The Pennsylvania State University
The Graduate School

TOPICS IN THE STANDARD MODEL OF PARTICLE PHYSICS EXTENSION AND FOURTH-ORDER GRAVITY

A Dissertation in
Physics
by
Joseph R. Ochoa

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2011

The thesis of Joseph R. Ochoa was reviewed and approved* by the following:

Stephon Alexander
Professor of Physics
Dissertation Co-Adviser
Co-Chair of Committee

Martin Bojowald
Professor of Physics
Dissertation Co-Adviser
Co-Chair of Committee

Richard Robinett
Professor of Physics
Graduate Officer for Department of Physics

Victor Nistor
Professor of Mathematics

*Signatures are on file in the Graduate School.

Abstract

In this thesis I present two possible signatures of quantum gravitational phenomenology. The first part of this thesis relates to a Lorentz symmetry violating extension of the standard model of particle physics. Here I show that a Chern-Simons type extension of the quantum electrodynamic (QED) sector of the standard model (SM) leads to the generation of circular polarization for photons. The polarization of scattered photons are analyzed using quantum field theoretic techniques and through the application of a generalized Boltzmann transport equation. In addition to the previously studied optical activity or birefringence effects induced by the particular interaction studied here, the Lorentz invariance violating interaction in question also leads to the generation of circular polarization. The possibility for observation of the effects in the cosmic microwave background (CMB) is discussed, although the circular polarization effects are shown to be at a level which is always sub-dominate to the birefringence effects.

The second part of this thesis relates to a fourth-order modification to the general theory of relativity (GR) which has appeared as quantum corrections in the effective spectral action of noncommutative geometry (NCG). A term which is proportional to the square of the Weyl curvature is added to the Einstein-Hilbert action of GR and the the gravitational wave solutions of this modified theory are derived. The implications for the possibility of constraining the parameters of NCG through the analysis of data on the rate of orbit decay of binary pulsars is discussed.

Table of Contents

List of Figures

Acknowledgments

There are many people for whom I will be forever grateful and without which this thesis would not have been possible. First, I would like to especially thank my advisor, Stephon Alexander, for his years of guidance, encouragement and always stimulating discussions. Through his example, he has always challenged and inspired me to think creatively and to attempt to build a true intuition in the field of cosmology and physics in general. I would like to thank Martin Bojowald for his always helpful instruction, stimulating discussions and for guiding me through several summers of independent study early in my graduate school career.

I would also like to thank and gratefully acknowledge collaboration with Arthur Kosowsky, William Nelson, and Mari Sakellariadou. I have learned much through our collaborations, interactions and discussions.

I would like to give a special thanks to Abhay Ashtekar for many instances in which he gave guidance, useful suggestions and words of encouragement. Likewise, I would like to give my thanks to Richard Robinett for the guidance he has offered me and for always being available for useful chats while I have been at Penn State.

I would like to thank many of the graduate students and post-docs in the Penn State Department of Physics at large as well as in the Institute for Gravitation and the Cosmos with whom it has been my great pleasure to interact, both in discussions on physics and socially, during my graduate school career, including Eloisa Bentivegna, Tanja Bode, Miguel Campiglia, Daniel Costantino, Asher Evans, Jeff Fraser, Eric Hazlett, Adam Henderson, Nathan Johnson-McDaniel, Alok Laddha, David Landy, George Paily, Andrew Randono, Juan Reyes, Timothy Russin, Mitchell Shuster, David Simpson, David Sloan, Victor Taveras, Casey Tomlin, Artur Tsobanjan, Deepak Vaid, Mark Wilson, Edward Wilson-Ewing, Shaun Wood, and Nicolas Yunes.

To the support staff, including Randi Neshteruk, Megan Meinecke, and Carol Deering, all of whom have helped me immensely in navigating the ins and outs of university policy, I am also very grateful. To say that this thesis would not have been possible if not for their support is truly an understatement.

Finally, I would very much like to thank my parents, Rolando and Yolanda Ochoa, for their continued support, constant encouragement in my interests, and unwavering belief in me.

The work presented in this thesis was supported in part by the Eberly College of Science, the Bunton-Waller Assistantship, and the Alfred P. Sloan Foundation Scholarship. I am also very grateful to these sources for their generosity without which I would not have been able to keep the lights on much less complete this thesis.

Dedication

To my parents

Chapter 1

Introduction

While the quest for a consistent quantum theory of gravity is ongoing, much research has centered on studying possible observational and experimental signatures of quantum gravitational phenomenology. Although the energy scales at which quantum gravity effects are expected to normally dominate are well beyond those scales currently accessible to earth based laboratory experiments, much can be learned by searching for and constraining those effects which might possibly be present in the low energy regime. Along these lines, cosmological and astrophysical signals can play an important role in the search for quantum gravitational signatures. Naturally, future experimental constraints on a whole host of phenomena will indeed guide high energy theorist beyond the current standard models and towards a complete theory of quantum gravity. One must remember that it was with Kirchoff's suggestion to measure the energy density spectrum of the thermal black body that the quantum revolution of the early 20th century was born. With this historical lesson in mind we present here two examples of possible observational signatures of quantum gravitational corrections.

This thesis consists of a general introduction, two distinct parts and a general conclusion. While the topics covered can be considered to fall under the general heading of quantum gravitational phenomenology, each part covers distinct projects and is based on papers which have appeared in refereed journals [1–3]. The introduction serves as a means of presenting background information and is intended for a general physics audience. In some instances the introductory material is not meant to be exhaustive but only to serve as suitable background for later topics, in which cases an attempt has been made to point the reader towards useful references. The general conclusion to this thesis serves as a means of summarizing the conclusions to each part and give a overview of possible future work.

Part I of this thesis describes the effect of a particular Lorentz symmetry violating interaction on the scattering of light and the implications of such an interaction for the anisotropies present in the cosmic microwave background (CMB) radiation. The particular interaction in question (detailed in Sec. 2.1 below) is most commonly encountered in the so-called particle physics standard-model extension (SME) [4–7] in which Lorentz invariance violating interactions are systematically considered within the formalism of effective quantum field theory. With an eye towards CMB anisotropies, we restrict our analysis to the Lorentz invariance violating extension of the quantum electrodynamic (QED) sector of the particle physics standard model.

The detection and analysis of the anisotropies in the CMB have played a major role in the modern understanding of inhomogeneities in the early universe. In the standard cosmology, primordial quantum fluctuations of physical fields lead to inhomogeneities in the early universe which are believed to ultimately lead to the large scale structure of the universe observed today. These primordial inhomogeneities leave an imprint on temperature and polarization fluctuations observed in the CMB whose study helps us understand the origins and nature of these inhomogeneities.

In the analysis of the polarization induced in the CMB, a standard analysis shows (see Subsection 1.1.3 below) that although linear polarization is generated in the CMB, no circular polarization is generated by the dominant process of Thomson scattering. The implications for first generation CMB sky survey anisotropy experiments is naturally then to focus on the detection of those primary sources of anisotropies, that is, those anisotropies associated with temperature and linear polarization fluctuations. Difficulties in experimental systematics of CMB circular polarization detection have also contributed to the focus on linear polarization detection. Generally, constraints on the extent to which the CMB is circularly polarized have been absent from the capabilities of CMB anisotropy experiments since the early constraints were placed [8, 9].

It was with this in mind that Stephon Alexander made the suggestion to search for an interaction that would generate circular polarization, a signal that if detected (or on which tighter constrained were placed) may inform us about physics beyond the standard models. My search for such an interaction ultimately led to the previously stated published work [1] in collaboration with Alexander and Arthur Kosowsky, whose techniques for analyzing standard CMB anisotropies [10] I implemented in the search for the novel signal. The results of our investigation show that through direct observation of the extent to which the CMB is circularly polarized, additional constraints and/or consistency checks may be placed on the violation of Lorentz invariance implied by the particular interaction studied here.

Part II of this thesis centers on the modifications to the gravitational wave solutions of Einstein's general theory of relativity (GR) due to a modification to the Einstein-Hilbert action by the addition of a term proportional to the square of the Weyl curvature tensor (see Subsection 1.2.1 below for the precise definition of the Weyl tensor) and is based on work done in collaboration with William Nelson and Mari Sakellariadou and originally published here [2, 3].

One of the novel predictions of Einstein's general theory of relativity (GR) is the existence of freely propagating gravitational waves in spacetime, that is to say, wave modes of the spacetime metric itself. These gravitational waves are propagating solutions to Einstein's equations in the absence of matter. While the direct detection of gravitational waves is as yet still on the horizon there is already indirect evidence for the gravitational radiation reaction process through observations made on the rate of damping of the orbits of binary stellar systems. In the coming years, gravitational wave astronomy using ground based interferometric detectors such as LIGO [11], Advanced LIGO [12], VIRGO [13], GEO600 [14] and TAMA300 [15] as well as the space borne detector LISA [16] will continue the search for direct detection of a signal of gravitational waves.

The modified theory of gravity under consideration in this thesis most suitably falls under the heading of fourth-order gravity [17], as the resulting equations of motion are fourth-order in derivatives of the spacetime metric degrees of freedom. The motivation for considering the particular modification to the Einstein-Hilbert action in question is due to its presence as a quantum correction in the effective action of noncommutative geometry (NCG), a candidate framework for the unification of gravity theory and the standard model of particle physics [18–20]. A

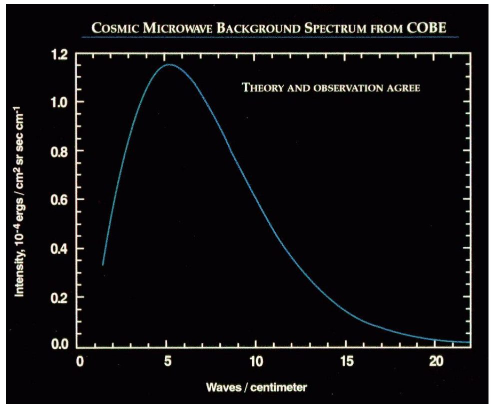

Figure 1.1. The black body spectrum of the CMB as measured by the COBE satellite. Note that the plotted curve represents the data collected. Image credit: NASA/COBE Science Team.

brief introduction to NCG is covered in Subsection 1.2.3 of this introduction with further details on NCG presented in Sec. 3.2. The implications for the gravitational wave signal arising from astrophysical sources is computed and discussed in detail throughout Chapter 3.

1.1 Generating Circular Polarization in the Cosmic Microwave Background

1.1.1 CMB basics

The Cosmic Microwave Background (CMB) was first discovered by Penzias and Wilson in 1965 [21], garnering them the 1978 Nobel Prize in Physics. A remnant of the hot big bang, the CMB is a nearly isotropic thermal radiation permeating the observable universe. The radiation has a black body spectrum with average temperature of $T = 2.725 \pm 0.002K$ [22], see Fig. 1.1. Anisotropies in the temperature or intensity of the CMB were first detected in detail by the COBE satelite [23]. Futher temperature and polarization anisotropies have been measured by the Wilkinson Microwave Anisotropy Probe (WMAP) [24–26]. Here I give a brief introduction to some of the physics behind the anisotropies in the CMB; for a more complete review see for example [27] or [28] and references therein.

1.1.2 Temperature Anisotropies

Immediately following the big bang, matter and photons form a tightly coupled fluid. As the universe undergoes cosmological expansion, temperatures decrease to scales at which stable Hydrogen, Helium and other light atoms are able to form. This epoch in the cosmological evolution is often referred to as recombination. Once this stage in the cosmological evolution is reached, roughly 13.7 billion years ago, the photons, now no longer able to ionize matter, begin to freely stream through the universe; this is the so-called last scattering or decoupling of the CMB photons. Strictly speaking, the CMB photons do however interact again with matter following decoupling; the free streaming of CMB photons is interrupted during a stage of reionization of the intergalactic medium roughly 400 million to 1 billion years after the big bang [27, 29].

Prior to last scattering, scalar (energy) density perturbations, as well as vector and tensor (gravitational wave) perturbations, representing inhomogeneities in the universe result in quadrapole anisotropies in the intensity of radiation. Although all three types of perturbations generally lead to quadrapole anisotropies in the temperature of CMB radiation, it is only the scalar perturbations which can lead to structure formation via gravitational instability and also result in the strongest signal in CMB anisotropies. Seen from our vantage point from earth these primordial scalar perturbation inhomogeneities appear as temperature anisotropies in the CMB as displayed in Fig. 1.2. An immediate consequence of the presence of quadrapole anisotropies in the intensity of radiation prior to last scattering is the generation of linear polarization in the CMB radiation during the epoch of decoupling.

1.1.3 Polarization and Thomson Scattering

The fact that the CMB must be linearly polarized given the presence of quadrapole temperature anisotropies can most easily be seen by considering the form of the scattering amplitude for the dominant process of Thomson scattering of the CMB photons off of free electrons during decoupling. The Thomson differential cross

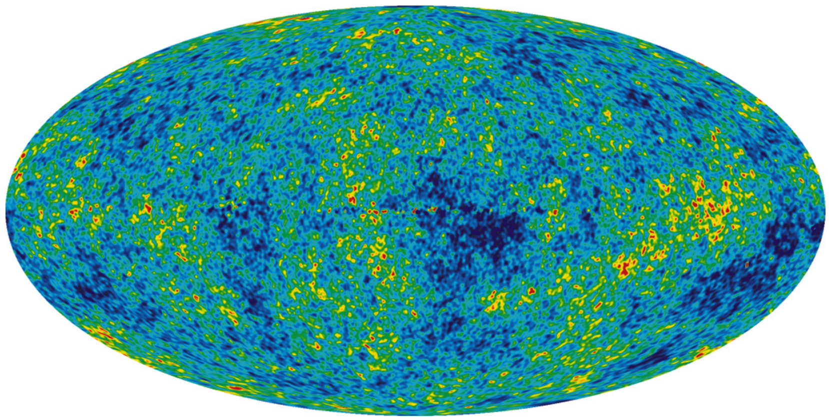

Figure 1.2. The sky map survey of the temperature anisotropies in the CMB as measured by the 5 year WMAP. Image credit: NASA/WMAP Science Team.

section has the form [30],

$$\frac{d\sigma_T}{d\Omega} \propto | \, \epsilon^{(s)} \cdot \epsilon^{(0)} \, |^2, \tag{1.1}$$

where $\epsilon^{(0)}$ is the incident polarization direction and $\epsilon^{(s)}$ is the scattered polarization direction. The scattering amplitude is maximized for scattered radiation propagating in a direction perpendicular to the incoming polarization direction. For the moment, consider fixing the direction of the scattered photon's momentum and consider the contribution to the scattering cross section from all incoming directions. Assume that the incoming radiation is unpolarized. If the incoming radiation intensity is isotropic then incident radiation from directions which are separated by 90° would compensate for each other resulting in an unpolarized scattered photon. However, if the incoming radiation intensity exhibits a quadrapole anisotropy, characterized by intensity peaks separated by 90°, then in this case linear polarization is generated in the scattered radiation. The detection of a polarization anisotropy in the CMB has been verified by several collaborations [25, 26, 31–33] and more detailed polarization data is expected from the PLANCK collaboration [34, 35] in the coming years.

Another prediction based solely on the assumption of Thomson scattering as the dominant process during last scattering is the expected absence of circular polarization in the CMB. To see explicitly that circular polarization is not generated by the Thomson scattering process, following a similar construction as the one given in [30], let us describe the initial polarization state of an incoming photon as,

$$\epsilon_{\parallel}(t) = \epsilon_{\parallel}^{(0)} sin(\omega t - \delta_1) \tag{1.2}$$

$$\epsilon_{\perp}(t) = \epsilon_{\perp}^{(0)} sin(\omega t - \delta_2) \tag{1.3}$$

Where $\epsilon_{\parallel}$ and $\epsilon_{\perp}$ are the independent polarizations which are parallel and perpendicular, respectively, to the scattering plane. The scattering plane being the plane spanned by the incoming and (observed) outgoing photon momenta. Here we will assume that $\epsilon_{\parallel}^{(0)}$,$\epsilon_{\perp}^{(0)}$, δ_1, and δ_2 are approximately constant functions in time on a time-scale of ω^{-1} but for arbitrary time dependence one would simply time average over a complete cycle in the subsequent calculation without changing the overall analysis in regards to the absence of circular polarization in the scattered radiation. The incoming light can be described via the Stokes parameters in the linear polarization basis as

$$I^{(0)} = (\epsilon_{\parallel}^{(0)})^2 + (\epsilon_{\perp}^{(0)})^2, \tag{1.4}$$

$$Q^{(0)} = (\epsilon_{\parallel}^{(0)})^2 - (\epsilon_{\perp}^{(0)})^2, \tag{1.5}$$

$$U^{(0)} = 2\epsilon_{\parallel}^{(0)}\epsilon_{\perp}^{(0)} cos(\Delta), \tag{1.6}$$

$$V^{(0)} = 2\epsilon_{\parallel}^{(0)}\epsilon_{\perp}^{(0)} sin(\Delta), \tag{1.7}$$

where $\Delta = \delta_1 - \delta_2$. Note that $I^{(0)}$ and $V^{(0)}$ are independent of any coordinate transformations in the plane normal to the photon's momentum. However, $Q^{(0)}$ and $U^{(0)}$ are in fact dependent on the particular choice of coordinates in the plane normal to the photon momentum; $Q^{(0)}$ and $U^{(0)}$ get mixed under a rotation of coordinates while the quantity $(Q^{(0)})^2 + (U^{(0)})^2$ remains invariant. For a photon scattered at an angle Θ with respect to the incoming photon, the form of the Thomson scattering differential cross section requires that the amplitudes of the scattering photon $\epsilon_{\parallel}^{(s)}$ and $\epsilon_{\perp}^{(s)}$ are in the ratio of $cos(\Theta) : 1$. The independent polarizations of the scattering photon will then be

$$\epsilon_{\parallel}^{(s)}(t) = \left[\frac{3}{2}\sigma_T\right]^{\frac{1}{2}} \epsilon_{\parallel}^{(0)} cos(\Theta)\ \ sin(\omega t - \delta_1)\,, \tag{1.8}$$

$$\epsilon_{\perp}^{(s)}(t) = \left[\frac{3}{2}\sigma_T\right]^{\frac{1}{2}} \epsilon_{\perp}^{(0)} sin(\omega t - \delta_2)\,, \tag{1.9}$$

where σ_T is the total cross section for Thomson scattering. The Stokes parameters for the outgoing photon are by definition

$$I^{(s)} = (\epsilon_{\parallel}^{(s)})^2 + (\epsilon_{\perp}^{(s)})^2\,, \tag{1.10}$$
$$Q^{(s)} = (\epsilon_{\parallel}^{(s)})^2 - (\epsilon_{\perp}^{(s)})^2\,, \tag{1.11}$$
$$U^{(s)} = 2\epsilon_{\parallel}^{(s)}\epsilon_{\perp}^{(s)} cos(\Delta)\,, \tag{1.12}$$
$$V^{(s)} = 2\epsilon_{\parallel}^{(s)}\epsilon_{\perp}^{(s)} sin(\Delta)\,. \tag{1.13}$$

Using Eq. (1.8) and Eq. (1.9) in the above definitions for the outgoing photon Stokes parameters we have

$$I^{(s)} \propto \left[(\epsilon_{\parallel}^{(0)})^2 cos^2(\Theta) + (\epsilon_{\perp}^{(0)})^2\right]\,, \tag{1.14}$$
$$Q^{(s)} \propto \left[(\epsilon_{\parallel}^{(0)})^2 cos^2(\Theta) - (\epsilon_{\perp}^{(0)})^2\right]\,, \tag{1.15}$$
$$U^{(s)} \propto \left[2\epsilon_{\parallel}^{(0)}\epsilon_{\perp}^{(0)} cos(\Theta) cos(\delta)\right]\,, \tag{1.16}$$
$$V^{(s)} \propto \left[2\epsilon_{\parallel}^{(0)}\epsilon_{\perp}^{(0)} cos(\Theta) sin(\delta)\right]\,. \tag{1.17}$$

Let us now express the right hand side of the above equations in terms of the incoming Stokes parameters by using the following rearrangement of variables

$$(\epsilon_{\parallel}^{(0)})^2 = \frac{1}{2}\left[I^{(0)} + Q^{(0)}\right]\,, \tag{1.18}$$
$$(\epsilon_{\perp}^{(0)})^2 = \frac{1}{2}\left[I^{(0)} - Q^{(0)}\right]\,, \tag{1.19}$$

which upon substitution gives

$$I^{(s)} \propto \left[I^{(0)}\big(1 + cos^2(\Theta)\big) - Q^{(0)}\big(1 - cos^2(\Theta)\big)\right]\,, \tag{1.20}$$
$$Q^{(s)} \propto \left[I^{(0)}\big(1 - cos^2(\Theta)\big) + Q^{(0)}\big(1 + cos^2(\Theta)\big)\right]\,, \tag{1.21}$$

$$U^{(s)} \propto \left[U^{(0)} cos(\Theta) \right], \tag{1.22}$$

$$V^{(s)} \propto \left[V^{(0)} cos(\Theta) \right], \tag{1.23}$$

We can see from the above expression Eq. 1.23 that $V^{(s)}$ is independent of the incoming stokes parameters $I^{(0)}$, $Q^{(0)}$, and $U^{(0)}$. In particular, if the incoming radiation contains no circularly polarized degree of freedom, *i.e.* $V^{(0)} = 0$, then the scattered radiation will similarly not be circularly polarized. Recall that it is only $Q^{(0)}$ and $U^{(0)}$ that are dependent on the coordinate representation in the plane normal to the photon momentum and in fact the V Stokes parameter is independent of any choice of coordinate representation. So the the conclusion of this particular analysis is that in Thomson scattering no circular polarization is generated. It is important to stress that the statement that the CMB will not contain any circularly polarized anisotropies is predicated on the assumption that Thomson scattering is the only relevant process on the surface of last scattering, in addition to the assumption that there was no net circular polarization before recombination begins. This is equivalent to the statement that the evolution equation for the V stokes parameters has "no source terms," that is, no terms involving the other three stokes parameters in order to generate circular polarization through the scattering process.

1.1.4 Evolution of Perturbations in FLRW Spacetime

As the calculations presented in Part I are an application of a generalized Boltzmann transport equation applied to the evolution of the CMB photons under the influence of a non-standard interaction, it is instructive to review here the evolution of standard CMB anisotropies using similar techniques. The techniques as applied to the evolution of anisotropies of the CMB were originally developed in [10].

Consider any phase space distributional function $f \equiv f(t, \vec{x}, \vec{k})$, that is to say a phase space density function on some choice of spatial coordinates $\vec{x}$, momentum $\vec{k}$ and with possibly explicit time dependence t. This distribution function should be thought in general to encode physical properties of some radiation ensemble, *e.g.* number density or polarization density. The generalized Boltzmann transport equation for this arbitrary distributional function f is expressed in the generic form

$$Lf = C\,, \tag{1.24}$$

where L is some differential operator, which we will refer to as the Liouville operator, and the right hand side term C, which we will refer to as the collision term, accounts for possible interactions of the radiation with intervening scatterers. The Boltzmann equation has application in the atmospheric scattering of light [30], the neutrino mixing problem [36, 37] and the scattering of CMB photons during recombination [10] just to name only a few.

The Liouville operator, L, determines the propagation of the ensemble in the absence of scattering and in the general relativistic case depends on the underlying spacetime metric. For example, in the cosmologically relevant case one may take metric perturbations about a flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime as follows

$$g_{00} = 1 + 2\Phi(t,\vec{x})\,, \tag{1.25}$$

$$g_{0i} = 0\,, \tag{1.26}$$

$$g_{ij} = -a^2(t)\{(1-\Psi(t,\vec{x}))\delta_{ij} + h_{ij}\}\,. \tag{1.27}$$

Here $a(t)$ is the usual scale factor and we have chosen the longitudinal gauge [38] wherein Φ and Ψ are scalar perturbations (in the Newtonian limit Φ coincides with the Newtonian gravitational potential), h_{ij} are tensor or gravitational wave perturbations, subjected to the transverse traceless gauge conditions ${h^i}_i = 0$ and $\partial^j h_{ji} = 0$, and we have ignored vector perturbations of the spacetime metric. If we split the distributional function f into a homogeneous part and a inhomogeneous part in the following manner

$$f(t,\vec{x},k,\hat{k}) = f^{(0)}(t,k) + f^{(1)}(t,\vec{x},k,\hat{k})\,, \tag{1.28}$$

the Lioville operator to zeroth order in metric perturbations takes the form

$$Lf^{(0)} = \frac{\partial f^{(0)}}{\partial t} - \frac{\dot{a}k}{a}\frac{\partial f^{(0)}}{\partial k} = 0\,, \tag{1.29}$$

which implies $f^{(0)}(t,k) = f^{(0)}(ak)$, *i.e.* the homogeneous part of f scales uniformly with the cosmological expansion. However, to first order in metric perturbations the Lioville operator on $f^{(1)}$ becomes

$$Lf^{(1)} = \frac{\partial f^{(1)}}{\partial t} + \frac{\partial f^{(1)}}{\partial x^i}\frac{\hat{k}^i}{a} - \frac{\dot{a}k}{a}\frac{\partial f^{(1)}}{\partial k} + \frac{\partial f^{(0)}}{\partial k}k\left[\frac{\partial \Psi}{\partial t} - \hat{k}^i\frac{\partial \Phi}{\partial x^i} - \frac{1}{2}\hat{k}^i\hat{k}^j\frac{\partial h_{ij}}{\partial t}\right]. \quad (1.30)$$

Eq. 1.30 clearly demonstrates how spacetime metric perturbations induce anisotropies in the distribution f. With the explicit form of the Lioville operator (to first order in metric perturbations) in hand one may begin to analyze the effects of scalar and tensor perturbations individually on the ensemble in question. Of course, in the case of the CMB photons, the collision term C is determined by the Compton scattering interaction which can be computed explicitly using quantum field theoretic perturbation theory (again, see [10]). The bulk of Part I of this thesis will concentrate on analyzing the effect of a Lorentz violating interaction on the polarization of CMB photons by explicitly calculating the corresponding collision term which arises in the presence of this non-standard interaction. Before ending this section we next briefly review the standard-model extension (SME), the framework in which our Lorentz symmetry violating interaction is most commonly encountered.

1.1.5 The Standard-Model Extension

The SME [4–7] has been proposed as a phenomenological model designed to incorporate, in addition to the standard model of particle physics, all possible Lorentz symmetry violations and as well as violations of CPT symmetry, the combination of the discrete symmetries charge conjugation (C), parity (P) and time (T) inversions. Such a model has been motivated by considerations of spontaneous breaking of Lorentz symmetry in certain string theories [39], non-standard optical effects in loop quantum gravity [40–42], and indications of Lorentz invariance violation in other candidate quantum gravity theories [43]. The SME incorporates these violations of Lorentz and CPT symmetry within an effective quantum field theoretic framework. Schematically, the SME can be described by the Lagrange density

$$\mathcal{L}_{\text{SME}} = \mathcal{L}_{\text{SM}} + \delta\mathcal{L}_{\text{LIV}}, \quad (1.31)$$

where $\mathcal{L}_{\mathrm{SM}}$ contains all the terms associated with the standard model (SM) of particle physics and $\delta\mathcal{L}_{\mathrm{LIV}}$ accounts for all possible Lorentz invariance violating (LIV) and CPT violating terms. One may further restrict the set of interaction in $\delta\mathcal{L}_{\mathrm{LIV}}$ to only those which are renormalizable in which case the full SME is renormalizable by construction. In the spirit of considering the SM as some low energy effective description of a more fundamental theory, the viewpoint taken in the SME is that possible quantum gravitationally induced Lorentz violation would appear as effective corrections to the highly phenomenologically successful SM. Typically, the LIV terms involve background fields which act to break particle Lorentz transformations (referred to as active transformations in a Lorentz invariant theory) while maintaining coordinate invariance (referred to as passive transformations in Lorentz invariant theories).

Studies of various Lorentz symmetry violating interactions in the SME have led to many high precision tests of Lorentz invariance involving photons [44–50], mesons [51–57], baryons [58–61], muons [62, 63] and electrons [64, 65]. Due to the work on which Part I of this thesis is based the 'vacuum' generation of circular polarization of light has been added to the list of possible signatures of Lorentz symmetry violation [1].

1.2 Gravitational Waves in Noncommutative Geometry

1.2.1 General Relativity Basics

Here I give a brief introduction to the basics of GR, throughout I will assume the speed of light, $c = 1$. In the Einstein-Hilbert formulation of GR the basic variables are the metric $g_{\mu\nu}$, $(\mu, \nu = 0, 1, 2, 3)$ and a covariant derivative operator $\bigtriangledown_\mu$ on a manifold M. The metric is related to the square of the line element via

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \tag{1.32}$$

and gives a notion of distance in the manifold M. Where as in special relativity the square of the line element is fixed with the special (flat) form as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \,, \tag{1.33}$$

that is, the manifold M is taken as flat Minkowski space, in GR the metric is a dynamical entity and the spacetime manifold M is allowed to be curved. The curvature of M is encoded in the Riemann curvature tensor which is defined as [38]

$$[\nabla_\mu, \nabla_\nu]\omega_\rho \equiv \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \right)\omega_\rho = R_{\mu\nu\rho}{}^\sigma \omega_\sigma \,, \tag{1.34}$$

where ω_μ is any dual vector field. The Reimann tensor satisfies the following properties:

$$R_{\mu\nu\rho}{}^\sigma = -R_{\nu\mu\rho}{}^\sigma \,, \tag{1.35}$$
$$R_{[\mu\nu\rho]}{}^\sigma = 0 \,, \tag{1.36}$$
$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} \qquad (\text{for } \nabla_\rho \, g_{\mu\nu} = 0) \,, \tag{1.37}$$
$$\nabla_{[\mu} R_{\nu\rho]\sigma}{}^\lambda = 0 \,, \tag{1.38}$$

where square brackets [...] denote antisymmetrization. The last property above, Eq. 1.38, is known as the Bianchi identity.

It is useful to next define the ‘trace part’ of the Reimann tensor known as the Ricci tensor, $R_{\mu\nu}$, given here by the equation[1]

$$R_{\mu\nu} = R^\rho{}_{\mu\nu\rho} \,. \tag{1.39}$$

The Ricci tensor is a symmetric rank-2 tensor, *i.e.* $R_{\mu\nu} = R_{\nu\mu}$. Continuing with the pattern, the next useful quantity to consider is the trace of the Ricci tensor known as the Ricci scalar curvature, R, given by the equation

$$R = R_\mu{}^\mu \,. \tag{1.40}$$

[1]Note that here we stray slightly from standard conventions in the definition of the Ricci tensor, our choice differs by a minus sign relative to that of [38]; for more on the particulars behind this choice see Appendix B.

Finally, the 'trace free' part of the Reimann tensor is known as the Weyl curvature tensor and is given by the equation

$$C_{\mu\lambda\nu\kappa} = R_{\mu\lambda\nu\kappa} + (g_{\mu[\nu}R_{\kappa]\lambda} - g_{\lambda[\nu}R_{\kappa]\mu}) - \frac{1}{3}g_{\mu[\nu}g_{\kappa]\lambda}R\,, \tag{1.41}$$

which, given the symmetry properties of the Reimann tensor above, can easily be shown to have vanishing trace.

The dynamics of the metric (and thus the manifold M) are determined via an action principle, with the full action of gravity and matter given by the equation

$$S = S_{\text{EH}} + S_{\text{matter}}\,, \tag{1.42}$$

where the Einstein-Hilbert action, S˙EH, is given by the equation

$$S_{\text{EH}} = \frac{1}{16\pi G}\int_M \sqrt{-g}R\,, \tag{1.43}$$

and where R is again the Ricci scalar curvature, g is the determinant of the metric $g_{\mu\nu}$ and G is Newton's gravitational constant. Variation of the action S with respect to the inverse metric $g^{\mu\nu}$ leads to the Einstein equation

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,, \tag{1.44}$$

where $G_{\mu\nu}$ is known as the Einstein tensor and is given by the equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\,. \tag{1.45}$$

The matter stress energy tensor $T_{\mu\nu}$ is defined (once an action for the matter degrees of freedom is specified) by the equation

$$T_{\mu\nu} = -\frac{1}{8\pi\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{1.46}$$

Given some information on the matter configuration, the Einstein equation can in principle be used to find the 10 independent components of the metric $g_{\mu\nu}$ and thus solve for the dynamics of the spacetime manifold M and the gravitational interactions of the matter content. GR makes strikingly distinct predictions from

Newton's theory of gravity, including black holes, dynamic cosmological solutions and gravitational waves.

1.2.2 Gravitational Waves in GR Basics

In this section I will review the standard procedure for the linearization of Einstein's equation in order to demonstrate the gravitational wave solution of GR, (*c.f.* [38] or [66]). For definiteness, I will follow the standard procedure while perturbing the spacetime metric $g_{\mu\nu}$ about a flat metric, $\eta_{\mu\nu}$, where

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}\,, \qquad \text{and} \qquad g^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu}\,, \tag{1.47}$$

and

$$\gamma = \gamma^{\mu}{}_{\mu} = \eta^{\mu\nu}\gamma_{\mu\nu}\,; \tag{1.48}$$

note that all tensor indices, except for those of the metric, $g_{\mu\nu}$, and the inverse metric, $g^{\mu\nu}$, are raised and lowered using the background metric $\eta_{\mu\nu}$. To linear order in metric perturbations $\gamma_{\mu\nu}$ the Einstein tensor is given by the equation

$$G^{\mu\nu} = +\frac{1}{2}\partial_\lambda\partial^\lambda\bar{\gamma}^{\mu\nu} + \mathcal{O}(\gamma^2)\,. \tag{1.49}$$

where $\mathcal{O}(\gamma^2)$ denotes any combinations which are of second-order in $\gamma_{\mu\nu}$ and we have defined the 'trace reverse' of $\gamma_{\mu\nu}$ defined by the equation

$$\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma\,, \tag{1.50}$$

The action of GR is invariant under general coordinate transformations, also known as diffeomorphism transformations. The action of an infinitesimal diffeomorphism generated by a vector field ξ^μ transforms the metric perturbations in the following manner:

$$\gamma^{\rm old}_{\mu\nu} \xrightarrow{\xi_\mu} \gamma^{\rm new}_{\mu\nu} = \gamma^{\rm old}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,. \tag{1.51}$$

The spacetimes described by the $\gamma^{\rm old}_{\mu\nu}$ or $\gamma^{\rm new}_{\mu\nu}$ are physically equivalent. Due to this invariance under diffeomorphisms we are free to impose the so called Lorenz gauge on the metric perturbations, *i.e.*,

$$\partial^{\mu}\bar{\gamma}_{\mu\nu} = 0\,, \tag{1.52}$$

which restrict the perturbations to be transverse. The Lorentz gauge however does not uniquely fix the gauge freedom. We still are free to perform gauge transformations, generated by any ξ_{μ}, that satisfy the equation

$$\partial_{\mu}\partial^{\mu}\xi_{\nu} = 0\,, \tag{1.53}$$

since this still preserves the gauge condition, Eq. (1.52). We can use this transformation, Eq. (1.53), to set $\gamma^{\text{new}} = 0$ and $\gamma_{0i}^{\text{new}} = 0$ $(i = 1, 2, 3)$ by solving the corresponding equations for ξ_i and their time derivatives on some initial surface $t = t_0$ where no sources are present and further extending into a source free region (*i.e.*, $T^{\mu\nu} = 0$). After performing these gauge fixing procedures we have expressed the metric perturbation, $\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu}$, in the so-called 'transverse-traceless' gauge. Furthermore, if $T_{00} = 0$ throughout the spacetime manifold M, we are also free to set $\gamma_{00} = 0$; this additional condition on γ_{00} together with the transverse-traceless condition is known as the radiation gauge.

The source free equations of motion for the spatial part of $\gamma_{\mu\nu}$ are now simply expressed as

$$-\frac{1}{2}\Box\gamma_{ij} = 0\,, \tag{1.54}$$

where $\Box = \partial^{\mu}\partial_{\mu}$ is the wave operator in 4-dimensions. It should be clear at this point that the solution propagates at the speed of light. If we assume propagation in the $x^3 = z$-direction, the general solution to Eq. 1.54 in the radiation gauge takes the form

$$\gamma_{ij} = \Lambda_{ij} f(t - z)\,. \tag{1.55}$$

In Equation (1.55) above, f is any arbitrary function of the combination $t - z$, and Λ_{ij} is some constant matrix which, due to the gauge fixing we can express in the form

$$\Lambda_{ij} = \begin{pmatrix} \Lambda_+ & \Lambda_\times & 0 \\ \Lambda_\times & -\Lambda_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.56}$$

where Λ_+ and $\Lambda_\times$ are the two independent polarization modes of the gravitational wave.

