefore, there is an atom of trade at (y,b) for all y close to x. Therefore, at (x,y) where $z \in (\mu, m)$, the seller is indifferent between trading with probability $x - f_+(y,z))$ where $y < x$ is close to x. We first show that $f_+(y,.)$ cannot be constant in $[\mu, \min\{m, \hat{T}(f_+(y,\mu),\mu)\})$.

For $z \in (\mu, m)$, let

$$F_L(x,z) = (x - f_+(y,\mu))p(f_+(y,\mu),z) + V(f_+(y,\mu),z)$$

We first note that $F_L(x,\mu) > \delta^s F_L(x, \mu_{+s}(\mu))$ for all $s \in (0, T(m, \min\{m, \hat{T}(f_+(y,\mu),\mu)\}))$, $m \in [\mu, \min\{m, \hat{T}(f_+(y,\mu),\mu)\})$. Suppose not. Then,

$$\begin{aligned}
F_L(x,\mu_{+s}(\mu)) &= (x - f_+(y,\mu)p(f_+(y,\mu),\mu_{+s}(\mu)) + V(f_+(y,\mu),\mu_{+s}(\mu)) \\
\implies \delta^s V(x,\mu_{+s}(\mu)) &= \delta^s(x - f_+(y,\mu)p(f_+(y,\mu),\mu_{+s}(\mu)) + \\
&\quad \delta^s V(f_+(y,\mu),\mu_{+s}(\mu)) \\
\implies V(x,\mu) &= \delta^s(x - f_+(y,\mu)p(f_+(y,\mu),\mu_{+s}(\mu)) + V(f_+(y,\mu),\mu)
\end{aligned}$$

Since $V(x,\mu) = F_L(x,\mu) = p(f_+(y,\mu),\mu)(x - f_+(y,\mu)) + V(f_+(y,\mu),\mu)$, we have $p(f_+(y,\mu),\mu) = \delta^s p(f_+(y,\mu),\mu_{+s}(\mu))$, but this contradicts the fact that prices are decreasing in time.

Next, we show that $f_+(y,.)$ cannot be increasing in time.

Lemma 18. *$f_+(y,.)$ cannot be strictly increasing*

Proof. Let $z_n \uparrow z$ and WLOG let $f_+(y,z) = \lim_{n\to\infty} f_+(y,z_n)$. By continuity of the payoff function, $\lim_{n\to\infty} V(y,z_n) = V(y,z)$, i.e.,

$$\lim_{n\to\infty} p(y,z_n)(y - f_+(y,z_n)) + V(f_+(y,z_n),z_n) = p(y,z)(y - f_+(y,z)) + V(f_+(y,z),z)$$
$$\implies \lim_{n\to\infty} p(y,z_n) = p(y,z)$$

This, in particular, implies that $\lim_{n\to\infty} \overline{T}(f_+(y,z_n),z_n) = \overline{T}(f_+(y,z),z)$. So for all $\varepsilon > 0$, there exists N s.t. for all $n > N$, $|\overline{T}(f_+(y,z),z) - \overline{T}(f_+(y,z_n),z)| < \varepsilon$. Suppose m is s.t. $|\delta^{\overline{T}(f_+(y,z),z)}(1-\mu_{\overline{T}(f_+(y,z),z)}) - \delta^{\overline{T}(f_+(y,z_m),z)}(1-\mu_{\overline{T}(f_+(y,z_m),z)})| < v_H(1-\delta^{\tilde{T}(f_+(y,z),z)})/2$. The reservation price at $(f_+(y,z_m),y)$ is $P(f_+(y,z_m),y) = v_H - \tilde{\delta}^{\overline{T}(f_+(y,z_m),z)}(v_H - v_L)(1-z)$. Therefore,

