The Pennsylvania State University

The Graduate School

College of Education

THE EFFECTS OF TEACHERS' KNOWLEDGE AND UNDERSTANDING OF ADDITION AND SUBTRACTION WORD PROBLEMS ON STUDENT UNDERSTANDING

A Dissertation in

Curriculum and Instruction

by

Jana Lunt

© 2010 Jana Lunt

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

May 2010

The dissertation of Jana Lunt was reviewed and approved* by the following:

Glendon W. Blume
Professor of Education
Dissertation Adviser
Chair of Committee
Graduate Program Coordinator

Rose Mary Zbiek
Professor of Education

James F. Nolan, Jr.
Hermanowicz Professor of Education

Bernard J. Badiali
Associate Professor of Education

*signatures are on file in the Graduate School.

ABSTRACT

This study examines the influence of teacher understanding on student understanding through teacher practice. Three elementary school teachers participated in a university course that discussed mathematical and pedagogical knowledge regarding addition and subtraction word problems. Data were gathered from the course, teacher interviews, classroom observations, and student interviews. The data from this exploratory study were analyzed qualitatively to describe the nature of the teachers' understandings, the ways teachers used their understandings in their practice, and the nature of their students' understandings. This study reveals that there were aspects of student understanding directly influenced by teachers' understandings, such as an expanded understanding of mathematical operations, the nature and quality of written and verbal representations of students' understanding, the ability to progress to more sophisticated levels of problem solving, and students' understandings of mathematical concepts that are more pervasive in mathematics than solving word problems. In addition there were aspects of teacher practice influenced by teachers' understandings that may have led to student understanding, such as, creating new tasks, the ability to comprehend and implement the lesson objectives, the ability to understand and address specific students'needs, the ability to purposefully target questions asked of students, requiring students to solve problems in multiple ways, providing examples with detailed explanation and interpretation, and allowing students to solve problems using their own valid interpretations. This study hypothesizes that teachers used their understanding to create and implement tasks at a high level of cognitive demand, maintaining that demand while the students implemented the tasks, which affected student understanding.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ACKNOWLEDGEMENTS

I would like to express my appreciation and love for my husband, Dave Lunt, who supported me through this process. I am so proud of us for doing this together. I would also like to thank my adviser, Dr. Glendon Blume, for his influence and encouragement. I learned so much through our conversations and work together; I always felt he was doing everything he possibly could to help me. My other committee members, Dr. Rose Mary Zbiek, Dr. James Nolan, and Dr. Bernard Badiali, also deserve my sincere gratitude. I appreciate their input and direction throughout my years at Penn State and am grateful for their contribution to my education and research.

I would also like to thank MJ Kitt, the mathematics coordinator for the local school district, who got this project off the ground by helping me find amazing teachers to participate in my study. I appreciate her interest in improving her own understandings and the quality of education for the students in this district. I also appreciate her desire to continue to increase teachers' mathematical understandings even after this project was finished.

The three teachers that were participants in the study constantly amazed me throughout our time together. I am so impressed by their willingness to learn and ability to use their understanding to change their practice. I learned a great deal through my interactions with them and I hope their enthusiasm is reflected in my own teaching.

Finally, I would like to acknowledge my daughter, Zola, who made this project more difficult – but makes everything worthwhile.

CHAPTER ONE

RATIONALE

Teachers have not always been acknowledged as being a mitigating factor in student learning. In 1966, James Coleman created a stir in education circles with his report which, among other claims, seemed to argue that teacher characteristics had little influence on the achievement of students (Campbell, Coleman, & Hobson, 1966). Factors such as socioeconomic status of the family and school effects were reported as having a greater influence on student achievement. Some in the education field took this report as evidence that the teacher did not make a substantial difference in how much students learn; others found this reasoning to be illogical and used this report as an impetus to begin researching the relationship between teacher knowledge and student achievement (Brophy & Good, 1986; Jencks, Smith, Blakeslee, & Anderson, 1972; Mosteller & Moynihan, 1972).

Research in the 1970s was also strongly influenced by the psychological persuasion of the time, which predominately incorporated a behaviorist perspective. Research programs other than Coleman’s were attempting to describe teaching in terms of teachers’ qualifications, such as coursework they had completed or certificates they held (Calderhead, 1996; Carter, 1990). Researchers investigated the relationships between these teacher characteristics and student learning, searching for patterns of effective teaching. Like the Coleman Report, they produced surprising and occasionally contradictory findings (Dunkin & Biddle, 1974). One complaint about the Coleman report, and other studies of this nature, was concern about the methodology used to conclude that teachers do not affect student achievement. It has been argued that because the Coleman report did not include data collected in teachers' classrooms, the data did not provide an accurate view of teachers and more detailed research was necessary (Shulman, 1986).

Instead of using teacher qualifications such as field of study or number of training programs attended, researchers began using data focusing on behaviors of teachers in the classroom. This major research program has since been called "process–product research" (Dunkin & Biddle, 1974; Mitzel, 1960). The intent of this program of research was that there is a relationship between the processes of teaching and the products of learning, specifically the achievement of students, and once the relationship was better understood instruction would improve and student achievement would improve (Anderson, Evertson, & Brophy, 1979). The majority of studies used general pedagogical behaviors, making assumptions about how the organization of instruction, teaching methods, and student–teacher interactions would affect student achievement (McDonald & Elias, 1976). Researchers were inside normal classrooms identifying and quantifying teacher behaviors.

For example, a series of studies done in New Zealand focused on process–product questions (Wright & Nuthall, 1970). They found a positive relationship between student achievement and behaviors such as percentage of closed questions asked by teachers or teachers giving praise to students. Other behaviors produced no significant correlation with student achievement, such as total teacher or pupil talk or starting lessons with review of the previous lesson. These are just a few examples out of many research programs; a thorough review of the process–product literature can be found in many sources including one in the *Handbook of Research on Teaching* by Brophy and Good (1986). This review formed conclusions about teacher behaviors such as pacing of instruction, content coverage, classroom management, whole-class versus small-group instruction, clarity of presentations, questioning, reacting to student responses, handling seatwork, and teacher's intentions or objectives. These studies were predominately quantitative studies that correlated the amount of a given teacher behavior with student achievement on a particular test or group of tests. From this body of work the field progressed by learning more about teacher effects on student achievement and began to dispel the notion that teachers do not make a difference in student learning.

There are many limits to the design of process–product research. A major limitation is the fact that a measurement of teacher knowledge, especially content-specific knowledge, is not taken into account in the many process–product research studies. If teachers do influence the achievement of students then the thoughts, beliefs, and knowledge of teachers is sure to be a factor. This, however, was not the end for research on the link between teachers and students, but only a beginning.

Since the 1970s research began to focus more on teachers' knowledge, beliefs, and decision making, and how those constructs influence student achievement. While investigating other frameworks, researchers began to realize that, aside from a teacher’s academic background or general pedagogical behaviors, the teacher’s individual characteristics were more important than initially recognized. This realization may have been influenced by new research on cognitive psychology (Borko & Putnam, 1996). The cognitive psychologists’ perspective that human beings are capable of and responsible for constructing their own reality led to research that focused on human knowledge in both the teacher and the student (Calderhead, 1996; Winne & Marx, 1977).

This line of reasoning led to research programs investigating decision making and problem solving by teachers (Elbaz, 1981; Leinhardt, Weidman, & Hammond, 1987). In these types of studies teachers were interviewed about their choices and the reasoning behind their choices. For example, research about practical knowledge or craft knowledge investigates practicing teachers’ attempts to describe the knowledge they draw on and develop while teaching, knowledge they need specifically for teaching (Elbaz, 1981; Grimmett & Mackinnon, 1992; Hiebert, Gallimore, & Stigler, 2002; Leinhardt, 1990; Meijer, 2002). Knowledge about classroom management strategies, how to engender class participation, or how to keep lessons running smoothly are possible aspects of knowledge studied in this line of research (Berliner, 1986; Doyle, 1986; Leinhardt & Greeno, 1986).

There are also many limitations to the way this research was designed. Most salient to my study is the fact that measurement of subject-specific teaching knowledge is not taken into account when investigating general teacher decision making. Although these studies contribute to acknowledging teachers as taking a major role in the achievement of students, they still do not include a teacher’s understanding of the subject matter being taught. And again, if teachers influence the achievement of students, then the teachers' content knowledge and teachers' knowledge of how to teach that content are likely to be significant factors.

Lee Shulman (1986a, 1986b) brought subject matter to the forefront within education research with his work in the mid-1980s. He recognized the need to include teacher content knowledge in research. His most important contribution may be his focus on a type of teacher knowledge he named *pedagogical content knowledge*. Pedagogical content knowledge, as the name implies, consists of teaching knowledge needed for teaching a specific subject matter. Shulman (1986a) wrote:

> Although research on teacher cognition may have yielded less than was anticipated in its first decade, it remains an area of immense promise. Changes in both teaching and teacher education will become operational through the minds and motives of teachers. Understanding how and why teachers plan for instruction, the explicit and implicit theories they bring to bear in their work, and the conceptions of subject matter that influence their explanations, directions, feedback and correctives, will continue as a central feature of research on teaching. (p. 14)

Subsequently more researchers began studying what kind of subject-matter knowledge teachers need, and what kind of knowledge affects the understandings and achievement of students.

One major research program that has stemmed from Shulman's work is the research by Deborah Ball and colleagues. Ball has worked to establish what it is teachers "do" and what kind of knowledge teachers need in order to successfully teach mathematics. She has then empirically tested her findings (Ball, 1991; Ball, 2000; Ball, 2003; Ball, Hill, & Bass, 2005; Ball, Lubienski, & Mewborn, 2001). She utilizes what she calls a "practice-based theory of mathematical knowledge for teaching" (Ball, 2001, p. 452). In her research she studies instruction in the classroom and then analyzes the data to discover what mathematical knowledge is needed for teachers to teach effectively. This mathematical knowledge for teachers includes and builds on content knowledge and pedagogical content knowledge. Once Ball believed she was able to identify teachers' knowledge, she and her colleagues developed and tested measures of mathematical knowledge for teachers. From her studies we have a better idea of the knowledge teachers need and we can further conclude that teacher knowledge is positively correlated with student achievement.

Part of Ball's research is qualitative in nature. Ball and colleagues have used a longitudinal study to develop their theory of mathematical knowledge as it is entailed by and used in teaching (Ball & Cohen, 1999). Using data collected in observation, student homework and class work, and teacher's lesson plans and reflections, Ball explored teaching, which she interpreted as everything teachers do to support student learning. This qualitative work focused on teacher knowledge in an effort to answer Shuman's call for developing and refining the knowledge of content needed by teachers. Ball's qualitative work is ongoing as she continues to refine the subcategories she has developed for content knowledge and pedagogical content knowledge.

A second part of Ball’s research is quantitative in nature. Ball’s measures of knowledge for teaching take the form of multiple choice tests, which have been developed using thousands of teachers. This work has further helped to refine, test, and articulate the subcategories of knowledge for teaching (Ball, Thames, & Phelps, 2008). In addition, Ball’s team has used quantitative research to test the effects of teachers’ mathematical knowledge for teaching on student achievement (Hill, Rowan, & Ball, 2005). Using measures they developed and refined through previous qualitative and quantitative work, they found that teacher’s mathematical knowledge was significantly related to student achievement gains. Their purpose was “to demonstrate the independent contribution of teachers’ mathematical knowledge for teaching to student achievement, independent of other possible measures of teacher quality such as teacher certification, educational coursework, and experience” (Hill et al., 2005, p. 375).

Ball’s results show that content knowledge for teachers is multidimensional and she represents those dimensions with subcategories of content and pedagogical content knowledge. She recognizes the need for further exploration of these categories, given that her work is a work in progress. Furthermore, she recognizes the need to explore how this improved understanding of mathematical knowledge for teaching influences student achievement (Ball et al., 2008).

If we proceed under the assumption that teacher knowledge does affect student achievement, then the question of how teacher knowledge affects student achievement is still left to be answered. A preservice elementary school teacher may be told what types of knowledge a teacher needs and that this knowledge affects what students will learn, but he or she should also understand how and why a teacher's understanding of this knowledge is important for students. The mathematics education field has a good enough understanding of what mathematical knowledge teachers should possess so that investigation of how this knowledge affects students' understandings is possible. If we assume that teacher understanding affects practice, and that teacher practice affects what students learn, then we can begin investigating these relationships. How does a teacher use their understanding in ways that affect the student and what the student understands about mathematics? How does the quality of a teacher's understanding of content knowledge and/or pedagogical content knowledge affect the quality of student understanding?

Before further explaining my study, it is necessary to briefly describe how I use the terms *knowledge* and *understanding*, although a more in-depth description will be provided in the literature review. In many articles and studies these words seem to be used interchangeably, or at the very least, without clarification. I will use the words *understanding* and *knowledge* in the way in which I believe most of the literature suggests. Understanding describes a person's, or group of peoples', particular internal grasp of a concept. One's understanding is a person's way of organizing and integrating knowledge (Davis, 2006). It is a continuum, meaning that one can, for example, have a partial understanding, an inferior understanding, a superior understanding, or a thorough understanding. On the other hand, I will use the word *knowledge* in two different ways. First, knowledge can be thought of as the body of facts that exists external to any one person (Fenstermacher, 1994). The mathematics education research field has developed a body of knowledge called teacher knowledge, which consists of facts and concepts, that is separate and external to a person's understanding of that knowledge. The abstract collection of facts is called teacher knowledge, and one particular teacher may have a degree of understanding of this collection of facts, and that would be the teacher's *understanding* of that teacher knowledge. Second, I also want to be able to claim that a person "knows" something, not only that they understand something. What a person "knows" is demonstrated by what they can declare or state. A teacher's "knowing" represents pieces of information a teacher may or may not possess. A teacher's understanding of what they know determines to what degree that knowledge is useful or detailed. Knowledge is what exists external to individuals; knowing and understanding are the ways an individual interacts with that knowledge.

Ball's research that investigates what teachers do and what knowledge is needed in the process has been very influential, but there is a lack of recognition of how a teacher's understanding of mathematics could affect a student's understanding of mathematics. Now that we have studies that show that teacher knowledge affects student achievement, it may be possible to have a more informative description of how a teacher's understanding affects students' understandings. Although there are likely to be many factors that contribute to this relationship, and a straightforward answer is likely not forthcoming, much can be learned by qualitatively investigating this relationship.

My study begins to investigate this relationship. I have gone back to teachers in practice to investigate how a teacher's understanding of mathematical knowledge, both content knowledge and pedagogical content knowledge, influences what a student learns. In three cases, I describe qualitatively the nature of a teacher's understanding of a particular mathematical topic and investigate how that understanding affects the teacher's practice while teaching that topic. How does a teacher use his or her understanding in decision making and day-to-day teaching? In addition, on an exploratory basis, I examine the link between the teacher understanding and student understanding in order to hypothesize about the relationships and connections between them.

Because so little work has been done qualitatively in this area, I use the word *hypothesize*. I will be making suggestions about the connections between teacher and student understanding. Connections between teacher understanding and teacher practice seem to be more tangible and relatively easier to discern, because there is more straightforward and accessible data that describes this relationship (Simon, 1995). But because teacher practice influences student understanding, and because the relationship between teacher understanding and student understanding is important, some progress needs to be made in understanding this relationship. Because it is a first step in this direction for mathematics education research, my study is exploratory in nature and will seek out possible connections that will need to be explored further. Since teacher knowledge is so expansive and varied, I limit this study to exploring one mathematical topic in light of only a few aspects of teacher knowledge.

In this study I will examine the understanding of three teachers regarding the content knowledge and pedagogical content knowledge research has shown to be integral in teaching addition and subtraction word problems. Analyzing these three cases, I hope to be able to draw some conclusions as to the connections between the teachers' understandings and the students' understandings. In the following paragraphs I will explain my research questions and how they relate to the research regarding content knowledge and pedagogical content knowledge.

Research Questions

1) How does a teacher's understanding of the structure of addition and subtraction word problems affect student understanding of addition and subtraction?
    a. How does a teacher's understanding of the structure of addition and subtraction word problems affect her practice?
    b. How do these practices influence student understanding of addition and subtraction?
2) How does a teacher's understanding of how students typically solve addition and subtraction word problems affect student understanding of addition and subtraction?
    a. How does a teacher's understanding of the way students solve addition and subtraction word problems affect her practice?

b. How do these practices influence student understanding of addition and subtraction?

Overview of Study

There are many studies that examine student achievement, in which student achievement is usually measured by scores on tests over a period of time (Begle, 1972; Croninger, Rice, Rathbun, & Nishio, 2007). In my study student achievement is investigated qualitatively, by describing the nature and depth of the students' understandings. Although it is important for students to be able to score well on tests, and using such tests is a legitimate way to investigate student achievement, a student's understandings should also be investigated and described qualitatively so that teachers and preservice teachers can be better informed about what such understanding entails and the types of understandings students possess or are able to possess.

Previously, teacher knowledge has been investigated qualitatively in relation to student achievement, but the knowledge was usually focused on general pedagogical knowledge, such as expert/novice literature that investigates constructs such as teacher planning, classroom management, and teacher reflection (Borko & Livingston, 1989). Instead of investigating general teaching knowledge in relation to overall student gains in achievement, this study investigates three cases of teachers' understanding of specific pedagogical and content knowledge in relation to the nature of student understanding.

I chose to use addition and subtraction as the mathematical topic for this study because of the large amount of previous research on teacher knowledge of addition and subtraction that was available, largely due to the work of the Cognitively Guided Instruction (CGI) project (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Carpenter, Fennema, Franke, Levi, & Empson 1999; Carpenter, Moser, Bebout, 1988). Through the CGI work we have an extensive body of research describing the content and pedagogical content knowledge teachers need to teach addition and subtraction. This knowledge includes an understanding of how students think about problems, how to model and represent problems, and a progression of the difficulty level of problems. There are few other elementary mathematical topics, if any, for which we have access to so much teacher content and pedagogical content knowledge. Therefore, I chose this mathematical topic as a base for my research study. I use the subject-matter knowledge for teachers that CGI has claimed is relevant and explore how that type of knowledge affects teacher practice and student understanding.

In addition, there exists previous research that suggests teachers without this understanding had a negative effect on student achievement on addition and subtraction problems. In a study by Carpenter, Hiebert, and Moser (1983), children's problem solving abilities and understanding actually declined after instruction. This study was early in the work of Carpenter and colleagues, and they have has since delineated what knowledge a teacher should have when teaching addition and subtraction. Therefore, this previous study and Carpenter's further work suggest that teacher knowledge of addition and subtraction has the potential to affect student understanding. It will be important to see how a teacher who possesses such knowledge might positively influence student understanding.

The first question was chosen because the structure of the addition and subtraction word problems is such a crucial aspect of teacher knowledge for this topic (Carpenter, et al., 1999). The type of problem (Join, Separate, Part–Part–Whole, Compare) and the types of knowns and unknowns in the problem are two components that designate the structure of the word problem. CGI claims that these two components are part of a child’s natural understanding of addition and subtraction word problems. This first research question represents one vital aspect of a teacher’s content knowledge and with it I will begin to explore how a teacher’s understanding of content knowledge of addition and subtraction word problems can influence student understanding.

The second research question represents a significant aspect of research by Carpenter and colleagues on teacher’s pedagogical content knowledge. The CGI work has developed a comprehensive description of the relative ease and difficulty of different types of addition and subtraction word problems (Carpenter et al., 1999). It has uncovered how students model and represent the different types, and how they are likely to solve word problems. This question was chosen so I could explore how an understanding of this knowledge could affect the decisions a teacher makes, which can in turn influence student understanding.

In the next chapter I will synthesize the literature that provided the conceptual and theoretical framework for my study. Following the literature synthesis, I will describe the methodology used to carry out the study. The fourth chapter describes the common results I found from the teachers and students in my study. The subsequent three chapters detail the results specific to the three individual teachers I observed and interviewed for the study, namely their understandings and their practice. The three chapters about the individual teachers contain my interpretation of how each teacher's understandings influenced her practice and her students' understandings. The last chapter will discuss my results and offer a hypothesis describing a relationship among teacher understanding, teacher practice, and student understanding.

CHAPTER TWO

LITERATURE REVIEW

This chapter will provide the conceptual and theoretical groundwork for how I perceive knowledge and understanding, teacher knowledge, student achievement, and the possible effects teacher knowledge has on student achievement. In addition I will discuss research that details what teacher understanding and student understanding might entail in the context of addition and subtraction at the elementary school level.

Knowledge Versus Understanding

The way I define the word knowledge is similar to the type of knowledge described by the ancient Greek word *episteme* (Eisner, 2002; Fenstermacher, 1994). *Episteme* can be thought of as scientific knowledge, declarative knowledge, propositional knowledge, informational knowledge and/or formal knowledge, all of which are overlapping types of concrete knowledge. It refers to knowledge that is true and certain. Knowledge is straightforward and it is measurable. I think of knowledge as facts that one either knows or does not know. Paper-and-pencil tests often serve to establish whether a student possesses knowledge of how to solve specific addition and subtraction problems, but even if that is a good measure of a person's knowledge, he or she may posses a piece of knowledge without understanding it.

Understanding is also measurable, but not as easily measured as knowledge, because it is more complex than reciting facts or data. Understanding is not a single concept, "but a family of interrelated abilities" that develop over time in an individual (Wiggins & McTighe, 1998, p. 5). Understanding involves "sophisticated insights and abilities, reflected in varied performances and contexts" (Wiggins & McTighe, 1998, p. 5). It is characterized by internal representations that get progressively more cohesive and structured (Hiebert & Carpenter, 1992). Understanding requires the person who posseses this understanding to be able to do certain things, and measurement of understanding relies on the degree to which the person is able to do those certain things. Understanding is a continuum (Davis, 2006), and therefore a person can have different types of understandings such as a broad understanding, a good understanding, a weak understanding, a poor understanding, or a "profound" understanding (Ma, 1999).

A person's *knowledge* consists of the facts he possesses. A person's *understanding* is a measure of the degree to which those facts are cohesive and structured to the individual. In order to be able to measure the understanding of the participants of this study, it was necessary to establish indicators of understanding that I could analyze. Because understanding requires the person to whom that understanding is attributed to be able to act on their knowledge, research regarding different ways people show understanding, or indicators of understanding, are necessary for data collection.

Wiggins and McTighe (1998) describe six indicators of understanding: *explanation, interpretation, application, perspective, empathy, and self-knowledge*. Understanding depends on the degree to which a person can *explain* or provide information about the required events, actions, or ideas. Investigating a teacher's understanding of mathematics, one might ask the teacher to provide explanations for actions or explanations about mathematical concepts. Understanding also depends upon the degree to which a person can provide *interpretation* of events, actions, or concepts. Explaining might require recognition of the details of a mathematical concept, whereas interpreting might require recognition of why a mathematical concept works.

Application is another important indicator of understanding. Someone who understands a concept should be able to use it in unfamiliar settings and diverse contexts. For example, a student who understands how to solve a Part–Part–Whole problem with the whole unknown, should be able to solve other problems of the same type. Understanding also depends upon the ability of a person to study a concept with *perspective*. This means the ability to critically investigate an idea, event, or action from a distance. Having *perspective* means being able to point out the assumptions in a problem, describe what needs to be justified, and decide what is reasonable within the problem.

Understanding a concept depends on one's ability to *empathize* with another person's view of the concept. A person's understanding of an idea, event, or action depends on the degree to which they can see multiple points of view, or multiple ways of thinking about the same concept. A teacher who understands a mathematical concept can better see how a student is thinking about that concept and tailor his or her words to facilitate student learning. *Self-knowledge* is Wiggins and McTighe's last indicator of understanding. Having *self-knowledge* means being able to see one's strengths and weaknesses within a particular understanding, or recognizing what one does or does not understand. A teacher's understanding depends on the degree to which she sees how personal biases inform but also weaken understanding. For example, a teacher who is aware of strengths and weakness in how she views addition and subtraction may pay attention to her weakness when struggling to explain a problem to students. She may ask herself, "What am I prone to misunderstand because of prejudice, habit, or style?" (Wiggins & McTighe, 1998, p. 57) Understanding can grow when one understands the limits of one's own understanding.

Sierpinska (1994) also identifies four indicators, which she calls mental operations, involved in understanding: *identification, discrimination, generalization,* and *synthesis.* She describes *identification* as the discovery or realization that a concept, previously in the background of consciousness, needs to be understood. As learning takes place, more objects that were in the background of one's mind come to the foreground through a reorganization of consciousness. For Sierpinska *discrimination* is the ability or act of recognizing differences between two objects or concepts. With understanding, one should be able to recognize characteristics of the object or concept trying to be understood, and also be able to recognize how the object or concept is distinct from others. In this research *generalization* is a mental operation in which the learner is able to recognize an object or concept as a particular case of another situation; and *synthesis* is the mental search for a unifying theme among multiple generalizations.

When I searched for the participants' understandings in my study, I chose to refer to both Sierpinska's work and the work of Wiggins and McTighe because they both offered valuable descriptions of indicators of understanding. The two groups of researchers offered different insights into in indicators of understanding, so I chose to include elements from both. With a broad definition of understanding I hope to be able to capture any hint or nuance of the students' understandings. Before interviewing and observing I focused my questions on aspects of the mental operations I thought I would be able to discern from and measure during interviews, namely, students' ability to *explain, interpret, discriminate, apply,* and *generalize*. However, during the data collection process I was always searching for any indication or action that signified understanding, including ones in addition to those aspects listed above. In a subsequent section about student understanding I will more fully describe why I chose those particular indicators of understanding and why I think they constitute a synthesis of the research on understanding.

Student Achievement: A Look at Understanding

There are many methods that can be used to describe and measure students' achievement. A common method in education literature is the use of some form of academic achievement test, such as national or state tests. For example, studies may use released items from *The National Assessment of Educational Progress* (NAEP), SAT items, or the Iowa Test of Educational Development (ITED) (Schoen, Cebulla, Finn, & Fi, 2003). There are also many research teams that construct their own quantitative measures to study student achievement (Cramer, Post, & delMas, 2002; Parker & Gerber, 2000). Often researchers will give a pretest and then posttest, compare scores, and then use this number as a means to measure growth (Hill, Rowan et al., 2005). Using well-constructed tests is a beneficial way to quantitatively measure a change in students' knowledge as an indicator of student achievement.

Instead of using standard tests to decide how much students know, researchers can also use techniques such as interviews to identify characteristics of students' understandings. For example, Cramer, Post, and delMas (2002) used student interviews to identify differences in students' thinking about fractions. The interview questions reflected the researchers' conceptions of the understandings that students beginning to study fraction should have. By analyzing answers to interview questions, Cramer and colleagues were able to better describe students' understandings of fractions.

My study also uses a qualitative approach to identifying student achievement. Instead of quantifying the amount of knowledge students have gained, I investigate the quality of student mathematical understanding of addition and subtraction. I determine the nature of teachers' understandings and see how they use that understanding to influence the nature of students' understandings. I investigate the nature of student mathematical understanding when it is influenced by teachers' mathematical knowledge for teaching. I am choosing to investigate this relationship qualitatively so I can try to discern the nature of the understandings that students have gained.

In order to effectively analyze student mathematical understanding, I needed to articulate how I would investigate my participants' understandings. To do this I referred both to Sierpinska's work and the work of Wiggins and McTighe as described previously. Each work describes indicators of understanding and mental operations involved in understanding mathematics. I chose to search for students' ability to *identify, explain, interpret, discriminate, apply,* and *generalize*. The questions I asked the students I observed attempted to address these indicators of understanding. My direct questioning did not focus on several categories in each framework, not because I did not view them as indicating understanding, but because they either overlap with other categories named or are less measurable in such young students because those students may not have reached a high enough level of understanding for those aspects to be demonstrated. I did not exclude them entirely, because I still searched for those indications of understanding, but I did not focus my interview questions on those particular indicators.

I believe the indicators of *synthesis, perspective, empathy,* and *self-knowledge* are difficult to measure in primary-grade students. These young students do not have the cognitive development to operate with that level of abstraction. These abilities require a more sophisticated use of the information with which they are working (Biggs & Collis, 1982; Collis, Romberg, & Jurdak, 1986).

I am interested in students' abilities to *identify, explain, interpret, discriminate, apply,* and generalize—I am not interested only in students' ability to get the correct answer. I am not investigating their ability to perform on a specific set of problems; I am investigating their ability to demonstrate these specific indicators of understanding.

Teacher Knowledge

The following paragraphs serve to examine research that investigates what teacher knowledge is and what aspects of teacher knowledge are useful in measuring student achievement. I use the word knowledge because I will be discussing teachers' knowledge abstractly, as a body of information, as opposed to a particular teacher's knowledge or understanding. I am interested in understanding the kinds of knowledge teachers utilize in their practice, such as decision making, lesson planning, and student interaction, and how that knowledge affects students and their achievement.

Since the 1980s there has been a great deal of research dedicated to exploring teachers' knowledge. Many different types of knowledge have been labeled and investigated. This review will focus on three of these types of knowledge: content knowledge, pedagogical content knowledge, and general pedagogical knowledge. I will describe how the literature treats these three types of knowledge and how I choose to define them. I will articulate my reasons for focusing on these categories of knowledge instead of other categories, and examine how these types of knowledge are related. I will also discuss how the differences between knowledge and beliefs can be difficult to negotiate, and how the combination of all of these ideas contributes to teacher effectiveness and student achievement.

Content knowledge. Although content knowledge may seem like an obvious choice for an important type of teacher knowledge, this type of research did not flourish until the 1980s (Borko & Putnam, 1996). Researchers continue to be involved in providing distinctions concerning how much and what kind of understanding of content knowledge is necessary. Researchers also continue to investigate how to reliably and validly measure teachers' content knowledge.

One approach to describing teachers' content knowledge is to access the number of courses, degrees, and certifications a teacher has completed (Ball, Lubienski et al., 2001; Carpenter, Fennema et al., 1988). This approach assumes that all teachers who take a given mathematics class will learn the same amount and same kind of mathematics. It also assumes that the more mathematics a teacher knows the better that teacher will be, resulting in students who know more mathematics. Although the methodology for this approach is easy to implement, the results have been surprising, if not confusing, to many researchers.

An early and well-known study of this type was a meta-analysis by Edward Begle (1979). After comparing the relationship between the number of mathematics courses teachers had taken beyond calculus and the performance of these teachers' students, Begle found positive main effects in only 10% of the cases. Even more surprising was the finding of negative main effects in 8% of the cases. There are many possible explanations for this result, such as teachers were not learning their courses' required content, or that only a limited number of courses were having an effect on teacher knowledge. But the most obvious result was the realization that quantifying the number of mathematics courses a teacher has taken does not explain the effectiveness of a mathematics teacher. There are many other studies that use these types of measures of knowledge (e.g., Monk, 1994), and some with conflicting results, but this approach to measuring teachers' content knowledge does not tell the whole story and other insights are important for advancing the field's understanding of the knowledge needed for teaching. Although certification and the types of classes teachers take should certainly not be ignored, more thought is needed to determine what kind of mathematics a teacher needs to know and what it means for teachers to know mathematics. It is possible that these types of measures are useful to measure what teachers know but not what they understand, which might be more applicable for investigating how teachers influence students.

A second approach to measuring and examining content knowledge involves a more detailed analysis of teachers' mathematical knowledge instead of investigating second-order measures like degrees and certificates. These studies use specific mathematical topics rather than broader mathematical knowledge (Ball et al., 2001). This second approach utilizes qualitative research methods and emphasizes tools such as interviewing mathematics teachers to understand the nature of the mathematical knowledge. Researchers have sought to establish what constitutes conceptual mathematical knowledge for teachers and students, as well as what kind of understanding of mathematical knowledge is needed specifically for teaching (Ball, 1991; Shulman, 1986b).

Shulman's (1986b) work provides the most commonly used conception of content knowledge, and the definition I will continue to use.

> [Content Knowledge] requires understanding the structures of the subject matter…The structures of a subject include both the substantive and the syntactic structures. The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts. The syntactic structures of a discipline are the set of ways in which truth or falsehood, validity or invalidity, are established. (p. 9)

This framework for content knowledge has been further developed by other researchers. Procedural fluency, knowledge *of* mathematics, and knowledge *about* mathematics, and a person's conceptual understanding of knowledge are some of the concepts influencing the development of this framework, which I will explore in the following paragraphs.

Content knowledge includes knowledge *of* mathematics and knowledge *about* mathematics (Ball, 1991). Ball contributes to the framework of content knowledge for teachers by distinguishing between knowledge *of* mathematics and knowledge *about* mathematics. Knowledge *of* mathematics is the part of subject matter knowledge that is most easily recognized as necessary; it includes knowledge of particular topics, procedures, and concepts. Another critical dimension, however, is knowledge *about* mathematics. This includes:

> Understandings about the nature of knowledge in the discipline—where it comes from, how it changes, and how truth is established. Knowledge about mathematics also includes what it means to "know" and "do" mathematics, the relative centrality of different

> ideas, as well as what is arbitrary or conventional versus what is necessary or logical…. (p. 12)

This second component of content knowledge is not as easily found in students or in teachers, but is an important aspect of a person's understanding of content knowledge.

Knowledge *of* the mathematics and teachers' mental organizations of the knowledge they have *about* mathematics seem to have an effect on the way they teach, and therefore have an impact on students (Fennema & Franke, 1992). For example, Ms. Jackson was an elementary school teacher interviewed and observed by researchers in the Cognitively Guided Instruction (CGI) project (Carpenter et al., 1989). She was judged to have a much richer conceptual understanding of addition and subtraction than of fractions. Her understanding of content knowledge had a significant impact on her decisions and actions in the classroom. Her methods of instruction, the central activities of her class, and the type of problems she used differed according to her understanding of content knowledge. When teaching addition and subtraction, she implemented more problem solving and introduced a wide variety of problems to the class. Ms. Jackson allowed her students to direct the discourse in her classroom and students often wrote and solved their own problems. However, when teaching fractions, Ms. Jackson used only one problem type, directed the discourse in the classroom more often, and was not as able to answer student questions. There was less problem solving and less discussion. Although the direct effects of these actions on the students were not measured, it seems reasonable to assume there was some impact. More research is necessary to know exactly how students are impacted with this type of variability in content knowledge.

A rich and integrated understanding of content knowledge both *of* and *about* mathematics has an effect on teaching. Teachers with an understanding of knowledge *of* and *about* mathematics have an understanding enabling them to focus on important ideas, conceptual explanations, and connections among topics within the curriculum (Borko & Putnam, 1996; Stein, Baxter, & Leinhardt, 1990). The teacher's understanding of content knowledge affects her practice, which can potentially influence students.