1.2.3 Noncommutative Geometry Basics

Here I'll give a very brief introduction to some of the concepts in noncommuntative geometry; for a more details see Sec. 3.1 or [18, 19, 67]. Noncommutative geometry has been proposed as a possible means towards unification of GR and the standard model of particle physics [20]. In a loose sense, the noncommutative geometry replaces the algebra of smooth functions $\mathcal{F}$ on an ordinary manifold with a noncommutative algebra $\mathcal{A}$ to, in a certain sense, describe a noncommutative manifold.

Consider again the full action of gravity and the standard model given by the equation

$$S = S_{EH} + S_{SM}. \tag{1.57}$$

The action S has a family of symmetries. The Einstein Hilbert action S_{EH} has gauge symmetries given by the group of diffeomorphism $Diff(M)$ that is, the group of general coordinate transformations on M. While the standard model of particle physics action S_{SM} has a group of (internal) gauge symmetries $\mathcal{G}$ given up to currently accessible energy scales as

$$\mathcal{G} = SU(3) \times SU(2) \times U(1). \tag{1.58}$$

The full group of symmetries of S are then given by

$$\mathcal{U} = \mathcal{G} \rtimes Diff(M), \tag{1.59}$$

i.e. the semidirect product of $\mathcal{G}$ with $Diff(M)$. If one were to look for an ordinary manifold that had as it's diffeomorphism group the full group $\mathcal{U}$, there would be no such ordinary manifold. But in fact this is not the case if one looks at noncommutative spaces where the object whose diffeomorphism group is $\mathcal{U}$ can be considered as a generalization of an ordinary manifold, e.g. a noncommutative manifold. And so the task of unification of the standard model of particle physics with gravity becomes, in the framework of noncommutative geometry, in a sense equivalent to writing down a noncommutative space whose algebra can be mapped to the desired full symmetry group, *e.g.* $\mathcal{U}$ above.

Using techniques in noncommutative geometry it has been proposed [67] that one may arrive at a full unified theory of gravity and the standard model with additional non-standard terms appearing in the resulting effective action. The gravity sector is modified such that nonstandard terms appear in additional to the standard Einstein-Hilbert term Eq. (1.43). Due to the additional terms suggested by the effective action of noncommutative geometry the linearized equation of motion for gravitational waves is modified. Where as in ordinary GR the equation for gravitational wave perturbations is a standard second-order wave equation Eq. (1.54), in the modified theory the gravitational wave perturbations are determined by a fourth-order wave equation. In collaboration with Nelson and Sakellariadou, I was able to find the solution to this fourth-order wave equation using Green's function techniques which led to the published work [2] and is the subject of Part II of this thesis.

Part I

Lorentz Invariance Violating Extension of Quantum Electrodynamics

Chapter 2

The Generation of Circular Polarization in the Cosmic Microwave Background

The standard cosmological model, which includes only Compton scattering photon interactions at energy scales near recombination, results in zero primordial circular polarization of the cosmic microwave background[1]. In this chapter we consider a particular renormalizable and gauge-invariant standard model extension coupling photons to an external vector field via a Chern-Simons term, which arises as a radiative correction if gravitational torsion couples to fermions. We compute the transport equations for polarized photons from a Boltzmann-like equation, showing that such a coupling will source circular polarization of the microwave background. For the particular coupling considered here, the circular polarization effect is always negligible compared to the rotation of the linear polarization orientation, also derived using the same formalism. We note the possibility that limits on microwave background circular polarization may probe other photon interactions and related fundamental effects such as violations of Lorentz and gauge invariance.

[1]This chapter is based on the work by the author in collaboration with Alexander and Kosowsky and originally published here [1]. Portions of Sec. 2.1 were originally written by Alexander and portions of the analysis in Sec. 2.9 were written and originally performed by Kosowsky and double-checked by the author.

2.1 Introduction

One of the great successes of the standard cosmology is the prediction and measurement of the temperature anisotropies in the cosmic microwave background radiation. Most of these photons have freely propagated since the epoch of last scattering roughly 14 billion years ago and encode the initial conditions for structure formation. Measurements are now consistent to high precision with the simplest cosmological models with an initial power-law spectrum of adiabatic perturbations. Linear polarization of the microwave background fluctuations is also a generic result of these models; recent detections of the linear polarization power spectrum and of the cross-correlation between linear polarization and temperature are also consistent with the same cosmological models.

In general, radiation can have linear polarization, with two degrees of freedom (a polarization amplitude and orientation) as well as circular polarization, with a single degree of freedom. It is well known that if an initially unpolarized photon field evolves solely via Compton scattering from free electrons plus free streaming, the resulting radiation field can have linear but not circular polarization. In the tight-coupling regime prior to last scattering when Compton scattering is rapid compared to the cosmological expansion time scale, the cosmic radiation field will be unpolarized. As the universe cools and the free electrons become bound into neutral hydrogen, a small linear polarization is generated from the balance of free-streaming and Compton scattering during this recombination process, but the resulting microwave background radiation today has circular polarization which is identically zero. In this chapter, we consider a generic class of interactions between photons and an external field which can produce circular polarization. The interactions have been considered in other contexts and are general enough to be expected in broad classes of theories beyond the standard model of particle physics. The same interaction can also arise if non-zero spacetime torsion impacts the microwave background radiation. The goal of this chapter is two-fold. First, we provide an explicit calculation showing how circular polarization can be generically sourced in the microwave background, with the relevant evolution equations. Second, we demonstrate what the underlying microphysics might look like.

Consider the following extension of the photon sector of quantum electrodynamics:

$$\begin{aligned} \mathcal{L}' &= \mathcal{L}_{\text{MAXWELL}} + \mathcal{L}_T \\ &\equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g\,\epsilon^{\mu\nu\alpha\beta}A_\mu T_\nu F_{\alpha\beta} \end{aligned} \tag{2.1}$$

where $\mathcal{L}_T$ is CPT odd and violates Lorentz invariance and g is the coupling constant of the interaction. Several authors have investigated such a Lorentz-invariance violating extension of QED for a constant 4-vector T^μ (see *e.g.* [4], [5], [44] and references therein). These so-called *Standard Model Extensions* have been shown to be renormalizable while maintaining gauge invariance [49]. We consider here only the flat-spacetime interaction term $\mathcal{L}_T$ for simplicity; this will be a good approximation in any cosmological context (see Refs. [68] and [69] for the curved spacetime generalization, which includes an extra factor of the square root of the metric determinant). If T_μ is fixed as a constant, the Lagrangian density in Eq. (2.1) is $U(1)$ gauge-invariant up to a boundary term; therefore in these cases the action is gauge invariant. It is well known that such an extension should result in optical activity in the propagation of electromagnetic radiation [70–73]; specifically, a modification to the dispersion relations of free electromagnetic radiation results in a rotation of the plane of linear polarization during propagation. We make no explicit assumption about whether T_μ is spacelike or timelike, although the timelike case appears pathological since it leads to a violation of causality and unitarity [47] . The magnitude of optical activity of electromagnetic radiation has been constrained by analysis of observational data from cosmological sources and from the microwave background radiation [26, 45, 70, 74–80]. Such a term may arise as a radiative correction following the coupling of gravitational torsion with fermionic matter [81, 82]. The same term has also been associated with the cancellation of gauge anomalies in QED when the background field T_μ is allowed to couple to the axial current (see *e.g.* [83]).

In this chapter, we show that in addition to the well-known polarization rotation, such a term may also generate circular polarization, although for the specific case of Eq. (2.1) the circular polarization is always negligible compared to the polarization rotation. The generation of circular polarization following the optical activity produced by T_μ parallels the Faraday conversion and Faraday rotation effects for propagation in magnetized plasmas; for a discussion in the context of the microwave background, see [84]. The observation of circularly polarized microwave background radiation could be evidence of Lorentz-invariance violation and thus physics beyond the standard model; conversely, limits on circular polarization may constrain a certain class of standard model extensions. For a related analysis using an axion-like pseudoscalar coupling to the electromagnetic field, see Ref. [85], who also find a nonzero circular polarization and rotation of linear polarization.

In Sec. 2.2, we review the usual description of polarized electromagnetic radiation in terms of Stokes parameters; linear polarization is described by the Q and U parameters, while circular polarization is described by a non-zero V parameter. Section 2.3 reviews the construction of the Boltzmann-type equation for the photon number density, starting from the quantum-mechanical evolution of the photon density matrix. In Secs. 2.4 and 2.8, we calculate the evolution of the Stokes parameters to first and second order, respectively, in the interaction term $\mathcal{L}_T$, deriving the evolution equation for the V polarization, which is generated from linear polarization due to the interaction term. We conclude in Sec. 2.9 with estimates of the size of the V polarization in the microwave background for given interactions along with the magnitude of linear polarization rotation. Appendix A addresses the issue of gauge invariance.

2.2 Stokes Parameters

The polarization state of light is most easily described by making use of the Stokes parameters. For a complete review see, *e.g.*, Refs. [86–88] or any optics text. Here we review the basic construction of the Stokes parameters in the classical and quantum mechanical contexts in order to motivate the quantum field theoretical construction. Consider a classical electromagnetic plane wave with electric field given by the following components

$$E_1(t) = a_1 \sin(\omega t - \epsilon_1) \quad \text{and} \quad E_2(t) = a_2 \sin(\omega t - \epsilon_2) \tag{2.2}$$

where we assume, for simplicity, that the wave is nearly monochromatic with frequency ω, such that a_1, a_2, ϵ_1, and ϵ_2 only vary on time scales long compared to

ω^{-1}. The Stokes parameters in the linear polarization basis are then defined as

$$I \equiv \left\langle (a_1)^2 + (a_2)^2 \right\rangle , \tag{2.3}$$

$$Q \equiv \left\langle (a_1)^2 - (a_2)^2 \right\rangle , \tag{2.4}$$

$$U \equiv \langle 2a_1 a_2 \cos\delta \rangle , \tag{2.5}$$

$$V \equiv \langle 2a_1 a_2 \sin\delta \rangle , \tag{2.6}$$

where $\delta \equiv \epsilon_2 - \epsilon_1$ and the brackets signify a time average over a time long compared to ω^{-1}. The I parameter measures the intensity of the radiation, while the parameters Q, U, and V each carry information about the polarization of the radiation. Unpolarized radiation is described by $Q = U = V = 0$. The linear polarization of the radiation is encoded in Q and U, while the parameter V is a measure of elliptical polarization with the special case of circular polarization ocurring when $a_1 = a_2$ and $\delta = \pm\pi/2$. From here on we will simply refer to V as the measure of circular polarization, which is technically correct if $Q = 0$. Note that while I and V are coordinate independent, Q and U depend on the orientation of the coordinate system used on the plane orthogonal to the direction of propagation. Under a rotation of the coordinate system by an angle ϕ, the parameters Q and U transform according to

$$Q' = Q\cos(2\phi) + U\sin(2\phi),$$

$$U' = -Q\sin(2\phi) + U\cos(2\phi),$$

while the angle defined by

$$\Phi = \frac{1}{2}\arctan\left(\frac{U}{Q}\right)$$

goes to $\Phi - \phi$ following a rotation by the angle ϕ. Therefore, Q and U only define an orientation and not a particular direction in the plane: after a rotation by π they are left unchanged. Physically, this is simply a manifestation of the oscillatory behavior of the electric field. These properties indicate that Q and U are part of a second-rank symmetric trace-free tensor P_{ab}, i.e. a spin-2 field in the plane orthogonal to the direction of propagation. Such a tensor can be represented as

$$P_{ij} = \begin{pmatrix} P & 0 \\ 0 & -P \end{pmatrix}, \tag{2.7}$$

in an orthonormal eigenbasis, where $P = (Q^2+U^2)^{1/2}$ is often called the magnitude of linear polarization.

In quantum mechanics we can express the state of a photon $\mathcal{A}$ as

$$|\mathcal{A}\rangle = \sum_i a_i |\epsilon_i\rangle , \tag{2.8}$$

where $|\epsilon_i\rangle$ $(i = 1, 2)$ span the polarization state space and a_i are in general complex. The projection operators

$$\hat{I} \equiv |\epsilon_1\rangle \langle\epsilon_1| + |\epsilon_2\rangle \langle\epsilon_2| , \tag{2.9}$$

$$\hat{Q} \equiv |\epsilon_1\rangle \langle\epsilon_1| - |\epsilon_2\rangle \langle\epsilon_2| , \tag{2.10}$$

$$\hat{U} \equiv |\epsilon_1\rangle \langle\epsilon_2| + |\epsilon_2\rangle \langle\epsilon_1| , \tag{2.11}$$

$$\hat{V} \equiv i\,|\epsilon_2\rangle \langle\epsilon_1| - i\,|\epsilon_1\rangle \langle\epsilon_2| \tag{2.12}$$

have expectation values in single photon states which give the classical Stokes parameters, Eqs. (2.3)–(2.6). In a general mixed state, the density matrix ρ on the polarization state space encodes the intensity and polarization of the photon ensemble. For example,

$$\langle\hat{Q}\rangle = \frac{\mathrm{tr}(\rho\hat{Q})}{\mathrm{tr}(\rho)} = \frac{1}{\mathrm{tr}(\rho)}\mathrm{tr}\left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \frac{\rho_{11} - \rho_{22}}{\mathrm{tr}(\rho)}.$$

Similar relations hold for the other "Stokes operators" such that the density matrix can be represented as

$$\rho = \frac{\mathrm{tr}(\rho)}{2}\begin{pmatrix} 1+Q & U-iV \\ U+iV & 1-Q \end{pmatrix}, \tag{2.13}$$

where $Q = \langle\hat{Q}\rangle$, $U = \langle\hat{U}\rangle$, and $V = \langle\hat{V}\rangle$.

2.3 The Photon Boltzmann Equation

We now review the construction of the evolution equation for the photon number operator under the influence of some perturbation to a free theory. The following formalism was developed to study neutrino mixing and damping [36, 37]. It has also been applied to describe the generation of linear polarization in the microwave background due to Compton scattering during recombination [10], generalizing an earlier kinetic equation treatment of microwave background temperature fluctuations [89].

Consider an ensemble of free photons. We will assume that the interaction $\mathcal{L}_T$ is slowly "turned on" and that the interactions of the photons with the external field T_μ are localized such that the photons can be considered free (with respect to the interaction $\mathcal{L}_T$) both before and after each point interaction. That is, we make the usual assumptions of scattering theory. We will not consider any possible interference effects which might occur between $\mathcal{L}_T$ and any other interaction.

The free photon field in the Coulomb (radiation) gauge can be expressed as

$$\hat{A}_\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k^0} \left[\hat{a}_s(k)\epsilon_{s\mu}(k)e^{-ik\cdot x} + \hat{a}_s^\dagger(k)\epsilon_{s\mu}^*(k)e^{ik\cdot x}\right], \tag{2.14}$$

where $\epsilon_{s\mu}(k)$ are the photon polarization 4-vectors and s, which takes the values 1 and 2, indexes the orthogonal transverse polarizations. The free creation and annihilation operators satisfy the canonical commutation relation

$$\left[\hat{a}_i(k), \hat{a}_j^\dagger(k')\right] = (2\pi)^3 2k^0 \delta_{ij}\delta^{(3)}(\mathbf{k} - \mathbf{k}') \tag{2.15}$$

where $k^0 = |\mathbf{k}|$.

We will be interested in the evolution of the polarization state of a photon ensemble, which is completely characterized by the density matrix ρ_{ij} defined via

$$\langle \hat{a}_i^\dagger(k)\hat{a}_j(k')\rangle = (2\pi)^3 2k^0 \delta^{(3)}(\mathbf{k} - \mathbf{k}')\rho_{ij}(\mathbf{k}). \tag{2.16}$$

The number operator $\hat{\mathcal{D}}_{ij}(\mathbf{k}) = \hat{a}_i^\dagger(\mathbf{k})\hat{a}_j(\mathbf{k})$, according to Eq. (2.16), is related to $\rho_{ij}(\mathbf{k})$ as

$$\langle \hat{\mathcal{D}}_{ij}(\mathbf{k})\rangle = (2\pi)^3 2k^0 \delta^{(3)}(0)\rho_{ij}(\mathbf{k}). \tag{2.17}$$

The infinite factor $\delta^{(3)}(0)$ is a remnant of the infinite quantization volume. As we show explicitly below, it cancels from all final expressions. Motivated by the construction in the quantum mechanical system above, we can project out quantities analogous to the classical Stokes parameters:

$$\mathrm{tr}(\sigma_0\rho(\mathbf{k})) \rightarrow I \propto \langle\hat{\mathcal{H}}_0(\mathbf{k})\rangle \tag{2.18}$$

$$\mathrm{tr}(\sigma_z\rho(\mathbf{k})) \rightarrow Q \propto \langle\hat{\mathcal{D}}_{11}(\mathbf{k})\rangle - \langle\hat{\mathcal{D}}_{22}(\mathbf{k})\rangle \tag{2.19}$$

$$\mathrm{tr}(\sigma_x\rho(\mathbf{k})) \rightarrow U \propto \langle\hat{\mathcal{D}}_{12}(\mathbf{k})\rangle + \langle\hat{\mathcal{D}}_{21}(\mathbf{k})\rangle \tag{2.20}$$

$$\mathrm{tr}(\sigma_y\rho(\mathbf{k})) \rightarrow iV \propto \langle\hat{\mathcal{D}}_{12}(\mathbf{k})\rangle - \langle\hat{\mathcal{D}}_{21}(\mathbf{k})\rangle \tag{2.21}$$

where σ_0 is the 2×2 identity matrix, σ_i are the Pauli matrices and the trace is over polarization indices. $\hat{\mathcal{H}}_0(\mathbf{k})$ is the free energy density operator of the mode with wavenumber $\mathbf{k}$.

We ignore any correlations such as $\langle\hat{a}_i(k)\hat{a}_j(k')\rangle$ and $\langle\hat{a}_i^\dagger(k)\hat{a}_j^\dagger(k')\rangle$ which might be generated via the interaction $\mathcal{L}_T$. In essence we are assuming that the background field T_μ varies slowly enough in time so that physical two-photon states are neither created nor destroyed by the interaction $\mathcal{L}_T$. If we define ω_T as a characteristic energy scale of the background field T_μ and ω as the frequency of a particular mode associated with free oscillations of the creation and annihilation operators, then we are requiring that $\omega_T \ll \delta\omega/\omega \ll \omega$ [90]. Note that $\delta\omega/\omega$ is the order at which mixing occurs between $\hat{a}_i$ and $\hat{a}_i^\dagger$, and such a mixing will result in a variation $\delta\langle\hat{a}_i(k)\hat{a}_j(k')\rangle \simeq (\delta\omega/\omega)\langle\hat{a}_i^\dagger(k)\hat{a}_j(k')\rangle$, which again we neglect. Naively, we then expect the following formalism to fail for low-frequency photon modes, although at precisely what scale the approximation breaks down depends on the characteristic scale of the background field T_μ.

The evolution of the number operator $\hat{\mathcal{D}}_{ij}(\mathbf{k})$ can be computed using a perturbative expansion in the interaction strength. The following reviews the construction detailed in [37]. Recall that the time evolution of any Heisenberg picture operator is defined by

$$\frac{d}{dt}\hat{\mathcal{D}}_{ij} = i\big[\hat{H}, \hat{\mathcal{D}}_{ij}\big]. \tag{2.22}$$

If the full Hamiltonian can be split into a free and interacting part, $\hat{H} = \hat{H}_{\text{free}} + \hat{H}_{\text{int}}$, then Eq. (2.22) becomes.

$$\frac{d}{dt}\hat{\mathcal{D}}_{ij} = i\big[\hat{H}_{\rm free}, \hat{\mathcal{D}}_{ij}\big] + i\big[\hat{H}_{\rm int}, \hat{\mathcal{D}}_{ij}\big]. \tag{2.23}$$

A first order perturbative approximation for the evolution of the number operator is found replacing all operators on the right hand side of Eq. (2.23) by their free theory counterparts, e.g.

$$\left\langle \frac{d}{dt}\hat{\mathcal{D}}_{ij} \right\rangle \simeq i\langle[\hat{H}^0_{\rm int}, \hat{\mathcal{D}}^0]\rangle, \tag{2.24}$$

where $\hat{\mathcal{O}}^0$ corresponds to the operator $\hat{\mathcal{O}}$ evaluated in terms of the operators of the free theory. The above assumes that $[\hat{H}^0_{\rm free}, \hat{\mathcal{D}}^0] = 0$. We will refer to the term on the right hand side of Eq. (2.24) as the refractive term, or the forward scattering term. To determine a second order perturbative approximation, we will use the fact that we can expand any operator to first order in interactions as

$$\hat{\xi}(t) \simeq \hat{\xi}^0(t) + i\int_0^t dt'[\hat{H}^0_{\rm int}(t-t'), \hat{\xi}^0(t)], \tag{2.25}$$

with the initial conditions $\hat{\xi}(0) = \hat{\xi}^0(0)$. The expansion Eq. (2.25) can be verified by explicitly taking the time derivative of both sides and seeing that one recovers the Heisenberg equation to first order in interactions. We now expand $[\hat{H}_{\rm int}, \hat{\mathcal{D}}]$ as in Eq. (2.25), insert the result into Eq. (2.23), and upon the evaluation of all operators in terms of the free theory operators arrive at

$$\left\langle \frac{d}{dt}\hat{\mathcal{D}}_{ij} \right\rangle(t) \simeq i\langle[\hat{H}^0_{\rm int}(t), \hat{\mathcal{D}}^0]\rangle - \int_0^t dt'\langle[\hat{H}^0_{\rm int}(t-t'), [\hat{H}^0_{\rm int}(t), \hat{\mathcal{D}}^0]]\rangle. \tag{2.26}$$

The second term in Eq. (2.26) will be referred to as the damping term or the non-forward scattering term. In terms of ρ, the evolution equation reads

$$\begin{aligned}(2\pi)^3\delta^3(0)2q^0\frac{d}{dt}\rho_{ij}(0,\mathbf{q}) \;&=\; i\langle[\hat{H}^0_{\rm int}(0), \hat{\mathcal{D}}^0_{ij}(\mathbf{q})]\rangle \\ &\quad -\frac{1}{2}\int_{-\infty}^{\infty} dt' \;\langle[\hat{H}^0_{\rm int}(t'), [\hat{H}^0_{\rm int}(0), \hat{\mathcal{D}}^0_{ij}(\mathbf{q})]]\rangle,\end{aligned} \tag{2.27}$$

where, as mentioned above, all factors of $\delta^3(0)$ will cancel from the final expressions. In going from Eq. (2.26) to Eq. (2.27), we have assumed the time step t in Eq. (2.26) is both small relative to the characteristic time scale of the evolution of ρ and large relative to the duration of a single interaction. This allows us to take $t \to \infty$ and set $\rho(t) = \rho(0)$ [37]. We have then replaced the integral $(\int_0^\infty dt)$ with $(\frac{1}{2}\int_{-\infty}^\infty dt)$, the difference being a principle part integral which is a second order correction to the refractive term. Equation (2.27) can be viewed as a *Generalized Boltzmann Equation* for the phase space function ρ. In this approximation, we have a set of differential equations for the components ρ_{ij} at $t = 0$; if the interactions are "forgotten" between intermediate collisions (an assumption known as molecular chaos in the derivation of the standard Boltzmann equation), then the differential equations will be valid for all times over which the interaction is relevant. The Liouville terms on the left side will incorporate any effects which result from a departure of the spacetime metric from a flat metric, including any weak inhomogeneities due to the presence of gravitational perturbations about a homogeneous cosmology. The case $H_{\rm int}(t) = \int d\mathbf{x}\bar{\psi}\gamma^\mu A_\mu \psi$ as in full QED, where ψ is a spinor field associated with the electron, recovers the radiative transfer equations of Chandrasekhar [30] in the appropriate limits (see [10] for more details).

As we will see below, this construction allows linearizing the right side of Eq. (2.27) in ρ. We can expand the photon density matrix about a uniform unpolarized distribution (ignoring any small inhomogeneities) as

$$\rho_{ij}(t,k,\hat{k}) = \rho_{ij}^{(0)}(t,k) + \rho_{ij}^{(1)}(t,k,\hat{k}), \tag{2.28}$$

where $\rho_{11}^{(0)} = \rho_{22}^{(0)}$ and $\rho_{12}^{(0)} = \rho_{21}^{(0)} = 0$. As a consistency check, the right side of Eq. (2.27) should vanish when evaluated in terms of $\rho^{(0)}$ so that in for example an FRW background of zero spatial curvature with scale factor $a(t)$ we have [10]

$$\frac{d}{dt}\rho_{11}^{(0)} = \frac{\partial \rho_{11}^{(0)}}{\partial t} - \frac{\dot{a}}{a}k\frac{\partial \rho_{11}^{(0)}}{\partial k} = 0, \tag{2.29}$$

the solution of which is $\rho_{11}^{(0)}(t,k) = \rho_{11}^{(0)}(ka)$, recovering the uniform redshift due to cosmological expansion.

In order to make contact with the measurable Stokes parameters, we define the normalized brightness perturbations [10]

$$\Delta_I \equiv \left[\frac{q}{4}\frac{\partial\rho^{(0)}_{11}(q)}{\partial q}\right]^{-1}(\rho^{(1)}_{11}+\rho^{(1)}_{22}), \tag{2.30}$$

$$\Delta_Q \equiv \left[\frac{q}{4}\frac{\partial\rho^{(0)}_{11}(q)}{\partial q}\right]^{-1}(\rho^{(1)}_{11}-\rho^{(1)}_{22}), \tag{2.31}$$

$$\Delta_U \equiv \left[\frac{q}{4}\frac{\partial\rho^{(0)}_{11}(q)}{\partial q}\right]^{-1}(\rho^{(1)}_{12}+\rho^{(1)}_{21}), \tag{2.32}$$

$$\Delta_V \equiv -i\left[\frac{q}{4}\frac{\partial\rho^{(0)}_{11}(q)}{\partial q}\right]^{-1}(\rho^{(1)}_{12}-\rho^{(1)}_{21}), \tag{2.33}$$

where $q = ka$ is the comoving photon momentum and we have expanded the density matrix ρ in a linear polarization basis.

2.4 The First-Order Interaction Term

We now evaluate the right side of the Generalized Boltzmann Equation (2.27) for an interaction Hamiltonian which is linear in the Hamiltonian density

$$\hat{\mathcal{H}}_T = -g\epsilon^{\mu\nu\alpha\beta} : \hat{A}_\mu T_\nu \hat{F}_{\alpha\beta} :, \tag{2.34}$$

where $\hat{A}_\mu$ for the free theory is given by Eq. (2.14) and $\hat{F}_{\mu\nu} = 2\partial_{[\mu}\hat{A}_{\nu]}$ is the free electromagnetic field strength operator. We will treat T_μ as a classical background field, the dynamics of which are not influenced by the electromagnetic interaction Eq. (2.34) and are assumed to be externally prescribed. The symbol $:\ \cdots\ :$ denotes normal ordering of the enclosed operator products.

Following a local $U(1)$ gauge transformation $\delta_G A_\mu = \partial_\mu\lambda$ of A_μ, the resulting change in the Hamiltonian density Eq. (2.34) is given by

$$\begin{aligned}\delta_G\mathcal{H}_T &= -g\epsilon^{\mu\nu\alpha\beta}(\partial_\mu\lambda)T_\nu F_{\alpha\beta} \\ &= g\epsilon^{\mu\nu\alpha\beta}\lambda(\partial_{[\mu}T_{\nu]})F_{\alpha\beta} - g\epsilon^{\mu\nu\alpha\beta}\partial_\mu(\lambda T_\nu F_{\alpha\beta}),\end{aligned} \tag{2.35}$$

where the square brackets denote anti-symmetrization and we have used $\partial_{[\mu}F_{\alpha\beta]} = 0$. If $\partial_{[\mu}T_{\nu]} = 0$ then the interaction Hamiltonian is gauge invariant up to a boundary term. We assume that the external field $T_\mu(x)$ does indeed satisfy this condition allowing us to maintain gauge invariance, which is detailed in the second-order calculation in Appendix A. Furthermore, we assume that $T_\mu(x)$ is a pseudo-vector: under a parity transformation the external field transforms as

$$T_\mu(t, -\mathbf{x}) = \begin{cases} -T_\mu(t, \mathbf{x}), & \mu = 0 \\ +T_\mu(t, \mathbf{x}), & \mu = 1, 2, 3 \end{cases} \tag{2.36}$$

and under time reversal the external field transforms as

$$T_\mu(-t, \mathbf{x}) = \begin{cases} +T_\mu(t, \mathbf{x}), & \mu = 0 \\ -T_\mu(t, \mathbf{x}), & \mu = 1, 2, 3. \end{cases} \tag{2.37}$$

We will find it useful to consider the Fourier transform of T_μ,

$$\tilde{T}_\mu(p) = \int d^4x \;\; T_\mu(x) e^{-ip\cdot x}. \tag{2.38}$$

In momentum space the gauge invariance restrictions can be expressed by the condition

$$p_{[\mu}\tilde{T}_{\nu]}(p) = 0. \tag{2.39}$$

In order to determine whether circular polarization can be sourced at some order of Eq. (2.35), we will not need to impose any further restrictions on $T_\mu(x)$ aside from those listed above. Specifically, our calculations do not assume that $T_\mu(x)$ is either timelike or spacelike. As indicated by the investigation in [47], violations of causality and unstable solutions may arise for the case of timelike $T_\mu(x)$, and the results here must be interpreted with care in this case.

In section 2.5 below we detail the calculation of the first-order interaction Hamiltonian which is linear in the Hamiltonian density Eq. (2.34), as well as the refractive and damping terms of Eq. (2.27) due to first-order processes.

2.5 First-Order Calculation

In this appendix we determine the contribution to the time evolution of the photon density matrix due to processes which are first order in the interaction Hamiltonian density Eq. (2.34). We first detail the calculation of the refractive term and then the damping term of Eq. (2.27). The first-order interaction Hamiltonian is simply defined by

$$\hat{H}^{(1)}_{\text{int}}(t) = \int d^3\mathbf{x}\ \hat{\mathcal{H}}_T\,. \tag{2.40}$$

In the approximation employed here, all operators in the collision terms of Eq.-(2.27) are from the free theory. From this point on, all operators represent free theory operators and we drop the '0' superscript used in Sec. 2.3. Ignoring any processes in which two physical photons are either annihilated or created leads to the following expression for the interaction Hamiltonian:

$$\begin{aligned} {}^1\hat{H}_{\text{int}}(t) &= 2g\int d^3\mathbf{x}\,dp\,d\mathbf{k}\,d\mathbf{k}'\,\epsilon^{\mu\nu\alpha\beta}\ \tilde{T}_\nu(p)(-ik'_\alpha) \\ &\quad\times\Big(\hat{a}^\dagger_s(k)\hat{a}_r(k')\epsilon^*_{s\mu}(k)\epsilon_{r\beta}(k')e^{i(p+k-k')\cdot x} \\ &\quad - \hat{a}^\dagger_r(k')\hat{a}_s(k)\epsilon^*_{r\beta}(k')\epsilon_{s\mu}(k)e^{i(p+k'-k)\cdot x}\Big)\,, \end{aligned} \tag{2.41}$$

where we have used the expression for $\hat{A}_\mu$ given in Eq. (2.14) and we have used the shorthand notation

$$\int dp \equiv \int \frac{d^4p}{(2\pi)^4}\,, \qquad\qquad \int d\mathbf{k} \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k^0}\,.$$

Performing the spatial integral and relabeling dummy momentum variables and spacetime indices in the second term of Eq. (2.41) gives

$$\begin{aligned} {}^1\hat{H}_{\text{int}}(t) &= -2ig\int dp\,d\mathbf{k}\,d\mathbf{k}'(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{p}-\mathbf{k}')e^{i(k_0+p_0-k'_0)t}\epsilon^{\mu\nu\alpha\beta}\tilde{T}_\nu(p) \\ &\quad\times\hat{a}^\dagger_s(k)\hat{a}_r(k')\epsilon^*_{s\mu}(k)(k+k')_\alpha\epsilon_{r\beta}(k')\,. \end{aligned} \tag{2.42}$$

Now perform the $\int d\mathbf{k}'$ integral, after which we have

$$\begin{aligned} {}^1\hat{H}_{\text{int}}(t) &= -2ig\int dp\,d\mathbf{k}\,\frac{\tilde{T}_\nu(p)e^{i(k_0+p_0-|\mathbf{k}+\mathbf{p}|)t}}{2\,|\mathbf{k}+\mathbf{p}|}\hat{a}^\dagger_s(k)\hat{a}_r(\tilde{k}) \\ &\quad\times\epsilon^*_{s\mu}(k)\big(\epsilon^{\mu\nu0\beta}(|\mathbf{k}|+|\mathbf{k}+\mathbf{p}|)+\epsilon^{\mu\nu j\beta}(2k_j+p_j)\big)\epsilon_{r\beta}(\tilde{k})\,, \\ &= -2ig\int dp\,d\mathbf{k}\,\frac{\epsilon^{\mu\nu\alpha\beta}\tilde{T}_\nu(p)e^{i(k_0+p_0-|\mathbf{k}+\mathbf{p}|)t}}{2\,|\mathbf{k}+\mathbf{p}|}\hat{a}^\dagger_s(k)\hat{a}_r(\tilde{k}) \\ &\quad\times\epsilon^*_{s\mu}(k)(k+\tilde{k})_\alpha\epsilon_{r\beta}(\tilde{k})\,, \end{aligned} \tag{2.43}$$

where $(\tilde{k})_\alpha = (|\mathbf{k}+\mathbf{p}|\,,\mathbf{k}+\mathbf{p})$. Equation (2.43) is our first-order interaction Hamiltonian. Now the commutator necessary for the refractive term of Eq. (2.27) is given by

$$\begin{aligned}\left[{}^1\hat{H}_{\rm int}(t),\hat{\mathcal{D}}_{uv}(\mathbf{q})\right] &= -2ig\int dp\,d\mathbf{k}\,\frac{\tilde{T}_\nu(p)e^{i(k_0+p_0-|\mathbf{k}+\mathbf{p}|)t}}{2\,|\mathbf{k}+\mathbf{p}|} \\ &\quad\times\epsilon^*_{s\mu}(k)\big(\epsilon^{\mu\nu\alpha\beta}(k+\tilde{k})_\alpha\big)\epsilon_{r\beta}(\tilde{k}) \\ &\quad\times(2\pi)^3 2q^0\left(\delta_{ur}\delta^3(\mathbf{q}-\mathbf{k}-\mathbf{p})\hat{a}^\dagger_s(k)\hat{a}_v(q)\right. \\ &\qquad\left.-\delta_{vs}\delta^3(\mathbf{q}-\mathbf{k})\hat{a}^\dagger_u(q)\hat{a}_r(\tilde{k})\right)\end{aligned} \tag{2.44}$$

where we have used the canonical commutation relations between the free creation and annihilation operators Eq. (2.15). Taking the expectation value of Eq. (2.44) and using the relationship between the number operator and density matrix given by Eq. (2.17), we arrive at the following expression:

$$\begin{aligned}i\left\langle\left[{}^1\hat{H}_{\rm int}(t),\hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right\rangle &= 2g\int dp\,d\mathbf{k}\,\frac{\tilde{T}_\nu(p)e^{i(k_0+p_0-|\mathbf{k}+\mathbf{p}|)t}}{2\,|\mathbf{k}+\mathbf{p}|} \\ &\quad\times\epsilon^*_{s\mu}(k)\epsilon^{\mu\nu\alpha\beta}(k+\tilde{k})_\alpha\epsilon_{r\beta}(\tilde{k}) \\ &\quad\times(2\pi)^6(2q^0)^2\delta^3(\mathbf{q}-\mathbf{k}-\mathbf{p})\delta^3(\mathbf{q}-\mathbf{k}) \\ &\quad\times\left(\delta_{ur}\rho_{sv}(\mathbf{q})-\delta_{vs}\rho_{ur}(\mathbf{q})\right)\,.\end{aligned} \tag{2.45}$$

