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COSMIC VEILING AND PRE-INFLATIONARY INITIAL CONDITIONS IN $R + R^2$ GRAVITY

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Abstract

In this thesis, we propose and explore the consequences of the principle of *cosmic veiling* for the pre-inflationary epoch of the universe. The principle is based on the requirement that there exists a consistent post-Planckian, but pre-inflationary epoch of the universe in which the framework of perturbative quantum field theory remains valid, and that the transition is smooth. Inflation then functions as a censor of the initial singularity by pushing back the Planck scale further into the past, serving a role similar to horizons of black holes that censor their singularities. The principle of cosmic veiling allows then a *maximum* amount of inflationary e-foldings to just barely censor the initial singularity. The initial quantum state is prepared at the transition between the quantum gravity and quantum field theory regimes, and is chosen to be compatible with cosmic veiling, that is, the initial state should not spoil the veiling by containing additional information from the Planck epoch, defined as the epoch when the curvature length scale is of the order of the Planck length.

An immediate consequence of this principle is, by allowing ourselves to push the domain of validity of quantum field theory to its maximal limit, we can set initial conditions at an earlier time. Classically, singularities are censored to prevent pathologies in the causual structure of the manifold. In conjunction with quantum mechanics, Hawking provided a mechanism through which one could peer at the singularity – only very barely – at the end of the black hole's life, through Hawking evaporation of black holes. Cosmic veiling is a similar, mild violation of the censorship of the initial singularity – we elevate the role of inflation as a censor that pushes back in time the Planck scale and require that the amount of inflation is bounded from *above* such that we can peer back far enough to observe some features from the primordial epoch. In this manner, the amount of inflation, which is already bounded from below by requiring sufficient e-foldings to be compatible with observation, we can postulate a very narrow "Goldilocks" band in which inflation lasts long enough to make our universe interesting, but not observationally excluded. We determine the effect on the cosmic microwave background and discuss the possibility of future observations.

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Acknowledgments

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To my parents and my family, who supported me from far away.

Chapter 1 Introduction

1.1 Canticle of veiling

At quantum threshold one can take
a principle of cosmic veil;
that inflation, by virtue ancilliary,
would Penrose censor, as customary,
the initial singularity.

Though only just, as gossamer.

The veiling on the background places
a special time, ’tween transition phases,
and by requiring the quantum state
avoids, of Planck, being au fait
thus posed the precepts of cosmic veil.

Ensuring features, as promiser.

Higher curv’ture can drive alone
the expansion to conclusions known.
But if at de Sitter-like attractor,
tease apart pre-inflationary factor,
the veil thus lifts for to behold:

the power adjust, astronomer.

1.2 A thinly-veiled proposal

Humanity's curiosity and innate inclination to craft stories makes us unique amongst living beings. Amongst these stories, none are more universal than our cosmic origins. We are like cosmic detectives, traversing the deep noire of the night sky, gathering clues to solve the universal mystery. Our space-age magnifying lenses are now able to probe deeply into the "scene of the crime" – the Cosmic Microwave Background (CMB) [1, 2], a relic of the nascent universe imprinted on the sky. And, from there, it's up to our keen gumshoe instincts to hone in on the puzzles we aim to solve.

What are the facts at the scene of the crime and what are our puzzles? For one, we observe that the universe is expanding – suggesting that the matter content was more densely packed together in the past. Look far enough into the past and we observe the CMB, a nearly uniform snapshot of black body photons at a temperature of 2.73 K [3–5]. The temperature fluctuations of this cosmic sea of photons are quite small, with conventional scale $\dfrac{\delta T}{T_{\mathrm{CMB}}} \approx 10^{-5}$. This is the first surprise – the universe we live in is shockingly homogeneous!

Like in any good mystery, however, there is a second act twist, a surprise within the surprise. Thermal fluctuations generally go as one over the square root of the degrees of freedom of the gas. However, the number of degrees of freedom in the early universe is so large, that purely canonical statistical fluctuations cannot explain the temperature deviation observed in the CMB. This means that the fluctuations are simultaneously incredibly small and incredibly large! Large, in this context, means that there must be some mechanism that adds power into the fluctuations. The usual culprit mechanism is inflation [6–8] – a period of time when the nascent universe expanded exponentially, taking the quantum fluctuations of the initial Fock gas and blowing them up to the scale we see today in the night sky [9]. There are several ingredients to this story – the mechanism to trigger inflation, how one prepares the initial state of the quantum fields, and the extent to which inflation lasts.

The usual framework for inflation includes the inflaton – a posited particle in a given potential that drives inflation by minimally coupling to gravity and subsequently to the rest of the Standard Model during a period of time called reheating. Introducing particles to explain phenomena is a heuristic that has been incredibly successful in the 20th century. When it comes to inflation, the standard framework is similarly wildly successful at matching observation; by positing an initial quantum vacuum in the far past of a de Sitter background, the framework generates a power spectrum that is very slightly red-tilted, and with power commensurate with observations. Which mysteries, if any, are left to be solved?

Unfortunately, the mechanisms that drive inflation are not unique, and there exists a veritable zoo of inflationary theories [10–13], with differing numbers of assumptions, parameters, and add-on effects which cannot be made falsifiable given the status of observation today. Details of the scenarios aside, the basic premise is to take quantum fluctuations and expand them to cover the entirety of the sky, which requires a principle to select a particular quantum state as initial state. Nominally, the simplest choice is to take an adiabatic vacuum state – that the ultraviolet modes of the state be in the ground state, the short-distance correlation functions approximate those of Minkowski space, the field modes be of positive frequency, and that the state has negligible particle creation by virtue of approximating the symmetry of the background [14, 15].

Consider, for a moment, this most standard set of inflationary assumptions, with the state of the perturbations being adiabatic at the end of inflation. If we take a background in which the universe is not inflationary infinitely into the past, we ought to observe particle production from this state as we wind the clock back and the universe becomes increasingly curved and hot [16], making backreaction a salient concern and leading us outside the regime of validity of quantum field theory in curved space-time. If we invert the time arrow on this thought process, were we to pick a random state from the quantum gravity regime, absent a mechanism or principle to set a measure on the configuration space of quantum initial states, we might be fooled into thinking that any hot state should lead into a reasonable adiabatic state for doing perturbative field theory – else we might not be able to do the perturbative theory at all!

Ultimately, the inflationary theory must, at some point, be an emergent result of a more fundamental theory of quantum gravity [17], consequently paring down the list of viable inflationary scenarios. With this "holy grail theory of everything" still on the horizon, in this thesis we propose and explore a different set of assumptions for driving inflationary expansion and choosing the initial quantum state. We propose a principle of *cosmic veiling*, which consists of the following statements:

1. Inflation functions as a censor of the initial singularity by pushing back the Planck scale further into the past, serving a role similar to horizons of black holes that censor their singularities. The principle of *cosmic veiling* is then to turn the full censorship into a gentler veiling. As we'll see, this principle will yield a *maximum* amount of e-foldings consistent with veiling.

2. By requiring a space-time background that is consistent with veiling, there must exist a post-Planckian, but pre-inflationary epoch of the universe in which the framework of perturbative field theory remains valid. Furthermore, the transition from the Planck era to this pre-inflationary regime is expected to be smooth.

3. The initial quantum state is prepared at the transition between the quantum gravity and quantum field theory regimes, and is chosen to be compatible with *cosmic veiling*, that is, the initial state should not spoil the veiling by containing additional information from the Planck epoch, defined as the epoch when the curvature length scale is of the order of the Planck length $\ell_P = \sqrt{G\hbar/c^3} \simeq 1.6 \times 10^{-35}\,\mathrm{m}$.

An immediate consequence of the first assumption is, by allowing ourselves to push the domain of validity of quantum field theory to its maximal limit, we can set initial conditions at an earlier time. Certainly, there exists frameworks other than semiclassical gravity for pre-inflationary cosmologies, such as those in Loop Quantum Cosmology (LQC), which have the added benefit of resolving the initial singularity [18–24]. The principle of cosmic veiling is inspired by Penrose's censorship of black holes in general relativity [25]. Classically, singularities are censored to prevent pathologies in the causual structure of the manifold. In conjunction with quantum mechanics, Hawking provided a mechanism through which one could peer at the singularity – only very barely – at the end of the black hole's life, through Hawking evaporation of black holes [26, 27]. Cosmic veiling is a similar, mild violation of the censorship of the initial singularity – we elevate the role of inflation as a censor that pushes the Planck scale back in time and require that the amount of inflation is bounded from *above* such that we can peer back far enough to observe some features from the primordial epoch. In this manner, the amount of inflation, which is already bounded from below by requiring sufficient e-foldings to be compatible with observation, we can postulate a very narrow "Goldilocks" band in which inflation lasts long enough to make our universe interesting, but not observationally excluded.

As we will see, the amount of pre-inflationary, non quasi-de Sitter e-foldings is very limited, primarily by virtue of this phase being radiation-like in its geometry, or fast-roll, despite being modeled as a matter-less universe. This, along with our first two assumptions, sets a special time – a threshold time from the Planck regime to a semiclassical one. Thus our third assumption comes in: we can set a cosmic veiling condition for the initial quantum state $\langle\psi_{\text{veil}}|$ at this time such that it carries the minimum amount of information from the Planck era. Consider for a moment the alternative – if there were imprints from the transition on the initial state, these primordial fluctuations could be enhanced by inflation and observed today in the CMB!

Taking cosmic veiling as a principle gives the primary ingredients requisite to set initial conditions for a given model of inflation. Since the initial state is set at the threshold of the Planck regime, with higher order curvature terms of gravity expected to play a significant role at this scale, the choice is made to work in the simplest possible case – a framework of inflation without a scalar inflaton field, in which inflation is driven purely by gravity [28, 29]. This model, originally proposed by Starobinsky in 1979 [30], provides a scheme in which the next leading order term to the Einstein-Hilbert action, αR^2, where α is a coupling constant of the theory, is already sufficient to drive inflation. As we will see, taking the purely gravitational point of view leads to a background that both gracefully enters and exits an inflationary phase.

The thesis is structured as follows:

In Chapter 1, we'll finish the Introduction by presenting a wide overview of the key picture in identifying cosmic veiling, namely how a quantity called the comoving-scale changes over different cosmological epochs. We'll identify the cosmic veiling conditions, and speak briefly of the background initial conditions.

In Chapter 2, we introduce the classical background for our framework, motivated by introducing higher curvature terms into the Einstein-Hilbert action, but ultimately simplified to the action due to Starobinsky: $S[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R+\alpha R^2)$. Three major regimes of classical solutions are explored and solved: the inflationary phase, quasi-de Sitter phase, the subsequent reheating phase, in which the geometry oscillates. Finally, the pre-inflationary phase, which is radiation-like in its geometry, does not have an exact closed form solution for $H(t)$, but does admit a relation between $\dot{H}$ and H that characterizes this regime, and introduces a relevant scale of the theory, H_c, the value of the Hubble parameter at the time the quasi-de Sitter and pre-inflationary solutions cross in the $(H, \dot{H})$ phase space diagram. The scale H_c will become instrumental in identifying the amount of inflationary e-foldings for a given classical trajectory.

In Chapter 3, we discuss the standard theory of quantization, taking a specific gauge to identify the physical degrees of freedom, and quantize only the resulting physical Hilbert space. Given the ADM ansatz with a lapse and shift, we can decompose the metric in Scalar-Vector-Tensor sectors and express a general perturbation to the metric along these sectors. Subsequently, we develop the second-order free action that governs the perturbations and select a gauge in which the are no Ricci curvature fluctuations on a given slicing, namely the comoving gauge. Finally, with solved constraints, we find the action for the perturbations in each sector, with the vectorial sector being trivial, and the scalar and tensor sectors both formulated in a general manner, with a characteristic function $Z_s(t)$ or $Z_t(t)$ that distinguishes them. As we will see, these functions and their Hubble-flow derivatives, encode the relative difference in the power spectrum amplitude for scalars and tensors, as well as the different in their spectral tilts.

In Chapter 4, we quantize the general, Fourier-passed action for the perturbations by computing the Poisson brackets from the classical theory and promoting them to canonical commutators. We build the Fock space starting with a vacuum annihilated by all annihilation operators, and developed by creation operators for each mode $\vec{k}$, and find a classically peaked representation of the field operator. Alongside the canonical commutation relation, the field representation yields a Wronskian relation for the mode functions, and with the action we determine the equation of motion for the mode functions, which closely resembles that of a harmonic oscillator with a friction term. We briefly discuss the observables that come from the mode functions, define the power spectrum, and present a fully solvable sample case: that of Bunch-Davies initial conditions.

In Chapter 5, once the mode equation is reformulated into a canonical form, with quantities expanded in small Hubble-flow parameters, and further expanded around a critical time, flagged by when the pivot mode k_* crosses the Hubble horizon, we find that the previously solved Bunch-Davies state functions as a zeroth order of this expansion. We briefly discuss the exact solutions to this canonical mode equation at given truncations, and then present a method that can be used to find the solution at arbitrary order in Hubble-flow parameters. The key results presented in the thesis, however, can be extracted already at linear order in ϵ_i's, which fortunately has Bessel functions as solutions of the mode equation. Some results at second order are presented without proof.

In Chapter 6, we present the results for observational quantities computed from the Starobinsky model, and discuss connections with observation. Here, we expand fully on the scaffolded development of the cosmic veiling principle. At first order, we identify the scale k_{feature} at which we expect to observe a pre-inflationary feature on the CMB, and find that, under the assumption that the pre-inflationary evolution does not spoil the results of the standard inflationary initial state at the scale k_*, there is a single-parameter expression that gives a near-minimal number of inflationary e-foldings compatible with observation, when the parameter is of order unity. We observe that, qualitatively, whenever this parameter is of order unity, the largest comoving scale in the primordial universe at the onset of inflation is of the same order as the largest comoving scale that we will ever observe in the modern universe. We first propose a principle in which the CMB, as the de facto probe for primordial perturbations, provides the precise comoving scale at the onset of inflation. Lastly, we posit cosmic veiling as a principle for setting initial conditions of the background, the initial state, and investigate the consequences. Among them, we find an expression for the number of inflationary e-foldings, of which the leading order is

$$N_{\text{veil}} \lesssim \log\left(\frac{1}{\alpha\Lambda}\right) \simeq 64\,, \tag{1.1}$$

with the coupling constants $\alpha \simeq 2.5 \times 10^5\,\ell_P^2$ of αR^2 and the cosmological constant $\Lambda \simeq 2.9 \times 10^{-122}\,\ell_P^{-2}$ which determine the large scale behavior of the universe in two completely different epochs! Given this principle of cosmic veiling, numerical simulations are run to identify the salient aspects of the power spectrum, among them the presence of power suppression at large scales, and finally we present an analytic approximation scheme that separates the pre-inflationary effect from the purely inflationary effect on the power spectrum by parametrizing deviations from squeezing, preserving the homogeneity and isotropy assumptions, via Bogoliubov coefficients, and discuss which effects, consistent with observation, might be imprinted on the CMB.

1.3 A brief co-moving history of the universe

At the heart of the explanation for the CMB lies the inflationary mechanism, which posits the early universe expands nearly exponentially in size. In order to motivate a pre-inflationary phase, a key element in the assumptions of cosmic veiling, we need to understand the role inflation plays in explaining the temperature fluctuations observed in the CMB. Figure 1.1 presents a co-moving history of the universe that can serve to illustrate the inflationary mechanism – in particular as a diagram of the homogeneous space-time background upon which we observe properties of anisotropic perturbations – as well as the open questions it leaves, and how cosmic veiling can fill in those gaps.

In what sense is Figure 1.1 a co-moving history? The horizontal scale ought to stand for time if this is indeed a history; in this case, the x-axis gives us our stand-in for time, the amount of e-foldings N. The way to think of e-folds is similar as thinking of folding a piece of paper and counting its thickness: fold once, and you get twofold the thickness. Fold once more, to get four times the thickness, and so on. E-folds give us the same geometric increase of size of the universe, but with the e as a root number rather than 2. These are e-folds of the scale factor $a(t)$, which gives us a sense of the size of the universe

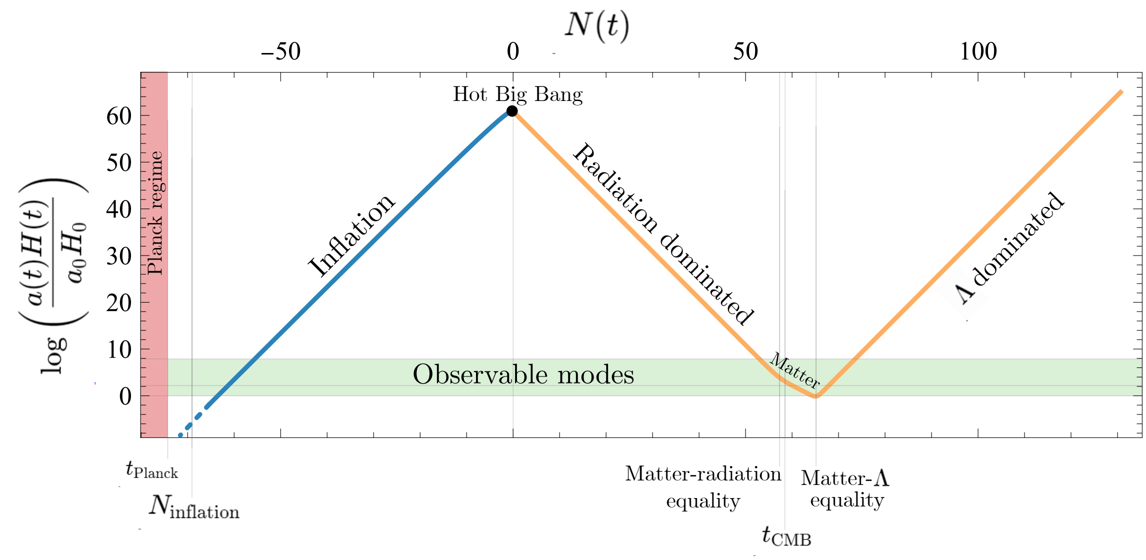

Figure 1.1. The co-moving history of the universe. Noted are some important epochs of the universe: inflation, radiation-dominated era, matter-dominated era, and $\Lambda-$dominated era. There are some noted special times, such as the time of the CMB, radiation-matter equality, and matter-Λ equality. There is a minimum of co-moving scale which corresponds to the largest scale ever observed in the universe. Also shown is the band of observable modes of the CMB, and how they cross the Hubble horizon several times over the history of the universe, leading to differing oscillatory and freezing behavior over epochs.

at a given time t. Specifically, if you have two freely moving bodies in space-time, such as galaxies, the scale factor captures the increasing distance between them over time. And if I know the scale factor at one point in time, I can relate it to the scale factor at another point in time by the amount of e-folds N that occurs in between:

$$a(t_1) = a(t_0)e^{N_{t_0 \to t_1}}. \tag{1.2}$$

The vertical scale of Figure 1.1 is the co-moving scale $a(t)H(t)$, where $H(t)$ is the Hubble parameter, derived from the scale factor:

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}. \tag{1.3}$$

Note that, while the scale factor is dimensionless, the Hubble factor has a dimensions of $[\text{time}]^{-1}$. This gives us a natural length scale alongside the speed of light, that of $c/H(t)$, which measured today gives us: $c/H_0 \simeq 14$billion light years. The scale $a(t)H(t)$ is then co-moving in the sense that this is the natural inverse length scale (in units with $c = 1$), or wavenumber scale, that a freely floating observer would measure as the universe expands.

Now, the history of this plot gives us the background, with several identifiable epochs of the universe - the radiation-dominated phase, followed by a short (in e-folds) matter-dominated phase, and finally the current cosmological constant-, or $\Lambda-$, dominated phase of the universe. During many of these eras, the co-moving scale decreases, with a notable time near the present at which we'd observe the largest co-moving scale the universe will *ever* see – remember that large scales corresponds to small co-moving scale $a(t)H(t)$ – just as the universe begins its accelerated expansion due to cosmological constant domination.

Now that we have the history of the background, we need to know how perturbations evolve on this background. While the specific nature of the perturbations depends on inflationary model, these are usually ultimately made manifest in temperature perturbations of the CMB. These perturbations are generally analyzed via spectral decomposition, that is, in terms of frequencies of the signal. Our diagram, helpfully, depicts also the band of observable modes of the CMB, randing in scale from about 5Mpc^{-1} to 1Gpc^{-1}. These modes evolve on the background according to the wave equation. In Minkowski space, this is simply the Klein-Gordon equation. When we consider cosmological spaces, there is an addition which depends on the Hubble factor:

$$\ddot{\phi} + 3H(t)\dot{\phi} + \frac{k^2}{a(t)^2}\phi = 0, \tag{1.4}$$

where ϕ is the mode function of the perturbation. This is, in essence, a damped harmonic oscillator, with $k/a(t)$ functioning as our frequency, and the Hubble factor taking the role of friction. However, it is worthwhile to note that this isn't really a friction, so much as it is a dilution! This equation has two regimes of interest – overdamped and underdamped solutions. In the underdamped case, the friction isn't strong enough to prevent oscillations, while in overdamped solutions, friction overtakes the oscillations, leading to solutions being frozen. So for a given wavenumber k, the relevant quantity that determines the freezing of the solution is $a(t)H(t)$. Note that this argument isn't quite precise - why would a solution freeze to a nonzero value in this scheme? A more careful treatment of this freezing phenomenon is given in Chapter 5.

Take, for example, a given band of modes k that is above the co-moving scale in the universe during the modern epoch. If we look at the past-evolution of the bundle of modes and choose quantum initial conditions $|\psi\rangle$ at a time when the modes are under the co-moving scale, since the modes are frozen, we would have no simple principle to select initial conditions that gives rise to the statistical distribution of anisotropies we observe in the CMB today. However, if we were to go further back in time, during the

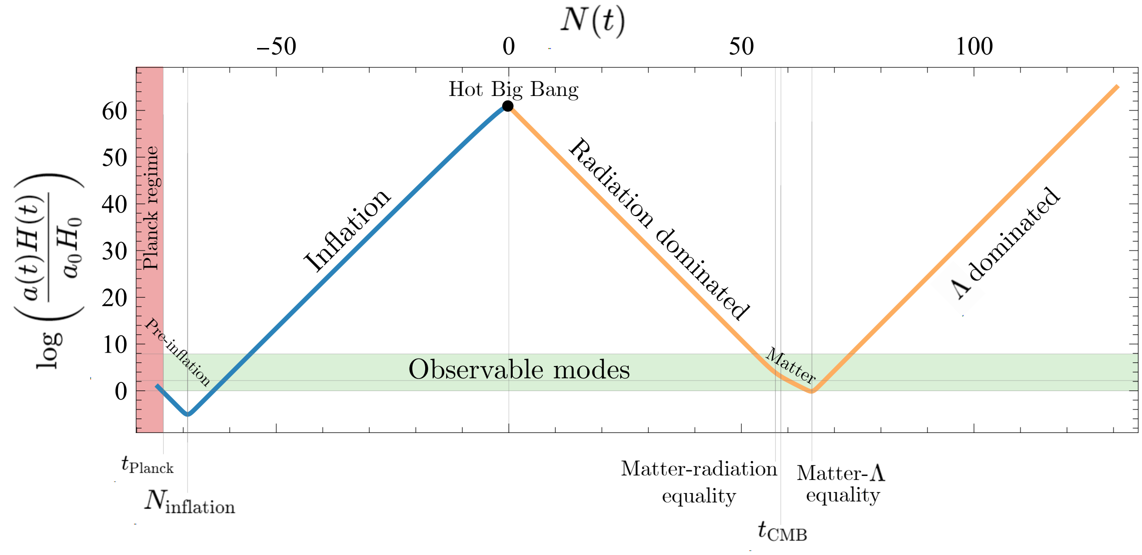

Figure 1.2. The co-moving history of the universe, now cosmically-veiled. The same important epochs of the universe are depicted, but now we have a pre-inflationary era that is post-Planckian, leading to a minimum of co-moving scale which corresponds to the largest scale in the primordial universe. The band of observable modes of the CMB now can have initial conditions set at a pre-inflationary time.

era of inflation, the band of modes would once again be above the co-moving scale, and so the modes are oscillatory. As it turns out, if we pick a vacuum state as our initial condition, evolve it through inflation, which at some point causes the band to become frozen, we find, after its unfreezing in the modern epoch, a statistical distribution of anisotropies that matches observation.

This is, of course, predicated on inflation having the depicted behavior that its co-moving scale increases over time. We'll see this derivation in Chapter 6. Altogether, inflation gives us a mechanism for explaining the statistics of temperature perturbations at late times in terms of simple initial conditions at a much earlier time. This is the standard story - so what are the open questions that inflation leaves? For one, the standard story doesn't specify what exactly is the mechanism that produces inflation. In addition, there's no maximum amount to the total amount of inflationary e-foldings – that is, inflation can last arbitrary long into the past absent some other mechanism. But is this infinite inflation compatible with quantum gravity? And if not, does a quantum gravity phase preceding inflation spoil the inflationary predictions?

The principle of cosmic veiling, inspired by censorship of black hole singularities by their event horizon, is a threshold condition that we posits we ought to observe that inflation veils rather than fully censors the initial singularity. By doing so, we require

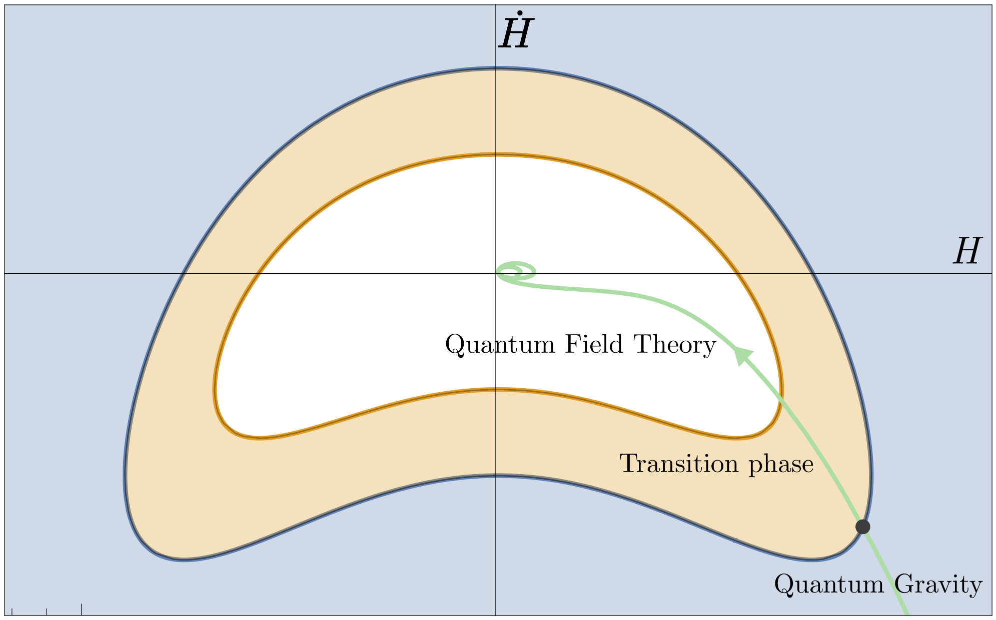

Figure 1.3. The quantum gravity to quantum field theory transition can be partitioned geometrically by the Kretschmann scalar. Shown above are $K = \text{const}$ slices on the $(H, \dot{H})$ plane. The outer surface represents the quantum gravity regime; any solution of the Friedmann equation which traverses into this regime is disallowed by our assumptions. The inner surface represents the surface of initial conditions that are considered, allowing ourselves a buffer in between where the transition to effective field theory may break down at some unspecified scale.

the existence of a post-Planckian, but pre-inflationary era of the universe in which the framework of perturbative field theory still holds, and that the transition between these regimes is smooth. Then, we can choose a vacuum state at this threshold – a simple initial condition, at an earlier, pre-inflationary time – and we can determine the consequences that such an initial state will have on observables. The change to our picture of the co-moving history of the universe is shown in Figure 1.2. We'll explore this modification further in Chapter 6.

In order to talk about a pre-inflationary, but post-quantum gravity era of the universe, that is, an era of time in which QFT in curved spacetime is able to make meaningful predictions, we first need to quantify the transition between these two periods, as shown in Figure 1.3. Working with FLRW spacetime absent of matter, the Kretschmann curvature scalar, $R_{abcd}R^{abcd}$ provides a curvature invariant that can already be computed with the FLRW metric without requiring any details of the inflationary theory. Using the metric $ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$, we find the Kretschmann scalar,

$$K = R_{abcd}R^{abcd} = 12(2H^4 + 2H^2\dot{H} + \dot{H}^2) \tag{1.5}$$

defines a series of contours in the $(H, \dot{H})$ plane when setting $K = \text{const}$. Regardless of physical theory, we expect quantum gravity effects to be significant at Planckian scales. The standard inflationary paradigm, using data from the Planck collaboration, puts an upper bound on the value of the Hubble parameter at the time of inflation:

$$\frac{H_*}{M_{\text{Planck}}} < 2.5 \times 10^{-5} \tag{1.6}$$

Thus we define a possible QG-to-QFT transition band, in terms of length scale, between the values of $L_{\text{Planck}} < L_{\text{QG-QFT transition}} < 10^4 L_{\text{Planck}}$.

It is within this band that we seek to impose pre-inflationary – but post-Planckian – initial conditions that allow us to use the well-studied framework of QFT in curved spacetime and probe for pre-inflationary effects that could be imprinted on the cosmic microwave background.

Chapter 2
The classical space of initial conditions

2.1 Introduction

We will work primarily in the modified gravity model proposed by Starobinsky in 1979. This model, developed prior to the inflationary hypothesis, was devised with with goal of investigating higher curvature effects on the early universe. At the time, there were no direct methods to work this theory into observations, so when Starobinsky came up with the potential typically used in single field inflaton theories, he was not simply coming up with a parametrization that would best fit data. While one can use this as a principle – gravity with additional curvature terms – as a starting point for doing inflation with a dummy scalar field, we insist on taking Starobinsky's action seriously. This becomes relevant in situations where we have to consider observables. After all, just as choosing a gauge informs the questions that we ask of observables, so too does what we genuinely consider to be the physical field or physical scalar and tensor perturbations. Moreover, taking this action seriously gives us a natural, intuitive way to ask questions about a pre-inflationary, but post-Planckian regime of the universe. It is fair to insist on including all higher curvature terms, after all, there's no a priori reason to truncate the gravitational theory at Einstein-Hilbert, and while higher curvature terms could certainly be considered, our starting point is the quartic order in derivatives action, organized by number of derivatives and including all possible ways the derivatives can appear while keeping the action diffeomorphism invariant: [31]

$$S[g_{\mu\nu}] = \int (c_0 + c_2 R + c_4 R^2 + c_4' R_{\mu\nu} R^{\mu\nu} + c_4'' R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + ...)\sqrt{-g} d^4x \tag{2.1}$$

Note that the coupling constants c_4, c_4', c_4'' are not independent. The Gauss-Bonnet term, which is a surface term in 3+1 gravity, constrains the relationship between the coefficients:

$$\mathcal{E} = 4R_{\mu\nu}R^{\mu\nu} - R^2 - R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \nabla_\mu C^\mu \tag{2.2}$$

Since the variation of this term yields does not contribute to the equation of motion, we can adjust the original formulation of our action by eliminating one of the three terms c_4, c_4', c_4'' using the Weyl tensor and restore the historical names of the coupling constants:

$$S[g_{\mu\nu}] = \frac{1}{16\pi G}\int(-2\Lambda + R + \alpha R^2 + \beta W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + ...)\sqrt{-g}d^4x \tag{2.3}$$

where G is Newton's constant and α is a coupling constant to the higher curvature term. Notice that since R has units of 1/length, α has units of area. A couple of remarks about this action are in order. The coupling constant β is currently largely unconstrained, while analysis of the inflationary phase in this theory and subsequent matching to observation will show a value of α of the order $10^{10}G\hbar$. Clearly any effective field theory should be expected to break down at the Planck scale. One might have concerns regarding a large order of magnitude of the coupling constant α that might affect the domain of validity of the effective field theory. Indeed, observations within the solar system provide weak constraints of the magnitude of α, limiting it to values below $10^{18}m^2$ or $10^{87}G\hbar$. As back-of-the-envelope calculation, consider how large a black hole might have to be in order to observe α. Well, α must be compatible with the effective field theory, and thus small compared to the bounds of the solar system. If α is 5 orders of magnitude removed from Planck length, might we observe a black hole with this energy scale? Well, 5 orders of magnitude removed from the Planck mass gives us about a 1 gram black hole. In comparison to a solar mass black hole, around $10^{30}kg$, the contribution of α is suppressed by 10^{-33}, so there's not much chance of seeing α in the solar system, so it can thankfully be relegated to an early universe effect!

Further stringent constraints from Eot-Wash experiments with torsion balances, in addition to results of the Gravity Probe B mission, have led to a bound of $|\alpha| < 1.0 \times 10^{-9}m^2$ or $3.8 \times 10^{60}G\hbar$. [32–35] So within the solar system, α is allowed to be quite large without ruining the validity of the theory at the scale of the solar system.

In this thesis, we will be working under the explicit assumption that, just in the way that there exist a separation of scales which allows us to describe gravity within the solar system using only the Einstein-Hilbert action with a high degree of fidelity, so too can we describe an inflationary, and pre-inflationary epoch with only quadratic curvature terms.