Not only can a teacher's understanding of content knowledge be rich and integrated, it can also be an explicit way of knowing, as opposed to a tacit way of knowing (Ball, 1991). A teacher possessing an adequate conceptual understanding needs to be able to explain to a student why she cannot divide by zero or why, when dividing by fractions, she is supposed to "invert and multiply." Many skillful mathematicians can untangle complicated problems without being able to explicitly articulate the reasons for their actions, but a skillful teacher must have more than a tacit way of understanding, a good teacher is able to express his or her ways of understanding so that a student can learn. Similarly, being able to "decompress" or "unpack" one's understanding is an important characteristic of a teacher's understanding of content knowledge (Ball & Bass, 2000). Ball describes this decompression as the ability to break down his or her understanding so that basic components are accessible and visible. For teachers, this means being able to decompress their mathematical knowledge into elements visible to students. Often, in a context other than teaching, knowledge is useful in a compressed form and is desired in a final, compact state. Teachers, however, need to be able to work with students whose understandings are still growing and therefore a teacher's understanding is most useful when it can be broken down into its elemental forms.

Pedagogical content knowledge. As mentioned previously, Shulman (1986a, 1986b, 1987) significantly changed the course of education research when he brought subject matter to the forefront. His most influential contribution to the field was the development of a type of teacher knowledge he named *pedagogical content knowledge*. Pedagogical content knowledge, as the name implies, consists of teaching knowledge needed for teaching a specific subject matter. Whereas historically it was thought that a teacher needed little more than an understanding of the subject in order to teach it, Shulman identified the importance of understanding how to teach a particular subject (Shulman, 1986a). Pedagogical content knowledge includes knowledge of:

> the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others . . . what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (p. 3)

Interest in the difference between expert and novice teachers led researchers to understand that there were important differences not only among teachers' understandings of content knowledge but also among teachers' understandings of pedagogical content knowledge. These differences could lead to gains in student achievement (Bullough Jr., 2001). The realization of the importance of this type of knowledge has changed education research. Part of this change includes a focus on the teacher and the decisions the teacher makes in the classroom.

Although content knowledge and pedagogical content knowledge are clearly related, they are different types of knowledge (Borko & Putnam, 1996), and have been shown statistically to be different from one another (Ball et al., 2005). Teachers who took tests that reliably measure both types of knowledge, had scores that were significantly dissimilar from one type of knowledge to the other. A teacher, for instance, might have the content knowledge necessary to carry out a particular multiplication problem, but still not be able to analyze the errors in a student's multiplication problem. A teacher can posses the content knowledge necessary to mathematically understand how a nonstandard algorithm is correct, but still not have an understanding consisting of pedagogical content knowledge to know when to implement it, or how students might benefit from its application to mathematics teaching.

A study of pedagogical content knowledge, determining what pedagogical content knowledge is and how it is different from content knowledge, illustrates the difference between the two research questions I proposed, given that one question focuses on content knowledge and the other focuses on pedagogical content knowledge. Researchers have attempted to categorize different types of pedagogical content knowledge (Barnett & Hodson, 2001; Graeber, 1999; Grossman, 1990). Some of these categories have included a teacher's overarching conception of the purposes for teaching a subject matter, knowledge of student understandings and misunderstandings of a subject matter, knowledge of curriculum and materials, and knowledge of strategies and representations useful for teaching a particular subject matter. It is not general knowledge that is necessary when teaching any subject, nor is it subject matter knowledge a nonteaching mathematician would need. Primarily, this knowledge requires a grasp of how to instruct students most effectively within a given subject matter.

Adequate distinction between teacher beliefs and teacher knowledge, two elements that can easily become intertwined, is also necessary. Studies about teacher beliefs in mathematics education research have sometimes utilized personal theories of teaching and learning (Calderhead, 1996; Thompson, 1992). To add to the complication, it has been noted that teachers treat their beliefs as knowledge, which makes it even more difficult to distinguish between the two (Grossman, Wilson, & Shulman, 1989). In addition, beliefs can be held with a varying degree of conviction. Moreover, what might have been judged as a belief at one time might be accepted as knowledge in light of new theories. This transference goes both ways. New theories can interpret something once considered to be teacher knowledge to be, in fact, belief (Thompson, 1992). Of course, research regarding teachers' beliefs is important in its own right, but serves a different purpose from research on teachers' knowledge, and thus it is important to be able to distinguish between the two.

The potential effects on student achievement make pedagogical content knowledge an important domain to define and measure in teachers. Measuring pedagogical content knowledge requires asking a different set of questions from that used for measuring content knowledge, but it is reasonable to imagine a set of questions that focus on the nature of teachers' pedagogical content knowledge. Such questions would include an evaluation of teachers' understandings of the knowledge that students bring to learning a concept, common misconceptions, stages of learning a concept, representations and explanations for important aspects of a concept, techniques for assessing student understanding, and/or knowledge of instructional strategies.

General pedagogical knowledge. This study questions how teacher understanding affects student understanding through teacher practice. The research questions address teacher understanding of two types of teacher knowledge, content knowledge and pedagogical content knowledge. There are other types of knowledge, such as general pedagogical knowledge, which will not be included in the study. The next paragraphs briefly explain the nature of this knowledge and why it is not included in this study.

General pedagogical knowledge is unique from content knowledge and pedagogical content knowledge because, as the name implies, it is not dependent on the subject matter being taught. Imagine a teacher who possesses a strong conceptual understanding of mathematics, is well-versed in the knowledge necessary for teaching mathematics, but is still unable to be an effective elementary school mathematics teacher. Is this possible? With a sufficient grasp on the first two types of knowledge, can one still be ineffective as a teacher? I believe there could be a teacher such as this, who did not understand how children learn, or how to communicate well with children. A teacher could possess the first two types of knowledge but still be unable to manage a classroom and smoothly conduct a lesson. The type of knowledge for teaching that these skills require is general pedagogical knowledge; the type of knowledge a teacher would need, regardless of subject being taught, to effectively operate in a classroom.

Since the 1980s, the focus of research on teaching and teachers has shifted away from general knowledge to content-dependent knowledge (Borko & Putnam, 1996). Although the development of teachers' knowledge with respect to a specific content has been vital to the growth of the profession, general pedagogical knowledge is still an important and interesting part of teacher knowledge. This type of knowledge is extremely varied and extensive, making it difficult to define, categorize, or measure. Essentially any type of knowledge a teacher has or needs that is not content dependent could be placed in this category. Classroom management strategies and instructional strategies are a large part of general pedagogical knowledge that have been studied by many researchers (Berliner, 1986; Borko & Putnam, 1996; Doyle, 1986; Leinhardt & Greeno, 1986). Teachers use their knowledge to make decisions regarding managing and organizing a classroom in order to promote thinking and learning. They must have strategies for establishing rules, monitoring and pacing events, and reacting to misbehavior.

More often than not, understanding of general pedagogical knowledge is implicit in teachers, making it difficult to measure or analyze (Borko & Putnam, 1996; Meijer, 2002). The fact that general pedagogical knowledge is often situational only makes measuring this type of knowledge more impractical. Much of teachers' understandings of general pedagogical knowledge is situated because it is learned within their own experience (Fennema & Franke, 1992). It is not easy to investigate which types of decisions teachers make using their pedagogical knowledge are generalizable, useful, or the kind of knowledge that should be measured, analyzed, and passed on to future generations of teachers. In addition, until general pedagogical knowledge is characterized accordingly it will be difficult to study how this knowledge affects student achievement.

Conclusions on teacher knowledge. My research focuses on content knowledge and pedagogical content knowledge, or using Deborah Ball's terminology, the mathematical knowledge needed for teaching (Ball et al., 2008). Because I hope to build on Ball's work, it is useful to understand her contribution to research on teacher knowledge. The figure that follows describes Ball's recent understanding of "Mathematical Knowledge for Teaching" (Ball, Phelps, & Thames, 2008).

Mathematical Knowledge for Teaching

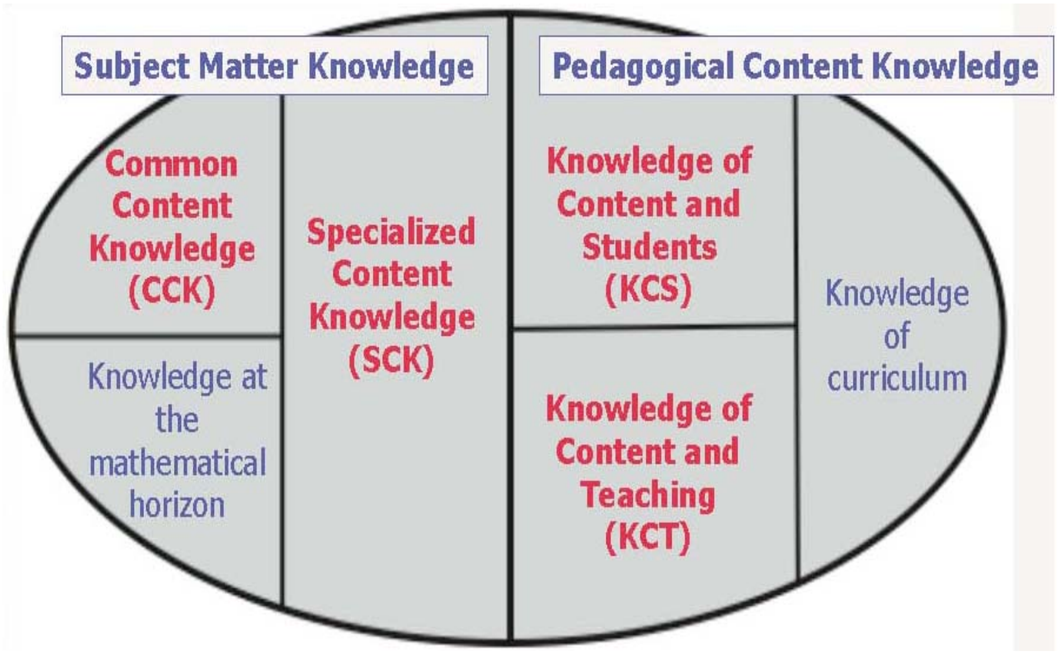

Figure 1. Mathematical Knowledge for Teaching (Ball et. al., 2008).

Within the domain of subject matter knowledge, Ball has classified Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), and Knowledge at the Mathematical Horizon. Common Content Knowledge (CCK) involves knowing how to solve mathematical problems and is the knowledge and skill also used in other settings, not specifically for teaching mathematics. Teachers need to understand how to solve problems, use correct notation, and read mathematical texts, which are activities that people besides mathematics teachers need to master. Specialized Content Knowledge (SCK) is content knowledge used specifically by teachers. Examples include responding to students' "why" questions, connecting a topic being taught to topics from previous and future years, and evaluating the plausibility of students' claims (Ball et al., 2008). SCK includes exploring the structure of problems, such as the differences between partitive and quotitive division problems or the differences between Join and Separate addition and subtraction problems (Carpenter et al., 1999). The third domain in subject matter knowledge Ball provisionally calls Knowledge at the Mathematical Horizon. This type of knowledge she claims we know very little about but may include knowledge of how all mathematical topics are related.

Pedagogical Content Knowledge also contains three subdomains. Knowledge of Content and Students (KCS) is knowledge that combines knowing about students and knowing about content. KCS involves knowing how students think about particular mathematics topics, how they solve problems, and conceptions or misconceptions they may have. Similarly, Knowledge of Content and Teaching (KCT) combines knowledge of a mathematical idea and knowledge of how to teach that idea. Knowing how to sequence instruction, knowing which examples to use, knowing which student contribution needs further elaboration, all involve knowledge that is an interaction between pedagogy and subject matter. Finally, Ball provisionally places Knowledge of Curriculum under pedagogical content knowledge. This is another subdomain about which we know very little. Although Shulman initially defines this type of knowledge and Ball includes it in her work, the importance and details of this knowledge is yet to be determined. Ball admits that her categorization is not definitive and hopes that further research will aid in more accurately articulating the types of knowledge, as well as add to the kinds of teacher knowledge. For now, these distinctions aided in my decision of the types of knowledge on which my research questions could focus.

Teaching is an artistry (Eisner, 2002), in which teachers are artists equipped with tools to encourage success. Teacher knowledge is an important part of the tools for teacher success, and although teaching is individualistic, situated, and contextual, there are tools such as teacher knowledge that can be analyzed and developed to encourage more successful teaching. Previous research shows a link between content knowledge and student achievement, and pedagogical content knowledge and student achievement. Most of these studies have been quantitative in nature, showing a correlation between teacher knowledge and student achievement. Further research, qualitative in nature, may give the field more of a sense of how this relationship works, exposing how teachers use their understanding, why their understanding is important, or how students are affected by the understandings of their teachers.

Evidence of Teacher Influence on Student Achievement

In addition to the quantitative studies that suggest that teacher knowledge affects student achievement (Hill et al., 2005), there are more descriptive examples in education literature of teachers drawing on their knowing and understanding in order to influence student achievement. Science education and literacy are both examples of fields of study in which there exists research that links teacher knowledge, teacher practice, and student achievement. These studies lay the groundwork for studying the connection between teacher understanding and student understanding in mathematics education.

Through a professional development course in literacy, it has been shown that teachers were able to increase their understandings, use those understandings to change and improve their teaching, and improve student achievement as well (Smith et al., 2001). Teachers were taught the relevant aspects of the early-literacy research base during professional development. Teachers who were able to gain an understanding of the applicable content knowledge and pedagogical content knowledge were able to change their teaching in two ways. First, the overall content the teachers taught changed. For the literacy teachers this included explicit instruction in phonemic awareness and letter sounds. The researchers explain this as a “macro” change in early literacy instruction, changing the “big ideas” that the teachers deemed critical to instruction. The teachers incorporated the literacy content they learned during professional development to the teaching in their own classroom. Second, the teachers made “micro” changes to instruction. These smaller changes that were a result of their recent increased understanding include an increased awareness of the effects that specific aspects of instruction had on student learning. For example, teachers paid more attention to the wording they used while teaching and tried to become more consistent over time with their wording. These changes in teacher practice, which stemmed from increased teacher understanding, had a positive effect on student achievement. Over the course of four years students continued to make large gains on the achievement test given.

A similar result was found when teachers took a course about the role of morphemes in students’ reading and spelling (Hurry, Nunes, Bryant, & Pretzlik, 2005). Through attending the course, teachers' knowing and understanding of content and pedagogical content knowledge was increased. This increased understanding was reflected in their practice as seen by the employment of different teaching methods and the focus on pertinent content. In turn, students made significant gains in spelling on various achievement tests when compared to a control group. These examples of literacy research demonstrate that it is possible to increase teachers' understandings, and that teachers can use their understandings to change classroom practice. Furthermore, changes in teacher understanding and classroom practice can improve student learning (McCutchen et al., 2002). My study attempts to build on this research by extending the ideas to mathematics education. Similarly, I would like to discover how understanding changes practices when teaching mathematics. I would also like to expand this line of research, investigating the cause of changes in practice and student understanding by qualitatively describing the nature of student understanding and how it has been impacted by teacher understanding.

Science education has a body of research detailing "conceptual change" in students, in which a teacher uses his or her knowing and understanding to strategically teach students with misconceptions about scientific principles. Understanding student misconceptions is an aspect of pedagogical content knowledge, and therefore relevant to my study about the effects of teacher understanding on student understanding. Without dealing with causation, this literature gives evidence that teachers use their knowing and understanding to make decisions in teaching that affect what students understand about science.

According to conceptual change literature, significant demands are placed on the quality of the teacher's understanding if the students are to better understand the principles being taught (Neale, Deborah, & Johnson, 1990; Scott, Asoko, & Driver, 1997). Teachers need to be aware of students' current misconceptions about the topic under consideration as well as likely pathways that could lead to a better understanding. Teachers need to be sensitive to the progress different students are making and be able to construct learning tasks that will support and encourage the change in students conceptual understanding. Also, teachers need to understand the topic well enough to be able to respond to the students' points of view. From empirical evidence this study shows that teaching for conceptual change necessitates a thorough understanding of the subject matter being taught and significant pedagogical content knowledge, including knowledge of students and knowledge of teaching (Anderson & Smith, 1987).

If teachers are relying on that knowledge and teaching for conceptual change then certain characteristics will be apparent in teaching (Neale et al., 1990; Smith et al., 1993):

1. Instruction is directed toward the contradiction of significant, general schemes students have about scientific principles. Instruction is directed toward the development of more powerful and accurate schemes.
2. Instruction is linked conceptually to prior lessons and students' informal experiences and knowledge.

3. Instruction elicits children's conceptions of natural phenomena through their own predictions and explanations.
4. Appropriate activities provide children opportunities to test predictions, discover evidence, and contrast explanations and conceptions.
5. Students represent their own thinking in several modes such as writing, speaking, drawing, and graphing. Students share and debate these representations with other students and the teacher.
6. Students summarize experiences and their results are related to prior and future lessons.

Evidence suggests that if teachers have the necessary content and pedagogical content knowledge and are able to utilize that knowledge using the enumerated teaching strategies in the classroom, then student achievement (as measured by pre- and posttests) will be significant (Eylon & Linn, 1988; Neale et al., 1990; Smith et al., 1993).

Teaching for conceptual change specifically focuses on changing students' scientific misconceptions and not just teaching in general. However, it seems reasonable that this research can be thought of in terms of teaching mathematics generally, in which addressing misconceptions is a part of teaching but not the sole focus. The previous teaching characteristics are certainly generalizable from science to mathematics education. This research indicates that if teachers have a certain degree of understanding of content and pedagogical content knowledge, they can use that knowledge in practice to cause or facilitate students to think differently about a subject through their practice. This encourages a belief that a teacher's knowing and understanding in mathematics has a link to student understanding in mathematics, through teacher practice.

Knowledge and Understanding in Addition and Subtraction

Research about content knowledge and pedagogical content knowledge in addition and subtraction word problems is extensive, which is my primary reason for choosing this mathematical topic. Thomas Carpenter and colleagues' work with Cognitively Guided Instruction (CGI) (Carpenter et al., 1988; Carpenter et al., 1989; Carpenter et al., 1996; Carpenter et al., 1999; Carpenter & Moser, 1984), is a widely accepted body of research that explicitly establishes the kind of content knowledge and pedagogical content knowledge that benefits teachers of primary grades. They have contributed substantially to the field, specifically in understanding addition and subtraction word problems and how children think about them.

Classifying addition and subtraction word problems. The CGI project claims that it is important for teachers to base their understanding of mathematics in the context of how children think about mathematics. They indicate that students intuitively solve word problems by modeling the action and the relationships given in the problem. Therefore, it is useful to classify problems based on the type of action or relationship within the problem, as opposed to classifying problems based on the operation used to solve the problem. Categorizing problem types in this way captures the differences between problems that children solve differently and provides a framework to identify relative difficulty of problems (Carpenter, Fennema, & Franke, 1996).

Figure 2 outlines the classification scheme used by CGI for addition and subtraction word problems (Carpenter et al., 1988).

Type	Problem		
	Result Unknown	Change Unknown	Start Unknown
Join	Connie had 5 marbles. Jim gave her 8 more. How many does Connie have altogether?	Connie has 5 marbles. How many more marbles does she need to win to have 13 altogether?	Connie had some marbles. Jim gave her 5 more marbles. Now she has 13 marbles. How many marbles did Connie have to start with?
Separate	Connie had 13 marbles. She gave 5 marbles to Jim. How many marbles does she have left?	Connie had 13 marbles. She gave some to Jim. Now she has 5 marbles left. How many marbles did Connie give to Jim?	Connie had some marbles. She gave 5 to Jim. Now she has 8 marbles left. How many marbles did Connie have to start with?
	Whole Unknown	Part Unknown	
Part–Part–Whole	Connie has 5 red marbles and 8 blue marbles. How many marbles does she have?	Connie has 13 marbles. Five are red and the rest are blue. How many blue marbles does Connie have?	
	Difference Unknown	does Connie have than Jim?	Compared Set Unknown
Compare	Connie has 13 marbles. Jim has 5 marbles. How many more marbles		Jim has 5 marbles. Connie has 8 more marbles than Jim. How many marbles does Connie have?

Referent Unknown	Connie has 13 marbles. She has 5 more than Jim.	How many marbles does Jim have?

Figure 2. Classification of Word Problems (Carpenter et al., 1988).

There are four types of word problems in this classification scheme, Join, Separate, Part–Part–Whole and Compare problems. The first two types of problems, Join and Separate, involve a direct or implied action. Join problems include problems in which a set is increased by a particular amount. We can examine the example used previously:

> Connie had 5 marbles. Jim gave her 8 more marbles. How many does Connie have altogether?

There are three quantities working together in Join problems, a start quantity, a change quantity, and a result quantity. The start quantity is the number of marbles with which Connie begins the problem. In Join problems the action takes place over time and there is a change. In this problem, the change quantity is the number of marbles Jim gives Connie. The result quantity is the number of marbles with which Connie ends the problem. Three distinct types of Join problems are generated when the unknown quantity is varied. Although all three types of problems are Join problems, they are solved quite differently when the change or start quantity is unknown as opposed to the result quantity being unknown.

Separate problems also have an action taking place over time. In this type of problem the start quantity is decreased instead of increased. For example, instead of Jim giving marbles to Connie, Connie is giving marbles to Jim (or taking away her own marbles). Similar to Join problems, there are also three quantities involved, a start quantity, a change quantity (the quantity removed), and a result quantity. By varying which quantity is unknown, three distinct Separate problems are created.

Part–Part–Whole and Compare problems involve static relationships. Unlike the Join or Separate problems, there is no action indicated and no change over time. Part–Part–Whole problems describe relationships between two subsets of one given set. Consider the given example:

> Connie has 5 red marbles and 8 blue marbles. How many marbles does she have?

This example has two subsets, red marbles and blue marbles, within the one given set of Connie's marbles. Both subsets play equivalent roles in the problem and only two types of Part–Part–Whole problems exist. Either the whole is unknown or one of the parts is unknown. A Part–Part–Whole problem can either give two of the parts and ask for the whole, or give the whole and one of the parts asking for the other part.

Compare problems describe the relationship between two distinct disjoint sets instead of a relationship between subsets. One set is compared to another set, and therefore two of the quantities can be labeled the Referent set and the Compared set. The third quantity is the Difference between the two sets, or the amount by which one exceeds the other. Consider the example:

> Connie has 13 marbles. Jim has 5 marbles. How many more marbles does Connie have than Jim?

No action is taking place so it is neither a Join nor a Separate problem. There is a relationship between two distinct sets instead of just one set of marbles, so it is not a Part–Part–Whole problem but a Compare problem. In this example Connie's marbles is the Referent set and Jim's marbles is the Compared set. The number of marbles by which Connie's set exceeds Jim's set is the Difference. With Compare problems any of the three quantities can be unknown, so there are three different types of Compare problems.

Modeling addition and subtraction word problems. The different types of addition and subtraction word problems are reflected in the way students solve problems. Most children invent their own modeling and counting strategies for solving word problems that are related to the structure of the problem (Carpenter et al., 1993). At first, children find solutions using direct modeling. They use physical objects (including fingers) to represent the quantities in the problem, and their own actions with the objects model the action in the word problem. There are many direct modeling strategies as described by the CGI group, including *Joining All, Joining To, Separating From, Separating To, Matching,* and *Trial and Error*.[1]

When *Joining All*, a strategy used for Join (Result Unknown) or Part–Part–Whole (Whole Unknown) word problems, two sets of objects are constructed, joined together, and then the student counts the union of the sets. When a student uses the *Joining To* strategy, most commonly for Join (Change Unknown) problems, he or she makes a set representing the initial quantity and then adds objects to it until the new collection of objects reaches the amount of the total given in the problem. In this case the number of objects added to the initial amount is the answer. These problems can be more difficult because the student must be able to keep track of the number of objects added on to the initial amount (Carpenter et al., 1999).

---

[1] Strategies are italicized to distinguish them from problem types, which are given in regular type.

Separating From is a strategy used most often when solving Separate (Result Unknown) problems. Children will construct a set of objects representing the initial quantity, remove the objects representing the change quantity, and then count the objects remaining. However, when the change quantity is unknown, the student uses a slightly different separating action. To solve a Separate (Change Unknown) problem the student constructs the initial quantity and objects are removed from the larger set until the number of objects that remain is equal to the result quantity. This is called *Separating To* and is more difficult because the child is unable simply to count objects as they are being physically removed. The child must be checking back to the initial set to determine whether the correct number of objects remain (Carpenter et al., 1999).

The *Matching* strategy is most commonly used in Compare (difference unknown) problems. When using this strategy, students construct two distinct sets of objects and then use a one-to-one correspondence between the two sets until one is exhausted. The objects that remain uncounted represent the answer. Occasionally children also use *Trial and Error* to solve problems, especially those difficult to model such as Start Unknown problems, in which the initial quantity is not apparent. Table 1 summarizes the direct modeling strategies (Carpenter et al., 1999).

Table 1

Direct Modeling Strategies

Problem	Strategy Description
Join (Result Unknown) Ellen had 3 tomatoes. She picked 5 more tomatoes. How many tomatoes does Ellen have now?	***Joining All*** A set of 3 objects and a set of 5 objects are constructed. The sets are joined and the union of the two sets is counted
Join (Change Unknown) Chuck had 3 peanuts. Clara gave him some more peanuts. Now Chuck has 8 peanuts. How many peanuts did Clara give him?	***Joining To*** A set of 3 objects is constructed. Objects are added to this set until there is a total of 8 objects. The answer is found by counting the number of objects added.
Separate (Result Unknown) There were 8 seals playing. 3 seals swam away. How many seals were still playing?	***Separating From*** A set of 8 objects is constructed. 3 objects are removed. The answer is the number of remaining objects.
Separate (Change Unknown) There were 8 people on the bus. Some people go off. Now there are 3 people on the bus. How many people got off the bus?	***Separating To*** A set of 8 objects is counted out. Objects are removed from it until the number of objects remaining is equal to 3. The answer is the number of objects removed.
Compare (Difference Unknown) Megan has 3 stickers. Randy has 8 stickers. How many more stickers does Randy have than Megan?	***Matching*** A set of 3 objects and a set of 8 objects are matched 1-to-1 until one set is used up. The answer is the number of unmatched objects remaining in the larger set.
Join (Start Unknown) Deborah had some books. She went to the library and got 3 more books. Now she has 8 books altogether. How many books did she have to start with?	***Trial and Error*** A set of objects is constructed. A set of 3 objects is added to the set, and the resulting set is counted. If the final count is 8 then the number of objects in the initial set is the answer. If it is not 8, a different initial set is tried.

Table 1. Direct Modeling Strategies (Carpenter et al., 1999).

Counting strategies for addition and subtraction word problems. After children have become adept at modeling with physical objects they begin to develop more efficient strategies. They realize they do not necessarily have to physically construct and count all of the quantities in the word problem. At this point children begin to incorporate counting strategies such as *Counting On From First, Counting On From Larger, Counting On To, Counting Down,* and *Counting Down To* (Carpenter et al., 1999).

When *Counting On From First*, in problems such as Join (Result Unknown) and Part–Part–Whole (Whole Unknown), children begin counting from the first addend in the problem, instead of starting from “one” and counting up through both of the numbers. The sequence of counting ends when the number of counting steps in the second addend is finished. For example, in the Join (Result Unknown) problem in which Ellen has 3 tomatoes and then picks 5 more, a child might start with 3 and count, “3 [pause], 4, 5, 6, 7, 8.” When *Counting On From Larger* the child is aware that it is more efficient to start with the larger number, regardless of the order in which they appear in the word problem. In the previous example the student using this strategy would start with 5 instead of 3, because it is easier to count on only three more numbers. Both strategies require some method of keeping track of the number of counting steps being counted on, so that the child stops counting when necessary. Most children use fingers, some use counters or tallies, and a substantial number give no visible evidence of how they keep track (Carpenter et al., 1999). When solving Join (Change Unknown) problems, the answer is determined by the number of steps in the counting sequence instead of the final number reached. A child will begin counting with the smaller number given, and count until reaching the larger number. The answer is the number of steps in between. This strategy is called *Counting On To.*

Children also use counting strategies when solving Separate (Result Unknown) problems, but in this case counting backward, or *Counting Down,* is usually necessary. For example, a child may start counting at the larger number and count down the number of steps given in the problem, keeping track of the number of steps with their fingers or tallies. The answer is the final number reached when counting. Children may also count backwards in a Separate (Change Unknown) problem, but in this case a child will start counting from the larger number and count until they reach the smaller number. The answer in these types of problems is the number of steps the child had to count and most likely kept track of with fingers or tallies. This strategy is called *Counting Down To*. Table 2 (Carpenter et al., 1999) summarizes the counting strategies described.

Table 2

Counting Strategies

Problem	Strategy Description
Join (Result Unknown) Ellen had 3 tomatoes. She picked 5 more tomatoes. How many tomatoes does she have now?	***Counting On From First*** The counting sequence begins with 3 and continues on 5 more counts. The answer is the last number in the counting sequence.
Join (Result Unknown) Ellen had 3 tomatoes. She picked 5 more tomatoes. How many tomatoes does she have now?	***Counting On From Larger*** The counting sequence begins with 5 and continues on 3 more counts. The answer is the last number in the counting sequence.
Join (Change Unknown) Chuck had 3 peanuts. Clara gave him some more peanuts. Now Chuck has 8 peanuts. How many peanuts did Clara give to him?	***Counting On To*** A forward counting sequence starts from 3 and continues until 8 is reached. The answer is the number of counting words in the sequence.
Separate (Result Unknown) There were 8 seals playing. 3 seals swam away. How many seals were still playing?	***Counting Down*** A backward counting sequence is initiated from 8. The sequence continues for 3 more counts. The last number in the counting sequence is the answer.
Separate (Change Unknown) There were 8 people on the bus. Some people got off. Now there are 3 people on the bus. How many people got off the bus?	***Counting Down To*** A backward counting sequence starts from 8 and continues until 3 is reached. The answer is the number of words in the counting sequence.

Table 2. Counting Strategies (Carpenter et al., 1999).

Levels of development. When children are learning addition and subtraction they generally use direct modeling strategies and progress to using more counting strategies. This change does not happen instantaneously, especially because some counting strategies are more difficult than others. For example, because it is more difficult to *Count Down*, some children use this strategy less often and may continue to use modeling strategies for Separate (Result Unknown) problems, despite using counting strategies for other problems. On the other hand, because direct modeling using *Matching* can become tedious, children tend to jump to a counting strategy when doing a Compare word problem. When children become more familiar with counting strategies they can often use them for problems for which the counting does not directly correspond to the action in the problem. But most often the strategy selected, even by older children, models the action suggested in the word problem.

Most children enter kindergarten with some ability to use Direct Modeling even with little or no instruction (Carpenter et al., 1993, Carpenter et al., 1983). Over time most children pass through three levels of problem solving (Carpenter et al., 1999). First, they use modeling exclusively; second, modeling strategies are replaced slowly by counting strategies; and third, children begin to rely on number facts. Children go through these phases at various rates and times. A child may be able to recall a number fact for a few problems but may need to use direct modeling for many others.

Within modeling and counting strategies there are levels of difficulty among problem types. Join (Result Unknown), Part–Part–Whole (Whole Unknown), and Separate (Result Unknown) problems are the only types that can be solved without thinking ahead about how to use the strategy to solve the problem. Therefore, *Joining All* and *Separating From* are easier than *Joining To* because students only need to consider one step at a time. Similarly *Counting On* and *Counting Down,* the counting strategy counterparts, are generally easier than *Counting On To.* The *Joining To* strategy, mainly used to solve Join (Change Unknown) problems, requires the students to think about the entire problem and is more difficult. Compare (Difference Unknown) can be even more difficult than Join (Change Unknown) unless the context of the problem gives clear, straightforward cues to help students think about matching. Any problem with Start Unknown is difficult for students to understand and model, because as the name suggests students are not given the number with which to begin. In addition, Part–Part–Whole (Part Unknown) problems are difficult because, although students have a starting number, there is no action to represent, therefore students struggle when modeling the problem.

As children learn more about number, addition, and subtraction they begin to use strategies more flexibly. Developing an understanding of part–whole relationships allows students to see problems in more broad categories (either a part is unknown or the whole is unknown), and counting strategies become more generalizable. For the Join (Result Unknown), Separate (Start Unknown), and Part–Part–Whole (Whole unknown) problems the parts are known and the whole is not. For Separate (Result and Change Unknown), Join (Change and Start Unknown), and Part–Part–Whole (Part Unknown) the whole and one of the parts are known, but the other part is unknown. Also, development of the principle that actions can be reversed allows children to be more flexible in their choice of strategy. Removal of objects can be undone by adding objects to a set. Reversing the action can be used to solve difficult problems such as Start Unknown problems.

Literature Discussion

The purpose of the literature review was to establish the framework for the study and a basis for the research methods employed. Through this review I have identified the lenses through which I am viewing knowledge and understanding, teacher knowledge, and student achievement. I argue that content knowledge and pedagogical content knowledge are essential constructs for analyzing teacher knowledge, and there is evidence that constructs from both of these bodies of knowledge are involved in influencing student understanding. In my study I strive to uncover how teacher knowledge affects student understanding using a qualitative perspective. I have chosen to examine teacher and student understanding of addition and subtraction word problems, due to the extensive research that has identified the content knowledge and pedagogical content knowledge for teachers in this area. I examine how a teacher's understanding of the structure of addition and subtraction word problems, a substantial part of their content knowledge in this subject, affects students' understandings of addition and subtraction through its effects on teacher practice. I also examine how a teacher's understanding of the strategies students use to solve addition and subtraction word problems, part of their pedagogical content knowledge, affects students' understandings of addition and subtraction through teacher practice.

CHAPTER THREE

RESEARCH METHODS

This qualitative study is designed as a collection of case studies (Merriam 1998). Three cases were studied - three teachers in their respective classrooms. Through these cases I will explore the relationship of how teacher understanding affects student understanding through teacher practice. To investigate this relationship I have studied three teachers' understandings of word problems, their classroom practices, and their students' understandings. Although this study is related to quantitative studies such as those by Ball and colleagues (Ball et al., 2005), it is meant to be complementary to rather than the same as those types of studies. Whereas a quantitative method is usually large scale and may determine whether there is a correlation between factors, my study did not begin with a hypothesis to test. This study analyzes the connections between teacher knowing and understanding, teacher practice, and student understanding in three cases. Therefore a quantitative study or a mixed methods study was not appropriate.

Teachers in the study were selected from a course I taught at the Pennsylvania State University. To encourage elementary school teachers to enroll in the course I enlisted the help of the mathematics coordinator for the local school district. I chose to use the local school district because I believed the teachers would be more likely to register for the course. I did not see this school district as having important distinguishing characteristics from other school districts that would have been likely to substantially affect my study. The mathematics coordinator distributed invitations to what she considered to be the most effective

20 first-, second-, and third-grade teachers in the district, in hopes of soliciting teachers with as much understanding as possible from the outset of the study. These grades were chosen because they would be most likely to produce data on students and teachers using and modeling basic addition and subtraction word problems. From the 20 invitations to the course there were four responses from teachers, two first-grade and two second-grade teachers. Two of those teachers were interested in taking the course from the university, and two were interested in being part of a professional development group but were not interested receiving course credit. The teachers were considered to be experts in their field by their peers and supervisors. They were all encouraged to join the group. The two teachers not participating for course credit were informed of the purposes of the content of the course and agreed to submit all written material for data collection purposes. There were no benefits offered to the participants other than course credit.