Perform the $\int d\mathbf{k}$ integral gives

$$\begin{aligned}i\left\langle\left[{}^1\hat{H}_{\rm int}(t),\hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right\rangle &= 2g\int dp\,\frac{\tilde{T}_\nu(p)e^{i(q_0+p_0-|\mathbf{q}+\mathbf{p}|)t}}{4\,|\mathbf{q}|\,|\mathbf{q}+\mathbf{p}|} \\ &\quad\times\epsilon^*_{s\mu}(q)\epsilon^{\mu\nu\alpha\beta}(q+\tilde{q})_\alpha\epsilon_{r\beta}(\tilde{q}) \\ &\quad\times(2\pi)^3(2q^0)^2\delta^3(-\mathbf{p})\left(\delta_{ur}\rho_{sv}(\mathbf{q})-\delta_{vs}\rho_{ur}(\mathbf{q})\right)\,,\end{aligned} \tag{2.46}$$

where as before we have defined $(\tilde{q})_\alpha = (|\mathbf{q}+\mathbf{p}|\,,\mathbf{q}+\mathbf{p})$ so that the above can be expressed as

$$\begin{aligned}i\left\langle\left[{}^1\hat{H}_{\rm int}(t),\hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right\rangle &= 2g\epsilon^*_{s\mu}(\mathbf{q})(2q^0)\left(\delta_{ur}\rho_{sv}(\mathbf{q})-\delta_{vs}\rho_{ur}(\mathbf{q})\right) \\ &\quad\times\int dp\,(2\pi)^3\delta^3(-\mathbf{p})\frac{\tilde{T}_\nu(p)e^{i(q_0+p_0-|\mathbf{q}+\mathbf{p}|)t}}{2\,|\mathbf{q}+\mathbf{p}|}\end{aligned} \tag{2.47}$$

$$\times\big(\epsilon^{\mu\nu 0\beta}(|\mathbf{q}| + |\mathbf{q}+\mathbf{p}|) + \epsilon^{\mu\nu j\beta}(2q_j + p_j)\big)\epsilon_{r\beta}(\mathbf{q}+\mathbf{p})\,.$$

Define the quantity

$$\begin{aligned}(2\pi)^3\delta^3(0)\mathcal{A}^{\mu\beta}(q) &= (2\pi)^3\delta^3(0)\mathcal{A}^{[\mu\beta]}(q)\,, \\ &= \int dp\,(2\pi)^3\delta^3(-\mathbf{p})\frac{\tilde{T}_\nu(p)}{2\,|\mathbf{q}+\mathbf{p}|} \\ &\qquad \times\big(\epsilon^{\mu\nu 0\beta}(|\mathbf{q}| + |\mathbf{q}+\mathbf{p}|) + \epsilon^{\mu\nu j\beta}(2q_j + p_j)\big)\,,\end{aligned} \tag{2.48}$$

where we have anticipated the fact that the delta-distribution factor will be present once an appropriate form for the external field $T_\mu(x)$ is chosen. The refractive term of Eq. (2.27) due to first order processes can then be expressed as

$$\begin{aligned} i\left\langle\left[{}^1\hat{H}_{\rm int}(0), \hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right\rangle &= 4gq^0(2\pi)^3\delta^3(0)\epsilon^*_{s\mu}(\mathbf{q})\epsilon_{r\beta}(\mathbf{q}) \\ &\qquad \times\left(\delta_{ur}\rho_{sv}(\mathbf{q}) - \delta_{vs}\rho_{ur}(\mathbf{q})\right)\mathcal{A}^{\mu\beta}(q)\,. \end{aligned} \tag{2.49}$$

Next we consider the damping term of Eq. (2.26), the integrand of which involves the double commutator

$$\begin{aligned} &\left[{}^1\hat{H}_{\rm int}(t-t'), \left[{}^1\hat{H}_{\rm int}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right] = (2ig)^2\int dl\,dp\,d\mathbf{k}_2\,d\mathbf{k}_1 \\ &\times\frac{\tilde{T}_\nu(p)e^{i((k_1)_0+p_0-|\mathbf{k}_1+\mathbf{p}|)t}}{2\,|\mathbf{k}_1+\mathbf{p}|}\frac{\tilde{T}_\lambda(l)e^{i((k_2)_0+l_0-|\mathbf{k}_2+\mathbf{l}|)(t-t')}}{2\,|\mathbf{k}_2+\mathbf{l}|} \\ &\times\epsilon^*_{s\mu}(k_1)\epsilon^{\mu\nu\alpha\beta}(k_1+\tilde{k}_1)_\alpha\epsilon_{r\beta}(\tilde{k}_1)\epsilon^*_{n\sigma}(k_2)\epsilon^{\sigma\lambda\rho\kappa}(k_2+\tilde{k}_2)_\rho\epsilon_{m\kappa}(\tilde{k}_2) \\ &\times(2\pi)^6 2q^0\Big\{\delta_{ur}\delta^3(\mathbf{q}-\mathbf{k}_1-\mathbf{p})\big[2(k_1)^0\delta_{ms}\delta^3(\mathbf{k}_2+\mathbf{l}-\mathbf{k}_1)\hat{a}^\dagger_n(k_2)\hat{a}_v(q) \\ &\qquad\qquad -2q^0\delta_{nv}\delta^3(\mathbf{k}_2-\mathbf{q})\hat{a}^\dagger_s(k_1)\hat{a}_m(\tilde{k}_2)\big] \\ &\qquad -\delta_{vs}\delta^3(\mathbf{q}-\mathbf{k}_1)\big[2q^0\delta_{mu}\delta^3(\mathbf{k}_2+\mathbf{l}-\mathbf{q})\hat{a}^\dagger_n(k_2)\hat{a}_r(\tilde{k}_1) \\ &\qquad\qquad -2(k_2)^0\delta_{nr}\delta^3(\mathbf{k}_2-\mathbf{k}_1-\mathbf{p})\hat{a}^\dagger_u(q)\hat{a}_m(\tilde{k}_2)\big]\Big\}\,, \end{aligned} \tag{2.50}$$

where we have used the canonical commutation relations between the free creation and annihilation operators, Eq. (2.15), and have defined $(\tilde{k}_1)_\alpha = (|\mathbf{k}_1+\mathbf{p}|\,, \mathbf{k}_1+\mathbf{p})$ and $(\tilde{k}_2)_\alpha = (|\mathbf{k}_2+\mathbf{l}|\,, \mathbf{k}_2+\mathbf{l})$. Now the $\int d\mathbf{k}_1$ and the $\int d\mathbf{k}_2$ integrals can be performed, giving

$$\left[{}^1\hat{H}_{\text{int}}(t-t'),\left[{}^1\hat{H}_{\text{int}}(t),\hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right] = (2ig)^2 2q^0 \int dl\, dp\, \tilde{T}_\lambda(l)\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\nu(p)\epsilon^{\mu\nu\alpha\beta}$$
$$\times\Bigg(\frac{e^{i(|\mathbf{q}-\mathbf{p}|+p_0-|\mathbf{q}|)t}\epsilon^*_{s\mu}(\tilde{q}_1)(\tilde{q}_1+q)_\alpha\epsilon_{r\beta}(q)\delta_{ur}}{4\,|\mathbf{q}-\mathbf{p}|\,|\mathbf{q}|} \tag{2.51}$$
$$\times\Bigg[2\,|\mathbf{q}-\mathbf{p}|\,\frac{e^{i(|\mathbf{q}-\mathbf{p}-\mathbf{l}|+l_0-|\mathbf{q}-\mathbf{p}|)(t-t')}}{4\,|\mathbf{q}-\mathbf{p}-\mathbf{l}|\,|\mathbf{q}-\mathbf{p}|}\epsilon^*_{n\sigma}(\tilde{q}_2)(\tilde{q}_{-p-l}+\tilde{q}_{-p})_\rho\epsilon_{m\kappa}(\tilde{q}_{-p})\delta_{ms}\hat{a}^\dagger_n(\tilde{q}_2)\hat{a}_v(q)$$
$$-2q^0\frac{e^{i(q_0+l_0-|\mathbf{q}+\mathbf{l}|)(t-t')}}{4\,|\mathbf{q}|\,|\mathbf{q}+\mathbf{l}|}\epsilon^*_{n\sigma}(q)(q+\tilde{q}_3)_\rho\epsilon_{m\kappa}(\tilde{q}_3)\delta_{nv}\hat{a}^\dagger_s(\tilde{q}_1)\hat{a}_m(\tilde{q}_3)\Bigg]$$
$$-\frac{e^{i((q)_0+p_0-|\mathbf{q}+\mathbf{p}|)t}\epsilon^*_{s\mu}(q)(q+\tilde{q}_4)_\alpha\epsilon_{r\beta}(\tilde{q}_4)\delta_{vs}}{4\,|\mathbf{q}+\mathbf{p}|\,|\mathbf{q}|}$$
$$\times\Bigg[2q^0\frac{e^{i(|\mathbf{q}-\mathbf{l}|+l_0-|\mathbf{q}|)(t-t')}}{4\,|\mathbf{q}-\mathbf{l}|\,|\mathbf{q}|}\epsilon^*_{n\sigma}(\tilde{q}_5)(\tilde{q}_5+q)_\rho\epsilon_{m\kappa}(q)\delta_{mu}\hat{a}^\dagger_n(\tilde{q}_5)\hat{a}_r(\tilde{q}_4)$$
$$-2\,|\mathbf{q}+\mathbf{p}|\,\frac{e^{i(|\mathbf{q}+\mathbf{p}|+l_0-|\mathbf{q}+\mathbf{p}+\mathbf{l}|)(t-t')}}{4\,|\mathbf{q}+\mathbf{p}|\,|\mathbf{q}+\mathbf{p}+\mathbf{l}|}\epsilon^*_{n\sigma}(\tilde{q}_{+p})(\tilde{q}_{+p}+\tilde{q}_6)_\rho\epsilon_{m\kappa}(\tilde{q}_6)\delta_{nr}\hat{a}^\dagger_u(q)\hat{a}_m(\tilde{q}_6)\Bigg]\Bigg),$$

where $\tilde{q}_1=\tilde{q}_{-p}$, $\tilde{q}_2=\tilde{q}_{-p-l}$,$\tilde{q}_3=\tilde{q}_{+l}$, $\tilde{q}_4=\tilde{q}_{+p}$,$\tilde{q}_5=\tilde{q}_{-l}$ and $\tilde{q}_6=\tilde{q}_{+p+l}$

After taking the expectation value, this becomes

$$\left\langle\left[{}^1\hat{H}_{\text{int}}(t-t'),\left[{}^1\hat{H}_{\text{int}}(t),\hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right]\right\rangle = (2ig)^2\int dl\, dp\, \tilde{T}_\lambda(l)\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\nu(p)\epsilon^{\mu\nu\alpha\beta}$$
$$\times(2\pi)^3\delta^3(\mathbf{l}+\mathbf{p})e^{i(p_0+l_0)t}$$
$$\times\Bigg(\frac{e^{-i(|\mathbf{q}|+l_0-|\mathbf{q}+\mathbf{l}|)t'}\epsilon^*_{s\mu}(\tilde{q}_3)(\tilde{q}_3+q)_\alpha\epsilon_{r\beta}(q)\delta_{ur}}{2\,|\mathbf{q}+\mathbf{l}|}$$
$$\times\Big[\epsilon^*_{n\sigma}(q)(q+\tilde{q}_3)_\rho\epsilon_{m\kappa}(\tilde{q}_3)\delta_{ms}\rho_{nv}(\mathbf{q}) \tag{2.52}$$
$$-\epsilon^*_{n\sigma}(q)(q+\tilde{q}_3)_\rho\epsilon_{m\kappa}(\tilde{q}_3)\delta_{nv}\rho_{sm}(\mathbf{q}+\mathbf{l})\Big]$$
$$-\frac{e^{-i(|\mathbf{q}+\mathbf{p}|+l_0-|\mathbf{q}|)t'}\epsilon^*_{s\mu}(q)(q+\tilde{q}_4)_\alpha\epsilon_{r\beta}(\tilde{q}_4)\delta_{vs}}{2\,|\mathbf{q}+\mathbf{p}|}$$
$$\times\Big[\epsilon^*_{n\sigma}(\tilde{q}_4)(\tilde{q}_4+q)_\rho\epsilon_{m\kappa}(q)\delta_{mu}\rho_{nr}(\mathbf{q}+\mathbf{p})$$
$$-\epsilon^*_{n\sigma}(\tilde{q}_4)(\tilde{q}_4+q)_\rho\epsilon_{m\kappa}(q)\delta_{nr}\rho_{um}(\mathbf{q})\Big]\Bigg).$$

Interchanging l and p in the first set of terms gives

$$\begin{aligned}
\left\langle\left[{}^1\hat{H}_{\rm int}(t-t'),\left[{}^1\hat{H}_{\rm int}(t),\hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right]\right\rangle &= (2ig)^2\int dl\,dp\,(2\pi)^3\delta^3(\mathbf{l}+\mathbf{p})e^{i(p_0+l_0)t} \\
\times\Bigg(&\frac{e^{-i(|\mathbf{q}|+p_0-|\mathbf{q}+\mathbf{p}|)t'}p_\lambda\phi(p)\epsilon^{\sigma\lambda\rho\kappa}l_\nu\phi(l)\epsilon^{\mu\nu\alpha\beta}\epsilon^*_{s\mu}(\tilde{q}_4)(\tilde{q}_4+q)_\alpha\epsilon_{r\beta}(q)\delta_{ur}}{2\,|\mathbf{q}+\mathbf{p}|} \\
&\times\Big[\epsilon^*_{n\sigma}(q)(q+\tilde{q}_4)_\rho\epsilon_{m\kappa}(\tilde{q}_4)\delta_{ms}\rho_{nv}(\mathbf{q}) \\
&\quad-\epsilon^*_{n\sigma}(q)(q+\tilde{q}_4)_\rho\epsilon_{m\kappa}(\tilde{q}_4)\delta_{nv}\rho_{sm}(\mathbf{q}+\mathbf{p})\Big] \\
&-\frac{e^{-i(|\mathbf{q}+\mathbf{p}|+l_0-|\mathbf{q}|)t'}\tilde{T}_\lambda(l)\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\nu(p)\epsilon^{\mu\nu\alpha\beta}\epsilon^*_{s\mu}(q)(q+\tilde{q}_4)_\alpha\epsilon_{r\beta}(\tilde{q}_4)\delta_{vs}}{2\,|\mathbf{q}+\mathbf{p}|} \\
&\times\Big[\epsilon^*_{n\sigma}(\tilde{q}_4)(\tilde{q}_4+q)_\rho\epsilon_{m\kappa}(q)\delta_{mu}\rho_{nr}(\mathbf{q}+\mathbf{p}) \\
&\quad-\epsilon^*_{n\sigma}(\tilde{q}_4)(\tilde{q}_4+q)_\rho\epsilon_{m\kappa}(q)\delta_{nr}\rho_{um}(\mathbf{q})\Big]\Bigg)\,.
\end{aligned} \tag{2.53}$$

After relabeling some spacetime and polarization indices in first set of terms, the above becomes

$$\begin{aligned}
\left\langle\left[{}^1\hat{H}_{\rm int}(t-t'),\left[{}^1\hat{H}_{\rm int}(t),\hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right]\right\rangle &= (2ig)^2\int dl\,dp\,(2\pi)^3\delta^3(\mathbf{l}+\mathbf{p})e^{i(p_0+l_0)t} \\
\times&\frac{\tilde{T}_\nu(p)\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\lambda(l)\epsilon^{\mu\nu\alpha\beta}\epsilon^*_{n\sigma}(\tilde{q}_4)(\tilde{q}_4+q)_\rho\epsilon_{m\kappa}(q)\epsilon^*_{s\mu}(q)(q+\tilde{q}_4)_\alpha\epsilon_{r\beta}(\tilde{q}_4)}{2\,|\mathbf{q}+\mathbf{p}|} \\
&\times\Big\{e^{-i(|\mathbf{q}|+p_0-|\mathbf{q}+\mathbf{p}|)t'}\delta_{um}\Big[\delta_{rn}\rho_{sv}(\mathbf{q})-\delta_{sv}\rho_{nr}(\mathbf{q}+\mathbf{p})\Big] \\
&\quad-e^{-i(|\mathbf{q}+\mathbf{p}|+l_0-|\mathbf{q}|)t'}\delta_{vs}\Big[\delta_{mu}\rho_{nr}(\mathbf{q}+\mathbf{p})-\delta_{nr}\rho_{um}(\mathbf{q})\Big]\Big\}\,.
\end{aligned} \tag{2.54}$$

Now integrate over $\int_{-t}^{t}dt'$ $(t\to\infty)$ and define $\Delta q = |\mathbf{q}+\mathbf{p}|-|\mathbf{q}|$ to arrive at

$$\begin{aligned}
D_{uv} &\equiv \int_{-\infty}^{\infty}dt'\left\langle\left[{}^1\hat{H}_{\rm int}(t'),\left[{}^1\hat{H}_{\rm int}(0),\hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right]\right\rangle \\
&= (2ig)^2\int dl\,dp\,\frac{(2\pi)^4\delta^3(\mathbf{l}+\mathbf{p})}{2\,|\mathbf{q}+\mathbf{p}|}\tilde{T}_\nu(p)\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\lambda(l)\epsilon^{\mu\nu\alpha\beta} \\
&\qquad\tilde{\epsilon}_{n\sigma}(\tilde{q}_4+q)_\rho(q+\tilde{q}_{+p})_\alpha\tilde{\epsilon}_{r\beta} \\
&\qquad\times\Big\{\delta(p_0-\Delta q)\Big[\delta_{rn}\delta_{um}\epsilon_{m\kappa}\epsilon_{s\mu}\rho_{sv}(\mathbf{q})-\delta_{um}\delta_{sv}\epsilon_{m\kappa}\epsilon_{s\mu}\rho_{nr}(\mathbf{q}+\mathbf{p})\Big]
\end{aligned} \tag{2.55}$$

$$+\delta(l_0+\Delta q)\bigg[\delta_{rn}\delta_{vs}\epsilon_{m\kappa}\epsilon_{s\mu}\rho_{um}(\mathbf{q})-\delta_{um}\delta_{sv}\epsilon_{m\kappa}\epsilon_{s\mu}\rho_{nr}(\mathbf{q}+\mathbf{p})\bigg]\bigg\}\,,$$

where for convenience we have defined D_{uv} above, as well as the abbreviations $\epsilon_{r\mu}=\epsilon_{r\mu}(\mathbf{q})$ and $\tilde{\epsilon}_{r\mu}=\epsilon_{r\mu}(\mathbf{q}+\mathbf{p})$. Note that since ρ is expressed in a linear polarization basis all polarization vectors above have been assumed to be real.Now in the $\int d\mathbf{l}$ integral above, the relevant factor can be simplified as

$$\begin{aligned}
&\tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\int d\mathbf{l}\delta(\mathbf{l}+\mathbf{p})\tilde{T}_\lambda(l)(q+\tilde{q}_{+p})_\rho\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\nu(p)(q+\tilde{q}_{+p})_\alpha\epsilon^{\mu\nu\alpha\beta}\\
&=\tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\bigg\{\tilde{T}_i(l_0,-\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|]\epsilon^{\sigma i0\kappa}-\tilde{T}_0(l_0,-\mathbf{p})(2q_i+p_i)\epsilon^{\sigma i0\kappa}\bigg\}\\
&\qquad\qquad\times\bigg\{\tilde{T}_j(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]\epsilon^{\mu j0\beta}-2\tilde{T}_0(p_0,\mathbf{p})q_j\epsilon^{\mu j0\beta}\bigg\}\,,\\
&=\tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\bigg\{\tilde{T}_i(l_0,\mathbf{p})[(|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|)+l_0]\epsilon^{\sigma i0\kappa}+2\tilde{T}_0(l_0,\mathbf{p})q_i\epsilon^{\sigma i0\kappa}\bigg\}\\
&\qquad\qquad\times\bigg\{\tilde{T}_j(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]\epsilon^{\mu j0\beta}-2\tilde{T}_0(p_0,\mathbf{p})q_j\epsilon^{\mu j0\beta}\bigg\}\,,
\end{aligned}$$

where in getting to the final expression we have used $p_{[\mu}\tilde{T}_{\nu]}(p)=0$, the fact that $\epsilon_{n\mu}(k)$ can be fixed as purely spatial for any momentum k, and the assumed behavior of $T_\mu(x)$ under a parity transformation, Eq. (2.36). Now perform the appropriate energy integrals made trivial by the presence of delta-distributions in Eq. (2.55). The relevant factor from the second term of Eq. (2.55) becomes

$$\begin{aligned}
&\tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\int dl_0\delta(l_0+\Delta q)\epsilon^{\sigma i0\kappa}\bigg\{\tilde{T}_i(l_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|+l_0]+2\tilde{T}_0(l_0,\mathbf{p})q_i\bigg\}\\
&\qquad\qquad\times\epsilon^{\mu j0\beta}\bigg\{\tilde{T}_j(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]-2\tilde{T}_0(p_0,\mathbf{p})q_j\bigg\}\,,\\
&=\tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\bigg\{\tilde{T}_i(-\Delta q,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-\Delta q]\epsilon^{\sigma i0\kappa}+2\tilde{T}_0(-\Delta q,\mathbf{p})q_i\epsilon^{\sigma i0\kappa}\bigg\}\\
&\qquad\qquad\times\bigg\{\tilde{T}_j(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]\epsilon^{\mu j0\beta}-2\tilde{T}_0(p_0,\mathbf{p})q_j\epsilon^{\mu j0\beta}\bigg\}\,,\\
&=\tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\bigg\{-2\tilde{T}_i(\Delta q,\mathbf{p})\,|\mathbf{q}|\,\epsilon^{\sigma i0\kappa}+2\tilde{T}_0(\Delta q,\mathbf{p})q_i\epsilon^{\sigma i0\kappa}\bigg\}\\
&\qquad\qquad\times\bigg\{\tilde{T}_j(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]\epsilon^{\mu j0\beta}-2\tilde{T}_0(p_0,\mathbf{p})q_j\epsilon^{\mu j0\beta}\bigg\}\,,
\end{aligned}$$

$$= \tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\Big\{ - 2\tilde{T}_{\lambda}(\Delta q, \mathbf{p})q_{\rho}\epsilon^{\sigma\lambda\rho\kappa}\Big\} \tag{2.56}$$
$$\times\Big\{\tilde{T}_j(p_0, \mathbf{p})[|\mathbf{q}| + |\mathbf{q}+\mathbf{p}| - p_0]\epsilon^{\mu j 0\beta} - 2\tilde{T}_0(p_0, \mathbf{p})q_j\epsilon^{\mu j 0\beta}\Big\},$$

where in arriving at the final expression above we have used the assumed behavior of $T_\mu(x)$ under a time reversal transformation Eq. (2.37). Now perform the p_0 integral present in the first term of Eq. (2.55) and arrive at

$$\tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\int dp_0\delta(p_0 - \Delta q)\epsilon^{\sigma i 0\kappa}\Big\{\tilde{T}_i(l_0, \mathbf{p})[|\mathbf{q}| + |\mathbf{q}+\mathbf{p}| + l_0] + 2\tilde{T}_0(l_0, \mathbf{p})q_i\Big\}$$
$$\times\epsilon^{\mu j 0\beta}\Big\{\tilde{T}_j(p_0, \mathbf{p})[|\mathbf{q}| + |\mathbf{q}+\mathbf{p}| - p_0] - 2\tilde{T}_0(p_0, \mathbf{p})q_j\Big\}$$
$$= \tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\epsilon^{\sigma i 0\kappa}\Big\{\tilde{T}_i(l_0, \mathbf{p})[|\mathbf{q}| + |\mathbf{q}+\mathbf{p}| + l_0] + 2\tilde{T}_0(l_0, \mathbf{p})q_i\Big\}$$
$$\times\Big\{2\tilde{T}_\nu(\Delta q, \mathbf{p})q_\alpha\epsilon^{\mu\nu\alpha\beta}\Big\}.$$

Next change the integration variable $l_0 \to -p_0$ remaining in the first term of Eq. (2.55) and the factor above becomes

$$\tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\Big\{ - \tilde{T}_i(p_0, \mathbf{p})[|\mathbf{q}| + |\mathbf{q}+\mathbf{p}| - p_0]\epsilon^{\sigma i 0\kappa} + 2\tilde{T}_0(p_0, \mathbf{p})q_i\epsilon^{\sigma i 0\kappa}\Big\} \tag{2.57}$$
$$\times\Big\{2\tilde{T}_\nu(\Delta q, \mathbf{p})q_\alpha\epsilon^{\mu\nu\alpha\beta}\Big\},$$

where we have again used Eq. (2.37) . Following some relabeling of spacetime and polarization indices, the factor Eq. (2.57) becomes identical to the factor Eq. (2.56). Thus Eq. (2.55) can be expressed as

$$D_{uv} = -(2ig)^2\epsilon_{m\kappa}\epsilon_{s\mu}\int \frac{dp}{2\,|\mathbf{q}+\mathbf{p}|}\tilde{T}_\lambda(\tilde{p})q_\rho\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\nu(p)(2q+\tilde{p})_\alpha\epsilon^{\mu\nu\alpha\beta}\tilde{\epsilon}_{n\sigma}\tilde{\epsilon}_{r\beta}$$
$$\times\Big(\delta_{nr}\delta_{us}\rho_{mv}(\mathbf{q}) + \delta_{rn}\delta_{vs}\rho_{um}(\mathbf{q}) \tag{2.58}$$
$$-\delta_{us}\delta_{mv}\rho_{rn}(\mathbf{q}+\mathbf{p}) - \delta_{um}\delta_{sv}\rho_{nr}(\mathbf{q}+\mathbf{p})\Big),$$

where we have defined the 4−vector $\tilde{p} = (\Delta q, \mathbf{p})$ and recall that $\Delta q = |\mathbf{q}+\mathbf{p}| - |\mathbf{q}|$.

The linear combinations necessary to describe the evolution of the independent polarization degrees of freedom are

$$
\begin{aligned}
D_{11} \pm D_{22} = &-(2ig)^2 \int \frac{dp}{2\left|\mathbf{q}+\mathbf{p}\right|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)(2q+\tilde{p})_\alpha \\
&\times\Bigg\{ (\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{1\beta} + \tilde{\epsilon}_{2\sigma}\tilde{\epsilon}_{2\beta}) \Big[2\epsilon_{1\kappa}\epsilon_{1\mu}\rho_{11}(\mathbf{q}) \pm 2\epsilon_{2\kappa}\epsilon_{2\mu}\rho_{22}(\mathbf{q}) \\
&\qquad + (\epsilon_{2\kappa}\epsilon_{1\mu} \pm \epsilon_{1\kappa}\epsilon_{2\mu})[\rho_{12}(\mathbf{q}) + \rho_{21}(\mathbf{q})] \Big] \\
&\quad - \big(\epsilon_{1\kappa}\epsilon_{1\mu} \pm \epsilon_{2\kappa}\epsilon_{2\mu}\big) \Big[2\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{1\sigma}\rho_{11}(\mathbf{q}+\mathbf{p}) + 2\tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{2\sigma}\rho_{22}(\mathbf{q}+\mathbf{p}) \\
&\qquad + (\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{2\sigma} + \tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{1\sigma})[\rho_{12}(\mathbf{q}+\mathbf{p}) + \rho_{21}(\mathbf{q}+\mathbf{p})] \Big] \Bigg\}, \qquad (2.59)
\end{aligned}
$$

$$
\begin{aligned}
D_{12} \pm D_{21} = &-(2ig)^2 \int \frac{dp}{2\left|\mathbf{q}+\mathbf{p}\right|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)(2q+\tilde{p})_\alpha \\
&\times\Bigg\{ (\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{1\beta} + \tilde{\epsilon}_{2\sigma}\tilde{\epsilon}_{2\beta}) \Big[\epsilon_{1\kappa}\epsilon_{2\mu}[\rho_{11}(\mathbf{q}) \pm \rho_{11}(\mathbf{q})] + \epsilon_{2\kappa}\epsilon_{1\mu}[\rho_{22}(\mathbf{q}) \pm \rho_{22}(\mathbf{q})] \\
&\qquad + (\epsilon_{1\kappa}\epsilon_{1\mu} + \epsilon_{2\kappa}\epsilon_{2\mu})[\rho_{12}(\mathbf{q}) \pm \rho_{21}(\mathbf{q})] \Big] \\
&\quad - (\epsilon_{2\kappa}\epsilon_{1\mu} \pm \epsilon_{1\kappa}\epsilon_{2\mu}) \Big[\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{1\sigma}[\rho_{11}(\mathbf{q}+\mathbf{p}) \pm \rho_{11}(\mathbf{q}+\mathbf{p})] \\
&\qquad + \tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{2\sigma}[\rho_{22}(\mathbf{q}+\mathbf{p}) \pm \rho_{22}(\mathbf{q}+\mathbf{p})] \\
&\qquad + (\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{2\sigma} \pm \tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{2\beta})[\rho_{12}(\mathbf{q}+\mathbf{p}) \pm \rho_{21}(\mathbf{q}+\mathbf{p})] \Big] \Bigg\}. \qquad (2.60)
\end{aligned}
$$

If we define the quantity

$$
(2\pi)^3\delta^{(3)}(0)\mathcal{B}_1^{\mu\kappa}(q) \equiv \int \frac{dp}{2\left|\mathbf{q}+\mathbf{p}\right|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)(2q+\tilde{p})_\alpha \epsilon^{\mu\nu\alpha\beta} \tilde{\epsilon}_{n\sigma}\delta_{nr}\tilde{\epsilon}_{r\beta} \qquad (2.61)
$$

and the integral operator

$$
\begin{aligned}
(2\pi)^3\delta^{(3)}(0)\mathcal{B}_2[q;\rho_{uv}] \equiv &\ \epsilon_{m\kappa}\epsilon_{s\mu} \int \frac{dp}{2\left|\mathbf{q}+\mathbf{p}\right|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)(2q+\tilde{p})_\alpha \epsilon^{\mu\nu\alpha\beta} \\
&\times \tilde{\epsilon}_{n\sigma}\tilde{\epsilon}_{r\beta}\big(\delta_{us}\delta_{mv}\rho_{rn}(\mathbf{q}+\mathbf{p}) + \delta_{um}\delta_{sv}\rho_{nr}(\mathbf{q}+\mathbf{p})\big), \qquad (2.62)
\end{aligned}
$$

Eq. (2.58) can be expressed in the general form

$$D_{uv} = (2ig)^2(2\pi)^3\delta^{(3)}(0)\Big\{\mathcal{B}_2[q;\rho_{uv}(q)] \\ -\epsilon_{m\kappa}\left(\epsilon_{u\mu}\rho_{m\nu}(\mathbf{q}) + \epsilon_{v\mu}\rho_{um}(\mathbf{q})\right)\mathcal{B}_1^{\mu\kappa}(q)\Big\}. \tag{2.63}$$

2.6 Analysis of First Order Effects

Quoting the results of Sec. 2.5 for the interaction Hamiltonian associated with first-order processes we have

$$^1\hat{H}_{\rm int}(t) = -2ig\int dp\, d\mathbf{k}\,\frac{\tilde{T}_\nu(p)e^{i(k_0+p_0-|\mathbf{k}+\mathbf{p}|)t}}{2\,|\mathbf{k}+\mathbf{p}|}\hat{a}_s^\dagger(k)\hat{a}_r(\tilde{k})\epsilon^*_{s\mu}(\mathbf{k}) \\ \times\Big(\epsilon^{\mu\nu0\beta}(|\mathbf{k}|+|\mathbf{k}+\mathbf{p}|) + \epsilon^{\mu\nu j\beta}(2k_j+p_j)\Big)\epsilon_{r\beta}(\mathbf{k}+\mathbf{p}). \tag{2.64}$$

The refractive term of Eq. (2.27) for this first order interaction Hamiltonian is given by

$$i\left\langle\left[^1\hat{H}_{\rm int}(0),\hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right\rangle = 4gq^0(2\pi)^3\delta^3(0)\epsilon_{s\mu}(\mathbf{q})\mathcal{A}^{\mu\beta}(q)\epsilon_{r\beta}(\mathbf{q}) \\ \times\left(\delta_{ur}\rho_{sv}(\mathbf{q}) - \delta_{vs}\rho_{ur}(\mathbf{q})\right), \tag{2.65}$$

where $\mathcal{A}^{\mu\beta}$ is defined via

$$(2\pi)^3\delta^3(0)\mathcal{A}^{\mu\beta}(q) = (2\pi)^3\delta^3(0)\mathcal{A}^{[\mu\beta]}(q) \\ = \int\frac{dp}{2\,|\mathbf{q}+\mathbf{p}|}(2\pi)^3\delta^3(-\mathbf{p})\tilde{T}_\nu(p)\epsilon^{\mu\nu\alpha\beta}(2q_\alpha+\tilde{p}_\alpha), \tag{2.66}$$

and we have for convenience defined $\tilde{p}\equiv(\Delta q,\mathbf{p})$, $\Delta q\equiv|\mathbf{q}+\mathbf{p}|-|\mathbf{q}|$.

As detailed in Sec. 2.5 the damping term of Eq. (2.27) due to first order processes is

$$D_{uv} \equiv \int_{-\infty}^{\infty}dt'\left\langle\left[^1\hat{H}_{\rm int}(t'),\left[^1\hat{H}_{\rm int}(0),\hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right]\right\rangle \\ = -(2ig)^2\epsilon_{m\kappa}\epsilon_{s\mu}\int\frac{dp}{2\,|\mathbf{q}+\mathbf{p}|}\tilde{T}_\lambda(\tilde{p})q_\rho\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\nu(p)(2q+\tilde{p})_\alpha\epsilon^{\mu\nu\alpha\beta}\tilde{\epsilon}_{n\sigma}\tilde{\epsilon}_{r\beta} \tag{2.67}$$

$$\times\bigg(\delta_{nr}\delta_{us}\rho_{mv}(\mathbf{q})+\delta_{rn}\delta_{vs}\rho_{um}(\mathbf{q})-\delta_{us}\delta_{mv}\rho_{rn}(\mathbf{q}+\mathbf{p})-\delta_{um}\delta_{sv}\rho_{nr}(\mathbf{q}+\mathbf{p})\bigg),$$

where we have defined $\epsilon_{r\mu}\equiv\epsilon_{r\mu}(\mathbf{q})$ and $\tilde{\epsilon}_{r\mu}\equiv\epsilon_{r\mu}(\mathbf{q}+\mathbf{p})$.

Once the density matrix is expanded as given in Eq. (2.28), it is straightforward to see that the refractive term, Eq.(2.65), vanishes when evaluated in terms of $\rho^{(0)}(|\mathbf{q}|)$. For the damping term, Eq. (2.67), we must perform an expansion of $\rho^{(0)}(|\mathbf{q}+\mathbf{p}|)=\rho^{(0)}(|\mathbf{k}|)$, in the p integral of Eq. (2.67), about $|\mathbf{q}|$:

$$\rho^{(0)}(|\mathbf{k}|)=\rho^{(0)}(|\mathbf{q}|)+\mathcal{O}\bigg(\frac{d\rho^{(0)}}{d\,|\mathbf{k}|}(|\mathbf{q}|)\bigg), \tag{2.68}$$

which is a suitable approximation as long as $\tilde{T}(p^0,\mathbf{p})$ has support solely over $|\mathbf{p}|\ll|\mathbf{q}|$, where $|\mathbf{q}|$ is the energy of the scattering photons. Then to lowest order in $\rho^{(0)}$ the damping term vanishes and we have

$$\frac{d}{dt}\rho^{(0)}=0+\mathcal{O}\bigg(g^2\frac{d\rho^{(0)}(|\mathbf{q}|)}{d\,|\mathbf{q}|}\bigg). \tag{2.69}$$

In terms of the Stokes brightness perturbations defined in Eqs. (2.30)–(2.33), to first order in g the evolution of the polarization of the photon ensemble becomes

$$\frac{d}{dt}\Delta_I = 0, \tag{2.70}$$
$$\frac{d}{dt}\Delta_Q = -g\alpha(q)\Delta_U, \tag{2.71}$$
$$\frac{d}{dt}\Delta_U = g\alpha(q)\Delta_Q, \tag{2.72}$$
$$\frac{d}{dt}\Delta_V = 0, \tag{2.73}$$

where we have defined the quantity

$$\alpha(q)\equiv 4\epsilon_{1\mu}(q)\mathcal{A}^{\mu\beta}(q)\epsilon_{2\beta}(q). \tag{2.74}$$

For processes which are first order in the Hamiltonian density Eq. (2.34), according to Eq. (2.73) no circular polarization is generated to $\mathcal{O}(g)$ in our approximation. In fact, it is easy to see that to $\mathcal{O}(g)$ (the refractive term), Eqs. (2.70)–(2.73) reproduce the well-known effect of optical activity of the electromagnetic radiation, rotating the plane of linear polarization during propagation [70–73]. This is a useful check of the calculations.

