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COMPATIBLE DECOMPOSITIONS OF THE CASSELMAN ALGEBRA AND THE REDUCED C^*-ALGEBRA OF A REAL REDUCTIVE LIE GROUP

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Abstract

For a real reductive group G, we investigate the structure of the Casselman algebra $\mathcal{S}(G)$ and its similarities to the structure of the reduced group C^*-algebra $C_r^*(G)$. We show that the two algebras are assembled from very similar elementary components in a compatible way. We prove that the two algebras have the same K-theory when restricted to a finite set of K-types. This is a refinement of the Connes-Kasparov isomorphism.

Contents

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Chapter 1 Introduction

1.1 From representation theory to operator algebras

One of Harish-Chandra's greatest achievements was his explicit decomposition of the space $L^2(G)$ into irreducible representations, where G is a noncompact, real reductive group, for example $G = \mathrm{SL}(n, \mathbb{R})$ (see [HC76]). In the case of compact groups, Harish-Chandra's result is a combination of the Peter-Weyl theorem and Weyl's character formula, and $L^2(G)$ decomposes as a direct sum according to the irreducible representations of G. For example, when $G = \mathrm{SO}(3)$, the decomposition is intimately related to the study of spherical harmonics (via the identification $S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2)$). In the real reductive case, $L^2(G)$ decomposes as a direct integral of the so-called "tempered representations," which Harish-Chandra essentially classified. He calculated explicitly the way in which a given function $f \in L^2(G)$ decomposes into parts associated to the tempered representations. This decomposition is known as Harish-Chandra's Plancherel theorem. Of course, this theorem greatly generalizes the classical Plancherel theorem in Fourier analysis.

One may hope to use the power of operator algebras to simplify the techniques required to produce this decomposition. A first step is Segal's abstract Plancherel theorem for "Type I" groups (see [Seg50]). At the very least, the abstract Plancherel theorem serves as a strong motivation for Harish-Chandra's Plancherel theorem. The role here of operator algebras is expressed well in the introduction to [Wal92, Chapter 14], which provides a mostly self-contained treatment of abstract representation theory geared towards studying real reductive groups.

In brief, for a locally compact group G, it turns out that the unitary irreducible representations (up to unitary equivalence) of G are in bijection with irreducible representations of the "full" group C^*-algebra, $C^*(G)$ (moreover, there are natural topologies where this bijection becomes a homeomorphism). For real reductive groups, the tempered representations (those appearing in the decomposition of $L^2(G)$) are in bijection with the irreducible representations of the "reduced" group C^*-algebra $C_r^*(G)$. Therefore, the hope is that (unitary) representation theory of the group may be investigated through these operator algebras.

One of the earliest works in this direction was due to Godement [God52]. Godement attempted to simplify various results of Harish-Chandra by a systematic use of operator algebra theory (particularly Banach algebras such as certain L^1 algebras). However, despite some promising early successes, less was accomplished than was initially hoped.

Eventually, the theory of C^*-algebras proved to be convenient for summarizing results in representation theory, particularly the results of Harish-Chandra. In [CCH16], Clare, Crisp and Higson investigated the structure theory of $C_r^*(G)$ and the "tempered dual," effectively combining the main results of the theory of tempered representations into a single C^*-algebraic statement, as follows.

First, there are discrete atoms corresponding to a family of representations called the discrete series, which are the irreducible representations that appear as direct summands of $L^2(G)$ (the understanding of their importance, and their explicit determination, was another big achievement of Harish-Chandra; see [HC65, HC66]). Note that such representations may not exist, which is the case for $\mathrm{SL}(3, \mathbb{R})$, for example.

The tempered representations then appear as (direct summands of) representations constructed from discrete series representations of certain subgroups of G. These representations have a continuous parameter, and form families called the principal series. If σ is the discrete series representation (of some subgroup of G), we write here $\mathfrak{v}_\sigma$ for the (real) vector space that the continuous parameter resides in. For each σ, the corresponding principal series representations have a common Hilbert space that G acts on, which we denote by $\mathcal{H}_\sigma$. The C^*-algebra acts on these representations as compact operators "continuously" with respect to the parameter, so that we obtain a map

$$C_r^*(G) \to C_0(\mathfrak{v}_\sigma, \mathcal{K}(\mathcal{H}_\sigma)),$$

where $\mathcal{K}(\mathcal{H}_\sigma)$ denotes the compact operators on $\mathcal{H}_\sigma$. Moreover, there is an action on the target C^*-algebra by a finite group, which we denote by W_σ, and the action of $C_r^*(G)$ on the principal series is invariant under this W_σ-action, so that we obtain a map

$$C_r^*(G) \to C_0(\mathfrak{v}_\sigma, \mathcal{K}(\mathcal{H}_\sigma))^{W_\sigma}.$$

One of the main results of [CCH16] is that such representations give everything. That is, we obtain the decomposition

$$C_r^*(G) \xrightarrow{\cong} \bigoplus_\sigma C_0(\mathfrak{v}_\sigma, \mathcal{K}(\mathcal{H}_\sigma))^{W_\sigma}.$$

This implicitly classifies the tempered dual $\widehat{G}_r$ as a topological space.

In summary, and as illustrated by the Clare-Crisp-Higson decomposition above, C^*-algebras have provided a convenient framework to describe and summarize results in representation theory. However, real contributions to representation theory from C^*-algebra theory had to wait for the development of K-theory, which has allowed one to gain new (geometric) information regarding these algebras, and therefore to say something about representations of the group. Development of operator K-theory ultimately led to the Baum-Connes conjecture, and in particular the Connes-Kasparov isomorphism.

1.1.1 The Connes-Kasparov Isomorphism

The Connes-Kasparov isomorphism, which was a primary motivation for the work in this dissertation, is a calculation of the K-theory of the reduced group C^*-algebra of a connected Lie group. See [BCH94, (4.20)], [GAJV19, Section 2.4] (for a general locally compact group, there is an analogous, still conjectural statement, known as the Baum-Connes conjecture; see [BCH94]). Under suitable conditions, the statement is that the K-theory group $K_0(C_r^*(G))$ is either 0, or is isomorphic to the representation ring of a maximal compact subgroup of G, denoted $R(K)$. When the K_0-group is 0, the K_1-group will be isomorphic to this representation ring.

The Connes-Kasparov isomorphism has seen various proofs over the past few decades. The first was a short announcement of a proof by Wassermann [Was87], though with few details. The idea was to make full use of the representation theory of real reductive groups in order to compute the K-theory of the reduced group C^*-algebra directly. This was followed by Lafforgue's proof using Banach algebra techniques and his Banach KK-theory ([Laf02]).

More recent proofs employ Vogan's classification and his theory of minimal K-types (see, for example, [Vog77, Vog81]). One such proof is due to Alexandre Afgoustidis, using the Mackey bijection, which he established for general real reductive groups (see [Afg19, Afg21]). Another proof is by Clare-Higson-Song-Tang [CHST23], which provides the details along Wasserman's announcement.

1.2 The Casselman algebra

It is also helpful to study other (convolution) algebras, which may not be C^*-algebras. A prominent example is the *Casselman algebra* $\mathcal{S}(G)$, consisting of "very rapidly decreasing" functions on the group (if G is considered as a Nash manifold, then this algebra is the corresponding Schwartz space; see [AG08]). An important result, known as the Casselman-Wallach theorem (see [BK14]), states that the irreducible, *nonunitary* (admissible) representations of G are in bijection with the (admissible) smooth dual of $\mathcal{S}(G)$.

In this dissertation, I investigate the structure of $\mathcal{S}(G)$ and prove that it is very similar to that $C_r^*(G)$, as described by the Clare-Crisp-Higson decomposition. At a first glance this may not seem plausible: elements $C_r^*(G)$ consist of (generalized) functions on G that are roughly in $L^2(G)$ (cf. [Cow78]), while the Casselman algebra is a much smaller Fréchet algebra. Nevertheless, I show that both algebras are assembled from essentially the same basic parts, in the same way, and that these two algebras have the same K-theory, where K-theory for Fréchet algebras has been defined by Phillips [Phi91].

1.3 Main Results

Let G be a real reductive group, and let K be a fixed maximal compact subgroup of G. Let F be a fixed finite subset of the dual $\widehat{K}$ of K (isomorphism classes of irreducible representations of K).

In what follows, we write $\mathcal{S}(G, F)$ and $C_r^*(G, F)$ for "truncated" versions of the algebras $\mathcal{S}(G)$ and $C_r^*(G)$ respectively. Roughly speaking, these are functions given by truncated Fourier polynomials with respect to some maximal compact subgroup K of G, and F is some finite subset of the dual $\widehat{K}$ of K. The example to keep in mind is $G = \mathrm{SL}(2, \mathbb{R})$ and $K = \mathrm{SO}(2)$, so that we really are truncating the (double) Fourier series to some finite set of Fourier modes.

The two main theorems of this dissertation are given below. We shall be informal;

precise statements are given in Chapter 4. The following theorem is a summary of four theorems stated in Section 4.4.

Theorem 1.3.1 (Structure of the Casselman algebra)**.** *The Casselman algebra $\mathcal{S}(G, F)$ is built, via extensions and Morita equivalences, from elementary components which are algebras of holomorphic matrix-valued functions invariant under the action of a finite group. The C^*-algebra $C_r^*(G, F)$ may be assembled in an analogous manner by algebras of invariant continuous matrix-valued functions.*

Moreover, the respective elementary components between $\mathcal{S}(G, F)$ and $C_r^*(G, F)$ are isomorphic in K-theory via an Oka principle.

With this structure theorem, and with a simple homotopy argument, we will also prove the following theorem in Section 5.2.

Theorem 1.3.2 (Refined Connes-Kasparov isomorphism)**.** *Let F be a ball of K-types. The inclusion map $\mathcal{S}(G, F) \to C_r^*(G, F)$ induces an isomorphism in K-theory:*

$$K_*(\mathcal{S}(G, F)) \xrightarrow{\cong} K_*(C_r^*(G, F)).$$

It is mathematical folklore that the isomorphism $K(\mathcal{S}(G)) \cong K(C_r^*(G))$ is equivalent to the Connes-Kasparov isomorphism. In brief, the proof of Connes-Kasparov by Lafforgue [Laf02] involves a homotopy of a certain representation (coming on the so-called "γ-element" due to Kasparov [Kas88]). The machinery in Lafforgue's proof (involving his Banach KK-theory) requires that this homotopy is in the space of isometric Banach space representations.

We note that this homotopy does not always exist through the tempered dual. However, Lafforgue manages to prove such a homotopy exists through isometric Banach space representations, although this homotopy is difficult to construct. The method ultimately leads to a calculation of $K(L^1(G))$ which is shown to be the representation ring $R(K)$ for the relevant K-theory index (so we would be left with showing $L^1(G) \to C_r^*(G)$ induces an isomorphism in K-theory). In fact, slightly stronger methods to that outlined above lead to the calculation of the K-theory of an algebra shown to have the same K-theory as $C_r^*(G)$ by elementary methods, which ultimately proves the Connes-Kasparov isomorphism.

Now, if the method of Lafforgue was used to compute $K(\mathcal{S}(G))$, then the homotopy is only required to be in the space of continuous representations, making this part of the proof elementary. We are then left with comparing the K-theories of $\mathcal{S}(G)$ and $C_r^*(G)$. It is in this sense that the isomorphisms in Theorem 1.3.2 refine the Connes-Kasparov isomorphism, and that the isomorphisms can be used to check the original Connes-Kasparov isomorphism.

1.4 Oka principle

The isomorphism in K-theory between $\mathcal{S}(G)$ and $C_r^*(G)$ can be thought of as a manifestation of an Oka principle. In the theory of several complex variables, Grauert [Gra57a, Gra57b, Gra58] proved that topological vector bundles on Stein spaces can be given a holomorphic structure, unique up to homotopy. This has an interpretation in K-theory (due to Novodvorskii [Nov67]), namely that the K-theory of a commutative Banach algebra is isomorphic to the topological K-theory of its Gelfand spectrum. See [BH21] for an exposition and references.

It was originally envisaged that the K-theory isomorphisms in Theorem 1.3.2 would be consequences of such an Oka principle in noncommutative geometry (a key paper in this direction is [Bos90]). In the end, this was not what happened, but nevertheless the Oka principle offers an interesting perspective on the main results of this dissertation, and may be the key to future simplifications of the work.

In [BH21], my advisor, Professor Nigel Higson, and I gave a simple proof of Novodvorskii's theorem. The original proof uses Grauert's Oka principle. In fact, Novodvorskii's theorem can be thought of as a sort of "abelianized" version of Grauert's result. But this theorem of Grauert is highly nontrivial, and our aim was to use the cohomological nature of K-theory to greatly simplify Novodvorskii's proof. In particular, we provide a completely accessible, almost self-contained account to those familiar with the basics of Banach algebra K-theory and analysis of several complex variables.

One particularly useful tool in our proof of this theorem is the "range plus range" theorem, which can be thought of as an analogue in K-theory of Cartan's "Theorem B" for Stein manifolds. The statement is that, given a pullback square of Banach algebras

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array},$$

if the maps $B \to D$ and $C \to D$ have dense range, and if in addition the sum map $B \oplus C \to D$ is surjective (so that the "range plus range" is all of D), then the above square leads to a Mayer-Vietoris sequence

$$\cdots \to K_i(A) \to K_i(B) \oplus K_i(C) \to K_i(D) \to K_{i+1}(A) \to \cdots .$$

Going back to the proof of Novodvorskii's theorem, provided that the hypotheses of the range plus range theorem hold (which is nontrivial), this theorem allows us to divide our space (the spectrum) in half, which is a process that may repeat until we take the limit and reduce to a lower-dimensional space. This is analogous to the proof of the Jordan-Brauer separation theorem (which leads to Brauer's fixed-point theorem).

Back to the matter at hand, we can interpret the isomorphism $K_*(\mathcal{S}(G)) \cong K_*(C_r^*(G))$ as a kind of Oka principle. As mentioned, the dual of $C_r^*(G)$ is the tempered dual of G, while the (admissible) dual of $\mathcal{S}(G)$ is the admissible dual of G, which in a very rough sense is a "complexification" of the tempered dual. Moreover, it turns out that $\mathcal{S}(G)$ can be thought of as (operator-valued) holomorphic functions on the admissible dual. We can then think of the map $\mathcal{S}(G) \to C_r^*(G)$ as a restriction map from holomorphic functions on the nonunitary principal series to continuous functions on the tempered dual.

1.5 Idea of the proofs of the main theorems

1.5.1 Morita equivalence of Fréchet algebras

As mentioned, the reduced group C^*-algebra and the Casselman algebra appear quite different in size, in that the Casselman algebra is a significantly smaller subalgebra of the group C^*-algebra. However, it turns out that much of this gap in size can be covered by a Fréchet algebra "Morita equivalence" based on the notion of Banach algebra Morita equivalence due to Lafforgue (see [Par09]).

A Morita equivalence for general rings is essentially an equivalence of their module categories. For C^*-algebras, there is a notion of strong Morita equivalence, defined via the existence of an imprimitivity bimodule (an "invertible" Hilbert bimodule between the two algebras). In [BGR77], Brown-Green-Rieffel show that two C^*-algebras A, B are strongly Morita equivalent if and only if the algebras are the "full corners" of a larger

C^*-algebra, i.e. the diagonals of a "block matrix" C^*-algebra, of the form

$$\begin{bmatrix} A & E \\ F & B \end{bmatrix},$$

with the condition that the two-sided ideals generated by A and B respectively (inside of this larger algebra) are dense.

Lafforgue developed a notion of (strong) Morita equivalence for Banach algebras, along with his Banach KK-theory. This was unpublished, but was written down by Paravicini in [Par09]. The sort of Morita equivalence defined by Lafforgue is analogous to the full corner condition given by Brown-Green-Rieffel. Moreover, with Lafforgue's Banach KK-theory, it can be shown that a Morita equivalence of Banach algebras in this sense leads to equivalent KK-theory.

In [Phi91], Phillips develops K-theory for Fréchet algebras, generalizing the K-theory for Banach algebras and σ-C^*-algebras (known as representable K-theory in the latter case). The Fréchet algebras are assumed to be an inverse limit of Banach algebras, where each connecting map has dense range. Phillips replaces the compact operators (which is normally used for stablization in the process of defining K-theory) with a smooth version that is a nuclear Fréchet space. He is also able to prove many of the typical properties of K-theory, for example the 6-term exact sequence and Bott periodicity. He is also able to relate the K-theory of the Fréchet algebra to the K-theory of the Banach algebras appearing in the inverse limit, giving a theorem analogous to the Milnor $\varprojlim^1$-sequence.

This leads to the Morita equivalence for Fréchet algebras. As mentioned, Morita equivalence of Banach algebras leads to isomorphic K-theories. By using the Milnor $\varprojlim^1$-sequence, and by defining an analogous notion of "full corner" Morita equivalence, it is possible to extend this result to Fréchet algebras. However, while there is a notion of Fréchet algebra KK-theory (see [Cun05, Cun97]), it is not known whether Morita equivalence leads to isomorphic KK-theory in this case.

1.5.2 Paley-Wiener Theorem

How do we obtain such a Morita equivalence for the Casselman algebra? Looking at the Clare-Crisp-Higson decomposition for the C^*-algebra, we can think of the decomposition as a description of the Fourier image of $C_r^*(G)$ (that is, as a sort of generalized Riemann-Lebesgue lemma). Indeed, there is a notion of Fourier transform by use of principal series representations, where the “frequency” parameter (from the classical Fourier transform) in this setting is the continuous parameter of the principal series. As in the Clare-Crisp-Higson decomposition, our Fourier transform becomes a continuous family of operators on the principal series. As the Casselman algebra is a subalgebra of $C_r^*(G)$, this means the Fourier transform applies to $\mathcal{S}(G)$, and we can try to understand the Fourier image. This problem is known as the Paley-Wiener problem.

In harmonic analysis, the classical Paley-Wiener theorem aims to describe the image of $C_c^\infty(\mathbb{R}^n)$ under Fourier transform. The conclusion is that the Fourier transform of a smooth compactly supported function extends to a holomorphic function on $\mathbb{C}^n$ satisfying a certain Schwartz decay property. Moreover, this property, and holomorphicity, classifies the Fourier image of $C_c^\infty(\mathbb{R}^n)$. The Paley-Wiener problem aims to generalize this theorem to general real reductive groups, namely to classify the Fourier image of $C_c^\infty(G)$. Similar to the classical theorem, the Fourier image consists of (operator-valued) holomorphic functions (with respect to the continuous parameter) that satisfy certain Schwartz-like decay analogous to the classical case. However, there are additional algebraic conditions on these functions that are a consequence of the representation theory (specifically, the reducibility of the principal series).

One of the first more general Paley-Wiener theorems was for Riemannian symmetric spaces [Hel84]. That is, for spaces of the form G/K, where G is a reductive group and K is a maximal compact subgroup. An example of such a space is the hyperbolic plane. The theorem for such spaces is due to Helgason (see [Hel84, Section IV.7]), and Gangolli [Gan71]. The proof makes use of a “contour shift” technique, which has proven to be a very effective technique in proving Paley-Wiener theorems for groups. In particular, the first proof of a Paley-Wiener theorem for general real reductive groups was given by Arthur [Art83], generalizing Campoli's result in [Cam80] for groups with real rank 1. In Campoli's paper, residues appear during the contour shift which are dealt with by showing they are certain matrix entries of discrete series representations. Arthur's paper is similar in spirit, although it is much more involved due to higher dimensional poles, as well as the more difficult representation theory involved for higher rank groups.

Helgason's contour shift technique, and the Arthur-Campoli formulation of the Paley-Wiener theorem, is also the inspiration for generalizations of the Paley-Wiener theorem to reductive symmetric spaces, i.e. spaces of the form G/H for certain subgroups H of G. Such theorems are due to van den Ban and Schichtkrull (see [vdBS99, vdBS05]). The residue techniques are somewhat similar in spirit, using a residue calculus they developed in [vdBS00], although once again the harmonic analysis (and representation theory) involved becomes more difficult in the symmetric space case.

In a different vein, an algebraic proof of a Paley-Wiener theorem was developed by Delorme [Del82] for groups with one conjugacy class of Cartan subgroups. This made use of minimal K-types, and in particular a certain generalization of the Harish-Chandra isomorphism given in [Del84] (a very interesting paper in its own right). Over two decades later, Delorme states and prove a Paley-Wiener theorem for general groups in [Del05], again starting from the generalized Harish-Chandra isomorphism given in [Del84], together with other results developed over the years.

Similar to $C_c^\infty(G)$, it turns out that the Fourier transform for elements of $\mathcal{S}(G)$ are holomorphic and satisfies a certain Schwartz condition with respect to the continuous parameter of the principal series. The techniques Delorme uses can be made to generalize to $\mathcal{S}(G)$ in the K-finite case, and moreover these techniques give us a lot of information regarding the structure of $\mathcal{S}(G)$. It is these techniques we exploit in this dissertation to relate $\mathcal{S}(G)$ to $C_r^*(G)$.

1.5.3 Main step in the proof

With a good understanding of the Fourier image of $\mathcal{S}(G)$ (via techniques developed for the Paley-Wiener theorem) and of $C_r^*(G)$ (via Clare-Crisp-Higson), we are able to compare these two algebras and find that they are assembled from similar elementary components in a compatible way. The techniques used only apply to K-finite versions of these groups. More specifically, given a finite subset $F \subset \widehat{K}$, there is a (K-finite) function $p_F \in C(K)$ which acts on any K-representation by projection onto the K-types in F (that is, the part of the representation in which K acts only by the representations in F). Now $\mathcal{S}(G)$ and $C_r^*(G)$ are $K \times K$-representations, and we write

$$\mathcal{S}(G,F) = p_F \mathcal{S}(G) p_F, \quad C_r^*(G,F) = p_F C_r^*(G) p_F.$$

That is, $\mathcal{S}(G,F)$ (resp $C_r^*(G,F)$) is the projection of $\mathcal{S}(G)$ (resp. $C_r^*(G)$) onto the $K \times K$-types in $F \times F$.

Now, the representations of K may be equipped with a notion of length, due to Vogan [Vog79]. When $R > 0$ and when F is the set of K-types with length at most R, in this dissertation I prove that the inclusion $\mathcal{S}(G,F) \hookrightarrow C_r^*(G,F)$ induces an isomorphism

$$K_*(\mathcal{S}(G,F)) \xrightarrow{\cong} K_*(C_r^*(G,F)). \tag{1.5.1}$$

Here, we are using the K-theory of Fréchet algebras defined by Phillips [Phi91] (the K-functor is written as RK there).

The proof of the isomorphism (1.5.1) is largely based on techniques due to Delorme in [Del05]. When we restrict the Fourier transform operators to the K-types in F, the restricted operators become finite-dimensional matrices. If our collection of K-types consists of only the "minimal K-types" of some principal series (based on the theory of Vogan [Vog81, Vog79]), then Delorme provides a complete description for the Fourier image with respect to that principal series (in the compactly supported case, see [Del05, (1.38)] and also [Del84]).

In other K-type blocks, it turns out that we can take a certain ideal of $\mathcal{S}(G)$ such that, if we quotient out by this ideal, these other blocks can be "factored through" the minimal K-type block. In this way we can reduce to the minimal K-type case through a "Morita equivalence" described earlier. This ideal (and the "factoring" technique) was used by Delorme in an "induction on length" argument to prove his Paley-Wiener theorem; see [Del05, Propositions 1 and 2]. In terms of the decomposition for $C_r^*(G)$ given in [CCH16], the ideal consists of the direct sum over $[P,\sigma]$ "lower" than a given principal series (this needs to be suitably defined based on minimal K-types). In this way we obtain a filtration for $C_r^*(G)$, and if we intersect with $\mathcal{S}(G)$ we obtain a filtration on (the K-finite version of) $\mathcal{S}(G)$. This filtration is very much analogous to the filtration by Afgoustidis in [Afg19]. In fact, the corresponding filtration on $C_r^*(G)$ is very similar (and likely the same).

1.6 Example: $G = \mathrm{SL}(2,\mathbb{R})$

To better illustrate this, let us consider the example $G = \mathrm{SL}(2,\mathbb{R})$. In this case, $K = \mathrm{SO}(2)$ and so $\widehat{K} = \mathbb{Z}$. If $F = \{2, 0, -2\}$, we have

$$\mathcal{S}(G,F) = \left\{ f = \begin{bmatrix} f_{2,2}(z) & (z+1)f_{2,0}(z) & (z^2-1)f_{2,-2}(z) \\ (z-1)f_{0,2}(z) & f_{0,0}(z) & (z-1)f_{0,-2}(z) \\ (z^2-1)f_{-2,2}(z) & (z+1)f_{-2,0}(z) & f_{-2,-2}(z) \end{bmatrix} \right\}.$$

where the functions $f_{i,j}(z)$ are even, holomorphic functions on $\mathbb{C}$ with uniform Schwartz decay on vertical strips in $\mathbb{C}$. For the C^*-algebra, we have

$$C_r^*(G,F) = C_0(i\mathbb{R}, M_3(\mathbb{C}))^{\mathbb{Z}/2} \oplus \mathbb{C} \oplus \mathbb{C},$$

where the action of the nontrival element $w \in \mathbb{Z}/2$ on $C_0(i\mathbb{R}, M_3(\mathbb{C}))$ is given by

$$(w \cdot f)(z) = \begin{bmatrix} \frac{z+1}{1-z} & & \\ & 1 & \\ & & \frac{z+1}{1-z} \end{bmatrix} f(-z) \begin{bmatrix} \frac{1-z}{z+1} & & \\ & 1 & \\ & & \frac{1-z}{z+1} \end{bmatrix}.$$

The first term, $C_0(i\mathbb{R}, M_3(\mathbb{C}))^{\mathbb{Z}/2}$, comes from the action of $C_r^*(G)$ on the spherical (unitary) principal series, while the two copies of $\mathbb{C}$ correspond to the discrete series representations $D_{2,+}$ and $D_{2,-}$ (which have minimal K-types 2 and -2 respectively).

The map $\mathcal{S}(G,F) \to C_r^*(G,F)$ under these identifications is given by

$$f \mapsto f|_{i\mathbb{R}} \oplus f_{2,2}(1) \oplus f_{-2,-2}(1).$$

In order to see that this map induces a K-theory isomorphism, take the ideals

$$\mathcal{J}_1 = \left\{ \begin{bmatrix} (z^2-1)f_{2,2}(z) & (z+1)f_{2,0}(z) & (z^2-1)f_{2,-2}(z) \\ (z-1)f_{0,2}(z) & f_{0,0}(z) & (z-1)f_{0,-2}(z) \\ (z^2-1)f_{-2,2}(z) & (z+1)f_{-2,0}(z) & (z^2-1)f_{-2,-2}(z) \end{bmatrix} \right\},$$

$$\mathcal{J}_2 = \left\{ \begin{bmatrix} (z^2-1)f_{2,2}(z) & (z+1)f_{2,0}(z) & (z^2-1)f_{2,-2}(z) \\ (z-1)f_{0,2}(z) & f_{0,0}(z) & (z-1)f_{0,-2}(z) \\ (z^2-1)f_{-2,2}(z) & (z+1)f_{-2,0}(z) & f_{-2,-2}(z) \end{bmatrix} \right\}.$$

so that we have $0 \subset \mathcal{J}_1 \subset \mathcal{J}_2 \subset \mathcal{S}(G,F)$ with $\mathcal{S}(G,F)/\mathcal{J}_2 \cong \mathbb{C}$ and $\mathcal{J}_2/\mathcal{J}_1 \cong \mathbb{C}$. The last claim is that $\mathcal{J}_1 \to C_0(i\mathbb{R}, M_3(\mathbb{C}))^{\mathbb{Z}/2}$ induces an isomorphism in K-theory. The key idea is that $\mathcal{J}_1$ is "Morita equivalent" (in the sense of Lafforgue) to its middle entry, i.e. the space $\mathsf{PW}(\mathbb{C})^{\mathbb{Z}/2} := \{[f_{0,0}(z)]\}$ (this is the K-type block corresponding to 0, the minimal K-type of the spherical principal series). Similarly, $C_0(i\mathbb{R}, M_3(\mathbb{C}))^{\mathbb{Z}/2}$ is Morita equivalent to its middle entry, i.e. the space $C_0(i\mathbb{R})^{\mathbb{Z}/2}$ of even C_0 functions. The restriction map $\mathsf{PW}(\mathbb{C})^{\mathbb{Z}/2} \to C_0(i\mathbb{R})^{\mathbb{Z}/2}$ here is easily shown to induce an isomorphism in K-theory by a simple homotopy argument.