Purpose and Content of the University Course

In this section I will outline the structure of the course. For more details about what transpired in the course and teachers' reactions to the course, see Appendix A. The course met and functioned mostly as a discussion group with all four elementary school teachers, the mathematics coordinator, and myself. Starting in late September, the course met ten times for a total of over eighteen hours; during this time the class discussed how children think about addition and subtraction word problems as researched by the Cognitively Guided Instruction (CGI) program.

The purpose of this course was to improve and evaluate the teachers' understandings of addition and subtraction word problems. During this course, in which we discussed the content knowledge and pedagogical content knowledge that is important for a teacher who is teaching addition and subtraction word problems, I had the opportunity to identify teachers who have a substantial understanding of this knowledge. The teachers enrolled in the course had the opportunity to learn and demonstrate their understandings of addition and subtraction word problems. The following paragraphs will outline the topics discussed in the course and the type of understandings teachers had if they continued to participate in my research study.

During the first group meeting I informed them of my ideas for the roles of the participants in the group. My role was to bring to the group my understanding of the knowledge I had researched had gathered, specifically through the CGI program, about addition and subtraction word problems. I explained that the purpose of first half of the course was for me to present to the group my strengths, which was my understanding of the different types of addition and subtraction word problems and the different ways children think about and model those problems, and the teachers' role was to learn and try to understand this information. I then explained to the teachers that the second half of the course was to draw upon their own strengths as expert elementary school teachers. Their role was to deliberate and decide how this knowledge could be used to change and enhance their own teaching. I informed them that I was not an expert elementary school teacher and would not be making these decisions for them. My role was to facilitate their learning about the research regarding this topic and support them as they struggled with how to improve their teaching with this improved understanding.

During the next two course meetings the class learned how to make distinctions among different types of addition and subtraction word problems. According to CGI, there are important distinctions among word problems that are reflected in the ways that children think about the problems, as well as how they solve them. Teachers need to learn the kinds of distinctions children make and classify word problems accordingly. In this course teachers were asked to focus on the type of action or the type of relationship described in the problem, because this represents the way that children think about word problems. Children solve word problems differently depending on the action or relationship given. This action or relationship also determines the level of difficulty of the addition or subtraction word problem. From this course, teachers needed to develop and examine their understandings of this type of classification system.

The University course used discussion-based meetings in order to draw on the experience of the elementary school teachers. Initially I asked the teachers to write three “different” addition word problems. From these problems we discussed what made them “different” from each other and what differences would be significant to the nature of the problem. From this discussion we were able to establish what we meant when we referred to the “structure” of a problem. The teachers, over the first three course meetings were able to identify the different structures that the CGI group focused on. Supplemental problems and quizzes for data collection purposes were given to the teachers. The following is an example of the type of task given to the elementary school teachers for discussion:

Read the following seven problems. Do you see any similarities or differences? Can you categorize the seven problems?

a. Lucy has five fish. She wants to buy eight more fish. How many fish would Lucy have then?

b. TJ had 13 chocolate chip cookies. At lunch she ate 5 of them. How many cookies did TJ have left?

c. Danielle has seven trolls in her collection. How many more does she have to buy to have eleven trolls?

d. Max spent nine dollars on a video game. He has seven dollars left. How much money did he start with?

e. Willy has twelve crayons. Lucy has seven crayons. How many more crayons does Willy have than Lucy?

f. Eleven children were playing in the sandbox. Some children went home. There were three children still playing in the sandbox. How many went home?

g. Molly has 9 dolls. Five of them have blonde hair and the rest have brown. How many dolls have brown hair?

Figure 3. Sample Task for Classification.

Tasks of this nature were successful in drawing out discussion based on the action in the problem and the types of knowns and unknowns in the problem. From tasks such as this teachers began to identify distinctions between Join, Separate, Part–Part–Whole, and Compare addition and subtraction word problems. Teachers were able to demonstrate the ability to focus on the basic structure of problems, such as actions and relationships, instead of just the mathematical operation *they* would use to solve the problem or key words within the problem.

A teacher with a well-developed understanding of the classification of addition and subtraction word problems was able to complete the following figure with the correct type of word problems.

Type	Problem		
	Result Unknown	Change Unknown	Start Unknown
Join			
Separate			
	Whole Unknown	Part Unknown	
Part- Part-Whole			
	Difference Unknown	Compared Set Unknown	Referent Unknown
Compare			

Figure 4. Classification Chart.

During work such as this, the elementary school teachers were not only asked to demonstrate what they knew by filling in the figure but they also were given the opportunity to *explain*[2] their understanding of the classification system. They were given the opportunity to *interpret* how and why this classification system is useful. Teachers were also asked to *discriminate* between different types of addition and subtraction word problems, *apply* their understandings to classify and solve problems they generate themselves, and *generalize* their knowledge by

[2] Indicators of understanding will be italicized in this chapter.

articulating what constitutes a class of problems and articulating the types of subgroups that can be generated within a class of problems.

In addition to understanding how to classify addition and subtraction word problems according to characteristics that affect children's thinking, teachers needed to obtain from this course a better understanding of how students use direct modeling to solve addition and subtraction word problems. Although not all children solve problems in exactly the same way, research shows that there are generalizable, cohesive patterns children use in solving addition and subtraction word problems (Carpenter et al., 1999). Learning how children directly model word problems was not addressed completely independently from the classification of word problems, as direct modeling is reflected in the classification types, but it was the focus of two of the group's meetings (see Table 4).

The elementary school teachers in the class were given physical objects to use to model the word problems. They were urged to think about how children might solve the problems *without* already knowing the answer, instead of using the objects to display the answer they already knew to be true. Again, the action or relationships given in the problem was the focus of the discussion. A teacher with a solid understanding of direct modeling was able to use and explain strategies such as: *Joining All, Joining To, Separating From, Separating To, Matching,* and *Trial and Error*. Not only should participating teachers be able to *explain* the strategies, they should be able to *interpret* the strategies and classes of problems to determine how and why the strategies relate to the structures of problems within the classification system, *discriminating* between which strategies are useful in solving the different classes of problems.

The teachers also watched video clips of classroom episodes during which children solved addition and subtraction word problems. The video clips were part of the DVD that accompanied the book *Children's Mathematics* (Carpenter et al., 1999). Teachers were asked to identify the type of problem the students were solving and to identify the strategy being used. The group would also discuss why this strategy was appropriate and how the students' strategies might evolve in the future. Teachers were also asked to create their own classroom episodes for which they generated word problems and discussed how students might solve those problems. Tasks such as this gave the elementary school teachers opportunity during the course to *apply* their understandings to problems they generate themselves, and *generalize* their knowledge to see particular instances as cases of a larger group of instances. Teachers in the course developed their understandings *of* mathematics and mathematics pedagogy through discussions and tasks completed in class.

Participants also explored how children's strategies change over time. Because children's strategies become more efficient and abstract as they learn, teachers need to be aware of how the children's strategies change. In this course the teachers discovered how children's direct modeling strategies change from using physical objects to using the counting strategies discussed in the previous chapter. A teacher who understood the material was able to demonstrate and explain counting strategies such as *Counting On from First*, *Counting On from*

Larger, Counting On To, Counting Down, and *Counting Down To*. Not only did I expect participating teachers to be able to *explain* the strategies, I expected them to be able to *interpret* how the strategies relate to the direct modeling and the classification system and which strategies are useful in solving the different kinds of problems.

Teachers participating in the study, by the end of the course, were able to make distinctions or *discriminate* among direct modeling and using counting strategies. They understood that when using direct modeling, the physical objects used in modeling represented the objects given in the problem. When using a counting strategy, the child realizes that the objects no longer need to be modeled and he can focus on the counting sequence in order to solve the problem. For example, a child can use his fingers for direct modeling by using individual fingers to represent the objects. Or, during a counting strategy, he can use his fingers to keep track of the count of the problem. The use of fingers does not correspond to only counting strategies or direct modeling, because it depends on how the child is thinking about and representing the problem. Teachers' understandings of the problem and children's thinking about the problem will allow them to *apply* their knowledge correctly.

Ultimately, the purpose of the course was for teachers to gain and display their understandings of how students think about addition and subtraction word problems, the classification of problems, the direct modeling of problems, the counting strategies used for solving problems, and the relationships among these topics. This understanding should be integrated and cohesive. Teachers should understand the difficult level of the different addition and subtraction word problems and the levels of development of the strategies to solve the word problems. Eventually children come to rely on number facts to solve problems (Carpenter et al., 1996), but a good teacher should understand the process children generally go through to get to that level of expertise. This type of understanding required the participants to think *of* and *about* the mathematics and pedagogy of addition and subtraction word problems.

The elementary school teachers in the course were asked to write in a journal, recording what they had learned about the mathematics and how to teach the mathematics. The journal served as a way for them to articulate their answers to several questions, namely: What mathematical knowledge did I gain? How am I thinking differently about the mathematics? What pedagogical knowledge of mathematics did I gain? How am I thinking differently about teaching the mathematics we have discussed? How does this relate to the teaching I do, or will do, in my classroom? What questions do I have? Along with audio-recordings of classroom discussions and copies of classroom work, journals were be a source for describing and evaluating teachers' understandings of the material.

As the instructor for this course I had substantial input as to what my participants learned about addition and subtraction word problems. It may appear to make this study an intervention study, but I argue that it is not. It does have characteristics of an intervention study, and I believe I could analyze the data using that as my focus, but my purpose is not to discover the effects of my intervention. The purpose of this study is to discover the relationship between teacher understanding and student understanding by investigating three cases, regardless of how they obtained their understanding. Ideally I would have had access to participants who possessed the type of knowledge described, and were utilizing that knowledge in their classroom, but because of my subject pool I was compelled to increase the teachers' understandings in order to better describe the relationship.

During the last half of the course teachers were given the opportunity to explore and evaluate lesson plans involving addition and subtraction word problems. Because all of the teachers enrolled in the class were using *Investigations* (Russell, Economopoulos, Murray, & Mokros, 2004) as their curriculum, they explored and evaluated their lesson plans involving this curriculum. The teachers collaborated together, across grade levels, to decide how they would incorporate their knoweldge into the their lesson plans. They worked together to rewrite almost all the word problems in the curriculum. My role during this time was to support their understandings, given that this was a time of continued learning. At no time did I tell them how to incorporate their new understandings in their teaching and, in fact, I told them they did not have to make changes. I also helped them focus their discussions by asking them questions such as:

- What is the goal for this lesson?
- How does it fit with the research we have discussed? How is it appropriate or inappropriate?
- How does *Investigations* deal with addition and subtraction word problem structure?

- How would you change or add to it?
- How do you anticipate students' responses to the given questions?
- How will you respond if a student says…?

The manner in which teachers use the curriculum provided me with an important avenue for investigating their understandings and how they utilized their understandings in practices such as planning lessons, choosing tasks, and anticipating student responses.

The teachers eagerly fulfilled their role as teaching experts and took charge of using their new knowledge immediately in their teaching (see Appendix A for more details). For example, toward the end of the course teachers began to be curious about the abilities of their students to solve word problems before the students were given instruction from the teachers. During one of the course meetings all of the teachers together decided to give their students a "pretest" in order to gauge their ability to solve these "different" kinds of problems. (See Apendix A for more details on the nature and results of this pretest.) In addition, not only did the teachers rewrite all the problems for teaching the first unit, the teachers themselves asked whether the group could meet an additional time to rewrite addition and subtraction word problems in the second half of the school year. Two months after what I thought would be our last course meeting I received an e-mail from the teachers requesting that our group meet again so that the second section of the curriculum containing addition and subtraction word problems, which we had not yet discussed, could be revamped. The teachers requested and were able to get a half day from their teaching schedules to meet with me and the math coordinator to work on the curriculum for our last university course meeting. The meetings took place over a period of eight months, beginning in September of 2008 and ending in May of 2009.

The following is a table outlining the schedule and focus of the meetings:

Table 3

Course Meetings

Course meetings	**Topics of Discussion**	**Focus of Teachers' Understandings**
Class one	Nature of course and roles of participants. Plan future meetings.	
Class two	Problem structure	Teachers focus on Join, and Part–Part–Whole. Teachers discover different quantities that can be known and unknown.
Class three	Problem structure	Teachers focus on Separate and Compare. Continued discussion of knowns and unknowns.
Class four	Problem solving strategies	Teachers discuss direct modeling strategies.
Class five	Problem solving strategies	Teachers discuss counting strategies.
Class six	Curriculum changes	Teachers focus on first-grade curriculum, rewriting word problems. Decide to give pretest.
Class seven	Curriculum changes	Teachers focus on second-grade curriculum, rewriting word problems. Discuss results of pretest.
Class eight	Conclusions of first round and Implications	Post-observation recap. Teachers individually discuss what they learned, what they would do different, etc.
Class nine	Curriculum changes	Rewrite word problems for second half of first and second-grade curriculum.
Class ten	Conclusion	Draw conclusions regarding implementation.

Table 3. Course Meetings.

Selection of Participants

Although four teachers attended the group meetings and I began to observe all four teachers, ultimately I used three teachers in my study to explore how a teacher's understanding can potentially be used to make decisions in and out of the classroom and how those decisions may be reflected in the students' understandings. All four teachers demonstrated an increase in their understandings of content and pedagogical content knowledge, but due to extenuating circumstances, I selected three teachers as participants. One of the teachers had a substitute teacher during a large part of my observation process, which reduced my ability to observe student–teacher interactions in her classroom. I decided to concentrate my efforts on the classrooms in which I was able to collect the most data, and because all four teachers taught this topic on many of the same days I was not always able to observe all four teachers. So, due to scheduling conflicts and substitute teachers, three teachers were chosen to be observed and interviewed.

Three teachers gave me some variety and variability in my study, which I might not have had if I had only observed one or two teachers. I also wanted to be able to observe frequently and interview my teachers with some depth, so I did not seek additional teacher participants. The three teachers I selected clearly exhibited understandings of the content and pedagogical content knowledge as described by current research. This study is not meant to illustrate what is representative of all elementary school classrooms, rather, it is meant to describe possible avenues for teacher understanding to affect student understanding through practice. Teacher understanding and student understanding may vary from classroom to classroom, but I am asserting that when teacher understanding exists, teachers can use that understanding to affect student understanding through their practice.

Teachers were asked to continue with the study after the course if they were able to demonstrate a significant degree of understanding of the material discussed in the course. All four teachers were able to do that (see Appendix A for more details). Any understanding the teachers demonstrated was understanding that could have potentiallly influenced their practice, which could have potentially influence their students' understanding. In order to continue in the study, teachers needed to be able to demonstrate through their work in the course some level of understanding of the structure of addition and subtraction word problems and the ways in which students solve addition and subtraction word problems. Teachers' ability to use and adapt curriculum materials according to what they had learned in the course was also a way to measure teacher understanding. If a teacher was able to demonstrate her ability to use her understanding in the implementation and adaption of curriculum materials, then I considered her to be a potentially useful participant in the study. Although the three teachers demonstrated a range of abilities and understandings, any demonstration of understanding was a potential way for me to study how this particular understanding might influence student understanding. The teachers with the deepest understandings provided the richest opportunity for me to study the connections between teacher understanding, teacher practice, and student understanding.

Classroom Observation

After the course had met four times I began observing the teachers participating in the study. I observed them one week before they began teaching the unit on addition and subtraction word problems. The purpose of this was to build rapport with the students so they would feel comfortable with my presense in the classroom and to test data collection strategies before the unit in which I was interested began. For example, I used this time to consider which students I wanted to observe and interview. In order to focus my data collection I concentrated on observing three or four students per teacher. The teachers were involved in my selection of students to observe and interview. The way in which teachers grouped their students also affected the type of students I chose to observe. All three of the teachers had students sitting together in groups, and therefore I chose one of the groups in the classroom as my participants. None of the teachers grouped their students by ability and there was a variety in the understandings of the students I observed. When I asked the teachers which table they thought best for me to observe, I suggested finding a table the teachers considered a middle or "average" group of students. I believed this group of students might give me the greatest opportunity to see students learning and benefitting from teacher understanding. An "average" group of students might demonstrate some struggle with the content and therefore might require interaction with the teacher but might also be able to articulate what they were thinking and understanding. I thought it would not be as beneficial to observe students who did not spend time interacting with the teacher, either because they were working by themselves or because they were spending the majority of their time with the teacher's aide and not the teacher who was participating in my study.

I also used the time when I observed the teachers teaching a different mathematical topic to decide the best strategy for me to observe students and teacher, including where I should sit and where the audio-recorder should be placed. I also used this time to practice collecting data. For example, if the class was studying a unit on geometry, I tried to find instances of students understanding the geometry concepts. I then tried to work backward to see what teacher practice may have influenced their understandings and what teacher knowledge may have influenced that practice. This included investigating the tasks that had been given to the student and any dialogue the student had had with the teacher or even other students. I also questioned the student to try to uncover how and why the student was thinking about the particular concept. I practiced observing how both teachers and students manifested their understandings of geometry verbally and nonverbally. I was then able to use this information to better collect data when the class began a unit on addition and subtraction word problems, which did not happen until after the university course had met five times.

Any time the teachers taught addition and subtraction word problems in their classrooms, from September 9th until May 19th, I was in class to observe. I also audio-recorded these teaching sessions. In addition I took field notes, which included teacher and student interactions, writing on the board, or other relevant data not captured by the audio-recording. Copies of lesson plans and classwork were also collected from the teacher. The field notes and recordings were used to form a record, as complete as possible, of the classroom proceedings.

These classroom observations were intended to generate ideas about how teachers use their understandings of mathematics and mathematics teaching on a day-to-day basis, and how this understanding influences what students understand.

The field notes, lesson plans, other additional documents such as student work, and transcribed audio-recordings were used to analyze the understanding teachers have, how this understanding affects the teachers' decision making, and how this understanding can potentially influence student understanding. I also examined the students' understandings of the mathematics by listening to class discussions, teacher and student discussions, and by accessing student work.

The teachers I observed all used *Investigations* in their classrooms. One of the teachers I observed is a first-grade teacher. There are two main units in the first-grade curriculum this teacher used that focused on or involved addition and subtraction word problems, the second and the sixth modules for the year. The second unit is entitled "Building Number Sense," and was taught in November of 2008. This unit is designed to be about eight weeks long. During these eight weeks the teacher had students solving addition and subtraction word problems during nine class periods that I observed. The sixth unit is entitled, "Number Games and Story Problems" and was taught in May of 2009. This unit is also designed to be about eight weeks long and also utilizes addition and subtraction word problems during some of the lessons. There was very little difference between the two units, both of which focused on Join and Separate (Result Unknown) problems. During these eight weeks the teacher had students solving addition and subtraction word problems during six class periods. One of these classes was taught by a substitute so I only observed five of the classes. The tasks in the curriculum played a very small role in the study, as the teachers decided to write their own tasks and did not follow the plans in *Investigations*. The weeks during which the teachers decided to teach addition and subtraction word problems was indicated by the curriculum, although, even then, the teachers did not restrict themselves to the exact number of lessons specified. The specific manner in which each teacher dealt with the curriculum will be addressed in the results chapters of the study.

Two of the teachers I observed are second-grade teachers. There are two units in the *Investigations* curriculum the teachers used that focus on addition and subtraction word problems, also the second and sixth modules of the year. The second module, entitled, “Coins, Coupons and Combinations,” is designed to be about seven weeks long and was taught in November and December of 2008. During this time I observed one teacher teaching addition and subtraction word problems on eight different days and the other teacher on seven days. The sixth unit entitled, “Putting Together and Taking Apart,” is designed to be about seven weeks long and was taught in April of 2009. During these seven weeks the lessons were focused on solving addition and subtraction word problems during five class periods that I observed. Again, there was very little difference between the two units, both of which focused on Join and Separate (Result Unknown) problems. And again, the curriculum played very little role because the teachers chose to create their own tasks.

During periods of time in which I was not observing classrooms I analyzed data I had collected and revised data collection materials. While analyzing first round data I generated observations to test and add to during the next rounds of data collection. I also revised and improved student interview tasks, teacher interview questions, and observation methods based on my experience collecting and analyzing the first round of data. The time in between rounds of collecting data was used to improve the type, amount, and quality of data I collected on the subsequent round.

Interviews

The elementary school teachers in the study were interviewed the same day, usually dircectly after, classroom observations. They were asked about how they used their understandings of addition and subtraction word problems and why they made specific decisions before and during class time. Events that meritted attention during interviews include the teacher making changes to lesson plans, claims she made about student understanding during class, the teacher intervening in student work, decisions she made about students modeling solution strategies, or just conversations the teacher had with a student. Teacher interviews further illuminated the type of understanding the teachers had and how they were able to use that understanding in their teaching. Interview questions focused on choices made during the class period, and included questions such as, "I am curious about the conversation you had with *(student X)*. Tell me more about it. Why did you say *(statement Y)*?" Or, "I noticed you had the students do *(problem Z)*, was there a reason you chose that problem at that time?"

During teacher interviews we discussed curriculum materials such as classroom assignments and the structure of the end-of-unit assessment. I asked them to assess the appropriateness of assignments and tests and later discussed the way they graded them. I tried to ascertain what they valued in students' responses and what their focus was in the students' understandings of addition and subtraction word problems. I also asked teachers to talk about how they might work with a student who gave a strong explanation and a student with mixed performance.

The students I chose to observe in each class were all interviewed using mathematics tasks similar to those they encountered during their regular class work. Most student interviews were informal and took place during class time. Student interviews allowed me to determine the nature of the students' mathematical understandings. To evaluate student understanding, I used the same framework discussed in Chapter 2 and the preceeding paragraphs, namely, I investigated students' ability to *identify, explain, interpret, discriminate, apply,* and *generalize* within the context of understanding addition and subtraction word problems.

When interviewing students I asked them to *explain* how they solved a word problem, or several word problems, completed during the class period. I also asked them why their process worked and how they knew to follow the steps they chose to take, to learn more about their *interpretation*. Once I was familiar with their understandings of the given problem I asked them to solve other problems. Some of the problems were similar to ones solved in class, therefore giving the students a chance to *apply* their knowledge in a familiar context. However, some of the problems the students were asked to solve had a different structure. All the problems I asked the students to solve in interviews were addition and subtraction word problems.

For example, if the student completed a Compare problem with the Difference Unknown in class, and I asked him questions about it, I might also have given the student another problem that was a Compare problem with difference unknown. While the student was solving these problems I would question students, searching for instances of students identifying commonalities in the types of problems or commonalities in the way they explained or interpreted the problem. These types of problems might also give students the opportunity to apply what they have learned in previous problems.

I might also have given another problem that may have been a *Join* problem with change unknown, and still another that may have been a *Separate* problem with change unknown. Changing the structure of a follow-up problem allows the student to apply what they have learned to other types of problems and discriminate between problems. It also allows me to compare and contrast the types of explanations and interpretations the students offer. The questions were necessarily appropriate for the ability of the student, but regardless, the student was asked to solve various types of addition and subtraction word problems.

After attempting to solve these additional problems, the students were asked to *discriminate* among problems. Specifically I asked the student to talk about similarities and differences between problems. They were asked to describe the quantities in the problems and the relationships between those quantities. I searched for instances of students' ability to *generalize* particular problems into cases of a larger group of problems. I was interested to see the factors on which the students focused and how the student's understanding and focus compared to the teacher's understanding. It is possible that the students were able to learn from the conversations I had with them, because my conversations did ask them to reflect on their interactions with their teacher and their own thinking about the topic.

However, my intention in our interactions was not to teach them but to discover the nature of their existing understanding.

Examples of Possible Connections

Before I began collecting data I did some thought experiments to try to determine the types of data I might collect and the possible connections I could see among teacher understanding, teacher practice, and student understanding. The following are three examples of how I thought these concepts could interact with each other.

I contend that a teacher who understands the structures outlined will be able to unpack the concept, seeing more details involved in understanding the concept, than a teacher who does not have that understanding. Part of the work of teachers is identifying and working toward the mathematical goals of the lesson (Ball, 2003). Teachers with this understanding may be able to set more refined goals for lessons that can allow their students to come to a broader and more in-depth understanding of addition and subtraction word problems. The students might be able to solve a larger variety of problems, more difficult problems, and recognize addition and subtraction as more than just “combine” and “take-away.”

A second implication of understanding the structure of addition and subtraction word problems might be a teacher’s ability to encourage students to investigate word problems more “mathematically.” Part of teacher practice is teaching students what counts as “mathematics” and mathematical practice (Ball, 2003). In this case teachers can increase students’ strategic competence by asking students to focus on the structure of word problems in order to figure out how to solve the problems, instead of having students focus on properties that are less mathematical, such as "key words" in the problem (Kilpatrick, Swafford, & Findell; 2001). Students may learn to study the problems given, determine what they know, determine what they do not know, and examine the way in which the problem is set up so that they can solve it. This mathematical thinking can replace simply guessing at an operation or just knowing what operation to use because the preceding 10 problems the student completed used the same operation.

Teachers participating in the study also understand the different methods children use in solving addition and subtraction word problems. Part of teacher practice is building correspondences between a model and a concept. Because teachers in this study understand how children directly model the word problems and how their modeling corresponds to the structure of the problem, a teacher should be able to aid in directing students to focus on correspondences between the model and the concept of addition and subtraction. Students might be able to understand that the manipulatives used represent the objects in the problem. Students might understand that the model represents the action or relationships in the problem instead of just memorizing a procedure using the manipulatives.

I anticipated that after the data collection process I would be working backwards through the connections between teacher understanding and student understanding. I thought I would be able to observe some understanding a student gained about addition and subtraction word problems. After seeing an indication of student understanding I would search for teacher practices that influenced that understanding, for example, what types of problems the student was working on, what types of interaction the student had with the teacher about this understanding and so on. Finally, through classroom observation and interview, I hoped to determine what kind of teacher understanding led to the particular teacher practice.

Analysis

The analysis was conducted qualitatively using the data I collected in classroom observations, field notes, documents from the teachers, and interviews. I tried to identify the participants' understanding during several passes through the data. First, each teacher was investigated individually. I assessed the nature of the teacher's understanding of addition and subtraction word problems by studying the audio-transcripts and notes from the course, the interviews with her, and observations from her teaching. All comments made by the teacher were investigated to see what could be claimed as part of the teacher's understanding. Any data I could identify that might indicate some aspect of teacher understanding was included in the analysis. These data points were categorized among the groups named Content Knowledge, Pedagogical Content Knowledge, or Other (for data points that I was not sure where to place or if I felt they were a combination of the previous two categories). The indicators of understanding outlined by Sierpinska (1994), and Wiggins and McTighe (1998) informed my characterization of the nature of their understandings. To help me decide whetehr and how teachers were understanding, I investigated whether the teachers were identifying, explaining, interpreting, discriminating, applying, or generalizing a concept.

After putting the teachers' understandings into the categories of Content Knowledge, Pedagogical Content Knowledge, and Other I reviewed the data points searching for indicators of understanding in which I found multiple pieces of evidence. In order to claim that a teacher had a certain type of understanding I searched for evidence of that understanding which manifested itself repeatedly during the data collection process. If I was only able to infer characteristics of a teacher's understanding from one or two instances I considered that unreliable data and did not include it in my description of the teacher. I also searched the data for evidence that teachers did not have the understanding to make sure the assertions were made with as little bias as possible. Once I gained a better understanding of the teachers' understandings I organized the categories into subcategories, based on types of understanding I determined were repeated in the data, to better describe what the teachers were understanding.

Next, I searched for the understanding of addition and subtraction for the teacher's individual students. For each student I investigated the classroom observations, interviews, worksheets and other written work chronologically to determine the nature of the student's understanding and to see how that understanding may have changed over time. I studied each student's involvement in class discussions, dialogues with the teacher and other students, written work, and interviews for data that would suggest the nature of the student's understanding of additon and subtraction. Again, I searched for the students' attempts to identify, explain, interpret, discriminate, apply, or generalize to allow me characterize the nature of their understandings. After searching for any data element that might represent some aspect of a students' understanding, I organized this data into categories, based on the types of students' understandings that naturally appeared repeatedly in the data, that I was able to infer from multiple observations. If only one student showed one particular understanding I did not convey that in my portrait of students' understandings in the classroom. I sought out student understanding that I found to appear consistently within a participant's data during the data collection process. I did not search for instances in which teacher understanding did not contribute to the students' understanding or instances in which the students' understanding was hindered because of a teacher's lack of understanding. I did not believe this type of data would contribute to answering my research questions and I was not interested in creating another deficit study.

After compiling notes on the qualitative nature of the students' understandings I studied those notes in conjunction with my data on the teacher's understanding to determined how the teacher used her understanding of addition and subtraction word problems and how her understanding may have influenced the students' understandings of addition and subtraction. In most instances I worked backwards, taking a particular understanding a student possessed, I investigated the data to find possible instances when the teacher may have influenced that understanding through teacher practice. In addition I hypothesized how the teacher's knowledge could have affected the teacher's practice. I used teacher interviews and course discussions to support my theories on how a teacher's knowledge influenced the teacher's practice and students' understandings. Multiple instances students showing evidence of a particular understanding, coupled with mulitple instances of a particular practice influencing that understanding, meritted a connection worthy to put forth.

In this study I was searching for evidence of student understanding and trying to identify and explain what led to that understanding, in terms of teacher understanding and practice. There were, of course, many instances in which the students did not seem to understand even when the teacher had understanding and was using her understanding in her practice. I do not see this as counterevidence, because there are so many instances in which understanding may not be observable leading up to a moment of observed understanding. My research investigates the positive instances of understanding and tries to find commonalities in those instances. Further research is necessary to investigate the cases I would be consider to be counterevidence.

During the next pass through the data I investigated what understandings and practices the teachers had in common with each other. Because the teachers had similar backgrounds coming from the same school district and had a shared experience in my course, there was much I could say about the teachers as a group, without repeating the information for each individual teacher when writing the results. I thought it would be useful to first write about the teachers as a group and then delve into their individual differences and experiences. For this phase of my analysis I examined the data and searched for multiple instances of a type of understanding or practice that the teachers shared. In order to claim a certain indicator of understanding was shared by all teachers, the instances that demonstrated the understanding had to be prevalent in the data collection for all three teachers.

Based on the research methods described in this chapter, I collected and analyzed the data from three teachers and their students. Although the teachers shared many experiences and developed their understandings together, there were noticeable differences in their teaching practice. These differences illuminated the different aspects of the teachers' understandings and also affected the types of student understanding I was able to observe. The three chapters that follow are the results of my analysis. In these chapters I will describe aspects of each teacher's unique understanding that I synthesized from the data. I will not give an exhaustive list of all of their mathematical understandings. The aspects of their understandings on which I will elaborate are those I thought I was able to infer to be influencing their practice in teaching addition and subtraction word problems and subsequently their students' understandings. I will also detail how the teachers used their new mathematical understandings in their own classrooms. In addition I will hypothesize how each teacher's understanding of addition and subtraction word problems could have affected her students' understandings.

CHAPTER FOUR

JULIE: USING STRATEGIES TO UNDERSTAND STRUCTURE

Julie[3] is one of the three cases of teachers using their understanding in a different way to influence their students' understandings. In this chapter I will explore Julie's understanding, her practice, and possible ways her understanding and practice could have affected her students' understandings. Specifically, I will explain how Julie's understanding of structure and solution strategies influenced her to focus her practice on students' use of solution strategies. I suggest that this practice influenced students' understanding of the strategies and the structure of the addition and subtraction word problems. When describing Julie and her students' understandings I will refer to the indicators of understanding detailed in the literature review, namely, the ability to identify, explain, interpret, discriminate, apply, and generalize (Sierpinska, 1994; Wiggins & McTighe, 1998).

Julie is a case of a teacher asking her students to use their solution strategies with connections to the structure of a problem. She showed an understanding of solving strategies, and privileged[4] the use of strategies with connections to structure in her practice. She was able to use this understanding to influence what mathematics students learned and the way students learned mathematics in her classroom.

---

[3] All names used are pseudonyms.

[4] Kendal and Stacey describe *privileging* as "a construct to describe a teacher's individual way of teaching and includes decisions about what is taught and how it is taught" (Kendal & Stacey, 2001, p. 145).

Julie's Understanding

One explicit aspect of Julie's understanding that stood out in comparison to the two other teachers' understandings was her grasp on the different types of direct modeling and counting strategies. Julie was able to identify the different types of strategies when, for example, watching video excerpts of students performing the strategies. She was able to explain how to perform each of the strategies and which strategies were useful for different problem types. Julie was also able to interpret why each strategy made sense as an appropriate strategy for a given problem structure. Although all three teachers were able to identify, explain, interpret, and discriminate among these strategies after instruction, Julie's understanding was very strong prior to instruction.

Julie suggested that her familiarity with the solution strategies may have resulted from first-graders' tendency to use these strategies more often than second-graders use them. This was evidenced in my observations as well, during which second-graders used recalled facts at a higher rate than modeling and counting strategies. Julie demonstrated more than just a factual knowledge of the list of counting and modeling strategies. She took many opportunities to apply her understanding in the classroom with her students. She understood that the steps in the strategies corresponded to the nature of word problem structure, making it a more useful understanding for her teaching.

Julie's understanding of modeling and counting strategies seemed to be strengthened by her understanding of the structure of addition and subtraction word problems. During many discussions about modeling a problem she was very adept at identifying the structure of the problem and using her interpretation of that structure to inform her problem-solving decisions. Julie often used her explanation of whether or not there was any action inherent in the problem as a guide to her interpretation. For example, when talking about a Part–Part–Whole problem (Whole Unknown) in which there are three red beads and two blue beads on a string she said:

> It's not 'I have three red beads and I bought four more,' it's just … they are there. It could be like, they are in a row. You are not ever putting anything together. They could be together but you are not asking the students to join two groups together. (Julie, Course Day 1, 117-121)

Julie identified the problem type, explained the action in the problem (also discriminating the problem from others by saying what it was "not"), and interpreted why this problem was a Part–Part–Whole problem. In this example, and many others like it, Julie focused on the structure of the problem and visualized the physical representations of the quantities in the problem, where they are located, their relationships to each other, and how they were moving or not moving. She used this information to discriminate among problem types such as Part–Part–Whole and Join problems.

Julie was able to generalize her observations about problems involving action as well. She thought that addition and subtraction word problems with action were easier for students to solve and understand. According to Julie, "It's actually the action, I think, that makes the problem easier for students. So they can more easily visualize what you want them to do because of the action." Although she understood Join and Separate problems were the easiest for students to visualize, and therefore the easiest to model, she also applied this understanding that students need to be able to visualize problems to other problem types.