$$
\begin{aligned}
&V(f_+(y,z),z) - [P(f_+(y,z_m),z)(f_+(y,z) - f_+(y,z_m)) + V(f_+(y,z_m),z)]\\
&\le \delta^{\tilde{T}(f_+(y,z),z)}[p(f_+(y,z),\tilde{\mu}(f_+(y,z),z))(f_+(y,z) - f_+(y,z_m)) + V(f_+(y,z_m),\tilde{\mu}(f_+(y,z),z))]\\
&\quad -[P(f_+(y,z_m),z)(f_+(y,z) - f_+(y,z_m)) + V(f_+(y,z_m),z)]\\
&\le (f_+(y,z) - f_+(y,z_m))(\delta^{\tilde{T}(f_+(y,z),z)}p(f_+(y,z),\tilde{\mu}(f_+(y,z),z)) - P(f_+(y,z_m),z)\\
&= (f_+(y,z) - f_+(y,z_m))[\delta^{\overline{T}(f_+(y,z),z)}(v_H - v_L)(1-\mu_{\overline{T}(f_+(y,z),z)})\\
&\quad -\delta^{\overline{T}(f_+(y,z_m),z)}(1-\mu_{\overline{T}(f_+(y,z_m),z)})(v_H - v_L) - v_H(1-\delta^{\tilde{T}(f_+(y,z),z)})] < 0
\end{aligned}
$$

which contradicts the optimality of the seller's strategy at $(f_+(y,z),z)$.

□

Therefore there exists $\varepsilon > 0$ s.t. for all $y \in (x, x+\varepsilon)$, $V(y,\mu) = W(y,\mu)$. This in turn means that $V(x,\mu) = W(x,\mu)$. We next show that if there is a gap at (g,μ) then there cannot be limit delay at (f,μ) for $f > g$. Suppose there is silent delay at (g,μ) and limit delay at (f,μ) for $f > g$. Let $T'(f,g;\mu)$ denote the minimum time taken for the state to transition from f to g. For every $t < \hat{T}(g,\mu)$, there is an atom of trade that precedes the gap at $(g,\mu_{+t}(\mu)$. Let $r(m) = \inf\{x > g | \tilde{T}(x,m) > 0\}$. Note that there is necessarily limit delay at (x,m) for $x > r(m)$ close to $r(m)$. Moreover, $r(.)$ is decreasing.

Claim 4. *Let $m \in (\mu, \hat{\mu}(g,\mu))$. If $x < r(m)$, then $f_{+t}(x,m) \geq r(\mu_{+t}(m))$ for all $t \in [0, \hat{T}(g,\mu)]$.*

Proof. Note that by definition of $r(.)$, $T'(x,g) > 0$. Suppose $f_{+t}(x,m) < r(\mu_{+t}(m))$ for some $t \in (0, \hat{T}(g,\mu))$, then the price at time t is $p(g, \mu_{+t}(m))$. Therefore the all prices following the initial price at (x,m) are strictly less than $v_H(1-\delta^t) + \delta^t p(g, \mu_{+t}(m)) = p(g,m)$. Since there is limit delay between g and x, the payoff would be lower than the payoff from charging $p(g,m)$, yielding a contradiction. □

Corollary 1. $r(.)$ *is strictly decreasing*

Proof. That $r(.)$ is weakly decreasing follows from the previous claim. Suppose for some $m > m'$, $r(m) > r(m')$. Then $f_{+T(m',m)}(r(m), m') \leq r(m')$. Note that equality cannot hold because the seller would be better off offering $p(g,m')$. Therefore, we get a contradiction. Suppose $r(.) = l$ over some interval $[k_1, k_2]$. For $x > l$ close to l and $\varepsilon > 0$ small, the payoff at (x, k_1) is atmost

$$
\begin{gathered}
(x - l + \varepsilon)v_H + \delta^{T(k_1,k_2)} V(l-\varepsilon, k_2) \\
< V(l-\varepsilon, k_1)
\end{gathered}
$$

Since, $V(x,k_1) \geq V(g,k_1)$, we get a contradiction. □

First we note that $V(r(m),m) = W(r(m),m)$ and let $m' = \mu_{+\hat{T}(g,\mu)}(\mu)$. Next, we have

$$
W(r(m),m) = (r(m)-g)[v_H(1-\delta^{\hat{T}(g,m)} + \delta^{\hat{T}(g,m)} p(g,m')] + W(g,m)
$$