Chapter 2 | Preliminaries

2.1 Real reductive groups

The primary references regarding real reductive groups (and their representations) that we will use are [Kna01, Wal88, Vog81]. A helpful introductory book is [Kna02]. We will also follow the notation in these books (particularly [Vog81] and [Wal88]). Many definitions are taken from these books, and we will provide specific references for the interested reader.

We will typically refer to groups by capital letters; particularly the letter G. The group G will be the main object of study in this dissertation, and we will usually consider various subgroups of G. We assume that our group is *linear*, meaning that G a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$ (or equivalently that G is a Lie group with a faithful finite-dimensional representation).

Given a Lie group G, we will write $\mathfrak{g}$ for the *complexified* Lie algebra of G; the real Lie algebra is denoted $\mathfrak{g}_0 = \mathrm{Lie}(G)$. We also write G_0 for the identity component of G (in the disconnected setting). In the case of linear groups, the exponential map $\exp : \mathfrak{g}_0 \to G$ is given by the matrix exponential: $\exp(X) = e^X$, once $\mathfrak{g}_0$ is identified as a set of matrices.

The definition of a real reductive group varies among the sources we use. A basic requirement is that the Lie algebra, $\mathfrak{g}_0$, should be reductive (for linear groups, a related assumption is that the group should be closed under conjugate transpose). The other assumptions typically control the disconnectedness of the group, for example there should be finitely many components, or that a maximal compact subgroup K meets every component of G. We will use the definition given in [Kna82], which may be considered as a (possibly) slightly stricter version of Vogan's definition [Vog81].

To state the definition, we need some notation. We recall that, if $\mathfrak{g}_0$ is the Lie algebra of group G, and if $\mathfrak{h}_0 \subset \mathfrak{g}_0$ is a Lie subalgebra, then there exists a connected subgroup $H \subset G$ with Lie algebra $\mathfrak{h}_0$. Such subgroups are called analytic subgroups (indeed, for complex Lie algebras the subgroup is a complex submanifold). Let $\mathfrak{gl}(n, \mathbb{C})$ denote the Lie algebra of $\mathrm{GL}(n, \mathbb{C})$ (that is, the set of all complex $n \times n$ matrices). Let $G_{\mathbb{C}}$ be the analytic subgroup of $\mathrm{GL}(n, \mathbb{C})$ corresponding to $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$. Let $Z_{\mathrm{GL}(n,\mathbb{C})}(G)$ denote the centralizer of G in $\mathrm{GL}(n, \mathbb{C})$.

Definition 2.1.1. A (linear) *real reductive group* is a linear group G with the following assumptions.

(a) The Lie algebra, $\mathfrak{g}_0$, is reductive.

(b) G has finitely many connected components.

(c) We have
$$G \subset G_{\mathbb{C}} \cdot Z_{\mathrm{GL}(n,\mathbb{C})}(G). \tag{2.1.1}$$

These assumptions are made in [Kna82], and in [CD84, Del05]. The point of the last assumption is to limit the disconnectedness of G. As a consequence of the last assumption, our groups automatically satisfy the following assumptions given in [Vog81]:

(d) For $g \in G$, the map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{Ad}_g(X) = gXg^{-1}$ is an inner automorphism of $\mathfrak{g}$ (as a complex Lie algebra).

(e) If $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ (see Section 2.2.1), and if H denotes the centralizer of $\mathfrak{h}_0$ in G, then H is abelian.

Groups satisfying (d) are said to be of inner type (see [Wal88, 2.2.8]), and many results in [Wal88] (and several other references) require this assumption. Vogan uses (e) as a weaker replacement of the assumption (2.1.1).

These groups are in the Harish-Chandra class (see, for example, [Kna02, Section VII.2]), and they satisfy the assumptions of Knapp-Stein [KS80] and Vogan [Vog79]. These are the assumptions made in [Del05], which is a key reference for the results in this dissertation. It is also possible to arrange the inclusion $G \subset \mathrm{GL}(n, \mathbb{C})$ so that G is closed under conjugate transpose in $\mathrm{GL}(n, \mathbb{C})$ (this may be done by averaging the standard inner product on $\mathbb{C}^n$ by a fixed maximal compact subgroup and the corresponding compact form of G, and then by choosing an orthonormal basis with respect to this inner product), so that G satisfies Wallach's definition [Wal88, 2.1.1]. Therefore, we will always assume $G \subset \mathrm{GL}(n, \mathbb{C})$ satisfies this, and we will define the Cartan involution θ on G by $\theta(g) = (g^*)^{-1}$ (see Section 2.2.3).

Our definition of real reductive groups includes the connected linear semisimple Lie groups (see [Vog81, Example 0.1.4 (a)] and [Wal88, Lemma 2.1.3]). The classical groups, such as $\mathrm{SL}(n, \mathbb{R})$, and $\mathrm{Sp}(2n, \mathbb{R})$, are such examples. The real points of an algebraic linear reductive group over $\mathbb{R}$ also satisfy our definition (see [CD84, Section 1.2]).

In the case when G is connected, the assumption (2.1.1) is automatic. For convenience, we will assume that G is connected for the main results in this dissertation. However, for proofs of the various representation-theoretic statements listed in this chapter it is often convenient to extend and prove the results for disconnected G. Also, the results of this dissertation apply to disconnected groups, too.

2.2 Structure theory

2.2.1 Cartan subalgebras and root systems

For simplicity, we define a Cartan subalgebra $\mathfrak{h}$ of a complex (reductive) Lie algebra $\mathfrak{g}$ to be a maximal abelian Lie subalgebra of $\mathfrak{g}$ whose elements consist of diagonalizable elements ([Wal88, 0.2]). The *root space decomposition* of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [X, H] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$. It turns out that $\mathfrak{g}_\alpha$ is either 0 or one-dimensional, and the set $\Delta(\mathfrak{g}, \mathfrak{h}) = \{\alpha \in \mathfrak{h}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq 0\}$ is finite. The elements of $\Delta(\mathfrak{g}, \mathfrak{h})$ are called the *roots* of $\mathfrak{g}$ (with respect to $\mathfrak{h}$).

A standard reference for Cartan subalgebras and root systems for complex reductive Lie algebras is [Kna02, Sections II.1-6] (see also [Wal88, Section 0.2]). Given a Cartan subalgebra, we write $W(\mathfrak{g}, \mathfrak{h})$ to denote the Weyl group, which is a finite subgroup of $O(\mathfrak{h}^*)$, generated by root reflections s_α on $\mathfrak{h}^*$, that preserves the set $\Delta(\mathfrak{g}, \mathfrak{h})$. We will also write $\Delta(\mathfrak{g}, \mathfrak{h})^+$ to denote a system of positive roots in $\Delta(\mathfrak{g}, \mathfrak{h})$, and write $\Phi(\mathfrak{g}, \mathfrak{h})$ for the corresponding simple roots.

A real Lie algebra $\mathfrak{h}_0$ is a (real) Cartan subalgebra of $\mathfrak{g}_0$ if the complexification $\mathfrak{h}$ is a

Cartan subalgebra of $\mathfrak{g}$ ([Wal88, 2.3.1]). We write H for the centralizer of $\mathfrak{h}_0$ in G. The subgroup H is called a *Cartan subgroup* of G (see [Wal88, 2.3.6] and [Vog81, Definition 0.1.5]).

2.2.2 Maximal compact subgroup and K-types

There exists one maximal compact subgroup up to conjugacy. We write K for a choice of maximal compact subgroup. In our setting, a typical choice for K is $G \cap U(n)$. We write $\widehat{K}$ for the (isomorphism classes of) irreducible unitary representations of K. We will write $\gamma \in \widehat{K}$ to denote a (fixed) representative (γ, V_γ) of an element of $\widehat{K}$. As G is assumed to be connected, so is K. Therefore, the elements of $\widehat{K}$ are classified by their so-called "highest weights". That is, the dominant (globally) integral weights of $\mathfrak{k}$.

The theory of irreducible representations of a (connected) compact group is standard, and can be found in many introductory textbooks. See, for example, [Kna02, Chapter V]. For the reader interested in the disconnected case, see [KV95, Section IV.2].

As notation, we write T_K to denote a maximal torus of K (that is, a maximal (connected, compact) abelian subgroup of K), and write $\mathfrak{t}_K$ for the complexified Lie algebra. Then $\mathfrak{t}_K$ is a Cartan subalgebra of $\mathfrak{k}$, and as usual we denote by $\Delta(\mathfrak{k}, \mathfrak{t}_K)$ the roots of $\mathfrak{k}$, by $\Delta(\mathfrak{k}, \mathfrak{t}_K)^+$ the positive roots of $\mathfrak{k}$, and we write $\Phi(\mathfrak{k}, \mathfrak{t}_K)$ for the corresponding simple roots.

2.2.3 Cartan decomposition and restricted roots

Let G be a real reductive group. There is an involution $\theta : G \to G$ such that K consists of the fixed points of θ (in our case, we use $\theta(g) = (g^*)^{-1}$). This involution is known as the *Cartan involution*. Accordingly, we define θ on $\mathfrak{g}$ by differentiation (in our case, $\theta X = -X^*$ for $X \in \mathfrak{g}$), and we obtain the *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{p} = \{X \in \mathfrak{g} : \theta X = -X\}$. In our case, $\mathfrak{p}$ consist of symmetric matrices.

Let $\mathfrak{a}_0$ be a subspace of $\mathfrak{p}_0$ which is maximal with respect to being an abelian Lie subalgebra of $\mathfrak{g}_0$ (for brevity, we say that $\mathfrak{a}_0$ is a maximal abelian subalgebra of $\mathfrak{p}_0$, even though $\mathfrak{p}_0$ is not a Lie algebra). As $\mathfrak{p}_0$ consists of self-adjoint matrices, $\mathfrak{a}_0$ consists of diagonalizable matrices, and so the action of $\mathfrak{a}_0$ on $\mathfrak{g}_0$ is diagonalizable. Therefore, we may decompose $\mathfrak{g}_0$ as

$$\mathfrak{g}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \bigoplus_{\alpha \in \mathfrak{a}_0^*} \mathfrak{g}_{0,\alpha},$$

where $\mathfrak{m}_0$ is the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_0$, and $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_0 : [X, H] = \alpha(H)X$ for all $H \in \mathfrak{a}_0\}$. The set $\Delta(\mathfrak{g}_0, \mathfrak{h}_0) = \{\alpha \in \mathfrak{h}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq 0\}$ is finite. Unlike in the case of Cartan subalgebras, $\mathfrak{g}_{0,\alpha}$ may be more than one-dimensional. The elements of $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$ are called the *restricted roots* of $\mathfrak{g}_0$ (relative to $\mathfrak{a}_0$), and are named this because (for a Cartan subalgebra $\mathfrak{h}$ containing $\mathfrak{a}_0$)

$$\Delta(\mathfrak{g}_0, \mathfrak{a}_0) = (\Delta(\mathfrak{g}, \mathfrak{h})|_{\mathfrak{a}_0}) \setminus \{0\}.$$

2.2.4 Iwasawa decomposition

Once and for all, we fix a choice of maximal abelian subalgebra $\mathfrak{a}_{\min,0}$ of $\mathfrak{p}_0$ (the notation will be helpful when we discuss parabolic subgroups). We also fix an element $H \in \mathfrak{a}_{\min,0}$ such that $\alpha(H) \neq 0$ for all $\alpha \in \Delta(\mathfrak{g}_0, \mathfrak{a}_{\min,0})$. We define

$$\Delta^+ = \Delta(\mathfrak{g}_0, \mathfrak{a}_{\min,0})^+ = \{\alpha \in \Delta(\mathfrak{g}_0, \mathfrak{a}_{\min,0}) : \alpha(H) > 0\}.$$

This set contains exactly half of the restricted roots, which we call the *(standard) positive restricted roots* for $\mathfrak{g}_0$. If α is positive, then $-\alpha$ is not positive (i.e., "negative"). Additionally, if α, β are positive roots and $\alpha + \beta$ is a root, then $\alpha + \beta$ is positive.

We define

$$\mathfrak{n}_{\min,0} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

Now let $A_{\min} \subset G$ denote the connected subgroup with Lie algebra $\mathfrak{a}_{\min,0}$, and let $N_{\min} \subset G$ denote the connected subgroup with Lie algebra $\mathfrak{n}_{\min,0}$. We then obtain the decomposition

$$G = KA_{\min}N_{\min}.$$

The above is known as the *Iwasawa decomposition.*

2.2.5 KAK decomposition and a norm on G

The choice of maximal abelian subalgebra $\mathfrak{a}_{\min,0}$ of $\mathfrak{p}_0$ is unique up to conjugation by an element of K. Accordingly,

$$\mathfrak{p}_0 = \bigcup_{k \in K} \mathrm{Ad}_k(\mathfrak{a}_{\min,0}).$$

As $K \cdot \exp \mathfrak{p}_0 = G$, we have

$$G = KA_{\min}K.$$

Now, the Lie algebra $\mathfrak{g}_0$ comes equipped with a natural invariant bilinear form, given by

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \, \mathrm{ad}_Y).$$

When $\mathfrak{g}$ is semisimple, the above form is nondegenerate (this is a criterion for semisimplicity). When $\mathfrak{g}$ is reductive, we have the decomposition

$$\mathfrak{g} = Z_{\mathfrak{g}}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}],$$

where $Z_{\mathfrak{g}}(\mathfrak{g})$ is the center of $\mathfrak{g}$, and $[\mathfrak{g}, \mathfrak{g}]$ is semisimple. In this case, we define $B(\cdot, \cdot)$ as before on $[\mathfrak{g}, \mathfrak{g}]$, and define $B(\cdot, \cdot)$ arbitrarily on $Z_{\mathfrak{g}}(\mathfrak{g})$ such that it is nondegenerate. The bilinear form B is chosen so that it is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. If we define

$$\langle X, Y \rangle = -B(X, \theta Y), \tag{2.2.1}$$

then $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathfrak{g}$, and we may define $\|X\|^2 = \langle X, X \rangle$.

If $g \in G$, and if we write $g = k_1 e^H k_2$ with $k_1, k_2 \in H$ and $H \in \mathfrak{a}_{\min,0}$, then we define

$$\|g\| = e^{\|H\|}.$$

The above definition is independent of the choice of decomposition of g. This is a "norm" on G in the sense of [Wal88, 2.A.2]. That is, the norm satisfies the following properties:

- $\|g\| = \|g^{-1}\|$,
- $\|g_1 g_2\| \leq \|g_1\| \|g_2\|$,
- $\{g \in G : \|g\| \leq r\}$ is compact for each $r > 0$,
- $\|k_1 e^{tX} k_2\| = \|e^X\|^t$ for each $t \geq 0$, $X \in \mathfrak{p}_0$, $k_1, k_2 \in K$.

2.2.6 Measures

The inner product (2.2.1) defines a metric on G and therefore a volume form. This gives a smooth, left-invariant measure on G. Moreover, real reductive groups are unimodular, so that this measure is actually bi-invariant. However, many subgroups we are interested in (namely, the parabolic subgroups) will not be unimodular.

At times it is necessary to normalize the measure on G (achieved simply by multiplying the measure by a constant). First, we will always normalize the measure on K so that $\mathrm{Vol}(K) = 1$. In addition, it may be necessary to normalize the measure dg in order to obtain certain integration formulas, such as

$$dg = a^{2\rho}\, dk\, da\, dn$$

according to the Iwasawa decomposition $G = KAN$ (see [Kna02, Proposition 8.43]). See [Kna02, Section VIII.4] and [Wal88, Section 2.4] for other integration formulas. In the above, ρ is the half sum of positive restricted roots.

Such integration formulas become useful when we want to "integrate" a representation (for example, allowing $C_c^\infty(G)$ or $\mathcal{S}(G)$ to act on a G-representation), as well as in defining certain intertwining operators.

2.3 Representation theory

The following is a brief outline of the sections to follow.

The representation theory of real reductive groups play an important role in the work given in this dissertation. The Fourier transform for such groups make use of representations of the group, in particular the principal series. We need to understand the harmonic analysis of such groups in order to analyze the structure of function spaces of the group.

For real reductive groups, there is an important family of representations called the principal series. These representations are built from a construction known as parabolic induction, where representations are built from representations of parabolic subgroups (in the case of matrix groups, this typically consists of block upper-triangle matrices). Let P be a (cuspidal) parabolic subgroup of G, let $P = MAN$ be its Langlands decomposition, and let σ be a square-integrable representation of M. Associated to the pair (P, σ) is a

Hilbert space $\mathcal{H}_\sigma$ and a series of G-representations $(\pi^P_{\sigma,\lambda}, \mathcal{H}_\sigma)$ for each $\lambda \in \mathfrak{a}^*$ (where $\mathfrak{a}$ is the complexification of the Lie algebra $\mathfrak{a}_0$ of A), called the (nonunitary) principal series. We denote the corresponding $(\mathfrak{g}, K)$-modules by $(\pi^P_{\sigma,\lambda}, I_\sigma)$.

Such principal series are not always irreducible, and indeed there are family of intertwining operators on the principal series, known as the Knapp-Stein family of intertwining operators. These integral operators that intertwining one principal series to another principal series. A large part of the representation theory of real reductive groups hinges on understanding these operators, for example their invertibility.

The principal series representations play an important role in the classification of representations. Specifically, it is known that every irreducible representation is a subrepresentation of some (minimal) principal series representation. Moreover, the Langlands classification states that every irreducible representation is the unique irreducible quotient of a certain principal series representation based on “Langlands data” (specifically, every such representation is the image of one of the Knapp-Stein intertwining operators).

The Langlands classification reduces the study of non-tempered representations to the tempered ones, which ultimately is reduced to the discrete series representations. This was eventually understood, leading to the Knapp-Zuckerman classifications. A more algebraic approach was developed by Vogan, leading the Vogan-Zuckerman classification. Vogan develops an invariant of representations called minimal K-types, and uses this in his classification.

2.4 G-Representations and $(\mathfrak{g}, K)$-modules

A G-representation is a topological vector space V equipped with a group homomorphism (the "action")

$$\pi : G \to \mathrm{GL}(V)$$

such that $G \times V \to V$ given by $(g, v) \mapsto \pi(g)v$ is continuous. We may also write $g \cdot v$ for $\pi(g)v$. If G is compact, we may alternatively use the term “G-module” for a G-representation.

In this dissertation, whenever possible, we will only consider Hilbert space representations (where the space V is a Hilbert space). In this case, a G-representation $\mathcal{H}$ is *unitary* is $\pi(g) : \mathcal{H} \to \mathcal{H}$ acts by unitary operators on G. Two unitary G-representations are *unitarily equivalent* if the action of G is intertwined by a unitary operator between the corresponding Hilbert spaces. For Hilbert space representations, the map π is uniformly bounded on compact sets.

A $(\mathfrak{g}, K)$-module is a vector space V equipped with a $\mathfrak{g}$-action and a K-action such that

- $k \cdot (X \cdot v) = \mathrm{Ad}_k(X) \cdot (k \cdot v)$ for all $X \in \mathfrak{g}, k \in K, v \in V$.
- Given $v \in V$, $W_v := \mathrm{span}(K \cdot v)$ is finite dimensional, and the action of K on W_v is continuous.
- Given $Y \in \mathfrak{k}$, $Y \cdot v = \frac{d}{dt}|_{t=0} \exp(tY) \cdot v$.

See [Wal88, Section 3.3].

Let V be a $(\mathfrak{g}, K)$-module. Given $\gamma \in \widehat{K}$, we write $V(\gamma)$ for the γ-isotypical subspace of V. That is,

$$V(\gamma) = \mathrm{im}(\mathrm{Hom}_K(V_\gamma, V) \otimes V_\gamma \to V).$$

Later, we will define functions p_γ which act by projection onto γ-isotypical subspaces, in which case will write $p_\gamma V$ instead of $V(\gamma)$. When $V(\gamma)$ is nonzero, we say that γ *occurs in* V (or "is a K-type of V") with *multiplicity* $\dim \mathrm{Hom}_K(V_\gamma, V)$.

A $(\mathfrak{g}, K)$-module is *weakly admissible* if, for each $\gamma \in \widehat{K}$, $V(\gamma)$ is finite-dimensional. A $(\mathfrak{g}, K)$-module is *admissible* if it is weakly admissible and finitely-generated. Note that some references describe admissible modules as Harish-Chandra modules, while weakly admissible modules are simply called admissible modules.

Now, given a G-representation $\mathcal{H}$, we write $\mathcal{H}^\infty$ to denote the *smooth vectors* in $\mathcal{H}$. An element $v \in \mathcal{H}$ is a smooth vector if the map

$$g \mapsto \pi(g)v$$

is a smooth map from G to $\mathcal{H}$. The smooth vectors are dense in $\mathcal{H}$ (see [Wal88, Theorem 1.6.2]).

Every G-representation gives rise to a $(\mathfrak{g}, K)$-module via the following construction. If $\mathcal{H}$ is a G-representation, we say $v \in \mathcal{H}$ is K-*finite* if $\dim \mathrm{span}(K \cdot v) < \infty$. We write $\mathcal{H}_K$ for the space of smooth, K-finite vectors in $\mathcal{H}$. The vector space $\mathcal{H}_K$ has a natural structure of a $(\mathfrak{g}, K)$-module by differentiating the action of G.

We say that the G-representation $\mathcal{H}$ is (weakly) admissible if $\mathcal{H}_K$ is (weakly) admissible. If $\mathcal{H}$ is weakly admissible, then the K-finite vectors are automatically smooth. Moreover, if $\mathcal{H}$ is admissible, then the K-finite vectors are analytic in the sense of [Wal88, 1.6.6] (see [Wal88, Theorem 3.4.9]).

2.5 Principal series

2.5.1 Parabolic subgroups

Let $\mathfrak{a}_{\min,0}$ denote our fixed choice of maximal abelian subalgebra of $\mathfrak{p}_0$, and let $A_{\min} = \exp(\mathfrak{a}_0)$ be the corresponding connected subgroup of G. Let $M_{\min}$ denote the centralizer of $A_{\min}$ in K. Finally, let

$$\mathfrak{n}_{\min,0} = \bigoplus_{\alpha \in \Delta(\mathfrak{g}_0, \mathfrak{a}_0)^+} \mathfrak{g}_\alpha,$$

and let $N_{\min} = \exp(\mathfrak{n}_{\min,0})$ be the corresponding connected subgroup of G. We set

$$P_{\min} = M_{\min} A_{\min} N_{\min}.$$

The subgroup $P_{\min}$ of G is a *minimal parabolic subgroup*. Note that we have made a choice for $\mathfrak{a}_{\min,0}$, and $P_{\min}$ depends on this choice. For this reason, we call $P_{\min}$ the *standard* minimal parabolic subgroup. Conjugates of this subgroup are also minimal parabolic subgroups, and general *parabolic subgroups* are subgroups containing a minimal one. For this dissertation, we specify that parabolic subgroups must contain $A_{\min}$ (as in [Del05]). The *standard* parabolic subgroups are those containing $P_{\min}$.

Let P be a parabolic subgroup. We write $L_P = P \cap \theta(P)$, the *Levi subgroup* of P. The *Levi decomposition* is $P = L_P N_P$, where N_P is the unipotent radical of P. There are subgroups M_P and A_P such that $L_P = M_P A_P$, so that

$$P = M_P A_P N_P.$$

The above decomposition is known as the *Langlands decomposition* of P. Specifically, M_P is the compactly generated subgroup of L_P (i.e. $M_P = {}^0L_P$), while A_P is the split component of L_P (see [Wal88, 2.2.1]). In this dissertation, the parabolic subgroup P is often unambiguous and we will suppress the subscript P appearing in the Langlands decomposition (so that we write $P = MAN$).

If P is a parabolic subgroup, we write Δ_P^+ for the roots appearing in the decomposition

$$\mathfrak{n}_0 = \bigoplus_{\alpha \in \Delta_P^+} \mathfrak{g}_{0,\alpha}.$$

We write

$$\mathfrak{a}_{0,P,+}^* = \{\lambda \in \mathfrak{a}_0^* : \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Delta_P^+\} \subset \mathfrak{a}_0^*$$

for the corresponding *(open) Weyl chamber*. In addition, we write $\mathfrak{a}_{P,+}^*$ for elements $\lambda \in \mathfrak{a}^*$ such that $\operatorname{Re} \lambda \in \mathfrak{a}_{0,P,+}^*$. These definitions are taken from notation in [Del84, 1.3], except for the notation $\mathfrak{a}_{P,+}^*$ (which is nonstandard).

Finally, we write $\log : A \to \mathfrak{a}_0$ for the inverse of the exponential map. Given $\lambda \in \mathfrak{a}^*$ and $a \in A$, we use the notation

$$a^\lambda = e^{\lambda(\log a)}.$$

2.5.2 Real parabolic induction and the principal series

Let P be a parabolic subgroup. Given a (Hilbert) representation (π, V_π) of P, there is a corresponding *induced* representation of G, denoted $\operatorname{Ind}_P^G \pi$. This representation, denoted $\operatorname{Ind}_P^G \pi$ is defined as the Hilbert space completion of

$$\{f : G \xrightarrow{C^\infty} V_\pi^\infty : f(gp) = \delta_P(p)^{-1/2} \pi(p)^{-1} f(g) \, \forall p \in P\},$$

(where $\delta_P(p)$ denotes the modular function of P, and V_π^∞ denotes the smooth vectors in V_π) with respect to the inner product $\langle f_1, f_2 \rangle_{L^2(K)} = \int_K \langle f_1(k), f_2(k) \rangle_{V_\pi} \, dk$. The action of G on $f \in \operatorname{Ind}_P^G(\pi)$ is given by

$$(g \cdot f)(g_2) = f(g^{-1} g_2).$$

See [Wal88, 1.5.1], although this reference uses the decomposition $G = PK$ instead of $G = KP$ that we use here.

When π is a unitary P-representation, the induced representation $\operatorname{Ind}_P^G \pi$ is a unitary G-representation ([Wal88, 1.5.3 (2)]). For this reason, $\operatorname{Ind}_P^G \pi$ is also called the unitary induced representation of π (to contrast with the definition which does not involve the adjustment factor $\delta_P(p)$).

Let $P = MAN$ be the Langlands decomposition of P. We will consider representations of P which are trivial on N, and are of the form $\pi = \sigma \otimes e^{\lambda}$, where (σ, V_σ) is a unitary representation of M, and $\lambda \in \mathfrak{a}^*$, so that e^{λ} is a (possibly nonunitary) character of A. In this case, the space $\mathrm{Ind}_P^G(\pi)$ is the completion of

$$\{f : G \xrightarrow{C^\infty} V_\sigma^\infty : f(gman) = a^{-(\lambda+\rho_P)}\sigma(m)^{-1}f(g) \forall m \in M, a \in A, n \in N\},$$

where $\rho_P = \frac{1}{2}\sum_{\alpha\in\Delta_P^+}(\dim \mathfrak{g}_{0,\alpha})\alpha \in \mathfrak{a}_0^*$ is the "half sum of positive roots" with respect to P. See [Wal88, 5.2.1], or [Kna01, p. 168] (which is better aligned with our conventions).