It was clear in Julie's teaching that she wanted her students to use their understandings and interpretation of the structure of the problem to decide how to model and solve the problem. The following is a typical example that shows Julie's ability to identify the structure of the problem and her understanding that the interpretation of the structure leads to a problem solving strategy. It also demonstrates that Julie understands that her students need to visualize the problem and model the relationships inherent in the problem. This example involves a lower achieving student trying to model a Separate (Result Unknown) problem in which someone has 12 apples and he eats 6; and the student is supposed to find how many apples are left. Julie works hard to get the student, Sasha, to see how the story is reflected in the direct modeling, but Sasha keeps getting 6 more blocks out to represent what is being eaten, instead of removing 6 out of the original 12 blocks that represent the apples.

Julie: Okay so let's go to the beginning of the story. It's always good to start from the beginning isn't it? Steve and Rosa had 12 apples. So show me their apples. Let's pretend that these are refrigerators. Show me their apples. So how many apples do they have at the start of the story? [Sasha gets 12 blocks from a large pile to represent the apples.]

Julie: And the story said that they eat some apples. So where are they going to get the apples that they are going to eat? [Sasha gets an additional 6 blocks from the large pile.]

Julie: Where did they come from? Here are the 12 apples. They can't eat apples from over here. They have to eat one of their 12 apples. They can't eat somebody else's apples. They have to eat their own apples. These are the apples they can't eat. They belong to somebody else. They have to eat from their own apples, that's the way it works. So show me what apples they can eat. How many are they eating again?

Sasha: 6. [Sasha begins counting out 6 cubes from the large pile, instead of from her pile of 12 cubes, again.]

Julie: Where did the six cubes come from? Did you get 6 more cubes? We have to remember we have the 12 cubes in the refrigerator. Alright. Here are the 12 apples in the refrigerator. And when they ate the 6 apples where did the 6 apples come from?

Sasha: They came from these 12.

Julie: These 12. So did you need to put more unifix cubes on here?

Sasha: Yes. No.

Julie: How many unifix cubes are on this paper right now? You got these 12 here. Do you have other unifix cubes on here?

Sasha: No.

Julie: Why not?

Throughout her interviews and her practice, she demonstrates her pedagogical understanding that all of her students, no matter their abilities, need to understand the nature of the problem on which they are working in order to understand the strategies they use to solve the problem. Although she struggles to assist Sasha in explaining what is going on in the problem, she wants her students to be able to interpret different structures of problems, even if it is difficult for them, so that they can apply their understandings and solve the problem with appropriate strategies. Daily examples, such as this, were evidence that Julie privileged teaching her solution strategies in her classroom with connections to the structure of the word problem.

Julie wanted her students to be able to understand the structure of problems because she understood that the operation that was chosen to solve a problem did not indicate the nature of the problem. She identified problems that could be considered addition in which there was no joining. Also, Julie identified joining problems that could be solved with subtraction. She explained that subtraction was not merely the act of separating. When asked if subtraction and separating were the same thing, she said:

> No I don't think it is. But I think that that's part of the thing that makes these problems, different problems, hard for kids, because it's not always just separating. You could have differences…It's almost like subtraction is used as a strategy at that point. It doesn't always reflect the meaning of the problem but it could be used as a strategy in order to solve it. (Julie, Course Day two, 56-60)

This appears to be evidence that Julie was expanding her conceptions about the structure of problems and the nature of addition and subtraction. Julie vocalized a very important aspect of her new understanding, she explained that calling a word problem a subtraction problem does not give it meaning but is only a way to solve a word problem.

When Julie was asked about how understanding the CGI materials can improve a teacher's practice, she again focused on the modeling dimension of the materials. She understood the importance of students being able to model all the different types of problems. Julie said:

> Because I find, even if first-grade kids don't want to use the tools that are there, it is important to get the cubes or whatever out and ask them to show you what's happening, because it's by actually looking and seeing what's happening that they can begin to process what they need to do to solve the problem. (Julie, Course Day 2, 50-54)

Julie's practice is filled with examples of identifying problem types, strategies that correspond to the problem types, and encouraging students to interpret the relationships in the problem through their modeling of the problem.

Julie also demonstrated a strong understanding that the counting strategies are related to the modeling strategies but are more abstract and require a more sophisticated understanding. For example, prior to any of my instruction, when asked how to solve a Join (Change Unknown) problem she was able to identify and explain the *Joining To* approach, the direct modeling approach CGI claims is most appropriate for that type of problem. But then she also was able to identify the abilities of her students and said that most of her students would not need to use the blocks, but would instead probably just use their fingers and then proceeded to apply her understanding that Join (Change Unknown) problem could also be solved using a corresponding counting strategy and demonstrated the *Counting On To* strategy, which she discriminated from the modeling strategy by saying it was similar to direct modeling but required a higher level of understanding.

One of the understandings Julie demonstrated during the course was her explicit understanding that students work through stages of problem solving and there are various levels of difficulty involved in using direct modeling, counting strategies, and fact recall. Julie had put her class into three ability groups for mathematics prior to our meeting. Although she had not formally verbalized the specific abilities of the groups she was aware of differences among the students and used those differences to form the groups. She knew her higher level students would use more recall and her lowest students would use more modeling. After giving the Join (Change Unknown) problem as the pretest for the class, she was pleased to see that her groups were constructed appropriately, meaning, those students who used modeling to solve the problem were in her lowest group, those she had thought were her middle group did, in fact, generally use counting strategies, and the students she ranked the highest did generally use number fact recall. She was able to make distinctions among students who needed to model, students who needed to use their fingers, students who could mentally count up and keep track of two different numbers in their head, and students who knew and used some number facts. After the pretest Julie was very adept at explaining these stages of problem-solving strategies students typically exhibit and could identify where her particular students were in this progression.

Julie also showed interest in her students developing increasingly sophisticated strategies for solving addition and subtraction word problems. She understood the level of abstraction used in the various strategies and how a student needed to be able to reason in order to use a certain strategy. Many times while teaching she identified the student's current ability level and then used that understanding to inform how she interacted with the student to support his progress to the next level. For example, a student who was *Counting All* was asked, "Is there a quicker way to do that? If I said 'show me five fingers' would you have to count out one, two, three, four, five?" She tried to encourage the student to think about *Counting On* from five instead of counting out both numbers being added together. In an interview, Julie explained that she could see the student was using *Counting All* as a strategy, but she believed the student was capable of using more sophisticated strategies. Examples such as this show her ability to identify the strategies being used by her students, her interpretation of the difficulty level of these strategies, and her desire for her students to progress through these stages. She was pleased when this student began to use *Counting On* later that day.

Julie expressed that it was important for her to assess an individual student's ability level and then use interaction that was appropriate for the particular student's needs. She demonstrated this by giving different suggestions to students depending on the student's understanding and ability. For a given problem she might suggest for one student to get out blocks or a number line, but on the same problem encourage a different student to solve the problem without physical manipulatives. In interviews she explained the level of abstraction necessary for different types of solving strategies and described specific students in her class who were developmentally prepared for the various types of strategies. In her practice she used her understanding of problem-solving strategies discriminately to facilitate student learning.

Finally, Julie demonstrated her understanding that using a strategy such as direct modeling, even if a student is consistently using counting strategies, can provide meaningful learning opportunities on a problem with which a student is struggling. Julie sometimes suggested that a student might be able to go back to modeling to better grasp the idea, instead of always pushing the students to use a more abstract strategy. This was evidenced many times in observations. For example, when a student was struggling with drawing, Julie encouraged her to use unifix cubes. To another child she said, “Sometimes when things get tricky it helps to have cubes” because he was finding a Part–Part–Whole (Part Unknown) problem difficult to solve. She understands when and how direct modeling can be a productive method for solving difficult problems.

Julie’s Practice

In this section I detail how Julie’s practice - the tasks she chooses, the time she spends on the tasks, her lesson objectives, the questions she asks her students, the mathematics on which she focuses, and her ability to understand her students and their needs - was affected by her understanding of addition and subtraction word problems. Because much of this understanding is new, it changes her practice from previous years. Although the change in practice was an indication of her use of understanding, change was not necessarily the point of the research study. Ideally, all teachers would have been using this understanding prior to the study, and if this had been the case, I would be still have been able to document how the understanding affected practice.

After meeting four times with our University course, Julie reread the *Investigations* curriculum and reported that the first section's objective was for students to learn about Join word problems. She was now unhappy with the lesson plan that contained only Join (Result Unknown) problems and decided that she would begin the first lesson with a Join (Change Unknown) problem instead. Julie claimed that her students already understood Join (Result Unknown) problems and therefore those would be too easy. She wanted to challenge them from the beginning of the lesson, instead of initially posing a problem they already knew how to do. She gave the students the first problem as they were all sitting together on the carpet in front of the class and then allowed them to work on solutions with a partner. She asked her students to picture what was going on in the problem and encouraged them to explain the problem in their own words before solving it. Julie then asked many students to share their solution strategies. She repeated this process next with a Join (Result Unknown) problem and then a Join (Start Unknown) problem.

With Julie's new understanding she thought it would be more beneficial for her students to see all types of Join problems. She explained that this method would more likely allow her students to understand the relationship between addition and subtraction. She asserted that some teachers and students might think all Join problems were addition problems and wanted to encourage her students to see that one could use addition and subtraction to solve different types of Join problems. Julie now thought that lessons that focused on one type of structure, but with various types of unknowns, would encourage students to think about the structure of the problem and then what operation to use to solve it, instead of “just taking the two numbers in the problem and adding them together.” She was excited to have her students compare and contrast the problems they did within one day, and used this first day as an introduction to Join problems.

Similarly, she wanted to follow this pattern on the next day but with Separate problems. She followed the *Investigations* suggestion of doing Separate problems on the second day, but this year, because of her new understanding, she began with a Separate (Change Unknown) problem instead of Result Unknown. Julie asked her students to picture what was happening in the story. She often asked them “Are there more at the beginning or at the end?” to press them to think about the action involved, which she understood to be important. In the interview following her teaching she revealed that although she felt her students were able to think about this problem successfully, she decided she wanted to give them more practice with Change Unknown problems, and so the second problem she gave her students was also a Separate (Change Unknown) problem. Julie was pleased with her students’ progress, but did not have enough time to let them try a Separate (Start Unknown) problem that day.

For the next two weeks (six class meetings) the students worked at stations in the classroom. Julie decided to stay at the station where students worked on the addition and subtraction word packet (see Appendix B) that the teachers had developed in our course together: a collection of problems, one and each side of a piece of paper, stapled together. She wanted to be sure that she, instead of an aide she had used in the past, was the one to see how the students were solving the problems. The students worked at their own pace through the problems and Julie was there to answer questions and make suggestions. By the end of this time most students had finished the packet and had attempted all the different types of problems.

After the students work on addition and subtraction word problems in the fall, Julie had some time to re-evaluate what she wanted to do with her students when they began to work on word problems again in the spring. For the most part, her lesson plans did not change, except that she decided to separate her students, who would all be working together on the same problems, into two groups. She decided to work with a small group of her most advanced students, on the same problems as the other group, because she indicated that she thought they could work at a faster pace and discuss the details of structure and strategy in more depth. For example, Julie thought the students had a good introduction to the types of word problems but wanted to spend more time talking about similarities and differences among the different word problem structures. According to Julie, this was not part of her practice in the past, but her understanding that there were important structural differences between different types of addition and subtraction word problems inspired her to encourage this understanding in her students. She wanted all of her students to "take the problem apart." She wanted to ask them, "How are you going to go about getting the answer here? Is it the same way you go about getting the answer here [referring to another problem]?" These questions focus on the strategies the students would use to solve problems. In an interview she explained that if students could discriminate among the different strategies they were using, the strategies would assist the students in discriminating among the problem structures. Julie indicated that she wanted the students to think about how a Join (Start Unknown) problem, for example, is similar to a Join (Result Unknown) problem, with the type of unknown varying. She did not think the class had enough of these types of discussions during the first group of lessons.

When the class began the next session on addition and subtraction word problems in the spring, Julie took this small group of students and had them work on a Part–Part–Whole (Whole Unknown), a Part–Part–Whole (Part-Unknown) and a Join (Result Unknown) problem the first day. She stated that she chose these problems because she wanted them to be able to compare the action in the Join problem to the inaction in the Part–Part–Whole problems. In addition, she chose two different Part–Part–Whole problems so the students could see that the unknown for which they were solving could vary in the problem. She understood the relationships between the quantities and the varying types of unknowns in problems and used this understanding to inform her choice of problems that she hoped would elicit meaningful conversation from her students. Julie's understanding shaped the objectives of her lessons. Her objective during the first data collection cycle was to introduce her class to all types of problems; her objective now was to have the students discriminate among the different types.

During the next 4 days the whole class worked at the stations set up in the classroom, one of which was the packet of problems the group of teachers wrote during the interim (see Appendix B). Julie kept the top six students as a group that rotated through the three stations in the classroom. She continued to work with them as a group when they came to her station of addition and subtraction word problems. After three days of classroom work the higher achieving group of students had finished the packet. Julie had a chance to study the problems given and identified that there were certain problems on which this group of students was not successful. From working with the students and assessing their packets, she noticed that Compare (Referent Unknown) problems were difficult for students. She saw that they struggled with this type of unknown even more than with the also-difficult Compare (Compared Set Unknown). Because she understood the differences between Compare (Referent Unknown) and Compare (Compared Set Unknown) problems she was able to design a final packet for this group of students that isolated the problem type with which they struggled, Compare (Referent Unknown) problems. She added to this packet a Separate (Start Unknown) problem that she also claimed was relatively difficult for the students and a Separate (Change Unknown) problem for variety. The top group of students was the only one that had enough time to work on these four additional problems.

After Julie had finished teaching addition and subtraction word problems in the spring she reported that she was happy with the progress her students had made.

She was pleased with the more challenging packet of problems she had made for her students at the end of the year. This addition caused her to think about other changes she might make in her teaching next year.

Julie thought it would be beneficial for her lowest group of students to work on a packet that was different from the others. In this packet Julie planned to include several of the same kind of problems consecutively to give this group of students a chance to see patterns in the structures. For example, the students might work on three Join (Result Unknown) problems, three Join (Change Unknown) problems, and then Join (Start Unknown) problems. Julie believed this format might give the lower students a better chance at understanding each type of structure before moving on to another type. She also wanted her higher achieving students to work with larger numbers when doing all the types of word problems. She thought she would still format her class the same, giving group instruction at first and then allowing the students to work on their packets at their own pace at a station. However, when teaching next year, all the packets will not be the same from the beginning. The lower group of students will have more of the same type of problems consecutively, as described, and the higher group of students would have problems with numbers of greater magnitude to work with to challenge them.

She also expressed that she wanted her students to work more on addition and subtraction word problems between the two chapters in the fall and spring. She reported that she wanted her students to continually develop their understandings of the problems, and she also said the students would simply enjoy working on more addition and subtraction word problems. Julie planned to work with the mathematics coordinator for the school district to write problems the students could do at other mathematics stations during the school year.

Also, Julie wanted her students to continue working on these types of problems for the rest of the school year, even though they were not typically part of the curriculum, possibly doing a word problem every morning before regular instruction. She claimed the students were beginning to feel at ease with the different structures of problems and now had goals for her students to improve their strategies to solve problems. Julie said, "So I'd like to push them to start thinking about numbers and they are not going to do that if they don't have problems to work with." She wanted to use concepts involved in addition and subtraction word problems as a tool for students to better understand the procedures used in adding and subtracting. Julie saw the counting strategies her students were using as strategies they could use to solve many kinds of mathematics problems.

> They are comfortable with different problem structures. So, that was more my learning goal before, make sure they could do these different kind of problems. Now that I have got that I feel like my learning goal can change more, and do like strategies. Push them to use bigger numbers and different strategies and things like that. (Julie, Course Day 6, 977-981)

Julie wanted more of her students to use sophisticated problem solving strategies, beyond direct modeling, including counting strategies, number recall, and standard algorithms. Julie identified that further work on addition and subtraction word problems, now that they understood the structures better, could improve the students' efficiency and procedural fluency.

Student Understanding in Julie's Classroom

It is very difficult to assert why a child understands a concept or from where that understanding came, given that it is difficult enough just to make the assertion that a child understands a concept. There are many mitigating factors other than the teacher's understanding and practice that can affect a child's understanding. Yet, it is important to try to qualitatively describe possible ways in which a teacher's understanding may have influenced a student's understanding. The purpose of this section is to describe scenarios in Julie's classroom during which teacher and student understanding could be connected through teacher practice. In the majority of cases I tended to work backward through the study's timeline. During a classroom observation or student interview I would hear or see something that indicated to me some aspect of student understanding. Next, I tried to indicate teaching practices that may have contributed to the understanding the student gained. Finally, through teacher interview and observation I tried to connect the teacher's practice to her understanding. The following paragraphs describe, for Julie, specific examples of possible ways teacher understanding can affect teacher practice which can in turn affect student understanding.

Student understanding of counting and modeling strategies. Even at the very beginning of observation I was impressed with how well students seemed to understand the counting strategies they were using. An overwhelming majority of the students were able to verbally explain the strategy in detail. For example, Sara, on the second day of class, explained how she found an answer to a Separate (Change Unknown) problem in which there were 12 birds at a feeder, some flew away, and there were 8 birds left. When asked how many flew away, she said, "12, 11, 10, 9, 8" while holding up a finger for each number she said aloud after the initial number 12. When asked how she found the answer she said, "I held 12 in my head and then counted backward, keeping track on my fingers." I was impressed at many students' abilities to explain the strategy they were using, for not only *Counting On* and *Counting Down* but also *Counting On To* and *Counting Down To*

There was a noticeable contrast from Julie's classroom to the second-grade classrooms in this aspect. Although it is difficult to pinpoint the reason for this contrast, it is true that Julie, on more occasions than the other teachers, spent time reviewing and giving examples of how to use counting and modeling strategies. This may be because the teachers thought that second-grade students use fewer of these strategies than first-grade students. With Julie, the understanding was easier to capture than it was with the other teachers, because she expressed it so often. Julie's repeated and consistent focus on the solution strategies shows her choice of privileging the teaching of these strategies in her classroom. The following excerpt is from the first day of observing Julie's class, but it is typical of the way she would address students' responses to questions about how they would solve problems.

Student: I remember that you started with four and [I went] one, two, three, four, so I needed six and so, five, six that I knew had to be there. And I went then I checked my fingers one, two, three, four, five, six, seven, I mean six.

Julie: So it sounds to me like you did it and then you checked it. So Lily took that four and she held it up in her head and did you use your fingers when you were doing that, Erin, or did you just keep track in your head?

Lily: I just kept track in my head and then I knew [inaudible]

Julie: Oh, so first she counted up in her head, and then we went “five and six” and said that that was two, and then she checked by putting out her fingers and counting and seeing if indeed it was two. So, Bill, how did you do it?

Bill: Um I started out with four and then I counted up and [inaudible] and I said five, six, then I added to four and then I got six.

Julie: Now, when you were counting up like that, could you also do that in your head? Were you keeping track of the numbers in your head? Okay, so you sort of used this strategy that Lily used, but then you didn’t check with your fingers. You were pretty sure you were okay?

Bill: Yes. (Julie, Day 1, 112-134)

Julie made sure to return to the children’s explanations and ask those students who were explaining to be clear about the strategies they were using.

Julie claimed that “students at this age are developing their own addition and subtraction strategies…it is helpful for them to see many examples, from me and from other students, so they can begin to do it themselves” (Julie, Course Day 5, 864-878). When asked how she chose which strategies to show for examples to her students she explained that she used strategies that “would be natural” for a typical student.” Her understanding about children’s strategies encouraged her to explicitly explain these strategies in her practice and ask her students to explain in detail as well. This practice may have influenced her students’ understandings of the strategies. Although this is difficult to prove, it is interesting to note that the students seemed to use the same vocabulary in their explanations as Julie. For example, students like Sara most often said they “held [a number] in their head” and then proceeded to count on or count back. This is the same language Julie used when describing her solution strategies.

Julie’s students also showed a surprising ability to apply their verbally stated understanding of the counting strategies to a written explanation of the counting strategies. Patterned after the way Julie diagrammed counting strategies on paper, Julie’s students had a similar approach to explaining their strategies on paper. This was true of most students in Julie’s class, not only the four I closely followed. On all of the problems in the packets Julie asked her students to explain in writing how they solved the problem, showing the thinking and the strategy they used. The following is a typical example of student work in Julie’s classroom and shows the common way students illustrate their thinking, patterned after Julie’s demonstrations at the board. A student, Sara, is solving a Part–Part–Whole (Whole Unknown) problem in which she needed to add the numbers eight and seven to find the whole (see Figure 5). Before Sara wrote her answer to the problem I watched her solve the problem by counting on, individually raising five fingers on one hand and then two fingers on the other. She then wrote how she solved the problem as follows:

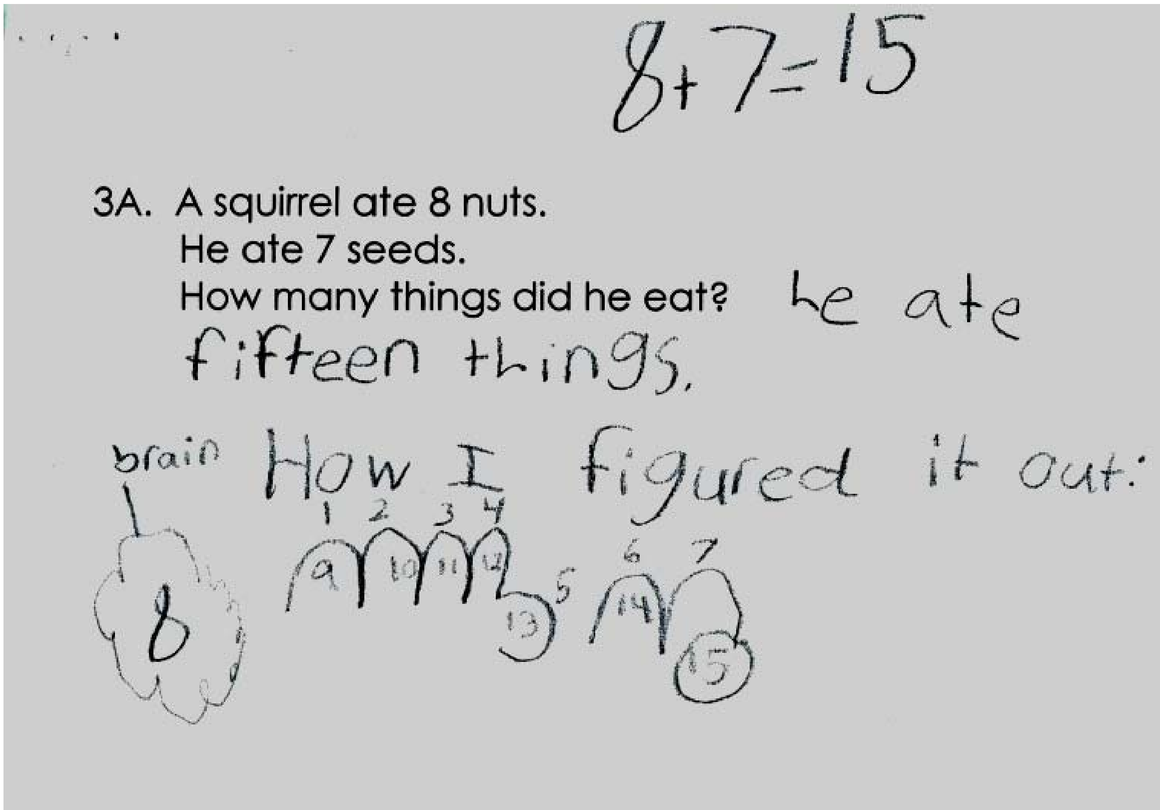

Figure 5. Sara's work.

When I asked Sara to explain how she solved the problem she said she "held eight in her head and counted up seven." Because the ways the students wrote the solution to the problems are so similar to each other, and to how Julie writes solutions on the board, I was afraid they might have memorized a strategy. Searching for more signs of memorization, I asked Sara to interpret the marks she had written on her paper. When I asked her why she circled the eight she answered, "that is the number I started with, the number I held in my head," and then wrote "brain" next to the number. She interpreted the rest of her writings by saying she drew five fingers and then needed two more to make seven. She labeled the seven fingers she was counting up ("these are the fingers I was counting") and then labeled the numbers she actually counted up ("these are the numbers I said in my head”). She then circled her final answer of 15. Sara was clearly able to explain the parts of the problem without memorization. Julie’s students that I observed were able to demonstrate this ability to describe their work with Join (Result Unknown), Join (Change Unknown), Separate (Result Unknown), Separate (Change Unknown), Separate (Start Unknown), Compare (Compared Set Unknown), Compare (Difference Unknown), and Part–Part–Whole (Part Unknown) problems. Their ability to explain and interpret their written descriptions of their counting strategies in this manner applied to *Counting On, Counting On To, Counting Down,* and *Counting Down To*.

In addition, one student was able to apply his work with *Counting On* to a more sophisticated strategy of counting up by more than one number at time. Instead of “hopping,” as Julie and her students said, by one number at a time, one student used this same written strategy but allowed the “hops” to represent more numbers, depending on how “high” the student wanted to go. For example, this student was solving a Part–Part–Whole (Part Unknown) problem and was trying to get from 13 to 21. The student wrote the number 13, and then made the hopping mark to 18, then 20, and then 21. The student then added 5, 2, and 1 to get 8. Julie was pleased to see this approach and explained it to the other students by comparing this strategy to those the other students were already using. “[He] is using hopping like we have been doing before, but he is hopping up by more than one number, as many numbers as he wants.” This is one example of a student applying his understanding that one could find the difference between two numbers by “hopping” from one number up to the next to solve problems in a more sophisticated way. It appeared to be an indication of a deeper understanding of the counting strategy taught by Julie.

Julie's students showed a particular ease at explaining and interpreting their verbal and written solution strategies that I did not see in other classrooms. In an interview at the end of the first data collection cycle I asked Julie why her students seemed to show this strength. She revealed that she had worked with the students to develop this understanding the first month of the school year before I began observing.

> Julie: We will talk about a problem and then I'll ask kids "How did you do it?" And they explain it to me, and then I demonstrate, "Well if I ask you to show your work this is what you could do." So a lot of the kids do that. Well, I saw there was 12 there, and then I counted up, so I showed them several times, you know so they've seen it before, that this would be a good way to show that kind of work. (Julie, Course Day 6, 995-1001)

In a course meeting another teacher admitted that often her students struggle with understanding when to count up or when to count down, or whether to count the initial starting number or ending number. Julie tried to explain her teaching practice to the fellow teacher.

> So I start at 23, so okay, let's put your finger on 23. And I've occasionally started them off verbally to get them moving in the beginning. So, if you gave one away, now how many would you have? Then they say, now I am down to 22. So that's pretty much what we had to start out, but I apparently have done it enough times with them... They've been using that strategy

> since the beginning of the year, and I have been modeling during morning meeting a lot what to do. So probably just that they've seen it, and the kids who are ready to do it, who aren't using the cubes, are remembering how to do that. (Julie, Course Day 7, 995-1044)

Julie used her understanding that the details of the modeling and counting strategies need to be correctly modeled for the students influenced her practice. It seemed probable that this practice had influenced her students' understandings of addition and subtraction word problems. The evidence seemed to be in the strategies and even word choices her students used that are so similar to those Julie used. Julie understood, and it seemed likely in the students I observed, that modeling the desired performance helped her students apply the strategies to their own work with explanation and interpretation.

I have previously asserted that Julie understood the difficulty level and stages of students' strategies, and that this understanding had affected her teaching practice. This practice had also affected her students' understandings and ability to progress through the stages. The following paragraphs trace one student's progress, as an example of the effects of the practice on students. This student was chosen because she was one of the most articulate of the four students I observed, and because, according to Julie, her work was representative of the way typical students progress, although she may have done so at a faster rate than others.

During the first observation cycle (seven days) Julie's student Jami used a variety of strategies to solve the addition and subtraction word problems. The first two days she mainly used counting strategies and even, Julie noticed, when Jami used a recalled fact she still reported using a counting strategy on her paper. On the second day of observation Julie encouraged the class, and Jami individually, to use recalled facts if that strategy was available to them. Julie believed students like Jami had greater understanding of the strategies than they were demonstrating. Julie stressed that they did not need to report a counting strategy if one was not used. I observed that for the rest of my data collection, when Jami used a recalled fact, that is what she reported. There was a rise in the number of recalled facts that she used, giving her more opportunities to demonstrate her ability to identify and explain this strategy. For the rest of the first observation period she mainly used a combination of recalled facts and counting strategies to solve her addition and subtraction word problems, the exception being only two problems on which she used direct modeling.

During the second observation cycle Julie encouraged Jami and the three other students in her group to try to use a strategy that was "easy for you and [was] the most efficient." Julie did not insist on a particular strategy, she just encouraged them to think about the strategies they were using and to chose one they could perform efficiently. Beginning on the first day of the second observation cycle Jami quit using counting strategies and began using number facts. She was not usually able to add and subtract the numbers directly, so she would use her understanding that numbers could be broken apart and then she would add or subtract parts together to get the answer. For example, on the first day when Jami needed to add 14 and 13 she broke the numbers into 10 and 4 and 10 and 3. She then added the two 10s together and then added the 3 and 4 together. Finally she found the answer by adding 20 and 7. Jami solved the rest of the addition and subtraction word problems that week by using this strategy of breaking numbers apart and using recalled facts. She had many opportunities to explain and apply this strategy during observations and interviews.

On the third day of the second data collection cycle Julie again encouraged Jami to progress in her problem-solving understanding. Judi encouraged Jami to use more sophisticated number sense instead of always breaking the numbers up into smaller numbers. For example, when adding 19 and 8, instead of breaking the 19 into 10 and 9, she encouraged Jami to just add one from the 8 onto the 19 and then add the remaining 7. This type of modeling by Julie allowed Jami to increase her number sense and progress in her understanding. Jami continued to demonstrate this ability when I observed her during the remaining two days. For example, the next day Jami was working on the following Join (Start Unknown) problem:

> Alex made some tacos. Sue made 12 more. Now they have 22 tacos. How many did Alex make?

Jami reported in an interview that she began by adding 12 and 12 together, but then realized that was too much. She took one away from each 12 and realized 11 and 11 would make 22. She then told me that she knew one of the numbers has to be 12, so she added 1 to one of the elevens to get the 12 she needed, and took one away from the other 11 to compensate and got 10. Therefore, she knew 10 and 12 would make 22, and 10 was her final answer. This type of reasoning, along with her ability to explain, interpret, and apply it, became characteristic for Jami. The timing of Jami's progression seems to indicate that it was in response to Julie prodding her to use more sophisticated solution strategies than drawing pictures or counting on fingers. Jami had many opportunities to see high-level performance modeled for her from which to draw, both from Julie and other students.

Student understanding of structure. The students were very adept at retelling and explaining the addition and subtraction problems in their own words, which suggested to me that they had a basic level of understanding about the structure of the word problems. The way the students were able to go back to the problem to check their answers, however, did indicate that they understood how the quantities in the problem were related. Julie repeatedly asked them whether their answers made sense with the problem and would ask them "what in the problem lets you know" how to solve it. She understood that her students would benefit from increasing their ability to use the problem-solving strategies with connections to the structure of the word problems. She did not just want students to follow solution strategies, she wanted their strategies to be influenced by the structure of the problem. Julie's ability to interpret the structure of the problem was evident in her teaching when she continuously asked her students to think about the action and relationships within the addition and subtraction word problems. Julie's attention to the structure of the word problem affected the students in that daily they were pressed to answer targeted questions asked by their teacher that required them to explain the structure of problems. From the data I am able to infer that Julie's practice of questioning her students is a linking mechanism for teacher and student understanding. Julie would ask the students to retell the problem or "story" and then ask them how the story helped them know what numbers and operation to use when solving it. The students were used to answering questions such as “What let’s you know that…” when Julie would specifically ask them to use the story itself for clues to highlight the structure of the problem. This aspect of Julie’s practice gave the students experience in interpreting the word problems with her.

Although obviously not every student had this understanding all the time, it was a level of understanding that was demonstrated on a consistent basis. A simple and common example of the students thinking about the structure of the problem is Sara solving a Separate (Start Unknown) problem on the last day of the second observation cycle. In this problem there were some balloons, 16 flew away, and she was left with 18 balloons, and she was supposed to find how many balloons there were initially.

Jana: So, Sara, on this balloon problem how come you decided to add them together?

Sara: Because that will get me the answer.

Jana: Even though some of the balloons flew away. You still added them together?

Sara: That would get me to where I started. (Sara, Observation Day 13, 38-45)

In her way, Sara knew that she needed to find the start, and she chose to use addition to find the answer to the separate problem. This demonstrated the student’s ability to recognize the problem as a Start Unknown problem and interpret this structure to find a way to solve the problem, adding quantities together to find the start. Sara is able to discriminate between Separate problems and subtraction problems, an understanding that is not always apparent in students or teachers, but is apparent in Julie's understanding as well. Julie's practice of asking the students targeted questions about the structure of problems, consistently asking them to use the relationships in the problem to make sense of it, appears to have positively affected the students' ability to interpret and solve the addition and subtraction word problems. Questions such as, "How did you know you were supposed to…," asked the students to identify quantities and relationships in the problem and interpret them in order to make sense of and solve the word problem.

It is easier to investigate how Julie's understanding affected her students' understandings during a teacher–student interaction in which the student is not solving the problem correctly and Julie intervenes, than in one in which the student is solving the problem correctly. The following is a description of Julie and Jami's interaction over a period of three observation days at the very end of the second observation cycle. On the first of the three days Jami attempted the two Compare (Referent Unknown) problems in the packet and solved them incorrectly. The first problem is the balloon problem Chad explained previously.

> Tom has nine red balloons. He has three more red balloons than blue balloons. How many blue balloons does he have?

The following is an excerpt from Julie and Jami's dialogue in class two observation days later, after Julie had studied Jami's packet.

Julie: So what does the nine represent?

Jami: He has nine red balloons.

Julie: He has nine red balloons and what does the three represent?

Jami: It represents the…Oh!

Julie: So what does he have more of? The red or blue? Tell me what you are thinking.

Jami: I know that nine minus three equals six.

Julie: How come you are subtracting them?

Jami: Because this might help me. I know that six plus three equals nine.

Julie: Okay so how come six makes more sense than twelve does?

Jami: Because he has three more red balloons.

Julie: So he has more red balloons than blue balloons. So what does the six represent?

Jami: The six blue balloons.

Julie: Blue balloons. So he has nine red and since he has three more red he must have less blue. So that's six. Cool, good job. (Julie and Jami, 090518, 1-28)

In this conversation Julie uses scaffolding. She targets her questions to encourage Jami to think about the structure of this problem without diminishing the difficulty of the problem. Julie then tried to press Jami to think about the numbers in the problem, the objects to which those numbers refer, and the relationships among the referents within the problem. She encouraged Jami to interpret why she subtracted instead of added based on those relationships.