Since we'll be working within the framework of flat FLRW slices, the description of the dynamics of the background will have the contribution by the Weyl tensor be automatically zero, as this theory is conformally flat. In addition, the scale of Λ is negligible at this curvature, so for the time being we will work solely from the action from Starobinsky, [30]

$$S[g_{\mu\nu}] = \frac{1}{16\pi G}\int (R+\alpha R^2)\sqrt{-g}d^4x \tag{2.4}$$

Once we have the action, since the equations of motion are stationary points of the action, we vary the action once and impose equality to zero to find the equations of motion of the background, where here we introduce the abstract functional derivative with the tensor $F^{\mu\nu}$, a tool that will help us later on when we take on the second-order action of the perturbations:

$$F^{\mu\nu} = \frac{\delta}{\delta g_{\mu\nu}}S = -\frac{1}{16\pi G}\left(G^{\mu\nu} + \Lambda g^{\mu\nu} + \alpha H^{\mu\nu} + 4\beta B^{\mu\nu}\right) \tag{2.5}$$

with the tensor $G^{\mu\nu}$ being the usual Einstein tensor, and the variation of R^2 giving the tensor $H^{\mu\nu}$:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{2.6}$$

$$H_{\mu\nu} = 2RG_{\mu\nu} - (\nabla_\mu\nabla_\nu R + \nabla_\nu\nabla_\mu R) + 2(g^{\alpha\beta}\nabla_\alpha\nabla_\beta R + \frac{1}{4}R^2)g_{\mu\nu} \tag{2.7}$$

In a frame where the inflaton field is taken as physical, the first variation of the action with respect to the metric and the field would yield the ordinary Einstein's equations and the Klein-Gordon equation, respectively. Here instead, assuming minimal coupling of matter fields to gravity, the zeroth order modified Eintein's equation becomes: [36] [37]

$$G_{\mu\nu} + \alpha\mathcal{H}_{\mu\nu} = 8\pi G T_{\mu\nu} = 0 \tag{2.8}$$

where the last equality reinforces that we'll be working in a universe with trivial matter content. It is worth noting at this point, once again in comparison to a frame with a physical inflaton field, one would field a potential with a plateau at $\phi \gg b$ given generically by:

$$V(\phi) = V_0\left(1 - e^{-\frac{\phi}{b}}\right)^2 \tag{2.9}$$

where an expansion near $\phi \sim 0$ and identification with the quadratic potential allows us to define an associated mass $V_0 = \frac{1}{2}M^2b^2$. Note the features of the theory – there is a width to the potential, as well as a height of the plateau. Moreover, the way in which one prepares the field, how it exponentially approaches slow-roll, whether it was always in that state or whether the slow-roll is preceded by a period of fast-roll, these are all considerations of the theory. In a framework in which we take seriously gravity-driven inflation, we are left with only one significant physical parameter – the value of α. This scale makes itself evident in its importance in the classical space of initial conditions we consider, that of the $(H, \dot{H})$ plane, where there is an attractor line at $\dot{H} = \frac{1}{36\alpha}$, corresponding to quasi-de Sitter inflationary evolution.

There's some remarks to be made here considering the configuration space of initial conditions. The word "attractor" here is charged; it implies a likelihood of inflation occurring. In addition, it makes it clear that we are not dealing with a canonical phase space in $(H, \dot{H})$, as an attractor in a phase space would violate the Liouville theorem. Converting our configuration space to a proper canonical phase space would have certain advantages, such as allowing us to define a integration measure on the space of initial conditions. With added information, either in the form of an independently-developed wavefunction on the space of initial conditions or more simply taking the assumption of ergodicity – that the universe can explore the space of initial conditions in a uniform manner, giving rise to a random set of initial conditions – it would be possible to argue for the likelihood of inflation from the logic of typicality [38]. We do not make any such arguments for typicality. Instead, we have a principle – cosmic veiling – that bounds from above the number of inflationary e-foldings, and we set out to find initial conditions compatible with this principle. For all such initial conditions, the line at $\dot{H} = -1/36\alpha$ is an attractor.

Since inflationary solutions must go through this attractor line, we require only one other parameter to specify an initial condition. The only relevant scale in the theory is the evolutionary scale, in terms of length that is $1/\sqrt{\alpha}$, so we will posit the transition scale $H_c = \frac{\mu}{\sqrt{\alpha}}$, where we will see observations exclude $\mu < 1.8$.

Working with a metric that is FLRW with flat spatial sections,

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 \tag{2.10}$$

we obtain modified Friedmann and Raychaudhuri equations:

$$H^2 + 6\alpha(6H^2\dot{H} - \dot{H}^2 + 2H\ddot{H}) = 0 \tag{2.11}$$

$$3H^2 + 2\dot{H} + 6\alpha(18H^2\dot{H} + 9\dot{H}^2 + 12H\ddot{H} + 2\dddot{H}) = 0 \tag{2.12}$$

where $H(t)$ is the Hubble parameter defined to be $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$. The Raychaudhuri equation, in this instance, is not independent of the Friedmann equation but it is linearly dependent on the Friedmann equation and its first derivative. It is useful to introduce quantities to speak efficiently about slow-roll and e-foldings. We already have one of them, the Hubble parameter, which gives the logarithmic rate of change of the scale factor $a(t)$. We introduce a second parameter, ϵ, which itself determines how quickly $H(t)$ changes:

$$\epsilon_1(t) = -\frac{\dot{H}}{H^2} \tag{2.13}$$

Note that ϵ is dimensionless as $H(t)$ has units of one over time. Here, the Hubble parameter can be thought of as the length scale associated to the extrinsic curvature of our sections. These sections change in time, so the Hubble parameter is the time scale over which it changes, or if we add in the speed of light c, the length scale. We can define this length scale, the Hubble radius, as $l_H(t) = \frac{1}{H(t)}$, and look at its rate of change $\dot{l}_H$, which is dimensionless:

$$\dot{l}_H = \frac{d}{dt}\frac{1}{H(t)} = -\frac{\dot{H}}{H^2} = \epsilon \tag{2.14}$$

So if we ask how quickly the Hubble radius changes, it's exactly ϵ_1! Moreover, it's clear that the Hubble rate measures a velocity, $\dot{a}(t) = a(t)H(t)$, with $H(t) > 0$ leading to expansion as $a(t)$ by definition is positive. If we take the definition of ϵ_1 and solve for $\ddot{a}(t)$, we find the condition for accelerating expansion:

$$\ddot{a} = (1 - \epsilon_1)aH^2 \tag{2.15}$$

So to check if the expansion is accelerating or decelerating, simply check whether ϵ_1 is bigger or smaller than 1, with a smaller value leading to acceleration. The quantity that is really relevant when we talk about how the scale factor changes is its logarithmic change, which we can give a name to, the number of e-foldings $N(t)$, which we can write by either taking a reference scale factor and take a ratio or conversely use to write $a(t)$ as a number of e-folds from that reference scale:

$$N(t) = \log\frac{a(t)}{a_0} \quad \text{or} \quad a(t) = e^{N(t)}a_0 \tag{2.16}$$

One of the nice things that comes from this definition is the relationship of the number of e-foldings to the Hubble rate, where $H(t)$ turns out to be precisely $\dot{N}(t)$, or allowing us to write e-folds as an integral in $H(t)$:

$$N(t) = \int_{t_0}^{t} H(t')dt' \tag{2.17}$$

So long as $H(t)$ remains positive, that is, we live in an expanding universe, we can ask how long time has passed in terms of number of e-foldings. Keen-eyed readers will have noticed that we've chosen to index the slow-roll parameter; it will be useful for us to introduce further Hubble-flow parameters once we begin speaking about the equation of motion of the perturbations. Each slow-roll parameter is recursive, each with a logarithmic derivative in time, made dimensionless by introducing the only relevant time scale $H(t)$:

$$\epsilon_1(t) = -\frac{\dot{H}}{H^2}, \qquad \epsilon_2(t) = \frac{\dot{\epsilon}_1}{H\epsilon_1}, \qquad \epsilon_3(t) = \frac{\dot{\epsilon}_2}{H\epsilon_2}, \qquad ... \tag{2.18}$$

In the literature, one often finds a second-order slow-roll parameter, $\delta = \frac{\ddot{H}}{\dot{H}H}$ that further characterizes the slow-roll of the inflaton in a given potential. The relationship between this parameter and our Hubble flow parameters is given by:

$$\epsilon_2(t) = 2\epsilon_1(t) + \delta(t) \tag{2.19}$$

Notice that whenever $\delta(t)$ is negligible, that is, $\ddot{H}$ is small, or the jerk of the scale factor is small, both Hubble flow parameters are roughly of the same order. Indeed, assuming we are in a regime in which each progressive derivative of $H(t)$ is smaller than the previous one, we will find that every Hubble-flow parameter is of the same order of magnitude. This will be useful for us in expanding our perturbation equation of motion at a given order. For now, it is sufficient to know that, in the background, we will use $\epsilon_1 < 1$ as the definition of an inflationary phase, which we distinguish from a more specific quasi-de Sitter phase in which $|\epsilon_i| \ll 1$ for all i, so that the Hubble rate is approximately constant. It can be intructive to re-express the Friedmann equation in terms of Hubble flow parameters, where we have placed all non-Einstein-Hilbert terms on the left, and the Einstein-Hilbert contribution on the right:

$$\left(1 - \frac{1}{2}\epsilon_1 + \frac{1}{3}\epsilon_2\right) 36\epsilon_1\alpha H^2 = 1. \tag{2.20}$$

Now given all these tools, we can identify 3 separate regimes in the Friedmann equations. We will visit them one by one, starting with the quasi-de Sitter phase, looking at how it ends gracefully into an oscillatory, reheating phase, and finally push the bounds of quantum field theory in curved spacetime and push further into the past, looking for the behavior of the pre-inflationary phase.

2.2 The quasi-de Sitter phase

To identify the quasi-de Sitter phase, or $R + \alpha R^2$ phase, we will take all the ϵ_i in the Friedmann equation to be small and of the same order, so that $\mathcal{O}(\epsilon_i) = \mathcal{O}(\epsilon_1)$. Neglecting terms of order $\mathcal{O}(\epsilon_1^2)$, we find:

$$36\epsilon_1 \alpha H^2 = 1 \tag{2.21}$$

This yields a simple equation of motion:

$$\dot{H} \approx -\frac{1}{36\alpha} \quad \Rightarrow H(t) \approx H(t_0) - \frac{1}{36\alpha}(t - t_0) \tag{2.22}$$

where t_0 is some reference time. In this phase, $\ddot{H}$ is negligible, so the slow-roll parameter δ is of higher order, and the relation reduces to $\epsilon_2 \approx \epsilon_1$.

Given the totality of the Friedmann equations, one can describe the classical trajectories on the $(H, \dot{H})$ configuration space by looking at the vector field flow of the Friedmann equation:

$$V_{\text{Friedmann}} = \left(\dot{H}, \frac{-36\alpha H^2 \dot{H} + 6\alpha \dot{H}^2 - H^2}{12\alpha H}\right) \tag{2.23}$$

Figure 2.1 shows a clear attractor line to which classical paths converge. This corresponds to a solution in which $\ddot{H} = 0$, which in turn allows us to solve the Friedmann equation for $\dot{H}$:

$$\dot{H} = \frac{18\alpha H^2 - \sqrt{6}\sqrt{54\alpha^2 H^4 + \alpha H^2}}{6\alpha}, \tag{2.24}$$

When expanded in series around $H \to \infty$, we find $\dot{H} \to -\frac{1}{36\alpha}$, confirming the quasi-de Sitter phase is defined entirely in terms of this attractor line. Thus, any trajectory that undergoes slow-roll inflation must cross this attractor line, which in turn is defined by the sole dimensionful parameter of the theory, α. In order to identify an initial condition in this space, we require two numbers, presumably $H, \dot{H}$. The latter can already be set this this attractor line – a requirement that any trajectory we investigate must be slow-roll inflationary. We then posit an additional relevant scale, written in terms of the only scale of the theory, which marks a transition scale from inflationary to pre-inflationary

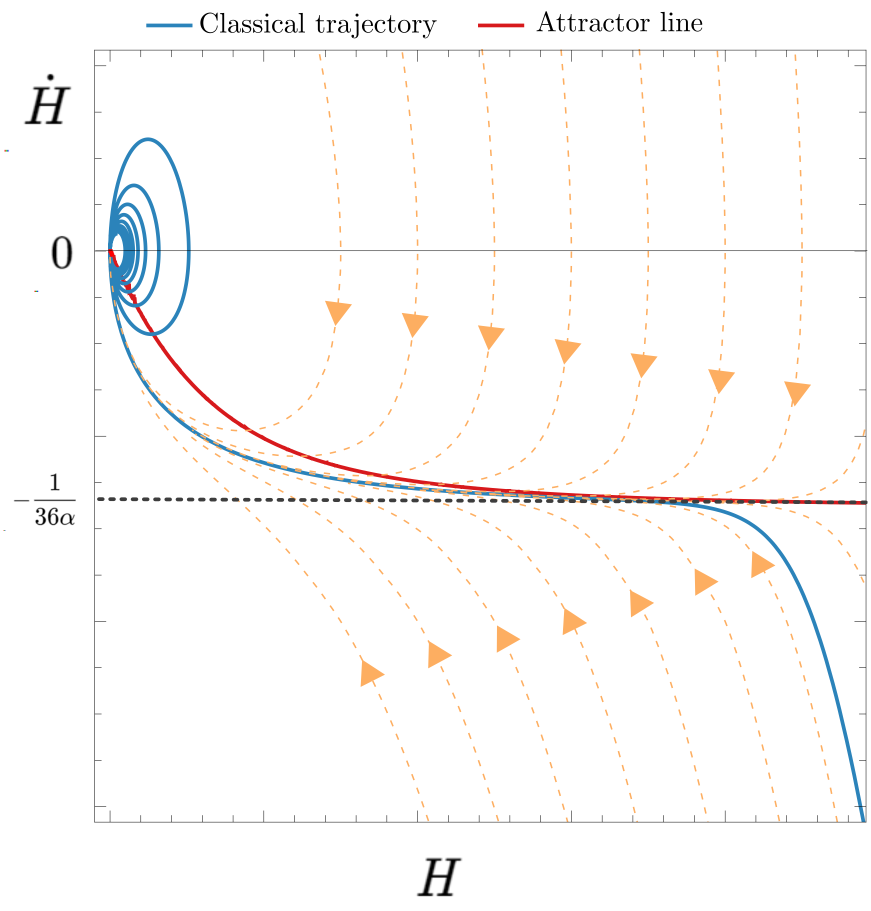

Figure 2.1. The vector field flow of classical solutions to the Friedmann equation. Shown in red is the attractor line. One specific solution (blue) is shown in totality, evolving into the reheating phase.

behavior. We call the quantity "H-cross", $H_c \equiv \frac{\mu}{\sqrt{\alpha}}$, where μ is an undetermined parameter of order unity. Note that this choice of initial conditions, without introducing an additional scale, selects only trajectories that approach the attractor line from below. While we won't directly consider trajectories that approach from above, note that some of these correspond to universes which have an instantaneous Minkowski-like point at $H = 0, \dot{H} = 0$, thought with nontrivial $\ddot{H}$, and which either become more and more quantum in nature or feature a non-quantum bounce where the space-time curvature reaches a maximum before going back to a quasi de Sitter regime.

Using this set of initial conditions as our pinpoint where our classical trajectory must pass through, we can calculate the total number of inflationary efoldings for a generic trajectory. Identifying the Hubble rate and time of crossing H_c, $t_\times$ as the integration constants H_0,t_0 in the equation governing slow-roll inflation, and defining the end of inflation to occur when $\epsilon_1(t_{\mathrm{end}}) \equiv 1$, the expression for inflationary efoldings is:

$$N_{\mathrm{infl}} = \int_{t_\times}^{t_{\mathrm{end}}} H(t)dt = 18H_c^2\alpha \tag{2.25}$$

There is a requisite minimal number of efoldings in order to match observations – a time interval of long enough duration is required for all observable modes k to be oscillatory at early times – that informs the value of μ. Assuming this minimal number of around 60 efolds and substituting in H_c, we find $\mu > 1.8$. We will see that, in order to observe pre-inflationary modifications to the power spectrum that are compatible with observation, μ cannot be much larger than 1.8.

2.3 The reheating phase

Reheating, or the $R + \alpha R^2 +$ matter phase, occurs at the end of inflation. Referring back to the Friedmann equation, reheating occurs at a time when the Hubble-flow parameters are no longer small and instead become comparable to unity and then larger. During this phase, the sign of $\dot{H}$ changes, so ϵ_1 oscillates between being positive and negative, and goes to infinity at times when H goes to zero. In this theory, absent of matter, this leads to oscillations that are no longer well-defined by only two initial conditions, as the origin of the $(H, \dot{H})$ plane is degenerate, and requires an additional input of $\ddot{H}$ to specify the behavior of the trajectory. In addition, since $H, \dot{H}$ are not canonically conjugate to each other, we should not expect that the Louisville flow volume will be conserved by classical trajectories. When one introduces matter into this framework, the expectation is that the coupling of matter to $R + \alpha R^2$ gravity will energize matter sectors, leading to a nontrivial energy density content and curing the degeneracy of the oscillations.

During reheating, the Friedmann equation is approximately given by:

$$\left(\frac{1}{2}\epsilon_1 + \frac{1}{3}\epsilon_2\right) 36\epsilon_1\alpha H^2 = 1 \tag{2.26}$$

This can be straightforwardly be solved for $H(t)$, yielding:

$$H(t) = C\cos^2\left(\frac{t - t_{\mathrm{reheat}}}{\sqrt{24\alpha}}\right) \tag{2.27}$$

It is possible to improve the approximation further by keeping the initially neglected term and plugging back into the full Friedmann equation with the ansatz $H(t) =$

$f(t)\cos^2\left(\frac{t-t_{\rm reheat}}{\sqrt{24\alpha}}\right)$, where $f(t)$ is an unspecified damping function. Under the condition that the damping is slow, we find $f(t) = \left(1+\frac{1}{8}\sin\left(\frac{t-t_{\rm reheat}}{\sqrt{6\alpha}} + \frac{1}{8\sqrt{6\alpha}}\left(t - t_{\rm reheat}\right)\right)\right)^{-1}$.

There is a separate story about matter during this period of time – how it becomes thermalized, changes in time, how the particle zoo of the Standard Model drop out the relativistic primordial gas according to their masses, however, that is not the focus of this thesis. Instead, we'll study perturbations during inflation, and how those perturbations become imprinted in the CMB.

2.4 The pre-inflationary phase

If we rewind back far enough in time, during a pre-inflationary phase of the universe, the curvature of the universe becomes large enough to neglect the Einstein-Hilbert term in the action, leaving behind a pure curvature squared action. This action is scale invariant, and so we'll see the scale α drop out from out equations of motion. As a consequence of one of our key assumptions – that there exists a separation of scales from the Planck era to the R^2 era to the Einstein-Hilbert and Λ-dominated era – is that this curvature squared, or scale-free phase of the universe is well described by quadratic terms in the action without requiring higher derivative terms and in which the framework of quantum field theory in curved space-time is still an effective perturbative theory that accurately describes the universe.

We're interested in determining the behavior of $H(t)$ during this pre-inflationary era, building toward describing at timeline of events from the very earliest non-Planckian times. During this epoch, the magnitude of the curvature is sufficiently large to disregard the R term in the Starobinsky action, and so we find the following Friedmann equation for the pre-inflationary dynamics:

$$\left(1-\frac{1}{2}\epsilon_1+\frac{1}{3}\epsilon_2\right)\epsilon_1 H^2 = 0 \quad or \quad 6H^2\dot{H} - \dot{H}^2 + 2H\ddot{H} = 0 \tag{2.28}$$

We can solve the equation of motion in a couple of steps. In this case, it's helpful to use a change of variables $H = w^2$ and integrate once to get an exact first order differential equation:

$$4w^3(3w^2\dot{w}+\ddot{w}) = 0 \qquad \Rightarrow \qquad 3w^2 + \frac{\ddot{w}}{\dot{w}} = 3w^2 + \frac{d\dot{w}}{dw} = 0 \tag{2.29}$$

$$\Rightarrow \qquad \dot{w} = -w^3 + \text{const}\,. \tag{2.30}$$

The first order equation, with restored H and $\dot{H}$ variables becomes:

$$\dot{H} = -2\sqrt{H}(H^{3/2} - \text{const}^{3/2}) \tag{2.31}$$

The constant term gives the value of $\dot{H}$ when H is 0. Given that there's an attractor line at $\dot{H} = \frac{1}{36\alpha}$ that any such pre-inflationary trajectory must cross from below, this solution won't in practice ever reach the horizontal axis. However, the value of α we'll determine is around 10^{10} in Planck units, so a clever selection of the constant of integration can closely approximate the time when the pre-inflationary and quasi-de Sitter solutions cross. For this reason, and since the total number of inflationary efoldings is one of our parameters, it makes sense to set const $= H_c$.

$$\dot{H} = -2\sqrt{H}(H^{3/2} - H_\times^{3/2}) \tag{2.32}$$

We can further solve the remaining first order equation by approximation. We can rewrite our equation by defining the parameter $\xi(t) = \frac{H_c}{H(t)}$, we can integrate once more:

$$\dot{\xi}(t) = 2H_c\,(1 - \xi(t)^{3/2}) \quad \Rightarrow \quad \frac{d\xi}{2H_c\,(1 - \xi(t)^{3/2})} = dt \tag{2.33}$$

and so in the regime in which $\xi \ll 1$, we can integrate once more to find the approximate solution:

$$t(\xi) \approx \frac{\xi}{2H_c} + \frac{\xi^{5/2}}{5H_c} \tag{2.34}$$

Once you revert back to H variable, it's clear that a lowest order solution is $H(t) = \frac{1}{2t}$. We can find the next order solution by positing an ansatz with a constant and square root term, $H_{\text{ansatz}} = \frac{1}{2t} + a_1 + a_2 t^{1/2}$. We then make sure the ansatz is self-consistent, by plugging back into Equation 2.34 and expanding out in a series, and ensuring that the series is equal to t order by order. Doing so imposes a consistency relation which sets the coefficients a_1 and a_2:

$$a_1 = 0 \tag{2.35}$$

$$-2a_2 + \frac{4}{5}\sqrt{2}H_c^{3/2} = 0 \quad \Rightarrow \quad a_2 = \frac{2}{5}\sqrt{2}H_c^{3/2} \tag{2.36}$$

This gives our approximate equation for the Hubble parameter during the pre-inflationary era:

$$H(t) \approx \frac{1}{2t} + \frac{2}{5}\sqrt{2t}H_c^{3/2} \tag{2.37}$$

Notice that, akin to that of a cosmological radiation-dominated phase, at lowest order $H(t) = \frac{1}{2t}$. It's important to note that the pre-inflationary era is not radiation dominated, after all the matter sector at this pre-Hot Big Bang time is trivial. This era is merely radiation-like in the sense that the expansion of the universe is akin to that of FLRW with dominant radiation sector, but in this case, it is purely driven by higher curvature gravity. In cosmological backgrounds with matter, one can write the energy density and pressure in terms of kinetic and potential energy:

$$\rho(t) = K(t) + V(t) \tag{2.38}$$

$$P(t) = K(t) - V(t) \tag{2.39}$$

Speficically, radiation dominated backgrounds can be identified by the equation of state $P = \frac{1}{3}\rho$. Putting together both of these ideas, we find that radiation, and thus radiation-like backgrounds, can be thought of as being kinetic dominated:

$$K = 2V \tag{2.40}$$

This idea is borne out in our gravity-driven background, by looking at the first Hubble flow parameter:

$$\epsilon_1(t) = 2\left(1 - \left(\frac{H_c}{H(t)}\right)^{3/2}\right) \tag{2.41}$$

Recall that $1/H(t)$ gives us a length scale, so in the Planck limit where $H(t)$ is large, we find $\epsilon_1 \to 2$. As it becomes smaller, ϵ_1 becomes smaller as well, eventually becoming unity and starting inflation, though at this time it is a fast-roll. At some point, $H(t)$ comes close enough to H_c that the quasi-de Sitter approximation takes over, and we officially start slow-roll inflation. In this model, pre-inflationary classical solutions of this kind enter inflation gracefully. Figure 2.2 gives us an idea of the timeline of events for a typical trajectory. Worth noting is that the onset of inflation occurs *prior* to the crossing time. That is, the onset where $\epsilon = 1$ that signals accelerating expansion occurs before we formally can describe the trajectory in terms of the inflationary, quasi-de Sitter evolution. However, this accelerated expansion is fast-roll, so not a lot of e-foldings develop, and the bulk of inflationary e-foldings occur during the quasi-de Sitter, slow-roll evolution after the time t_{cross}.

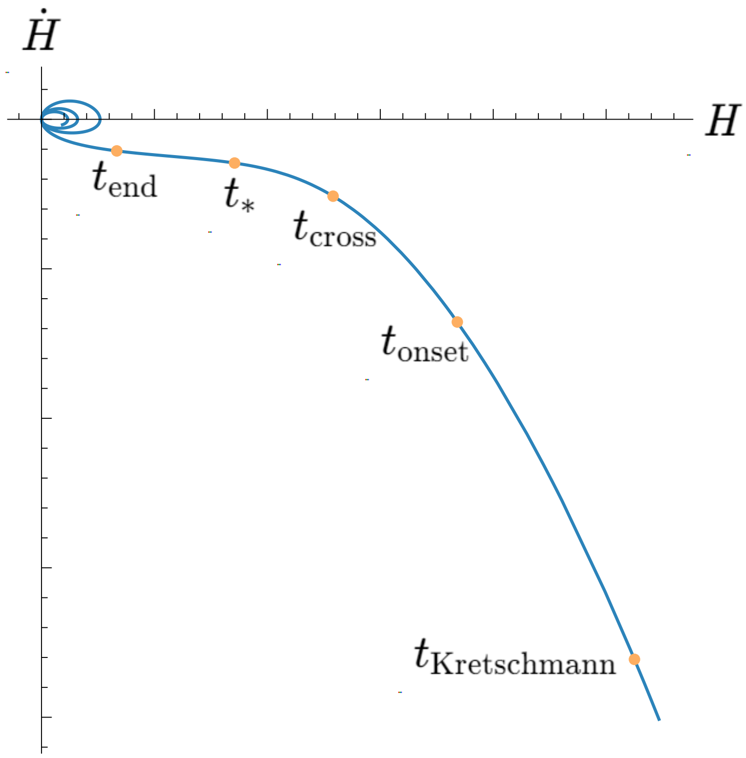

Figure 2.2. Shown is a typical trajectory evolving in time and major milestones of its evolution. Formal solutions are evolved from the Kretschmann time, though there must exist a past-facing extension towards the initial singularity. The onset of inflation occurs first, when $\epsilon = 1$ the first time during evolution, followed by the crossing time when the pre-inflationary and inflationary solutions meet. Sometime later, the pivot mode k_* crosses the Hubble horizon at time t_*, and finally inflation ends, leading to the oscillatory behavior of reheating.

2.5 The quantum gravity threshold

How far into the past can we extend our solution? Certainly, we should not trust our perturbative field theory at the Planck scale, where quantum gravity effects are expected to be significant. While the boundary where quantum field theory in curved space-time framework ceases to accurately describe the physics of the system may be fuzzy, we can at least set a hard boundary using the Kretschmann curvature scalar, as curvature is the relevant physical scale that best probes Planck effects in our pure gravity theory and Kretschmann is a simple curvature invariant that can be computed in FLRW regardless of any additional details:

$$K_{\text{Planck}} = R_{abcd}R^{abcd} = 12(2H^4 + 2H^2\dot{H} + \dot{H}^2) = 1/\ell_P^4 \tag{2.42}$$

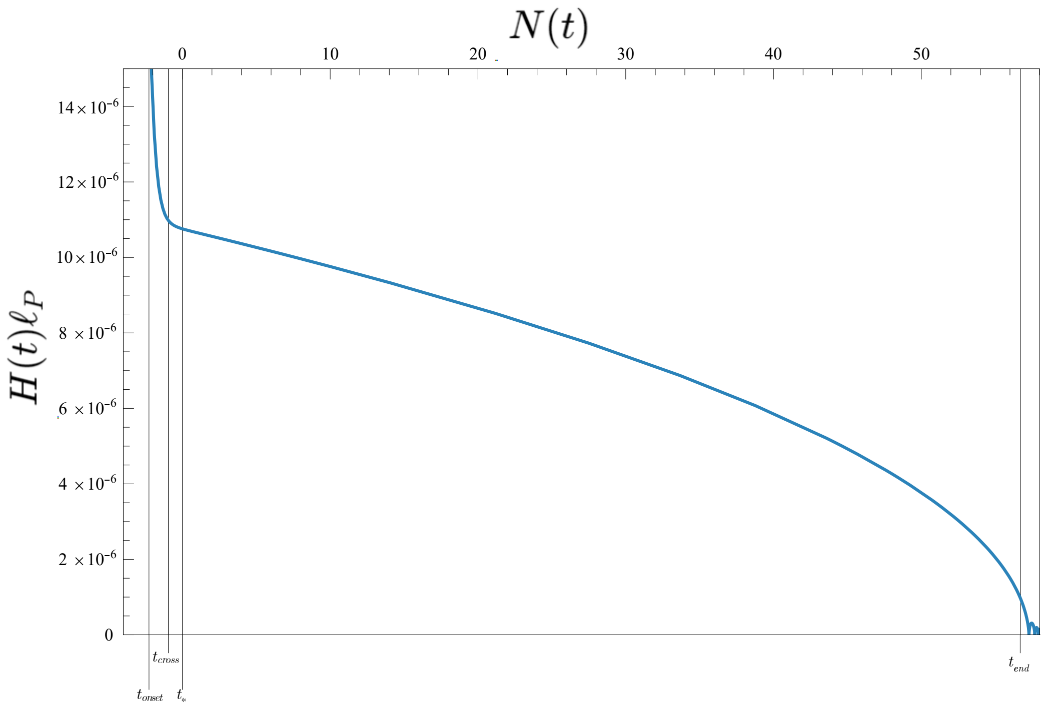

Figure 2.3. Evolution of the Hubble factor $H(N)$ across these 3 regimes. A sharp "knee" indicates the transition from the pre-inflationary to quasi-de Sitter phase, and oscillations mark the beginning of reheating. Scale of Hubble factor given in units of Planck. The zero of e-foldings is set at the time t_*.

This boundary, set by ℓ_P, defines a cardiod-like shape in the $(H, \dot{H})$ plane. The choice of our initial condition, namely the selection of a given H_c with the selection of a trajectory that crosses the attractor line from below, precludes us from initial conditions in the upper plane of the space. An additional aggravating factor is that some upper plane initial conditions will actually first exit the cardiod, and so we cannot take seriously any trajectory that first heads into the quantum gravity regime before entering the cardiod at a later time. Similarly, we do not explore initial conditions with $H < 0$ initially, that is, universes that are contracting. In order to make one last classification of background trajectories, we need to describe the Kretschmann curve a bit more detail.

The relevant branch of the Kretschmann curve we wish to look at is the one in which $H > 0$ and $\dot{H} < 0$. This branch, near the bottom of the well of the cardiod, is given by:

$$\dot{H} = -\frac{1}{6}\left(6H^2 + \frac{3 - 36H^4\ell_P^4}{\ell_P^2}\right) \tag{2.43}$$

The bottom of the well in this branch can be identified with $H_{\min} = \frac{1}{2^{3/4}3^{1/4}L_{\text{Planck}}}$.

Note that, this branch evaluate at H_{min}, we find $\dot{H} = -\frac{1}{\sqrt{6}l^2}$. Extracting the slow-roll parameter at this point gives $\epsilon = 2$, that is, the bottom of the well corresponds to an exact radiation-like phase! At this point, we can begin talking about the number of inflationary efolds you might generate from a given trajectory. As noted in Figure 2.3 there is a clear change in behavior of $H(t)$ between the pre-inflatioanry and inflationary solution that is sharp - so we can find the intersection of said trajectory and the Kretschmann curve by combining 2.43 and 2.32, with the latter containing H_c. Thus, we can write the number of inflationary efolds from the equation 2.25. The result more instructive when expanded in series around the bottom of the well:

$$N_{\text{slow-roll}} \approx \frac{\sqrt{6}\alpha(6HL_{\text{Planck}} - 2^{1/4}3^{3/4})^{4/3}}{\ell_P^2} \tag{2.44}$$

Notice that at H_{min}, the number of inflationary efoldings is identically zero. That is, the bottom of the well, exact radiation-like solution, corresponds to a trajectory that goes directly from pre-inflationary behavior to reheating! We can take its derivative with respect to H to get a sense of how sensitive the number of efolds is to the chosen trajectory:

$$\frac{dN_{\text{slow-roll}}}{dH} \approx \frac{8\sqrt{6}\alpha(6HL_{\text{Planck}} - 2^{1/4}3^{3/4})^{1/3}}{\ell_P^2} \tag{2.45}$$

The important value to note here is the dependence on α/ℓ_P. Since the scale α of inflation is set around 10^10 that of Planck, one does not need to deviate much from the bottom of the well to generate a lot of efoldings. One final remark: certain trajectories that intersect the Kretschmann curve to the left of the bottom of the well tend to intersect the origin of the $(H, \dot{H})$ plane, and then loop back and approach the attractor line from above. We won't consider these models, ones in which the universe momentarily becomes Minkowski and then goes into an $\epsilon < 0$ inflation phase.

