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BLACK HOLE ENTROPY, CONSTRAINTS, AND SYMMETRY IN QUANTUM GRAVITY

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Abstract

This thesis addresses two major problems in the area of quantum gravity. The first regards an extension of the statistical mechanical derivation of the Bekenstein-Hawking entropy from loop quantum gravity in [12]. Let us review what was accomplished in [12]. In [12], equilibrium black holes were modeled by isolated horizons. We recall the following terminology: When the intrinsic geometry of an isolated horizon is spherically symmetric, we call it "type I". When the intrinsic geometry of an isolated horizon is only axisymmetric, we call it "type II". The quantum geometry and entropy of *type I* isolated horizons were investigated in [12]. The first part of this thesis generalizes the investigations in [12] to the type II case, thereby encompassing distortions and rotations of the horizon. In particular, the leading term in the entropy of large horizons is again given by one fourth of the horizon area, using the *same* value of the Barbero-Immirzi parameter as was used in the type I case.

The second problem addressed in this thesis regards how to define 'symmetric state' in quantum gravity (as well as quantum field theory more generally). The question is approached via a discussion on the relation between symmetry reduction before and after quantization of a field theory. A toy model field theory is used: the axisymmetric Klein-Gordon field. We consider three possible notions of symmetry at the quantum level: invariance under the group action, and two notions derived from imposing symmetry as a system of constraints a la Dirac. One of the latter two turns out to be the most appropriate notion of symmetry in the sense that it satisfies a number of physical criteria, including the commutativity of quantization and symmetry reduction. Somewhat surprisingly, the requirement of invariance under the symmetry group action is *not* appropriate for this purpose. A generalization of the physically selected notion of symmetry to loop quantum gravity is presented and briefly discussed.

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for my mother and father

Chapter 1

Introduction

At the beginning of the 20th century, two revolutions in physics came about. One was quantum mechanics, and the other general relativity. The first was the result of an attempt to understand an array of experimental phenomena that defied foundational assumptions of known physics at that time. The most elementary notions had to be revised, and physicists such as Planck, Bohr, Heisenberg, Schrödinger, Einstein and others took up the challenge to piece together and understand the radically new structures and laws being uncovered before them. The result was quantum mechanics, formulated systematically first by Schrödinger and Heisenberg and then developed further and applied to new domains by others after them. Quantum theory succeeded in explaining, with precision, a wide range of facts that had no explanation otherwise. Examples of such facts include the values of the specific heat of certain materials, the minutest details of atomic spectra, and the structure of the periodic table of the elements. Thus, quantum mechanics was a remarkable success. However, quantum mechanics involved a radical revision of basic notions: it forced us to replace our classical understanding of the world with wave-functions and operators, with arbitrary superpositions of "classical states" allowed. Because of the fundamental nature of these revisions, there are strong reasons to believe that quantum mechanics must underlie *everything*. That is, in order for physics to be consistent, it seems that all areas of physics must be eventually described by quantum theory. When a satisfactory classical theory is available for a given class of phenomena, there are time-tested strategies for teasing out the structure of the corresponding quantum theory from clues in the Hamiltonian and

Lagrangian frameworks — that is, strategies for *quantization.*

General relativity, the other revolution, was the culmination and completion of what Einstein had begun in special relativity. In special relativity the prospect of an absolute frame of rest in physics was banished, and space and time were inextricably intertwined in a single geometry. In general relativity, the absolute reference for non-acceleration was, remarkably, also banished; motion took on a purely relational character. This was made possible through the discovery that gravity is geometry: the geometry of spacetime is dynamical, and gravity is a manifestation of its curvature. Furthermore, the field equation of general relativity contained within itself in seamless fashion the law of inertia, Newton's law of gravity (in an appropriate limit), (local) conservation of energy and momentum of all matter, and all of Euclidean geometry (in an appropriate limit). It was a theory of which Max Born said

> The theory appeared to me then, and still does, the greatest feat of human thinking about nature, the most amazing combination of philosophical penetration, physical intuition, and mathematical skill .... It appealed to me like a great work of art ... [1]

At the theory's birth, its compelling character was apparent to more than one of the physics luminaries of the day — even before any compelling *observational* evidence for its radical suggestions were present. The fact that most of its radical suggestions have proven true, and that it has survived 90 years of increasingly high-precision observational scrutiny should impress anyone. However, general relativity, in spite of its simplicity, beauty, and power, has resisted attempts at quantization. That is, a theory of *quantum gravity* remains elusive.

This thesis addresses two major problems in the area of quantum gravity. It is thus organized into two parts. The first extends the generality of a statistical mechanical calculation of black hole entropy from loop quantum gravity, while the second investigates the notion of 'symmetric state' in quantum gravity, as well as in quantum field theory more generally. The first part is based on work done jointly with Abhay Ashtekar and Chris Van Den Broeck and was first reported in [2]. The second is based on the work of the author [3].

Both problems are addressed in the approach to quantizing first class constrained Hamiltonian systems pioneered by Bergmann and Dirac. This approach draws upon a remarkable compatibility between first class constrained systems and quantum mechanics. Specifically, the two, classically distinct roles of first class constraints as restrictions on initial data and generators of gauge transformations converge into one in quantum theory: Imposition of the quantum constraints and division by the corresponding gauge become identical operations (see, *e.g.* [4]). The tight relation among constraints and gauge present classically is thus 'explained' by the underlying quantum mechanics. In general relativity, the consequences of this approach to quantization are striking: The quantum constraints of general relativity force the wave-function of the universe to be *spatially diffeomorphism-invariant* and invariant under time evolution [1]. This gives Einstein's insight that individual space-time points have no physical reality of their own, so to speak, mathematical *teeth* in quantum theory.

Loop quantum gravity (LQG) is the only modern approach to quantum gravity that holds fast not only to Einstein's deep insight that *gravity is geometry*, but also to Bergmann and Dirac's deep insights into the ramifications of first class quantum constraints. In addition, it holds to Einstein's more general principle of (what is now called) *background independence*: the proposition that, with geometry no longer non-dynamical, no non-dynamical field should ever appear in physics [5].

1.1 The entropy calculation and its extension

In the late 1970's, Bekenstein, Hawking and others compellingly argued that black holes are thermodynamic objects [6, 7]. They proposed a temperature $\kappa/8\pi$ and entropy $a/4$ be assigned to a black hole, where κ denotes what is called the "surface gravity", and a the area of the event horizon. These quantities are typically referred to as the "Hawking temperature" and "Bekenstein-Hawking entropy", respectively. A challenge for any theory of quantum gravity is to give a statistical mechanical account of this thermodynamics. Black hole thermodynamics in this way becomes an indirect window onto quantum gravity.

In particular, any quantum theory of gravity should be able to reproduce the Bekenstein-Hawking entropy through a statistical mechanical counting of micro-

[1] more precisely, invariant under time evolution for all choices of lapse and shift with sufficient fall-off at infinity.

scopic degrees of freedom. This challenge was partially met in string theory and loop quantum gravity around the same time, in the late 90's [8, 9, 10, 11, 12]. Each of the calculations had its strengths and weaknesses. Within string theory, detailed calculations were carried out for a sub-class of extremal black holes known as BPS states. These are far from astrophysically realistic, but the exact coefficient of $\frac{1}{4}$ in the Bekenstein-Hawking entropy is obtained, which is impressive. A subsequent version of the calculation based instead on AdS-CFT correspondence did not require the assumption of BPS states; however, it obtained the wrong numerical coefficient in the expression for the entropy, being off by a factor of $\left(\frac{4}{3}\right)^{1/4}$ [13].

In the original loop quantum gravity calculation, on the other hand, no extremality was required. The charges on the horizon could be arbitrary. Furthermore, stationarity of geometry was imposed only on the horizon geometry itself in the form of *isolated horizon* boundary conditions. On the other hand, loop quantum gravity possesses a free parameter, known as the Barbero-Immirzi parameter γ, which trickles down into the spectrum of the area operator, and through the area operator, it enters the expression for the entropy. Consequently, loop quantum gravity does not predict the numerical coefficient in the expression for the entropy. *Rather*, by stipulating that the numerical factor be $\frac{1}{4}$, one fixes the parameter γ (see equation (3.48)). This is non-trivial, since, in order to fix γ, it suffices to consider the entropy of a single black hole. With the value of γ so fixed, one then automatically obtains the Bekenstein-Hawking entropy for black holes of arbitrary electric charge and area, as well black holes with minimally and non-minimally coupled scalar fields at the horizon [12, 14].

The loop quantum gravity calculation in [12] made the assumption that the *intrinsic geometry of the horizon* is spherically symmetric. The first part of the present thesis significantly extends the analysis by requiring only axisymmetry at the horizon. This is a significant generalization, as now rotating and distorted black holes are encompassed in the calculation. One should note that this relaxation from spherical symmetry at the horizon to axisymmetry is much more than a relaxation from the Schwarzschild family to the Kerr family of horizon geometries. The space of possible axisymmetric horizon geometries is far richer than the space of Kerr horizon geometries: the former is infinite dimensional whereas the latter is only two dimensional.

To define the phase space we will quantize and the final ensemble whose entropy is to be calculated, certain *multipoles* of isolated horizons are needed. These multipoles are diffeomorphism invariant, completely determine the horizon geometry up to diffeomorphism, and have other physically desirable properties.

The material in this thesis addressing the entropy calculation is organized as follows. Chapter 2 gives an overview of isolated horizons and their multipoles, largely based on [15]. Motivations for the definitions are discussed, and basic results are mentioned. Chapter 3 starts with the appropriate classical phase space, proceeds with its quantization, discusses the quantum geometry available at horizon, and finally defines an ensemble and calculates the entropy.

1.2 Symmetric states

In chapter 4, we turn to the notion of symmetry in quantum field theory.

The usual approach to symmetry is to impose symmetry on the wave-function: A state is symmetric if it is invariant under the action of the symmetry group. In chapter 4, we suggest an alternative approach. The essential argument is: if we want to impose the quantum analogue of symmetry of the classical *fields*, then we should be imposing symmetry on the *field operators*, not the wavefunction. Concretely, the approach is to formulate the classical symmetry condition in the form of a (at least formally) first class system of constraint functions which are then imposed as operator constraints in the quantum theory.

This second, new approach to symmetry in quantum field theory has a number of physically desirable properties that the more usual definition of symmetry in terms of invariance under the group action does not have. These are demonstrated and discussed in chapter 4 using, for simplicity, a free scalar field theory. However, an extension of the results to loop quantum gravity is discussed at the end. [2]

Beyond its nice properties, the introduction of this new definition of symmetry is critical for two reasons. First, in the case of a spatially compact universe, all

[2]It should be said that the notion of symmetry introduced by Bojowald and Kastrup [16] is related to the present approach. The present approach was inspired by and seeks to improve upon their approach.

$$\begin{tikzcd} \boxed{\text{Classical theory}} \arrow[r, "\text{quantization}"] \arrow[d, "\text{symmetry reduction}"'] & \boxed{\text{Full quantum theory}} \arrow[d, "\text{symmetry reduction}"'] \\ \boxed{\text{Reduced classical theory}} \arrow[r, "\text{quantization}"] & \text{match ?} \end{tikzcd}$$

Figure 1.1. The commutativity question: do quantization and symmetry reduction commute? The answer depends crucially on the meaning of 'symmetry reduction' at the quantum level.

physical states are invariant under all possible diffeomorphisms. In particular, all states will be invariant under any possible spatial symmetry group action one can specify. Thus, the standard definition of symmetry becomes trivial, and hence meaningless, in this case. However, the new approach to symmetry is *not* trivial in this case.

Secondly, the introduction of this new approach to symmetry is crucial for the issue of commutativity of symmetry reduction and quantization. Consider the commutative diagram in figure 1.2. Commutativity of the diagram obviously depends to a large extent on how the quantum symmetry reduction is done. In the scalar field theory case (primarily considered in this thesis), it is the *new* approach to quantum symmetry that achieves commutativity of the diagram. The standard approach to quantum symmetry does not lead to commutativity (in any natural sense). The hope is that the new approach to symmetry is the correct direction to go in understanding better the relation between LQG and the symmetry reduced models inspired by LQG. These symmetry reduced models include loop quantum cosmology and the models of gravitational collapse spearheaded by Bojowald and others, models which are beginning to address the resolution of classical singularities in quantum gravity.

The presentation in chapter 4 differs from that in [3] in that coherent states now play a more central role, and dependence on kinematical and dynamical linearity is largely eliminated.

Chapter 2

Multipole Moments of Isolated Horizons

2.1 Introduction

Multipole moments play an important role in Newtonian gravity and Maxwellian electrodynamics. Conceptually, there are two distinct notions of multipole moments —source multipoles which encode the distribution of mass (or charge-current), and field multipoles which arise as coefficients in the asymptotic expansions of fields. In Newtonian gravity, the first set is of direct interest to equations of motion of extended bodies while the second determines the gravitational potential outside sources. However, via field equations one can easily relate the two sets of multipoles. In the Maxwell theory, the rate of change of the source dipole moment is directly related to the energy flux measured at infinity. Because of such useful properties, there has been considerable interest in extending these notions to general relativity.

Results of the Maxwell theory were extended to the weak field regime of general relativity —i.e., linearized gravity— quite some time ago. Already in 1916 Einstein obtained the celebrated formula relating the rate of change of the quadrupole moment of the source to the energy flux at infinity [17]. In the fifties, Sachs and Bergmann extended the relation between the source multipoles and asymptotic fields [18]. However, in the framework of *exact* general relativity, progress has been slow. In the seventies, Geroch, Hansen and others [19] restricted themselves to the stationary context and introduced field multipoles by analyzing the asymptotic structure of suitable geometric fields constructed from the metric and equations they satisfy near spatial infinity. As in electrodynamics — and, in contrast to the situation in the Newtonian theory — they found that there are *two* sets of multipoles, the mass multipoles $\mathbb{M}_{(n)}$ and the angular momentum multipoles $\mathbb{J}_{(n)}$. In static situations, all the angular momentum multipoles $\mathbb{J}_{(n)}$ vanish and the mass multipoles $\mathbb{M}_{(n)}$ are constructed from the norm of the static Killing field which, like the Newtonian gravitational potential, satisfies a Laplace-type equation outside sources. In the (genuinely) stationary context, the $\mathbb{J}_{(n)}$ are non-zero and are analogous to the magnetic multipoles in the Maxwell theory. In the Newtonian theory, since the field multipoles are defined as coefficients in the $1/r$ expansion of the gravitational potential, knowing all multipoles one can trivially reconstruct the potential outside sources. In general relativity, there are considerable coordinate ambiguities in performing asymptotic expansions of the metric. Therefore, Geroch and Hansen were led to define their multipoles using other techniques. Nonetheless, Beig and Simon [20] have established that the knowledge of these multipoles does suffice to determine the space-time geometry near infinity. Their construction also shows that, if two stationary space-times have the same multipoles, they are isometric in a neighborhood of infinity.

In the Geroch-Hansen framework, one works on the 3-dimensional manifold of orbits of the stationary Killing field and multipole moments $\mathbb{M}_{(n)}$ and the $\mathbb{J}_{(n)}$ arise as symmetric, trace-free tensors in the tangent space of the point Λ at spatial infinity of this manifold. Now, the vector space of n-th rank, symmetric traceless tensors on $\mathbb{R}^3$ is naturally isomorphic with the vector space spanned by the linear combinations $\sum_{m=-n}^{m=n} a_{nm} Y_n^m(\theta, \phi)$ of spherical harmonics on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. Therefore, each $\mathbb{M}_{(n)}$ and $\mathbb{J}_{(n)}$ uniquely defines a set of numbers, M_{nm} and J_{nm} with $m \in \{-n, -n+1, \ldots, n-1, n\}$. Finally, let us consider stationary space-times which are also axisymmetric (e.g., the Kerr space-time). Then $M_{n,m} = M_n \delta_{m,0}$ and $J_{nm} = J_n \delta_{m,0}$, whence multipoles are completely characterized by two sets of numbers, M_n, J_n with $n = 0, 1, \ldots$. This fact will be useful to us because most of our analysis will be restricted to axisymmetric isolated horizons.

In stationary space-times, then, the situation with field multipoles is well developed. The status of source multipoles, on the other hand, has not been so satisfactory. Dixon developed a framework to define source multipoles [21] but the program did not reach the degree of maturity enjoyed by the field multipoles.

In this chapter, we will focus on the problem of defining the analogs of source multipoles for black holes in equilibrium. The isolated horizon framework provides a suitable quasi-local arena for describing such black holes [22, 23, 24, 25, 26]. Thus, our task is to introduce an appropriate definition of multipoles which capture distortions of the horizon geometry and of the distribution of angular momentum currents on the horizon itself, and explore properties of these multipoles. Can the multipoles provide a diffeomorphism invariant characterization of the horizon geometry? Can they suffice to determine space-time geometry in the neighborhood of the horizon in stationary space-times? These are attractive possibilities. But since the horizon lies in a genuinely strong field region, a priori it is not obvious that a useful notion of multipoles can be introduced at all. Indeed, since the problem of defining and analyzing multipoles has turned out to be so difficult already for relativistic fluids, at first glance it may seem hopelessly difficult for black holes. However, a tremendous simplification occurs because black holes are purely geometric objects; one does not have to resolve the messy and difficult issues related to the details of matter sources. We will see that this simplification makes it possible to carry out a detailed analysis and satisfactorily address the issues raised above.

We will restrict our detailed analysis to axisymmetric (or type II) isolated horizons, where the symmetry restriction applies only to the horizon geometry (and to the pull-back of the Maxwell field to the horizon) and not to the entire space-time. The material is organized as follows. In section 2.2, we recall the relevant facts about isolated horizons and axisymmetric geometric structures. In section 2.3 we define the multipoles M_n and J_n and show that these two sets of numbers provide a complete characterization of the isolated horizon geometry. For the specific case of Maxwell sources, we introduce another pair, Q_n, P_n of electromagnetic multipoles and show that they suffice to determine the pull-back of the Maxwell tensor as well as that of its dual. Finally, the initial value formulation based on two intersecting null surfaces [27] implies that, if there is a stationary Killing field in a neighborhood of the horizon, multipoles also suffice to determine the near horizon geometry [28]. In section 2.4, we reformulate the main ideas of the chapter in terms of tetrad and connection variables. Finally, in section 2.5, we summarize what we have done in defining multipoles for isolated horizons, and note the range of applications of the framework, the application to quantum gravity used in this thesis being one of them.

Unless otherwise stated, in this chapter all manifolds and fields will be assumed to be smooth.

2.2 Preliminaries

2.2.1 Isolated horizons

In this sub-section we briefly recall the relevant notions pertaining to isolated horizons [22, 24, 25, 26]. This discussion will also serve to fix our notation.

Let us begin with the basic definitions [24]. A *non-expanding horizon* Δ is a null, 3-dimensional sub-manifold of the 4-dimensional space-time $(\mathcal{M}, g_{ab})$, with topology $\mathbb{S}^2 \times \mathbb{R}$, such that:
i) the expansion θ_ℓ of its null normal ℓ vanishes; and,
ii) Field equations hold on Δ with stress energy, T_{ab}, satisfying the very weak requirement that $-T^a{}_b\ell^b$ is a future-directed, causal vector. (Throughout, ℓ^a will be assumed to be future pointing.)

Before discussing their consequences, let us note two facts about these assumptions: i) if the expansion vanishes for one null normal ℓ^a, it vanishes for all null normals $f\ell^a$; and, ii) the condition on the stress energy is satisfied by all the standard matter fields provided they are minimally coupled to gravity. (Non-minimally coupled matter can be incorporated by a small modification of this condition. See, e.g., [29].) Since Δ is a null 3-surface, its intrinsic 'metric' q_{ab} has signature 0,+,+. The definition ensures that the flux of the matter 4-momentum across Δ vanishes and $\mathcal{L}_\ell q_{ab} \,\widehat{=}\, 0$, *where* $\widehat{=}$ *denotes equality restricted to points of* Δ. In particular, the area of the 2-sphere cross-sections of the horizon is constant on Δ. We will denote this area by a_Δ. The definition also implies that the space-time derivative operator ∇ naturally induces a unique derivative operator D on Δ. Furthermore, $D_a l^b = \omega_a l^b$ for some globally defined 1-form ω_a on the horizon. This 1-form will play an important role. It is referred to as the "rotation 1-form," as it encodes information about rotation.

The pair (q_{ab}, D) is referred to as the intrinsic *geometry* of Δ. The notion that the black hole itself is in equilibrium is captured by requiring that this geometry is time independent:

An *isolated horizon* $(\Delta, [\ell])$ is a non-expanding horizon Δ equipped with an equivalence class $[\ell^a]$ of null normals ℓ^a, where $\ell^a \sim \ell'^a$ if and only if $\ell'^a = c\ell^a$ for a positive constant c, such that $[\mathcal{L}_\ell,\, D] \,\hat{=}\, 0$ on Δ.[1]

Since Δ is a null surface, given any one null normal ℓ^a in $[\ell]$, we have $\ell^a D_a \ell^b = (k)_\ell \ell^b$ for some $(k)_\ell$. (Thus, $(k)_\ell = \ell^a \omega_a$.) The requirement $[\mathcal{L}_\ell,\, D] \,\hat{=}\, 0$ further implies that $(k)_\ell$ is constant on Δ. $(k)_\ell$ is referred to as the *surface gravity* of Δ with respect to ℓ^a. Note that if $\ell'^a = c\ell^a$ then $(k)_{\ell'} = c(k)_\ell$. Thus, the *value* of surface gravity refers to a specific null normal; it is not a property of $(\Delta, [\ell])$. However, one *can* unambiguously say whether the given isolated horizon is *extremal* (i.e., has $(k)_\ell = 0$) or *non-extremal* (i.e., has $(k)_\ell \neq 0$).

The isolated horizon definition extracts from the notion of the Killing horizon just that 'tiny' part which turns out to be essential for black hole mechanics [24, 26] and, more generally, to capture the notion that the horizon is in equilibrium, allowing for dynamical processes and radiation in the exterior region [22]. Indeed, Einstein's equations admit solutions with isolated horizons in which there is radiation arbitrarily close to the horizons [30]. Finally, note that the definition uses conditions which are *local* to Δ. Hence, unlike event horizons one does not require the knowledge of full space-time; the notion is not 'teleological'.

Of particular interest to our analysis is the case when the only matter present *at* the horizon is a Maxwell field. In this case, only the *pull-backs* B_{ab} and E_{ab} to Δ, respectively of the Maxwell field F_{ab} and its dual ${}^\star F_{ab}$ are needed in the isolated horizon analysis. *We will refer to the quadruplet* $(q_{ab}, D, B_{ab}, E_{ab})$ *as the Einstein-Maxwell geometry of the isolated horizon.*

Remark: Note that the notion of an isolated horizon $(\Delta, [\ell])$, can be formulated intrinsically, using only those fields which define its geometry, without reference to the full space-time metric or curvature. Specifically: i) the condition ℓ^a is a null

[1] Given (Δ, q_{ab}, D), one can show that, generically, there is an unique equivalence class $[\ell]$ of null normals such that $(\Delta, [\ell])$ is an isolated horizon. However, our analysis will not be restricted to this case.

normal to Δ is captured in the property that q_{ab} has signature 0,+,+ with $q_{ab}\ell^b \,\widehat{=}\, 0$; ii) the condition $\theta_\ell \,\widehat{=}\, 0$ can be replaced by $\mathcal{L}_\ell q_{ab} \,\widehat{=}\, 0$; iii) the requirement on the stress-energy (in the definition of a non-expanding horizon) refers only to fields B_{ab} and E_{ab}; and iv) the condition $[\mathcal{L}_\ell,\, D] \,\widehat{=}\, 0$ refers only to D (and not to the full space-time connection). *This is why the quadruplet* $(q_{ab}, D, B_{ab}, E_{ab})$*, defined intrinsically on* Δ*, was singled out to introduce the notion of Einstein-Maxwell horizon geometry.* All equations that are required in the derivation of the laws of mechanics of isolated horizons in Einstein-Maxwell theory [24, 26] as well the main results of this chapter refer just to these fields.

Next, let us examine symmetry groups of isolated horizons. A *symmetry* of $(\Delta, \ell^a, q_{ab}, D, B_{ab}, E_{ab})$ is a diffeomorphism on Δ which preserves q_{ab}, D, B_{ab} and E_{ab} and at most rescales ℓ^a by a positive constant. It is clear that diffeomorphisms generated by ℓ^a are symmetries. So, the symmetry group G_Δ is at least 1-dimensional. The question is: Are there any other symmetries? At infinity, we generally have a universal symmetry group (such as the Poincaré or the anti-de Sitter) because all metrics under consideration approach a fixed metric (Minkowskian or anti-de Sitter) there. In the case of the isolated horizons, generically we are in the strong field regime and space-time metrics do not approach a universal metric. Therefore, the symmetry group is not universal. However, there are only three universality classes:

i) Type I: the isolated horizon geometry is spherical; in this case, G_Δ is four dimensional;

ii) Type II: the isolated horizon geometry is axisymmetric; in this case, G_Δ is two dimensional;

iii) Type III: the diffeomorphisms generated by ℓ^a are the only symmetries; G_Δ is one dimensional.

Note that these symmetries refer *only to* the horizon geometry. The full space-time metric need not admit any isometries even in a neighborhood of the horizon. Physically, type II horizons are the most interesting ones. They include the Kerr-Newman horizons as well as their generalizations incorporating distortions (due to exterior matter or other black holes) and hair. Our main results refer to the type II case.

Finally, one can use the field equations to isolate the *freely specifiable data* which determines the Einstein-Maxwell geometry of an isolated horizon [25, 31]. The analysis is naturally divided in to two cases: $(k)_\ell \neq 0$ and $(k)_\ell = 0$. Fix any 2-sphere cross-section $\tilde{\Delta}$ of Δ and denote by $\tilde{q}^a{}_b$ the natural projection operator on $\tilde{\Delta}$. Then, on any type II horizon, we have the following.[2]

- In the *non-extremal case*, the free data consists of the projections $\tilde{q}_{ab}, \tilde{\omega}_a, \tilde{B}_{ab}, \tilde{E}_{ab}$ to $\tilde{\Delta}$ (using $\tilde{q}^a{}_b$) of $q_{ab}, \omega_a, B_{ab}, E_{ab}$ on Δ. (Recall that ω_a is defined by $D_a \ell^b = \omega_a \ell^b$.) That is, given the free data on $\tilde{\Delta}$, using the projections of the field equations on $\tilde{\Delta}$, one can reconstruct the full Einstein-Maxwell geometry $(q_{ab}, D, B_{ab}, E_{ab})$ on Δ.

- In the *extremal* case, remarkably, the fields q_{ab}, ω_a, B_{ab} and E_{ab} turn out to be all *universal*; they are the same as those in the extremal Kerr-Newman geometry [31]! The free data consists of $(\tilde{S}_{ab})$, a symmetric, second rank tensor on $\tilde{\Delta}$, determined by D via $\tilde{S}_{ab} = \tilde{q}_a{}^c \tilde{q}_b{}^d D_a n_b$ where n_a is the covariant normal to $\tilde{\Delta}$ within Δ, satisfying $\ell^a n_a = -1$. (Thus, $\tilde{S}_{ab}$ is the analog of the 'extrinsic curvature' of $\tilde{\Delta}$ within Δ.)

These results will play an important role in section 3. Since, as summarized above, the free data for the non-extremal and extremal cases are so different, appropriate definitions of multipoles in the two cases would correspondingly have to be quite different. Because of this, and because it is the non-extremal case that is of astrophysical interest, throughout most of this chapter and all of the next, we will restrict ourselves to non-extremal isolated horizons.

Specifically, in section 3 of this chapter, for the non-extremal case, we will construct multipoles from the free data and show that they provide a diffeomorphism invariant characterization of the free data and hence, via field equations, of the horizon geometry.[3]

[2] Proofs of these assertions are sketched in Appendix A of the paper [15].

[3] This general philosophy is rather similar to that used by Janis and Newman to introduce a notion of multipoles using 'free data' on null surfaces [32]. However, since our analysis is restricted to isolated horizons rather than general null surfaces, our results are free of coordinate ambiguities.

2.2.2 Axisymmetric structures

In this sub-section we will recall a few facts about axisymmetric geometries on $\mathbb{S}^2$. While these results are usually just assumed in general relativity, we chose to include this material because we were not able to find a reference where the reasoning is spelled out. This discussion will also make our assumptions explicit.

Let $\mathbb{S}$ be a manifold with the topology of a 2-sphere, equipped with a metric $\tilde{q}_{ab}$. We will denote by $\tilde{\epsilon}_{ab}$ the alternating tensor on $\mathbb{S}$ compatible with $\tilde{q}_{ab}$, by a the area of $\mathbb{S}$ and by R its radius, defined by $a = 4\pi R^2$. We will say that $(\mathbb{S}, \tilde{q}_{ab})$ is *axisymmetric* if it admits a Killing field ϕ^a with closed orbits which vanishes *exactly* at two points of $\mathbb{S}$. The two points will be referred to as *poles*. We will now show that such metric manifolds carry invariantly defined coordinates and discuss properties of the metric coefficients in these coordinates.

Since $\mathcal{L}_\phi \tilde{\epsilon}_{ab} = 0$, there exists a unique (globally defined) function ζ on $\mathbb{S}$ such that

$$\tilde{D}_a \zeta = \frac{1}{R^2}\, \tilde{\epsilon}_{ba} \phi^b \quad \text{and} \quad \oint_{\mathbb{S}} \zeta \,\tilde{\epsilon} = 0\,. \tag{2.1}$$

It is clear that $\mathcal{L}_\phi \zeta \,\hat{=}\, 0$ on all of $\mathbb{S}$, while $\tilde{D}_a\zeta$ vanishes *only* at the poles. Hence, ζ is a monotonic function on the 1-dimensional manifold $\hat{\mathbb{S}}$ of orbits of ϕ^a. Now $\hat{\mathbb{S}}$ is a closed interval and the end points correspond to the two poles. Hence ζ monotonically increases from one pole to another. We will say that ζ assumes its minimum value at the *south pole* and the maximum at the *north pole*.

Next, let us introduce a vector field ζ^a on $\mathbb{S}' = (\mathbb{S} -$ poles) via:

$$\tilde{q}_{ab}\zeta^a \phi^b = 0 \quad \text{and} \quad \zeta^a \tilde{D}_a \zeta = 1 \tag{2.2}$$

Then it follows that

$$\zeta^a = \frac{R^4}{\mu^2}\, \tilde{q}^{ab}\, \tilde{D}_b \zeta\,, \tag{2.3}$$

where $\mu^2 = \tilde{q}_{ab}\phi^a\phi^b$ is the squared norm of ϕ^a. Hence integral curves of ζ^a go from the south pole to the north (and ζ^a diverges as one approaches the poles). Using ζ^a, we can define a preferred affine parameter ϕ of ϕ^a as follows: Fix any one integral curve I of ζ^a in $\mathbb{S}'$ and set $\phi = 0$ on I. Thus, on I, we have $\mathcal{L}_\zeta \phi = 0$. Now, since ϕ^a is a Killing field and ζ^a is constructed uniquely from $(\tilde{q}_{ab}, \phi^a)$, it follows that $\mathcal{L}_\phi \zeta^a = 0$. Hence, we conclude: $\mathcal{L}_\phi(\mathcal{L}_\zeta \phi) = 0$. Since $\mathcal{L}_\zeta \phi = 0$ on I, it now follows that ϕ = const on *every* orbit of ζ^a. This now implies that the affine parameter ϕ of ϕ^a has the same range on every orbit of ϕ^a.[4] Without loss of generality, using the rescaling freedom in ϕ^a, we will assume that ϕ^a is such that $\phi \in [0, 2\pi)$ on $\mathbb{S}$.

Thus, starting from geometry, we have constructed two coordinates ζ, ϕ on $\mathbb{S}$ such that $\phi^a \equiv \partial/\partial\phi$ and $\zeta^a \equiv \partial/\partial\zeta$ are orthogonal. Equations (2.1) and (2.3) now imply that the metric has the form:

$$\tilde{q}_{ab} = R^2(f^{-1} D_a\zeta\, D_b\zeta + f D_a\phi\, D_b\phi) \quad \text{and} \quad \tilde{q}^{ab} = \frac{1}{R^2}(f\zeta^a\zeta^b + f^{-1}\phi^a\phi^b) \tag{2.4}$$

where $f = \mu^2/R^2$. The fact that the area of $\mathbb{S}$ is $4\pi R^2$ now implies that the range of ζ is necessarily $[-1, 1]$. Conversely, given any 'coordinates' ζ', ϕ' in $[-1, 1] \times \mathbb{R}/2\pi\mathbb{Z}$ in which the metric can be expressed in (the primed version of) the form (2.4), it is easy to show that $\zeta' = \zeta$ and ϕ' is at most a rigid shift of ϕ.

Using the expression (2.4), one can also show that ζ increases from -1 at the south pole to $+1$ at the north pole uniformly in proportion to (physical) area. This will be used in section 3.3 to write down an explicit formula for ζ which we then quantize.

Functions ζ, ϕ serve as 'coordinates' on $\mathbb{S}$ modulo usual caveats: they are ill-defined at the poles and ϕ has a 2π discontinuity on the integral curve I of ζ^a. We have to ensure that the metric $\tilde{q}_{ab}$ is smooth in spite of these coordinate problems. The discontinuity at I causes no problems. However, poles do require a careful treatment because the norm μ of ϕ^a —and hence f— vanishes there. Smoothness of $\tilde{q}_{ab}$ at the poles (i.e. absence of conical singularities) imposes a non-trivial condition on f:

$$\lim_{\zeta \to \pm 1} f'(\zeta) = \mp 2 \tag{2.5}$$

where 'prime' denotes derivative with respect to ζ. On a metric 2-sphere, we have $f = 1 - \zeta^2$ and we can bring the metric to the standard form simply by setting $\zeta = \cos\theta$. In the general case, f has the same values and first derivatives at the poles as on a metric 2-sphere. Using l'Hopital's rule, one can show that this fact suffices to ensure that the metric (2.4) is smooth at the poles.

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[4]To make this argument precise, one should work on the universal covering of the orbits of ϕ^a on $\mathbb{S}'$ but the additional steps are straightforward.

Finally, we note a property of these axisymmetric metrics which will be useful in section 2.3.1. A simple calculation shows that the scalar curvature $\tilde{\mathcal{R}}$ of $\tilde{q}_{ab}$ is given by:

$$\tilde{\mathcal{R}}(\zeta,\phi) = -\frac{1}{R^2}\, f''(\zeta)\,. \tag{2.6}$$

By integrating it twice with respect to ζ *and using the boundary conditions* $(f|_{\zeta=-1}) = 0$ and $(f'|_{\zeta=-1}) = 2$, one can reconstruct the function f from the scalar curvature:

$$f = -R^2\left[\int_{-1}^{\zeta} \mathrm{d}\zeta_1 \int_{-1}^{\zeta_1} \mathrm{d}\zeta_2\, \tilde{\mathcal{R}}(\zeta_2)\right] \;+\; 2(\zeta+1) \tag{2.7}$$

Thus, thanks to the preferred coordinates admitted by an axisymmetric geometry on $\mathbb{S}$, given the area a of $\mathbb{S}$ and the scalar curvature $\tilde{\mathcal{R}}$, the metric $\tilde{q}_{ab}$ is completely determined.

Remark: Coordinates (ζ,ϕ), determined by the axisymmetry of $\tilde{q}_{ab}$, also enable us to define a *canonical* round, 2-sphere metric $\tilde{q}^{\circ}_{ab}$ on $\mathbb{S}$:

$$\tilde{q}^{\circ}_{ab} = R^2(f_{\circ}^{-1} D_a\zeta\, D_b\zeta + f_{\circ} D_a\phi\, D_b\phi) \tag{2.8}$$

where $f_{\circ} = 1-\zeta^2$. Note that $\tilde{q}^{\circ}_{ab}$ has the *same area element* as $\tilde{q}_{ab}$. This round metric captures the extra structure made available by axisymmetry in a coordinate invariant way. The availability of $\tilde{q}^{\circ}_{ab}$ enables one to perform a natural spherical harmonic decomposition on $\mathbb{S}$. This fact will play a key role throughout section 2.3.

2.3 Multipoles of type II isolated horizons

This section is divided into four parts. In the first two, we restrict ourselves to non-extremal, type II isolated horizons with no matter fields on them. We begin in the first part by defining a set of multipoles, I_n, L_n, starting from the horizon geometry. These two sets of numbers provide a convenient diffeomorphism invariant characterization of the horizon geometry, so that we refer to I_n, L_n as *geometric multipoles.* In the second part, we rescale these moments by appropriate dimensionful factors to obtain mass and angular momentum multipoles M_n, J_n. Finally, in the third part we discuss electromagnetic multipoles.

2.3.1 Geometric Multipoles

Let $(\Delta, [\ell])$ be a non-extremal, type II isolated horizon with an axial Killing field ϕ^a. In this sub-section, we will ignore matter fields on Δ and concentrate just on the horizon geometry defined by (q_{ab}, D). Fix a cross-section $\tilde{\Delta}$ of Δ. Then, as summarized in section 2.2.1, the free data that determine the horizon geometry consists of the pair $(\tilde{q}_{ab}, \tilde{\omega}_a)$ where $\tilde{q}_{ab}$ is the intrinsic metric on $\tilde{\Delta}$ and $\tilde{\omega}_a$ is the projection on $\tilde{\Delta}$ of the 1-form ω_a on Δ (defined by $D_a \ell^b = \omega_a \ell^b$). However, there is some gauge freedom associated with our choice of the cross-section $\tilde{\Delta}$ [25]. We will first spell it out and then define multipoles using gauge invariant fields.

For simplicity of presentation, let us fix a null normal ℓ^a in $[\ell^a]$. Then, $\tilde{\Delta}$ can be regarded as a leaf of a foliation $u = \text{const}$, where u is such that $\ell^a D_a u \,\widehat{=}\, 1$. For notational simplicity, we will set $n_a = -D_a u$ so that n_a is the covariant normal to the foliation satisfying $\ell^a n_a = -1$. The projection operator $\tilde{q}_a{}^b$ on the leaves of this foliation is given by $\tilde{q}_a{}^b = \delta_a^b + n_a \ell^b$. Hence, $\tilde{q}_{ab} = q_{ab}$ and $\tilde{\omega}_a = \omega_a + (k)_\ell n_a$ as tensor fields on Δ. Since $\mathcal{L}_\ell q_{ab} \,\widehat{=}\, 0$ and $\mathcal{L}_\ell \omega_a \,\widehat{=}\, 0$ on any isolated horizon, and since $\mathcal{L}_\ell n_a \,\widehat{=}\, 0$ from definition of n_a, it follows that $(\tilde{q}_{ab}, \tilde{\omega}_a)$ on any one leaf is mapped to that on any other leaf under the natural diffeomorphism (generated by ℓ^a) relating them. Let us now consider a cross-section $\tilde{\Delta}'$ which does *not* belong to this foliation. Let $u' = \text{const}$ denote the corresponding foliation. Set $f = u - u'$. Then, regarded as tensor fields on Δ the two sets of free data are related by

$$\tilde{q}'_{ab} = \tilde{q}_{ab} \quad \text{and} \quad \tilde{\omega}'_a = \tilde{\omega}_a + (k)_\ell \, D_a f \tag{2.9}$$

Thus, under the natural diffeomorphism (defined by the flow of ℓ^a) between $\tilde{\Delta}$ and $\tilde{\Delta}'$, $\tilde{q}_{ab}$ is mapped to $\tilde{q}'_{ab}$ but $\tilde{\omega}_a$ is *not* mapped to $\tilde{\omega}'_a$; the difference is a gradient of a function. This is the gauge freedom in the free data.

It is therefore natural to consider, in place of $\tilde{\omega}_a$, its curl. From the isolated horizon framework, it is known that for any null tetrad $\ell^a, n^a, m^a, \bar{m}^a$ such that $\ell^a \in [\ell]$, the Weyl components Ψ_0 and Ψ_1 vanish on Δ whence Ψ_2 is gauge invariant [24, 25], and the curl of $\tilde{\omega}_a$ is given just by $\text{Im}\Psi_2$:

$$D_{[a}\tilde{\omega}_{b]} = \text{Im}\Psi_2 \, \epsilon_{ab} \tag{2.10}$$

where ϵ_{ab} is the natural area element on Δ (which satisfies $\epsilon_{ab}\ell^a \,\hat{=}\, 0$ and $\mathcal{L}_\ell \epsilon_{ab} \,\hat{=}\, 0$). Thus, the gauge invariant content of $\tilde{\omega}_a$ is coded in $\mathrm{Im}\Psi_2$.

The second piece of free data is the metric $\tilde{q}_{ab}$ on $\tilde{\Delta}$. In section 2.2.2 we showed that using an invariant coordinate system (ζ, ϕ), one can completely determine $\tilde{q}_{ab}$ in terms of a number, the area a, and a function, its scalar curvature $\tilde{\mathcal{R}}$. If, as assumed in this sub-section, the cosmological constant is zero and there are no matter fields on Δ, then $\tilde{\mathcal{R}} = -4\mathrm{Re}\Psi_2$ [23]. Hence the gauge invariant part of the free data that determines the horizon geometry is neatly coded in the Weyl component Ψ_2.

It is therefore natural to define multipoles using a complex function Φ_Δ on Δ:

$$\Phi_\Delta := \frac{1}{4}\tilde{\mathcal{R}} - i\mathrm{Im}\Psi_2\,. \tag{2.11}$$

(Thus, in absence of matter on Δ, $\Phi_\Delta = -\Psi_2$ while in presence of matter it is given by $\Phi_\Delta \,\hat{=}\, -\Psi_2 + (1/4)R_{ab}\tilde{q}^{ab} - (1/12)R$, where R_{ab} is the Ricci tensor and R the scalar curvature of the 4-metric at the horizon.) Since all fields are axisymmetric, using the natural coordinate ζ on $\tilde{\Delta}$, *we are led to define multipoles as*:

$$I_n + iL_n := \oint_{\tilde{\Delta}} \Phi_\Delta\, Y_n^0(\zeta)\ \mathrm{d}^2\tilde{V} \tag{2.12}$$

or,

$$I_n := \frac{1}{4}\oint_{\tilde{\Delta}} \tilde{\mathcal{R}}\, Y_n^0(\zeta)\,\mathrm{d}^2\tilde{V} \quad \text{and} \quad L_n := -\oint_{\tilde{\Delta}} \mathrm{Im}\Psi_2\, Y_n^0(\zeta)\ \mathrm{d}^2\tilde{V}.$$

Here Y_n^0 are the $m=0$ spherical harmonics, subject to the standard normalization:

$$\oint_{\tilde{\Delta}} Y_n^0 Y_m^0\,\mathrm{d}^2\tilde{V} = R_\Delta^2\,\delta_{n,m}\,, \tag{2.13}$$

where R_Δ is the horizon radius defined through its area a_Δ via $a_\Delta = 4\pi R_\Delta^2$. Thus, given any horizon geometry, we can define two sets of numbers, I_n and L_n. Now, from (2.1) and (2.10), ζ and $\mathrm{Im}\Psi_2$ are completely determined, in a covariant manner, by the metric $\tilde{q}_{ab}$, the rotation one-form $\tilde{\omega}_a$, and the rotational Killing field ϕ^a. Therefore, it is immediate that if (Δ, q_{ab}, D) and (Δ', q'_{ab}, D') are related by a diffeomorphism, we have $I_n = I'_n$ and $L_n = L'_n$; the two sets of numbers are diffeomorphism invariant. If the isolated horizon were of type I, q_{ab} would be spherically symmetric and $\mathrm{Im}\Psi_2$ would vanish [24]. Then, the only non-zero multipole would be I_0 which, by the Gauss-Bonnet theorem, has a universal value ($\sqrt{\pi}$; see below). Given a generic type II horizon, as we saw in section 2.2.2, the axisymmetric structure provides a canonical 2-sphere metric q°_{ab} (which, in the type I case, coincides with the physical metric). The physical geometry has distortion and rotation built in it. The round metric q°_{ab} serves as an invariantly defined 'background' against which one can measure distortions and rotations. Multipoles I_n, L_n provide a diffeomorphism invariant characterization of these. More precisely, they encode the difference between the physical horizon geometry (q_{ab}, D) and the fiducial, type I geometry determined by $(\tilde{q}^{\circ}_{ab}, \tilde{\omega}_a = 0)$.

Finally, note that I_n and L_n cannot be specified entirely freely but are subject to certain algebraic constraints. The first comes from the Gauss-Bonnet theorem which, in the axisymmetric case, follows from the boundary condition (2.5) on f' and the expression (2.6) of the scalar curvature in terms of f:

$$I_0 = \frac{1}{4} \oint_{\tilde{\Delta}} \tilde{\mathcal{R}}\, Y_0^0(\zeta)\, \mathrm{d}^2\tilde{V} = \sqrt{\pi}\,. \tag{2.14}$$

The second comes directly from the relation (2.10) between $\mathrm{Im}\Psi_2$ and curl of $\tilde{\omega}_a$:

$$L_0 = -\frac{1}{\sqrt{4\pi}} \oint_{\tilde{\Delta}} \mathrm{Im}\Psi_2\, \mathrm{d}^2\tilde{V} = 0 \tag{2.15}$$

The third constraint comes again from (2.5) and (2.6):

$$I_1 := \frac{\sqrt{3}}{8\sqrt{\pi}} \oint_{\tilde{\Delta}} \tilde{\mathcal{R}}\, \zeta\, \mathrm{d}^2\tilde{V} = 0 \tag{2.16}$$

We will show in section 2.3.2 that (2.15) implies that, as one would physically expect, the 'angular momentum monopole' necessarily vanishes and (2.16) implies that the mass dipole vanishes, i.e., that our framework has automatically placed us in the 'center of mass frame of the horizon'. Next, because Φ_Δ is smooth, these moments have a certain fall-off. Let us assume that Φ_Δ is C^k (i.e., the space-time metric is C^{k+2}). Then as n tends to infinity, I_n and L_n must fall off in such a way that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{k+1} |n^m (I_n - iL_n)|^2 < \infty \tag{2.17}$$

Finally, there is a constraint arising from the fact that f is non-negative and vanishes only at the poles. (This property of f is essential for regularity of the metric.) Using (2.6) and the definition (2.12) of I_n, one can express f in terms of I_n. Unfortunately, the resulting restriction on multipoles is quite complicated:

$$\begin{aligned} f(\zeta) = 1 - \zeta^2 \ &+ \ \sum_{n=2}^{\infty} \frac{2}{\sqrt{\pi(2n+1)}} \big[- \frac{1}{2n+3} P_{n+2}(\zeta) \\ &+ \ \frac{2(2n+1)}{(2n+3)(2n-1)} P_n(\zeta) - \frac{1}{2n-1} P_{n-2}(\zeta) \big] I_n \ \geq 0 \end{aligned} \tag{2.18}$$

and vanishes only if $\zeta = \pm 1$, where P_n are the Legendre polynomials.