We may restrict the functions to K. We define

$$\mathcal{H}_\sigma = \{f : K \xrightarrow{L^2} V_\sigma : f(km) = \sigma(m)^{-1}f(k) \,\forall m \in M_P \cap K\}.$$

The action of g on $f \in \mathcal{H}_\sigma$ is defined by

$$(\pi_{\sigma,\lambda}^P(g)\varphi)(k) = a_P(g^{-1}k)^{-(\lambda+\rho_P)}\sigma(m_P(g^{-1}k))^{-1}\varphi(k_P(g^{-1}k)), \tag{2.5.1}$$

where for each $g \in G$ we decompose $g = k_P(g)m_P(g)a_P(g)n_P(g)$ according to the decomposition $G = KP = KMAN$ (the elements $k_P(g)$ and $m_P(g)$ are only unique up to an element of $K \cap M$; see [Wal88, 4.5.5]). With this action, the restriction map $\mathrm{Ind}_P^G(\sigma \otimes e^{\lambda}) \to \mathcal{H}_\sigma$ is an equivalence of G-representations. The representation $(\pi_{\sigma,\lambda}^P, \mathcal{H}_\sigma)$ is known as the *compact picture* of $\mathrm{Ind}_P^G(\sigma \otimes \lambda)$ (the noncompact picture involves restriction to $\theta(N)$). See, for example, [Kna01, Section VII.1].

We note that $\mathcal{H}_\sigma$ is independent of λ (in fact, $\mathcal{H}_\sigma = \mathrm{Ind}_{K\cap M}^K \sigma$); we only vary the action with respect to λ. In fact, the family $(\pi_{\sigma,\lambda}, \mathcal{H}_\sigma)_{\lambda\in\mathfrak{a}^*}$ is a *holomorphic family* of G-representations (in the sense of [vdBS14]; see also [DS04, Appendix A]). This family of representations is known as the *principal series*.

We will write I_σ to denote the K-finite vectors in $\mathcal{H}_\sigma$, so that $(\pi_{\sigma,\lambda}^P, I_\sigma)$ is the corresponding $(\mathfrak{g}, K)$-module of $(\pi_{\sigma,\lambda}^P, \mathcal{H}_\sigma)$. We remark that elements of I_σ are smooth functions $f : K \to V_\sigma$ whose image is contained in a finite dimensional subspace of K-finite vectors in V_σ (see [Wal88, Lemma 5.2.2]).

2.6 Harish-Chandra isomorphism

See [Wal88, Section 3.2]. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and fix a positive root system $\Delta(\mathfrak{g},\mathfrak{h})^+$. Accordingly, there is a decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is the sum of the positive root spaces (resp. negative root spaces). By the Poincare-Birkhoff-Witt theorem, there is a decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (\mathfrak{n}^-\mathcal{U}(\mathfrak{g}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{n}^+)$$

By projection, and because $\mathcal{U}(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$, we obtain a map (in fact, an algebra homomorphism)

$$q : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*].$$

Now, let ρ denote the half sum of positive roots. We define the map

$$\mathsf{HC} : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]$$

by $\mathsf{HC}(u)(\lambda) = q(u)(\lambda - \rho)$ for $u \in \mathcal{U}(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$.

Theorem 2.6.1 (Harish-Chandra; see [Wal88, 3.2.3])**.** *The map* HC *restricts to an isomorphism*

$$\mathsf{HC} : \mathcal{Z}(\mathfrak{g}) \xrightarrow{\cong} \mathbb{C}[\mathfrak{h}^*]^{W(\mathfrak{g},\mathfrak{h})}.$$

The above isomorphism is known as the *Harish-Chandra isomorphism.* Now, for each $\lambda \in \mathfrak{h}^*$, we obtain a character

$$\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$$

given by $\chi_\lambda(p) = \mathsf{HC}(p)(\lambda)$. By the Harish-Chandra isomorphism, every character χ is of the form χ_λ for some $\lambda \in \mathfrak{h}^*$, unique up to the action of $W(\mathfrak{g},\mathfrak{h})$.

2.7 Discrete series

An irreducible unitary representation of G is said to be *square-integrable* if one (hence all) of its matrix coefficients is square-integrable (see [Wal88, 1.3.2]). Each square-integrable representation has an embedding into $L^2(G)$ as a G-representation. In the case of real reductive groups, the family of square-integrable representations (up to unitary equivalence) is called the *discrete series*, because they appear as discrete atoms in a Plancherel formula for $L^2(G)$.

Theorem 2.7.1 (Harish-Chandra; see [Wal88, Theorem 7.7.1])**.** *A real reductive group G has a discrete series representation if and only if the rank of G (the dimension of a Cartan subalgebra of $\mathfrak{g}$) equals the rank of K, or equivalently if G contains a Cartan subgroup contained entirely in K.*

Moreover, Harish-Chandra has provided a parametrization of the discrete series. As G has the same rank as K, there is a compact Cartan subgroup H of G. Given a discrete series representation π, we denote by $\Lambda_\pi \in i\mathfrak{h}_0^*$ the Harish-Chandra parameter of π. It turns out that the infinitesimal character of π is precisely χ_{Λ_π} (see [Kna01, Theorem 9.20 (a)]).

The most convenient description of Λ_π (in the context of this dissertation) comes from the work of Schmidt [Sch75]. There is a "lowest" (or "minimal") K-type of π which in particular has multiplicity one. If μ is the highest weight of this K-type, and if ρ_G and ρ_K denote the respective half-sum of positive roots, then

$$\Lambda_\pi = \mu - \rho_G + 2\rho_K.$$

See also [Kna01, Theorem 9.20 (b)]. The K-type μ is lowest in the sense that every other K-type of π is the sum of μ with some integer combination of positive roots (see [Kna01, Theorem 9.20 (c)]).

Definition 2.7.2. A parabolic subgroup $P = MAN$ is *cuspidal* if there exists a Cartan subgroup $T \subset M$ contained entirely within $K \cap M$.

Definition 2.7.3. We will write $\widehat{M}_d$ to denote the isomorphism classes of square-integrable representations of M ([Wal88, 1.3.2]). When we write $\sigma \in \widehat{M}_d$, we refer to a fixed representative of the corresponding isomorphism class. We refer to such elements as *discrete series representations* of M.

Given $\sigma \in \widehat{M}_d$, we use $\Lambda_\sigma \in i\mathfrak{t}_0^*$ to denote the Harish-Chandra parameter of $\sigma|_{M_0}$, where M_0 denotes the connected component of M at the identity (see [Kna01, Theorem 9.20]).

Definition 2.7.4. A *cuspidal pair* is a pair (P, σ) consisting of a cuspidal parabolic subgroup P together with some $\sigma \in \widehat{M}_d$.

The discrete series is important to the tempered representations (the representations weakly embeddable in $L^2(G)$).

Theorem 2.7.5 (See [Wal88, Proposition 5.2.5]). *Suppose V is a tempered, irreducible $(\mathfrak{g}, K)$-module. There is a parabolic subgroup P, a discrete series representation σ of M, and some unitary $\lambda \in i\mathfrak{a}_0^*$ such that V is a direct summand of $I_{\sigma,\lambda}^P$.*

In the future, when we refer to a "principal series representation," we will mean representations induced from cuspidal pairs (rather than from arbitrary unitary representations σ).

2.8 Hecke algebra

The following uses notation and results from [KV95, Chapter 1].

Definition 2.8.1. The Hecke algebra $R(K)$ of K is the space of K-finite smooth functions on K.

We note that (locally K-finite) K-modules give rise to (approximately unital) $R(K)$-modules, and vice-versa. Here, a K-module E is locally K-finite if it satisfies $E_K = E$. See [KV95, Theorem 1.57].

Definition 2.8.2. Given $\gamma \in \widehat{K}$, define $p_\gamma \in C^\infty(K)$ by

$$p_\gamma(k) = (\dim V_\gamma)\overline{\mathrm{Tr}(\gamma(k^{-1}))}.$$

Given a K-module (π, E), then $\pi(p_\gamma)$ is precisely the projection onto the γ-isotypical component of E.

Given a finite subset $F \subset \widehat{K}$, we set

$$p_F = \sum_{\gamma \in F} p_\gamma.$$

We will make use of the projections p_F extensively. In particular, given a K-module E, we will write $p_F E$ to denote the projection of E onto the K-types in F (instead of more common notation such as $E(F)$, which appears for example in [Wal88]).

It is readily checked that the functions p_γ are projections in $R(K)$ for each $\gamma \in \widehat{K}$. From [KV95, (1.37) and Proposition 1.39],

$$R(K) \cong \bigoplus_{\gamma \in \widehat{K}} p_\gamma R(K) \cong \bigoplus_{\gamma \in \widehat{K}} \mathrm{End}(V_\gamma),$$

where V_γ denotes a vector space representative of $\gamma \in \widehat{K}$.

Definition 2.8.3. The Hecke algebra $R(\mathfrak{g}, K)$ of G is the convolution algebra of K-finite distributions of G which are supported in K.

By [KV95, Corollary 1.71], there is an isomorphism of algebras

$$R(K) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g}) \xrightarrow{\cong} R(\mathfrak{g}, K) \tag{2.8.1}$$

given by $T \otimes u \mapsto T *_K u$. Here, we identify $u \in \mathcal{U}(\mathfrak{g})$ with the distribution $\tilde{u} \cdot \delta_e$ supported on the identity $\{e\}$, where $\tilde{u}$ is the left-invariant differential operator corresponding to u.

We remark that the category of $(\mathfrak{g}, K)$-modules is equivalent to the category of approximately unital $R(\mathfrak{g}, K)$-modules ([KV95, Theorem 1.117]). In terms of Fourier theory, $R(\mathfrak{g}, K)$ corresponds to polynomials with respect to the continuous parameter of the principal series.

Lemma 2.8.4 (See [Del84, Proposition 1])**.** *Let (P, σ) be a cuspidal pair. For $h \in R(\mathfrak{g}, K)$ and $v, w \in I_\sigma$, the map $\lambda \mapsto \langle \pi^P_{\sigma,\lambda}(h)v, w\rangle$ is a polynomial function in $\lambda \in \mathfrak{a}^*$.*

2.9 Minimal K-types

Given a K-type, $\gamma \in \widehat{K}$, we will write

$$\|\gamma\|^2 = \langle \gamma + 2\rho_c, \gamma + 2\rho_c\rangle.$$

This is known as the length, or "norm," of γ, and the concept is due to Vogan (see for example [Vog77]). Given a $(\mathfrak{g}, K)$-module V, the *minimal K-types* of V are those K-types appearing in V of smallest length (out of the K-types in V). This is a particularly important invariant of the representation, leading to the classification of admissible $(\mathfrak{g}, K)$-modules, known as Vogan-Zuckerman classification. In particular, we have the following important theorem of Vogan.

Theorem 2.9.1 (Vogan, [Vog79, Theorem 1.1]). *Let V be an irreducible admissible $(\mathfrak{g}, K)$-module. Then the minimal K-types occur in V with multiplicity* 1.

Now, let (P, σ) be a cuspidal pair. We write $(\pi^P_{\sigma,\lambda}, I_\sigma)$ for the principal series.

Definition 2.9.2. The set of minimal K-types of I_σ is denoted by $A(\sigma)$. We refer to these as the minimal K-types for σ. We write $\|\sigma\|$ to denote the length of any element of $A(\sigma)$.

It turns out that every K-type is the minimal K-type of some principal series representation. We can make a stronger statement.

Definition 2.9.3. Let $(P_1 = L_1N_1, \sigma_1)$ and $(P_2 = L_2N_2, \sigma_2)$ be cuspidal pairs. We say that the cuspidal pairs are *G-conjugate* if there exists some $g \in G$ such that $gL_1g^{-1} = L_2$, and such that $g\sigma_1g^{-1}$ is unitarily equivalent to σ_2.

Two G-conjugate pairs is also known as associate pairs (see, for example, [CCH16]).

Theorem 2.9.4 (Vogan, [Vog79, Theorems 7.16 and 7.17]). *The elements of $A(\sigma)$ occur with multiplicity one in I_σ. Moreover, the sets $A(\sigma)$ partition $\widehat{K}$, and if (P', σ') is another pair such that $A(\sigma') = A(\sigma)$, then the pairs (P, σ) and (P', σ') are G-conjugate.*

2.10 Intertwining operators

There is an important family of intertwining operators on the principal series. Let $\sigma \in \widehat{M}_d$, and let $\lambda \in \mathfrak{a}^*$. Fix two parabolic subgroups $P = MAN_P$ and $Q = MAN_Q$, both of which have the same Levi subgroup $L = MA$. We obtain two different representations: $(\pi^P_{\sigma,\lambda}, \mathcal{H}_\sigma)$ and $(\pi^Q_{\sigma,\lambda}, \mathcal{H}_\sigma)$. These two representations are related as follows. Given $v \in \mathcal{H}_\sigma$, we define for $u \in K$,

$$(A(Q, P, \sigma, \lambda)v)(u) = \int_{\theta(N_P)\cap N_Q} a_P(\overline{n})^{-\lambda-\rho} v(uk_P(\overline{n}^{-1}))d\overline{n}.$$

It turns that out that, when $\lambda \in \mathfrak{a}^*_{P,+}$, the above integral converges, and we obtain an intertwining operator

$$A(Q, P, \sigma, \lambda) : (\pi^P_{\sigma,\lambda}, \mathcal{H}_\sigma) \to (\pi^Q_{\sigma,\lambda}, \mathcal{H}_\sigma)$$

In general, however, the above integral may not converge. Instead, it turns out that, for $v, w \in \mathcal{H}_\sigma$, the map $\lambda \mapsto \langle A(Q,P,\sigma,\lambda)v, w\rangle_{L^2(K)}$ extends meromorphically to $\mathfrak{a}^*$ as a function of λ, and so we obtain a family of intertwining operators for generic λ.

Theorem 2.10.1 (See [KS80])**.** *Fix two parabolic subgroups $P = MAN_P$ and $Q = MAN_Q$ with the same Levi subgroup MA. Let $\sigma \in \widehat{M}_d$. If $\lambda \in \mathfrak{a}^*_{P,+}$ and $v \in I_\sigma$, the integral*

$$(A(Q,P,\sigma,\lambda)v)(u) = \int_{\theta(N_P)\cap N_Q} a_P(\overline{n})^{-\lambda-\rho} v(uk_P(\overline{n}^{-1}))d\overline{n}$$

converges for each $u \in K$, and defines an element $A(Q,P,\sigma,\lambda)v \in I_\sigma$. Moreover, the map $v \mapsto A(Q,P,\sigma,\lambda)v$ defines an intertwining operator

$$A(Q,P,\sigma,\lambda) : (\pi^P_{\sigma,\lambda}, I_\sigma) \to (\pi^Q_{\sigma,\lambda}, I_\sigma).$$

Finally, for each $v, w \in I_\sigma$, the map $\lambda \mapsto \langle A(Q,P,\sigma,\lambda)v, w\rangle$ extends to a meromorphic function in λ, and in this way we obtain intertwining operators $A(Q,P,\sigma,\lambda)$ for generic $\lambda \in \mathfrak{a}^*$.

Definition 2.10.2. The family of operators $A(Q,P,\sigma,\lambda)$ from Theorem 2.10.1 is called the *family of (unnormalized) Knapp-Stein intertwining operators*. When σ is unambiguous we will write $A(Q,P,\lambda)$ instead of $A(Q,P,\sigma,\lambda)$.

We now describe a normalization of the intertwining operators, which results in a family of operators that are rational in the continuous parameter. This differs from the normalization found in [KS80], and we instead normalize with respect to some fixed minimal K-type of I_σ (this idea can be found in [Del84]).

Definition 2.10.3. Fixing a minimal K-type $\mu_0 \in A(\sigma)$, $A(Q,P,\sigma,\lambda)$ acts by a scalar on $p_{\mu_0} I_\sigma$ (since μ_0 has multiplicity 1), which we denote by $c_{\mu_0}(Q,P,\sigma,\lambda)$. The *normalized* intertwining operator is

$$\mathcal{A}(Q,P,\lambda) = \mathcal{A}(Q,P,\sigma,\lambda) = c_{\mu_0}(Q,P,\sigma,\lambda)^{-1} A(Q,P,\sigma,\lambda).$$

Theorem 2.10.4 (See [Del05, (1.12)])**.** *The operator $\mathcal{A}(Q,P,\lambda)$ is unitary on $i\mathfrak{a}^*_0$, and is independent of λ on minimal K-types. When P, Q, R are parabolic subgroups with a common Levi subgroup, we have*

$$\mathcal{A}(R,Q,\lambda)\mathcal{A}(Q,P,\lambda) = \mathcal{A}(R,P,\lambda), \quad \mathcal{A}(P,Q,\lambda)\mathcal{A}(Q,P,\lambda) = \mathrm{Id}_{I_\sigma}$$

as meromorphic functions of λ. *Additionally, for each finite subset* $F \subset \widehat{K}$, *the operator* $\mathcal{A}(Q, P, \lambda)$ *is rational in* λ *when restricted to* $p_F I_\sigma$.

2.10.1 Intertwining operators via the Weyl group action

Finally, we describe an action of the Weyl group on the principal series via intertwining operators. Let $W(\mathfrak{g}_0, \mathfrak{a}_0) = W(A) = N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0)$ denote the (restricted) Weyl group corresponding to A.

Given $w \in N_K(\mathfrak{a}_0)$ and $\sigma \in \widehat{M}_d$, we define

$$w \cdot \sigma(m) = \sigma(w^{-1} m w).$$

We will write W_σ for the set of $w \in W(\mathfrak{g}_0, \mathfrak{a}_0)$ such that for any representative $m_w \in N_K(\mathfrak{a}_0)$ of w, the representation $m_w \cdot \sigma$ is unitarily equivalent to σ. In other words, W_σ is the stabilizer of σ with respect to the action of $W(\mathfrak{g}_0, \mathfrak{a}_0)$ on $\widehat{M}_d$.

We note that only one representative needs to stabilize σ, since $Z_K(\mathfrak{a}_0)$ preserves the isomorphism class of σ. We will always identify $w \in W_\sigma$ with a fixed choice of representative in $N_K(\mathfrak{a}_0)$.

If $w \in W_\sigma$, we write

$$\sigma(w) : V_\sigma \xrightarrow{\cong} V_{w \cdot \sigma}$$

for the corresponding unitary equivalence. The notation $\sigma(w)$ is based on the fact ([KS80, Lemma 7.9]) that it is possible to extend the action σ of M on V_σ to the group generated by M and w, unique up to a constant of modulus 1 (we have taken this notation from [Del84, p. 124]; see also [KS80, Proposition 7.10]). We then define

$$\sigma(w) : I_\sigma \to I_{w \cdot \sigma}$$

pointwise, so that $(\sigma(w)v)(k) := \sigma(w)(v(k))$ for $v \in I_\sigma$ and $k \in K$. Up to a scalar of modulus one, this is independent of the choice of representative for w.

The operator $\sigma(w)$ acts as multiplication by a scalar on $p_{\mu_0} I_\sigma$, where μ_0 is the fixed minimal K-type of I_σ that we used to normalize the intertwining operators in Definition 2.10.3. Now, we may modify $\sigma(w)$ by a scalar of modulus one, so we choose $\sigma(w)$ such that this scalar is 1 (see [Del05, p. 999]).

Definition 2.10.5. Given $w \in W_\sigma$, define the map

$$T(w) : I_\sigma \to I_{w\cdot\sigma}$$

by $(T(w)\varphi)(k) = \varphi(kw)$, which intertwines $\pi^P_{\sigma,\lambda}$ and $\pi^{wPw^{-1}}_{w\cdot\sigma,w\lambda}$.

Given $w \in W_\sigma$, we define

$$\mathcal{A}(P, w, \lambda) = \mathcal{A}(P, w, \sigma, \lambda) = \sigma(w)^{-1}T(w)\mathcal{A}(w^{-1}Pw, P, \sigma, \lambda),$$

which intertwines $\pi^P_{\sigma,\lambda}$ and $\pi^P_{\sigma,w\lambda}$.

As an immediate consequence of Theorem 2.10.4 and the definitions, $\mathcal{A}(P, w, \lambda)$ acts as a scalar, independent of λ, on minimal K-types, and acts trivially on $p_{\mu_0} I_\sigma$. Also, $\mathcal{A}(P, w, \lambda)$ is unitary when $\lambda \in i\mathfrak{a}^*$.

Theorem 2.10.6 (See [Del05, (1.14) (iv)])**.** *If $w, w' \in W_\sigma$, then*

$$\mathcal{A}(P, w'w, \sigma, \lambda) = \mathcal{A}(P, w', \sigma, w\lambda)\mathcal{A}(P, w, \sigma, \lambda).$$

2.11 Langlands classification and Vogan classification

We will say that a *Langlands triple* is a triple (P, σ, λ) consisting of a parabolic subgroup $P = MAN$, a tempered representation σ of M, and some $\lambda \in \mathfrak{a}^*_+$.

Theorem 2.11.1 (Langlands classification; see [Wal88, 5.4.1])**.** *Let (P, σ, λ) be a Langlands triple. Then $J^P_{\sigma,\lambda} = I_\sigma / \ker(\mathcal{A}(\theta(P), P, \sigma, \lambda))$ is the unique irreducible quotient of $(\pi^P_{\sigma,\lambda}, I_\sigma)$. Such a quotient is called a Langlands quotient.*

Moreover, if (P', σ', λ') is another Langlands triple, and if $J^{P'}_{\sigma',\lambda'}$ is equivalent to $J^P_{\sigma,\lambda}$, then $P = P'$, $\lambda = \lambda'$, and σ is unitarily equivalent to σ'.

Finally, all admissible representations is the Langlands quotient of some Langlands data (P, σ, λ).

We will need, in particular, Vogan-Zuckerman classification on the unitary principal series. The statement we use is [Del05, (1.7)], but the reference is [Vog81, Chapter 6]. In the following theorem, we use $I^P_{\sigma,\lambda}$ to denote the $(\mathfrak{g}, K)$-module $(\pi^P_{\sigma,\lambda}, I_\sigma)$.

Theorem 2.11.2 (Vogan classification)**.** *Let (P, σ) be a cuspidal pair, and let $\lambda \in \overline{\mathfrak{a}^*_{P,+}}$. There exists a partition*

$$A(\sigma) = A(\sigma)_1 \sqcup \cdots \sqcup A(\sigma)_l \tag{2.11.1}$$

of the set $A(\sigma)$, and a corresponding unique decomposition

$$I^P_{\sigma,\lambda} \cong I^P_{\sigma,\lambda}[A(\sigma)_1] \oplus \cdots \oplus I^P_{\sigma,\lambda}[A(\sigma)_l], \tag{2.11.2}$$

where $I^P_{\sigma,\lambda}[\mu_i]$ are subrepresentations, each with a unique irreducible quotient $J^P_{\sigma,\lambda}[A(\sigma)_i]$ whose minimal K-types is precisely the set $A(\sigma)_i$.

In particular, every irreducible subquotient of $I^P_{\sigma,\lambda}$ contains a minimal K-type of I_σ.

Given $\mu \in A(\sigma)$, then if $\mu \in A(\sigma)_i$ we will write $I^P_{\sigma,\lambda}[\mu] = I^P_{\sigma,\lambda}[A(\sigma)_i]$ and $J^P_{\sigma,\lambda}[\mu] = J^P_{\sigma,\lambda}[A(\sigma)_i]$. If (Q, σ) is another cuspidal pair with the same Levi subgroup as P, and if $\lambda \in \overline{\mathfrak{a}^*_{P,+}} \cap \overline{\mathfrak{a}^*_{Q,+}}$, then $J^P_{\sigma,\lambda}[\mu] = J^Q_{\sigma,\lambda}[\mu]$ for each $\mu \in A(\sigma)$.

The partition given in (2.11.1) comes from the action of the so-called "R-group", which is a particular subgroup of W_σ. As a brief description, the R-group R_σ is the quotient of W_σ by the subgroup W^0_σ of elements w such that $\mathcal{A}(P, w, 0)$ is the identity on I_σ. The R-group is abelian (every element has order 2), and its dual, $\widehat{R}_\sigma$, acts simply transitively on $A(\sigma)$. Given $\lambda \in \overline{\mathfrak{a}^*_{P,+}}$, we set

$$R_\sigma(\lambda) = \{r \in R_\sigma : r \cdot \lambda \in W^0_\sigma \cdot \lambda\},$$

and we set $\widehat{R}_\sigma(\lambda)$ to be the annihilator of $R_\sigma(\lambda)$ in $\widehat{R}_\sigma$. Then the partition (2.11.1) consists of the orbits of the action of $\widehat{R}_\sigma(\lambda)$ on $A(\sigma)$. This explicit description of the partition (in terms of the R-group) forms the complete Vogan classification.

See Section 3.2 for a more thorough discussion of this group.

2.12 Phillip's K-theory for Fréchet algebras

One of the main goals of this dissertation is to compare the K-theory of the reduced group C^*-algebra with the K-theory of the Casselman algebra. However, the latter is a Fréchet algebra, and so we need a K-theory for Fréchet algebras. Moreover, the typical (for example, algebraic) K-theory is not suitable to get a decent theory with properties we are familiar with, such as Bott periodicity.

Instead, we will use the K-theory of Phillips [Phi91]. Typically, operator K-theory (such as that used for C^*-algebras and Banach algebras) use the compact operators, or even the algebra $\bigcup_n M_n(\mathbb{C})$ of all finite matrices, to stabilize the algebra. However, these are not satisfactory for Fréchet algebras, because tensoring by these algebras may not result in another Fréchet algebra (and ultimately would not allow us to prove the familiar properties of K-theory).

For Fréchet algebras, it is appropriate to use a "smooth version" of the compact operators,

$$\mathcal{K}_\infty = \{f : \mathbb{Z}^2 \to \mathbb{C} : \sum_{m,n} (1 + |m| + |n|)^N |f(m,n)| < \infty \, \forall N \in \mathbb{N}\},$$

for the purpose of stabilization. The above is a nuclear Fréchet algebra, so that if A is a Fréchet algebra, then so is $\mathcal{K}_\infty \otimes A$.

With this new algebra, we may define K-theory in the usual way. That is, if A is a Fréchet algebra, we may define $K_0(A)$ as the set of homotopy classes of idempotents $p \in M_2((\mathcal{K}_\infty \otimes A)^+)$ such that $p - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathcal{K}_\infty \otimes A)$, and we define $K_1(A)$ as the homotopy classes of invertible $u \in (\mathcal{K}_\infty \otimes A)^+$ such that $u - 1 \in \mathcal{K}_\infty \otimes A$. In [Phi91], Phillip's uses the notation $RK_*(A)$ (for representable K-theory), and $K_*(A)$ to denote the typical K-theory definition. In this dissertation, we will simply use $K_*(A)$ in both cases.

In [Phi91, Section 7], Phillips shows that this definition reduces to the usual definitions for the special cases of A. For example, if A is a Banach algebra, then $K_*(A)$ is isomorphic to the typical operator K-theory of A. Another important example is when A is a Fréchet algebra whose set $\mathrm{GL}_1(A)$ of invertible elements of A is open, and such that the inverse map $a \mapsto a^{-1}$ is continuous on $\mathrm{GL}_1(A)$ (this is known as a "good" algebra in [Bos90]). For a "good" Fréchet algebra, we may use the typical definition of K-theory (involving finite matrices in A). Finally, if A is a commutative unital Fréchet algebra, then the Fréchet algebra K-theory of A is isomorphic to the representable (topological) K-theory of its maximal ideal spectrum (this is a generalization of Novodvorskii's theorem [Nov67] to Fréchet algebras).

The K-theory for Fréchet algebras satisfies several typical properties of K-theory. Let A be a Fréchet algebra. We define

$$S(A) = \{f \in C([0,1], A) : f(0) = f(1) = 0\}.$$

Then Phillips proves Bott periodicity: there is an isomorphism (the Bott map)

$$\beta : K_0(A) \xrightarrow{\cong} K_1(SA).$$

He also proves the standard cohomological property of K-theory. If $J \subset A$ is a closed two-sided ideal, and if $Q = A/J$, then we obtain a 6-term exact sequence

$$\begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(Q) \\ \uparrow & & & & \downarrow \\ K_1(Q) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J). \end{array}$$

As usual, if we define $K_{i+1}(A) = K_i(SA)$, then the above leads to a long exact sequence in K-theory.