Let $m_+ = \mu_{+t}(m)$. Noting that $v_H(1-\delta^{\hat{T}(g,m)}) + \delta^{\hat{T}(g,m)} p(g,m') = v_H(1-\delta^t) + \delta^t p(g,m_+)$, we have,

$$
\delta^t W(r(m),m_+) - \delta^t W(r(m_+),m_+) = (r(m)-r(m_+))p(g,m) + (r(m_+)-g)v_H(1-\delta^t)
$$

Dividing by t and taking $t \to 0$

$$
\begin{aligned}
-\frac{\partial W(r(m),m)}{\partial f}\frac{dr(m)}{d\mu}\frac{d\mu}{dt} &= -\frac{dr(m)}{d\mu}\frac{d\mu}{dt}p(g,m) + (r(m)-g)rv_H \\
\implies -\frac{dr(m)}{d\mu}\frac{d\mu}{dt}\left[\frac{\partial W(r(m),m)}{\partial f} - p(g,m)\right] &= (r(m)-g)rv_H \qquad (2.12)
\end{aligned}
$$

Since $r(.)$ denote the indifference points, the rate of trade is given by $\frac{dr(.)}{dt}$. Since $r(m_t) \to g$ as $t \to \hat{T}(g, m_t)$, we have that $p(r(m), m) = p(g, m)$, which contradicts Equation (2.12).

Therefore, if there is a at (g, μ), there cannot be any limit delay at (f, μ) for $f > g$. But this once again, yields a contradiction as $m \to \hat{\mu}(g, \mu)$, $V(f, m) < W(f, m)$.

Chapter 3

Bargaining with Unobservable Investments in Outside Options

3.1 Introduction

In this paper, we study a bargaining game between a seller who has an outside option and a buyer who has private information about his valuation. A costless investment must be made by the seller in each period in order to maintain the outside option. In this setting, we study the optimal investment strategy for the seller.

I construct an equilibrium in which the seller stops investing on the equilibrium path. Under some conditions, this equilibrium gives the seller a higher payoff than an equilibrium in which she maintains investment in the outside option. In equilibrium, the seller makes an offer and if this offer is rejected, exits with a very high probability in the second period. In equilibrium, if the seller does not exit after the first offer is rejected, the buyer believes it to be unlikely that the seller has access to the outside option, and is therefore unwilling to accept a high price in the second period. This rationalizes the seller's decision to exit at the beginning of the second period.

3.2 Model

Time is discrete and indexed by $t = 0, 1, ...$ A seller(she) bargains with a buyer(he) whose value is privately known to him. The buyer's type depends on the realization of a random variable $q \sim U[0, 1]$ and is given by

$$v(q) = v_i \quad q \in (q_{i+1}, q_i] \tag{3.1}$$

for $i = 1, 2, 3$, with $q_4 = 0$ and $q_1 = 1$. The seller makes an unobservable investment in each period to maintain her outside option. The outside option is available to the seller in any given period, if she has invested in all previous periods. If, in any period, the seller ceases to invest in the outside option, the outside option becomes unavailable to her in all future periods. The seller's investment decision is unobservable to the buyer. In worker-firm negotiations, for example, this would mean that the worker chooses whether or when to (irreversibly) stop searching for other employment opportunities. Further, the firm does not observe the worker's search process. We describe the timeline below. The seller's payoff from exercising her outside option is $\alpha v(q)$, where q is the buyer's type and $\alpha < 1$.