The relevant linear combinations of the damping term Eq. (2.67) which source the polarization of the photon ensemble to $\mathcal{O}(g^2)$ and due to first order processes are

$$
\begin{aligned}
D_{11} + D_{22} = &-(2ig)^2 \int \frac{dp}{2\,|\mathbf{q}+\mathbf{p}|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)(2q+\tilde{p})_\alpha \\
\times\Bigg\{ &(\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{1\beta} + \tilde{\epsilon}_{2\sigma}\tilde{\epsilon}_{2\beta}) \Big[2\epsilon_{1\kappa}\epsilon_{1\mu}\rho^{(1)}_{11}(\mathbf{q}) + 2\epsilon_{2\kappa}\epsilon_{2\mu}\rho^{(1)}_{22}(\mathbf{q}) + (\epsilon_{2\kappa}\epsilon_{1\mu} \\
&\qquad + \epsilon_{1\kappa}\epsilon_{2\mu})[\rho^{(1)}_{12}(\mathbf{q}) + \rho^{(1)}_{21}(\mathbf{q})] \Big] \\
&- (\epsilon_{1\kappa}\epsilon_{1\mu} + \epsilon_{2\kappa}\epsilon_{2\mu}) \Big[2\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{1\sigma}\rho^{(1)}_{11}(\mathbf{q}+\mathbf{p}) + 2\tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{2\sigma}\rho^{(1)}_{22}(\mathbf{q}+\mathbf{p}) \\
&\qquad + (\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{2\sigma} + \tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{1\sigma})[\rho^{(1)}_{12}(\mathbf{q}+\mathbf{p}) + \rho^{(1)}_{21}(\mathbf{q}+\mathbf{p})] \Big] \Bigg\} \\
&+ \mathcal{O}\left(g^2 \frac{d\rho^{(0)}(|\mathbf{q}|)}{d\,|\mathbf{q}|} \right),
\end{aligned}
$$

$$
\begin{aligned}
D_{11} - D_{22} = &-(2ig)^2 \int \frac{dp}{2\,|\mathbf{q}+\mathbf{p}|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)(2q+\tilde{p})_\alpha \\
\times\Bigg\{ &(\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{1\beta} + \tilde{\epsilon}_{2\sigma}\tilde{\epsilon}_{2\beta}) \Big[2\epsilon_{1\kappa}\epsilon_{1\mu}\rho^{(1)}_{11}(\mathbf{q}) - 2\epsilon_{2\kappa}\epsilon_{2\mu}\rho^{(1)}_{22}(\mathbf{q}) \\
&\qquad + (\epsilon_{2\kappa}\epsilon_{1\mu} - \epsilon_{1\kappa}\epsilon_{2\mu})[\rho^{(1)}_{12}(\mathbf{q}) + \rho^{(1)}_{21}(\mathbf{q})] \Big] \\
&- (\epsilon_{1\kappa}\epsilon_{1\mu} - \epsilon_{2\kappa}\epsilon_{2\mu}) \Big[2\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{1\sigma}\rho^{(1)}_{11}(\mathbf{q}+\mathbf{p}) + 2\tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{2\sigma}\rho^{(1)}_{22}(\mathbf{q}+\mathbf{p}) \\
&\qquad + (\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{2\sigma} + \tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{1\sigma})[\rho^{(1)}_{12}(\mathbf{q}+\mathbf{p}) + \rho^{(1)}_{21}(\mathbf{q}+\mathbf{p})] \Big] \Bigg\} \\
&+ \mathcal{O}\left(g^2 \frac{d\rho^{(0)}(|\mathbf{q}|)}{d\,|\mathbf{q}|} \right),
\end{aligned}
$$

$$
\begin{aligned}
D_{12} + D_{21} = &-(2ig)^2 \int \frac{dp}{2\,|\mathbf{q}+\mathbf{p}|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)(2q+\tilde{p})_\alpha \\
\times\Bigg\{ &(\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{1\beta} + \tilde{\epsilon}_{2\sigma}\tilde{\epsilon}_{2\beta}) \Big[2\epsilon_{1\kappa}\epsilon_{2\mu}\rho^{(1)}_{11}(\mathbf{q}) + 2\epsilon_{2\kappa}\epsilon_{1\mu}\rho^{(1)}_{22}(\mathbf{q})
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad +(\epsilon_{1\kappa}\epsilon_{1\mu}+\epsilon_{2\kappa}\epsilon_{2\mu})[\rho^{(1)}_{12}(\mathbf{q})+\rho^{(1)}_{21}(\mathbf{q})]\Big]\\
&-(\epsilon_{2\kappa}\epsilon_{1\mu}+\epsilon_{1\kappa}\epsilon_{2\mu})\Big[2\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{1\sigma}\rho^{(1)}_{11}(\mathbf{q}+\mathbf{p})+2\tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{2\sigma}\rho^{(1)}_{22}(\mathbf{q}+\mathbf{p})\\
&\qquad\qquad +(\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{2\sigma}+\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{2\beta})[\rho^{(1)}_{12}(\mathbf{q}+\mathbf{p})+\rho^{(1)}_{21}(\mathbf{q}+\mathbf{p})]\Big]\Big\}\\
&+\mathcal{O}\left(g^2\frac{d\rho^{(0)}(|\mathbf{q}|)}{d\,|\mathbf{q}|}\right),
\end{aligned}
$$

$$
\begin{aligned}
D_{12}-D_{21} = -(2ig)^2\int\frac{dp}{2\,|\mathbf{q}+\mathbf{p}|}\tilde{T}_\lambda(\tilde{p})q_\rho\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\nu(p)(2q+\tilde{p})_\alpha\\
\times\Big\{(\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{1\beta}+\tilde{\epsilon}_{2\sigma}\tilde{\epsilon}_{2\beta})(\epsilon_{1\kappa}\epsilon_{1\mu}+\epsilon_{2\kappa}\epsilon_{2\mu})[\rho^{(1)}_{12}(\mathbf{q})-\rho^{(1)}_{21}(\mathbf{q})]\\
-(\epsilon_{2\kappa}\epsilon_{1\mu}-\epsilon_{1\kappa}\epsilon_{2\mu})(\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{2\sigma}-\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{2\beta})[\rho^{(1)}_{12}(\mathbf{q}+\mathbf{p})-\rho^{(1)}_{21}(\mathbf{q}+\mathbf{p})]\Big\}\\
+\mathcal{O}\left(g^2\frac{d\rho^{(0)}(|\mathbf{q}|)}{d\,|\mathbf{q}|}\right).
\end{aligned}
$$

Given the above expressions it is easy to see that no mixing occurs between Δ_V and the set $\{\Delta_I, \Delta_Q, \Delta_U\}$ as a result of the damping term Eq. (2.67). Therefore, in our approximation, no circular polarization is generated by first order processes up to $\mathcal{O}(g^2)$.

2.7 Second-Order Calculation

In this section we explicitly calculate the relevant quantities describing the evolution of the photon density matrix due to processes which are second order in the interaction Hamiltonian Eq. (2.34). The second-order scattering matrix operator is

$$
\begin{aligned}
\hat{S}^{(2)} &= -\frac{1}{2}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dt' T\{\hat{H}^{(1)}_{\rm int}(t)\hat{H}^{(1)}_{\rm int}(t')\}\,,\\
&\equiv -i\int_{-\infty}^{\infty}dt\hat{H}^{(2)}_{\rm int}(t)\,, \qquad (2.75)
\end{aligned}
$$

where $\hat{H}^{(1)}_{\text{int}}(t)$ is the first-order interaction Hamiltonian, Eq. (2.40). We will denote the interaction Hamiltonian operator which has a non-zero overlap with a single photon lying in both the initial and final scattering states as ${}^{2}H_{\text{int}}(t)$. Applying Wick's theorem to simplify the time ordered product in Eq. (2.75) gives (ignoring vacuum terms)

$$
\begin{aligned}
-i\hat{H}^{(2)}(t) = &-\frac{1}{2}(2g)^2 \int d^3\mathbf{x}d^4y\, \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}T_\nu(x)T_\sigma(y) \qquad (2.76)\\
\times : \Big(&\overbracket{\hat{A}_\mu(x)\partial_\alpha\hat{A}_\beta(x)\hat{A}_\rho}(y)\partial_\lambda\hat{A}_\kappa(y) + \overbracket{\hat{A}_\mu(x)\partial_\alpha\hat{A}_\beta(x)\hat{A}_\rho(y)\partial_\lambda\hat{A}_\kappa}(y) \\
&+\hat{A}_\mu(x)\partial_\alpha\overbracket{\hat{A}_\beta(x)\hat{A}_\rho}(y)\partial_\lambda\hat{A}_\kappa(y) + \hat{A}_\mu(x)\partial_\alpha\overbracket{\hat{A}_\beta(x)\hat{A}_\rho(y)\partial_\lambda\hat{A}_\kappa}(y)\Big) :,
\end{aligned}
$$

where all partial derivatives are understood as acting solely on the function immediately to the right; the variable being differentiated is in the argument of this function. We denote the contraction of two operators $\hat{A}$ and $\hat{B}$ by

$$
\overbracket{\hat{A}\hat{B}}\,.
$$

To simplify Eq. (2.76), we use the free-theory photon propagator in the Feynman gauge,

$$
D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\mu\nu}e^{-ik\cdot(x-y)}}{k^2+i\epsilon}\,. \tag{2.77}
$$

(See Appendix A below for a demonstration of gauge invariance, where we explicitly consider a different gauge-fixed photon propagator). In order to deal with the derivative couplings, we interpret the time ordering as T^* ordering; specifically, we require that derivative couplings act outside of the time ordering operation [91]. For convenience, define the operators

$$
\hat{A}^+_\mu(x,p) = \hat{a}_s(p)\epsilon_{s\mu}(p)e^{-ip\cdot x}\,, \tag{2.78}
$$

$$
\hat{A}^-_\mu(x,p) = \hat{a}^\dagger_s(p)\epsilon^*_{s\mu}(p)e^{ip\cdot x}\,. \tag{2.79}
$$

Equation (2.76) then becomes

$$
-i\hat{H}^{(2)}(t) = -\frac{1}{2}(2g)^2 \int d^3\mathbf{x}d^4y \int \frac{d^4k}{(2\pi)^4}\frac{d^3\mathbf{p}_1}{(2\pi)^3 2p^0_1}\frac{d^3\mathbf{p}_2}{(2\pi)^3 2p^0_2}
$$

$$
\begin{aligned}
&\times\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}T_\nu(x)T_\sigma(y)\left(\frac{-ie^{-ik\cdot(x-y)}}{k^2+i\epsilon}\right)\\
\times\ :\Big[\ & g_{\mu\rho}(-ip_{1\alpha})\big(A^+_\beta(x,p_1)-A^-_\beta(x,p_1)\big)(-ip_{2\lambda})\big(A^+_\kappa(y,p_2)-A^-_\kappa(y,p_2)\big)\\
&+(ik_\lambda)g_{\mu\kappa}(-ip_{1\alpha})\big(A^+_\beta(x,p_1)-A^-_\beta(x,p_1)\big)\big(A^+_\rho(y,p_2)+A^-_\rho(y,p_2)\big)\\
&+(-ik_\alpha)g_{\beta\rho}\big(A^+_\mu(x,p_1)+A^-_\mu(x,p_1)\big)(-ip_{2\lambda})\big(A^+_\kappa(y,p_2)-A^-_\kappa(y,p_2)\big)\\
&+(-ik_\alpha)(ik_\lambda)g_{\beta\kappa}\big(A^+_\mu(x,p_1)+A^-_\mu(x,p_1)\big)\big(A^+_\rho(y,p_2)+A^-_\rho(y,p_2)\big)\ \Big]:\ .
\end{aligned}
\tag{2.80}
$$

Picking out the non-vanishing overlap of $\hat{H}^{(2)}(t)$ on single-photon initial and final scattering states and calling this ${}^2\hat{H}_{\rm int}(t)$ gives

$$
\begin{aligned}
{}^2\hat{H}_{\rm int}(t) &= -\frac{i}{2}(2g)^2\int d^3\mathbf{x}d^4y\int dk\,d\mathbf{p}_1\,d\mathbf{p}_2\\
&\quad\times\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}T_\nu(x)T_\sigma(y)\left(\frac{-ie^{-ik\cdot(x-y)}}{k^2+i\epsilon}\right)\\
\times\ \Big[\ & g_{\mu\rho}(-ip_{1\alpha})(-ip_{2\lambda})\big(-\hat{A}^-_\kappa(y,p_2)\hat{A}^+_\beta(x,p_1)-\hat{A}^-_\beta(x,p_1)\hat{A}^+_\kappa(y,p_2)\big)\\
&+g_{\mu\kappa}(ik_\lambda)(-ip_{1\alpha})\big(\hat{A}^-_\rho(y,p_2)\hat{A}^+_\beta(x,p_1)-\hat{A}^-_\beta(x,p_1)\hat{A}^+_\rho(y,p_2)\big)\\
&+g_{\beta\rho}(-ik_\alpha)(-ip_{2\lambda})\big(-\hat{A}^-_\kappa(y,p_2)\hat{A}^+_\mu(x,p_1)+\hat{A}^-_\mu(x,p_1)\hat{A}^+_\kappa(y,p_2)\big)\\
&+g_{\beta\kappa}(-ik_\alpha)(ik_\lambda)\big(\hat{A}^-_\rho(y,p_2)\hat{A}^+_\mu(x,p_1)+\hat{A}^-_\mu(x,p_1)\hat{A}^+_\rho(y,p_2)\big)\ \Big],
\end{aligned}
\tag{2.81}
$$

where we have again used the shorthand notation for the integral measures defined in Sec. 2.5. After some relabeling of spacetime indices, Eq. (2.81) becomes

$$
\begin{aligned}
{}^2\hat{H}_{\rm int}(t) &= -\frac{i}{2}(2g)^2\int d^3\mathbf{x}d^4y\int dk\,d\mathbf{p}_1\,d\mathbf{p}_2\,\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}T_\nu(x)T_\sigma(y) \qquad (2.82)\\
\times\ (-ig_{\mu\rho})\Big[\ &\left(\frac{e^{-ik\cdot(x-y)}}{k^2+i\epsilon}\right)\big(p_{1\alpha}(p_2-k)_\lambda-k_\alpha(p_2-k)_\lambda\big)\hat{A}^-_\kappa(y,p_2)\hat{A}^+_\beta(x,p_1)\\
&+\left(\frac{e^{-ik\cdot(y-x)}}{k^2+i\epsilon}\right)\big(p_{1\alpha}(p_2-k)_\lambda-k_\alpha(p_2-k)_\lambda\big)\hat{A}^-_\beta(x,p_1)\hat{A}^+_\kappa(y,p_2)\Big],
\end{aligned}
$$

using the property $D_{\mu\nu}(x-y)=D_{\mu\nu}(y-x)$. Now plug in Eq. (2.78) and Eq. (2.79), express $T_\mu(x)$ in terms of its Fourier transform, and perform the $\int d\mathbf{x}$ and $\int dy$

integrals to get

$$
\begin{aligned}
{}^2\hat{H}_{\rm int}(t) = &-\frac{i(2\pi)^7}{2}(2g)^2 \int dk\, d\mathbf{p}_1\, d\mathbf{p}_2\, \frac{\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)}{k^2+i\epsilon} \\
&\times(-ig_{\mu\rho})\big(p_{1\alpha}(p_2-k)_\lambda - k_\alpha(p_2-k)_\lambda\big) \\
\times &\Big[e^{-i((p_1)^0-(l_1)^0+k^0)t}\delta^3(\mathbf{p}_1-\mathbf{l}_1+\mathbf{k})\delta^4(p_2+l_2+k)\epsilon^*_{s\kappa}(p_2)\epsilon_{r\beta}(p_1)\hat{a}^\dagger_s(p_2)\hat{a}_r(p_1) \\
&+e^{i((p_1)^0+l_1{}^0+k_0)t}\delta^3(\mathbf{p}_1+\mathbf{l}_1+\mathbf{k})\delta^4(p_2-l_2+k)\epsilon^*_{s\beta}(p_1)\epsilon_{r\kappa}(p_2)\hat{a}^\dagger_s(p_1)\hat{a}_r(p_2)\Big].
\end{aligned} \tag{2.83}
$$

Next perform the k integral:

$$
\begin{aligned}
{}^2\hat{H}_{\rm int}(t) = &-\frac{i(2\pi)^3}{2}(2g)^2 \int d\mathbf{p}_1 d\mathbf{p}_2\, dl_1 dl_2\, \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(-ig_{\mu\rho}) \\
\times &\Big[\big((p_1+p_2+l_2)_\alpha(2p_2+l_2)_\lambda\big)e^{-i((p_1)^0-(l_1)^0-(p_2)^0-(l_2)^0)t} \\
&\quad\times\delta^3(\mathbf{p}_1-\mathbf{l}_1-\mathbf{p}_2-\mathbf{l}_2)\frac{\epsilon^*_{s\kappa}(p_2)\epsilon_{s\beta}(p_1)\hat{a}^\dagger_s(p_2)\hat{a}_r(p_1)}{(p_2+l_2)^2+i\epsilon} \\
&+\big((p_1+p_2-l_2)_\alpha(2p_2-l_2)_\lambda\big)e^{i((p_1)^0+l_1{}^0-(p_2)^0+l_2{}^0)t} \\
&\quad\times\delta^3(\mathbf{p}_1+\mathbf{l}_1-\mathbf{p}_2+\mathbf{l}_2)\frac{\epsilon^*_{s\beta}(p_1)\epsilon_{r\kappa}(p_2)\hat{a}^\dagger_s(p_1)\hat{a}_r(p_2)}{(p_2-l_2)^2+i\epsilon}\Big].
\end{aligned} \tag{2.84}
$$

We are now in the position to compute the commutator of ${}^2\hat{H}_{\rm int}(t)$ and $\hat{\mathcal{D}}_{uv}(\mathbf{q})$ necessary for the refractive term of Eq. (2.27). This commutator yields

$$
\begin{aligned}
\Big[{}^2\hat{H}_{\rm int}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})\Big] = &-\frac{i(2\pi)^3}{2}(2g)^2 \int d\mathbf{p}_1 d\mathbf{p}_2\, dl_1 dl_2\, \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa} \\
&\times\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(-ig_{\mu\rho})e^{i(l_1{}^0+l_2{}^0)t} \\
\times&\Big[\big((k_1)_\alpha(k_2)_\lambda\big)e^{-i((p_1)^0-(p_2)^0)t}\delta^3(\mathbf{p}_1-\mathbf{l}_1-\mathbf{p}_2-\mathbf{l}_2)\frac{\epsilon^*_{s\kappa}(p_2)\epsilon_{s\beta}(p_1)}{(p_2+l_2)^2+i\epsilon} \\
&\times(2\pi)^3 2q^0\left(\delta^3(\mathbf{q}-\mathbf{p}_1)\delta_{ur}\hat{a}^\dagger_s(p_2)\hat{a}_v(q)-\delta^3(\mathbf{q}-\mathbf{p}_2)\delta_{vs}\hat{a}^\dagger_u(q)\hat{a}_r(p_1)\right) \\
+&\big((k_3)_\alpha(k_4)_\lambda\big)e^{i((p_1)^0-(p_2)^0)t}\delta^3(\mathbf{p}_1+\mathbf{l}_1-\mathbf{p}_2+\mathbf{l}_2)\frac{\epsilon^*_{s\beta}(p_1)\epsilon_{r\kappa}(p_2)}{(p_2-l_2)^2+i\epsilon} \\
&\times(2\pi)^3 2q^0\left(\delta^3(\mathbf{q}-\mathbf{p}_2)\delta_{ur}\hat{a}^\dagger_s(p_1)\hat{a}_v(q)-\delta^3(\mathbf{q}-\mathbf{p}_1)\delta_{vs}\hat{a}^\dagger_u(q)\hat{a}_r(p_2)\right)\Big].
\end{aligned} \tag{2.85}
$$

Where $k_1 = p_1+p_2+l_2$, $k_2 = 2p_2+l_2$, $k_3 = p_1+p_2-l_2$, and $k_4 = 2p_2-l_2$. Take the expectation value of Eq. (2.85) above to get

$$\left\langle \left[{}^2\hat{H}_{\rm int}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right\rangle = -\frac{i(2\pi)^3}{2}(2g)^2 \int d\mathbf{p}_1 d\mathbf{p}_2\, dl_1 dl_2\, \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa} \tag{2.86}$$
$$\times \tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(-ig_{\mu\rho})e^{i({l_1}^0+{l_2}^0)t}$$
$$\times(2\pi)^6(2q^0)^2\delta^3(\mathbf{q}-\mathbf{p}_1)\delta^3(\mathbf{p}_2-\mathbf{q})\delta^3(\mathbf{p}_1+\mathbf{l}_1-\mathbf{p}_2+\mathbf{l}_2)$$
$$\times\left(\delta_{ur}\rho_{sv}(\mathbf{q})-\delta_{vs}\rho_{ur}(\mathbf{q})\right)$$
$$\times\Bigg[(k_1)_\alpha(k_2)_\lambda e^{-i((p_1)^0-(p_2)^0)t}\frac{\epsilon^*_{s\kappa}(p_2)\epsilon_{s\beta}(p_1)}{(p_2+l_2)^2+i\epsilon}$$
$$+(k_3)_\alpha(k_4)_\lambda e^{i((p_1)^0-(p_2)^0)t}\frac{\epsilon^*_{s\beta}(p_1)\epsilon_{r\kappa}(p_2)}{(p_2-l_2)^2+i\epsilon}\Bigg].$$

Now perform the $\mathbf{p}_1$ and $\mathbf{p}_2$ integrals, giving

$$\left\langle \left[{}^2\hat{H}_{\rm int}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right\rangle = -\frac{i(2\pi)^3}{2}(2g)^2 \int dl_1 dl_2\, \delta^3(\mathbf{l}_1+\mathbf{l}_2)\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa} \tag{2.87}$$
$$\times \tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(-ig_{\mu\rho})e^{i({l_1}^0+{l_2}^0)t}$$
$$\times\big(\delta_{ur}\rho_{sv}(\mathbf{q})-\delta_{vs}\rho_{ur}(\mathbf{q})\big)\Bigg[(2q+l_2)_\alpha(2q+l_2)_\lambda\frac{\epsilon^*_{s\kappa}(q)\epsilon_{s\beta}(q)}{(q+l_2)^2+i\epsilon}$$
$$+(2q-l_2)_\alpha(2q-l_2)_\lambda\frac{\epsilon^*_{s\beta}(q)\epsilon_{r\kappa}(q)}{(q-l_2)^2+i\epsilon}\Bigg].$$

Focus now on the following factors present in Eq. (2.87):

$$\epsilon^{\mu\nu\alpha\beta}{\epsilon_\mu}^{\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)\epsilon^*_{s\kappa}(q)\epsilon_{r\beta}(q)(2q\pm l_2)_\alpha(2q\pm l_2)_\lambda$$
$$=2\epsilon^{\mu\nu\alpha\beta}{\epsilon_\mu}^{\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)\epsilon^*_{s\kappa}(q)\epsilon_{r\beta}(q)(2q\pm l_2)_\alpha q_\lambda\,,$$

where we have used $l_{[\lambda}\tilde{T}_{\sigma]}(l)=0$ in arriving at the expression on the right. Now use the epsilon identity $\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\sigma\lambda\kappa}=-6\delta^{[\nu}_\sigma\delta^\alpha_\lambda\delta^{\beta]}_\kappa$ to perform the contractions above and arrive at

$$-2\bigg([\tilde{T}_1\cdot\tilde{T}_2][(2q\pm l_2)\cdot q][\epsilon_r\cdot\epsilon_s]-[\tilde{T}_2\cdot\tilde{T}_1][\epsilon_r\cdot q][(2q\pm l_2)\cdot\epsilon_s] \tag{2.88}$$
$$+[\tilde{T}_2\cdot\epsilon_r][(2q\pm l_2)\cdot\epsilon_s][\tilde{T}_1\cdot q]-[\tilde{T}_2\cdot\epsilon_r][(2q\pm l_2)\cdot q][\epsilon_s\cdot\tilde{T}_1]$$
$$+[(2q\pm l_2)\cdot\tilde{T}_2][q\cdot\epsilon_r][\tilde{T}_1\cdot\epsilon_s]-[\tilde{T}_2\cdot(2q\pm l_2)][\tilde{T}_1\cdot q][\epsilon_r\cdot\epsilon_s]\bigg).$$

To simplify this, use $\epsilon_r \cdot \epsilon_s = -\delta_{rs}$, which when contracted with $(\delta_{ur}\rho_{sv}(\mathbf{q}) - \delta_{vs}\rho_{ur}(\mathbf{q}))$ in Eq. (2.87) vanishes, as well as $\epsilon_r(q)\cdot q = 0$ and $q \cdot q = 0$. The remaining terms from the expression Eq. (2.88) then become

$$\begin{aligned}
&\mp 2[\tilde{T}(l_2^0,\mathbf{l}_2)\cdot\epsilon_r]\epsilon_{s\mu}q_\nu\Big((l_2)^\mu\tilde{T}^\nu(l_1^0,-\mathbf{l}_2)-(l_2)^\nu\tilde{T}^\mu(l_1^0,-\mathbf{l}_2)\Big)\\
&\qquad = \pm 2[\tilde{T}(l_2^0,\mathbf{l}_2)\cdot\epsilon_r]\epsilon_{si}q_0\Big((l_2)^i\tilde{T}^0(l_1^0,\mathbf{l}_2)+(l_2)^0\tilde{T}^i(l_1^0,\mathbf{l}_2)\Big)\\
&= \pm 2[\tilde{T}(l_2^0,\mathbf{l}_2)\cdot\epsilon_r]\epsilon_{si}\tilde{T}^i(l_1^0,\mathbf{l}_2)q_0\big(l_1^0+l_2^0\big)\,,
\end{aligned}$$

where we have used Eq. (2.36) as well as repeated use of $l_{[\lambda}\tilde{T}_{\sigma]}(l) = 0$. Inserting this expression back into Eq. (2.87), we have

$$\begin{aligned}
\Big\langle\Big[{}^2\hat{H}_{\text{int}}(t),\hat{\mathcal{D}}_{uv}(\mathbf{q})\Big]\Big\rangle &= -(2\pi)^3(2g)^2\big(\delta_{ur}\rho_{sv}(\mathbf{q})-\delta_{vs}\rho_{ur}(\mathbf{q})\big) \qquad (2.89)\\
&\times\int dl_1 dl_2\,\delta^3(\mathbf{l}_1+\mathbf{l}_2)e^{i({l_1}^0+{l_2}^0)t}[\epsilon_r\cdot\tilde{T}(l_2^0,\mathbf{l}_2)][\tilde{T}(l_1^0,\mathbf{l}_2)\cdot\epsilon_s][q\cdot(l_1+l_2)]\\
&\times\left[\frac{1}{(l_2)^2+2l_2\cdot q+i\epsilon}-\frac{1}{(l_2)^2-2l_2\cdot q+i\epsilon}\right].
\end{aligned}$$

We now define a quantity $\mathcal{T}_{ij}$ by

$$\begin{aligned}
(2\pi)^3\delta^3(0)\mathcal{T}_{ij}(q) &= -\int dl_1 dl_2\,(2\pi)^3\delta^3(\mathbf{l}_1+\mathbf{l}_2)\tilde{T}_j(l_2^0,\mathbf{l}_2)\tilde{T}_i(l_1^0,\mathbf{l}_2)[q\cdot(l_1+l_2)]\\
&\times\left[\frac{1}{(l_2)^2+2l_2\cdot q+i\epsilon}-\frac{1}{(l_2)^2-2l_2\cdot q+i\epsilon}\right]. \qquad (2.90)
\end{aligned}$$

Equation (2.89) can be expressed conveniently in terms of this quantity:

$$\begin{aligned}
i\Big\langle\Big[{}^2\hat{H}_{\text{int}}(0),\hat{\mathcal{D}}_{uv}(\mathbf{q})\Big]\Big\rangle &= i(2\pi)^3\delta^3(0)(2g)^2\epsilon_s^\mu(q)\epsilon_r^\nu(q)\mathcal{T}_{\mu\nu}(q) \qquad (2.91)\\
&\times\big(\delta_{ur}\rho_{sv}(\mathbf{q})-\delta_{vs}\rho_{ur}(\mathbf{q})\big)\,,
\end{aligned}$$

where we have exploited the fact that $\epsilon_s^i(q)\epsilon_r^j(q)\mathcal{T}_{ij}(q) = \epsilon_s^\mu(q)\epsilon_r^\nu(q)\mathcal{T}_{\mu\nu}(q)$ since $\epsilon(q)$ is purely spatial. Explicitly, the components of $i\Big\langle\Big[{}^2\hat{H}_{\text{int}}(t),\hat{\mathcal{D}}_{uv}(\mathbf{q})\Big]\Big\rangle$ read as

$$\begin{aligned}
i\Big\langle\Big[{}^2\hat{H}_{\text{int}}(t),\hat{\mathcal{D}}_{11}(\mathbf{q})\Big]\Big\rangle &= i(2\pi)^3\delta^3(0)(2g)^2\mathcal{T}_{\mu\nu}(q) \qquad (2.92)\\
&\times\Big(-\epsilon_1^\mu(q)\epsilon_2^\nu(q)\rho_{12}(\mathbf{q})+\epsilon_2^\mu(q)\epsilon_1^\nu(q)\rho_{21}(\mathbf{q})\Big)\,,
\end{aligned}$$

$$i\left\langle\left[{}^2\hat{H}_{\rm int}(t),\hat{\mathcal{D}}_{12}(\mathbf{q})\right]\right\rangle = i(2\pi)^3\delta^3(0)(2g)^2\mathcal{T}_{\mu\nu}(q) \times\left([\epsilon_1^\mu(q)\epsilon_1^\nu(q)-\epsilon_2^\mu(q)\epsilon_2^\nu(q)]\rho_{12}(\mathbf{q})-\epsilon_2^\mu(q)\epsilon_1^\nu(q)[\rho_{11}(\mathbf{q})-\rho_{22}(\mathbf{q})]\right), \quad (2.93)$$

$$i\left\langle\left[{}^2\hat{H}_{\rm int}(t),\hat{\mathcal{D}}_{21}(\mathbf{q})\right]\right\rangle = i(2\pi)^3\delta^3(0)(2g)^2\mathcal{T}_{\mu\nu}(q) \times\left(-[\epsilon_1^\mu(q)\epsilon_1^\nu(q)-\epsilon_2^\nu(q)\epsilon_2^\mu(q)]\rho_{21}(\mathbf{q})+\epsilon_1^\mu(q)\epsilon_2^\nu(q)[\rho_{11}(\mathbf{q})-\rho_{22}(\mathbf{q})]\right), \quad (2.94)$$

$$i\left\langle\left[{}^2\hat{H}_{\rm int}(t),\hat{\mathcal{D}}_{22}(\mathbf{q})\right]\right\rangle = i(2\pi)^3\delta^3(0)(2g)^2\mathcal{T}_{\mu\nu}(q) \times\left(\epsilon_1^\mu(q)\epsilon_2^\nu(q)\rho_{12}(\mathbf{q})-\epsilon_2^\mu(q)\epsilon_1^\nu(q)\rho_{21}(\mathbf{q})\right). \quad (2.95)$$

2.8 Analysis of Second-Order Interaction Effects

We now move on to calculate the contribution to the evolution of the photon density matrix from scattering processes which are second order in the interaction Hamiltonian density operator Eq. (2.34). Details of the calculation of the second-order interaction Hamiltonian and the corresponding refractive term are presented in Sec 2.7. The second-order interaction Hamiltonian is given by

$$\begin{aligned}{}^2\hat{H}_{\rm int}(t) = &-\frac{i(2\pi)^3}{2}(2g)^2\int\frac{d^3\mathbf{p}_1d^3\mathbf{p}_2dl_1dl_2}{(2\pi)^6 2p_1^0 2p_2^0}\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(-ig_{\mu\rho})\\ &\times\Bigg[\Big((p_1+p_2+l_2)_\alpha(2p_2+l_2)_\lambda\Big)e^{-i((p_1)^0-(l_1)^0-(p_2)^0-(l_2)^0)t}\delta^3(\mathbf{p}_1-\mathbf{l}_1-\mathbf{p}_2-\mathbf{l}_2)\\ &\times\frac{\epsilon^*_{s\kappa}(p_2)\epsilon_{s\beta}(p_1)\hat{a}_s^\dagger(p_2)\hat{a}_r(p_1)}{(p_2+l_2)^2+i\epsilon}\\ &+\Big((p_1+p_2-l_2)_\alpha(2p_2-l_2)_\lambda\Big)e^{i((p_1)^0+{l_1}^0-(p_2)^0+{l_2}^0)t}\delta^3(\mathbf{p}_1+\mathbf{l}_1-\mathbf{p}_2+\mathbf{l}_2)\\ &\times\frac{\epsilon^*_{s\beta}(p_1)\epsilon_{r\kappa}(p_2)\hat{a}_s^\dagger(p_1)\hat{a}_r(p_2)}{(p_2-l_2)^2+i\epsilon}\Bigg].\end{aligned} \quad (2.96)$$

The refractive term of Eq. (2.27) due to this interaction Hamiltonian is

$$\left\langle\left[{}^2\hat{H}_{\rm int}(0),\hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right\rangle = (2\pi)^3\delta^3(0)4g^2\big(\delta_{ur}\rho_{sv}(\mathbf{q})-\delta_{vs}\rho_{ur}(\mathbf{q})\big)\epsilon_s^\mu(q)\epsilon_r^\nu(q)\mathcal{T}_{\mu\nu}(q),$$

where we have defined $\mathcal{T}_{ij}(q)$ via

$$(2\pi)^3\delta^3(0)\mathcal{T}_{ij}(q) = -\int dl_1dl_2\,(2\pi)^3\delta^3(\mathbf{l}_1+\mathbf{l}_2)\tilde{T}_j(l_2^0,\mathbf{l}_2)\tilde{T}_i(l_1^0,\mathbf{l}_2)[q\cdot(l_1+l_2)]$$

$$\times\left[\frac{1}{(l_2)^2 + 2l_2\cdot q + i\epsilon} - \frac{1}{(l_2)^2 - 2l_2\cdot q + i\epsilon}\right]. \qquad (2.97)$$

Note that the interaction Hamiltonian Eq. (2.96) and the refractive term Eq. (2.97) were computed using the photon propagator in the Feynman gauge Eq. (2.77), but a demonstration of gauge invariance of these results is detailed in Appendix A. For an unpolarized photon ensemble, it is easy to see that the right side of Eq. (2.97) vanishes when evaluated in terms of $\rho^{(0)}$ defined in Eq. (2.28) above. The contribution to the perturbed density matrix, $\rho^{(1)}$, to second order in the interaction coupling g is

$$\frac{d}{dt}\rho^{(1)}_{uv}(\mathbf{q}) \;=\; \frac{2ig^2}{q^0}\left(\delta_{ur}\rho^{(1)}_{sv}(\mathbf{q}) - \delta_{vs}\rho^{(1)}_{ur}(\mathbf{q})\right)\epsilon^{\mu}_{s}(q)\epsilon^{\nu}_{r}(q)\mathcal{T}_{\mu\nu}(q). \qquad (2.98)$$

Here we have used Eq. (2.27) and have ignored the damping term, which is of $\mathcal{O}(g^4)$. Expressing Eq. (2.98) explicitly in terms of photon density matrix components gives the evolution equations

$$\frac{d}{dt}\rho^{(1)}_{11}(\mathbf{q}) \;=\; \frac{2ig^2\mathcal{T}_{\mu\nu}}{q^0}\left(-\epsilon^{\mu}_{1}\epsilon^{\nu}_{2}\rho^{(1)}_{12}(\mathbf{q}) + \epsilon^{\mu}_{2}\epsilon^{\nu}_{1}\rho^{(1)}_{21}(\mathbf{q})\right), \qquad (2.99)$$

$$\frac{d}{dt}\rho^{(1)}_{22}(\mathbf{q}) \;=\; \frac{2ig^2\mathcal{T}_{\mu\nu}}{q^0}\left(\epsilon^{\mu}_{1}\epsilon^{\nu}_{2}\rho^{(1)}_{12}(\mathbf{q}) - \epsilon^{\mu}_{2}\epsilon^{\nu}_{1}\rho^{(1)}_{21}(\mathbf{q})\right), \qquad (2.100)$$

$$\frac{d}{dt}\rho^{(1)}_{12}(\mathbf{q}) \;=\; \frac{2ig^2\mathcal{T}_{\mu\nu}}{q^0}\left([\epsilon^{\mu}_{1}\epsilon^{\nu}_{1} - \epsilon^{\mu}_{2}\epsilon^{\nu}_{2}]\rho^{(1)}_{12}(\mathbf{q}) - \epsilon^{\mu}_{2}\epsilon^{\nu}_{1}[\rho^{(1)}_{11}(\mathbf{q}) - \rho^{(1)}_{22}(\mathbf{q})]\right), \qquad (2.101)$$

$$\frac{d}{dt}\rho^{(1)}_{21}(\mathbf{q}) \;=\; \frac{2ig^2\mathcal{T}_{\mu\nu}}{q^0}\left(\epsilon^{\mu}_{1}\epsilon^{\nu}_{2}[\rho^{(1)}_{11}(\mathbf{q}) - \rho^{(1)}_{22}(\mathbf{q})] - [\epsilon^{\mu}_{1}\epsilon^{\nu}_{1} - \epsilon^{\mu}_{2}\epsilon^{\nu}_{2}]\rho^{(1)}_{21}(\mathbf{q})\right). \qquad (2.102)$$

All polarization vectors ϵ^{μ}_{r} in the above expression depend on $\mathbf{q}$, the same photon momentum as in the argument of the photon density matrix. Using the Stokes brightness perturbations in Eqs. (2.30)–(2.33), we can now express the evolution of the polarization of the photon ensemble due to processes mediated by ${}^{2}\hat{H}_{\rm int}(t)$ as

$$\frac{d}{dt}\Delta_I(\mathbf{q}) \;=\; 0, \qquad (2.103)$$

$$\frac{d}{dt}\Delta_Q(\mathbf{q}) \;=\; -\frac{g^2}{q^0}\left(\zeta(\hat{\mathbf{q}})\Delta_V(\mathbf{q}) + i\psi(\hat{\mathbf{q}})\Delta_U(\mathbf{q})\right), \qquad (2.104)$$

$$\frac{d}{dt}\Delta_U(\mathbf{q}) = -\frac{g^2}{q^0}\left(\chi(\hat{\mathbf{q}})\Delta_V(\mathbf{q}) - i\psi(\hat{\mathbf{q}})\Delta_Q(\mathbf{q})\right), \tag{2.105}$$

$$\frac{d}{dt}\Delta_V(\mathbf{q}) = \frac{g^2}{q^0}\left(\chi(\hat{\mathbf{q}})\Delta_U(\mathbf{q}) + \zeta(\hat{\mathbf{q}})\Delta_Q(\mathbf{q})\right), \tag{2.106}$$

where we have defined the contractions

$$\zeta(\hat{\mathbf{q}}) \equiv -2\mathcal{T}_{\mu\nu}(\epsilon_1^\mu\epsilon_2^\nu + \epsilon_2^\mu\epsilon_1^\nu), \tag{2.107}$$

$$\chi(\hat{\mathbf{q}}) \equiv 2\mathcal{T}_{\mu\nu}(\epsilon_1^\mu\epsilon_1^\nu - \epsilon_2^\mu\epsilon_2^\nu) \tag{2.108}$$

$$\psi(\hat{\mathbf{q}}) \equiv 2\mathcal{T}_{\mu\nu}(\epsilon_1^\mu\epsilon_2^\nu - \epsilon_2^\mu\epsilon_1^\nu) \tag{2.109}$$

and the quantity $\mathcal{T}_{\mu\nu}$ is defined via the integral expression Eq. (2.97).