In the above, our Fréchet algebras are assumed to be locally multiplicatively convex, so that we may write them as an inverse limit of Banach algebras. As such we will write $A = \varprojlim A_n$, where A_n is an inverse system of Banach algebras, such that the maps $\pi_{n,m} : A_n \to A_m$ (for $n \geq m$) defining the inverse system all have dense range. This assumption is important in order to make reductions to the Banach algebra case. In the future, we will always write $A = \varprojlim A_n$ to represent A as an inverse limit of Banach algebra such that the connecting maps have dense range.

Now suppose $A = \varprojlim A_n$. Phillips has provided a "Milnor $\varprojlim^1$" sequence. We define

$$\varprojlim{}^1 K_i(A_n) = \operatorname{coker}(\delta : \prod_n K_i(A_n) \to \prod_n K_i(A_n)),$$

where, writing $(a_n) \in \prod_n K_i(A_n)$, we set $\delta(a_n) = (a_n - (\pi_{n+1,n})_*(a_{n+1}))$. Then there is a short exact sequence

$$0 \to \varprojlim{}^1 K_{1-*}(A_n) \to K_*(A) \to \varprojlim K_*(A_n) \to 0.$$

The above exact sequence gives us a direct way to prove isomorphisms in K-theory by a reduction to Banach algebras. In particular, if A_n all have the same K-theory, then $\varprojlim^1 K_{1-*}(A_n) = 0$. Therefore, by use of mapping cones, an isomorphism $K_*(A) \cong K_*(B)$ will result from compatible isomorphisms $K_*(A_n) \cong K_*(B_n)$. Among other uses, this will allow us to prove an extension of Lafforgue's Morita equivalence result (see [Par09]) to Fréchet algebras.

Chapter 3
Background on Delorme's Work

In this chapter, we will summarize various results of Delorme which are relevant to this dissertation. Delorme's statement and proof of the Paley-Wiener theorem, and in particular the techniques used in [Del05], are of particular importance to the main theorems in this dissertation. We will adapt these results from the compactly supported functions to the Casselman algebra, which will allow us to prove a structure theorem for this algebra. Finally, the result in K-theory for Fréchet algebras regarding Morita equivalence will allow us to use this structure theorem to prove a refined Connes-Kasparov isomorphism.

Let me provide a brief summary of the results shown in this chapter. In [Del84], Delorme proves a generalized Harish-Chandra isomorphism theorem, which he uses to prove a Paley-Wiener theorem for groups with one conjugacy class of Cartan subgroups (for example, complex groups). In that paper, he also proves an important theorem which unites certain subgroups of the Weyl group (called R-groups), defined in different ways by Knapp-Stein and Vogan. Delorme shows that Vogan's R-group satisfies certain axioms, and that when these axioms are satisfied, the group must be determined by intertwining operators and therefore be the same as the Knapp-Stein R-group. We emphasize that it is not at all obvious that these definitions produce the same R-group, as this group is defined via root systems which are not always the same between the two definitions.

In addition, towards his Paley-Wiener theorem for general real reductive groups, he proves a result with Souaifi in [DS04] which generalizes the subquotient theorem for a general *possibly reducible* (admissible) $(\mathfrak{g}, K)$-module. We note that such theorems typically only apply to irreducible admissible $(\mathfrak{g}, K)$-modules, where a version of Schur's lemma applies (see [Wal88, 0.5.1]). Instead, the proof of this theorem involves rather technical harmonic analysis (the paper is primarily about a theorem for general reductive symmetric spaces, and the result we cite uses the group case of this theorem). We recall that the Paley-Wiener space (the Fourier image of $C_c^\infty(G)$, or of $\mathcal{S}(G)$) is described as holomorphic functions with a certain decay condition, as well as with certain algebraic restrictions. The result in [DS04] is essential in producing enough of these algebraic restrictions to fully describe the Paley-Wiener space.

Note that Delorme uses the results listed here to prove the Paley-Wiener theorem specifically for $C_c^\infty(G)$. This dissertation is concerned with the Casselman algebra $\mathcal{S}(G)$, and so we will slightly extend some of his results to this case in Chapter 6. In this chapter, we will provide motivation for the proof of these results (geared toward $\mathcal{S}(G)$). For the definition of $\mathcal{S}(G)$, as well as the "Paley-Wiener space" for a complex vector space (denoted $\mathsf{PW}(V)$), see Definitions 4.1.1 and 4.3.3. In the analogy between $\mathcal{S}(G)$ and $C_c^\infty(G)$, the space $\mathsf{PW}(\mathbb{C}^n)$ is analogous to the Fourier image of $C_c^\infty(\mathbb{R}^n)$.

3.1 The work of Delorme and Flensted-Jensen

Let us begin with a theorem due to Delorme and Flensted-Jensen.

Theorem 3.1.1 ([DFJ91, Theorem 3 (b)])**.** *Let G be a connected, semisimple group. Let (π, V) be a G-representation, and fix $\gamma \in \widehat{K}$.*

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $F \in \mathsf{PW}(\mathfrak{h}^*)^{W(\mathfrak{g},\mathfrak{h})}$. There exists some $f \in p_\gamma \mathcal{S}(G) p_\gamma$ such that

$$\pi(f) = F(\Lambda_\pi) p_\gamma,$$

where Λ_π is the Harish-Chandra parameter of the infinitesimal character of π.

We also remark that a version of this theorem applies to semisimple symmetric spaces. See [DFJ91, Theorem 3 (a)]. Indeed, the above theorem can be regarded as the group case of the theorem for symmetric spaces. In addition, the analogous statement for compactly supported functions hold too, which is used in the traditional Paley-Wiener theorems. See [DFJ91, Theorems 1 and 2].

The above theorem is important to the Paley-Wiener theorem as it allows us to produce many functions in $\mathcal{S}(G)$. In particular, the above theorem roughly says that the Fourier image contains multiples of the identity in a particular K-type block of the Paley-Wiener space. Moreover, Delorme uses this theorem to fully understand the Fourier image of $\mathcal{S}(G)$ when restricted to the minimal K-types of a given principal series.

There are several different approaches to the proof of Theorem 3.1.1. The simplest method involves a reduction to the spherical (K-bi-invariant) case (that is, when γ is the identity). Indeed, any symmetric space G/H has a corresponding Riemannian symmetric space G^d/K^d, which was developed and used by Flensted-Jensen in [FJ80] (this is related to taking the compact form $\mathfrak{g}_u = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ of a reductive Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$). Moreover, the smooth, K-finite functions on G/H correspond precisely to the K^d-finite functions on G^d/K^d. By the Paley-Wiener theorem for spherical functions (due to Helgason and Gangolli [Hel84, Gan71]), we may choose a K^d-biinvariant function on G^d whose spherical transform is our given $F \in \mathsf{PW}(\mathfrak{h}^*)^{W(\mathfrak{g},\mathfrak{h})}$. By a simple averaging trick, and by the relationship between G^d/K^d and G/H, we may then produce an element of $p_\gamma C_c^\infty(G/H)p_\gamma$ with the desired properties. The same reduction applies to $\mathcal{S}(G)$; however we must either use a corresponding spherical Paley-Wiener theorem for such functions (see [TV71] or [Ank91]), or we can use a trick to reduce to the complex group case, where there is an explicit formula for the inverse spherical transform (see [FJ78]).

3.2 Definitions of the R-group

We now describe the contributions in [Del84]. First, we recall from [Vog81] that the Weyl group W_σ has a semidirect product decomposition of the form

$$W_\sigma = R_\sigma \rtimes W_\sigma^0$$

with various important properties which we list in Theorem 3.2.3.

The group R_σ is known as the R-group, and is of particular importance due to its connection with intertwining operators (roughly speaking, it is the part of the Weyl group which gives nontrivial intertwining operators, and therefore determines the reducibility of the principal series). Moreover, in the Vogan definition of the R-group, the character group $\widehat{R}_\sigma$ of R_σ acts simply transitively on the minimal K-types $A(\sigma)$, which is important to the Vogan-Zuckerman classification theorem.

There are two definitions of W_σ^0 and R_σ: one by Knapp-Stein [KS80, Section 13], and one by Vogan [Vog81, Section 4.3] (in the former, W_σ^0 is denoted as W'_σ). In both definitions, the subgroup W_σ^0 is the Weyl group of some smaller root system (called "useful" and "good" roots in the respective definitions), and R_σ is defined as the corresponding quotient W_σ/W_σ^0 (although it turns out that R_σ may be identified as a subgroup of W_σ).

The two approaches to defining R_σ appear rather different, and indeed it is not always the case that the root systems defining W_σ^0 between the two definitions are the same. Regardless, it is interesting to understand the relationship between the two.

In [Del84], Delorme proves that these two definitions produce the same groups. This is useful because it allows us to take important properties from both. In particular, the Vogan classification uses Vogan's definition of the R-group, while at other times we would like to use the property that the (normalized) intertwining operators corresponding to elements of W_σ^0 acts as the identity.

Definition 3.2.1. Fix a cuspidal pair (P, σ). Given $w \in W_\sigma$ and $\mu \in A(\sigma)$, let $a_\mu(w)$ be the scalar such that $\mathcal{A}(P, w, \sigma, 0)p_\mu = a_\mu(w)p_\mu$. Then, given $\mu, \nu \in A(\sigma)$, we set

$$\hat{r}_{\mu\nu}(w) = a_\mu(w)a_\nu(w)^{-1}.$$

Theorem 3.2.2 ([Del84, Theorem 1 (iv)])**.** *Under Vogan's definition of the R-group, for each $\mu, \nu \in A(\sigma)$, the function $\hat{r}_{\mu\nu}$ is trivial on W_σ^0 and therefore defines a character of R_σ.*

Moreover, under Vogan's definition of the action of $\widehat{R}_\sigma$ on $A(\sigma)$, the function $\hat{r}_{\mu\nu}$ is the unique element of $\widehat{R}_\sigma$ such that $\hat{r}_{\mu\nu} \cdot \nu = \mu$.

The theorem that follows is an "axiomatic" description of the R-group, and the action of $\widehat{R}_\sigma$ on $A(\sigma)$. In particular, the above theorem holds for any definition of the R-group and action which satisfies the properties listed in the next theorem. However, these properties are tailored to Vogan's definition, and it is really the last statement of the next theorem that equates Vogan's R-group to the Knapp-Stein R-group.

For Property 3 of the following Theorem, we recall that the minimal K-types of a principal series representation, say $I_{\sigma,\lambda}^P$, appear with multiplicity one by [Vog79, Theorem 1.1]. Therefore, for each $\mu \in A(\sigma)$, the representation $I_{\sigma,\lambda}^P$ has a unique irreducible subquotient $J_{\sigma,\lambda}[\mu] = J_{\sigma,\lambda}^P[\mu]$ containing μ. See also Theorem 2.11.2 for the classification theorem of Vogan.

Theorem 3.2.3 (See [Del84, Section 1.6 and Theorem 1])**.** *Let $(P = MAN, \sigma)$ be a cuspidal pair. Suppose we have a semidirect product decomposition $W_\sigma = R_\sigma W_\sigma^0$ of subgroups, and a simply transitive action of $\widehat{R}_\sigma$ on $A(\sigma)$. The group W_σ^0 and the action of $\widehat{R}_\sigma$ are uniquely determined by the following properties.*

1. *W_σ^0 is generated by reflections.*

2. R_σ is a product of copies of $\mathbb{Z}/2$ (that is, every element of R_σ has order 2).

3. Let $\lambda \in \overline{\mathfrak{a}^*_{P,+}}$. Given $\mu \in A(\sigma)$, let $J_{\sigma,\lambda}[\mu]$ denote the unique irreducible subquotient of $I_{\sigma,\lambda}[\mu]$ containing the K-type μ. If $\mu, \mu' \in A(\sigma)$, then $J_{\sigma,\lambda}[\mu]$ equals $J_{\sigma,\lambda}[\mu']$ if and only if μ and μ' are related by an element of $\widehat{R}_\sigma(\lambda)$: the annihilator of $\{r \in R_\sigma : r \cdot \lambda \in W^0_\sigma \lambda\} \subset R_\sigma$.

These properties are satisfied by Vogan's definition of the R-group (as defined in [Vog81]).

If W^0_σ and $\widehat{R}_\sigma$ satisfy these properties, then these groups can be described completely in terms of the (normalized) intertwining operators. Namely, W^0_σ is precisely the subgroup of elements w for which the normalized intertwining operator $\mathcal{A}(P, w, \sigma, 0)$ is the identity, and the action of $\widehat{R}_\sigma$ is characterized by the property that $\hat{r}_{\mu\nu} \cdot \nu = \mu$, where $\hat{r}_{\mu\nu}$ is defined as in Definition 3.2.1.

By [KS80, Theorem 13.4], the Knapp-Stein group W'_σ is determined by the intertwining operators, as described in the last statement of the above theorem. Therefore, as Vogan's group W^0_σ satisfies the properties in the above theorem, this means that the two groups must coincide (and therefore so must the R-groups). Finally, for this section (and this dissertation), we have assumed that the group is connected. However, it is possible to extend everything to the disconnected case, and we refer to [CD84, Section 4] and [Del05, Appendix] for the reduction to the connected case.

3.3 Generalized Harish-Chandra homomorphism

We now describe the other contribution of [Del84], considered the main theorem of the paper. We will rephrase the theorem in terms of the Hecke algebra $R(\mathfrak{g}, K)$, which replaces the algebra $\mathcal{U}(\mathfrak{g})^K$ when considering disconnected G. As such, the following theorem applies to disconnected G (although the reader will need to adjust the arguments given in [Del84] slightly to this case).

In what follows, we define $R(\mathfrak{g}, A(\sigma)) = p_{A(\sigma)} R(\mathfrak{g}, K) p_{A(\sigma)}$. By Lemma 2.8.4, for each cuspidal (P, σ), there is a map

$$\pi_\sigma : R(\mathfrak{g}, K) \to \mathbb{C}[\mathfrak{a}^*] \otimes \mathrm{End}(I_\sigma)$$

given by $\pi_\sigma(T)(\lambda) = \pi^P_{\sigma,\lambda}(T)$ for each $T \in R(\mathfrak{g}, K)$ and $\lambda \in \mathfrak{a}^*$. Note that $\pi^P_{\sigma,\lambda}(T)$ is can be written as a finite-dimensional matrix for each $T \in R(\mathfrak{g}, K)$, due to the fact that the elements of $R(\mathfrak{g}, K)$ are K-finite. If we multiply $p_{A(\sigma)}$ to the left and right, we restrict to a map

$$\pi_\sigma : R(\mathfrak{g}, A(\sigma)) \to \mathbb{C}[\mathfrak{a}^*] \otimes \mathrm{End}(p_{A(\sigma)} I_\sigma).$$

The following theorem describes the image of the above map, and is a simplified version of [Del84, Theorem 3].

Theorem 3.3.1 (See [Del84, Theorem 3])**.** *We have*

$$\pi_\sigma(R(\mathfrak{g}, A(\sigma))) = \left(\mathbb{C}[\mathfrak{a}^*] \otimes \mathrm{End}(p_{A(\sigma)} I_\sigma)\right)^{W_\sigma},$$

where the action of W_σ *on* $\mathbb{C}[\mathfrak{a}^*] \otimes \mathrm{End}(p_{A(\sigma) I_\sigma})$ *is given by*

$$(w \cdot p)(\lambda) = \mathcal{A}(P, w, \sigma, w^{-1}\lambda) p(w^{-1}\lambda) \mathcal{A}(P, w^{-1}, \sigma, \lambda).$$

The full statement of [Del84, Theorem 3] describes the (μ, ν)-blocks of $R(\mathfrak{g}, A(\sigma))$, where $\mu, \nu \in A(\sigma)$. Specfically, let $\hat{r}_{\mu\nu}$ be the unique element of $\widehat{R}_\sigma$ such that $\hat{r}_{\mu\nu} \cdot \nu = \mu$. Define

$$\mathbb{C}[\mathfrak{a}^*]^{W_\sigma^0}(\hat{r}_{\mu\nu}) = \{p \in \mathbb{C}[\mathfrak{a}^*] : p(w\lambda) = \hat{r}_{\mu\nu}(\overline{w}) p(\lambda)\},$$

where $\overline{w}$ is the image of w under the quotient map $W_\sigma \to W_\sigma / W_\sigma^0 = R_\sigma$. Then

$$\pi_\sigma(p_\mu R(\mathfrak{g}, K) p_\nu) = \mathbb{C}[\mathfrak{a}^*]^{W_\sigma^0}(\hat{r}_{\mu\nu}) \otimes \mathrm{Hom}(p_\nu I_\sigma, p_\mu I_\sigma).$$

This is considered a generalized Harish-Chandra homomorphism, due to the following. If we consider the spherical principal series, there is one minimal K-type: the trivial K-type, appearing with multiplicity one in the spherical principal series (so that $p_\mu I_\sigma = \mathbb{C}$). In this case, and assuming the group is connected, we obtain a homomorphism

$$\pi : \mathcal{U}(\mathfrak{g})^K \to \mathbb{C}[\mathfrak{a}^*]^{W(\mathfrak{g}_0, \mathfrak{a}_0)}.$$

This map is precisely the Harish-Chandra homomorphism HC for $\mathcal{U}(\mathfrak{g})^K$, for which the classical result is that this homomorphism is surjective. Roughly speaking, Theorem 3.3.1 describes what happens when we consider analogous homomorphisms for an arbitrary K-type (which is always a minimal K-type of some principal series).

We remark that the above theorem is proved by purely algebraic means (but still through use of deep representation theory such as Vogan classification). By use the theorem in [DFJ91], the above theorem allows us to completely the describe the Fourier image of $\mathcal{S}(G)$ when restricted to minimal K-types. That is to say, the only harmonic analysis required up to this point is the spherical Paley-Wiener theorem of Helgason and Gangolli, which can be further reduced to the classical Paley-Wiener theorem (on $\mathbb{R}^n$) by the reduction to the complex group case given in [FJ78].

Finally, let us consider what happens when $G = \mathrm{SL}(2, \mathbb{R})$. In this case, we will describe the Fourier image of $R(\mathfrak{g}, A(\sigma))$ when σ is one of the two representations of $M = \{\pm \mathrm{Id}\}$. In the spherical case, $\sigma = \sigma_0$ is the trivial representation, and $A(\sigma)$ consists of the trivial K-type which we denote by $0 \in \widehat{K} \cong \mathbb{Z}$. We then simply obtain

$$\pi_{\sigma_0}(R(\mathfrak{g}, \{0\})) = \mathbb{C}[z]^{\mathbb{Z}/2} = \mathbb{C}[z^2].$$

That is, we obtain all even polynomials. This is just the classical Harish-Chandra isomorphism for $\mathrm{SL}(2, \mathbb{R})$, and indeed $R(\mathfrak{g}, \{0\})$ is generated by the Casimir element.

When $\sigma = \sigma_1$ is the nontrivial ("odd") representation, then $A(\sigma) = \{1, -1\}$, and we obtain

$$\pi_{\sigma_1}(R(\mathfrak{g}, \{1, -1\})) = \left\{ \begin{bmatrix} p_{-1,-1}(z^2) & zp_{-1,1}(z^2) \\ zp_{-1,1}(z^2) & p_{1,1}(z^2) \end{bmatrix} : p_{i,j} \in \mathbb{C}[z] \right\}.$$

By use of [DFJ91], we can show that $\pi_{\sigma_1}(\mathcal{S}(G, \{-1\}))$ and $\pi_{\sigma_1}(\mathcal{S}(G, \{1\}))$ equal $\mathsf{PW}(\mathbb{C})^{\mathbb{Z}/2}$, and by multiplication with $\pi_{\sigma_1}(R(\mathfrak{g}, \{1, -1\}))$, we may describe $\pi_{\sigma_1}(\mathcal{S}(G, \{-1, 1\}))$ in an analogous fashion to $R(\mathfrak{g}, \{-1, 1\})$ (the polynomials $p_{i,j}(z^2)$ are replaced by elements of $\mathsf{PW}(\mathbb{C})^{\mathbb{Z}/2}$)

3.4 A subquotient theorem by Delorme and Souaifi

Theorem 3.4.1 (See [DS04, Theorem 3 (ii)]). *Let X be an admissible $(\mathfrak{g}, K)$-module whose K-types have length larger than R. Then X is a subquotient of a direct sum of (successive derivatives of) principal series representations, each of which contain only K-types of length larger than R.*

We note that, of course, this is a generalization of the classical Harish-Chandra subquotient theorem to possibly reducible representations. However, the proof of this theorem is highly technical, requiring deep aspects of the harmonic analysis for symmetric spaces, and the statement involving length of K-types requires Vogan classification.

We will define the notion of derivative for representations when it becomes relevant to do so, in Chapter 6. For now, we just note that the action $\pi_{\sigma,\lambda}(g)$ on I_σ is holomorphic in λ, and that the naive (partial) derivative will not give a representation due to the Leibniz rule, but instead the operator

$$\begin{bmatrix} \pi_{\sigma,\lambda}(g) & \partial_z \pi_{\sigma,\lambda}(g) \\ 0 & \pi_{\sigma,\lambda}(g) \end{bmatrix}$$

(where z is some coordinate for λ) on $I_\sigma \oplus I_\sigma$ does give a representation for each λ, which may be regarded as the "derivative" of the principal series.

Anyway, the above theorem is of particular importance to the Paley-Wiener theorem, as it supplies us with algebraic conditions that the Fourier transform of $\mathcal{S}(G)$ must satisfy. First, a simple corollary of the above theorem is the following.

Theorem 3.4.2. *Let $\phi \in C_c^\infty(G)$ (or $\mathcal{S}(G)$), and suppose that ϕ acts by 0 on any principal series representation whose K-types have length larger than R. Then ϕ acts by 0 on any admissible $(\mathfrak{g}, K)$-module V whose K-types have length larger than R.*

Indeed, the $(\mathfrak{g}, K)$-module V is a subquotient of some direct sum of (derivatives of) principal series representations whose K-types have length larger than R. As ϕ vanishes identically with respect to the continuous parameter of the (relevant) principal series, so must its derivatives. It follows that ϕ acts by 0 on V.

Now, suppose ϕ is as in the above theorem, and let $f(\lambda) = \pi^P_{\sigma,\lambda}(\phi)$ (which is a holomorphic function in λ). Then f will vanish on the kernel of the (normalized) intertwining operators $\mathcal{A}(Q, P, \sigma, \lambda)$ when $\|\sigma\| \geq R$, because this kernel does not contain any K-type in $A(\sigma)$ and so every K-type must have length more than R (this also works for the "derivatives" of such operators). Therefore, a simple argument shows that $\mathcal{A}(Q, P, \sigma, \lambda)$ must divide f. This result will be called the "Divisiblity Theorem", and is the basis for a theorem which we call the "Factoring Theorem". The Factoring Theorem will allow us to relate arbitrary K-type blocks of the Fourier image of f to the minimal K-type block (which we have already described in the previous section). In this fashion, we will be able to completely the describe the Fourier image of the (K-finite) Casselman algebra by an induction on the length of K-types, culminating in the structure theorem stated in Theorem 1.3.1. This is also how Delorme proves his Paley-Wiener theorem in the compactly supported case.

Chapter 4

A structure theorem for the Casselman algebra

4.1 The Casselman algebra and reduced group C^*-algebra

Definition 4.1.1. The Casselman algebra $\mathcal{S}(G)$ is the space

$$\mathcal{S}(G) = \{\phi \in C^\infty(G) \mid \ \|g\|^N (L_u R_v \phi)(g) \in L^1(G)\ \forall u, v \in \mathcal{U}(\mathfrak{g}), N \in \mathbb{N}\}.$$

Here, L_u (resp. R_v) denotes the left-regular (resp. right-regular) action of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ on $C^\infty(G)$.

The Casselman algebra is a Fréchet algebra with seminorms defined as follows. Fix an ordered basis $X_1, \ldots, X_{\dim G}$ of G. We set

$$\|\phi\|_{\mathcal{S}(G),N,k} = \sum_{|I|,|J| \leq k} \int_G (1 + \|g\|)^N |L_{X^I} R_{X^J} \phi| \, dg, \tag{4.1.1}$$

where I and J are multi-indices. Note that, because $\|gh\| \leq \|g\|\|h\|$ for $g, h \in G$,

$$(1 + \|g\|)^N \leq (1 + \|h^{-1}g\|)^N (1 + \|h\|)^N.$$

From here, it is not difficult to show that the above seminorms are submultiplicative.

Definition 4.1.2. The reduced group C^*-algebra $C_r^*(G)$ is the completion of $L^1(G)$ with respect to the norm

$$\|f\|_{C_r^*(G)} = \sup_{\|h\|_{L^2(G)}=1} \|f * h\|.$$

That is, $C_r^*(G)$ is the closure of $L^1(G)$ embedded into $\mathcal{B}(L^2(G))$ under the left-regular representation.

Note that $\mathcal{S}(G)$ is a subset of $L^1(G)$ and therefore a subset of $C_r^*(G)$.

We recall that there are functions $p_\gamma \in C^\infty(K)$ which act on K-modules by projection onto the γ-isotypical component of the module, and for $F \subset \widehat{K}$ we set $p_F = \sum_{\gamma \in F} p_\gamma$.

Definition 4.1.3. Given a finite subset $F \subset \widehat{K}$, we write

$$p_F = \sum_{\gamma \in F} p_\gamma.$$

Treating $\mathcal{S}(G)$ and $C_r^*(G)$ as $K \times K$-modules, we define

$$\mathcal{S}(G, F) = p_F \mathcal{S}(G) p_F, \quad C_r^*(G, F) = p_F C_r^*(G) p_F.$$

We now provide a more precise version of Theorem 1.3.2. We recall that for each $\mu \in \widehat{K}$ there is a number $\|\mu\|$ which we call the "length" of the K-type μ. A ball of K-types then consists of K-types with length at most R for some $R \geq 0$. The main theorem is as follows.

Theorem 4.1.4. *Let $R \geq 0$ and set $F = \{\gamma \in \widehat{K} : \|\gamma\| \leq R\}$. Then the inclusion map*

$$\mathcal{S}(G, F) \to C_r^*(G, F)$$

induces an isomorphism in K-theory.

.

4.2 Outline of the structure theorem

The primary aim of this dissertation is to prove a structure theorem for $\mathcal{S}(G, F)$, and to show that the inclusion $\mathcal{S}(G, F) \hookrightarrow C_r^*(G, F)$ induces an isomorphism

$$K_*(\mathcal{S}(G, F)) \xrightarrow{\cong} K_*(C_r^*(G, F)). \tag{4.2.1}$$

Here, we are using the K-theory of Fréchet algebras defined by Phillips [Phi91]; see Section 2.12 for a summary.

The proof of the isomorphism (4.2.1) is largely based on techniques due to Delorme [Del05] that are used in his characterization of the Fourier image of $C_c^\infty(G)$. These techniques can be adapted to $\mathcal{S}(G)$ with little change.

Here are the main steps in the argument.

Let P be a cuspidal parabolic subgroup of G, let $P = MAN$ be its Langlands decomposition, and let σ be a square-integrable representation of M. Recall that associated to the pair (P, σ) is a Hilbert space $\mathcal{H}_\sigma$ and a series of G-representations $(\pi^P_{\sigma,\lambda}, \mathcal{H}_\sigma)$ for each $\lambda \in \mathfrak{a}^*$, called the (nonunitary) principal series. The corresponding $(\mathfrak{g}, K)$-module is denoted by $(\pi^P_{\sigma,\lambda}, I_\sigma)$.