Given a set of multipoles $\{I_n, L_n\}$ satisfying (2.14)-(2.18), a horizon area a_Δ and a surface gravity $\kappa_\ell \neq 0$, through explicit construction, one can show there exists a horizon geometry (q, D) with the chosen multipoles, horizon area and surface gravity. Furthermore, this horizon geometry (q, D) is unique up to diffeomorphisms. We refer the reader to the paper [15] for proofs of these assertions.

Remark: We conclude by noting some simplifications that occur in the presence of additional symmetries. Certain space-time metrics, such as the Kerr solutions have a *discrete (spatial) reflection symmetry*, $(\zeta, \phi) \mapsto (-\zeta, \phi + \pi)$, under which $\Psi_2 \mapsto \Psi_2^\star$. Therefore, in the isolated horizon framework it is interesting to consider the case in which $\Phi_\Delta \mapsto \Phi_\Delta^\star$ and $\epsilon_{ab} \mapsto \epsilon_{ab}$ under the discrete diffeomorphism $\zeta \mapsto -\zeta$ on Δ. Then, since Y_n^0 are even/odd under reflections if n is even/odd, it follows that $I_n = 0$ for all odd n and $L_n = 0$ for all even n. Next, consider the case in which the isolated horizon is a Killing horizon of a *static* Killing field. Then, one can show that $\mathrm{Im}\Psi_2 \hat{=} 0$ [26]. Hence $L_n = 0$ for all n but in general I_n can be arbitrary, capturing possible distortions in the horizon geometry. Finally, consider *type I isolated horizons* on which the horizon geometry is spherically symmetric. Then, Φ_Δ and ϵ_{ab} are spherically symmetric. It is then obvious from properties of spherical harmonics $Y_n^0(\zeta)$ that for all $n > 0$, we have $I_n = L_n = 0$. Since L_0

always vanishes, in this case the only non-trivial multipole is $I_0 = \sqrt{\pi}$.

2.3.2 Mass and angular Momentum multipoles

As is obvious from their definition, I_n, L_n are all dimensionless. Therefore, it is difficult to attribute direct physical interpretation to them. In this sub-section we will argue that they can be rescaled in a natural fashion to obtain quantities which can be interpreted as mass and angular momentum multipoles M_n and J_n.

In the isolated horizon framework, the area a_Δ is defined geometrically. One then defines the horizon angular momentum J_Δ as a surface term in the expression of the generator of rotations, evaluated on a 2-sphere cross-section of the horizon. J_Δ is unambiguous because type II horizons come with an axial symmetry [26]. The horizon mass M_Δ is also defined using Hamiltonian methods as the generator of a preferred time translation. However, the preferred time translation varies from space-time to space-time. If $J_\Delta = 0$, the time-translation points along ℓ^a; if not, it is a suitable linear combination of ℓ^a and ϕ^a, which can be fixed only *after* one knows the value of J_Δ [26]. Thus, J_Δ is defined first before one can fix M_Δ. In the same spirit, we will first define the angular momentum multipoles J_n and *then* the mass multipoles M_n.

We begin by recalling a general fact about angular momentum. Fix a space-time $(\mathcal{M}, g_{ab})$ and a space-like 2-sphere S in it. Let φ be any vector field tangential to S. Then, by regarding S as the inner boundary of a partial Cauchy surface M, one can use the Hamiltonian framework to define a 'conserved' quantity J_S^φ

$$J_S^\varphi = -\frac{1}{8\pi G} \oint_S K_{ab} \varphi^a dS^b \tag{2.19}$$

where K_{ab} is the extrinsic curvature of M. In a general space-time, this quantity is independent of M if and only if φ is divergence free with respect to the natural area element of S. Thus, for each divergence-free φ on S, J_S^φ depends only on the 4-metric g_{ab} and the choice of S. J_S^φ can then be interpreted as the φ^a-component of a 'generalized angular momentum' associated with S. If S happens to be a cross-section of Δ, as one would expect, one can recast this expression in terms of the fields defined by the isolated horizon geometry, making no reference at all to the partial Cauchy surface M [26]:

$$J_S^{\varphi} \mathrel{\hat{=}} -\frac{1}{8\pi G}\oint_S \varphi^a \tilde{\omega}_a \, d^2V \mathrel{\hat{=}} -\frac{1}{4\pi G}\oint_S f\,[\mathrm{Im}\Psi_2]\, \mathrm{d}^2V\,. \tag{2.20}$$

Here f is a 'potential' for φ^a on Δ —given by $\varphi^a = \epsilon^{ab}D_b f$— which exists because $\mathcal{L}_\varphi \epsilon_{ab} \mathrel{\hat{=}} 0$. By the isolated horizon boundary conditions it follows that if φ^a is the restriction to S of a vector field on Δ satisfying $\mathcal{L}_\ell\, \varphi^a \mathrel{\hat{=}} 0$, then J_S^φ is independent of the 2-sphere cross-section S used in (2.20).

Thus, on any isolated horizon there is a well-defined, gauge invariant notion of a 'generalized angular momentum' J_S^φ, associated with any divergence free vector field φ^a satisfying $\mathcal{L}_\ell \varphi^a \mathrel{\hat{=}} 0$. $\mathrm{Im}\Psi_2$ plays the role of the 'angular momentum aspect'. Hence, it is natural to construct the angular momentum multipoles J_n by rescaling the L_n with appropriate dimensionful factors. This strategy is supported also by other considerations. First, since $\mathrm{Im}\Psi_2$ transforms as a pseudo-scalar under spatial reflections, we will automatically satisfy the criterion that the angular momentum multipoles should transform as pseudo tensors. Second, all angular momentum multipoles would vanish if and only if $\mathrm{Im}\Psi_2 \mathrel{\hat{=}} 0$ and this is precisely the condition defining non-rotating isolated horizons [24, 26]. Thus, the strategy has an overall coherence.

To obtain the precise expression, let us first recall the situation in magnetostatics in flat space-time. If the current distribution j^a is axisymmetric, the nth magnetic moment $\mathbf{M}_n$ is given by:

$$\mathbf{M}_n = \int r^n P_n(\cos\theta)\; \vec{\nabla}\cdot(\vec{x}\times\vec{j})\, d^3x\,, \tag{2.21}$$

where P_n are the Legendre polynomials. If the current distribution is concentrated on the sphere S defined by $r = R$, the expression simplifies to:

$$\mathbf{M}_n = R^{n+1}\oint_S (\tilde{\epsilon}^{ab}\tilde{D}_b P_n(\cos\theta))\; \tilde{j}_a\, d^2V\,, \tag{2.22}$$

where $\tilde{\epsilon}_{ab}$ is the alternating tensor on the $r = R$ 2-sphere and $\tilde{j}_b$ is the projection of j_b on this 2-sphere. Note that this expression refers just to the axisymmetric structure on the 2-sphere S and not to the flat space in which it is embedded.

Comparison of (2.20) with (2.22) suggests that we can think of the horizon slice S as being endowed with a surface 'current density'

$$(\tilde{j}_\Delta)_a = -\frac{1}{8\pi G}\,\tilde{\omega}_a \tag{2.23}$$

and define the angular momentum (or 'current') moments as:

$$\begin{aligned} J_n &= -\frac{R_\Delta^{n+1}}{8\pi G}\oint_S (\tilde{\epsilon}^{ab}\tilde{D}_b P_n(\cos\theta))\,\tilde{\omega}_a\, d^2V \\ &= -\sqrt{\frac{4\pi}{2n+1}}\,\frac{R_\Delta^{n+1}}{4\pi G}\oint_S Y_n^0(\zeta)\mathrm{Im}\Psi_2\, d^2V \\ &= \sqrt{\frac{4\pi}{2n+1}}\,\frac{R_\Delta^{n+1}}{4\pi G}\,L_n \end{aligned} \tag{2.24}$$

Let us now turn to the mass multipoles, M_n. When all J_n vanish, we should be left just with M_n. These are then to be obtained by rescaling the multipole moments I_n by appropriate dimensionful factors. In electrostatics, when the charge density is axisymmetric, the electric multipoles are defined by

$$\mathbf{E}_n = \int r^n P_n(\cos\theta)\,\rho\, d^3x\,. \tag{2.25}$$

When the charge is concentrated on the sphere S defined by $r = R$, the expression simplifies to:

$$\mathbf{E}_n = R^n \oint_S P_n(\cos\theta)\tilde{\rho}\, d^2V \tag{2.26}$$

where $\tilde{\rho}$ is the surface charge density. Again, the final expression refers only to the axisymmetric structure on the 2-sphere S and not to the flat space in which it is embedded. Hence we can take it over to type II horizons. What we need is a notion of a 'surface mass density'. Now, Hamiltonian methods have provided a precise definition of mass M_Δ of type II isolated horizons in the Einstein-Maxwell theory [24, 26]. The structure of geometric multipoles I_n now suggests that we regard M_Δ as being 'spread out' on the horizon, the 'surface density' $\tilde{\rho}_\Delta$ being uniformly distributed in the spherical case but unevenly distributed if the horizon is distorted. It is then natural to set[5]

$$\tilde{\rho}_\Delta = \frac{1}{8\pi}\, M_\Delta \tilde{\mathcal{R}} = -\frac{1}{2\pi}\, M_\Delta \mathrm{Re}\Psi_2\,. \tag{2.27}$$

This heuristic picture motivates the following definitions:

$$\begin{aligned} M_n \;&:=\; -\sqrt{\frac{4\pi}{2n+1}}\,\frac{M_\Delta R_\Delta^n}{2\pi}\oint_S Y_n^0(\zeta)\,\mathrm{Re}\Psi_2\, d^2V \\ &=\; \sqrt{\frac{4\pi}{2n+1}}\,\frac{M_\Delta R_\Delta^n}{2\pi}\, I_n\,. \end{aligned} \tag{2.28}$$

Here M_Δ is the isolated horizon mass, which is determined by the horizon radius R_Δ and angular momentum J_1 via:

$$M_\Delta = \frac{1}{2GR_\Delta}\sqrt{R_\Delta^4 + 4G^2 J_1}\,, \tag{2.29}$$

Since our definitions are based on analogies with source multipoles in the Maxwell theory, an important question is whether they have the physical properties we expect in general relativity. We have:

- As discussed in section 2.3.1, the geometrical multipoles L_0 and I_1 vanish. Hence it follows that the angular momentum monopole moment J_0 vanishes as one would expect on physical grounds, and the mass dipole moment M_1 vanishes implying that we are in the center of mass frame.

- By construction, the mass monopole M_0 agrees with the horizon mass M_Δ and, by inspection, the angular momentum dipole moment J_1 equals the horizon angular momentum J_Δ, calculated through Hamiltonian analysis [26].

- Restrictions imposed by symmetries on the horizon geometry follow immediately from the remark at the end of section 2.3.1:
  i) if the horizon geometry is such that $\Psi_2 \mapsto \Psi_2^*$ under the reflection $\zeta \mapsto -\zeta$, then $M_n = 0$ if n is odd and $J_n = 0$ if n is even. This is in particular the case for the Kerr family of isolated horizons.

[5]Note incidentally that the smaller the principal radii of curvature of the intrinsic geometry, the higher is $\tilde{\rho}_\Delta$. Thus, the situation has a qualitative similarity with the way charge is distributed on the surface of a conductor.

ii) If $\mathrm{Im}\Psi_2 \hat{=} 0$, then all angular momentum multipoles vanish. This is in particular the case if Δ is a Killing horizon in a static space-time.
iii) If the horizon geometry is spherically symmetric, $M_n = 0$ for all $n > 0$ and $J_n = 0$ for all n.

- Our multipoles M_n, J_n are constructed just from the knowledge of the horizon geometry; knowledge of the space-time metric in the exterior region is not required. In particular, there may well be matter sources outside the horizon, responsible for its distortion and the exterior geometry need not even be stationary or asymptotically flat. Even when the exterior *is* stationary and asymptotically flat, there is no a priori reason to expect that these 'source multipoles' would agree with the 'field multipoles' defined at infinity —unless symmetry principles are involved— because the gravitational field outside the horizon would also act as a source, contributing to the 'total' moments at infinity.

- As noted in the introduction to this chapter, in vacuum, stationary space-times, Hansen's field moments at infinity suffice to determine the geometry in the neighborhood of infinity. Is there an analogous result for the horizon multipoles? The answer turns out to be affirmative. Fix a cross section S of Δ and consider the future directed, inward pointing null vector field n^a which is orthogonal to S, normalized so that $\ell^a n_a = -1$. The null geodesics originating on S with tangent vector $-n^a$ generate a null surface $\mathcal{N}$. In the source-free Einstein theory, there is a well-defined initial value problem based on the double null surfaces, Δ and $\mathcal{N}$ [27]. The freely specifiable data consists of the pair $(\tilde{q}_{ab}, \tilde{\omega}_a)$ on Δ and the Newman-Penrose component Ψ_4 of the Weyl tensor on $\mathcal{N}$ [22].

  Now suppose that the space-time is analytic near Δ and admits a stationary Killing field t^a in a neighborhood of Δ which is time-like in the exterior region and becomes null on Δ. Then, as one might imagine, the initial value problem becomes highly constrained: Ψ_4 on $\mathcal{N}$ is determined by the horizon geometry [28]. Hence, the horizon multipoles suffice to determine the solution to Einstein's equation in a past neighborhood of $\Delta \cup \mathcal{N}$. Thus, qualitatively the result is the same as with the field multipoles.[6]

To summarize, the mass and angular momentum multipoles M_n, J_n have physically expected properties. This in turn strengthens the heuristic picture we used to fix the dimensionful rescalings of I_n, L_n. We first considered stationary, axisymmetric charge and current distributions with support on a 2-sphere in the Maxwell theory and expressed the electric and magnetic multipoles using only the axisymmetric structure on the 2-spheres without reference to Minkowski spacetime. We then noted that these structures are available also on type II horizons.[7] Additional structures made available on these horizons by geometric and Hamiltonian methods then led us to our definitions of M_n, J_n. The physical picture is that observers in the exterior region (between the horizon and infinity) can regard the horizon multipoles as arising from an effective (but fictitious) mass density $\tilde{\rho}_\Delta = -(1/2\pi)M_\Delta \mathrm{Re}\Psi_2$ and a current density $(\tilde{j}_\Delta)_a = -(1/8\pi G)\tilde{\omega}_a$ on Δ. This picture may be useful in physical applications.

We complete this subsection with explicit formulae for the multipoles in the Kerr case, to satisfy the curiosity of the reader:

$$\begin{aligned}
M_{2s} &= (-1)^s\sqrt{2\pi}\left(\prod_{m=0}^{2s}(2m+1)\right)\frac{M^2 R_\Delta^{2s}}{\sqrt{aR_\Delta}}P_{-\frac{5}{2}}^{-2s-\frac{1}{2}}\left(\frac{r_+}{R_\Delta}\right), && (2.30)\\
M_{2s+1} &= 0, && (2.31)\\
J_{2s} &= 0, && (2.32)\\
J_{2s+1} &= \frac{1}{2G}(-1)^s\sqrt{2\pi}\left(\prod_{m=0}^{2s+1}(2m+1)\right)\frac{M R_\Delta^{2s+2}}{\sqrt{aR_\Delta}}P_{-\frac{5}{2}}^{-2s-\frac{3}{2}}\left(\frac{r_+}{R_\Delta}\right), && (2.33)
\end{aligned}$$

where P_ν^μ denotes the appropriate associated Legendre function of the second kind (using the conventions of [33]).

[6] However, while the domain in which the solution is determined by field multipoles is known to be large [20], the available results on the double-null initial value problem only ensure the existence of the solution in a small, past neighborhood of $\Delta \cup \mathcal{N}$. But it is quite possible that the double-null initial value results can be significantly strengthened.

[7] Note that the spherical harmonics $Y_n^0(\zeta)$ used in the definition of geometric multipoles I_n, L_n refer to the unique round metric $\tilde{q}^\circ_{ab}$ defined by coordinates (ζ, ϕ) (see (2.8)). They are eigenstates of the Laplacian defined by $\tilde{q}^\circ_{ab}$ and not of the physical metric $\tilde{q}_{ab}$. Therefore, the extension of the Maxwell formulas is natural. As noted in section 2.2.2, $\tilde{q}^\circ_{ab}$ is *uniquely* determined by the axisymmetry of $\tilde{q}_{ab}$ and the two metrics have the *same area element*. Therefore, $\tilde{\rho}_\Delta$ and $(\tilde{j}_\Delta)_a$ can be interpreted as the mass and current densities in terms of *either* metric.

2.3.3 Extension: inclusion of Maxwell fields

Let us continue to restrict ourselves to the non-extremal case but allow Maxwell fields on Δ. Then the Einstein-Maxwell horizon geometry consists of the quadruplet $(q_{ab}, D, B_{ab}, E_{ab})$. The presence of matter fields on Δ cause a few minor modifications in the discussion of gravitational multipoles. We will first discuss these and then turn to the electromagnetic multipoles.

In the gravitational case, the definition of the basic complex field Φ_Δ on Δ continues to be the same in terms of $\tilde{\mathcal{R}}$ and $\mathrm{Im}\Psi_2$ but changes if we use $\mathrm{Re}\Psi_2$ in place of $\tilde{\mathcal{R}}$:

$$\Phi_\Delta := \frac{1}{4}\tilde{\mathcal{R}} - i\mathrm{Im}\Psi_2 = -\Psi_2 + \frac{1}{4}R_{ab}\tilde{q}^{ab}. \tag{2.34}$$

where the term involving the space-time Ricci tensor can be expressed in terms of B_{ab} and E_{ab} as:

$$R_{ab}\tilde{q}^{ab} = G\left(\tilde{B}_{ab}\tilde{B}^{ab} + \tilde{E}_{ab}\tilde{E}^{ab}\right). \tag{2.35}$$

The definition of mass M_Δ changes from (2.29) to

$$M_\Delta = \frac{\sqrt{(R_\Delta^2 + GQ_\Delta^2)^2 + 4G^2J_\Delta^2}}{2GR_\Delta}. \tag{2.36}$$

The subsequent definition of the geometric and physical multipole moments is the same as in section 2.3.1. The reconstruction of the free data from multipoles is also unaffected but there is an additional term involving B_{ab} and E_{ab} in the reconstruction of D from the free data (see Appendix A in [15]). Finally, there is a minor modification in the list of properties of M_n, J_n listed in section 2.3.2. This arises because, in presence of a Maxwell field on the horizon, the canonical angular momentum J_Δ obtained by Hamiltonian methods contains two terms, a gravitational one and an electromagnetic one [26]. The angular momentum dipole moment J_1 yields just 'the gravitational part' of J_Δ. There does not seem to be a natural generalization of the definition of angular momentum multipoles such that J_1 would agree with the full J_Δ.

Let us now turn to the electromagnetic fields. As noted in section 2.2.1, the electromagnetic free data consists of the projections $\tilde{B}_{ab}$ and $\tilde{E}_{ab}$ of B_{ab} and E_{ab} on a cross-section $\tilde{\Delta}$. Therefore, it is natural to define the electromagnetic counterpart of Φ_Δ:

$$\Phi_\Delta^{\rm EM} := -\frac{1}{4}\,\tilde\epsilon^{ab}\,[\tilde E_{ab} + i\tilde B_{ab}] \tag{2.37}$$

(which is just the Newman-Penrose component ϕ_1 of the Maxwell field), and define multipoles via:

$$\begin{aligned} Q_n &= \frac{R^n}{\sqrt{4\pi(2n+1)}}\oint_{\tilde\Delta} {\rm Re}(\Phi_\Delta^{\rm EM})\, Y_n^0(\zeta)\, {\rm d}^2\tilde V \\ P_n &= \frac{R^n}{\sqrt{4\pi(2n+1)}}\oint_{\tilde\Delta} {\rm Im}(\Phi_\Delta^{\rm EM})\, Y_n^0(\zeta)\, {\rm d}^2\tilde V\,. \end{aligned} \tag{2.38}$$

One can check that Q_0 and P_0 are the electric and magnetic charges of the horizon. Thus, heuristically ${\rm Re}(\Phi_\Delta^{\rm EM})/4\pi$ and ${\rm Im}(\Phi_\Delta^{\rm EM})/4\pi$ may be thought of as 'surface charge densities' on the horizon and charge multipoles capture the non-uniformity in the distributions of electric and magnetic charge densities.

Remark: If non-electromagnetic sources are also present, we can still define the gravitational (and electromagnetic) multipoles as above but the multipoles for other sources have to be defined case by case. Gravitational multipoles again determine the 'free data' for the horizon geometry (q_{ab}, D). However, to reconstruct the horizon geometry from this data, one needs to know those matter fields which determine the space-time scalar curvature R and the component $\tilde q^{ab}R_{ab}$ of the Ricci tensor, evaluated at the horizon.

2.4 Tetrads and connections

The presentation up until this point is logically complete. However, now we wish to rewrite the results of this chapter in terms of variables which will be of direct use in the next chapter. Specifically, for quantization, connection variables are necessary. In this section we will recast the constructions of this chapter, replacing the metric by a tetrad and connection. Throughout this presentation, we assume no matter is present at the horizon. This is both for simplicity and because this is the primary case considered in the next chapter.

2.4.1 Foliation of Δ and an adapted tetrad

Consider a spacetime $(\mathcal{M}, g_{ab})$ containing an isolated horizon Δ. First, we recall from [25] that there is a unique preferred foliation of Δ by 2-spheres, defined by the condition that on each leaf $\tilde{q}^{ab}\mathcal{D}_a\tilde{\omega}_b = 0$, where $\tilde{q}_{ab}$, $\tilde{\omega}_a$ denote the pull-back of q_{ab} and the rotation one-form ω_a to the leaf. The leaves of this preferred foliation are known as "good cuts".

We use these "good cuts" to define the notion of a null tetrad adapted to the horizon. First, recall that a (Newman-Penrose) null tetrad $(\ell^a, n^a, m^a, \overline{m}^a)$ is a quadruplet of vectors such that ℓ^a and n^a are real, m^a is complex, all of the vectors are null, and the only non-zero inner products between the vectors are $\ell^a n_a = -1$ and $m^a \overline{m}_a = 1$. When $(\ell^a, n^a, m^a, \overline{m}^a)$ is a null tetrad on a spacetime $(\mathcal{M}, g_{ab})$ with isolated horizon inner boundary Δ, we say that the tetrad is *adapted to the horizon* if ℓ^a belongs to the equivalence class $[\ell^a]$ of null normals on Δ, and n^a is orthogonal to the leaves of the preferred foliation of Δ. It follows that for such adapted tetrads, m^a is tangent to the preferred leaves.

Furthermore, each such adapted null tetrad admits a coordinate v on Δ such that $n_{\underleftarrow{a}} = -\partial_a v$. Let us show this. Let a null tetrad $(\ell^a, n^a, m^a, \overline{m}^a)$ adapted to the horizon be given, so that $\ell^a \in [\ell^a]$ and n_a is normal to the leaves of the preferred foliation. Define a coordinate v on Δ by setting v equal to an arbitrary constant v_o on some leaf S_o of the preferred foliation, and then "evolve" v to the rest of Δ via the equation $\ell^a\partial_a v = 1$. Using the fact that ℓ^a is a symmetry of both q_{ab} and ω_a one can show that the resulting coordinate v will be, in fact, constant on *all* leaves of the preferred foliation. It follows that $-\partial_a v$ is normal to the preferred slices; furthermore by construction $\ell^a(-\partial_a v) = -1$. But these two conditions alone suffice to determine the one-form $-\partial_a v$; as $n_{\underleftarrow{a}}$ also satisfies these two conditions, it follows $n_{\underleftarrow{a}} = -\partial_a v$.

Given a space-time geometry with isolated horizon Δ, the freedom in the choice of an adapted tetrad is characterized by $(\ell^a, n^a, m^a, \overline{m}^a) \mapsto (c\ell^a, c^{-1}n^a, e^{i\theta}m^a, e^{-i\theta}\overline{m}^a)$, $c \in \mathbb{R}^+$, $\theta \in \mathbb{R}$. Furthermore, for any fixed tetrad, the coordinate v is unique up to addition of a constant.

Finally, we introduce some conventions. Suppose S is a leaf of the preferred foliation of Δ. Then T^*S will be identified with the subset of $T^*\Delta|_S$ orthogonal to ℓ^a, and $T^*\Delta$, in turn, will be identified with the subset of $T^*\mathcal{M}|_\Delta$ orthogonal to n^a.

With a slight abuse of terminology, we will sometimes refer to elements of T^*S as being "tangent to S", and elements of $T^*\Delta$, "tangent to Δ." A single arrow below a space-time tensor will denote its pull-back to Δ, while a double underarrow will denote its pull-back to a given S.

2.4.2 The horizon data $(m^a, \tilde{V}_a)$

In this subsection we introduce variables at the horizon, to be referred to as the "horizon data". These will be needed in the next chapter. In order to introduce them, however, first we need to review the variables for the first order formulation of gravity used in quantization, as well as introduce some partial gauge fixing.

As before, let $\mathcal{M}$ denote the spacetime manifold. The basic space-time variables, of the first order formulation of gravity we are using, are a Lorentz connection ω_a^{IJ} and a tetrad e_I^a. (Note ω_a^{IJ} is distinct from ω_a — index structure will make clear the difference.) The internal indices $I, J, K, \ldots$ are associated with an internal four dimensional tetrad space; local rotations in this internal space constitute the new gauge freedom which is gained in using connection and tetrad variables. The 4-metric is determined by e_I^a in the usual fashion: $g_{ab} = e_a^I e_{bI}$. As we are using a first order formulation of gravity, prior to the equations of motion, ω_a^{IJ} is independent of the tetrad e_I^a. However, onshell, ω_a^{IJ} is equal to the Levi-Civita connection determined by e_I^a. We assume $\mathcal{M}$ has an $\mathbb{S}^2 \times \mathbb{R}$ inner boundary Δ, with Δ an isolated horizon.

Next, we introduce some partial gauge fixing and associated necessary extra structures. First, we fix a foliation of Δ by 2-spheres, and fix a vector field ℓ^a such that its flow preserves the foliation. These then uniquely determine a covector field n_a on Δ via the conditions (1.) n_a be normal to the fixed foliation, and (2.) $\ell^a n_a = -1$. Using these structures, we partially gauge fix diffeomorphism freedom by requiring the fixed foliation to coincide with the "good cuts" determined by the geometry. That is, we impose $\tilde{q}^{ab}\mathcal{D}_a\tilde{\omega}_b = 0$ on each leaf. In addition to this condition, we will also find it convenient to partially gauge fix the internal tetrad rotation freedom at Δ. To do this, we additionally fix a field of internal Newman-Penrose null tetrads $(\ell^I, n^I, m^I, \bar{m}^I)$ at Δ (so that $\ell^I n_I = -1$, $\bar{m}^I m_I = 1$ are the only non-zero inner products between the elements), with orientation $\epsilon_{IJKL} =$

$24i\ell_{[I}n_J m_K \bar{m}_{L]}$. We then impose the partial gauge-fixing conditions

$$\begin{aligned} e^a_I \ell^I &= \ell^a \\ e^I_a n_I &= n_a \end{aligned} \tag{2.39}$$

One can show this reduces the internal gauge rotation freedom at Δ to U(1), selecting a U(1) sub-bundle (over Δ) of the principal Lorenz bundle. This U(1) sub-bundle may be identified with the spin-bundle of any 2-sphere cross-section of Δ. To complete the Newman-Penrose tetrad, we define

$$m^a := e^a_I m^I. \tag{2.40}$$

The partial gauge fixing implies that $(\ell^a, n^a, m^a, \overline{m}^a)$ is a null tetrad adapted to the horizon, and hence m^a is tangent to the leaves of the fixed foliation. m^a determines for us a spin dyad (e^a_1, e^a_2) on each cross-section of the horizon via $m^a =: \frac{1}{\sqrt{2}}(e^a_1 + ie^a_2)$. Because of this, we will sometimes refer to m^a as a complex 'spin-dyad.' m^a is the first part of the horizon data.

The second part of the horizon data will come from the connection, and to introduce it we will need the following notation. First, given any internal 2-form X_{IJ}, recall the definition of its *self-dual* and *anti-self-dual* parts, $X^{(+)}_{IJ}$ and $X^{(-)}_{IJ}$, respectively:

$$X^{(\pm)}_{IJ} := \frac{1}{2}\left(X_{IJ} \mp \frac{i}{2}\epsilon_{IJ}{}^{KL}X_{KL}\right) \tag{2.41}$$

It will be convenient to generalize this to the notion of γ-*self-dual* part:

$$X^{(\gamma)}_{IJ} := \frac{1}{2}\left(X_{IJ} - \frac{\gamma}{2}\epsilon_{IJ}{}^{KL}X_{KL}\right). \tag{2.42}$$

For the cases $\gamma = \pm i$ this reduces to the self-dual and anti-self-dual parts. This definition of γ-self-dual part is useful in discussing the Ashtekar-Barbero formulation of gravity (to be used in quantization in the next chapter), in which case γ is taken to be the Barbero-Immirzi parameter (see section 3.2). In that context, the γ-self-dual part of ω_{aIJ} is the connection variable in the canonical theory obtained from the Legendre transform of the action. It will therefore be natural to construct the connection part of the horizon data from $\omega^{(\gamma)IJ}_a$.

Let us do so. The connection part of the horizon data, V_a, will be a $U(1)$ connection on the U(1) sub-bundle over Δ selected by the partial gauge fixing conditions described above. It will be convenient at the same time to define an analogous *complex* $U(1)$ connection $V^{(+)}$ on the *complexification* of this U(1) sub-bundle. The definitions are

$$V_a := -im_I\bar{m}_J\,\omega_{\underleftarrow{a}}^{(\gamma)IJ} \tag{2.43}$$

$$V_a^{(+)} := -im_I\bar{m}_J\,\omega_{\underleftarrow{a}}^{(+)IJ}. \tag{2.44}$$

where the underarrow denotes pull-back to Δ. Because of the partial gauge fixing that has been done, $\omega_a^{(\gamma)IJ}$ and $\omega_a^{(+)IJ}$ are completely determined by V_a and $V_a^{(+)}$, respectively. We introduce $V_a^{(+)}$ because it will be useful as an intermediate structure in some derivations. Let $\tilde{V}_a$ denote the pull-back of V_a to a given slice S. Then $(m^a, \tilde{V}_a)$ will be referred to as the *horizon data.* One can show $\tilde{V}_a$ is of the form

$$\tilde{V}_a = \frac{1}{2}(-\Gamma_a + \gamma\tilde{\omega}_a) \tag{2.45}$$

where $\Gamma_a := i\bar{m}^b\mathcal{D}_a m_b$ is the spin connection determined by m_a, with $\mathcal{D}_a$ the derivative operator determined by $q_{\underleftarrow{ab}}$ on the slice S, and $\tilde{\omega}_a$ is the pull-back of the rotation one-form to S. From this equation one can see that $(m^a, \tilde{V}_a)$ determines $\tilde{\omega}_a$ and hence $(\tilde{q}_{ab}, \tilde{\omega}_a)$, which in turn is the free data for the isolated horizon geometry (q, D) [25, end of section V]. Hence the name 'horizon data' is justified. We call $(m^a, \tilde{V}_a)$ "type I" or "type II" according to whether the associated horizon geometry is type I or type II.

Lastly, we note how, with the internal gauge fixing introduced above, the isolated horizon boundary conditions can be captured in terms of the U(1) connection $\tilde{V}$. First, in terms of the pull-back, $\tilde{V}_a^{(+)}$, of the complex connection $V_a^{(+)}$ defined in (2.44), the condition that the horizon be isolated may be captured in the equation

$$2\partial_{[a}\tilde{V}_{b]}^{(+)} = \Psi_2\epsilon_{ab}. \tag{2.46}$$

This follows from a slight modification of the discussion leading to (B23) in [11].

[8] Equation (2.45), applied to the case $\gamma = i$, yields $\tilde{V}_a^{(+)} = \frac{1}{2}(-\Gamma_a + i\tilde{\omega}_a)$. Thus,

$$\tilde{V}_a = \mathrm{Re}\tilde{V}_a^{(+)} + \gamma \mathrm{Im}\tilde{V}_a^{(+)} \tag{2.47}$$

so that if we define ${}^{\gamma}\Psi_2 := \mathrm{Re}\Psi_2 + \gamma\mathrm{Im}\Psi_2$, in terms of $\tilde{V}_a$, (2.46) becomes

$$2\partial_{[a}\tilde{V}_{b]} = {}^{\gamma}\Psi_2 \epsilon_{ab}. \tag{2.48}$$

With the internal gauge fixing in place reducing the internal gauge freedom to U(1) at the horizon, this equation is sufficient to imply isolation of the horizon. (This condition, or rather, the reformulation of it (2.61) will be important in chapter 3, where it will be imposed as an operator equation in quantum theory.)

2.4.3 Canonically associated type I data

Let S denote any 2-sphere cross-section of Δ. We will show that, for every type II horizon data set $(m^a, \tilde{V}_a)$, there is a canonically associated type I horizon data set $(\mathring{m}^a, \tilde{V}_a^{\circ})$. This type I data set will have an important role in the derivation of the symplectic structure and quantization in chapter 3.

To begin, we recall (from §2.2.2) there exists a natural coordinate system (ζ, φ), unique up to rigid shifts in φ, in which q takes the form

$$q_{ab} = R^2(f^{-1}\mathcal{D}_a\zeta\mathcal{D}_b\zeta + f\mathcal{D}_a\varphi\mathcal{D}_b\varphi) \tag{2.49}$$

for some function $f = f(\zeta)$, and where R is the area radius $a =: 4\pi R^2$. The canonical type I metric associated with q (as in equation (2.8)) is then

$$q^{\circ}_{ab} := R^2(f_{\circ}^{-1}\mathcal{D}_a\zeta\mathcal{D}_b\zeta + f_{\circ}\mathcal{D}_a\varphi\mathcal{D}_b\varphi) \tag{2.50}$$

where $f_{\circ} := 1 - \zeta^2$.

Now, a spin-dyad $\mathring{m}^a$ has more information than just that contained in q°: it also contains "gauge" information. Somehow we need to intertwine the gauge in-

[8](2.46) here follows directly from (B23) in [11]. However, the derivation in [11] made use of the sphericity of the type I horizon geometry. Nevertheless, it is easy to see that (B23) still holds even without assuming sphericity: the sphericity was only used to kill a certain term which gets killed anyway on being pulled back to the 2-sphere slice.

formation contained in m^a into our definition of $\mathring{m}^a$. This is accomplished by fixing natural reference spin-dyads $\underset{\hat{}}{m}^a$ and $\underset{\hat{}}{\mathring{m}}^a$, available due to the structure provided by axisymmetry:

$$\underset{\hat{}}{m}^a := \frac{1}{R\sqrt{2}}\left\{f^{\frac{1}{2}}\left(\frac{\partial}{\partial\zeta}\right)^a + if^{-\frac{1}{2}}\left(\frac{\partial}{\partial\varphi}\right)^a\right\} \tag{2.51}$$

$$\underset{\hat{}}{\mathring{m}}^a := \frac{1}{R\sqrt{2}}\left\{f_\circ^{\frac{1}{2}}\left(\frac{\partial}{\partial\zeta}\right)^a + if_\circ^{-\frac{1}{2}}\left(\frac{\partial}{\partial\varphi}\right)^a\right\} \tag{2.52}$$

$\underset{\hat{}}{m}^a$ so defined is compatible with the physical metric q, whereas $\underset{\hat{}}{\mathring{m}}^a$ is compatible with the type I metric q°. As m^a and $\underset{\hat{}}{m}^a$ are gauge-related, there exists a function α on Δ such that

$$m^a = e^{i\alpha}\underset{\hat{}}{m}^a. \tag{2.53}$$

α is precisely the gauge information contained in m^a. We transfer it over to our definition of $\mathring{m}^a$ by defining

$$\mathring{m}^a := e^{i\alpha}\underset{\hat{}}{\mathring{m}}^a. \tag{2.54}$$

Turning to the connection, we first note that, in the type I case, since $q^{ab}\mathcal{D}_a\tilde{\omega}_b = 0$ and $\partial_{[a}\tilde{\omega}_{b]} = 0$ [26], we have $\tilde{\omega}_a = 0$. We therefore define

$$\tilde{V}_a^\circ := -\frac{1}{2}\mathring{\Gamma}_a := -\frac{i}{2}\mathring{\bar{m}}^b\mathring{\mathcal{D}}_a\mathring{m}_b \tag{2.55}$$

which reduces to

$$\begin{aligned}\tilde{V}_a^\circ &= -\frac{1}{2}(\Gamma_a + \frac{1}{2}(f' - f_\circ')\mathcal{D}_a\varphi) \\ &= \tilde{V}_a - \frac{1}{4}(f' - f_\circ')\mathcal{D}_a\varphi - \frac{\gamma}{2}\omega_a \end{aligned} \tag{2.56}$$

where prime denotes derivative with respect to ζ. $(\mathring{m}^a, \tilde{V}_a^\circ)$ as defined above is then the type I horizon data canonically associated with the type II horizon data $(m^a, \tilde{V}_a)$.

Next, we show that $(m^a, \tilde{V}_a) \mapsto (\mathring{m}^a, \tilde{V}_a^\circ)$ is both diffeomorphism and U(1)-gauge covariant. This becomes crucial in the next chapter, when the diffeomorphism and Gauss constraints are imposed. The diffeomorphism covariance follows from the background independence of the construction. To address U(1)-gauge covariance, because $\tilde{V}^{\circ}_a$ is covariantly determined by $\mathring{m}^a$, it is sufficient to demonstrate the U(1)-gauge covariance of the map $m^a \mapsto \mathring{m}^a$. To prove this latter covariance, let us first write α, $\underaccent{\hat}{m}^a$, $\mathring{\underaccent{\hat}{m}}^a$, and $\mathring{m}^a$ as functions of m^a: $\alpha(m^a)$, $\underaccent{\hat}{m}^a(m^a)$, $\mathring{\underaccent{\hat}{m}}^a(m^a)$,$\mathring{m}^a(m^a)$. Now, because $\underaccent{\hat}{m}^a$ and $\mathring{\underaccent{\hat}{m}}^a$ depend only on the metric and the metric is U(1)-gauge invariant, $\underaccent{\hat}{m}^a$ and $\mathring{\underaccent{\hat}{m}}^a$ are also U(1)-gauge invariant. From the definition of $\alpha(m^a)$,

$$\begin{aligned} \left(e^{i\theta}m^a\right) &= e^{i\alpha(e^{i\theta}m^a)}\underaccent{\hat}{m}^a\left(e^{i\theta}m^a\right) && (2.57) \\ \text{and} \qquad m^a &= e^{i\alpha(m^a)}\underaccent{\hat}{m}^a(m^a). && (2.58) \end{aligned}$$

It thus follows

$$\alpha(e^{i\theta}m^a) = \alpha(m^a) + \theta. \tag{2.59}$$

Consequently

$$\begin{aligned} \mathring{m}^a(e^{i\theta}m^a) &= e^{i\alpha(e^{i\theta}m^a)}\mathring{\underaccent{\hat}{m}}^a\left(e^{i\theta}m^a\right) \\ &= e^{i\theta}e^{i\alpha(m^a)}\mathring{\underaccent{\hat}{m}}^a(m^a) = e^{i\theta}\mathring{m}^a(m^a) && (2.60) \end{aligned}$$

proving the desired covariance.

Furthermore, in the type I case — that is, when $(m^a, \tilde{V}_a)$ is type I — the canonically associated type I data $(\mathring{m}^a, \tilde{V}^{\circ}_a)$ is in fact the same as the physical data $(m^a, \tilde{V}_a)$. For, in that case $f_{\circ} = f$, implying $\mathring{\underaccent{\hat}{m}}^a = \underaccent{\hat}{m}^a$, which implies $\mathring{m}^a = m^a$, which in turn implies $\tilde{V}^{\circ}_a = \tilde{V}_a$.

To complete the discussion, we note that the horizon boundary condition (2.48) can be recast in terms of the type I connection $\tilde{V}^o$. To accomplish this, we substitute (2.56) into (2.48). On using (2.6),(2.10) and $\tilde{\mathcal{R}} = -4\mathrm{Re}\Psi_2$ (since we are in vacuum), the factors exactly conspire to give

$$\mathrm{d}\tilde{V}^{\circ} = -\frac{2\pi}{a_o}\,{}^2\epsilon. \tag{2.61}$$

which is none other than the familiar type I horizon boundary condition used in the prior calculation [11, 12]! This is of dramatic importance for the next chapter.

Figure 2.1. Left: axial foliation ξ, defined by the orbits of the axial symmetry field. Right: the coordinate ζ increases uniformly in proportion to physical area, starting at -1 at the south pole and ending at $+1$ at the north pole.

2.4.4 Reconstruction of $\tilde{V}_a$ using multipoles

Lastly, again for purposes in the next chapter, we show that, given the horizon area element ${}^2\!\epsilon$, the type I connection $\tilde{V}^\circ$, the multipoles I_n, L_n and a certain "axial" foliation of S (described next), one can fully reconstruct the physical type II connection $\tilde{V}$ at the horizon. Let us begin by specifying precisely the required foliation of S. First note that, given the physical geometry q_{ab} on S, the axial symmetry field ϕ^a Lie-dragging q_{ab} is generically unique. The orbits of this symmetry field give us a foliation of S into circular leaves; this is the axial foliation we need and we denote it by "ξ".

Let us demonstrate the reconstruction. Suppose we are given only the structures just mentioned. We will proceed to show that, using only relations universally holding for all type II isolated horizons, we will be able to deduce an expression for $\tilde{V}$ in terms of the given data, thus allowing reconstruction.

In section 2.2.2, it was noted that ζ increases uniformly in proportion to (physical) area from the south pole to the north pole (see figure 2.1). It follows that

$$\zeta(z) = -1 + 2\,\frac{a(z)}{a_S} \tag{2.62}$$

where a_S denotes the area of S and $a(z)$ denotes the area of the portion S_z of S bounded by the leaf of ξ labeled by z (and containing the south pole). From the definition of the multipoles, and the orthonormality of the functions $\{Y_n^0(\zeta)\}$, we can then obtain an expression for Ψ_2 in terms of the knowns thus far:

$$\Psi_2(\zeta) = \frac{-1}{R^2} \sum_{n=o}^{\infty} (I_n + iL_n)\, Y_n^0(\zeta), \tag{2.63}$$

Since $\tilde{\mathcal{R}} = -4\mathrm{Re}\Psi_2$, we can then use (2.7) to obtain

$$f(\zeta) := 4\left[R_\Delta^2 \int_{-1}^{\zeta} \mathrm{d}\zeta_1 \int_{-1}^{\zeta_1} \mathrm{d}\zeta_2\, \mathrm{Re}\Psi_2\,(\zeta_2)\right] + 2(\zeta+1)\,. \tag{2.64}$$

so that from (2.4),

$$\tilde{q}_{ab} = R^2(f^{-1} D_a\zeta\, D_b\zeta + f D_a\phi\, D_b\phi). \tag{2.65}$$

$\tilde{\omega}_a$ is then uniquely determined by the conditions $\partial_{[a}\tilde{\omega}_{b]} = \mathrm{Im}\Psi_2\epsilon_{ab}$ and $\tilde{q}^{ab}\mathcal{D}_a\tilde{\omega}_b = 0$. Finally, from (2.56), we can reconstruct the physical U(1) connection $\tilde{V}_a$, as desired:

$$\tilde{V}_a = \tilde{V}_a^\circ + \frac{1}{4}(f' - f_\circ')\mathcal{D}_a\varphi + \frac{\gamma}{2}\tilde{\omega}_a\,, \tag{2.66}$$

where the prime denotes derivative with respect to ζ.

As we will see in the next chapter, the above reconstruction will extend to quantum geometry.

2.5 Discussion

Source multipoles in Newtonian gravity characterize the way in which mass is distributed, higher multipoles providing a complete description of distortions, i.e., departures from sphericity. The multipole moments defined in this chapter do the same for horizon geometry. In this sense, they can be regarded as 'source multipoles' associated with a black hole, distinct from the 'field multipoles' defined at infinity. In Newtonian gravity and Maxwellian electrodynamics, there is a simple relation between the two because both these theories are 'Abelian': the field does not serve as its own source. In non-Abelian contexts such as Yang-Mills theory and general relativity, the field in the region between the source and infinity itself acts as an effective source. Hence one would expect there to exist two *distinct* sets of multipoles, one associated with the central gravitating body and the other associated with the entire system. The multipoles introduced in this chapter give us a notion of source multipoles associated with a black hole in equilibrium.

The horizon multipoles defined in this chapter are likely to have four sets of applications:

1. Since, in the stationary case, the multipoles determine the space-time geometry in a neighborhood of the horizon (see §2.3.2), they have potential use in formulating equations of motion for a test mass in the neighborhood of the horizon. Hansen's multipoles at infinity have successfully been used for a similar purpose, but in a neighborhood of infinity; it allowed one to construct an elegant framework in which one can "measure" the Hansen moments of a large black hole using gravitational radiation from the infall of a smaller black hole [34].

2. The multipoles may have use in formulating equations of motion for a small black hole moving in a background space-time (*i.e.* in the gravitational field produced by *other* more massive bodies). The law of geodesic motion for a body moving in a background space-time is only an approximation that holds when all internal structure of the moving body is ignored. When more precise equations of motion are desired, it is common to represent the internal structure of the moving body by multipole moments (see for example, [35]). As, heuristically, the coupling between the body and the background happens at the body itself rather than at infinity, one would expect source multipoles to be the most relevant for this purpose. Thus, when the moving body is a black hole, one expects use of something like the multipoles defined here to be more appropriate than other previously defined multipole moments in the literature.

3. The diffeomorphism invariance of the multipoles, and the fact that they bear physical meaning, makes them an ideal language for describing the physics in binary black hole numerical simulations. Furthermore, one can, without difficulty, extend the definition of the multipoles to dynamical situations [15, 36]. That is, not only can they be used in describing physics at times of approximate equilibrium in the horizons, but they can also be used to

describe physics in fully dynamical regimes. These applications of the multipoles have been carried out, *e.g.*, in [36].

4. Finally, the multipoles are important in the extension of the loop quantum gravity entropy calculation discussed in this thesis. The need for the multipoles here arises as follows. To calculate black hole entropy from first principles, one needs to construct an ensemble, where the 'macroscopic parameters' describing the system are fixed. To be physically meaningful — and to be well defined at all in quantum theory — these parameters have to be diffeomorphism invariant. For type I horizons, this is straightforward: there is only one gravitational parameter, which can be taken to be the horizon area (or mass). For type II horizons, one can not just work with mass and angular momentum because the horizon may be distorted by various types of hair. Even within the 4-dimensional Einstein-Maxwell theory where no-hair theorems hold, distortions can be caused by matter rings and other black holes. Even if the black hole itself is isolated, one can not automatically rely on uniqueness theorems and say that it must be a Kerr hole because it is physically unreasonable to require that the whole space-time be stationary. (Since one wishes to calculate entropy of a black hole in equilibrium, one should only need to require that the horizon geometry is time independent, not the whole universe.) To incorporate physically realistic situations, then, one needs a diffeomorphism invariant characterization of the horizon fields. Multipoles can now serve as the required parameters in the construction of the ensemble. It turns out that they can in fact be used to calculate entropy associated with type II horizons, as reported in [2] and described in the next chapter.