Chapter 3
Quantization and mode equations

3.1 Introduction

Let's briefly outline our strategy, which follows the techniques of effective field theory. [39] There are two general approaches to quantization of the action. On one hand, we could take a gauge invariant approach, and quantize the full kinematical Hilbert space. We'll take the approach of identifying a specific gauge to work in, identify appropriate gauge and Hamiltonian constraints, and look to only to quantize the physical Hilbert space. In order to find this action for tha gravitational perturbations, we'll need to find the second functional derivative of the action. When we first looked at the background, we saw that the first functional derivative with respect to the metric, when taken to be stationary points, yield the equations of motion of the background. Next, we'll look at the second functional derivative to get the equations of motion of the perturbations. In doing so, we'll need to identify gauge conditions, so we'll need to identify the physical degrees of freedom. To do so, we split the metric into an Scalar-Vector-Tensor (SVT) decomposition, which will allow us to identify scalar and tensor degrees of freedom, and their respective actions.

We'll end up solving our equations of motion mode-by-mode, so it'll be useful to speak in the language of Fourier transforms. We declare the following convention, for a given function $f(\vec{x})$ that is square-integrable, that is, $f(\vec{x}) \in L^2(\mathbb{R}^3)$, the Fourier and inverse Fourier transforms are given by:

$$\tilde{f}(\vec{k}) = \int f(\vec{x}) e^{-} - i\vec{k} \cdot \vec{x} d^3\vec{x} \tag{3.1}$$

$$f(\vec{x}) = \int \tilde{f}(\vec{k}) e^{+i\vec{k}\cdot\vec{x}} \frac{d^3\vec{k}}{(2\pi)^3} \tag{3.2}$$

with a compatible Dirac Delta defined by:

$$\int e^{-i\vec{k}\cdot\vec{x}} d^3\vec{x} = (2\pi)^3\delta^{(3)}(\vec{k}) \tag{3.3}$$

This is the Fourier transform that we are used to using in physics, but it's important to keep in mind why we are using it. The Fourier transform can be thought of as a basis of eigenfunctions of the Laplacian:

$$\Delta = \delta^{ij}\partial_i\partial_j, \quad \Delta e^{-i\vec{k}\cdot\vec{x}} = -\vec{k}^2 e^{-i\vec{k}\cdot\vec{x}} \tag{3.4}$$

Since we'll be working in flat spatial slices of FLRW, this Fourier transform is the choice that is adapted to the symmetries in question, namely translations and rotations in 3d Euclidean space. In a general curved space-time, with metric g_{ab} and compatible covariant derivative D_a, the covariant Laplacian doesn't just have δ_{ab}, but contains covariant derivatives:

$$\Delta = g^{ab}\nabla_a\nabla_b = \frac{1}{\sqrt{h}}\partial_a(\sqrt{h}h^{ab}\partial_b) \tag{3.5}$$

One could still ask for the eigenfunctions of the curved Laplacian, but in our case, the symmetry of our background is what allows us to select the ordinary Fourier transform, as opposed to harmonic analysis on spheres or hyperplanes. Thus, instead of working in the full metric $g_{\mu\nu}$ as it doesn't many manifest the symmetries of our space-time, we'll utilize the ADM formalism in order to decompose the 4d metric into a 3d metric plus a lapse and a shift. For the metric, we'll need to do an SVT decomposition; for example, in $\mathbb{R}^3$, one can take a reference vector and decompose any other vector into a part parallel to the reference vector, and thus affixed by a scalar value, and one orthogonal to the reference vector.

We start with the background, written in terms of the ADM ansatz with a lapse and shift:

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{3.6}$$

Starting with the shift, in the FLRW background, it's identically 0, so we can begin by writing out its perturbation. As the shift is a vector, it can be decomposed into a portion in the direction of $\vec{k}$ and a portion that is transverse, or in other words, a divergence-less and an irrotational part:

$$\delta N^\mu(\vec{k}, t) = \varepsilon(ik^\mu S(\vec{k}, t) + N^\mu_\perp(\vec{k}, t)) \tag{3.7}$$

with $\delta_{\mu\nu} k^\mu N_\perp^\nu(\vec{k}, t) = 0$. Note that we'll raise and lower indices with the background metric $\delta_{\mu\nu}$, since we think of this framework as a perturbative expansion and the perturbation would introduce higher orders than in the free theory. If you reverse the Fourier transform, it can be written in terms of a derivative of a scalar plus a vector with zero divergence:

$$\delta N_\mu(\vec{x}, t) = \varepsilon(\partial_\mu S(\vec{x}, t) + \delta N_{\mu_\perp}(\vec{x}, t)) \tag{3.8}$$

The perturbation of the scalar lapse is simple to write at first order:

$$N = 1 + \varepsilon \delta N \tag{3.9}$$

Note that we won't be able to set δN or δN^a to zero since they are Lagrange multipliers for the contraint equations and were we to set them to zero, we would miss equations.

We required the coordinates because the perturbation depends on the coordinates we use. The next step is to decompose the ADM metric into Scalar-Vector-Tensor (SVT) contributions. Consider the metric h_{ab}, which is the 3d metric tensor under 3d diffeomorphisms of the FLRW slices. The SVT decomposition of this metric then refers to ISO(3), all translations, reflections and rotations on 3d Euclidean space. Explicitly, we write the perturbation as:

$$h_{ab}(\vec{x}, t) = a(t)^2 \delta_{ab} + \delta h_{ab}(\vec{x}, t) \tag{3.10}$$

We can then write the decomposition of the perturbation h_{ab} with respect to the background metric δ_{ab}. To do so, we'll go to Fourier transform as it becomes easier to separate out derivatives. We ask: how can we decompose a tensor? We can write a tensor out in terms of a scalar, in this case the trace of the metric, times δ_{ab}, but we can also write a scalar since two momenta k_a, k_b can also give a tensor. We can also write a tensor in terms of a vector $k_a B_b$ where we ask the vector B_b to be transverse. Finally, the only thing that's left is something that's genuinely a tensor that is transverse and traceless. In other words, we have:

$$\begin{aligned}
\delta h_{ab}(\vec{k}, t) = & - 2\mathcal{R}(\vec{k}, t) a^2(t) \delta_{ab} && (3.11)\\
& - k_a k_b C(\vec{k}, t) && (3.12)\\
& + a(t)\left(i k_a B_b(\vec{k}, t) + i k_b B_a(\vec{k}, t)\right) && (3.13)\\
& + a(t)^2 \gamma_{ab}(\vec{k}, t) && (3.14)
\end{aligned}$$

with:

$$\text{Scalars: } \mathcal{R}, C \tag{3.15}$$

$$\text{Transverse vector: } B_a \text{ with } k^a B_a = 0 \tag{3.16}$$

$$\text{Transverse traceless tensor: } \gamma_{ab} \text{ with } k^a \gamma_{ab} \text{ and } \delta^{ab}\gamma_{ab} = 0 \tag{3.17}$$

In components where k points in the z direction, these are:

$$\vec{k} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}, \quad B_a = \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix}, \quad \gamma_{ab} = \begin{pmatrix} \gamma_+ & \gamma_\times & 0 \\ \gamma_\times & \gamma_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.18}$$

In this form it's straightforward to count: we have two scalars, two vector components, and two tensor components; six total components split into 3 groups. The trace $R(\vec{k}, t)$ has a traditional name – the curvature perturbation – while the rest don't, as we'll see them disappear once we impose constraints. Now written in position space, including the small parameter ε to track order:

$$\begin{aligned} h_{ab}(\vec{x}, t)dx^a dx^b &= \left(a(t)^2\delta_{ab} + \varepsilon\delta_{ab}\right) dx^a dx^b && (3.19)\\ &= a(t)^2(1 - 2\varepsilon\mathcal{R}(\vec{x}, t))\delta_{ab}dx^a dx^b && (3.20)\\ &+ \varepsilon\partial_a\partial_b C(\vec{x}, t)dx^a dx^b && (3.21)\\ &+ \varepsilon a(t)(\partial_a B_b(\vec{x}, t) + \partial_b B_a(\vec{x}, t))dx^a dx^b && (3.22)\\ &+ \varepsilon a(t)^2\gamma_{ab}(\vec{x}, t)dx^a dx^b && (3.23) \end{aligned}$$

Note that, if you want to track where the scale factors should be, x is defined to be a coordinate, so $a(t)dx$ is a physical length, and so when we take a derivative, we get a $\frac{1}{a(t)}\frac{\partial}{\partial x^a}$.

Let's make sense of the trace $\mathcal{R}(\vec{x}, t)$ which we called the curvature perturbation. Using this metric, we can compute the 3d Ricci tensor:

$${}^{(3)}R_{ab} = \varepsilon\left(\delta^{ab} + \partial_a\partial_b\right)\mathcal{R}(\vec{x}, t) - \varepsilon\frac{1}{2}\Delta\gamma_{ab}(\vec{x}, t) \text{ where } \Delta = \delta^{ab}\partial_a\partial_b \tag{3.24}$$

As expected given the flat FLRW background, at zeroth order in ε, our spatial sections are flat. The functions $C(\vec{x}, t)$ and $B_a(\vec{x}, t)$ don't contribute at linear order, and the tensorial part of the Ricci tensor is encoded entirely in $\Delta\gamma_{ab}(\vec{x}, t)$. When we further take the trace of the Ricci tensor, we find that the Ricci salar is the Laplacian of the curvature perturbation with a coefficient

$$^{(3)}R = h^{ab(3)}R_{ab} = \varepsilon \frac{4}{a(t)^2} \Delta \mathcal{R}(\vec{x}, t) \tag{3.25}$$

The curvature perturbation is not directly the physical perturbation, but it is directly related to the Ricci scalar. When we compute the power spectrum, it'll be the fluctuations of this quantity that we'll calculate and see imprinted in the CMB with the amplitude 10^{-5}.

We can already count out the vector contribution to the perturbations. From the shift we get a vector perturbation which is split into a scalar part plus a transverse part. We similarly get a transverse vector contribution from the perturbation of the metric tensor. In the Hamiltonian constraint, there is a vector constraint which can itself be written perturbatively in terms of a scalar part and a transverse part. The only other constraint, from gauge fixing, also provides a vector contribution similarly split. Thus, counting dynamical degrees of freedom, we find for vectors, the two perturbations are exactly constrained by two equations, meaning that there's no vector contribution to the perturbations.

3.2 The action at quadratic order, gauge fixing and constaints

Now that we've identified the degrees of freedom of the perturbations, we identify which of those are dynamical. The plan will be to fix a gauge, solve the constraints, and then write the action just for the cosomological perturbations which remain and are physical. Note that, when we use ε to track order, and then Taylor series and truncate the action at order 2, the free quadratic action will have ε set to 1. The average perturbation should be zero, but has square root variance, which has to be small compared to the background to ensure self-consistency.

It'll be useful to introduce the auxiliary field $\chi(\vec{x}, t)$:

$$\chi(\vec{x}, t) = 1 + 2\alpha R(\vec{x}, t) \tag{3.26}$$

Our action rewritten in terms of this auxiliary field becomes:

$$S[g_{\mu\nu}] = \frac{1}{16\pi G}\int (R+\alpha R^2)\sqrt{-g}d^4x \Rightarrow S[g_{\mu\nu},\chi] = \frac{1}{16\pi G}\int (\chi R - \frac{1}{4\alpha}(\chi-1)^2)\sqrt{-g}d^4x \tag{3.27}$$

We'll consider $R(\vec{x},t)$ and $\chi(\vec{x},t)$ to be independent from each other as we vary the action. We'll take once again a variation of the full metric, where we will use a bar to denote the background quantities:

$$g_{\mu\nu}(x) \equiv \bar{g}_{\mu\nu}(x) + \varepsilon\delta g_{\mu\nu}(x) \tag{3.28}$$

$$\chi(x) = \bar{\chi}(x) + \varepsilon\delta\chi(x) \tag{3.29}$$

We'll also require raising indices, thus we'll need the inverse metric. Note that, when we invert the metric, we are inverting the previous definitions, so we cannot define the former and say that the latter is purely linear as there must corrections:

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \varepsilon\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\delta g_{\alpha\beta} + \mathcal{O}(\varepsilon^2) \tag{3.30}$$

Given an action, the equations of motion are stationary points of the action. As before, we can define the following functional derivatives with respect to the metric $g_{\mu\nu}$ and auxiliary field χ:

$$F^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}} \quad K = \frac{\delta S}{\delta\chi} \tag{3.31}$$

When you impose equality to zero, one should recover the previous modified Friedmann equations of motion once χ has been substituted in. To go to next order, we vary a second time by taking once more the functional variation of the first derivative. Written out symbolically, we're looking to expand the action in a series in ε:

$$S[\bar{g}_{\mu\nu} + \varepsilon\delta g_{\mu\nu}, \bar{\chi} + \varepsilon\delta\chi] = S[\bar{g}_{\mu\nu},\bar{\chi}] + \varepsilon S_1[\delta g_{\mu\nu},\delta\chi] + \varepsilon^2 S_2[\delta g_{\mu\nu},\delta\phi] + \mathcal{O}(\varepsilon^3) \tag{3.32}$$

At order zero, we simply get the background action, which doesn't contribute to the perturbations as it's constant with respect to the perturbations. The linear order term is zero because we assume the background is a classical solution to the equations of motion, and thus provides a stationary point:

$$\varepsilon S_1[\delta g_{\mu\nu},\delta\chi] = \int \left(\varepsilon\frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \epsilon\frac{\delta S}{\delta\chi}\delta\chi\right) d^4x\bigg|_{\text{background}} \tag{3.33}$$

$$= \int (\varepsilon F^{\mu\nu}\delta g_{\mu\nu} + \epsilon K\delta\chi)\, d^4x\Big|_{\text{background}} \tag{3.34}$$

$$= \int \left(\varepsilon \bar{F}^{\mu\nu}\delta g_{\mu\nu} + \epsilon \bar{K}\delta\chi\right) d^4x \tag{3.35}$$

The goal is to compute the second order free action for the perturbations:

$$\varepsilon^2 S_2[\delta g_{\mu\nu}, \delta\chi] = \int \left(\frac{1}{2}\varepsilon^2 F_1^{\mu\nu}\delta g_{\mu\nu} + \frac{1}{2}\varepsilon^2 K_1\delta\chi\right) d^4x \tag{3.36}$$

where we define the additional functional derivatives F_1 and K_1 by expanding these in terms of ε around the background:

$$F^{\mu\nu}[\bar{g}_{\mu\nu} + \varepsilon\delta g_{\mu\nu}, \bar{\chi} + \varepsilon\delta\chi] = 0 + \varepsilon F_1 + \mathcal{O}(\varepsilon^2) \tag{3.37}$$

$$K[\bar{g}_{\mu\nu} + \varepsilon\delta g_{\mu\nu}, \bar{\chi} + \varepsilon\delta\chi] = 0 + \varepsilon K_1 + \mathcal{O}(\varepsilon^2) \tag{3.38}$$

Remember, at order zero, these must evaluate to zero because they correspond to the background functional derivatives $\bar{F}^{\mu\nu}$ and $\bar{K}$. Note that, for the order 2 Lagrangian, we get extra $\delta g_{\mu\nu}$ and $\delta\chi$ as part of the full functional differential of S_1, but now each of F_1 and K_1 themselves are linear in $\delta g_{\mu\nu}$ and $\delta\chi$, so we're going to end up with contributions $\delta g_{\mu\nu}\delta g_{\mu\nu}$, $\delta g_{\mu\nu}\delta\chi$, and $\delta\chi\delta\chi$. We have to take in all contributions are the same order of ε, so now with the mixed term, when we speak about perturbations, we cannot separate the two. Were we to work explicitly in a framework with gravity plus inflaton field, the scalar field of the metric and that of the inflaton would thus appear mixed together at the level of the quadratic action.

Before we simply plug in the variations into the second order Lagrangian, it's worthwhile taking some time to think about what we should expect. Given the SVT decomposition we've done, we have 3 different types of objects, and since our action is quadratic, we have up to 9 possible combinations. However, the action is a scalar quantity, which means that the way we combined objects must also yield a scalar, so for example, we should not expect a scalar-vector coupling to survive. That leaves only 3 possible options: scalar-scalar, vector-vector, and tensor-tensor. Note, if one wishes to investigate the theory at higher orders to account for interactions and non-Gaussianities, one should expect more complicated couplings to emerge. For now, we expect a quadratic action with a scalar-scalar coupling with 5 different scalars, a vector-vector coupling with 2 vectors, and a tensor-tensor coupling with just one tensor perturbation:

$$S_2[\delta g_{\mu\nu}, \delta\chi] = S_2^{(ss)}[\delta h_{\mu\nu}^{(s)}, \delta N, \delta N_\mu^{(s)}, \delta\chi] + S_2^{(vv)}[\delta h_{\mu\nu}^{(v)}, \delta N_\mu^{(v)}] + S_2^{(tt)}[\delta h_{\mu\nu}^{(t)}] \tag{3.39}$$

$$=S_2^{(s)}[\mathcal{R}, C, \delta N, S, \delta\chi] + S_2^{(v)}[B_a, \delta N_\perp^a] + S_2^{(t)}[\gamma_+, \gamma_\times] \tag{3.40}$$

3.3 The comoving gauge and constraints

Theories with gauge symmetries contain redundant degrees of freedom, creating the challenge of uniquely determining the physical state and choice of gauge-fixing conditions. In order to extract predictive results, tailored to the deriving the power spectrum, we will choose the comoving gauge. [36] The action that we start with is invariant under all coordinate transformations; at zeroth order, this is the symmetry:

$$x^\mu \to x^\mu + \xi^\mu(x) \tag{3.41}$$

When we chose the background, we picked cartesian coordinates that were comoving. To understand this notion of comoving, it is easier to first think in terms of the sister space-time with Einstein-Hilbert action plus a scalar field. In this space-time, we think of comoving in the sense that the energy-momentum tensor is simple, depending only on t, and having no energy-flux:

$$\bar{T}_0^0 = -\bar{\rho}(t) \quad \bar{T}_b^a = \bar{P}(t)\delta_b^a \quad \bar{T}_a^0 = 0 \tag{3.42}$$

However, we still have the freedom to write a linear-at-epsilon diffeomorphism, after all, as we're doing everything perturbatively, we can express even the gauge symmetry order by order. We can consider an SVT decomposition of the linear diffeomorphism $\xi^\mu(x)$, such that:

$$t \to t + \varepsilon\xi^0(\vec{x}, t) \tag{3.43}$$

$$x^a \to x^a + \varepsilon\delta^{ab}\left(\xi_b^{(v)}(\vec{x}, t) + \partial_a\xi^{(s)}(\vec{x}, t)\right) \tag{3.44}$$

In words, given an initial flat FLRW slicing of the space-time, we can either deform the spacetime in a timelike manner by putting in a small "bump" in the foliation, or if we deform only with respect to the spatial parts, we preserve the section but deform them in either a rotational, vectorial manner, or in a divergenceless, scalar way. A gauge fixing is then a specific choice of $\xi(\vec{x}, t)$, which can be set individually for both the scalar $\xi^0(\vec{x}, t), \xi^{(s)}(\vec{x}, t)$ and vectorial sectors $\xi^{(v)}(\vec{x}, t)$ separately.

Consider a slicing that is not the initial one but one closely related to it. Instead, choose a slicing that, one you introduce perturbations, re-slices the space-time in a way that it's not the background energy-momentum tensor that has no energy fluxes, but the perturbed one:

$$T^0_a = \bar{T}^0_a + \cancelto{0}{\delta T^0_a} = 0 \tag{3.45}$$

The background slicing already had vanishing flux, so by imposing this additional condition, you select a slicing on the basis of the matter that you have. Now we can more easily envision what this condition means in out purely gravitational space-time. The information of the field is encoded in the energy-momentum tensor, which translates to the Starobinsky tensor H_{ab}. We similarly require that:

$$H^0_a = \bar{H}^0_a + \cancelto{0}{\delta H^0_a} = 0 \tag{3.46}$$

The modified Einstein equations link together R_{ab} and H_{ab}, and indeed, our gauge choice is equivalent to $\delta\chi = \delta(1 + 2\alpha R(\vec{x}, t)) = 0 \Rightarrow \delta R = 0$. That is, given an initial foliation such that there were no Ricci curvature fluctuations, we choose a new foliation such that the slicing is comoving in the sense that the are still no Ricci curvature fluctuations.

A good choice of gauge makes asking the questions we're interested in a simpler task. This choice – that the curvature of the spatial sections is such that there's no Ricci curvature fluctuation – is not enough to completely fix the gauge. In addition, we'll choose $C = 0$, which won't affect the final result since our questions have answers that are independent of C, so this will solely simplify the work.

Next we have constraints, which are essentially the 00 and $0i$ components of the Einstein equations. Recall that $F^{\mu}_{(1)\nu}$ gives Einstein's equations, which are second order equations. However, some components, such as the space-space or the time-space ones are first order, so they are constraints for the initial data. If you write them in the language of the variational tensor $F^{\mu\nu}$, we have a scalar, Hamiltonian constraint, as well as a vectorial, 3d-diffeomorphism constraint, which can itself be decomposed into a scalar and transverse vector form:

$$0 = \mathcal{H} = F^0_0 \tag{3.47}$$

$$0 = \mathcal{H}_a = F^0_a \quad \rightarrow \quad \mathcal{H}^{(s)}_a = 0 \;\text{ and }\; \mathcal{H}^{(v)}_a = 0 \tag{3.48}$$

The last step is then to take the action and evaluate it on the solution of the gauge fixing and constraint equations. Whatever is left is then the reduced action for the dynamical degrees of freedom. We begin with the scalar sector. Beginning with the action at quadratic order for the scalar part, we have:

$$S_2^{(s)}[\mathcal{R}, C, \delta N, S, \delta\chi] = \int \left(\frac{1}{2} h^{(s)}_{\mu\nu} F^{\mu\nu}_{(1)} + \frac{1}{2}\delta\chi K_{(1)} \right) d^4x \tag{3.49}$$

where we will use the scalar part of the SVT-decomposed metric:

$$g_{\mu\nu} = \left(\begin{array}{c|c} -1 - 2\epsilon\delta N(t, \vec{x}) & \epsilon\partial_i S(t, \vec{x}) \\ \hline \epsilon\partial_i S(t, \vec{x}) & a(t)^2(1 - 2\epsilon\mathcal{R}(t, \vec{x})) \end{array} \right) \tag{3.50}$$

Now choose the surface deformation and 3d diffeomorphism such that the FLRW sections have a scalar component of the metric with $C = 0$ and vanishing Ricci scalar fluctuation. Next, we put the scalar part of the SVT decomposed metric and plug into the diffeomorphism constraint:

$${}^{(1)}F^0_i = G^0_i + \alpha H^0_i = 0, \tag{3.51}$$

where we can find the solution, written in Fourier transform, gives the perturbation of the lapse proportional to the time derivative of the curvature perturbation:

$$\delta N(\vec{k}, t) = -\frac{2\chi(t)\dot{\mathcal{R}}(\vec{k}, t)}{2H(t)\chi(t) + \dot{\chi}(t)} \tag{3.52}$$

Note that this isn't a choice – it is a consequence of of the selection of our physical variables. If we initially slice the space-time flat, and then fix the gauge, this constraint tells you how to re-slice correctly such that there's no Ricci scalar perturbation. Now we can consider the Hamiltonian constraint:

$${}^{(1)}F^0_0 = G^0_0 + \alpha H^0_0 = 0, \tag{3.53}$$

After passing to Fourier space, we find a condition of the scalar $S(\vec{k}, t)$, which comes from the shift:

$$S(\vec{k}, t) = \frac{4k^2 H(t)\mathcal{R}(\vec{k}, t)\chi(t)^2 + \dot{\chi}(t)(2k^2\mathcal{R}(\vec{k}, t)\chi(t) + 3a(t)^2\dot{\mathcal{R}}(\vec{k}, t)\dot{\chi}(t))}{k^2(2H(t)\chi(t) + \dot{\chi}(t))^2} \tag{3.54}$$

This gives the divergence part of the shift, deforming the slices in a way that depends on the curvature perturbation and its first derivative. Note that there's a factor of $1/k^2$. In position-space, this corresponds to an inverse Laplacian Δ^{-1}, which is a nonlocal quantity. That is, this expression appears to require us to know the curvature everywhere in space in order to know how to deform a small region! It's important to keep in mind that, though this is indeed a noncausal, nonlocal quantity, it's also not an observable, thus there's no need to give a physical interpretation to intermediate steps in the calculation.

Let's briefly review the state of our scalar degrees of freedom. We started with 5 different scalars in the SVT decomposed action, $\delta N, S, \mathcal{R}, C$, and $\delta\chi$. With our choice of gauge, we eliminated $C = 0$, and $\delta\chi = 0$. Then consequently, the solution to our constraint equations gave an expression for δN and S in terms of $\mathcal{R}$. So from this complicated action with 5 scalars and all possible mixings, we find at the end something that depends only on $\mathcal{R}$. The result is a quadratic action, with a kinetic term minus a potential term, just like that of a free field, but with a nontrivial, time-dependent function in front and a volume factor $a(t)^3$:

$$S_2[g_{\mu\nu}] = \int dt \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{a(t)^3}{8\pi G} \frac{3\chi(t)}{4} \left(\frac{\epsilon_\chi(t)}{1 - \epsilon_\chi(t)/2} \right)^2 \left(|\dot{\mathcal{R}}(\vec{k}, t)|^2 - \frac{k^2}{a(t)^2} |\mathcal{R}(\vec{k}, t)|^2 \right) \quad (3.55)$$

where we have introduced the slow-roll parameter for $\chi(t)$:

$$\epsilon_\chi(t) = -\frac{\dot{\chi}(t)}{H(t)\chi(t)} \quad (3.56)$$

Note that, since this action depends on $\epsilon_\chi(t)$, and thus on the time derivative of the Ricci scalar, $R(t)$. If we were doing this computation on de Sitter, this change would be zero and so there would be no second order action and thus no scalar perturbations. It's worthwhile to introduce a quantity that sums up the modification from a free scalar field and that will allow us to draw a parallel to the tensorial quadratic action:

$$Z_s(t) = \frac{3\chi(t)}{16\pi G} \left(\frac{\epsilon_\chi(t)}{1 - \epsilon_\chi(t)/2} \right)^2 \quad (3.57)$$

And so we may rewrite the action for scalar perturbations in a more compact manner:

$$S_2[g_{\mu\nu}] = \int dt \int \frac{d^3\vec{k}}{(2\pi)^3} a(t)^3 Z_s(t) \frac{1}{2} \left(|\dot{\mathcal{R}}(\vec{k}, t)|^2 - \frac{k^2}{a(t)^2} |\mathcal{R}(\vec{k}, t)|^2 \right) \quad (3.58)$$

As we will see later, this function $Z_s(t)$ will have a big effect on the behavior of the scalar perturbations, particularly during the pre-inflationary to quasi-de Sitter transition. For now, we'll merely note that this action is that of a harmonic oscillator with time dependent mass and frequency given by:

$$m(t) = \frac{d^3\vec{k}}{(2\pi)^3} a(t)^3 Z_s(t) \tag{3.59}$$

$$\omega(t) = \frac{|k|}{a(t)} \tag{3.60}$$

This procedure can be repeated to find the dynamical vector and tensor degrees of freedom. For the vectors, one starts with the SVT-decomposed metric written in background plus perturbation:

$$g_{\mu\nu} dx^\mu dx^\nu = (\bar{g}_{\mu\nu} + \epsilon h^{(v)}_{\mu\nu}) dx^\mu dx^\nu \tag{3.61}$$

$$= -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j + \epsilon a(t) \delta N_i^{(v)} dt dx^i + \epsilon a(t)(\partial_i B_j - \partial_j B_i) dx^i dx^j \tag{3.62}$$

We see that there are two vector quantities, the transverse vector part of the shift and the transverse vector part of the 3d metric. But we also have two vector constraints, one from the Hamiltonian constraint and one from the 3d diffeomorphism constraint. Thus, we expect that there will be no dynamical vector degrees of freedom. Indeed, the simplest way to show this is to choose the gauge fixing condition $B_i = 0$, and then plug in the vector sector of the perturbed metric into the diffeomorphism constraint:

$$^{(1)}F^0_i = G^0_i + \alpha H^0_i = 0 \Rightarrow \delta N_i^{(v)} = 0 \tag{3.63}$$

Thus in the gauge where B_i is zero, so too does the transverse vector from the shift also becomes zero, and we see that there are no propagating vector modes.

We can now turn to tensor perturbations, where we begin with the now-familiar starting point: the SVT-decomposed tensorial sector of the metric expressed in terms of a background plus perturbation:

$$g_{\mu\nu} dx^\mu dx^\nu = (\bar{g}_{\mu\nu} + \epsilon h^{(v)}_{\mu\nu}) dx^\mu dx^\nu \tag{3.64}$$

$$= -dt^2 + a(t)^2 (\delta_{ij} + \epsilon \gamma_{ij}(\vec{x}, t)) dx^i dx^j \tag{3.65}$$

where γ_{ij} is transverse and traceless:

$$\delta^{\alpha i} \partial_\alpha \gamma_{ij} = 0$$

$$\delta^{ij} \gamma_{ij} = 0$$

Remember that, at the quadratic level, we know that tensors aren't going to mix with scalars and vectors, so it is sufficient to put in only the tensor perturbation. Since we'll end up writing the action in momentum space, we next write the transverse traceless tensor in Fourier transform:

$$\gamma_{ij}(\vec{x},t) = \sum_{\sigma=+,\times} \int h_\sigma(\vec{x},t) e_{ij}^{(\sigma)}(\vec{k}) e^{+i\vec{k}\cdot\vec{x}} \frac{d^3\vec{k}}{(2\pi)^3} \tag{3.66}$$

with the choice in which $\vec{k} = (0,0,k)$, the linear polarization matrices are given by:

$$e_{ij}^{(+)} = \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.67}$$

$$e_{ij}^{(\times)} = \begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.68}$$

There is no tensorial diffeomorphism, thus the second order action is automatically gauge invariant. We can then plug the tensorial sector of the metric into the quadratic Lagrangian, deriving the reduced action for tensor perturbations:

$$S_2[g_{\mu\nu}] = \sum_{\sigma=+,\times} \int dt \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{a(t)^3(1+2\alpha R(t))}{16\pi G} \frac{1}{2} \left(|\dot{h}_\sigma(\vec{k},t)|^2 - \frac{k^2}{a(t)^2} |h_\sigma(\vec{k},t)|^2 \right) \tag{3.69}$$

We have a very similar structure to that of the reduced scalar action! There's a kinetic term, one per polarization, as well as a potential term with frequency $\frac{|k|}{a(t)}$, and a volume factor $a(t)^3$. We can make the two actions look even more similar by introducing writing the defining the tensor-characterizing function $Z_t(t)$:

$$Z_t(t) = \frac{\chi(t)}{16\pi G} = \frac{1+2\alpha R(t)}{16\pi G} \tag{3.70}$$

allowing us to write the second order action in a familiar form:

$$S_2[g_{\mu\nu}] = \sum_{\sigma=+,\times} \int dt \int \frac{d^3\vec{k}}{(2\pi)^3} a(t)^3 Z_t(t) \frac{1}{2} \left(|\dot{h}_\sigma(\vec{k},t)|^2 - \frac{k^2}{a(t)^2} |h_\sigma(\vec{k},t)|^2 \right) \tag{3.71}$$

Like with scalars, the characteristic function $Z_t(t)$ will modify the evolution of the perturbations from that of the free field during the pre-inflationary phase and quasi-de

Sitter phase. In particular, we will see the characteristic functions for scalars and tensors differ in terms of their expansions in ϵ_1, which in turn will lead to much larger power for scalars than for tensors in the power spectrum. In addition, the logarithmic derivatives of $Z(t)$ functions will show up, defined similarly to the Hubble flow parameters. From this point on, then, we differentiate between the slow-roll parameters for H, $\epsilon_{1H}, \epsilon_{2H}, ...$ and those for the characteristic functions Z, $\epsilon_{1Z}, \epsilon_{2Z}, ...$

Chapter 4 Quantization and the mode equation

4.1 Introduction

We have successfully decomposed the metric to put it in a form that is easy to work with, which allows us to bring it all together and look at the full perturbed metric. This is:

$$g_{\mu\nu} = \begin{pmatrix} -1-2\epsilon\delta N & \epsilon\delta_x S & \epsilon\delta_y S & \epsilon\delta_z S \\ \epsilon\delta_x S & a(t)^2(1-2\epsilon\mathcal{R}) & 0 & 0 \\ \epsilon\delta_y S & 0 & a(t)^2(1-2\epsilon\mathcal{R}) & 0 \\ \epsilon\delta_z S & 0 & 0 & a(t)^2(1-2\epsilon\mathcal{R}) \end{pmatrix} \tag{4.1}$$