- Seller decides whether to continue investing in outside option, if she has invested in all previous periods.
- Seller decides whether or not to opt out.
- If she doesn't opt out, she makes an offer to the buyer.
- The buyer chooses whether or not to accept the offer

3.3 Strategies

Let o^{t-1} denote the availability of the outside option at the beginning of period t, where $o^{t-1} = 1$ if the outside option is available at the beginning of period t and zero otherwise. We denote the seller's time t private history by $\{o^s\}_{s=1}^t$. A public history is a sequence of past price offers. A (pure) investment strategy is a function $\sigma_i^t : \mathcal{H}^t \times \{0, 1\} \to \{0, 1\}$ with the constraint that $\sigma_i^t(h^t, 0) = 0$ for any time t history where the seller has ceased to invest in the past. If the seller ceases to invest in period t, then $o^t = 0$. The seller's exit strategy is given by $\sigma_e^t : \mathcal{H}^t \times \{0, 1\} \to [0, 1]$ where $\sigma_e^t(h^t, 0) = 0$ for any time t history at any period t. For any $h^t \in \mathcal{H}^t$, $\sigma_e^t(h^t, 1)$ denotes the probability with which the seller opts out. A pure offer strategy for the seller is a function $\sigma_p^t : \mathbf{H}^t \times \{0, 1\} \to \mathbf{R}_+$.

We consider two classes of equilibria-the full investment equilibrium and the partial investment equilibrium. An equilibrium is a *full investment equilibrium* if $\sigma_i^t(h^t, 1) = 1$

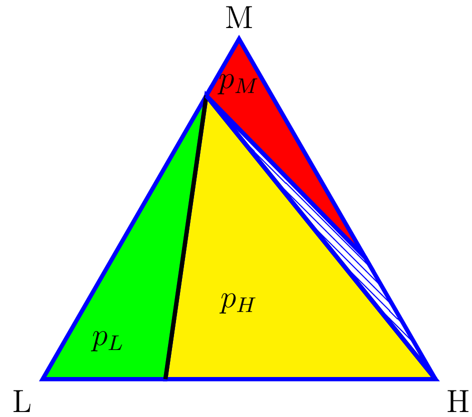

Figure 3.1: Partial Investment Equilibrium

for any history h^t at any time t. An equilibrium is a *partial investment equilibrium* if it is not a full investment equilibrium.

We construct a partial investment equilibrium that does better than a full investment equilibrium.

Since the equilibrium outcome of the full investment equilibrium we construct is identical to the equilibrium outcome when the seller always has the outside option, we omit the construction.

3.4 A Partial Investment Equilibrium: Example

We first illustrate the partial investment equilibrium with a three price example. Suppose there are three feasible prices: $\{p_L, p_M, p_H\}$.

We now look at a partial investment equilibrium which yields a higher payoff than any full investment equilibrium. Consider the following equilibrium play

- The seller destroys the outside option with a small probability ε
- Negotiation Stage: The seller's first offer is p_H which is accepted by all types with value H
- Exit Stage: At the beginning of the second period, the seller exits with some positive probability. The belief that the seller has ceased to invest in the outside option if she doesn't exit is $\dfrac{v_2 - p_M}{\delta(v_2 - p_L)}$

- Deadlock Stage: If the seller does not exit, she makes no offer for T^* periods[1]before making an offer p_M.
- Negotiation Stage II: At the end of T_1^* periods p_M is offered which is accepted by all types between $\hat{q}$ and q_3
- Exit Stage II: After p_M is offered, the seller who is able to exit does so, while the type that has destroyed its outside option offers p_L

Where T^* is the extent of delay that makes the seller indifferent between opting out and offering p_M and $\hat{q}$ is the belief that makes the seller indifferent between offering v_L and exercising her outside option. The above equilibrium play is supported by the following off path play

- If the proposer offers p_M when some types with valuation H are yet to accept an offer, all types till $\hat{q}$ accept the offer, following which there is suitable randomization
- If the proposer deviates by offering p_M during the deadlock stage, the respondent rejects the offer. In the next period, the proposer exits if she has an outside option and offers p_L otherwise
- If at any stage p_L is offered, it is accepted by all types

The blue checked region in Figure 3.1 represents the region of beliefs at which a partial investment equilibrium does better.

3.5 A Partial Investment Equilibrium

We now analyze the general model. Much of the intuition from the example in the previous section carries over to the general case. However, in the general model, the equilibrium does not feature delay.