Chapter 3

Entropy of rotating and distorted horizons in loop quantum gravity

3.1 Introduction

Since the work by Bekenstein, Hawking, *et al.* in the early seventies on black hole thermodynamics, various proposals have been made for a microscopic explanation of black hole entropy. Of these, the only ones concretely related to well-developed approaches to quantum gravity are those based on string theory and loop quantum gravity. The present work aims at an extension of the latter. The previous loop quantum gravity calculation [12] (see also [10] and earlier work cited therein), restricted itself to the case in which the intrinsic geometry of the black hole horizon is spherically symmetric – the "type I" case in the terminology of chapter 2. We will refer to this previous calculation as the "type I" calculation. In the present work, we extend the calculation to the inclusion of rotation and distortion of the horizon compatible with axisymmetry — the "type II" case.

Both of these calculations assume *isolated horizon* boundary conditions at the horizon. Isolated horizons (IHs) provide a quasi-local framework to describe black holes which are themselves in equilibrium but in space-times whose exterior regions may carry time-dependent fields and geometry [37]. The zeroth and the first laws of black hole mechanics of classical general relativity were first established for globally stationary black holes [6]. However, they extend to all IHs [24, 26].

The thermodynamic entropy is again given by 1/4th the horizon area (provided matter is minimally coupled to gravity). These results hold not just for the Kerr-Newman family but also for other astrophysically realistic black holes which may be distorted. It is natural to ask if a quantum gravity description of the IH geometry can lead to a statistical mechanical calculation of entropy of this diverse family of black holes.

In the globally stationary situation, black holes without external influences are completely characterized by their mass (or horizon area), spin and possible charges associated with gauge fields and dilatons (which, however, will be ignored in most of this presentation). The entropy of such a black hole with *fixed* mass and spin is given by 1/4 the horizon area. In the quasi-local context of IHs, mass and spin do not suffice to characterize a time independent horizon geometry. One needs an infinite set of multipoles [15] to capture the distortions in the mass and angular momentum distribution on the horizon induced, e.g., by external matter rings, which are ignored by fiat in the black hole uniqueness theorems.

In this chapter, we will begin by sketching the essential features of a Hamiltonian framework for the sector of general relativity consisting of space-times which admit an isolated horizon with fixed multipoles. Then, we will carry out a non-perturbative quantization using ideas from quantum geometry [38, 39] and finally calculate the number of microstates of the resulting quantum horizon. This strategy is the same as that used in [10, 11, 12, 40, 41] for the simplest (type I) isolated horizons.

It will turn out that the canonical structure of the type II phase space will be isomorphic in every relevant way to the canonical structure of the type I phase space. Most of this chapter in fact consists in classical analysis showing how this is the case. The (basic) quantization itself will then be taken directly from the quantization used in the type I analysis [12].

Throughout, the conventions of [38] are used.

3.2 Action and Hamiltonian framework

We begin by specifying the basic variables, action, and boundary conditions. From there we will derive a symplectic structure in the covariant phase space framework, as this turns out to be the easiest context for that task. Then we will perform the Legendre transform to obtain the canonical (*i.e.* 3+1) phase space, and analyze the gauge motions generated by the constraints. This will complete the preparations for the quantization to be presented in section 3.3.

3.2.1 General considerations

Our arena is a spacetime manifold $\mathcal{M}$ with an inner boundary Δ, topologically $\mathbb{S}^2 \times \mathbb{R}$. The basic variables are a Lorentz connection ω_a^{IJ} and a (co-)tetrad e_a^I. The action used is that given by Holst [42]:

$$S_H(e,\omega) = \frac{1}{4k}\int_{\mathcal{M}} \epsilon_{IJKL} e^I \wedge e^J \wedge \Omega^{KL} - \frac{1}{2k\gamma}\int_{\mathcal{M}} e^I \wedge e^J \wedge \Omega_{IJ} \qquad (3.1)$$

where Ω_{ab}^{IJ} is the curvature of ω_a^{IJ}, $k := 8\pi G$ and γ is referred to as the *Barbero-Immirzi parameter*. In terms of the γ-self-dual part of Ω_{ab}^{IJ} (defined in equation (2.42) of the last chapter),

$$S_H(e,\omega) = -\frac{1}{\gamma k}\int_{\mathcal{M}} e^I \wedge e^J \wedge \Omega_{IJ}^{(\gamma)}. \qquad (3.2)$$

This is the action which forms the starting point for loop quantum gravity [38]. For the case $\gamma = i$, it reduces to the self-dual action. When varied, this action yields the standard equations of motion for general relativity.

Next, we fix on the inner boundary Δ the same extra structures as in §2.4.2. That is, we first fix a foliation of Δ and a vector field ℓ^a whose flow preserves the foliation. This in turn determines a unique covector n_a orthogonal to the leaves of the foliation and satisfying $\ell^a n_a = -1$. Secondly, we fix a field of internal null tetrads $(\ell^I, n^I, m^I, \overline{m}^I)$ at the horizon.

We then define the space of histories (to be used in the action principle) to consist in the set of all pairs (ω_a^{IJ}, e_a^I) such that the following conditions hold.

1. Appropriate fall-off conditions are satisfied at spatial infinity.

2. The inner boundary Δ is a type II isolated horizon with the partial gauge fixings of §2.4.1 imposed:

(a) The leaves of the fixed foliation of Δ are required to coincide with the *good cuts* uniquely determined by geometry [25]. That is, on each leaf of the fixed foliation, we impose $\tilde{q}^{ab}\mathcal{D}_a\tilde{\omega}_b = 0$ where $\tilde{q}_{ab}$ is the pull-back of the metric to the leaf, $\mathcal{D}_a$ is the associated Levi-Civita derivative operator, and $\tilde{\omega}_a$ is the pull-back of the rotation one-form to the leaf (see chapter 2).

(b) $\ell^a = e^a_I \ell^I$ and $n^a = e^a_I n^I$, partially gauge fixing the internal rotation freedom at the horizon, reducing the gauge group to U(1) at the horizon.

3. Δ has *fixed area* a_o and *fixed multipoles* $\mathring{I}_n, \mathring{L}_n$[1].

(Note the axial symmetry vector field is *not* fixed.) Let the sub-space of these histories extremizing the action be denoted $\mathbf{\Gamma}_{cov}$. $\mathbf{\Gamma}_{cov}$ is then the covariant phase space.

Let us take the opportunity to remind the reader about the idea of a "covariant phase space." The covariant phase space of a theory is the space of solutions (on spacetime), equipped with a symplectic structure Ω_{cov} obtained from second variations of the action (see [43]). The covariant phase space is related to the more usual phase space of initial data (also called the "canonical" or "3+1" phase space) as follows. Suppose spacetime, $\mathcal{M}$, admits a Cauchy surface Σ. Let $(\mathbf{\Gamma}, \Omega)$ denote the phase space of initial data on Σ, defined with the usual symplectic structure encoding the relation between fields and momenta. Then, any solution in $\mathbf{\Gamma}_{cov}$ induces initial data on Σ, giving us a map $\mathbf{\Gamma}_{cov} \to \mathbf{\Gamma}$. Let η denote this map. Because Σ is a Cauchy surface, η is an isomorphism. In fact one can show, in general, η is furthermore a *phase space* isomorphism:

$$\eta^*\Omega = \Omega_{cov}. \tag{3.3}$$

Because of this, the covariant phase space viewpoint and the canonical phase space viewpoint are (in general) equivalent when a Cauchy surface is used to define the canonical phase space. In the case relevant for this chapter, however, we will

[1] In the type I case, all physical multipoles except M_o are zero, M_o is simply $\sqrt{a_o/16\pi}$, and Ψ_2 is given by $-2\pi/a_o$. Therefore to single out the relevant sector of general relativity, it suffices to fix just the horizon area a_o. This is precisely what was done in [10] although at that time the notion of multipoles was not available.

use only a *partial* Cauchy surface to define the canonical phase space. In this case, initial data on a given Σ then corresponds to *more* than one solution to the equations of motion. η is consequently no longer one to one, but is a projection. Fortunately, however, one still has $\eta^*\Omega = \Omega_{cov}$, so that the covariant and canonical frameworks are still, in this sense, consistent, which is all that is necessary for our purposes.

We use the covariant phase space framework to calculate symplectic structure. This is because we will have need of a surface term in the symplectic structure; the covariant phase space framework allows for a simpler, clearer derivation of this surface term. After the surface term is determined in the next few subsections, it will be transferred (in section 3.2.5) to the canonical picture, in preparation for quantization.

3.2.2 Symplectic structure

Let us determine an expression for the symplectic structure on Γ_{cov}. Let M_1 and M_2 denote surfaces in $\mathcal{M}$ to be understood as (partial-)Cauchy surfaces, with M_1 preceding M_2, and let S_1,S_2 denote their respective intersections with Δ (as in figure 3.1).

From variation of the action, one finds the symplectic current to be [38]

$$\boldsymbol{\omega}(\delta_1, \delta_2) = -\frac{2}{k}\delta_{[1}(e^I \wedge e^J) \wedge \delta_{2]}\omega_{IJ}^{(\gamma)} \tag{3.4}$$

where δ_1,δ_2 are tangent vectors to $\mathbf{\Gamma}_{cov}$. It turns out that one cannot simply use the naive symplectic structure

$$\mathbf{\Omega}_M(\delta_1, \delta_2) = \int_M \boldsymbol{\omega}(\delta_1, \delta_2) \tag{3.5}$$

because it is not preserved under time evolution. That is, even when δ_1, δ_2 are restricted to be tangent to the space of solutions, $\mathbf{\Omega}_{M_1}(\delta_1, \delta_2)$ in general fails to be equal to $\mathbf{\Omega}_{M_2}(\delta_1, \delta_2)$. Figure 3.1 shows the cause of this: symplectic current is escaping across the horizon. More precisely, from figure 3.1 and the fact that

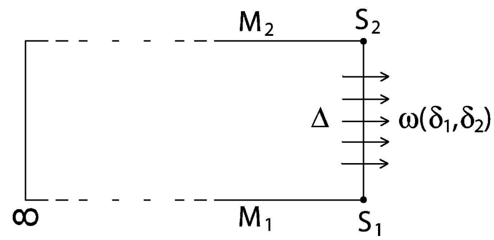

Figure 3.1. Symplectic current escapes through the horizon. M_1 and M_2 are partial Cauchy surfaces with inner boundaries S_1 and S_2, and outer boundaries at spatial infinity.

symplectic current is conserved, we have

$$\int_{M_2} \boldsymbol{\omega}(\delta_1, \delta_2) = \int_{M_1} \boldsymbol{\omega}(\delta_1, \delta_2) - \int_{\Delta} \boldsymbol{\omega}(\delta_1, \delta_2) \tag{3.6}$$

(The contribution from the integral at spatial infinity vanishes because of the fall-off imposed.) The solution to the problem is to use the isolated horizon boundary conditions to rewrite the Δ-integral: [2]

$$\int_{\Delta} \boldsymbol{\omega}(\delta_1, \delta_2) = \left(\oint_{S_2} - \oint_{S_1} \right) \lambda(\delta_1, \delta_2) \tag{3.7}$$

If we then define our symplectic structure to be

$$\boldsymbol{\Omega}(\delta_1, \delta_2) := \int_{M} \boldsymbol{\omega}(\delta_1, \delta_2) + \oint_{S} \lambda(\delta_1, \delta_2) \tag{3.8}$$

the symplectic structure will be conserved as required.

3.2.3 Surface term in the symplectic structure

By expanding out the basic variables e^I_a and ω^{IJ}_a in terms of $\ell^I, n^I, m^I, \bar{m}^I$ and $\ell_a, n_a, m_a, \bar{m}_a$ using Newman-Penrose coefficients, and making use of the isolated horizon boundary conditions, one can, through straightforward manipulation, show

[2] As one would expect, there is more than one way to split the Δ integral into two such boundary integrals. This leads to an ambiguity in the final symplectic structure one uses. Note this sort of ambiguity will *always* be present whenever one's spatial manifold has a boundary. Below, we will present one particularly natural way of splitting up the Δ integral.

that

$$\int_{\Delta} \boldsymbol{\omega}(\delta_1, \delta_2) = \frac{1}{k}\int_{\Delta}\left[\delta_1\, {}^2\!\epsilon \wedge \delta_2(-\kappa_{(\ell)} n) - \frac{i}{\gamma}\delta_1\, {}^2\!\epsilon \wedge \delta_2((\epsilon - \bar{\epsilon})n) - (1 \leftrightarrow 2)\right] \quad (3.9)$$

where $\kappa_{(\ell)} := \omega_a \ell^a$, ${}^2\epsilon = i m \wedge \bar{m}$ is the area 2-form on Δ, and ϵ is a Newman-Penrose coefficient satisfying

$$\mathcal{L}_{\ell} m_a = (\epsilon - \bar{\epsilon}) m_a. \quad (3.10)$$

Let S_0 denote any of the leaves of the foliation of Δ to the past of both S_1 and S_2. Recall from §2.4.1 that there exists a coordinate v on Δ, unique up to addition of a constant, such that $n_{\underleftarrow{a}} = -\partial_a v$. (Note $n_{\underleftarrow{a}} = -\partial_a v$ is equivalent to $\ell^a \partial_a v = 1$ and v is constant on slices of the fixed foliation.) For convenience in the calculations below, let us fix the freedom in the coordinate v on Δ by requiring it to be zero on S_0. Define potentials ψ and $\tilde{\psi}$ on Δ satisfying

$$\begin{aligned} \mathcal{L}_{\ell}\psi &= \kappa_{(\ell)} \\ \mathcal{L}_{\ell}\tilde{\psi} &= -i(\epsilon - \bar{\epsilon}) \end{aligned} \quad (3.11)$$

with initial conditions $\psi = \tilde{\psi} = 0$ on S_0.

Since $(-\ell^a, \bar{m}^a, m^a)$ is dual to $(n_a, m_a, \bar{m}_a)$,

$$\mathrm{d}\psi = -(\ell \cdot \mathrm{d}\psi) n + (\bar{m} \cdot \mathrm{d}\psi) m + (m \cdot \mathrm{d}\psi)\bar{m} \quad (3.12)$$

$$\mathrm{d}\tilde{\psi} = -(\ell \cdot \mathrm{d}\tilde{\psi}) n + (\bar{m} \cdot \mathrm{d}\tilde{\psi}) m + (m \cdot \mathrm{d}\tilde{\psi})\bar{m} \quad (3.13)$$

Since $\kappa_{(\ell)}$ is constant on Δ [26], and $\ell^a D_a v = 1$, it follows from the definition of ψ that ψ is constant on each leaf of the fixed foliation. Consequently $m \cdot \mathrm{d}\psi = \bar{m} \cdot \mathrm{d}\psi = 0$, and (3.12) reduces to

$$\mathrm{d}\psi = -(\ell \cdot \mathrm{d}\psi) n = -\kappa_{(\ell)} n \quad (3.14)$$

Equation (3.13) shows us that in every history, $\mathrm{d}\tilde{\psi} + (\ell \cdot \mathrm{d}\tilde{\psi}) n$ is tangent to the fixed foliation of Δ (i.e., is orthogonal to ℓ^a), whence $\delta_2(\mathrm{d}\tilde{\psi} + (\ell \cdot \mathrm{d}\tilde{\psi}) n)$ is tangent to the fixed foliation of Δ. Likewise ${}^2\epsilon$ is tangent to the fixed foliation of Δ throughout the space of histories, so that $\delta_1\, {}^2\epsilon$ is also tangent to the fixed foliation of Δ. But then

$$\delta_1{}^2\!\epsilon \wedge \delta_2(\mathrm{d}\tilde{\psi} + (\ell \cdot \mathrm{d}\tilde{\psi})n) = 0 \tag{3.15}$$

since the left hand side at each leaf S_v is then a three-form on the two-dimensional manifold S_v.

Using (3.14),(3.11) and (3.15) in (3.9) gives us

$$\int_\Delta \boldsymbol{\omega}(\delta_1, \delta_2) = \frac{1}{k}\int_\Delta \left[\delta_1{}^2\!\epsilon \wedge \delta_2(\mathrm{d}\psi) - \frac{1}{\gamma}\delta_1{}^2\!\epsilon \wedge \delta_2(\mathrm{d}\tilde{\psi}) - (1 \leftrightarrow 2)\right] \tag{3.16}$$

From the Cartan identity,

$$\mathcal{L}_\ell{}^2\!\epsilon = \mathrm{d}(\ell \lrcorner {}^2\!\epsilon) + \ell \lrcorner \mathrm{d}\,{}^2\!\epsilon, \tag{3.17}$$

and the fact that $\mathcal{L}_\ell{}^2\!\epsilon = 0$ and $\ell \lrcorner {}^2\!\epsilon = 0$, it follows that $\ell \lrcorner \mathrm{d}\,{}^2\!\epsilon = 0$, so that $\mathrm{d}\,{}^2\!\epsilon$ is degenerate. (Here $\lrcorner$ denotes contraction of a vector with the first index of a form.) But the space of three forms on Δ at each point is one-dimensional, so that this suffices to imply $\mathrm{d}\,{}^2\!\epsilon = 0$. Using this fact to rewrite (3.16),

$$\begin{aligned}\int_\Delta \boldsymbol{\omega}(\delta_1, \delta_2) &= \frac{1}{k}\int_\Delta \mathrm{d}\left[\delta_1{}^2\!\epsilon \wedge \delta_2\psi - \frac{1}{\gamma}\delta_1{}^2\!\epsilon \wedge \delta_2\tilde{\psi} - (1 \leftrightarrow 2)\right] \\ &= \frac{1}{k}\left(\oint_{S_2} - \oint_{S_1}\right)\left[\delta_1{}^2\!\epsilon\delta_2\psi - \frac{1}{\gamma}\delta_1{}^2\!\epsilon\delta_2\tilde{\psi} - (1 \leftrightarrow 2)\right] \end{aligned} \tag{3.18}$$

As ψ is constant on each leaf of the fixed foliation of Δ,

$$\oint_{S_1} \delta_1{}^2\!\epsilon\delta_2\psi = \delta_2\psi\delta_1 \oint_{S_1} {}^2\!\epsilon = 0 \tag{3.19}$$

whence (3.18) further reduces

$$\int_\Delta \boldsymbol{\omega}(\delta_1, \delta_2) = \frac{1}{\gamma k}\left(\oint_{S_2} - \oint_{S_1}\right)\left[\delta_1\tilde{\psi}\delta_2{}^2\!\epsilon - (1 \leftrightarrow 2)\right] \tag{3.20}$$

One may be tempted to stop the derivation at this point. For, from this equation, it is evident one could choose $\lambda(\delta_1, \delta_2) = \frac{1}{\gamma k}[\delta_1\tilde{\psi}\delta_2{}^2\!\epsilon - (1 \leftrightarrow 2)]$. However, such a choice of λ is not acceptable because $\tilde{\psi}$ appears in it. $\tilde{\psi}$ is well-defined in the covariant phase space framework, but it is not a well-defined function in terms of the basic variables when we go over to the canonical framework (§3.2.5). The reason for this is that the value of $\tilde{\psi}$ on a given slice S is not a function only of the basic variables e_a^I and ω_a^{IJ} on S, but rather depends on the values of these fields on the *entire* segment of Δ between S and S_0. That is, $\tilde{\psi}$ is not a function of the basic variables that is 'local in time'. Because of this, $\tilde{\psi}$ is ill-defined in the canonical framework, where the basic variables live on a single spatial slice representing only an instant of time. Therefore $\tilde{\psi}$ should be understood merely as an auxiliary variable introduced in the covariant phase space picture for convenience in the present calculation; in the end, the final expression for the symplectic structure should, preferably, not explicitly involve $\tilde{\psi}$. [3]

3.2.4 Surface symplectic structure in terms of canonical type I connection

Recall from section 2.4.3 that there is a canonical type I connection $\tilde{V}^\circ$ defined on slices of Δ, determined uniquely by the dynamical fields on the horizon. We will use this type I connection to rewrite the integral of the symplectic current over Δ and solve the problem mentioned at the end of the last subsection. We will be able to extract an expression for the surface symplectic structure which only depends locally (in time) on the dynamical fields, and hence will carry over unambiguously to the canonical framework in section 3.2.5.

We begin by substituting (2.61) into (3.20) and integrating by parts,

$$\int_\Delta \boldsymbol{\omega}(\delta_1, \delta_2) = \frac{1}{\gamma k}\frac{a_o}{2\pi}\left(\oint_{S_2} - \oint_{S_1}\right)\left[\delta_1 \tilde{V}^\circ \wedge \delta_2 \mathrm{d}\tilde{\psi} - (1 \leftrightarrow 2)\right] \tag{3.21}$$

In what follows it will be convenient to use the following notational convention: given any (possibly tensorial) field Ψ on Δ, we let $\Psi(v)$ denote the unique field on Δ satisfying $\Psi(v)\mid_{S_v} = \Psi\mid_{S_v}$ and $\mathcal{L}_\ell \Psi(v) = 0$. From (3.10) and (3.11) it is then not hard to show that

$$m_a(v) = e^{i\tilde{\psi}(v)} m_a(0) \tag{3.22}$$

[3] One can ignore the fact that $\tilde{\psi}$ is ill-defined in the canonical framework by, e.g., simply declaring $\tilde{\psi}$ to be a new degree of freedom at the horizon. Nevertheless, if one does this, there are other difficulties/ inelegancies that one encounters when trying to implement the quantization.

On each slice, $\underset{\frown}{m}_a$ is defined in a unique and covariant manner from q_{ab}. So $\mathcal{L}_\ell q_{ab} = 0$ implies $\mathcal{L}_\ell \underset{\frown}{m}_a = 0$. For the same reason $\mathcal{L}_\ell \underset{\frown}{\mathring{m}}_a = 0$. These together with (2.53) and (2.54) imply

$$\begin{aligned} m^a(v) &= e^{i(\alpha(v)-\alpha(0))} m^a(0) &(3.23)\\ \mathring{m}^a(v) &= e^{i(\alpha(v)-\alpha(0))} \mathring{m}^a(0) &(3.24)\end{aligned}$$

These together with (3.22) imply

$$\mathring{m}_a(v) = e^{i\tilde{\psi}(v)} \mathring{m}_a(0) \tag{3.25}$$

substituting this expression into (2.55) yields

$$\tilde{V}^\circ_a(v) = \tilde{V}^\circ_a(0) + \frac{1}{2}\mathrm{d}\tilde{\psi}(v) \tag{3.26}$$

Let v_1 and v_2 denote the coordinates for leaves S_1 and S_2, respectively. Substituting (3.26) into (3.21) gives

$$\int_\Delta \boldsymbol{\omega}(\delta_1, \delta_2) = \frac{2}{\gamma k}\frac{a_o}{2\pi}\left(\oint_{S_2} - \oint_{S_1}\right)\left[\delta_1 \tilde{V}^\circ \wedge \delta_2(\tilde{V}^\circ - \tilde{V}^\circ(0)) - (1 \leftrightarrow 2)\right] \tag{3.27}$$

Let

$$I := \frac{4}{\gamma k}\frac{a_o}{2\pi}\left(\oint_{S_2} - \oint_{S_1}\right)\delta_1 \tilde{V}^\circ \wedge \delta_2 \tilde{V}^\circ \tag{3.28}$$

We have

$$\int_\Delta \boldsymbol{\omega}(\delta_1, \delta_2) = I - \frac{2}{\gamma k}\frac{a_o}{2\pi}\left(\oint_{S_2} - \oint_{S_1}\right)\left[\delta_1 \tilde{V}^\circ \wedge \delta_2 \tilde{V}^\circ(0) - (1 \leftrightarrow 2)\right] \tag{3.29}$$

Plugging in (3.26),

$$\begin{aligned} \int_\Delta \boldsymbol{\omega}(\delta_1, \delta_2) &= I - \frac{2}{\gamma k}\frac{a_o}{2\pi}\left(\oint_{S_2} - \oint_{S_1}\right)\left[\delta_1 \tilde{V}^\circ(0) \wedge \delta_2 \tilde{V}^\circ(0) - (1 \leftrightarrow 2)\right] \\ &\quad - \frac{1}{\gamma k}\frac{a_o}{2\pi}\left(\oint_{S_2} - \oint_{S_1}\right)\left[\delta_1 \mathrm{d}\tilde{\psi} \wedge \delta_2 \tilde{V}^\circ(0) - (1 \leftrightarrow 2)\right] \\ &= I - \frac{1}{\gamma k}\frac{a_o}{2\pi}\left(\oint_{S_2} - \oint_{S_1}\right)\left[\delta_1 \mathrm{d}\tilde{\psi} \wedge \delta_2 \tilde{V}^\circ(0) - (1 \leftrightarrow 2)\right] \end{aligned}$$

$$
\begin{aligned}
&= I + \frac{1}{\gamma k}\frac{a_o}{2\pi}\left(\oint_{S_2} - \oint_{S_1}\right)\left[\delta_1\tilde{\psi}\wedge\delta_2 \mathrm{d}\tilde{V}^\circ(0) - (1\leftrightarrow 2)\right] \\
&= I + \frac{1}{\gamma k}\frac{a_o}{2\pi}\left(\oint_{S_2} - \oint_{S_1}\right)\left[\delta_1\tilde{\psi}\wedge\delta_2 \mathrm{d}\tilde{V}^\circ - (1\leftrightarrow 2)\right] \\
&= I - \int_\Delta \boldsymbol{\omega}(\delta_1,\delta_2) \qquad (3.30)
\end{aligned}
$$

where we have used $\mathcal{L}_\ell \tilde{V}^\circ(0) = 0$ (in virtue of the definition of $\tilde{V}^\circ(0)$) in the second step and $\mathcal{L}_\ell \mathrm{d}\tilde{V}^\circ = \mathcal{L}_\ell\left(-\frac{2\pi}{a_o}\,{}^2\!\epsilon\right) = 0$ (due to isolation) in the penultimate step. (Note it is $\tilde{V}^\circ$ and not $\tilde{V}^\circ(0)$ appearing in the penultimate step.) Thus

$$
\int_\Delta \boldsymbol{\omega}(\delta_1,\delta_2) = \frac{1}{2}I = \frac{2}{\gamma k}\frac{a_o}{2\pi}\left(\oint_{S_2} - \oint_{S_1}\right)\delta_1\tilde{V}^\circ\wedge\delta_2\tilde{V}^\circ \qquad (3.31)
$$

Whence we may define

$$
\boldsymbol{\Omega}(\delta_1,\delta_2) := \int_M \boldsymbol{\omega}(\delta_1,\delta_2) + \frac{2}{\gamma k}\frac{a_o}{2\pi}\oint_S \delta_1\tilde{V}^\circ\wedge\delta_2\tilde{V}^\circ \qquad (3.32)
$$

as our symplectic structure.

Let us summarize what has been done. We set out in subsection 3.2.2 to determine the symplectic structure for the phase space Γ_{cov}. In the process, it was found that a surface term in the symplectic structure was necessary in order for the symplectic structure to be conserved in time. In subsection 3.2.4, we recast the surface term to depend on the basic fields in a manner local in time. The result is the surface term above, written in terms of the canonical type I connection $\tilde{V}^\circ$, determined by the dynamical fields at the horizon. In this form, the symplectic structure can be carried over to the canonical (*i.e.* 3+1) framework, to be introduced in the next section.

3.2.5 Structure of the canonical phase space

For quantization, it is necessary to change to the canonical picture; we do so here. Let M denote a fixed submanifold in $\mathcal{M}$ to be thought of as a (partial-)Cauchy surface. Let us require that M be chosen to intersect Δ in a leaf of its fixed foliation. Let S denote this leaf. Let $\underleftarrow{\ \ }$ denote pull-back to M. As in [38], we fix a field τ^I on M, here required to approach $\frac{1}{\sqrt{2}}(\ell^I + n^I)$ at S. e^a_I is then partially gauge fixed by requiring $e^a_I\tau^I$ to be the future pointing normal to M at each point. Let i, j, k denote abstract indices associated with the internal subspace perpendicular to τ^I at each point. Let q^i_I denote the orthogonal projector onto this subspace at each point. As this subspace is three dimensional, it may be identified with the Lie algebra of SU(2). Make such an identification such that $\epsilon_{ijk} := q^I_i q^J_j q^K_k \tau^L \epsilon_{LIJK}$ is the structure tensor for the algebra. The canonical variables are then defined to be [38]:

$$A^i_a := -\epsilon^i{}_{IJ}\omega^{(\gamma)IJ}_{\underleftarrow{a}} \tag{3.33}$$

$$\Sigma^i_{ab} := \frac{1}{k\gamma}\epsilon^i{}_{jk}e^j_{\underleftarrow{a}}e^k_{\underleftarrow{b}} \tag{3.34}$$

or, in terms of the densitized triad $\tilde{E}^a_i$ induced on M, $\Sigma^i_{ab} = \frac{1}{k\gamma}\underset{\sim}{\eta}_{abc}\tilde{E}^{ci}$, where $\underset{\sim}{\eta}_{abc}$ is the Levi-Civita density on M. We then define our canonical phase space $\mathbf{\Gamma}$ to be the space of all (smooth) initial data (A^i_a, Σ^i_{ab}) on M compatible with the boundary conditions. The holonomies of the connection A^i_a take values in $SL(2,\mathbb{C}) \cong SU(2)^{\mathbb{C}}$ unless γ is real, in which case they take values in SU(2). The functional calculus on the space of connections is well-developed only when the structure group is compact; for this reason we must restrict to the case in which γ is real.

Define $r^i := \frac{1}{\sqrt{2}}q^i_I(\ell^I - n^I)$; then $\tilde{V}_a = \frac{1}{2}r_i A^i_a$. The meaning of the U(1) sub-bundle selected by the partial gauge fixing conditions in section 2.4.1 is now more transparent: it is the sub-bundle preserving r^i at each point. The gauge transformations living in this sub-bundle are local gauge rotations about r^i; it is then easy to see why such a bundle is U(1) and is naturally identifiable with the spin bundle over S.

In terms of these variables, the symplectic structure, determined in the foregoing subsections, is given by:

$$\mathbf{\Omega}(\delta_1, \delta_2) = -2\int_M (\delta_1 A^i \wedge \delta_2\Sigma_i - \delta_2 A^i \wedge \delta_1 \Sigma_i) + \frac{2}{\gamma k}\frac{a_o}{2\pi}\oint_S \delta_1\tilde{V}^\circ \wedge \delta_2\tilde{V}^\circ \tag{3.35}$$

where δ_1, δ_2 are any two tangent vectors to the phase space $\mathbf{\Gamma}$; and $\tilde{V}^\circ$ is the canonical type I spin connection associated with $(q, \tilde{V})$ via (2.66).

The constraints for the theory are given by [38]:

$$C_G(\Lambda^i) := \int_M \Lambda_i \mathrm{d}_A \Sigma^i \tag{3.36}$$

$$C_{Diff}(\vec{N}) := \int_M \left[(N \lrcorner F^i)\wedge\Sigma_i - (N\lrcorner A^i)\wedge \mathrm{d}_A\Sigma_i\right] \tag{3.37}$$

$$C(N) := \frac{\gamma}{2}\int_M N\left[\epsilon^{ij}{}_k(E_i\lrcorner F^k) - (1+\gamma^2)(E_i\lrcorner(K^i\wedge K^j))\right]\wedge\Sigma_j \tag{3.38}$$

where d_A denotes the covariant exterior derivative operator determined by $A_{ai}{}^j := \epsilon_{ik}{}^j A_a^k$, F_{ab}^i denotes the curvature of A_a^i, E_i^a is the undensitized triad, and $K_a^i := E^{bi}K_{ab}$ with K_{ab} the extrinsic curvature of M.

To determine the canonical gauge in the framework, we need to determine for which smearings $\Lambda^i, \vec{N}, N$ the smeared constraints are differentiable functions on the phase space and therefore generate Hamiltonian vector fields δ_Λ, $\delta_{\vec{N}}$, and δ_N. That is, we need to determine for which smearings $\Lambda^i, \vec{N}, N$ there exist $\delta_\Lambda, \delta_{\vec{N}}, \delta_N$ such that $\delta C_G(\Lambda) = \mathbf{\Omega}(\delta,\delta_\Lambda)$, $\delta C_{Diff}(\vec{N}) = \mathbf{\Omega}(\delta, \delta_{\vec{N}})$, and $\delta C(N) = \mathbf{\Omega}(\delta,\delta_N)$ for all δ tangent to $\mathbf{\Gamma}$. Since the bulk term in the symplectic structure is the same as in standard general relativity, we know that in the bulk — and hence, by continuity, on S as well — any such $\delta_\Lambda, \delta_{\vec{N}}, \delta_N$ must correspond to the usual gauge motions generated by the constraints. δ_Λ must correspond to local SU(2) gauge rotations, $\delta_{\vec{N}}$ to diffeomorphisms, and δ_N to time evolution (on shell). We furthermore know that these gauge motions, if they are to exist, must preserve the phase space $\mathbf{\Gamma}$, and in particular must preserve the boundary conditions and partial gauge fixings imposed at the horizon. These two facts lead to the following restrictions on Λ^i, $\vec{N}$, and N, if $\delta_\Lambda, \delta_{\vec{N}}, \delta_N$ are to generate motions:

1. Λ^i must be proportional to r^i at the horizon; that is, at the horizon it must take values in the Lie subalgebra corresponding to the U(1) sub-bundle selected by the partial gauge fixing conditions.

2. $\vec{N}$ must be tangent to S.

3. N must vanish on S.

The restriction on N deserves further explanation. If N were not to vanish on S, the flow generated by δ_N would not preserve the boundary condition that S be the intersection of M with an isolated horizon. This can be seen as follows. On shell, recall δ_N generates time evolution along the time-like vector field $N\tau^a$, where τ^a is the unit future time-like normal to the partial Cauchy surface M. At the horizon, τ^a, being time-like, points into the horizon. Hence, if N were to not vanish on S, the flow generated by δ_N would carry S into the horizon — S would "fall into" the black hole and so cease to be the location of an isolated horizon. Hence, if δ_N is to define a well-defined flow on our phase space $\mathbf{\Gamma}$, N must vanish on S.

We know the items in the above list are minimum restrictions on Λ^i, $\vec{N}$, and N if $C_G(\Lambda)$, $C_{Diff}(\vec{N})$, and $C(N)$ are to generate gauge motions. One can check that, in fact, these are all of the restrictions. [4] Thus, the allowed smearings of the constraints are identical to the allowed smearings in the type I case, whence the notions of canonical gauge are the same. In particular, the canonical gauge transformations for $\tilde{V}_a = \frac{1}{2} r_i A_a^i$ are again precisely U(1) gauge rotations and diffeomorphisms. The U(1)-gauge and diffeomorphism covariance of the map $\tilde{V} \mapsto \tilde{V}^\circ$ proved in §2.4.3 now becomes important: It tells us that, furthermore, U(1)-gauge rotations and diffeomorphisms are *also* the precise canonical gauge for $\tilde{V}^\circ$. This will be important in the quantization.

Note that the surface term is the symplectic structure of a *Chern-Simons theory* for a U(1) connection $\tilde{V}^\circ$, with level $k = a_o/4\pi\gamma\ell_{\rm Pl}^2$. Because variations of only $\tilde{V}^\circ$ — rather than $\tilde{V}$ — appear, (2.61) provides the most convenient way to incorporate the key boundary condition that $\mathrm{d}\tilde{V}$ is given by (2.48), where Ψ_2 has the given set of multipoles.

Because the surface symplectic structure, the horizon boundary condition, and the notions of canonical gauge are identical to those in the type I calculation, the constructions in [12] can be carried over basically unchanged to the present calculation. Let us briefly review the quantization strategy adopted in [12]. We define a 'bulk phase space' $\mathbf{\Gamma}_B$ and a 'surface phase space' $\mathbf{\Gamma}_S$ as follows. $\mathbf{\Gamma}_B$ is isomorphic with $\mathbf{\Gamma}$ as a manifold but it is equipped with a symplectic structure $\mathbf{\Omega}_B$ given just by the volume term in (3.35). $\mathbf{\Gamma}_S$ is the space of U(1) connections $\tilde{V}^\circ$ on S equipped with the Chern-Simons symplectic structure — i.e., the surface

[4] In verifying this statement for $\vec{N}$, crucial use must be made of the fact that all the multipoles (and area) have been fixed.

term in (3.35). Then, we have natural maps

$$p_B\colon \mathbf{\Gamma} \to \mathbf{\Gamma}_B, \qquad p_S\colon \mathbf{\Gamma} \to \mathbf{\Gamma}_S$$

given by

$$p_B(A,\Sigma) = (A,\Sigma) \qquad p_S(A,\Sigma) = \tilde{V}^\circ$$

so that

$$\mathbf{\Omega}_{\rm grav} = p_B^*\mathbf{\Omega}_B + p_S^*\mathbf{\Omega}_S.$$

We will see that this structure on the kinematical space of classical states is faithfully mirrored in the kinematical space of quantum states.

Remark. Note that the definition of Γ_S is determined by the surface term in the symplectic structure. We include in Γ_S only those surface fields that appear in $\mathbf{\Omega}_S$, so that $\mathbf{\Omega}_S$ is non-degenerate on Γ_S. Non-degeneracy of $\mathbf{\Omega}_S$ is *needed* for quantizing the surface phase space. One might say $\mathbf{\Omega}_S$ "grabs on to" exactly the appropriate fields at the horizon.

Heuristically this is what symplectic structure *always* does: it determines the physical degrees of freedom of a system. This is evident, for example, in the covariant phase space formulation of a first class constrained system (such as general relativity): It is precisely the degenerate directions of the symplectic structure which correspond to unphysical degrees of freedom. This general observation might lead one to suggest the following interpretation of the degrees of freedom which will be quantized in the surface phase space. The boundary term in the symplectic structure 3.35 came from integrating the symplectic current on the horizon, and this was done in lieu of integrating over the interior of the black hole in both Cauchy surfaces in subsection 3.2.2. It is thus natural to interpret the degrees of freedom associated with the boundary term as representing the degrees of freedom neglected in this approximation: That is, perhaps the surface phase space degrees of freedom should be understood to represent the *internal* degrees of freedom of the black hole.

3.3 Quantum horizon geometry

Let us summarize the situation presented by the classical analysis. Each type II horizon geometry defines a canonical type I geometry. Furthermore, the Hamiltonian theory of the sector of general relativity admitting type II isolated horizons *with fixed area and multipoles*, at the horizon makes direct reference only to the type I connection $\tilde{V}^{\circ}$. The surface symplectic structure in terms of $\tilde{V}^{\circ}$ is the same as in the type I case, and the notions of canonical gauge for $\tilde{V}^{\circ}$ are also the same as in the type I case. Therefore, the situation is formally identical to that present in the type I case, and the quantization can be carried out by taking over the mathematical constructions from [12]. However, the physical interpretation of states and operators has to be made in terms of the physical type II geometries now under consideration.

Let us first summarize the mathematical structure from [12]. To begin with, there is a kinematical Hilbert space $\mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_S$ where $\mathcal{H}_B$ is built from suitable functions of generalized connections in the bulk and $\mathcal{H}_S$ from suitable functions of generalized surface connections on S. [5] The bulk Hilbert space $\mathcal{H}_B$ describes the 'polymer excitations' of the bulk geometry. Each excitation which punctures S endows it with a certain quantum of area. The surface Hilbert space $\mathcal{H}_S$ consists of states of the level k, U(1) Chern-Simons theory for the connection $\tilde{V}^{\circ}$ on the punctured S. To ensure that S is indeed the desired horizon, only those states in $\mathcal{H}$ are selected which satisfy the operator analog of (2.61), called the quantum horizon boundary condition.[6] This operator equation on permissible states allows the connection and the triad to fluctuate but demands that they do so in tandem. As emphasized in [12], this operator equation is stringent and admits a sufficient number of solutions only because of a surprising agreement between an infinite set of eigenvalues of a quantum geometry operator in the bulk with an infinite set of eigenvalues of a Chern-Simons operator on S. The subspace $\mathcal{H}_{\text{kin}}$ of $\mathcal{H}$ on which this condition is met is then the Hilbert space of kinematic states describing

[5] In the construction of $\mathcal{H}_S$ in [12], critical use was made of the fact that U(1)-gauge transformations and diffeomorphisms are the canonical gauge for W. As already pointed out, this is again true for $\tilde{V}^{\circ}$ in the present context, so, again, the construction of $\mathcal{H}_S$ in [12] can be carried over without change.

[6] In addition, the states have quantum area and multipoles close to the macroscopic classical values. The multipole operators are defined below.

quantum geometry in the sector of general relativity now under consideration.

We now want to extract the quantum geometry of the horizon. For this, first recall the axial foliation ξ of S discussed in section 2.4.4, determined by the orbits of the axial symmetry field at the horizon. ξ varies non-trivially over phase space. However, ξ is a pure gauge degree of freedom; that is, the group of diffeomorphisms acts transitively on the space of possible ξ's. To extract the physics of the situation in the quantum theory, we simply fix ξ. This may be thought of as "gauge-fixing" for the purpose of reconstructing the quantum horizon geometry.

Regarding the legitimacy of this: It would perhaps be more satisfactory if the gauge-fixing could be done in a more standard manner, *i.e.* if one quantized a classical theory in which ξ is fixed. The problem is that if one *actually* gauge fixes ξ to be equal to some background ξ_0, that reduces the group of diffeomorphisms we divide out by at the quantum level. This reduction of the gauge group becomes messy when we take into consideration the handling of the extra structure used in the quantization of Γ_S, discussed in §IV.C and §V.B.2 of [12]. [7] In fact, if we do this, the final entropy we calculate is highly ambiguous. One viewpoint is that the mess is due to the fact that, by fixing ξ, we have broken diffeomorphism invariance, and diffeomorphism invariance is "sacred." The point, however, is that while we do not want to gauge fix ξ in the full technical sense of the word, we do need ξ in the intermediate step to build certain physical operators. Our justification for fixing ξ in constructing the operators is that we will be dividing out by diffeomorphisms eventually anyway so that the choice of a fixed ξ ultimately does not really matter. The most important operators we will build using ξ are the multipole operators; they will carry over to the physical Hilbert space, and once we are at the level of the physical Hilbert space, ξ isn't even a degree of freedom – it has been washed out by dividing by diffeomorphisms.

For convenience, let us furthermore introduce a coordinate ζ_0 labeling the leaves of ξ. Nothing we do is going to depend on the choice of this coordinate. To describe the quantum horizon geometry, the first step is to introduce an operator analog of the preferred coordinate ζ which played a key role in the classical theory. (2.62)

[7] The gauge group of the canonical theory no longer acts transitively on the set of possible such extra structures, so that the choice of the extra structure becomes a continuous-infinite dimensional quantization ambiguity.

suggests the definition

$$\hat{\zeta}(z) = -1 + 2\,\frac{\hat{a}(z)}{\hat{a}_S}\,, \tag{3.39}$$

where $\hat{a}_S$ and $\hat{a}(z)$ are the area operators associated with S and S_z, the 'southern' portion of S bounded by the leaf of ξ labeled by $\zeta_0 = z$. To make the action of this operator explicit, let us first note [12] that the Hilbert space $\mathcal{H}_{\rm kin}$ (on which $\hat{\zeta}(z)$ acts) can be decomposed as a direct sum,

$$\mathcal{H}_{\rm kin} = \oplus_{\mathcal{P},\vec{j}}\mathcal{H}^{\mathcal{P},\vec{j}}, \tag{3.40}$$

where $\mathcal{P}$ denotes a finite set of punctures and $\vec{j}$ is a set of half integers labeling the punctures. In each state in $\mathcal{H}^{\mathcal{P},\vec{j}}$ the Ith puncture is endowed with a quantum of area of magnitude $8\pi\gamma\sqrt{j_I(j_I+1)}\ell_{\rm Pl}^2$. Each $\mathcal{H}^{\mathcal{P},\vec{j}}$ is an eigenspace of the $\hat{\zeta}$ operator, with eigenvalue:

$$\zeta^{\mathcal{P},\vec{j}}(z) = -1 + 2\frac{\sum_{I'}\sqrt{j_{I'}(j_{I'}+1)}}{\sum_I\sqrt{j_I(j_I+1)}}\,, \tag{3.41}$$

where the sum in the numerator ranges over all punctures on S_z while the sum in the denominator ranges over all punctures on S. (The presence of $\hat{a}_S$ — rather than a_o — in the denominator ensures that eigenvalues of $\hat{\zeta}$ range from -1 to 1 as required in the definition of $Y_n^0(\hat{\zeta})$.) In the classical theory, the knowledge of ζ and multipoles $\mathring{I}_n, \mathring{L}_n$ suffices to determine the horizon geometry (q, V). The idea is to mimic that strategy. However, care is needed because the eigenvalues of $\hat{\zeta}$ are discontinuous functions: they jump at each z value where the leaf of ξ contains a puncture. (This is depicted schematically in figure 3.2.) This makes the quantum geometry 'rough'.

Fix a state in $\mathcal{H}^{\mathcal{P},\vec{j}}$. To make the nature of the quantum geometry in this state explicit, let us introduce a set of smooth functions $\zeta_{(k)}(z)$ on S which converge to the eigenvalue $\zeta^{\mathcal{P},\vec{j}}(z)$ in the sup norm as k tends to infinity. In addition, fix a coordinate φ compatible with ξ. (φ is *only* used in constructing the family of type I and type II metrics introduced below, and the operator $\hat{V}$. If one does not wish to construct these operators, one need not fix φ. Fixing φ constitutes another level of gauge fixing.) Each $\zeta_{(k)}$ then defines via (2.8) a round metric $\mathring{q}_{(k)}$. Using the

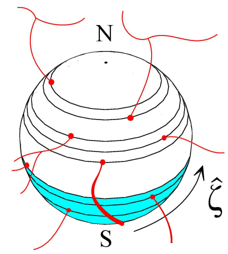

Figure 3.2. The eigenvalues of $\hat{\zeta}$ are discontinuous at leaves with punctures, but are constant elsewhere. As with the classical coordinate ζ, and as is necessary for subsequent steps in quantization, all the eigenvalues start at -1 at the south pole, increasing to 1 at the north pole.

fixed multipoles, for each k, we can also define a smooth function $\Psi_2^{(k)}$:

$$\Psi_2^{(k)}(\zeta_{(k)}) = -\frac{1}{R_o^2}\sum_{n=o}^{\infty}(\mathring{I}_n + i\mathring{L}_n)\,Y_n^0(\zeta_{(k)}). \tag{3.42}$$

Using $\mathrm{Re}\Psi_2^{(k)}$ in (2.64), we also obtain a sequence of functions $f^{(k)}$ through (2.64), and via (2.49) we then obtain a sequence of axisymmetric metrics $q_{(k)}$. As k tends to infinity, $\Psi_2^{(k)}$, $f^{(k)}$ and $f_\circ^{(k)}$ have well-defined limits $\Psi_2, f, f_\circ$ which, however, are discontinuous functions on S. However, $q_{(k)}, \mathring{q}_{(k)}$ do *not* admit limits even in the distributional sense because the metric coefficients are quadratic in $d\zeta_{(k)}/dz$ and these functions tend to Dirac distributions in the limit. This is not surprising because polymer quantum geometry does not naturally admit metric operators. Nonetheless, one can regard the family $q_{(k)}$ as providing an intuitive visualization of the quantum metric on the horizon in the following sense. First, a type II metric is completely determined by multipoles and the function ζ, and in the above construction multipoles are fixed and the $\zeta_{(k)}$ tend to the physical ζ uniformly. Second, every type II metric determines the multipoles I_n and for the family $q_{(k)}$ these are precisely the given $\mathring{I}_n$.