The metric can now be converted to an operator on a Hilbert space, with an expectation value that is the classical FLRW $-1 + a(t)^2\delta_{\mu\nu}$, but now it also has fluctuations. It has a lapse, shift, curvature perturbation and tensor modes and seems like it has many operators, but as we have seen, it's actually just two operators: the curvature perturbation and the tensor operator:

$$\hat{g}_{\mu\nu} = \left(\begin{array}{c|c} -1-2\delta\hat{N}(\vec{x},t) & \partial_i\hat{S}(\vec{x},t) \\ \hline \partial_i\hat{S}(\vec{x},t) & a(t)^2((1-2\hat{\mathcal{R}}(\vec{x},t))\delta_{ij} + \hat{\gamma}_{ij}(\vec{x},t)) \end{array}\right) \tag{4.2}$$

$$= \left(\begin{array}{c|c} -1-\frac{2}{H(t)}\frac{d}{dt}\hat{R} & \frac{1}{H(t)}\partial_i\hat{R} + \epsilon(t)a(t)^2\Delta^{-1}\frac{d}{dt}\hat{R} \\ \hline \frac{1}{H(t)}\partial_i\hat{R} + \epsilon(t)a(t)^2\Delta^{-1}\frac{d}{dt}\hat{R} & a(t)^2((1-2\hat{\mathcal{R}}(\vec{x},t))\delta_{ij} + \hat{\gamma}_{ij}(\vec{x},t)) \end{array}\right) \tag{4.3}$$

Note that, though we have access to this quantized metric, it's worth keeping in mind that it's not an observable – indeed, it's an explicitly nonlocal quantity. We can take components of the metric from the 3d metric and from the time-time component, and look at their commutator. One depends on $\dot{\mathcal{R}}$ and one has $\mathcal{R}$ as well, which don't commute. Thus different components of the metric are noncommutative, which implies a Heisenberg uncertainty relation for quantum geometry!

$$[\hat{g}_{ij}, \hat{g}_{tt}] \neq 0 \tag{4.4}$$

That is, even in perturbative quantum gravity on a cosmological background, you cannot find a state that has section with exactly zero curvature perturbation and a lapse that is also exactly zero. We can push this even further. Recall that our second order action for the perturbations is essentially the action of a harmonic oscillator with time-dependent mass and frequency, so finding Poisson brackets and passing to commutators is straightforward. We start by first defining a general action for the perturbations in an agnostic fashion:

$$S[\phi] = \int dt \int \frac{d^3\vec{k}}{(2\pi)^3} a(t)^3 Z(t) \frac{1}{2} \left(|\dot{\phi}(\vec{k},t)|^2 - \frac{\vec{k}^2}{a(t)^2} |\phi(\vec{k},t)|^2 \right) \tag{4.5}$$

where $Z(t)$ could be either $Z_s(t)$ or $Z_t(t)$, and the field $\phi(\vec{k},t)$ could be either the tensor or scalar curvature perturbation. The procedure is standard: begin with the classical theory, compute the Poisson brackets in phase space and promote them to canonical commutators. The canonical momentum conjugate to the field is given from the variation of the action with respect to $\dot{\phi}(\vec{k},t)$:

$$\pi(\vec{k},t) = \frac{\delta S}{\delta \dot{\phi}(\vec{k},t)} = \frac{1}{2} a(t)^3 Z(t) \dot{\phi}(-\vec{k},t) \tag{4.6}$$

Given the Lagrangian, we can establish the associated Poisson bracket relation, picking up an additional factor of $(2\pi)^3$ when expressed in momentum space:

$$\{\phi(\vec{k},t), \pi(\vec{k}',t)\} = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \tag{4.7}$$

The Hamilton equation for momentum allows us to express the Poisson bracket in terms of the field and its time derivative, which we can then promote to commutators by introducing a factor of $i\hbar$:

$$[\hat{\phi}(\vec{k},t), \frac{d}{dt}\hat{\phi}(\vec{k}',t)] = i \frac{2\hbar}{a(t)^3 Z(t)} (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \tag{4.8}$$

Finally, note that regardless of whether we are speaking about $Z_s(t)$ or $Z_t(t)$, both functions contain $\frac{1}{G}$, which means that the commutator will have a factor of $G\hbar$, the Planck area! We are doing quantum gravity! In fact, we have Heisenberg uncertainty relations for quantum geometry. To be more explicit, recall that the curvature perturbation is connected to the 3d Ricci tensor and 3d Ricci scalar, which we can also promote to operators:

$$^{(3)}\hat{R}(\vec{x},t) = \frac{4}{a(t)^2}\Delta\hat{\mathcal{R}}(\vec{x},t) \tag{4.9}$$

$$^{(3)}\hat{R}_{ab}(\vec{x},t) = (\partial_a\partial_b + \delta_{ab})\hat{\mathcal{R}}(\vec{x},t) - \frac{1}{2}\Delta\hat{\gamma}_{ab}(\vec{x},t) \tag{4.10}$$

The next step in quantization is to decompose the field into a sum of mode functions, each with its own associated momentum. A Fock space is built from the ground up, starting with a Fock vacuum $|0\rangle$ that is a Gaussian state annihilated by all annihilation operators, and further expanded with creation operators $\hat{b}^\dagger$ for each mode $\vec{k}$. The algebraic structure of the space is then determined by the commutation relation between creation and annihilation operators; in particular, for a bosonic Fock space, we have:

$$[\hat{b}(\vec{k}),\hat{b}^\dagger(\vec{k}')] = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}'), \quad [\hat{b}(\vec{k}),\hat{b}(\vec{k}')] = 0, \quad \hat{b}(\vec{k})\,|0\rangle = 0 \;\; \forall\vec{k} \tag{4.11}$$

By construction, the creation and annihilation operators don't depend on time, so all time dependence is encoded in the mode functions. Then, given the vacuum reference state $|0\rangle \in \mathcal{H}_0$, one can build a one particle state by acting the creation operator on the vacuum, and summing up over all momenta:

$$|1,f\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3} f(\vec{k})\hat{b}^\dagger(\vec{k})\,|0\rangle \in \mathcal{H}_1 \tag{4.12}$$

where $f(\vec{k})$ is the mode function associated to the momentum $\vec{k}$. The Fock space can then be built from a direct sum of spaces each built from copies of the single-particle state: $\mathcal{F} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus S(\mathcal{H}_1 \otimes H_1) \oplus ...$, where the S helps us denote states with identical particles, that is, a bosonic Fock space. The field operator can then be written linearly in $\hat{b}(\vec{k})$ and $\hat{b}^\dagger(\vec{k}')$, ensuring that $\hat{\phi}(\vec{x},t)$ has only Gaussian correlations.

$$\hat{\phi}(\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3}\left(u(\vec{k},t)\hat{b}(\vec{k}) + \hat{u}^*(\vec{k},t)\hat{b}^\dagger(\vec{k})\right)e^{+i\vec{k}\vec{x}} \tag{4.13}$$

Given this representation, we can quickly check that the field is classically peaked, with perturbations that are quantum, so we ought to have an expectation value for the vacuum that is vanishing:

$$\langle 0| \hat{\phi}(\vec{x},t) |0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3} \langle 0| \left(u(\vec{k},t)\hat{b}(\vec{k}) + \hat{u}^*(\vec{k},t)\hat{b}^\dagger(\vec{k})\right) |0\rangle\, e^{+i\vec{k}\cdot\vec{x}} = 0 \tag{4.14}$$

Meanwhile, the first non-trivial computation occurs at the level of the two-point correlation function, which will allow us to ascribe physical meaning to the mode functions:

$$\langle 0| \hat{\phi}(\vec{x},t)\hat{\phi}(\vec{x}',t') |0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d^3\vec{k}'}{(2\pi)^3} e^{+i\vec{k}\cdot\vec{x}} e^{+i\vec{k}'\cdot\vec{x}'} \tag{4.15}$$

$$\langle 0| \left(u(\vec{k},t)\hat{b}(\vec{k}) + u^*(\vec{k},t)\hat{b}^\dagger(\vec{k})\right)\left(u(\vec{k}',t')\hat{b}(\vec{k}') + u^*(\vec{k}',t')\hat{b}^\dagger(\vec{k}')\right) |0\rangle \tag{4.16}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} u(\vec{k},t)u^*(\vec{k},t') e^{+i\vec{k}\cdot(\vec{x}'-\vec{x}')} \tag{4.17}$$

Since our state is Gaussian, by Wick's theorem, every higher-order correlation function can be computed from the two-point correlation. The mode functions, then, encode all information about the vacuum state, capturing its vanishing expectation value and the two-point correlation, thus its dispersion. We can simplify our mode functions further; after all, we began with the assumption that the space-time is homogeneous and isotropic, so there mustn't be a preferred directly, and the two-point functions must only depend on the distance between the points: $G(\vec{x},t;\vec{y},t) = G(|\vec{x}-\vec{y}|,t)$. Thus, the mode functions $u(\vec{k},t)$ must depend only on the magnitude of k, simplifying the field representation:

$$\hat{\phi}(\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3} \left(u(k,t)\hat{b}(\vec{k}) + \hat{u}^*(k,t)\hat{b}^\dagger(\vec{k})\right) e^{+i\vec{k}\vec{x}} \tag{4.18}$$

We can plug in this field representation into the canonical commutation relation, and using the fundamental ones with the creation and annihilation operators, we find that the mode functions must satisfy:

$$u(k,t)\dot{u}^*(k,t) - \dot{u}(k,t)u^*(k,t) = \frac{i\hbar}{a(t)^3 Z(t)} \tag{4.19}$$

This is the Wronskian of the mode functions. By ensuring that the Wronskian is nonzero, the mode functions are made linearly independent, which in turn implies that the mode function $u(\vec{k},t)$ must be complex. A straightforward treatment of the Hamilton equations yield a linear equation of motion for the field, which in turn becomes an equation of motion that the mode functions must also satisfy:

$$\ddot{u}(\vec{k},t) + \frac{\frac{d}{dt}\left(a(t)^3 Z(t)\right)}{a(t)^3 Z(t)} H(t)\dot{u}(\vec{k},t) + \frac{\vec{k}^2}{a(t)^2} u(\vec{k},t) = 0 \tag{4.20}$$

This appears to be an equation of motion for a harmonic oscillator with a friction term. It's important to remember that this isn't friction, but rather dilution due to the expansion of the universe.

Let's speak briefly about observables. After all, our end goal is to make connection with observation, in particular with the power spectrum of the field – but how exactly is this power spectrum defined? Well, the field is not the object that is usually measured, as the field is defined at a point. Instead, one usually measures a smearing of the filed over a given region, or alternatively, one measures the Fourier modes of the filed via spectral decomposition. Instead of looking for observables labeled by a point in space, we instead look for observables labeled by a function f that captures modes of the field across a particular frequency band:

$$\hat{\phi}_f(t) = \int d^3\vec{x} f(\vec{x})\hat{\phi}(\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3} \tilde{f}(k)\hat{\phi}(\vec{k},t) \tag{4.21}$$

For us, this filter function $f(k)$ characterizes the properties of the thermalized plasma of the CMB. We can check that this observable of the field is still classically peaked with expectation value zero:

$$\langle 0|\,\hat{\phi}_f(t)\,|0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3} \tilde{f}(k)\,\langle 0|\,\hat{\phi}(\vec{k},t)\,|0\rangle = 0 \tag{4.22}$$

The two-point correlation of the Fock vacuum, however, is non-trivial and will allow us to ascribe physical meaning to the mode function.

$$\langle 0|\left(\hat{\phi}_f(t)\right)^2|0\rangle = \tag{4.23}$$

$$= \int d^3\vec{x}\int d^3\vec{y} f(\vec{x})f(\vec{y})\,\langle 0|\,\hat{\phi}(\vec{x},t)\hat{\phi}(\vec{y},t)\,|0\rangle \tag{4.24}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3}|\tilde{f}(k)|^2|u(k,t)|^2 \tag{4.25}$$

In turn, this yields a simple expression for the variance of the field observable:

$$\Delta\phi_f = \sqrt{\langle\phi_f^2\rangle - \langle\phi_f\rangle^2} = \int \frac{d^3\vec{k}}{(2\pi)^3}|\tilde{f}(k)|^2|u(k,t)|^2 \tag{4.26}$$

The mode function $u(\vec{k},t)$ can then be understood as follows: given a measurement made in a Fourier band, the modulus squared of the mode function $|u(\vec{k},t)|^2$ gives the amplitude of the variance of that band. This is what we refer to as the power spectrum – the quantity that gauges the strength of field fluctuation in a particular frequency band. Once again, the spacetime is isotropic, we can make a further simplification by integrating the variance over a spherical shell, leaving only a single integral over the magnitude of $\vec{k}$:

$$\langle 0| \left(\hat{\phi}_f(t)\right)^2 |0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3} |\tilde{f}(k)|^2 |u(k,t)|^2 \tag{4.27}$$

$$= \int_0^\infty \frac{4\pi}{(2\pi)^3} k^2 dk |u(k,t)|^2 |\tilde{f}(k)|^2 \tag{4.28}$$

$$= \int_0^\infty \frac{k^3}{2\pi^2} |u(k,t)|^2 |\tilde{f}(k)|^2 \tag{4.29}$$

$$= \int_0^\infty P(k,t) |\tilde{f}(k)|^2 \tag{4.30}$$

where we have defined the power spectrum:

$$P(k,t) = |u(k,t)|^2 \frac{k^3}{2\pi^2} . \tag{4.31}$$

4.2 Bunch Davies initial conditions

It's worthwhile to remark on the importance of initial conditions, as well as how quantities in this quantization framework are related to each other. Now that we're equipped with an equation of motion to solve, which is of second order, we need a choice of two initial conditions $u(k,t_0)$ and $\dot{u}(k,t_0)$ that satisfy the canonical Wronskian in order to find a solution. But these mode functions fully characterize the correlation functions, which in turn characterize the Fock vacuum $|0\rangle$. That is, all of these quantities are related – a choice of mode functions gives us a choice of vacuum $|0\rangle$ in terms of correlation functions $G(|\vec{x}-\vec{y}|,t_0)$ and $\frac{d}{dt}G(|\vec{x}-\vec{y}|,t_0)\Big|_{t=t_0}$.

Let's take a moment to return to our picture of the evolution of the universe. We already have everything we need from quantum field theory, so we can look at how it connects to observations of the CMB, and then we'll talk about zero order Bunch Davies solutions.

Ordinarily, give a Minkowski vacuum, we can say what is the state and compute its fluctuations. For the CMB, we said these fluctuations were the anisotropies that are observed, effectively arguing that the fluctuations are of quantum origin. Thus, we need to provide a state for said quantum fields. If we wanted to prescribe the state during the hot Big Bang, there's unfortunately no notion of vacuum for the observable band of modes at the time. Thus, the idea is to go further into earlier times, to a phase in which $a(t)H(t)$ is increasing, so the modes are oscillatory and we can choose a vacuum at this time. This is a principle of our *choosing*. Then, there is a process that changes the vacuum in a way we have to determine. Note that it doesn't matter exactly where in the inflationary regime we put the vacuum because if the phase in which we put initial conditions evolves slowly enough, the vacuum is adiabatic. We could require that the state is a vacuum just before a given mode crosses the Hubble horizon, and use that as a principle, but there's no need to restrict ourselves so much. We'll start with this assumption, but later we'll parametrize solutions as squeezing of the vacuum.

Let's look at what our equation of motion would look like during slow-roll inflation. Recall that we have two requirements of this phase in order to conform to observation: we need a phase with increasing $a(t)H(t)$ so that the spectral tilt is red, which implies an accelerating scale factor, and we need this phase to last sufficiently long that all observable modesn are covered. Let's assume then, that we have such a slow-roll inflation, such that $\epsilon_1(t) \ll 1$. Note that the logarithmic derivative that appears in the equation of motion as a "friction" can be expressed in terms of a small deviation from $3H(t)$ including a "slow-roll" of the function $Z(t)$:

$$\frac{\frac{d}{dt}\left(a(t)^3 Z(t)\right)}{a(t)^3 Z(t)} H(t) = \left(3 - \epsilon_Z(t)\right) H(t) \tag{4.32}$$

where we have defined the slow-roll parameter for the characteristic function $Z(t)$:

$$\epsilon_Z(t) = -\frac{\dot{Z}(t)}{H(t)Z(t)} \,. \tag{4.33}$$

This is a dimensionless parameter, similar to $\epsilon_1(t)$, that measures how much $Z(t)$ changes over one Hubble time. Let's assume this parameter is similarly small during slow-roll, so $\epsilon_Z(t) \ll 1$. Note that the following equation of motion has no approximations in $\epsilon_1(t)$ or $\epsilon_Z(t)$ yet; this is exact up until this point:

$$\ddot{u}(\vec{k}, t) + \left(3 - \epsilon_Z(t)\right) H(t)\dot{u}(\vec{k}, t) + \frac{\vec{k}^2}{a(t)^2} u(\vec{k}, t) = 0 \,. \tag{4.34}$$

What kind of behavior can we expect? We have an oscillator with a "friction" term, which is approximately constant during slow-roll, but $a(t)$ changes rapidly in time -

exponentially so in exact de Sitter. For a given mode k, there is an early time when we can neglect the friction and a late time in which you can neglect the frequency. That is, we are going from an oscillatory solution into a frozen one. The sense in which it is freezing is a specific one: that in which, in an oscillator where the friction is like a mass that is increasing exponentially while the length is also increasing exponentially but with twice the coefficient due to the $a(t)^2$. When this occurs, we'll find that we freeze to something finite and nonzero.

There's a key time in between the two regimes, when both the friction and frequency are of the same order and you cannot neglect either. This occurs when $a(t)H(t) \sim k$, that is, when the mode k crosses the Hubble horizon. We'll call the freezing time t_k when $a(t)H(t)$ is exactly k, for a given k. The relevant physics we want to capture is this transitional period. We will end up doing approximations in $\epsilon_1(t)$, so it's worthwhile to start with a baseline, zeroth order approximation, which corresponds to the Bunch-Davies vacuum. Thus, we'll start by saying all ϵ's are small and set them to zero and solve everything up to the end. Then, we'll build a perturbation theory order by order around this zeroth order solution.

Since our slow-roll quantities are zero, the Hubble parameter and $Z(t)$ don't change and we can set them to be constants:

$$\epsilon_1(t) = 0 \quad \Rightarrow \quad H(t) = H_0 \quad \Rightarrow \quad a_{\mathrm{dS}}(t) = a_0 e^{H_0 t} \text{ de Sitter} \tag{4.35}$$

$$\epsilon_Z(t) = 0 \quad \Rightarrow \quad Z(t) = Z_0 \tag{4.36}$$

Were we to take a specific theory on inflation, $Z(t)$ would clearly be a function of $H(t)$, so we're not really free to set it to a specific constant if $H(t)$ is de Sitter. For now, we'll treat treat them as separate, so that Z_0 is independent, and later we can check for consistency. Our equation of motion simplifies so that we have constant friction, with exponential behavior in both the frequency and the Wronskian:

$$\ddot{u}(k,t) + 3H_0\dot{u}(k,t) + \frac{k^2}{a_{\mathrm{dS}}(t)^2}u(k,t) = 0 \tag{4.37}$$

$$u(k,t)\dot{u}^*(k,t) - \dot{u}(k,t)u^*(k,t) = \frac{i\hbar}{a_{\mathrm{dS}}(t)^3 Z_0} \tag{4.38}$$

Here we'll introduce a trick that we'll use again when we do the series expansion. There's two parts to this technique: the first is a reparametrization of time, followed by a rescaling. This change of variables is written in the following manner:

$$u(k,t) = \frac{v(k, x(t))}{\sqrt{m(t)}} \tag{4.39}$$

We'll show that one can determine $x(t)$ and $m(t)$ such that we can put the equation of motion and the canonical Wronskian in the following canonical forms:

$$v''(x) + \left(\frac{k^2}{k_*^2} - \frac{2}{x^2}\right) v(x) = 0 \tag{4.40}$$

$$v(x)v'^*(x) - v'(x)v^*(x) = i \tag{4.41}$$

where we have simplified the notation, dropping the explicit k dependence, with the understanding that these equation apply to each mode k individually, and where we established a prime meaning derivative with respect to x. There's a couple of things we ought to note. By writing this into a canonical form, we've eliminated the first derivative term, simplifying our analysis of the relevant term, $\left(\frac{k^2}{k_*^2} - \frac{2}{x^2}\right)$. There could be a generic function of x inside this expression, but we'll require that the constant part is always k^2 over a reference k_*^2, which one often refers to as the pivot mode. In the literature, one finds different values of this pivot mode, but the important thing to remember is that this is a *choice*, so one ought to always ensure they are well-informed as to which pivot mode a particular study utilizes. If the power spectrum is exactly scale invariant, we should find that our final answer doesn't depend on the scale, which we should be able to read by having k_* disappear from our answer. The function of x that appears here is not free – it's something that's generated from a calculation given the imposition of constant k^2/k_*^2 term. This same logic will be utilized when we take on the full non-Bunch-Davies theory, but the function inside the parenthesis will be more complicated.

In order to determine the new time x and rescaling $\mu(x)$, we start by taking our ansatz $u(t) = \frac{v(x(t))}{\sqrt{\mu(t)}}$ into the equation of motion for $u(t)$ and then impose that there should not be a $v'(x)$ term in the equation of motion for $v(x(t))$. We find the first derivative term to be:

$$v'(x(t)) \left(\frac{3\dot{a}(t)\dot{x}(t)}{a(t)\sqrt{\mu(t)}} - \frac{\dot{x}(t)\dot{\mu}(t)}{\mu(t)^{3/2}} + \frac{\ddot{x}(t)}{\sqrt{\mu(t)}} \right) \tag{4.42}$$

We can then solve and integrate for $\mu(t)$:

$$\mu(t) = c_0 a(t)^3 \dot{x}(t) \tag{4.43}$$

We can set the constant of integration c_0 by imposing that the commutation relation for $v(x(t))$ takes on a canonical form. Plugging in our ansatz for $u(t)$ once again, we find:

$$\frac{v(x(t))v'^*(x(t))\dot{x}(t)}{\mu(t)} - \frac{v^*(x(t))v'(x(t))\dot{x}(t)}{\mu(t)} = -\frac{i\hbar}{Z_0 a(t)^3} \tag{4.44}$$

$$\frac{v(x)v'^*(x)}{a(t)^3 c_0} - \frac{v^*(x)v'(x)}{a(t)^3 c_0} = -\frac{i\hbar}{Z_0 a(t)^3} \tag{4.45}$$

$$\Rightarrow c_0 = \frac{Z_0}{\hbar} \tag{4.46}$$

Putting all of this together, we can write out a canonical, simplified version of our equation of motion in the form $v''(x) + Q(x)v(x) = 0$, where the to-be-determined function $Q(x)$ will contain the relevant physics, that is, whether our system is oscillating or frozen. In order to determine $x(t)$, we must impose that $Q(x)$ has a constant term in the form k^2/k_*^2. Plugging in our ansatz and found solutions into the equation of motion, we find the following $Q(x)$, which gives us a simple differential equation for $x(t)$:

$$Q(x) = \frac{k^2}{a(t)^2\dot{x}(t)^2} - \frac{3a'(t)^2}{4a(t)^2\dot{x}(t)^2} - \frac{3\ddot{a}(t)}{2a(t)\dot{x}(t)^2} + \frac{3\ddot{x}(t)^2}{4\dot{x}(t)^4} - \frac{\dddot{x}(t)}{2\dot{x}(t)^3} \tag{4.47}$$

$$\Rightarrow \dot{x}(t) = \frac{k_*}{a(t)} \tag{4.48}$$

We can solve this simply by integrating in time. Note that in this case, this is precisely conformal time, which gives us the following expression for $x(t)$:

$$x(t) = \int \frac{k_*}{a(t)} dt = k_*\eta(t) = -\frac{k_*}{a_{\mathrm{dS}}(t)H_0} \tag{4.49}$$

Note that this $x(t)$ gives exactly the desired properties we spoke of earlier! When $x < -1$, $k_* > a_{\mathrm{dS}}(t)H_0$ and we have oscillatory behavior. When $0 > x > -1$, $k_* < a_{\mathrm{dS}}(t)H_0$, the mode is frozen. The time $x = -1$ is exactly when the pivot mode k_* crosses the Hubble horizon. There are also some things to note about the behavior of x as a time. When $x \to -\infty$, this indicates the far past, while the far future is given by $x- > 0$. This is not the whole lifetime of the universe, however! This is solely a model in which we're approximating the universe during this inflationary patch, and shouldn't be trusted for all time. Now, we can write down $\mu(x)$, as it's given directly from the expression for $x(t)$, and ultimately write the full parametrization of $u(t)$:

$$\mu(t(x)) = k_* a_{\mathrm{dS}}(t)\frac{Z_0}{\hbar} = \frac{Z_0}{\hbar}\frac{k_*^3}{(-H_0 x)^2} \tag{4.50}$$

$$\Rightarrow u(t) = \sqrt{\frac{\hbar}{Z_0}} \frac{-H_0 x}{k_*^{3/2}} v(x) \tag{4.51}$$

Now assuming the de Sitter ansatz for the scale factor, $a(t) = a_0 e^{H_0 t}$, we get a simple equation to solve, precisely those in equation 4.40! The solution is a pair of complex conjugated functions that give the Bunch-Davies basis:

$$v_0(x) = \frac{1}{\sqrt{2k/k_*}} \left(1 - \frac{i}{\frac{k}{k_*} x}\right) e^{-i \frac{k}{k_*} x} \tag{4.52}$$

$$v_0^*(x) = \frac{1}{\sqrt{2k/k_*}} \left(1 + \frac{i}{\frac{k}{k_*} x}\right) e^{+i \frac{k}{k_*} x} \tag{4.53}$$

As a basis, we can write any solution as a linear combination of $v_0(x)$ and $v_0^*(x)$ with Bogoliubov coefficients $\alpha(k)$ and $\beta(k)$:

$$v(x) = \alpha(k) v_0(x) + \beta(k) v_0^*(x) \quad \text{with} \quad |\alpha|^2 - |\beta|^2 = 1 \tag{4.54}$$

What makes this basis special is that it's the solution that has adiabatic correlation functions in the far past. One can take the modulus squared, and see that when we take $x \to -\infty$, we find a constant:

$$|v_0(x)|^2 = \left(1 + \frac{1}{\left(\frac{k}{k_*} x\right)^2}\right) \frac{1}{2k/k_*} \xrightarrow[x \to -\infty]{} \frac{1}{2k/k_*} \tag{4.55}$$

For de Sitter, this gives us a notion of in-vacuum, though in this case in-vacuum in the far past really means just a bit before the mode crosses the Hubble horizon. This gives us a sense of what we're going to call an adiabatic vacuum. Asking if a mode function is adiabatic is not something that is immediate to recognize due to the space-time changing in time. However, we can compute correlation functions from them, which contain the modulus squared $|v(x)|^2$, which itself depends on time with $x(t) \sim e^{-H_0 t}$, so this is a fast time dependence. In the far past, we find the modulus squared goes to a constant, and this is the notion which we call adiabatic – the correlation function is constant in the far past, that is, it is unchanging in time up to corrections. Remember the power spectrum contains the modulus squared of $u(t)$, not $v(x)$, but as $v(x)$ diverges as $1/x$ at late times (that is, $x \to 0$), $u(t)$ itself constains x as well, as so $u(t)$ freezes. Note that you have to fix k, otherwise regardless of how far into the past we go, one could always find a k such that you find the modulus squared to be non-adiabatic.

We can plug in $u(t)$ and $v(x)$ with these change in variables, to compute the Bunch-Davies power spectrum as a function of x and determine its late-time behavior:

$$P(k) = |u(k,t)|^2 \frac{k^3}{2\pi^2} \tag{4.56}$$

$$= |v(k,x)|^2 \frac{\hbar}{Z_0} \frac{(-H_0 x)^2}{k_*^3} \frac{k^3}{2\pi^2} \tag{4.57}$$

$$= \left(1 + \left(\frac{kx}{k_*}\right)^2\right) \frac{\hbar H_0^2}{4\pi^2 Z_0} \tag{4.58}$$

$$\xrightarrow[x\to 0^-]{} \frac{\hbar H_0^2}{4\pi^2 Z_0} \tag{4.59}$$

At late times, we see the time dependence disappear and the power spectrum amplitude goes to a constant, despite the fact that the universe keeps expanding. We see also that the power doesn't depend on k – this is scale invariance. That is, there is the same amount of power in every interval of $\log k$. Another way to say it is that the power spectrum is flat or has zero tilt. In this expression, we observe too the importance of the characteristic functions $Z(t)$. How these functions may depend on ϵ_1 as a series may affect the power by either suppressing it or enhancing it. In addition, both of these functions contain Newton's gravitational constant G, so we'll see the Planck area $G\hbar$ in some form in the power spectrum, confirming that this is indeed a quantum gravity phenomenon.

The method thus presented gives a sense for the technique we'll use for higher order corrections. If you have the most general, second order, linear differential equation, $f(t)\ddot{u}(t) + g(t)\dot{u}(t) + \omega(t)u(t) = 0$, there always exists a change of variables that can bring it to a canonical form $v''(x) + Q(x)v(x) = 0$. In addition, we want to bring the Wronskian also into a canonical form, and then put $Q(x)$ in a special form where we extract a particular constant k^2/k_*^2, and expand the rest of the terms in an expansion $1/x, 1/x^2, 1/x^3, \ldots.$ As it happens, these all have names; Bessel functions corresponds to $1/x^2$, Whittaker functions to $1/x$. Our approach is similar, but we'll do this order by order since, as we'll see, the expansion corresponding to quasi de-Sitter at second order in ϵ_1 will include a logarithmic function which doesn't have a close-form solution, necessitating a slightly different approach.

Chapter 5
Non Bunch-Davies initial conditions and the power spectrum

5.1 Introduction

We've seen how to generate the power spectrum when we assume the space-time is de Sitter, $H(t)$ is constant, and our initial conditions for the mode functions are Bunch-Davies in-vacuum. The standard way to getting to next order is to assume instead of a constant $H(t)$ it instead changes slowly, and our equations of motion for the modes end up being Bessel equations that we can solve. This method works well if the order at which things change slowly is linear in ϵ_1, but fails for higher orders. Instead, we'll develop a theory of perturbations from scratch, and apply said theory and look for special cases. The framework is as follows: we'll begin with qualitative analysis of the mode equation and put it into a canonical form through Mukhanov-Sasaki variables. Then, we prepare all our relevant quantities in terms of an expansion in Hubble-flow parameters. Next, expand around the critical time, the time when the pivot mode k_* crosses the Hubble horizon, which we call the pivot time, and then write the canonical equation of motion for the mode function in terms of a potential that has a leading term that is constant, a zeroth order term that gives back Bunch-Davies, and then a series which we control and truncate to a desired order. The endpoint is to find $Q(x)$ as a series:

$$Q(x) = \frac{k^2}{k_*^2} + \delta Q(x) \tag{5.1}$$

Both the scale factor and the characteristic functions $Z(t)$ are fixed once we choose the FLRW background. The equation of motion for the modes is linear, with a "friction" term and an angular frequency. Given initial conditions, what kind of behavior do we expect? The inflationary background is close to de Sitter, with a $Z(t)$ that is close to being constant. We're interested in an inflationary phase that transitions between two regimes: one at early times in which we describe an oscillator that is under-damped and we can neglect friction and prepare the state in the vacuum, and one at late times when we have an over-damped oscillator, and large friction that prevents oscillation of the modulus squared $|u(k,t)|^2$.

When we choose initial conditions and look at late times, under the assumption that friction goes to a constant $3H_0$ and $\omega(t) \to 0$ (since $\omega \sim 1/a$), then $|u(k,t)|^2$ goes to a constant. Thus we expect the solution is going to be a constant – the power spectrum amplitude – plus an exponential approach to a constant that is compatible with the canonical commutation relations. Note that we're not taking the initial condition at $t \to -\infty$. We merely need a phase of the universe prior to freezing – what happens before either a pre-inflationary or purely quantum gravity era, is not needed for this part of the analysis. First we assume that we begin with an in-vacuum, and once we determine the mode functions for that in-vacuum, we can modify the state by writing a new initial state as a squeezed state using Bogoliubov coefficients.

The analytic procedure we used in Chapter 4 allowed us to move all complications in the equation of motion from the friction term and the complicated commutation relations into x and $Q(x)$. We will then take $a(t)$ and $H(t)$, expand them in a Hubble-flow expansion, and then look at $Q(x)$ in terms of this expansion. Since we know the de Sitter solution, we can then find the general solution order by order. This is extremely helpful to do in analytical approximations in a slow-roll expansion, but it is a poor route to take if using numerical computation. This is due to having to compute x and $Q(x)$ out of background quantities, which include lots of derivatives and introduces noise and error into the computation. When we present numerics to compare and verify the predictions from the analytic approximations, the numerics will be done in proper time in the original scaling of $u(k,t)$.

The language of "slow-roll" is tailored to having a potential, a scalar field, and the image of a ball in a potential with friction so that the ball rolls slowly either because the kinetic energy is small or the friction is large. For us, this idea captures a property of the background, which can be phrased independent of a potential, and independent of the mechanism that drives inflation – slow-roll means that the Hubble factor changes slowly and the characteristic functio $Z(t)$ changes slowly. Changing slowly in this case means with respect to proper time, which is dimensionful. We want to use dimensionless quantities so that we can use them to organize our expansion, so we require logarithmic rates of change. We have done this before by defining the Hubble flow parameters. The idea was that, in order to capture the change of H, that is, $\dot{H}$, we first want it logarithmic so we compute $\dot{H}/H$. The dot carries a dimension of time, so we put in another H, and chose a convention with a minus sign so that if H is decreasing, ϵ_1 is positive. In plain terms, one H has to do with the quantity being derived, and the other with the dot to make the parameter dimensionless. We can use this same iterative process to define Z-flow functions:

$$\epsilon_{1Z} = -\frac{\dot{Z}(t)}{H(t)Z(t)} \tag{5.2}$$

$$\epsilon_{2Z} = -\frac{\dot{\epsilon}_{1Z}}{H(t)\epsilon_{1Z}} \tag{5.3}$$

$$\epsilon_{3Z} = -\frac{\dot{\epsilon}_{2Z}}{H(t)\epsilon_{2Z}}... \tag{5.4}$$

Everytime we take a time derivative of $a(t)$ or $H(t)$ or $Z(t)$ we can replace them with these Hubble- and Z-flow functions. Once we write everything in terms of these, we can make all our work symbolic and automatic. Since we'll want an expansion in ϵ, we simply put a λ in front of each parameter, and do a series in λ to a desired order. Note that these are constructred such that each of them are of the same order; ϵ_2 expressed as an expansion in ϵ_1 would be $\epsilon_2 = c_1\epsilon_1 + c_2\epsilon_1^2 + ...$ Once one fixes the scale factor, all coefficients in the expansion are determined and you can express each one in terms of the previous one. At this point, we are making the minimal assumption that these parameters are all small and independent.