Proposition 1. *Suppose $v_1/v_2 < v_2/v_3$. Then for any α in $(v_1/v_2, v_2/v_3)$ there exist functions $q_2(\alpha)$ and $q_3(q_2, \alpha)$ s.t. for $q_2 \in (q_2(\alpha), 1)$ and $q_3 \in (q_3(q_2, \alpha), \overline{q}_1(q_2, \alpha))$, there is a partial investment equilibrium for Δ small enough, that yields a limit payoff higher than the limit payoff from any full investment equilibrium.*

[1] We take a sequence of Δ such that T^* is an integer

Why might a partial investment equilibrium fare better than a full investment equilibrium? When the seller retains her outside option with probability one, she is tempted to leverage her outside option to extract rents from the lower buyer types. This implies that high type buyers would rather wait for prices to fall in the future than accept a high price today. Therefore, the seller is unable to extract a high price from the high types. However, when the seller underinvests with a small probability, the buyer becomes skeptical about the availability of the outside option over time. This means that the low buyer types are less likely to accept a high offer. This, in turn enables the seller to exit early in the game. Because the seller exits early in the game, she is able to extract rents from the high type.

3.6 Proof

Let $q_2(\alpha) \equiv (1-\alpha)v_1/\alpha(v_2-v_1)$, $\overline{q}_1(q_2,\alpha) = \overline{q}_1{}^2$, $q_3(q_2,\alpha) = \max\{\overline{q}_1 - \frac{v_1(1-q_2)}{v_3-v_1}, \frac{\overline{q}_1(q_2,\alpha)(v_2(1-\alpha))}{v_3-\alpha v_2}\}$, $\underline{\alpha} = v_1/v_2$ and $\overline{\alpha} = v_2/v_3$.

Let $v_1/v_2 < \alpha < v_2/v_3$. We verify that $\overline{q}_1 \in (q_3, q_2)$. First, $q_2 > (1-\alpha)v_1/\alpha(v_2-v_1)$ implies that $\overline{q}_1$ is between zero and q_2. Next, note that $\overline{q}_1 > q_3(q_2,\alpha)$ since $v_2(1-\alpha) < v_3-\alpha v_2$ and $\overline{q}_1 > 0$. So it is possible to choose a q_3 in the relevant range, i.e., $\overline{q}_1(q_2,\alpha) > q_3$. So, $\overline{q}_1 \in (q_3, q_2)$. Next, since $\alpha v_3 < v_2$, $\overline{q}_1 v_2 + \alpha W(\overline{q}_1) > q_3\alpha v_3 + (\overline{q}_1 - q_3)\alpha v_2 + \alpha W(\overline{q}_1) = \alpha W(0)$. So the limit payoff in any retain equilibrium is $v_2\overline{q}_1 + \alpha W(\overline{q}_1)$.

We construct a partial investment equilibrium (for Δ small enough) in which the limit payoff is $v_3 q_3 + \alpha W(q_3)$. We first verify that this payoff is higher than any full investment equilibrium payoff. We see that the partial investment payoff is higher if $\overline{q}_1 < \frac{(v_3-\alpha v_2)q_3}{v_2(1-\alpha)}$, i.e., $q_3 > \frac{\overline{q}_1(q_2,\alpha)(v_2(1-\alpha))}{v_3-\alpha v_2}$.

Let Δ be small enough that $S_1^F(q_\Delta) = \phi$ (where $S_1^F(.)$ is as defined in the existence section), $v_2(\overline{q}_1(\Delta) - q_3 + \delta\alpha W(\overline{q}_1(\Delta)) - \alpha W(q_3) > 0$ and $\overline{q}_1(\Delta) - q_3 < \frac{v_1(1-q_2)}{v_3-v_1}$ (this is possible since $\overline{q}_1 - q_3 < \frac{v_1(1-q_2)}{v_3-v_1}$). Since the reservation price schedule is flat between cutoffs and since $t(q) - q$ is atleast $\frac{v_1(1-q_2)}{v_3-v_1}$, this ensures that the price offer is the same at q_Δ and q_3 in the absence of outside options. Let μ_Δ be s.t. $(v_2 - \delta(1-\mu_\Delta)(v_2 -$

[2] i.e., the belief which makes the seller indifferent between offering v_1 and exiting.