At first one might be confused by the fact that all the metrics $q^{(k)}$ so constructed are axisymmetric, whereas all we began with was a finite set of punctures, which is certainly not axisymmetric. However, one should remember the type II geometry is determined not just by the punctures, but *also* by the fixed multipoles. One needs both. Therefore, one should not expect any 'naive' relation between the location of the punctures and the physical, type II quantum geometry.

As one might expect from the type I analysis [12], the quantum connection operator can be defined more directly. Let $\hat{V}^\circ$ denote the quantum connection on $\mathcal{H}_S$ corresponding to the classical $\tilde{V}^\circ$; this operator comes directly from the geometric quantization of $\mathcal{H}_S$. Using (2.66) we can then define an operator $\hat{V}$ on $\mathcal{H}_{\rm kin}$ corresponding to the classical *'physical'* connection $\tilde{V}$:

$$\hat{V} = \hat{V}^\circ - \frac{1}{4}(f' - f'_\circ)(\hat{\zeta})\, D_a\varphi + \frac{i}{2}\tilde{\omega}_a(\hat{\zeta})\,, \tag{3.43}$$

where $f', f'_\circ$ and $\tilde{\omega}$ are all defined by the limiting procedure described above. $\hat{V}$ is a well-defined quantum connection: One can show that its holonomies along arbitrary (analytic) edges on S, including those which may have a puncture at their end points, are well-defined. $\hat{V}^\circ$ is flat everywhere except at the punctures in the sense that the holonomy around a closed loop not enclosing any puncture is identity. This is not the case with $\hat{V}$. The distortion in the quantum horizon geometry manifests itself through these non-trivial holonomies.

Finally, we can define multipoles operators. Taking the limit $k \to \infty$ of (3.42) we obtain the Ψ_2 operator corresponding to the fiducial foliation of S:

$$\hat{\Psi}_2(z) = -\frac{1}{R_o^2}\sum_{n=o}^{\infty}(\mathring{I}_n + i\mathring{L}_n)\, Y_n^0(\hat{\zeta}(z)). \tag{3.44}$$

(The numerical coefficient is left $1/R_o^2$ — rather than $4\pi/\hat{a}_S$ — to ensure the agreement with the definition of $\hat{\Psi}_2$ used in quantum horizon boundary condition of the type I analysis, where $\hat{\Psi}_2$ is taken to be the c-number $-\frac{2\pi}{a_o}$.) From (2.12),

$$I_n + iL_n = -\frac{a_S}{2}\int_{-1}^{1}\Psi_2 Y_{n,0}(\zeta)\mathrm{d}\zeta \tag{3.45}$$

Quantum multipoles can be defined by replacing ζ with $\hat{\zeta}$ in the above expression. However some care involving a regularization in terms of $\zeta_{(k)}$ is needed to give precise meaning to the integrand. Because the eigenvalues of $\hat{\zeta}$ depend on position in a discontinuous manner at the punctures, the eigenvalues of $\mathrm{d}\hat{\zeta}$ will have δ-functions at the punctures. But the other elements of the integrand, as they are functions of (the undifferentiated) $\hat{\zeta}$, will be *discontinuous* at the punctures.

Consequently, the meaning of the expression is ambiguous: we have delta functions multiplied into discontinuous functions. The $\zeta_{(k)}$ regularization provides a simple way out of this difficulty, so that we define

$$\hat{I}_n + i\hat{L}_n = -\lim_{k\to\infty} \frac{\hat{a}_S}{2} \int_{-1}^{1} \mathring{\Psi}_2(\hat{\zeta}_{(k)}) Y_{n,0}(\hat{\zeta}_{(k)}) \mathrm{d}\hat{\zeta}_{(k)} = \frac{\hat{a}_S}{a_o} \left(\mathring{I}_n + i\mathring{L}_n \right) \tag{3.46}$$

giving us the final expression for the multipole operators.

Remark. The fact that in the end we have *operators* corresponding to the multipoles is similar to what happens in the original type I quantization [12]: There, area was constant on the original phase space being quantized, but nevertheless area ended up being a non-trivial operator in the final quantum theory.

The reason this happened, essentially, was due to the fact that the surface degrees of freedom and bulk degrees of freedom were separated and quantized separately. Prior to imposition of the quantum boundary condition, the fixing of the area was *only* reflected in the surface quantum theory, not in the bulk. The fact that area was fixed was "communicated" to the bulk only through the imposition of the quantum boundary condition enforcing the proper relation between surface and bulk degrees of freedom. This condition is an operator equation which allows bulk and surface fields to fluctuate and only asks that they do so in tandem. As a consequence, after the quantization a restricted but still non-trivial horizon area operator remained.

Something similar happens here with the multipoles as well. The fact that multipoles are fixed is initially only reflected in the surface quantum theory — in the fact that, quantum mechanically, $\tilde{V}^\circ$ is the only connection degree of freedom intrinsic to the surface. (If multipoles were not fixed, $\tilde{V}^\circ$ would *not* be sufficient to embody all of the connection degrees of freedom intrinsic to S. [8]) This fixing of the multipoles is then communicated to the bulk only after the imposition of the quantum boundary condition. Once again, it is the indirect manner of the imposition of fixed multipoles in the bulk which allows non-trivial (though

[8]In the quantization [12], which we are here again using, connection degrees of freedom intrinsic to the horizon are deliberately excluded from the bulk quantum theory: The wave function in the bulk is only allowed to depend on holonomies along edges no open portion of which lies in S. Thus, the surface quantum theory alone represents these degrees of freedom, and it does so by W (here $\tilde{V}^\circ$). But $\tilde{V}^\circ$ only determines all connection degrees of freedom intrinsic to S (*viz.*, $\tilde{V}$) if the multipoles have been fixed.

restricted) operators corresponding to multipoles to exist in the quantum theory. [9]

Concretely, of course, one can understand the non-triviality of the multipole operators as descending from the non-triviality of the area operator: the presence of $\hat{a}_S$ in the numerator of (3.46) comes from the presence of the area element in the definition (2.12); it is through the presence of the area element in (2.12) that the non-triviality of $\hat{I}_n$, $\hat{L}_n$ arises. One is forced to use the physical area element operator because there is no fixed background area element. The presence of a_o in the denominator of (3.46) comes from the definition (3.44) of $\hat{\Psi}_2$.

3.4 Entropy

The calculation of entropy can be taken over from the type I analysis in a straightforward fashion. (Indeed, most of the above discussion of quantum operators encoding type II horizon quantum geometry is not needed for calculating the entropy.) We first impose the quantum Einstein equations following the same procedure as in [12]. This procedure is again applicable without change because the notions of canonical gauge in the present context — including the canonical gauge for $\tilde{V}^\circ$ — are identical to those in the type I case. Denote the resulting Hilbert space by $\mathcal{H}_{\rm phy}$. To incorporate the fact that we are interested in the horizon states of a black hole with fixed parameters, let us construct a micro-canonical ensemble consisting of states in $\mathcal{H}_{\rm phy}$ for which the eigenvalues of $\hat{a}_S$,$\hat{I}_n$,$\hat{L}_n$ lie in a small interval around a_o, $\mathring{I}_n$, $\mathring{L}_n$ and count the Chern-Simons surface states in this ensemble. [10] From (3.46) it is easy to see that, for this ensemble, the relative fluctuations in the multipoles will be equal to the relative fluctuations in the area. Since eigenstates of $\hat{a}_S$

[9]The situation is also qualitatively similar to geometric quantization of a spinning particle (though less so). There, the physical phase space is obtained by fixing the value of the total angular momentum classically — it is a 2-sphere whose radius is determined by the fixed value of the angular momentum squared. The angular momentum operator is defined on the resulting Hilbert space but its eigenvalues are different from the fixed, classical value.

[10]It has been suggested by some that an alternative area operator be used to define the relevant ensemble. Classically, there is more than one expression for the horizon area, such as $\oint_S \sqrt{\det q}$ or $k\gamma \oint_S \mid \Sigma \cdot r \mid$. When quantized, these lead to different area operators. However, of the possible area operators one could define, our viewpoint is that the standard loop quantum gravity area operator specialized to the surface S is the most robust as far as its physical meaningfulness. The other possible area operators have as their starting point expressions which have the interpretation of area at the horizon only, whence they have significantly weaker support interpretationally.

are also eigenstates of $\hat{I}_n, \hat{L}_n$ and eigenvalues of $\hat{I}_n, \hat{L}_n$ are completely determined by $\mathring{I}_n, \mathring{L}_n$ and a_S, the counting is the same as in the type I case [40, 41]. Hence the entropy $S_{\rm hor}$ is again given by

$$S_{\rm hor} = \frac{a_{\rm hor}}{4\ell_{\rm Pl}^2} - \frac{1}{2}\ln(\frac{a_{\rm hor}}{\ell_{\rm Pl}^2}) + o\ \ln(\frac{a_{\rm hor}}{\ell_{\rm Pl}^2}) \tag{3.47}$$

provided γ is chosen as in the type I analysis [41],

$$\gamma = 0.23753295796592\ldots \tag{3.48}$$

We will conclude with a comment on inclusion of matter fields. If matter is minimally coupled to gravity, as in the type I case, there are no matter surface terms in the symplectic structure, whence there are no independent *surface* degrees of freedom associated with these matter fields. Furthermore, the gravitational symplectic structure continues to be given by (3.35) whence the analysis summarized here undergoes only inessential changes. For an example showing how the analysis goes through in the minimally coupled case, see appendix A, in which the incorporation of the Maxwell field is discussed at length. In the case of non-minimally coupled matter (which is only touched upon here in this thesis), the gravitational symplectic structure does change. However, by introducing multipoles also for the matter fields, one can extend the analysis of [14] and show that, even in the non-minimally coupled case, the classically expected entropy expression [29] is recovered again for the same value of the Barbero-Immirzi parameter.

3.5 Discussion

We have mapped the entropy calculation problem for the type II case to the type I case: in terms of the variable $\tilde{V}_a^\circ$, the surface term in the symplectic structure is just Chern-Simons, and the relation between $\tilde{V}^\circ$ and the bulk variables is the same as in the type I case – that is, the appropriate "quantum boundary condition" we impose at the quantum level is still the same. Therefore we are able to do the quantization in the same manner as was done in the type I case, and get the same entropy. The difference between the type I and type II cases lies in the physical interpretation of $\tilde{V}^\circ$. In the type I case, the curvature $\mathrm{d}\tilde{V}^\circ$ is directly related to the curvature of the intrinsic geometry of the horizon. Consequently, in the type I case, the concentrations of $\mathrm{d}\tilde{V}^\circ$ at the punctures have a natural interpretion as conical singularities in the horizon geometry [12]. In the type II case, on the other hand, in order to obtain a physical interpretation, we introduced the $\hat{\zeta}$ operator, $\hat{\Psi}_2$ operator, and multipole operators.

A remark is in order as to the method of quantization we chose. It would be preferable to not fix multipoles prior to quantization. However, constructing operators corresponding to multipoles in full loop quantum gravity would be a highly non-trivial task. Multipoles are moments of Ψ_2, and Ψ_2 is a particular component of the curvature. One quantizes expressions involving curvature by rewriting them in terms of holonomies around loops and taking the limit as the size of the loops approaches zero. Unfortunately, in loop quantum gravity there is considerable ambiguity in the way such a limit is taken. This is the main source of ambiguity in the proposed definitions of the operators for the Hamiltonian constraint in loop quantum gravity, for example (see, e.g., section (6) in [38] and references cited therein). Nevertheless, one can hope these issues will be cleared up at some point.

It is worthwhile to note how much of an extension the present work represents. The space of type II isolated horizons is infinite dimensional, encompassing the (2 dimensional) family of Kerr isolated horizons as well as all possible distortions of such horizons compatible with axisymmetry. A vast range of astrophysically realistic black holes are therefore covered.

Chapter 4

Quantum field theory and its symmetry reduction

4.1 Introduction

In full loop quantum gravity, the Hamiltonian constraint is difficult to solve, and there are many ambiguities. It is possible that considerable progress could be made by restricting oneself to symmetric situations. Since most of the work in the classical theory is carried out in the context of homogeneous and isotropic models, and perturbations thereon, in quantum theory one should expect symmetric models to similarly play an important role. A major thrust in this direction is loop quantum cosmology (LQC) [44], which attempts to describe the homogeneous and isotropic sector of quantum gravity. Other symmetry reduced models in loop quantum gravity have also been constructed, for example, for the purpose of better understanding quantum black holes [45].

These models, however, are obtained by first symmetry reducing at the *classical* level and *then* quantizing. For these models, therefore, an issue of primary importance is the relation to the full theory, loop quantum gravity. The issue of the relation of LQC to the full theory is particularly important as possible predictions testable by cosmological observations are starting to be made based on LQC [46]. It is important to know to what extent tests of such possible predictions will in fact be tests of full loop quantum gravity.

In dealing with either LQC or related models, the underlying hope is that quantization and symmetry reduction commute in a suitable sense. The question of commutation of symmetry reduction and quantization is an old one. However, it is often not appreciated that the question of whether commutation is achieved depends in a critical way on what one means by the "symmetric sector" of the full quantum theory. One would like the "symmetric sector" of the full quantum theory (defined in some physically well-motivated way) to be isomorphic to the reduced-then-quantized theory.[1] If one can achieve such an isomorphism, not only at the level of Hilbert space structure, but also at the level of dynamics, one will have achieved full commutation of symmetry reduction and quantization. One may also have partial commutation: only the kinematical structure of the reduced theory may be isomorphic to the "symmetric sector" of the full theory. Nevertheless, even in such a situation one can ask if there is some choice of Hamiltonian operator in the reduced theory that "best" represents the information contained in the Hamiltonian of the full theory.

We will address all of these issues, but in the simple context where the full theory is well understood: the axisymmetric, free Klein-Gordon field in Minkowski space. We choose to consider a spatial symmetry (axisymmetry) in order to imitate the situations of primary interest, namely, LQC and related models, all of which are based on spatial symmetries. The analysis of the Klein-Gordon field presented here will suggest a method of generalizing the results to LQG and other theories. The programme of generalization for the case of LQG is sketched in the conclusions.

Above it was noted that there is an "ambiguity" in the notion of symmetry in quantum theory. At first this may seem surprising. Nevertheless, there are in fact at least two possible approaches to defining a notion of symmetry at the quantum level:

1. Demanding invariance under the action of the symmetry group
2. Taking a system of constraints that classically isolates the symmetric sector, and then imposing these constraints as one would in constrained quantization.

[1]The phrase 'reduced-then-quantized theory' is of course ambiguous. What is meant here, roughly, is the reduced theory, considered as a theory on the reduced spatial manifold (pp.80,139), quantized *using the same quantization methods as in the full theory.*

In the case of the axisymmetric Klein-Gordon theory, the notion of symmetry in sense 1 above is straightforward: a state is axisymmetric if it is annihilated by $\hat{\mathbb{L}}_z$, the operator corresponding to the total angular momentum in the z direction, since this is the generator of the action of rotations about the z axis.

However, we will find two distinct, but natural ways of implementing notion 2, corresponding to two different ways of reformulating the symmetry constraints as a first class system — to be referred to as reformulations 'c' and 'b'. (The motivation for these designations will be given below; this is a change from the terminology in [3].) In this chapter we thus actually consider three distinct notions of symmetry:

1. Requiring invariance under the action of the symmetry group ($\hat{\mathbb{L}}_z \Psi = 0$).

2. Imposition of $\mathcal{L}_\phi \hat{\varphi}(x) \Psi = 0$. (Constraint reformulation 'c'.)

3. Imposition of $a([\mathcal{L}_\phi f, \mathcal{L}_\phi g])\Psi = 0$. (Constraint reformulation 'b'.)

where ϕ^a is the axial Killing field on Minkowski space generating rotation about the z axis. In 3, $[f, g]$ denotes the phase space point determined by the initial data $\varphi = f$, $\pi = g$, with $a([f, g])$ denoting the associated annihilation operator (see next section). 'c' stands for "configuration", reflecting that symmetry is imposed only on the configuration variable φ; this symmetry is furthermore "exact" in the sense that fluctuations of φ from axisymmetry are zero. 'b' stands for "balanced", referring to the fact that symmetry is imposed on configuration and momenta in a more balanced way — fluctuations from symmetry are more evenly distributed between configuration and momenta (§4.5.5). We will refer to the three notions of symmetry in the above list by the names "invariance symmetry", "configuration symmetry", and "balanced symmetry" respectively. The latter two will also sometimes be referred to as "c-symmetry" and "b-symmetry" for short. A state satisfying one of these conditions of symmetry will likewise be referred to as "invariant", "c-symmetric", or "b-symmetric".

In the rest of this chapter, these three notions of symmetry will be explained, justified, characterized and compared in detail. Simpler, less central results will be stated without proof. The c-symmetric sector (with appropriate choice of inner product) and b-symmetric sector of the quantum theory will turn out to be naturally isomorphic to the reduced-then-quantized Hilbert space $\mathcal{H}_{red}$. Thus c-symmetry and b-symmetry as notions of symmetry achieve commutation of quantization and reduction. Furthermore

- c-symmetry and b-symmetry are strictly stronger than invariance symmetry. That is, if a state is c-symmetric or b-symmetric, it is also invariant, but not conversely (§4.4.5, 4.5.4).
- The space of c-symmetric states is the space of wavefunctions with support only on symmetric configurations. [2] It is thus the analogue of the notion of symmetry used by Bojowald in quantum cosmology.
- The space of b-symmetric states is equal to the span of the set of coherent states associated with the symmetric sector of the classical theory.
- In a precise sense, b-symmetric states are those in which all non-symmetric modes are unexcited.
- For b-symmetric states, fluctuations away from axisymmetry are minimized in a precise sense.
- The quantum Hamiltonian preserves the space of invariant states and the space of b-symmetric states, but not the space of c-symmetric states. Thus, balanced symmetry achieves full commutation of reduction and quantization, whereas configuration symmetry achieves commutation only at the level of Hilbert space structure.

Because of the last four items on this list, we argue that balanced symmetry should be preferred over configuration symmetry as an embedding of the reduced theory.

We then motivate and discuss a prescription for carrying arbitrary operators in the full theory over to the reduced theory. Finally, in the conclusions, we summarize the results and discuss application to loop quantum gravity. For convenience of the reader we have collected definitions of mathematical symbols in appendix E.

We conclude with a conceptually important point. One may object that commutation of reduction and quantization is achieved (in method 'b') only because

[2] Although this seems obvious at first, the rigorous formulation of this statement is more non-trivial to prove.

we have chosen to use an "indirect" notion of symmetry, rather than the obvious notion of invariance under the symmetry group. But, in fact, in quantum gravity, if the question of commutation is even to be *posed* (in a non-trivial way), one *must* use a notion of symmetry other than invariance symmetry. For, in quantum gravity, after the diffeomorphism constraint is solved, the action of any spatial symmetry is trivial, and so invariance symmetry becomes a vacuous notion. The reason for this is that the symmetry group becomes a subgroup of the gauge group of the canonical theory.[3] But if this is the case in quantum gravity, perhaps, then, one should not be surprised if also in other theories invariance symmetry is inappropriate for commutation questions. Indeed, in the Klein-Gordon case at hand, not only is invariance symmetry less desirable in that it does not achieve commutation, but, as the list of results indicates, b-symmetry satisfies many physical criteria which invariance symmetry does not.

4.2 Preliminaries: review of quantization of the Klein-Gordon field

First, let us review those aspects of the treatment [47] of the quantization of the free Klein-Gordon theory that will be used in the rest of this chapter (and in the associated appendices). This section will also serve to fix notation.

4.2.1 Classical theory

Let Σ denote a fixed Cauchy surface: a spatial hyperplane in Minkowski space. Let q_{ab} denote the induced Euclidean metric on Σ. The phase space Γ is a vector space parametrized by two smooth real scalar fields $\varphi(x)$ and $\pi(x)$ on Σ with Schwartz fall-off at infinity. The symplectic structure is simply

$$\Omega([\varphi,\pi],[\varphi',\pi']) = \int_\Sigma (\pi\varphi' - \varphi\pi')\mathrm{d}^3x \tag{4.1}$$

[3]As discussed in the conclusions, this situation can be exactly mimicked in the axisymmetric Klein-Gordon case by simply declaring the three components of the total angular momentum to be constraints, so that the canonical gauge group is just the group of SO(3) rotations about the origin.

so that the Poisson brackets between the basic variables are

$$\{\varphi(x), \pi(y)\} = \delta^3(x,y) \tag{4.2}$$

and $\{\varphi(x), \varphi(y)\} = \{\pi(x), \pi(y)\} = 0$. In a word, Γ is the cotangent bundle over $\mathcal{S}(\Sigma)$, the space of Schwartz functions on Σ.

The Hamiltonian of the scalar field with mass m is

$$\mathbb{H} = \frac{1}{2}\int_\Sigma \{\pi^2 + (\vec{\nabla}\varphi)\cdot(\vec{\nabla}\varphi) + m^2\varphi^2\}\mathrm{d}^3x \tag{4.3}$$

From this Hamiltonian, one derives the equations of motion to be

$$\dot{\varphi} = \pi \tag{4.4}$$

$$\dot{\pi} = \Delta\varphi - m^2\varphi \tag{4.5}$$

where Δ is the Laplacian on Σ. Let $\Theta := -\Delta + m^2$. Choosing the complex structure

$$J[\varphi, \pi] = [-\Theta^{-\frac{1}{2}}\pi, \Theta^{\frac{1}{2}}\varphi] \tag{4.6}$$

we turn Γ into a complex vector space. The Hermitian inner product thereby determined on Γ is then

$$\begin{aligned} \langle[\varphi,\pi],[\varphi',\pi']\rangle &:= \frac{1}{2}\Omega(J[\varphi,\pi],[\varphi',\pi']) - \mathrm{i}\frac{1}{2}\Omega([\varphi,\pi],[\varphi',\pi']) \\ &= \frac{1}{2}(\Theta^{\frac{1}{2}}\varphi,\varphi') + \frac{1}{2}(\Theta^{-\frac{1}{2}}\pi,\pi') - \frac{\mathrm{i}}{2}(\pi,\varphi') + \frac{\mathrm{i}}{2}(\varphi,\pi') \end{aligned} \tag{4.7}$$

where for f, g functions on Σ, we define $(f,g) := \int_\Sigma fg\mathrm{d}^3x$. Completing Γ with respect to this Hermitian inner product gives the one particle Hilbert space h.

In constructing the Hilbert space for the field theory, one then has two possible approaches: the Fock and Schrödinger approaches.

4.2.2 Fock quantization

In the Fock approach, the full Hilbert space is constructed as

$$\mathcal{H} := \mathcal{F}_s(h) := \bigoplus_{n=0}^{\infty} \bigotimes_s^n h \tag{4.8}$$

where $\overset{n}{\otimes}_s h$ denotes the symmetrized tensor product of n copies of h. [4]

For each n, the inner product on h induces a unique inner product on $\overset{n}{\otimes} h$ via the condition

$$\langle \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n, \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n \rangle_{\overset{n}{\otimes} h} = \langle \psi_1, \phi_1 \rangle \langle \psi_2, \phi_2 \rangle \cdots \langle \psi_n, \phi_n \rangle \tag{4.9}$$

for all $\{\psi_i\}, \{\phi_i\} \in h$. This in turn induces an inner product on $\overset{n}{\otimes}_s h$.

Let $A, B, C \ldots$ denote abstract indices associated with the one particle Hilbert space h. Let prime denote topological dual. Then, for each n, define the complex conjugation map $\overset{n}{\otimes} h \mapsto (\overset{n}{\otimes} h)'$, $\psi^{A_1 \ldots A_n} \mapsto \overline{\psi}_{A_1 \ldots A_n}$ by

$$\overline{\psi}_{A_1 \ldots A_n} \phi^{A_1 \ldots A_n} := \langle \psi, \phi \rangle_{\overset{n}{\otimes} h}. \tag{4.10}$$

A given member $\Psi \in \mathcal{H} = \mathcal{F}_s(h)$ takes the form

$$\Psi = (\psi, \psi^{A_1}, \psi^{A_1 A_2}, \psi^{A_1 A_2 A_3}, \ldots) \tag{4.11}$$

with each component $\psi^{A_1 \ldots A_n}$ satisfying $\psi^{A_1 \ldots A_n} = \psi^{(A_1 \ldots A_n)}$. The inner product on $\mathcal{H}$ is then defined by

$$\langle \Psi, \Phi \rangle = \sum_{n=0}^{\infty} \overline{\psi}_{A_1 \ldots A_n} \phi^{A_1 \ldots A_n}. \tag{4.12}$$

Given an element $\xi^A = [\varphi, \pi]^A \in h$, one has associated creation and annihilation operators which act on $\mathcal{H}$ by

$$a^\dagger(\xi)\Psi \;:=\; (0, \psi \xi^{A_1}, \sqrt{2}\xi^{(A_1}\psi^{A_2)}, \sqrt{3}\xi^{(A_1}\psi^{A_2 A_3)}, \ldots) \tag{4.13}$$

[4] That is, $\overset{n}{\otimes} h$ is the space of all continuous multilinear maps $\overset{n}{\times} h' \to \mathbb{C}$; $\overset{n}{\otimes}_s h$ is then the space of all members of $\overset{n}{\otimes} h$ invariant under arbitrary permutations of arguments.

$$a(\xi)\Psi \;:=\; (\overline{\xi}_A\psi^A, \sqrt{2}\overline{\xi}_A\psi^{AA_1}, \sqrt{3}\overline{\xi}_A\psi^{AA_1A_2}, \ldots) \tag{4.14}$$

One can check

$$[a(\xi), a^\dagger(\eta)] = \langle \xi, \eta \rangle\, \mathbb{1} \tag{4.15}$$

The unique normalized state annihilated by all the annihilation operators is the vacuum; it is given by

$$\Psi_0 := (1, 0, 0, 0, \ldots) \tag{4.16}$$

In terms of the creation and annihilation operators, the representation of the smeared field operators is given by

$$\hat{\varphi}[f] \;:=\; \mathrm{i}\{a([0, f]) - a^\dagger([0, f])\} \tag{4.17}$$

$$\hat{\pi}[g] \;:=\; -\mathrm{i}\{a([g, 0]) - a^\dagger([g, 0])\} \tag{4.18}$$

With these definitions, one can check

$$[\,\hat{\varphi}[f], \hat{\pi}[g]\,] = \mathrm{i}\int_\Sigma \mathrm{d}^3x f g \equiv \mathrm{i}(f, g) \tag{4.19}$$

with all other commutators zero, so that (4.17) indeed gives a representation of the Poisson algebra of smeared field variables (4.2).

It is useful to note that, by using the fact that $a^\dagger(\xi)$ is linear and $a(\xi)$ is anti-linear in ξ, one can invert (4.17) to obtain an expression for the creation and annihilation operators in terms of the field operators:

$$\begin{aligned} a([f, g]) &= \frac{1}{2}\hat{\varphi}[\Theta^{\frac{1}{2}} f - \mathrm{i}g] + \frac{1}{2}\hat{\pi}[\Theta^{-\frac{1}{2}} g + \mathrm{i}f] \\ a^\dagger([f, g]) &= \frac{1}{2}\hat{\varphi}[\Theta^{\frac{1}{2}} f + \mathrm{i}g] + \frac{1}{2}\hat{\pi}[\Theta^{-\frac{1}{2}} g - \mathrm{i}f] \end{aligned} \tag{4.20}$$

These expressions can then be carried over to the classical theory to obtain functions on Γ which are classical analogues of the creation and annihilation operators. Upon simplifying the expressions for these classical analogues, one obtains the remarkably simple result

$$a([f, g]) \;=\; \langle [f, g], [\varphi, \pi] \rangle \tag{4.21}$$

$$a^{\dagger}([f,g]) \;=\; \langle[\varphi,\pi],[f,g]\rangle \tag{4.22}$$

The Poisson brackets among these classical analogues exactly mimic the commutators of the quantum counterparts.

Next, we quantize the Hamiltonian. Rewriting the classical Hamiltonian (4.3),

$$\begin{aligned}\mathbb{H} &= \frac{1}{2}\int_{\Sigma}(\pi^2-\varphi\Delta\varphi+m^2\varphi^2)\mathrm{d}^3x \\ &= \frac{1}{2}\int_{\Sigma}(\pi^2+\varphi\Theta\varphi)\mathrm{d}^3x \end{aligned}\tag{4.23}$$

From (4.4) and (4.6), we obtain the one particle Hamiltonian operator on h,

$$\begin{aligned}\hat{H}[\varphi,\pi] &:= J\frac{\mathrm{d}}{\mathrm{d}t}[\varphi,\pi] \\ &= [\Theta^{\frac{1}{2}}\varphi,\Theta^{\frac{1}{2}}\pi]\end{aligned}\tag{4.24}$$

In terms of this, the classical Hamiltonian can be expressed as

$$\mathbb{H}=\langle[\varphi,\pi],\hat{H}[\varphi,\pi]\rangle \tag{4.25}$$

Let $\{\xi_i=[f_i,g_i]\}$ denote an arbitrary orthonormal basis of h. Then

$$\begin{aligned}\mathbb{H} &= \sum_{i,j}\langle[\varphi,\pi],\xi_i\rangle\langle\xi_i,\hat{H}\xi_j\rangle\langle\xi_j,[\varphi,\pi]\rangle \\ &= \sum_{i,j}\langle\xi_i,\hat{H}\xi_j\rangle a^{\dagger}(\xi_i)a(\xi_j)\end{aligned}\tag{4.26}$$

Which is an expression that can be taken directly over to the quantum theory, using normal ordering:

$$\hat{\mathbb{H}}=\sum_{i,j}\langle\xi_i,\hat{H}\xi_j\rangle a^{\dagger}(\xi_i)a(\xi_j) \tag{4.27}$$

4.2.3 Schrödinger quantization

As mentioned, the classical phase space Γ has a cotangent bundle structure $T^*\mathcal{C}$ over some appropriately defined configuration space $\mathcal{C}$.

In the case of a finite number of degrees of freedom, the standard way to quantize a cotangent bundle $T^*\mathcal{C}$ is via a Schrödinger representation – that is, a representation of the field operators on an $L^2(\mathcal{C}, \mathrm{d}\mu)$ for some appropriately chosen measure μ.

In the field theory case, however, the measures one is interested in using are usually not supported on the classical configuration space $\mathcal{C}$, but rather on some appropriate distributional-like extension $\overline{\mathcal{C}}$. This extension is referred to as the *quantum configuration space.*

In the case of the free Klein-Gordon field in Minkowski space, the appropriate quantum configuration space can be taken to be the space of tempered distributions $\mathcal{S}'(\Sigma)$ on Σ [48]. $\mathcal{S}(\Sigma)$ denotes the space of Schwartz functions equipped with the appropriate topology [49], and the prime indicates the topological dual. From here on φ will denote an element of $\mathcal{S}'(\Sigma)$.

The appropriate measure is the Gaussian measure heuristically given by the expression

$$\text{“}\mathrm{d}\mu = \exp\left\{-\frac{1}{2}(\varphi, \Theta^{\frac{1}{2}}\varphi)\right\}\mathcal{D}\varphi\text{”} \tag{4.28}$$

where $\mathcal{D}\varphi$ is the fictitious translation-invariant “Lesbesgue” measure on $\mathcal{S}'(\Sigma)$ scaled such that $\int_{\mathrm{d}} \mu = 1$. To define the measure rigorously, one can specify its *Fourier transform.* The Fourier transform χ_μ of a measure μ is defined by

$$\chi_\mu(f) := \int_{\varphi \in \mathcal{S}'(\Sigma)} \mathrm{e}^{\mathrm{i}\varphi(f)} \mathrm{d}\mu \tag{4.29}$$

for $f \in \mathcal{S}(\Sigma)$. The Fourier transform giving rise to (the rigorous version of) the measure in (4.28) is

$$\chi_\mu(f) = \exp\left\{-\frac{1}{2}(f, \Theta^{-\frac{1}{2}}f)\right\} \tag{4.30}$$

For further details, see [48]. $\mathcal{H} = L^2(\mathcal{S}'(\Sigma), \mathrm{d}\mu)$ is then the Hilbert space of states in the quantum field theory.

For this thesis it will also be necessary to introduce a certain dense subset of $\mathcal{H}$ — the space of *cylindrical functions.* A function $\Psi : \mathcal{S}'(\Sigma) \to \mathbb{C}$ is called *cylindrical* if $\Psi[\varphi] = F(\varphi(e_1), \ldots, \varphi(e_n))$ for some $\{e_1, \ldots, e_n\} \subseteq \mathcal{S}(\Sigma)$ (referred to as “probes”) and some smooth function $F : \mathbb{R}^n \to \mathbb{C}$ (with growth at most exponential [5]). More specifically, such a Ψ is said to be *cylindrical with respect to* the "probes" $e_1, \ldots, e_n$. Let the space of cylindrical functions be denoted Cyl. $\mathcal{H}$ is the Cauchy completion of Cyl.

Next, the representation of the field observables on $\mathcal{H}$ is

$$\begin{aligned}
(\hat{\varphi}[f]\Psi)[\varphi] &:= \varphi[f]\Psi[\varphi] && (4.31)\\
(\hat{\pi}[g]\Psi)[\varphi] &:= \left[\text{Self-adjoint part of } -\mathrm{i}\int_\Sigma \mathrm{d}^3x g \frac{\delta}{\delta\varphi}\right]\Psi[\varphi] \\
&= -\mathrm{i}\int_\Sigma \mathrm{d}^3x \left(g\frac{\delta}{\delta\varphi} - \varphi\Theta^{\frac{1}{2}}g\right)\Psi[\varphi] && (4.32)
\end{aligned}$$

Note that the action is well defined on Cyl and preserves Cyl. Thus a finite linear combination of finite products of $\hat{\varphi}[f]$ and $\hat{\pi}[g]$ is densely defined on $\mathcal{H}$.

We then use equations (4.20) to define creation and annihilation operators in the Schrödinger picture. Substituting (4.31) and (4.32) into these expressions and simplifying, we obtain

$$\begin{aligned}
a([f,g]) &= \frac{1}{2}\int_\Sigma \mathrm{d}^3x \left(f - \mathrm{i}\Theta^{-\frac{1}{2}}g\right)\frac{\delta}{\delta\varphi} && (4.33)\\
a^\dagger([f,g]) &= \hat{\varphi}[\Theta^{\frac{1}{2}}f + \mathrm{i}g] - \frac{1}{2}\int_\Sigma \mathrm{d}^3x \left(f + \mathrm{i}\Theta^{-\frac{1}{2}}g\right)\frac{\delta}{\delta\varphi} && (4.34)
\end{aligned}$$

(In (4.33), the $\hat{\varphi}$ terms exactly cancel, leaving only a $\delta/\delta\varphi$ term.) The unique normalized state in the kernel of all of the annihilation operators is

$$\Psi_0[\varphi] \equiv 1 \qquad (4.35)$$

The availability of a vacuum state and creation and annihilation operators in the Schrödinger picture allows one to construct a mapping from the Fock Hilbert space into the Schrödinger Hilbert space. One finds that the mapping is unitary, so that the Fock and Schrödinger descriptions of the theory are equivalent.

Let us next consider the Hamiltonian operator on $\mathcal{H}$. Substituting (4.33),(4.34)

[5] That is, we require $\lim_{t\to\infty} \frac{F(t\vec{u})}{e^{\alpha t}}$ to be finite for all $\alpha \in \mathbb{R}^+$ and $\vec{u} \in \mathbb{R}^n$. This ensures that the cylindrical functions belong to $\mathcal{H}$ (*i.e.* that they be square integrable).

into (4.27) and simplifying, one obtains

$$\hat{\mathbb{H}} = \int_{\Sigma^2} \mathrm{d}^3x \mathrm{d}^3y A(x,y)\varphi(y)\frac{\delta}{\delta\varphi(x)} - \int_{\Sigma^2} \mathrm{d}^3x\mathrm{d}^3y B(x,y)\frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)} \tag{4.36}$$

where

$$A(x,y) = \frac{1}{2}\sum_{i,j}\langle\xi_i, \hat{H}\xi_j\rangle(f_j - \mathrm{i}\Theta^{-\frac{1}{2}}g_j)(x)(\Theta^{\frac{1}{2}}f_i + \mathrm{i}g_i)(y) \tag{4.37}$$

$$B(x,y) = \frac{1}{4}\sum_{i,j}\langle\xi_i, \hat{H}\xi_j\rangle(f_i + \mathrm{i}\Theta^{-\frac{1}{2}}g_i)(x)(f_j - \mathrm{i}\Theta^{-\frac{1}{2}}g_j)(y) \tag{4.38}$$

where, as before, $\{\xi_i \equiv [f_i, g_i]\}$ is an orthonormal basis of the one particle Hilbert space. By integrating $A(x,y)$ and $B(x,y)$ against test functions, one can show that $A(x,y)$ is the integral kernel of $\Theta^{\frac{1}{2}}$ and $B(x,y) = \frac{1}{2}\delta^3(x,y)$. Thus, [6]

$$\hat{\mathbb{H}} = \int_{\Sigma} \mathrm{d}^3x \left\{ (\Theta^{\frac{1}{2}}\varphi)(x)\frac{\delta}{\delta\varphi(x)} - \frac{1}{2}\frac{\delta^2}{\delta\varphi(x)\delta\varphi(x)} \right\} \tag{4.39}$$

The rigorous meaning of this expression is not immediately obvious. However, note that for any cylindrical function $\Psi[\varphi] = F(\varphi(e_1), \ldots, \varphi(e_n))$,

$$\frac{\delta}{\delta\varphi(x)}\Psi[\varphi] = \sum_{i=1}^{n} e_i(x)(\partial_i F)(\varphi(e_1), \ldots, \varphi(e_n)). \tag{4.40}$$

Therefore, the action of $\hat{\mathbb{H}}$ (4.39) on the space of cylindrical functions, Cyl, is well defined. Furthermore one can check that $\hat{\mathbb{H}}$ preserves Cyl. In proving this, the fact that there is no term quadratic in φ in (4.39) is important.[7] Since Cyl is dense in $\mathcal{H}$, we may take Cyl to be the domain of $\hat{\mathbb{H}}$; with this domain, one can show that $\hat{\mathbb{H}}$ is essentially self-adjoint. Let us sketch one way to show this. Let Cyl_{poly} denote the space of all functions of the form $\Psi[\varphi] = F(\varphi(e_1), \ldots, \varphi(e_n))$ with F

[6] To our knowledge, expression (4.39) has not appeared in the literature.

[7] The reason the φ^2 term is absent is that our quantum measure is Gaussian. Thus there is a tight relation between kinematics (the choice of measure, and hence the representation of the quantum algebra) and dynamics (the Hamiltonian operator). This is the origin of the usual statement that in quantum field theory "dynamics dictates the choice of kinematics"!

polynomial. Then $\mathrm{Cyl}_{poly} \subset \mathrm{Cyl}$ and Cyl_{poly} is dense in $\mathcal{H}$. One can show that

$$\mathrm{Cyl}_{poly} \subseteq (\hat{\mathbb{H}} \pm i)[\mathrm{Cyl}] \tag{4.41}$$

so that, with domain Cyl, the range of $\hat{\mathbb{H}} \pm i$ is dense in $\mathcal{H}$. It follows from the corollary to theorem VIII.3 in [49] that $\hat{\mathbb{H}}$ with domain Cyl is essentially self-adjoint. Thus, $\hat{\mathbb{H}}$ with domain Cyl has a unique self-adjoint extension. It is this self-adjoint extension that we henceforth take to be the meaning of $\hat{\mathbb{H}}$.

4.3 Different methods of imposing symmetry

4.3.1 Classical analysis

As mentioned in the introduction, incorporation of symmetry by requiring invariance under the action of the symmetry group is straightforward in the present context: it corresponds to requiring a state to be annihilated by the operator $\hat{\mathbb{L}}_z$ corresponding to the z component of the total angular momentum. However, selection of the symmetric sector via imposition of a system of constraints deserves further explanation.

Classically the condition for symmetry takes the form of the constraints

$$\mathcal{L}_\phi \varphi = 0 \quad \text{and} \quad \mathcal{L}_\phi \pi = 0 \tag{4.42}$$

If we smear the constraints, they take the form

$$\varphi[\mathcal{L}_\phi f] = 0 \quad \text{and} \quad \pi[\mathcal{L}_\phi f] = 0 \tag{4.43}$$

for all test functions f in $\mathcal{S}(\Sigma)$, the space of Schwartz functions. The form of the smearings $\mathcal{L}_\phi f$ and $\mathcal{L}_\phi g$ comes from an integration by parts. More generally, the significance of the form $\mathcal{L}_\phi f$ for test functions is the following. Let $\mathcal{S}(\Sigma)_{inv}$ denote the space of elements of $\mathcal{S}(\Sigma)$ Lie dragged by ϕ^a. One can show that the space all test functions of the form $\mathcal{L}_\phi f$ is precisely the orthogonal complement of $\mathcal{S}(\Sigma)_{inv}$ in $\mathcal{S}(\Sigma)$ (with respect to the usual inner product). Thus, if we set $\mathcal{S}(\Sigma)_\perp := \{\mathcal{L}_\phi f \mid f \in \mathcal{S}(\Sigma)\}$, then $\mathcal{S}(\Sigma) = \mathcal{S}(\Sigma)_{inv} \oplus \mathcal{S}(\Sigma)_\perp$. This gives us another way to view the above set of smeared constraints: They are the non-symmetric components of the fields; by requiring these to vanish, we impose symmetry.

Therefore, what we would ideally like to do in the quantum theory is impose

$$\hat{\varphi}[\mathcal{L}_\phi f]\Psi = 0 \quad \text{and} \quad \hat{\pi}[\mathcal{L}_\phi f]\Psi = 0 \tag{4.44}$$

for all $f \in \mathcal{S}(\Sigma)$. However, the proposed system of constraints is second class, and, as Dirac taught us, such systems of constraints cannot be consistently imposed in quantum theory in this fashion. One will find that the unique solution to these constraints is the zero vector.

To get around this difficulty, the strategy is to reformulate the constraints (4.43) as an equivalent first class system. We consider two such reformulations:

1. the set of constraints $\{\varphi[\mathcal{L}_\phi f]\}_{f\in\mathcal{S}(\Sigma)}$
2. the set of constraints $\{a([\mathcal{L}_\phi f, \mathcal{L}_\phi g])\}_{f,g\in\mathcal{S}(\Sigma)}$

We will refer to these as constraint set 'c' and constraint set 'b', respectively. 'c' stands for 'configuration' and refers to the fact that symmetry is imposed only on the configuration variable. 'b' stands for 'balanced' and refers to the fact that configuration and momenta are treated in a more balanced way. $a([f,g])$ here is the classical analogue of the annihilation operator as given in (4.21). Thus, constraint set 'b' consists in complex linear combinations of the constraints in (4.43). Each of the constraint sets 'c' and 'b' forms a first class system. Although the constraint set 'c' is obtained by simply dropping all the constraints on momenta, nevertheless as explained below 'c' is in a certain sense (relevant for quantum theory) equivalent to the full set of constraints.

We should also mention that other proposals for imposing second class constraints have been made in the past, such as that proposed in Klauder's 'universal procedure' for imposing constraints [50]. There is in fact a relation between approach 'b' and Klauder's approach: The former is a special case of the latter with some natural choices made. This is discussed later on in section 4.6 of this thesis. In addition approach 'b' has similarities to the method of imposing second class constraints discussed in [51], as was noticed after this work was completed.

Let us introduce some notation. Let $\Gamma = \{[\varphi, \pi]\}$ be the full classical phase space. Let

$$\Gamma_{inv} := \{[\varphi,\pi] \in \Gamma \mid \mathcal{L}_\phi \varphi = 0 \text{ and } \mathcal{L}_\phi \pi = 0\}$$

$$\Gamma_c := \{[\varphi,\pi] \in \Gamma \mid \mathcal{L}_\phi \varphi = 0\}$$

$$\Gamma_b := \{[\varphi,\pi] \in \Gamma \mid a([\mathcal{L}_\phi f, \mathcal{L}_\phi g]) \mid_{[\varphi,\pi]} = 0 \quad \forall f, g \in \mathcal{S}(\Sigma)\}$$

So that Γ_c is the constraint surface associated with constraint set 'c', and Γ_b is the constraint surface associated with constraint set 'b'.

Analysis of constraint set 'c'

Since the constraint set 'c' is obtained by dropping constraints from the full set (4.43), it is not surprising that Γ_c is larger than Γ_{inv}. However, the symplectic structure induced on Γ_c via pull-back, $\Omega_c := i^*\Omega$, is degenerate – as we should expect since constraint set 'c' is first class. The degenerate directions are just the "gauge" generated by the constraints $\varphi[\mathcal{L}_\phi f]$, namely $\pi(x) \mapsto \pi(x) + \mathcal{L}_\phi f$. If we divide out by this "gauge," the resulting manifold, $\hat{\Gamma}_c$ is naturally isomorphic to Γ_{inv}.

One may object: this notion of "gauge" is not *physical* gauge; it is gauge generated by constraints that we have imposed completely by hand. This is true, but the point is that when a constraint is imposed at the quantum level, you automatically divide out by the corresponding "gauge" *whether or not the gauge is "physical"*.

At the quantum level, we will find that the solution to constraint set 'c', when equipped with an appropriate inner product, is naturally isomorphic to the Hilbert space one obtains when first reducing and then quantizing. The fact that $\hat{\Gamma}_c$ is naturally isomorphic to Γ_{inv} is the imprint of this fact on the classical theory.

A final important note about constraint set 'c' is that its elements *do not* weakly Poisson-commute with the total Hamiltonian for the free scalar field (4.3). This foreshadows the fact that in the quantum theory, the total Hamiltonian operator will not preserve the solution space to constraint set 'c'.

Analysis of constraint set 'b'

First, it is important to note that the classical observable $a([f,g])$, when expanded out as $a([f,g]) = \langle [f,g], [\varphi,\pi] \rangle$, is a complex linear combination of the constraints (4.43). In fact, in rewriting the full constraint set (4.43) as constraint set 'b', *no constraints have been dropped.* Rather, one has reduced the number of constraints by half by simply taking complex linear combinations of the original constraints. So, $\Gamma_b = \Gamma_{inv}$.

It is easy to see how this works in a simpler example. Suppose we are working in a theory in which $\{x_1, x_2, x_3, p_1, p_2, p_3\}$ are the basic variables, and we want to impose the second class system of constraints $x_3 = 0$, $p_3 = 0$. The analogue of reformulation 'c' in this context would be to just drop the $p_3 = 0$ constraint. The analogue of reformulation 'b' would be to replace the two constraints with the single constraint $z_3 := x_3 + ip_3 = 0$. Obviously z_3, being only a single constraint, makes up a first class system of constraints. Nevertheless, classically, $z_3 = 0$ is completely equivalent to $x_3 = 0$ and $p_3 = 0$. This is one of the strengths of reformulation strategy 'b': the reformulation is classically completely equivalent to the original set of constraints, but is now a first class system so that it can be imposed consistently in quantum theory.