The circular polarization brightness Δ_V is sourced by terms which are proportional to either Δ_Q or Δ_U: a linearly polarized photon ensemble in the presence of the interaction Eq. (2.34) will acquire circular polarization due to processes which are of second order in the interaction. As long as the interaction acts, both Stokes brightnesses Δ_Q and Δ_U are rotated with Δ_V. Note that the above equations do not depend on the time component of $T_\mu(x)$ since the polarization vectors are purely spatial.

These equations are valid for a spatially constant field $T^\mu(x)$; extending the results to a spatially-varying field is messy but straightforward, resulting in additional convolution integrals.

It is convenient to define the tensors

$$A_\mu{}^\nu(k) \equiv \epsilon_{1\mu}(k)\epsilon_1^\nu(k) - \epsilon_{2\mu}(k)\epsilon_2^\nu(k), \tag{2.110}$$

$$S_\mu{}^\nu(k) \equiv \epsilon_{1\mu}(k)\epsilon_2^\nu(k) + \epsilon_{2\mu}(k)\epsilon_1^\nu(k). \tag{2.111}$$

Under a rotation of the polarization vectors in the plane transverse to the photon momentum by an angle ϕ, these projection tensors transform as

$$(A_\mu{}^\nu \pm iS_\mu{}^\nu) \to e^{\mp 2i\phi}(A_\mu{}^\nu \pm iS_\mu{}^\nu). \tag{2.112}$$

The quantities $(A_\mu{}^\nu \pm iS_\mu{}^\nu)$ are conventionally referred to as functions with spin weight $s = \pm 2$, respectively, on the plane transverse to $\mathbf{k}$ [92, 93] (see also [94] for the application of spin-weighted quantities to the description of polarization fields on the sky). Under this same rotation of the polarization vectors the Stokes parameters Δ_Q and Δ_U transform as

$$(\Delta_Q \pm i\Delta_U) \rightarrow e^{\mp 2i\phi}(\Delta_Q \pm i\Delta_U), \tag{2.113}$$

while Δ_I and Δ_V, being independent of the polarization vector directions, are of spin weight $s = 0$. In terms of these complex quantities, Eqs. (2.103)–(2.106) take the forms

$$\begin{aligned}
\frac{d}{dt}\Delta_I &= 0, && (2.114)\\
\frac{d}{dt}(\Delta_Q \pm i\Delta_U) &= \pm i\frac{2g^2}{k^0}\mathcal{T}_\nu\mathcal{T}^\mu({A_\mu}^\nu \pm i{S_\mu}^\nu)\Delta_V, && (2.115)\\
\frac{d}{dt}\Delta_V &= -i\frac{g^2}{k^0}\mathcal{T}_\nu\mathcal{T}^\mu\Big[({A_\mu}^\nu - i{S_\mu}^\nu)(\Delta_Q + i\Delta_U) && (2.116)\\
&\qquad -({A_\mu}^\nu + i{S_\mu}^\nu)(\Delta_Q - i\Delta_U)\Big].
\end{aligned}$$

In this form, it is clear that the source terms of the above equations transform in the appropriate manner. In particular, both terms on the right hand side of Eq. (2.116) are of spin weight $s = 0$.

2.9 Conclusions and Discussion

The evolution equations Eq. (2.103) to Eq. (2.106), along with Eq. (2.70) to Eq. (2.73), are the central result of this chapter. Other source terms associated with the usual Compton scattering effects will also appear on the right sides. While the polarization brightnesses will be zero prior to recombination, during recombination Δ_Q and Δ_U become non-zero, with an amplitude a factor of 20 smaller than the intensity fluctuations Δ_I. It is easy to see by inspection of the evolution equations that at that point, Δ_Q and Δ_V will rotate into each other with a characteristic angular frequency $\omega_{QV} = g^2\zeta/k^0$, Δ_U and Δ_V will rotate into each other with a characteristic frequency $\omega_{UV} = g^2\chi/k^0$, and Δ_Q and Δ_U will rotate into each other with a characteristic frequency $\omega_{QU} = g\alpha$, along with an exponential decay or growth of Δ_Q and Δ_U associated with the first order damping effects. All of these source terms are active whenever the interaction Eq. (2.34) is nonzero, in contrast to the conventional Compton scattering terms, which are only significant when the photons propagate through ionized regions of the universe.

The rotation between Δ_Q and Δ_U can be constrained from current measurements in a straightforward way. Linear polarization on the sky is conveniently expressed in a different basis, corresponding to the "gradient" and "curl" pieces of the polarization tensor field [95, 96], also known as the E/B decomposition [94]. This decomposition is useful because scalar density perturbations in the universe, which evolve into the structures we see today via gravitational instability, generate only E-mode polarization. Subsequent rotation of the polarization plane as the wave propagates rotates E-mode into B-mode. Current limits on the amplitude of B-mode polarization (see Refs. [31, 97, 98] for some recent linear polarization measurements) can be translated into limits on the total rotation of linear polarization between the time of last scattering and today; see Refs. [80, 99] for corresponding limits on magnetic fields due to Faraday rotation. Precise limits on the interaction studied here from the first-order rotation effect Eq. (2.71) and Eq. (2.72) can be obtained similarly, and will be computed elsewhere. But we know that the total amount of rotation must be small at frequencies between 50 GHz and 150 GHz where good measurements of the primordial linear polarization have been made. Given this observational constraint on linear polarization rotation, can some realistic cosmological model generate detectable circular polarization via Eqs. (2.104)–(2.106)? First, note that ω_{QU} has dimensions of $[gT]$, and that for this rotation to be below current limits,

$$\omega_{QU} \simeq gT \ll H_0; \tag{2.117}$$

otherwise, as the microwave background photons propagate a Hubble distance from last scattering until today, we would have substantial rotation of E-mode into B-mode polarization. We are aware of no other laboratory or theoretical constraints on this class of interactions.

Now note that any first order damping effects will contribute an extra factor of length compared to $\alpha(q)$, arising from an additional time integral over the field $T_\mu(x)$; if the field is active for all times, this leads to roughly a factor of H_0^{-1}. Then the time scale for exponential growth or decay of linear polarization is, by dimensional estimate, $\omega_{QU}(\omega_{QU}/H_0)$, which is small compared to ω_{QU}: we can always neglect the exponential growth or decay of linear polarization compared to its rotation.

For generation of V polarization, ω_{QV} and ω_{UV} both have dimensions $[g^2T^2/k^0]$, differing only by a geometric factor related to the propagation directions of the photons and the direction of the field $T^\mu(x)$. So a dimensional estimate for both is $\omega_{QV} \simeq \omega_{UV} \simeq \omega_{QU}(\omega_{QU}/k^0)$. A typical microwave background photon today will have a frequency of $k^0 \simeq 100\,\mathrm{GHz}$ or $10^{11}\,\mathrm{s}^{-1}$, while the characteristic size of ω_{QU} at the observational limit is H_0, a huge mismatch in scales. So in the case considered here, the generation of circular polarization is always vastly subdominant to the rotation of the linear polarization, and can be neglected.

The calculation presented here demonstrates generally that, given additional interactions beyond Compton scattering, circular polarization is not necessarily zero, and elaborates the framework for calculating it for a given microphysical interaction. Other interactions may well induce circular polarization without optical activity from the linear interaction term, and for these cases circular polarization could be the most constraining probe. We have also only considered a constant field $\mathcal{T}^\mu$ for simplicity; calculations for a non-constant field are messier but straightforward, involving convolutions over the field and photon distributions. Spatial or temporal variations in the field could change the relative importance of the optical activity and circular polarization generation effects. In particular, the torsion field necessarily couples to fermions via the interaction $\mathcal{L}_{TF} = g_1\mathcal{T}_\mu\bar{\psi}\gamma^\mu\psi$ as well as a torsion-induced four-fermion interaction $\mathcal{L}_{FF} = g_2\bar{\psi}\gamma^5\gamma^\mu\psi\bar{\psi}\gamma_5\gamma_\mu\psi$ [100–102], where g_1 and g_1 are renormalized couplings. It would be interesting to study the coupled system of torsion and fermions subject to our formalism for evaluating V, since backreaction effects from the torsion-fermion interaction could enhance the amplitude and modify the scale of V; we leave this to future work.

We encourage experimenters to make measurements testing the standard lore that circular polarization of the cosmic microwave background radiation should be identically zero, and theorists to consider the effects of any non-standard photon couplings on microwave background polarization as photons propagate over cosmological distances.

Part II

Fourth Order Gravity

Chapter 3

Gravitational Waves in the Effective Action of NonCommutative Geometry

The spectral triple approach to noncommutative geometry allows one to develop the entire standard model (and supersymmetric extensions) of particle physics from a purely geometry stand point and thus treats both gravity and particle physics on the same footing[1]. The bosonic sector of the theory contains a modification to Einstein-Hilbert gravity, involving a nonconformal coupling of curvature to the Higgs field and conformal Weyl term (in addition to a nondynamical topological term). In this chapter we derive the weak field limit of this gravitational theory and show that the production and dynamics of gravitational waves are significantly altered. In particular, we show that the graviton contains a massive mode that alters the energy lost to gravitational radiation, in systems with evolving quadrupole moment. We explicitly calculate the general solution and apply it to systems with periodically varying quadrupole moments, focusing in particular on the the well know energy loss formula for circular binaries.

[1]This chapter is based on work by the author in collaboration with Nelson and Sakellariadou originally published here [2]; portions of Sec. 3.1 were written by Sakellariadou and Nelson, the analytic calculation in Sec. 3.7 was originally performed by Nelson and double-checked by the author, and the numerical analysis carried out in Sec. 3.7 was performed by Nelson.

3.1 Introduction

Noncommutative Geometry (NCG) is a gravitational theory which, even in its simplest form, can explain the Standard Model of particle physics, and account for all current experimental data, in a rather simple and certainly elegant way. The simple — in the sense that it generalizes the continuum Riemaniann manifold by considering its product by a discrete two points space — NCG proposal should be certainly replaced by a less trivial noncommutative space as one reaches Planckian energy scales. Nevertheless, this is the proposal we have at hand, and given its success in accounting for the Standard Model of particle physics, it offers a valid theoretical framework to address early universe issues. Alternatively, one can use experimental high energy physics data and astrophysical observations/measurements in order to test this NCG proposal and constrain its parameters. This is the approach used in this study.

One should indeed view this NCG proposal as an effective theory, which can however offer a valuable information about any NCG approach. In what follows we derive explicitly the weak field limit of this gravitational theory and then show that the production and dynamics of gravitational waves are both considerably modified from those obtained within the familiar General Relativity approach.

More precisely, in Section 3.2 we give a short introduction to the noncommutative geometry spectral action, the framework within which we will then focus. In Sec. 3.3 we first state in detail the conventions and signature we use and we analyze the issue of gauge conditions. We then analyze linear perturbations around a Minkowski background metric and we solve the noncommutative geometry gravitational wave equation in terms of the retarded Green's function. We find that gravitational waves are only sourced from systems with a nontrivial quadrupole moment, as within General Relativity, while the NCG theory contains massive as well as massless gravitons. In Sec. 3.7 we concentrate first on some simple and then on some physical examples. Using the requirement that the mass of the gravitons must be positive (and real), we can fix the sign of the couplings in the NCG spectral action approach. We then calculate the energy loss for a circular binary system and compare it to the results obtained from standard General Relativity. We conclude that the amplitude of modifications within NCG is small, nevertheless the NCG approach leads to some distinctive features which we analyze. We round up with our conclusions in Sec. 3.8.

3.2 Noncommutative Geometry Spectral Action

In the NonCommutative Geometry [18, 19] approach, the Standard Model (SM) of electroweak and strong interactions is considered as a phenomenological model, which dictates the geometry of space-time, so that the associated Maxwell-Dirac action functional produces the SM with all known experimental results. The outcome of this approach is a geometric space defined by the product, $\mathcal{M} \times \mathcal{F}$, of a continuum compact Riemaniann manifold, $\mathcal{M}$, and a tiny discrete finite noncommutative space, $\mathcal{F}$, composed of only two points. Such an almost commutative space is the simplest extension of the more familiar commutative space upon which General Relativity is formulated. Certainly one should not expect the validity of this simplistic approach to hold at the Planck scale, which is the scale at which all notion of classical geometry loses its meaning.

The metric dimension of the product geometry $\mathcal{M} \times \mathcal{F}$ is 4, the same as the ordinary space-time manifold. Thus, the metric dimension of the noncommutative space $\mathcal{F}$ is zero, while for noncommutative spaces one must distinguish between the metric dimension and the KO-dimension. The internal space $\mathcal{F}$ has KO-dimension 6 to allow fermions to be simultaneously Weyl and chiral, while it is discrete to avoid the infinite tower of massive particles that are produced in string theory.

The noncommutative nature of $\mathcal{F}$ is given by the real spectral triple $(\mathcal{A}, \mathcal{H}, D)$ that generalizes Riemannian geometry to the noncommutative setting; $\mathcal{A}$ is an involution of operators on the finite-dimensional Hilbert space $\mathcal{H}$ of Euclidean fermions, and D is a self-adjoint unbounded operator in $\mathcal{H}$. The choice of Hilbert space has no importance, since all separable infinite-dimensional Hilbert spaces are isomorphic. The algebra $\mathcal{A}$, related to the gauge group of local gauge transformations, is the algebra of coordinates. A space is described by the algebra of coordinates, which in the context of NCG is represented as operators on a Hilbert space. Since real coordinates are represented by self-adjoint operators, all information about a space within NCG is encoded in the algebra of coordinates $\mathcal{A}$. By assuming that the algebra constructed in $\mathcal{M} \times \mathcal{F}$ is symplectic-unitary, $\mathcal{A}$ must be of the form

$$\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}) \ ; \tag{3.1}$$

$k = 2a$, $\mathbb{H}$ is the algebra of quaternions. The choice $k = 4$ is the first value that produces the correct number of fermions in each generation, *i.e.*, $k^2 = 16$ fermions in each of the three generations [103].

The operator D corresponds to the inverse of the Euclidean propagator of fermions, and is given by the Yukawa coupling matrix which encodes the masses of the elementary fermions and the Kobayashi–Maskawa mixing parameters. The commutator $[D, a]$, with $a \in \mathcal{A}$, plays the rôle of the differential quotient da/ds, with ds the unit of length. The familiar geodesic formula

$$d(x, y) = \inf \int_\gamma ds \ , \tag{3.2}$$

where the infimum is taken over all possible paths connecting x to y, which is used to determine the distance $d(x, y)$ between two points x and y within Riemannian geometry, is replaced by

$$d(x, y) = \sup\{|f(x) - f(y)| \colon f \in \mathcal{A}, ||[D, f]|| \leq 1\} \ , \tag{3.3}$$

where D is the inverse of the line element ds, within the noncommutative spectral geometry.

The fermions of the SM provide the Hilbert space $\mathcal{H}$ of a spectral triple for the algebra $\mathcal{A}$, while the bosons of the SM, including the Higgs boson, are obtained through inner fluctuations of the Dirac operator of the product $\mathcal{M} \times \mathcal{F}$ geometry. Hence, the Higgs boson, which generates the masses of elementary particles through spontaneous symmetry breaking, becomes just a gauge field corresponding to a finite difference. Note that the corresponding mass scale specifies the inverse size of the discrete geometry $\mathcal{F}$.

Applying the spectral action principle, according to which the action functional on spectral triples depends only on the spectrum of the line element, *i.e.*, the inverse of the Dirac operator, to the inner fluctuations of the product geometry $\mathcal{M} \times \mathcal{F}$, one recovers the SM coupled to gravity in the Euclidean form. Thus, the NCG spectral action approach — limited to the classical level even though it can *a priori* be quantized — offers an elegant geometric interpretation of the SM , the most successful phenomenological model of particle physics.

To be more precise, the SM Lagrangian — including mixing and Majorana mass terms for neutrinos, minimally coupled to gravity — can be successfully recovered from the asymptotic expansion of the spectral action functional

$$\mathrm{Tr}\Big(f\Big(\frac{D}{\Lambda}\Big)\Big) \ , \tag{3.4}$$

where f is a positive even function of the real variable and Λ fixes the energy scale. Note that D/Λ is dimensionless since the Dirac operator, being a differential operator, has dimensions of mass. The physical Lagrangian is thus obtained from the asymptotic expansion in the energy scale Λ of the spectral action functional, Eq. (3.4). More precisely, using heat kernel methods one can write the square of the Dirac operator in terms of the inverse metric, the unit matrix and two matrix functions computed from D and show that the trace, Eq. (3.4) above, can be expanded in a power series as a function of the inverse scale Λ and it can thus be written in terms of the geometrical Seeley-deWitt coefficients a_n, as [104]

$$\sum_{n=0}^{\infty} F_{4-n}\Lambda^{4-n}a_n \ , \tag{3.5}$$

where the function F is defined such that $F(D^2) = f(D)$. Defining the moments

$$f_k = \int_0^\infty f(u)u^{k-1}\mathrm{d}u \quad , \quad \text{for} \quad k > 0 \ , \tag{3.6}$$

and $f_0 = f(0)$, one finds

$$\begin{aligned} F_4 &= 2f_4 \\ F_2 &= 2f_2 \\ F_0 &= f_0 \\ F_{-2n} &= \Big[(-1)^n\Big(\frac{\mathrm{d}}{2u\mathrm{d}u}\Big)^n f\Big](0) \quad \text{for} \quad n \geq 1 \ , \end{aligned} \tag{3.7}$$

while the coefficients a_n are known for any second order elliptic differential operator.

The coupling with fermions can be obtained by including an additional fermionic term

$$\frac{1}{2} < J\psi, D\psi > \ , \tag{3.8}$$

in Eq. (3.4), where J is the real structure on the spectral triple and ψ is a spinor in the Hilbert space $\mathcal{H}$ of the quarks and leptons.

The spectral action approach leads naturally to the merging of the three coupling constants at the unification scale, $g_2 = g_3 = \sqrt{5/3}g_1$, it provides neutrino masses and mixing as well as the see-saw mechanism, and it predicts a heavy Higgs mass.

The spectral action, Eq. (3.4), can be expanded in powers of the scale Λ in the form

$$\mathrm{Tr}\left(f\left(\frac{D}{\Lambda}\right)\right) \sim \sum_{k\in \mathrm{DimSp}} f_k \Lambda^k {\int\!\!\!\!\!\!-} |D|^{-k} + f(0)\zeta_D(0) + \mathcal{O}(1) \ , \tag{3.9}$$

where f_k are the momenta of the function f given in Eq. (3.6), the noncommutative integration is defined in terms of residues of zeta functions, and the sum is over points in the *dimension spectrum* of the spectral triple.

The physical Lagrangian that one obtains in this approach, contains, in addition to the full SM Lagrangian, the Einstein-Hilbert action with a cosmological term, a topological term related to the Euler characteristic of the space-time manifold, a conformal Weyl term and a conformal coupling of the Higgs field to gravity. Note that the coefficients of the gravitational terms depend on the Yukawa parameters of the particle physics content. Within the NCG spectral action, one works in Euclidean rather than Lorentzian signature, assuming that one can get back to the Minkowski signature through Wick rotation.

One then sets the parameters of the NCG spectral action at the (unification) scale Λ; predictions at lower energies are recovered by running the parameters down through Renormalization Group Equations (RGE). Hence, the spectral action at the unification scale Λ offers a framework to investigate early universe cosmological models [3, 105–108], while extrapolations to lower energies can be obtained via, firstly, RGE and secondly, inclusion of nonperturbative effects in the spectral action.

Adopting Euclidean signature, the gravitational part of the asymptotic formula for the bosonic sector of the NCG spectral action, including the coupling between the Higgs field ϕ and the Ricci curvature scalar R, is [20]

$$\mathcal{S}_{\rm grav}^{\rm E} = \int \left(\frac{1}{16\pi G} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \tau_0 R^\star R^\star - \xi_0 R |\mathbf{H}|^2 \right) \sqrt{g} {\rm d}^4 x \ . \tag{3.10}$$

Note that $\mathbf{H}$ is a rescaling $\mathbf{H} = (\sqrt{a f_0}/\pi)\phi$ of the Higgs field ϕ to normalize the kinetic energy; the moment f_0 is physically related to the coupling constants at unification and the coefficient a is related to the fermion and lepton masses and lepton mixing.

In the above action, Eq. (3.10), the first two terms only depend upon the Riemann curvature tensor; the first is the Einstein-Hilbert term with the second one being the Weyl curvature term. The third term

$$R^\star R^\star = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}_{\rho\sigma} \ ,$$

is the topological term that integrates to the Euler characteristic and hence is non-dynamical. The spectral action contains one more term that couples gravity with the SM, namely the last term in Eq. (3.10), which should always be present when one considers gravity coupled to scalar fields. This coupling can have significant consequences at high energies, such as in the early universe [3, 105–108], however in this chapter we will be concerned with the low energy weak curvature regime where this term is small.

Neglecting the nonminimal coupling between the Higgs field and the Ricci curvature, the equations of motion derived from the Lorentzian version of spectral action above read [105]

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - 32\pi G \alpha_0 \left[2 C^{\mu\lambda\nu\kappa}_{\ \ ;\lambda;\kappa} + C^{\mu\lambda\nu\kappa} R_{\lambda\kappa} \right] = 8\pi G T^{\mu\nu}_{\rm matter} \ , \tag{3.11}$$

implying that the NCG corrections vanish [105] for Friedmann-Lemaître-Robertson-Walker (FLRW) cosmologies. [The reader is directed to Subsection 3.3.1 for a definition and discussion of the Lorentzian conventions used.]

We will be concerned with linear perturbations around a Minkowski background metric in the synchronous gauge, so that the perturbed metric reads

$$g_{\mu\nu} = {\rm diag} \left(\{a(t)\}^2 \left[-1, \left(\delta_{ij} + h_{ij}\left(x\right) \right) \right] \right) \ , \tag{3.12}$$

where $a(t)$ is the cosmological scale factor. Throughout this chapter we work in a flat background and hence $a(t) = 1$ and $\dot{a} \equiv \mathrm{d}t/\mathrm{d}t = 0$. The remaining gauge freedom can be completely fixed by setting $\nabla_i h^{ij} = 0$. [A detailed discussion of this gauge fixing is given in Subsection 3.3.3 below.]

After taking care of sign conventions and the proper gauge conditions, in Sec. 3.3 below we show that, the linearized equations of motion derived from the NCG spectral action, for such perturbations, read

$$\left(\Box - \beta^2\right) \Box h^{\mu\nu} = \beta^2 \frac{16\pi G}{c^4} T^{\mu\nu}_{\text{matter}} \ , \tag{3.13}$$

where $T^{\mu\nu}_{\text{matter}}$ is taken to lowest order in $h^{\mu\nu}$. This implies that it is independent of $h^{\mu\nu}$ and satisfies the conservation equations

$$\frac{\partial}{\partial x^\mu} T^\mu_{\ \nu} = 0 \ . \tag{3.14}$$

It is important to note that β, defined as

$$\beta^2 \equiv -1/(32\pi G \alpha_0) \ , \tag{3.15}$$

in Eq. (3.13), turns out to play the rôle of a mass and hence has to be real and positive, implying that $\alpha_0 < 0$. In the following we will see that, for $\alpha_0 > 0$, the gravitational waves evolve according to a Klein-Gordon like equation with a tachyonic mass, and hence the background, which in our case is Minkowski space, is unstable. We can thus restrict to $\alpha_0 < 0$ for Minkowski space to be a (stable) vacuum of the theory.

3.3 Perturbation equations

In order to write down the linearized equations of motion, we will first discuss our conventions for the metric signature and the Ricci tensor. We then go on to write down the full linearized equation of motion and conclude this section by imposing the proper gauge conditions.

3.3.1 Conventions for $R_{\mu\nu}$ and signature

Throughout part II, we are using conventions in which the signature is $(-,+,+,+)$ and the Ricci tensor is defined as $R_{\mu\nu} = R^{\rho}{}_{\mu\nu\rho}$, with $R_{\mu\nu\rho}{}^{\sigma}\omega_{\sigma} = [\nabla_{\mu}, \nabla_{\nu}]\omega_{\rho}$. In General Relativity such choices are merely conventions, which are relatively unimportant (provided of course that one is consistent), here however the situation is very different. The Lorentzian version of the NCG action, Eq. (3.10), that we use reads

$$\mathcal{S}^{\rm L}_{\rm grav} = \int \left(\frac{1}{16\pi G} R + \alpha_0 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \tau_0 R^{\star}R^{\star} - \xi_0 R|\mathbf{H}|^2 \right) \sqrt{-g}\,{\rm d}^4x \ . \quad (3.16)$$

It is thus clear that the conventions used to define, for example, $R_{\mu\nu}$ will radically alter the theory, unless one also alters the (signs) of the couplings. Specifically, consider using the opposite convention for the Ricci tensor, which introduces a negative sign on all terms depending on $R_{\mu\nu}$, but *not* on terms depending on $R_{\mu\nu\rho\sigma}$. Since our action now contains terms of both kinds (*i.e.*, R and $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$), this change of convention introduces a relative sign change. This can simply be compensated for by changing $\alpha_0 \to -\alpha_0$. However, without this change the action is very different. An exactly similar change happens if we considered a different choice of convention for the signature or the sign of $R_{\mu\nu\rho\sigma}$, as these both introduce a sign change in the R term, but not the $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ term.

As shown in Refs. [105–108], the presence of the non-minimal coupling of curvature to the Higgs field can have significant effects of the cosmological dynamics and one may wonder whether the sign ambiguities discussed here may affect these results. Fortunately, from Eq. (3.16) it is clear that the relative sign between the Einstein-Hilbert term (R) and the nonminimal coupling ($R|\mathbf{H}|^2$) is independent of any convention (since they both contain R). In a cosmological setting (*i.e.*, for FLRW geometries) the Weyl term vanishes and hence, in homogeneous cosmologies, the only NCG affects come from the nonminimal coupling [105], allowing such issues to be avoided.

Since the underlying NCG theory is only developed for the Euclidean signature, it does not provide a guide for the Wick rotation to the Lorentzian space. Hence the choice of the sign of the couplings, appropriate for a particular choice of convention, can only be made by testing the *physical* consequences of the theory.

In the following section we will show that gravitational waves offer an excellent probe of the couplings in this theory, but even without an in-depth analysis, from Eq. (3.13) one can immediately see that the coupling α_0 must be negative (in the conventions used here). If it were not, the β parameter would be complex and this would correspond to a tachyonic mode of the graviton (we refer the reader to a discussion below). This would indicate that Minkowski space-time is unstable to small perturbations. If we rule out such a possibility on physical grounds (or require that the Lorentzian version of the NCG action should admit Minkowski as a stable vacuum) then we can restrict $\alpha_0 < 0$ *with the conventions used here.* Consider for example the consequence of changing the signature, so that $\Box \rightarrow -\Box$. Such a change of convention essentially changes the sign of β^2 and the conclusions would be reversed[2].

3.3.2 Linearized Equations of Motion

Variation of the gravitational part action Eq. (3.16) *w.r.t.* the metric $g_{\mu\nu}$ leads to the following addition to the Einstein tensor of General Relativity (GR),

$$G^{\mu\nu}_{\text{NCG}} = -\frac{1}{2\kappa} G^{\mu\nu}_{\text{Einstein}} + 2\alpha_0 \left(2 \bigtriangledown_\lambda \bigtriangledown_\kappa C^{\mu\kappa\nu\lambda} + C^{\mu\kappa\nu\lambda} R_{\kappa\lambda} \right) , \tag{3.17}$$

where as usual

$$G^{\mu\nu}_{\text{Einstein}} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \ .$$

Given the convention used here to define the Ricci tensor the Weyl Tensor is explicitly given as,

$$C_{\mu\lambda\nu\kappa} = R_{\mu\lambda\nu\kappa} + (g_{\mu[\nu} R_{\kappa]\lambda} - g_{\lambda[\nu} R_{\kappa]\mu}) - \frac{1}{3} g_{\mu[\nu} g_{\kappa]\lambda} R \, . \tag{3.18}$$

Using the contracted Bianchi identity

$$\bigtriangledown^\kappa R_{\mu\lambda\nu\kappa} = -(\bigtriangledown_\lambda R_{\mu\nu} - \bigtriangledown_\mu R_{\lambda\nu}) \, , \tag{3.19}$$

[2]The concerned reader should note that exactly the same situation arises in standard Klein-Gordon equation, where the sign of the mass term is changed under a change of signature.

and its remaining trace

$$\bigtriangledown^{\kappa} R_{\lambda\kappa} \;=\; \frac{1}{2} \bigtriangledown_{\lambda} R \,, \tag{3.20}$$

we can arrive at the following expression

$$\begin{aligned} 2 \bigtriangledown^{\lambda} \bigtriangledown^{\kappa} C_{\mu\lambda\nu\kappa} \;=\;& -C_{\lambda\mu\kappa\nu} R^{\lambda\kappa} - \bigtriangledown^{\lambda} \bigtriangledown_{\lambda} \left(R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} R\right) \\ & + \frac{1}{3} \bigtriangledown_{\mu} \bigtriangledown_{\nu} R - 2 R_{\mu\rho} R^{\rho}{}_{\nu} + \frac{2}{3} R R_{\nu\mu} \\ & + \frac{1}{2} g_{\mu\nu} \left(R_{\kappa\lambda} R^{\lambda\kappa} - \frac{1}{3} R^2\right) \;. \end{aligned} \tag{3.21}$$

Notice that the expression Eq. (3.21) above for $2 \bigtriangledown_{\lambda} \bigtriangledown_{\kappa} C^{\mu\lambda\nu\kappa}$ shows that the $C^{\mu\kappa\nu\lambda} R_{\kappa\lambda}$ term in Eq. (3.17) exactly cancels in favor of terms which are of second order of solely the Ricci tensor and/or Ricci scalar.

We now follow the standard procedure of perturbing about a flat metric, where

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \quad , \quad g^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu} \;, \tag{3.22}$$

and

$$\gamma = \gamma^{\mu}{}_{\mu} = \eta^{\mu\nu} \gamma_{\mu\nu} \;; \tag{3.23}$$

all tensor indices are raised and lowered using the background metric $\eta_{\mu\nu}$ (except for the indices of $g_{\mu\nu}$ and $g^{\mu\nu}$). To first order in metric perturbations we then have

$$\begin{aligned} & 2\Big\{ \bigtriangledown_{\lambda} \bigtriangledown_{\kappa} C^{\mu\kappa\nu\lambda} \Big\} = \\ & \partial_{\lambda}\partial^{\lambda} \Big(\partial^{\kappa} \partial^{(\nu} \bar{\gamma}^{\mu)}{}_{\kappa} - \frac{1}{2} \partial_{\kappa} \partial^{\kappa} \bar{\gamma}^{\mu\nu} - \frac{1}{6} \eta^{\mu\nu} \partial^{\sigma} \partial^{\kappa} \bar{\gamma}_{\sigma\kappa} \Big) \\ & -\partial_{\lambda}\partial^{\nu} \Big(\partial^{\kappa} \partial^{(\lambda} \bar{\gamma}^{\mu)}{}_{\kappa} - \frac{1}{2} \partial_{\kappa} \partial^{\kappa} \bar{\gamma}^{\lambda\mu} \Big) + \frac{1}{6} \partial^{\mu} \partial^{\nu} \partial^{\lambda} \partial^{\kappa} \bar{\gamma}_{\lambda\kappa} \\ & -\frac{1}{6} \Big(\eta^{\mu\nu} \partial_{\kappa} \partial^{\kappa} - \partial^{\mu} \partial^{\nu} \Big) \partial_{\lambda} \partial^{\lambda} \gamma + \mathcal{O}(\gamma^2) \,. \end{aligned} \tag{3.24}$$

where $\mathcal{O}(\gamma^2)$ denotes any second order combinations of $\gamma_{\mu\nu}$ and we have defined

$$\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \gamma \,, \tag{3.25}$$

i.e., the *trace reverse* of $\gamma_{\mu\nu}$.

Similarly, to linear order in metric perturbations the Einstein tensor is simply

$$G^{\mu\nu}_{\text{Einstein}} = +\frac{1}{2}\partial_\lambda\partial^\lambda\bar{\gamma}^{\mu\nu} + \mathcal{O}(\gamma^2)\ . \tag{3.26}$$

3.3.3 Gauge Conditions

In calculating [105] the linearized equations of motion, the traceless transverse gauge was imposed on the metric perturbations $h_{\mu\nu}$; here we explicitly show that this is indeed a valid choice. As before, we denote metric perturbations that have not been gauge fixed by $\gamma_{\mu\nu}$ and reserve $h_{\mu\nu}$ for the final, gauge fixed perturbations that correspond to the physical gravitational waves.

As always we have a freedom due to diffeomorphism invariance of the action to restrict the gauge of the metric perturbations. Explicitly, under a diffeomorphism generated by ξ_μ the metric perturbations $\gamma_{\mu\nu}$ transform as

$$\gamma^{\text{old}}_{\mu\nu} \xrightarrow{\xi_\mu} \gamma^{\text{new}}_{\mu\nu} = \gamma^{\text{old}}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,. \tag{3.27}$$

Without loss of generality, one can impose the *Lorentz gauge conditions*

$$\partial^\mu\bar{\gamma}_{\mu\nu} = 0\,, \tag{3.28}$$

restricting the perturbations to be transverse, where we introduced the "trace reverse" of $\gamma_{\mu\nu}$, as in Eq. (3.25).