Julie also helped Jami with the second Compare (Referent Unknown) problem that same day. The problem was:

> Peg caught some fish. Jack caught 19 fish. Jack caught 13 less fish than Peg. How many fish did Peg catch?

In this problem the Referent is also unknown, but it is slightly different in that it uses the word “less” instead of “more.” Julie uses scaffolding with Jami so that she will think through the problem similarly to how she did on the previous problem.

Julie: Did you read through it? Okay, so tell me what you understand about that.

Jami: Peg caught some fish and Jack caught 13 less fish than Peg. Okay. So I am trying to figure out how many fish Peg caught.

Julie: Okay. So, Jack caught 19. And he caught 13 less fish than Peg. So how many did Peg catch? Any ideas on how to do that one? Who caught more fish? Or who caught less fish?

Jami: Jack.

Julie: Jack caught less than Peg. Which one do we know? Do we know Jack or do we know Peg? We know how many one person caught. What do we know?

Jami: Jack’s.

Julie: We know that Jack caught 19 fish. So we’re trying to figure out how many Peg caught. So did Peg catch more or less than Jack.

Jami: More.

Julie: Okay how do you know that?

Jami: Because Jack caught 13 less than Peg.

Julie: He caught less and Peg has to catch more. Right? How many more did Peg catch?

Jami: 13. Okay. I think I got an idea. [Jami decomposes 13 into 10 and 3, adds 10 and 19 to get 29, and then adds 29 and 3 to get 32.]

Julie: What do you think?

Jami: He got 32.

Julie: Okay, so does that make sense with the problem? Does that sound like about 13 more? (Julie and Jami, 090518, 30-72)

In both of these Compare problems Julie uses scaffolding to redirect Jami to think about the relationships between the quantities, and specifically asks her to think about who has more and who has less, independently of the words “more” and “less” in the problems. Prior to the course she would not have even given her students one of these problems, and according to Julie, her questioning was not as “involved” because “it did not need to be.” She found she had to ask students more targeted questions about the nature of the problem and the relationships in the problem to press her students to think about the word problem structure. In previous years the students only had primarily two structures in their problems, Join (Result Unknown) and Separate (Result Unknown). Because Julie could now explain and discriminate among more types of problems, she was asking more of her students. Julie attempted to improve their ability to interpret problems by targeting the questions on the structure of the problems.

It was not until the next day that I observed Jami solving a Compare (Referent Unknown) problem on her own and was able to infer more about what Jami understood. Jami and some of the other students had finished their initial packets and Julie had noticed many students’ lack of success on the Compare (Referent Unknown) problems. Therefore, as she explained to me, she wrote a few more problems for them to work on the next day, two of them being the following Compare (Referent Unknown) problems, to see whether they made any progress in their understandings.

> Joe has 19 apples. He has 12 more apples than oranges. How many oranges does Joe have?
>
> There were 32 dogs at the park. There were 14 more dogs than cats. How many cats were at the park?

Jami initially adds the 19 and 12 in the first problem, but then without any intervention decides to subtract the two numbers. After her work I interviewed Jami.

Jana: So you think the subtraction one is right? So how come there can't be 31 oranges?

Jami: Because she has 12 more apples. There are more apples than oranges.

Jana: Okay. So tell me what you are thinking about this one. Just read it first. There were 32 dogs in the park. There are 14 more dogs than cats. How many cats? Okay.

Jami: Break up the 32 into 20 and 12 and then minus 10 from 20 and then minus 4 from 12.

Jana: You did that really well in your head. So how come on this one you are minusing instead of adding.

Jami: Because that will get me the answer. Because this [pointing to her original incorrect work] will get me 46. But there are 14 more dogs.

After Julie's intervention, as Jami solved the Compare problems on her own she was following Julie's example by reasoning about which quantity should be more and which quantity should be less. She was able to refer back to the relationships within the problem to correct her mistake on the first problem. She was able to improve her understanding from Julie's scaffolding the day before to solve these two problems. Although her work on these two problems does not tell us an extensive amount about her understanding, we see that some explanation, interpretation, and application on Jami's part were possible. It is probable that with even more interaction between Jami and Julie, Jami's understanding would increase.

In addition to their ability to explain, interpret, and apply, Julie's students also began to show evidence of their ability to discriminate among problem types. Although it was difficult for the students to explain similarities and differences between problems other than the operation that was used ("they are both addition problems" etc.) there were several instances in which a student was able to describe, based on structure, how one type of problem was different from another type of problem.

On the sixth day of observation Jami was asked to compare the following two problems:

> Rosa has 11 library books. 4 of them were funny. How many books were not funny?
>
> Kim saw 20 ducks on the pond. Then 9 flew away. How many ducks were still on the pond?

At first Jami stated that the library problem was not like the duck problem, but struggled to interpret why. She said they were both subtraction, but she understood that they were still different. Finally she said they were different because, "library books don't fly away!" She apparently understood there was something inherently different about the two problems regarding structure. I think this statement showed that Jami was discriminating between the two problems based on the action or inaction within the problem.

In a different example Julie had a discussion with a group of students on the first day of the second data collection cycle. Julie had chosen to give her students three different types of word problems, a Part–Part–Whole (Whole Unknown), then a Join (Result Unknown), and finally a Part–Part–Whole (Part Unknown) problem to encourage them to investigate the structures of problems. After the students had a chance to solve all three problems Julie asked them whether the Join (Result Unknown) problem was like the Part–Part–Whole (Whole Unknown) problem they did. Lily was able to discriminate between the two by saying they were different because in the Part–Part–Whole problem "I know they are already there, all 13 and 14 are already on the branches." The objects were already present at the onset of the problem, whereas in the Join problem "more came." Lily used the lack of action in the Part–Part–Whole problem to discriminate it from the action of the Join problem.

On the following day, Jami also explained an example of what makes two problems similar. She correctly solved the following Part–Part–Whole (Part Unknown) problem.

> There are some blue beads on a string. There are 15 red beads on the string. There are 25 beads altogether. How many blue beads are there?

When I asked her if this problem was like any of the other problems she had worked on that day she said it was “kind of like” the Part–Part–Whole (Whole Unknown) problem, “because there’s like some things…and then it’s like…altogether.” Although it was difficult for Jami to express further what she meant by “things,” I believe Jami was referring to the parts of the problem as “some things” and “altogether” was the whole. I think this example shows one of Julie’s students generalizing two problems. Jami seemed to see these two problems as particular cases of the Part–Part–Whole group of problems.

Again, it is difficult to say what caused students to be able to discriminate among different types of addition and subtraction problems. Julie used her understanding in the way that she presented different problems to the students, purposefully matching pairs of word problems (such as one with action one without, or two of the same type but with different unknowns) to give students the opportunity to compare and contrast. Julie would ask questions of the students to encourage them to focus on these possible similarities and differences. For example, with each word problem she would ask students what was known, what was unknown, and what was “going on in the problem.” These types of questions asked by Julie gave students the opportunity to understand the different types of structures, which would enable them to discriminate among the types of addition and subtraction word problems.

Student understanding of strategy and structure. There were examples from the data in which the students demonstrated an understanding of strategy and structure in order to successfully investigate the problem. In each of the examples the students' understandings of "fact families" often plays a part in their discussion. They had learned about fact families from Julie earlier in the year.[5] The following examples are evidence that students used understanding of strategy and structure to think about how addition and subtraction are related.

Julie and the other teachers were able to explain that most addition and subtraction word problems could be solved in a variety of ways, and in fact, many problems could be solved using addition or subtraction. Julie was very clear that a given problem could be both an "addition problem" and a "subtraction problem." This type of conclusion may have led Julie to think of addition and subtraction as a way to solve problems, or "strategy," and less of a way categorize problems. In her practice Julie did not title the word problems she gave her students as addition or subtraction problems. Each time she gave them a word problem she described it as a story that they needed to solve. She was pleased to see students solving the problems in different ways and would emphasize to the students that different members of the class were solving a problem with different operations. When Chad used an equation such as 21 – 13 = ? in his work and another student used 13 + ? = 21 for the same problem, Chad said they were they were similar and Judy remarked, "Like a fact family!"

---

[5] A fact family is a group of equations that fit together because they are all related, such as 3 + 2 = 5, 2 + 3 = 5, 5 – 3 = 2, 5 – 2 = 3. It was not until I was analyzing the data that I realized how often the students recalled fact families, and unfortunately I was not able to gather much information about this specific understanding from the students or Julie during the data collection process. I do know that Julie used fact families frequently in her practice before I began data collection.

This type of practice may have been part of the reason why students so readily used both addition and subtraction in their problem solving. For example, on the last day of the first data collection cycle Jami was working on a Separate (Start Unknown) problem. First she wrote the equation, _ – 4 = 3, which showed a correct interpretation of the structure of the problem. She also explained the structure saying, “Tim started out with some toy cars, he gave four away, and now he has three. The blank is first because that is what I don’t know.” Jami then proceeded to solve the problem by saying, “I know that four plus three is seven.” She then wrote seven in the blank spot and circled her answer. Jami knew that a subtraction sign best described the action in the problem and used that to write her initial equation, but she also knew that she could use her recalled addition fact to solve that subtraction problem.

The students’ ability to use addition and subtraction together in the same problem was not uncommon. Chad demonstrated a similar understanding at the end of the second data collection cycle when solving the following Join (Start Unknown) problem.

> A frog ate some flies. Then it ate 12 crickets. It ate 24 bugs altogether.
> How many flies did the frog eat?

Chad began by indicating that he wanted to solve the problem by taking away the 12 crickets from the total number of bugs, and wrote the equation 24 – 12 = ?. Despite having solved many problems such as this using subtraction, he decided to use addition to solve this problem. He said he knew “12 plus 10 is 22, 12 plus 11 is 23, and so 12 plus 12 is 24.” Chad used what Julie refers to as the “missing addend" approach to solve his initial subtraction problem, demonstrating an understanding that he could add on from the subtrahend to the minuend to find the difference in a subtraction problem. He was able to interpret the problem in a way that went beyond labeling it as an addition or subtraction problem.

Eventually the students were able to explicitly use fact families when solving addition and subtraction word problems. On the second day of the second data collection cycle Jami was again solving a Separate (Start Unknown) problem. The following is an excerpt from my dialogue with Jami.

Jana: It says Ken had some pennies. He lost ten of his pennies. Now he has six left. How many pennies did he start with?

Jami: Sixteen. Six plus ten pennies equals sixteen. So sixteen minus ten equals six.

Jana: Okay so how did you know to add the six and the ten together?

Jami: Because it's kind of a fact family.

Jami was able to explain her understanding that if 6 plus 10 equals 16 than 16 minus 10 equals 6. This indicates some understanding that there exists an inverse relationship between addition and subtraction. She is clearly not thinking of this particular problem as either addition or subtraction, but instead using the operations as a strategy to solve the problem.

Jami was able to apply this understanding to other types of problems and on other days of observation as well. For example, on the following day she worked on a Compare (Difference Unknown) problem. During an interview she explained how she used the fact families. Figure 6 illustrates the problem and her work.

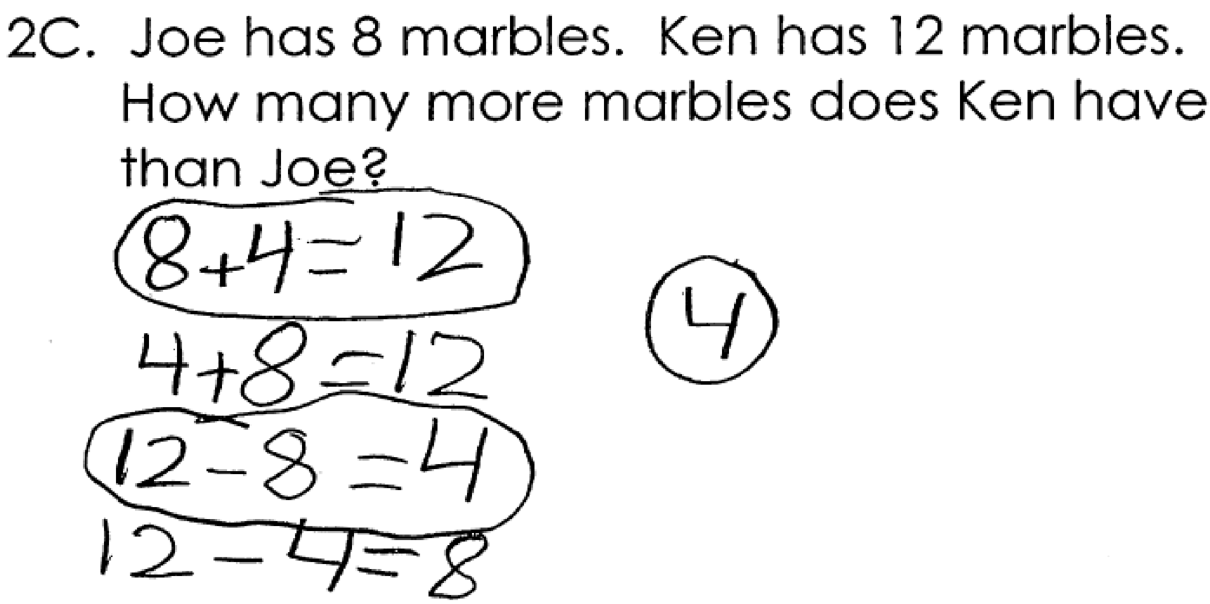

Figure 6. Jami's work.

Jami: So I did 8 plus 4 to get 12. 4 plus 8 equals 12 and then 12 minus 8 equals 4. So 12 minus 4 equals 8.

Jana: How come you wrote all of those?

Jami: Because it kind of helped.

Jana: Okay.

Jami: Because the fact family all equal 12 or minus from 12.

Jana: Okay, so how come you circled these two right here? [Jami had circled 8 + 4 = 12 and 12 – 8 = 4.]

Jami: Because they are the most helpful ones.

Jana: How come these two are the most helpful?

Jami: Because this one is kind of like an answer.

Jana: The 8 plus 4 equals 12? What do you mean "it's kind of like an answer?"

Jami: Because it tells you the answer. And then 12 minus 8 kind of is like this because it tells you the answer again. [Jami writes and circles the number 4 to the side of the equations as her answer.]

In this example Jami is using the fact family, and therefore the relationships between the three numbers, to reason about the Compare problem. She is concerned about the structure of the problem, and not just the numbers, and decides that two of the equations best describe the situation. She is able to interpret and solve the problem with both an addition and a subtraction equation. Jami is able to connect her understandings of structure, strategy, and addition and subtraction. Julie was aware that these word problems should not be thought of in terms of addition or subtraction problems, but that those operations could be used as strategies to solve the word problems. In her practice it was clear that sometimes a problem could be solved using either operation, due to her examples and to her acknowledgement and praise of other students' examples. The students were accustomed to thinking of the problems in terms of the relationships in the problem, and this focus likely attributed to the students using addition and subtraction as strategies to solve the problem instead of as a way to describe the nature of the problem.

Summary of Influences of Julie's Understanding on Student Understanding

The most obvious change in Julie's, and the other teachers', practice was the complete revamping of the word problems given to the students. Although this does not explain causality for the students' understandings it is an abrupt change in the teachers' practice that gave students experience with a deeper curriculum and, with that, opportunity for more depth in their understandings. This, of course, was just the first step for Julie.

For Julie it was important to model correct examples of how to use the strategies that were a part of her understanding. She repeatedly and explicitly explained how these strategies worked and how they related to the addition and subtraction word problems the students were attempting. Julie also gave the students in her class countless opportunities to model correct strategies as well. Although it is difficult to establish in this study what caused students to understand the strategies Julie was providing them, it is noteworthy that her students' written and verbally stated strategies consistently mimicked Julie's examples. Julie tried to use strategies that described as literally as possible the thoughts and actions of the students when solving the addition and subtraction word problems.

Julie was aware of each students' level of solving ability with these addition and subtraction word problems, and encouraged students to progress to more sophisticated strategies when ready. Julie expected her students to continue to develop more sophisticated strategies, she showed them examples of more sophisticated strategies, and encouraged other students to do so as well. Devising more sophisticated strategies to solve problems was a highly valued practice in Julie's classroom, one that regularly received recognition and praise.

Julie especially encouraged students to use multiple operations to solve a word problem. She did not treat problems as if they could only be solved with one operation, and took many opportunities to solve a problem with addition and subtraction. She would often ask whether students could solve problems in a different way and would point out explicitly when one person used addition and one person used subtraction to solve a problem. Julie became more flexible in her understanding of addition and subtraction by viewing these operations as strategies to solve problems (Rittle-Johnson & Star, 2007; Star & Rittle-Johnson, 2008). Her practice likely contributed to the flexibility of students' thinking, specifically students viewing the operation used to solve the problem as a strategy instead of a defining characteristic of the problem.

When discussing the word problems, Julie targeted her questions in ways that asked students to explain and interpret the structure of the problem. Julie asked students to think about the numbers in the problem, the objects to which those numbers refer, and the relationships among the referents within the problem instead of focusing on getting an answer or which operation to use to solve the problem. Students were able to interpret the problem, solve the problem, and check their work, using their understanding about the structure of the word problems.

Julie used her understanding of the structures of problems to better understand and carry out the objectives for her lessons. Instead of her objectives being confined to the outline given in *Investigations*, she was able to decide whether her lessons were to introduce a certain type of problem, discriminate among different types of problems, or develop strategies to solve problems. Julie constructed her lessons in ways that elicited meaningful conversation from her students, according to her objectives for the lesson. If she wanted them to think about different possible unknowns, she gave them two Join problems with different unknowns. If she wanted them to compare problems with action to those that did not, she gave a Part–Part–Whole and a Join problem. Julie would target her questions during the lesson based on these objectives. These practices gave students the opportunity to understand the different types of structures, which would enable them to discriminate among the types of addition and subtraction word problems.

With Julie's understanding of the mathematical content she was able to better understand her students and their needs. She gained deeper insight into her students' weaknesses, deeper than just knowing whether they could or could not find an answer to a problem. Julie was able to take a group of students, such as the higher achieving students, and determine the kinds of problems with which they struggled and design more problems for them. She assisted them in their learning by asking them targeted questions, pointed at the weaknesses she saw in their understandings. Julie's understanding and practice allowed the students to develop their understandings and apply them in the opportunities they were given.

It is difficult to determine what causes these practices to affect student understanding. By examining this case of how teacher understanding affects student understanding through her practice, I have highlighted examples of this type of interaction.

CHAPTER FIVE

PAM: UNDERSTANDING THROUGH EQUATION WRITING

This chapter describes Pam's unique understandings that I was able to uncover during the course, interviews, and classroom observations. It is not an exhaustive list of her understandings, but it includes the aspects that I perceived to have influenced her practice in teaching addition and subtraction word problems and subsequently may have influenced her students' understandings. This chapter also describes how Pam used her new mathematical understanding to teach second grade and indicates how her understanding of addition and subtraction word problems may have affected her second-grade students' understandings. When describing Pam's understanding and that of her students, I will continue to refer to the indicators of understanding described by Sierpinska and Wiggins, such as identification, explanation, interpretation, discrimination, application, and generalization (Sierpinska, 1994; Wiggins & McTighe, 1998).

Pam is a case of a teacher using equations to encourage her students to better understand the solving strategies for and problem structure of addition and subtraction word problems. Equation writing is essential in her understanding of content and pedagogy, and she privileges equation writing in her practice. This is evidenced in the ways she teaches her students and her students' understanding of mathematics.

Pam's Understanding

As Pam learned more about the different possible structures of addition and subtraction word problems she became increasingly unhappy with the way she had been teaching. She thought that her teaching had been monotonous and unchallenging for the students. She was especially unhappy because she believed her students were not learning to think mathematically but were only following procedures. She explained that her students seemed to barely read the problems because the story did not matter much, since the students knew they were "just going to take the two numbers and add them or subtract them," depending on the operation they were using that day.

Pam identified that before this study her second-grade students were only repeating what they learned in first grade about addition and subtraction word problems and the only way to challenge them was with larger numbers. She was able to distinguish between having lesson goals that challenged students with large numbers, and goals that challenged students to understand different types of word problems. With her previous knowledge, she explained, she was only able to do the former. Consequently, she was anxious to apply her new understanding of the different structures of problems in her teaching in such a way that students had to read the story problem and understand it, then apply their understandings to solve the problem. She explained that the structure of the problem was important, and the students' interpretations of the structure should dictate how they solve the problem.

Pam began to interpret addition and subtraction differently as a result of the course discussions. She explained that all the structures fell under "the umbrella of addition," and that word problems could look "different" but still be addition problems. She identified that "there's no reason why we can't spend more time with them to make them more fluent, not just in the [procedures of addition and subtraction], but in understanding the problem. We are not really making them very fluent in math problems" (Pam, Course Day 4, 965-966). Pam explained that understanding the different structures of word problems was essential for being "fluent" in addition and subtraction word problems.

Pam understood that there were eleven kinds of addition and subtraction word problems, instead of just the two or three she had been using, and wanted her practice to reflect this understanding. She emphasized, from the first day of teaching, that she was going to give students many different kinds of word problems, and the students' job was to not only solve the problems but also to discriminate among problems, explain differences, and interpret why those differences were important. Pam was aware that students were not accustomed to thinking about the structure of a problem but were accustomed to "just looking at something and going, 'I do not get it'" (Pam, Course Day 7, 506-507). She explained that this way of thinking about problems was different and that she was going to have to "teach them how they have to go back and look at it."

Pam used mathematical equations to understand and model the word problems. When we were discussing the differences between the different structures during the course, Pam explained that one could illustrate some of the differences between problems with a mathematical equation. Specifically Pam discriminated between using a subtraction sign when given a Separate problem and an addition sign when given a Join problem. She also discriminated among the places where a question mark would be placed, the notation she used for signifying the unknown in the problem. For example, in a Join (Start Unknown) problem the question mark, Pam explained, belonged at the beginning of the equation. Pam explained that if a student could write an equation that properly represented the word problem then that would demonstrate that the student understood “what the problem was asking.”

Essential to Pam’s understanding was that the question mark could be in different parts of the problem and that each such case would represent a differently structured word problem. In Join and Separate problems there was a distinct beginning, middle, and end to the problem, and so the question mark should be placed accordingly. During the first cycle of observations Pam was still solidifying her own ideas about how students should write the equations. In an interview after class Pam said:

> One of the problems that we did, I think it was a Start Unknown for Joining. And even though the problem was written with the information missing from the first number, they still set it up with the number they knew first, plus the question mark, with the answer. And I panicked and I was like, well who cares about that. As long as they didn’t do that for subtraction. That would have gotten them. But they didn’t do that. (Pam, Observation Day 5, 436-445)

Pam has identified the equation for modeling the Join (Start Unknown) problem that most directly reflects the problem’s structure but was aware that another equation, although less informed by the structure of the problem, would lead to a correct answer. She was not concerned about her students’ thinking as long as they did not use the incorrect equation for subtraction problems, which would lead to an incorrect answer.

By the second observation cycle Pam had changed her mind. She explained to her students that she was going to be "picky" about which equation they wrote. Even if it would give them the right answer, she wanted the equation to represent the structure of the problem. Because she wanted the equation to show how the students understood the problem, she explained that she needed them to use equations that they understood best represented the problem.

Pam: This problem was different than the last one. How is it different?

Student: Your question mark is in the middle.

Pam: Okay, so you had your question mark in the middle. Why did you have it set up that way? Reread the problem. Kim and Jay counted animals in the park. They counted some robins and then 37 squirrels. There were 60 animals altogether. How many robins did they see? So Jim how did you set up your problem then?

Jim: 37 plus question mark equals 60.

Pam: Did anybody set it up differently than Jim? Eddie how did you set it up?

Eddie: Question mark plus 37 equals 60.

Pam: Okay, and I think I noticed that Eddie and Jim ended up with the same answer. Does it matter that Jim has the question mark here? And Eddie has his here?

Class: No.

Pam: Why not? You all seem pretty confident that it doesn't matter. What if I change the equations slightly? Watch what I am doing. [Pam changes the addition sign in both equations to subtraction.]

Pam: Do you see this? Okay. So all I did was change the operation we were doing, right? First we were doing addition, now I made them subtraction. And suddenly it matters, doesn't it? Okay, I am going to say there is a correct spot. One of these is correct and the other one I would say is incorrect because I am a little bit strict about it. So if you look at that problem again, which one of these is correct? It matters what you write down. One of these two is correct. Heather?

Heather: I think it's that one [Heather pointed to ? + 37 = 60] because it says first there are some robins then it says they counted 37 squirrels.

Pam: This question mark is standing for what word?

Heather: Some

Pam: Some, good job picking that out. Some robins and 37 squirrels equals 60 animals altogether. So even though I bet Ethan and Carter had the same exact answer, I am going to be a little bit strict from now on. In the future I am going to check because it does matter what you write down. I really want you to pay attention to where you put that question mark.

Pam discriminated between equations that would give a correct answer to the problem and equations that reflected the structure of the problem. She explained to her students that although many equations will yield the correct answer for some word problems, they do not all correctly reflect the structure of the problem. She identified that if it is important for students to use the equation that correctly represents the word problem, then she needed her students to be thinking about the location of each number from the outset to generate good habits.

Pam explained that if she wanted her students to be "fluent" in these types of problems, then it was more important for them to understand the problem than to merely get the correct answer. Pam demonstrated that she was more concerned about her students' understandings when she said, "I could see if they had an understanding of what was asked of them based on an equation they started with and that's where my eyes went directly, not at their answer but how did they start" (Pam, Course Day 8, 334-338). She understood the importance of the structure of the problem and applied that understanding to how she interpreted the students' work.

Pam explained how her understanding had changed by discussing how her expectations of her students had changed.

> I have never done this [equation writing] because I never really needed to reinforce understanding. It's never been an issue that I thought, okay, how am I going to know if they are understanding? And I think that's what led me to think I always have a hard time judging. That's one avenue to get them thinking about what is going on in the problem, and that sometimes the first part of the story is missing, sometimes the middle, sometimes the end. And they really made that connection between what is going on in the story.

> Because in the whole group they really verbalized it, and symbolized it with mathematical symbols, and that is a really good tool for them. They are only in second grade and they are learning to symbolize English sentences. (Pam, Course Day 6, 559-568)

Previously Pam had not been teaching her students about the different structures of word problems. She found the material she was teaching before to be straightforward and even boring for her students, and did not need to use equation writing as a tool to see whether they were understanding the structure of the word problems. Pam identified the fact that she had previously had a difficult time interpreting the students' understandings because her mathematical understanding was not specific enough. She understood that it had transformed the way students interpret and represent the addition and subtraction word problems.

Learning about the structure of addition and subtraction word problems encouraged Pam to define her concepts of addition and subtraction in a different way. She explained that changing the unknown in a given problem emphasized a relationship between addition and subtraction. For example, comparing a Join (Result Unknown) problem and a Join (Change Unknown) problem, Pam said that she could "see within one type of problem how the operations were opposites of each other" (Pam, Course Day 2, 1023-1024). She said that although she would use addition to solve the Result Unknown problem, she could use subtraction to solve the Change Unknown problem, even though they were both Join problems. Furthermore, she said she would likely still use addition to solve the Change Unknown problem, thinking about what she could "add-on" to get the result. Julie used this manipulation of the variables and types of operations to describe how addition and subtraction were opposites. Pam identified that there is not just one way to think about addition and subtraction, and that they even can be used to solve the same problem. She interpreted the operations, like Julie, as “strategies” that can be used at the students’ discretion, and not a defining characteristic of the problem. Thinking of operations in this way increased her flexibility in problem solving and strategic competence (Kilpatrick, Swafford, & Findell; 2001). She also acknowledged that fact families would aid in understanding that addition and subtraction were opposites of each other and that if the students were “flexible with numbers” they could use fact families to solve problems.

Pam’s understanding was illustrated in her teaching when she encouraged her students to use addition or subtraction to solve a problem. For example, when talking to the class she identified that one student was using subtraction to solve a Join (Start Unknown) problem, whereas another student solved it using addition. Pam emphasized using the most efficient strategy, which she acknowledged might not be the same for each student.

Even when one student chose a particular operation to use to solve the problem, Pam often asked her students to check their work using the “opposite” operation. Pam explained, “Remember we talked about how addition and subtraction are opposites yesterday. It’s also a good way to check yourself if you are doing an addition problem, you might subtract to check for the answer” (Pam, Observation Day 7, 127-130). There were many examples of Pam using the opposite operation with individual students to help them check their understandings.

Pam's student, Niki, subtracted 10 from 39 and got 19. Although Niki seemed unsure of the validity of the answer, she was hesitant about how to proceed. Pam used addition to ask her to check her work and understand the relationship among the numbers. "So that does not make sense to you. If you add these numbers, 19 plus 10, what does that give you?...This number is 39. These numbers [19 and 10] should equal 39. So 39 take away 10..." (Pam, Observation Day 7, 221-243).

There were many times in our course discussions that Pam would write an addition equation to represent a word problem, but then use subtraction to solve the problem, or solve her subtraction equation using addition. She explained that an equation for a word problem should use a particular operation because of the structure, but that addition or subtraction could be used as a strategy to solve it. Pam explained this view of addition and subtraction in the following paragraph, after another teacher had asked her whether she was "going to do addition or subtraction when you start teaching."

> I guess I am confused about the question. I planned on doing Join, Result Unknown, Change Unknown, and Start Unknown on the first day. Whether or not they chose... I don't know if they will use subtraction as a strategy to solve it. So I don't know what will happen when the numbers get harder or when they start to write them out. I do not know if subtraction will introduce itself, then on the next day they do Separating Result Unknown, Change Unknown, and Start Unknown. (Pam, Course Day 5, 801-809)

Pam realized that the important characteristic of the problem was its underlying structure, not the operation one uses to solve it. She identified the structure of the problems she was going to give her students and then explained that the students could choose the strategy (or operation) they wanted to use to solve the problem. Pam clearly discriminated between the structure of the problem and the operation used to solve the problem.

Pam's understanding of the direct modeling and counting strategies detailed by CGI is very difficult to describe because they occurred so infrequently in her classroom. The vast majority of her students used number facts to solve problems. Like Julie's advanced students, they used their number sense to break apart numbers and combine numbers in solving the word problems. She was open to students using a variety of strategies, depending on what was "efficient" for them. She did not discourage using modeling but did not encourage it either, because she did not see it as a necessary strategy for her students.

Pam's Practice

In this section I detail how Pam's practice was affected by her understanding of addition and subtraction word problems, such as, the tasks she chooses, the types of representations she utilizes, the mathematics on which she focuses, and the questions she asks.

Like Julie, Pam chose to follow the outline of the *Investigations* unit by beginning with Join problems, and instead of only using Result Unknown problems she decided to give all three types of Join problems. She explained that she used relatively smaller numbers in these problems to see whether the students struggled with understanding the structure of the problems and to identify whether they "had the strategies" to solve them. Her learning objectives for her students were for them to compare and contrast all three types of Join problems. She also did not want them to stumble because of difficult numbers. Her goal was "to spend more time with [different problems] and show them all the different things they could be doing with [Join]…before I jump into [Separate] because we don't even get a chance to talk about how these problems are related" (Pam, Course Day 5, 316-320). She had a small group of six less-advanced students who worked with an aide in class. They solved the same types of word problems as the rest of the class but had even smaller numbers. Pam knew this group of students would work at a slower pace and would use more modeling than the other students. She wanted them to be able to work together at their own rate and have a chance to get more one-on-one assistance.

Pam altered her curriculum by starting with a Join (Change Unknown) problem. She followed the *Investigations* lesson by asking the students to picture the problem in their mind, and explained to me that this step was now necessary because the problem actually challenged her students. The class then went over different ways they could solve the problem, including different methods of adding or subtracting that would make sense with the structure of the problem. Next Pam gave her students a Join (Result Unknown) problem and again asked the students to focus on the structure of the problem. Before Pam asked the students to solve the problem she immediately asked, "Is this problem different than the last one? How are these problems different?" A student responded, "It didn't give you what you have to add. But in the other one it did tell you what you have to add." Another student added, “In number one they tell you how many there is in the end, but in number two they don’t tell you what you have in the end.” Because Pam was able to discriminate between these two Join problems she was able to juxtapose the problems to purposefully elicit a meaningful conversation from her students. By asking these types of questions first she emphasized the importance of understanding the structure of the problem over even finding the answer.

Pam privileged equation writing on the first day, and every day thereafter. She asked her students to always begin solving problems by writing the correct equations first. “Just start from the beginning using the numbers in here. If I was going to start…using what they give you in the problem. What would I write down to get me started? This might help you when solving problems” (Pam, Observation Day 1, 382-385). A student responded “13 plus 12 equals something.” Pam then asked the students to write an equation for the Join (Change Unknown) problem and compare the two equations. The students were able to interpret the two word problems distinctly and represented them with different equations, with the unknowns in the appropriate places.

Pam explained that now she needed to investigate her students’ understanding of the structure of the problem, something she had never needed to do before, and equation writing was one way to make this possible. She explained that the task of writing equations would be a catalyst for her students to think more about the structure of the problem, and would also be a way for her students to demonstrate their understandings. Pam asked her students to write an equation for a Join (Start Unknown) problem she gave them. After the students wrote the equation ? + 22 = 32, she reminded them that the “question mark can move to lots of different places.” She asked them to “write the equation so I know you know what you are solving for.” In addition to requiring students to write equations individually, Pam took the opportunity to discriminate among problems with her students. She asked them why problems were similar or different and what made them more or less challenging. In her practice she now focused her questioning on the relationships in the problem, and what was known and unknown in the problem. Pam continued with this focus on the second day of data collection, but used Separate problems instead of Join problems. After completing all three types of Separate problems her students began the packet of questions the teachers in the study had compiled for the second-graders (see Appendix B).

Pam asked her students to solve every problem in two ways after writing the equation. She did this hoping that students would use more than one strategy to solve problems. She wanted students to increase their own strategic competence and experiment with addition and subtraction and to invent strategies tied to their own understandings. She wanted these strategies to be based on the structures of the problems, so she asked questions such as, “What is happening in the problem? What do we know? What do we need to find out? What does the equation look like? What does this represent? What in the story told you to do it this way?”

During the second data collection cycle, day six of Pam’s observation, Pam asked the students to write their own word problem for homework. Pam used this as a measure for the students understanding. She was hopeful that with her new understanding and her new practices, her students would write better equations, and would be able to write equations that were not just for Join (Result Unknown) problems.