But one may object: how is this possible? You cannot change the fact that a certain constraint submanifold is first or second class merely by reformulating it in terms of different constraints because first-class and second-class character are *geometrical* properties of the constraint submanifold [52]. This is indeed true. Our underlying constraint submanifold is still *geometrically* a second-class constraint surface. We have merely allowed it to be *formally* expressed as a first class system by allowing our constraints to be complex. But fortunately, for a system of constraints to be consistently implementable in quantum theory, it is sufficient that they be only formally first class – i.e., that their Poisson brackets with each other vanish weakly.

Another fact that is important to note is that all the elements of constraint set 'b' weakly Poisson-commute with the full Hamiltonian $\mathbb{H}$. This points to the fact that, in quantum theory, the full Hamiltonian operator $\hat{\mathbb{H}}$ *will* preserve the solution space to constraint set 'b'.

4.3.2 Setting up the quantum analysis

Recall that Cyl denotes the space of cylindrical functions on $\mathcal{S}'(\Sigma)$. Let Cyl^* denote its algebraic dual.

Let

$$\mathcal{H}_{inv} := \{\Psi \in \mathcal{H} \mid \hat{\mathbb{L}}_z \Psi = 0\} \tag{4.45a}$$

$$\mathrm{Cyl}^*_{inv} := \{\eta \in \mathrm{Cyl}^* \mid \hat{\mathbb{L}}_z^* \eta = 0\} \tag{4.45b}$$

$$\mathrm{Cyl}^*_c := \{\eta \in \mathrm{Cyl}^* \mid \hat{\varphi}[\mathcal{L}_\phi f]^* \eta = 0 \quad \forall f \in \mathcal{S}(\Sigma)\} \tag{4.45c}$$

$$\mathcal{H}_b := \{\Psi \in \mathcal{H} \mid a([\mathcal{L}_\phi f, \mathcal{L}_\phi g])\Psi = 0 \quad \forall f, g \in \mathcal{S}(\Sigma)\} \tag{4.45d}$$

$\mathcal{H}_{inv}$ and Cyl^*_{inv} are the sets of elements in $\mathcal{H}$ and Cyl^* fixed by the natural action of rotations about the z-axis, whence they are implementations of "invariance symmetry," the first notion of symmetry mentioned in the introduction. (Cyl^*_{inv} has been introduced simply for the purpose of comparison with Cyl^*_c.)

Cyl^*_c is the solution space for constraint set 'c' at the quantum mechanical level. Constraint set 'c' forces its solutions to have support only on symmetric configurations, as we shall see below. The space of symmetric configurations has measure zero with respect to the quantum measure μ on $\mathcal{S}'(\Sigma)$. Since μ characterizes the inner product in $\mathcal{H}$, all solutions to 'c' in $\mathcal{H}$ thus have norm zero, whence one must go to the larger space Cyl^* to find non-trivial solutions.

In addition, one should note that the characterization of Cyl^*_c as the space of functions with support only on symmetric configurations makes Cyl^*_c the analogue of the notion of symmetry used by Bojowald to embed loop quantum cosmology and other symmetry reduced models into full loop quantum gravity [16].

$\mathcal{H}_b$ is the solution space for constraint set 'b' at the quantum mechanical level.

4.4 Analysis of Cyl^*_c

4.4.1 Two preliminary lemmas

We will denote the group of rotations about the z-axis by $\mathcal{T} \subset \mathrm{Diff}(\Sigma)$. In the reduced theory, the spatial manifold is taken to be $B := \Sigma/\mathcal{T}$, and the quantum configuration space $\mathcal{S}'(B)$. Let $P : \Sigma \to B$ denote canonical projection. Let

$\mathcal{S}'(\Sigma)_{inv}$ and $\mathcal{S}(\Sigma)_{inv}$ denote the $\mathcal{T}$-invariant subspaces of $\mathcal{S}'(\Sigma)$ and $\mathcal{S}(\Sigma)$, respectively. $\mathcal{S}(\Sigma)_{inv}$ is then naturally identifiable with $\mathcal{S}(B)$; we make this identification. Define $I : \mathcal{S}'(\Sigma)_{inv} \to \mathcal{S}'(B)$ by $[I(\alpha)](f) := \alpha(P^* f)$. Let $\pi : \mathcal{S}(\Sigma) \to \mathcal{S}(B)$ denote group averaging with respect to the action of $\mathcal{T}$. We here use "group averaging" in a more general sense than usual in that we are not group averaging "states." It will be convenient in this chapter to let "group averaging" have this more general meaning of averaging elements of any vector space over the action of a group. One can show the pull-back $\pi^* : \mathcal{S}'(B) \to \mathcal{S}'(\Sigma)_{inv}$ is the inverse of I, so that

Lemma 4.1. *I is an isomorphism.*

Thus $\mathcal{S}'(\Sigma)_{inv}$ and $\mathcal{S}'(B)$ are naturally isomorphic. Because of this, henceforth we will simply identify these two spaces. That is, the isomorphism I will sometimes not be explicitly written. In addition, we will sometimes implicitly use the fact that I is compatible with the structure of the cylindrical functions. Let Cyl_{red} denote the space of cylindrical functions in the reduced theory. We then have

Lemma 4.2. *If $\Phi \in \mathrm{Cyl}$, then $\Phi \circ I^{-1} \in \mathrm{Cyl}_{red} \subseteq \mathcal{H}_{red}$, and the map $\Phi \mapsto \Phi \circ I^{-1}$ is onto Cyl_{red}.*

4.4.2 Cyl^*_c = set of elements of Cyl^* with support on $\mathcal{S}'(\Sigma)_{inv}$

It will be useful to next prove the precise way in which Cyl^*_c is the space of all elements of Cyl^* with support in $\mathcal{S}'(\Sigma)_{inv}$.

Define $\mathrm{Cyl}_\sim := \{\Psi \in \mathrm{Cyl} \mid \mathrm{Supp}\,\Psi \cap \mathcal{S}'(\Sigma)_{inv} = \emptyset\}$. Then we say $\eta \in \mathrm{Cyl}^*$ has support on $\mathcal{S}'(\Sigma)_{inv}$ if η is zero on $\mathrm{Cyl}_\sim$.

Lemma 4.3. *For $\Psi \in \mathrm{Cyl}$, $\Psi \in \mathrm{Cyl}_\sim$ iff Ψ is of the form $\sum_{i=1}^n \varphi(\mathcal{L}_\phi f_i)\Phi_i$ for some $\{f_i\} \subset \mathcal{S}(\Sigma)$ and some $\{\Phi_i\} \subset \mathrm{Cyl}$.*

Proof.

($\Leftarrow$) Obvious.

($\Rightarrow$) Suppose $\Psi \in \mathrm{Cyl}_\sim$. As an element of Cyl, $\Psi[\varphi]$ depends on φ only via a finite number of "probes" (see section 4.2). There is an ambiguity in how one chooses the probes; what is important is the finite dimensional subspace of $\mathcal{S}(\Sigma)$ spanned by these probes. Let V denote this finite dimensional subspace. We may then choose any set of probes spanning V to represent Ψ as a cylindrical function. Using the decomposition $\mathcal{S}(\Sigma) = \mathcal{S}(\Sigma)_{inv} \oplus \mathcal{S}(\Sigma)_{\perp}$ (see section 4.3.1), we demand that our choice of probes spanning V be a set of the form $\{\mathcal{L}_\phi f_1, \dots \mathcal{L}_\phi f_n, e_1, \dots, e_m\}$ where $\mathcal{L}_\phi f_1, \dots \mathcal{L}_\phi f_n$ are all in $\mathcal{S}(\Sigma)_{\perp}$ and $e_1, \dots, e_m$ are all in $\mathcal{S}(\Sigma)_{inv}$.

Then Ψ may be written

$$\Psi[\varphi] = F(\varphi(\mathcal{L}_\phi f_1), \dots, \varphi(\mathcal{L}_\phi f_n), \varphi(e_1), \dots, \varphi(e_m)) \tag{4.46}$$

for some smooth F. Because $\Phi \in \mathrm{Cyl}_\sim$ it follows that $F(0, \dots, 0, y_1, \dots, y_m) = 0$ for all $y_1, \dots, y_m$. For each $i \in \{1, \dots, n\}$, define

$$G_i(x_1, \dots, x_n, y_1, \dots, y_m) := \frac{F(0, \dots, 0, x_i, \dots, y_m) - F(0, \dots, 0, x_{i+1}, \dots, y_m)}{x_i} \tag{4.47}$$

Since F is smooth, it follows that all the G_i are smooth. The G_i's thus determine elements of Cyl:

$$\Phi_i[\varphi] := G_i(\varphi(\mathcal{L}_\phi f_1), \dots, \varphi(\mathcal{L}_\phi f_n), \varphi(e_1), \dots, \varphi(e_m)) \tag{4.48}$$

One can also show

$$F \equiv \sum_{i=1}^{n} x_i G_i \tag{4.49}$$

It therefore follows that

$$\Psi[\varphi] = \sum_{i=1}^{n} \varphi(\mathcal{L}_\phi f_i) \Phi_i \tag{4.50}$$

proving the desired form. □

The following theorem then easily follows.

Theorem 4.4. *Given* $\eta \in \mathrm{Cyl}^*$,

$$\eta \in \mathrm{Cyl}^*_c \qquad \textit{iff} \qquad \eta(\Psi) = 0 \quad \forall \Psi \in \mathrm{Cyl}_\sim$$

That is,

$$\eta \in \mathrm{Cyl}^*_c \qquad \textit{iff} \qquad \mathrm{Supp}\, \eta \subseteq \mathcal{S}'(\Sigma)_{inv} \tag{4.51}$$

Proof.

(⇒)

Suppose $\hat{\varphi}(\mathcal{L}_\phi f)^*\eta = 0$ for all f. Then $\eta(\varphi(\mathcal{L}_\phi f)\Phi) = 0$ for all $f \in \mathcal{S}(\Sigma)$ and $\Phi \in \mathrm{Cyl}$, whence

$$\eta\left(\sum_{i=1}^{n} \varphi(\mathcal{L}_\phi f_i)\Phi_i\right) = 0 \tag{4.52}$$

for all $\{f_i\} \subset \mathcal{S}(\Sigma)$ and $\{\Phi_i\} \subset \mathrm{Cyl}$. The above lemma then implies η is zero on $\mathrm{Cyl}_\sim$.

(⇐)

Suppose $\eta \in \mathrm{Cyl}^*$ is zero on $\mathrm{Cyl}_\sim$. Then, in particular, $\eta(\varphi(\mathcal{L}_\phi f)\Phi) = 0$ for all $f \in \mathcal{S}(\Sigma)$ and $\Phi \in \mathrm{Cyl}$, whence $\eta \in \mathrm{Cyl}^*_c$. □

Thus, in a precise sense, Cyl^*_c is the subspace of Cyl^* consisting in elements with support only on symmetric configurations.

4.4.3 Embedding of $\mathcal{H}_{red}$ into Cyl^*_c: $\mathcal{H}_c$

Next we construct an anti-linear embedding of $\mathcal{H}_{red}$ into Cyl^*_c. For $\Phi \in \mathrm{Cyl}$ and $\Psi \in \mathcal{H}_{red}$, define [8]

$$(\mathfrak{C}\Psi)(\Phi) := \int_{\varphi'_s \in \mathcal{S}'(\Sigma)_{inv}} \overline{\Psi[\varphi'_s]}\Phi[\varphi'_s]\mathrm{d}\mu_{red} = \langle \Psi, \Phi|_{\mathcal{S}'(\Sigma)_{inv}}\rangle_{\mathcal{H}_{red}} \tag{4.53}$$

so that $\mathfrak{C} : \mathcal{H}_{red} \to \mathrm{Cyl}^*$.

Lemma 4.5. $\mathfrak{C}$ *is one-to-one.*

Proof.

It is sufficient to show the kernel of $\mathfrak{C}$ is trivial. Suppose $\mathfrak{C}\Psi = 0$. Then for all $\Phi' \in \mathrm{Cyl}$

$$\begin{aligned} \int_{\varphi_s \in \mathcal{S}'(\Sigma)_{inv}} \overline{\Psi[\varphi_s]}\Phi'[\varphi_s]\mathrm{d}\mu_{red} &= 0 \quad \forall \Phi' \in \mathrm{Cyl} \\ \Rightarrow \int_{\varphi_s \in \mathcal{S}'(\Sigma)_{inv}} \overline{\Psi[\varphi_s]}\Phi[\varphi_s]\mathrm{d}\mu_{red} &= 0 \quad \forall \Phi \in \mathrm{Cyl}_{red} \end{aligned} \tag{4.54}$$

[8] The complex conjugation on Ψ (and hence anti-linearity of the embedding) is incorporated here in order to make the connections drawn in section 4.7 simpler.

But since Cyl_{red} is dense in $\mathcal{H}_{red} = L^2(\mathcal{S}'(\Sigma)_{inv}, \mathrm{d}\mu_{red})$, it follows $\Psi = 0$, proving the desired trivial kernel. $\square$

Thus, $\mathfrak{E}$ gives an embedding of $\mathcal{H}_{red}$ (as a vector space) into Cyl^*. Furthermore,

Lemma 4.6. *The image of* $\mathfrak{E}$ *is contained in* Cyl^*_c.

Proof.

For all $f \in \mathcal{S}(\Sigma)$ and $\Phi \in \mathrm{Cyl}$,

$$\{\hat{\varphi}[\mathcal{L}_\phi f]^* \mathfrak{E}(\Psi)\}[\Phi] = \mathfrak{E}(\Psi)(\hat{\varphi}[\mathcal{L}_\phi f]\Phi) = \int_{\varphi_s \in \mathcal{S}'(\Sigma)_{inv}} \overline{\Psi[\varphi_s]} \varphi_s(\mathcal{L}_\phi f)\Phi[\varphi_s] = 0 \tag{4.55}$$

since $\varphi_s(\mathcal{L}_\phi f) = 0$. The lemma follows. $\square$

Thus $\mathfrak{E}$ gives a linear embedding of $\mathcal{H}_{red}$ in Cyl^*_c. Let the image of this embedding be denoted $\mathcal{H}_c$. We equip $\mathcal{H}_c$ with the natural inner product

$$\langle \mathfrak{E}\Psi_1, \mathfrak{E}\Psi_2 \rangle := \langle \Psi_1, \Psi_2 \rangle \tag{4.56}$$

induced from $\mathcal{H}_{red}$ via the isomorphism.

4.4.4 Nonpreservation of $\mathcal{H}_c$ by $\hat{\mathbb{H}}$

Lastly, we make some remarks as to the (dual) action of the Hamiltonian $\hat{\mathbb{H}}$ and the lack of preservation of the c-symmetric sector by $\hat{\mathbb{H}}^*$. As mentioned earlier, $\hat{\mathbb{H}}$ in (4.39) preserves Cyl; thus it has a dual action $\hat{\mathbb{H}}^*$ on Cyl^*. We will show that, as expected from the classical analysis (see section 4.3.1) , $\hat{\mathbb{H}}^*$ does not preserve $\mathcal{H}_c$.

Proposition 4.7. $\mathcal{H}_c$ *is not preserved by* $\hat{\mathbb{H}}^*$.

Proof.

We show this by showing that the image of the reduced theory vacuum, $\eta_0 := \mathfrak{E}\overset{r}{\Psi}_0$, in $\mathcal{H}_c$, is mapped out of $\mathcal{H}_c$ by $\hat{\mathbb{H}}^*$.

Suppose by way of contradiction that $\hat{\mathbb{H}}^*\eta_0 = \eta$ for some $\eta = \mathfrak{E}\Psi \in \mathcal{H}_c$.

Let $f \in \mathcal{S}(\Sigma)$ be chosen non-axisymmetric, and define

$$\Phi[\varphi] := F(\varphi(\mathcal{L}_\phi f)) \tag{4.57}$$

where $F : \mathbb{R} \to \mathbb{C}$ is for now left unspecified. Now, for $\varphi_s \in \mathcal{S}(\Sigma)_{inv}$,

$$\begin{aligned}(\hat{\mathbb{H}}\Phi)[\varphi_s] &= (\Theta^{\frac{1}{2}}\varphi_s, \mathcal{L}_\phi f)F'(\varphi_s(\mathcal{L}_\phi f)) - \frac{1}{2}(\mathcal{L}_\phi f, \mathcal{L}_\phi f)F''(\varphi_s(\mathcal{L}_\phi f)) \\ &= -\frac{1}{2}(\mathcal{L}_\phi f, \mathcal{L}_\phi f)F''(0) \end{aligned} \tag{4.58}$$

So,

$$\begin{aligned}(\hat{\mathbb{H}}^*\eta_0)(\Phi) &:= \eta_0(\hat{\mathbb{H}}\Phi) \\ &= \int_{\varphi_s \in \mathcal{S}'(\Sigma)_{inv}} \overline{\Psi_0[\varphi_s]}(\hat{\mathbb{H}}\Phi)[\varphi_s]\mathrm{d}\mu_{red} \\ &= -\frac{1}{2}(\mathcal{L}_\phi f, \mathcal{L}_\phi f)F''(0)\int_{\varphi_s \in \mathcal{S}'(\Sigma)_{inv}} \overline{\Psi_0[\varphi_s]}\mathrm{d}\mu_{red} \\ &= -\frac{1}{2}(\mathcal{L}_\phi f, \mathcal{L}_\phi f)F''(0). \end{aligned} \tag{4.59}$$

And

$$\begin{aligned}\eta(\Phi) &= \int_{\varphi_s \in \mathcal{S}'(\Sigma)_{inv}} \overline{\Psi[\varphi_s]}\Phi[\varphi_s]\mathrm{d}\mu_{red} \\ &:= F(0)\int_{\varphi_s \in \mathcal{S}'(\Sigma)_{inv}} \overline{\Psi[\varphi_s]}\mathrm{d}\mu_{red} \end{aligned} \tag{4.60}$$

But $F''(0)$ and $F(0)$ are independently specifiable. Thus, one can always choose the function F (and hence Φ) such that expressions (4.59) and (4.60) are not equal. It follows that

$$\hat{\mathbb{H}}^*\eta_0 \neq \eta \tag{4.61}$$

yielding a contradiction. Thus $\hat{\mathbb{H}}^*$ maps $\eta_0 \in \mathcal{H}_c$ out of $\mathcal{H}_c$. □

4.4.5 c-symmetry is stronger than invariance symmetry

Theorem 4.8. $\mathcal{H}_c \subsetneq \mathrm{Cyl}^*_{inv}$ [9]

[9] Ideally one would have liked to prove the stronger result $\mathrm{Cyl}^*_c \subsetneq \mathrm{Cyl}^*_{inv}$: but in fact one does not even have $\mathrm{Cyl}^*_c \subseteq \mathrm{Cyl}^*_{inv}$ (see appendix B for proof). One has to restrict to $\mathcal{H}_c$ before one has a subspace of Cyl^*_{inv}. This reminds us of the importance of restricting to appropriately defined normalizable states before expecting certain properties to hold.

Proof.

We first prove $\mathcal{H}_c \subset \mathrm{Cyl}^*_{inv}$. Suppose $\eta \in \mathcal{H}_c$ so that $\eta = \mathfrak{E}(\Psi)$ for some $\Psi \in \mathcal{H}_{red}$. For all $g \in \mathfrak{T}$ and $\Phi \in \mathrm{Cyl}$,

$$\begin{aligned}(g \cdot (\mathfrak{E}\Psi))(\Phi) &:= (\mathfrak{E}\Psi)(g^{-1} \cdot \Phi) \\ &= \int_{\varphi_s \in \mathcal{S}'(\Sigma)} \overline{\Psi[\varphi_s]}\Phi[g \cdot \varphi_s] \mathrm{d}\mu_{red} \\ &= \int_{\varphi_s \in \mathcal{S}'(\Sigma)} \overline{\Psi[\varphi_s]}\Phi[\varphi_s] \mathrm{d}\mu_{red} \\ &= (\mathfrak{E}\Psi)(\Phi) \end{aligned} \quad (4.62)$$

whence $\mathfrak{E}\Psi \in \mathrm{Cyl}^*_{inv}$, proving $\mathcal{H}_c \subset \mathrm{Cyl}^*_{inv}$.

Next, to show $\mathcal{H}_c \subsetneq \mathrm{Cyl}^*_{inv}$, we construct an element of Cyl^*_{inv} that is not in $\mathcal{H}_c$.

To facilitate explicit calculation, let us choose

$$f(\rho, z, \phi) := H(\rho, z) \sin \phi \quad (4.63)$$

where $H(\rho, z)$ is any non-negative, non-zero, smooth function of compact support such that all derivatives of H vanish at $\rho = 0$ (to ensure smoothness of f at the axis). Define $\alpha \in \mathcal{S}'(\Sigma)$ by

$$\alpha(h) := \int_\Sigma (hf) \mathrm{d}^3 x \quad (4.64)$$

Then define $\eta \in \mathrm{Cyl}^*$ by

$$\eta(\Phi) := \int_{g \in \mathfrak{T}} \Phi[g \cdot \alpha] \mathrm{d}g. \quad (4.65)$$

so that $\eta \in \mathrm{Cyl}^*_{inv}$.

To show $\eta \notin \mathrm{Cyl}^*_c$, we construct an element Φ of $\mathrm{Cyl}_\sim$ such that $\eta(\Phi) \neq 0$.

Let $F(x) := x^2$, so that F is smooth, zero only at zero, and positive everywhere else. Define $\Phi \in \mathrm{Cyl}$ by

$$\Phi[\varphi] := F(\varphi(\mathcal{L}_\phi f)) \quad (4.66)$$

so that Φ is in $\mathrm{Cyl}_\sim$. We have

$$\eta(\Phi) = \frac{1}{2\pi} \int_{\phi'=0}^{2\pi} F(\alpha(g(-\phi') \cdot \mathcal{L}_\phi f)) \mathrm{d}\phi' \quad (4.67)$$

where we have parametrized the group of rotations $\mathcal{T}$ in the usual way by $\phi' \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$. Working out the expression further, we get

$$\eta(\Phi) = \frac{1}{2\pi} \int_{\phi'=0}^{2\pi} F(-a\pi \sin \phi')\mathrm{d}\phi' \tag{4.68}$$

where $a := \int_B H(\rho, z)^2 \rho \mathrm{d}\rho \mathrm{d}z > 0$. Since the above integrand is positive almost everywhere, $\eta(\Phi) > 0$.

Thus $\eta \notin \mathrm{Cyl}_c^*$, proving in particular $\eta \notin \mathcal{H}_c$, so that $\mathcal{H}_c \subsetneq \mathrm{Cyl}_{inv}^*$. □

4.5 Analysis of $\mathcal{H}_b$

We next analyze the structure and properties of $\mathcal{H}_b$, helping us to further grasp its physical meaning in different ways.

4.5.1 Description of $\mathcal{H}_b$ in terms of coherent states

Next recall that, in free Klein-Gordon theory, with each $\xi \in h$ one has an associated (normalized) *coherent state* $\Psi_\xi^{coh} \in \mathcal{H}$ defined by

$$\Psi_\xi^{coh} = \mathrm{e}^{\hat{\Lambda}(\xi)}\Psi_0 \tag{4.69}$$

where $\hat{\Lambda}(\xi) := a^\dagger(\xi) - a(\xi)$. As a side note, for $\xi = [\varphi', \pi']$, $e^{\hat{\Lambda}(\xi)} = e^{i(\hat{\varphi}[\pi'] - \hat{\pi}[\varphi'])}$; from this one can see $e^{\hat{\Lambda}(\xi)}$ is a sort of 'phase space translation operator.' In 4.69, one is just starting with the vacuum and 'translating' its expectation values to different points in phase space.

In the case where $\xi \in \Gamma \subset h$, Ψ_ξ^{coh} has the interpretation of being the quantum state that "best approximates" the classical state ξ. The expectation values of field operators determined by Ψ_ξ^{coh} are precisely the values of the fields in ξ, and uncertainties in appropriate field components are minimized.

It is not hard to show that Ψ_ξ^{coh} satisfies the usual property of being a simultaneous eigenstate of the annihilation operators:

$$a(\eta)\Psi_\xi^{coh} = \langle \eta, \xi \rangle \Psi_\xi^{coh} \tag{4.70}$$

Theorem 4.9.

$$\mathcal{H}_b = \text{span}\{\Psi_\xi^{coh}\}_{\xi \in h_{inv}}$$

Proof.

($\supseteq$)

For each $\xi \in h_{inv}$

$$a(\eta)\Psi_\xi^{coh} = \langle \eta, \xi \rangle \Psi_\xi^{coh} = 0 \quad \forall \eta \in h_{inv}^\perp \tag{4.71}$$

so that $\Psi_\xi^{coh} \in \mathcal{H}_b$, whence $\text{span}\{\Psi_\xi^{coh}\}_{\xi \in h_{inv}} \subseteq \mathcal{H}_b$.

($\subseteq$)

Suppose $\Psi \in \mathcal{H}_b$. Since the coherent states span $\mathcal{H}$, there exists a linearly independent set of coherent states $\{\Psi_{\xi_i}^{coh}\}_{i \in I}$ and non-zero coefficients $\{\lambda_i\}_{i \in I} \subseteq \mathbb{C}$ such that

$$\Psi = \sum_{i \in I} \lambda_i \Psi_{\xi_i}^{coh} \tag{4.72}$$

Since $\Psi \in \mathcal{H}_b$, for all $\eta \in h_{inv}^\perp$,

$$0 = a(\eta)\Psi = \sum_{i \in I} \lambda_i \langle \eta, \xi_i \rangle \Psi_{\xi_i}^{coh}. \tag{4.73}$$

By the linear independence of $\{\Psi_{\xi_i}^{coh}\}_{i \in I}$, and since the coefficients λ_i are all non-zero, it follows that for each $i \in I$,

$$\langle \eta, \xi_i \rangle = 0 \quad \forall \eta \in h_{inv}^\perp \tag{4.74}$$

so that

$$\xi_i \in h_{inv} \quad \forall i \in I \tag{4.75}$$

whence $\Psi \in \text{span}\{\Psi_\xi^{coh}\}_{\xi \in h_{inv}}$. $\square$

Note that because Γ_{inv} is dense in h_{inv} and $\xi \mapsto \Psi_\xi^{coh}$ is continuous [10] one can

[10] The continuity of $\xi \mapsto \Psi_\xi^{coh}$ can be seen from the relation

$$\| \Psi_\xi^{coh} - \Psi_{\xi_i}^{coh} \|^2 = 2 - 2\cos(\text{Im}\langle \xi, \xi_i \rangle) e^{-\frac{1}{2}\|\xi - \xi_i\|^2}$$

If $\xi_i \to \xi$, from the continuity in ξ_i of the right hand side of the above equation, $\Psi_{\xi_i}^{coh} \to \Psi_\xi^{coh}$.

replace h_{inv} with Γ_{inv} in the statement of the above theorem. The theorem then expresses $\mathcal{H}_b$ as a span of coherent states associated with the axisymmetric sector of the strictly classical theory.

4.5.2 Isomorphism with $\mathcal{H}_{red}$

More importantly, $\mathcal{H}_b$ is naturally isomorphic to $\mathcal{H}_{red}$. Let us prove this in the following way. Let $\overset{r}{\Psi}_0$ denote the vacuum in the reduced theory. For $\xi \in h_{inv}$, define

$$\overset{r}{\Psi}{}^{coh}_{\xi} := e^{\hat{\Lambda}_{red}(\xi)} \overset{r}{\Psi}_0 \tag{4.76}$$

where $\hat{\Lambda}_{red}(\xi) := a^{\dagger}_{red}(\xi) - a_{red}(\xi)$. $\{\overset{r}{\Psi}{}^{coh}_{\xi}\}_{\xi \in h_{inv}}$ gives a family of coherent states in the reduced theory. This family of coherent states satisfies the completeness relation

$$\int_{\xi \in \Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \mid \overset{r}{\Psi}{}^{coh}_{\xi} \rangle \langle \overset{r}{\Psi}{}^{coh}_{\xi} \mid = 1\!\!1_{\mathcal{H}_{red}} \tag{4.77}$$

where the measure $\mathrm{d}\nu^{o}_{red}$ is a particular rigorization of the 'Lesbesgue measure' on Γ_{inv}. More specifically, $\mathrm{d}\nu^{o}_{red}$ is a *generalized measure*: it defines an integration functional, but not a set function. An advantage of generalized measures (in this context) is that they can integrate functions directly on the *classical* phase space Γ_{inv}, without need for introduction of a larger distributional extension of Γ_{inv}. For the precise definition of $\mathrm{d}\nu^{o}_{red}$, and further details, see appendix C.

Lemma 4.10. *For all $\xi, \xi' \in h_{inv}$,*

$$\langle \overset{r}{\Psi}{}^{coh}_{\xi}, \overset{r}{\Psi}{}^{coh}_{\xi'} \rangle = \langle \Psi^{coh}_{\xi}, \Psi^{coh}_{\xi'} \rangle \tag{4.78}$$

where $\{\Psi^{coh}_{\xi}\}_{\xi \in h}$ is the family of coherent states in the full theory.

Proof.

The calculation of $\langle \Psi^{coh}_{\xi}, \Psi^{coh}_{\xi'} \rangle = e^{i\mathrm{Im}\langle \xi, \xi' \rangle} e^{-\frac{1}{2}\|\xi - \xi'\|^2}$ (as in appendix B.2) uses only the relations

$$[a(\xi), a^{\dagger}(\xi')] = \langle \xi, \xi' \rangle \qquad \forall \xi, \xi' \in h \tag{4.79}$$

$$a(\xi)\Psi_0 = 0 \qquad \forall \xi \in h \tag{4.80}$$

$$\langle \Psi_0, \Psi_0 \rangle = 1 \tag{4.81}$$

In the reduced theory, exactly the same relations are satisfied:

$$[a_{red}(\xi), a^{\dagger}_{red}(\xi')] = \langle \xi, \xi' \rangle \qquad \forall \xi, \xi' \in h \tag{4.82}$$

$$a_{red}(\xi)\Psi_0 = 0 \qquad \forall \xi \in h \tag{4.83}$$

$$\langle \overset{r}{\Psi}_0, \overset{r}{\Psi}_0 \rangle = 1 \tag{4.84}$$

Thus

$$\langle \overset{r}{\Psi}{}^{coh}_{\xi}, \overset{r}{\Psi}{}^{coh}_{\xi'} \rangle = e^{i\mathrm{Im}\langle \xi, \xi' \rangle} e^{-\frac{1}{2}\|\xi - \xi'\|^2} \tag{4.85}$$

as well. $\square$

The reduced theory completeness relation (4.77) and lemma 4.10 are the only properties of $\{\Psi^{coh}_{\xi}\}$ and $\{\overset{r}{\Psi}{}^{coh}_{\xi}\}$ that we will need for the present subsection.

We next use the measure appearing in (4.77) to define a map $i : \mathcal{H}_{red} \to \mathcal{H}_b$ by

$$\iota \overset{r}{\Psi} := \int_{\xi \in \Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \langle \overset{r}{\Psi}{}^{coh}_{\xi}, \overset{r}{\Psi} \rangle \Psi^{coh}_{\xi} \tag{4.86}$$

From its definition, and lemma 4.10, it should not surprising that ι is isometric:

Lemma 4.11.

$$\langle \iota\Psi_1, \iota\Psi_2 \rangle = \langle \Psi_1, \Psi_2 \rangle$$

for all $\Psi_1, \Psi_2 \in \mathcal{H}_{red}$.

Proof.

$$\begin{aligned}
\langle \iota\Psi_1, \iota\Psi_2 \rangle &= \int_{\xi \in \Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \int_{\xi' \in \Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \langle \Psi_1, \overset{r}{\Psi}{}^{coh}_{\xi} \rangle \langle \Psi^{coh}_{\xi}, \Psi^{coh}_{\xi'} \rangle \langle \overset{r}{\Psi}{}^{coh}_{\xi'}, \Psi_2 \rangle \\
&= \int_{\xi \in \Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \int_{\xi' \in \Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \langle \Psi_1, \overset{r}{\Psi}{}^{coh}_{\xi} \rangle \langle \overset{r}{\Psi}{}^{coh}_{\xi}, \overset{r}{\Psi}{}^{coh}_{\xi'} \rangle \langle \overset{r}{\Psi}{}^{coh}_{\xi'}, \Psi_2 \rangle \\
&= \langle \Psi_1, \Psi_2 \rangle
\end{aligned} \tag{4.87}$$

$\square$

Furthermore ι is one-to-one and onto. We prove this by showing

$$\Pi\Psi := \int_{\xi\in\Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \langle \Psi^{coh}_{\xi}, \Psi\rangle \overset{r}{\Psi}{}^{coh}_{\xi} \tag{4.88}$$

is an inverse:

Proposition 4.12.

$$\iota \circ \Pi = \mathbb{1}, \quad \Pi \circ \iota = \mathbb{1} \tag{4.89}$$

Proof.

We prove these equalities by proving the equality of their matrix elements. Since $\{\Psi^{coh}_{\xi}\}_{\xi\in\Gamma_{inv}}$ spans $\mathcal{H}_b$ and $\{\overset{r}{\Psi}{}^{coh}_{\xi}\}_{\xi\in\Gamma_{inv}}$ spans $\mathcal{H}_{red}$, it is sufficient to check the matrix elements with respect to the (symmetric) coherent states Ψ^{coh}_{ξ} and $\overset{r}{\Psi}{}^{coh}_{\xi}$, $\xi \in \Gamma_{inv}$. That is, it is sufficient to check that

$$\langle \Psi^{coh}_{\xi_1}, \iota \circ \Pi \Psi^{coh}_{\xi_2}\rangle = \langle \Psi^{coh}_{\xi_1}, \Psi^{coh}_{\xi_2}\rangle \tag{4.90}$$

for all $\xi_1, \xi_2 \in \Gamma_{inv}$ and

$$\langle \overset{r}{\Psi}{}^{coh}_{\xi_1}, \Pi \circ \iota \overset{r}{\Psi}{}^{coh}_{\xi_2}\rangle = \langle \overset{r}{\Psi}{}^{coh}_{\xi_1}, \overset{r}{\Psi}{}^{coh}_{\xi_2}\rangle \tag{4.91}$$

for all $\xi_1, \xi_2 \in \Gamma_{inv}$. These two equations follow by repeated application of the completeness relation (4.77) and lemma 4.10. □

Thus, ι gives an isometric embedding of $\mathcal{H}_{red}$ into $\mathcal{H}$, with $\mathcal{H}_b$ as the image.

4.5.3 Preservation by $\hat{\mathbb{H}}$

$\mathcal{H}_b$ is also preserved by the quantum time evolution. This can be viewed as a consequence of the covariance of the family $\{\Psi^{coh}_{\xi}\}$ with respect to time evolution, that is, of the fact that $\Psi^{coh}_{e^{it\hat{H}}\xi} = e^{it\hat{\mathbb{H}}}\Psi^{coh}_{\xi}$ for all t. More generally, the term 'covariant' has the following meaning. Suppose $e^{it\hat{X}}$ is a given canonical transformation on $h \supset \Gamma$. Let $e^{it\hat{\mathbb{X}}}$ denote the quantization of this transformation, so that $e^{it\hat{\mathbb{X}}}$ is a unitary operator on $\mathcal{H}$. Then we say $\{\Psi^{coh}_{\xi}\}$ is *covariant* with respect to the transformation $e^{it\hat{X}}$ if $\Psi^{coh}_{e^{it\hat{X}}\xi} = e^{it\hat{\mathbb{X}}}\Psi^{coh}_{\xi}$. In fact, the family of coherent states

$\{\Psi_\xi^{coh}\}$ is covariant with respect to a much larger class of transformations than time evolution. Because the broader covariance property of $\{\Psi_\xi^{coh}\}$ will also be useful in the next subsection, we prove the full, broader property.

Suppose $e^{it\hat{X}}$ is a one parameter family of transformations on $h \supset \Gamma$, so that $\hat{X} : h \to h$ is the operator generating this family. $\hat{X}$ will be symmetric with respect to $\langle , \rangle$ if $e^{it\hat{X}}$ preserves both the symplectic structure and complex structure J. This is true in particular for both time evolution and any flow arising from a family of spatial isometries. Let us assume $\hat{X}$ is symmetric with respect to $\langle , \rangle$ and that it has a unique self-adjoint extension. From the geometrical formulation of quantum mechanics [53], it is well known that the generator, in the *phase space* sense, of the family of transformations $e^{it\hat{X}}$, is the phase space function

$$\langle [\varphi, \pi], \hat{X}[\varphi, \pi] \rangle. \tag{4.92}$$

The generator of the corresponding flow in the quantum field theory is the quantization of this phase space function. Imitating the quantization of $\hat{\mathbb{H}}$ in section 4.2.2, one obtains

$$\hat{\mathbb{X}} = \sum_{i,j} \langle \xi_i, \hat{X}\xi_j \rangle a^\dagger(\xi_i) a(\xi_j) \tag{4.93}$$

With this established, we prove

Lemma 4.13.

$$[\hat{\mathbb{X}}, a^\dagger(\xi)] = a^\dagger(\hat{X}\xi)$$

Proof.

$$\begin{aligned}
[\hat{\mathbb{X}}, a^\dagger(\xi)] &= \sum_{i,j} \langle \xi_i, \hat{X}\xi_j \rangle [a^\dagger(\xi_i) a(\xi_j), a^\dagger(\xi)] \\
&= \sum_{i,j} \langle \xi_i, \hat{X}\xi_j \rangle \langle \xi_j, \xi \rangle a^\dagger(\xi_i) \\
&= \sum_{i} \langle \xi_i, \hat{X}\xi \rangle a^\dagger(\xi_i) \\
&= a^\dagger \left(\sum_i \xi_i \langle \xi_i, \hat{X}\xi \rangle \right) \\
&= a^\dagger(\hat{X}\xi).
\end{aligned} \tag{4.94}$$

□

Lemma 4.14.

$$e^{it\hat{\mathbb{X}}}\hat{\Lambda}(\xi)e^{-it\hat{\mathbb{X}}} = \hat{\Lambda}\left(e^{it\hat{\mathbb{X}}}\xi\right)$$

Proof.

The complex conjugate of the last lemma yields

$$[\hat{\mathbb{X}}, a(\xi)] = -a(\hat{X}\xi) \tag{4.95}$$

so that

$$\begin{aligned} [i\hat{\mathbb{X}}, \hat{\Lambda}(\xi)] &= ia^\dagger(\hat{X}\xi) + ia(\hat{X}\xi) \\ &= a^\dagger(i\hat{X}\xi) - a(i\hat{X}\xi) \\ &= \hat{\Lambda}(i\hat{X}\xi). \end{aligned} \tag{4.96}$$

The Baker-Hausdorff formula applied to $e^{it\hat{\mathbb{X}}}\hat{\Lambda}(\xi)e^{-it\hat{\mathbb{X}}}$ then gives the desired result.
□

Theorem 4.15.

$$e^{it\hat{\mathbb{X}}}\Psi^{coh}_{\xi} = \Psi^{coh}_{e^{it\hat{X}}\xi}$$

Proof.

Let

$$f(s) := e^{it\hat{\mathbb{X}}}\Psi^{coh}_{s\xi} := e^{it\hat{\mathbb{X}}}e^{\hat{\Lambda}(s\xi)}\Psi_0 \tag{4.97}$$

and

$$g(s) := \Psi^{coh}_{e^{it\hat{X}}s\xi} := e^{\hat{\Lambda}(e^{it\hat{X}}s\xi)}\Psi_0. \tag{4.98}$$

Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}f(s) &= e^{it\hat{\mathbb{X}}}\hat{\Lambda}(\xi)\Psi_0 \\ &= e^{it\hat{\mathbb{X}}}\hat{\Lambda}(\xi)e^{-it\hat{\mathbb{X}}}\Psi_0 \\ &= \hat{\Lambda}(e^{it\hat{X}\xi})\Psi_0 \end{aligned}$$

$$= \frac{\mathrm{d}}{\mathrm{d}s} g(s) \tag{4.99}$$

for all s. This combined with $f(0) = g(0)$ implies $f(s) = g(s)$ for all s; in particular, $f(1) = g(1)$, implying the desired result. $\square$

In particular, for the case of time evolution ($\hat{X} = \hat{H}$, $\hat{\mathbb{X}} = \hat{\mathbb{H}}$),

$$e^{it\hat{\mathbb{H}}} \Psi_{\xi}^{coh} = \Psi_{e^{it\hat{H}}\xi}^{coh} \tag{4.100}$$

Similarly, in the reduced theory,

$$e^{it\hat{\mathbb{H}}_{red}} \overset{r}{\Psi}{}_{\xi}^{coh} = \overset{r}{\Psi}{}_{e^{it\hat{H}_{red}}\xi}^{coh} = \overset{r}{\Psi}{}_{e^{it\hat{H}}\xi}^{coh}. \tag{4.101}$$

The proof is exactly the same as for the full theory.

(4.100) and (4.101) imply that $\mathcal{H}_b$ is preserved by $\hat{\mathbb{H}}$ and furthermore the Hamiltonian operator thereby induced on $\mathcal{H}_b$, and thence on $\mathcal{H}_{red}$, is precisely $\hat{\mathbb{H}}_{red}$. That is,

Theorem 4.16. *For all* $\overset{r}{\Psi} \in \mathcal{H}_{red}$,

$$\hat{\mathbb{H}} \left(\iota \overset{r}{\Psi} \right) = \iota \left(\hat{\mathbb{H}}_{red} \overset{r}{\Psi} \right) \tag{4.102}$$

Proof.

For all $\xi \in h_{inv}$,

$$\begin{aligned} e^{it\hat{\mathbb{H}}} \left(\iota \overset{r}{\Psi}{}_{\xi}^{coh} \right) &= e^{it\hat{\mathbb{H}}} \Psi_{\xi}^{coh} \\ &= \Psi_{e^{it\hat{H}}\xi}^{coh} \\ &= \iota \overset{r}{\Psi}{}_{e^{it\hat{H}}\xi}^{coh} \\ &= \iota \left(e^{it\hat{\mathbb{H}}_{red}} \overset{r}{\Psi}{}_{\xi}^{coh} \right) \end{aligned} \tag{4.103}$$

Proving (4.102) holds for all $\overset{r}{\Psi} = \overset{r}{\Psi}{}_{\xi}^{coh}$ with $\xi \in h_{inv}$. But $\{\overset{r}{\Psi}{}_{\xi}^{coh}\}_{\xi \in h_{inv}}$ spans $\mathcal{H}_{red}$, whence (4.102) holds for all $\overset{r}{\Psi} \in \mathcal{H}_{red}$. $\square$

As a side note, if all one wants is preservation of $\mathcal{H}_b$, a weakened version of

(4.100) suffices. Namely, suppose instead all we knew was that our family of coherent states were such that there exists $A : \mathbb{R} \times \Gamma \to \Gamma$ such that

$$e^{it\hat{\mathbb{H}}}\Psi^{coh}_{\xi} = \Psi^{coh}_{A(t,\xi)}. \tag{4.104}$$

If, with respect to the symmetry group, the Hamiltonian $\hat{\mathbb{H}}$ is invariant and $\xi \mapsto \Psi^{coh}_{\xi}$ is covariant, it is not hard to show that $A(t,\xi)$ will also be covariant. These properties then suffice to imply $\mathcal{H}_b = \mathrm{span}\{\Psi^{coh}_{\xi}\}_{\xi\in\Gamma_{inv}}$ is preserved by $\hat{\mathbb{H}}$. A family of coherent states that satisfies (4.104) is called *dynamical* [54, 55, 56].

4.5.4 b-symmetry is stronger than invariance symmetry

As mentioned, the map $\xi \mapsto \Psi^{coh}_{\xi}$ is covariant with respect to the action of all one parameter groups of spatial isometries. That this is true in particular for our symmetry group $\mathcal{T}$ is sufficient to imply $\mathcal{H}_b \subseteq \mathcal{H}_{inv}$.

Proposition 4.17.

$$\mathcal{H}_b \subseteq \mathcal{H}_{inv}$$

Proof.

For all $\xi \in h_{inv}$ and $g \in \mathcal{T}$,

$$g \cdot \Psi^{coh}_{\xi} = \Psi^{coh}_{g\cdot\xi} = \Psi^{coh}_{\xi} \tag{4.105}$$

so that $\Psi^{coh}_{\xi} \in \mathcal{H}_{inv}$ for all $\xi \in h_{inv}$. But $\{\Psi^{coh}_{\xi}\}_{\xi\in h_{inv}}$ spans $\mathcal{H}_b$, so that $\mathcal{H}_b \subseteq \mathcal{H}_{inv}$. □

Furthermore, $\mathcal{H}_b \subsetneq \mathcal{H}_{inv}$, as we will prove. We proceed by explicitly constructing an element of $\mathcal{H}_{inv}$ that is not in $\mathcal{H}_b$.

Let $\hat{L}_z := -J\mathcal{L}_{\phi}$, the one particle generator of rotations about the z-axis. The most general form for an eigenstate of $\hat{L}_z$ with eigenvalue L_z is

$$([\varphi, \pi] =) \left[\mathrm{Re}\left\{A(\rho,z)e^{iL_z\phi}\right\}, \Theta^{\frac{1}{2}}\mathrm{Im}\left\{A(\rho,z)e^{iL_z\phi}\right\}\right]. \tag{4.106}$$

Because $A(\rho,z)$ is freely specifiable, one can always construct an eigenstate of $\hat{L}_z$ (for any eigenvalue) that is normalizable. Let ξ_+ be a normalizable eigenstate of

$\hat{L}_z$ with eigenvalue $L_z > 0$ and let ξ_- be any normalizable eigenstate of $\hat{L}_z$ with eigenvalue $-L_z$. Let

$$\Psi := a^\dagger(\xi_+)a^\dagger(\xi_-)\Psi_0. \tag{4.107}$$

Claim: Ψ is in $\mathcal{H}_{inv}$ but not $\mathcal{H}_b$
Proof.

$$\begin{aligned}
\hat{\mathbb{L}}_z a^\dagger(\xi_+)a^\dagger(\xi_-)\Psi_0 &= [\hat{\mathbb{L}}_z, a^\dagger(\xi_+)]a^\dagger(\xi_-)\Psi_0 + a^\dagger(\xi_+)[\hat{\mathbb{L}}_z, a^\dagger(\xi_-)]\Psi_0 + a^\dagger(\xi_+)a^\dagger(\xi_-)\hat{\mathbb{L}}_z\Psi_0 \\
&= a^\dagger(\hat{L}_z\xi_+)a^\dagger(\xi_-)\Psi_0 + a^\dagger(\xi_+)a^\dagger(\hat{L}_z\xi_-)\Psi_0 \\
&= L_z\Psi - L_z\Psi \\
&= 0
\end{aligned} \tag{4.108}$$

So $\Psi \in \mathcal{H}_{inv}$. To show $\Psi \notin \mathcal{H}_b$, we first note that, because $\xi_+ \notin h_{inv}$, there exists $\eta \in h_\perp$ such that $\langle \eta, \xi_+ \rangle \neq 0$. We have

$$a(\eta)a^\dagger(\xi_+)a^\dagger(\xi_-)\Psi_0 = \langle \eta, \xi_+ \rangle a^\dagger(\xi_-)\Psi_0 + \langle \eta, \xi_- \rangle a^\dagger(\xi_+)\Psi_0 \tag{4.109}$$

The first term is non-zero, and the $a^\dagger(\xi_+)\Psi_0$ factor in the second term is linearly independent of the first term. Thus, the sum must also be non-zero. So $a(\eta)\Psi \neq 0$, whence $\Psi \notin \mathcal{H}_b$. □

4.5.5 Minimization of fluctuations from axisymmetry

A last notable property of $\mathcal{H}_b$ is that the fluctuations from axisymmetry in its members are under complete control and are in a certain sense *minimized*, whereas in $\mathcal{H}_{inv}$ it is difficult to prove anything general regarding fluctuations from axisymmetry.