Choosing this gauge (and dropping the label *new* on $\gamma_{\mu\nu}$), Eq. (3.24) simplifies to

$$2\Big\{\bigtriangledown_\lambda\bigtriangledown_\kappa C^{\mu\lambda\nu\kappa}\Big\} = -\frac{1}{2}\partial_\kappa\partial^\kappa\Big(\partial_\lambda\partial^\lambda\bar{\gamma}^{\mu\nu} + \frac{1}{3}\big(\eta^{\mu\nu}\partial_\lambda\partial^\lambda - \partial^\mu\partial^\nu\big)\gamma\Big)\,. \tag{3.29}$$

Combining the above result with the Einstein contribution, $G^{\mu\nu}_{\text{Einstein}}$, to the equation of motion, the left-hand side of Eq. (3.11) is, to first order in $\gamma_{\mu\nu}$, given by

$$\begin{aligned} G^{\mu\nu}_{\text{NCG}} &= -G^{\mu\nu}_{\text{Einstein}} - \frac{1}{\beta^2}\Big\{2\bigtriangledown_\kappa\bigtriangledown_\lambda C^{\mu\lambda\nu\kappa}\Big\}\,, \\ &= -\frac{1}{2}\partial_\kappa\partial^\kappa\bar{\gamma}^{\mu\nu} + \frac{1}{2\beta^2}\partial_\kappa\partial^\kappa\Big(\partial_\lambda\partial^\lambda\bar{\gamma}^{\mu\nu} + \frac{1}{3}\big(\eta^{\mu\nu}\partial_\lambda\partial^\lambda - \partial^\mu\partial^\nu\big)\gamma\Big)\,. \end{aligned} \tag{3.30}$$

However, the Lorentz gauge does not uniquely fix all the gauge freedom. More precisely, we still are free to perform gauge transformations, generated by any ξ_μ, that satisfy

$$\partial_\mu \partial^\mu \xi_\nu = 0 \ , \tag{3.31}$$

since this still preserves the gauge condition, Eq. (3.28), as can be checked directly.

We can use this transformation, Eq. (3.31), to set (in the new frame) $\gamma = 0$ and $\gamma_{0i} = 0$ ($i = 1, 2, 3$) by solving the corresponding equations for ξ_i and their time derivatives on some initial surface $t = t_0$ where no sources are present and further extending into a source free region (*i.e.*, $T^{\mu\nu} = 0$) [38]. After performing these gauge transformations the source free equations of motion (*i.e.*, the left-hand side of Eq. (3.11)) read

$$-\frac{1}{2}\partial_\kappa \partial^\kappa \bar{\gamma}_{\mu\nu} + \frac{1}{2\beta^2}\partial_\kappa \partial^\kappa \partial_\lambda \partial^\lambda \bar{\gamma}_{\mu\nu} \quad = \quad 0 \ , \tag{3.32}$$

where we have made repeated use of the fact that $\partial_\mu \partial^\mu \xi_\nu = 0$.

Up to this point, the gauge restrictions $\gamma = 0$ and $\gamma_{0i} = 0$ ($i = 1, 2, 3$) are the same as those typically used in General Relativity, which is to be expected since all we have used is the diffeomorphism invariance of the action. However, in determining whether we can set $\gamma_{00} = 0$, the equations of motion are used and hence one might expect that this gauge condition will be different than that of General Relativity. To confirm its validity note that Eq. (3.28) implies

$$\frac{\partial \gamma_{00}}{\partial t} = 0 \, . \tag{3.33}$$

Now, using the equation of motion in the presence of matter source we arrive at

$$\begin{aligned} \bigtriangledown^2 \gamma_{00} - \frac{1}{\beta^2} \bigtriangledown^2 \left(\bigtriangledown^2 \gamma_{00} \right) \quad &= \quad -\frac{2\kappa}{c^4} T_{00} \ , \\ \left(1 - \frac{1}{\beta^2} \bigtriangledown^2 \right) \bigtriangledown^2 \gamma_{00} \quad &= \quad -\frac{2\kappa}{c^4} T_{00} \, . \end{aligned} \tag{3.34}$$

Recall that General Relativity is recovered in this setting by taking $\beta \to \infty$. Thus, one can see that in this limit the equation simplifies to $\bigtriangledown^2 \gamma_{00} = -(16\pi G)/c^4 T_{00}$, which fixes γ_{00} to be a constant (assuming the space-time is asymptotically flat) away from the source. Finally, a redefinition (gauge transformation) allows us to set $\gamma_{00} = 0$.

From Eq. (3.34) we see that, away from the source, $\nabla^2\gamma_{00} = 0$ is still a solution and hence we can fix $\gamma_{00} = 0$, however this is no longer the only solution to Eq. (3.34). In particular, away from the source one could fix γ_{00} via,

$$\Big(1 - \frac{1}{\beta^2}\nabla^2\Big)\gamma_{00} = 0 \ , \tag{3.35}$$

which clearly solves Eq. (3.34). This would result in a modification of what is often referred to as the *radiation gauge*. In the following, we choose $\gamma_{00} = 0$ so as to be able to directly compare our results to the standard ones obtained within General Relativity.

In cases where sources are present, the NCG equation of motion with gravity and normal matter is

$$\partial_\kappa\partial^\kappa\bar{\gamma}^{\mu\nu} - \frac{1}{\beta^2}\partial_\kappa\partial^\kappa\Big[\partial_\lambda\partial^\lambda\bar{\gamma}^{\mu\nu} + \frac{1}{3}\big(\eta^{\mu\nu}\partial_\lambda\partial^\lambda - \partial^\mu\partial^\nu\big)\gamma\Big] = -\frac{2\kappa}{c^4}T^{\mu\nu} \ . \tag{3.36}$$

Since $T_{\mu\nu} \neq 0$ we are not free to impose the traceless condition of the radiation gauge[3]. But the explicit presence of the trace γ in Eq. (3.36) above can be eliminated by formally defining the tensor $\bar{h}_{\mu\nu}$ as,

$$\bar{h}_{\mu\nu} = \bar{\gamma}_{\mu\nu} - \frac{1}{3\beta^2}\mathcal{O}^{-1}\big(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\big)\gamma \, , \tag{3.37}$$

where the operator $\mathcal{O}$ is given by the equation

$$\mathcal{O} = \Big(1 - \frac{\Box}{\beta^2}\Big) \ . \tag{3.38}$$

This is a modification of the *trace reverse* of the metric perturbations that is usually used, however it performs the same task, namely removing the trace from the equations of motion, Eq. (3.36). As it can be easily checked, the Lorentz gauge condition, Eq. (3.28), is satisfied by $\bar{h}_{\mu\nu}$ as long as it is also satisfied by $\bar{\gamma}_{\mu\nu}$. Note

[3]Nor would this be possible if one had chosen the modified radiation gauge implied by Eq. (3.35).

that the trace of $\bar{h}_{\mu\nu}$ is

$$\bar{h} = -\Big(1 + \frac{\mathcal{O}^{-1}\Box}{\beta^2}\Big)\gamma \;, \tag{3.39}$$

so we see that indeed, this reproduces the *trace reverse* of $\gamma_{\mu\nu}$ in the limit $\beta \to \infty$. Clearly then, when we are away from a source, we can impose that $\gamma = 0$ and this implies that $\bar{h} = 0$.

In terms of $\bar{h}_{\mu\nu}$ the equation of motion, Eq. (3.36), is

$$\Big(1 - \frac{1}{\beta^2}\Box\Big)\Box\bar{h}^{\mu\nu} \quad = \quad -\frac{16\pi G}{c^4}T^{\mu\nu} \;. \tag{3.40}$$

Dropping the over-bars, this is exactly Eq. (3.13).

3.4 Fourth Order Green's Function

The general physical solution to Eq. (3.40) is given by the equation

$$h^{\mu\nu} = 2\beta^2\kappa \int \mathrm{d}S(x')G_{\mathrm{R}}(x,x')T^{\mu\nu}(x')\,, \tag{3.41}$$

where the Green's function $G_{\mathrm{R}}(x,x')$ satisfies the fourth-order partial differential equation:

$$\Big(\Box - \beta^2\Big)\Box G_{\mathrm{R}}(x,x') \quad = \quad 4\pi\delta^{(4)}(x-x')\,, \tag{3.42}$$

where the operators $\Box$ above are acting on x. In order to find a solution $G_{\mathrm{R}}(x,x')$ to Eq. (3.42) consider two distributions g_1 and g_2 which satisfy the following second-order equations:

$$\left(\Box - \beta^2\right) g_1 \quad = \quad 4\pi\delta^{(4)}\left(x-x'\right) \;, \tag{3.43}$$

$$\Box g_2 \quad = \quad 4\pi\delta^{(4)}\left(x-x'\right) \;. \tag{3.44}$$

Then one can easily verify that the combination

$$G_{\mathbf{R}}(x,x') = \frac{1}{\beta^2}\left(g_1 - g_2\right) \;, \tag{3.45}$$

will be a solution to Eq. (3.42). Physically we are interested in the retarded Green's function solution to Eq. (3.42), that is of the form

$$G_{\mathrm{R}}(x,x') = \Theta(t-t')g(x-x')\,, \tag{3.46}$$

where $\Theta(z)$ is the Heavyside step function. So g_1 and g_2 in Eq. (3.45) above must each be retarded solutions to Eq. (3.43) and Eq. (3.44), respectively. The other three combinations of retarded and advanced solutions violate causality and will not be considered further. So we have reduced the problem of finding the retarded Green's function solution of the fourth-order, Eq. (3.42), to finding the retarded Green's function solutions of the two second-order differential equations, Eq. (3.43) and Eq. (3.44).

3.5 Fourth Order Green's Function Calculation

In what follows, we shall explicitly detail the Green's function solution to the fourth-order wave equation, *i.e.* Eq. (3.42). Those not interested in the details of the calculation may wish to proceed to the next section.

We begin with the following fourth-order wave equation

$$\left(\Box-\beta^2\right)\Box G_{\mathrm{R}}(x,x') \quad = \quad 4\pi\delta^{(4)}(x-x')\,. \tag{3.47}$$

Lorentz symmetry of the background restricts G_{R} to be solely a function of $x-x'$. Given the following inverse Fourier transforms:

$$G_{\mathrm{R}}(x,x') = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\,\tilde{G}_{\mathrm{R}}(k)e^{ik\cdot(x-x')}\,, \tag{3.48}$$

and

$$\delta(x-x') = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\,e^{ik\cdot(x-x')}\,, \tag{3.49}$$

where $k\cdot z = -\omega z_0 + \mathbf{k}\cdot\mathbf{z}$, the Fourier transform $\tilde{G}_{\mathrm{R}}(k)$ must satisfy

$$\tilde{G}_{\mathrm{R}}(k) = \frac{4\pi}{\left[(\omega+i\epsilon)^2-\mathbf{k}^2-\beta^2\right]\left[(\omega+i\epsilon)^2-\mathbf{k}^2\right]}\,, \tag{3.50}$$

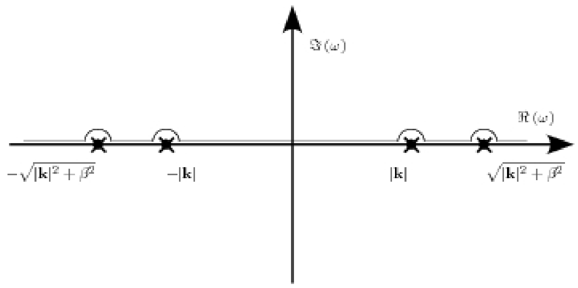

Figure 3.1. The retarded Green's function is defined by extending the contour of the integral into the positive half of the imaginary plane around each of the four poles, $\omega = \pm|\mathbf{k}|$ and $\omega = \pm\sqrt{|\mathbf{k}|^2+\beta^2}$.

in order to solve Eq. (3.47).

Upon performing the inverse Fourier transform to determine the coordinate expression for $G_{\rm R}(x,x')$, the following pole prescription uniquely determines the retarded Green's function (the reader is referred to Fig. 3.1). The contours are traversed above the poles in the complex ω plane: for $t-t'>0$ we close the contour in the lower half plane picking up the residue of the poles; for $t-t'<0$ we close the contour in the upper half plane, thus enclosing no poles.

Note that $\tilde{G}_{\rm R}(k)$ can be rewritten as

$$\tilde{G}_{\rm R}(k) = 4\pi\left\{\frac{1}{\beta^2\left[(\omega+i\epsilon)^2-\mathbf{k}^2-\beta^2\right]} - \frac{1}{\beta^2\left[(\omega+i\epsilon)^2-\mathbf{k}^2\right]}\right\}, \quad (3.51)$$

$$\equiv \frac{1}{\beta^2}\left[\tilde{g}_1(k)-\tilde{g}_2(k)\right], \quad (3.52)$$

which is in fact simply the Fourier transform of Eq. (3.45). We will analyze each term in Eq. (3.52) separately.

First we need to define the following quantities:

$$T \equiv t-t',$$
$$\mathbf{R} \equiv \mathbf{r}-\mathbf{r}',$$
$$\tilde{k} \equiv |\mathbf{k}|^2+\beta^2. \quad (3.53)$$

In what follows, we will set the speed of light $c=1$. The $\tilde{g}_1(k)$ term can be integrated as follows:

$$\begin{aligned} g_{1\mathrm{R}}(x-x') &= \frac{1}{(2\pi)^4}\int \mathrm{d}\omega \mathrm{d}^3\mathbf{k}\,\tilde{g}_1(k) \\ &= \frac{1}{4\pi^3}\int \mathrm{d}\omega \mathrm{d}^3\mathbf{k}\,\frac{e^{-i\omega T}e^{i\mathbf{k}\cdot\mathbf{R}}}{\left[((\omega+i\epsilon)+\sqrt{\tilde{k}})((\omega+i\epsilon)-\sqrt{\tilde{k}})\right]}\,, \end{aligned} \tag{3.54}$$

where the $+i\epsilon$ above is simply a mnemonic for the retarded Green's function pole prescription. The contour integral in the complex ω plane results in $-2\pi i \sum Res$ and we have

$$g_{1\mathrm{R}}(x-x') = \frac{i\Theta(T)}{4\pi^2}\int \mathrm{d}^3\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{R}}\left\{\frac{e^{iT\sqrt{\tilde{k}}}-e^{-iT\sqrt{\tilde{k}}}}{\sqrt{\tilde{k}}}\right\}. \tag{3.55}$$

Upon performing the angular integral, one finds

$$g_{1\mathrm{R}}(x-x') = \frac{\Theta(T)\beta}{2\pi|\mathbf{R}|}\left[\mathcal{I}^+ - \mathcal{I}^-\right]\ , \tag{3.56}$$

where

$$\mathcal{I}^{\pm} = \int_0^\infty \frac{k\mathrm{d}k}{\sqrt{k^2+1}}\left(e^{i\beta(|\mathbf{R}|k\pm T\sqrt{k^2+1})} + e^{-i\beta(|\mathbf{R}|k\pm T\sqrt{k^2+1})}\right)\,. \tag{3.57}$$

We will first focus on the solution which is interior to the light cone, *i.e.*, $T > |\mathbf{R}|$. In the first term in the integral, Eq. (3.56) above, we perform the following change of variables:

$$\begin{aligned} |\mathbf{R}|k + T\sqrt{k^2+1} &= \tau\cosh v\,, \\ Tk + |\mathbf{R}|\sqrt{k^2+1} &= \tau\sinh v\,, \\ k &= -\frac{|\mathbf{R}|}{\tau}\cosh v + \frac{T}{\tau}\sinh v\,, \\ \frac{\mathrm{d}k}{\sqrt{k^2+1}} &= \mathrm{d}v\,, \end{aligned} \tag{3.58}$$

where $v \in [v_0, \infty)$, $v_0 \equiv \operatorname{arcosh}(T/\tau)$ and we have defined $\tau = \sqrt{T^2 - |\mathbf{R}|^2}$.

In the second term in the integral in Eq. (3.56) above, we will perform a different change of variables given by the equations

$$
\begin{aligned}
|\mathbf{R}|k - T\sqrt{k^2+1} &= -\tau\cosh\bar{v}\,, \\
Tk - |\mathbf{R}|\sqrt{k^2+1} &= \tau\sinh\bar{v}\,, \\
k &= \frac{|\mathbf{R}|}{\tau}\cosh\bar{v} + \frac{T}{\tau}\sinh\bar{v}\,. \\
\frac{\mathrm{d}k}{\sqrt{k^2+1}} &= \mathrm{d}\bar{v}\,,
\end{aligned}
\tag{3.59}
$$

where for these change of variables $\bar{v} \in [-v_0, \infty)$. Note that, the variable $\bar{v}$ spans the interval $[-v_0, 0]$, while k correspondingly spans the interval $[0, |\mathbf{R}|/\tau]$. After these changes of variables we arrive at the following:

$$
\begin{aligned}
g_{1\mathrm{R}}(x-x') &= \frac{\Theta(T)\beta}{2\pi|\mathbf{R}|}\Bigg\{\int_{v_0}^{\infty}\mathrm{d}v\big[-\frac{|\mathbf{R}|}{\tau}\cosh v + \frac{T}{\tau}\sinh v\big]\big[e^{i\beta\tau\cosh v} + e^{-i\beta\tau\cosh v}\big] \\
&\qquad - \int_{-v_0}^{\infty}\mathrm{d}\bar{v}\Big[\frac{|\mathbf{R}|}{\tau}\cosh\bar{v} + \frac{T}{\tau}\sinh\bar{v}\Big]\Big[e^{i\beta\tau\cosh\bar{v}} + e^{-i\beta\tau\cosh\bar{v}}\Big]\Bigg\}, \\
&= \frac{\Theta(T)\beta}{2\pi|\mathbf{R}|}\int_{-\infty}^{\infty}\mathrm{d}v\Big[-\frac{|\mathbf{R}|}{\tau}\cosh v + \frac{T}{\tau}\sinh v\Big]\Big[e^{i\beta\tau\cosh v} + e^{-i\beta\tau\cosh v}\Big] \\
&= -\frac{\Theta(T)\beta}{2\pi\tau}\int_{-\infty}^{\infty}\mathrm{d}v\Big[e^{i\beta\tau\cosh v - v} + e^{-i\beta\tau\cosh v - v}\Big]\,,
\end{aligned}
\tag{3.60}
$$

where the last equality follows by symmetry; it should be understood that the limits in the integrals above are such that the integrals are convergent.

We note the following integral representations [109] of the Hankel functions of order α of the first and second kind, respectively:

$$
\begin{aligned}
H_\alpha^{(1)}(x) &= \frac{e^{-i\alpha\frac{\pi}{2}}}{\pi i}\int_{-\infty-i\epsilon}^{\infty+i\epsilon}\mathrm{d}v\,e^{ix\cosh v - \alpha v}\,, \\
H_\alpha^{(2)}(x) &= -\frac{e^{+i\alpha\frac{\pi}{2}}}{\pi i}\int_{-\infty+i\epsilon}^{\infty-i\epsilon}\mathrm{d}v\,e^{-ix\cosh v - \alpha v}\,,
\end{aligned}
$$

related to the Bessel functions of first and second kind via the relations:

$$H_\alpha^{(1)}(x) = \mathcal{J}_\alpha(x) + i\mathcal{Y}_\alpha(x)\,, \tag{3.61}$$

$$H_\alpha^{(2)}(x) = \mathcal{J}_\alpha(x) - i\mathcal{Y}_\alpha(x)\,, \tag{3.62}$$

where $\mathcal{J}_\alpha(x)$ and $\mathcal{Y}_\alpha(x)$ are Bessel functions of the first and second kind respectively; *i.e.*, the two linearly independent solutions to Bessel's equation:

$$x^2\frac{d^2F}{dx^2}+x\frac{dF}{dx}+(x^2-\alpha^2)F=0\,. \tag{3.63}$$

In particular, we require the Hankel functions of order $\alpha=1$, which are given as

$$\begin{aligned} H_1^{(1)}(x) &= -\frac{1}{\pi}\int_{-\infty-i\epsilon}^{\infty+i\epsilon}\mathrm{d}v\,e^{ix\cosh v-v}\,,\\ H_1^{(2)}(x) &= -\frac{1}{\pi}\int_{-\infty+i\epsilon}^{\infty-i\epsilon}\mathrm{d}v\,e^{-ix\cosh v-v}\,. \end{aligned}$$

Thus, one can express the integral for $g_{1\mathrm{R}}$, Eq. (3.60), in terms of these Hankel functions as

$$\begin{aligned} g_{1\mathrm{R}}(x-x') &= \frac{\Theta(T)\beta}{2\pi\tau}\left\{\pi H_1^{(1)}(\beta\tau)+\pi H_1^{(2)}(\beta\tau)\right\}\,,\\ &= \Theta(T)\frac{\beta\mathcal{J}_1(\beta\tau)}{\tau}\qquad\text{for }\;T>|\mathbf{R}|\,. \end{aligned} \tag{3.64}$$

Note that, $\mathcal{J}_1(\beta\tau)$ is the Bessel function of first kind of order 1 and we have used Eq. (3.61) and Eq. (3.62) to arrive at the final expression, Eq. (3.64), above.

Looking now at the exterior of the light cone, *i.e.*, $|\mathbf{R}|>T$, we make the following change in variables in the first term in the integral of Eq. (3.56) above:

$$\begin{aligned} |\mathbf{R}||\mathbf{k}|+T\sqrt{|\mathbf{k}|^2+1} &= \xi\sinh v\,,\\ T|\mathbf{k}|+|\mathbf{R}|\sqrt{|\mathbf{k}|^2+1} &= \xi\cosh v\,,\\ k &= \frac{|\mathbf{R}|}{\xi}\sinh v-\frac{T}{\xi}\cosh v\,,\\ \frac{\mathrm{d}|\mathbf{k}|}{\sqrt{|\mathbf{k}|^2+1}} &= \mathrm{d}v\,. \end{aligned}$$

where as before $v\in[v_0,\infty)$, but now $v_0\equiv\operatorname{arsinh}(T/\xi)$ and we have defined $\xi=\sqrt{|\mathbf{R}|^2-T}$.

In the second term in the integral of Eq. (3.56) we perform the following change of variables:

$$
\begin{aligned}
|\mathbf{R}||\mathbf{k}|-T\sqrt{|\mathbf{k}|^2+1} &= \xi \sinh \bar{v}\,, \\
-T|\mathbf{k}|+|\mathbf{R}|\sqrt{|\mathbf{k}|^2+1} &= \xi \cosh \bar{v}\,, \\
k &= \frac{|\mathbf{R}|}{\xi}\sinh \bar{v} + \frac{T}{\xi}\cosh \bar{v}\,, \\
\frac{\mathrm{d}|\mathbf{k}|}{\sqrt{|\mathbf{k}|^2+1}} &= \mathrm{d}\bar{v}\,,
\end{aligned}
$$

where $\bar{v} \in [-v_0, \infty)$.

Upon using these change of variables we arrive at

$$
g_{1\mathrm{R}}(x-x') = -2\frac{\Theta(T)}{|\mathbf{R}|}\frac{T}{\xi}\delta(\sqrt{|\mathbf{R}|^2-T^2}\,)\,, \tag{3.65}
$$

which vanishes since we are explicitly considering the $|\mathbf{R}|> T$ region and thus the delta function vanishes.

We still have yet to determine the singular part of the Green's function on the light cone $|\mathbf{R}|= T$. To do this we will repeat the formalism established in Ref. [110]. Note that, we very well could have determined the full Green's function $g_{1\mathrm{R}}(x-x')$ and not just the part on the light cone via the formalism in Ref. [110]. In fact, a full review of the formalism will serve as a useful check on the calculation of the smooth part of the Green's function determined above via Fourier transform.

To begin, we first integrate the Green's function equation, Eq. (3.47), over a space-time volume which contains the source event $x' = 0$, namely

$$
\int_{\partial V} (g_1)^{;\mu}\mathrm{d}\Sigma_\mu - \beta^2 \int_V g_1 = 4\pi\,, \tag{3.66}
$$

where Gauss' theorem was used to arrive at the first term above and $\mathrm{d}\Sigma_\mu$ is the surface element of the boundary ∂V. Assuming $\int_V f_1$ vanishes as the integration volume vanishes, we are left with

$$
\lim_{V\to 0}\int_{\partial V} (g_1)^{;\mu}\mathrm{d}\Sigma_\mu = 4\pi\,. \tag{3.67}
$$

Now introduce the coordinates (w, χ, θ, ϕ) given by the following expressions

$$
\begin{aligned}
t &= w \cos\chi\,, \\
x &= w \sin\chi \sin\theta \cos\phi\,, \\
y &= w \sin\chi \sin\theta \sin\phi\,, \\
z &= w \sin\chi \cos\theta\,,
\end{aligned}
$$

such that the line element $\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$ of the flat background takes the form

$$
\begin{aligned}
\mathrm{d}s^2 &= -\cos 2\chi \mathrm{d}w^2 + 2w \sin 2\chi \mathrm{d}w \mathrm{d}\chi \\
&\quad + w^2 \cos 2\chi \mathrm{d}\chi^2 + w^2 \sin^2\chi \mathrm{d}\Omega^2\,,
\end{aligned} \tag{3.68}
$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$. In these coordinates, the surface ∂V is given by choosing constant w, and the Synge world function σ is

$$
\sigma = -\frac{1}{2} w^2 \cos 2\chi \;; \tag{3.69}
$$

notice that for time-like events $-2\sigma = \tau^2$ where τ is as previously defined.

The following quantities will be useful for what follows:

$$
\begin{aligned}
\sqrt{-g} &= w^3 \sin^2\chi \sin\theta\,, \\
g^{ww} &= -\cos 2\chi\,, \\
g^{w\chi} &= \frac{\sin 2\chi}{w}\,, \\
g^{\chi\chi} &= \frac{\cos 2\chi}{w^2}\,,
\end{aligned}
$$

where g is the determinant of the metric $g_{\alpha\beta}$. The only nonzero component of $\mathrm{d}\Sigma_\alpha$ is

$$
\mathrm{d}\Sigma_w = w^3 \sin^2\chi \mathrm{d}\chi \mathrm{d}\Omega\,, \tag{3.70}
$$

where $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$.

In these coordinates, the retarded Green's function is given by the following:

$$g_{1\mathrm{R}} \;=\; \Theta(w\cos\chi)g(\sigma)\;, \tag{3.71}$$

where $g(\sigma)$ is an as yet undetermined, possibly distributional, function. We will only need the following gradient of $g_{1\mathrm{R}}$ (omitting the label $_\mathrm{R}$)

$$\begin{aligned}(g_1)^{;w} &= g^{w\mu}(g_1)_{;\mu}\\ &= g^{ww}(g_1)_{;w} + g^{w\chi}(g_1)_{;\chi}\,. \end{aligned} \tag{3.72}$$

A straight-forward calculation leads to

$$\begin{aligned}(g_1)^{;w} &= -\delta(w\cos\chi)\cos\chi g(\sigma)\\ &\quad +w\Theta(w\cos\chi)g'(\sigma)\,, \end{aligned} \tag{3.73}$$

where the prime on $g(\sigma)$ denotes differentiation with respect to σ. We then have

$$\begin{aligned}\int_{\partial V}(g_1)^{;\mu}\mathrm{d}\Sigma_\mu &= \int_{\partial V} w^3\sin^2\chi\mathrm{d}\chi\mathrm{d}\Omega\big\{w\Theta(w\cos\chi)g'(\sigma)\\ &\qquad -\delta(w\cos\chi)\cos\chi g(\sigma)\big\}\\ &= 4\pi w^4\int_0^{\frac{\pi}{2}}\sin^2\chi\mathrm{d}\chi g'(\sigma)\,, \end{aligned} \tag{3.74}$$

where the Heaviside function has restricted the limits of χ integration such that $\cos\chi \geq 0$ and the delta term vanishes.

Changing integration variable from χ to σ in the integral above we arrive at the following condition on $g(\sigma)$:

$$\lim_{\epsilon\to 0}\epsilon\int_{-\epsilon}^{\epsilon}\mathrm{d}\sigma\,\Xi\left(\frac{\sigma}{\epsilon}\right)g'(\sigma) = 1\,, \tag{3.75}$$

where

$$\begin{aligned}\epsilon &\equiv \frac{1}{2}w^2\,,\\ \Xi\left(\frac{\sigma}{\epsilon}\right) &\equiv \sqrt{\frac{1+\frac{\sigma}{\epsilon}}{1-\frac{\sigma}{\epsilon}}}\,. \end{aligned}$$

We now propose the following ansatz for $g(\sigma)$:

$$\begin{aligned} g(\sigma) &= V(\sigma)\Theta(-\sigma) + A\delta(\sigma) + B\delta'(\sigma) \\ &\quad + C\delta''(\sigma) + D\delta'''(\sigma) + \ldots\,, \end{aligned} \tag{3.76}$$

where $V(\sigma)$ is a smooth function of σ and $A, B, \ldots$ are constants. Inserting the ansatz Eq. (3.76) into Eq. (3.75) gives

$$\begin{aligned} 1 &= \lim_{\epsilon \to 0} \epsilon \left\{ \int_{-\epsilon}^{\epsilon} \mathrm{d}\sigma\, \Xi\left(\frac{\sigma}{\epsilon}\right) V'(\sigma)\Theta(-\sigma) - \Xi\,(0)\, V(0) \right. \\ &\quad \left. - \frac{A}{\epsilon}\dot{\Xi}\,(0) + \frac{B}{\epsilon^2}\ddot{\Xi}\,(0) - \frac{C}{\epsilon^3}\dddot{\Xi}\,(0) + \frac{D}{\epsilon^4}\Xi^{(4)}\,(0) + \ldots \right\}. \end{aligned} \tag{3.77}$$

The first two terms on the left-hand side of Eq. (3.77) above vanish, since $V(\sigma)$ is assumed to be smooth. The limit exist as long as $B = C = \ldots = 0$ and the condition Eq. (3.75) will be satisfied iff $A = -1$, since $\dot{\Xi}\,(0) = 1$. To fully determine the smooth part of $g(\sigma)$ one needs then only to solve the homogeneous equation (*i.e.*, $x \neq x'$):

$$(\Box - \beta^2)g(\sigma) = 4g'(\sigma) + 2\sigma g''(\sigma) - \beta^2 g(\sigma) \;=\; 0\,,$$

from which it is straight forward to verify that (the reader is referred to Ref. [110] for further details)

$$V(\sigma) = \frac{\beta \mathcal{J}_1(\beta\sqrt{-2\sigma})}{\sqrt{-2\sigma}}\,. \tag{3.78}$$

Returning to the original coordinates (t, r, θ, ϕ) we then have

$$\begin{aligned} g_{1\mathrm{R}}(x - x') &= -\delta(\frac{1}{2}\tau^2)\Theta(T) + \frac{\beta\mathcal{J}_1(\beta\tau)}{\tau}\Theta\left(\frac{1}{2}(T^2 - |\mathbf{R}|^2)\right)\Theta(T)\,, \\ &= -2\delta(\tau^2)\Theta(T) + \frac{\beta\mathcal{J}_1(\beta\tau)}{\tau}\Theta\left(T - |\mathbf{R}|\right)\Theta(T)\,, \end{aligned}$$

where we have used some standard properties of the Dirac delta function and the Heavyside step function.

To determine $g_{2\mathrm{R}}(x-x')$ we need to consider the integral

$$g_{2\mathrm{R}}(x-x') = \frac{1}{4\pi^3}\int \mathrm{d}\omega \mathrm{d}^3\mathbf{k}\frac{e^{-i\omega T}e^{i\mathbf{k}\cdot\mathbf{R}}}{\big[(\omega+i\epsilon+|\mathbf{k}|)(\omega+i\epsilon-|\mathbf{k}|)\big]}\,, \tag{3.79}$$

where the pole prescription is again that of the retarded Green's function. A straight-forward calculation leads to

$$\begin{aligned} g_{2\mathrm{R}}(x-x') &= \frac{\Theta(T)}{|\mathbf{R}|}\big\{\delta((T+|\mathbf{R}|)-\delta(T-|\mathbf{R}|)\big\}\,, \\ &= -2\Theta(T)\delta(\tau^2)\,. \end{aligned} \tag{3.80}$$

Finally, the Green's function which satisfies the fourth-order wave equation Eq.-(3.42) is given by

$$\begin{aligned} G_{\mathrm{R}}(x-x') &= \frac{1}{\beta^2}\big(g_{1\mathrm{R}}(x-x')-g_{2\mathrm{R}}(x-x')\big)\,, \\ &= \frac{\beta\mathcal{J}_1(\beta\tau)}{\beta\tau}\Theta(T-|\mathbf{R}|)\Theta(T)\,. \end{aligned} \tag{3.81}$$

Note that the Green's function above is subject to the initial value condition:

$$\frac{\partial G_{\mathrm{R}}(x-x')}{\partial t}\Big|_{t=0} = 0\ .$$

3.6 Solutions to the Fourth Order Wave Equation

As detailed in the previous section the Green's function for the fourth-order wave equation we are interested in here (*i.e.* Eq. (3.42)) are determined by the Green's functions for the two second-order wave equations, Eq. (3.43) and Eq. (3.44) given respectively as

$$g_{1\mathrm{R}} = \left\{\frac{\beta\mathcal{J}_1(\beta\tau)}{\tau}\Theta(cT-|\mathbf{R}|)-2\delta(\tau^2)\right\}\Theta(T)\,, \tag{3.82}$$

$$g_{2\mathrm{R}} = -2\delta(\tau^2)\Theta(T)\,, \tag{3.83}$$

where as before we have defined

$$\begin{aligned} T &\equiv t - t' \\ \mathbf{R} &\equiv \mathbf{r} - \mathbf{r}' \\ \tau &\equiv \sqrt{(cT)^2 - |\mathbf{R}|^2}\ , \end{aligned}$$

and $\mathcal{J}_1(x)$ is the first order Bessel function of the first kind. Thus, Eq. (3.45) implies that

$$G_{\mathrm{R}}(x, x') = \frac{\mathcal{J}_1(\beta\tau)}{\beta\tau}\Theta(cT - |\mathbf{R}|)\Theta(T)\,. \tag{3.84}$$

Note the absence of the delta function singularities on the light cone, consistent with the general analysis detailed in Ref. [111].

Using Eq. (3.41) one finds that the field is given by

$$\begin{aligned} h^{\mu\nu}(\mathbf{r}, t) = \frac{4G\beta}{c^4}\int \mathrm{d}\mathbf{r}'\mathrm{d}t' \frac{\Theta(T)}{\sqrt{(cT)^2 - |\mathbf{R}|^2}} \\ \times \mathcal{J}_1\left(\beta\sqrt{(cT)^2 - |\mathbf{R}|^2}\right) T^{\mu\nu}(\mathbf{r}', t') \\ \times\Theta(cT - |\mathbf{R}|)\ . \end{aligned} \tag{3.85}$$

One thus finds that the field is sourced only by regions *within* our past light code (*i.e.*, $cT > |\mathbf{R}|$), which is expected for the propagation of a (positive-real) massive field. Notice that if $\beta^2 < 0$, corresponding to $\alpha_0 > 0$, we find that an observed field is sourced from regions with space-like separation. This is due to the fact that $\beta^2 < 0$ corresponds to a tachyon (complex mass) mode of the gravitational wave.

If we consider the far-field limit, *i.e.* $|\mathbf{r}| \approx |\mathbf{r} - \mathbf{r}'|$, we can write the spatial components of this field as

$$\begin{aligned} h^{ik}(\mathbf{r}, t) \approx \frac{2G\beta}{3c^4}\int_{-\infty}^{t - \frac{1}{c}|\mathbf{r}|} \frac{\mathrm{d}t'}{\sqrt{c^2(t - t')^2 - |\mathbf{r}|^2}} \\ \times\mathcal{J}_1\left(\beta\sqrt{c^2(t - t')^2 - |\mathbf{r}|^2}\right)\ddot{D}^{ik}(t')\ , \end{aligned} \tag{3.86}$$

where we have, as usual, introduced the quadrupole moment,

$$D^{ik}\left(t\right) \equiv \frac{3}{c^2}\int \mathrm{d}\mathbf{r}\; x^i x^k T^{00}(\mathbf{r},t)\;, \tag{3.87}$$

the second time derivative of which is given by

$$\int \mathrm{d}\mathbf{r} T^{ik}\left(\mathbf{r},t\right) = -\frac{1}{6}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[D^{ik}(t)\right]\;. \tag{3.88}$$

In conclusion, just as in the case of General Relativity, we find that the gravitational waves are only sourced from systems with a nontrivial quadrupole moment. This is essentially due to the conservation of the energy momentum tensor, which is unaltered in this theory. However, the propagation of gravitational waves is significantly altered, by the presence of additional massive modes.