> Your homework is to write an additional story problem. You have to write and solve a story problem about a combining situation. You are going to write the equation first in class. When you go home there is going to be that equation that you are going to write a story problem for. Some might have the question mark in the middle, at the end, or the beginning. What would be an equation that would be something that would challenge you? You also have to solve it. Are you going to have a story problem where your question mark is first, in the middle, in the end? Did you use the right language? Show me something that is going to show your advanced thinking. (Pam, Observation Day 6, 517-557)

Pam had given this assignment in the past but had been unsatisfied with the results. As she said, "problems they write are really junky. There were five birds and seven birds how many birds were there? You know they don't even think this could look really different." Pam explained that the students did not place much emphasis on the parts of the word problem and did not use these problems to develop their thinking. This year Pam wanted her students to discriminate between the different types of problems that the students wrote, using an additional method to enhance the students' thinking about structure. She hoped that with the variety of problems done in class the students would apply their concepts of structure in the written word problems. She asked the students to write the equation first and then write the word problem. She explained that this would give them a concrete representation of the structure they needed to incorporate in their writing of the story.

Pam's increased understanding not only caused her to change her practice by giving her students different types of word problems and asking different questions, it also increased her expectations of the students' abilities. During an interview after the last day of observation I asked Pam how her expectations changed on the assessment problems she was yet to give them. The following excerpt shows her expectations for one of the problems.

Pam: He leaves with $.75 and when he gets to school he realizes there's a hole in his pocket and finds he only has $.47 left. How much did he lose? And I think what will be interesting is the different interpretations of this problem that I am going to get that maybe I would not have had in the past.

Jana: So what would you have gotten in previous years?

Pam: What they would've done? They would've done 75 minus 47 equals question mark. Or they might have done 47 plus something equals 75. Now there is variety in how they'll interpret it. Even when I am checking the workbook. I have to keeping thinking about it because I wouldn't necessarily do it… when people teach an algorithm combined with a story problem I feel like there are only one, maybe two, ways to go about it. [The students] can't demonstrate how well they understand addition and subtraction. It's a subtraction situation

> but how many of them will think of it as addition. And that's not wrong. Or even like you could think of it as 75 and I lost something to get 47, or 75 and I take out 47, what's left…Or I had 47 now and at some point I have 75. How much was in between? (Pam, Observation Day 9, 79-117)

Instead of merely expecting her students to find an answer, she is now expecting her students to identify the types of knowns and unknowns, interpret the problem in their own way, and use a variety of strategies to solve the problem.

One other notable change was her confidence in judging the abilities of her students, and in particular, her confidence in her decisions to get some students help in special school programs. In the past she claimed her attitude was, "better safe than sorry," when it came to deciding whether the students needed extra attention. The following explains her beliefs at the time of the interview.

> I feel like I really, because I know so much about the story problems and things that they are doing, I really feel confident in making that decision and knowing and being able to talk to parents. 'This is why. This is what I see. This is what they are doing,' and other years I've just, I don't know, I have not been that confident in it. And I have been more proactive now. I have communicated with them this week about, 'I'm seeing this and I am really concerned about this. Are they having trouble with the homework at home, because they aren't in school or they are in school?'

Pam understood the types of addition and subtraction problems and the relationships within those problems and this increased her confidence in her ability to understand her students' understanding and what they need to make progress in their development. She was able to be more specific about the learning goals for her students and therefore was able to be more specific about how the students were progressing. Her understanding increased her ability and desire to communicate with parents about their children.

Student Understanding in Pam's Classroom

This section will describe possible connections between teacher and student understanding as mediated through teacher practice. The following paragraphs describe examples of possible ways student understanding was affected by teacher practice, which was affected by the teacher's understanding.

Student Understanding of Strategies. The second-grade students with whom Pam worked rarely used direct modeling when solving addition and subtraction word problems; in fact, they did not often use counting strategies either. Although I never saw Pam discouraging her students from using these strategies, she did not necessarily encourage them either. When I began observing her classroom, Pam's students seemed to prefer other strategies and, according to Pam, they had "moved beyond" counting and modeling and "did not need them as much" (Pam, Course Day 4, 364-366). Pam's students used their understandings of breaking apart numbers and recalled facts to add and subtract. For example, on the second day of observation, Heather's work when solving a Join (Start Unknown) problem in which she needed to find what to add to 24 to get 43 demonstrates how students typically solved problems. She wrote the equation ? + 24 = 43 and then began adding numbers to 24. She added 10 first, writing, 24 + 10 = 34, and then another 10, writing, 34 + 10 = 44. She then realized 44 was too large and subtracted 1, writing, 44 – 1 = 43. To find her answer she added the two 10s and then subtracted 1.

Pam's students occasionally showed even more sophisticated understandings of addition and subtraction in their problem solving. When Niki solved a Join (Change Unknown) problem on the third day of observation she wrote the equation 32 + ? = 50. She then added 30 + 20 = 50 and then 32 + 18 = 50. Pam asked her about her thinking.

Pam: You did 30 + 20 = 50 and then?

Niki: Because this [the number in the problem] is 32, this [the number in her equation] has to be 32. So I added 2 more. Then I took 2 less from the 20 because if I added 2 more to here [the first addend] I have to take away from there [the second addend].

Pam: How did you know the 50 would not change?

Niki: Because I added 2 more and took away 2 more so it would be the same thing.

Niki showed surprising mathematical insight with her generalization and her nascent understanding that she could add to one addend and subtract from the other addend and maintain the same sum. I am not claiming that Pam specifically had an understanding of invariance or the sophisticated generalizing Niki may have been approaching. I do not have any direct evidence of Pam's understanding in these areas. There is evidence that Pam did understand that numbers can be decomposed and composed in a variety of ways and that compensation can be utilized in adding and subtracting two numbers. The students used these strategies daily in class as well. I suggest that Pam's understanding may have opened the doors for Niki's understanding and Pam's practices enabled Niki to come to more sophisticated understandings. Although Pam did not show evidence of thinking about this mathematics prior to Niki's work, she had enough understanding to see the validity of Niki's arguments and encouraged Niki in her work. Pam might have had an implicit understanding that became explicit when she allowed her students to let it surface. Pam created an environment in which student understanding could grow because of her own understanding.

Niki showed another more sophisticated understanding about the nature of addition and subtraction when solving a Separate (Result Unknown) problem on the end-of-unit assessment. She wrote the equation 32 – 7 = ? and then wrote 30 – 5 = 26, and then 32 – 7 = 26. Niki had a computational error in her work, because the equations should have equaled 25, but she realized that she could use what she knew about 30 – 5, added two to both numbers, and concluded that the difference would remain the same. Pam encouraged these solution strategies in group discussions. When Pam would notice students using these types of strategies at their desks, she would make note of it and then purposefully asked those students to explain their thinking to the rest of the class. Pam explained that she thought it was important for students to be able to use addition and subtraction creatively before learning an algorithm to solve problems and gave students time to construct their own strategies before teaching them any algorithms. The students had not learned any algorithm to add and subtract two-digit numbers prior to doing these word problems and were performing the operation using whatever method they preferred. By not teaching the students a standard algorithm for adding and subtracting, this forced the students to develop their own understanding of the operations and the properties associated with the operations if they were going to progress. These students were beginning to show understandings of properties such as, $x + \text{a} - \text{a} = x$, or if $x - y = z$ then $(x - \text{a}) - (y - \text{a}) = z$. Pam also gave the students opportunities to progress by expecting them to discover two different ways to solve each problem and demonstrating examples repeatedly in the classroom, either through her own examples or those of other students. I claim that these practices affected the students' understandings of addition and subtraction and led many students to the described understandings of the properties of addition and subtraction.

Every day that I observed Pam there were examples of her teaching students that they could use addition and subtraction as strategies to solve problems and that a given problem does not have to be an "addition problem" or a "subtraction problem." For Pam, her understanding of this generalization had originated before my study began, in that she explained that she always preferred solving subtraction problems with addition. But, through the study, her understanding was further developed. As Pam learned about the different structures of addition and subtraction word problems she would often comment about how they could be solved with either operation and that students should be allowed to solve problems according to their own strengths.

Because Pam asked her students to solve each problem in two different ways, and often encouraged them to use each operation to solve the problem, there were many examples of students using addition and subtraction to solve the same problem. Heather, on the second day of observation, solved a Separate (Change Unknown) problem initially using subtraction. She wrote the equation 27 – ? = 19 and then subtracted 7 and then 1 from 27 to get 19. Heather solved it a second way by adding 1 to 19 and then 7 to 20 to get 27, and a final answer of 8. Pam noticed Heather and a partner solving her problem using both addition and subtraction and took the opportunity to explain the method to her class.

Not only was Pam encouraging her students to use subtraction as a strategy to solve word problems but she also explained that they could use addition as a strategy to solve problems with a subtraction sign. I claim that conversations such as these may have contributed (a) to the abundance of examples of students' ability to explain how addition and subtraction are "opposites" of each other, (b) their understandings that, after writing a subtraction equation, they could use addition as a strategy to solve the equation, and (c) their understandings that subtraction can be used to solve equations that contain an addition sign. Pam's practice of explicitly asking her students to solve the word problems in two different ways, and sometimes with two different operations, contributed to the students' ability to interpret word problems using both operations. Pam's understanding that addition and subtraction could be used to solve a variety of problems influenced her practice, which I believe influenced her students' ability to see the operations as strategies used to solve problems instead of designating the nature of the problem.

Student Understanding of Structure. Pam's understanding that there existed different structures of addition and subtraction word problems influenced her to ask her students daily to discriminate among word problems. Having the students learn about the structure of all types of problems through discussions of similarities and differences among problems was one of Pam's stated primary objectives. There was evidence that some students were able to understand that addition and subtraction word problems could take many forms.

Heather, during the first day of observation, was able to talk about differences between a Join (Result Unknown) problem and a Join (Change Unknown) problem. When Pam asked her how the Result Unknown was different from the Change Unknown Heather said, "For this one [Change Unknown] it didn't give you what you have to add, but in the other it did tell you what you have to add." Pam's consistent emphasis on where the question mark was in the equation likely influenced Heather's attention to what parts of the problem were given and what part of the problem was missing. Pam consistently targeted her questions so that her students were asked to identify that for which they were solving in the problem, and the different ways that could change in the problems. There were many examples of Heather and other students explaining whether they were "looking for the beginning" or the middle or the end of a problem, or "looking for a part" or the "whole thing."

As well as discriminating between problems, students also were able to explain the details of a problem, even difficult ones such as Compare problems. Eddie was working on the following Compare (Referent Unknown) on the last day of the second observation cycle when I had the opportunity to interview him and observe him work through an explanation that demonstrated an understanding of the nature of the problem.

> Pete caught some fish. Sally caught 13 more fish than Pete. Sally caught 51 fish. How many fish did Pete catch?

Eddie initially wrote ? + 13 = 51 – 13 =?. He was unsure whether he wanted to add to get 51 or subtract 13 from 51.

Jana: Eddie, can you explain your equation here for number nine? Tell me what you were thinking and what you wrote down.

Eddie: Well it's something minus 13...it's 51 minus 13 equals 17.

Jana: What about something plus 13?

Eddie: Wait a second. [Eddie erases ? + 13]

Jana: So it's just 51 minus 13 equals something. So how come you erased this part, why is it just this part over here?

Eddie: Because you already know that Sally caught 13 more fish.

Jana: Okay, so you know Sally caught 13 more fish than Pete.

Eddie: And it was 51

Jana: What was 51?

Eddie: The fish that Sally caught.

Jana: The fish that Sally caught was 51 and she got 13 more. So you subtracted.

Eddie was able to explain the parts of the problem and thought about whether he wanted to use addition or subtraction to solve the problem. It is unclear whether he understood that either operation would yield a correct response, but it is clear that he can identify who caught more fish and how to find the missing set. Pam's practice of having the students visualize the problem and explain the parts of the problem, for every problem done together in class, may have influenced Eddie's ability to identify and explain the nature of the word problem.

The nature of Pam's questions also contributed to the students' understandings, beyond her merely asking them to explain each problem. She encouraged correct explanations by leading them through it, asking what the numbers referred to in the problem and the relationship among the referents in the problem. She not only expected students to be able to explain the nature of the problem initially but she also guided them to the understandings by asking them targeted questions that she hoped would lead to understanding. The following are examples of questions she asked during the first day of observation. Where does that number come from? What does it mean in the problem? Why did you start with that number? How did you know that…? How are these problems different? If I wanted to write what they were asking me here in the form of an equation what would I write down without putting in the answer yet? You guys told me the problems were different, so how do their equations look different? What do these two have in common? So, where is the part that you don't know? Where does the question mark go? Pam privileged equation writing in her classroom, but she did not do it in a way that encouraged memorization. She asked her students to think about the quantities in the problem and their relationships, providing an avenue for them to focus on the important mathematics in the problem.

The use of equations to identify, explain, and discriminate among word problems was clearly privileged in Pam's classroom. Pam's practice, influenced by her understanding, is reflected in her students' understandings. It was important for Pam that her students could show their work written in mathematical symbols, not just explained verbally. She would often say things such as, "Let's write that down first with numbers. The way we can show that is …" to try to get students to translate their ideas into mathematical sentences.

The following example illustrates Pam's conception of the importance of being able to express mathematically the nature of the problem. Niki was working on a Separate (Change Unknown) problem in which she struggled at first to write an equation.

Pam: Make sure you show me an equation you are starting with. If you can, look at what we talked about up there, because you did a really nice job explaining with your words out loud. Don't forget; show me the equation that you start with. What's the first equation? Where does the question mark go? [Niki writes 38 – ? = 17]. How did you know what the equation was going to look like?

Niki: Well, it was the wind blew some away and there were 17 left.

Pam: So how come you knew the question mark was going to go there, instead of maybe here or here [pointing to the 38 and the 17]?

Niki: I didn't know how many blew away.

Pam: Okay, so you know it's going to be the question mark. How come the question mark wasn't where the 17 is?

Niki: Because it told you how many were left.

Pam: Oh okay. Sounds good. Now try to show me an equation for this one.

(Pam, Observation Day 2, 400-425)

Niki was able to identify and explain the action in the problem and what all of the numbers represented. She was able to apply that understanding and wrote an equation that represented the structure of the problem. This is a succinct example of a student who was successful without much of an intervention from Pam, but there were many similar instances in which Pam asked the same type of questions of a struggling student. I suggest that this type of questioning by Pam, led Niki and other students to develop, over time, their understandings of structure, demonstrated through their ability to write equations for the word problems.

Pam's practice influenced the students' ability to write equations. Pam was concerned about the way students wrote equations for word problems and she wanted the word problems to reflect the meaning of the story as much as possible. Her students' understandings of this were strengthened during the time I observed them. For example, Heather became more adept at writing equations as time passed. On the first day of observations she wrote 24 + ? = 43 for a Join (Start Unknown) problem. I suspect that Heather understood the question but was not being specific about listing them in order. Pam responded, "This [question mark] is for what you don't know, right Heather? So in my parking lot I saw some cars. I also saw 24 trucks. So where is the part that you don't know?" Pam's questioning targeted the fact that the question mark represents the unknown, and that the unknown had a certain location in the equation that represented the relationship within the problem. After Pam encouraged her to read the problem and translate it more directly with mathematical symbols, Heather changed her equation to ? + 24 = 43. During the rest of my data collection I observed Heather solve three more Join (Start Unknown) problems. Each time she wrote the equation with the unknown in the beginning and was able to interpret the word problem correctly, explaining not only what the referents were in the equation for each part of the problem but why she wrote them in the specific places in the equation. I believe this aspect of Heather's understanding was influenced by Pam's targeted questions and expectations.

Heather applied this ability to interpret Join (Start Unknown) problems with mathematical equations to other types of problems as well. On the first day of the second cycle of observations Heather was working on the following Part–Part–Whole (Part Unknown) problem.

> Kim and Jay counted animals in the park. They counted some robins and 37 squirrels. There were 60 animals altogether. How many robins did they see?

Heather first wrote her equation as 37 + ? = 60. While explaining her thinking to me she changed her mind, however.

Jana: Can you tell me how you set it up right here?

Heather: I did 37 because it says we counted some robins and 37 squirrels. So it would be 37 and then some number because they just said some number. So then you don't know that number and then you take, oh wait, so we don't know that number so I put a question mark there.

Jana: So tell me, when you were talking you said ‘oh wait.’ What were you thinking right there?

Heather: The question mark went there because that came first, the number we do not know. [Writes ? + 37 = 60].

Jana: Will that change your answer at all if you change the equation?

Heather: No because it will still be the same answer. (Heather, Observation Day 6, 243-258)

Heather was happier with the second equation because of the wording in the problem; as the problem is stated, the unknown comes first, so she wanted that reflected in her equation. She understood invariance and the properties of addition well enough to realize she did not need to solve the problem again because her answer would be the same.

Pam asked her students to write equations for all the different types of word problems. She was aware that some word problems can have more than one equation that makes sense with the problem. She allowed her students to interpret the word problems and explain them in their own way. On the seventh day of observation two of her students were working on the following Compare (Difference Unknown) problem.

> One second-grade class flew 28 kites. A third-grade class flew 53 kites. How many less kites did the second-graders fly?

Eddie solved the problem by writing 53 – ? = 28. He then began to use a type of *Counting Down To* method by taking numbers away from 53 until he reached 28. The sum of the numbers he took away from 53 gave him his answer. He explained that he had to find “how many less” kites 28 was from 53 and so he had to count down. Heather also solved this problem but used the equation 53 – 28 = ?. She subtracted 28 in parts from 53 and reached the same answer. Pam encouraged both students to use the methods they had chosen and saw them as equally valid. She had the students share their answers with each other and the students were not concerned that their equations were different or that they solved them differently, in fact they were pleased they had found multiple ways to solve the problem. Although she could have been more explicit about why certain word problems had more specific equations that would represent them, and which stories fit these criteria, she was able to explain to her students that sometimes problems could have different equations that represent them equally well. I think this is an example of a situation in which Pam’s practice may not have directly caused students to be able to see a problem represented with multiple equations, but her practice did allow that understanding in her students to occur. She understood the mathematics enough to comprehend what her students were doing and incorporated it into her own understanding. She encouraged her students to interpret the problems in their own way, and when they discovered problems that had various interpretations she was able to see that this was possible and desirable. She encouraged them to continue in their thinking.

It was interesting to hear Pam’s ideas of how her students were progressing compared to the progress of students in her class during the prior year. The following excerpt comes from an interview I had with Pam on the last day of data collection. Pam had remarked that her students seemed to have remembered a great deal from the beginning of the year to the end of the year.

Pam: I am surprised, yes. I am shocked that we are not, especially with subtraction. I definitely remember in the spring, it falling apart with subtraction again, and it was like starting from square one [in previous years]. And I feel like yesterday and today, some things here and there, but not like it was before. They are going into it, and the connections…that subtraction can be thought of with addition because we do that all the time, and they are really doing it.

Jana: What do you think the difference is?

Pam: I mean, I definitely spent more quality time with story problems this year than last year. And they have been exposed to more types…I feel that expanded their thinking from the fall. It was not totally confusing them to see something else. They are just going right through them; it is not even different for them anymore. And I think the time I spent was much more quality. You know, I feel okay about these kids kind of going along and looking at all of those strategies. I definitely think creating the different kinds of story problems, I think that they were ready for in the fall made them more prepared for this in the spring. I feel like, where do I take them from here?

Pam understood that her students were doing much more than memorizing problems. She understood that through her practice she had expanded their thinking, giving them opportunities to explore how to explain, interpret, and solve a variety of addition and subtraction word problems. She was excited and eager to see what the students were capable of in the future.

Summary of Influences of Pam's Understanding on Student Understanding

Pam understood that students need to build on their own knowledge and develop their individual understandings of addition and subtraction and this influenced her practice and her students' understandings. By not teaching the students a standard algorithm for adding and subtracting, she allowed—and basically coerced—her students to develop their own strategies as shortcuts for modeling and counting strategies. She gave her students opportunities and praise for demonstrating more sophisticated ways to solve problems, and often took the opportunity to demonstrate her strategies to her students. In her practice, Pam asked her students to solve all their word problems in two different ways, which specifically asked them to make connections between strategies using the multiple ways. These connections could be between things such as representations, operations, or solution strategies. Because of the flexibility in her understanding she asked the students to solve problems in multiple ways and then compare and contrast those strategies. This practice gave the students further incentive and avenues to develop their flexibility, which increased their conceptual and procedural understanding (Rittle-Johnson & Star, 2007; Star & Rittle-Johnson, 2008). By demonstrating examples repeatedly, students were not completely left on their own, but were given a path they could follow in building their understandings.

Part of the students' understandings of the strategies used to solve word problems was that the operations of addition and subtraction can be used as strategies, and the two operations could sometimes be used to solve the same problem. The students I observed showed consistent understandings that the operations were opposites of each other and could also be used to check a student's work, or as a second method to solve a problem. Pam's practice of explicitly asking her students to solve the word problems in two different ways, and sometimes with two different equations, contributed to the students' ability to interpret word problems using both operations. There were times when Pam asked students to use a particular sign to represent an action inherent in the word problem, but she did not control how they used the operations as strategies to solve any of the word problems. She distinguished between using the operations to model problems and using the operations as strategies to solve problems. This influenced her students' ability to see the operations as strategies used to solve problems instead of only designating the nature of the problem.

The nature of Pam's questioning in the classroom contributed to the students' understandings of structure. Beyond Pam simply asking them to explain each problem, she encouraged more explicit interpretations by leading them, asking them questions such as what the numbers refer to in the problem and the relationship among the referents in the problem. She did not simply expect students to be able to explain the nature of the problem initially; she guided them by asking them targeted questions that she hoped would lead to understanding. Pam did this repeatedly, over the entire observation period, which established the focus of her teaching and the understandings in her students. I claim that the emphasis and focus of her questions led to student understanding because they were consistently being asked to explain, interpret, discriminate and apply that type of understanding. Pam had specific learning goals for her students to understand the structure of the problems, and used her questioning to aid her students in reaching those goals.

Part of the students' understandings of structure was their ability to discriminate among different problem types, an objective of which Pam was clear before instruction began. In her practice Pam used equation writing as a tool for students to articulate the relationships in the problem. Pam's consistent emphasis on where the question mark was in the equation was part of her plan to press students to discriminate, showing what parts of the problem involved values that were known and what part of the problem involved a value that was unknown. Pam consistently targeted her questions so that her students were asked to identify the different possible knowns and unknowns in the problem. Pam's practice of having the students visualize the problem, explain the parts of the problem, and write an equation that represent the problem, before even solving the problem, may have influenced the students' ability to discriminate among problem types.

It was important to Pam that her students could translate their ideas about the nature of the problem into mathematical sentences. Pam asked her students to write an equation for every word problem they solved. She asked them to identify the quantities in the problem and how they should be represented in the problems. She gave them examples to follow. Daily, Pam asked students to give their own examples, with explanations for how they wrote their equations, including why they decided to represent the quantities accordingly and where to place the representations in the equation. I suggest that this type of questioning and practice led students to develop, over time, their understandings of structure through equation writing.

Throughout this study I have been trying to uncover ways in which a teacher's understanding influences student understanding through her practice. Most of my observations have been concerned with ways in which teachers proactively try to influence the students. I assert that in Pam's classroom, I was able to infer from observations, numerous situations in which Pam's practice did not necessarily cause student understanding but allowed it to happen because she did not try to change or hinder it. Pam was aware, and mentioned in interviews, that certain word problems did not necessarily have one equation that would best represent them, but this understanding was not explicit in her teaching. Therefore, Pam's practice did not cause students to be able to see a problem represented with multiple equations, but it did not inhibit her students from obtaining this understanding. Because of her understanding, and through her practice, she encouraged her students to interpret the problems in their own ways, and when they discovered problems that had various applicable interpretations, she was able to verify the validity of the argument and encouraged them to continue in their thinking.

CHAPTER SIX

CARMEN: UNDERSTANDING PROBLEM SOLVING

Carmen's individual understanding and practice that I observed during the course, interviews, and classroom observations is described in greater detail in this chapter. This chapter also describes how Carmen used her new mathematical understanding in practice and how her understanding may have affected her second-grade students' understandings. When detailing Carmen's understanding and that of her students, I will continue to refer to the indicators of understanding described by Sierpinska and Wiggins, namely identification, explanation, interpretation, discrimination, application, and generalization (Sierpinska, 1994; Wiggins & McTighe, 1998).

Carmen is a case of a teacher using addition and subtraction word problems as a means to teach students about problem solving in general. This chapter gives evidence of Carmen's understanding and practice that led to this conclusion. I will also describe how this practice affected her students' understanding.

Carmen's Understanding

Carmen identified on the first day of the course that it was important to classify the different types of problems based on "what kind of thinking is involved for the students." She explained that "structure is based on the action, or what the child has to do to solve the problem, since there is not always an action in the problem." Carmen explained that when she said "what the child has to do" she was referring to the way the students would model the word problem. Carmen was very aware that she needed to determine what her students were thinking about the problem and press them to think more effectively about the structure and strategies used to solve addition and subtraction word problems.

Carmen understood that the tasks she designed needed to build off of the students' knowledge, letting the students guide the practice in the classroom. As she said:

> We feel like we have to help them get the right answer all the time. Really the right answer is not their understanding of the problem that I care about, so you know that's interesting to me to let something go. And I think that is why I'd be able to say I don't care about how we are solving the problem. I want to know what you understand from this. What is it asking? What is it saying? What is it telling you? And we will get out the approaches on how to solve that later, but just understanding what is there first is the important thing. (Carmen, Class Day 6, 293-301)

Carmen was able to distinguish between understanding the structure of the problem and the strategy used to solve the problem. She wanted the understandings of the students to determine the pace of the course, and the first step was for them to explain and interpret the structure of the word problem. Carmen recognized that her students have the ability, on their own, to "tune in to the structure of the problem." She understood that pedagogically it was important to let them learn and struggle to find ways to solve all of the types of problems. Carmen's privileged this process of students learning how to solve problems on their own in developing strategic competence because she used problem solving as a measure of their success.

Carmen's understanding and practice changed together through the observation period. She allowed her understanding and practice, including daily lesson plans, to evolve as she gathered information about her students' understandings. Although the other teachers came into teaching with a lesson plan, loosely based (at least initially) on *Investigations*, Carmen changed and formed her lesson plans daily, according to what she learned from the students that day in class. Therefore, instead of describing the rest of her understanding and practice separately, I will describe how the two evolved together.

Carmen's Practice and Understanding: Observation Cycle One

On the fourth day of our course (October 28th), five weeks before Carmen began teaching this topic, she decided to begin teaching her class with a problem other than a Join problem. She said, "How about forget the Join altogether and why not just throw a different one out." Carmen explained that even beginning with a Join (Change Unknown) problem was not difficult enough for her students and would not challenge them. She did not want to start them with "a combine situation" such as problems the students were accustomed to solving. By the next course meeting she had solidified her ideas about how she would begin teaching. She explained to the other teachers:

> I think of myself when I learned, you had an example and then this is how you do it. There is the example and this is how you do it. And then you just get to be applying the strategy. I guess I am playing

> with just putting any old type up there. What's your approach? It is not like you are teaching them how to do it. It is just, what is your approach? So does it really matter that they are Join or that they are Comparing? (Carmen, Course Day 5, 819-826)

Carmen explained that her emphasis was on the students learning to solve problems, and the addition and subtraction word problems were a conduit for that learning. Therefore, it did not matter which type of problem the students were asked to solve, as long as it required them to think about the relationships in the problem and use their interpretation of that structure to solve the problem. Before she even began teaching, it was clear that Carmen intended to privilege problem solving in her classroom.

On the first day of teaching Carmen gave her students a Part–Part–Whole (Part Unknown) problem and a Compare (Difference Unknown) problem. After the first day of teaching Carmen explained that the Compare problem was more difficult than the Part–Part–Whole problem, and also explained that the different types of Compare problems would be even more of a challenge. She also identified that the students were using strategies without understanding how they related to the problem. For example, to aid in their use of counting strategies, they would use a hundreds chart on which they would color in the two numbers given in the problem and then not know whether they were supposed to count the number filled in or start with the next number. Carmen was able to identify the students' current strategies and how students understood these problems and then changed her practice accordingly.

The second day, instead of giving more of any type of problem, Carmen decided to focus on what she considered to be the most difficult type of addition and subtraction word problems, Compare problems. She was hopeful that if her students could learn to explain and interpret the structure of these problems, they would be able to apply their understandings to solve the other types as well. She also decided to emphasize that the students needed to make sure the strategy used to solve the problem corresponded with the structure of the problem. Carmen gave her students a Compare (Comparison Set Unknown) problem and a Compare (Referent Unknown) problem.

Carmen decided the Referent Unknown problems were the most difficult and that students struggled with understanding whether they were supposed to count on or back. For example, the students were asked to solve the following problem:

> Ben went on 33 rides at the fair. Ben rode 13 more than Ryan did.
>
> How many rides did Ryan ride?

Carmen explained that it was difficult for students to identify who rode more and who rode less. Carmen also learned that students would read the word "more" in the problem and naturally be inclined to count on, despite the fact that in Referent Unknown problems, the opposite is necessary.

Carmen explained her practice to the other teachers during one of the course meetings.

> I decided from that that my kids needed more practice with comparing. So I decided the next day to do more comparing, and that first day I wasn't real happy with the way I myself presented things.

> I still felt like I was missing something. And I did not really like the way things had gone. So I went back and we did the compare problem and I started focusing a lot more about what is happening in this word problem. Tell me what is happening. I don't care about the numbers so much; just tell me the story that is behind it.

Carmen explained a problem-solving technique she thought would direct her students to think about structure, having them explain the story in their own words, describing the relationships in the problem without using the numbers in the problem.

> And that is pretty much what I focused on then with the compare problems, because then the compare problems that I focused on had the opposite wording in it. So that if you said someone caught 22 fish and then they caught 14 more fish than the other person… So they struggled with that a little bit again, until I went back to make them just talk about what's happening in the problem. And then when I did that, it was, the light bulb started to go off. And as soon as I said, go back to the problem. Read it. What's happening? Don't worry about the numbers. Just tell me what is happening in your story. When they could tell me the story without the numbers, then it seemed to give them more focus and they knew what to do and how to approach it. So it is real interesting for me, because I have never focused on those things before. I always say, 'well what's the important information.' And I started off doing that. But I don't feel

> like it was really helping the kids. They are starting to think a little bit more about it and now they are much more confident when I say well does that make sense. Why does it make sense? They are able to verbalize it a lot better than what they were at the beginning. So I guess I've learned that it's really important to get them to understand what's going on and to have them go back when they are finished too, because it was interesting that when they knew they made a mistake, when you asked them if they did it and they said "yeah, but wait a minute. That doesn't make any sense." And I said, "Well what are you going to do about it. Why doesn't' it make sense?" They were able to tell me why. "Well because he has the most and he doesn't have the most now." So they were making sense of that.

Carmen focused the students on finding out what happened in the problem, not just picking out the important numbers as she previously had done, but thinking about the structure of the problem. Her practice of asking students to retell the story without the numbers served to focus the students on the relationships in the problem. This method was intended to ask students to think about the mathematical relationships in the problem, and detour children from trying to memorize a way to answer questions. All four students I observed that day were able to solve the Compare (Comparison Set Unknown) problem but struggled with the Compare (Referent Unknown) problem.

On the third day of teaching Carmen again focused on Compare problems and gave the students two Compare (Referent Unknown) problems, one using the word *more* and one using the word *less*. She designed a task that she understood would target students' attention directly on the relationships in the Compare problems. Carmen specifically explained that she wanted to give the students two problems in which the only difference was isolating what she considered the current challenge, the students figuring out whether they needed to count up or count back. One of the first questions she would ask her students after they read the problem was, "who has more" or "who has less?" She continued to refer them back to the problem. Carmen also was influenced by her practice the day before and began repeatedly asking students whether the answer "made sense." She understood that checking to see that an answer "made sense" was an essential part of problem solving.

Next, Carmen decided that the students needed a more concrete way to understand the relationships in the problem and to see whether the answer they arrived at made sense with those relationships. Carmen developed a series of questions that she created for each word problem and asked the students to answer. On the fourth day of teaching she again gave the students Compare (Referent Unknown) problems with "more" and "less" and then asked them the following questions. Who caught more fish? (This question was written to be applicable to the story; more fish is just an example.) What clue tells you this? Does your answer make sense? Why does your answer make sense? She gave students room to write their answers after each question. This list of questions is further evidence of Carmen's privileging problem solving in her classroom. Carmen hoped that this series of questions would continually redirect the students' thinking to the learning objectives that she had established, explanation and interpretation of structure, and development of their problem-solving abilities.

Carmen thought that her students were making progress in visualizing the structure of the problem, and on days five and six of teaching, the last two days of this cycle of observations, Carmen believed that the students were ready to try other types of problems. She gave her students the packet the teachers had developed and let them work on any problems they chose (see Appendix B).

Student Understanding: Observation Cycle One

At first the students almost exclusively used a hundreds chart (a chart with numbers from 1 to 100, with 10 numbers in each row) to solve problems. Carmen was aware that they were not using the charts with much understanding and with little reference back to the original problem. For example, they were asked to solve this Compare problem.

> Penny jumped 61 times. Martha jumped 27 times less than Penny did. How many times did Martha jump?

The vast majority of the students would color in the square containing the numbers 27 and 61 and then try to decide what to do from there, even in cases such as this, in which filling in the number 27 does not make much sense, because that represented the difference between the two numbers of jumps. They would also struggle with deciding whether they should "count" 61 or 27 when they counted between the numbers.

Carmen encouraged her students to refer back to the relationships in the problem so that their strategies would be influenced by the structure of the problem.

She would ask them what the numbers represented and why it would make sense to "count" them or not. For example, on the previous problem she had the following conversation with a student.

Carmen: If you count here now, he is counting up from 27 to 61, should I count that 27 or should I not count that 27?

Student: Not count.

Carmen: Why not?

Student: They both jumped 27.

Carmen: Do the rest of you agree? Let's make this a little easier. Let's say Penny jumped 11 times and Mike jumped 7. How many extra jumps? What can we do to figure that out? Do we count this number or not? Now I circled what Mike jumped. Seven jumps. I drew a line underneath what Penny jumped, 11 times. And the question is how many more times did Penny jump than Mike? Did they both jump at least 7?

Student: Yes. Then one of them jumps more.

Carmen: Right, but they both jump this many, so would I have to count that number or not count that number?

Student: They at least jumped 7. He made a mistake when he stopped. You don't count that because he stopped.

Carmen: So what is extra? What is the next jump that is an extra jump?

Student: 8.

Carmen: So do you count that number or do you not count that number?

Student: He wrecked on the 8.

Carmen: He wrecked on the 8, so where should I start counting?

Student: 8.

Carmen: Are we sure? [Many students still seem unsure.] Penny and Mike, come over here. Alright, now, when we count, you are going to jump. You are going to stop, only jump how many times?

Mike: 7.

Carmen: And you are only going to jump…?

Penny: 11.

Carmen: So let's see both of you jump.

Students: 1, 2, 3, 4, 5, 6, 7,

Carmen: But Penny is going to keep going. Now did I count this time? I am not going to count until now, the next jump right so…

Students: 8, 9, 10, 11

Carmen: How many extra jumps did she have?