How do we write conformal time in terms of this Hubble flow expansion? In FLRW, we can write the metric in proper time or as a flat space up to a conformal factor:

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 = \Omega(\eta)^2(-d\eta^2 + d\vec{x}^2) \tag{5.5}$$

Equating the two gives the definition of conformal time η such that $d\eta/dt = 1/a(t)$. Now we want to find the change of variable from t to η for a given scale factor, without knowing the scale factor. We'll need to integrate, which could be complicated, as it's nonlocal and has a memory, but we can trade in these complications for extra time derivatives. At order zero, in exact de Sitter, the integral is simple:

$$\begin{aligned} a_{\mathrm{dS}}(t) &= a_0 e^{H_0 t} \\ H_{\mathrm{dS}}(t) &= H_0 \end{aligned} \qquad \Rightarrow \qquad \eta_{\mathrm{dS}}(t) = \int^t \frac{dt'}{a_{\mathrm{dS}}(t')} = -\frac{1}{a_{\mathrm{dS}}(t) H_{\mathrm{dS}}(t)} \tag{5.6}$$

Now suppose we're merely close to de Sitter. In this case, we can parametrize the corrections in terms of Hubble flow parameters, writing the following ansatz:

$$\eta(t) = -\frac{1}{a(t)H(t)}(1 + b_1 \epsilon_{1H}(t)) \tag{5.7}$$

We can determine b_1 by imposing that $d\eta/dt = 1/a(t)$, which gives $b_1 = 1$. If we want to go to higher order, we need to introduce all permutations of how second-order Hubble-flow parameters could appear, so our ansatz becomes:

$$\eta(t) = -\frac{1}{a(t)H(t)}(1 + \epsilon_{1H}(t) + c_{11}\epsilon_{1H}(t)^2 + c_{22}\epsilon_{2H}(t)^2 + c_{12}\epsilon_{1H}(t)\epsilon_{2H}(t)) \tag{5.8}$$

Once more imposing $d\eta/dt = 1/a(t)$ allows us to determine the coefficients, and we find the conformal time at second order is:

$$\eta(t) = -\frac{1}{a(t)H(t)}(1 + \epsilon_{1H}(t) + \epsilon_{1H}(t)^2 - \epsilon_{1H}(t)\epsilon_{2H}(t) + ...) \tag{5.9}$$

The next step is to find the change of variable from t to x. But we already know x is given by the conformal time, $x(t) = k_* \eta(t)$, so we have the change of variable. In particular, we can solve for the scale factor in terms of x :

$$a(t) = -\frac{k_*}{H(t)x(t)}(1 + \epsilon_{1H}(t) + \epsilon_{1H}(t)^2 - \epsilon_{1H}(t)\epsilon_{2H}(t) + ...) \tag{5.10}$$

Finally, we can take this and plug into our expression for $Q(x)$:

$$\begin{aligned} Q(x) &= \frac{k^2}{k_*^2} - \frac{2}{x} + \frac{-3\epsilon_{1H}(x) + \frac{3}{2}\epsilon_{1Z}(x)}{x^2} && (5.11) \\ &+ \frac{1}{4x^2}\Big(-16\epsilon_{1H}(x)^2 + 10\epsilon_{1H}(x)\epsilon_{1Z}(x) - \epsilon_{1Z}(x)^2 && (5.12) \\ &+ 15\epsilon_{1H}(x)\epsilon_{2H}(x) - 2\epsilon_{1Z}(x)\epsilon_{2Z}(x)\Big) + \mathcal{O}(\epsilon^3) && (5.13) \end{aligned}$$

We have a constant term plus a first correction that corresponds to Bunch-Davies. We then have a linear correction in Hubble- and Z-flow functions as well as higher order corrections, but these still have a dependence on x. It we were to set x to be constant, it would be a simple equation; in fact, the solution would be Bessel functions. However, x is not constant and we want to treat it correctly.

Previously, we introduced the pivot mode k_*. If we find the power spectrum at the pivot mode, we can ask how it changes around that value. If the power spectrum is log-log linear, then we need only the tilt to fully describe that change. In particular, we're interested just a band around k_*. So we choose a k_*, presumably one that from the point of view of observation we can measure to good precision. Then, we look at a neighborhood of that value and read the tilt and its running. This pivot mode also fixes a time – the time t_* when the pivot mode transitioned from underdamped to overdamped. The transition in $Q(x)$ for the mode k_* occurs when $Q(x)|_{k=k_*} = 0$, which occurs near $x \sim -1$. We invert this idea and choose this as a definition; $x_* = -1$ gives the definition of our pivot mode. We can then find k_* in terms of corrections in ϵ. By definition, the freezing time t_* is now given by $x(t_*) = x_* = -1$. Then, we say k_* is $a(t_*)H(t_*)$ up to corrections:

$$-\frac{k_*}{a(t_*)H(t_*)}(1+\mathcal{O}(\epsilon)) = -1 \quad \Rightarrow \quad k_* = a(t_*)H(t_*)(1+\mathcal{O}(\epsilon)) \tag{5.14}$$

We have to track everything at a given order of ϵ, so we cannot throw away these corrections! Now we can turn to the expansion term in $Q(x)$. Each $\epsilon(x)$ is going to be expanded, and our goal is to write everything in a Taylor series around x_*. In this case, we're not interested in the difference from x_*, but rather the ratio x/x_*. Doing an expansion in terms of $\log x/x_*$ maintains the desired properties of x, and ultimately, we'll be using log-log plots in the power spectrum anyways. We begin by writing $H(x)$ in terms of a Taylor series in $\log x$:

$$H(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!}\left(\log\frac{x}{x_*}\right)^n = H_* + c_1 \log\frac{x}{x_*} + \frac{1}{2}\left(\log\frac{x}{x_*}\right)^2 + ... \tag{5.15}$$

We can determine the coefficients as follows. Since we know $\dot{x}$ by definition, we can invert the relationship and write $t'(x)$ and $t''(x)$:

$$t'(x) = \frac{dt}{dx} = \left(\frac{dx}{dt}\right)^{-1} = \frac{1}{\dot{x}} \tag{5.16}$$

$$t''(x) = \frac{d}{dx}\frac{dt}{dx} = \frac{dt}{dx}\frac{d}{dt}\left(\frac{1}{\dot{x}}\right) = \frac{1}{\dot{x}}\frac{d}{dt}\left(\frac{1}{\dot{x}}\right) \tag{5.17}$$

We can plug this back into the definition of the Taylor series coefficients, and get expressions for each coefficient:

$$c_1 = \frac{d}{d\left(\log \frac{x}{x_*}\right)} H(x)\bigg|_{x\to x_*} = \frac{x}{\dot{x}}\dot{H}\bigg|_{x\to x_*} \tag{5.18}$$

$$= (1 + \epsilon_{1H*} + ...)\epsilon_{1H*}H_* \tag{5.19}$$

$$c_2 = \left(1 - x\frac{\ddot{x}}{\dot{x}^2}\right)\frac{x}{\dot{x}}\dot{H} + \frac{x^2}{\dot{x}^2}\ddot{H} \tag{5.20}$$

$$= (1 + ...)\left(\epsilon_{1H*}^2 + \epsilon_{1H*}\epsilon_{2H*}\right)H_* \tag{5.21}$$

Put together, we find an expression for $H(x)$ as a log series. In turn, we can follow the same procedure to write $Z(x)$, and the Hubble- and Z-flow functions, which we write out in the following unfortunate wall of equations:

$$a(x) = \left(1 + \epsilon_{1H*} + \epsilon_{1H*}^2 - \epsilon_{1H*}\epsilon_{2H*} + \left(-\epsilon_{1H*} - 2\epsilon_{1H*}^2 + \epsilon_{1H*}\epsilon_{2H*}\right)\log\frac{x}{x_*}\right. \tag{5.22}$$

$$\left. - \epsilon_{1H*}\epsilon_{2H*}\left(\log\frac{x}{x_*}\right)^2 + ...\right)\left(-\frac{k_*}{H_* x}\right) \tag{5.23}$$

$$H(x) = \left(1 + \left(\epsilon_{1H*} + \epsilon_{1H*}^2\right)\left(\log\frac{x}{x_*}\right) + \left(\epsilon_{1H*}^2 + \epsilon_{1H*}\epsilon_{2H*}\right)\left(\log\frac{x}{x_*}\right)^2 + ...\right)H_* \tag{5.24}$$

$$Z(x) = \left(1 + \left(\epsilon_{1Z*} + \epsilon_{1Z*}^2\right)\left(\log\frac{x}{x_*}\right) + \left(\epsilon_{1Z*}^2 + \epsilon_{1Z*}\epsilon_{2Z*}\right)\left(\log\frac{x}{x_*}\right)^2 + ...\right)Z_* \tag{5.25}$$

$$\epsilon_{1H}(x) = \epsilon_{1H*} + \epsilon_{1H*}\epsilon_{2H*}\log\frac{x}{x_*} + ... \tag{5.26}$$

$$\epsilon_{1Z}(x) = \epsilon_{1Z*} + \epsilon_{1Z*}\epsilon_{2Z*}\log\frac{x}{x_*} + ... \tag{5.27}$$

We see that these are organized both as a series in ϵ and a log series, and that these aren't the same expansion; the former deals with the deviation from exact de Sitter while the latter is about investigating the relevant time near freezing, so we can understand the mechanism. It's important to remember also the qualitative behavior we're trying to preserve. The mode function $v(x)$ satisfies an equation that goes from stable to unstable and is going to diverge, meanwhile $\mu(x)$ is also going to diverge, but $|u(k,t)|^2$ must go to a constant. Thus the divergence in the numerator and denominator of the $u(k,t)$ ansatz have to cancel exactly, and so we have to carefully keep track of everything order by order. Fortunately, this can also function as a check on our calculations; if things don't cancel, that's a sign that something has gone awry.

Now that we have everything, we can plug it all into $Q(x)$ and write it out as a log expansion:

$$Q(x) = \frac{k^2}{k_*} - \frac{2 - \lambda b_{1*} + \lambda^2 b_{2*} + \lambda^2 c_{2*} \log \frac{x}{x_*} + ...}{x^2} \tag{5.28}$$

where

$$b_{1*} = \frac{3}{2}\left(2\epsilon_{1H*} - \epsilon_{1Z*}\right) \tag{5.29}$$

$$b_{2*} = 4\epsilon_{1H*}^2 - 4\epsilon_{1H*}\epsilon_{2H*} + \frac{1}{4}\epsilon_{1Z*}^2 + \frac{1}{2}\epsilon_{1Z*}\epsilon_{2Z*} - \frac{5}{2}\epsilon_{1H*}\epsilon_{1Z*} \tag{5.30}$$

$$c_{2*} = 3\epsilon_{1H*}\epsilon_{2H*} - \frac{3}{2}\epsilon_{1Z*}\epsilon_{2Z*} \tag{5.31}$$

The parameter λ here is solely to help us keep track of orders. The first order λ term can be generated simply by evaluating our previous expression for $Q(x)$ at the time t_*, 5.11. However, these epsilons change in time, which give contributions to the λ^2 and logarithmic term, and we also get contributions from ϵ^2's showing up. At order λ^0, we recover the exact Bunch-Davies solution. At λ^1, the solution turns out to be exact Bessel functions. However, for λ^2, the log term complicates computations, and so we'll be using a Green function method to find the approximate solution. If we merely wanted the power spectrum amplitude and tilt at leading order, the $\lambda-$linear expression is sufficient, however, we'll be looking to predict the power spectrum at order ϵ^2. As a remark, note that our definition of the pivot mode also has a correction in orders of epsilon, so we need to make sure we match orders when computing the full power spectrum.

5.2 Green function method

The tools required to to up to second order can be found in a paper by Stewart and Gong (2001/2005). There have been different proposals to approach the second order power spectrum, but we find this one to be the most likely to be of use at higher orders as well. In this case, we have an equation that we want to solve, and we can use Green function methods to solve said differential equation with a parameter, perturbatively in that parameter. The strategy is to first begin by standardizing our expressions for $Q(x)$ in terms of an expansion in the parameter λ, and to write down an ansatz for a solution of the differential equation as a series in λ:

$$Q(x) = Q_0(x) + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} q_n(x) v(x) = v_0(x) + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} w_n(x) \tag{5.32}$$

In this case, remember that λ^0 case corresponds to Bunch-Davies, so we are expanding our solutions around Bunch-Davies, so the solution $v_0(x)$ is already known. We can then plug in this ansatz into the equation of motion and organize everything order-by-order:

$$v_0''(x) + Q_0(x)v_0(x) = 0 \tag{5.33}$$

$$w_1''(x) + Q_0(x)w_1(x) = -q_1(x)v_0(x) \tag{5.34}$$

$$w_2''(x) + Q_0(x)w_2(x) = -q_2(x)v_0(x) - 2q_1(x)w_1(x) \tag{5.35}$$

We see the zeroth order equation gives the already-solved Bunch-Davies case. Subsequent equations have the same form on the left-hand side, but with a different source on the right. We could write it in general as:

$$w_n''(x) + Q_0(x)w_n(x) = J_n(x) \tag{5.36}$$

To solve this class of equations, we say that the solution $w_n(x)$ is the Green function of the source:

$$w_n(x) = \int_{-\infty}^{x} G(x,s)J_n(s)ds \tag{5.37}$$

Recall that the Green function is the solution to the differential equation with a delta function as a source, which we then impose conditions to ensure that it is causal:

$$G''(x,s) + Q_0(x)G(x,s) = \delta(x-s) \tag{5.38}$$

with causal conditions:

$$\begin{cases} G(s-\varepsilon, s) = 0 \\ G'(s-\varepsilon, s) = 0 \quad \varepsilon > 0 \end{cases} \tag{5.39}$$

Causal in this case means that it depends on the source: just before the source there's nothing and we get trivial Green function and first derivative. In fact, we want it to be zero for all times before the source. After the source, $\delta(x-s)$ is zero, and so we want the solution to be the homogeneous solution, of which we have already found a basis of solutions: $v_0(x)$ and $v_0^*(x)$. Once we know the Green function for the delta function source, we can integrate it, and find a general solution. We can write the Green function as a linear combination of our basis with coefficients that depend on s, the location of the source:

$$G(x,s) = A(s)v_0(x) + B(s)v_0^*(x) \quad \text{for } x > s \tag{5.40}$$

We determine the coefficients $A(s)$ and $B(s)$ by imposing the continuity of $G(x,s)$ across the source, and a jump of $G'(x,s)$ across the source. The statemetn of continuity is given by:

$$\lim_{\varepsilon\to 0^+}\left(G(s+\varepsilon,s)-G(s-\varepsilon,s)\right)=0 \tag{5.41}$$

If $G(x,s)$ were to be discontinuous, the first derivative would contain a $\delta(x-s)$, and thus the second derivative would contain $\delta'(x-s)$. Since the differential equation doesn't have a $\delta'(x-s)$, the two couldn't possibly match, thus the condition that $G(x,s)$ must be continuous. The condition that we have a jump across the source is the following:

$$\lim_{\varepsilon\to 0^+}\left(G'(s+\varepsilon,s)-G'(s-\varepsilon,s)\right)=1 \tag{5.42}$$

The proof of this conditions is as follows. Take the equation of motion and integrate it across the source. The right-hand side delta function becomes 1, while the left-hand side contains an oscillatory term which over the integral can be averaged out to be the function's value times the width of the integral, 2ε, and so we're left with an integral of $G''(x,s)$:

$$\int_{s-\varepsilon}^{s+\varepsilon} G''(x,s)dx+Q(s)G(s,s)2\varepsilon=1 \tag{5.43}$$

$$\Rightarrow G'(s+\varepsilon,s)-G'(s-\varepsilon,s)=1 \quad \text{in the limit } \varepsilon\to 0 \tag{5.44}$$

where the last line is obtained by sending ε to zero. Imposing these conditions of continuity and jump, we have:

$$\begin{cases}\text{Continuity} \Rightarrow A(s)v_0(s)+B(s)v_0^*(s)=0\\ \text{Jump} \Rightarrow A(s)v_0'(s)+B(s)v_0'^*(s)=1\end{cases} \tag{5.45}$$

This gives us two equations and two unkonwns, so we can solve them to get $A(s)$ and $B(s)$. We find the answer is a quantity divided by a determinant, which corresponds precisely to the canonical Wronskian:

$$A(s)=\frac{-v_0^*(s)}{v_0(s)v_0'^*(s)-v_0'(s)v_0^*(s)}=iv_0^*(s) \tag{5.46}$$

$$B(s)=\frac{+v_0(s)}{v_0(s)v_0'^*(s)-v_0'(s)v_0^*(s)}=-iv_0(s) \tag{5.47}$$

This gives us a simple solution for the general form of the causal Green function:

$$G(x,s) = \theta(x-s)i\left(v_0^*(s)v_0(x) - v_0(s)v_0^*(x)\right) \tag{5.48}$$

where $\theta(x-s)$ is the Heavyside step function which ensures that it is causal. At order zero, we have the reference equation of motion and commutation relations, with in-vacuum solution that is Bunch-Davies 4.52. Now we have a first order correction with source $J_1(x) = -q_1(x)v_0(x) = \frac{b_1}{x^2}v_0(x)$. The solution is then:

$$w_1(x) = \int_{-\infty}^{x} G(x,s)J_1(s)ds \tag{5.49}$$

We primarily interested in the late time behavior, that is, the $x \to 0$ limit. In this limit, we find that $w_1(x)$ diverges. Our goal is to determine the divergent part, as that is the relevant quantity that ensures $|u(k,x)|^2$ freezes to the correct value. The technique is then to split the integral and identify which part is the divergent contribution. In this case, we can see the integrand diverges at 0, so we take the integrand and expand it around zero and give it a name, $\Omega_0(x,s)$:

$$\Omega_0(x,s) = \text{Series}[G(x,s)J_1(s) \text{ around } (x,0,0),(s,0,0)] \tag{5.50}$$

Now instead of solving the full integral for $w_1(x)$, we instead compute the divergent part of the integral from -1 to x, and then we pay it as a separate integral, and a final part that with the rest of the range of the integral:

$$\begin{aligned} w_1(x) = &\int_{-1}^{x} \Omega_0(x,s)ds \;\leftarrow\; \text{divergent as } x \to 0 & (5.51)\\ &+ \int_{-1}^{x} \left(G(x,s)J_1(s) - \Omega_0(x,s)\right)ds \;\leftarrow\; \text{finite} & (5.52)\\ &+ \int_{-\infty}^{-1} G(x,s)J_s(s)ds \;\leftarrow\; \text{finite} & (5.53)\end{aligned}$$

All together, we find the following expression for $w_1(x)$ keeping only the divergent part explicit:

$$w_1(x) = \frac{1}{\sqrt{2k/k_*}}\frac{ib_1}{3}\frac{1}{\frac{k}{k_*}x}\left(-2+\gamma_E+\log\left(-2\frac{k}{k_*}x\right)\right) + \text{ finite terms as } x \to 0 \tag{5.54}$$

We can then compute the first order correction to the mode function $v(x)$. Note that, expanded in the limit $x \to 0$, we can neglect the 1 and the exponential, while for $w_1(x)$

we have already kept only the terms that are needed, with whatever is left being finite terms in the limit $x \to 0$:

$$
\begin{aligned}
v(x) = & v_0(x) + w_1(x) && (5.55)\\
= & \frac{1}{\sqrt{2k/k_*}} \left(\cancel{1} - \frac{i}{\frac{k}{k_*} x} \right) e^{-i \frac{k}{k_*} x} && (5.56)\\
& + \lambda \frac{1}{\sqrt{2k/k_*}} \frac{i b_1}{3} \frac{1}{\frac{k}{k_*} x} \left(-2 + \gamma_E + \log \left(-2 \frac{k}{k_*} x \right) \right) && (5.57)\\
& + \text{finite terms as } x \to 0 && (5.58)
\end{aligned}
$$

We can then compute the modulus squared with the leading order correction:

$$
|v(x)|^2 = \frac{1}{2 \left(\frac{k}{k_*} \right)^3 x^2} \left(1 - \frac{2}{3} \lambda b_1 (-2 + \gamma_E) - \frac{2}{3} \lambda b_1 \log \left(-2 \frac{k}{k_*} x \right) + ... \right) \tag{5.59}
$$

$$
= \frac{1}{2 \left(\frac{k}{k_*} \right)^3 x^2} \left(1 - (2\epsilon_{1H*} - \epsilon_{1Z*})(-2 + \gamma_E) - (2\epsilon_{1H*} - \epsilon_{1Z*}) \log \left(-2 \frac{k}{k_*} x \right) + ... \right) \tag{5.60}
$$

Note that we have divergences that go as $1/x^2$ as well as logarithmic. There will also in principle be divergences that are higher order in λ, for instance terms that go as $\log^2$. Recall the definition of the power spectrum; the ansatz for $u(k, t)$ depends on both $v(k, x)$ and $\mu(x)$. We can use the expressions we found for the scale factor, as well as the Hubble- and Z-flow parameters in 5.22 and put them all together for find a formula for $\mu(x)$ as a log series:

$$
\begin{aligned}
\mu(x) &= \frac{k_* a(x)^2 Z(x)}{\hbar} && (5.61)\\
&= \left(1 + 2\epsilon_{1H*} + (-2\epsilon_{1H*} + \epsilon_{1Z*}) \left(\log \frac{x}{x_*} \right) + ... \right) \frac{k_*^2 Z_*}{\hbar H_*^2 x^2} && (5.62)
\end{aligned}
$$

Just as we did with $|v(x)|^2$, we find that there is a quantity that diverges as $1/x^2$ as we as additional log divergences. Clearly, the x^{-2} will cancel with each other, but it's rather remarkable that, if the coefficients on the log term weren't exactly the same for both $\mu(x)$ and $|v(x)|^2$, then the divergences would not cancel! Now we can simply take the ratio and present the result for the late-time power spectrum: [40]

$$P(k) = \lim_{t\to\infty} |u(k,t)|^2 \frac{k^3}{2\pi^2} = \lim_{x\to 0^-} \frac{|v(x)|^2}{\mu(x)} \frac{k^3}{2\pi^2} \tag{5.63}$$

$$= \left(1 - 2(1+C_0)\epsilon_{1H*} + C_0\epsilon_{1Z*} + (-2\epsilon_{1H*} + \epsilon_{1Z*}) \log\frac{k}{k_*} + ...\right) \frac{\hbar H_*^2}{4\pi^2 Z_*} \tag{5.64}$$

where C_0 is a number given by:

$$C_0 = \gamma_E + \log 2 - 2 \approx -0.729 \tag{5.65}$$

All the starred quantities are evaluated at the freezing time. Since both $H(t)$ and $Z(t)$ change in time, this technique allows us to express everything in terms of the value at the reference time plus their change in terms of flow parameters. Remember that the reference time is conventional, so when we compare to observatin we have to declare it. Let's take stock of our results. When we did the zeroth order Bunch-Davies case, we found the power spectrum with relatively little work – the power was just given by the front coefficient! To get the first order correction, we had to go through a complicated framework that keeps track order-by-order in ϵ. The constant piece in this case is not that important as it's a small correction to unity, we have the power spectrum amplitude as the value of the power at the reference k_*:

$$A_* \equiv P(k_*) = (1 - 2(1+C_0)\epsilon_{1H*} + C_0\epsilon_{1Z*}) \frac{\hbar H_*^2}{4\pi^2 Z_*} \tag{5.66}$$

However, the logarithmic term is the important one since it contains all the dependence on k! This is where we get the tilt; cosmological perturbations plus inflation gives us a tilt that is small! The spectral tilt is the log derivative of the power spectrum; in a log-log plot, $P(k)$ would look like a line at linear order in $\log\frac{k}{k_*}$. This line can then be described by its value at a given reference position plus the slope at that position. The spectral tilt in this case, at linear order, is:

$$\theta_* \equiv k\frac{d}{dk} \log P(k)\bigg|_{k=k_*} = -2\epsilon_{1H*} + \epsilon_{1Z*} \tag{5.67}$$

Chapter 6
Pre-inflationary initial conditions

Up until now, our theory of quantum perturbations has been largely model-agnostic; our results simply required that the space-time in question was at least close to de Sitter so that we could write out the perturbations in terms of an expansion around the Bunch-Davies in-vacuum. Now, we declare the model. Once more, we are working in a purely gravity-driven inflationary background, the action of which is given by Starobinsky:

$$S[g_{\mu\nu}] = \frac{1}{16\pi G}\int (R+\alpha R^2)\sqrt{-g}d^4x\,. \tag{6.1}$$

In Chapter 3, we worked on decomposing this action into a scalar-vector-tensor (SVT) decomposition, identified the dynamical degrees of freedom, and wrote down a second-order action for the perturbations:

$$\frac{\frac{d}{dt}\left(a(t)^3 Z(t)\right)}{a(t)^3 Z(t)}H(t) = \left(3-\epsilon_Z(t)\right)H(t) \tag{6.2}$$

where the characteristic function $Z(t)$ determined the behavior of scalar and tensor perturbations:

$$Z_s(t) = \frac{3\chi(t)}{16\pi G}\left(\frac{\epsilon_\chi(t)}{1-\epsilon_\chi(t)/2}\right)^2 \tag{6.3}$$

$$Z_t(t) = \frac{\chi(t)}{16\pi G} = \frac{1+2\alpha R(t)}{16\pi G} \tag{6.4}$$

Let's consider what we require from the model. The quasi-de Sitter approximation gives us a relationship between H_*, ϵ_{1H*} and α, since:

$$\epsilon_{1H}(t_*) = -\frac{\dot{H}_*}{H_*^2} \approx \frac{1}{36\alpha H_*^2} \tag{6.5}$$

Thus we need two parameters to determine the remaining one. Our expressions for the power spectrum, however, depend on ϵ_{1Z*} and Z_*; however, these functions are geometric in nature, and so can be determined by the specific background. In fact, we wish to express these characteristic functions in terms of solely the Hubble-flow parameter ϵ_{1H}; doing so allows us to keep two observational quantities, the power spectrum amplitude and the spectral tilt, and two unknowns, say H_* and ϵ_{1H*}. In principle, we could use the approximation scheme from Chapter 5, calculating Z_* and ϵ_{1Z*} at a given order in Hubble-flow parameters. However, inevitably this would introduce higher Hubble-flow parameters – but we don't have a background-agnostic manner of identifying them. We can say something about ϵ_{1H} at the onset of inflation, and even its relationship with α and H_* during inflation, but nothing of ϵ_{2H} without specifying a background. So let's specify a background! We're going to make use of another approximation scheme, one that relies entirely on ϵ_{1H}. Note that this new approximation is solely to determine the *static* values of ϵ_{1Z} and Z at the time t_*, and that this approximation is in slow-roll parameters – whereas the Chapter 5 approximation was a *time* based approximation using x as a clock, around the pivot time t_*.

Our approximation begins with an ansatz for $H(t)$ during the quasi-de Sitter phase, treated more carefully to include higher order ϵ_{1H} terms:

$$H(t) \approx \frac{1}{36\alpha\epsilon_{1H}(t)}(1 + b_1\epsilon_{1H}(t) + b_2\epsilon_{1H}^2(t) + ...) \tag{6.6}$$

The background Friedmann equation constraints $\dot{\epsilon}_{1H}(t)$, which can be solved exactly in terms of $H(t)$ and $\epsilon_{1H}(t)$:

$$\dot{\epsilon}_{1H}(t) = \frac{1 - 36\alpha H(t)^2\epsilon_{1H}(t) + 18\alpha H(t)^2\epsilon_{1H}(t)^2}{12\alpha H(t)} \tag{6.7}$$

We can impose the consistency relation that $\epsilon_{\text{ansatz}}(t) = \epsilon_{1H}(t)$ up to second order in ϵ_{1H}. This yields the following series expansion for $H_{\text{ansatz}}(\epsilon_{1H})$:

$$H_{\text{ansatz}}(\epsilon_{1H}) = \frac{1}{6\sqrt{\alpha\epsilon_{1H}}} - \frac{\epsilon_{1H}}{72\alpha} + \mathcal{O}(\epsilon_{1H}^{3/2}) \tag{6.8}$$

In order to track how many derivatives of ϵ_{1H} we require, we first consider the scalar sector as it will have more derivatives. The Ricci scalar contains $\dot{H}(t)$, which goes as ϵ_{1H}, and thus so does $\chi(t) = 1 + 2\alpha R(t)$. Then, $\epsilon_\chi(t)$, which is the log derivative of $\chi(t)$, will contain $\dot{\epsilon}_{1H}$, and thus so will $Z_s(t)$. Finally, the scalar Z-flow function is itself a log derivative, and thus will have one more derivative of ϵ_{1H}. That is, we need only find an expression for $\ddot{\epsilon}_{1H}(t)$ in this approximation to compute all the relevant quantities. Taking the derivative of $\dot{\epsilon}_{1H}(t)$ from 6.7 and substituting back in expressions for $\dot{H}(t)$ and $\dot{\epsilon}_{1H}(t)$, we find $\ddot{\epsilon}_{1H}(t)$ in terms of $H(t)$ and $\epsilon_{1H}(t)$:

$$\ddot{\epsilon}_{1H}(t) = -\frac{1}{4\alpha} + \frac{\epsilon_{1H}(t)}{3\alpha} + 9H(t)^2\epsilon_{1H}(t) + \mathcal{O}(\epsilon_{1H}^2) \tag{6.9}$$

Introducing the ansatz for $H(t)$ as a series in ϵ_{1H} gives a simple expression for $\ddot{\epsilon}_{1H}$ in a likewise series, and we find that the second derivative can be safely ignored at linear order:

$$\ddot{\epsilon}_{1H} \sim \mathcal{O}(\epsilon_{1H}^2) \tag{6.10}$$

The relevant quantities can then be computed straightforwardly from this collection of expressions of $H(\epsilon_{1H})$ and its derivatives in terms of ϵ_{1H}, $\dot{\epsilon}_{1H}$, and $\ddot{\epsilon}_{1H}$. We present the result up to first order in ϵ_{1H} :

$$Z_s(\epsilon_{1H}) = \frac{\epsilon_{1H}}{2G\pi} + \mathcal{O}(\epsilon_{1H}^2) \tag{6.11}$$

$$\epsilon_{1Z_s}(\epsilon_{1H}) = -2\epsilon_{1H} + \mathcal{O}(\epsilon_{1H}^2) \tag{6.12}$$

$$Z_t(\epsilon_{1H}) = \frac{5}{144G\pi} + \frac{1}{24G\pi\epsilon_{1H}} + \frac{\epsilon_{1H}}{108G\pi} + \mathcal{O}(\epsilon_{1H}^2) \tag{6.13}$$

$$\epsilon_{1Z_t}(\epsilon_{1H}) = 2\epsilon_{1H} + \mathcal{O}(\epsilon_{1H}^2) \tag{6.14}$$

With the input from the model, we can then express the power spectrum amplitude and its tilt and – critically – we can evaluate whether the model exhibits red or blue tilt. Already from our expressions, we can note that ϵ_{1Z} for scalars is negative so added to $-2\epsilon_{1H}$ already gives a red tilt! Meanwhile, for tensors, we find that the tilt is zero at linear order in ϵ_{1H}:

$$\theta_{s*} = -2\epsilon_{1H*} + \epsilon_{1Z*} = -4\epsilon_{1H*} + \mathcal{O}(\epsilon_{1H*}^2) \tag{6.15}$$

$$\theta_{t*} = \mathcal{O}(\epsilon_{1H*}^2) \tag{6.16}$$

As for the amplitude, the scalar power is *enhanced* by the presence of ϵ_{1H*} while in contrast, the tensor power is *suppressed* by ϵ_{1H*}! Recall that for tensors, the power spectrum gets an extra 2 from the two polarizations as well as an additional 2 from the trace of the polarization tensor. The expression given accounts already for this factor of

4:

$$A_{s*} \equiv P_s(k_*) = \frac{G\hbar H_*^2}{2\pi\epsilon_{1H*}} - \frac{(17+24C_0)G\hbar H_*^2}{12\pi} + \frac{(85+72C_0)G\hbar H_*^2\epsilon_{1H*}}{72\pi} \tag{6.17}$$

$$+\mathcal{O}(\epsilon_{1H*}^2) \tag{6.18}$$

$$\tag{6.19}$$

$$A_{t*} \equiv P_t(k_*) = \frac{24G\hbar H_*^2\epsilon_{1H*}}{\pi} - \frac{4(17+24C_0)G\hbar H_*^2\epsilon_{1H*}}{\pi} + \mathcal{O}(\epsilon_{1H*}^2) \tag{6.20}$$

For scalars, the enhanced, leading term, is the most important one, with the next order term being smaller by a factor of ϵ_{1H*}. In the numerical simulations run, we find that ϵ_{1H*} becomes as small as around 1%. Observations from Planck, presented at a reference scale of $k_* = 0.002\text{Mpc}^{-1}$ suggests a scalar amplitude around 10^{-9}, which means if $\epsilon_{1H*} \sim 1\%$, then the tensor amplitude will be around 10^{-14}! The origin of this discrepancy in power can be traced back, in this frame, to the characteristic functions $Z(t)$. When we look at the effective Hubble scale that the modes see in their equation of motion Equation 4.32, $\left(1 - \frac{1}{3}\epsilon_Z(t)\right) H(t)$, we see one very important difference between the scalar and tensor behavior shown in Figures 6.1 and 6.2.