Given $\phi \in \mathcal{S}(G, F)$ and $v \in p_F I_\sigma$, we define $\pi^P_{\sigma,\lambda}(\phi)v \in p_F I_\sigma$ by

$$\pi^P_{\sigma,\lambda}(\phi)v = \int_G \phi(g)\pi^P_{\sigma,\lambda}(g)v \, dg.$$

The above defines an "integrated" representation of $\mathcal{S}(G, F)$ on $p_F I_\sigma$. Importantly, the map $\lambda \mapsto \pi^P_{\sigma,\lambda}(\phi)$ is a holomorphic function from $\mathfrak{a}^*$ to the finite-dimensional space $\mathrm{End}(p_F I_\sigma)$. Moreover, if we set (for a finite-dimensional normed vector space V)

$$\begin{aligned} \mathsf{PW}(\mathfrak{a}^*, V) = \{f : \mathfrak{a}^* \to V : \quad & f \text{ is holomorphic and} \\ & \sup_{|\operatorname{Re}\lambda| \le k} (1 + |\lambda|)^N \|f(\lambda)\| < \infty \text{ for all } N, k \in \mathbb{N}\}, \end{aligned}$$

then we obtain from $\pi^P_{\sigma,\lambda}$ a continuous map

$$\pi_\sigma : \mathcal{S}(G, F) \to \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma)).$$

Let $A(\sigma)$ denote the set of minimal K-types of I_σ. We recall that, by a deep theorem of Vogan [Vog79], the set $A(\sigma)$ determines the pair (P, σ) up to G-conjugacy, and that the sets $A(\sigma)$ partition $\widehat{K}$ (this is stated more precisely in Section 4.4). Accordingly, we can totally order the G-conjugacy classes $[P, \sigma]$ using the sets $A(\sigma)$ and the common lengths of their elements. Choose representatives (P_n, σ_n) so that

$$[P_1, \sigma_1] < [P_2, \sigma_2] < \cdots$$

We then define ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_N = \mathcal{S}(G, F)$$

by the property that $\pi_{\sigma_m}(J_n) = 0$ for $m > n$. Thus, J_1 consists of functions $\phi \in \mathcal{S}(G, F)$

which vanish on every principal series other than the spherical principal series I_{σ_1} (whose minimal K-type is the trivial K-type), while J_2 consists of functions that vanish on all principal series other than I_{σ_1} and I_{σ_2}, and so on. By definition, π_{σ_n} is injective on the subquotient $\mathcal{J}_n/\mathcal{J}_{n-1}$.

We define "Morita equivalence" for Fréchet algebras $\mathcal{A}$ in the narrow sense that if p is a projection in (the "multiplier algebra" of) $\mathcal{A}$ such that $\overline{\mathcal{A}p\mathcal{A}} = \mathcal{A}$, then $\mathcal{A} \sim p\mathcal{A}p$. We will prove in Section 5.1 that if $\mathcal{A} \sim p\mathcal{A}p$, then the inclusion $p\mathcal{A}p \hookrightarrow \mathcal{A}$ induces an isomorphism in K-theory. Making use of Delorme's techniques (adapted to $\mathcal{S}(G)$), we obtain a "Morita equivalence"

$$\mathcal{J}_n/\mathcal{J}_{n-1} \sim \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_{A(\sigma_n)}I_{\sigma_n}))^{W_{\sigma_n}},$$

where the finite group W_{σ_n} acts on $\mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_{A(\sigma_n)}I_{\sigma_n}))$ in a fairly simple way (in particular, the action is mostly induced by an action on $\mathfrak{a}^*$).

We define $J_n \subset C_r^*(G, F)$ similarly, and we have (using results of [CCH16])

$$J_n/J_{n-1} \sim C_0(i\mathfrak{a}_0^*, \mathrm{End}(p_{A(\sigma_n)}I_{\sigma_n}))^{W_{\sigma_n}}.$$

Moreover, the inclusion

$$\mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_{A(\sigma)}I_\sigma)))^{W_\sigma} \hookrightarrow C_0(i\mathfrak{a}_0^*, \mathrm{End}(p_{A(\sigma)}I_\sigma))^{W_\sigma}$$

induces an isomorphism in K-theory by a simple homotopy argument.

It follows that the inclusion

$$\mathcal{J}_n/\mathcal{J}_{n-1} \hookrightarrow J_n/J_{n-1}$$

induces an isomorphism in K-theory. The isomorphism (4.2.1) is established by a series of 6-term exact sequence and five-lemma arguments.

To summarize, we apply techniques of Delorme [Del05] and the results of Clare-Crisp-Higson [CCH16] to decompose $\mathcal{S}(G, F)$ and $C_r^*(G, F)$ into elementary components, which are Morita equivalent to fairly simple function spaces. These have isomorphic K-theory by a simple homotopy argument, which can be regarded as a simple application of the Oka principle.

4.3 Fourier transform and the Paley-Wiener space

In order to prove Theorem 1.3.1, we use a notion of Fourier transform on real reductive groups which apply to elements of $\mathcal{S}(G, F)$.

Definition 4.3.1. A cuspidal pair is a pair (P, σ) consisting of a cuspidal parabolic subgroup $P = MAN$ and a discrete series representation $\sigma \in \widehat{M}_d$ of M.

Definition 4.3.2. Fix a finite set $F \subset \widehat{K}$. Given a cuspidal pair (P, σ), define

$$\pi_\sigma = \pi_\sigma^P : \mathcal{S}(G, F) \to C(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma))$$

by

$$\pi_\sigma(\phi)(\lambda)v = \pi_{\sigma,\lambda}^P(\phi)v = \int_G \phi(g)\pi_{\sigma,\lambda}^P(g)v\, dg,$$

for each $\lambda \in \mathfrak{a}^*$ and $v \in p_F I_\sigma$. The above integral converges from the proof of Lemma 4.3.4 below.

Definition 4.3.3. Given a Euclidean vector space V_0 with complexification V, the *Paley-Wiener space* of V is defined to be

$$\begin{aligned}\mathsf{PW}(V) = \{f : V \to \mathbb{C} : \quad & f \text{ is holomorphic and} \\ & \sup_{\|\operatorname{Re}\lambda\| \le k} (1 + |\lambda|)^N |f(\lambda)| < \infty \text{ for all } N, k \in \mathbb{N}\}.\end{aligned}$$

The space $\mathsf{PW}(V)$ is a Fréchet algebra with respect to the norms

$$\|f\|_{\mathsf{PW}(V),N,k} = \sup_{|\operatorname{Re}\lambda| \le k} (1 + |\lambda|)^N |f(\lambda)|.$$

Note that we will often regard $\mathfrak{a}^*$ as the complexification of $\mathfrak{a}_0^*$, which is Euclidean by use of the inner product (2.2.1).

Lemma 4.3.4. *Fix a cuspidal pair (P, σ). For each $\phi \in \mathcal{S}(G, F)$ and vectors $v, w \in p_F I_\sigma$, the map $\lambda \mapsto \langle \pi_{\sigma,\lambda}^P(\phi)v, w\rangle$ defines an element of $\mathsf{PW}(\mathfrak{a}^*)$. That is,*

$$\pi_\sigma(\mathcal{S}(G, F)) \subset \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma)).$$

Moreover, the map $\pi_\sigma : \mathcal{S}(G, F) \to \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma))$ is a continuous homomorphism between Fréchet algebras.

Proof. The proof for $C_c^\infty(G)$ in place of $\mathcal{S}(G)$ is given in [Del05, Lemma 1]. We shall adapt the argument given there. We use the following estimate from [Del05, (1.25)] (which we have relaxed slightly):

$$\|\pi^P_{\sigma,\lambda}(g)\| \leq \|g\|^{2|\operatorname{Re}\lambda|}.$$

Then we see that, for $\varphi, \psi \in I^P_{\sigma,\lambda}$,

$$|\langle \pi^P_{\sigma,\lambda}(\phi)v, w\rangle| \leq \int_G |\phi(g)| \|g\|^{2|\operatorname{Re}\lambda|} \|v\| \|w\| \, dg.$$

Therefore,

$$\sup_{|\operatorname{Re}\lambda|\leq k} |\langle \pi^P_{\sigma,\lambda}(\phi)v, w\rangle| \leq \int_G |\phi(g)| \|g\|^{2k} \|v\| \|w\| \, dg,$$

which is finite by the definition of $\mathcal{S}(G)$ (in particular, $\pi^P_{\sigma,\lambda}(\phi)$ is well-defined). From the definition (2.5.1) of $\pi^P_{\sigma,\lambda}(g)$, we see that $\langle \pi_{\sigma,\lambda}(\phi)v, w\rangle$ is a holomorphic function in λ.

Now fix $N \in \mathbb{N}$. Set $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ (recall Definition 2.7.2), which is a Cartan subalgebra of $\mathfrak{g}$. Let $W(\mathfrak{g}, \mathfrak{h})$ denote the corresponding Weyl group. From [Del05, (1.27)] there exist $Q_1, \dots, Q_r \in \mathbb{C}[\mathfrak{h}^*]^{W(\mathfrak{g},\mathfrak{h})}$ such that

$$(1 + |\nu|^2)^N \leq |Q_1(\nu)| + \cdots + |Q_r(\nu)| \tag{4.3.1}$$

for $\nu \in \mathfrak{h}^*$. Let $\mathcal{Z}(\mathfrak{g})$ denote the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. Choosing $z_1, \dots, z_r \in \mathcal{Z}(\mathfrak{g})$ corresponding to Q_i via the Harish-Chandra isomorphism (see [Wal88, Theorem 3.2.3]), then

$$\pi^P_{\sigma,\lambda}(L_{z_i}\phi) = \pi^P_{\sigma,\lambda}(z_i)\pi^P_{\sigma,\lambda}(\phi) = Q_i(\Lambda_\sigma + \lambda)\pi^P_{\sigma,\lambda}(\phi),$$

where $\Lambda \in i\mathfrak{t}_0^*$ is as in Definition 2.7.3.

Applying (4.3.1) to $\nu = \Lambda_\sigma + \lambda$, we have

$$\sup_{|\operatorname{Re}\lambda|\leq k} (1 + |\Lambda_\sigma|^2 + |\lambda|^2)^N |\langle \pi^P_{\sigma,\lambda}(\phi)v, w\rangle| \\ \leq \sum_{i=1}^r \sup_{|\operatorname{Re}\lambda|\leq k} |\langle \pi^P_{\sigma,\lambda}(L_{z_i}\phi)v, w\rangle| < \infty$$

As σ is fixed, the above is equivalent to the condition defining $\mathsf{PW}(\mathfrak{a}^*)$. From the definition of the topologies defined for $\mathcal{S}(G)$ and $\mathsf{PW}(\mathfrak{a}^*, \operatorname{End}(p_F I_\sigma))$, the above estimate proves that π_σ is continuous. The fact that π_σ is an algebra homomorphism follows from the identity

$$\pi^P_{\sigma,\lambda}(\phi_1 * \phi_2) = \pi^P_{\sigma,\lambda}(\phi_1)\pi^P_{\sigma,\lambda}(\phi_2)$$

for $\phi_1, \phi_2 \in \mathcal{S}(G)$, which holds for any G-representation. □

By the Plancherel formula, the map $\bigoplus_{(P,\sigma)} \pi_\sigma$ is injective on $\mathcal{S}(G, F)$ (in fact, only the standard minimal parabolic subgroup is needed in the direct sum). This map is known as the Fourier transform, and an interesting problem is to characterize the Fourier image as functions on the various $\mathfrak{a}^*$ with particular properties. Such a characterization is known as a Paley-Wiener theorem. For $C_c^\infty(G)$, the Fourier image was first characterized by Arthur [Art83], and later characterized in a different way by Delorme [Del05] (both characterizations turn out to be the same *a priori*; see [vdBS14]).

We will make use of techniques that Delorme developed in [Del05], and adapt these to $\mathcal{S}(G)$. These techniques make use of several deep results in representation theory, including the theory of Knapp-Stein intertwining operators, Vogan's minimal K-types, and Vogan-Zuckerman classification. The results used are summarized in the first sections of [Del05] and [Del84].

4.3.1 Polynomial division of rapidly decreasing functions

We will need some lemmas regarding polynomial division of rapidly decreasing functions. This will be important when we want to study divisibility of certain functions in $\mathsf{PW}(\mathfrak{a}^*)$. The following lemmas are analogues of [CD90, Lemma B.1, Theorem B.1], and [CD84, Lemmas 7,8].

We begin with a lemma due to Ehrenpreis [Ehr70, Theorem 1.4].

Lemma 4.3.5 ([CD90, Lemma B.1 (i)])**.** *Given a complex polynomial p on $\mathbb{C}^n$, there exist constants $c, m \geq 0$ such that if h is holomorphic on the closed polydisc $\Delta_\rho(z_0) = \{z \in \mathbb{C}^n : \|z - z_0\| \leq \rho\}$ (where $\|z\| = \sup_j |z_j|$), then*

$$|h(z_0)| \leq c\rho^{-m} \sup_{z\in\Delta_\rho(z_0)} |p(z)h(z)|.$$

We say that a nonzero polynomial p *divides* a holomorphic function f if f/p extends to a holomorphic function.

Lemma 4.3.6. *Let V_0 be a Euclidean vector space with complexification V. Let $f \in \mathsf{PW}(V)$ and $p \in \mathbb{C}[V]$. If p divides f, then $f/p \in \mathsf{PW}(V)$.*

Proof. Compare to [CD90, Lemma B.1]. We identify V with $\mathbb{C}^n$ via the inner product. Set $h = f/p$, which extends to an entire function on $\mathbb{C}^n$. Setting $\rho = 1$ in Lemma 4.3.5, for each N and $k \geq 0$,

$$\sup_{\|\operatorname{Re} z\| \leq k} (1 + \|z\|)^N |h(z)| \leq c \sup_{\|\operatorname{Re} z\| \leq k} (1 + \|z\|)^N \sup_{z' \in \Delta_1(z)} |f(z')|.$$

When $z' \in \Delta_1(z)$, then $1 + \|z\| \leq 2(1 + \|z'\|)$ and $\|\operatorname{Re} z'\| \leq \|\operatorname{Re} z\| + 1$ by the triangle inequality. Therefore,

$$\sup_{\|\operatorname{Re} z\| \leq k} (1 + \|z\|)^N |h(z)| \leq 2^N c \sup_{\|\operatorname{Re} z\| \leq k+1} (1 + \|z\|)^N |f(z)| < \infty. \tag{4.3.2}$$

This proves that $h \in \mathsf{PW}(V)$. □

Theorem 4.3.7. *Fix a Euclidean vector space V_0 with complexification V, and a finite group W generated by reflections on V_0. There exist homogeneous complex polynomials $p_i \in \mathbb{C}[V]$ such that*

$$\mathsf{PW}(V) = \sum_i p_i \, \mathsf{PW}(V)^W$$

Moreover, the above sum is free in the sense that the decomposition $f = \sum p_i f_i$ (where $f \in \mathsf{PW}(V), f_i \in \mathsf{PW}(V)^W$) is unique.

This analogue of this theorem for compactly supported smooth functions is due to Rais [Rai83]. See also [CD90, Theorem B.1]; the proof of Theorem 4.3.7 is almost identical, and so we shall not provide complete details.

Proof sketch. According to [HC58, Lemma 8], $\mathbb{C}[V]$ is free over $\mathbb{C}[V]^W$ with homogeneous basis $p_1, \ldots, p_{|W|} \in \mathbb{C}[V]$.

Write $W = \{w_1, \ldots, w_{|W|}\}$. Given $f \in \mathsf{PW}(V)$, for each $\lambda \in \mathbb{C}$ we obtain a $|W| \times |W|$ linear system

$$f(w_j \lambda) = \sum_{i=1}^{|W|} p_i(w_j \lambda) f_i(\lambda), \quad j = 1, \ldots, |W|.$$

If $D(\lambda) = \det([p_i(w_j \lambda)])$ denotes the determinant of this system, then $D(\lambda)$ is a polynomial which is nonzero because of the polynomial case above. By Cramer's rule, we can find (unique) functions $g_i(\lambda) \in \mathsf{PW}(V)$ such that

$$D(\lambda)f(w_j\lambda) = \sum_i p_i(w_j\lambda)g_i(\lambda).$$

Now, approximating f by truncated Taylor series polynomials q_n, there are unique W-invariant polynomials $f_{n,i}$ such that $q_n = \sum p_i f_{n,i}$. Moreover, $g_{n,i}(\lambda) = D(\lambda)f_{n,i}(\lambda)$ converge to g_i uniformly on compact sets as $n \to \infty$. The estimate (4.3.2) (applied to $f = g_{n,i}$ and $F = f_{n,i}$) is then used to prove that D divides each g_i, and that $f_{n,i}$ converge to $f_i = g_i/D$ uniformly on compact sets. By Lemma 4.3.6, we have $f_i \in \mathsf{PW}(V)^W$. Uniqueness of the f_i follows from uniqueness of the $f_{n,i}$. □

Let $P = MAN$ be a cuspidal parabolic subgroup. Let $T \subset M$ be a θ-stable Cartan subgroup contained in K, with Lie algebra $\mathfrak{t}_0$. We set $\mathfrak{h} = \mathfrak{a} + \mathfrak{t}$. We need a particular action of the restricted Weyl group $W(\mathfrak{g}_0, \mathfrak{a}_0)$ on $i\mathfrak{t}_0^*$ due to Knapp. Its important property is its relationship to the action of $W(\mathfrak{g}_0, \mathfrak{a}_0)$ on $\widehat{M}_d$.

Theorem 4.3.8 ([Kna82, Theorem 3.7, Theorem 4.10])**.** *Let M_0 denote the connected component of M at the identity. There exists an action of $W(\mathfrak{g}_0, \mathfrak{a}_0)$ on $i\mathfrak{t}_0^*$ such that, given $\sigma \in \widehat{M}_d$ and Harish-Chandra parameter Λ_σ of $\sigma|_{M_0}$ (recall Definition 2.7.3), if $(w \cdot \sigma)|_{M_0}$ is unitarily equivalent to $\sigma|_{M_0}$ then $w \cdot \Lambda_\sigma = \Lambda_\sigma$. For each $w \in W(\mathfrak{g}_0, \mathfrak{a}_0)$, there exists a representative $k \in N_K(\mathfrak{a}_0)$ of w such that w acts by Ad_k on $i\mathfrak{t}_0$.*

See also the discussion before [CD84, Lemma 7]. Note that, in particular, W_σ acts trivially on Λ_σ. Theorem 4.3.8 allows us to define an action of $W(\mathfrak{g}_0, \mathfrak{a}_0)$ (in particular, W_σ^0) on $\mathfrak{h}^*$. Moreover, because this action comes from Ad_k for some $k \in K$, for each $w \in W(\mathfrak{g}_0, \mathfrak{a}_0)$ there exists some $w' \in W(\mathfrak{g}, \mathfrak{h})$ such that w acts on $\mathfrak{h}^*$ the same way as w'.

Lemma 4.3.9. *Fix $\sigma \in \widehat{M}_d$. Let $\Lambda_\sigma \in i\mathfrak{t}_0^*$ be the Harish-Chandra parameter of $\sigma|_{M_0}$ (recall Definition 2.7.3) Then*

$$\mathsf{PW}(\mathfrak{a}^*)^{W_\sigma^0} = \{\lambda \mapsto f(\Lambda_\sigma + \lambda) : f \in \mathbb{C}[\mathfrak{a}^*]^{W_\sigma^0}\,\mathsf{PW}(\mathfrak{h}^*)^{W(\mathfrak{g},\mathfrak{h})}\}.$$

Proof. Compare to [CD84, Lemmas 7,8]. We first note that

$$\{\lambda \mapsto f(\Lambda_\sigma + \lambda) : f \in \mathsf{PW}(\mathfrak{h}^*)^{W_\sigma^0}\} = \mathsf{PW}(\mathfrak{a}^*)^{W_\sigma^0}. \tag{4.3.3}$$

Indeed, choose any W_σ^0-invariant $F \in \mathsf{PW}(\mathfrak{t}^*)$ such that $F(\Lambda_\sigma) = 1$. Then any $f \in$

$\mathsf{PW}(\mathfrak{a}^*)^{W_\sigma^0}$ is the restriction of $(\lambda_1 \oplus \lambda) \mapsto F(\lambda_1)f(\lambda)$ to $\Lambda_\sigma + \mathfrak{a}^*$, where $\lambda_1 \in \mathfrak{t}^*, \lambda \in \mathfrak{a}^*$.

Next, by Theorem 4.3.7 we see that

$$\mathsf{PW}(\mathfrak{h}^*) = \mathbb{C}[\mathfrak{h}^*]\,\mathsf{PW}(\mathfrak{h}^*)^{W(\mathfrak{g},\mathfrak{h})}.$$

Now note that elements of $\mathsf{PW}(\mathfrak{h}^*)^{W(\mathfrak{g},\mathfrak{h})}$ are also W_σ-invariant by the discussion before this lemma. Therefore, averaging the decomposition $f = \sum p_i f_i$ by the action of W_σ^0, we have

$$\mathsf{PW}(\mathfrak{h}^*)^{W_\sigma^0} = \mathbb{C}[\mathfrak{a}^*]^{W_\sigma^0}\,\mathsf{PW}(\mathfrak{h}^*)^{W(\mathfrak{g},\mathfrak{h})}. \tag{4.3.4}$$

The lemma follows from (4.3.3) and (4.3.4). □

4.4 A structure theorem relating $\mathcal{S}(G, F)$ and $C_r^*(G, F)$

We recall that two cuspidal pairs $(P = L_P N_P, \sigma), (Q = L_Q N_Q, \tau)$ are G-conjugate if there is an element of G which conjugates L_P to L_Q and conjugates σ to τ. Note that a G-conjugacy class of cuspidal pairs is also known as an associate class (see [CCH16, Definition 5.2]).

Because a given G-conjugacy class of cuspidal pairs $[P, \sigma]$ only depends on the Levi subgroup L, we may also write $[L, \sigma]$. Moreover, because the Levi subgroup is implicitly specified by the representation σ, we may simply write $[\sigma]$ for such a class.

These classes can be used to index the series appearing in Harish-Chandra's Plancherel formula. In [CCH16], these classes index the decomposition for $C_r^*(G)$, and we will use these in an analogous manner for $\mathcal{S}(G)$. Recall the following theorem of Vogan which says (in particular) that we can determine the class $[\sigma]$ by the set $A(\sigma)$.

Theorem 4.4.1 (See [Vog79, Theorem 7.17])**.** *The sets $A(\sigma)$ partition $\widehat{K}$, and two such sets $A(\sigma), A(\tau)$ are equal if and only if $[\sigma] = [\tau]$.*

As an immediate consequence, we see that the value $\|\sigma\|$ (that is, the value $\|\mu\|$ for any $\mu \in A(\sigma)$) is independent of the cuspidal pair (P, σ) up to G-conjugacy. We fix a total order on G-conjugacy classes $[\sigma]$ such that if $\|\sigma\| < \|\tau\|$ then $[\sigma] < [\tau]$. We fix representatives (P_n, σ_n) for each G-conjugacy class, so that

$$[\sigma_1] < [\sigma_2] < \cdots .$$

Let π_n denote π_{σ_n}, and p_n denote $p_{A(\sigma_n)}$. Often, n will be fixed and we will write $\sigma = \sigma_n$.

Definition 4.4.2. For each $n \in \mathbb{N}$, define the ideal $\mathcal{J}_n \subset \mathcal{S}(G, F)$ by

$$\mathcal{J}_n = \bigcap_{m>n} \ker\left(\pi_m : \mathcal{S}(G,F) \to \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_F I_{\sigma_m}))\right).$$

We define ideals $J_n \subset C_r^*(G, F)$ analogously. We also set $\mathcal{J}_0 = J_0 = 0$.

By Theorem 4.4.1, for each finite set $F \subset \widehat{K}$, we have $p_F \mathcal{J}_n p_F = \mathcal{S}(G,F)$ and $p_F J_n p_F = C_r^*(G,F)$ for large enough n.

Fix $R \geq 0$ and set $F = \{\gamma \in \widehat{K} : \|\gamma\| \leq R\}$. For each $n \in \mathbb{N}$, we have the injection

$$\overline{\pi}_n : \mathcal{J}_n/\mathcal{J}_{n-1} \hookrightarrow \pi_n(\mathcal{S}(G,F)).$$

Therefore, as an algebra we may identify $\mathcal{J}_n/\mathcal{J}_{n-1}$ with the image of $\mathcal{J}_n$ under π_n. Similarly, we may identify J_n/J_{n-1} with $\pi_n(J_n)$.

The main theorem will be a consequence of the following four theorems describing these subquotients. We let $w \in W_\sigma$ act on $\mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_{A(\sigma)} I_\sigma))$ by

$$(w \cdot \phi)(\lambda) := \mathcal{A}(P, w, w^{-1}\lambda)\phi(w^{-1}\lambda)\mathcal{A}(P, w^{-1}, \lambda).$$

Note that this defines an action by Theorem 2.10.6. The above similarly defines an action of W_σ on $C_0(i\mathfrak{a}_0^*, \mathrm{End}(p_{A(\sigma)} I_\sigma))$.

Theorem 4.4.3. *For each $n \in \mathbb{N}$ with $A(\sigma_n) \subset F$,*

$$(\mathcal{J}_n/\mathcal{J}_{n-1}) p_n (\mathcal{J}_n/\mathcal{J}_{n-1}) = \mathcal{J}_n/\mathcal{J}_{n-1}.$$

Theorem 4.4.4. *For each $n \in \mathbb{N}$ with $A(\sigma_n) \subset F$,*

$$\overline{(J_n/J_{n-1}) p_n (J_n/J_{n-1})} = J_n/J_{n-1}.$$

Theorem 4.4.5. *For each $n \in \mathbb{N}$ with $A(\sigma_n) \subset F$,*

$$\pi_n(p_n \mathcal{J}_n p_n) = \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_n I_{\sigma_n}))^{W_{\sigma_n}}.$$

Theorem 4.4.6. *For each $n \in \mathbb{N}$ with $A(\sigma_n) \subset F$,*

$$\pi_n(p_n J_n p_n) = C_0(i\mathfrak{a}_0^*, \mathrm{End}(p_n I_{\sigma_n}))^{W_{\sigma_n}}.$$

The four theorems above comprise the statement made in Theorem 1.3.1. To summarize, the Casselman algebra $\mathcal{S}(G,F)$ is assembled via extensions of the algebras $\mathcal{J}_n/\mathcal{J}_{n-1}$, which are Morita equivalent to the elementary components $p_n\mathcal{J}_n p_n$ (and analogously for $C_r^*(G,F)$). We will see that, thanks to Theorems 4.4.5 and 4.4.6, the elementary components between $\mathcal{S}(G,F)$ and $C_r^*(G,F)$ are homotopic, and therefore have equal K-theory.

Theorem 4.4.3 is the most technical, and is a key consequence of Delorme's techniques. We will leave this theorem to Section 5.3, and prove the other three theorems here.

We need to rewrite Theorem 4.4.5 in a form compatible with [Del05, (1.38)]. This will also allow us to understand the action of W_σ on $\mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_n I_\sigma))$. Recall from Theorem 3.2.3 that W_σ decomposes as a semidirect product

$$W_\sigma = R_\sigma W_\sigma^0,$$

where W_σ^0 is characterized by the property that $\mathcal{A}(P_n, w, \lambda)$ is the identity, and (among other properties) R_σ is a product of copies of $\mathbb{Z}/2$. Moreover, the characters $\widehat{R}_\sigma$ act on $A(\sigma)$ simply transitively. We also recall from Definition 3.2.1 and Theorem 3.2.2 that the intertwining operators $\mathcal{A}(P, w, \lambda)$ acts by scalars $a^\mu(w)$ on $p_\mu I_\sigma$ for each $\mu \in A(\sigma)$, and $\hat{r}_{\mu\nu}(w) := a^\mu(w)(a^\nu(w))^{-1}$ is a character of R_σ, and moreover is the unique element of $\widehat{R}_\sigma$ such that $\hat{r}_{\mu\nu} \cdot \nu = \mu$.