Student: 4

Carmen worked to direct the students' attention back to the structure of the problem. In this example she used scaffolding, pressing the students to answer questions that established the number of times they both jumped, and which numbers represented the "extra" jumps. She wanted the numbers the students discussed to relate back to the jumps in the problem. She used her understanding in her practice to ask students to focus on not just the numbers, but to what they refer and how they relate to the nature of the problem.

After the third day of teaching the students began to use the chart more effectively. The students I observed no longer filled in the two numbers from the problem immediately. They read the problem and then only filled in the numbers that were appropriate. They could explain why they counted certain numbers and not others. For example, on day three Rich explains, "Because she needs to know how many more, so I counted from here to here. Yeah, 24 through here, and I counted the 57 because that's more, you need to know how many more and it is 34." He is thinking about the relationship between the numbers 23 and 57 and used that relationship to decide how much more 57 is than 23. It was not until the end of the week that students began to consistently use successful strategies to solve problems. There were many computational or counting errors, but Carmen was pleased that by the end of the week they were using their understandings of the structure of the problem to choose appropriate solution strategies.

Carmen explained that some of the students were struggling with understanding the structure of the problem, evidenced by their struggle with the hundreds chart. As a result, she encouraged them to use modeling to understand the structure and to solve the problem. Carmen was helping Penny with a Compare (Referent Unknown) problem on the second day of instruction in which Penny had to find how many rides Ryan went on if Ben went on 33 rides and rode on 13 more than Ryan did. Carmen tried to allow Penny to think about the structure by decreasing the size of the numbers. "What if I said Ben went on 10 rides? Let's just pretend. Ben went on 10 rides, okay. And he went on 3 more rides than Ryan. Can you figure that one out?" Penny still struggled so Carmen said, "What if he rode 1 more ride than Ryan?" Penny incorrectly answered 11, counting on instead of counting back. Realizing that Penny did not understand the nature of the problem, as she described later in an interview, she asked Penny to draw out 10 tally marks, each representing one of the rides Ben went on.

Carmen: Now Ben went on one more ride than Ryan did. He had one more, how many did Ryan go on?

Penny: There's one more.

Carmen: There's one extra, show me the one extra. Just circle it. So how many rides did he ride?

Penny: 9.

Carmen then had Penny use this modeling strategy with larger numbers.

Carmen: Okay, so now let's go on and make it a little harder. 10 rides, and 3 of them. He rode 3 extra rides.

Penny: [incorrectly counts out to 13]

Carmen: But wait, he went on 3 extras.

Penny: Oops.

Carmen: Would it make sense to put those 3 on?

Penny: No.

Carmen: Why not? Who rode the most rides?

Penny: Ben.

Carmen: And that's what you just wrote, Ben right? So can we add more rides to it if he went on the most?

Penny: No.

Carmen: No he went on 3 more rides than what Ryan did.

Penny: 1, 2, 3 [circles 3 of the original 10 tally marks.]

Carmen: So how many did Ryan ride?

Penny: 7.

Carmen: All we're doing is using bigger numbers. So here how many rides did Ben go on?

Penny: 33.

Carmen: So what are the most rides you can have?

Penny: 33.

Carmen: Okay he went on 13 more rides than Ryan did. Did you have a strategy you can try to figure out? (Carmen, Observation Day 2, 216-316)

It was difficult to tell from this interaction how much Penny was understanding. However, the next day during an interview with me, Penny directly referred to the strategy she used with Carmen. She was solving a Compare (Referent Unknown) problem and said she "wants to use the same strategy as yesterday." She made her tally marks, circled the ones she did not need because the Comparison set had "more" than the Referent Set, and counted the rest. Although she made a counting error, she remembered and used the strategy facilitated by Carmen the day before. Similarly, Penny used the same modeling strategy to correctly solve a Compare (Referent Unknown) problem on day four of my observations. Carmen understood that it is important to be able to interpret the structure of the problem through the use of a solution strategy. This enabled Penny to develop a tool to explain and interpret the Compare problems on which she was working. Eventually, toward the end of the second observation cycle, Penny was able to progress further and could solve the problems using a counting strategy and even fact recall.

Carmen's focused her questioning on understanding the problem and not just solving it. She asked her students to be able to explain the problem, without using numbers, a task designed to focus the students on the relationships among quantities. It is likely that her students were influenced by this practice as they began to develop the ability to explain the structure and relationships between quantities in the addition and subtraction word problems. For example, when Carmen gave the students two Compare (Referent Unknown) problems on the third day of teaching, Rich worked to understand how the words "more" and "less" were related to the quantities in the following two problems.

> Noelia had some Webkinz. Anna had 53 Webkinz. Anna had 27 more Webkinz than Noelia. How many Webkinz did Noelia have?
>
> James bought some marbles. Adam bought 81 marbles. Adam had 36 less than James. How many marbles did James have?

On the first problem Rich incorrectly added 27 on to 53. Carmen asked Rich to read the problem again and explain it without using the numbers. Then when asked how he might solve the problem, Rich identified that he needed to count backwards instead. To make sure Rich understood the reasoning, and was not just guessing the opposite strategy, Carmen asked him what part of the story told him he needed to count backwards. Rich replied, "Because she has less than Anna." Rich was able to identify that Noelia had less from his interpretation of the fact that Anna had 27 more.

On the second problem Rich read the problem and then Carmen asked him to explain the problem.

Carmen: Tell me what you know.

Rich: James has 36 more than him.

Carmen: So what do you do?

Rich: Count 36 more.

Carmen: Where do you start?

Rich: 81

Carmen: How did you know which way to go, up or down?

Rich: Because James has more.

Carmen: How do you know?

Rich: Because it says Adam has less, which means James has more.

(Carmen, Observation Day 3, 754-765)

Rich utilized the same patterns that Carmen had used in her teaching practice, identifying which quantity was more and which was less, and then solving the problem accordingly. Rich was identifying the structure of the problem, including the relationships between quantities, and then using his interpretation of those relationships to solve the problem. The next school day the same reasoning appeared in Rich's work. Without intervention from Carmen, Rich was able to solve the same type of problems. During interviews with me he used the explanations of the relationships between quantities he had developed previously with Carmen, and his interpretations of those relationships led him to successfully solve several Compare problems.

Carmen explained in an interview that her interactions with her students, such as these, led her to design the series of questions that she asked her students after each problem on the following teaching day, Who has more? What clue tells you this? Does your answer make sense? Why does your answer make sense? She explained that these types of questions directed Rich's attention to the structure of the problem and she wanted to use the same pattern for her other students. Carmen actively designed questions for her students to redirect their thinking toward the relationships in the problem. This focus of Carmen's questioning targeted important information that likely led to students' meaningful understandings of the relationships in the problem.

After so much time spent on Compare, and especially Compare (Referent Unknown) problems, Carmen was very interested to see what the students would do when given the other types of problems without further instruction. She was pleased to see that the students were able to apply their ability to read, explain, and interpret to many of the other problems in order to solve them. For example, on the last day of this observation cycle, Adam solved a Separate (Start Unknown) and Join (Start Unknown) problem without hesitation. He was able to explain the relationships between the quantities and find an appropriate strategy to solve the problem ("I need to count back 18 from 39 to see how many she started with"). Similarly, Rich was also able to solve a Join (Start Unknown) problem on day six of observation, the last day in this cycle. To figure out how many leaves the problem started with, he explained that he needed to go to 36, the number she has now, and subtract 17. When Rich was asked why that would work, he replied, "because that is what she started with." Rich claimed the Join (Result Unknown) problem he tried next was "the easiest problem so far," and used *Counting On* to find the correct answer.

Lynn was also able to apply what she had learned about Compare problems to solve the following Separate (Start Unknown) problem.

> Erika has some horses in a stall at a competition. Judges from the competition came and took 16 of them. She has 48 horses left. How many horses did she start with?

Lynn made 48 tally marks on her paper and then added 16 more marks to get a total of 64. When asked how she decided how to solve the problem Lynn said, "Because they took them away, and you have to figure out what they did, well, how many she had first. And if they didn't take them away she would have them there, the same amount she started with (Lynn, Observation Day 5, 143-148)." As explained in interviews throughout her teaching, Carmen made a conscious decision to concentrate her teaching on the most difficult type of problem first, the Compare (Referent Unknown) problem. She believed that if students could explain and interpret the structure of these problems, they would be able to do this as well on the other types of problems. Carmen recognized the structure in the addition and subtraction word problems and then used that understanding to create a productive instructional sequence for those problems. While implementing that sequence, she chose questions to ask of students that highlighted the problems' structure. She used her understanding in her daily interaction with her students, by focusing the questions and class discussions on understanding the nature of the problems being solved. Carmen was pleased with the results and, in general, found that students were able to understand many different types of addition and subtraction word problems after struggling with Compare problems.

Not all of Carmen's students solved all of the types of problems perfectly the last two days of first observation cycle, but the most common mistake was computational errors. For example, when Lynn was solving a Join (Start Unknown) problem she correctly decided to count back, but counted back one fewer number than what was correct. Similarly, when Penny solved a Separate (Change Unknown) problem she correctly applied her the modeling strategy she had developed with Carmen days earlier for Compare problems. Penny said she was going to, "put 46 in my head, and count back to 22. While I'm doing it I'm going to make tally marks for every time I count, so I know how many." The marks that she made recorded the change quantity. Unfortunately, she was supposed to count back to 27 instead of 22, but her only error was likely misreading the problem or forgetting which number with which she was working.

Carmen identified that her students were making mostly computational errors after this first cycle of teaching. Although she was surprised that her students were not able to more effectively carry out the procedures, she was pleased that they seemed to understand the natures of the problems and were applying their understandings to choose correct strategies to solve the problems. She believed the students could work on their procedural understandings more in the future, with many types of problems.

Carmen's Practice and Understanding: Observation Cycle Two

During the first cycle of observations Carmen wanted her students to think about the structure in the Compare problems, so she had them answer questions for each problem such as: Who had more? What part of the story tells you this? Does your answer make sense? And, Why does it make sense? On the first day of the second round of observations, Carmen explained in an interview that this form of questioning was only appropriate for Compare problems with one of the sets unknown. Carmen planned to give her students all the types of problems starting from the first day of this second cycle of observations, and she identified that she did not want the success of her students' transition to be left to chance. So, she constructed another checklist—designed to be applicable for all problem types—that she wanted her students to follow when solving problems. Although she had the checklist prepared before class, she developed it with the students so they would take ownership of it and phrase it in their own words. As they solved each problem the students were to make a check when they completed each of the following: 1) Read the problem. 2) Look for clues in the problem, find the important information. 3) Think of some strategies. 4) Solve the problem. 5) See if it makes sense. Again, during this second observation cycle, Carmen is explicitly privileging problem solving in her practice. Although this checklist is clearly similar to what appears in the work of George Polya (1985), she made no reference to using his work.

The students began with a Join (Start Unknown) problem, one Carmen chose because she understood that it was one of the more difficult types of problems other than Compare problems, with which she did not want to begin again. She asked the students to explain what was happening in the problem, including the quantities the students did and did not know. Carmen asked the students not only to explain what was important in the story but also to interpret why the details were important. After asking students for strategies to solve the problem, Carmen would ask the students whether their answers made sense. She would have the students reread the problem, with the answer included, so that the students could discover whether the answer made sense in the problem. For example, if the problem began, "Tony had some cookies…," and the student found the answer to be ten cookies, then the student would reread the problem as, "Tony had ten cookies…" to see whether the answer fit logically into the problem. After the students worked on the Join (Start Unknown) problem as a class, they began work on the packet the teachers had constructed that contained addition and subtraction word problems of all the types identified by CGI. They were instructed to use the checklist to find answers to problems.

The second day of teaching followed a similar pattern, but the class did a Separate (Change unknown) problem together before continuing on with their packets. The students were again instructed to use their checklist to think through the problems. Carmen explained in an interview that she was concerned her students might try to "guess at a strategy" (specifically guessing whether to use addition or subtraction, which Carmen described as possible strategies) instead of thinking through which operation to use. To encourage her students to mindfully discriminate among strategies she would ask them questions such as, "How would we know to do 64 minus 38? Why would that work? What would that tell you if I took what I started with and I subtracted what I have left…what is that going to tell us? Does that make sense?"

Carmen's practice during the last three days followed the same pattern. She began by having the entire class work with a Compare (Difference Unknown), Compare (Reference Unknown), or a Separate (Start Unknown) problem, respectively, and then the students worked on their packets at an individual pace. During the entire course of data collection, working on these types of word problems was the only task she assigned the students.

Carmen explained to me in an interview that her students were able to choose correct strategies to solve problems but they were again struggling to carry out the procedures using the chosen strategies correctly. Although solving the problems correctly was not her focus initially, during the last 3 days she began to put more emphasis on demonstrating correct strategies with her students. She hoped that through practice her students would learn how to effectively use the strategies they chose and that it would "start to click." Carmen also explained that she hoped students would learn proper procedures by watching her and other students demonstrate correct strategies.

For instance, a typical example of how Carmen would work with students as a class to understand not only the structure of problems but also how to carry out the procedures was illustrated on the second to last day of observations. Carmen was solving the following Compare (Referent Unknown) problem with her students.

> Brenden caught some salamanders. Tyson caught 17 more salamanders than Brenden. Tyson caught 31 salamanders. How many salamanders did Brenden catch?

Carmen went through the checklist with her students. After reading the problem the class searched for the important information they needed from the problem. The students thought of some strategies that might be appropriate for the problem, and the class solved the problem together. A student suggested using the hundreds chart.

Student: Go to what Tyson caught and then count down to 17."

Carmen: Why do we start at 31?

Student: Could we start at 17?

Carmen: What are we comparing?

Student: How many salamanders did Brenden catch?

Carmen: The 31 tells us what Ty caught, so we have to start there to see how many Brenden caught. Who caught the most?

Students: Tyson

Carmen: How many more did he catch?

Students: 17.

Carmen: So If we are at 31

Student: Take away 17.

Carmen: How would I do that? (Carmen, Observation Day 10, 294-310)

Carmen demonstrated how to take away 17 on the chart and then also showed the students how to explain it with writing on their paper. She also asked the students whether their procedure and the answer made sense with the problem. Carmen wanted her students to be able to explain and interpret the nature of the problem and also solve the problems with understanding.

On the last day Carmen explained to me in an interview that over half the students seemed to have clear understandings of the structure of the problems and could carry out the procedures correctly. She suspected the others were still struggling to get an accurate answer despite their understandings of structure. To test this hypothesis she had the students who had finished the first packet work on problems from a second packet she made on her own. This packet contained a Join (Change Unknown) and (Start Unknown, Separate (Change Unknown), Compare (Difference Unknown), and Part–Part–Whole (Part Unknown) problem. However, it only used numbers that combined to be 100, in other words, numbers with which Carmen thought the students would be very comfortable (See Appendix C). Out of the four students I had observed, two of them were able to solve problems from the initial packet and this second packet successfully. The other two students fit the category about which Carmen was concerned. These two students got correct answers to all the problems in this second packet that contained the numbers with which the students were more comfortable. During interviews they were still able to explain the nature of the problems. They were able to apply their understandings they had gained in the previous weeks to solve these problems quickly and efficiently. From this test Carmen concluded that these students did understand the structure of the problems and were struggling to correctly carry out procedures with numbers that were difficult for them.

Through these experiences Carmen learned that students need to continue to practice thinking about relationships within the problems in order to retain and build their understandings. Carmen discussed this during an interview on the last day of observation.

Carmen: [My intern and I] thought they did well after we did it the first time, and then we kind of left it for a while, which just…next year I think I'll change things around a bit and do a word problem a day. Make it part of my calendar thing, just do different types, have that be a little routine that this is something that we do and keep following a procedure. I think they get the procedure really well. You know helping them go through.

Jana: By procedure you mean the steps of problem solving.

Carmen: Yes. But you know, I think the best thing I have learned is if you do not keep doing it, keep after it, they are not going to, they didn't know as much as we thought. (Carmen, Observation Day 11 Interview, 52-73

Student Understanding: Observation Cycle Two

This section details the understandings exhibited by the students during this second cycle of observation. On the first day of this cycle Carmen introduced her checklist of questions to help students solve the variety of addition and subtraction word problems. From the beginning there were instances of students using the checklist to navigate through a problem. Even on this first day Rich said, "When I first sat down I didn't know what to do. When I started to read [the checklist], I started to figure it out, and it got easy. Because, first I sat down and I checked these off, and then I did it. This is what happens." The checklist was designed by Carmen to help the students stay organized and focused on the structure in order to solve problems.

The following dialogue is a typical example of how Carmen interacted with her students during this second observation cycle, using the checklist to press her students to better understand the structure of the addition and subtraction word problems.

Adam: [Reads the problem.] On Monday he checked out some books from the library. He returned 17 books on Tuesday. He still had 23 books check checked out at this house. How many books did he check out on Monday?

Carmen: Okay, so tell me in your own words what happened.

Adam: He read some books from the library. Returned 17 of them. And then he still had 23 more books at his house.

Carmen: Okay, so if we keep going on our check list, like you are doing, we figure out what's the important information there.

Adam: How many he returned to the library.

Carmen: Okay, that's one important information. What else is important?

Adam: How many does he still have at his house?

Carmen: Okay, and anything else?

Adam: And it said he had some books. Oh, there is something else. He had some there to start with.

Carmen had previously asked students to focus on the details of the story without talking about the numbers in the story. In previous years, when students were asked for the important information, they would repeat the numbers, without talking about objects to which they referred. Now Carmen's students were habitually talking about the quantities in the problem and their relationships instead of pointing out the numbers.

Carmen: Okay, so now how are we going to figure out how many he started off with?

Adam: He returned 17 books. He still had 23 books at his house. How many did he check out on Monday? We don't know. This one is hard.

Carmen: This one is a little tough, isn't it? So think about what is happening. When he first went and got

Adam: Some books. Then he gave back 17. And he had some left. 23.

Carmen: How do we figure out how to start? What did he start out with?

Adam: He started out with, I don't know.

Carmen: What could we try?

Adam: Take away maybe?

Carmen: Okay try it. See if it works.

Adam: He still has 6 books left at his house. Because he took 17 in and he has 1, 2, 3, 4, 5, 6 left.

Carmen: Okay. Let's see if that works.

Adam: [Adam reads story with his answer of 6 embedded in the story.] Nate checked out 6 books from the library on Monday. [Shakes head.]

Carmen: What's off? Right away you shook your head, why?

Adam: Because that didn't sound right. Because if he checked out 6 books. And then returned 17, that's less than that one. That doesn't make any sense.

Using Carmen's checklist, Adam returned to the original problem to see whether his incorrect answer made sense. After reading it, he decided that his answer did not make sense.

Carmen: Okay. Good. I love the way you are going back and checking to see if it makes sense. What else could we do? That didn't work the taking away thing didn't work did it? So what else could we do to figure it out?

Adam: Maybe let's try adding. What a minute. It would have to be subtraction because he put 17 back. He didn't check them out.

Carmen: But what is that 17 coming away from?

Adam identified the action in the problem as separating, so he is again gravitating towards using subtraction in the problem. Carmen attempted to confirm his ideas that there is separating in the problem, but tried to allow him to focus on the relationship between the 17 and the original number of books.

Adam: The 23, so we can minus…

Carmen: Is that what is happening?

Adam: No.

Carmen: What's the 17 coming away from?

Adam: The some books that he had.

Carmen: So let's try to write this as a math problem, so if we want to write this as a subtraction problem like you are saying. The 'some books' would be what?

Adam: You don't know.

Carmen: So what could you put there if you wanted to write that down? [writes a question mark] Then what?

Carmen focused her questions on the relationships in the problem, "What is the 17 coming away from?" Carmen began to show Adam the equation that would model the relationships, but she did not get very far before Adam had another idea about how to make sense of the problem.

Adam: Oh. What a minute. I think adding them might work.

Carmen: Why?

Adam: Because you add that to this 23.

Carmen: You add the ones he returned, with the ones he has left? And that will give us what?

Adam: more that we need.

Adam realized that he needed to begin this problem with a larger number of books, if he is going to take some away and still have some books left over.

This thinking may be an application of Carmen's pattern of asking "Who has more?" when doing the Compare problems previously.

Carmen: So let's see if it works. Try it.

Adam: See that would give us 40.

Carmen: Okay well let's see if that works. [Adam reads problem to himself.] Now why does that work? You said it makes sense. Why does that work?

Adam: Because he returned some books.

Carmen: Okay, so he returned 17 books. And what else?

Adam: He had 23 books left.

Carmen: So if you add those two number together that will give you how many you started off with? Does that make sense?

Adam: Yes.

Carmen: That does, doesn't it? Think about that. If you took, here I have some. And I give some back to you and I have some left. Well if I add those together I get what I started with, right? (Carmen, Observation Day 10, 41-213)

Carmen and Adam used the checklist and focused their dialogue on the relationships among the quantities in the problem. Carmen made sure Adam understood what the numbers in the problem represented and targeted the structure of the problem. With Carmen's scaffolding, Adam showed some ability to think about the action and the relationships in the problem. He was able to check his work and see whether it makes sense.

On the following day Adam showed evidence of learning from his work with Carmen. As a class they worked on the following problem, that is also a Separate (Start Unknown) problem, chosen by Carmen because students "struggled with one the day before."

> On Monday, Anna went to the carnival and bought some ride tickets. She went on 19 rides. On Tuesday she still has 22 tickets left for rides. How many ride tickets did she buy?

As a class they went through the checklist of steps, and Carmen asked Adam how he solved the problem. He said:

> If we add 22 and 19 together that would give us how many tickets she had on Monday. Because if you add them together, you would know how many she bought. Because before she took those apart she had the answer that equals 22 plus 19. (Adam, Observation Day 11, 60-76)

Adam seemed to be using the same type of reasoning he had used on the previous day with Carmen. It was also interesting that immediately following this class discussion Carmen approached me with excitement that Adam seemed to learn from their previous interaction.

> It was funny that Adam didn't even have to think about it. Yesterday it was so hard for him. And a couple others I noticed, that was a problem so I picked that same type of problem. And today it was like, I got it, and there wasn't even any question. When I sat and

> talked with them when they were discussing strategies Adam knew right away. (Carmen, Observation Day 11, 195-199)

Adam was able to explain the Separate (Start Unknown) problem. He was able to explain the structure of the problem, what the numbers refer to, and how the referents relate together. This understanding came at least in part through his interactions with Carmen on the previous day. Carmen's practice of pressing for answers to questions regarding the meaning of the problem structure and her practice of asking her students to make sense of the problems through the checklist of questions influenced Adam's ability to identify the types of knowns and unknown in the problem, explain the relationship between the quantities, and interpret those relationships in order to find the answer to the problem.

Carmen's learning goals for her students focused on understanding the nature of the addition and subtraction problems. She wanted her students to be able to explain the structure of the problems and use their interpretation of the problems to decide how to solve them. Halfway through the second cycle of observations Carmen identified that her students, in general, were successful at understanding the types of problems they were given and could even choose appropriate strategies to solve the problems. She realized, however, that they were less successful at carrying out the strategies. Because of Carmen's understanding she placed greater importance on understanding structure and that is reflected in what the students understood. Carmen argued that her students' procedural understandings were less important at this time and that it was better for her to focus on their conceptual understandings. She asked questions of her students that targeted this conceptual understanding and spent a greater amount of time encouraging them to explain the relationships in the problem and interpreting how the relationships fit together than she did asking them to find the correct answer.

By the last day of observations Carmen had studied the students' packets of addition and subtraction word problems and summarized that half the students were still making mistakes adding and subtracting despite having a solid understanding of the structure of the problems. She had the students who finished the packets solve the additional packet she designed the night before, which contained more word problems, but with easier numbers. Every student I observed was able to successfully complete, with explanations and interpretations, all the problems in this packet, even though two of the four were struggling to find correct answers in the previous problems. From this exercise it seemed that Carmen's hypothesis was correct. The students were able to understand the nature of the problems, and even chose correct strategies to solve the problems, but struggled to carry out the procedures. I believe the students' understandings were affected by Carmen's practice. She was explicitly less concerned about, gave less time to, and endowed with less importance the correct answer to the problems. She was more concerned that the students develop their strategic competence and were able to make sense of a problem and choose a strategy that also made sense. Computational errors were of little concern to Carmen; she made little notice when students' answers differed from the correct answer by a small amount.

When Carmen was asked how her students' understandings had changed during the current year in comparison to other years, her answer included how her practice had influenced the change.

> I have them tell me the story to begin with now. Where I say, what's happening? Before they even begin the problem I ask them, "Tell me what's going on there." It's more the overall general picture of what's going on. It gets more generalities instead of just the specifics. (Carmen, Course Day 11, 183-214)

Upon further questioning I conclude that by "specifics" Carmen refers to the numbers in and solution to the problem. She identifies that her current students are able to explain a general picture of what is happening in the story, not just identify the important numbers and find an answer. I again asked Carmen how this affected the students' understandings, and the following was her reply.

> I think now they are able to know, well, if I am starting off with this, and you know someone took something away from that, well then my answer, what I am looking for, has to be less than that or it has to be more than that. So they know a direction that they need to go, or they can make sense of what their answer should be. They may not use the right approach but then they are able to say, "Well wait a minute. That's not right." And they never did that before. I mean it

> was always they get the answer, done. That's it. So it's teaching that those two parts. There's that beginning part where you really have to understand what's happening before you know what to do. And then the ending part of going back to say, does it make sense. And even if it's not the right answer, I don't care about that. It's the approach that they go to get it. (Carmen, Course Day 11, 183-214)

Carmen's focus was on the students' ability to explain and interpret word problems prior to choosing an appropriate strategy to solve the word problem. She thought it was important for her students to gain this type of understanding before she became concerned about their procedural fluency. Carmen would focus on procedural fluency subsequently, giving her students further time and practice solving addition and subtraction word problems. She planned to continue using these types of problems in her daily practice.

Summary of Influences of Carmen's Understanding on Student Understanding

In her practice Carmen actively designed questions for her students to redirect their thinking toward the relationships in the problem so that they could become successful problem solvers. She would ask students to retell the story without using the numbers in the problem, only focusing on the relationships within the problem. She also designed questions to be asked while solving the problems, such as, Who has more? What clue tells you this? Does your answer make sense? Why does your answer make sense? These targeted questions focused on important information that likely led students to meaningful understandings of the relationships in the problem. Carmen's practice of targeting those relationships in her questioning, and her practice of asking her students to make sense of the problems through the checklist of questions influenced students' ability to identify the types of knowns and unknown in the problem, explain the relationship between the quantities, and interpret those relationships in order to find the answer to the problem. She developed other questions during the second observation cycle that encouraged them to solve more general types of problems: 1) Read the problem. 2) Look for clues in the problem, find the important information. 3) Think of some strategies. 4) Solve the problem. 5) See if it makes sense. By "look for clues," Carmen was asking students to go back to the problem to find indications of the structure of the problem. She used her understanding in her daily interaction with her students as well as her lesson planning, by focusing the questions and class discussions on solving problems by understanding the nature of the problems and not just the numbers in the problem. Her students demonstrated the ability to interpret the different types of word problems, find appropriate ways to solve the problems, and check their answers to see whether, in fact, they did make sense within the problem.

Carmen identified that her students were using aids such as a hundreds chart with little comprehension and with little reference to the structure of the word problem. In her practice she consistently directed the students' attention back to the relationships in the problem. She used her understanding in her practice to press students to focus not just on the numbers but to what they referred and how they related to the structure of the problem. She would often decrease the magnitude of a number to allow her students to concentrate on the structure of the problem. She would target her questions about using the charts back to the structure of the problem. Eventually the students began to use the aids more effectively. There were still many computational errors, but the students were using their understandings of the structure of the problem to choose appropriate strategies to solve their addition and subtraction word problems.

For students who struggled to understand the structure of a word problem, even after Carmen's redirection and targeted questions, Carmen encouraged the use of modeling to understand and solve the addition and subtraction word problems. Carmen not only demonstrated repeatedly but along the way asked questions and gave comments to foster students' ability to identify, explain, and interpret the steps of modeling the word problems. Some of her students demonstrated the ability to use and understand modeling as a tool, and then were able to move beyond modeling to counting strategies and fact recall with a similar understanding. Carmen spent less time on finding the correct answer to the problems than she did on being able to explain and interpret each type of problem and choose an appropriate strategy to solve the problems according to its structure. At this time, she was explicitly less concerned about computational errors than a conceptual understanding of the word problems. Her understanding affected her learning goals for her students, which I believe affected their strategic competence. The students were able to understand the nature of the problems, and even chose correct strategies to solve the problems, but struggled to carry out the details of the strategies. Carmen asked her students to make sense of the problem, choose an appropriate strategy that corresponded to their interpretation of the problem, and check to see whether their answer was logical. At this point of instruction it was of little concern if the students' answers were not correct, as long as the strategy used to find the answer made sense within the structure of the problem. This consistent focus in her practice was evident in the students' understandings, as they often made computational errors but were successful in the ways Carmen deemed most important. Carmen intended to address understanding of structure and strategy first, then address getting correct answers.

CHAPTER SEVEN

CONCLUSION

In this chapter I present the conclusions I was able to draw after conducting a cross- participant analysis of the data I collected describing how teacher understanding affects student understanding. I will describe the aspects of student understanding that the teachers were able to influence as a result of their own understandings and practices. I will also describe the aspects of teachers' practice that were key factors in the ways that teachers can influence student understanding. These descriptions will illustrate my hypothesis that teachers used their understanding to set up tasks that were of a high cognitive demand (Stein, Smith, Henningsen, & Silver, 2000), maintain that cognitive demand through the implementation of those tasks, and as a result, affect student understanding. Finally, I will discuss the limitations of this study, additional research questions that could be addressed in the future, and the implications of this study for teachers' practice as well as for teacher education.

Aspects of Student Understanding Influenced by Teacher Understanding

As I discuss aspects of student mathematical understanding influenced by the teachers' understandings, I will utilize the indicators of understanding I have adopted for this study - students' ability to identify, explain, interpret, discriminate, apply, and generalize. The three teachers increased the students' ability to explain and interpret the problems, make sense of the problem, choose an appropriate strategy that corresponded to their interpretation of the problem, discriminate among problem types, and apply their understandings to solving other problems. In this section I will describe how these abilities affected the students' mathematical understandings.

There was an extensive amount of evidence in all three classrooms that demonstrated that teacher understanding can lead to an expanded student understanding of mathematical operations. In these instances I believe that the teachers' flexible understandings enabled the students to interpret the mathematical operation as a strategy used to solve the problem instead of a defining characteristic of the problem. The teachers used their strategic competence (Kilpatrick et al, 2001) in asking the students to explain that problems can be solved in multiple ways, including by the use of different operations. Some students began to show an ability to interpret the operations as "opposites" of each other. Showing evidence of the development of strategic competence, some students applied this understanding by explaining that the opposite operation could be used to check their own work or to solve a problem using a second method. The teachers I observed showed the ability to broaden students' interpretations of addition and subtraction instead of limiting it. The teachers' flexible understandings of the addition and subtraction operations influenced their practice, which I believe influenced their students' ability to explain and interpret the operations as strategies to be used to solve problems instead of only designating the nature of the problem.

Student understanding of the written representations of their work was one of the most notable influences of the teachers' understandings. Because of multiple ways the teachers were able to represent and interpret their representations, the students understood that addition and subtraction word problems could be represented with devices such as equations, pictures, diagrams, or tally marks. They influenced how the students were able to interpret the representations and how they used the representations to explain and interpret the problems given to them. The teachers influenced how the students were able to discriminate between problems using the written representations and how the students could apply their use of written representations to other problems. This was most obvious in Julie's and Pam's classrooms, in which both teachers spent a great deal of time modeling this behavior and encouraging students to mathematically represent on paper what they were thinking.

As Pam developed her own understanding of the mathematical topic, she decided that equation writing would be a vital tool to encourage understanding in her students. Pam asked students to write an equation to represent each mathematical problem they were asked to solve. She used equation writing to ask students to identify the type of problem they were solving. The students were able to successfully use equations to explain the parts of the problem and interpret the relationships within the problem. In her practice Pam decided to use equation writing as a tool for students to observe and articulate the differences in structures. Pam's consistent emphasis on where the question mark was in the equation was part of her plan to enable students to discriminate, showing what parts of the problem were given and what part of the problem was missing. Pam's students were able to translate their ideas about the nature of the problem into mathematical sentences.

Julie understood that many students struggle with understanding, using, and explaining the counting strategies they use to solve problems. She was able to identify that students should represent their solution strategies with diagrams that accurately describe what they are thinking. Julie patterned her own solving practices after her understanding of how students solve problems. She introduced them to a tool to allow them to explain and effectively use their written strategies by demonstrating many examples in which she connected her written work back to the structure of the problems. Her students were able to identify the steps of the strategies they wrote. They were able to explain what they were writing, and interpret why it could be used to solve each problem. Julie asked them to apply their understandings of the written representations to solving a variety of problems.

Similarly, the teachers influenced the students' ability to explain verbally their mathematical thinking. I view this indicator of understanding as similar to that of their written representations, although it uses a different form of representation. The influence on students' ability to give verbal explanations of their thinking was demonstrated in each of the teachers' classrooms. The students' written and verbally stated representations of their thinking consistently reflected the many examples the teachers had provided for them. All three teachers used written and verbal representations that described, as literally as possible, the thoughts and actions of the students when solving the problems. These practices may have contributed to why the teachers' abilities to explain the structure or interpret the solution strategies seemed to be incorporated by their students.

The ability of students to understand more sophisticated levels of problem solving was also influenced by the teachers' mathematical and pedagogical understandings. The teachers understood the students' current ability levels and they also understood the more sophisticated strategies, so they developed tasks to build on the students' understanding. The teachers understood the importance of students developing their individual interpretations of the mathematical topic. Their mathematical understanding was flexible enough to allow students to explore the mathematics. Practices such as refraining from teaching a standard algorithm until the students were ready allowed students to develop their own explanations and strategies as shortcuts for direct modeling and counting strategies. Teachers gave students many opportunities to model more sophisticated ways to solve problems. The teachers were able to identify when students were using aids without the ability to explain or interpret their meaning, and used their understandings to enrich their practice so that students used manipulatives with understanding. All three teachers encouraged their students to use less sophisticated methods for solving problems when they believed a student lacked the ability to interpret the nature of the problem, which the teachers hoped would lead to a deeper mathematical understanding for the student.

The final aspect of student understanding that was influenced by teacher understanding is students' understandings of mathematical concepts that are more pervasive in mathematics than the specific topic they were discussing. Students in this study were able to learn more about concepts such as recognizing invariance, generalizing, justifying, and focusing on the structure of the problem instead of just how to solve a problem. My data collection did not focus on invariance or justifying, so my interview questions were also not necessarily focused on these types of abilities, but there clearly was some evidence of teachers influencing students in these areas. In this study, these young students were developing the building blocks for identifying and explaining these important mathematical practices and powerful mathematical concepts.