Let us be more precise. Recall ideally one may wish to impose $\mathcal{L}_\phi\hat{\varphi}(x)\Psi = 0$ and $\mathcal{L}_\phi\hat{\pi}(x)\Psi = 0$, but that, in this form, this is not possible. Therefore we imposed instead a complex linear combination of these constraints (in approach 'b'). Nevertheless, the resulting states $\Psi \in \mathcal{H}_b$ are still such that

$$\begin{aligned} \langle \Psi, \mathcal{L}_\phi \hat{\varphi}(x) \Psi \rangle &= 0 & (4.110) \\ \langle \Psi, \mathcal{L}_\phi \hat{\pi}(x) \Psi \rangle &= 0 & (4.111) \end{aligned}$$

that is, the expectation values of $\hat{\varphi}(x)$ and $\hat{\pi}(x)$ are axisymmetric. The easiest way to see this is actually to first note that $\mathcal{H}_b \subset \mathcal{H}_{inv}$ and then show that (4.110) and (4.111) hold for *all* members of $\mathcal{H}_{inv}$. One can show this using the fact that for all rotations g, $\hat{\varphi}(g \cdot x) = U_g \hat{\varphi}(x) U_g^{-1}$. Thus both $\mathcal{H}_b$ and $\mathcal{H}_{inv}$ consist in states giving rise to axisymmetric field expectation values.

The difference between $\mathcal{H}_b$ and $\mathcal{H}_{inv}$ comes, however, when we consider *fluctuations* from axisymmetry. For any operator $\hat{O}$ on $\mathcal{H}$ and $\Psi \in \mathcal{H}$, the "fluctuation" in $\hat{O}$ determined by Ψ is defined by

$$\Delta_\Psi \hat{O} := \sqrt{\langle \Psi, \hat{O}^2 \Psi \rangle - \langle \Psi, \hat{O} \Psi \rangle^2} \tag{4.112}$$

Smearing the symmetry constraint operators against a test function f, we get $\hat{\varphi}[\mathcal{L}_\phi f]$ and $\hat{\pi}[\mathcal{L}_\phi f]$. For $\Psi \in \mathcal{H}_b$, with unit norm, one can show the uncertainties in the non-axisymmetric modes are given by

$$\begin{aligned} \Delta_\Psi \hat{\varphi}[\mathcal{L}_\phi f] &= \sqrt{\frac{1}{2} \int \mathrm{d}^3 x (\mathcal{L}_\phi f) \Theta^{-\frac{1}{2}} \mathcal{L}_\phi f} & (4.113) \\ \Delta_\Psi \hat{\pi}[\mathcal{L}_\phi f] &= \sqrt{\frac{1}{2} \int \mathrm{d}^3 x (\mathcal{L}_\phi f) \Theta^{\frac{1}{2}} \mathcal{L}_\phi f} & (4.114) \end{aligned}$$

The easiest way to prove this is to first prove it for $\Psi = \Psi^{coh}_\xi$ ($\xi \in h_{inv}$). This can be done by rewriting $\hat{\varphi}[\mathcal{L}_\phi f]$ and $\hat{\pi}[\mathcal{L}_\phi f]$ in terms of creation and annihilation operators; $\Delta_{\Psi^{coh}_\xi} \hat{\varphi}[\mathcal{L}_\phi f]$ and $\Delta_{\Psi^{coh}_\xi} \hat{\pi}[\mathcal{L}_\phi f]$ can then be easily calculated using the commutation relations among the creation and annihilation operators.

Specializing (4.113) and (4.114) to the case of f an eigenfunction of Θ, we furthermore get

$$\Delta_\Psi \hat{\varphi}[\mathcal{L}_\phi f] \Delta_\Psi \hat{\pi}[\mathcal{L}_\phi f] = \frac{1}{2} \tag{4.115}$$

saturating Heisenberg's uncertainty principle. [11]

4.6 Additional observations, using kinematical linearity

In the last section, we proved properties of $\mathcal{H}_b$ starting from the form of $\mathcal{H}_b$ in terms of coherent states (theorem 4.9). This was done in order to avoid crucial use of the linearity of the Klein-Gordon theory, and so emphasize methods of proof that will more easily extend to non-linear theories. However, there are a couple observations which require a reformulation of $\mathcal{H}_b$ that does use linearity in an essential way. We go over these observations here.

First, the full theory Hilbert space $\mathcal{H}$, using its Fock structure, can be rewritten as follows:

$$\begin{aligned}
\mathcal{H} = \mathcal{F}_s(h) &= \oplus_{n=0}^{\infty} \overset{n}{\otimes}_s h = \oplus_{n=0}^{\infty} \overset{n}{\otimes}_s (h_{inv} \oplus h_\perp) \\
&\overset{\text{nat.}}{\cong} \oplus_{n=0}^{\infty} \oplus_{j=0}^{n} \left\{ \left(\overset{j}{\otimes}_s h_{inv} \right) \otimes \left(\overset{n-j}{\otimes}_s h_\perp \right) \right\} \\
&= \oplus_{n=0}^{\infty} \oplus_{m=0}^{\infty} \left\{ \left(\overset{n}{\otimes}_s h_{inv} \right) \otimes \left(\overset{m}{\otimes}_s h_\perp \right) \right\} \\
&\overset{\text{nat.}}{\cong} \left\{ \oplus_{n=0}^{\infty} \overset{n}{\otimes}_s h_{inv} \right\} \otimes \left\{ \oplus_{m=0}^{\infty} \overset{m}{\otimes}_s h_\perp \right\} \\
&= \mathcal{F}_s(h_{inv}) \otimes \mathcal{F}_s(h_\perp)
\end{aligned} \tag{4.116}$$

Thus if we define $\mathcal{H}_\perp := \mathcal{F}_s(h_\perp)$,

$$\mathcal{H} \overset{\text{nat.}}{\cong} \mathcal{H}_{red} \otimes \mathcal{H}_\perp \tag{4.117}$$

This is the decomposition of $\mathcal{H}$ into a tensor product of symmetric and non-symmetric modes. We note, using the natural isomorphism in (4.117), if $a_{red}^\dagger(\cdot)$, $a_{red}(\cdot)$ and $a_\perp^\dagger(\cdot)$, $a_\perp(\cdot)$ denote the creation and annihilation operators on $\mathcal{H}_{red}$ and

[11] There exists a complete basis of eigenfunctions f of Θ. However, technically $\hat{\varphi}[f]$ and $\hat{\pi}[g]$ are well defined as operators only when f and g are in $\mathcal{S}(\Sigma)$ — and no eigenstates of Θ are in $\mathcal{S}(\Sigma)$. Therefore, *prima facie* the spreads $\Delta_\Psi \hat{\varphi}[\mathcal{L}_\phi f]$ and $\Delta_\Psi \hat{\pi}[\mathcal{L}_\phi f]$ are not defined for eigenfunctions f. Nevertheless, *the right hand sides* of equations (4.113) and (4.114) *are* well defined for f an eigenfunction of Θ, so that we can take the spreads to be simply defined by these expressions in that case.

$\mathcal{H}_\perp$, respectively, one can show, for $\xi_s \in h_{inv}$ and $\xi_\perp \in h_\perp$,

$$\begin{aligned} a^\dagger(\xi_s + \xi_\perp) &= a^\dagger_{red}(\xi_s) \otimes a^\dagger_\perp(\xi_\perp) \\ a(\xi_s + \xi_\perp) &= a_{red}(\xi_s) \otimes a_\perp(\xi_\perp). \end{aligned} \tag{4.118}$$

Let $\overset{\perp}{\Psi}_0 = (1, 0, 0, \dots)$ denote the vacuum of $\mathcal{H}_\perp = \mathcal{F}_s(h_\perp)$. Then, using (4.118), we show that, in terms of (4.117), $\mathcal{H}_b$ takes the form

Theorem 4.18.

$$\mathcal{H}_b = \{\overset{r}{\Psi} \otimes \overset{\perp}{\Psi}_0 \mid \overset{r}{\Psi} \in \mathcal{H}_{red}\}. \tag{4.119}$$

Proof.

$$\begin{aligned} \mathcal{H}_b &= \{\Psi \in \mathcal{H} \mid a(\eta)\Psi = 0 \quad \forall \eta \in h_\perp\} \\ &= \left\{\overset{r}{\Psi} \otimes \overset{\perp}{\Psi} \in \mathcal{H}_{red} \otimes \mathcal{H}_\perp \mid a(\eta)\overset{r}{\Psi} \otimes \overset{\perp}{\Psi} = 0 \quad \forall \eta \in h_\perp\right\} \\ &= \left\{\overset{r}{\Psi} \otimes \overset{\perp}{\Psi} \in \mathcal{H}_{red} \otimes \mathcal{H}_\perp \mid \overset{r}{\Psi} \otimes \left(a_\perp(\eta)\overset{\perp}{\Psi}\right) = 0 \quad \forall \eta \in h_\perp\right\} \\ &= \left\{\overset{r}{\Psi} \otimes \overset{\perp}{\Psi} \in \mathcal{H}_{red} \otimes \mathcal{H}_\perp \mid \overset{\perp}{\Psi} = \overset{\perp}{\Psi}_0 \equiv 1\right\} \\ &= \left\{\overset{r}{\Psi} \otimes \overset{\perp}{\Psi}_0 \in \mathcal{H}_{red} \otimes \mathcal{H}_\perp\right\} \\ &= \mathcal{H}_{red} \otimes \overset{\perp}{\Psi}_0 \end{aligned} \tag{4.120}$$

□

Additionally, if we let $\{\xi_i\}_{i \in I}$ be an orthonormal basis of h_{inv} and $\{\xi_i\}_{i \in J}$ an orthonormal basis of $h_\perp$,

$$\begin{aligned} \hat{\mathbb{H}} &= \sum_{i,j \in I \cup J} \langle \xi_i, \hat{H}\xi_j \rangle a^\dagger(\xi_i) a(\xi_j) \\ &= \sum_{i,j \in I} \langle \xi_i, \hat{H}\xi_j \rangle a^\dagger(\xi_i) a(\xi_j) + \sum_{i,j \in J} \langle \xi_i, \hat{H}\xi_j \rangle a^\dagger(\xi_i) a(\xi_j) \\ &= \left\{\sum_{i,j \in I} \langle \xi_i, \hat{H}\xi_j \rangle a^\dagger_{red}(\xi_i) a_{red}(\xi_j)\right\} \otimes \mathbb{1} + \mathbb{1} \otimes \left\{\sum_{i,j \in J} \langle \xi_i, \hat{H}\xi_j \rangle a^\dagger_\perp(\xi_i) a_\perp(\xi_j)\right\} \end{aligned} \tag{4.121}$$

so that if we define

$$\hat{\mathbb{H}}_\perp := \sum_{i,j\in J} \langle \xi_i, \hat{H}\xi_j \rangle a_\perp^\dagger(\xi_i) a_\perp(\xi_j) \tag{4.122}$$

then

$$\hat{\mathbb{H}} = \hat{\mathbb{H}}_{red} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\mathbb{H}}_\perp \tag{4.123}$$

$\hat{\mathbb{H}}_{red}$ is then the Hamiltonian for the symmetric modes, and $\hat{\mathbb{H}}_\perp$ the Hamiltonian for the non-symmetric modes. The unique lowest-eigenvalue eigenstate of $\hat{\mathbb{H}}_\perp$ is the vacuum $\overset{\perp}{\Psi}_0$. We thus see that theorem 4.18 exhibits $\mathcal{H}_b$ as the space of states in which *all non-symmetric modes are unexcited.* This is probably the description of $\mathcal{H}_b$ that would most appeal to the intuition of a particle physicist, because a particle physicist's intuition is so often linked to Fock space structure. However, as remarked at the start of the section, it is a description that uses crucially the kinematical linearity of the theory, a linearity that will not be present in more general situations. In particular such linearity is not present in the case of loop quantum gravity. We thus did not introduce this description until now because of it lacks use in extending the ideas of this chapter to the physically more interesting, kinematically non-linear theories.

Also, it should be remarked that with the above reformulation of $\mathcal{H}_b$, many of the properties of $\mathcal{H}_b$ proven in the last section are now easier to show. Again, we proved the results in the last section using coherent states because we wanted to emphasize methods of proof which will be amenable to extension to the non-linear case.

Lastly, the above reformulation of $\mathcal{H}_b$ allows us to draw a connection between b-symmetry and Klauder's approach to imposing second class constraints. Because $\overset{\perp}{\Psi}_0$ is the unique eigenstate of $\mathbb{1} \otimes \hat{\mathbb{H}}_\perp$ with eigenvalue zero, theorem 4.18 implies

$$\mathcal{H}_b = \mathrm{Ker}\left(\mathbb{1} \otimes \hat{\mathbb{H}}_\perp\right) \tag{4.124}$$

Thus, $\mathbb{1} \otimes \hat{\mathbb{H}}_\perp$ by itself could have been taken as the sole constraint. One can cast $\hat{\mathbb{H}}_\perp$ as a quadratic combination of the *original second class set of self-adjoint constraint operators* (equation (4.44)). This in turn makes this way of imposing the (exact) constraints (4.44) an instance of Klauder's universal procedure for imposing constraints (with 'δ' being set to zero; see [50]).

Let us show this. First let $\{f_i\}$ denote any basis of $\mathcal{S}(\Sigma)_\perp$ orthonormal with respect to $(\cdot, \Theta^{\frac{1}{2}}\cdot)$. Then $\{\xi_i := (f_i, \Theta^{\frac{1}{2}} f_i)\}$ is an orthonormal basis of $h_\perp$ (as one can check). Define

$$\hat{\eta}_{[i,0]} \quad := \quad \hat{\varphi}[f_i] \tag{4.125}$$

$$\hat{\eta}_{[i,1]} \quad := \quad \hat{\pi}[f_i]. \tag{4.126}$$

so that $\{\hat{\eta}_{[i,A]}\}$ is a basis of the full original set of constraint operators (4.44). Define the matrix

$$M^{[i,A],[j,B]} := \alpha^{AB}\langle \xi_i, \hat{H}\xi_j\rangle = \alpha^{AB}(f_i, \Theta f_j) \tag{4.127}$$

where $\alpha^{AB} := \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. It is not hard to check that M is Hermitian and positive definite. We have

$$\mathbb{1} \otimes \hat{\mathbb{H}}_\perp = \sum_{[i,A],[j,B]} M^{[i,A][j,B]}\hat{\eta}_{[i,A]}\hat{\eta}_{[j,B]} \tag{4.128}$$

casting $\mathbb{1} \otimes \hat{\mathbb{H}}_\perp$ in the desired form.

4.7 Viewpoint on $\mathcal{H}_c$ using 'squeezed states'

Recall the natural embedding $\mathfrak{E} : \mathcal{H}_{red} \hookrightarrow \mathrm{Cyl}^*$ defined (in §4.4.3) by $(\mathfrak{E}\Psi)(\Phi) = \langle \Psi, \Phi|_{\mathcal{S}'(\Sigma)_{inv}}\rangle$. One can rewrite this as follows. For $\varphi_o \in \mathcal{S}'(\Sigma)$, we can define the distribution $\delta_{\varphi_o} \in \mathrm{Cyl}^*$ by

$$\delta_{\varphi_o}(\Phi) := \Phi(\varphi_o) \tag{4.129}$$

Then

$$\mathfrak{E}\Psi := \int_{\varphi_s \in \mathcal{S}'(\Sigma)_{inv}} \mathrm{d}\mu_{red}\overline{\Psi[\varphi_s]}\delta_{\varphi_s} = \int_{\varphi_s \in \mathcal{S}'(\Sigma)_{inv}} \mathrm{d}\mu_{red}\overline{\delta_{\varphi_s}(\Psi)}\delta_{\varphi_s} \tag{4.130}$$

This form bears striking resemblance to the map $\iota : \mathcal{H}_{red} \to \mathcal{H}$ as defined in subsection 4.5.2. Using (4.130), we can rewrite $\mathcal{H}_c$ as

$$\mathcal{H}_c \quad := \quad \mathrm{Im}\mathfrak{E}$$

$$= \quad (\langle , \rangle_{\mathcal{H}_c}\text{-normalizable subspace of}) \quad \mathrm{span}\{\delta_\varphi\}_{\varphi \in \mathcal{S}'(\Sigma)_{inv}} \qquad (4.131)$$

This in turn is reminiscent of the formulation of $\mathcal{H}_b$ in terms of coherent states (4.5.1). In what follows we make these similarities precise by

1. using the standard embedding $\mathcal{H} \hookrightarrow \mathrm{Cyl}^*$ (using the inner product on $\mathcal{H}$) to think of the 'b' embedding as an embedding in Cyl^*
2. constructing a family of embeddings of $\mathcal{H}_{red} \hookrightarrow \mathrm{Cyl}^*$ intermediate between $\mathcal{H}_c$ and $\mathcal{H}_b$ using *squeezed* states. [12]

Heuristically, squeezed states are modified coherent states in which the uncertainty is not evenly distributed between configuration and momentum. This is only a heuristic statement for two reasons. First, the configuration and momentum variables have different dimensions, so that 'equality' of uncertainty in configuration and momentum is not a priori defined. This is true even in the case of the simple harmonic oscillator. In field theory, one has an additional source of ambiguity: uncertainties in configuration and momenta depend on the smearing of the configuration and momenta, and the Heisenberg bound can at most be saturated only for certain smearings. Furthermore, even for smearings which saturate the bound, the distribution of uncertainty between configuration and momenta depends on the smearing (even when the state is coherent/squeezed). This makes the notion of 'equal distribution of uncertainty between configuration and momentum' ill-defined at an even deeper level in field theory. [13]

Nevertheless, one can define 'squeezed' coherent states; Let us do so by constructing the squeezed states explicitly. For every choice of positive, self-adjoint operator $\hat{\mathcal{O}} : \mathcal{S}(\Sigma) \to \mathcal{S}(\Sigma)$, we have a corresponding Gaussian measure, given heuristically by

$$\text{“}\mathrm{d}\mu_{\hat{\mathcal{O}}} = \exp\left\{ -\frac{1}{2}(\varphi, \hat{\mathcal{O}}^{-1}\varphi) \right\} \mathcal{D}\varphi\text{”} \qquad (4.132)$$

[12] General squeezed states will live in Cyl^* — *i.e.*, be non-normalizable — because, generally, squeezed states belong to a different representation of the quantum observable algebra (in the algebraic QFT sense). Normally, one might want to restrict consideration to normalizable squeezed states; however, here it is necessary to include non-normalizable squeezed states in order to make the desired connection with $\mathcal{H}_c$, which lives in Cyl^*.

[13] When dynamics are brought in, one still cannot unambiguously say what 'evenly distributed uncertainty' means, but nevertheless there is then at least a *natural* choice of distribution of uncertainties — namely the unique one leading to dynamical coherent states (see discussion below).

or rigorously by the Fourier transform

$$\chi_{\hat{\mathcal{O}}}(f) = \exp\left\{-\frac{1}{2}(f, \hat{\mathcal{O}} f)\right\}. \tag{4.133}$$

$d\mu_{\hat{\mathcal{O}}}$ is said to be the Gaussian measure with *variance* $\hat{\mathcal{O}}$. (As for all Gaussian measures in this work, $d\mu_{\hat{\mathcal{O}}}$ is normalized such that $\int d\mu_{\hat{\mathcal{O}}} = 1$.) $d\mu_{\hat{\mathcal{O}}}$ in turn defines for us a "vacuum" $\overset{\hat{\mathcal{O}}}{\eta}_0$ in Cyl*:

$$\overset{\hat{\mathcal{O}}}{\eta}_0(\Phi) := \int_{\varphi \in \mathcal{S}'(\Sigma)} \Phi[\varphi] d\mu_{\hat{\mathcal{O}}}. \tag{4.134}$$

for all $\Phi \in$ Cyl. Since $a^\dagger(\cdot)$ and $a(\cdot)$ (defined in the standard Fock space, *i.e.* with $\hat{\mathcal{O}} = \Theta^{-\frac{1}{2}}$) preserve Cyl, they have dual action on Cyl*, and hence so does the generator $\hat{\Lambda}(\xi) := a^\dagger(\xi) - a(\xi)$ of translations in phase space. Thus, for any $\hat{\mathcal{O}}$ and $\xi \in h$, we define the associated *squeezed coherent state* in Cyl*

$$\overset{\hat{\mathcal{O}}}{\eta}{}^{coh}_{\xi} := e^{\hat{\Lambda}(\xi)^*} \overset{\hat{\mathcal{O}}}{\eta}_0. \tag{4.135}$$

The choice $\hat{\mathcal{O}} = \Theta^{-\frac{1}{2}}$ gives the standard coherent states introduced and used in section 4.5.

Define $\hat{\mathcal{O}}_\lambda := \lambda \Theta^{-\frac{1}{2}}$ and label the associated vacuum $\overset{\lambda}{\eta}_0$ and coherent states $\overset{\lambda}{\eta}{}^{coh}_{\xi}$. Then

$$\overset{1}{\eta}_0 = (\Psi_0)^* \tag{4.136}$$

$$\overset{1}{\eta}{}^{coh}_{\xi} = (\Psi^{coh}_{\xi})^* \tag{4.137}$$

are the standard coherent states used in section 4.5. For the limit $\lambda \to 0$ we define

$$\begin{aligned} \overset{0}{\eta}_0(\Phi) &:= \lim_{\lambda \to 0} \overset{\lambda}{\eta}_0(\Phi) \\ &= \lim_{\lambda \to 0} \int \Phi[\varphi] \exp\left\{-\frac{1}{2}\lambda^{-1}(\varphi, \Theta^{-\frac{1}{2}}\varphi)\right\} \mathcal{D}\varphi \\ &= \Phi[0] \end{aligned} \tag{4.138}$$

so that $\overset{0}{\eta}_0 = \delta_0$. [14] Furthermore, for $\xi = [\varphi', \pi']$, we have for all $\Phi \in \text{Cyl}$,

$$\begin{aligned}
\overset{0}{\eta}{}^{coh}_{\xi}(\Phi) &= \left(e^{\hat{\Lambda}(\xi)^*}\delta_0\right)(\Phi) = \delta_0\left(e^{\hat{\Lambda}(\xi)}\Phi\right) = \left(e^{\hat{\Lambda}(\xi)}\Phi\right)[0] \\
&= \left(e^{i(\hat{\varphi}[\pi'] - \hat{\pi}[\varphi'])}\Phi\right)[0] = \left(e^{-\frac{i}{2}(\pi',\varphi')}e^{i\hat{\varphi}[\pi']}e^{-i\hat{\pi}[\varphi']}\Phi\right)[0] \\
&= e^{-\frac{i}{2}(\pi',\varphi')}\left(e^{i\hat{\varphi}[\pi']}\right)[0]\left(e^{-i\hat{\pi}[\varphi']}\Phi\right)[0] \\
&= e^{-\frac{i}{2}(\pi',\varphi')} \cdot 1 \cdot \Phi[0 + \varphi'] = e^{-\frac{i}{2}(\pi',\varphi')}\Phi[\varphi']
\end{aligned} \tag{4.139}$$

where the Baker-Campbell-Hausdorff formula was used in the second line. Thus

$$\overset{0}{\eta}{}^{coh}_{\xi} = e^{-\frac{i}{2}(\pi',\varphi')}\delta_{\varphi'} \tag{4.140}$$

so that the $\lambda \to 0$ limit gives us the 'coherent states' from which $\mathcal{H}_c$ is constructed.

Before introducing a family of embeddings interpolating between the 'c' and 'b' embeddings, it is necessary to introduce the *Gaussian* generalized measure $\mathrm{d}\nu_{red}$ associated with the Lesbesgue generalized measure $\mathrm{d}\nu^o_{red}$:

$$\mathrm{d}\nu_{red} = e^{-\langle\xi,\xi\rangle}\mathrm{d}\nu^o_{red}. \tag{4.141}$$

Furthermore, let $\mathrm{d}\tilde{\mu}_{red}$ denote the Gaussian generalized measure on $\mathcal{S}(\Sigma)$ with variance $\Theta^{\frac{1}{2}}$. (All Gaussian measures and Gaussian generalized measures in this work are normalized to be probability — *i.e.* to be such that they integrate the constant function 1 to 1.) Let $\mathrm{d}\mu_{red}$ denote the Gaussian measure used in the Schrödinger representation of the reduced theory (D.9) (and in the definition of $\mathfrak{E}$ (4.53)), but now reinterpreted as a generalized measure on $\mathcal{S}(\Sigma)$. One can check

$$\mathrm{d}\nu_{red} = \mathrm{d}\mu_{red} \times \mathrm{d}\tilde{\mu}_{red}. \tag{4.142}$$

With these structures introduced, we define the family of embeddings

$$\mathfrak{E}_\lambda\Psi := \int_{\xi\in\Gamma_{inv}} \mathrm{d}\nu_{red}\, e^{\lambda\langle\xi,\xi\rangle}\, \overline{\overset{\lambda}{\eta}{}^{coh}_{\xi}(\Psi)}\overset{\lambda}{\eta}{}^{coh}_{\xi}. \tag{4.143}$$

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[14] Thus the limit $\lambda \to 0$ of the *measures* $\mathrm{d}\mu_{\hat{\mathcal{O}}_\lambda}$ is no longer Gaussian, but is a Dirac measure.

The 'c' and 'b' embeddings now become the $\lambda = 0$ and $\lambda = 1$ cases, respectively, in this family. More precisely, $\mathfrak{E}_0 = \mathfrak{E}$ and $\mathfrak{E}_1 = {}^* \circ \iota$. The latter of these equations is obvious, once it is recalled $\mathrm{d}\nu_{red} e^{\langle \xi, \xi \rangle} = \mathrm{d}\nu^o_{red}$. The former is perhaps not quite obvious. Let us prove the former, using the fact that $\mathrm{d}\nu_{red}$· decomposes as $\mathrm{d}\mu_{red} \times \mathrm{d}\tilde{\mu}_{red}$, with $\mathrm{d}\tilde{\mu}_{red}$ a probability measure. We have

$$\begin{aligned}
\mathfrak{E}_0 \Psi &= \int_{\xi \in \Gamma_{inv}} \mathrm{d}\nu_{red} \overline{\overset{0}{\eta}{}^{coh}_\xi (\Psi)} \overset{0}{\eta}{}^{coh}_\xi \\
&= \int_{[\varphi', \pi'] \in \Gamma_{inv}} \mathrm{d}\nu_{red} \overline{\delta_{\varphi'}(\Psi)} \delta_{\varphi'} \\
&= \int_{[\varphi', \pi'] \in \mathcal{S}(\Sigma) \times \mathcal{S}(\Sigma)} \mathrm{d}\mu_{red} \mathrm{d}\tilde{\mu}_{red} \overline{\delta_{\varphi'}(\Psi)} \delta_{\varphi'} \\
&= \int_{\varphi' \in \mathcal{S}(\Sigma)} \mathrm{d}\mu_{red} \overline{\delta_{\varphi'}(\Psi)} \delta_{\varphi'} = \mathfrak{E}\Psi
\end{aligned} \tag{4.144}$$

Lastly, we use this new way of looking at $\mathcal{H}_c$ to make an observation regarding its non-preservation by $\hat{\mathbb{H}}$ (*i.e.* by the quantum dynamics). The statement made at the end of subsection 4.5.3 applies here: If $\left\{ \overset{\hat{\mathcal{O}}}{\eta}{}^{coh}_\xi \right\}$ as a family of coherent states is preserved by the quantum evolution (*i.e.* is *dynamical*) in a manner covariant with respect to the symmetry group, then the associated 'symmetric sector' $\left(\mathrm{span} \left\{ \overset{\hat{\mathcal{O}}}{\eta}{}^{coh}_\xi \right\}_{\xi \in \Gamma_{inv}} \right)$ will be preserved by quantum time evolution. However, of *all* the families $\left\{ \overset{\hat{\mathcal{O}}}{\eta}{}^{coh}_\xi \right\}$ considered in equation (4.135), *only one* is dynamical: the standard family given by the choice $\hat{\mathcal{O}} = \Theta^{-\frac{1}{2}}$. The fact that $\hat{\mathbb{H}}^*$ failed to preserve $\mathcal{H}_c$ in subsection 4.4.4 can now be viewed as due to the fact that $\{\delta_{\varphi'}\}$ *is non-dynamical as a family of 'coherent states.'*

4.8 Carrying operators from $\mathcal{H}$ to $\mathcal{H}_{red}$

We have finished investigating the properties of 'c' and 'b' symmetry in the present simple model.

One of the nice properties of $\mathcal{H}_b$ is that the Hamiltonian preserves it, so that the Hamiltonian has a well- defined restriction to $\mathcal{H}_b$ which can then be carried over to $\mathcal{H}_{red}$ via the natural isomorphism. The operator thereby induced on $\mathcal{H}_{red}$ is the same as the Hamiltonian in the reduced theory, so that $\mathcal{H}_b$ gives a fully dynamical embedding of the reduced theory.

However, in more general situations, even if the Hamiltonian preserves a given choice of "symmetric sector" in a given theory, other operators of interest may not. It is therefore of interest to investigate the possibility of a general rule for carrying over *any* operator $\hat{\mathcal{O}}$ on the full theory Hilbert space $\mathcal{H}$ to an operator $\hat{\mathcal{O}}_{red}$ on the reduced theory Hilbert space $\mathcal{H}_{red}$ that somehow "best approximates the information contained in $\hat{\mathcal{O}}$." We will motivate and suggest such a prescription for a completely general theory, and then look at applications to example operators in the model theory considered in this chapter. We assume only that we are given some embedding ι of the reduced theory, $\mathcal{H}_{red}$ into the full theory $\mathcal{H}$. [15]

For $\hat{\mathcal{O}}$ Hermitian (*i.e.*, symmetric), a list of physically desirable criteria for the corresponding $\hat{\mathcal{O}}_{red}$ might include

1. $\hat{\mathcal{O}}_{red}$ is Hermitian.
2. $\langle \iota\Psi_1, \hat{\mathcal{O}}\iota\Psi_2\rangle = \langle \Psi_1, \hat{\mathcal{O}}_{red}\Psi_2\rangle$
3. $\Delta_{\iota\Psi_1}\hat{\mathcal{O}} = \Delta_{\Psi_1}\hat{\mathcal{O}}_{red}$

for all Ψ_1, Ψ_2 in $\mathcal{H}_{red}$. That is, one might want Hermicity, matrix elements and fluctuations to be preserved.

Fortunately the second of these criteria uniquely determines $\hat{\mathcal{O}}_{red}$:

$$\hat{\mathcal{O}}_{red} := \iota^{-1} \circ \mathcal{P} \circ \hat{\mathcal{O}} \circ \iota. \tag{4.145}$$

where $\mathcal{P} : \mathcal{H} \to \iota[\mathcal{H}_{red}]$ denotes orthogonal projection. This is perhaps what one would first write down as a possible prescription. The point, however, is that this prescription is not *ad hoc*: it is uniquely determined by a physical criterion, namely (2) in the above list. Furthermore,

[15] This is yet another advantage of 'b' symmetry, at least in the present simple model: the states are normalizable, and it is only in this case that the general prescription described here will apply. In the case of 'c' symmetry, even though the Hamiltonian does not preserve $\mathcal{H}_c$, one might have still hoped to induce a Hamiltonian operator on $\mathcal{H}_{red}$ from that on $\mathcal{H}$ via some other manner, such as the one described here; but it is not at all obvious how to do that due to the non-normalizability of c-symmetric states. The combination of the Hamiltonian not preserving $\mathcal{H}_c$ and $\mathcal{H}_c$ not having any normalizable elements thus frustrates attempts to use c-symmetry to compare the dynamics in the full and reduced theories in any systematic way.

Theorem 4.19. *The prescription defined in (4.145) satisfies all three of the desired properties,* [16] *except that the last property is replaced by*

$$\Delta_{\iota\Psi_1}\hat{\mathcal{O}} \geq \Delta_{\Psi_1}\hat{\mathcal{O}}_{red} \tag{4.146}$$

with equality holding iff $\hat{\mathcal{O}}$ preserves $\iota[\mathcal{H}_{red}]$. [17]

Let us look at some example operators in the Klein-Gordon theory considered in the present chapter. The example of $\hat{\mathbb{H}}$ has already been remarked upon. We proceed, then, to look at the basic configuration and momentum operators $\hat{\varphi}[f], \hat{\pi}[g]$. It is convenient to split these operators into parts. Define as operators on $\mathcal{H}$,

$$\hat{\tilde{\varphi}}_s[f] := \hat{\varphi}[f_s] \qquad \hat{\tilde{\pi}}_s[g] := \hat{\pi}[g_s] \tag{4.148}$$

$$\hat{\tilde{\varphi}}_\perp[f] := \hat{\varphi}[f_\perp] \qquad \hat{\tilde{\pi}}_\perp[g] := \hat{\pi}[g_\perp] \tag{4.149}$$

Note that the latter pair of operators are just the symmetry constraint operators. The tildes on these four operators are to distinguish them from the related operators on $\mathcal{H}_{red}$ and $\mathcal{H}_\perp$. For the configuration operators we have $\hat{\tilde{\varphi}}_s[f]\Psi[\varphi] = \varphi_s[f]\Psi[\varphi]$ and $\hat{\tilde{\varphi}}_\perp[f]\Psi[\varphi] = \varphi_\perp[f]\Psi[\varphi]$. In terms of the corresponding operators on $\mathcal{H}_{red}$ and $\mathcal{H}_\perp$,

$$\hat{\tilde{\varphi}}_s[f] := \hat{\varphi}_s[f] \otimes \mathbb{1} \qquad \hat{\tilde{\pi}}_s[g] := \hat{\pi}_s[g] \otimes \mathbb{1} \tag{4.150}$$

$$\hat{\tilde{\varphi}}_\perp[f] := \mathbb{1} \otimes \hat{\varphi}_\perp[f] \qquad \hat{\tilde{\pi}}_\perp[g] := \mathbb{1} \otimes \hat{\pi}_\perp[g] \tag{4.151}$$

$\hat{\tilde{\varphi}}_s[f]$ and $\hat{\tilde{\pi}}_s[g]$ both preserve $\mathcal{H}_b$, whereas $\hat{\tilde{\varphi}}_\perp[f]$ and $\hat{\tilde{\pi}}_\perp[g]$ do not. Nevertheless, on carrying these operators over to the reduced theory using (4.145) we get exactly

[16] Even though Hermicity of $\hat{\mathcal{O}}$ implies $\hat{\mathcal{O}}_{red}$ is Hermitian, *self-adjointness* of $\hat{\mathcal{O}}$ does not imply self-adjointness of $\hat{\mathcal{O}}_{red}$. This fact is discussed on page 19 of [57].

[17] As a side note, this result fully extends to non-Hermitian operators if we replace condition (1) with $(\hat{\mathcal{O}}_{red})^\dagger = (\hat{\mathcal{O}}^\dagger)_{red}$, define the spread of a non-Hermitian operator by

$$\Delta_\Psi\hat{\mathcal{O}} := \sqrt{\langle\Psi, \tfrac{1}{2}(\hat{\mathcal{O}}^\dagger\hat{\mathcal{O}} + \hat{\mathcal{O}}\hat{\mathcal{O}}^\dagger)\Psi\rangle - \mid\langle\Psi, \hat{\mathcal{O}}\Psi\rangle\mid^2}. \tag{4.147}$$

and give as the condition for equality in (4.146) the condition that *both* $\hat{\mathcal{O}}$ and $\hat{\mathcal{O}}^\dagger$ preserve $\iota[\mathcal{H}_{red}]$.

what one would expect:

$$(\hat{\hat{\varphi}}_s[f])_{red} = \hat{\varphi}_s[f] \tag{4.152}$$
$$(\hat{\hat{\pi}}_s[g])_{red} = \hat{\pi}_s[g] \tag{4.153}$$

but

$$(\hat{\hat{\varphi}}_\perp[f])_{red} = 0 \tag{4.154}$$
$$(\hat{\hat{\pi}}_\perp[g])_{red} = 0. \tag{4.155}$$

The inclusion of orthogonal projection in prescription (4.145) is essential in getting the last couple of equations above. Even though the symmetry constraint operators $\hat{\hat{\varphi}}_\perp[f] = \hat{\varphi}[f_\perp]$ and $\hat{\hat{\pi}}_\perp[g] = \hat{\pi}[g_\perp]$ do not annihilate $\mathcal{H}_b$, nevertheless, as one would hope, their corresponding operators induced on $\mathcal{H}_{red}$ via (4.145) are identically zero.

Lastly, one can also look at the angular momentum operator $\hat{\mathbb{L}}_z$. In the reduced classical theory the z-angular momentum is identically zero, so that one would expect the corresponding operator in the reduced model to be identically zero as well. Indeed,

$$(\hat{\mathbb{L}}_z)_{red} := \iota^{-1} \circ \mathcal{P} \circ \hat{\mathbb{L}}_z \circ \iota = 0 \tag{4.156}$$

as follows from $\mathcal{H}_b \subset \mathcal{H}_{inv}$.

4.9 Summary and outlook

4.9.1 Physical meaning(s) of $\mathcal{H}_b$

It is notable that $\mathcal{H}_b$ has a number of characterizations with completely distinct physical meaning all pointing to ways in which $\mathcal{H}_b$ embodies the notion of "symmetry." They are

1. $\mathcal{H}_b$ is the solution space to a set of constraints whose classical analogues isolate the axisymmetric sector of the classical phase space;

2. $\mathcal{H}_b$ is the span of the coherent states associated with the axisymmetric sector

of the classical theory;

3. $\mathcal{H}_b$ is the space of states in which all non-symmetric modes are unexcited (§4.6).

The first two of these in a clear way point to $\mathcal{H}_b$ as the "quantum analogue of the *classical* axisymmetric sector". The idea of invariance under the group action (leading to $\mathcal{H}_{inv}$), on the other hand, is the quantum analogue of classical axisymmetry in a slightly more indirect sense. It is *invariance* under the quantum analogue of *classical rotation about the z axis.* It is a subtle but clear distinction. Another way to state this distinction is that in $\mathcal{H}_{inv}$ we are imposing '$\mathcal{L}_\phi \Psi = 0$' [18], whereas in $\mathcal{H}_b$ we are imposing (an appropriate complex linear combination of) the conditions $\mathcal{L}_\phi \varphi(x) = 0$, $\mathcal{L}_\phi \pi(x) = 0$. In $\mathcal{H}_{inv}$ we are imposing axisymmetry on the *wave-function* whereas in $\mathcal{H}_b$ we are imposing axisymmetry on the *field operators.*

One can see the distinction in yet another way as well. Recall in the classical theory that the total angular momentum is given by the expression

$$\mathbb{L}_z = \int_\Sigma \varphi(\mathcal{L}_\phi \pi) \mathrm{d}^3 x. \tag{4.157}$$

Therefore, classically the condition $\mathbb{L}_z = 0$ is weaker than the condition that $\mathcal{L}_\phi \varphi = 0$ and $\mathcal{L}_\phi \pi = 0$. Likewise, as shown in subsections 4.4.5 and 4.5.4, quantum mechanically $\hat{\mathbb{L}}_z \Psi = 0$ is weaker than (an appropriate reformulation of) $\mathcal{L}_\phi \hat{\varphi}(x)\Psi = 0$ and $\mathcal{L}_\phi \hat{\pi}(x)\Psi = 0$. Again, it is $\mathcal{H}_b$ (and $\mathcal{H}_c$) that is playing the role of the quantum analogue(s) of classical axisymmetry.

Furthermore, as was seen in section 4.5.5, one can grasp the difference between $\mathcal{H}_{inv}$ and $\mathcal{H}_b$ in terms of fluctuations from axisymmetry. Expectation values for

[18] For t a tensor field on Σ,

$$\mathcal{L}_\phi t = \frac{\mathrm{d}}{\mathrm{d}\phi'} g(\phi') \cdot t \bigg|_{\phi'=0}$$

where $g(\phi') \cdot t$ denotes the push-forward — *i.e.* natural left action — of the rotation $g(\phi')$ on t. Likewise for $\Psi \in \mathcal{H}$ we define

$$\text{`}\mathcal{L}_\phi \Psi\text{'} = \frac{\mathrm{d}}{\mathrm{d}\phi'} g(\phi') \cdot \Psi \bigg|_{\phi'=0} = i\hat{\mathbb{L}}_z \Psi.$$

The use of '$\mathcal{L}_\phi \Psi$' in '$\mathcal{L}_\phi \Psi = 0$' is solely in order to highlight the analogy with and distinction from $\mathcal{L}_\phi \varphi(x) = 0$ and $\mathcal{L}_\phi \pi(x) = 0$.

field operators are axisymmetric both for states in $\mathcal{H}_{inv}$ and for states in $\mathcal{H}_b$. However, the standard deviation, or "fluctuations", of $\hat{\varphi}[\mathcal{L}_\phi f]$ and $\hat{\pi}[\mathcal{L}_\phi g]$ from zero are completely controlled in $\mathcal{H}_b$ — they are the *same* for all members of $\mathcal{H}_b$, and are in a certain sense minimized, as discussed in section 4.5.5. In $\mathcal{H}_{inv}$, one does not have this control.

Lastly, it is $\mathcal{H}_b$ and $\mathcal{H}_c$ that achieve commutation of symmetry reduction and quantization, the former at the full level of dynamics. $\mathcal{H}_{inv}$ does not achieve commutation at any level.

4.9.2 Future directions: sketch of application to LQG

4.9.2.1 Motivation and strategy

As pointed out earlier, the embedding of symmetry reduced theories into full loop quantum gravity suggested by Bojowald and Kastrup is analogous to the embedding $\mathcal{H}_c$ in the Klein-Gordon model considered here. Nevertheless, in the Klein-Gordon model, we saw that, for multiple reasons, the embedding $\mathcal{H}_b$ is preferable to $\mathcal{H}_c \subseteq \mathrm{Cyl}^*_c$:

1. $\hat{\mathbb{H}}$ preserves $\mathcal{H}_b$ whereas $\hat{\mathbb{H}}^*$ does not preserve $\mathcal{H}_c$. Consequently, it is only $\mathcal{H}_b$ that gives us an embedding of *both* Hilbert space structure and dynamics.

2. Fluctuations from axisymmetry in $\mathcal{H}_b$ are more evenly distributed between configuration and momentum variables, and are in a certain sense minimized.

3. $\mathcal{H}_b$ is the span of the set of coherent states associated with the symmetric sector of the classical theory — a particularly elegant characterization that brings out a physical content not fully shared by $\mathcal{H}_c$.

It would be ideal, then, if one could extend the notion of balanced symmetry (embodied in $\mathcal{H}_b$) to the case of LQG. The most obvious avenue for this is to use the characterization in theorem 4.9 — that of the span of semi-classical states associated with the symmetric sector of the classical theory. For, ideas on semi-classical states in LQG have already been introduced [58, 59, 60]. Indeed, one of the results in [61] seems to partially support this strategy. There it was found that one had to restrict precisely to coherent symmetric states before one could reproduce in full LQG a result known in the reduced theory — namely, the boundedness of the inverse volume operator. [19]

However, there is a freedom in the choice of semiclassical states used in defining $\mathcal{H}_b^{LQG}$. In the next two subsections we will consider how this freedom can be used to reproduce additional characteristics of balanced symmetry.

4.9.2.2 Solving a set of quantum constraints that classically isolates the symmetric sector

First, characterization (1) listed in section 4.9.1 is easily reproduced by using complexifier coherent states. To see this, let P denote the SU(2) principal bundle for the theory, with base space Σ. Let $\mathcal{S}$, a subgroup of the automorphisms of P, be the symmetry group of interest. Then if we define $\mathcal{H}_b^{LQG}$ to be the span of complexifier coherent states associated with symmetric field configurations, all states Ψ in $\mathcal{H}_b^{LQG}$ will satisfy

$$(\widehat{\Phi_\alpha^* A^{\mathbb{C}}(e) - A^{\mathbb{C}}(e)})\Psi = \left(U_\alpha \hat{A}^{\mathbb{C}}(e) U_\alpha^{-1} - \hat{A}^{\mathbb{C}}(e)\right)\Psi = 0 \qquad (4.158)$$

for all edges e and all $\alpha \in \mathcal{S} \subset \mathrm{Aut}(P)$. Here Φ_α and U_α denote the action of α on the kinematical phase space and kinematical Hilbert space, respectively. $\hat{A}^{\mathbb{C}}(\cdot)$ are the "annihilation operators" defined in [58] depending on a particular choice of complexifier. The classical constraints under the hat on the left hand side select

[19] The following is a side note. [61] nevertheless found that, on more general states approximately invariant under the action of the symmetry group (translations and rotations) on large scales, the inverse volume operator is *unbounded*. From this they conclude that "the boundedness of the inverse scale factor in isotropic and homogeneous LQC does not extend to the full theory even when restricting LQG to those states which one would use to describe a maximally homogeneous and isotropic situation (modulo fluctuations)"(pp.4-5).

However, in light of the present research, as written, this statement is not just. For, as was pointed out earlier, in quantum gravity, the notion of symmetry given simply by invariance under the action of the symmetry group becomes trivial once one goes to the level of solutions to the diffeomorphism constraint. Therefore, the symmetry restriction used in [61] to make the statement of unboundedness is, strictly speaking, empty of physical content. Rather, as has been a main point of this chapter, when comparing a full quantum theory with a corresponding symmetry reduced theory, the notion of symmetric sector in the full theory should be more restrictive than the one defined by invariance under the symmetry group action. And, on the choice of "symmetric sector" suggested by the present research, [61] did find boundedness.

uniquely, at the classical level, the ($\mathcal{S}$-)symmetric sector.[20] So, again like $\mathcal{H}_b$ in the scalar field case, $\mathcal{H}_b^{LQG}$ will solve a set of constraints that, at the classical level, uniquely select the appropriate classical symmetric sector.