By the above, each $w = w^0 r \in W_\sigma$ (where $w^0 \in W_\sigma^0$, $r \in R_\sigma$) acts on $f \in p_\mu \, \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_{A(\sigma)} I_\sigma)) p_\nu$ by

$$(w \cdot f)(\lambda) = \hat{r}_{\mu\nu}(r) f((w^0)^{-1}\lambda).$$

Therefore, if $\mathsf{PW}(\mathfrak{a})^{W_\sigma^0}(\hat{r}_{\mu\nu})$ denotes the space of W_σ^0-invariant functions f such that $f(r\lambda) = \hat{r}_{\mu\nu}(r) f(\lambda)$ for each $r \in R_\sigma$, then

$$p_\mu \, \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_{A(\sigma)} I_\sigma))^{W_\sigma} p_\nu = \mathsf{PW}(\mathfrak{a}^*)^{W_\sigma^0}(\hat{r}_{\mu\nu}) \otimes \mathrm{Hom}(p_\nu I_\sigma, p_\mu I_\sigma). \tag{4.4.1}$$

In particular, Theorem 4.4.5 is equivalent to the identification

$$p_\mu \pi_n(\mathcal{J}_n) p_\nu = \mathsf{PW}(\mathfrak{a}^*)^{W_\sigma^0}(\hat{r}_{\mu\nu}) \otimes \mathrm{Hom}(p_\nu I_\sigma, p_\mu I_\sigma). \tag{4.4.2}$$

Our discussion also proves that the action of W_σ on $\mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_n I_\sigma))$ is induced by its action on $\mathfrak{a}^*$ and a diagonal action on $\mathrm{End}(p_n I_\sigma)$ (with respect to the entries).

Proof of Theorem 4.4.5. The proof of (4.4.2) is identical to that of [Del05, (1.38)] when adapted to $\mathcal{S}(G)$. To adapt the proof to $\mathcal{S}(G)$, we use [DFJ91, Theorem 3] instead of [DFJ91, Theorem 2], and we use Lemmas 4.3.6 and 4.3.9 in place of [Del05, (1.21), (1.37)]. As (4.4.2) is equivalent to the theorem, we are done. □

To prove Theorems 4.4.4 and 4.4.6, we need the following. Recall that $(\pi^P_{\sigma,\lambda}, \mathcal{H}_\sigma)$ denotes the (principal series) Hilbert space representation of G whose $(\mathfrak{g}, K)$-module is $(\pi^P_{\sigma,\lambda}, I_\sigma)$.

Theorem 4.4.7 (See [CCH16, Propositions 5.17, 6.7 and Theorem 6.8])**.** *Define* $\pi_n : C^*_r(G) \to C_0(i\mathfrak{a}^*_0, \mathcal{B}(\mathcal{H}_\sigma))$ *by* $\pi_n(\phi)(\lambda) = \pi^{P_n}_{\sigma_n,\lambda}(\phi)$. *The* C^**-algebra homomorphism*

$$\bigoplus_{n\in\mathbb{N}} \pi_n : C^*_r(G) \to \bigoplus_{n\in\mathbb{N}} \pi_n(C^*_r(G))$$

is an isomorphism. Consequently, the map $J_n/J_{n-1} \to \pi_n(C^*_r(G,F))$ *is an isomorphism of* C^**-algebras. Moreover, for each cuspidal pair* (P,σ)*,*

$$\pi_\sigma(C^*_r(G)) = C_0(i\mathfrak{a}^*_0, \mathcal{K}(\mathcal{H}_\sigma))^{W_\sigma}, \tag{4.4.3}$$

where $\mathcal{K}(\mathcal{H}_\sigma)$ *denotes compact operators on* $\mathcal{H}_\sigma$*, and each* $w \in W_\sigma$ *acts on* $f \in C_0(i\mathfrak{a}^*_0, \mathcal{K}(\mathcal{H}_\sigma))$ *via* $(w \cdot f)(\lambda) = \mathcal{A}(P, w, w^{-1}\lambda) f(w^{-1}\lambda)\mathcal{A}(P, w^{-1}, \lambda)$.

Theorem 4.4.6 follows by multiplying p_n to the left and right in (4.4.3).

Proof of Theorem 4.4.4. It suffices to prove that

$$\mathcal{I} := \pi_n(C^*_r(G) p_n C^*_r(G))$$

is dense in $\pi_n(C^*_r(G))$, because then $p_F \mathcal{I} p_F = \pi_n((J_n/J_{n-1})p_n(J_n/J_{n-1}))$ is dense in $p_F \pi_n(C^*_r(G)) p_F = \pi_n(J_n/J_{n-1})$.

Suppose $\mathcal{I}$ is not dense in $\pi_n(C^*_r(G))$. Then there is an irreducible representation of $\pi_n(C^*_r(G))$ which vanishes on $\mathcal{I}$ (this is a consequence of [Dix77, Proposition 2.11.2 (i)]).

By Theorem 4.4.7, every irreducible representation of $\pi_n(C^*_r(G))$ is an irreducible subquotient of the representation $(\pi^{P_n}_{\sigma_n,\lambda}, \mathcal{H}_{\sigma_n})$ for some $\lambda \in i\mathfrak{a}^*_0$. Vogan's classification (2.11.2) implies that a minimal K-type must be contained in this subquotient, and therefore this representation does not vanish on $\mathcal{I}$, which is a contradiction. □

Chapter 5
Proofs of the main theorems

5.1 Fréchet algebra K-theory and Morita equivalence

5.1.1 Mapping cones

We recall the notion of mapping cones for Fréchet algebras and the corresponding 6-term exact sequence in K-theory. Here we use the K-theory and results of Phillips [Phi91], and we will write K_i instead of RK_i.

Definition 5.1.1. The mapping cone of a continuous Fréchet algebra homomorphism $f : A \to B$ is the Fréchet algebra

$$\mathsf{MC}(f) = \{(\gamma, a) \in C([0,1], B) \oplus A : \gamma(0) = f(a),\ \gamma(1) = 0\}.$$

Lemma 5.1.2. *The map $f : A \to B$ induces an isomorphism in K-theory if and only if $\mathsf{MC}(f)$ has zero K-theory.*

Proof. We have a short exact sequence

$$0 \to S(B) \to \mathsf{MC}(f) \to A \to 0,$$

where $S(B)$ is the suspension of B,

$$S(B) = \{\phi : [0,1] \to B : \phi(0) = \phi(1) = 0\}.$$

By Theorems 6.1 and 5.5 of [Phi91], we obtain to a 6-term exact sequence

$$\begin{array}{ccccc} K_0(\mathsf{MC}(f)) & \longrightarrow & K_0(A) & \longrightarrow & K_0(B) \\ \uparrow & & & & \downarrow \\ K_1(B) & \longleftarrow & K_1(A) & \longleftarrow & K_1(\mathsf{MC}(f)). \end{array}$$

The lemma follows immediately from this exact sequence. □

5.1.2 Morita equivalence

We recall a theorem regarding Morita equivalence for Banach algebras, due to Lafforgue and recorded by Paravicini [Par09].

Definition 5.1.3. A Banach algebra (or Fréchet algebra) A is said to be *non-degenerate* if the multiplication map $A \times A \to A$ has dense range.

We define the *multiplier algebra* $M(A)$ of a Banach algebra A to be the algebra of double centralizers of A. That is, $M(A)$ consists of pairs (L, R) of homomorphisms $A \to A$ which satisfy $aL(b) = R(a)b$ for $a, b \in A$. Elements of $M(A)$ act on the left of A via L and on the right via R. That is, if $T = (L, R) \in M(A)$, then $Ta = L(a)$ and $aT = R(a)$.

Definition 5.1.4. Let A be a Banach algebra, and let p be an idempotent in the multiplier algebra $M(A)$. Then p is said to be a *full idempoten*t if ApA is dense in A.

Theorem 5.1.5 (cf. [Par09, Proposition 4.5 and Theorem 4.25])**.** *Let A be a non-degenerate Banach algebra and let $p \in M(A)$ be a full idempotent. The inclusion map $pAp \hookrightarrow A$ induces an isomorphism in K-theory.*

Remark 5.1.6. In the generality of [Par09, Proposition 4.5 and Theorem 4.25], it is not explicitly stated that the isomorphism between $K_*(pAp)$ and $K_*(A)$ is induced by inclusion. However, we can apply Paravicini's theorem to the mapping cone MC of the inclusion map to see that $K_*(\mathsf{MC}) \cong K_*(p\,\mathsf{MC}\,p)$. As $p\,\mathsf{MC}\,p$ is the cone of pAp, which is a contractible algebra, it follows that MC vanishes in K-theory, which implies Theorem 5.1.5.

Now let A be a non-degenerate Fréchet algebra. We follow the convention in [Phi91]. That is, we assume that A is an inverse limit of Banach algebras A_n, such that the associated homomorphisms $\pi_{m,n} : A_m \to A_n$ and $\pi_n : A \to A_n$ have dense range. Note that A_n is non-degenerate for each n, because $A_n A_n$ contains $\pi_n(AA)$, which is dense in $\pi_n(A)$ and hence A_n. We will write $A = \varprojlim A_n$ to present a Fréchet algebra A as an inverse limit of Banach algebras A_n under this convention.

Definition 5.1.7. Let $A = \varprojlim A_n$ be a Fréchet algebra. A full idempotent p of A will refer to a pair of idempotent continuous linear maps $p_L, p_R : A \to A$ such that

1. $ap_L(b) = p_R(a)b$ for all $a, b \in A$,
2. For each n, there exists $p_n \in M(A_n)$ such that
   (i) $p_n \pi_n(a) = \pi_n(p_L(a))$ and $\pi_n(a)p_n = \pi_n(p_R(a))$.
   (ii) $\pi_{m,n}(p_m a) = p_n \pi_{m,n}(a)$ and $\pi_{m,n}(ap_m) = \pi_{m,n}(a)p_n$.
3. $Ap_L(A)$ is dense in A.

As usual, p acts on the left of A by p_L, and on the right by p_R.

Theorem 5.1.8. *Let $A = \varprojlim A_n$ be a Fréchet algebra, where $\{A_n\}$ is an inverse system of Banach algebras such that the homomorphisms $\pi_n : A \to A_n$ have dense range. If p is a full idempotent of A, then the inclusion map $pAp \to A$ induces an isomorphism in K-theory.*

Proof. Note that $A_n p_n A_n$ is dense in A_n, because the former contains $\pi_n(ApA)$, which is dense in $\pi_n(A)$ and hence dense in A_n. By Theorem 5.1.5, this means that $K_*(p_n A_n p_n) \cong K_*(A_n)$.

Let MC be the mapping cone of $pAp \to A$, and let MC_n be the mapping cone of $p_n A_n p_n \to A_n$. Then $\pi_{m,n}$ induces a homomorphism $\pi_{m,n} : \mathsf{MC}_m \to \mathsf{MC}_n$ and similarly π_n induces a homomorphism $\pi_n : \mathsf{MC} \to \mathsf{MC}_n$. Moreover, $\mathsf{MC} = \varprojlim \mathsf{MC}_n$. According to [Phi91, Theorem 6.5], we have the short exact sequence

$$0 \to \varprojlim{}^1 K_{1-*}(\mathsf{MC}_n) \to K_*(\mathsf{MC}) \to \varprojlim K_*(\mathsf{MC}_n) \to 0.$$

As $K_*(\mathsf{MC}_n) = 0$ by Theorem 5.1.5, the above sequence implies $K_*(\mathsf{MC}) = 0$. □

5.2 Proof of Theorem 1.3.1, assuming Theorem 4.4.3

We now turn to the proof of Theorem 1.3.1, which states that the inclusion map $\mathcal{S}(G,F) \to C_r^*(G,F)$ induces an isomorphism in K-theory. It suffices to prove that the mapping cone of this inclusion vanishes in K-theory. The proof begins by studying the inclusion maps $\mathcal{J}_n/\mathcal{J}_{n-1} \to J_n/J_{n-1}$. In this section, we make use of Theorem 4.4.3, whose proof we have deferred to the next section.

We will apply the results of Section 5.1 regarding Fréchet algebra K-theory and Morita equivalence. In order to do so, we must first complete the space $\mathcal{J}_n/\mathcal{J}_{n-1}$ to suitable Banach algebras. We recall that the topology of $\mathcal{S}(G)$ is generated by the seminorms

$$\|\phi\|_{\mathcal{S}(G),N,k} = \sum_{|I|,|J|\leq k} \int_G (1+\|g\|)^N |L_{X^I} R_{X^J} \phi(g)| dg,$$

for some fixed choice of orthonormal basis $X_1, \ldots, X_{\dim G}$ of $\mathfrak{g}$. Now, the action of K is continuous with respect to the seminorms $\|\cdot\|_{\mathcal{S}(G),N,k}$, and the kernel of $\|\cdot\|_{\mathcal{S}(G),N,k}$ is a K-submodule of $\mathcal{S}(G)$. Therefore, p_F extends to a multiplier of the corresponding completions $\mathcal{S}_{N,k}(G)$. We may also complete $\mathcal{J}_n/\mathcal{J}_{n-1}$ with respect to the corresponding (sub)quotient seminorms, and p_F extends to these completions too. In particular, Theorem 4.4.3 implies that p_F is a full idempotent of $\mathcal{J}_n/\mathcal{J}_{n-1}$.

On the Fourier transform side, recall that we also have norms on $\mathsf{PW}(\mathfrak{a}^*, V)$ for a fixed finite-dimensional normed space V. Given $k > 0$ and each $N \in \mathbb{N} \cup \{0\}$, we set $X = \{\lambda \in \mathfrak{a}^* : |\operatorname{Re}\lambda| \leq k\}$ and

$$\|f\|_{N,k} = \|f\|_{N,X} = \sup_{\lambda \in X} (1+|\lambda|)^N \|f(\lambda)\|_V. \tag{5.2.1}$$

These norms generate the topology of $\mathsf{PW}(\mathfrak{a}^*, V)$.

Lemma 5.2.1. *Let W be a finite subgroup of the orthogonal group $O(\mathfrak{a}_0^*)$, and suppose the action of W on $\mathsf{PW}(\mathfrak{a}^*, V)$ has the form*

$$(w \cdot f)(\lambda) = D(w) f(w^{-1}\lambda),$$

where $D(w) \in \operatorname{Aut}(V)$ is independent of λ. Then the completion of $\mathsf{PW}(\mathfrak{a}^*, V)^W$ with respect to the norm $\|\cdot\|_{N,X}$ defined by (5.2.1) is isomorphic to

$$\mathsf{PW}_N(X,V)^W = \{f : X \to V : \ (1+|\lambda|)^N f \in C_0(X),\ D(w) f(w^{-1}\lambda) = f(\lambda),$$

f is continuous on X, holomorphic on the interior of $X\}$.

The above lemma is a consequence of the next two lemmas.

Lemma 5.2.2. *With notation as in Lemma 5.2.1, the space* $\bigcap_{M=0}^{\infty} \mathsf{PW}(X,V)^W$ *is dense in* $\mathsf{PW}_N(X,V)^W$ *with respect to* $\|\cdot\|_{N,X}$.

Proof. Fix an orthonormal basis $\{e_i\}$ on $\mathfrak{a}_0^*$ and set $\lambda^2 = \sum_i \langle \lambda, e_i \rangle^2 \in \mathbb{C}$. That is, if we identify $\mathfrak{a}^*$ with $\mathbb{C}^n$ and write $\lambda = (\lambda_1, \ldots, \lambda_n)$, then $\lambda^2 = \sum_i \lambda_i^2$. This quantity is independent of the basis chosen. Note that $\lambda^2 \leq 0$ when $\lambda \in i\mathfrak{a}_0^*$, and that $(w \cdot \lambda)^2 = \lambda^2$ for $w \in W$.

Suppose $f \in \mathsf{PW}_N(X,V)^W$. Now, $g_s(\lambda) = e^{s\lambda^2} f(\lambda)$ is in $\bigcap_M \mathsf{PW}_M(X,V)^W$ for each $s > 0$. For any $\varepsilon > 0$ and any compact set $Y \subset X$ we may choose $s > 0$ such that $|e^{s\lambda^2} - 1| < \varepsilon$ for $\lambda \in Y$, so that

$$(1 + |\lambda|)^N \|g_s(\lambda) - f(\lambda)\|_V < \|f\|_{N,X} \varepsilon$$

for $\lambda \in Y$. If we choose Y large enough so that $(1+|\lambda|)^N \|f(\lambda)\|_V < \varepsilon$ when $\lambda \notin Y$, then we see that g_s approximates f in $\|\cdot\|_{N,X}$. This proves the lemma. $\square$

Lemma 5.2.3. *With notation as in Lemma 5.2.1, the space* $\mathsf{PW}(\mathfrak{a}^*, V)^W$ *is dense in* $\bigcap_{M=0}^{\infty} \mathsf{PW}_M(X,V)^W$ *with respect to* $\|\cdot\|_{N,X}$ *for each* $N \geq 0$.

Proof. As in the previous lemma, we fix an orthonormal basis $\{e_i\}$ on $\mathfrak{a}_0^*$ and set $\lambda^2 = \sum_i \langle \lambda, e_i \rangle^2$.

Fix $f \in \bigcap_M \mathsf{PW}_M(X,V)^W$. We define, for each $t > 0$, $\varphi_t(\lambda) = (\pi t)^{(\dim \mathfrak{a})/2} e^{t\lambda^2}$ and

$$h_t(\lambda) = \int_{i\mathfrak{a}_0^*} f(x) \varphi_t(\lambda - x) dx. \tag{5.2.2}$$

Note that $\int_{i\mathfrak{a}^*} \varphi_t(x) dx = 1$ for each $t > 0$.

First, we prove that $h_t \in \mathsf{PW}(\mathfrak{a}^*, V)$. Indeed, for $M > 0$, via $(1+|\lambda|)^M \leq (1 + |\lambda - x|)^M (1+|x|)^M$,

$$\int_{i\mathfrak{a}_0^*} (1+|\lambda|)^M \|f(x)\|_V |\varphi_t(\lambda - x)| dx \leq C(\lambda) \|f\|_{N,X} \|\varphi\|_{N+2\dim\mathfrak{a}, |\operatorname{Re}\lambda|} < \infty,$$

where $C(\lambda) = \int_{i\mathfrak{a}_0^*} (1 + |\lambda - x|)^{-2\dim\mathfrak{a}} dx$, which is bounded in λ when $|\operatorname{Re}\lambda|$ is bounded. It also follows that h_t is holomorphic on $\mathfrak{a}^*$ because we can pass the derivative through the integral. This proves that $h_t \in \mathsf{PW}(\mathfrak{a}^*, V)$.

Now we prove that h_t approximates f. Fix $\varepsilon \in (0,1)$ and choose $\delta \in (0,\varepsilon)$ such that

$$\|(1+|\lambda|)^N f(z) - (1+|\lambda - x|)^N f(\lambda - x)\|_V < \varepsilon$$

whenever $|x| < \delta$, $x \in i\mathbb{R}$, and $\lambda \in X$. Then, for each $\lambda \in X$ and $|x| < \delta$, by noting $(1+|\lambda|)^N = ((1+|\lambda - x|) + (|\lambda| - |\lambda - x|))^N$ and applying binomial expansion, we get

$$(1+|\lambda|)^N \|f(\lambda) - f(\lambda - x)\|_V < (1 + 2^N \|f\|_{N,X})\varepsilon.$$

We will prove an estimate for $(1+|\lambda|)^N \|f(\lambda) - h_t(\lambda)\|_V$ when λ is in the interior of X. In this case, the integrand $f(x)\varphi_t(\lambda - x)$ is a holomorphic function of x on a domain containing $s\lambda + i\mathfrak{a}_0^*$ for each $s \in [0,1]$. Therefore, we may shift the contour, so that

$$\int_{i\mathfrak{a}_0^*} f(x)\varphi_t(\lambda - x)dx = \int_{\lambda + i\mathfrak{a}_0^*} f(x)\varphi_t(\lambda - x)dx = \int_{i\mathfrak{a}_0^*} f(\lambda - x)\varphi_t(x)dx.$$

Our previous estimate proves

$$\int_{\substack{x \in i\mathbb{R},\\ |x| < \delta}} (1+|\lambda|)^N \|f(\lambda) - f(\lambda - x)\|_V |\varphi_t(x)| dx < (1 + 2^N \|f\|_{N,X})\varepsilon.$$

Now, for large $t > 1$, the fact that $(1+|x|)^N \varphi_1(x)$ is integrable implies

$$\int_{\substack{x \in i\mathbb{R},\\ |x| > \delta}} (1+|x|)^N |\varphi_t(x)| dx \le \int_{\substack{x \in i\mathbb{R},\\ |x| > t\delta}} (1+|x|)^N |\varphi_1(x)| dx < \varepsilon.$$

Therefore, using $(1+|\lambda|) \le (1+|\lambda - x|)(1+|x|)$,

$$\int_{\substack{x \in i\mathbb{R},\\ |x| > \delta}} (1+|\lambda|)^N \|f(\lambda) - f(\lambda - x)\|_V |\varphi_t(x)| < 2\|f\|_{N,X}\varepsilon.$$

We have shown, for large t,

$$\|(1+|\lambda|)^N (f - h_t)(\lambda)\|_{N,X} < (1 + 2^N \|f\|_{N,X})\varepsilon.$$

The lemma follows by averaging h_t with respect to the action of W. $\square$

Lemma 5.2.1 follows immediately from Lemmas 5.2.2 and 5.2.3.

In what follows, we will apply the above lemma to $V = \mathrm{End}(p_n I_\sigma)$. We have seen that W_σ acts on $\mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_n I_\sigma))$ as in the lemma.

Proposition 5.2.4. *The inclusion of mapping cones*

$$\begin{aligned} &\mathsf{MC}(p_n(\mathcal{J}_n/\mathcal{J}_{n-1})p_n \to p_n(J_n/J_{n-1})p_n) \\ &\qquad\qquad \hookrightarrow \mathsf{MC}(\mathcal{J}_n/\mathcal{J}_{n-1} \to J_n/J_{n-1}) \end{aligned}$$

induces an isomorphism in K-theory.

Proof. By Theorems 4.4.3 and 4.4.7, together with Theorem 5.1.8, the inclusions $p_n(\mathcal{J}_n/\mathcal{J}_{n-1})p_n \hookrightarrow \mathcal{J}_n$ and $p_n(J_n/J_{n-1})p_n \hookrightarrow J_n$ induce isomorphisms in K-theory.

Now write

$$\begin{aligned} \mathsf{MC} &= \mathsf{MC}(\mathcal{J}_n/\mathcal{J}_{n-1} \to J_n/J_{n-1}), \\ \mathsf{MC}_{\text{reduced}} &= \mathsf{MC}(p_n(\mathcal{J}_n/\mathcal{J}_{n-1})p_n \to p_n(J_n/J_{n-1})p_n). \end{aligned}$$

If we apply the 6-term exact sequence appearing in the proof of Lemma 5.1.2 to MC and $\mathsf{MC}_{\text{reduced}}$, we obtain a morphism of exact sequences

$$\begin{array}{ccccccccc} \cdots \to & K_i(p_n J_n/J_{n-1} p_n) & \to & K_{i+1}(\mathsf{MC}_{\text{reduced}}) & \to & K_{i+1}(p_n \mathcal{J}/\mathcal{J}_{n-1} p_n) & \to \cdots \\ & \downarrow \cong & & \downarrow & & \downarrow \cong & \\ \cdots \to & K_i(J_n/J_{n-1}) & \to & K_{i+1}(\mathsf{MC}) & \to & K_{i+1}(\mathcal{J}/\mathcal{J}_{n-1}) & \to \cdots . \end{array}$$

The proposition follows from the five-lemma. □

Theorem 5.2.5. *The map $\mathcal{J}_n/\mathcal{J}_{n-1} \to J_n/J_{n-1}$ induces an isomorphism in K-theory.*

Proof. Set $\sigma = \sigma_n$. We assume $A(\sigma_n) \subset F$, as otherwise (by our assumptions on F) $\mathcal{J}_n/\mathcal{J}_{n-1} = 0$ and $J_n/J_{n-1} = 0$. By the previous proposition and Theorem 5.1.8, it suffices to show that the mapping cone of $p_n(\mathcal{J}_n/\mathcal{J}_{n-1})p_n \to p_n(J_n/J_{n-1})p_n$ has zero K-theory. By Theorems 4.4.5 and 4.4.4, we must show that the restriction map

$$\mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_n I_\sigma))^{W_\sigma} \to C_0(i\mathfrak{a}_0^*, \mathrm{End}(p_n I_\sigma))^{W_\sigma} \tag{5.2.3}$$

induces an isomorphism in K-theory.

We consider tubes X of the form $\{\lambda \in \mathfrak{a}^* : \|\operatorname{Re}\lambda\| < k\}$ for some $k > 0$. If we define $\mathsf{PW}_N(X, \operatorname{End}(p_n I_\sigma))^{W_\sigma}$ as in Lemma 5.2.1, we have

$$\mathsf{PW}(\mathfrak{a}^*, \operatorname{End}(p_n I_\sigma))^{W_\sigma} = \varprojlim_{X,N} \mathsf{PW}_N(X, \operatorname{End}(p_n I_\sigma))^{W_\sigma}.$$

Also, for any fixed tube X, we have

$$C_0(i\mathfrak{a}_0^*, \operatorname{End}(p_n I_\sigma))^{W_\sigma} = \varinjlim_{l} \mathsf{PW}_0(2^{-l}X, \operatorname{End}(p_n I_\sigma))^{W_\sigma}, \tag{5.2.4}$$

where the direct limit is in the category of Banach algebras and contractive morphisms (note that $\mathsf{PW}_0(X, \operatorname{End}(p_N I_\sigma))^{W_\sigma}$ consists of W_σ-invariant C_0 functions on X which are holomorphic on the interior of X).

We first show that $\mathsf{PW}_N(X, \operatorname{End}(p_n I_\sigma))^{W_\sigma}$ is independent of N up to K-theory isomorphism. Indeed, the inclusion maps

$$\mathsf{PW}_{N+1}(X, \operatorname{End}(p_n I_\sigma))^{W_\sigma} \hookrightarrow \mathsf{PW}_N(X, \operatorname{End}(p_n I_\sigma))^{W_\sigma}$$

have dense range, and we claim that the image is holomorphically stable. Fix $f \in \mathsf{PW}_{N+1}(X, \operatorname{End}(p_n I_\sigma))^{W_\sigma}$ and suppose it has a quasi-inverse $h \in \mathsf{PW}_N(X, \operatorname{End}(p_n I_\sigma))^{W_\sigma}$. Then for each $\lambda \in X$, the operator $1 + f(\lambda)$ is invertible and

$$h(\lambda) = -f(\lambda)(1 + f(\lambda))^{-1}.$$

Now, as f vanishes at infinity (on X), $\det(1 + f)$ is bounded away from 0. Cramer's rule implies that $(1 + f)^{-1}$ is bounded on X, hence the function $-[(1 + |\lambda|)^{N+1} f](1 + f)^{-1}$ is bounded on X. Hence, $h \in \mathsf{PW}_{N+1}(X, \operatorname{End}(p_n I_\sigma))^{W_\sigma}$. This proves holomorphic stability, and Karoubi density implies that they have the same K-theory.

We now show that $\mathsf{PW}_0(X, \operatorname{End}(p_n I_\sigma))^{W_\sigma}$ is independent of X up to homotopy of Banach algebras. More precisely, we claim that the restriction map

$$\operatorname{rest} : \mathsf{PW}_0(X, \operatorname{End}(p_n I_\sigma))^{W_\sigma} \to \mathsf{PW}_0(X/2, \operatorname{End}(p_n I_\sigma))^{W_\sigma}$$

has homotopy inverse

$$\alpha_1 : \mathsf{PW}_0(X/2, \operatorname{End}(p_n I_\sigma))^{W_\sigma} \to \mathsf{PW}_0(X, \operatorname{End}(p_n I_\sigma))^{W_\sigma},$$

given by $(\alpha_1 f)(\lambda) = f(\lambda/2)$. The composition $\alpha_1 \circ \mathrm{rest}$ is the restriction to $t = 1$ of the map

$$\alpha_\bullet \circ \mathrm{rest} : \mathsf{PW}_0(X, \mathrm{End}(p_n I_\sigma))^{W_\sigma} \times [0,1] \to \mathsf{PW}_0(X, \mathrm{End}(p_n I_\sigma))^{W_\sigma},$$

given by $\alpha_t \circ \mathrm{rest}(f)(\lambda) = f(\lambda/(1+t))$. Of course, when $t = 0$, the above map is the identity on $\mathsf{PW}_0(X, \mathrm{End}(p_n I_\sigma))^{W_\sigma}$. Similarly, the map

$$\mathrm{rest} \circ \alpha_\bullet : \mathsf{PW}_0(X/2, \mathrm{End}(p_n I_\sigma))^{W_\sigma} \times [0,1] \to \mathsf{PW}_0(X/2, \mathrm{End}(p_n I_\sigma))^{W_\sigma},$$

given by $(\mathrm{rest} \circ \alpha_t)(f)(\lambda) = f(\lambda/(1+t))$, defines a homotopy between $\mathrm{rest} \circ \alpha_1$ and the identity on $\mathsf{PW}_0(X/2, \mathrm{End}(p_n I_\sigma))^{W_\sigma}$.