Specifically, for scalar modes, we see a dip below the x-axis for $\left(1 - \frac{1}{3}\epsilon_Z(t)\right) H(t)$, which acts as a negative friction coefficient in the mode equation, adding power to scalar modes, whereas such behavior is not observed in the tensor $H_{\text{effective}}$.

Note that ϵ_{1H} is a geometric quantity ultimately computed from the metric tensor. It can be helpful to translate these results in terms of number of e-foldings – after all, the number of e-foldings in either the Jordan or Einstein frame is the same, allowing us to have a more intuitive sense of the power spectrum amplitude and tilt. The number of e-foldings since the time the pivot mode k_* crosses the Hubble horizon can be expressed by:

$$N_{\text{inflation}} = \int_{t*}^{t_{\text{end}}} H(t)dt = \int_{*}^{\text{end}} \frac{H(\epsilon_{1H})}{\dot{\epsilon}_{1H}} d\epsilon_{1H} \tag{6.21}$$

Recall that during quasi-de Sitter, at lowest order, ϵ_{1H} is monotonically increasing, so it makes for a reasonable clock. Plugging in our ansatz for the background and integrating, we find the following expansion of e-folds in terms of ϵ_{1H}:

$$N_* = -\frac{1}{2} + \frac{1}{2\epsilon_{1H*}} \tag{6.22}$$

It's easy to reverse this relationship and then write our power spectrum quantities in terms of series of number of efoldings since the time t_*. Moreover, we can mirror the

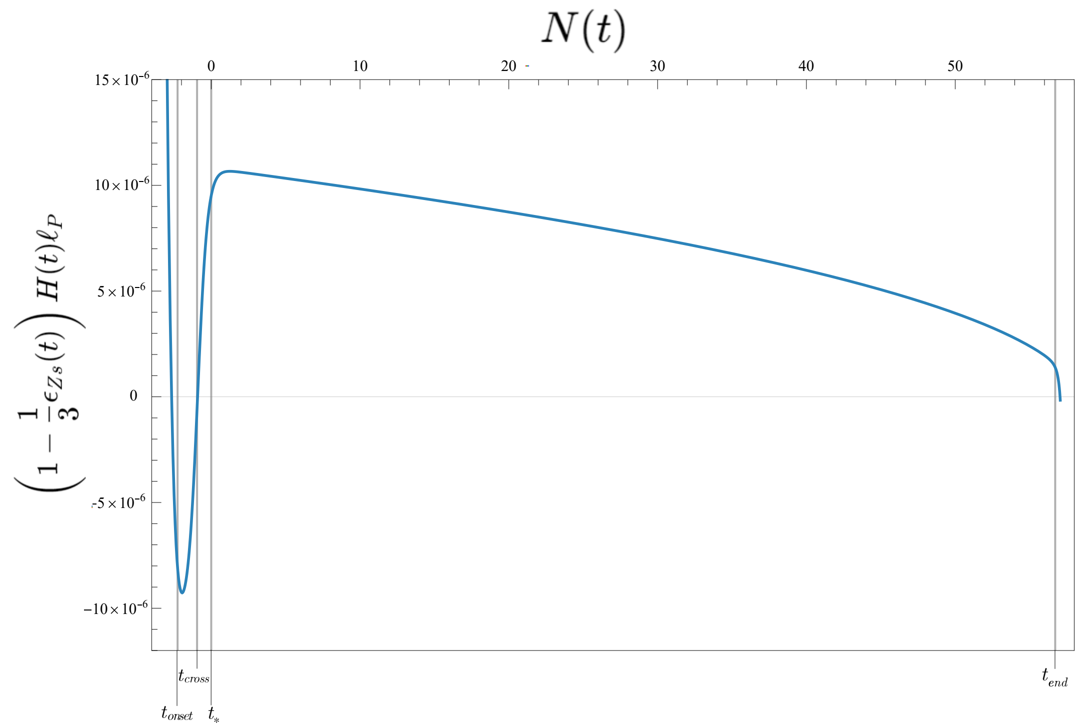

Figure 6.1. The quantity $\left(1-\frac{1}{3}\epsilon_Z(t)\right)H(t)$, which appears in the friction term in 4.32 becomes negative for scalar modes, effectively acting as a negative friction and adding power into the modes.

expression that gives us the total number of e-foldings from H_c and ask only that it gives us the amount of e-foldings since the mode k_* crossed the horizon, $N_* = 18\alpha H_*^2$. In turn, we can rewrite the power spectrum quantities in terms of the number of e-foldings and constants of the theory:

$$\theta_{s*} = -\frac{2}{N_*} + \mathcal{O}(N_*^{-2}) \tag{6.23}$$

$$\Rightarrow n_{s*} = \theta_{s*} + 1 = 1 - \frac{2}{N_*} + \mathcal{O}(N_*^{-2}) \tag{6.24}$$

$$\theta_{t*} = \mathcal{O}(N_*^{-2}) \tag{6.25}$$

$$A_{s*} = \frac{G\hbar N_*^2}{18\pi\alpha} - \frac{(17+24C_0)G\hbar N_*}{216\pi\alpha} + \frac{(85+72C_0)G\hbar}{2592\pi\alpha} \tag{6.26}$$

$$+\mathcal{O}(N_*^{-2}) \tag{6.27}$$

$$\tag{6.28}$$

$$A_{t*} = \frac{12G\hbar N_*^2}{18\pi\alpha} - \frac{2(17+24C_0)G\hbar H_*^2}{18\pi\alpha} + \mathcal{O}(N_*^{-2}) \tag{6.29}$$

Given the central value data from the Planck 2018 collaboration [41], evaluated at

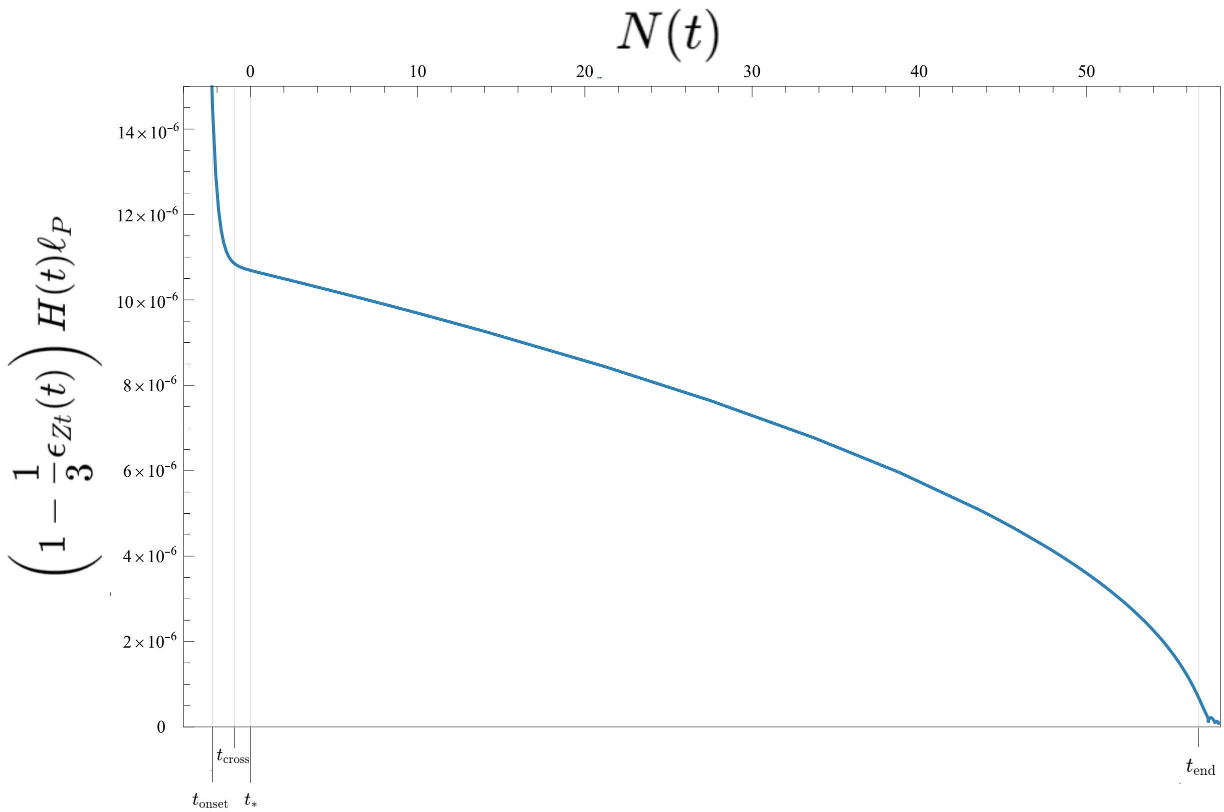

Figure 6.2. The quantity $\left(1-\frac{1}{3}\epsilon_Z(t)\right)H(t)$, which appears in the friction term in 4.32, remains positive for all time for tensor modes.

the mode $k_* = 0.05\text{Mpc}^{-1}$, namely:

$$A_s = 2.099 \times 10^{-9} \tag{6.30}$$

$$n_s = 0.9649 \tag{6.31}$$

we find the following best-fit values for the number of e-foldings N_* and the coupling constant α:

$$N_* = 56.98 \text{ e-foldings} \tag{6.32}$$

$$\alpha = 2.73 \times 10^{10}\ell_P^2 \tag{6.33}$$

as well as values for the geometric quantities, H_* and ϵ_{1H*}:

$$H_* = 1.07 \times 10^{-5}\frac{1}{\ell_P} \tag{6.34}$$

$$\epsilon_{1H*} = 8.70 \times 10^{-2} \tag{6.35}$$

The tensor-to-scalar ratio r, which is often used in the literature to characterize inflationary models in an observational space, can now be readily computed from the leading order terms of the power amplitudes. Expressed as a series up to second order, this is:

$$r = 48\epsilon_{1H*}^2 + \mathcal{O}(\epsilon_{1H*}^3) \tag{6.36}$$

Or in terms of number of e-foldings:

$$r = \frac{12}{N_*^2} + \mathcal{O}(N_*^{-3}) \tag{6.37}$$

A few remarks are in order. While the analysis in Chapter 5 using the Green function method can be used to find the power spectrum up to arbitrary order in ϵ_i, it quickly becomes computationally difficult to go to higher than linear orders. The result presented, in a background-agnostic manner, was given at linear order in ϵ_i, so the results presented here are likewise at linear order. When we do so, however, we find a quadratic order tensor-to-scalar ratio! Should we trust this result? Certainly, we can say that the tensor-to-scalar ratio is at lowest order quadratic. A paper by de Felice which approaches the solution of the mode equation in a different manner – through Bessel functions – finds the same coefficient. In addition, we find the coefficient to be consistent with our numerical simulations.

In the literature, one often sees a consistency relation for single scalar field inflation which relates the tensor-to-scalar ratio r to the tensor spectral tilt n_T:

$$r \approx -8n_T \tag{6.38}$$

It's a rather simple equation, which comes from noting that $r = 16\epsilon$ and $n_T \approx 2\epsilon$ in the inflaton framework. At linear order, it seems, we cannot say much about the consistency relation. Numerical results, however, suggest that the consistency relation is satisfied in this framework – which in turn suggests the following expression for n_T at quadratic order:

$$n_T \approx -6\epsilon_{1H*} \approx -\frac{3}{2N_*^2} \tag{6.39}$$

We'll see numeric results and consistency checks later on in this chapter.

For now, we turn to analytic, pre-inflationary modifications to the power spectrum.

6.1 Pre-inflationary effect on the power spectrum

Let's review our pre-inflationary initial conditions. Certainly, we can use Friedmann equations and the ΛCDM model to get an idea of the evolution of the universe in terms of e-foldings. Starting with the Friedmann equation, and for simplicity assuming a perfectly flat universe:

$$H(t) = H_0 \sqrt{\frac{\Omega_r}{a(t)^4} + \frac{\Omega_m}{a(t)^3} + \Omega_\Lambda} \tag{6.40}$$

which, since $a(N) = e^{(N - N_{\text{today}})}$ setting $a_{\text{today}} = 1$, we can write Hubble during the post-reheating universe as:

$$H(N) = H_0 \sqrt{\frac{\Omega_r}{e^{(4(N - N_{\text{today}}))}} + \frac{\Omega_m}{e^{(3(N - N_{\text{today}}))}} + \Omega_\Lambda} \tag{6.41}$$

Similarly, we can find a numerical expression for $H(N)$ during a pre-inflationary and inflationary time. Starting with the Starobinsky equation, we need only two initial conditions to find a unique solution. These are chosen at the crossing time, when the pre-inflationary approximation of $H(t)$ crosses the attractor line $\dot{H} = -\frac{1}{36\alpha}$, which can be rewritten in terms of an e-fold derivative, $\frac{dH}{dN} = \frac{dH}{dt}\left(\frac{dN}{dt}\right)^{-1} = \frac{\dot{H}}{H}$. With the additional choice of $H_c = \frac{\mu}{\sqrt{\alpha}}$, with μ remaining a parameter we can tune to see how much pre-inflationary evolution is available between the observation boundary and the quantum gravity theoretical boundary. We can now put together the evolution of the universe, in particular looking at the physical Hubble rate $a(t)H(t)$ during these two epochs on the same log-log plot, along with the band of observable modes.

A few remarks are in order. There's a clear minimum value that μ must take on – around $\mu \sim 1.8$, otherwise the pivot mode k_* would not have entered the Hubble horizon in time to coincide with observations. Modes above the Hubble horizon oscillate, so as a principle, we could set the initial state to be that of the Bunch-Davies vacuum, while those below are frozen. It is sensible to use this principle – whatever are the details of the transition between full quantum gravity and quantum field theory, the transition must be gentle enough that inflationary evolution is not disrupted. A choice of vacuum assures that is the case.

We would prefer to utilize the same principle with all modes, so we can push the value of μ a little to include all modes, and use the same principle. However, at this point, we could keep pushing the value of μ so that modes are oscillatory even further in the past – meaning at $\mu > 1.8$, there is room for observable scalar modes to be affected by pre-inflationary effects! We can instead use as our principle setting a $pre-inflationary$ vacuum state that evolves naturally and provides a new initial state at the onset of inflation. As we'll see in this Chapter, this'll be a squeezed vacuum that can be fully characterized by Bogoliubov coefficients. The effect of this squeezing is to *suppress* modes that are near the bottom of the well – the longest wavelength modes we observe – while leaving high k modes largely untouched. This is a boon – allowing a modification of the power spectrum of the CMB at large scales, which, as it's limited by cosmic variance, still presents a challenge in fully understanding its features, with some hints that there may be deviation from scale invariance. [42]

For larger values of μ, the physical length scale in the far past becomes much smaller than all observable modes – at this point, we begin to lose the effects of power suppression, leaving behind the usual Bunch-Davies power spectrum. Even further increasing μ gives us a background in which our framework is no longer even valid – as soon as the transition from pre-inflationary to inflationary occurs in the band in which we no longer trust quantum field theory in curved space-time, we necessarily must move to a full theory of quantum gravity. Thus, in terms of this framework, we have a hard lower bound on μ, a hard upper bound on μ, and moreover, a more restrictive, soft upper bound on μ, where, if we hope to be able to resolve observations within our lifetimes, we cannot have a value of μ that strays far from 1.8.

In order to be aligned with observations, any pre-inflationary effect must leave the power spectrum nearly scale invariant at high k – so when do we begin to see pre-inflationary features? Consider in our diagram, a k when we transition from pre-inflationary to inflationary evolution, named k_{feature}. We seek to write this critical k in terms of our inflationary e-folds parameter μ. For the sake of simplicity, take an inflationary background that is purely de Sitter. In this background, the slope of the inflationary section of the log-log plot is:

$$\frac{d(\log(\frac{a(t)H(t)}{a_0 H_0}))}{dN} = \frac{d\log(e^{H_0 t})}{dN} = \frac{d(H_0 t)}{dN} = \frac{dN}{dN} = 1 \tag{6.42}$$

Recall that $N_{\text{infl}} = 18\mu^2$ captures how many total inflationary efoldings there are, while N_* is the amount of inflationary efoldings since the pivot mode crossed the Hubble horizon. Putting together these ideas, using the slope of the log-log graph, we can write the following equality:

$$\log\left(\frac{k_*}{k_{\text{feature}}}\right) = N_{\text{infl}} - N_* \tag{6.43}$$

$$\Rightarrow k_{\text{feature}} = e^{-(N_{\text{infl}}-N_*)} k_* \tag{6.44}$$

$$\Rightarrow k_{\text{feature}} = e^{-(18\mu^2-N_*)} k_*, \tag{6.45}$$

Thus, if we require that $k_{\text{feature}} < k_*$, that is, features should only be observed when we change the dynamics from the standard framework below the pivot mode k_*, then $18\mu^2 > N_*$, or $\mu > \sqrt{\frac{N_*}{18}}$, which for $N_* \approx 60$ gives $\mu \approx 1.8$, giving a hard lower bound on μ that is of order unity. We can also reverse this logic – under the assumption that the pre-inflationary evolution doesn't affect the quasi-de Sitter result at the scale k_*, and given that this result predicts the inflationary energy scale at α, taking $\mu \approx 1$ as a principle gives a number of efoldings compatible with observation that can be written in terms of the relevant inflationary scale, $H_\times \approx \frac{1}{\sqrt{\alpha}}$ with $N_{\text{infl}} = 18H_\times^2\alpha$.

As a small aside, we can posit an upper bound of total pre-inflationary e-foldings given that any reasonable classical trajectory must maintain the usual quasi-de Sitter result. Consider once more the log plot of the physical length scale $a(t)H(t)$ against e-foldings. We can calculate the derivative of the plot during the pre-inflationary era:

$$\frac{d\log\frac{a(t)H(t)}{a_0H_0}}{dN} = \frac{1}{a(t)H(t)}\left(\dot{a}(t)H(t) + a\dot{H}(t)\right)\frac{dt}{dN} \tag{6.46}$$

$$= \frac{1}{H(t)}(H(t)^2 + \dot{H}(t))\frac{dt}{dN} \tag{6.47}$$

$$= \frac{1}{H(t)^2}(H(t)^2 + \dot{H}(t)) = 1 - \epsilon_{1H}(t) \tag{6.48}$$

where the second-to-last equality comes from $\frac{dN}{dt} = H(t)$. This also gives us a simple manner of identifying extrema of comoving scale: whenever $\epsilon_{1H}(t) = 1$. Since during the period of pre-inflation the space-time behaves like a radiation background, we have $\epsilon_H = 2$ during this period, giving us a plot derivative of -1, that is, the logarithmic difference in physical scale directly gives us the amount of e-folds that have passed. The maximum amount of physical scale that we could have during pre-inflation is from the Planck scale up to the α scale of inflation. That is, we have a maximum number of pre-inflationary e-foldings. During this radiation-like phase, the expressions for the scale factor and Hubble rate are quite simple, and a quick substitution allows us to rewrite the scale factor in terms of Hubble as $a(t) = \frac{a_0}{\sqrt{2H(t)}}$, or $a(t)H(t) = \frac{a_0}{\sqrt{2}}\sqrt{H(t)}$. Putting these ideas together, we find the expression for the maximum number of pre-inflationary e-foldings:

$$\begin{aligned}
N_{\text{pre-inflationary}} &= \log\frac{a_{\text{Planck}}H_{\text{Planck}}}{a_0H_0} - \log\frac{a_\alpha H_\alpha}{a_0H_0} && (6.49)\\
&= \log\frac{a_{\text{Planck}}H_{\text{Planck}}}{a_\alpha H_\alpha} && (6.50)\\
&= \log\frac{\sqrt{H_{\text{Planck}}}}{\sqrt{H_\alpha}} && (6.51)\\
&= \log\frac{\sqrt{L_\alpha}}{\sqrt{\ell_P}} && (6.52)\\
&= \log\frac{\sqrt{10^5\ell_P}}{\sqrt{\ell_P}} = \frac{1}{2}\log 10^5 \sim 5 \text{ e-folds}\,. && (6.53)
\end{aligned}$$

There is a curious coincidence that occurs near $\mu \sim 1$; namely, the primordial universe largest comoving scale at the onset of inflation is roughly the same as the post-Hot Big Bang universe's largest comoving scale that occurs near the matter-dark energy transition. Can we turn this observation into a principle? It's worth remembering that the CMB is a probe – the surface of last scattering we observe bears the temperature gradients and photon polarizations caused by early universe density variations and gravitational waves. As any probe, it has a physical size, the scale of which is given by the comoving scale at the time of reionization. Certainly, this is a drawn-out process that doesn't occur all at the same time, but using the usual redshift for the CMB, $z_{CMB} = 1100$, we can find how many e-folds ago this occurred:

$$\frac{1}{a(t)} = 1 + z \Rightarrow \frac{1}{a_0e^N} = 1 + z \tag{6.54}$$

$$N_{CMB} = -\log(1 + z_{CMB}) \approx -7 \tag{6.55}$$

In cosmology, we often write down the Friedmann equation for the modern universe in terms of the energy content of the universe: radiation, matter, curvature and the cosmological constant:

$$H(t) = H_0\sqrt{\frac{\Omega_r}{a(t)^4} + \frac{\Omega_m}{a(t)^3} + \frac{\Omega_k}{a(t)^2} + \Omega_\Lambda} \tag{6.56}$$

where the nominal values are given by today's Hubble rate $H_0 = 67.4$km/s/Mpc, unitless density parameters $\Omega_r = 9.0 \times 10^-5$, $\Omega_m = 0.32$, $\Omega_\Lambda = 0.68$, and where for simplicity we'll take a flat universe with $\Omega_k = 0$. Rewriting the Friedmann equation in terms of efoldings makes it easy to write the expression for the scale factor as well:

$$H(N) = H_0\sqrt{\Omega_r e^{-4(N-N_{\text{today}})} + \Omega_m e^{-3(N-N_{\text{today}})} + \Omega_\Lambda} \tag{6.57}$$

$$a(N) = e^{(N-N_{\text{today}})} \tag{6.58}$$

where we have set $a_0 = a_{\text{today}} = 1$. With these, we can determine the comoving scale at the time of the CMB:

$$k_{CMB} = a(N_{CMB})H(N_{CMB}) \approx 0.0048\text{Mpc}^{-1} \tag{6.59}$$

Using the equation 6.43, we can find the corresponding parameter μ that gives k_{CMB} as the largest comoving mode at the onset of inflation:

$$\mu = \frac{1}{3\sqrt{2}}\sqrt{N_{\text{infl}} - \log\frac{k_*}{k_{\text{CMB}}}} \approx 1.8 \tag{6.60}$$

We've seen already that $\mu \approx 1.8$ is compatible with observations, giving us just enough inflation so that pre-inflationary features might appear at the largest comoving scales. That is, we could take the following principle for setting our initial condition: *The largest comoving scale at the time of the CMB, which serves as a probe for primordial perturbations, should be the same size comoving scale at the time of the onset of inflation.* Note that we state this as a principle – we do not provide a mechanism for this occur.

This idea, however, does still rely on a particular measurement apparatus – the CMB – as well as a special time in the universe, namely the Recombination of electrons and protons into electrically neutral atoms. So let's consider one last principle to set just the right amount of inflation: censorship.

In classical general relativity, singularities must be censored lest they create causal paradoxes such as the closed time-like curves or create pathologies where considering a Cauchy surface, rendering the initial value formulation ill-defined. Singularities, then, must be classically censored by a horizon. When considering quantum mechanical effects in conjunction with general relativity, Hawking radiation leads to a case in which a naked singularity is present only at the end of the black hole's life. Our proposal, then, is a very mild violation of this censorship, but for the initial cosmological singularity. The inflationary framework already censors the Planck phase, functioning like the classical horizon of a black hole. By lasting for a long enough time, every mode that we observe today must've become unfrozen at some point in the past, whereas any mode that remained frozen during inflation, and thus bearing direct imprints from the Planck era, is permanently inaccessible to us in the future, purely by virtue of the matter-energy makeup of our universe. Inflation, then, is a past-facing censor analogue to the black hole horizon future-facing censor. Is there, then, an analogue of Hawking evaporation?

We propose thusly the principle for setting a *maximum* number of inflationary e-foldings: inflation should last just long enough to barely censor the initial singularity. What happens when there's just enough inflation, so that the Planck regime goes from a hard censor to a soft blurring, from a cosmic umbra to a cosmic penumbra, from an iron curtain to a gossamer veil?

6.2 Cosmic veiling

Before we look at the consequences of this cosmic veiling, let's establish its condition. Figure 6.3 shows examples of two backgrounds which are not cosmically veiled. We've established that we ought to observe features at the scale where the background goes from pre-inflationary to inflationary. The top picture gives us an example of a naked initial singularity - the feature is observed at small angular scales and thus details from the Planck era ought to be imprinted in the sky today. We don't observe this, so this scenario is excluded.

The bottom picture presents a cosmically censored scenario. Inflation lasts long enough to push any features that might show up past the largest angular scale we'd ever see in our universe, censoring any Planck effects completely. This is a possible scenario, but ultimately wouldn't present anything interesting for observation.

Cosmic veiling condition ought to occur when the largest co-moving scale in the primordial universe matches the largest co-moving scale observed in the modern epoch, as shown in Figure 6.4. In this case, the scale at which features arise is available to us, though they are difficult to probe as they correspond to the largest angular scales of the universe, which ultimately have challenging to determine statistically significant behavior due to cosmic variance.

Let $N \equiv 0$ at the end of inflation, when $\epsilon = 1$. Define N_{re} to be the amount of e-foldings at the end of reheating, and onset of the radiation dominated phase. Note that there isn't a definite hard transition from reheating to radiation – simply as the temperature of the universe rises, the equation of state that governs the evolution of the universe will change from $w = 0$ (matter) to $w = \frac{1}{3}$ (radiation) in a smooth manner. We can separate the log co-moving scale into a log ratio of scale factors and log ratio of

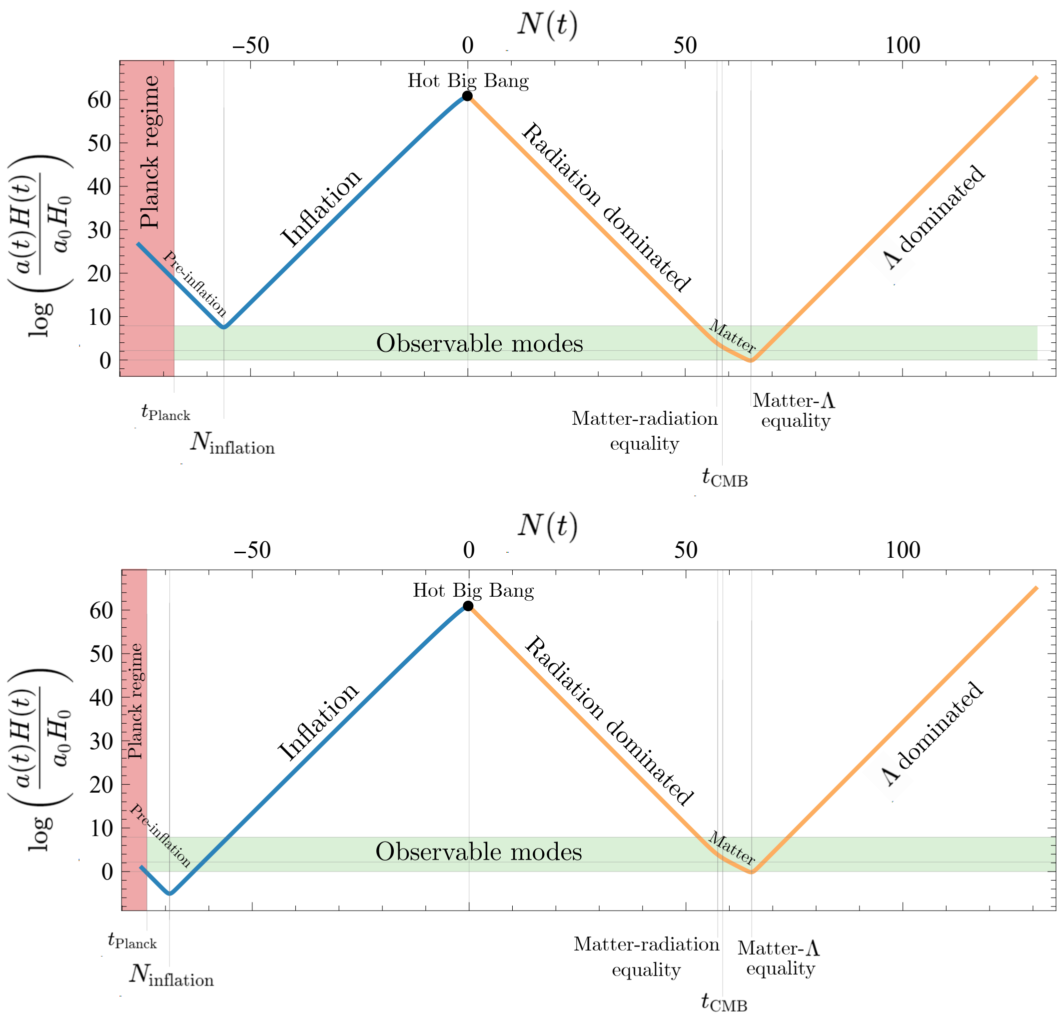

Figure 6.3. Pictured are two examples of non-cosmically veiled backgrounds. The co-moving scale aH is shown as a function of e-folds across the lifetime of the universe. The primordial co-moving scale that precedes the Hot Big Bang at some point must cross into the quantum gravity regime at the Planck scale. The top figure shows a background that is disallowed by observation, while the bottom figure shows a background that is cosmically censored.

Hubble parameter.

$$\Delta = \log \frac{a_{\max} H_{\max}}{a_{\min} H_{\min}} \tag{6.61}$$

$$= \log \frac{a_{\max}}{a_{\min}} + \log \frac{H_{\max}}{H_{\min}} \tag{6.62}$$

Recall that during inflation, the slope on the $\log aH - -N$ plane is nearly 1, so this target value, which we call Δ, is also related to the amount of inflationary e-foldings. The log of ratio of scale factors is simply the amount of e-folds from the maximum of

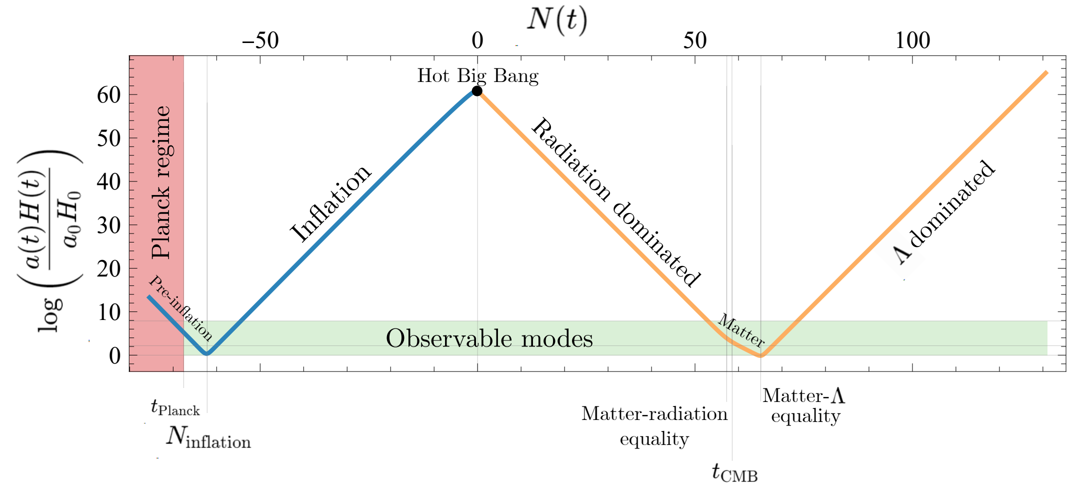

Figure 6.4. Pictured is a cosmically veiled background. Cosmic veiling occurs when the largest scales in the primordial and modern universe are nearly the same value. This means that the scale at which additional features arise are available to us. The comoving scale aH is shown as a function of e-folds across the lifetime of the universe. Two solutions are joined – the primordial comoving scale prior to the Hot Big Bang, and the comoving scale afterwards, during the radiation-, matter, and finally $\Lambda-$dominated universe. The red area is the full quantum gravity regime at the Planck scale. The band of observable modes k that are accessible to us by the CMB is shown in green.

co-moving scale to its minimum, $\Delta N_{\text{max}\to\text{min}}$. This value can be separated into a part we understand well and an uncertain amount of e-folds. The primary uncertainty comes from the amount of reheating e-foldings which is not well known. On the other hand, given the ΛCDM model of the universe, we have a good understanding of how the universe will evolve after the Hot Big Bang. Thus, we can separate $\Delta N_{\text{max}\to\text{min}}$ into $N_{\text{reheat}} + N_{\Lambda\text{CDM}}$ (see Figure 6.5).