3.7 Examples

As a simple pedagogical example let us consider the case of $\ddot{D}^{ik} \approx$ constant, for which one can explicitly perform the integral in Eq. (3.86) to find

$$\begin{aligned} h^{ik}\left(\mathbf{r},t\right) \;\approx\; & -\frac{2G}{3c^2|\mathbf{r}|}\ddot{D}^{ik}\Big|_{\text{fixed}} \\ & \times\left[\sinh\left(\beta|\mathbf{r}|\right) - \cosh\left(\beta|\mathbf{r}|\right) + 1\right]\;. \end{aligned} \tag{3.89}$$

Thus, one recovers the standard result in the $\beta \to \infty$ limit, namely

$$\begin{aligned} \lim_{\beta\to\infty} h^{ik}\left(\mathbf{r},t\right) &= \lim_{\beta\to\infty}\left[-\frac{2G}{3c^4|\mathbf{r}|}\ddot{D}^{ik}\Big|_{\text{fixed}}\left(1-e^{-\beta|\mathbf{r}|}\right)\right] \\ &= {}^{(\mathrm{GR})}h^{ik}\left(\mathbf{r},t\right)\lim_{\beta\to\infty}\left(1-e^{-\beta|\mathbf{r}|}\right), \end{aligned} \tag{3.90}$$

where ${}^{(\mathrm{GR})}h^{ik}$ denotes the field in the General Relativistic case. This is expected, since the $\beta \to \infty$ limit corresponds to taking $\alpha_0 \to 0$ and, as it can be seen from either Eq. (3.10) or Eq. (3.13), one then recovers the Einstein-Hilbert result.

The next simplest example is a system with a periodically varying quadrupole moment, *i.e.*

$$\ddot{D}^{ij}\left(t\right) = A^{ij}\cos\left(\omega^{ij}t + \phi^{ij}\right)\;, \tag{3.91}$$

where A^{ij} is a constant in time, ω^{ij} is the frequency of the oscillations of the ij component and ϕ^{ij} is the phase, both of which we consider to be time independent. No summation over i, j is implied. Using this in Eq. (3.86) one finds

$$\begin{aligned} \dot{h}^{ij} &= \frac{4G\beta A^{ij}\omega^{ij}}{3c^4}\left[\sin\left(\omega^{ij}t+\phi^{ij}\right) f_c\left(\beta|\mathbf{r}|,\frac{\omega^{ij}}{\beta c}\right)\right. \\ &\left. + \cos\left(\omega^{ij}t+\phi^{ij}\right) f_s\left(\beta|\mathbf{r}|,\frac{\omega^{ij}}{\beta c}\right)\right] , \end{aligned} \tag{3.92}$$

where again no summation is implied and we have defined the functions,

$$\begin{aligned} f_{\mathrm{s}}(x,z) &\equiv \int_0^\infty \frac{\mathrm{d}s}{\sqrt{s^2+x^2}}\mathcal{J}_1(s)\sin\left(z\sqrt{s^2+x^2}\right) , \\ f_{\mathrm{c}}(x,z) &\equiv \int_0^\infty \frac{\mathrm{d}s}{\sqrt{s^2+x^2}}\mathcal{J}_1(s)\cos\left(z\sqrt{s^2+x^2}\right) . \end{aligned} \tag{3.93}$$

These functions are highly oscillatory, with somewhat different behavior for $z > 1$ and $z < 1$. Because they have a typical frequency of the order of z, which is (in general) different to ω^{ij}, the wave-form of the gravitational radiation, Eq. (3.86) and its time derivative, Eq. (3.92) can experience beat phenomena. In particular, interference between the various functions can result in a significant enhancement of the amplitude.

As a specific example, of significant physical interest, consider a pair of masses m_1 and m_2, in a circular binary system, under the assumption that the internal structure of the bodies can be neglected. For such a system, orbiting in the xy-plane, one finds that the only nonzero components of the quadrupole are [66],

$$\begin{aligned} \dddot{D}^{xx}(t) &= 12\mu|\rho|^2\sin\left(2\psi(t)\right)\omega^3 \\ &= -\dddot{D}^{yy}(t) , \\ \dddot{D}^{xy}(t) &= -12\mu|\rho|^2\cos\left(2\psi(t)\right)\omega^3 , \\ D^{zz} &= -\mu|\rho|^2 , \end{aligned} \tag{3.94}$$

where $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass of the system, $|\rho|$ is the magnitude of the separation vector between the bodies, which is constant for circular orbits, ψ

is the angle of the bodies relative to the x-axis and $\omega = \dot{\psi}$ is the orbital frequency, which for this simple system is a constant given by

$$\omega \equiv \dot{\psi} = |\rho|^{-3/2}\sqrt{G\left(m_1 + m_2\right)} \ . \tag{3.95}$$

Using Eq. (3.86) one finds

$$\begin{aligned}\dot{h}^{ij}\dot{h}_{ij} = & \frac{9\mu^2|\rho|^2\omega^4G^2\beta^2}{c^6} \\ & \times\left[f_{\rm c}^2\left(\beta|\mathbf{r}|,\frac{2\omega}{\beta c}\right) + f_{\rm s}^2\left(\beta|\mathbf{r}|,\frac{2\omega}{\beta c}\right)\right] \ .\end{aligned} \tag{3.96}$$

Following the standard approach (*see e.g.*, Ref. [66]), one finds that the rate of energy loss from a system, in the far field limit, is given by[4]

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \approx \frac{c^2}{20G}|\mathbf{r}|^2\dot{h}_{ij}\dot{h}^{ij} \ . \tag{3.97}$$

This allows us to explicitly test the approximation against binary pulsar measurements, for which the energy loss has been very well characterized [113].

For a quantitative fit to the data one would have to extend this example to non-circular orbits and also account for tidal and other near field effects. However, such calculations rapidly become rather involved and even within this simple (and important) system one can derive some general consequences.

The functions in Eq. (3.93) are highly resonant at $z = 1$, which corresponds to an orbital frequency $\omega = \beta c/2$, however they are readily calculated for both $z < 1$ and $z > 1$. In these regions the functions given in Eq. (3.93) can be evaluated numerically and fitted to an explicit functional forms. For $\omega < \beta c/2$ this gives

$$\begin{aligned}& \left[f_{\rm c}\left(\beta|\mathbf{r}|,\frac{2\omega}{\beta c}\right)\right]^2 + \left[f_{\rm s}\left(\beta|\mathbf{r}|,\frac{2\omega}{\beta c}\right)\right]^2 \\ \approx & \frac{1}{\left(\beta|\mathbf{r}|\right)^2}\exp\left(\frac{C}{\beta|\mathbf{r}|\left(1-\frac{2\omega}{c\beta}\right)}\mathcal{J}_1\left(\beta|\mathbf{r}|-\frac{2\omega}{c\beta}\right)\right) \ ,\end{aligned} \tag{3.98}$$

[4]For an explicit verification that the rate of energy loss Eq. (3.97) is not altered through a modification to the effective gravitational wave stress-energy tensor in the modified gravity theory considered here see [112].

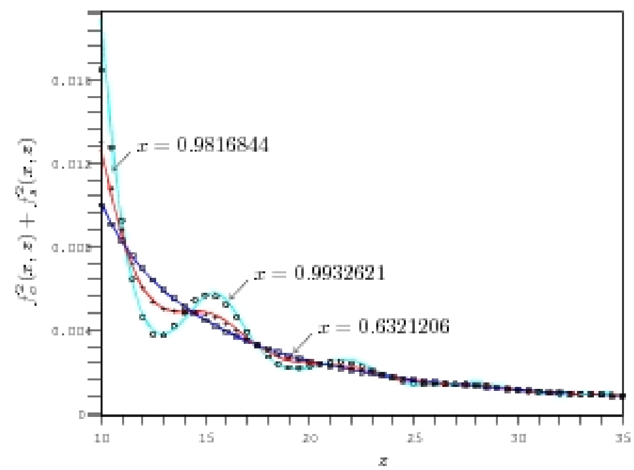

Figure 3.2. The points are the numerical evaluation of $f_{\rm c}^2(x,z) + f_{\rm s}^2(x,z)$ (for three different values of $z < 1$) and the lines are plots of the corresponding fitted function given in Eq. (3.98). Notice that the fitting function breaks down as we approach $z \to 1$, which corresponds to $2\omega \to \beta c$.

where $C \approx 0.175$ is approximately a constant except as $2\omega \to \beta c$. In Fig. 3.2 we illustrate some examples of this approximation and show that the approximation is good even for $2\omega/\beta c \approx 0.99$.

While for $\omega > \beta c/2$ one finds

$$\begin{aligned} & \left[f_{\rm c}\left(\beta|\mathbf{r}|, \frac{2\omega}{\beta c}\right)\right]^2 + \left[f_{\rm s}\left(\beta|\mathbf{r}|, \frac{2\omega}{\beta c}\right)\right]^2 \\ \approx \quad & \frac{4}{(\beta|\mathbf{r}|)^2} \sin^2\left(\beta|\mathbf{r}|\left(\tilde{f}\left(\frac{2\omega}{\beta c}\right)\right)^{-1}\right) \ , \end{aligned} \tag{3.99}$$

where for the function $\tilde{f}$ is approximately

$$\begin{aligned} \tilde{f}\left(\frac{2\omega}{\beta c}\right) \ \approx \ & 4\sqrt{\left(\frac{2\omega}{\beta c}\right)^2 - 1} \\ & +2\exp\left(-\sqrt{\left(\frac{2\omega}{\beta c}\right)^2 - 1}\right) \ . \end{aligned} \tag{3.100}$$

Figure 3.3 shows some examples of the quality of this fit, which, just as for Eq. (3.98), is best away from $\omega = \beta c/2$, however remains very good even as one

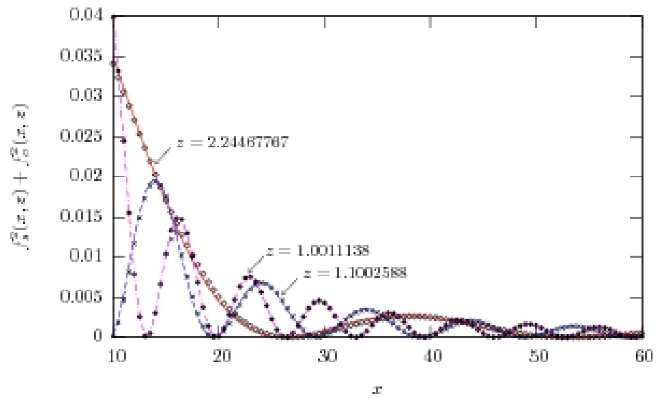

Figure 3.3. The points are the numerical evaluation of $f_\mathrm{c}^2(x,z) + f_\mathrm{s}^2(x,z)$ (for three different values of $z > 1$) and the lines are plots of the corresponding fitted function given in Eq. (3.99). Notice that the fitting function remain a good approximation even as $z \to 1$, which corresponds to $2\omega \to \beta c$.

approaches this limit. It is important to note that Eq. (3.99) does not approximate the General Relativity solution for $\beta \to \infty$, since, in this limit, all (finite) orbital frequencies satisfy $\omega < \beta c/2$.

Using the approximation given in Eq. (3.98), one can check that, for slow orbital frequencies, the expected result of General Relativity is indeed recovered in the $\beta \to \infty$ limit. Specifically, we can expand Eq. (3.97) in the large distance (large $|\mathbf{r}|$) and small orbital frequency (*i.e.*, $2\omega \ll c\beta$) limit, to find the first order corrections to the standard result of General Relativity, namely

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \approx \frac{32G\mu^2\rho^2\omega^6}{5c^5}\left[1 + \frac{C}{\beta|\mathbf{r}|\left(1 - \frac{2\omega}{\beta c}\right)}\mathcal{J}_1\left(\beta|\mathbf{r}| - \frac{2\omega}{\beta c}\right) + \dots\right] . \qquad (3.101)$$

Thus, the $\beta \to \infty$ (*i.e.*, $\alpha_0 \to 0$) limit reproduces the General Relativity result, as it should. Also note that any deviation from the standard result is suppressed by the distance to the source, at least for orbital frequencies small compared to βc. Although the amplitude of the deviation from the standard result is small, there are two interesting features: firstly, the existence of a maximum frequency βc and secondly, the fact that the deviation is oscillatory.

The maximum frequency comes from the fact that, in addition to a natural speed c, this theory has a natural length, given by $\beta^{-1} = \sqrt{-\alpha_0 G}$. This natural length scale comes from the first two moments of the test function used to define the spectral action, Eq. (3.7). Physically, one can think of this as the scale at which noncommutative effects become dominant. This is extremely suggestive of an underlying maximum frequency, which would have rather significant consequences for the particle physics sector of the theory, in particular for renormalization. However, it must be remembered that here we are working with one simple system and even in this case Eqs. (3.98)–(3.99) are numerical approximations. Nonetheless, the existence of a maximum frequency in this system allows NCG effects to significantly enhance the production of gravitational radiation. This is particularly important given the suppression with $1/|\mathbf{r}|$, that is present in Eq. (3.101).

The presence of the Bessel function in Eq. (3.101) means that the amplitude of the deviation from the standard result of General Relativity will oscillate both with changing distances and changing frequencies. This allows for a myriad of possible observational signatures, such as distinct beats of the observed energy loss of binary pulsars, correlated to their changing orbital frequency and the distance to the binary. The fact that a similar phenomena occurs for the gravitational wave itself, Eq. (3.86), implies that there would also be a beat structure in direct detection observations. In the case of a binary pair the amplitude of the beat will be heavily suppressed compared to that of the carrier wave and thus it is likely to remain below observational sensitivity, except in systems with very large orbital frequency.

3.8 Conclusions and Disscussion

NonCommutative Geometry is a natural extension to our familiar notions of Riemannian geometry, that has the additional benefit of producing the action of all the Standard Model fields in addition to gravity terms, purely through geometrical considerations. Thus NCG treats both gravity and matter on an equal footing and provides us with concrete relationships between matter and gravitational couplings. The gravitational sector of (the asymptotic expansion of) this theory produces modifications to General Relativity and in this chapter we explore the ramifications of these modifications on the formation and evolution of gravitational waves.

We have shown that the theory contains both massive and massless gravitons and that the requirement that the mass of these gravitons be positive fixes the sign of one of the couplings in the theory (for a given choice of sign conventions). We also show that both these modes are sourced by the quadrupole moment of a system (just as in standard GR) and that the retarded Green's function is not restricted to the past light cone of the observer (unlike GR), as one would expect for a system with massive modes. We have explicitly calculated the energy loss for a circular binary system and compared the results to those of standard GR. We have demonstrated that the amplitude of these NCG modifications is suppressed by the distance between the observer and the source of the gravitational waves and hence will typically be small.

Despite the extremely small amplitude of deviations from standard results, we have shown that NCG produces several distinctive features. Firstly, the amplitude of the energy lost by a binary pair can be higher or lower than the expected value, depending on the orbital period of the pair and the distance to the observer. This opens up the possibility that the observed energy loss from such a pair would be seen to oscillate as the binary moves with respect to the Earth. While such effects are likely to be beyond current observational resolution, they allow for an unexpected beat phenomenon, which would be a concrete signature of NCG.

In addition, we have shown that the amplitude of these effects is (approximately) proportional to $(1 - 2\omega/c\beta)^{-1}$, where ω is the orbital frequency of the binary. Thus, it would appear that the NCG corrections to the energy loss by the binary can become arbitrarily large as the frequency of the binary approach the critical frequency βc. In such a regime, the weak field approach taken here would no longer be valid (and numerical approximations break down), so one would not trust systems very close to this limit, however it is certainly true that astrophysical constraints on the parameters of the theory will be significantly improved for objects with a very rapidly changing quadrupole moment. A precise understanding of such systems is likely to require detailed knowledge of various astrophysical effects (radiation and particle production, tidal stripping etc.) as well as analytic solutions to the graviton field in the large frequency regime.

Finally, the form of Eq. (3.92) suggests that similar behaviour may be present in other systems, with periodic, or almost periodic, variations in the (mass) quadrupole moment. For laboratory systems, the gravitational radiation predicted by General Relativity is negligible, however if the NCG enhancement were sufficiently large, this may no longer be true. Laboratory systems regularly have very large oscillation frequencies (e.g., lattice vibrations in solids can easily exceed 10^{12}Hz) which would experience anomalous damping, if the system was producing significant amounts of gravitational radiation. This opens up the (remote) possibility that the noncommutative nature of space-time might be probed in the laboratory.

One can immediately use the results of this chapter to examine circular binary systems, in order to constrain the value of β [3]. Similarly, one can include eccentricity which may result in more restrictive constraints on the theory. An alternative avenue would be to use the gravitational wave-forms given here to deduce the consequences for direct gravity wave searches (LIGO, VIRGO, LISA, *etc*). In particular, to extend these result to the large field regime and look for modifications to the *chirp* that develops at the end of in-spiral events.

Part III

Conclusions

Chapter 4

General Conclusions and Discussion

While I have considered two very distinct projects in this thesis they may loosely fall under the heading of quantum gravitational phenomenology. The need for the study and cataloging of any possible physical signatures of quantum gravity effects should be apparent; the theoretical task of formulating a consistent theory of quantum gravity is undoubtedly a difficult one.

In addition to reproducing the well known optical activity effects induced by the Lorentz violating interaction, Eq. 2.1, I have demonstrated that the very same interaction may generate circular polarization in the anisotropies of CMB photons at second-order in the interaction coupling. While the generation of circular polarization, V, is sub-dominant to the mixing of the linear polarization parameters Q and U as discussed in Sec. 2.9, I believe such a signal should be understood experimentally and encourage experimenters to verify the hierarchy of signal strengths and to place tighter physical constraints. Furthermore, following the publication of our original work on the subject, subsequent study by other investigators [114, 115] seem to indicate that the generation of circular polarization is a generic consequence of many Lorentz violating interactions in extended QED models. Together with the work presented here, these studies will hopefully add further encouragement to experimenters to look in detail at any possible signal of circularly polarized CMB radiation, in addition to more well known standard sources, such as primordial and intergalactic magnetic fields as well as atmospheric effects.

On the theoretical front, future work should be done to try and build microphysical and/or cosmological models which attempt to explain the possible presence of Lorentz violating interactions of the type studied in this thesis. That such effects may have explanations in physics lying within the 'desert' between the electroweak energy scale and the natural quantum gravitational Planck scale should be left open as a real possibility.

The benefits of broadening at both low and high energies our understanding of the role of Lorentz symmetry, a symmetry that is at the core of all of modern physics, will undoubtedly be many. There are open theoretical questions as to whether one should even expect to see signs of Lorentz violation at low energy [116] and as to the role that such a signature can play in quantum gravitational phenomenology [117]. With the ongoing theoretical pursuits to solve the quantum gravity problem continuing, any phenomenology should indeed be welcomed as a useful tool. Ultimately the question of CMB circular polarization will decided by experiment in the near future as higher precision data is collected of CMB anisotropies.

Likewise, the near future will see the many gravitational wave detectors up and running and the first signals of gravitational waves will unarguably be the final linchpin in supporting the predictions of Einstein's theory of general relativity. As we have shown, the effective spectral action of NCG makes a prediction distinct from GR for the propagating gravitational wave degrees of freedom. Although surely not a unique feature of NCG theories, there should be confidence that gravitational wave signals may hold some key to solving the problem of quantum gravity [118]. Second and third generation gravitational wave detectors will surely be a useful tool in, just as Einstein's theory of general relativity supplanted Newton's theory of gravity, guiding physics to what may come next.

While the direct detection of gravitational waves will surely be helpful in placing constraints on the parameters of NCG, another prospect for future work is to consider the role that the modifications to gravitational wave solutions derived here may have on the correlations of anisotropies in the CMB radiation. As is well known, gravitational wave perturbations lead to the very particular $B-$mode polarization in the CMB [10,28]. In fact, the uncovering of such modes of polarization and their correlations with temperature and $E-$mode polarization anisotropies in the CMB can be considered an indirect confirmation for the existence of primordial gravitational waves and of course gravitational radiation in general. It would certainly be interesting to investigate the implications that the modified gravitational wave solutions of Chapter 3 would have on the $B-$mode signal and correlations in the CMB.

Finally, future work can also be done on analyzing spherically symmetric solutions of the gravitational sector of the effective spectral action of NCG Eq. (3.10) and thus determine any modifications which might be present to the standard Schwarzschild solution of GR. Parameterizing the metric in order to impose spherical symmetry leads to equations of motion which do not seem to be particularly amendable to analytic methods and thus numerical investigation certainly seems appropriate. Such a modified Schwarzschild solution may lead to the placing of tighter constraints on the parameters of NCG than was possible in the analysis of Chapter 3.

Part IV

Appendices

Appendix **A**

Gauge Invariance of Second Order Scattering Effects Due to T_μ

Here we verify that the calculation in section 2.7 is gauge invariant by explicitly using a different gauge-fixed photon propagator, namely

$$D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-ie^{-ik\cdot(x-y)}}{k^2+i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{A.1}$$

Isolating the contribution to $\left\langle \left[{}^2\hat{H}_{\rm int}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right\rangle$ due to the term linear in $(1-\xi)$ gives

$$\begin{aligned}\left\langle \left[{}^2\hat{H}_{\rm int}(t)_\xi, \hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right\rangle = &-\frac{i(2\pi)^3}{2}(2g)^2 \int dl_1 dl_2\, \delta^3(\mathbf{l}_1+\mathbf{l}_2)\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa} \\ &\times \tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(i(1-\xi))e^{i({l_1}^0+{l_2}^0)t} \\ \times\big(\delta_{ur}\rho_{sv}(\mathbf{q}) - \delta_{vs}\rho_{ur}(\mathbf{q})\big)&\left[\frac{(q+l_2)_\mu(q+l_2)_\rho}{(q+l_2)^2}(2q+l_2)_\alpha(2q+l_2)_\lambda\frac{\epsilon^*_{s\kappa}(q)\epsilon_{s\beta}(q)}{(q+l_2)^2+i\epsilon}\right. \\ &\left.+\frac{(q-l_2)_\mu(q-l_2)_\rho}{(q-l_2)^2}(2q-l_2)_\alpha(2q-l_2)_\lambda\frac{\epsilon^*_{s\beta}(q)\epsilon_{r\kappa}(q)}{(q-l_2)^2+i\epsilon}\right],\end{aligned} \tag{A.2}$$

where we have performed all the integrals similar to those in arriving at Eq. (2.87). Simplify the above by making repeated use of the anti-symmetry of the epsilon tensor:

$$\left\langle \left[{}^2\hat{H}_{\rm int}(t)_\xi, \hat{\mathcal{D}}_{uv}(\mathbf{q})\right]\right\rangle = -\frac{i(2\pi)^3}{2}(2g)^2 \int dl_1 dl_2\, \delta^3(\mathbf{l}_1+\mathbf{l}_2)\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}$$

$$
\begin{aligned}
&\times \tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(i(1-\xi))e^{i({l_1}^0+{l_2}^0)t} \\
&\times\big(\delta_{ur}\rho_{sv}(\mathbf{q})-\delta_{vs}\rho_{ur}(\mathbf{q})\big)\bigg[\frac{(q+l_2)_\mu(q+l_2)_\rho}{(q+l_2)^2}q_\alpha q_\lambda\frac{\epsilon^*_{s\kappa}(q)\epsilon_{s\beta}(q)}{(q+l_2)^2+i\epsilon} \\
&\qquad+\frac{(q-l_2)_\mu(q-l_2)_\rho}{(q-l_2)^2}q_\alpha q_\lambda\frac{\epsilon^*_{s\beta}(q)\epsilon_{r\kappa}(q)}{(q-l_2)^2+i\epsilon}\bigg], \\
&= -\frac{i(2\pi)^3}{2}(2g)^2\int dl_1 dl_2\,\delta^3(\mathbf{l}_1+\mathbf{l}_2)\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(i(1-\xi))e^{i({l_1}^0+{l_2}^0)t} \\
&\times\big(\delta_{ur}\rho_{sv}(\mathbf{q})-\delta_{vs}\rho_{ur}(\mathbf{q})\big)\bigg[\frac{(l_2)_\mu(l_2)_\rho}{(q+l_2)^2}q_\alpha q_\lambda\frac{\epsilon^*_{s\kappa}(q)\epsilon_{s\beta}(q)}{(q+l_2)^2+i\epsilon}+ \\
&\qquad\frac{(l_2)_\mu(l_2)_\rho}{(q-l_2)^2}q_\alpha q_\lambda\frac{\epsilon^*_{s\beta}(q)\epsilon_{r\kappa}(q)}{(q-l_2)^2+i\epsilon}\bigg], \\
&= -\frac{i(2\pi)^3}{2}(2g)^2\int dl_1 dl_2\,\delta^3(\mathbf{l}_1+\mathbf{l}_2)(i(1-\xi))\big(\delta_{ur}\rho_{sv}(\mathbf{q})-\delta_{vs}\rho_{ur}(\mathbf{q})\big)e^{i({l_1}^0+{l_2}^0)t} \\
&\times\bigg(\epsilon^{\mu\nu\alpha\beta}(l_2)_\mu\tilde{T}_\nu(l_1)q_\alpha\bigg)\bigg(\epsilon^{\rho\sigma\lambda\kappa}(l_2)_\rho\tilde{T}_\sigma(l_2)q_\lambda\bigg)\bigg[\frac{\epsilon^*_{s\kappa}(q)\epsilon_{s\beta}(q)}{(q+l_2)^2(q+l_2)^2+i\epsilon} \\
&\qquad+\frac{\epsilon^*_{s\beta}(q)\epsilon_{r\kappa}(q)}{(q-l_2)^2(q-l_2)^2+i\epsilon}\bigg], \\
&= 0.
\end{aligned}
$$

This expression vanishes because $\epsilon^{\rho\sigma\lambda\kappa}(l_2)_\rho\tilde{T}_\sigma(l_2)=0$, as we have required in order to arrive at Eq. (2.89). Therefore, the final evolution equations for the photon density matrix ρ will indeed be independent of the gauge parameter ξ.

Appendix B

The Ricci Sign Conventions

In this appendix we review the sign conventions which are used in Chapter 3 of this work. The Riemann tensor is given by

$$\left[\bigtriangledown_\mu, \bigtriangledown_\nu\right]\omega_\rho = R_{\mu\nu\rho}{}^\sigma \omega_\sigma \,.$$

The Einstein-Hilbert action of GR is defined by

$$I_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\,R\,.$$

The Weyl tensor is given as:

$$C_{\mu\lambda\nu\kappa} = R_{\mu\lambda\nu\kappa} + (g_{\mu[\nu}R_{\kappa]\lambda} - g_{\lambda[\nu}R_{\kappa]\mu}) - \frac{1}{3}g_{\mu[\nu}g_{\kappa]\lambda}R\,.$$

The contracted Bianchi identity is given as

$$\bigtriangledown^\kappa R_{\mu\lambda\nu\kappa} = -(\bigtriangledown_\lambda R_{\mu\nu} - \bigtriangledown_\mu R_{\lambda\nu})\,.$$

Using the contracted Bianchi identity above, Eq. (B.1), yields the following relationship:

$$\bigtriangledown^\lambda \bigtriangledown^\kappa R_{\mu\lambda\nu\kappa} \;\; = \;\; -(\bigtriangledown^\lambda \bigtriangledown_\lambda R_{\mu\nu} - \bigtriangledown^\lambda \bigtriangledown_\mu R_{\lambda\nu})\,.$$

The following is independent of convention for definition of the Ricci tensor since the sign from convention cancels from each term as expected,

$$\bigtriangledown^{\kappa} R_{\lambda\kappa} \;=\; \frac{1}{2} \bigtriangledown_{\lambda} R\,.$$

The following calculation explicitly shows that $\bigtriangledown^{\lambda} \bigtriangledown^{\kappa} C_{\mu\lambda\nu\kappa}$ is independent of sign convention of the Ricci tensor (to demonstrate this I do not fix the sign). The following relationships will be useful:

$$\begin{aligned}
\bigtriangledown_{\lambda} \bigtriangledown_{\mu} R^{\rho}{}_{\nu} &= \bigtriangledown_{\mu} \bigtriangledown_{\lambda} R^{\rho}{}_{\nu} + R_{\lambda\mu\nu}{}^{\kappa} R^{\rho}{}_{\kappa} - R_{\lambda\mu\sigma}{}^{\rho} R^{\sigma}{}_{\nu}\,, && \text{(B.1)}\\
\bigtriangledown_{\lambda} \bigtriangledown_{\mu} R^{\lambda}{}_{\nu} &= \bigtriangledown_{\mu} \bigtriangledown_{\lambda} R^{\lambda}{}_{\nu} + R_{\lambda\mu\nu}{}^{\kappa} R^{\lambda}{}_{\kappa} - R_{\lambda\mu\rho}{}^{\lambda} R^{\rho}{}_{\nu} && \text{(B.2)}\\
&= \bigtriangledown_{\mu} \bigtriangledown_{\nu} R + R_{\lambda\mu\nu\kappa} R^{\lambda\kappa} \pm R_{\mu\rho} R^{\rho}{}_{\nu}\,, && \text{(B.3)}\\
\bigtriangledown_{\lambda} \bigtriangledown_{\mu} R^{\nu}{}_{\nu} &= \bigtriangledown_{\mu} \bigtriangledown_{\lambda} R^{\nu}{}_{\nu} + R_{\lambda\mu\nu}{}^{\kappa} R^{\nu}{}_{\kappa} - R_{\lambda\mu\sigma}{}^{\nu} R^{\sigma}{}_{\nu}\,, && \text{(B.4)}\\
\bigtriangledown_{\lambda} \bigtriangledown_{\mu} R^{\nu}{}_{\nu} &= \bigtriangledown_{\mu} \bigtriangledown_{\lambda} R^{\nu}{}_{\nu}\,, && \text{(B.5)}
\end{aligned}$$

and

$$R_{\lambda\mu\kappa\nu} = C_{\lambda\mu\kappa\nu} \pm \left(g_{\lambda[\kappa} R_{\nu]\mu} - g_{\mu[\kappa} R_{\nu]\lambda}\right) \mp \frac{1}{3} g_{\lambda[\kappa} g_{\nu]\mu} R\,. \qquad \text{(B.6)}$$

$$\begin{aligned}
\bigtriangledown^{\lambda} \bigtriangledown^{\kappa} C_{\mu\lambda\nu\kappa} &= \bigtriangledown^{\lambda} \bigtriangledown^{\kappa} R_{\mu\lambda\nu\kappa} \mp \bigtriangledown^{\lambda} \bigtriangledown^{\kappa} \left(g_{\mu[\nu} R_{\kappa]\lambda} - g_{\lambda[\nu} R_{\kappa]\mu}\right)\\
&\quad \pm \frac{1}{3} g_{\mu[\nu} g_{\kappa]\lambda} \bigtriangledown^{\lambda} \bigtriangledown^{\kappa} R\,,\\
\bigtriangledown^{\lambda} \bigtriangledown^{\kappa} C_{\mu\lambda\nu\kappa} &= \bigtriangledown^{\lambda} \bigtriangledown^{\kappa} R_{\mu\lambda\nu\kappa} \mp \frac{1}{2} \bigtriangledown^{\lambda} \bigtriangledown^{\kappa} \left(g_{\mu\nu} R_{\kappa\lambda} - g_{\mu\kappa} R_{\nu\lambda} - g_{\lambda\nu} R_{\kappa\mu} + g_{\lambda\kappa} R_{\nu\mu}\right)\\
&\quad \pm \frac{1}{3} g_{\mu[\nu} g_{\kappa]\lambda} \bigtriangledown^{\lambda} \bigtriangledown^{\kappa} R\,,\\
\bigtriangledown^{\lambda} \bigtriangledown^{\kappa} C_{\mu\lambda\nu\kappa} &= \pm\left(\bigtriangledown^{\lambda} \bigtriangledown_{\lambda} R_{\mu\nu} - \bigtriangledown^{\lambda} \bigtriangledown_{\mu} R_{\lambda\nu}\right)\\
&\quad \mp \frac{1}{2} \Big[g_{\mu\nu} \bigtriangledown^{\lambda} \bigtriangledown^{\kappa} R_{\kappa\lambda} - g_{\mu\kappa} \bigtriangledown^{\lambda} \bigtriangledown^{\kappa} R_{\nu\lambda}\\
&\qquad - g_{\lambda\nu} \bigtriangledown^{\lambda} \bigtriangledown^{\kappa} R_{\kappa\mu} + g_{\lambda\kappa} \bigtriangledown^{\lambda} \bigtriangledown^{\kappa} R_{\nu\mu} \Big]\\
&\quad \pm \frac{1}{6} \left(g_{\mu\nu} g_{\kappa\lambda} - g_{\mu\kappa} g_{\nu\lambda}\right) \bigtriangledown^{\lambda} \bigtriangledown^{\kappa} R\,,
\end{aligned}$$

$$
\begin{aligned}
\triangledown^{\lambda}\triangledown^{\kappa}C_{\mu\lambda\nu\kappa} &= \pm\Big(\triangledown^{\lambda}\triangledown_{\lambda}R_{\mu\nu} - \triangledown^{\lambda}\triangledown_{\mu}R_{\lambda\nu} - \frac{1}{2}\Big[g_{\mu\nu}\triangledown^{\lambda}\frac{1}{2}\triangledown_{\lambda}R - \triangledown^{\lambda}\triangledown_{\mu}R_{\nu\lambda} \\
&\qquad - \triangledown_{\nu}\frac{1}{2}\triangledown_{\mu}R + \triangledown^{\lambda}\triangledown_{\lambda}R_{\nu\mu}\Big] \\
&\quad \frac{1}{6}g_{\mu\nu}\triangledown^{\lambda}\triangledown_{\lambda}R - \frac{1}{6}\triangledown_{\nu}\triangledown_{\mu}R\Big), \\
\triangledown^{\lambda}\triangledown^{\kappa}C_{\mu\lambda\nu\kappa} &= \pm\Big(\frac{1}{2}\triangledown^{\lambda}\triangledown_{\lambda}R_{\mu\nu} - \frac{1}{2}\triangledown^{\lambda}\triangledown_{\mu}R_{\lambda\nu} - \frac{1}{4}g_{\mu\nu}\triangledown^{\lambda}\triangledown_{\lambda}R + \frac{1}{4}\triangledown_{\nu}\triangledown_{\mu}R \\
&\quad \frac{1}{6}g_{\mu\nu}\triangledown^{\lambda}\triangledown_{\lambda}R - \frac{1}{6}\triangledown_{\nu}\triangledown_{\mu}R\Big), \\
\triangledown^{\lambda}\triangledown^{\kappa}C_{\mu\lambda\nu\kappa} &= \pm\Big(\frac{1}{2}\triangledown^{\lambda}\triangledown_{\lambda}R_{\mu\nu} - \frac{1}{2}\triangledown^{\lambda}\triangledown_{\mu}R_{\lambda\nu} - \frac{1}{12}g_{\mu\nu}\triangledown^{\lambda}\triangledown_{\lambda}R \\
&\quad +\frac{1}{12}\triangledown_{\nu}\triangledown_{\mu}R\Big), \\
\triangledown^{\lambda}\triangledown^{\kappa}C_{\mu\lambda\nu\kappa} &= \pm\Big(\frac{1}{2}\triangledown^{\lambda}\triangledown_{\lambda}\big(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\big) - \frac{1}{2}\triangledown^{\lambda}\triangledown_{\mu}R_{\lambda\nu} + \frac{1}{12}\triangledown_{\nu}\triangledown_{\mu}R\Big).
\end{aligned}
$$

Using the Eqs. (B.1)–(B.5), yields

$$
\begin{aligned}
\triangledown^{\lambda}\triangledown^{\kappa}C_{\mu\lambda\nu\kappa} &= \pm\Big(\frac{1}{2}\triangledown^{\lambda}\triangledown_{\lambda}\big(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\big) \\
&\quad -\frac{1}{2}(\frac{1}{2}\triangledown_{\mu}\triangledown_{\nu}R + R_{\lambda\mu\nu\kappa}R^{\lambda\kappa} \pm R_{\mu\rho}R^{\rho}{}_{\nu}) + \frac{1}{12}\triangledown_{\nu}\triangledown_{\mu}R\Big), \\
&= \pm\Big(\frac{1}{2}\triangledown^{\lambda}\triangledown_{\lambda}\big(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\big) - \frac{1}{6}\triangledown_{\mu}\triangledown_{\nu}R \\
&\quad -\frac{1}{2}R_{\lambda\mu\nu\kappa}R^{\lambda\kappa} \mp \frac{1}{2}R_{\mu\rho}R^{\rho}{}_{\nu}\Big), \\
&= \pm\Big(\frac{1}{2}\triangledown^{\lambda}\triangledown_{\lambda}\big(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\big) - \frac{1}{6}\triangledown_{\mu}\triangledown_{\nu}R \\
&\quad +\frac{1}{2}R_{\lambda\mu\kappa\nu}R^{\lambda\kappa} \mp \frac{1}{2}R_{\mu\rho}R^{\rho}{}_{\nu}\Big).
\end{aligned}
$$