$P(t(\overline{q}_1(\Delta)))))(\overline{q}_1(\Delta)-q_3)+\delta W(\overline{q}_1(\Delta))-\alpha W(q_3)=0$. Existence of such a μ_Δ is guaranteed by the fact that the left hand side is strictly negative for $\mu=0$ and strictly positive for $\mu=1$ and the left hand side is continuous in μ. Consider the following on-path play

- In the first period, the seller ceases to invest with a small probability ε less than one
- She then makes the offer $p(0)=v_3-\delta(1-\varepsilon)(v_3-p(q_3))$, where $p(q_3)$ is the offer made at q_3 (to be described shortly). This offer is accepted by all types till q_3
- In the second period, she exits with probability $1-\mu_\Delta\varepsilon/(1-\mu_\Delta)(1-\varepsilon)$. If she doesn't exit, she makes an offer $p(q_3)=v_2-\delta(1-\mu_\Delta)(v_2-P(t(\overline{q}_1(\Delta))))$. This offer is accepted by all types till $\overline{q}_1(\Delta)$
- In the third period, the seller exits with probability 1 if she has the outside option and continues to bargain otherwise.

The above on path play is supported by the following off path behavior. Let q_1 be such that

$$p(0)(q_3-q_1)+\delta W(q_3)=[v_2-\delta^2(1-\mu_\Delta)(v_2-P(t(\hat{q}_\Delta)))](\overline{q}_1(\Delta)-q_1)+\delta W(\overline{q}_1(\Delta)) \quad (3.2)$$

Note that as $\delta\to 0$, $v_2-\delta^2(1-\mu_\Delta)(v_2-P(t(\hat{q}_\Delta)))$ goes to $\alpha v_2<\lim_{\Delta\to 0}p(0)$, and so q_1 goes to q_3.

- At any stage if an offer of $P(q')$ is made for $q'>\overline{q}_1(\Delta)$ all types till q' accept the offer and play proceeds as prescribed by the Coasian equilibrium.
- If the belief state is q_3, $\mu=\mu_\Delta$ and the offer is between $p(q_3)$ and $\lim_{q\downarrow\overline{q}_1(\Delta)}P(q)$, all types till $\overline{q}_1(\Delta)$ accept the offer, following which there is suitable randomization (both in opting out behavior and next period offers).
- If $q<q_3$, and the offer is greater than $v_3(1-\delta)+p(0)\delta$, it is rejected
- If $q<q_3$ and an offer between $v_3(1-\delta)+p(0)\delta$ and $p(0)$ is made, all types till q_1 accept the offer, following which there is suitable randomization.
- If the offer is between $p(0)$ and $v_2-\delta^2(1-\varepsilon)(v_2-P(t(\overline{q}_1(\Delta))))$ all types till q_3 accept the offer, following which play proceeds as on path

- For offers between $v_2 - \delta^2(1 - \varepsilon)(v_2 - P(t(\overline{q}_1(\Delta))))$ and $v_2 - \delta^2(1 - \mu_\Delta)(v_2 - P(t(\overline{q}_1(\Delta))))$ all types till q_3 accept the offer following which the seller stops investing with positive probability and then exits in a way that makes value v_2 buyers indifferent between accepting and rejecting. The belief upon staying in the game at q_3 is μ_Δ and play proceeds as on path

- For any offer between $v_2 - \delta^2(1 - \varepsilon)(v_2 - P(t(\hat{q}_\Delta)))$ and $\lim_{q \downarrow \overline{q}_1(\Delta)} P(q)$, all types till $\overline{q}_1(\Delta)$ accept the offer, following which there is suitable randomization (with investment being suspended if necessary).

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