4.9.2.3 Preservation by quantum dynamics

Perhaps more importantly, one would like to reproduce the property that $\mathcal{H}_b$ is preserved by the Hamiltonian (in the case of LQG, a constraint in the bulk). It is not obvious how to do this; nevertheless we mention some possibilities. Perhaps complexifier coherent states could again be used, with the complexifier being tailored to the dynamics in some way; or perhaps one needs a different approach. As already indicated in section 4.5.3, what one needs is essentially *'temporally stable'* or *'dynamical'* coherent states if $\mathcal{H}_b$ is to be preserved by the Hamiltonian constraint.

The problem of constructing temporally stable coherent states is discussed in [54, 55, 56]. In [54] and [55], two general schemes are given for constructing stable families of coherent states. Unfortunately in both of these schemes, the label space for the coherent states is no longer necessarily the classical phase space, Γ, whence it is not obvious whether it is possible or appropriate to use such coherent states in constructing $\mathcal{H}_b^{LQG}$ in the manner described above. In [56], on the other hand, Γ is retained as the label space for the coherent states, but they conclude that *exactly* stable families of coherent states do not always exist, but rather, for interacting theories, one in general expects only *approximately* stable families. However, this statement is made for a fixed set of 'fundamental operators' used to characterize semi-classicality; it is not clear it holds if the choice of 'fundamental operators' is not so fixed.

We leave investigation along these lines to future research. The main reason for desiring $\mathcal{H}_b^{LQG}$ to be preserved by the Hamiltonian constraint is that then a constraint operator $\hat{C}(x)_{red}$ is induced on $\mathcal{H}_b^{LQG}$, making $(\mathcal{H}_b^{LQG}, \hat{C}(x)_{red})$ a closed system that could be compared, for example, with loop quantum cosmology. However, even if $\mathcal{H}_b^{LQG}$ is not preserved by dynamics, $\mathcal{H}_b^{LQG}$ is still valuable in that

[20] assuming the $A^{\mathbb{C}}(e)$ for all edges e separate points in the kinematical phase space. This is gauranteed to be true at least locally on the phase space and is hoped to hold globally for complexifiers of physical interest [58].

it gives us a notion of 'symmetric sector'. This notion of 'symmetric sector' can in principle be transferred to the physical Hilbert space (as described below), at which point preservation by constraints is no longer an issue. Comparison with LQC might then be attempted directly at the level of the physical Hilbert space [62].

4.9.2.4 Gauge fixing and symmetry at the physical Hilbert space level

Next let us discuss two issues related to constraints. First, as just touched upon, is the question of how one might obtain from $\mathcal{H}_b^{LQG}$ a 'symmetric sector' in the final *physical* Hilbert space of LQG. Let $\mathcal{H}_{kin}$ denote the kinematical Hilbert space of the theory, let $\mathcal{H}_{Diff}$ denote the solution to the Gauss and diffeomorphism constraints, and let $\mathcal{H}_{Phys}$ denote the space solving the Hamiltonian constraint as well. We have already suggested how to define the "b-symmetric sector" in $\mathcal{H}_{kin}$. To obtain a notion of symmetric sector in $\mathcal{H}_{Diff}$, the obvious strategy is to group average the "b-symmetric states" in $\mathcal{H}_{kin}$. This strategy is natural in light of [63] and the fact that we are using the definition of the symmetric sector inspired by theorem 4.9. Furthermore, if one follows the master constraint programme [64], one can use the master constraint to group average[21] states from $\mathcal{H}_{Diff}$ to $\mathcal{H}_{Phys}$ and so transfer the notion of symmetric sector to $\mathcal{H}_{Phys}$.

The second issue related to constraints is that of gauge-fixing the symmetry group — that is, choosing a symmetry group which is not invariant under conjugation by diffeomorphisms and gauge transformations. Such a choice of symmetry group is made in LQC, for example. We note the following: on group averaging the symmetric sector over gauge transformations and diffeomorphisms, any such gauge-fixing will be washed out. This can be seen as follows. Let $\mathcal{H}_b^G$ denote the "symmetric sector" of $\mathcal{H}_{kin}$ corresponding to the subgroup G of the automorphism group of the principal bundle. If the only "background" used in the construction of $\mathcal{H}_b^G$ is the choice of group G, then, for any automorphism α of P, we will have covariance:

$$U_\alpha[\mathcal{H}_b^G] = \mathcal{H}_b^{\alpha \cdot G \cdot \alpha^{-1}}. \tag{4.159}$$

[21] or, more or less equivalently, use the zero eigenvalue spectral projection operator for the master constraint

Now, if we had not gauge fixed, our symmetric sector would consist in the span of all $\mathcal{H}_b^{\alpha\cdot G\cdot\alpha^{-1}}$ for α in the automorphism group. This follows from the fact that we are defining the quantum symmetric sector as the span of coherent states associated with the classical symmetric sector. Thus, from the above equation, it is clear that on group averaging over the automorphism group, one will obtain the same subspace of $\mathcal{H}_{Diff}$ whether one gauge fixes the symmetry group or doesn't.

Indeed, this situation can be mimicked in the Klein-Gordon toy model by simply declaring, for example, that $\mathbb{L}_x, \mathbb{L}_y, \mathbb{L}_z$ be constraints. This is a first class system, and the gauge group generated is the full group of SO(3) rotations about the origin. In this context, the group of rotations about the z-axis is then

1. a subgroup of the full canonical gauge group.

2. furthermore a gauge-fixed group. It is not left invariant by conjugation by the rest of the canonical gauge group.

These two properties precisely mimic the situation in loop quantum cosmology. In this toy model, one has the possibility check that certain nice properties of $\mathcal{H}_b$ are preserved by the group averaging procedure, such as the minimization of fluctuations from axisymmetry. This could possibly be done by group averaging the kinematical symmetry constraints $\{\hat{\varphi}[\mathcal{L}_\phi f], \hat{\pi}[\mathcal{L}_\phi g]\}$ to obtain operators on the physical Hilbert space. One could then calculate the fluctuations of these operators from zero for the proposed symmetric sector in the physical Hilbert space.

4.9.2.5 Isomorphism with a reduced quantum model

We conclude with a final note on how all of this is, or might be, related to LQC and similar reduced models. What has mainly been discussed thus far is how one should define the notion of "symmetric sector" in LQG appropriate for comparison with reduced models. It is not at all clear, however, whether one should expect the "symmetric sector" so defined to be isomorphic to the Hilbert space in the corresponding model quantized a la Bojowald.[22] If it is not, we argue that the

[22] There is a fundamental difference between the configuration algebra underlying the full theory and the configuration algebra underlying LQC and other related models. Specifically: in such models, only holonomies along edges adapted to the symmetry are included in the algebra. This makes isomorphism with $\mathcal{H}_b^{LQG}$ seem less likely unless perhaps $\mathcal{H}_b^{LQG}$ is modified in some way.

physics of the "symmetric sector" defined along the lines suggested in this section should be considered the more fundamental description. Perhaps one could even formulate the physics of this sector in such a way that one could easily calculate corrections to predictions made using LQC and related models.

One thing should be remarked upon, however: the work presented in this chapter clarifies that the standard criticism [65] of minisuperspace models stating that "fluctuations in non-symmetric modes are unphysically set to zero", violating the uncertainty principle, is not just. Rather, if one stays strictly within the confines of the reduced theory, any question about fluctuations in non-symmetric modes has no a priori answer: one must first embed the reduced theory into the full theory. It is only then that the question has meaning. As was seen in the present toy model, in one embedding, 'c', one indeed has no fluctuations in at least the configuration part of the non-symmetric modes (the fluctuations in the momentum parts are not defined). In embedding 'b', on the other hand, one has non-zero fluctuations in both the configuration and momenta of the non-symmetric modes, and the uncertainty principle is furthermore not violated, as must be the case for elements of $\mathcal{H}$.

As this objection is the most common one made regarding the trustworthiness of minisuperspace models, perhaps the statement is that LQC, and other minisuperspace models, are more trustworthy that previously assumed. At least one cannot make a general statement that they are not trustworthy. The degree of trustworthiness, however, probably depends on the particular minisuperspace under consideration. The constructions in this chapter should aid in assessing this question by providing a guideline for the relation between full and reduced theories in the ideal case.

Appendix A

The entropy calculation: inclusion of Maxwell fields

As noted in [12] and elsewhere, a key fact which any statistical derivation of the Bekenstein-Hawking entropy must account for is that the entropy depends only on *geometrical fields*. Therefore, it is important to consider how the entropy calculation presented in chapter 3 is modified when matter is present at the horizon. In the present thesis, we only discuss in detail the inclusion of Maxwell fields, but inclusion of dilaton and non-minimally coupled scalar fields can be handled as well, by modifying the arguments for the type I case [12, 66] appropriately.

The Maxwell space-time field variable is a U(1) connection which we denote by $\mathbf{A}$; let $\mathbf{F} = \mathrm{d}\mathbf{A}$ denote the corresponding field strength. The action is

$$S_{EM} = -\frac{1}{16\pi}\int_{\mathcal{M}} \sqrt{-g} F_{ab} F^{ab} \mathrm{d}^4 x = \frac{1}{8\pi}\int_{\mathcal{M}} F \wedge {}^{\star}F \tag{A.1}$$

In addition to appropriate fall-off conditions as infinity (the details of which are not important for this section), we impose at the horizon the partial gauge-fixing condition $\mathcal{L}_{\ell}\underleftarrow{\mathbf{A}} = 0$. That is, we require that $\mathbf{A}$ be in 'a gauge adapted to the horizon', as in [26].

Let us now consider the required modifications to the derivation in chapter 3. First, because the macroscopic degrees of freedom at the horizon are not only gravitational, but electromagnetic, it will be necessary to fix *both* the geometric and electromagnetic multipoles. This will give us the necessary control in defining an ensemble for which to calculate the entropy. Let $\{\mathring{Q}_n, \mathring{P}_n\}$ denote the chosen fixed values of the electromagnetic multipoles.

There are two key facts which allow the argument in chapter 3 to go through without affecting the entropy: (1.) The form of the horizon boundary condition (2.61), which embodies the condition that S be an isolated horizon, does not change, and (2.) there is no surface term in the Maxwell symplectic structure. The rest of the appendix is organized into two sections. The first section elaborates on the two key points just mentioned, while the second summarizes the quantization, the construction of electromagnetic multipole operators for defining an ensemble, and the calculation of the entropy.

A.1 Elaboration on two key facts

The quantum boundary condition remains the same

One can check that, in the presence of matter, the horizon boundary condition (2.46) on the *complex* U(1) connection, $\tilde{V}^{(+)}$, generalizes to

$$\begin{aligned} \mathrm{d}\tilde{V}^{(+)} &= \left(\Psi_2 - \Phi_{11} - \frac{R}{24}\right)\Sigma\cdot r \\ &= \left(-\Phi_\Delta - \frac{R}{6}\right)\Sigma\cdot r \end{aligned} \tag{A.2}$$

where $\Phi_{11} := \frac{1}{4}R_{ab}(\ell^a n^b + m^a\overline{m}^b)$, and Φ_Δ is defined in (2.11). This again follows from a slight modification of the discussion in appendix B of [11]. Since we here only wish to consider Maxwell fields as matter, this simplifies to

$$\mathrm{d}\tilde{V}^{(+)} = -\Phi_\Delta\Sigma\cdot r. \tag{A.3}$$

So that in terms of $\tilde{V}$,

$$\mathrm{d}\tilde{V} = -{}^\gamma\Phi_\Delta\Sigma\cdot r. \tag{A.4}$$

where ${}^\gamma\Phi_\Delta := \mathrm{Re}\Phi_\Delta + \gamma\mathrm{Im}\Phi_\Delta$. As before, however, we are ultimately interested in reformulating this in terms of the type I connection $\tilde{V}^\circ$, as this is the horizon variable most convenient for quantization. Substituting (2.56) into the above equation, and using (2.6) and (2.10), again the factors exactly conspire to yield

$$\mathrm{d}\tilde{V}^{\circ} = -\frac{2\pi}{a_o}\Sigma \cdot r \tag{A.5}$$

so that the formulation of the quantum horizon boundary condition is (perhaps surprisingly) unaffected by the presence of Maxwell fields.

The Maxwell symplectic structure possesses no surface term

The action for the Maxwell theory is

$$S_{EM} = -\frac{1}{16\pi}\int_{\mathcal{M}} \sqrt{-g} F_{ab}F^{ab}\mathrm{d}^4x = \frac{1}{8\pi}\int_{\mathcal{M}} F \wedge {}^{\star}F \tag{A.6}$$

variation of the action yields

$$\delta\left(\frac{1}{8\pi}\int_{\mathcal{M}} F \wedge {}^{\star}F\right) = -\frac{1}{4\pi}\int_{\mathcal{M}} \delta A \wedge \mathrm{d}F + \frac{1}{4\pi}\int_{\partial\mathcal{M}} \delta A \wedge {}^{\star}F. \tag{A.7}$$

From the boundary term in the variation, we can read off the symplectic potential current density as

$$\theta_{EM}(\delta) = \frac{1}{4\pi}\delta A \wedge {}^{\star}F \tag{A.8}$$

giving the symplectic current density

$$\omega_{EM}(\delta_1, \delta_2) = \frac{1}{4\pi}\left(\delta_1{}^{\star}F \wedge \delta_2 A - \delta_2{}^{\star}F \delta_1 A\right) \tag{A.9}$$

When pulled back to a spatial hypersurface, ${}^{\star}F$ becomes the electric field two-form E_{ab}; thus, the above symplectic current density is the expected one, with E and A conjugate. There will be a boundary term in the symplectic structure iff the integral of the symplectic current density on Δ fails to vanish on the space of histories. We will show that, in fact, the integral of the $\omega_{EM}(\delta_1, \delta_2)$ on Δ does vanish identically, so that no boundary term is required. Let us go over the proof.

First, as noted in [11], the isolated horizon boundary conditions imply that the Maxwell Newman-Penrose coefficient ϕ_0 vanishes at the horizon. It follows that

$${}^{\star}F_{\underleftarrow{ab}} \mathrel{\hat{=}} -2\mathrm{Re}\phi_1 \epsilon_{ab} \tag{A.10}$$

where $\epsilon_{ab} = 2im_{[a}\overline{m}_{b]}$ is the horizon area two-form on Δ. Pulling back (A.9) to Δ

and substituting in (A.10), [1]

$$\underset{\leftarrow}{\omega}_{EM}(\delta_1,\delta_2) = \frac{1}{2\pi}\left(\delta_1 A \wedge \delta_2\{(\mathrm{Re}\phi_1)\,{}^2\!\epsilon\} - (1 \leftrightarrow 2)\right). \tag{A.11}$$

Now, $\delta^d_{\underset{\leftarrow}{a}} = -\ell^d n_a + m^d \overline{m}_a + \overline{m}^d m_a$. Writing (A.11) with indices explicit, and inserting δ,

$$\omega_{EM}(\delta_1,\delta_2)_{\underset{\leftarrow}{abc}} = \frac{3}{2\pi}\left(\delta_1 A_d(-\ell^d n_{[a} + m^d \overline{m}_{[a} + \overline{m}^d m_{[a})\delta_2\{(\mathrm{Re}\phi_1)\epsilon_{bc]}\} - (1 \leftrightarrow 2)\right). \tag{A.12}$$

As noted in §3.2.3, since ${}^2\!\epsilon$ is tangent to the fixed foliation of Δ throughout the space of histories, $\delta_2\,{}^2\!\epsilon$ is tangent to the fixed foliation of Δ. Thus,

$$\omega_{EM}(\delta_1,\delta_2) = -\frac{1}{2\pi}\left[\delta_1(A_d \ell^d n) \wedge \delta_2\{(\mathrm{Re}\phi_1)\,{}^2\!\epsilon\} - (1 \leftrightarrow 2)\right]. \tag{A.13}$$

Integrating this over Δ, and using the fact that n is minus the differential of the affine coordinate v on Δ,

$$\int_\Delta \omega_{EM}(\delta_1,\delta_2) = \frac{1}{2\pi}\int \mathrm{d}v \int_{S_v}\left[\delta_1(A\cdot\ell)\delta_2\{(\mathrm{Re}\phi_1)\,{}^2\!\epsilon\} - (1 \leftrightarrow 2)\right]. \tag{A.14}$$

Now, as in [26], we note that from the Cartan identity, and the fact that $\mathcal{L}_\ell \underset{\leftarrow}{\mathbf{A}} = 0$, we have

$$\begin{aligned} 0 &\,\widehat{=}\, \mathcal{L}_\ell \underset{\leftarrow}{\mathbf{A}} \,\widehat{=}\, \ell \lrcorner \mathrm{d}\mathbf{A} + \mathrm{d}(\ell\cdot\mathbf{A}) \\ &\,\widehat{=}\, \mathrm{d}(\ell\cdot\mathbf{A}) \end{aligned} \tag{A.15}$$

whence $\ell\cdot\mathbf{A}$ is constant on Δ (the 'analogue of the zeroth law of black hole mechanics for the electromagnetic field'). Thus,

$$\begin{aligned} \int_{S_v}\delta_1(A\cdot\ell)\delta_2\{(\mathrm{Re}\phi_1)\,{}^2\!\epsilon\} &\,\widehat{=}\, \delta_1(A\cdot\ell)\delta_2\int_{S_v}(\mathrm{Re}\phi_1)\,{}^2\!\epsilon \\ &\,\widehat{=}\, 4\pi\delta_1(A\cdot\ell)\delta_2 Q \end{aligned} \tag{A.16}$$

[1]When indices are suppressed, we write the area form as ${}^2\!\epsilon$ to remind the reader of its rank, to distinguish it from volume forms of other ranks.

Which is zero since the electric charge Q is fixed. Therefore, from (A.14),

$$\int_{\Delta} \omega_{EM}(\delta_1, \delta_2) = 0 \tag{A.17}$$

whence there is no surface term in the symplectic structure for the Maxwell degrees of freedom. Because of this, as per the argument at the end of section 3.2.5, the Maxwell degrees of freedom will make no contribution to the final entropy of the black hole.

A.2 Summary of quantization and entropy

Using the above results, let us summarize the quantization and the calculation of the entropy. The gravitational phase space is unchanged with the inclusion of matter. The symplectic structure, with its surface term remains unchanged. Furthermore, as noted above, the horizon boundary condition to be imposed in quantum theory remains unchanged. Hence, the entire discussion leading up to the construction of $\mathcal{H}_{\rm kin}$ in section 3.3 remains valid, without change.

Let us next address the electromagnetic degrees of freedom. Recall that part of the isolated horizon boundary conditions entails that all equations of motion hold at the horizon. Hence, even though the main isolated horizon boundary conditions are purely geometrical in character, they have implications for matter fields via Einstein's equations. In fact, in part because the electromagnetic multipoles are fixed, the non-gauge electromagnetic degrees of freedom intrinsic to the horizon are *completely determined* by the geometric degrees of freedom. To see this, first note that by inverting (2.38), one can express ϕ_1 in terms of the fixed electromagnetic multipoles, ${}^2\!\epsilon$ and ζ. Then recall, as noted earlier, that the Newman-Penrose coefficient ϕ_0 vanishes on isolated horizons. It follows that

$$\begin{aligned} {}^\star\underleftarrow{F} &\,\widehat{=}\, -2(\mathrm{Re}\phi_1)\,{}^2\!\epsilon \\ \underleftarrow{F} &\,\widehat{=}\, 2(\mathrm{Im}\phi_1)\,{}^2\!\epsilon \end{aligned} \tag{A.18}$$

Thus, with the electromagnetic multipoles fixed, ${}^\star\underleftarrow{F}$ and $\underleftarrow{F}$ are entirely determined by the geometric degrees of freedom at the horizon, as claimed. Because of this, as in [12], we eliminate these electromagnetic degrees of freedom in favor of the geometric degrees of freedom *prior to quantization.*

Nevertheless, there are still pure gauge degrees of freedom of the Maxwell field intrinsic to the horizon. The strategy of solving the constraints (A.18) in favor of the geometric degrees of freedom prior to quantization therefore should not be understood as the reason why the Maxwell fields do not contribute to the entropy. Rather, the decisive reason for the non-contribution is the fact that there is no surface term in the symplectic structure of the Maxwell field, as already noted. Because of this, as well as because we solve the constraints (A.18) prior to quantization, the details of the quantization of the Maxwell field beyond what has already been said will not matter for the purposes of calculating black hole entropy, as we shall see. We therefore do not go into such details.

Denote the results of the kinematical quantizations described above by $\mathcal{H}^{grav}_{kin}$ and $\mathcal{H}^{EM}_{kin}$, respectively. the kinematical Hilbert space for the full system is then

$$\mathcal{H}_{kin} = \mathcal{H}^{EM}_{kin} \otimes \mathcal{H}^{grav}_{kin}. \tag{A.19}$$

As in section 3.3, one can at this point introduce operators describing the physical quantum geometry of the horizon: $\hat{\zeta}(z)$, $\hat{\Phi}_\Delta(z)$, and the geometric multipole operators $\hat{I}_n, \hat{L}_n$. Note that instead of a $\hat{\Psi}_2(z)$ operator, we construct a $\hat{\Phi}_\Delta(z)$ operator, as this is now the field directly related to the definition of the geometric multipoles. As before, after regularizing in a manner analogous to that used before, one obtains

$$\hat{I}_n + i\hat{L}_n = \frac{\hat{a}_S}{a_o}\left(\mathring{I}_n + i\mathring{L}_n\right). \tag{A.20}$$

With electromagnetic fields now present, we can additionally construct operators corresponding to ϕ_1 and the electromagnetic multipoles. The construction is exactly analogous to the construction of the operators $\hat{\Phi}_\Delta$ and $\hat{I}_n + i\hat{L}_n$ in an obvious way. In the end we have

$$\hat{\phi}_1(z) = \sum_{n=0}^{\infty} \frac{\sqrt{4\pi(2n+1)}}{R_o^{n+2}}(\mathring{Q}_n + i\mathring{P}_n)Y_n^0(\hat{\zeta}(z)) \tag{A.21}$$

$$\hat{Q}_n + i\hat{P}_n = \frac{\hat{a}_S}{a_o}\left(\mathring{Q}_n + i\mathring{P}_n\right). \tag{A.22}$$

As with the geometric multipole operators, because the electromagnetic multipole operators are diffeomorphism invariant they will carry over to operators on the physical Hilbert space, so that they can be used to characterize the ensemble of physical states described below.

Finally, let us sketch the method of solving the quantum Einstein equation, noting what differs from the presentation in [12]. First, substituting the expression for $\mathcal{H}_{kin}^{grav}$ in [12] into (A.19), we have

$$\mathcal{H}_{kin} = \bigoplus_{\mathcal{P},\vec{m},\vec{a}:-2\vec{m}=\vec{a} \mod k} \mathcal{H}_{kin}^{EM} \otimes \mathcal{H}_{V,grav}^{\mathcal{P},\vec{m}} \otimes \mathcal{H}_S^{\mathcal{P},\vec{a}} \tag{A.23}$$

where $\mathcal{P}$ is a finite point set in S, $\vec{m}$ is an assignment of a half integer to each point, and $\vec{a}$ is an assignment of an integer modulo k to each point. $\vec{m}$ determines eigenvalues of the flux operators $\int_U \hat{\Sigma} \cdot r$ on the horizon, and $\vec{a}$ determines eigenvalues of the $U(1)$ holonomy operators on the surface Hilbert space $\mathcal{H}_S$. $\mathcal{H}_{V,grav}^{\mathcal{P},\vec{m}}$ and $\mathcal{H}_S^{\mathcal{P},\vec{a}}$ are the corresponding eigensubspaces (see [12] for details).

To solve the diffeomorphism constraint, we must divide out by the group $\mathcal{D}$ of all spatial diffeomorphisms preserving the inner boundary S. Following [12], we do this as follows. For every positive integer n, we fix a set of n punctures $\mathcal{P}_n$. Define

$$\mathcal{H}_S^{n,\vec{a}} := \mathcal{H}_S^{\mathcal{P}_n,\vec{a}}. \tag{A.24}$$

Let $\mathcal{D}_{\mathcal{S}}$ denote the group of spatial diffeomorphisms that are identity on S. We define

$$\mathcal{H}_V^{n,\vec{m}} := \frac{\mathcal{H}_{kin}^{EM} \otimes \mathcal{H}^{\mathcal{P}_n,\vec{m}}}{\mathcal{D}_{\mathcal{S}}}. \tag{A.25}$$

The solution to the diffeomorphism constraint can then be written

$$\mathcal{H}^{\text{Diff}} = \bigoplus_{n,\vec{m},\vec{a}:-2\vec{m}=\vec{a} \mod k} \mathcal{H}_V^{n,\vec{m}} \otimes \mathcal{H}_S^{n,\vec{a}}. \tag{A.26}$$

The division by $\mathcal{D}$ here is achieved by a combination of dividing out by $\mathcal{D}_{\mathcal{S}}$ and then gauge fixing the remaining diffeomorphism freedom Diff(S). Finally, coming to the Hamiltonian constraint, recall from section 3.2.5 that the Hamiltonian constraint generates gauge only when smeared against a lapse that vanishes at the horizon. Thus, as in [12], the Hamiltonian constraint should not be imposed on the horizon, but only in the bulk. Let $\tilde{\mathcal{H}}_V^{n,\vec{m}}$ denote the subspace of $\mathcal{H}_V^{n,\vec{m}}$ solving the Hamiltonian constraint in the bulk. The final physical Hilbert space for the system can then be written

$$\mathcal{H}^{phys} = \bigoplus_{n,\vec{m},\vec{a}:-2\vec{m}=\vec{a} \mod k} \tilde{\mathcal{H}}_V^{n,\vec{m}} \otimes \mathcal{H}_S^{n,\vec{a}}. \tag{A.27}$$

This is the same form as in [12], except that here $\tilde{\mathcal{H}}_V^{n,\vec{m}}$ includes electromagnetic degrees of freedom. It will be convenient to define

$$\mathcal{H}_V^{phys} := \bigoplus_{n,\vec{m}} \tilde{\mathcal{H}}_V^{n,\vec{m}} \tag{A.28}$$

and

$$\mathcal{H}_S^{phys} := \bigoplus_{n,\vec{a}} \mathcal{H}_S^{n,\vec{a}} \tag{A.29}$$

so that

$$\mathcal{H}^{phys} \subset \mathcal{H}_V^{phys} \otimes \mathcal{H}_S^{phys} \tag{A.30}$$

We next define the micro-canonical ensemble of interest to consist of states in $\mathcal{H}^{phys}$ for which the eigenvalues of $\hat{a}_S$, $\hat{I}_n$, $\hat{L}_n$, $\hat{Q}_n$, $\hat{P}_n$ lie in a suitable small interval around a_o, $\mathring{I}_n$, $\mathring{L}_n$, $\mathring{Q}_n$ and $\mathring{P}_n$. Note that, because of equations (3.46) and (A.22), the relative fluctuations of all of these operators will be equal to each other in such an ensemble.

In order to reduce to the type I case, and so make use of the previous result in calculating the entropy, we note that $\hat{I}_n$, $\hat{L}_n$, $\hat{Q}_n$, $\hat{P}_n$ are all functions of $\hat{a}_S$, so that, if we wish, we can define $\mathcal{H}^{bh}$ using the eigenstates of $\hat{a}_S$ alone, as in the type I case. Let $\mathcal{H}^{bh}$ denote (the span of) the states included in this ensemble — say, the span of all eigenstates of $\hat{a}_S$ with eigenvalues in the range $(a_o - \delta, a_o + \delta)$ for some δ much smaller than a_o, but sufficiently large such that $(a_o - \delta, a_o + \delta)$ includes at least one eigenvalue of $\hat{a}_S$. Because of the infinite degrees of freedom in the bulk, $\mathcal{H}^{bh}$ is infinite dimensional. However, it is only the surface states which we wish to count, so, as in [12] we define

$$\mathcal{H}_S^{bh} := \left\{ \Psi_S \in \mathcal{H}_S^{phys} \mid \exists \Psi_V \in \mathcal{H}_V^{phys} s.t.\, \Psi_V \otimes \Psi_S \in \mathcal{H}^{bh} \right\} \tag{A.31}$$

$\mathcal{H}_S^{bh}$ is finite dimensional, and the entropy of the black hole is given by

$$S_{bh} = \ln(\dim \mathcal{H}_S^{bh}) \tag{A.32}$$

We next argue that, in fact, the entropy obtained in the present context will be the same as that obtained in [12] (and hence chapter 3). We do this by first noting that $\mathcal{H}^{bh}$ depends on the spaces $\{\tilde{\mathcal{H}}_V^{n,\vec{m}}\}$ only through the spectra $\left\{\text{Spect}\left(\hat{a}_S|_{\tilde{\mathcal{H}}_V^{n,\vec{m}}}\right)\right\}_{n,\vec{m}}$. This can be seen from the explicit expression

$$\mathcal{H}_S^{bh} = \bigoplus_{\substack{n,\vec{a}:\ \exists \vec{m} s.t. -2\vec{m}=\vec{a} \mod k \\ \text{and Spect}\left(\hat{a}_S|_{\tilde{\mathcal{H}}_V^{n,\vec{m}}}\right)\cap(a_o-\delta,a_o+\delta)\neq\emptyset}} \mathcal{H}_S^{n,\vec{a}} \tag{A.33}$$

which one can prove. We next make here the same assumption about the Hamiltonian constraint made in section VI.A of [12]. This assumption can be recast as the assumption

$$\text{Spect}\left(\hat{a}_S|_{\tilde{\mathcal{H}}_V^{n,\vec{m}}}\right) = \text{Spect}\left(\hat{a}_S|_{\mathcal{H}_V^{n,\vec{m}}}\right) \tag{A.34}$$

for all n, $\vec{m}$. With this assumption, it is easy to show that the spectra $\left\{\text{Spect}\left(\hat{a}_S|_{\tilde{\mathcal{H}}_V^{n,\vec{m}}}\right)\right\}_{n,\vec{m}}$ are the same here as they are in [12]. The two facts primarily relevant in showing this are (1.) $\hat{a}_S$ kinematically acts only on the gravitational sector, and (2.) $\hat{a}_S$ is diffeomorphism invariant. Because of this equality of spectra, $\mathcal{H}_S^{bh}$ here is identical to $\mathcal{H}_S^{bh}$ in [12]. It follows that, with Maxwell fields included, the black hole entropy found in chapter 3 does not change, and one again obtains the Bekenstein-Hawking entropy with the same value of the Barbero-Immirzi parameter (3.48).

Appendix B

Extra proofs left out of chapter 4

B.1 $\mathrm{Cyl}^*_c \not\subseteq \mathrm{Cyl}^*_{inv}$

Definition 1. Given a set of vectors $\mathcal{U}$ in a topological vector space $\mathcal{V}$, the *Hamel span* of $\mathcal{U}$, denoted $\mathrm{span}_H\mathcal{U}$, is the set of all finite linear combinations of elements of $\mathcal{U}$.

That is, limits are not taken in the Hamel span. The purpose of this definition is to distinguish from "$\mathrm{span}\,\mathcal{U}$", which, in the main text, is understood to mean the *closure* of the set of all finite linear combinations. The reason for calling it 'Hamel span' becomes clear when we recall the definition of 'Hamel basis':

Definition 2. Given a topological vector space $\mathcal{V}$, a *Hamel basis* $\mathcal{U}$ is a linearly independent set such that $\mathrm{span}_H\mathcal{U} = \mathcal{V}$.

It will be convenient to also introduce the notion of 'Hamel complement':

Definition 3. Given a subspace $\mathcal{U}$ of a topological vector space $\mathcal{V}$, a *Hamel complement of* $\mathcal{U}$ *in* $\mathcal{V}$ is any subspace $\mathcal{W}$ of $\mathcal{V}$ such that $\mathcal{U} \cap \mathcal{W} = \{0\}$ and

$$\mathrm{span}_H\mathcal{U} \cup \mathcal{W} = \mathcal{V} \tag{B.1}$$

Let $\mathrm{Cyl}_\sim := \{\Phi \in \mathrm{Cyl} \mid \mathrm{Supp}\Phi \cap \mathcal{S}'(\Sigma)_{inv} = \emptyset\}$. Note, as was proven in subsection 4.4.2,

$$\mathrm{Cyl}^*_c = \{\alpha \in \mathrm{Cyl}^* \mid \alpha(\Phi) = 0 \quad \forall \Phi \in \mathrm{Cyl}_\sim\} \tag{B.2}$$

Lemma B.1. *Suppose* $\Theta \in \mathrm{Cyl}$ *is not in* $\mathrm{Cyl}_\sim$. *Let* $g \in \mathcal{T}$. *Then* $g \cdot \Theta \propto \Theta$ *implies* $g \cdot \Theta = \Theta$.

Proof.

Suppose $g \cdot \Theta = \lambda\Theta$. Since $\Theta \in \mathrm{Cyl}_\sim$, there exists $\alpha \in \mathcal{S}'(\Sigma)_{inv}$ such that $\Theta[\alpha] \neq 0$. So

$$\begin{aligned}(g \cdot \Theta)[\alpha] &= \lambda\Theta[\alpha] \\ \Theta[g^{-1} \cdot \alpha] &= \lambda\Theta[\alpha] \\ \Theta[\alpha] &= \lambda\Theta[\alpha]\end{aligned}$$

whence $\lambda = 1$. □

Theorem B.2. $\mathrm{Cyl}_c^* \not\subseteq \mathrm{Cyl}_{inv}^*$

Proof.

We will prove this by constructing an element of Cyl_c^* that is not in Cyl_{inv}^*. For $e \in \mathcal{S}(\Sigma)$, define $\Phi_e \in \mathrm{Cyl}$ by

$$\Phi_e[\varphi] := \varphi(e) \tag{B.3}$$

Let e_1 be any element of $\mathcal{S}(\Sigma)_{inv}$. Define $\Phi_1 := \Phi_{e_1}$. Then Φ_1 is not in $\mathrm{Cyl}_\sim$ and is $\mathcal{T}$-invariant. Let e_2 be any element of $\mathcal{S}(\Sigma)$ whose symmetric and non-symmetric parts are both non-zero. Define $\Phi_2 := \Phi_{e_2}$. Then Φ_2 is not in $\mathrm{Cyl}_\sim$ and is not $\mathcal{T}$-invariant[1]. Let $g \in \mathcal{T}$ be such that $g \cdot \Phi_2 \neq \Phi_2$. Let $\mathcal{K}$ be a Hamel basis of $\mathrm{Cyl}_\sim$.

Case 1. $\{\Phi_1, \Phi_2, g \cdot \Phi_2\}$ *is linearly independent.*

> Then $\{\Phi_1, \Phi_2, g\cdot\Phi_2+\Phi_1\}$ is also linearly independent. Extend $\{\Phi_1, \Phi_2, g\cdot \Phi_2 + \Phi_1\} \cup \mathcal{K}$ to a Hamel basis $\{\Phi_1, \Phi_2, g \cdot \Phi_2 + \Phi_1\} \cup \mathcal{K} \cup \mathcal{L}$ of Cyl.
>
> Let $\mathcal{U} := \mathrm{span}_H\{\Phi_2, g\cdot\Phi_2+\Phi_1\}\cup\mathcal{K}\cup\mathcal{L}$. Then $\mathcal{U}$ is a Hamel complement of $\{\Phi_1\}$ in Cyl. Furthermore, $\Phi_2 \in \mathcal{U}$ but $g \cdot \Phi_2 \notin \mathcal{U}$. [2]

[1]For the proof, any Φ_1 not in $\mathrm{Cyl}_\sim$ and $\mathcal{T}$-invariant, and any Φ_2 not in $\mathrm{Cyl}_\sim$ and not $\mathcal{T}$-invariant will do.

[2]Proof of latter: Suppose by way of contradiction that $g \cdot \Phi_2 \in \mathcal{U}$. Then, by $\mathcal{U}$'s linearity, $\Phi_1 = (g \cdot \Phi_2 + \Phi_1) - g \cdot \Phi_2$ is also in $\mathcal{U}$, yielding a contradiction.

Case 2. $\{\Phi_1, \Phi_2, g \cdot \Phi_2\}$ *is linearly dependent.*

Since Φ_1 is $\mathcal{T}$-invariant, but Φ_2 is not, $\{\Phi_1, \Phi_2\}$ is linearly independent. Thus

$$g \cdot \Phi_2 = a\Phi_1 + b\Phi_2. \tag{B.4}$$

for some a, b. Since $g \cdot \Phi_2$ is not $\mathcal{T}$-invariant, $b \neq 0$. Furthermore, since $g \cdot \Phi_2 \neq \Phi_2$, by the above lemma B.1, $g \cdot \Phi_2$ is not proportional to Φ_2, so that $a \neq 0$. Complete $\{\Phi_1, \Phi_2\} \cup \mathcal{K}$ into a Hamel basis $\{\Phi_1, \Phi_2\} \cup \mathcal{K} \cup \mathcal{L}$ of Cyl. Let $\mathcal{U} := \mathrm{span}_H\{\Phi_2\} \cup \mathcal{K} \cup \mathcal{L}$. Then $\mathcal{U}$ is a Hamel complement of $\{\Phi_1\}$ in Cyl. Furthermore, $\Phi_2 \in \mathcal{U}$, whereas $g \cdot \Phi_2 \notin \mathcal{U}$. [3]

In either case, we have a Hamel complement $\mathcal{U}$ of $\{\Phi_1\}$ such that $\Phi_2 \in \mathcal{U}$ but $g \cdot \Phi_2 \notin \mathcal{U}$.

Define $\eta \in \mathrm{Cyl}^*$ by

$$\begin{aligned} \eta(\Phi_1) &= 1 \\ \eta(\Phi) &= 0 \quad \forall \Phi \in \mathcal{U} \end{aligned}$$

Then, since $\mathrm{Cyl}_\sim \subseteq \mathcal{U}$, $\eta \in \mathrm{Cyl}^*_c$. But $\eta(\Phi_2) = 0$ whereas $\eta(g \cdot \Phi_2) \neq 0$, whence $\eta \notin \mathrm{Cyl}^*_{inv}$.

Thus $\mathrm{Cyl}^*_c \not\subseteq \mathrm{Cyl}^*_{inv}$. □

B.2 Other lemmata regarding coherent states

Let us first recall some relevant conventions and notation:

- h = one particle Hilbert space.
- $\mathcal{H} := (\oplus_s)_{n=0}^\infty \otimes^n h$.
- $\Psi_0 := (1, 0, 0, \ldots)$ denotes the vacuum.

[3] Proof of latter: suppose by way of contradiction that $g \cdot \Phi_2 \in \mathcal{U}$. Then from $\mathcal{U}$'s linearity, $\Phi_1 = \frac{1}{a}(g \cdot \Phi_2 - b\Phi_2)$ is also in $\mathcal{U}$, yielding a contradiction.

- For $\xi \in h$, $a^\dagger(\xi)$ and $a(\xi)$ denote the creation and annihilation operators associated with ξ. With the conventions in thesis, $[a^\dagger(\xi), a(\eta)] = \langle \xi, \eta \rangle$.

For $\xi \in h$, $\Psi^{coh}_\xi := e^{\hat{\Lambda}(\xi)}\Psi_0$ is the coherent state associated with ξ, where $\hat{\Lambda}(\xi) := a^\dagger(\xi) - a(\xi)$. Because of the skew-hermicity of $\hat{\Lambda}(\xi)$, $e^{\hat{\Lambda}(\xi)}$ is unitary, whence each Ψ^{coh}_ξ is unit norm. Also, note that $\hat{\Lambda}(\xi)$ is *real* linear in ξ, but not complex linear in ξ.

Lemma B.3. *Overlap between coherent states. For all $\xi, \eta \in h$,*

$$\langle \Psi^{coh}_\xi, \Psi^{coh}_\eta \rangle = e^{i\mathrm{Im}\langle \xi,\eta\rangle} e^{-\frac{1}{2}\|\xi-\eta\|^2} \tag{B.5}$$

Proof.

$$\langle \Psi^{coh}_\xi, \Psi^{coh}_\eta \rangle = \langle \Psi_0, e^{-\hat{\Lambda}(\xi)} e^{\hat{\Lambda}(\eta)} \Psi_0 \rangle \tag{B.6}$$

Now,

$$\begin{aligned} [\hat{\Lambda}(\xi), \hat{\Lambda}(\eta)] &= -[a^\dagger(\xi), a(\eta)] - [a(\xi), a^\dagger(\eta)] \\ &= -2i\mathrm{Im}\langle \xi, \eta \rangle \end{aligned} \tag{B.7}$$

So that, using the Baker-Campbell-Hausdorff formula, and the real-linearity of $\hat{\Lambda}(\xi)$ in ξ,

$$e^{-\hat{\Lambda}(\xi)} e^{\hat{\Lambda}(\eta)} = e^{i\mathrm{Im}\langle \xi,\eta\rangle} e^{\hat{\Lambda}(\eta-\xi)} \tag{B.8}$$

whence

$$\langle \Psi^{coh}_\xi, \Psi^{coh}_\eta \rangle = e^{i\mathrm{Im}\langle \xi,\eta\rangle} \langle \Psi_0, e^{\hat{\Lambda}(\eta-\xi)} \Psi_0 \rangle \tag{B.9}$$

Again, using Baker-Campbell-Hausdorff,

$$e^{\hat{\Lambda}(\eta-\xi)} = e^{a^\dagger(\eta-\xi)} e^{-a(\eta-\xi)} e^{-\frac{1}{2}\|\eta-\xi\|^2} \tag{B.10}$$

so that

$$\begin{aligned} \langle \Psi^{coh}_\xi, \Psi^{coh}_\eta \rangle &= e^{i\mathrm{Im}\langle \xi,\eta\rangle} e^{-\frac{1}{2}\|\eta-\xi\|^2} \langle \Psi_0, e^{a^\dagger(\eta-\xi)} e^{-a(\eta-\xi)} \Psi_0 \rangle \\ &= e^{i\mathrm{Im}\langle \xi,\eta\rangle} e^{-\frac{1}{2}\|\eta-\xi\|^2} \langle \Psi_0, e^{a^\dagger(\eta-\xi)} \Psi_0 \rangle \\ &= e^{i\mathrm{Im}\langle \xi,\eta\rangle} e^{-\frac{1}{2}\|\eta-\xi\|^2} \end{aligned} \tag{B.11}$$

□

Lemma B.4. *For all* $\xi, \eta \in h$,

$$\| \Psi_{\xi}^{coh} - \Psi_{\eta}^{coh} \|^2 = 2 - 2\cos(\mathrm{Im}\langle \xi, \eta \rangle) e^{-\frac{1}{2}\|\xi - \eta\|^2} \tag{B.12}$$

Proof.

$$\begin{aligned} \| \Psi_{\xi}^{coh} - \Psi_{\eta}^{coh} \|^2 &:= \langle \Psi_{\xi}^{coh} - \Psi_{\eta}^{coh}, \Psi_{\xi}^{coh} - \Psi_{\eta}^{coh} \rangle \\ &= 2 - 2\mathrm{Re}\langle \Psi_{\xi}^{coh}, \Psi_{\eta}^{coh} \rangle \\ &= 2 - 2\cos(\mathrm{Im}\langle \xi, \eta \rangle) e^{-\frac{1}{2}\|\xi - \eta\|^2} \end{aligned} \tag{B.13}$$

□

Theorem B.5. $\xi \mapsto \Psi_{\xi}^{coh}$ *is continuous with respect to the topologies on* h *and* $\mathcal{H}$ *(induced by their respective norms).*

Proof.

Suppose $\xi_i \to \xi$ in h. From lemma B.4,

$$\| \Psi_{\xi}^{coh} - \Psi_{\xi_i}^{coh} \|^2 = 2 - 2\cos(\mathrm{Im}\langle \xi, \xi_i \rangle) e^{-\frac{1}{2}\|\xi - \xi_i\|^2} \tag{B.14}$$

The continuity of the right hand side in ξ_i then implies $\Psi_{\xi_i}^{coh} \to \Psi_{\xi}^{coh}$. □

B.3 Proof of (slight generalization of) theorem (4.19)

Let ι denote an embedding of the Hilbert space $\mathcal{H}_{red}$ into the Hilbert space $\mathcal{H}$. Let $\langle , \rangle$ and $\| \cdot \|$ denote the inner product and induced norm on $\mathcal{H}$ and let $\langle , \rangle_{red}$ and $\| \cdot \|_{red}$ denote the inner product and induced norm on $\mathcal{H}_{red}$, so that

$$\langle \psi_1, \psi_2 \rangle_{red} = \langle \iota\psi_1, \iota\psi_2 \rangle \tag{B.15}$$

and

$$\| \psi \|_{red} = \| \iota\psi \| \tag{B.16}$$

for all $\psi_1, \psi_2, \psi \in \mathcal{H}_{red}$.

Let $P : \mathcal{H} \to \iota[\mathcal{H}_{red}]$ denote orthogonal projection. For any operator $\hat{\mathcal{O}}$ on $\mathcal{H}$, define a corresponding operator $\hat{\mathcal{O}}_{red}$ on $\mathcal{H}_{red}$ by

$$\hat{\mathcal{O}}_{red} := \iota^{-1} \circ P \circ \hat{\mathcal{O}} \circ \iota. \tag{B.17}$$

Furthermore, let us generalize the definition of the variance of an operator to the case of a non-Hermitian operator:

$$\Delta_\Psi \hat{\mathcal{O}} := \sqrt{\left\langle \Psi, \frac{1}{2}(\hat{\mathcal{O}}^\dagger \hat{\mathcal{O}} + \hat{\mathcal{O}}\hat{\mathcal{O}}^\dagger)\Psi \right\rangle - | \langle \Psi, \hat{\mathcal{O}}\Psi \rangle |^2} \tag{B.18}$$

Given these definitions, we have the following.

Theorem B.6. *For any operator $\hat{\mathcal{O}}$ on $\mathcal{H}$,*

1. $\hat{\mathcal{O}}^\dagger_{red} = (\hat{\mathcal{O}}^\dagger)_{red}$
2. $\langle \iota\psi_1, \hat{\mathcal{O}}\iota\psi_2 \rangle = \langle \psi_1, \hat{\mathcal{O}}_{red}\psi_2 \rangle_{red}$ *for all* $\psi_1, \psi_2 \in \mathcal{H}_{red}$
3. $\Delta_\psi \hat{\mathcal{O}}_{red} \leq \Delta_{\iota\psi}\hat{\mathcal{O}}$ *for all* $\psi \in \mathcal{H}_{red}$*, with equality holding iff* $\hat{\mathcal{O}}\iota\psi \in \iota[\mathcal{H}_{red}]$ *and* $\hat{\mathcal{O}}^\dagger\iota\psi \in \iota[\mathcal{H}_{red}]$ [4]

Proof.