Using Eq. (B.6), yields

$$
\begin{aligned}
\triangledown^{\lambda}\triangledown^{\kappa}C_{\mu\lambda\nu\kappa} &= \pm\Big(\frac{1}{2}\triangledown^{\lambda}\triangledown_{\lambda}\big(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\big) - \frac{1}{6}\triangledown_{\mu}\triangledown_{\nu}R \\
&\quad +\frac{1}{2}\big(C_{\lambda\mu\kappa\nu} \pm (g_{\lambda[\kappa}R_{\nu]\mu} - g_{\mu[\kappa}R_{\nu]\lambda}) \mp \frac{1}{3}g_{\lambda[\kappa}g_{\nu]\mu}R\big)R^{\lambda\kappa}
\end{aligned}
$$

$$
\begin{aligned}
& \mp\frac{1}{2}R_{\mu\rho}R^{\rho}{}_{\nu}\Big)\,, \\
= \quad & \pm\Big(\frac{1}{2}\bigtriangledown^{\lambda}\bigtriangledown_{\lambda}\big(R_{\mu\nu}-\frac{1}{6}g_{\mu\nu}R\big)-\frac{1}{6}\bigtriangledown_{\mu}\bigtriangledown_{\nu}R \\
& +\frac{1}{2}\big(C_{\lambda\mu\kappa\nu}\pm\frac{1}{2}(g_{\mu\nu}R_{\kappa\lambda}-g_{\mu\kappa}R_{\nu\lambda}-g_{\lambda\nu}R_{\kappa\mu}+g_{\lambda\kappa}R_{\nu\mu}) \\
& \mp\frac{1}{6}(g_{\mu\nu}g_{\kappa\lambda}-g_{\mu\kappa}g_{\nu\lambda})R\big)R^{\lambda\kappa} \\
& \mp\frac{1}{2}R_{\mu\rho}R^{\rho}{}_{\nu}\Big)\,, \\
= \quad & \pm\Big(\frac{1}{2}\bigtriangledown^{\lambda}\bigtriangledown_{\lambda}\big(R_{\mu\nu}-\frac{1}{6}g_{\mu\nu}R\big)-\frac{1}{6}\bigtriangledown_{\mu}\bigtriangledown_{\nu}R \\
& +\frac{1}{2}C_{\lambda\mu\kappa\nu}R^{\lambda\kappa}\pm\frac{1}{4}(g_{\mu\nu}R_{\kappa\lambda}R^{\lambda\kappa}-R^{\lambda}{}_{\mu}R_{\nu\lambda}-R^{\kappa}{}_{\nu}R_{\kappa\mu}+RR_{\nu\mu}) \\
& \mp\frac{1}{12}(g_{\mu\nu}R-R_{\mu\nu})R\mp\frac{1}{2}R_{\mu\rho}R^{\rho}{}_{\nu}\Big)\,, \\
= \quad & \pm\Big(\frac{1}{2}\bigtriangledown^{\lambda}\bigtriangledown_{\lambda}\big(R_{\mu\nu}-\frac{1}{6}g_{\mu\nu}R\big)-\frac{1}{6}\bigtriangledown_{\mu}\bigtriangledown_{\nu}R\mp R_{\mu\rho}R^{\rho}{}_{\nu} \\
& +\frac{1}{2}C_{\lambda\mu\kappa\nu}R^{\lambda\kappa}\pm\frac{1}{4}g_{\mu\nu}R_{\kappa\lambda}R^{\lambda\kappa}\pm\frac{1}{4}RR_{\nu\mu} \\
& \mp\frac{1}{12}g_{\mu\nu}R\pm\frac{1}{12}R_{\mu\nu})R\Big)\,, \\
= \quad & \pm\Big(\frac{1}{2}\bigtriangledown^{\lambda}\bigtriangledown_{\lambda}\big(R_{\mu\nu}-\frac{1}{6}g_{\mu\nu}R\big)-\frac{1}{6}\bigtriangledown_{\mu}\bigtriangledown_{\nu}R\mp R_{\mu\rho}R^{\rho}{}_{\nu} \\
& +\frac{1}{2}C_{\lambda\mu\kappa\nu}R^{\lambda\kappa}\pm\frac{1}{4}g_{\mu\nu}R_{\kappa\lambda}R^{\lambda\kappa}\pm\frac{1}{4}RR_{\nu\mu}\mp\frac{1}{12}g_{\mu\nu}R^{2}\pm\frac{1}{12}R_{\mu\nu}R\Big), \\
= \quad & \pm\Big(\frac{1}{2}\bigtriangledown^{\lambda}\bigtriangledown_{\lambda}\big(R_{\mu\nu}-\frac{1}{6}g_{\mu\nu}R\big)-\frac{1}{6}\bigtriangledown_{\mu}\bigtriangledown_{\nu}R\mp R_{\mu\rho}R^{\rho}{}_{\nu} \\
& +\frac{1}{2}C_{\lambda\mu\kappa\nu}R^{\lambda\kappa}\pm\frac{1}{3}RR_{\nu\mu}\pm\frac{1}{4}g_{\mu\nu}\big(R_{\kappa\lambda}R^{\lambda\kappa}-\frac{1}{3}R^{2}\big)\Big)\,, \\
= \quad & \frac{1}{2}\Big(\pm\bigtriangledown^{\lambda}\bigtriangledown_{\lambda}\big(R_{\mu\nu}-\frac{1}{6}g_{\mu\nu}R\big)\mp\frac{1}{3}\bigtriangledown_{\mu}\bigtriangledown_{\nu}R-2R_{\mu\rho}R^{\rho}{}_{\nu} \\
& \pm C_{\lambda\mu\kappa\nu}R^{\lambda\kappa}+\frac{2}{3}RR_{\nu\mu}+\frac{1}{2}g_{\mu\nu}\big(R_{\kappa\lambda}R^{\lambda\kappa}-\frac{1}{3}R^{2}\big)\Big)\,.
\end{aligned}
$$

Finally we arrive at the equation

$$
2\bigtriangledown^{\lambda}\bigtriangledown^{\kappa}C_{\mu\lambda\nu\kappa} = \pm\bigtriangledown^{\lambda}\bigtriangledown_{\lambda}\big(R_{\mu\nu}-\frac{1}{6}g_{\mu\nu}R\big)\mp\frac{1}{3}\bigtriangledown_{\mu}\bigtriangledown_{\nu}R-2R_{\mu\rho}R^{\rho}{}_{\nu} \quad \text{(B.7)}
$$

$$\pm C_{\lambda\mu\kappa\nu}R^{\lambda\kappa} + \frac{2}{3}RR_{\nu\mu} + \frac{1}{2}g_{\mu\nu}\big(R_{\kappa\lambda}R^{\lambda\kappa} - \frac{1}{3}R^2\big)\,.$$

Or upon rearrangement of terms we have the equation

$$\begin{aligned} 2\bigtriangledown^\lambda\bigtriangledown^\kappa C_{\mu\lambda\nu\kappa} \mp C_{\lambda\mu\kappa\nu}R^{\lambda\kappa} &= \pm\bigtriangledown^\lambda\bigtriangledown_\lambda\big(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\big) \mp \frac{1}{3}\bigtriangledown_\mu\bigtriangledown_\nu R - 2R_{\mu\rho}R^\rho{}_\nu \\ &+\frac{2}{3}RR_{\nu\mu} + \frac{1}{2}g_{\mu\nu}\big(R_{\kappa\lambda}R^{\lambda\kappa} - \frac{1}{3}R^2\big)\,. \end{aligned} \tag{B.8}$$

We can see from Eq. (B.8) that the quantity $2\bigtriangledown^\lambda\bigtriangledown^\kappa C_{\mu\lambda\nu\kappa}$ is independent of the sign convention of $R_{\mu\nu}$ w.r.t. $R^\lambda{}_{\mu\kappa\nu}$, remembering that the Ricci convention flips the sign of the Ricci tensor.

In terms of the (trace reversed) metric perturbations the first order contribution of $2\bigtriangledown^\lambda\bigtriangledown^\kappa C_{\mu\lambda\nu\kappa}$ is given by the following above regardless of Ricci convention, that is:

$$2\Big\{\bigtriangledown_\lambda\bigtriangledown_\kappa C^{\mu\kappa\nu\lambda}\Big\}^{(1)} = \partial_\lambda\partial^\lambda\Big(\partial^\kappa\partial^\nu\bar{\gamma}^{\mu)}{}_\kappa - \frac{1}{2}\partial_\kappa\partial^\kappa\bar{\gamma}^{\mu\nu} - \frac{1}{6}\eta^{\mu\nu}\partial^\sigma\partial^\kappa\bar{\gamma}_{\sigma\kappa}\Big) \tag{B.9}$$

$$-\partial_\lambda\partial^\nu\Big(\partial^\kappa\partial^{(\lambda}\bar{\gamma}^{\mu)}{}_\kappa - \frac{1}{2}\partial_\kappa\partial^\kappa\bar{\gamma}^{\lambda\mu}\Big) + \frac{1}{6}\partial^\mu\partial^\nu\partial^\lambda\partial^\kappa\bar{\gamma}_{\lambda\kappa} \tag{B.10}$$

$$-\frac{1}{6}\Big(\eta^{\mu\nu}\partial_\kappa\partial^\kappa - \partial^\mu\partial^\nu\Big)\partial_\lambda\partial^\lambda\gamma\,.$$

The first order Einstein tensor however changes sign :

$$\overset{(1)}{G}{}^{\mu\nu}_{\rm E} = \mp\frac{1}{2}\Box\bar{\gamma}_{\mu\nu}\,,$$

where the upper sign is again for the convention $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ (*c.f.* [38]) and the lower sign is for the convention $\tilde{R}_{\mu\nu} = R^\rho{}_{\mu\nu\rho}$ (used here).

If we are using the first convention (*i.e.* $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$) we should start with the following action (shown schematically) with the Ricci scalar term and the Weyl squared term with the same relative sign :

$$S_{\rm grav} = \int d^4x\sqrt{-g}\big(\frac{1}{2\kappa}R + |\alpha_0|C^2\big)\,, \tag{B.11}$$

where $\kappa = 8\pi G$. The variation of the action above w.r.t. $\delta g_{\mu\nu}$ leads to the following equation of motion:

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{\rm grav}}{\delta g_{\mu\nu}} = -\frac{1}{2\kappa}G_{\rm E}^{\mu\nu} + 2|\alpha_0|\Big(2\,\nabla_\lambda\nabla_\kappa C^{\mu\kappa\nu\lambda} - C^{\mu\kappa\nu\lambda}R_{\kappa\lambda}\Big)\,, \tag{B.12}$$

To first order in the (trace reversed) metric pertubations the above equation becomes

$$\frac{1}{\sqrt{-g}}\frac{\delta \overset{(1)}{S}_{\rm grav}}{\delta g_{\mu\nu}} = +\frac{1}{4\kappa}\Box\bar{\gamma}^{\mu\nu} - |\alpha_0|\Box\Big(\Box\bar{\gamma}^{\mu\nu} + \frac{1}{3}\big(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu\big)\gamma\Big)\,. \tag{B.13}$$

From which we can clearly see that we have stable massive modes.

If however we choose the Ricci convention $\tilde{R}_{\mu\nu} = R^\rho{}_{\mu\nu\rho}$ the gravitational action which leads to stable massive modes is

$$S_{\rm grav} = \int d^4x\sqrt{-g}(+\frac{1}{2\kappa}\tilde{R} - |\alpha_0|C^2)\,, \tag{B.14}$$

the variation of which is the equation

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{\rm grav}}{\delta g_{\mu\nu}} = -\frac{1}{2\kappa}\tilde{G}_{\rm E}^{\mu\nu} - 2|\alpha_0|\Big(2\,\nabla_\lambda\nabla_\kappa C^{\mu\kappa\nu\lambda} + C^{\mu\kappa\nu\lambda}\tilde{R}_{\kappa\lambda}\Big)\,. \tag{B.15}$$

Note that the sign in the $C^{\mu\kappa\nu\lambda}R_{\kappa\lambda}$ term has flipped relative to the first case in order that the entire $|\alpha_0|$ term be independent of the Ricci sign convention as demonstrated above. To first order in perturbations the above equation becomes

$$\frac{1}{\sqrt{-g}}\frac{\delta \overset{(1)}{S}_{\rm grav}}{\delta g_{\mu\nu}} = -\frac{1}{4\kappa}\Box\bar{\gamma}^{\mu\nu} + |\alpha_0|\Box\Big(\Box\bar{\gamma}^{\mu\nu} + \frac{1}{3}\big(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu\big)\gamma\Big)\,,$$

which again admits stable massive modes.

To conclude our Riemann and Ricci conventions used throughout Chapter 3 are as follows:

$$\big[\nabla_\mu,\nabla_\nu\big]\omega_\lambda = R_{\mu\nu\lambda}{}^\sigma\omega_\sigma\,,$$

and

$$R_{\mu\nu} = R^\rho{}_{\mu\nu\rho}\,.$$

Bibliography

[1] Alexander, S., J. Ochoa, and A. Kosowsky (2009) "Generation of Circular Polarization of the Cosmic Microwave Background," *Phys.Rev.*, **D79**, p. 063524, 0810.2355.

[2] Nelson, W., J. Ochoa, and M. Sakellariadou (2010) "Gravitational Waves in the Spectral Action of Noncommutative Geometry," *Phys.Rev.*, **D82**, p. 085021, 1005.4276.

[3] ——— (2010) "Constraining the Noncommutative Spectral Action via Astrophysical Observations," *Phys.Rev.Lett.*, **105**, p. 101602, 1005.4279.

[4] Colladay, D. and V. A. Kostelecký (1997) "CPT violation and the standard model," *Phys. Rev.*, **D55**, pp. 6760–6774, hep-ph/9703464.

[5] ——— (1998) "Lorentz-violating extension of the standard model," *Phys. Rev.*, **D58**, p. 116002, hep-ph/9809521.

[6] Kostelecký, V. and R. Lehnert (2001) "Stability, causality, and Lorentz and CPT violation," *Phys.Rev.*, **D63**, p. 065008, hep-th/0012060.

[7] Bluhm, R. and V. Kostelecký (2005) "Spontaneous Lorentz violation, Nambu-Goldstone modes, and gravity," *Phys.Rev.*, **D71**, p. 065008, hep-th/0412320.

[8] Lubin, P., P. Melese, and G. Smoot (1983) "Linear and Circular Polarization of the Cosmic Microwave Background," *Astrophys.J.*, **273**, pp. L51–L54.

[9] Partridge, R., J. Nowakowski, and H. Martin (1988) "Linear Polarization Fluctuations in the Cosmic Microwave Background," *Nature*, **331**, pp. 146–147.

[10] Kosowsky, A. (1996) "Cosmic microwave background polarization," *Annals Phys.*, **246**, pp. 49–85, astro-ph/9501045.

[11] LIGO, `http://www.ligo.caltech.edu/`.

[12] LIGO, A., `http://www.advancedligo.mit.edu/`.

[13] VIRGO, `http://www.virgo.infn.it/`.

[14] gEO600, `http://www.geo600.org/`.

[15] TAMA300, `http://tamago.mtk.nao.ac.jp/`.

[16] LISA, `http://lisa.nasa.gov/`.

[17] Stelle, K. (1978) "Classical Gravity with Higher Derivatives," *Gen. Rel. Grav.*, **9**, pp. 353–371.

[18] Connes, A. (1994) *Noncommutative Geometry*, Academic Press, New York, NY.

[19] Connes, A. and M. Marcolli (2008) *Noncommutative Geometry, Quantum Fields and Motives*, Hindustan Book Agency, India.

[20] Chamseddine, A. H., A. Connes, and M. Marcolli (2007) "Gravity and the standard model with neutrino mixing," *Adv.Theor.Math.Phys.*, **11**, pp. 991–1089, `hep-th/0610241`.

[21] Penzias, A. A. and R. W. Wilson (1965) "A Measurement of excess antenna temperature at 4080- Mc/s," *Astrophys. J.*, **142**, pp. 419–421.

[22] Mather, J. C., D. J. Fixsen, R. A. Shafer, C. Mosier, and D. T. Wilkinson (1999) "Calibrator Design for the COBE Far Infrared Absolute Spectrophotometer (FIRAS)," *Astrophys. J.*, **512**, pp. 511–520, `astro-ph/9810373`.

[23] Smoot, G. F., C. Bennett, A. Kogut, E. Wright, J. Aymon, et al. (1992) "Structure in the COBE differential microwave radiometer first year maps," *Astrophys.J.*, **396**, pp. L1–L5.

[24] Spergel, D. et al. (2003) "First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Determination of cosmological parameters," *Astrophys.J.Suppl.*, **148**, pp. 175–194, `astro-ph/0302209`.

[25] ——— (2007) "Wilkinson Microwave Anisotropy Probe (WMAP) three year results: implications for cosmology," *Astrophys.J.Suppl.*, **170**, p. 377, `astro-ph/0603449`.

[26] Komatsu, E. et al. (2009) "Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation," *Astrophys.J.Suppl.*, **180**, pp. 330–376, `0803.0547`.

[27] Mukhanov, V. (2005) *Physical Foundations of Cosmology*, Univ. Pr., Cambridge, UK.

[28] Hu, W. (2008) "Lecture Notes on CMB Theory: From Nucleosynthesis to Recombination," * Brief entry *, 0802.3688.

[29] Kaplinghat, M., M. Chu, Z. Haiman, G. Holder, L. Knox, et al. (2003) "Probing the reionization history of the universe using the cosmic microwave background polarization," *Astrophys.J.*, **583**, pp. 24–32, astro-ph/0207591.

[30] Chandrasekhar, S. (1960) *Radiative Transfer*, Dover, New York.

[31] Ade, P. et al. (2008) "First season QUaD CMB temperature and polarization power spectra," *Astrophys.J.*, **674**, pp. 22–28, 0705.2359.

[32] Pryke, . C. et al. (2009) "Second and third season QUaD CMB temperature and polarization power spectra," *Astrophys.J.*, **692**, pp. 1247–1270, * Brief entry *, 0805.1944.

[33] Hinderks, . J. et al. (2009) "QUaD: A High-Resolution Cosmic Microwave Background Polarimeter," *Astrophys.J.*, **692**, pp. 1221–1246, * Brief entry *, 0805.1990.

[34] Bouchet, F. (2007) "The Planck satellite: Status and perspectives," *Mod.Phys.Lett.*, **A22**, pp. 1857–1863.

[35] Ade, P. et al. (2011) "Planck Early Results: The Planck mission," * Temporary entry *, 1101.2022.

[36] Raffelt, G., G. Sigl, and L. Stodolsky (1993) "NonAbelian Boltzmann equation for mixing and decoherence," *Phys.Rev.Lett.*, **70**, pp. 2363–2366, hep-ph/9209276.

[37] Sigl, G. and G. Raffelt (1993) "General kinetic description of relativistic mixed neutrinos," *Nucl.Phys.*, **B406**, pp. 423–451.

[38] Wald, R. (1984) *General Relativity*, University of Chicago Press, Chicago, IL.

[39] Kostelecký, V. and S. Samuel (1989) "Spontaneous Breaking of Lorentz Symmetry in String Theory," *Phys.Rev.*, **D39**, p. 683.

[40] Gambini, R. and J. Pullin (1999) "Nonstandard optics from quantum space-time," *Phys.Rev.*, **D59**, p. 124021, gr-qc/9809038.

[41] ALFARO, J., H. A. MORALES-TECOTL, and L. F. URRUTIA (2000) "Quantum gravity corrections to neutrino propagation," *Phys.Rev.Lett.*, **84**, pp. 2318–2321, gr-qc/9909079.

[42] ——— (2002) "Loop quantum gravity and light propagation," *Phys.Rev.*, **D65**, p. 103509, hep-th/0108061.

[43] JACOBSON, T., S. LIBERATI, and D. MATTINGLY (2006) "Lorentz violation at high energy: Concepts, phenomena and astrophysical constraints," *Annals Phys.*, **321**, pp. 150–196, astro-ph/0505267.

[44] KOSTELECKÝ, V. A., C. D. LANE, and A. G. M. PICKERING (2002) "One-loop renormalization of Lorentz-violating electrodynamics," *Phys. Rev.*, **D65**, p. 056006, hep-th/0111123.

[45] CARROLL, S. M. and G. B. FIELD (1997) "Is there evidence for cosmic anisotropy in the polarization of distant radio sources?" *Phys.Rev.Lett.*, **79**, pp. 2394–2397, astro-ph/9704263.

[46] JACKIW, R. and V. KOSTELECKÝ (1999) "Radiatively induced Lorentz and CPT violation in electrodynamics," *Phys.Rev.Lett.*, **82**, pp. 3572–3575, hep-ph/9901358.

[47] ADAM, C. and F. R. KLINKHAMER (2001) "Causality and CPT violation from an Abelian Chern-Simons like term," *Nucl.Phys.*, **B607**, pp. 247–267, hep-ph/0101087.

[48] KOSTELECKÝ, V., R. LEHNERT, and M. J. PERRY (2003) "Spacetime - varying couplings and Lorentz violation," *Phys.Rev.*, **D68**, p. 123511, astro-ph/0212003.

[49] KOSTELECKÝ, V. A. and A. G. M. PICKERING (2003) "Vacuum photon splitting in Lorentz-violating quantum electrodynamics," *Phys. Rev. Lett.*, **91**, p. 031801, hep-ph/0212382.

[50] LEHNERT, R. (2003) "Threshold analyses and Lorentz violation," *Phys.Rev.*, **D68**, p. 085003, gr-qc/0304013.

[51] ACKERSTAFF, K. ET AL. (1997) "A Study of B meson oscillations using hadronic Z^0 decays containing leptons," *Z.Phys.*, **C76**, pp. 401–415, hep-ex/9707009.

[52] ABE, K. ET AL. (2001) "Measurement of $B_d^0 - \bar{B}_d^0$ mixing rate from the time evolution of dilepton events at the υ_{4S}," *Phys.Rev.Lett.*, **86**, pp. 3228–3232, hep-ex/0011090.

[53] Aubert, B. et al. (2003) "Limits on the lifetime difference of neutral B mesons and on CP, T, and CPT violation in B0 anti-B0 mixing," hep-ex/0303043.

[54] Link, J., M. Reyes, P. Yager, J. Anjos, I. Bediaga, et al. (2003) "Charm system tests of CPT and Lorentz invariance with FOCUS," *Phys.Lett.*, **B556**, pp. 7–13, hep-ex/0208034.

[55] Kostelecký, V. and R. Potting (1995) "CPT, strings, and meson factories," *Phys.Rev.*, **D51**, pp. 3923–3935, revised version, hep-ph/9501341.

[56] Colladay, D. and V. Kostelecký (1995) "Tests of direct and indirect CPT violation at a B factory," *Phys.Lett.*, **B344**, pp. 259–265, hep-ph/9501372.

[57] Kostelecký, V. (2001) "Formalism for CPT, T, and Lorentz violation in neutral meson oscillations," *Phys.Rev.*, **D64**, p. 076001, hep-ph/0104120.

[58] Humphrey, M., D. Phillips, E. Mattison, R. Vessot, R. Stoner, et al. (2003) "Testing Lorentz and CPT symmetry with hydrogen masers," *Phys.Rev.*, **A68**, p. 063807, physics/0103068.

[59] Kostelecký, V. and C. D. Lane (1999) "Constraints on Lorentz violation from clock comparison experiments," *Phys.Rev.*, **D60**, p. 116010, hep-ph/9908504.

[60] Lane, C. D. (2005) "Probing Lorentz violation with Doppler-shift experiments," *Phys.Rev.*, **D72**, p. 016005, hep-ph/0505130.

[61] Cane, F., D. Bear, D. Phillips, M. Rosen, C. Smallwood, et al. (2004) "Bound on Lorentz and CPT violating boost effects for the neutron," *Phys.Rev.Lett.*, **93**, p. 230801, physics/0309070.

[62] Hughes, V., M. Grosse Perdekamp, D. Kawall, W. Liu, K. Jungmann, et al. (2001) "Test of CPT and Lorentz invariance from muonium spectroscopy," *Phys.Rev.Lett.*, **87**, p. 111804, hep-ex/0106103.

[63] Bluhm, R., V. Kostelecký, and C. D. Lane (2000) "CPT and Lorentz tests with muons," *Phys.Rev.Lett.*, **84**, pp. 1098–1101, hep-ph/9912451.

[64] Dehmelt, H., R. Mittleman, J. van Dyck, R.S., and P. Schwinberg (1999) "Past electron positron g - 2 experiments yielded sharpest bound on CPT violation," *Phys.Rev.Lett.*, **83**, pp. 4694–4696, hep-ph/9906262.

[65] Bluhm, R., V. Kostelecký, and N. Russell (1999) "CPT and Lorentz tests in hydrogen and anti-hydrogen," *Phys.Rev.Lett.*, **82**, pp. 2254–2257, hep-ph/9810269.

[66] Landau, L. and E. Lifshitz (1975) *The Classical Theory of Fields*, Elsevier, Oxford.

[67] Connes, A. (2006) "Noncommutative geometry and physics," *AIP Conf. Proc.*, **861**, pp. 47–69.

[68] Kostelecký, V. A. (2004) "Gravity, Lorentz violation, and the standard model," *Phys. Rev.*, **D69**, p. 105009, hep-th/0312310.

[69] Kant, E. and F. R. Klinkhamer (2005) "Maxwell-Chern-Simons theory for curved spacetime backgrounds," *Nucl.Phys.*, **B731**, pp. 125–139, hep-th/0507162.

[70] Carroll, S. M., G. B. Field, and R. Jackiw (1990) "Limits on a Lorentz and Parity Violating Modification of Electrodynamics," *Phys.Rev.*, **D41**, p. 1231.

[71] Lue, A., L.-M. Wang, and M. Kamionkowski (1999) "Cosmological signature of new parity violating interactions," *Phys.Rev.Lett.*, **83**, pp. 1506–1509, astro-ph/9812088.

[72] Balaji, K., R. H. Brandenberger, and D. A. Easson (2003) "Spectral dependence of CMB polarization and parity," *JCAP*, **0312**, p. 008, hep-ph/0310368.

[73] Lepora, N. F. (1998) "Cosmological birefringence and the microwave background," gr-qc/9812077.

[74] Kostelecký, V. A. and M. Mewes (2001) "Cosmological constraints on Lorentz violation in electrodynamics," *Phys. Rev. Lett.*, **87**, p. 251304, hep-ph/0111026.

[75] Cabella, P., P. Natoli, and J. Silk (2007) "Constraints on CPT violation from WMAP three year polarization data: a wavelet analysis," *Phys.Rev.*, **D76**, p. 123014, 0705.0810.

[76] Feng, B., M. Li, J.-Q. Xia, X. Chen, and X. Zhang (2006) "Searching for CPT Violation with Cosmic Microwave Background Data from WMAP and BOOMERANG," *Phys.Rev.Lett.*, **96**, p. 221302, astro-ph/0601095.

[77] Kahniashvili, T., R. Durrer, and Y. Maravin (2008) "Testing Lorentz Invariance Violation with WMAP Five Year Data," *Phys.Rev.*, **D78**, p. 123009, 0807.2593.

[78] Xia, J.-Q., H. Li, X.-l. Wang, and X.-m. Zhang (2008) "Testing CPT Symmetry with CMB Measurements," *Astron.Astrophys.*, **483**, pp. 715–718, 0710.3325.

[79] Xia, J.-Q., H. Li, G.-B. Zhao, and X. Zhang (2008) "Testing CPT Symmetry with CMB Measurements: Update after WMAP5," *Astrophys.J.*, **679**, pp. L61–L63, * Brief entry *, 0803.2350.

[80] Kahniashvili, T., Y. Maravin, and A. Kosowsky (2009) "Faraday rotation limits on a primordial magnetic field from Wilkinson Microwave Anisotropy Probe five-year data," *Phys.Rev.*, **D80**, p. 023009, 0806.1876.

[81] De Sabbata, V. and M. Gasperini (1980) "On the Maxwell Equations in a Riemann-Cartan Space," *Phys.Lett.*, **A77**, pp. 300–302.

[82] ——— (1981) "Gauge Invariance, Semiminimal Coupling, and Propagating Torsion," *Phys.Rev.*, **D23**, pp. 2116–2120.

[83] Harvey, J. A., C. T. Hill, and R. J. Hill (2008) "Standard Model Gauging of the Wess-Zumino-Witten Term: Anomalies, Global Currents and pseudo-Chern-Simons Interactions," *Phys.Rev.*, **D77**, p. 085017, 0712.1230.

[84] Cooray, A., A. Melchiorri, and J. Silk (2003) "Is the cosmic microwave background circularly polarized?" *Phys.Lett.*, **B554**, pp. 1–6, astro-ph/0205214.

[85] Finelli, F. and M. Galaverni (2009) "Rotation of Linear Polarization Plane and Circular Polarization from Cosmological Pseudo-Scalar Fields," *Phys.Rev.*, **D79**, p. 063002, * Brief entry *, 0802.4210.

[86] McMaster, W. H. (1954) "Polarization and the Stokes Parameters," *Am.J.Phys.*, **22**, p. 351.

[87] ——— (1961) "Matrix Representation of Polarization," *Rev. Mod. Phys.*, **33**, p. 8.

[88] Jackson, J. (1999) *Classical Electrodynamics*, Wiley, New York, NY.

[89] Dodelson, S. and J. M. Jubas (1995) "Reionization and its imprint on the cosmic microwave background," *Astrophys.J.*, **439**, pp. 503–516, astro-ph/9308019.

[90] Raffelt, G., G. Sigl, and L. Stodolsky (1992) "Quantum statistics in particle mixing phenomena," *Phys.Rev.*, **D45**, pp. 1782–1788.

[91] G.Sterman (1993) *An Introduction to Quantum Field Theory*, Cambridge University Press, Cambridge, UK.

[92] Newman, E. and R. Penrose (1966) "Note on the Bondi-Metzner-Sachs group," *J.Math.Phys.*, **7**, pp. 863–870.

[93] Goldberg, J., A. MacFarlane, E. Newman, F. Rohrlich, and E. Sudarshan (1967) "Spin s spherical harmonics and edth," *J.Math.Phys.*, **8**, p. 2155.

[94] Zaldarriaga, M. and U. Seljak (1997) "An all sky analysis of polarization in the microwave background," *Phys.Rev.*, **D55**, pp. 1830–1840, `astro-ph/9609170`.

[95] Kamionkowski, M., A. Kosowsky, and A. Stebbins (1997) "A Probe of primordial gravity waves and vorticity," *Phys.Rev.Lett.*, **78**, pp. 2058–2061, `astro-ph/9609132`.

[96] ——— (1997) "Statistics of cosmic microwave background polarization," *Phys.Rev.*, **D55**, pp. 7368–7388, `astro-ph/9611125`.

[97] Nolta, M. et al. (2009) "Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Angular Power Spectra," *Astrophys.J.Suppl.*, **180**, pp. 296–305, `0803.0593`.

[98] Bischoff, . C. et al. (2008) "New Measurements of Fine-Scale CMB Polarization Power Spectra from CAPMAP at Both 40 and 90 GHz," *Astrophys.J.*, **684**, pp. 771–789, * Brief entry *, `0802.0888`.

[99] Kosowsky, A., T. Kahniashvili, G. Lavrelashvili, and B. Ratra (2005) "Faraday rotation of the Cosmic Microwave Background polarization by a stochastic magnetic field," *Phys.Rev.*, **D71**, p. 043006, `astro-ph/0409767`.

[100] Shapiro, I. L. (1998) "Torsion: Theory and possible observables," Review paper to appear in Contemporary Fundamental Physics. Edited by Valeri Dvoeglazov. Nova Science Publishers, `hep-th/9811072`.

[101] Perez, A. and C. Rovelli (2006) "Physical effects of the Immirzi parameter," *Phys.Rev.*, **D73**, p. 044013, `gr-qc/0505081`.

[102] Alexander, S. and N. Yunes (2008) "Chern-Simons Modified Gravity as a Torsion Theory and its Interaction with Fermions," *Phys.Rev.*, **D77**, p. 124040, `0804.1797`.

[103] Chamseddine, A. H. and A. Connes (2007) "Conceptual Explanation for the Algebra in the Noncommutative Approach to the Standard Model," *Phys.Rev.Lett.*, **99**, p. 191601, `0706.3690`.

[104] ——— (2006) "Scale invariance in the spectral action," *J.Math.Phys.*, **47**, p. 063504, `hep-th/0512169`.

[105] Nelson, W. and M. Sakellariadou (2010) "Cosmology and the Noncommutative approach to the Standard Model," *Phys.Rev.*, **D81**, p. 085038, 0812.1657.

[106] ——— (2009) "Natural inflation mechanism in asymptotic noncommutative geometry," *Phys.Lett.*, **B680**, pp. 263–266, 0903.1520.

[107] Marcolli, M. and E. Pierpaoli (2009) "Early Universe models from Noncommutative Geometry," 0908.3683.

[108] Buck, M., M. Fairbairn, and M. Sakellariadou (2010) "Inflation in models with Conformally Coupled Scalar fields: An application to the Noncommutative Spectral Action," *Phys.Rev.*, **D82**, p. 043509, 1005.1188.

[109] Abramowitz, M. and I. Stegun (1964) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Press, New York, NY.

[110] Poisson, E. (2004) "The Motion of point particles in curved space-time," *Living Rev.Rel.*, **7**, p. 6, gr-qc/0306052.

[111] Prakash, S. (1950) "Solution of generalized wave equations of order higher than 2," *Proceedings Mathematical Sciences*, **31**, pp. 240–260.

[112] Stein, L. C., N. Yunes, and S. A. Hughes (2011) "Effective Gravitational Wave Stress-energy Tensor in Alternative Theories of Gravity," *Phys.Rev.*, **D83**, p. 064038, 1012.3144.

[113] Will, C. M. (2005) "The Confrontation between general relativity and experiment," *Living Rev.Rel.*, **9**, p. 3, an update of the Living Review article originally published in 2001, gr-qc/0510072.

[114] Bavarsad, E., M. Haghighat, Z. Rezaei, R. Mohammadi, I. Motie, et al. (2010) "Generation of circular polarization of the CMB," *Phys.Rev.*, **D81**, p. 084035, 0912.2993.

[115] Motie, I. and S.-S. Xue (2011) "Euler-Heisenberg Lagrangian and photon circular polarization," * Temporary entry *, 1104.3555.

[116] Collins, J., A. Perez, D. Sudarsky, L. Urrutia, and H. Vucetich (2004) "Lorentz invariance and quantum gravity: an additional fine-tuning problem?" *Phys.Rev.Lett.*, **93**, p. 191301, gr-qc/0403053.

[117] Collins, J., A. Perez, and D. Sudarsky (2006) "Lorentz invariance violation and its role in quantum gravity phenomenology," hep-th/0603002.

[118] Yunes, N. (2008) *At The Interface: Gravitational Waves as Tools to Test Quantum Gravity and Probe the Astrophysical Universe*, Ph.D. thesis, The Pennsylvania State University.

Vita

Joseph R. Ochoa

Joseph R. Ochoa was born in Fort Lauderdale, Fl on March 19, 1979, the last of the six children of Rolando Ochoa and Yolanda Marcos. He obtained a B.A. in Physics from the University of California, Berkeley in 2002. Since 2004 he has been at the Pennsylvania State University and has worked in Stephon Alexander's research group since 2006. His honors include the Bunton-Waller Assistantship, the Excellence in Teaching Award at Penn State in 2005, and an Alfred P. Sloan Foundation Scholarship. In 2011 he began a one-year visiting assistant professor in physics appointment at Haverford College. His publications include:

- S. Alexander, A. Kosowsky, J. Ochoa, *Generation of Circular Polarization of the Cosmic Microwave Background*, Phys. Rev. D 79, 063524 (2009).
- W. Nelson, J. Ochoa, M. Sakellariadou, *Gravitational Waves in the Spectral Action of Noncommutative Geometry*, Phys. Rev. D 82, 085021 (2010).
- W. Nelson, J. Ochoa, M. Sakellariadou, *Constraining the Noncommutative Spectral Action via Astrophysical Observations*, Phys. Rev. Lett. 105, 101602 (2010).