Given an evolution of the present universe given by 6.57 and measuring from the Hot Big Bang up to the minimum of co-moving scale in modern epoch, neglecting the contribution of radiation since $\Omega_r \ll \Omega_\Lambda, \Omega_m$, we find an expression including the number of e-folds $N_{\Lambda\text{CDM}}$:

$$\frac{d}{dN}\log\left(a(N)H(N)\right) = 0 \tag{6.63}$$

$$\Rightarrow N_{\Lambda\text{CDM}} + N_{\text{reheat}} \approx N_{\text{today}} - \frac{1}{3}\log\left(\frac{2\Omega_\Lambda}{\Omega_m}\right). \tag{6.64}$$

We see that $N_{\Lambda\text{CDM}}$ is in our recent past, with $\frac{1}{3}\log\left(\frac{2\Omega_\Lambda}{\Omega_m}\right) \approx 0.5$ e-foldings ago, or in

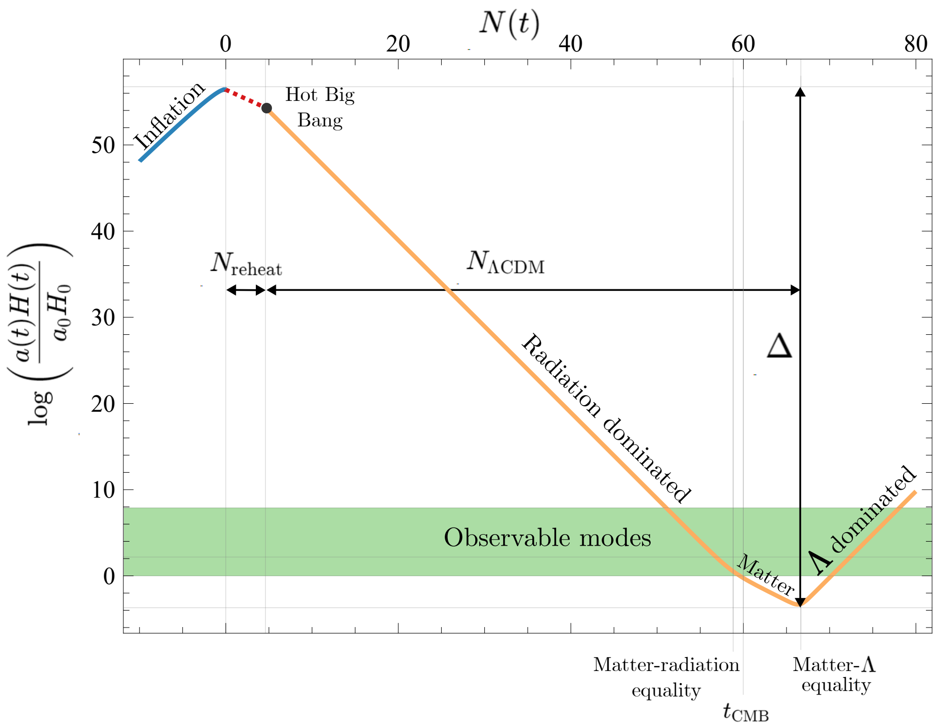

Figure 6.5. A close up picture of the co-moving scale during the modern epoch of the universe. This figure reveals an uncertainty in the amount of e-foldings that occur from the end of inflation to the present day – that of reheating e-foldings (red dotted line). The amount of e-foldings $N_{\Lambda\text{CDM}}$ is well understood in the context of the ΛCDM model. The value Δ which gives us the log difference in co-moving scale from its maximum to minimum also corresponds to the amount of inflationary e-foldings for a cosmically-veiled background.

terms of redshift, we have:

$$z = \frac{a_{\text{today}}}{a(t)} - 1 = e^{\Delta N} - 1 \approx \sqrt{e} - 1 \approx 0.64 \tag{6.65}$$

At this point, it is easy to calculate H_{min} and a_{min}:

$$H_{\text{min}} = H(N_{\Lambda\text{CDM}}) = H_0\sqrt{3\Omega_\Lambda} \tag{6.66}$$

$$a_{\text{min}} = \left(\frac{\Omega_m}{2\Omega_\Lambda}\right)^{1/3} \tag{6.67}$$

Our ignorance is now encoded in the value of N_{today}. We can push this uncertainty onto N_{reheat} by defining the value of the Hubble parameter at the onset of the Hot Big Bang to be the one given at the end of reheating, that is, impose $H(N_{\text{reheat}}) \equiv H_{\text{BB}}$. Further, at the onset of the Hot Big Bang, we can make a further assumption that the matter and vacuum energy content of the universe is negligible compared to the radiation energy content, allowing the expression, from 6.57:

$$N_{\text{today}} = N_{\text{reheat}} + \frac{1}{2}\log\frac{H_{\text{BB}}}{H_0\Omega_r} \tag{6.68}$$

Recall that the vacuum density parameter is given by $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$, allowing us to rewrite N_{today} without the value of the Hubble parameter today:

$$N_{\text{today}} = N_{\text{reheat}} + \frac{1}{4}\log\frac{3H_{\text{BB}}^2\Omega_\Lambda}{\Lambda\Omega_r} \tag{6.69}$$

Together with Equation 6.63 , we get then a refined expression for $N_{\Lambda\text{CDM}}$:

$$N_{\Lambda\text{CDM}} \approx \frac{1}{4}\log\left(\frac{H_{\text{BB}}^2}{\Lambda}\right) + \frac{1}{12}\log\left(\frac{27}{16}\frac{\Omega_m^4}{\Omega_r^3\Omega_\Lambda}\right) \tag{6.70}$$

Now we are able to sharpen and focus on the notion of constraining the total amount of reheating e-foldings. Consider a reheating phase described by an equation of state w_{reheat} where w_{reheat} lies between its matter value and radiation value: $0 \le w_{\text{reheat}} \le \frac{1}{3}$. The expression for the scale factor in terms of the equation of state can be generically written as $a(t) = t^{\frac{2}{3(w+1)}}$. Alongside $a(N) = e^N$, where N here is measured from the maximum of $\log aH$, we can rewrite the Hubble factor in terms of e-folds and the equation of state:

$$H(N) = \frac{2}{3}\frac{e^{-\frac{3N}{2}(1+w_{\text{reheat}})}}{1+w_{\text{reheat}}} \tag{6.71}$$

For small values of w_{re}, we can expand the expression for the logarithm of the comoving scale and find a relatively simple expression, up to first order in w_{reheat} :

$$\log(aH) = -w_{\text{reheat}} - \frac{1}{2}N(1+3w_{\text{reheat}}) - \log\frac{3}{2} \tag{6.72}$$

Note that the sign next to e-folds will always be negative for positive values of w_{reheat} – whenever the matter content is non-vacuum or non-inflationary, the comoving scale contracts. For the special cases in which the equation of state describes matter or radiation, we have the following result:

$$\log(aH) = -\frac{1}{2}N - \log\frac{3}{2} \qquad \text{(matter)} \tag{6.73}$$

$$\log(aH) = -N - \frac{1}{3} - \log\frac{3}{2} \qquad \text{(radiation)} \tag{6.74}$$

We see a general form of the expression that is $\log(aH) = -\eta(N)N + \text{const}$, where we've made the contraction of the comoving scale explicit and with $\frac{1}{2} \leq \eta(N) \leq 1$ is a function that describes how the coefficient of N, that is, the slope in the $\log aH - N$ diagram, changes, depending on the variable equation of state w_{reheat} during reheating. Now we are free to manipulate this expression in order to determine bounds for H_{BB}:

$$\log H = -\log a - \eta(N)N + \text{const} \tag{6.75}$$
$$= -N - \eta(N)N + \text{const} \tag{6.76}$$
$$= -(1 + \eta(N))N + \text{const} \tag{6.77}$$
$$\Rightarrow H(N) = e^{-(1+\eta(N))N} H_{\text{max}}, \tag{6.78}$$

where H_{max} is the value of $H(N)$ at $N = 0$, the point where the comoving scale is maximum. This quantity is in turn easily computed, as quasi-de Sitter inflation was defined by $\dot{H} \approx -\frac{1}{36\alpha}$, ending when $\epsilon = 1$, thus giving:

$$H_{\text{max}} = \frac{1}{6\sqrt{\alpha}} \tag{6.79}$$

Assuming the evolution of the comoving scale is smooth during reheating, bounds can be placed on the value of H_{BB} with 6.78. Taking the two extremes in which reheating is either a pure matter dominated or radiation dominated phase, we find the following lower bound from radiation-modeled reheating and upper bound from matter-modeled reheating:

$$e^{-2N_{\text{reheat}}} H_{\text{max}} \leq H_{\text{BB}} \leq e^{-\frac{3}{2}N_{\text{reheat}}} H_{\text{max}} \tag{6.80}$$

In either case, H_{BB} is smaller than H_{max} for a nontrivial amount of reheating e-foldings, while an equality is only achieved when reheating is instantaneous, so $N_{\text{reheat}} = 0 \Rightarrow H_{\text{BB}} = H_{\text{max}}$. Putting this together with 6.70, we can put together an expression for the log change in comoving scale from the end of inflation to the minimum value it ever reaches, which we define by the quantity Δ:

$$\Delta = \log \frac{a_{\text{max}} H_{\text{max}}}{a_{\text{min}} H_{\text{min}}} \tag{6.81}$$
$$= \log \frac{a_{\text{max}}}{a_{\text{min}}} + \log \frac{H_{\text{max}}}{H_{\text{min}}} \tag{6.82}$$
$$= -N_{\Lambda\text{CDM}} - N_{\text{reheat}} + \log\left(\frac{1}{6\sqrt{\alpha\Lambda}}\right) \tag{6.83}$$

$$= -N_{\text{reheat}} - \frac{1}{4}\log\left(\frac{H_{\text{BB}}^2}{\Lambda}\right) - \frac{1}{2}\log(\alpha\Lambda) - \log 6 - \frac{1}{12}\log\left(\frac{27}{16}\frac{\Omega_m^4}{\Omega_r^3\Omega_\Lambda}\right) \tag{6.84}$$

Now we can plug in our result relating the Hubble scale at the end of inflation to that at the onset of the Hot Big Bang, where for the sake of simplicity, we've taken $\eta(N)$ to now be simply η as an unknown average value, but which must lie between the extremes of matter and radiation, that is, $\frac{1}{2} \leq \eta \leq 1$:

$$H_{\text{BB}} = e^{-(\eta+1)N_{\text{reheat}}} H_{\max} = e^{-(\eta+1)N_{\text{reheat}}} \frac{1}{6\sqrt{\alpha}} \tag{6.85}$$

$$\Rightarrow -\frac{1}{4}\log\left(\frac{H_{\text{BB}}^2}{\Lambda}\right) = \frac{1}{2}(\eta+1)N_{\text{reheat}} + \frac{1}{4}\log(\alpha\Lambda) + \frac{1}{2}\log 6 \tag{6.86}$$

which gives our final expression for the log difference of comoving scales Δ:

$$\Delta = -\frac{1}{2}(1-\eta)N_{\text{reheat}} - \frac{1}{4}\log(\alpha\Lambda) - \frac{1}{2}\log 6 - \frac{1}{12}\log\frac{27}{16} + \frac{1}{12}\log\frac{\Omega_r^3\Omega_\Lambda}{\Omega_m^4} \tag{6.87}$$

$$= \frac{1}{4}\log\left(\frac{1}{\alpha\Lambda}\right) - \frac{1-\eta}{2}N_{\text{reheat}} - \frac{1}{12}\log\left(4 \times 3^9 \frac{\Omega_m^4}{\Omega_r^3\Omega_\Lambda}\right) \tag{6.88}$$

Some remarks are in order. Firstly, having made explicit the expected signs of every term, we can evaluate the relative sizes of each of these contributions. The first part, which depends on the energy scale of inflation and the modern, accelerating expansion of the universe, yields the largest contribution to Δ! Using approximate values in Planck units gives an approximate value:

$$\frac{1}{4}\log\left(\frac{1}{\alpha\Lambda}\right) \approx \frac{1}{4}\log\left(\frac{1}{10^{10}10^{-122}}\right) \approx \frac{1}{4}\log e^{(2.3\times 112)} \approx 64.5 \tag{6.89}$$

Recall that this Δ corresponds precisely to the amount of inflationary e-foldings, since the derivative of $\log aH$ with respect to N is exactly 1 during inflation. Thus the leading term in the calculation, which was motivated purely from an argument from cosmic censorship, gives an amount of inflation that is very near to the current minimum amount of inflation compatible with observation. The question remains: do the following terms significantly change this value, and how large can N_{reheat} become before being disallowed by current observation? Beginning with the matter content term, given nominal values, we find:

$$\frac{1}{12}\log\left(4 \times 3^9 \frac{\Omega_m^4}{\Omega_r^3\Omega_\Lambda}\right) \approx 2.9 \tag{6.90}$$

This is an order of magnitude smaller than the leading term, though still provides enough of a decrease in e-foldings to be relevant to our calculation. We ought to sharpen our notion of our minimal number of e-foldings to be compatible with observation.

Recall the results of the power spectrum spectral tilt in terms of the number of e-folds N_* since the pivot mode k_* crossed the Hubble horizon 6.23:

$$n_{s*} = 1 - \frac{2}{N_*} \tag{6.91}$$

Solving for the number e-folds, we find $N_* = \frac{2}{1-ns} \approx 57$ e-foldings, given the central value for the spectral tilt as given in the Planck collaboration 2018 paper on the constraints on inflation. The worst case scenario for the Δ is to have $\eta = \frac{1}{2}$, dominated by matter, in which case we find our bound for Δ is given by:

$$\Delta - N_* \approx 3.92 - \frac{N_{\text{reheat}}}{4} \Rightarrow N_{\text{reheat}} \approx 15.7 \tag{6.92}$$

That is, as long as the amount of e-folds that occurred during reheating is less than 16, we can use the principle of cosmic censorship to impose a maximum amount of total inflationary e-foldings without coming into tension with observation.

We can sanity-check this result by taking the temperature of the CMB today and blueshift it by the predicted e-folds into the past and seeing if it matches our expectation of the energy scale at the Hot Big Bang. We know temperature in cosmological space-times scales with redshift, which allows us to write the temperature at the time of the Hot Big Bang in terms of e-folds since reheating:

$$T(t) = T_0\,(1+z) = \frac{T_0}{a(t)} = T_0 e^{-N} \tag{6.93}$$

The time of the minimum of co-moving scale is very near to today, so we'll simply take the amount of e-foldings $N_{\Lambda\text{CDM}}$, and approximate the Hot Big Bang energy scale with the energy scale α and check self-consistency:

$$\begin{aligned} N_{\Lambda\text{CDM}} &= \frac{1}{4}\log\left(\frac{H_{\text{BB}}^2}{\Lambda}\right) + \frac{1}{12}\log\left(\frac{27}{16}\frac{\Omega_m^4}{\Omega_r^3\Omega_\Lambda}\right) && (6.94)\\ &= \frac{1}{4}\log\left(\frac{1}{36\alpha\Lambda}\right) + \frac{1}{12}\log\left(\frac{27}{16}\frac{\Omega_m^4}{\Omega_r^3\Omega_\Lambda}\right) && (6.95)\\ &= \frac{1}{4}\log\left(\frac{1}{\alpha\Lambda}\right) - \frac{1}{4}\log 36 + \frac{1}{12}\log\left(\frac{27}{16}\frac{\Omega_m^4}{\Omega_r^3\Omega_\Lambda}\right) && (6.96)\\ &\simeq 65 && (6.97)\end{aligned}$$

Then, taking the temperature today, rounded to 3K or 0.26meV, we can determine the temperature at the time of the Hot Big Bang:

$$T(t_{\mathrm{HBB}}) = T_0 e^{-N_{\Lambda\mathrm{CDM}}} \simeq 4 \times 10^{15}\ \mathrm{GeV} \tag{6.98}$$

which indeed gives the expected energy scale during reheating.

Next, we explore the consequences when cosmic censorship becomes cosmic veiling and we get to peer slightly into the effects that come from just enough inflation.

6.3 Numerical evolution of background and quantum initial conditions

The paradigm proposed for setting initial conditions consists not only of a choice of background initial condition – the choice consistent with cosmic veiling, but also of a choice of initial state. The choice of initial state is stated thusly: *The initial state $|\psi_{veil}\rangle$ should be such that it carries the minimum amount of information from the quantum gravity epoch.* That is, given that the cosmic veiling condition fixes a time – there's a maximum amount of inflationary e-foldings in order to have veiling, plus the limited amount of pre-inflationary e-foldings – the choice of state should be a vacuum state that doesn't spoil the veiling. Since we expect the transition of the background from quantum gravity to effective field theory to be smooth, so too should we impose that the transition does not imprint any quantum gravity information on the initial state that could be blown up and observed in the sky today.

Once we have made this choice of initial state, we can evolve it with the chosen background and generate the power spectrum of the theory. However, the evolution is non-trivial, the transition from pre-inflationary to inflationary epoch is sharp, with a prominent "knee" featured in the the $(H, \dot{H})$ phase space diagram. It is necessary for us then to evolve the initial state numerically in order to find the resulting power spectrum. We begin by describing the background evolution and subsequent procedure for evolving the quantum initial condition.

Given the definition of the Kretschmann scalar 2.42, we impose the pre-inflationary characteristic equation of $\dot{H}(t)$ 2.32, which allows us to write the Kretschmann scalar in terms of solely $H(t)$ and the crossing time Hubble factor H_c:

$$K = 24(2H_c^3 H(t) - 2H_c^{3/2} H(t)^{5/2} + H(t)^4) \tag{6.99}$$

This expression of the Kretschmann curve is essentially the intersection of the Kretschmann boundary with a given pre-inflationary trajectory, the latter of which depends directly on the total amount of inflationary e-foldings, set by H_c. Replacing H_c directly with e-foldings via $N_{\text{infl}} = 18\alpha H_c^2$ gives us a surface of initial conditions that depend on an initial value of H and N_{infl} as a parameter. The initial value of H can then be set by selecting a particular scale for the Kretschmann surface, $K = \frac{1}{L_{\text{initial}}^2}$, which can be naturally selected to be the Planck scale as a maximal boundary for the effective field theory. If one wished to be more conservative, the scale could be set at $10\ell_P$ or $100\ell_P$, but the results here will hold regardless, unless one were to approach the scale of inflation far enough to exclude pre-inflationary evolution.

With a selection of the initial Hubble parameter H, its first derivative is determined by the pre-inflationary characteristic equation, alongside the desired number of e-foldings, and thus we have a set of background initial conditions that depend solely on the input parameter of inflationary e-foldings.

There is one more requisite ingredient before solving the Friedmann equation. At this point, the value of α could be taken as a parameter of the theory, and thus yield a two-parameter family of solutions alongside N_{infl}. However, for simplicity, we can also take α to be the one that is determined already by observation, given only the power spectrum expression from the quasi-de Sitter framework. We will see in the next section that this is indeed a reasonable approximation, as the pre-inflationary evolution has the fortunate quality of leaving the power spectrum at small scales essentially unchanged, and thus shouldn't change the results garnered from observation.

With this nominal value, which we derive later as $\alpha \approx 2.27 \times 10^{10}\ \ell_P^2$, one can solve the Friedmann equation, in proper time, with the stated background initial conditions. The solution for $H(t)$ that's generated numerically has a past extension which is short and becomes intractable due to the singularity, and a future extension which describes several oscillations of reheating. The numerics allow us to set a timeline of the evolution of the background trajectory – given a proper time at the numerical singularity of $t_{\text{sing}} = 0$, the timescales of salient events are:

$$t_{\text{Kretschmann}} \sim 100\, t_{\text{Planck}} \tag{6.100}$$

$$t_{\text{inflation onset}} \sim 3.4 \times 10^4\, t_{\text{Planck}} \tag{6.101}$$

$$t_* \sim 2.9 \times 10^5\, t_{\text{Planck}} \tag{6.102}$$

$$t_{\text{inflation end}} \sim 9.1 \times 10^6\, t_{\text{Planck}} \tag{6.103}$$

The full expression for the mode equation depends on up to second degree Hubble flow parameters, which contain up to fourth derivatives of $H(t)$. We can reduce this noise by using the Friedmann equation to solve for $\ddot{H}(t)$ and $\dddot{H}(t)$ in terms of $H(t) and \dot{H}(t)$, essentially imposing that the classical trajectory is not just one in which the classical initial condition has $H(t)$ and $\dot{H}(t)$ set by the Friedmann equation, but also its subsequent derivatives. In effect, this requirements reduces the noise in the derivatives.

The next section will provide details on deriving the first order in ϵ_i's equation of motion for the modes, and their solutions. For the time being, we present them here sans justification:

$$v_{\text{QdS}}(x) = \sqrt{x}J_\nu\left(\frac{kx}{k_*}\right) - iA\sqrt{x}Y_\nu\left(\frac{kx}{k_*}\right) \tag{6.104}$$

with parameters ν and γ given by:

$$\nu_{\text{QdS}} \approx \frac{3}{2} + \epsilon 1H - \frac{\epsilon_{1Z}}{2} \tag{6.105}$$

$$\gamma_{\text{QdS}} = \sqrt{1 + 2\epsilon_{1H}} \tag{6.106}$$

Transforming back from $v(x)$ to $u(t)$ mode functions in this case is trivial – we have access to numerical interpolations of $a(t)$ and $H(t)$ which give $x(t)$ for all relevant times, and the conversion mass-analogue function that translates between v and u can be simplified and approximated into a straightforward expression:

$$m_{\text{pre}}(x) \approx \frac{a(t_0)^3 Z(t_0) x}{2t_0\hbar} + \mathcal{O}(x^4), \tag{6.107}$$

The derivation of this expression will be presented in the next section. The only difference then will be the time at which this function is evaluated – in the next section it will be evaluated at the transition of pre-inflationary to quasi-de Sitter regimes, but at this moment, we evaluate it at the same time we've set our initial conditions – consummate with the Kretschmann boundary. With these pieces in place, the ansatz for the mode functions can be evolved with the quantum equation of motion to yield an interpolation of the mode functions at all future times, for each mode k, which are selected with a logarithmic binning to cover the space of relevant modes from $\frac{1}{2}10^{-1}k_*$ to $50k_*$. Only the late time behavior is needed of these modes, as well as their behavior in aggregate, so the mode functions are evaluate at late times, $10t_{\text{inflation end}}$, and taken the modulus squared of. The power spectrum for scalars can then be generated, as shown in Fig. 6.6.

Similarly, the power spectrum for tensors is generated, with the only difference being

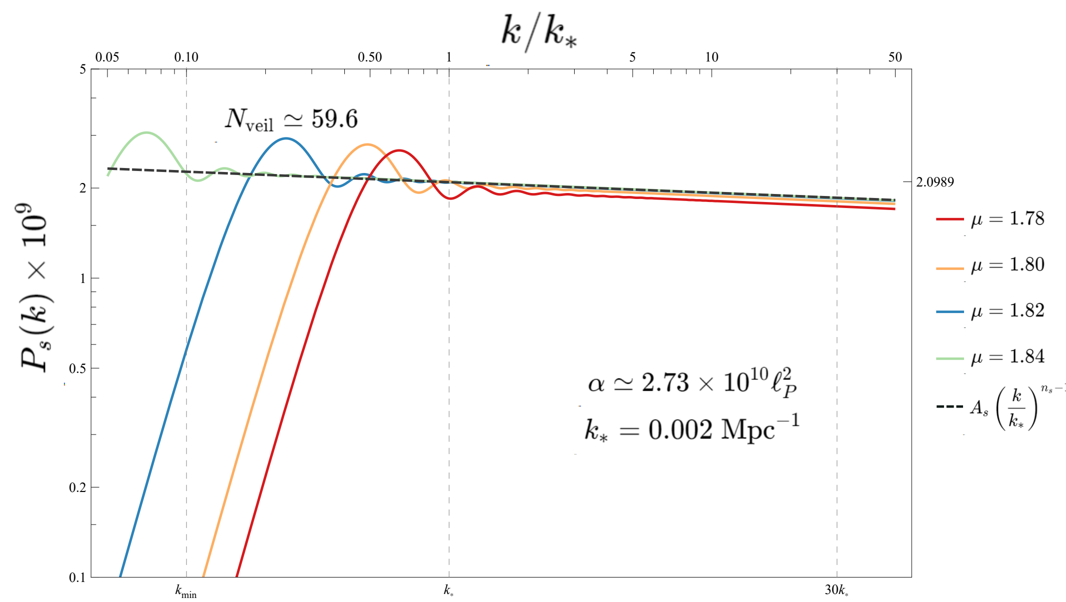

Figure 6.6. Numerical power spectrum for scalars. The nominal values of μ taken here are: $1.78, 1.80, 1.82, 1.84$. The nearly-scale invariant result from observation without pre-inflationary evolution shown in the black dashed line.

the the $Z(t)$ functions and their log-derivatives taken to be the tensorial ones in Fig. 6.7.

6.4 Analytic approximation and interpretation of results

We begin once more with the mode equation, this time not adapted to quasi-de Sitter but to a pre-inflationary regime. Since ϵ_i's are not necessarily small during this epoch, we need to return to the full mode equation:

$$\ddot{u}(\vec{k},t) + (3 - \epsilon_Z(t))\, H(t)\dot{u}(\vec{k},t) + \frac{\vec{k}^2}{a(t)^2}u(\vec{k},t) = 0 \tag{6.108}$$

Once again, we introduce the variable $x(t) = \frac{k_*}{a(t)H(t)}$, and rewrite the mode equation in terms of x The change of variable will require translating from time derivatives to $x-$derivatives:

$$\dot{x} = H(t)x(t)(-1 + \epsilon_{1H}(t) \tag{6.109}$$

$$\ddot{x} = -H(t)^2x(t)(-1 + \epsilon_{1H}(t)(1 + \epsilon_{2H}(t))) \tag{6.110}$$

The chain rule gives us $\dot{u} = U'(x)\dot{x}$ and $\ddot{u} = U''(x)\dot{x}^2 + U'(x)\ddot{x}$, and we normalize the

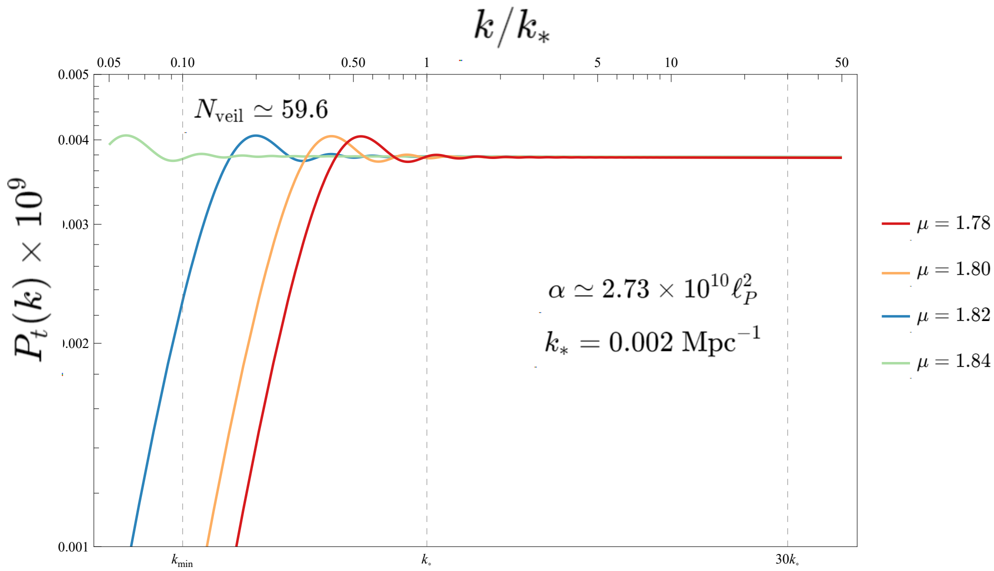

Figure 6.7. Numerical power spectrum for scalars. The nominal values of μ taken here are: $1.78, 1.80, 1.82, 1.84$.

mode equation so the second derivative term has coefficient of unity:

$$\frac{k^2 U(x)}{k_*^2(-1+\epsilon_{1H}(t))^2} + \frac{(-2+\epsilon_{1Z}(t)-\epsilon_{1H}(t)(-2\epsilon_{1Z}(t)+\epsilon_{2H}(t)))}{x(-1+\epsilon_{1H}(t))^2}U'(x) + U''(x) = 0 \quad (6.111)$$

The Wronskian is similarly modified by going to the x variable, and will provide us one more change of variable to bring the Wronskian into a canonical form:

$$U^*(x)U'(x) - U(x)U^{*'}(x) = -\frac{ix^2\hbar H(t)^2}{k_*^3 Z(t)(-1+\epsilon_{1H}(t))} \quad (6.112)$$

We define the function $m(x) = \left(\frac{x^2\hbar H(t)^2}{k_*^3 Z(t)(-1+\epsilon_{1H}(t))}\right)^{-1}$, which serves as a mass-analogue for the oscillator, and redefine the mode variables:

$$U(x) = \frac{v(x)}{\sqrt{m(x)}} \quad (6.113)$$

$$U^*(x) = \frac{v^*(x)}{\sqrt{m(x)}} \quad (6.114)$$

With this, the equation of motion no longer contains a first derivative term, allowing us to write it in an exact canonical form:

$$\frac{1}{4k_* x^2(-1+\epsilon_{1H}(t))^4} v_*(x)\Big(-8k_s^2 + 4k^2x^2 - 2k_*^2\epsilon_{1H}(t)^3(-2+\epsilon_{1Z}(t)+\epsilon_{2H}(t)) \tag{6.115}$$

$$-k_*^2\epsilon_{1Z}(t)(-6+\epsilon_{1Z}(t)+2\epsilon_{2Z}(t)^2)+ \tag{6.116}$$

$$\epsilon_{1H}(t)^2\Big(-16k_*^2 + 4k^2x^2 + k_*^2\Big(10\epsilon_{1Z}(t)+\epsilon_{1Z}(t)^2+2\epsilon_{1Z}(t)\epsilon_{2Z}(t) \tag{6.117}$$

$$+4\epsilon_{2H}(t)+\epsilon_{2H}(t)^2-2\epsilon_{2H}(t)\epsilon_{3H}(t)\Big)\Big)+ \tag{6.118}$$

$$2\epsilon_{1H}(t)\Big(10k_*^2 - 4k^2x^2 + k_*^2\Big(-7\epsilon_{1Z}(t)+\epsilon_{1Z}(t)^2+2\epsilon_{1Z}(t)\epsilon_{2Z}(t)\Big) \tag{6.119}$$

$$+\epsilon_{2H}(t)(-1+\epsilon_{2H}(t)+\epsilon_{3H}(t))\Big)\Big) + v_*''(x) = 0 \tag{6.120}$$

As we've discussed previously, we're only interested in the solution up to linear order in ϵ_i's; expanding this equation of motion, assuming all Hubble flow parameters are of the same order of magnitude, yields:

$$v_*(x)\left(\frac{k^2}{k_*^2} - \frac{2}{x^2} + \left(\frac{2k^2}{k_*^2} - \frac{3}{x^2}\right)\epsilon_{1H}(t) + \frac{3\epsilon_{1Z}(t)}{2x^2}\right) + v_*''(x) = 0 \tag{6.121}$$

We can check this equation against known solutions. For example, we take take the perfect de Sitter limit, in which Hubble flow parameters are trivial, which gives the simplified, familiar equation of motion:

$$v_*(x)\left(\frac{k^2}{k_*^2} - \frac{2}{x^2}\right) + v_*''(x) = 0 \tag{6.122}$$

The linear-order equation of motion presents only a slightly more complicated Bessel equation. In particular, we can identify the relevant frequency function:

$$Q(x) \sim \frac{k}{k_*} - \frac{2}{x^2} + \left(\frac{2k^2}{k_*^2} - \frac{3}{x^2}\right)\epsilon_1(t) + \frac{3\epsilon_{Z_1}(t)}{2x^2} \tag{6.123}$$

which in turn can be compared to that of the canonical Bessel:

$$Q(x) = \gamma^2 - \frac{\nu^2 - \frac{1}{4}}{x^2} \tag{6.124}$$

with relevant coefficients given by the Hubble flow parameters:

$$\gamma^2 = (1+3\epsilon_{H_*})\left(\frac{k}{k_*}\right)^2 + O\left(\frac{1}{x}\right) \tag{6.125}$$

$$\nu^2 = \frac{3}{2} + \epsilon_{H_*} - \frac{1}{2}\epsilon_{Z_{1*}} + O(\epsilon^2) \tag{6.126}$$

When we put the equation of motion in this simpler, canonical form, we can easily guess an ansatz for the basis of solutions, which Hankel functions:

$$v_*(x)\left(\frac{k^2}{k_*^2} - \frac{\nu^2 - 1/4}{x^2}\right) + v_*''(x) = 0 \tag{6.127}$$

$$\Rightarrow v_{\text{ansatz}}(x) = A\sqrt{x}J_\nu\left(\frac{kx}{k_*}\right) - iA\sqrt{x}Y_\nu\left(\frac{kx}{k_*}\right) \tag{6.128}$$

$$v_{\text{ansatz}}(x) = A\sqrt{x}J_\nu\left(\frac{kx}{k_*}\right) + iA\sqrt{x}Y_\nu\left(\frac{kx}{k_*}\right) \tag{6.129}$$

The change in variables is also intended to put the Wronskian in a canonical form:

$$v_*(x)v'(x) - v(x)v^{*'}(x) = i \tag{6.130}$$

Inserting our ansatz into the Wronskian sets the undetermined coefficient to $A = \sqrt{\pi}2$. This gives us the general framework requisite to solve the equation of motion both during the pre-inflationary and the quasi-de Sitter regimes. We proceed to give an outline of the strategy.