Using the direct limit (5.2.4), and continuity in K-theory (see [BH21, Theorem 3.3]), we now see that the restriction map (5.2.3) induces an isomorphism in K-theory. □

We now prove that the inclusion $\mathcal{S}(G,F) \to C_r^*(G,F)$ induces an isomorphism in K-theory. We shall do so by a series of six-term exact sequence arguments and five lemma arguments. More precisely, for each $n \in \mathbb{N}$, we prove that the mapping cone

$$\mathsf{MC}_n = \mathsf{MC}\left(\mathcal{J}_n \to J_n\right)$$

vanishes in K-theory. When $n = 0$, we have $\mathcal{J}_0 = J_0 = 0$. For $n > 0$, the short exact sequence of Fréchet algebras

$$0 \to \mathsf{MC}_{n-1} \to \mathsf{MC}_n \to \mathsf{MC}(\mathcal{J}_n/\mathcal{J}_{n-1} \to J_n/J_{n-1}) \to 0,$$

leads (via [Phi91, Theorem 6.1]) to the 6-term exact sequence

$$\begin{array}{ccccc}
K_0(\mathsf{MC}_{n-1}) & \longrightarrow & K_0(\mathsf{MC}_n) & \longrightarrow & K_0(\mathsf{MC}(\mathcal{J}_n/\mathcal{J}_{n-1} \to J_n/J_{n-1})) \\
\uparrow & & & & \downarrow \\
K_1(\mathsf{MC}(\mathcal{J}_n/\mathcal{J}_{n-1} \to J_n/J_{n-1})) & \longleftarrow & K_1(\mathsf{MC}_n) & \longleftarrow & K_1(\mathsf{MC}_{n-1}).
\end{array}$$

By Theorem 5.2.5 and the above exact sequence, we see that $K_*(\mathsf{MC}_n) \cong K_*(\mathsf{MC}_{n-1})$. Inductively it follows that $K_*(\mathsf{MC}_n) = 0$. As $\mathcal{J}_n = \mathcal{S}(G,F)$ and $J_n = C_r^*(G,F)$ for large enough n, this concludes the proof of Theorem 1.3.2.

5.3 Proof of Theorem 4.4.3

We now prove Theorem 4.4.3, which states that

$$\mathcal{J}_n/\mathcal{J}_{n-1} = (\mathcal{J}_n/\mathcal{J}_{n-1})p_n(\mathcal{J}_n/\mathcal{J}_{n-1}).$$

We shall reduce this theorem to a "Factoring Theorem", which we then prove in the next section. The Factoring Theorem is the analogue for $\mathcal{S}(G)$ of results of Delorme [Del05], particularly [Del05, Proposition 1].

As usual, given a finite set $F \subset \widehat{K}$ we write

$$R(\mathfrak{g}, F) = p_F R(\mathfrak{g}, K) p_F.$$

To prove Theorem 4.4.3, the important point is that the matrices $\pi_n(\mathcal{J}_n)$ can be reduced (via polynomials) to matrices on only the minimal K-types, stated below.

Theorem 5.3.1 ("Factoring Theorem", cf. [Del05, Proposition 1]). *For each $n \in \mathbb{N}$*

$$\pi_n(\mathcal{J}_n) = \pi_n(R(\mathfrak{g}, F)p_n\mathcal{S}(G, F)p_n R(\mathfrak{g}, F)).$$

The inclusion $\supseteq$ follows from the fact that $p_{A(\sigma_n)}\mathcal{S}(G,F)p_{A(\sigma_n)} \subset \mathcal{J}_n$, and that $\mathcal{J}_n$ is an $R(\mathfrak{g}, F)$-bisubmodule of $\mathcal{S}(G, F)$. The difficulty lies in the inclusion $\subseteq$, which we prove in Chapter 6. This theorem is due to Delorme [Del05, Proposition 1] in the $C_c^\infty(G)$ case.

We recall (4.4.1), which (combined with Theorem 4.4.5) implies that for $\mu, \nu \in A(\sigma)$,

$$p_\mu \pi_n(\mathcal{J}_n) p_\nu = \mathsf{PW}(\mathfrak{a}^*)^{W_\sigma^0}(\hat{r}_{\mu\nu}) \otimes \mathrm{Hom}(p_\nu I_\sigma, p_\mu I_\sigma). \tag{5.3.1}$$

Let us briefly recall the notation in the above equation. The group W_σ decomposes as a semidirect product of subgroups

$$W_\sigma = R_\sigma W_\sigma^0,$$

where R_σ is a product of copies of $\mathbb{Z}/2$. Also, there is a simply transitive action of $\widehat{R}_\sigma$ on $A(\sigma)$, and we write $\hat{r}_{\mu\nu}$ to denote the unique element of $\widehat{R}_\sigma$ such that $\hat{r}_{\mu\nu} \cdot \nu = \mu$. Accordingly, we define $\mathsf{PW}(\mathfrak{a}^*)^{W_\sigma^0}(\hat{r}_{\mu\nu})$ to consist of all W_σ^0-invariant f such that

$$f(w\lambda) = \hat{r}_{\mu\nu}(w) f(\lambda)$$

for any $w \in W_\sigma$ (we have extended $\hat{r} \in \widehat{R}_\sigma$ to W_σ via the decomposition $W_\sigma = R_\sigma W_\sigma^0$.)

We now proceed toward the proof of Theorem 4.4.3, which states that

$$(\mathcal{J}_n/\mathcal{J}_{n-1})p_n(\mathcal{J}_n/\mathcal{J}_{n-1}) = \mathcal{J}_n/\mathcal{J}_{n-1}.$$

Lemma 5.3.2. *We have*

$$\mathsf{PW}(\mathfrak{a}^*) \cdot \mathsf{PW}(\mathfrak{a}^*) = \mathsf{PW}(\mathfrak{a}^*).$$

Proof. Let A act on $\mathsf{PW}(\mathfrak{a}^*)$ by $(a \cdot f)(\lambda) = e^{\lambda(\log a)} f(\lambda)$. Integrating this representation, the Casselman algebra, $\mathcal{S}(A)$, of A acts on $\mathsf{PW}(\mathfrak{a}^*)$ as multiplication by the (Euclidean) Fourier transform, and the lemma is implied by the statement that $\mathcal{S}(A) \cdot \mathsf{PW}(\mathfrak{a}^*) = \mathsf{PW}(\mathfrak{a}^*)$. This is now a consequence of [BK14, Remark 2.19]. □

Lemma 5.3.3. *We have*

$$\mathsf{PW}(\mathfrak{a}^*) = \mathbb{C}[\mathfrak{a}^*]\, \mathsf{PW}(\mathfrak{a}^*)^{W(\mathfrak{a})}.$$

This is a consequence of Theorem 4.3.7.

Lemma 5.3.4. *For each cuspidal pair (P, σ) and each $\mu, \nu \in A(\sigma)$,*

$$\mathsf{PW}(\mathfrak{a}^*)^{W_\sigma^0}(\hat{r}_{\mu\nu}) \cdot \mathsf{PW}(\mathfrak{a}^*)^{W_\sigma} = \mathsf{PW}(\mathfrak{a}^*)^{W_\sigma^0}(\hat{r}_{\mu\nu}). \tag{5.3.2}$$

Proof. Because $1 \in \mathbb{C}[\mathfrak{a}^*]$, Lemmas 5.3.2 and 5.3.3 imply

$$\mathsf{PW}(\mathfrak{a}^*)\, \mathsf{PW}(\mathfrak{a}^*)^{W(\mathfrak{a})} = \mathsf{PW}(\mathfrak{a}^*)\mathbb{C}[\mathfrak{a}^*]\, \mathsf{PW}(\mathfrak{a}^*)^{W(\mathfrak{a})} = \mathsf{PW}(\mathfrak{a}^*)\, \mathsf{PW}(\mathfrak{a}^*) = \mathsf{PW}(\mathfrak{a}^*),$$

and we obtain (5.3.2) by averaging by W_σ^0 and projecting onto the $\widehat{R}_\sigma$-isotypical component $\hat{r}_{\mu\nu}$ (note that this projection commutes with multiplication by $\mathsf{PW}(\mathfrak{a}^*)^{W(\mathfrak{a})}$). □

Proof of Theorem 4.4.3. We will prove that

$$\overline{\pi}_n(\mathcal{J}_n/\mathcal{J}_{n-1}) = \overline{\pi}_n(\mathcal{J}_n/\mathcal{J}_{n-1})p_n\overline{\pi}_n(\mathcal{J}_n/\mathcal{J}_{n-1}),$$

which implies the theorem because $\overline{\pi}_n$ is an injective algebra homomorphism. Note that this is equivalent to the statement

$$\pi_n(\mathcal{J}_n) = \pi_n(\mathcal{J}_n)p_n\pi_n(\mathcal{J}_n).$$

Set $\sigma = \sigma_n$. It suffices to prove that

$$\pi_n(\mathcal{S}(G, A(\sigma))) = \pi_n(\mathcal{S}(G, A(\sigma))R(\mathfrak{g}, A(\sigma))\mathcal{S}(G, A(\sigma))), \tag{5.3.3}$$

because then, by Theorem 5.3.1 and the fact that $p_n \in R(\mathfrak{g}, A(\sigma))$,

$$\begin{aligned}\pi_n(\mathcal{J}_n)p_n\pi_n(\mathcal{J}_n) &= \pi_n(R(\mathfrak{g}, F)\mathcal{S}(G, A(\sigma))R(\mathfrak{g}, A(\sigma))\mathcal{S}(G, A(\sigma))R(\mathfrak{g}, F)) \\ &= \pi_n(R(\mathfrak{g}, F)\mathcal{S}(G, A(\sigma))R(\mathfrak{g}, F)) = \pi_n(\mathcal{J}_n).\end{aligned}$$

We make use of the explicit formula for $\mu, \nu \in A(\sigma)$,

$$\pi_n(p_\mu \mathcal{S}(G, A(\sigma))p_\nu) = \mathsf{PW}(\mathfrak{a}^*)^{W_\sigma^0}(\hat{r}_{\mu\nu}) \otimes \mathrm{Hom}(I_\sigma(\nu), I_\sigma(\mu))$$

which follows from Theorem 4.4.5 and (4.4.1). Applying (5.3.2) on matrices, we obtain

$$\pi_n(p_\mu \mathcal{S}(G, A(\sigma))p_\nu) = \pi_n(p_\mu \mathcal{S}(G, A(\sigma))p_\nu \mathcal{S}(G, A(\sigma))p_\nu).$$

As p_ν acts as the identity on $p_\nu R(\mathfrak{g}, A(\sigma))p_\nu$, we have

$$\pi_n(p_\mu \mathcal{S}(G, A(\sigma))p_\nu) = \pi_n(p_\mu \mathcal{S}(G, A(\sigma))p_\nu R(\mathfrak{g}, A(\sigma))p_\nu \mathcal{S}(G, A(\sigma))p_\nu).$$

This gives the inclusion $\subseteq$ of (5.3.3), and the other inclusion $\supseteq$ follows because $\pi_\sigma(\mathcal{S}(G, A(\sigma_n)))$ is closed under the left and right action of $R(\mathfrak{g}, A(\sigma_n))$. $\square$

Chapter 6

Proofs of Delorme's factoring and divisibility theorems for $\mathcal{S}(G, F)$

6.1 Outline and an example

We now turn to the proof of Theorem 5.3.1, which states that

$$\pi_n(\mathcal{J}_n) = \pi_n \left(R(\mathfrak{g}, F)\mathcal{S}(G, A(\sigma_n))R(\mathfrak{g}, F) \right).$$

As it will be important to consider arbitrary parabolic subgroups with some fixed Levi subgroup, we recall that we chose representatives (P_n, σ_n) for each G-conjugacy class. We will fix n such that $A(\sigma_n) \subset F$, and write $P_n = MAN$, $\sigma = \sigma_n$.

Definition 6.1.1. Two parabolic subgroups P and Q with Levi subgroup MA are *adjacent* if $\Delta_P^+ \cap -\Delta_Q^+$ has a unique reduced root. If α is this root, and if $\lambda \in \mathfrak{a}^*$, then λ_α will denote the projection of λ onto $\mathbb{C}\alpha \subset \mathfrak{a}^*$ with respect to the inner product (2.2.1).

When P and Q are adjacent, the operator $A(Q, P, \sigma, \lambda)$ (and its normalized version) depends only on λ_α. Indeed, $\theta(N_P) \cap N_Q$ can be regarded as the "$\theta(N)$" of $G_\alpha = Z_G(\ker \alpha)$, which has split rank 1 (see [Kna02, VII.6]). Then $a_P(\overline{n}) \in G_\alpha$, so that $a_P(\overline{n})^\lambda = a_P(\overline{n})^{\lambda_\alpha}$. Alternatively, this can be deduced by an induction in stages formula (see [Del05, (1.4)]).

Definition 6.1.2. We define $\mathsf{PW}_{\text{divis}}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma))$ to be the set of functions $f \in \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma))$ with the following divisibility properties:

1. For each parabolic subgroup P with Levi subgroup MA, there exists a (unique)

function $f^P \in \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma))$ such that

$$\mathcal{A}(P, P_n, \lambda) f(\lambda) = f^P(\lambda) \mathcal{A}(P, P_n, \lambda).$$

2. For each $w \in W_\sigma$, the map f^P satisfies

$$\mathcal{A}(P, w, \lambda) f^P(\lambda) = f^P(w\lambda) \mathcal{A}(P, w, \lambda).$$

3. Let P and Q be adjacent parabolic subgroups with Levi subgroup MA. The map $\lambda \mapsto f^P(\lambda)\mathcal{A}(Q, P, \lambda)^{-1}$, initially meromorphic on $\mathfrak{a}^*$, extends to a holomorphic function on a neighborhood of $\overline{\mathfrak{a}^*_{P,+}}$.

Properties 1 and 2 are satisfied by $\pi_\sigma(\phi)$ for any $\phi \in \mathcal{S}(G, F)$, due to the intertwining relations

$$\mathcal{A}(P, Q, \lambda)\pi^Q_{\sigma,\lambda}(\phi) = \pi^P_{\sigma,\lambda}(\phi)\mathcal{A}(P, Q, \lambda), \quad \mathcal{A}(P, w, \lambda)\pi^P_{\sigma,\lambda}(\phi) = \pi^P_{\sigma,w\lambda}(\phi)\mathcal{A}(P, w, \lambda).$$

An important step in the proof of Theorem 5.3.1 is to show that elements of $\pi_n(\mathcal{J}_n)$ satisfy Property 3.

In the following, we set $\sigma = \sigma_n$, and we assume F contains $A(\sigma)$.

Theorem 6.1.3 (Factoring Theorem)**.**

$$\mathsf{PW}_{\text{divis}}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma)) \subseteq \pi_\sigma(R(\mathfrak{g}, F)\mathcal{S}(G, A(\sigma))R(\mathfrak{g}, F)).$$

Theorem 6.1.4 (Divisibility Theorem)**.**

$$\pi_\sigma(\mathcal{J}_n) \subseteq \mathsf{PW}_{\text{divis}}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma)).$$

The following lemma implies that the above inclusions are equalities.

Lemma 6.1.5. *We have*

$$\pi_\sigma(R(\mathfrak{g}, F)\mathcal{S}(G, A(\sigma))R(\mathfrak{g}, F)) \subseteq \pi_\sigma(\mathcal{J}_n).$$

Proof. This is a consequence of the fact that $\pi_\tau(p_{A(\sigma)}) = 0$ when $[\tau] > [\sigma]$. ∎

The two theorems and the lemma imply that

$$\pi_\sigma(\mathcal{J}_n) = \mathsf{PW}_{\mathrm{divis}}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma)) = \pi_\sigma(R(\mathfrak{g}, F)\mathcal{S}(G, A(\sigma))R(\mathfrak{g}, F)),$$

which implies Theorem 5.3.1.

In the $C_c^\infty(G)$ case, the Divisibility and Factoring Theorems are essentially [Del05, Proposition 1, and (3.8)], and our proofs are almost identical. In fact, the only real difference is the use of polynomial division on $\mathsf{PW}(\mathfrak{a}^*)$ (see Lemma 4.3.6), whose proof is practically the same as in the $C_c^\infty(G)$ case (due to Clozel and Delorme [CD90]).

It is illuminating to view these two theorems in the context of the spherical principal series, where the minimal K-type is the trivial K-type, which is the case $n = 1$. We will consider the example $G = \mathrm{SL}(2, \mathbb{R})$.

Example 6.1.6. Let $G = \mathrm{SL}(2, \mathbb{R})$ and $K = \mathrm{SO}(2)$. The two relevant parabolic subgroups are the minimal parabolic subgroup P consisting of upper triangle matrices in G, and G itself. We set

$$M = \{\pm \mathrm{Id}\}, \quad A = \{\mathrm{diag}(e^t, e^{-t}) : t \in \mathbb{R}\}, \quad N = \left\{ \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \in \mathbb{R} \right\},$$

so that $P = MAN$ is the Langlands decomposition of P.

We identify $\widehat{K}$ with $\mathbb{Z}$. We then set $F = \{-2, 0, 2\} \subset \widehat{K}$. There are three relevant cuspidal pairs whose corresponding principal series contains K-types in F.

- (P, σ), where $\sigma = \sigma_1$ be the trivial representation of M. The corresponding principal series I_σ is known as the spherical principal series.
- (G, σ_2), where $\sigma_2 = (\pi_{D_{2,+}}, D_{2,+})$ is the discrete series whose minimal K-type is $2 \in \widehat{K}$. In this case, $\pi_{\sigma_2} = \pi_{D_{2,+}}$.
- (G, σ_3), where $\sigma_3 = (\pi_{D_{2,-}}, D_{2,-})$ is the discrete series whose minimal K-type is $-2 \in \widehat{K}$. In this case, $\pi_{\sigma_3} = \pi_{D_{2,-}}$.

We also will use the fact that we have an embedding of representations,

$$(\pi_{D_{2,+}} \oplus \pi_{D_{2,-}}, D_{2,+} \oplus D_{2,-}) \to (\pi^P_{\sigma,1}, I_\sigma).$$

Given $k_\theta = e^{i\theta} \in K$, we define $e_n(k_\theta) = e^{-in\theta}$. As a K-representation, I_σ consists of even K-types with multiplicity 1. That is, I_σ is spanned by e_n for even n. Accordingly, $p_F I_\sigma$ has ordered basis e_2, e_0, e_{-2}, while $p_F D_{2,+}$ has basis e_2 and $p_F D_{2,-}$ has basis e_{-2} (which we will identify inside of I_σ). Using this ordered basis, we identify $p_F I_\sigma$ with $\mathbb{C}^3$.

We will parametrize $\mathfrak{a}^*$ as follows. Set $H_s = \operatorname{diag}(1, -1) \in \mathfrak{a}_0$. We then identify $\mathfrak{a}^*$ with $\mathbb{C}$ via $\lambda \mapsto \lambda(H_s)$. Under our identifications, the positive root in $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$ is the usual $\alpha = 2$.

With the above identifications, elements of $\pi_\sigma(\mathcal{S}(G, F))$ identify with maps $(z \mapsto f(z)) \in \mathsf{PW}(\mathbb{C}, M_3(\mathbb{C}))$. We will now write down this image and verify Theorems 6.1.3 and 6.1.4 in this context.

It is simpler to compute $\pi_\sigma(R(\mathfrak{g}, F))$ by completely algebraic means. From the identification $R(\mathfrak{g}, F) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} R(K)$, and from the formulas given in [Var99, Proposition 31, p. 136], we can explicitly compute that

$$\pi_\sigma(R(\mathfrak{g}, F)) = \left\{ \begin{bmatrix} p_{2,2}(z^2) & (z+1)p_{2,0}(z^2) & (z^2-1)p_{2,-2}(z^2) \\ (z-1)p_{0,2}(z^2) & p_{0,0}(z^2) & (z-1)p_{0,-2}(z^2) \\ (z^2-1)p_{-2,2}(z^2) & (z+1)p_{-2,0}(z^2) & p_{-2,-2}(z^2) \end{bmatrix} \right\},$$

where $p_{i,j} \in \mathbb{C}[z]$ for $i, j \in F$. The above matrices are with respect to the ordered basis e_2, e_0, e_{-2} of $p_F I_\sigma$.

It can also be shown that $\pi_\sigma(\mathcal{S}(G, F))$ has the same form as above, with the $p_{i,j}(z^2)$ replaced by even functions $f_{i,j}(z)$ in $\mathsf{PW}(\mathbb{C})$. We see that the entries have particular algebraic relations. For example, the top right entry is an even function with guaranteed zeros at $z = \pm 1$.

In our case of $\mathrm{SL}(2, \mathbb{R})$, it is easy to see where the algebraic relations come from. First of all, the Weyl group $W_\sigma = W(\mathfrak{g}_0, \mathfrak{a}_0)$ consists of two elements, where the nontrivial element w acts on $\mathfrak{a}^* \cong \mathbb{C}$ via $z \mapsto -z$. The corresponding action of $\mathcal{A}(P, w, \sigma, z)$ (where we have normalized $A(P, w, \sigma, z)$ with respect to the trivial K-type, and restricted the operator to $p_F I_\sigma$) is identified as the matrix

$$\mathcal{A}(P, w, \sigma, z) = \begin{bmatrix} \frac{z-1}{z+1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{z-1}{z+1} \end{bmatrix}.$$

on $p_F I_\sigma$ with respect to the basis e_2, e_0, e_{-2}. The algebraic conditions follow from the intertwining relation

$$f(z) = \mathcal{A}(P, w, -z) f(-z) \mathcal{A}(P, w, z).$$

In other words, in this case it holds that $\pi_\sigma(\mathcal{S}(G, F)) = \mathsf{PW}(\mathfrak{a}^*, \operatorname{End}(p_F I_\sigma))^{W_\sigma}$ (we caution that this does not necessarily hold for more general groups).

Let us now describe the ideal $\pi_\sigma(\mathcal{J}_1)$. The ideal $\mathcal{J}_1$, which is $\ker(\pi_{\sigma_2}) \cap \ker(\pi_{\sigma_3})$, is given (under Fourier transform) by

$$\pi_\sigma(\mathcal{J}_1) = \left\{ \begin{bmatrix} (z^2-1) f_{2,2}(z) & (z+1) f_{2,0}(z) & (z^2-1) f_{2,-2}(z) \\ (z-1) f_{0,2}(z) & f_{0,0}(z) & (z-1) f_{0,-2}(z) \\ (z^2-1) f_{-2,2}(z) & (z+1) f_{-2,0}(z) & (z^2-1) f_{-2,-2}(z) \end{bmatrix} \right\},$$

where the functions $f_{i,j}$ are even functions in $\mathsf{PW}(\mathbb{C})$. Indeed, when $\phi \in \mathcal{J}_1$, we see that (identifying $D_{2,\pm}$ inside of $I_{\sigma,1}$)

$$f_{2,2}(1) = p_2 \pi^P_{\sigma,1}(\phi) p_2 = \pi_{D_{2,+}}(\phi) p_2 = 0,$$

and similarly we have $f_{-2,-2}(1) = 0$. Because both $f_{2,2}$ and $f_{-2,-2}$ are even, we can therefore factor out $(z^2 - 1)$ from these functions.

Now, let $\overline{P} = \theta(P)$ be the opposite parabolic subgroup (consisting of lower-triangular matrices in G). Again, we write $\mathcal{A}(\overline{P}, P, \lambda)$ for the normalization of $A(\overline{P}, P, \sigma, \lambda)$ with respect to the trivial K-type, restricted to $p_F I_\sigma$. The Divisibility Theorem states that $\pi_\sigma(\phi)(z)\mathcal{A}(\overline{P}, P, z)^{-1}$ is holomorphic for $\operatorname{Re} z \geq 0$. This is easily verified by the explicit description of $\pi_\sigma(\mathcal{J}_\sigma)$ and the fact that

$$\mathcal{A}(\overline{P}, P, z) = \begin{bmatrix} \dfrac{z-1}{z+1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\dfrac{z-1}{z+1} \end{bmatrix}.$$

The Factoring Theorem is the statement that the whole of $\pi_\sigma(\mathcal{J}_1)$ can be obtained by applying elements of the Hecke algebra on the left and right to elements of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & f_{0,0}(z) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and taking the span. The above is easily verified by our explicit calculation of $\pi_\sigma(R(\mathfrak{g}, F))$ and $\pi_\sigma(\mathcal{J}_1)$. Of course, from our calculation of $\pi_\sigma(\mathcal{J}_1)$ we can see directly that $\pi_\sigma(\mathcal{J}_1) = \pi_\sigma(\mathcal{J}_1)p_{\{0\}}\pi_\sigma(\mathcal{J}_1)$ (which is Theorem 4.4.3 in our case).

6.2 Proof of the Divisibility Theorem

The proof of the Divisibility Theorem relies on the theory of derivatives for holomorphic families of representations, developed by Delorme and Souaifi. See [DS04], and see [vdBS14] for a systematic treatment of this theory. The Divisibility Theorem is essentially a consequence of the following theorem of Delorme and Souaifi.

Theorem 6.2.1 (See [DS04, Theorem 3 (ii)])**.** *Let X be an admissible $(\mathfrak{g}, K)$-module whose K-types have length larger than R. Then X is a subquotient of a direct sum of (successive derivatives of) principal series representations, each of which contain only K-types of length larger than R.*

We define what is meant by differentiation below.

Definition 6.2.2 (See [Del05] and [vdBS14])**.** Let Ω be an open subset of $\mathbb{C}$. Given an operator-valued holomorphic function $A : \Omega \to \mathrm{End}(V)$, we define

$$\Delta_z A : \Omega \to \mathrm{End}(V \oplus V)$$

by the block matrix formula

$$\Delta_z A(z) = \begin{bmatrix} A(z) & A'(z) \\ 0 & A(z) \end{bmatrix}.$$

Lemma 6.2.3. *Let $A, B : \Omega \to \mathrm{End}(V)$ be holomorphic. Then*

$$\Delta_z(A(z)B(z)) = (\Delta_z A(z))(\Delta_z B(z)).$$

Proof. This follows from the Leibniz rule. □

We must also account for several variables. The following defines a partial derivative for operator-valued holomorphic functions.

Definition 6.2.4. Let $\Omega \subset \mathbb{C}^n$ be an open subset, and let $A : \Omega \to \operatorname{End}(V)$ be holomorphic. Given $w \in \mathbb{C}^n$, we define

$$\Delta_w A : \Omega \subset \mathbb{C} \to \operatorname{End}(V \oplus V)$$

by

$$\Delta_w A(z) = \Delta_{z'} A_{z,w}(0),$$

where $A_{z,w} : \Omega' \subset \mathbb{C} \to \operatorname{End}(V)$ is defined by $A_{z,w}(z') = A(z + z'w)$, and where $\Delta_{z'} A_{z,w}$ is defined as in the previous definition.

We also use the same definition for holomorphic functions on complex vector spaces. Given a coordinate system $(z_1, \ldots, z_n)$ of this vector space, we may use the notation Δ_{z_i} instead of Δ_{e_i}, where e_i is the corresponding basis.