Proof of (1)

For all $\psi_1, \psi_2 \in \mathcal{H}_{red}$,

$$\begin{aligned} \langle \psi_1, \hat{\mathcal{O}}_{red}\psi_2 \rangle_{red} &= \langle \iota\psi_1, \iota\hat{\mathcal{O}}_{red}\psi_2 \rangle \\ &= \langle \iota\psi_1, P \circ \hat{\mathcal{O}} \circ \iota\psi_2 \rangle \\ &= \langle \iota\psi_1, \hat{\mathcal{O}}\iota\psi_2 \rangle \end{aligned} \tag{B.19}$$

where the projection could be dropped because taking the inner product with $\iota\psi_1$

[4] Thus, in particular, if $\hat{\mathcal{O}}$ preserves $\mathcal{H}_{red}$ and is Hermitian, one always has equality.

automatically performs the projection for you. Continuing,

$$\begin{aligned}
\langle \psi_1, \hat{\mathcal{O}}_{red}\psi_2\rangle_{red} &= \langle \hat{\mathcal{O}}^\dagger \iota\psi_1, \iota\psi_2\rangle \\
&= \langle P \circ \hat{\mathcal{O}}^\dagger \iota\psi_1, \iota\psi_2\rangle \\
&= \langle \iota^{-1} \circ P \circ \hat{\mathcal{O}}^\dagger \circ \iota\psi_1, \psi_2\rangle_{red} \\
&= \langle (\hat{\mathcal{O}}^\dagger)_{red}\psi_1, \psi_2\rangle_{red}
\end{aligned} \tag{B.20}$$

whence $(\hat{\mathcal{O}}_{red})^\dagger = (\hat{\mathcal{O}}^\dagger)_{red}$.

Proof of (2)

For all $\psi_1, \psi_2 \in \mathcal{H}_{red}$,

$$\begin{aligned}
\langle \iota\psi_1, \hat{\mathcal{O}}\iota\psi_2\rangle &= \langle \iota\psi_1, P \circ \hat{\mathcal{O}}\iota\psi_2\rangle \\
&= \langle \psi_1, \iota^{-1} \circ P \circ \hat{\mathcal{O}}\iota\psi_2\rangle_{red} \\
&= \langle \psi_1, \hat{\mathcal{O}}_{red}\psi_2\rangle_{red}
\end{aligned} \tag{B.21}$$

Proof of (3)

For all $\psi \in \mathcal{H}_{red}$,

$$\begin{aligned}
(\Delta_\psi \hat{\mathcal{O}}_{red})^2 &= \frac{1}{2} \parallel \hat{\mathcal{O}}_{red}\psi \parallel_{red}^2 + \frac{1}{2} \parallel (\hat{\mathcal{O}}_{red})^\dagger \psi \parallel_{red}^2 - \mid \langle \psi, \hat{\mathcal{O}}_{red}\psi\rangle_{red} \mid^2 \\
&= \frac{1}{2} \parallel \hat{\mathcal{O}}_{red}\psi \parallel_{red}^2 + \frac{1}{2} \parallel (\hat{\mathcal{O}}^\dagger)_{red} \psi \parallel_{red}^2 - \mid \langle \psi, \hat{\mathcal{O}}_{red}\psi\rangle_{red} \mid^2 \\
&= \frac{1}{2} \parallel P \circ \hat{\mathcal{O}}\iota\psi \parallel^2 + \frac{1}{2} \parallel P \circ \hat{\mathcal{O}}^\dagger \iota\psi \parallel^2 - \mid \langle \iota\psi, \hat{\mathcal{O}}\iota\psi\rangle \mid^2 \\
&\leq \frac{1}{2} \parallel \hat{\mathcal{O}}\iota\psi \parallel^2 + \frac{1}{2} \parallel \hat{\mathcal{O}}^\dagger \iota\psi \parallel^2 - \mid \langle \iota\psi, \hat{\mathcal{O}}\iota\psi\rangle \mid^2 \\
&= (\Delta_{\iota\psi}\hat{\mathcal{O}})^2
\end{aligned} \tag{B.22}$$

with equality holding iff both projection operators P in the third line are not necessary: i.e., iff $\hat{\mathcal{O}}\iota\psi \in \iota[\mathcal{H}_{red}]$ and $\hat{\mathcal{O}}^\dagger\iota\psi \in \iota[\mathcal{H}_{red}]$.

□

Appendix C

The completeness relation for coherent states

C.1 Rigorous formulation of the completeness relation using generalized measures

Let us begin by stating, heuristically, the completeness relation we wish to prove: [1]

$$\int_{\xi \in \Gamma_{inv}} \mathcal{D}^2\xi |\overset{r}{\hat{\Psi}}{}^{coh}_{\xi}\rangle\langle\overset{r}{\hat{\Psi}}{}^{coh}_{\xi}| = \mathbb{1}_{\mathcal{H}_{red}} \tag{C.1}$$

where $\mathcal{D}^2\xi$ is a heuristic 'Lesbesgue measure' on Γ_{inv}. Of course $\mathcal{D}^2\xi$ does not really exist as a measure; however, the measure must be (a rigorization of) such a 'Lesbesgue measure' in order for the completeness relation to hold – a Gaussian measure cannot replace $\mathcal{D}^2\xi$. (Indeed, if a Gaussian measure replaces $\mathcal{D}^2\xi$ in (C.1), the left hand side will be zero, once things are made rigorous) The easiest way to construct such a 'Lesbesgue measure' is to first define a Gaussian measure, and then multiply it by a factor to get rid of the Gaussian dependence. But even this is not quite enough: for, the requisite Gaussian measure will require definition on $\mathcal{S}'(\Sigma)_{inv} \times \mathcal{S}'(\Sigma)_{inv}$, which is larger than h_{inv} [48]. Most of the time, when this measure is used to integrate (both in the completeness relation above, as well as in sections 4.5 and 4.7), the integrand is defined at most on h_{inv}. The cleanest way

[1] Of course a completeness relation analogous to (C.11) also holds in the full theory, but it is the completeness relation for the reduced theory that is needed in chapter 4.

to overcome this problem is to think in terms of *generalized measures* [67] — that is, think strictly in terms of the integration functional, without assuming it arises from a corresponding set function. Such a generalized measure can be defined in a way not involving the introduction of the larger space $\mathcal{S}'(\Sigma)_{inv} \times \mathcal{S}'(\Sigma)_{inv}$. In fact, it is not even necessary to have integrands defined on all of h_{inv}: we will be able to integrate functions even on the classical phase space Γ_{inv}. A further motivation for using generalized measures is that they will allow us to define an analogue of a Lesbesgue measure directly, which will be useful.

The term "generalized measure" was first introduced by Baez [67]. However, as noted below, there is a certain class of generalized measures which are equivalent to what are called *promeasures* or *cylindrical measures* [68, 69]. The ideas behind these trace back to Kolmogorov [69]. Some of the generalized measures in this thesis are of this type. In this thesis we use (a slight generalization of) the definition of Baez because of its simplicity and generality:

Definition 4. Let X be an arbitrary Hausdorff space, and let A_ν denote a given $*$-algebra of complex continuous functions on X. Let $\| \cdot \|$ denote a given norm on A_ν such that when A_ν is completed with respect to this norm, it becomes a C^*-algebra. Let $\int \mathrm{d}\nu : A_\nu \to \mathbb{C}$ be a linear functional, assumed to be bounded — that is, for some $C > 0$,

$$| \int \mathrm{d}\nu f | \leq C \parallel f \parallel \tag{C.2}$$

for all $f \in A_\nu$. Then we say $\int \mathrm{d}\nu\cdot$ is a *generalized measure* on X, or that $\left(X, A_\nu, \int \mathrm{d}\nu\cdot\right)$ is a *generalized measure space.*

We first introduce the requisite Gaussian generalized measure on Γ_{inv}, and then introduce an associated 'Lesbesgue' generalized measure on Γ_{inv}.

For the Gaussian generalized measure, which we will denote by $\mathrm{d}\nu_{red}$, the basic algebra of integrable functions $A_{\nu_{red}}$ is taken to be the *-algebra of cylindrical functions on Γ_{inv}, that is, the *-algebra of functions $\Phi : \Gamma \to \mathbb{C}$ of the form

$$\Phi[\xi] = F((\xi, \alpha_1), \ldots, (\xi, \alpha_n)) \tag{C.3}$$

for some $\alpha_1, \ldots, \alpha_n \in \Gamma_{inv}$ and $F : \mathbb{R}^n \to \mathbb{C}$ with growth at most exponential. The symmetric inner product $(\cdot,\cdot)$ here is defined by

$$([f,g],[f',g']) := (f,f') + (g,g') = \int_\Sigma f f' \mathrm{d}^3x + \int_\Sigma g g' \mathrm{d}^3 x. \tag{C.4}$$

for all $[f,g],[f',g'] \in \Gamma_{inv}$. The above *-algebra we denote by $\mathrm{Cyl}^\Gamma_{red}$. On $\mathrm{Cyl}^\Gamma_{red}$ we define a norm, which we call a 'Gaussian sup norm':

$$\parallel \Phi \parallel_G := \sup_{\xi\in\Gamma_{inv}} e^{-\langle\xi,\xi\rangle}|\Phi[\xi]|. \tag{C.5}$$

One can check this norm is well defined on all of $\mathrm{Cyl}^\Gamma_{red}$, and that the completion of $\mathrm{Cyl}^\Gamma_{red}$ with respect to this norm yields a C^*-algebra. Generalized measures with such a *-algebra of integrable functions are the generalized measures equivalent to specifying the *promeasures* or *cylindrical measures* mentioned earlier.

Such generalized measures can be uniquely defined by specifying the *Fourier transform*, defined by

$$\chi_{\nu_{red}}(\alpha) := \int_{\xi\in\Gamma_{inv}} \mathrm{d}\nu_{red} e^{i(\xi,\alpha)} \tag{C.6}$$

where $\alpha = [f,g] \in \Gamma_{inv}$. The Gaussian measure we will need is the one with variance $\hat{O}:\Gamma_{inv}\to\Gamma_{inv}$ defined by

$$\hat{O} : [\varphi,\pi] \mapsto [\Theta^{-\frac{1}{2}}\varphi, \Theta^{\frac{1}{2}}\pi]. \tag{C.7}$$

This is the variance directly related to the hermitian inner product by

$$\langle\xi,\xi\rangle = \frac{1}{2}(\xi,\hat{O}^{-1}\xi) \tag{C.8}$$

Thus, the Gaussian measure is given heuristically by

$$\text{“}\mathrm{d}\nu_{red} = e^{-\frac{1}{2}(\xi,\hat{O}^{-1}\xi)}\mathcal{D}^2\xi = e^{-\langle\xi,\xi\rangle}\mathcal{D}^2\xi\text{”} \tag{C.9}$$

(with “$\mathcal{D}^2\xi$ scaled such that $\int \mathrm{d}\nu_{red} = 1$”) and rigorously by the Fourier transform

$$\chi_{\nu_{red}}(\alpha) = e^{-\frac{1}{2}(\alpha,\hat{O}\alpha)} \tag{C.10}$$

for all $\alpha\in\Gamma_{inv}$. One can check that the generalized measure so-defined is bounded with respect to the norm $\|\cdot\|_G$ on the algebra $\mathrm{Cyl}^{\Gamma}_{red}$, as required in definition 4. (This $\mathrm{d}\nu_{red}$ is exactly the same as the measure $\mathrm{d}\mu$ in [70], except that here $\mathrm{d}\nu_{red}$ is for the reduced theory.)

We can now state rigorously the completeness relation we wish to prove:

$$\int_{\xi\in\Gamma_{inv}} \mathrm{d}\nu_{red} e^{\langle\xi,\xi\rangle} |\overset{r}{\Psi}{}^{coh}_{\xi}\rangle\langle\overset{r}{\Psi}{}^{coh}_{\xi}| = \mathbb{1}_{\mathcal{H}_{red}} \tag{C.11}$$

However, it will be more convenient, in sections 4.5.2 and 4.7, to formulate the completeness relation using a single 'Lesbesgue'-like generalized measure $\mathrm{d}\nu^{o}_{red}$ representing directly the combination '$\mathrm{d}\nu_{red}e^{\langle\xi,\xi\rangle}$' appearing in (C.11).

The basic algebra of integrable functions for $\mathrm{d}\nu^{o}_{red}$ is taken to be

$$A_{\nu^{o}_{red}} = e^{-\frac{1}{2}(\xi,\hat{O}^{-1}\xi)} \cdot \mathrm{Cyl}^{\Gamma}_{red}. \tag{C.12}$$

We equip this algebra with the usual sup norm $\|\cdot\|_{\infty}$. The integration functional is then defined by

$$\int \mathrm{d}\nu^{o}_{red}\tilde{\Phi} := \int \mathrm{d}\nu_{red} e^{\frac{1}{2}(\xi,\hat{O}^{-1}\xi)}\tilde{\Phi}. \tag{C.13}$$

The boundedness of this integration functional $\int \mathrm{d}\nu^{o}_{red}\cdot$ with respect to $\|\cdot\|_{\infty}$ follows from the aforementioned boundedness of $\int \mathrm{d}\nu_{red}\cdot$. One can check that the resulting generalized measure $\mathrm{d}\nu^{o}_{red}$ is translation invariant, and so warrants the name 'Lesbesgue.'[2] (Note: the translation invariance of $\mathrm{d}\nu^{o}_{red}$ will be very useful in the proof in the next section.) We can express the relation between $\mathrm{d}\nu_{red}$ and $\mathrm{d}\nu^{o}_{red}$ by

$$\mathrm{d}\nu_{red} = \mathrm{d}\nu^{o}_{red} e^{-\frac{1}{2}(\xi,\hat{O}^{-1}\xi)}. \tag{C.15}$$

[2] As a side note, if $\hat{O}$ had been replaced by a different variance operator $\hat{\mathcal{O}}$, a different 'Lesbesgue generalized measure' would have resulted. This is easy to see from the fact that for each choice of variance operator $\hat{\mathcal{O}}$, the resulting Lesbesgue generalized measure $\mathrm{d}\nu^{o}_{\hat{\mathcal{O}}}$ satisfies its own unique normalization condition

$$\int \mathrm{d}\nu^{o}_{\hat{\mathcal{O}}} e^{-\frac{1}{2}(\xi,\hat{\mathcal{O}}^{-1}\xi)} = 1. \tag{C.14}$$

Thus, there are in fact an infinity of such 'Lesbesgue generalized measures'.

C.2 Proof of the completeness relation

Let us prove the completeness relation.

Theorem C.1.
$$\int_{\xi\in\Gamma_{inv}} \mathrm{d}\nu^{o}_{red} |\overset{r}{\Psi}{}^{coh}_{\xi}\rangle\langle\overset{r}{\Psi}{}^{coh}_{\xi}| = \mathbb{1}_{\mathcal{H}_{red}} \tag{C.16}$$

Proof.

We proceed by showing that the left hand side and the right hand side have the same matrix elements with respect to arbitrary coherent states. Let $\eta = [f, g]$, $\eta' = [f', g'] \in h_{inv}$ be given. Let

$$M(\eta', \eta) := \langle\overset{r}{\Psi}{}^{coh}_{\eta'}| \left\{ \int_{\xi\in\Gamma_{inv}} \mathrm{d}\nu^{o}_{red} |\overset{r}{\Psi}{}^{coh}_{\xi}\rangle\langle\overset{r}{\Psi}{}^{coh}_{\xi}| \right\} |\overset{r}{\Psi}{}^{coh}_{\eta}\rangle \tag{C.17}$$

Using lemma B.3,

$$\begin{aligned}
M(\eta', \eta) &= \int_{\xi\in\Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \langle\overset{r}{\Psi}{}^{coh}_{\eta'}, \overset{r}{\Psi}{}^{coh}_{\xi}\rangle\langle\overset{r}{\Psi}{}^{coh}_{\xi}, \overset{r}{\Psi}{}^{coh}_{\eta}\rangle \\
&= \int_{\xi\in\Gamma_{inv}} \mathrm{d}\nu^{o}_{red} e^{i\mathrm{Im}\langle\eta',\xi\rangle} e^{-\frac{1}{2}||\eta'-\xi||^2} e^{i\mathrm{Im}\langle\xi,\eta\rangle} e^{-\frac{1}{2}||\xi-\eta||^2} \\
&= \int_{\xi\in\Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \exp\left\{ i\mathrm{Im}\langle\eta', \xi\rangle + \mathrm{Re}\langle\eta', \xi\rangle - \tfrac{1}{2} \parallel \eta' \parallel^2 - \tfrac{1}{2} \parallel \xi \parallel^2 \right. \\
&\qquad\qquad \left. + i\mathrm{Im}\langle\xi, \eta\rangle + \mathrm{Re}\langle\xi, \eta\rangle - \tfrac{1}{2} \parallel \eta \parallel^2 - \tfrac{1}{2} \parallel \xi \parallel^2 \right\} \\
&= e^{-\frac{1}{2}||\eta||^2 - \frac{1}{2}||\eta'||^2} \int_{\xi\in\Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \exp\left\{ - \parallel \xi \parallel^2 + \langle\eta', \xi\rangle + \langle\xi, \eta\rangle \right\} \\
&= e^{-\frac{1}{2}||\eta||^2 - \frac{1}{2}||\eta'||^2} \int_{\xi=[\varphi,\pi]\in\Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \exp\left\{ - \tfrac{1}{2} \left(\xi, \hat{O}^{-1}\xi\right) \right. \\
&\qquad + \tfrac{1}{2} \left(\xi, \hat{O}^{-1}\eta'\right) + \tfrac{i}{2}\left(f', \pi\right) - \tfrac{i}{2}\left(g', \varphi\right) \\
&\qquad \left. + \tfrac{1}{2} \left(\xi, \hat{O}^{-1}\eta\right) + \tfrac{i}{2}\left(\varphi, g\right) - \tfrac{i}{2}\left(\pi, f\right) \right\} \\
&= e^{-\frac{1}{2}||\eta||^2 - \frac{1}{2}||\eta'||^2} \int_{\xi\in\Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \exp\left\{ - \tfrac{1}{2} \left(\xi, \hat{O}^{-1}\xi\right) + \tfrac{1}{2} \left(\xi, \hat{O}^{-1}(\eta + \eta')\right) \right. \\
&\qquad \left. + \tfrac{i}{2}\left(\xi, [g - g', f' - f]\right) \right\}
\end{aligned} \tag{C.18}$$

Define $u := \frac{1}{2}(\eta + \eta')$. Completing the square in the last expression, we then have

$$\begin{aligned} M(\eta', \eta) \quad &= e^{-\frac{1}{2}\|\eta\|^2 - \frac{1}{2}\|\eta'\|^2} \int_{\xi \in \Gamma_{inv}} \mathrm{d}\nu^o_{red} \exp\Big\{ -\tfrac{1}{2}\Big(\xi - u, \hat{O}^{-1}(\xi - u)\Big) + \tfrac{1}{2}\Big(u, \hat{O}^{-1}u\Big) \\ &\qquad + \tfrac{i}{2}\Big(\xi, [g - g', f' - f]\Big)\Big\}. \end{aligned} \tag{C.19}$$

Performing the change of variables given by $\xi_{new} = \xi_{old} - u$, and using the translation invariance of $\mathrm{d}\nu^o$, we obtain

$$\begin{aligned} M(\eta', \eta) &= e^{-\frac{1}{2}\|\eta\|^2 - \frac{1}{2}\|\eta'\|^2} \int_{\xi \in \Gamma_{inv}} \mathrm{d}\nu^o_{red} \exp\Big\{ -\tfrac{1}{2}\Big(\xi, \hat{O}^{-1}\xi\Big) + \tfrac{1}{2}\Big(u, \hat{O}^{-1}u\Big) \\ &\qquad + \tfrac{i}{2}\Big(\xi, [g - g', f' - f]\Big) + \tfrac{i}{2}\Big(u, [g - g', f' - f]\Big)\Big\} \\ &= \exp\big\{-\tfrac{1}{2} \parallel \eta \parallel^2 - \tfrac{1}{2} \parallel \eta' \parallel^2 + \tfrac{1}{4} \parallel \eta + \eta' \parallel^2 + \tfrac{i}{4}([f + f', g + g'], [g - g', f' - f])\big\} \cdot \\ &\qquad \cdot \int_{\xi \in \Gamma_{inv}} \mathrm{d}\nu_{red} \exp\big\{\tfrac{i}{2}(\xi, [g - g', f' - f])\big\} \\ &= \exp\big\{-\tfrac{1}{2} \parallel \eta \parallel^2 - \tfrac{1}{2} \parallel \eta' \parallel^2 + \tfrac{1}{4} \parallel \eta + \eta' \parallel^2 + \tfrac{i}{2}(g, f') - \tfrac{i}{2}(f, g')\big\} \cdot \\ &\qquad \cdot \exp\Big\{-\tfrac{1}{8}\Big([g - g', f' - f], \hat{O}[g - g', f' - f]\Big)\Big\} \\ &= \exp\Big\{-\tfrac{1}{2} \parallel \eta \parallel^2 - \tfrac{1}{2} \parallel \eta' \parallel^2 + \tfrac{1}{4} \parallel \eta + \eta' \parallel^2 + i\mathrm{Im}\langle\eta', \eta\rangle \\ &\qquad - \tfrac{1}{8}\Big(g - g', \Theta^{-\frac{1}{2}}(g - g')\Big) - \tfrac{1}{8}\Big(f' - f, \Theta^{\frac{1}{2}}(f' - f)\Big)\Big\} \\ &= \exp\Big\{-\tfrac{1}{2} \parallel \eta \parallel^2 - \tfrac{1}{2} \parallel \eta' \parallel^2 + \tfrac{1}{4} \parallel \eta + \eta' \parallel^2 + i\mathrm{Im}\langle\eta', \eta\rangle - \tfrac{1}{8}\Big(\eta - \eta', \hat{O}^{-1}(\eta - \eta')\Big)\Big\} \\ &= \exp\big\{-\tfrac{1}{2} \parallel \eta \parallel^2 - \tfrac{1}{2} \parallel \eta' \parallel^2 + \tfrac{1}{4} \parallel \eta + \eta' \parallel^2 + i\mathrm{Im}\langle\eta', \eta\rangle - \tfrac{1}{4} \parallel \eta - \eta' \parallel^2\big\} \\ &= \exp\big\{-\tfrac{1}{2} \parallel \eta \parallel^2 - \tfrac{1}{2} \parallel \eta' \parallel^2 + \mathrm{Re}\langle\eta', \eta\rangle + i\mathrm{Im}\langle\eta', \eta\rangle\big\} \\ &= \exp\big\{-\tfrac{1}{2} \parallel \eta' - \eta \parallel^2 + i\mathrm{Im}\langle\eta', \eta\rangle\big\} \end{aligned} \tag{C.20}$$

In the second line, the substitution $\mathrm{d}\nu_{red} = \mathrm{d}\nu^o_{red} e^{-\frac{1}{2}(\xi, \hat{O}^{-1}\xi)}$ has been used. Note the last line is just $\langle \overset{r}{\Psi}{}^{coh}_{\eta'}, \overset{r}{\Psi}{}^{coh}_{\eta} \rangle$. We thus have

$$\langle \overset{r}{\Psi}{}^{coh}_{\eta'} | \left\{ \int_{\xi \in \Gamma_{inv}} \mathrm{d}\nu^o_{red} | \overset{r}{\Psi}{}^{coh}_{\xi} \rangle \langle \overset{r}{\Psi}{}^{coh}_{\xi} | \right\} | \overset{r}{\Psi}{}^{coh}_{\eta} \rangle = \langle \overset{r}{\Psi}{}^{coh}_{\eta'}, \overset{r}{\Psi}{}^{coh}_{\eta} \rangle \tag{C.21}$$

for all $\eta, \eta' \in h$, whence

$$\int_{\xi \in \Gamma_{inv}} \mathrm{d}\nu^o_{red} | \overset{r}{\Psi}{}^{coh}_{\xi} \rangle \langle \overset{r}{\Psi}{}^{coh}_{\xi} | = \mathbb{1}_{\mathcal{H}_{red}} \tag{C.22}$$

□

Appendix D

The reduced-then-quantized free scalar field theory

Let (ρ, z, ϕ) denote standard cylindrical coordinates on Σ such that the symmetry vector field ϕ^a is equal to $\frac{\partial}{\partial\phi}$. Let $B := \Sigma/\mathfrak{T}$ denote the reduced spatial manifold. Let $P : \Sigma \to B$ denote canonical projection, and let $q^{ab} := P_* g^{ab}$. B may be coordinatized by (ρ, z), which are then Cartesian coordinates for $q_{ab} := (q^{ab})^{-1}$. The configuration and momentum variables φ and π may then be represented by functions on B.

More specifically, for φ and π symmetric, we define [1]

$$\varphi_r(\rho, z) = \sqrt{2\pi}\varphi(\rho, z) \qquad \pi_r(\rho, z) = \sqrt{2\pi}\rho\pi(\rho, z) \tag{D.2}$$

(The $\sqrt{2\pi}$ factors are included for later convenience.) Let Γ_{red} denote the reduced phase space – the space of all possible $[\varphi_r, \pi_r]$ [2]. The symplectic structure induced

[1] The definition of π_r can be motivated by considering the weight one densitization of π, $\tilde{\pi} := (\det g)^{\frac{1}{2}}\pi$. Using the projection mapping $P : \Sigma \to B$, we can then define $\tilde{\pi}_r := \sqrt{2\pi}P_*\tilde{\pi}$, where the push-forward is defined by treating $\tilde{\pi}$ as a measure. If we then dedensitize $\tilde{\pi}_r$ using q_{ab}: $\pi_r := (\det q)^{-\frac{1}{2}}\tilde{\pi}_r$, then

$$\pi_r = \sqrt{2\pi}\rho\pi. \tag{D.1}$$

[2] Classically there are also boundary conditions which φ_r and π_r must satisfy at $\rho = 0$ in order to ensure smoothness. However, when going over to the quantum theory, because there is no surface term in the symplectic structure at $\rho = 0$, there are no separate degrees of freedom at $\rho = 0$, and the boundary conditions do not matter.

on Γ_{red} is simply

$$\Omega([\varphi_r,\pi_r],[\varphi'_r,\pi'_r]) = \int_B (\pi_r\varphi'_r - \varphi_r\pi'_r)\mathrm{d}\rho\mathrm{d}z \tag{D.3}$$

Thus we see that at least kinematically, in terms of (φ_r,π_r), the reduced theory is nothing other than a free Klein-Gordon theory on B with flat metric q_{ab}. [3] From the time evolution of (φ,π) in the full theory, the time-evolution of (φ_r,π_r) is

$$\begin{aligned} \dot{\varphi}_r &= \rho^{-1}\pi_r && \text{(D.4)} \\ \dot{\pi}_r &= \rho(\Delta_\Sigma - m^2)\varphi_r && \\ &= \rho(\Delta_B + \frac{1}{\rho}\frac{\partial}{\partial\rho} - m^2)\varphi_r && \text{(D.5)} \end{aligned}$$

where Δ_Σ denotes the Laplacian on Σ determined by g_{ab} and Δ_B denotes the Laplacian on B determined by q_{ab}. Let $\Theta := -\Delta_\Sigma + m^2 = -\Delta_B - \frac{1}{\rho}\frac{\partial}{\partial\rho} + m^2$. Note that from $(\Theta^q f,g)_\Sigma = (f,\Theta^q g)_\Sigma$ for arbitrary $q\in\mathbb{Q}$, it follows $(\rho\Theta^q f,g)_B = (f,\rho\Theta^q g)_B$.

Given a choice of parametrization of time, from [47], the naturally associated complex structure on the classical phase space is

$$J = -(-\mathcal{L}_\xi\mathcal{L}_\xi)^{-\frac{1}{2}}\mathcal{L}_\xi \tag{D.6}$$

where $\mathcal{L}_\xi$ denotes derivative with respect to the time evolution vector field ξ. From (D.4, D.5), one then calculates

$$J[\varphi_r,\pi_r] = [-\Theta^{-\frac{1}{2}}\rho^{-1}\pi_r, \rho\Theta^{\frac{1}{2}}\varphi_r]. \tag{D.7}$$

Following [47], the Hermitian inner product thereby determined on the classical phase space is

$$\langle[\varphi_r,\pi_r],[\varphi'_r,\pi'_r]\rangle = \frac{1}{2}(\rho\Theta^{\frac{1}{2}}\varphi_r,\varphi'_r)_B + \frac{1}{2}(\Theta^{-\frac{1}{2}}\rho^{-1}\pi_r,\pi'_r)_B - \frac{\mathrm{i}}{2}(\pi_r,\varphi'_r)_B + \frac{\mathrm{i}}{2}(\varphi_r,\pi'_r)_B \tag{D.8}$$

where $(f,g)_B := \int_B fg\mathrm{d}\rho\mathrm{d}z$ [4] . We take the quantum configuration space to be

[3]Note the role of the definition of φ_r and π_r in making this the case.

[4]Unless otherwise specified, from now on all integrations over B are understood to be with

$\mathcal{S}'(B)$, with quantum measure given, again following [47], by

$$\text{“d}\mu_{red} = \exp\left\{-\frac{1}{2}(\varphi, \rho\Theta^{\frac{1}{2}}\varphi)_B\right\}\mathcal{D}\varphi.\text{”} \tag{D.9}$$

More rigorously, the Fourier transform of the measure is given by

$$\chi_{\mu_{red}}(f) = \exp\left\{-\frac{1}{2}(f, \Theta^{-\frac{1}{2}}\rho^{-1}f)_B\right\} \tag{D.10}$$

We will denote the space of cylindrical functions in the reduced theory by Cyl_{red}. That is, Cyl_{red} is the space of functions $\Phi : \mathcal{S}'(B) \to \mathbb{C}$ of the form

$$\Phi[\alpha] = F(\alpha(f_1), \ldots, \alpha(f_n)) \tag{D.11}$$

for some $f_1, \ldots, f_n \in \mathcal{S}(B)$ and some smooth $F : \mathbb{R}^n \to \mathbb{C}$ with growth at most exponential.

The representation of the field observables $\varphi[f] := \int_B f\varphi$ and $\pi[g] := \int_B g\pi$ is given by

$$\begin{aligned} (\hat{\varphi}_r[f]\Psi)[\varphi_r] &= \varphi_r[f]\Psi[\varphi_r] && \text{(D.12)} \\ (\hat{\pi}_r[g]\Psi)[\varphi_r] &= -\mathrm{i}\int_B \left(g\frac{\delta}{\delta\varphi_r} - \varphi_r\rho\Theta^{\frac{1}{2}}g\right)\Psi[\varphi].^5 && \text{(D.13)} \end{aligned}$$

For a given point $[\varphi_r, \pi_r] = [f, g]$ in the classical phase space, we have the “classical observables” for the corresponding annihilation and creation operators:

$$\begin{aligned} a_{red}([f,g])\mid_{[\varphi_r,\pi_r]} &= \langle [f,g], [\varphi_r, \pi_r]\rangle \\ &= \frac{1}{2}(\varphi_r[\rho\Theta^{\frac{1}{2}}f - \mathrm{i}g] + \pi_r[\Theta^{-\frac{1}{2}}\rho^{-1}g + \mathrm{i}f]) && \text{(D.14)} \\ a^\dagger_{red}([f,g])\mid_{[\varphi_r,\pi_r]} &= \langle [\varphi_r, \pi_r], [f,g]\rangle \\ &= \frac{1}{2}(\varphi_r[\rho\Theta^{\frac{1}{2}}f + \mathrm{i}g] + \pi_r[\Theta^{-\frac{1}{2}}\rho^{-1}g - \mathrm{i}f]) && \text{(D.15)} \end{aligned}$$

respect to $\mathrm{d}^2x := \mathrm{d}\rho\mathrm{d}z$, and all integrations over Σ are understood to be with respect to $\mathrm{d}^3x := \rho\mathrm{d}\rho\mathrm{d}z\mathrm{d}\phi$.

[5] As noted earlier, $\frac{\delta}{\delta\varphi_r}$ is defined with respect to the volume form $\mathrm{d}\rho\mathrm{d}z$.

Quantizing by substituting in (D.12, D.13), we obtain

$$
\begin{aligned}
a^\dagger_{red}([f,g]) &= \varphi_r[\rho\Theta^{\frac{1}{2}}f + \mathrm{i}g] - \frac{\mathrm{i}}{2}\int_B \left\{\Theta^{-\frac{1}{2}}(\rho^{-1}g) - \mathrm{i}f\right\}\frac{\delta}{\delta\varphi_r} && \text{(D.16)}\\
a_{red}([f,g]) &= -\frac{\mathrm{i}}{2}\int_B \left\{\Theta^{-\frac{1}{2}}(\rho^{-1}g) + \mathrm{i}f\right\}\frac{\delta}{\delta\varphi_r} && \text{(D.17)}
\end{aligned}
$$

Lastly we quantize the (reduced) Hamiltonian. The reduced Hamiltonian is

$$
\mathbb{H}_{red} = \frac{1}{2}\int_B (\rho^{-1}\pi_r^2 + \rho(\vec{\nabla}\varphi_r)^2 + \rho m^2\varphi_r^2)\mathrm{d}\rho\mathrm{d}z \tag{D.18}
$$

This can be checked to be consistent with (D.3, D.4, D.5). We next rewrite the Hamiltonian,

$$
\begin{aligned}
\mathbb{H}_{red} &= \frac{1}{2}\int_B (\rho^{-1}\pi_r^2 + \rho(\vec{\nabla}\varphi_r)^2 + \rho m^2\varphi_r^2)\mathrm{d}\rho\mathrm{d}z\\
&= \frac{1}{2}\int_B \left(\rho^{-1}\pi_r^2 + \rho\varphi_r\left\{-\Delta_B - \rho^{-1}\frac{\partial}{\partial\rho} + m^2\right\}\varphi_r\right)\mathrm{d}\rho\mathrm{d}z\\
&= \frac{1}{2}\int_B (\rho^{-1}\pi_r^2 + \rho\varphi_r\Theta\varphi_r)\mathrm{d}\rho\mathrm{d}z && \text{(D.19)}
\end{aligned}
$$

From (D.4,D.5), we deduce the one particle Hamiltonian:

$$
\begin{aligned}
\hat{H}_{red}[\varphi_r,\pi_r] &= J\frac{\mathrm{d}}{\mathrm{d}t}[\varphi_r,\pi_r]\\
&= [\Theta^{\frac{1}{2}}\varphi_r, \rho\Theta^{\frac{1}{2}}\rho^{-1}\pi_r]^{\P} && \text{(D.20)}
\end{aligned}
$$

So,

$$
\mathbb{H}_{red} = \langle[\phi_r,\pi_r], \hat{H}_{red}[\phi_r,\pi_r]\rangle \tag{D.21}
$$

matching one's expectations. Let $\{\xi_i = [f_i, g_i]\}$ denote an arbitrary basis of Γ_{red}, orthonormal with respect to $\langle\cdot,\cdot\rangle$. Then,

$$
\begin{aligned}
\mathbb{H}_{red} &= \sum_{i,j}\langle[\phi_r,\pi_r],\xi_i\rangle\langle\xi_i,\hat{H}_{red}\xi_j\rangle\langle\xi_j,[\phi_r,\pi_r]\rangle\\
&= \sum_{i,j}\langle\xi_i,\hat{H}_{red}\xi_j\rangle a^\dagger_{red}(\xi_i)a_{red}(\xi_j) && \text{(D.22)}
\end{aligned}
$$

¶If $\xi = [\varphi_r, \pi_r]$ is thought of as representing the corresponding symmetric full theory initial data, $\hat{H}_{red}\xi = \hat{H}\xi$.

To quantize we use the normal ordering above and substitute in (D.16, D.17), to obtain

$$\mathbb{H}_{red} = \int_{x\in B, y\in B} \left\{ A(x,y)\varphi_r(y)\frac{\delta}{\delta\varphi_r(x)} - B(x,y)\frac{\delta^2}{\delta\varphi_r(x)\delta\varphi_r(y)} \right\} \tag{D.23}$$

where

$$A(x,y) := \frac{1}{2}\sum_{i,j}\langle\xi_i, \hat{H}_{red}\xi_j\rangle(f_j - \mathrm{i}\Theta^{-\frac{1}{2}}\rho^{-1}g_j)(x)(\rho\Theta^{\frac{1}{2}}f_i + \mathrm{i}g_i)(y) \tag{D.24}$$

$$B(x,y) := \frac{1}{4}\sum_{i,j}\langle\xi_i, \hat{H}_{red}\xi_j\rangle(\Theta^{-\frac{1}{2}}g_i - \mathrm{i}f_i)(x)(\Theta^{-\frac{1}{2}}\rho^{-1}g_j + \mathrm{i}f_j)(y) \tag{D.25}$$

By integrating against test functions, one can show $A(x,y)$ is the integral kernel of $\Theta^{\frac{1}{2}}$, and $B(x,y) = \frac{1}{2}\rho^{-1}\delta^2(x,y)$. It follows

$$\hat{\mathbb{H}}_{red} = \int_{x\in B} \left\{ (\Theta^{\frac{1}{2}}\varphi_r)(x)\frac{\delta}{\delta\varphi_r(x)} - \frac{1}{2}\rho^{-1}\frac{\delta^2}{\delta\varphi_r(x)^2} \right\}. \tag{D.26}$$

Appendix E

List of symbols and basic relations for chapter 4

For Klein-Gordon model:

$\overset{n}{\otimes}$	n-fold tensor product
$\overset{n}{\otimes}_s$	symmetrized n-fold tensor product
$(f,g) = (f,g)_\Sigma := \int_\Sigma fg\mathrm{d}^3x$	
$(f,g)_B := \int_B fg\mathrm{d}^2x$	
$\langle\cdot,\cdot\rangle$	inner product on h, $\mathcal{H}$, $\mathcal{H}_{red}$, or $\mathcal{H}_\perp$, depending on context
$[f,g]$	point in Γ defined by $\varphi = f$, $\pi = g$. (not to be confused with commutator; context makes clear which is intended)
$a(\cdot)$, $a^\dagger(\cdot)$	annihilation and creation operators in the full theory, or their classical counterparts, depending on the context
$a_r(\cdot) = a_{red}(\cdot)$ $a_r^\dagger(\cdot) = a_{red}^\dagger(\cdot)$	annihilation and creation operators in the reduced theory, or their classical counterparts, depending on the context
$B := \Sigma/\mathcal{T}$	spatial manifold for the reduced theory
Cyl, Cyl*	space of cylindrical functions on $\mathcal{S}'(\Sigma)$, and the algebraic dual
Cyl$_{red}$, Cyl$^*_{red}$	space of cylindrical functions on $\mathcal{S}'(\Sigma)_{inv}$, and the algebraic dual

$$\begin{aligned} \mathrm{Cyl} &\hookrightarrow \mathcal{H} \hookrightarrow \mathrm{Cyl}^* \\ \mathrm{Cyl}_{red} &\hookrightarrow \mathcal{H}_{red} \hookrightarrow \mathrm{Cyl}^*_{red} \end{aligned}$$

Cyl^*_{inv}	$\mathcal{T}$-invariant subspace of Cyl^*
$\mathrm{Cyl}^*_c \subset \mathrm{Cyl}^*$	quantum mechanical solution to constraint set 'c'
$\mathrm{d}^2x = \mathrm{d}\rho \mathrm{d}z$	
$\mathrm{d}^3x = \rho \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}z$	
$\mathrm{Diff}(\Sigma)$	group of diffeomorphisms of Σ
$\mathfrak{E} : \mathcal{H}_{red} \hookrightarrow \mathrm{Cyl}^*_c$	see §4.4.3
f_s, $f_\perp$	components of a given $f \in \mathcal{S}(\Sigma)$ with respect to the decomposition $\mathcal{S}(\Sigma) = \mathcal{S}(\Sigma)_{inv} \oplus \mathcal{S}(\Sigma)_\perp$
h	one particle Hilbert space of full theory
$\mathcal{H} := \mathcal{F}_s(h)$ $= L^2(\mathcal{S}'(\Sigma), \mathrm{d}\mu)$	full field theory Hilbert space
$h_\perp$	orthogonal complement of h_{inv} in h
$\mathcal{H}_\perp := \mathcal{F}_s(h_\perp)$	
h_{inv}	$\mathcal{T}$-invariant subspace of h
$\mathcal{H}_{inv}$	$\mathcal{T}$-invariant subspace of $\mathcal{H}$
$\mathcal{H}_{red} := \mathcal{F}_s(h_{inv})$ $= L^2(\mathcal{S}'(B), \mathrm{d}\mu_{red})$	reduced theory Hilbert space
$\mathcal{H} \overset{\text{nat.}}{\cong} \mathcal{H}_{red} \otimes \mathcal{H}_\perp$	
$\mathcal{H}_b \subset \mathcal{H}$	quantum mechanical solution to constraint set 'b'
$\mathcal{H}_b = \mathrm{Im}\, \iota$	
$\mathcal{H}_c := \mathrm{Im}\, \mathfrak{E}$	
$\hat{H}$, $\hat{H}_{red}$	one particle Hamiltonians in the full and reduced theories, respectively
$\mathbb{H}$, $\hat{\mathbb{H}}$	total Hamiltonian in the full theory and its quantization
$\mathbb{H}_{red}$, $\hat{\mathbb{H}}_{red}$	total Hamiltonian in the reduced theory and its quantization
$I : \mathcal{S}'(\Sigma)_{inv} \to \mathcal{S}'(B)$	is defined by $[I(\beta)](f) := \beta(P^* f)$; I is an isomorphism
J	complex structure on Γ; in appendix D: complex structure in the reduced theory

$\mathbb{L}_z, \hat{\mathbb{L}}_z$	z-component of the total angular momentum in the full theory, and its quantization
$\mathcal{L}_\phi$	Lie derivative with respect to ϕ
m	scalar field mass
$P : \Sigma \to B$	canonical projection
$\mathcal{S}(\Sigma)_\perp$	the orthogonal complement of $\mathcal{S}(\Sigma)_{inv}$ in $\mathcal{S}(\Sigma)$ (§4.3.1)
$\mathcal{S}(\Sigma)$, $\mathcal{S}(B)$	space of Schwartz functions on Σ, B
$\mathcal{S}(\Sigma)_{inv}$, $\mathcal{S}'(\Sigma)_{inv}$	$\mathfrak{T}$-invariant subspaces of $\mathcal{S}(\Sigma)$ and $\mathcal{S}'(\Sigma)$, respectively
$\mathcal{S}'(\Sigma)$, $\mathcal{S}'(B)$	space of tempered distributions on Σ, B

$$\begin{aligned} \mathcal{S}(\Sigma)_{inv} &\overset{\text{nat.}}{\cong} \mathcal{S}(B) \\ \mathcal{S}'(\Sigma)_{inv} &\overset{\text{nat.}}{\cong} \mathcal{S}'(B) \end{aligned}$$

$\mathrm{span}\{\cdot\}$	Cauchy completion of the set of all finite linear combinations
$\mathfrak{T} \subset \mathrm{Diff}(\Sigma)$	the group of rotations about z-axis
U_g	action of a given $g \in \mathfrak{T}$ on $\mathcal{H}$
x^1, x^2, x^3	Cartesian coordinates on Σ

Γ	full phase space
$\Gamma_{inv} \subset \Gamma$	$\mathfrak{T}$-invariant subspace of Γ
$\Delta = \Delta_\Sigma$	Laplacian on Σ
Δ_B	Laplacian on B
$\Delta_\Psi \hat{\mathcal{O}}$	variance ("fluctuation") in $\hat{\mathcal{O}}$ for the state Ψ
$\Theta := -\Delta + m^2$	
$\iota : \mathcal{H}_{red} \hookrightarrow \mathcal{H}$	see §4.5.2
$\hat{\Lambda}(\xi) := a^\dagger(\xi) - a(\xi)$	generator of phase space translations
μ	measure on $\mathcal{S}'(\Sigma)$ used in Schrödinger representation of full theory (§4.2.3)
μ_{red}	measure on $\mathcal{S}'(B)$ used in Schrödinger representation of reduced theory (appendix D)
ν_{red},ν^o_{red}	Gaussian and Lesbesgue generalized measures on Γ_{inv} (§C.1)
π	group averaging map on $\mathcal{S}(\Sigma)$ (§4.4.1)

π_s, $\pi_\perp$	components of π with respect to the decomposition $\mathcal{S}(\Sigma) = \mathcal{S}(\Sigma)_{inv} \oplus \mathcal{S}(\Sigma)_\perp$
ρ, ϕ, z	cylindrical coordinates on Σ
Σ	spatial hyperplane in Minkowski space
$\vec{\phi} := \frac{\partial}{\partial\phi}$	axial symmetry field
$\hat{\varphi}[f]$, $\hat{\pi}[g]$	basic smeared field operators in the full theory
$\hat{\varphi}_\perp[f]$, $\hat{\pi}_\perp[g]$	the operators on $\mathcal{H}_\perp$ corresponding to $\varphi_\perp[f]$ and $\pi_\perp[g]$. $\hat{\varphi}_\perp[f]$ is defined by multiplication and $\hat{\pi}_\perp[g]$ is the self-adjoint part of $-\mathrm{i}\int_\Sigma g\frac{\delta}{\delta\varphi_\perp}$
$\varphi_r = \varphi_{red} := (2\pi)^{\frac{1}{2}}\varphi_s$, $\pi_r = \pi_{red} := (2\pi)^{\frac{1}{2}}\rho\pi_s$	basic classical fields in the reduced theory; relation to fields in the full theory.
$\hat{\varphi}_r[f] = \hat{\varphi}_{red}[g]$ $\hat{\pi}_r[g[= \hat{\pi}_{red}[g]$	smeared field operators in the reduced theory
φ_s, $\varphi_\perp$	components of φ with respect to the decomposition $\mathcal{S}(\Sigma)=\mathcal{S}(\Sigma)_{inv}\oplus\mathcal{S}(\Sigma)_\perp$ or $\mathcal{S}'(\Sigma)=\mathcal{S}'(\Sigma)_{inv}\oplus\mathcal{S}'(\Sigma)_\perp$, according to the context
$\hat{\varphi}_s[f]$, $\hat{\pi}_s[g]$	the operators on $\mathcal{H}_{red}$ corresponding to the smeared functions $\varphi_s[f]$ and $\pi_s[g]$. $\hat{\varphi}_s[f]$ acts by multiplication and $\hat{\pi}_s[g]$ is the self-adjoint part of $-\mathrm{i}\int_\Sigma g\frac{\delta}{\delta\varphi_s}$.

$$
\begin{aligned}
\hat{\varphi}_s[f] &= (2\pi)^{-\frac{1}{2}}\hat{\varphi}_{red}[f_s] \\
\hat{\pi}_s[g] &= (2\pi)^{-\frac{1}{2}}\hat{\pi}_{red}[\rho^{-1}g_s] \\
\hat{\varphi}[f] &= \hat{\varphi}_s[f]\otimes \mathbb{1} + \mathbb{1}\otimes\hat{\varphi}_\perp[f] \\
\hat{\pi}[g] &= \hat{\pi}_s[g]\otimes \mathbb{1} + \mathbb{1}\otimes\hat{\pi}_\perp[g]
\end{aligned}
$$

Ψ_0, $\overset{r}{\Psi}_0$	vacuum in the full (§4.2) and reduced (§4.5.2) theory, respectively
$\Psi_\xi^{coh}, \overset{r}{\Psi}{}_\xi^{coh}$	coherent states in full and reduced theories ($\xi \in h$ or $\xi \in h_{inv}$, accordingly) (§4.5.1, §4.5.2)
$\Omega(\cdot,\cdot)$	symplectic structure on Γ; in appendix D: symplectic structure in the reduced theory

For LQG (§4.9.2):

$\mathrm{Aut}(P)$	group of automorphisms of P
P	SU(2) principal bundle over Σ

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