As we've stated, there's no closed form solution to the equations of motion that smoothly changes from pre-inflationary to quasi-de Sitter, which necessitated the numerical analysis; the evolution of the background is non-trivial due to the sharp evolution at the transition to inflation, the "knee" that appears in the $(H, \dot{H})$ phase space diagram. In order to interpret the numerics, we will work with an analytic approximation, transition from the pre-inflationary to quasi-de Sitter solutions, by gluing this instantanously at the transition. This is a very coarse approximation, to be sure! However, it does illustrate the more salient aspects of the numerical solution, namely the presence of power suppression at large scales, and the appearance of features at the previously calculated k_{feature} scale. At the gluing time, we interpret both the pre-inflationary and quasi-de Sitter Bessel ansatz as valid bases of solutions to the equation of motion – meaning we can write one in terms of the other, and then separate the pre-inflationary effect on the power spectrum from the usual slightly-red tilted result.

We begin by first finding the analytic solution at first order in Hubble flow parameters during the pre-inflationary period. We will need to rewrite $H_{\text{pre}}(t)$ in terms of the relevant

x variable. Recall that the pre-inflationary solution for the Hubble parameter is given in a small t series by:

$$H_{\text{pre}}(t) \approx \frac{1}{2t} + \frac{2}{5}H_c\sqrt{2H_c t} + \mathcal{O}(t^2) \tag{6.131}$$

One can solve the differential equation for $a(t)$ and expand around small time. At leading order, the constant of integration to $a_c/\sqrt{t_c}$ by the simple requirement that $a(t_c) = a_c$. Here t_c is arbitrary, but will eventually be taken to have the same physical meaning as H_c – the crossing time when the solutions are glued together. The expression for the scale factor is then:

$$a_{\text{pre}}(t) \approx a_c\sqrt{\frac{t}{t_c}} + \mathcal{O}(t^2) \tag{6.132}$$

With the scale factor, any of the relevant background quantities can be computed straightforwardly. In particular, $Z(t)$ is relevant in the mass-analogue function $m(x)$, and is found to take a similar form when expanded around early times:

$$Z_{\text{pre}}(t) \approx Z_c\sqrt{\frac{t_c}{t}} + \mathcal{O}(1) \tag{6.133}$$

Given the time-dependent expressions for $H(t)$ and $a(t)$, the series expansion at early times for $x(t)$ can be written, and subsequently the relationship inverted to rewrite all pre-inflationary quantities in terms of x:

$$x_{\text{pre}}(t) = \frac{k_*}{a_{\text{pre}}(t)H_{\text{pre}}(t)} \approx \frac{2k_*\sqrt{tt_c}}{a_c} + \mathcal{O}(t^2) \tag{6.134}$$

$$\Rightarrow t_{\text{pre}}(x) = \frac{a_c^2 x_{\text{pre}}^2}{4k_*^2 t_c} + \mathcal{O}(x^5) \tag{6.135}$$

Altogether, this allows us to write the mass-analogue function for pre-inflationary times,

$$m_{\text{pre}}(x) \approx \frac{a_c^3 Z_c x}{2t_c\hbar} + \mathcal{O}(x^4), \tag{6.136}$$

as well as plug all relevant quantities into the *exact* equation of motion 6.115 to determine a new approximation:

$$\left(\frac{k^2}{k_*^2} + \frac{1}{4x^2}\right)v(x) + v''(x) = 0 \tag{6.137}$$

This equation of motion still has a simple Bessel function solution with $\nu = 0$ and $\gamma = 1$, and the slow-roll parameters are not present in this case – they are not small parameters, thus it is not valuable to express them explicitly in the equation of motion, nor in the ansatz:

$$v_{\mathrm{pre}}(x) = \sqrt{x} J_0\left(\frac{kx}{k_*}\right) - i\sqrt{x} Y_0\left(\frac{kx}{k_*}\right) \tag{6.138}$$

The equation of motion expanded around linear order in Hubble flow parameters is already adapted to the quasi-de Sitter regime. Therefore, proposing an ansatz is straightforward:

$$v_{\mathrm{QdS}}(x) = \sqrt{x} J_\nu\left(\frac{kx}{k_*}\right) - i\sqrt{x} Y_\nu\left(\frac{kx}{k_*}\right) \tag{6.139}$$

with parameters ν and γ given by:

$$\nu_{\mathrm{QdS}} = \frac{1}{2}\sqrt{3(3+4\epsilon_{1H}-2\epsilon_{1Z})} \approx \frac{3}{2} + \epsilon_{1H} - \frac{\epsilon_{1Z}}{2} \tag{6.140}$$

$$\gamma_{\mathrm{QdS}} = \sqrt{1+2\epsilon_{1H}} \tag{6.141}$$

The remaining piece of the puzzle is determining the mass-analogue function $m_{\mathrm{QdS}}(x)$. Consider the Hubble factor in a background in which ϵ_{1H} is roughly constant during the quasi-de Sitter phase. In this case, consider the log derivative of $H(x)$:

$$x\frac{d\log H(x)}{dx} = \frac{x}{H(x)}\frac{dH(x)}{dx} = \frac{x}{H(x)}\frac{\dot{H}}{\dot{x}} \tag{6.142}$$

$$= -\frac{xH\epsilon_{1H}}{\dot{x}} \tag{6.143}$$

where the last equality comes from the definition of ϵ_{1H}. We can compute the time derivative $\dot{x}$ to find a simple result:

$$\dot{x} = -\frac{k_*}{a(t)} - \frac{k_*\dot{H}(t)}{a(t)H(t)^2} = -xH(t) - x\frac{\dot{H}(t)}{H(t)} = xH(x)(-1+\epsilon_{1H}) \tag{6.144}$$

$$\Rightarrow x\frac{d\log H(x)}{dx} = -\frac{\epsilon_{1H}}{-1+\epsilon_{1H}} \tag{6.145}$$

Expanding this expression to first order in Hubble flow parameters, and integrating both sides, we find a simple approximation for $H(x)$ during the quasi-de Sitter regime:

$$H(x) \approx x^{\epsilon_{1H*}} H_* \tag{6.146}$$

In fact, any quantity that has a slowly changing log derivative will have the same form, and since the Hubble flow parameters are defined precisely to have small log derivatives, we can similarly express ϵ_{1H} and ϵ_{1Z} near the crossing time:

$$\epsilon_{1H\times} = \epsilon_{1H*} x_c^{\epsilon_{2H*}} = \epsilon_{1H*} x_c^{-2\epsilon_{1H*}} \tag{6.147}$$

$$\epsilon_{1Z\times} = \epsilon_{1Z*} x_c^{\epsilon_{2Z*}} = -2\epsilon_{1H*} x_c^{-2\epsilon_{1H*}} \tag{6.148}$$

The mass-analogue function can then be written, at linear order in Hubble parameters:

$$m_{\mathrm{QdS}}(x) = \frac{k_*^3 x^{-2(1+\epsilon_{1H*}+\epsilon_{1Z*}} Z_*(1-\epsilon_{1H*})}{H_*^2 \hbar} \tag{6.149}$$

Now we have the ingredients requisite to glue the solutions at the threshold of inflation. At the time of matching, we have two equally valid, linearly independent bases of solutions of the equation of motion – the pre-inflationary and the quasi-de Sitter bases. Thus, we ought to be able to write one basis in terms of the other. Consider the quasi-de Sitter mode function can be written in terms of the pre-inflationary basis, where ρ, σ are complex coefficients:

$$v_{\mathrm{QdS}} = \rho v_{\mathrm{pre}} + \sigma v_{\mathrm{pre}}^* \tag{6.150}$$

The Wronskian gives us a sense of linear independence, generically, given two differentiable functions f, g as inputs:

$$\langle f, g \rangle = f \dot{g}^* - \dot{f} g^* \tag{6.151}$$

$$\langle f, f \rangle = i \tag{6.152}$$

This use of the Wronskian allows us to compute the coefficients ρ and σ. Using purely its linear properties, we find:

$$\langle v_{\mathrm{QdS}}, v_{\mathrm{pre}} \rangle = \langle \rho v_{\mathrm{pre}} + \sigma v_{\mathrm{pre}}^*, v_{\mathrm{pre}} \rangle \tag{6.153}$$

$$= \rho \langle v_{\mathrm{pre}}, v_{\mathrm{pre}} \rangle \tag{6.154}$$

$$= \rho\, i \tag{6.155}$$

At the same time, computing via the definition of the Wronskian yields:

$$\langle v_{\mathrm{QdS}}, v_{\mathrm{pre}} \rangle = v_{\mathrm{QdS}} \dot{v}_{\mathrm{pre}}^* - \dot{v}_{\mathrm{QdS}} v_{\mathrm{pre}}^* \tag{6.156}$$

$$\Rightarrow \rho = -i(v_{\mathrm{QdS}} \dot{v}_{\mathrm{pre}}^* - \dot{v}_{\mathrm{QdS}} v_{\mathrm{pre}}^*) \tag{6.157}$$

The other coefficient can be similarly computed:

$$\sigma = +i(v^*_{\mathrm{QdS}}\dot{v}^*_{\mathrm{pre}} - \dot{v}^*_{\mathrm{QdS}}v^*_{\mathrm{pre}}) \tag{6.158}$$

Equipped in this manner, we are able to determine an expression for the change to the power spectrum that comes from the pre-inflationary evolution. Recall that the mode function shows up in the power spectrum as a modulus squared:

$$P(k) \propto |v_{\mathrm{QdS}}(k)|^2 \tag{6.159}$$

Starting with $v_{\mathrm{QdS}} = \rho v_{\mathrm{pre}} + \sigma v^*_{\mathrm{pre}}$, we can manipulate and expand the modulus squared:

$$|v_{\mathrm{QdS}}|^2 = |\rho v_{\mathrm{pre}} + \sigma v^*_{\mathrm{pre}}|^2 \tag{6.160}$$

$$= (|\rho|^2 + |\sigma|^2)|v_{\mathrm{pre}}|^2 + \rho\sigma^* v_{\mathrm{pre}}v_{\mathrm{pre}} + \sigma\rho^* v^*_{\mathrm{pre}}v^*_{\mathrm{pre}} \tag{6.161}$$

At this point, we're seemingly stuck with a complicated expression. However, the power spectrum is always evaluated at late times, which allows us to make a simplifying assumption. The pre-inflationary mode function is a Hankel function – a combination of Bessels of type 1 and 2. The late time limit, in terms of x, is $x \to 0$, which for our Bessel functions means that the real part will go to 0, while the imaginary part diverges! Of course, we never reach the divergence, and in fact, it'll be cancelled in the modulus squared, but we can make use of the fact that, at late times, the mode function is purely imaginary. From this, we can rewrite:

$$|v_{\mathrm{QdS}}|^2 = (|\rho|^2 + |\sigma|^2)|v_{\mathrm{pre}}|^2 + \rho\sigma^* v_{\mathrm{pre}}v_{\mathrm{pre}} + \sigma\rho^* v^*_{\mathrm{pre}}v^*_{\mathrm{pre}} \tag{6.162}$$

$$= (|\rho|^2 + |\sigma|^2)|v_{\mathrm{pre}}|^2 + (\rho\sigma^* + \rho^*\sigma)(-\mathrm{Im}[v_{\mathrm{pre}}]^2) \tag{6.163}$$

$$= (|\rho|^2 + |\sigma|^2 - \rho\sigma^* - \rho^*\sigma)|v_{\mathrm{pre}}|^2 \tag{6.164}$$

$$= |\rho - \sigma|^2|v_{\mathrm{pre}}|^2 \tag{6.165}$$

The end result gives us a simple rule of thumb; the contribution to the power spectrum from the pre-inflationary evolution can be neatly summarised in the factor $|\rho - \sigma|^2$. It is the sufficient to characterize the behavior of this term in order to understand how the power spectrum is modified from the usual expression. This standard expression, of course, can be generated by taking the modulus squared of the quasi-de Sitter mode equation solution $|v_{\text{Qds}}|^2$, and mapping back to $U(x)$ by diving by $m(x)$:

$$|v_{\text{Qds}}|^2 = \frac{H_* \pi x^{3+2H_*-\epsilon_{1Z*}}\hbar}{4k_*^3 Z_*}\Bigg(J_{\frac{3}{2}+\epsilon_{1H*}-\frac{\epsilon_{1Z*}}{2}}\left(\frac{kx\sqrt{1+2\epsilon_{1H*}}}{k_*}\right)^2 \tag{6.166}$$

$$+Y_{\frac{3}{2}+\epsilon_{1H*}-\frac{\epsilon_{1Z*}}{2}}\left(\frac{kx\sqrt{1+2\epsilon_{1H*}}}{k_*}\right)^2\Bigg) \tag{6.167}$$

We needn't change time variable from x to t since we'll end up taking the late time limit regardless. Focusing solely on the form of the time-dependent, asymptotic term, we find at late times, $x \to 0$:

$$\lim_{x\to 0} x^{2\nu}\left(J_\nu(x\mu)^2 + Y_\nu(x\mu)^2\right) = \frac{4^\nu \gamma^{-2\nu}\Gamma(\nu)^2}{\pi^2} \tag{6.168}$$

where Γ is the Euler-Gamma function. All put together and expanded at linear order in $\epsilon_i's$, we find the usual expression for the power spectrum at linear order:

$$P(k) = \frac{H_*^2\hbar}{4\pi^2 Z_*} + \frac{1}{4\pi Z_*}H_*^2\hbar\Bigg(\epsilon_{1Z*}(-2+\gamma+\log 2) - \epsilon_{1H*}(-1+2\gamma+\log 4) \tag{6.169}$$

$$+(-2\epsilon_{1H*}+\epsilon_{1Z*})\log\frac{k}{k_*}\Bigg) \tag{6.170}$$

By taking as input the specific background which offers a background solution to the Friedmann equation from 6.11 and subsequent modification to the expressions for the power spectrum amplitude and spectral tilt that come from this specific solution, that is,

$$A_s = \frac{H_*^2 G\hbar}{2\pi\epsilon_{1H*}} \tag{6.171}$$

$$n_s = 1 - 4\epsilon_{1H*}, \tag{6.172}$$

we can then solve the system of equations and find the values for H_* and ϵ_{1H*}:

$$H_* = 1.17 \times 10^{-5}\ell_P^{-1} \tag{6.173}$$

$$\epsilon_{1H*} = 0.0089 \tag{6.174}$$

In turn, this allows us to use the same series expression for $H(\epsilon)$ and leverage the constant ϵ approximation of the choice of background solution in order to ultimately determine α from observation, by imposing that $H(\epsilon_{1H*}) = H_*$:

$$H_* = H(\epsilon_{1H*}) \sim \frac{1}{6\sqrt{\alpha\epsilon_{1H*}}} - \frac{\sqrt{\epsilon_{1H*}}}{72\sqrt{\alpha}} + \mathcal{O}(\epsilon_{1H*}^{3/2}) \tag{6.175}$$

$$\Rightarrow \alpha \approx 2.27 \times 10^1 0\ell_P^2 \tag{6.176}$$

This calculation allows us to seed our numerical simulations, since the numerics require the value of α to be known in order to compute the solution to the Friedmann equation.

Meanwhile, the full pre-inflationary factor is a complicated expression. For simplicity, rewriting the expression back into terms of ν_{QdS} and γ_{QdS}, and defining the argument of the Bessel functions to be $\xi = \frac{kx_c}{k_*}$, the pre-inflationary factor $\Pi(k)$ becomes:

$$\Pi(k) = \frac{k^2\pi^2 x_c^2}{16k_*^2}\Bigg(J_0(\xi)^2 J_{\nu-1}(\gamma\xi)^2 + 2\epsilon_{1H*}J_0(\xi)^2 J_{\nu-1}(\gamma\xi)^2 + 4\gamma J_0(\xi)J_1(\xi)J_{\nu-1}(\gamma\xi)J_\nu(\gamma\xi) \tag{6.177}$$

$$+4J_1(\xi)^2 J_\nu(\gamma\xi)^2 - 2J_0(\xi)^2 J_{\nu-1}(\gamma\xi)J_{\nu+1}(\gamma\xi) - 4\epsilon_{1H*}J_0(\xi)^2 J_{\nu-1}(\gamma\xi)J_{\nu+1}(\gamma\xi) \tag{6.178}$$

$$-4\gamma J_0(\xi)J_1(\xi)J_\nu(\gamma\xi)J_{\nu+1}(\gamma\xi) + J_0(\xi)^2 J_{\nu+1}(\gamma\xi)^2 + 2\epsilon_{1H*}J_0(\xi)^2 J_{\nu+1}(\gamma\xi)^2 \tag{6.179}$$

$$+J_{\nu-1}(\gamma\epsilon)^2 Y_0(\xi)^2 + 2\epsilon_{1H*}J_{\nu-1}(\gamma\xi)^2 Y_0(\xi)^2 - 2J_{\nu-1}(\gamma\xi)J_{\nu+1}(\gamma\xi)Y_0(\xi)^2 \tag{6.180}$$

$$-4\epsilon_{1H*}J_{\nu-1}(\gamma\xi)J_{\nu+1}(\gamma\xi)Y_0(\xi)^2 + J_{\nu+1}(\gamma\xi)^2 Y_0(\xi)^2 + 2\epsilon_{1H*}J_{\nu+1}(\gamma\xi)^2 Y_0(\xi)^2 \tag{6.181}$$

$$+4\gamma J_{\nu-1}(\gamma\xi)J_\nu(\gamma\xi)Y_0(\xi)Y_1(\xi) - 4\gamma J_\nu(\gamma\xi)J_{\nu+1}(\gamma\xi)Y_0(\xi)Y_1(\xi) + 4J_\nu(\gamma\xi)^2 Y_1(\xi)^2\Bigg) \tag{6.182}$$

How well does the pre-inflationary factor actually capture the modifications from the standard results for scalar and tensor power spectra, namely:

$$P_s(k) = A_s\left(\frac{k}{k_*}\right)^{n_s-1} \tag{6.183}$$

$$P_t(k) = A_t\left(\frac{k}{k_*}\right)^{n_t} \tag{6.184}$$

Figures 6.8 and 6.9 show the pre-inflationary factor against the power spectrum, which has been normalized over its standard expression. For scalar modes, there doesn't seem to be much agreement, while the tensor pre-inflationary factor appears to exactly capture the pre-inflationary modification to the power spectrum. The source of this discrepancy across scalar and tensor modes goes back to the friction-like term in the mode equation.

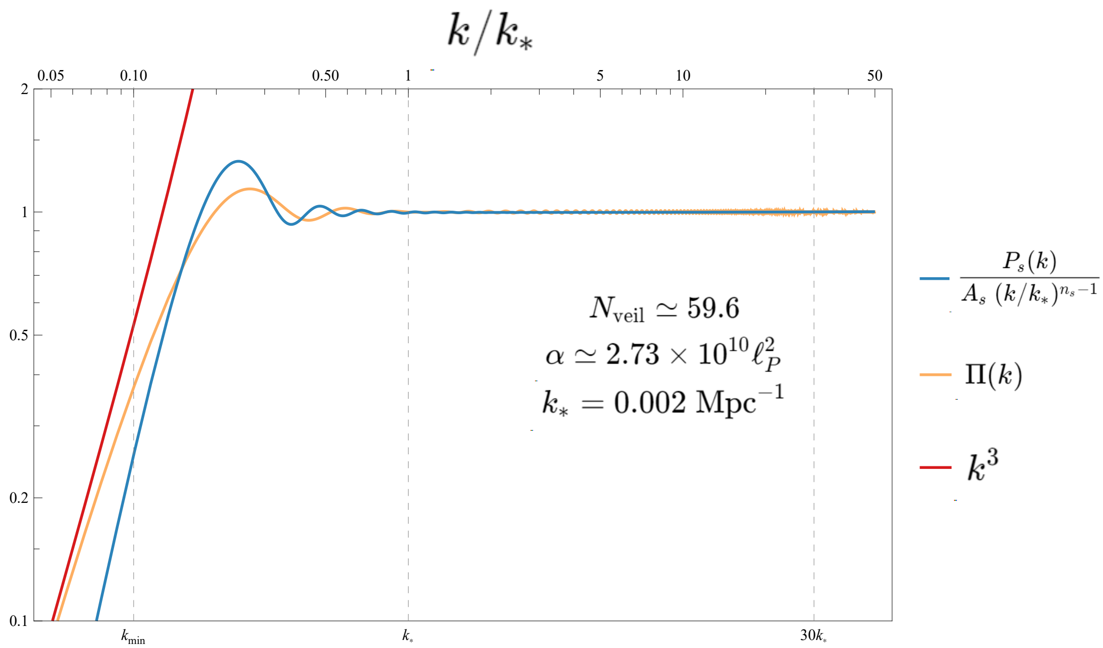

Figure 6.8. The scalar pre-inflationary factor can be compared to the full numerical power spectrum, which has been normalized over its standard expression, as well as it's low-k approximation. We see there's not a lot of agreement between these functions except at high values of k.

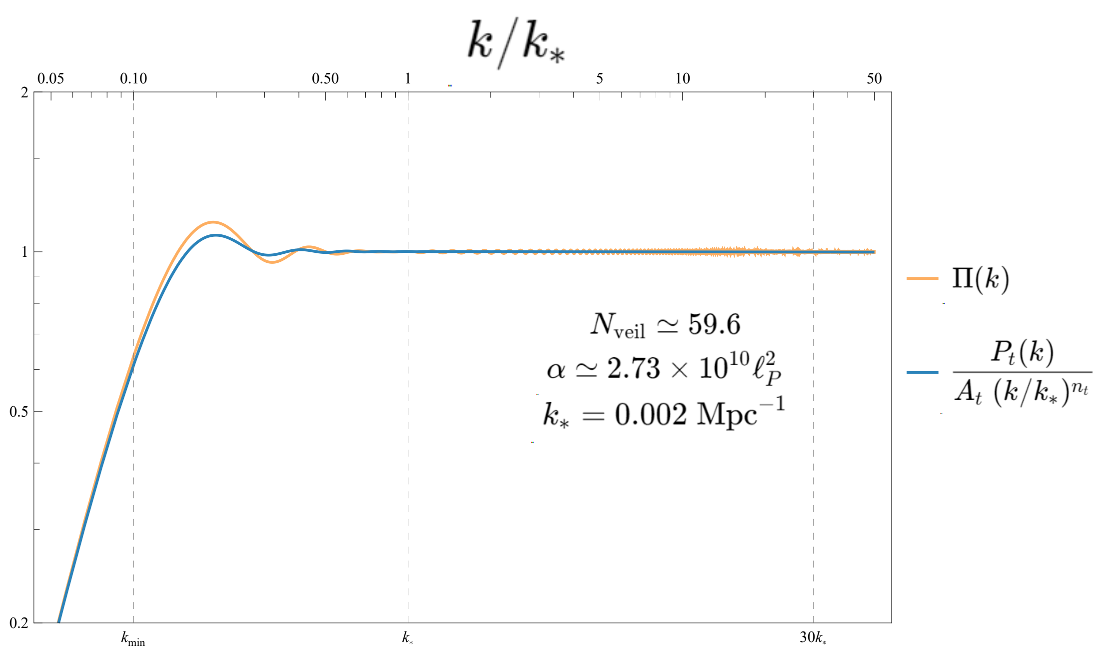

Figure 6.9. The tensor pre-inflationary factor, on the other hand, compares rather favorably with its power spectrum, once again normalized over its standard expression.

As shown in Figure 6.1, the scalar modes observe a non-trivial evolution in which the "friction" they experience momentarily becomes negative precisely around the time of the pre-inflationary to inflationary transition occurs. Meanwhile, in Figure 6.2, the "friction" experienced by tensor modes remains positive throughout its evolution. Recall that we choose a rather crude method of generating the pre-inflationary factor. Namely, we chose to instantaneously glue the pre-inflationary to the inflationary solution and from there tease apart the standard power spectrum from the effects of pre-inflation. That is, with the scalar modes non-trivial evolution, we shouldn't expect the pre-inflationary factor derived in this manner to capture all the features of the evolution.

We can do some simple checks of Equation 6.182. When we take the Bogoliubov coefficients, and look at their behavior in the UV, we find the following simple, series expanded expressions:

$$|\rho(k)|^2 = 1 + \frac{\epsilon_{1H*}^2}{4} + \mathcal{O}(\epsilon_{1H*}^4) \tag{6.185}$$

$$|\sigma(k)|^2 = \frac{\epsilon_{1H*}^2}{4} + \mathcal{O}(\epsilon_{1H*}^4) \tag{6.186}$$

When we take the de Sitter limit, we find $\rho = 1$ and $\sigma = 0$, that is, there is no particle production! This is expected, as a past-facing de Sitter solution would insist on staying in de Sitter infinitely into the past. Now, while the full expression is complicated, it's possible to glean some information by looking at the behavior of this function at high k and low k extrema. Expanding out $\Pi(k)$ around high values of k, and subsequently expanding at linear order in ϵ_i's, we find:

$$\Pi(k) \rightarrow_{k\to\infty} \frac{1+\epsilon_{1H*} - \epsilon_{1H*}\cos\left(\pi\epsilon_{1H*} - \frac{2kx_c\sqrt{1+2\epsilon_{1H*}}}{k_*} - \frac{\pi\epsilon_{1Z*}}{2}\right)}{\sqrt{1+2\epsilon_{1H*}}} \tag{6.187}$$

$$\approx 1 - \epsilon_{1H*}\cos\left(\frac{2kx_c}{k_*}\right) \tag{6.188}$$

This expression indicates the presence of oscillations at large values of k, which are not observed in the power spectrum. However, we ought to be skeptical of results of the pre-inflationary factor in this regime – after all, the process for obtaining this factor comes from the very coarse process of gluing solutions to the equation of motion instantaneously, to which large k modes should be more sensitive. Oscillations aside, we expect there to be no modification to the power spectrum at small scales. However, the low k regime is a different story. Applying the same expansion procedure at large scales, with redefining the argument $\xi = \frac{kx_c}{k_*}$, yields:

$$\Pi(k)_{k\to 0} = \frac{k^3}{k_*^3} \frac{2^{-2(\nu+2)} x_c^3 \gamma^{2\nu} \xi^{2\epsilon_{1H*}-\epsilon_{1Z*}}}{\Gamma\left(\frac{5}{2} + \epsilon_{1H*} - \frac{\epsilon_{1Z*}}{2}\right)^2} \Bigg(16 - 32\gamma_{\text{Euler}}\nu \tag{6.189}$$

$$+15\gamma_{\text{Euler}}^2\nu^2 + 4\pi^2\nu^2 - 16\nu^2\log(2\xi)^2 + 16\nu(-2 + 2\gamma_{\text{Euler}}\nu)\log\frac{\xi}{2}\Bigg) \tag{6.190}$$

The leading order term for $\Pi(k)$ goes as $k^{3+2\epsilon_{1H*}-\epsilon_{1Z*}}$, indicating that, at large scales, *the pre-inflationary factor introduces power suppression that scales roughly as* k^3! By the principle of cosmic veiling, if we are able to observe the veiled early universe, then we ought to observe power suppression; or by contrapositive, if there is no power suppression, then the universe is completely censored!

What are the connections we can make with the numerical result? The pre-inflationary factor $\Pi(K)$ depends on $k, k_*, x_c, \epsilon_{1H*}$ and ϵ_{1Z*} – whilst the numerics depend on the amount of inflationary e-foldings and α, the latter of which can be taken as a parameter, or in conjunction with the theory framework, be set by observation. Setting the crossing (gluing) time x_c allows us to compute $H(x_c)$ for a given choice of background dynamics, as well as subsequently determine ϵ_{1H*} and ϵ_{1Z*}, and the total number of inflationary e-foldings.

Thankfully, the approximations in 6.147 allow us to write the pre-inflationary factor *solely in terms of the gluing time x_c!* In turn, this gives us a single parameter family of solution of $\Pi(k)$, which we can then use to find the gluing time for which we do not observe the key feature of power suppression at large scales. When we compare the first peak of $\Pi(k)$ to the minimum observable mode k_{min}, we find a gluing time of $x_c \sim 27$. Putting this time in terms of inflationary e-foldings yields:

$$N_{\text{infl}} = 18H_c^2\alpha \approx 18H_*^2 x_c^{2\epsilon_{1H*}}\alpha \approx 59.6 \ \text{ e-folds} \tag{6.191}$$

When we compare this to our calculation for k_{feature}, with $N_* \sim 57$ (see 6.2) and $k_{\text{feature}} \sim 10^{-1}k_*$, we find $\mu \sim 1.82$ or $N_{\text{infl}} \sim 59.3$ e-foldings.

6.5 Conclusion

In the spirit of true crime, even after decades of investigation, it can be downright grueling to pinpoint all the details of mystery, especially when the original whodunnit dates back to the 1960's. We have set out with some essential assumptions:

1. A complete theory of quantum gravity should transition smoothly from the Planck regime to semiclassical gravity. Moreover, there exists a post-Planckian, but pre-inflationary era that admits the methods of perturbative field theory.

2. The principle of cosmic veiling allows us to characterize an interesting universe: one in which we may just barely observe primordial modifications to the power spectrum, but is not excluded by observation.

3. In addition, cosmic veiling sets a special time when these features can be seen – placing an upper bound on the total amount of inflationary e-foldings consistent with veiling. The initial quantum state at this transition threshold should contain no information from the Planck era, preserving cosmic veiling.

These assumptions allow us to reach surprisingly far! We take the minimal next step by letting gravity be the pure driver of inflation, and discover that, for a significant portion of the resulting configuration space, inflation is an attractor. Without a sense of measure on a proper canonical phase space, one cannot make any statements regarding the likelihood of inflation occurring, however. Perhaps, after determining the subset of initial conditions that is compatible with cosmic veiling, a measure that depends non-trivially on the coupling constants of the theory can be derived from the action.

Once the machinery of perturbative field theory has been applied to gravity and the exact equation of motion for the modes is obtained, we find ourselves at the mercy of the choice of approximation of the background. When we take quasi de Sitter, for example, the order to which we expand the background dictates whether the solution for the modes is a simple linear combination of Bessel function, or requires extra machinery via the Green's function method. There exists some generalizations of the Bessel equation which have closed-form, named solutions, such as the Whittaker function. The eminent issue in the exact mode equation in quasi de Sitter at second order is the presence of a logarithmic term. Whether there could be a fully understood solution to that particular differential equation would dictate the existence of a solution pathway similar to that of de Sitter, up to second order.

At second order, we don't observe much change to the usual power spectrum amplitude or spectral tilt. The tensor spectral tilt may well be of second order in Hubble parameters rather than perfectly flat, thus the second order expansion should confirm the consistency relation. Certainly, a second order treatment of the perturbations could also uncover additional effects in the pre-inflationary factor, or provided fortunate cancellations in its spurious UV oscillations. I think, more exciting still, is the idea of incorporating the

Weyl curvature into the calculation. [43] As we had previously stated, the background is conformally flat, so the Weyl terms does not contribute to the classical dynamics. However, Weyl curvature can and does appear when the full quadratic action for the tensor perturbations. Effects on the power spectrum amplitude and spectral tilt have been explored – but what other imprints might the presence of Weyl curvature have on the pre-inflationary evolution of the modes? However, it's important to note that our major results still apply even when we introduce Weyl curvature into the action! The combination $\alpha\Lambda$ that gives us the amount of veiling e-folds $N_{\rm veil}$ is derived using only background dynamics, which Weyl does not affect. Additionally, introducing Weyl doesn't change the scalar perturbations, so we can still use the observations of Planck to set the value of α.

A cosmically veiled universe is arguably one of the most interesting ones to be a part of. By having an upper bound on inflationary e-foldings, we limit the range over which the power spectrum can be nearly-scale invariant with a red tilt, as we would have power suppression at large scales. We may even live in a universe where new insights from observation might not be woefully far into the future. The next generation CMB detector, spearheaded by the CMB-S4 team [44], targets sensitivities in the tensor-to-scalar ratio r down to $r = 0.001$, put bounds on the energy scale of inflation, and probe large scale, low $\ell \sim 20$ amplitudes of the CMB. There are deep consequences resulting from either a detection or non-detection of tensor modes. On one hand, a detection would confirm the energy scale of inflation in the single field inflation framework, as well as open the door to new physics at the α scale. Alternatively, a non-detection could spell the death knell for the simpler inflationary frameworks, such as the one studied herein by Starobinsky, as they would become disfavored by observation.

Most tantalizing, cosmic veiling gives us a through line, connecting the nascent, primordial universe, to our current day. This connection, coming from the inflationary e-folding bound by veiling,

$$N_{\rm veil} \lesssim \log\left(\tfrac{1}{\alpha\Lambda}\right) \simeq 64\,, \tag{6.192}$$

brings together two fundamental constants, which in essence encapsulate the separation of scales in the second order Einstein-Hilbert action. Whether the cosmic veiling principle could be elevated to a consequence of another more fundamental mechanism is up for speculation; that together, they should bound the expansion of the early universe would be nothing short of an awe-inspiring mystery.

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- Cosmic veiling and pre-inflationary conditions in $R + \alpha R^2$ gravity
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- Differential photometry target selection, data collection and analysis

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