Using this definition of derivative, we may now differentiate the principal series (see [vdBS14] and [DS04]). Higher derivatives are obtained by successively applying Δ_w.

Theorem 6.2.5. *Elements of $\mathcal{J}_n$ act by* 0 *on any admissible $(\mathfrak{g}, K)$-module V whose K-types have length larger than $\|\sigma_n\|$.*

Proof. By definition of $\mathcal{J}_n$ and our total ordering on G-conjugacy classes $[\sigma]$, such elements $\phi \in \mathcal{J}_n$ act by zero on principal series representations whose K-types have length larger than $\|\sigma_n\|$. By our definition of Δ_z^N, it follows that ϕ acts by 0 on the corresponding (successive) derivative representations of these principal series.

According to Theorem 6.2.1, V is a subquotient of a direct sum of derivatives of principal series representations, each of whose K-types have length larger than $\|\sigma_n\|$. It follows that ϕ acts by 0 on V. □

Turning to the Divisibility Theorem, we need one more general lemma.

Lemma 6.2.6. *Let V be a finite-dimensional vector space. Let $f : \mathbb{C} \to \operatorname{End}(V)$ be holomorphic, and let $A : \Omega \to \operatorname{End}(V)$ be a rational function defined and holomorphic in a neighborhood $\Omega \subset \mathbb{C}$ of* 0*. If $\Delta_z^N f(0)$ vanishes on $\ker(\Delta_z^N A(0))$ for each N, then $f(z)A(z)^{-1}$ is holomorphic in a neighborhood of* 0*.*

Proof. Let $v \in V$ and suppose z^{N+1} divides $\mathcal{A}(z)v$. Then $\Delta_z^N A(0)v = 0$ and therefore $\Delta_z^N f(0)v = 0$. Unpacking the definition of Δ_z^N, this means that $(z \mapsto f(z)v)^{(k)}(0) = 0$ whenever $k \leq N$, and therefore z^{N+1} divides $f(z)v$. It follows that we can define $f(z)A(z)^{-1}v$ in a neighborhood of 0, which will be holomorphic in z. □

We must also deal with several variables. The following lemma generalizes the previous lemma to this case.

Lemma 6.2.7. *Let V be a finite-dimensional vector space. Let $f : \mathbb{C}^n \to \mathrm{End}(V)$ be holomorphic, and let $A : \Omega \to \mathrm{End}(V)$ be a rational function defined and holomorphic in a neighborhood $\Omega \subset \mathbb{C}$ of 0. Embed $\mathbb{C}$ into $\mathbb{C}^n$ via the first coordinate, for which we write $\lambda = (\lambda_1, \lambda') \in \mathbb{C}^n$, where $\lambda_1 \in \mathbb{C}$ and $\lambda' \in \mathbb{C}^{n-1}$.*

If $\Delta^N_{\lambda_1} f(0, \lambda')$ vanishes on $\ker(\Delta^N_{\lambda_1} A(0))$ for each N and for some $\lambda' \in \mathbb{C}^{n-1}$, then $\lambda \mapsto f(\lambda)A(\lambda_1)^{-1}$ is holomorphic in a neighborhood of $(0, \lambda') \in \mathbb{C}^n$.

This follows from the previous lemma. The point is that, if λ_1^N divides $A(\lambda_1)v$, then λ_1^N divides $f(\lambda)v$ (where $v \in V$).

Proof of the Divisibility Theorem. See [Del05, (3.8)]. Fix $\phi \in \mathcal{J}_n$ and set $f = \pi_{\sigma_n}(\phi)$. Let P and Q be adjacent parabolic subgroups containing MA. Note that (defining f^P as in Definition 6.1.2)

$$f^P(\lambda) = \pi^P_{\sigma,\lambda}(\phi).$$

We treat the unique reduced $\alpha \in \Delta^+_P \cap -\Delta^+_Q$ as the first coordinate of $\mathfrak{a}^*$, where we extend $\{\alpha\}$ to some basis. We note that $\mathcal{A}(Q, P, \lambda)$ depends only on λ_α (see Definition 6.1.1). We must show that $f^P(\lambda)\mathcal{A}(Q, P, \lambda_\alpha)^{-1}$ is holomorphic in a neighborhood of each $\lambda \in \overline{\mathfrak{a}^*_{P,+}}$.

According to Lemma 6.2.7, it suffices to prove that $\Delta^N_{\lambda_\alpha} f^P(\lambda)$ vanishes on the space $\ker(p_F \Delta^N_{\lambda_\alpha} \mathcal{A}(Q, P, \lambda_\alpha)) = p_F \ker(\Delta^N_{\lambda_\alpha} \mathcal{A}(Q, P, \lambda_\alpha))$ whenever $\mathrm{Re}\,\lambda_\alpha \geq 0$. Now, the $(\mathfrak{g}, K)$-module

$$\ker(\Delta^N_{\lambda_\alpha} \mathcal{A}(Q, P, \lambda_\alpha))$$

is a submodule of $(\Delta^N_{\lambda_\alpha} \pi^P_{\sigma_n,\lambda}, I^{\oplus N+1}_{\sigma_n})$. Moreover, because $\mathcal{A}(Q, P, \lambda_\alpha)$ is constant and nonzero on the minimal K-types of I_{σ_n}, the operators $\Delta^N_{\lambda_\alpha} \mathcal{A}(Q, P, \lambda_\alpha)$ are also constant and nonzero on the minimal K-types of $(I^P_{\sigma_n})^{\oplus N+1}$. Therefore, $\ker(\Delta^N_{\lambda_\alpha} \mathcal{A}(Q, P, \lambda_\alpha))$ does not contain any K-type in $A(\sigma_n)$, and therefore its K-types have length larger than $\|\sigma_n\|$. By Theorem 6.2.5, ϕ acts by 0 on $\ker(\Delta^N_{\lambda_\alpha} \mathcal{A}(Q, P, \lambda_\alpha))$. But ϕ acts precisely by $\Delta^N_{\lambda_\alpha} f^P(\lambda)$, so we are done. $\square$

6.3 Proof of the Factoring Theorem

Our aim is to prove that

$$\mathsf{PW}_{\mathrm{divis}}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma)) \subseteq \sum_{\mu,\nu\in A(\sigma)} \pi_\sigma(R(\mathfrak{g},F)p_\mu \mathcal{S}(G,A(\sigma))p_\nu R(\mathfrak{g},F)) \tag{6.3.1}$$

(recall Definition 6.1.2).

Let $u \in \mathsf{PW}_{\mathrm{divis}}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma))$. We recall that we have fixed a representative $(P_n = MAN, \sigma = \sigma_n)$ of $[\sigma]$, and that for any parabolic subgroup P with Levi subgroup MA, the function

$$u^P(\lambda) = \mathcal{A}(P, P_n, \lambda)u(\lambda)\mathcal{A}(P_n, P, \lambda)$$

defines an element of $\mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma))$.

Most details of the proof of the Factoring Theorem can be found in [Del05, Section 2], which treats $C_c^\infty(G)$ instead of $\mathcal{S}(G)$. As a result, we will state only what is necessary to cite Delorme's results. However, for the benefit of the reader, we briefly outline the details found in [Del05, Section 2].

We wish to find a decomposition

$$u = \sum \phi_i M_{ij} \tilde{\phi}_j,$$

where $\phi_i \in \pi_\sigma(R(\mathfrak{g},F)p_{\mu_i})$, $\tilde{\phi}_j \in \pi_\sigma(p_{\tilde{\mu}_j} R(\mathfrak{g},F)))$, $M_{ij} \in \pi_\sigma(p_{\mu_i}\mathcal{S}(\mathfrak{g},A(\sigma))p_{\tilde{\mu}_j})$, and $\mu_i, \tilde{\mu}_j \in A(\sigma)$. If we fix any choice of such $\phi_i, \tilde{\phi}_j$, then this becomes a linear algebra equation with respect to (the matrix components of) M_{ij}, over the field of meromorphic functions on $\mathfrak{a}^*$, where we are treating M_{ij} as matrices via

$$\pi_\sigma(p_{\mu_i}\mathcal{S}(G,A(\sigma))p_{\tilde{\mu}_j}) = \mathsf{PW}(\mathfrak{a}^*)^{W_\sigma^0}(\hat{r}_{\mu_i\tilde{\mu}_j}) \otimes \mathrm{Hom}(p_{\mu_i}I_\sigma, p_{\tilde{\mu}_j}I_\sigma)$$

as a consequence of Theorem 4.4.5 and (4.4.1).

By use of Cramer's rule, we obtain M_{ij} as meromorphic functions of the form p/q, where $p \in \mathsf{PW}(\mathfrak{a}^*)$, and q is a determinant term which depends on $\phi_i, \tilde{\phi}_j$. By understanding these determinants, we find that M_{ij} is holomorphic (hence in $\mathsf{PW}(\mathfrak{a}^*)$) for certain u. We then decompose u into pieces where the above is possible. We will have obtained $u = \sum \phi_i N_{ij}\tilde{\phi}_j$ where $N_{ij} \in \mathsf{PW}(\mathfrak{a}^*) \otimes \mathrm{Hom}(I_\sigma(\mu_j), I_\sigma(\mu_i))$. Finally, to obtain the W_σ-invariance, we must average using $\mathcal{A}(P_n, w, \lambda)$ for $w \in W_\sigma$, which will provide us with the desired decomposition of u.

Definition 6.3.1. Let $l = \dim p_F I_\sigma$.

By *left factoring data*, we refer to a list of tuples $(\mu_i, v_i, \phi_i)_{i=1}^l$ such that $\mu_i \in A(\sigma)$, v_i is a unit vector in $p_{\mu_i} I_\sigma$, and $\phi_i \in \pi_\sigma(R(\mathfrak{g}, F)p_{\mu_i})$. Corresponding to this data is the vector space $V = \bigoplus \mathbb{C} v_i$.

By *right factoring data*, we refer to a list of tuples $(\tilde{\mu}_j, \tilde{v}_j, \tilde{\phi}_j)_{j=1}^l$ with $\tilde{\mu}_j \in A(\sigma)$, $\tilde{v}_j$ a unit vector in $p_{\tilde{\mu}_j} I_\sigma$, and $\tilde{\phi}_j \in \pi_\sigma(p_{\tilde{\mu}_j} R(\mathfrak{g}, F))$. Correspondingly, we define $\widetilde{V} = \bigoplus \mathbb{C}\tilde{v}_j$.

Given a set of left factoring data (μ_i, v_i, ϕ_i) and parabolic subgroup P with Levi subgroup MA, define the operator

$$\Phi^P = \Phi^P(\lambda) : V \to p_F I_\sigma$$

by $\Phi^P(\lambda)(v_i) = \pi^P_{\sigma,\lambda}(\phi_i)v_i$. Given right factoring data $(\tilde{\mu}_j, \tilde{v}_j, \tilde{\phi}_j)$, we define

$$\widetilde{\Phi}^P = \widetilde{\Phi}^P(\lambda) : p_F I_\sigma \to \widetilde{V}$$

by $\widetilde{\Phi}^P(\lambda)(\psi) = \sum_j \langle \pi^P_{\sigma,\lambda}(\tilde{\phi}_j)\psi, \tilde{v}_j\rangle_{L^2(K)} \tilde{v}_j$.

Finally, given left and right factoring data, we define the "elementary matrices"

$$E_{ij} : \widetilde{V} \to V$$

by $E_{ij}(\tilde{v}_j) = v_i$ and $E_{ij}(\tilde{v}_k) = 0$ for $k \neq j$.

Lemma 6.3.2 (See [Del05, Lemmas 2, 3, and 9])**.** *Let P be a parabolic subgroup with Levi subgroup MA.*

1. *There exists a polynomial $b^P \in \mathbb{C}[\mathfrak{a}^*]$, nonzero on $-\overline{\mathfrak{a}^*_{P,+}}$, such that*

$$\det(p_F \mathcal{A}(\theta(P), P, \lambda)) = c \frac{b^P(\lambda)}{\overline{b}^P(-\lambda)},$$

   where $\overline{b}^P(\lambda) = \overline{b^P(\overline{\lambda})}$, and $c \in \mathbb{C}$ is a constant with modulus 1

2. *There exists polynomials $\Psi^P, \widetilde{\Psi}^P \in \mathbb{C}[\mathfrak{a}^*]$ such that*

$$\det \Phi^P(\lambda) = \Psi^P(\lambda)\overline{b}^P(-\lambda), \quad \det \widetilde{\Phi}^P(\lambda) = \widetilde{\Psi}^P(\lambda) b^P(\lambda).$$

Moreover, if Q is another parabolic subgroup, then Ψ^P and Ψ^Q are related by a nonzero constant (and similarly for $\widetilde{\Psi}^P$ and $\widetilde{\Psi}^Q$).

3. *The span of Ψ^P across all left factoring data equals $\mathbb{C}[\mathfrak{a}^*]^{W_\sigma^0}$. The span of $\tilde{\Psi}^P$ across all right factoring data equals $\mathbb{C}[\mathfrak{a}^*]^{W_\sigma^0}$.*

The above lemma is a specialization of [Del05, Lemmas 2, 3, and 9], which we have provided for context in order to state the next lemma, as well as to prove the Factoring Theorem.

Lemma 6.3.3. *Let $u \in \mathsf{PW}_{\text{divis}}(\mathfrak{a}^*, \mathrm{End}(p_F I_\sigma))$. Fix factoring data (μ_i, v_i, ϕ_i) and $(\tilde{\mu}_j, \tilde{v}_j, \tilde{\phi}_j)$ with corresponding $\Phi^P, \widetilde{\Phi}^P$. There exists functions*

$$M_{ij}^P = M_{ij}^P(\lambda) \in \mathsf{PW}(\mathfrak{a}^*)$$

such that, with $M^P = \sum_{i,j} M_{ij}^P E_{ij}$,

$$\Psi^P \widetilde{\Psi}^P u^P = \Phi^P M^P \widetilde{\Phi}^P. \tag{6.3.2}$$

This is the analogue of [Del05, Lemma 6]. If we ignore the rapidly decreasing condition (i.e. that $M_{ij}^P \in \mathsf{PW}(\mathfrak{a}^*)$), then this states that M_{ij}^P is a holomorphic function, which is [Del05, Lemma 7]. We do not prove that $M_{ij}^P \in \mathsf{PW}(\mathfrak{a}^*)^{W_\sigma^0}(\hat{r}_{\mu_i \mu_j})$ (stated analogously in [Del05, Lemma 6]), because this is not true in general. Instead, the issue of W_σ-invariance will be dealt with in the proof of the Factoring Theorem.

Proof. By [Del05, Lemma 7, p. 1013], if N^P denotes the solution to

$$u^P = \Phi^P N^P \widetilde{\Phi}^P,$$

then $\Psi^P \widetilde{\Psi}^P N^P$ is a holomorphic function (this only uses the divisibility properties listed in Definition 6.1.2). Now, according to Cramer's rule, there exists functions $p_{ij}^P \in \mathsf{PW}(\mathfrak{a}^*)$ such that, writing $N^P = \sum E_{ij} N_{ij}^P$, then

$$N_{ij}^P(\lambda) = \frac{p_{ij}^P(\lambda)}{\det \Phi^P(\lambda) \det \widetilde{\Phi}^P(\lambda)}.$$

Then the fact that $\Psi^P \widetilde{\Psi}^P N_{ij}^P$ is holomorphic implies that $\det \Phi^P(\lambda) \det \widetilde{\Phi}^P(\lambda)$ divides $\Psi^P \widetilde{\Psi}^P p_{ij}^P$. According to the Lemma 4.3.6 on polynomial division, this implies that $\Psi^P \widetilde{\Psi}^P N_{ij}^P \in \mathsf{PW}(\mathfrak{a}^*)$.

Now, $M_{ij} = \Psi^P \widetilde{\Psi}^P N_{ij}^P$ is the solution to $\Psi^P \widetilde{\Psi}^P u^P = \Phi^P M^P \widetilde{\Phi}^P$, so we have shown $M_{ij} \in \mathsf{PW}(\mathfrak{a}^*)$. $\square$

For completeness, and to benefit the reader, we will summarize the proof of [Del05, Lemma 7]. The claim is that $\Psi^P \widetilde{\Psi}^P N^P$ is holomorphic. It suffices to show that $\Psi^P \widetilde{\Psi}^P N^P$ is holomorphic on $\overline{\mathfrak{a}^*_{P,+}}$ for each P (this is because each N^P and N^Q are intertwined by $\mathcal{A}(Q, P, \lambda)$, which is nonzero and independent of λ when acting on minimal K-types).

By writing $\mathcal{A}(\theta(P), P, \lambda)$ as a product of $\mathcal{A}(Q, R, \lambda)$ for adjacent Q, R, Property 3 of Definition 6.1.2 implies that $u^P(\lambda)\mathcal{A}(\theta(P), P, \lambda)^{-1}$ extends to a holomorphic function on $\overline{\mathfrak{a}^*_{P,+}}$. Now, with notation as in the above proof,

$$u^P(\lambda)\mathcal{A}(\theta(P), P, \lambda)^{-1} = \Phi^P N^P \widetilde{\Phi}^P \mathcal{A}(\overline{P}, P, \lambda)^{-1}.$$

The rational function $\widetilde{\Phi}^P \mathcal{A}(\theta(P), P, \lambda)^{-1}$ turns out to be a polynomial in λ (we can commute the action of $\mathcal{A}(\theta(P), P, \lambda)^{-1}$ over to $\widetilde{V}$), and

$$\det\left(\widetilde{\Phi}^P \mathcal{A}(\theta(P), P, \lambda)^{-1}\right) = \widetilde{\Psi}^P(\lambda)\overline{b}^P(-\lambda).$$

Therefore, by Cramer's rule, and since $\overline{b}^P(-\lambda)$ is nonzero on $\overline{\mathfrak{a}^*_{P,+}}$,

$$N_{ij}^P = \frac{q_{ij}}{\Psi^P \widetilde{\Psi}^P},$$

where q_{ij} is holomorphic on $\overline{\mathfrak{a}^*_{P,+}}$. This proves that $\Psi^P \widetilde{\Psi}^P N^P$ is holomorphic on $\overline{\mathfrak{a}^*_{P,+}}$.

Proof of the Factoring Theorem. Fix an element $u \in \mathsf{PW}_{\text{divis}}(\mathfrak{a}^*, \operatorname{End}(p_F I_\sigma))$. In the following, we will only consider $P = P_n$ and omit the corresponding superscripts (for example, we write $\Phi = \Phi^{P_n}$). Using Lemma 6.3.2, choose several left and right factoring data $(\mu_i^{(m)}, v_i^{(m)}, \phi_i^{(m)})$ and $(\tilde{\mu}_j^{(r)}, \tilde{v}_j^{(r)}, \tilde{\phi}_j^{(r)})$ such that

$$\sum_m \Psi_m \equiv 1, \quad \sum_r \widetilde{\Psi}_r \equiv 1.$$

Let $M^{(m,r)}$ be the corresponding matrices as in Lemma 6.3.3, and let

$$E_{ij}^{(m,r)} \in \operatorname{Hom}(I_\sigma(\tilde{\mu}_j^{(r)}), I_\sigma(\mu_i^{(m)}))$$

be corresponding "elementary matrices" between left and right factoring data. Then

$$\begin{aligned} u &= \sum_{m,r} \Psi_m u \widetilde{\Psi}_r = \sum_{m,r} \Phi_m M^{(m,r)} \widetilde{\Phi}_r \\ &= \sum_{m,r,i,j} \pi_\sigma(\phi_i^{(m)}) M_{ij}^{(m,r)} E_{ij}^{(m,r)} \pi_\sigma(\tilde{\phi}_j^{(r)}). \end{aligned}$$

We have shown that

$$u \in \pi_\sigma(R(\mathfrak{g}, F) p_{A(\sigma)}) \cdot \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_{A(\sigma)} I_\sigma)) \cdot \pi_\sigma(p_{A(\sigma)} R(\mathfrak{g}, F)).$$

Now, u commutes with the action of W_σ given by $\mathcal{A}(P_n, w, \lambda)$ (this is Property 2 of Definition 6.1.2), and so if we average by this action we obtain

$$\begin{aligned} u &\in \left[\pi_\sigma(R(\mathfrak{g}, F) p_{A(\sigma)}) \cdot \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_{A(\sigma)} I_\sigma)) \cdot \pi_\sigma(p_{A(\sigma)} R(\mathfrak{g}, F))\right]^{W_\sigma} \\ &= \pi_\sigma(R(\mathfrak{g}, F) p_{A(\sigma)}) \cdot \mathsf{PW}(\mathfrak{a}^*, \mathrm{End}(p_{A(\sigma)} I_\sigma))^{W_\sigma} \cdot \pi_\sigma(p_{A(\sigma)} R(\mathfrak{g}, F)) \\ &= \pi_\sigma(R(\mathfrak{g}, F) \mathcal{S}(G, A(\sigma)) R(\mathfrak{g}, F)), \end{aligned}$$

where Theorem 4.4.5 is used in the second equality, and for the first equality we note that elements of $\pi_\sigma(R(\mathfrak{g}, F))$ commute with the action of W_σ. □

6.4 The Paley-Wiener theorem for $\mathcal{S}(G)$

We have mentioned that the Division and Factoring theorems are analogs of propositions which Delorme uses to prove the (K-finite) Paley-Wiener theorem for $C_c^\infty(G)$ (see [Del05, Theorem 1]). We will now show that these theorems can be used to prove the K-finite Paley-Wiener theorem for $\mathcal{S}(G)$. We first need a definition for the Paley-Wiener space.

As usual, we have representatives (P_n, σ_n) for the G-conjugacy classes of cuspidal pairs. We will now keep track of subscripts, so that $P_n = M_n A_n N_n$, and we write $\mathfrak{a}_n^*$ for the complexified Lie algebra of A_n.

Definition 6.4.1 (cf. [Del05, Definition 2])**.** Fix a finite subset $F \subset \widehat{K}$. We write $\mathcal{PW}(G, F)$ for the space of maps

$$f = \sum_n f_n \in \bigoplus_n \mathsf{PW}(\mathfrak{a}_n^*, \mathrm{End}(p_F I_{\sigma_n})),$$

where the direct sum is algebraic, such that f and its derivatives $\Delta^k f = \sum_n \Delta^k f_n$ preserve any invariant $(\mathfrak{g}, K)$-subspaces of (derivatives of) principal series representations, and commute with any intertwining operators between subquotients of such spaces (cf. [Del05, (3.2)]). We define $\mathcal{PW}(G, K)$ to be the union of the $\mathcal{PW}(G, F)$ over all finite sets F.

In particular, given $f \in \mathcal{PW}(G, F)$, the map $f_n \in \mathsf{PW}(\mathfrak{a}_n^*, \mathrm{End}(p_F I_{\sigma_n}))$ satisfies 1. and 2. of Definition 6.1.2. Also, thanks to [DS04], f can be made to act on any admissible $(\mathfrak{g}, K)$-module (this partly explains the assumption on f regarding intertwining operators on subquotients).

It is shown in [vdBS14, Section 4.3] that in the above definition we only require the derivatives of $f = (f_n)$ to preserve invariant subspaces of the corresponding sums of (derivatives of) principal series where f acts nontrivially (the requirement regarding intertwining operators follows from this). See [vdBS14] for a deeper analysis of this requirement, and for other equivalent definitions of the space $\mathcal{PW}(G, K)$.

Theorem 6.4.2. *The Fourier transform map*

$$\mathcal{F} : \mathcal{S}(G, K) \to \mathcal{PW}(G, K),$$

given by $\phi \mapsto \sum_n \pi_n(\phi)$*, is a bijection.*

Injectivity follows, for example, from the Plancherel formula.

To prove surjectivity, it suffices to show that $\mathcal{S}(G, F) \to \mathcal{PW}(G, F)$ is surjective for balls of K-types F. Let us define $\mathcal{J}_N^{\mathsf{PW}}$ to be the functions $f = \sum f_n \in \mathcal{PW}(G, F)$ such that $f_m = 0$ for $m > N$. It then suffices to show that

$$\mathcal{F} : \mathcal{J}_N \to \mathcal{J}_N^{\mathsf{PW}}$$

is surjective. We will prove this by induction (note that $\mathcal{J}_0^{\mathsf{PW}} = \mathcal{J}_0 = 0$).

Suppose that $\mathcal{F} : \mathcal{J}_{N-1} \to \mathcal{J}_{N-1}^{\mathsf{PW}}$ is surjective. The same argument given in the proof of the Divisibility Theorem shows that

$$\{f_N : f \in \mathcal{J}_N^{\mathsf{PW}}\} \subset \mathsf{PW}_{\mathrm{divis}}(\mathfrak{a}_N^*, \mathrm{End}(p_F I_{\sigma_N})).$$

(Here we have used the assumption that the derivatives of f preserve invariant subspaces in the definition of $\mathcal{PW}(G, F)$.) In fact, the above is the true analog of [Del05, Proposition 2] (our statement of the Divisibility Theorem is about the Fourier image of $\mathcal{S}(G)$). By the Factoring Theorem, this means that for $f \in \mathcal{J}_N^{\mathsf{PW}}$,

$$f_N \in \pi_N(\mathcal{J}_N).$$

Let us write $\phi_N \in \mathcal{J}_N$ such that $\pi_N(\phi_N) = f_N$. Then $f - \mathcal{F}(\phi_N) \in \mathcal{J}_{N-1}^{\mathsf{PW}}$. The result now follows by induction.

We have shown that $\mathcal{F}(\mathcal{S}(G,K)) = \mathcal{PW}(G,K)$. Of course, it follows immediately that

$$\mathcal{F} : p_\gamma \mathcal{S}(G,K) p_\delta \to p_\gamma \, \mathcal{PW}(G,K) p_\delta$$

is a bijection for any $\gamma, \delta \in \widehat{K}$, and so $\mathcal{F} : \mathcal{S}(G,F) \to \mathcal{PW}(G,F)$ is a bijection for any finite subset $F \subset \widehat{K}$ (not only for balls of K-types). Finally, if we provide $\mathcal{PW}(G,K)$ with the structure of a $(\mathfrak{g},K)$-module via $T \cdot f = \sum_n \pi_n(T) f_n$ for $T \in R(\mathfrak{g},K)$, and if treat $\mathcal{S}(G,K)$ as a $(\mathfrak{g},K)$-module via the left regular representation, then the Fourier transform map $\mathcal{F}$ is an isomorphism of $(\mathfrak{g},K)$-modules. A similar statement can be made for the right regular representation.

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Vita

Jacob Bradd

Education

- Penn State: PhD in Mathematics.
- University of Wollongong: Bachelor of Mathematics (Advanced) Honours.
- UOW: Bachelor of Mathematics and Applied Statistics (Majored in Mathematics and Statistics).

Publication List

- *On the Compatible Decomposition Between the Casselman Algebra and the Reduced Group C^*-algebra of a Real Reductive Group*. In progress.
  Preprint DOI: `arXiv:2010.09428`
- *Novodvorskii's Theorem and the Oka principle.*
  DOI: `10.1007/s40879-021-00462-z`

Talks given by location

- Weekend Workshop on Representation Theory and NCG in St. Louis.
- NCG, Index Theory and Representation Theory conference in Kyoto.
- William and Mary for the NCGAHS Conference.
- Talks given for the Student NCG Seminar at Penn State.
- University of Wollongong for the OANCG seminar.

Teaching Experience

- TA for Math 498 (Game Theory) for Leonid Vaserstein: Fall 2018, Spring 2019.
- Instructor for Math 17: Spring 2019, Fall 2019.
- Instructor for Math 26: Spring 2020.
- Grader for Math 535, 536 (Graduate Linear and Abstract Algebra).
- Instructor for Math 22: Fall 2021, Sections 3,6,12.
- Instructor for Math 220 (Matrices): Fall 2022, Sections 5, 15.

Awards at Penn State:
Pritchard Dissertation Fellowship (2023) and ZZRQ Award (2021-2022).