Aspects of Teacher Practice That Influenced Student Understanding

The first and most drastic change in the teacher's practice was the complete revamping of the problems given to the students. Although this does not establish causality for the students' understandings, it was an interesting and abrupt change in the teachers' practice that served as an indication of their understandings of the topic and gave students experience with a deeper curriculum. It is clear that the teachers' understandings affected their practice through the types of problems they chose to give to their students and through the sequencing of those problems. This was the first step for all three teachers in affecting what the students understood about addition and subtraction word problems.

The teachers did more than just change the problems they gave the students, they used their understandings to better comprehend and carry out the objectives for each of their lessons. They were able to make more specific lesson objectives and have those objectives guide their interactions with the students throughout their teaching. Instead of the objectives being confined to the outline given in *Investigations*, the three teachers were able to decide whether and how they needed to supplement the materials. They could decide whether the purpose of their lessons was to introduce and identify a certain type of problem, develop explanations and interpretations of problems, discriminate among different types of problems, or develop strategies to apply their understandings and to solve problems.

This may have influenced student understanding because the teachers were able to construct their lessons in ways that elicited meaningful conversation from students according to the objectives for the lesson. During student and teacher interactions, which cannot be planned out beforehand, the teachers were able to target their questions to the lesson objectives they had previously designed.

The teachers' understandings enabled them to better address the needs of their students, not only through the building of their lesson plans but also in daily interactions with students. According to the teachers, before they learned more about addition and subtraction word problems they were unhappy with how they previously had taught the lessons, but were unable to make changes because they did not understand the mathematics well enough. By their own admissions, they were only able to change the numbers in the problem to increase or decrease difficulty. Because of their increased mathematical understandings they were able to better understand their students' mathematical understandings and how they could construct tasks that would build on those understandings. The teachers gained insight into what students' weaknesses and strengths were in their explanations and interpretations, and could modify their practice to accommodate those accordingly. Teachers could work with a group of students, such as the higher achieving students or lower achieving students, and better determine what kinds of problems were difficult for them, determine what those difficulties were, address the issues, and design more problems for them to address their specific needs. This more tailored instruction likely influenced the nature of many of the students' mathematical understandings.

All three teachers identified that the questions they asked their students were more targeted and focused on the mathematics they wanted their students to understand than in previous years. For example, the teachers' understandings of structure were evident in their teaching when they continuously asked students to explain and interpret the relationships within the addition and subtraction word problems. The practice of questioning students is a linking mechanism for teacher and student understanding and is a source of insight as to the nature of the teacher's understandings and what might be reflected in student understanding. These teachers used scaffolding to press students to identify, explain, and interpret the numbers in the problem, the objects to which those numbers refer, and the relationships among the referents within the problem, instead of focusing on getting an answer or which operation to use to solve the problem. Teachers did not simply expect students initially to be able to explain things such as the nature of the problem; they guided them to the understandings by asking them targeted questions that they hoped would lead to understanding.

Carmen, in particular, explicitly designed questions for her students to redirect their thinking toward the relationships in the problem. She would ask students to explain the story without using the numbers in the problem, only focusing on the relationships within the problem. She also designed questions to interpret the problem, such as, Who has more? What clue tells you this? Does your answer make sense? Why does your answer make sense? Carmen's practice of targeting those relationships in her questioning influenced students' ability to identify the types of knowns and unknown in the problem, explain the relationship between the quantities, and interpret those relationships in order to find the answer to the problem.

The teachers all asked their students to solve problems in multiple ways. This practice affected the students' ability to identify and explain the properties of addition and subtraction and the strategies used in solving addition and subtraction word problems. The students were not allowed to rely on memorized strategies that were given to them. If the students wanted to progress, they had to develop their own strategies, with correct explanation and interpretation of those strategies. Asking the students to solve problems in multiple ways increased the flexibility of their understanding (Rittle-Johnson & Star, 2007; Star & Rittle-Johnson, 2008). It gave the students further incentive and avenues to develop their strategies and develop their ability to explain and interpret the mathematics they understood and used.

The teachers used multiple representations in their teaching. To some degree all the teachers asked their students to write equations, and Pam used that practice more extensively than the others. She pressed students to identify the quantities in the problem and how they should be represented in the equations. Pam asked them to translate their interpretations of the problems into mathematical sentences. She asked students to model high-level performance with explanations for how they wrote their equations, including interpreting why they decided to represent the quantities in the manner they chose and where to place the quantities in the equation. The teachers also encouraged the use of direct modeling to understand the nature of difficult problems. The teachers' understandings of the modeling strategy and their understandings of the importance of being able to explain the nature of the problem, assisted students to develop their ability to directly model, which helped them explain and interpret the problems on which they were working.

The teachers also influenced student understanding when they modeled solving mathematical problems with their students. Obviously it takes more than just showing a student how to solve a problem for that student to gain understanding, but the manner in which teachers demonstrated examples seemed to affect the students' understandings. The teachers spent time not only showing examples of ways to solve problems but also explaining how the strategies worked and how the strategies were contingent upon the nature of the problems they were attempting to solve. Although I am calling them teacher demonstrations, the students were actively involved in the discussions and were being led by the teachers and the teachers' goals for the demonstration. Students had countless opportunities to explain their own thinking as well, with feedback from the teachers. Because of the regular use of examples in the classroom, students were not completely left on their own to develop understandings, but were given a path they could follow in building their understandings.

Finally, students were encouraged to interpret problems in their own way, even if their equally valid interpretation was different from the teacher's interpretation. Because of their understandings, teachers were able to find validity in the arguments that students made that were different from their own, and encouraged the students to continue in their thinking. It takes teacher understanding to identify that there are many appropriate ways to think about mathematical problems, and applying that understanding to practice may express itself as knowing when not to intervene or influence how a student is interpreting a problem. Teachers can allow students to develop their correct interpretations, merely by knowing when to refrain from imposing their own ideas that might hinder students' productive thinking.

Teacher Understanding, Student Understanding, and Cognitive Demand

The previous paragraphs discussed the specific facets of student understanding that the teachers directly affected and the actions that the teachers took to influence student understanding. In the following paragraphs I will describe my hypothesis of how teachers use their understanding to affect student understanding by setting up tasks of high cognitive demand and maintaining that cognitive demand throughout implementation of those tasks.

Research regarding the design and implementation of tasks in the middle school has uncovered the construct of "cognitive demand," a lens to analyze the level of cognition needed by students to complete mathematical tasks (Stein, Grover, & Henningsen, 1996). The literature describes four categories of cognitive demand possible for tasks. Tasks that require "doing mathematics" and the use of "procedures with connections" are both types of tasks requiring a high level of cognitive demand; tasks that require the use of "procedures without connections" or "memorization" are tasks with lower-level demands (Stein, Smith, Henningsen, & Silver, 2000).

Even if tasks are designed to be cognitively demanding, that does not ensure that students will form a deeper understanding of mathematics (Cohen, 1990). Teachers can set up tasks or implement tasks in a way that decreases the cognitive demand. In a study by Stein and colleagues (1996), fifty-three percent of the tasks that were set up to require the use of “procedures with connections” declined in level of cognitive demand such that students were no longer making connections when using procedures. Of the tasks that required students to “do mathematics,” sixty-two percent were implemented in a way in which students were no longer performing at a level of high cognitive demand.

The following figure from Henningsen and Stein (1997) illustrates the three phases through which tasks pass: as written by the developers, as set up by the teacher, and as implemented by the students. The arrow I have inserted points to the component of the task framework that my research addresses.

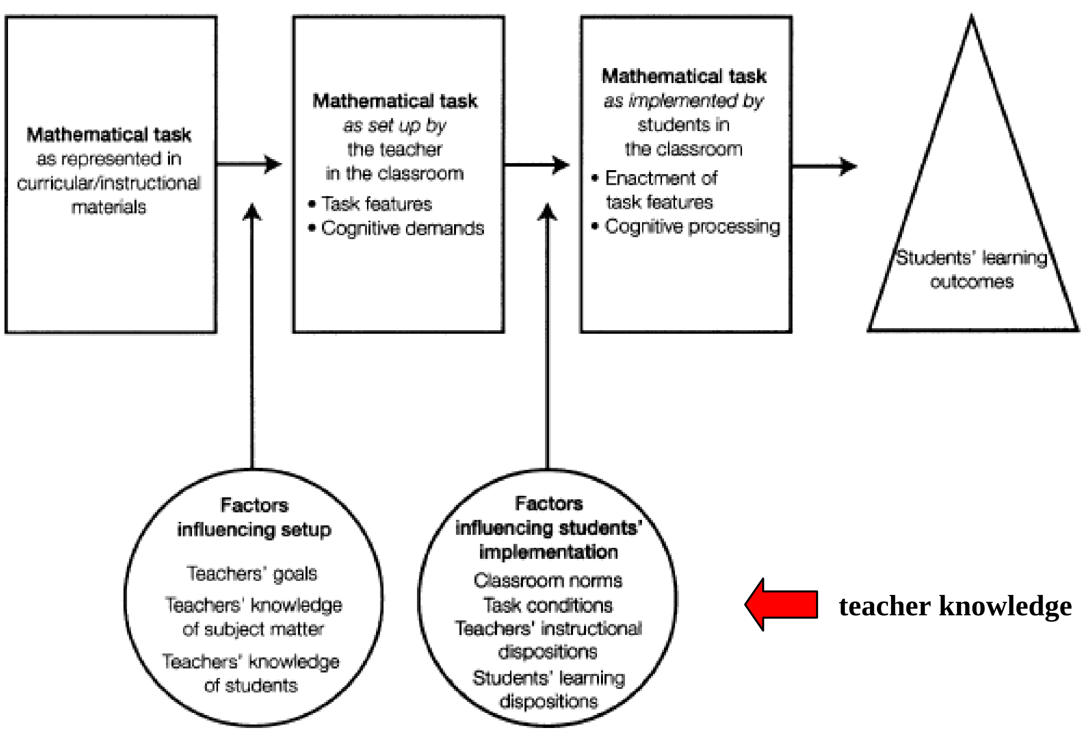

Figure 7. Framework for Mathematical Tasks (adapted from Henningsen & Stein, 1997, p.528)

Teacher knowledge of subject matter appears as a factor influencing the setup of the task, but it does not appear as a factor influencing the implementation of the task. I agree that teachers' instructional dispositions, which include attitudes and beliefs about content and pedagogy, determine the way a teacher desires to implement a task. I also claim that teachers' mathematical knowledge for teaching, including content and pedagogical content knowledge, determines the teacher's ability to implement that task at a high level of cognitive demand and maintain that level throughout implementation.

Teachers have a way of operating that is driven by what they believe, but their ability to implement according to their beliefs is influenced by what they understand. A cross-case analysis of the participants in my study showed that the teachers' practice all shared a common feature. Not only did they desire to implement tasks at a high level of cognitive demand, they each used their understanding to create situations in which students were asked to think about mathematics, using tasks that were at a high level of cognitive demand and maintaining that demand during implementation.

The teachers chose and set up tasks for their students with task features that Stein and colleagues claim ask students to think deeply about mathematics.

> The task features refer to aspects of tasks that mathematics educators have identified as important considerations for the engagement of student thinking, reasoning, and sensemaking: the existence of multiple solution strategies, the extent to which the task lends itself to multiple representations, and the extent to which the task demands

> explanation and/or justifications from the students. (Stein, Grover, & Henningsen, 1996, p. 461)

Task features that require students to use multiple solution strategies, multiple representations, and explanations and/or justifications encourage students to think deeply about mathematics My data shows that all three teachers consistently asked students to use multiple solution strategies. Students were asked to solve problems in two different ways and with different operations. They were asked to do problems in which they could utilize direct modeling, counting strategies, and fact recall. They were asked to use mathematical equations, number lines, hundreds charts, pictures and other representations for their work. Teachers also consistently asked students to explain and justify their thinking.

I claim that the teachers used their understanding to create the two types of higher level cognitive demand tasks, "using procedures with connections" and "doing mathematics." Tasks that use procedures with connections were chosen to "focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas" (Stein et al., 2000, p. 16). For example, Carmen asked her students to answer her list of problem-solving questions to encourage them to think about the relationships between the quantities in the word problems. The teachers used problems that asked students to represent the quantities and actions in the problem in multiple ways, such as with diagrams, manipulatives, symbols, and problem situations (Stein et al., 2000, p. 16). Teachers consistently asked students to make connections among the representations to develop their meaning, such as when Pam asked students to write mathematical equations to encourage students to understand the types of knowns and unknowns in the addition and subtraction word problems.

Teachers also used their understanding to create tasks of higher-level cognitive demand that fit into the category of “doing mathematics.” Students were often given problems that required nonalgorithmic thinking, and in fact, in all three classes, students were purposefully not given algorithms in order to develop more complex thinking. Teachers understood the complexity of the new addition and subtraction problems. They could also think flexibly about the problems, not defining them by a single operation, but admitting different operations depending on the strategies students chose. Therefore, they required the students to explore to understand the relationships in the problems. Another characteristic of “doing mathematics” is requiring students to “access relevant knowledge and experiences and make appropriate use of them in working through the task” (Stein et al., 2000, p. 16). Julie was an excellent example of a teacher teaching her students counting strategies to solve problems and then allowing students to access those strategies to solve problems they had not previously encountered. All three teachers chose tasks that required considerable cognitive effort so that some level of anxiety was experienced by the students. My data shows that teachers used their understanding to create tasks of a high cognitive demand, tasks with characteristics comparable to those in the literature that are deemed to be of high cognitive demand.

The teachers were also able to use their understanding to maintain that demand level over time. According to Stein and colleagues there are seven factors associated with maintenance of high-level cognitive demands (Stein et al., 2000).

These factors coincide with the aspects of teacher practice that I claimed to be influenced by teacher understanding. My data reveals examples of teachers demonstrating those factors in their classrooms, which played an essential role in their ability to maintain the high cognitive demand of the tasks.

Eighty-two percent of tasks that stayed at a high cognitive demand were tasks that built on students' prior knowledge (Henningsen & Stein, 1997). The participants in my study wrote a pretest for their students, structured using Change Unknown problems. They completely changed their curriculum, based on the results of the pretest they gave their students to assess student knowledge. The teachers also encouraged the students to use strategies to solve problems based on their understanding of the students' ability levels. Using their increased mathematical understandings, teachers were better able to determine the students' prior understandings and design activities that could improve those understandings—without their increased mathematical understanding they could only increase or decrease the size of the numbers. Julie, for example, was able to consider a group of students, determine their weakness on Compare (Referent Unknown) problems, and implement tasks to address that weakness.

Another important factor associated with maintaining tasks of high cognitive demand is "sustained press for justifications, explanations, and/or meaning through teacher questioning, comments, and/or feedback" (Stein et al., 2000, p. 27). My results indicate numerous examples of teachers asking for students to reason and defend their understanding. A prominent feature of their teaching was targeting questions to ask students to think about the mathematics the teachers understood to be important in the problem, such as, understanding relationships between quantities or interpreting the operations used to solve the problem.

Scaffolding of student thinking and reasoning is another significant factor to maintaining tasks' high level of cognitive demand. With each teacher I witnessed situations during which students were unable to complete a task and the teacher provided assistance through questioning and comments that enabled the student to complete the task without reducing the demand of the task. For example, when one student was unable to understand a Separate (Change Unknown) problem, Carmen scaffolded the student's thinking to press her to think about the relationships in the problems. Carmen helped her develop a strategy using tally marks to solve the problem. The student was eventually able to build off this scaffolding and solve these problems without Carmen's assistance.

Carmen also demonstrated a fourth factor, giving students a means of monitoring their own progress. One of the purposes of giving students the checklist to fill out for each problem was to enable students to guide their own progress in problem solving. She encouraged in students the ability to monitor each step toward solving the problem. The final step was for the students to ask themselves whether the answer made sense. If it did not make sense, she encouraged them to begin the process again.

Teachers who model high-level solution strategies also show effective maintenance of high cognitive demand. The teachers in my study did model ways to solve problems, explaining how strategies worked and how the strategies were appropriate based on the nature of the problems. They acknowledged that they were unable successfully to do this modeling before learning more about addition and subtraction word problems. Students were actively involved in the modeling discussions and were being led by the teachers and the teachers' goals for the demonstration. Part of these demonstrations involved drawing conceptual connections, another important factor in maintaining the task demand level. Pam, for example, would draw connections between addition and subtraction when demonstrating the use of the operations within one word problem.

Another highly integral factor for maintaining demand is giving students sufficient time to explore. Too little or too much time can be detrimental to preserving the demand of a task. I speculate that my participants gave students sufficient time to explore, however, my data is insufficient to make claims about this factor. Determining the appropriate amount of time to spend on a particular task was not something I addressed in my study, nor was collecting data on the amount of time students spent on tasks.

I also hypothesize that student understanding was affected by the practice of teachers' maintaining the cognitive demand of the tasks implemented. Comparing my data with the cognitive demand associated with high levels of thinking shows that students were able to think deeply about mathematics (Stein et al., 2000). Students were able to respond to the features of the task that called for using multiple solution strategies, using multiple representations, and giving explanations.

There is evidence that suggests that the students demonstrated the type of understandings encouraged by tasks with higher cognitive demand level that focused on "procedures with connections" (Stein et al., 2000). Students were able to develop deeper levels of understanding of mathematical concepts, such as their expanded understanding of the operations of addition and subtraction, through their use of procedures. Students were able to represent problems in multiple ways, such as modeling, equation writing, and creating their own word problems. I found that the teachers' practices greatly influenced the nature of students' written and verbal representations of their work. I also found that students were able to develop meaning by making connections among multiple representations. For example, the students in Pam's class were able to use equation writing, picture drawing, and creating of problem situations to better understand the relationships between types of knowns and unknowns in an addition and subtraction word problem. There were many examples of students showing cognitive effort, not following procedures mindlessly, and engaging with underlying conceptual ideas.

There is also evidence that students were able to demonstrate the type of understandings that "doing mathematics" tasks are intended to engender. Students showed nonalgorithmic thinking while thinking of multiple ways to solve the addition and subtraction problems. There is evidence of students beginning to explore aspects of mathematical concepts such as invariance, generalizing, justifying, and focusing on the structure of the problem instead of just how to solve a problem. There were countless examples of students accessing relevant knowledge and using it appropriately to work through a task. One such example was Heather, a student in Pam's class, who decided how to solve a problem based on her understanding of the relationships between the change quantity and the result quantity to find the starting quantity of a problem. All three teachers described how their students demonstrated more cognitive effort, including some levels of anxiety, due to the changes the teachers made in the curriculum during the year in which my study was conducted because of their increased mathematical understanding. The teachers were able to make the problems less predictable, and therefore the solution process less predictable, encouraging the students to use processes in which mathematicians engage when solving problems.

Finally, I suggest that student thinking at a high level of cognitive demand is an indication of student understanding. When comparing the attributes of student thinking at a high level of cognitive demand with the attributes of student understanding as outlined by this study, there appears to be a clear connection. The attributes of student understanding I have been investigating are students' ability to identify, explain, interpret, discriminate, apply, and generalize (Sierpinska, 1994; Wiggins & McTighe, 1998). These attributes of understanding displayed by the students are also indications of student thinking at a high level of cognitive demand when "doing mathematics" or using "procedures with connections," which ask students to pursue mathematical processes such as explaining reasoning, interpreting procedures and concepts, or applying understanding to tasks. Therefore, thinking at a high level of cognitive demand is a hallmark for understanding.

Investigating the similarities and differences in the cases of this study shed light on the ways that teacher understanding affects student understanding through teacher practice. I claim that, even though the teachers were teaching the same topic, and even had the same specific goals at the outset of teaching (to have their students understand these four types of addition and subtraction word problems), the manifestations of their teaching were uniquely affected by their mathematical understandings. I believe the teachers were all able to effectively teach this particular topic, generating some clear student understandings in each of the classrooms, but they accomplished their initial goals by privileging different practices in their classrooms. The teachers' understandings influenced the practices they privileged, which influenced the understandings of the students (Kendal & Stacy, 2001).

Although the teachers privileged different practices, by studying similarities across cases we learn something about the connection between teacher understanding and student understanding. There was evidence that teachers were able to use their understanding to create and choose tasks of a high level of cognitive demand. There was also evidence that when teachers' implemented tasks, maintaining that high level of cognitive demand, they used their understanding of mathematics and pedagogy. Affected by the teachers' practice, the students' also demonstrated the ability to implement tasks at a high level of cognitive demand. Consequently, this practice positively affected the students' understandings.

Implications

The results from this study could impact preservice teachers and researchers on at least three levels, considering this study from a more focused perspective to a broader perspective. From a narrow perspective, this study could impact the way teachers teach addition and subtraction word problems. The results of this study indicate that with a knowledgeable teacher who sets appropriate learning goals, students are capable of understanding addition and subtraction word problems of a variety of structures. Students bring to the classroom, and can also be taught, strategies to understand and solve Join, Separate, Part–Part–Whole, and Compare word problems. When given the opportunity, first- and second-grade students were able to consistently explain, interpret, and discriminate among, all four of these types of addition and subtraction word problems. Students are capable of expanding their understandings of the addition and subtraction operations and developing their understandings of properties of addition and subtraction as well as other mathematical topics such as invariance or compensation.

This could have a strong impact on the way teachers teach their students, even if this variety of problems is not currently included in the curriculum they are using. Teachers teaching this topic might find this study an incentive to expand their own understandings of addition and subtraction word problems. The three teachers in this study showed that with increased mathematical and pedagogical understandings, they were able to change their teaching so that the students were able to learn more than in previous years. Teachers who seek out professional development experiences to increase their understandings of this topic, and theoretically other mathematical topics, can use their understandings to influence what and how their students learn, supplementing their curriculum when necessary. Preservice teachers should be increasing their own understandings of operations, and learning about how students think about these operations, so deeper teacher understanding can lead to practices that might cause teachers to supplement curriculum or teach differently what is in the curriculum when they begin their teaching.

From a broader perspective, this study highlights teaching practices that can be effective when trying to positively influence student learning. Teachers can benefit from purposefully examining their own practices to investigate how their practices are influencing their students. Teachers might find the practices uncovered in this study to be beneficial in their own classrooms. Specifically, teachers should ask themselves, to what degree do I understand the mathematics associated with the learning objectives I have for my students? This study could prompt teachers to seek out professional development that increases their mathematical understanding. Like the participants of this study, teachers with a well-developed understanding of the mathematics should be able to identify the needs of students by articulating what they do and do not understand. Teachers should be aware of their efforts to redirect students' attention toward the foundational mathematical concepts teachers want them to obtain. Teachers at all levels should ask students to solve problems in multiple ways, through equation writing, modeling or other forms of representation. This study might encourage teachers to ensure that students understand the nature of the problems they are solving, as well as offer perspectives on how that might be accomplished.

I believe this study could influence teachers and researchers to investigate the practices elementary school teachers are privileging and how they are using those practices as an impetus for student learning. The construct of privileging has been used primarily to investigate practices involving technology in secondary education. I think this could be a lens to view how mathematics is taught in elementary schools. In this study I found that problem solving was the driving force behind Carmen's teaching of addition and subtraction word problems. Ultimately she wanted her students to become problem solvers, and used addition and subtraction as a means of achieving that end. This was an important goal for Carmen, and she used this goal to provide structure in her teaching. It influenced her work inside and outside of the classroom, including how she designed lessons, assessed student understanding, and engaged her students in dialogue. Equation writing, on the other hand, was a constant presence in Pam's classroom. She understood that equation writing would encourage her students to understand better the structure of the word problems, would demonstrate the students' understandings more effectively, and would be an important tool that they could use and develop in other mathematical areas. This study could influence researchers to bring the practices that teachers privilege to light, investigating the effects of different types of privileging on student learning.

Research indicates that teachers struggle to maintain the high cognitive demand of tasks during implementation. We have many examples in the literature of teachers and students decreasing the cognitive demand of tasks. This study illustrates a positive example of maintaining high cognitive demand. From this, researchers can gain a better understanding of how the literature on cognitive demand can be extended to tasks in the elementary school curriculum. Analyzing teacher practice and curriculum through this lens at the primary grades might give insight as to how to improve the curriculum or at least how to implement the curriculum to increase student understanding.

Limitations of the Study

In this project it was relatively easy to gather data regarding the understandings of the teachers through observation and interviews. It was also fairly straightforward to obtain information regarding how the understanding affected their practice. It was more difficult to establish how that practice influenced student understanding, because there are so many other factors impacting this process, and these students were sometimes difficult to question and understand. A major limitation of the study, however, is the difficulty in observing why and how certain practices of the teachers were affecting the students' understandings. There were times it seemed clear that a practice was very influential to student understanding, but it was difficult to say why it was influential to student understanding. I believe it is important to qualitatively describe the aspects of teacher knowledge that influence student understanding, and how teacher practices influence student understanding, but this particular study is limited in its ability to describe why these concepts are affecting student understanding. Because of the nature of the study, my data was very limited on this account. I believe further studies, using the conclusions that I have drawn here and with a slightly different methodology that I will describe subsequently, would more effectively be able to describe the causality.

In this study I was investigating teachers' understandings of structure and solution strategies for addition and subtraction word problems, how they use their understandings in their practice, and how that practice influences their students' understandings. One of the limitations of this study is concerned with the teachers' and students' understandings of solution strategies and the associated practices. My data were fairly limited with respect to observing teachers' and students' understandings of the solution strategies for addition and subtraction word problems because the teachers had already worked with the students on these strategies during the previous school year since these strategies were not new to the teachers. The modeling and counting strategies specified by the CGI material are not unique to solving addition and subtraction word problems, and they were used in previous units taught, and I was not able to observe teachers addressing students' understandings about this topic as much as I would have if I had begun observing them earlier in the year with other mathematical topics.

For example, it would have been useful if I had been able to observe Julie initially teach modeling and counting strategies to her first-grade students, because by the time I started observing they already were quite proficient with them. Although I was able to draw some conclusions about their understandings being related to her own, I think I would have been able to understand more about that connection if I had observed her teaching them earlier in the school year. Additionally, the second-grade students had already moved past using most of the strategies discussed by CGI, so I was not able to gather as much information about particular strategies in those classrooms either. I believe the data I was able to gather for the teachers' understandings and practice regarding structure was stronger and more complete.

There were other limitations regarding the amount of data I was able to collect about teachers' mathematical understandings. The teachers appeared to be influencing their students' mathematical understandings of things such as properties of addition and subtraction or rudimentary notions of invariance. But I did not recognize this connection until the later stages of data analysis and did not have enough data collected about the nature of the teachers' mathematical understandings of these concepts in order to make more solid claims. If I had observed the teachers earlier in the school year or if I had gathered more data about their mathematical understandings that was not directly tied to the CGI material, I may have been able to make stronger claims about the connections between the teachers' and students' understandings. I focused my data collection on teacher understanding too closely to the CGI material to make stronger conclusions about the connections between teacher and student understanding about concepts such as invariance, generalizing, or justifying.

After analyzing all the data and drawing my conclusions, I believe that a future study using a different mathematical topic might be more useful in investigating the connections between teacher understanding and student understanding. I used addition and subtraction word problems because CGI had done extensive, valuable research on the topic, and in that respect, it was a viable choice. However, there are aspects of the CGI research that made me question whether another topic would be more enlightening. According to Carpenter, CGI developed its categories for the structure of addition and subtraction word problems based on how children naturally think about these types of problems. Obviously the teacher still impacted the students' understandings, and I was still able to gather a great deal of pertinent data, but I think that it would be interesting to do a similar study with a mathematical topic for which research was less grounded in how students naturally tend to use strategies and view structure. There may be other mathematical topics for which the development of students' understandings may be more dependent on the teachers' practices than is the case with addition and subtraction word problems. Possible examples of these might be operations with fractions or long division.

Another drawback of using addition and subtraction word problems as the mathematical topic to investigate the relationships between teacher and student understandings is the age group of the students in the study. It is very difficult to qualitatively describe any person's mathematical understanding, but I would claim that it is more difficult do so with a first- or second-grade student. Of course, that does not imply this type of research should never be done with students with less well-developed communicative ability, but it might not have been as informative to conduct this type of research initially with these young students. I believe a necessary aspect of this line of research is uncovering how connections between teacher and student understanding take place, which means uncovering what types of questions the researcher should ask the students when observing and interviewing them. It was difficult for these students to communicate the details of their thoughts and especially difficult to understand the trail from their current thoughts to when those thoughts originated. Asking many questions to get to that information can be disturbing for a student, especially a student of this age and with this level of communication. It may have been enlightening to begin this sort of research with older students who had better developed communication skills and increased ability to describe the nature of their thoughts and the outside influences that may have affected their thoughts.

Further Questions

This study has just begun the work of qualitatively analyzing the connection between teacher understanding and student understanding. There are many more questions that further studies could answer, extending the initial steps that were taken in this study. Even a study that was similar to this study, especially if it used another mathematical topic and age group of students, would have the potential to expand upon my results.

A study of one teacher over an extended period of time would also be useful in investigating what I would consider to be counterevidence in my study. There are many situations in which the teacher possesses a solid understanding of mathematics and pedagogy, and the types of practices I have described exist in her teaching, but there is no evidence of student understanding. Why is there no evidence of student understanding? Does that mean the student is not being affected by those practices? What do teachers do in these types of situations, do they continue with the same types of practices or do they use other methods? Are there commonalities across situations? What are the other factors that may contribute to a student not understanding under these circumstances? All of these questions could contribute to the body of work begun in this study, but they are questions that cannot be answered currently with my study.

In addition, more work can be done to understand how teachers influence aspects of student understanding, such as their concepts of representation or expanded notion of operations. It would be important to investigate what teachers privileged, how this affected what students understood and other influences from the classroom context that may influence how students gained that understanding. It would also be interesting to investigate how students use a mathematical understanding, such as their expanded notion of operation, in other contexts besides the context in which they showed evidence of gaining the understanding. For example, how does an expanded understanding of addition affect the students' understandings of other operations such as multiplication? I believe that through this study I witnessed students learning important mathematical concepts, such as an expanded notion of operation, and it would be useful to investigate more about how this understanding was gained and how students were able to utilize this type of understanding encountering multiplication and division.

Finally, in addition to the majority of the discussion that focused on how the teachers' understandings' in this study directly influenced their practice and purposefully influenced the students' understandings, I briefly discussed how teachers were able to affect their students' understandings indirectly,. I believe it would be beneficial to investigate further how teachers can influence their students indirectly by being open to the learning that their students are experiencing. For example, what kinds of practices and classroom norms contribute to students learning mathematical concepts that a teacher does not previously show evidence of understanding? Simon (1995) documented that teachers' understandings can develop during their teaching; it would be interesting to investigate how this process affects the students' understandings. A teacher who has enough understanding to learn from the understandings of her students can positively influence those students who have a strong mathematical understanding, but a closer look at this process would illuminate the details of how student and teacher learning take place.

Closing Remarks

The results from this study are exploratory in nature, rather than confirmatory, given that the study was used only to generate hypotheses concerning the link between teacher understanding and student understanding. This study provides evidence that teachers can use their understanding in the classroom to influence student understanding if they begin with understanding of the content and how to teach content.

Although this study was situated in a mathematics classroom, it is possible that similar results could also be found using other disciplines and further studies may be able to illuminate how teacher understanding affects student understanding in those disciplines. Previous mathematics, science, and literacy quantitative research demonstrated that teachers were able to increase their understandings, use understandings in their practice, and increase students' scores. My research explores how and why the students knowing and understanding is affected by teacher understanding and through teacher practice.

I provided the teachers with generally the same knowledge generated by the CGI group, and, through their individual perspectives, they were able to develop their own unique understandings and use those understandings to teach their students in unique ways. All three teachers, in their own way, used their understanding to create tasks of high cognitive demand. They continued to use their understanding to maintain that cognitive demand while implementing those tasks. The nature of the students' understanding was a response to their work on tasks of this high level of cognitive demand.

REFERENCES

Anderson, C. W., & Smith, E. L. (1987). Teaching science. In V. Richardson-Koehler (Ed.), *Educators' handbook: A research perspective* (pp. 84–111). White Plains, NY: Longman Inc.

Anderson, L., Evertson, C., & Brophy, J. E. (1979). An experimental study of effective teaching in first-grade reading groups. *Elementary School Journal, 79*, 193–223.

Ball, D. L. (1991). Research on teaching mathematics: Making subject matter part of the equation. In J. Brophy (Ed.), *Advances in research on teaching: Vol. 2. Teachers' subject matter knowledge and classroom instruction* (pp. 1–48). Greenwich, CT: JAI Press.

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*, 241–247.

Ball, D. L. (2003). *What mathematical knowledge is needed for teaching mathematics*. Secretary's Summit on Mathematics. Washington, DC: U.S. Department of Education.

Ball, D. L. (2007). *Who knows mathematics well enough to teach it?* Holt, MI: Michigan Council of Teachers of Mathematics.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83–104). Westport, CT: Ablex.

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In G. Sykes & L. Darling-Hammond (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3–32). San Francisco, CA: Jossey Bass.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator 14*(22), 43–46.

Ball, D. L., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching (4th ed.)* (pp. 433–456). New York: Macmillan.

Ball, D. L., Phelps, G., & Thames, M. H.. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389–407.

Barnett, J., & Hodson, D. (2001). Pedagogical context knowledge: Toward a fuller understanding of what good science teachers know. *Science Teacher Education, 85*, 426–453.

Begle, E. G. (1972). *Teacher knowledge and student achievement in algebra* School Mathematics Study Group (Report No.9). Palo Alto, CA: Stanford University.

Begle, E. G. (1979). *Critical variables in mathematics education: Findings from a survey of the empirical literature.* Washington, DC: Mathematical Association of America and National Council of Teachers of Mathematics.

Berliner, D. C. (1986). In pursuit of the expert pedagogue. *Educational Researcher, 15*(7), 5–13.

Biggs, J. B., & Collis, K. F. (1982). *Evaluating the quality of learning: The SOLO taxonomy (structure of the observed learning outcome)*. New York: Academic Press.

Borko, H., & Livingston, C. (1989). Cognition and improvisation: Differences in mathematics instruction by expert and novice teachers. *American Educational Research Journal*, *26*, 473–498.

Borko, H., & Putnam, R. T. (1996). Learning to teach. In D. Berliner & R. Calfee (Eds.), *Handbook of educational psychology* (pp. 673–708). New York: Macmillan**.**

Brophy, J., & Good, T. L. (1986). Teacher behavior and student achievement. In M. Wittrock (Ed.), *Handbook of research on teaching*. New York: Macmillan Publishing Company.

Bullough, R. V., Jr. (2001). Pedagogical content knowledge circa 1907 and 1987: A study in the history of an idea. *Teaching and Teacher Education*, *17*, 655–666.

Calderhead, J. (1996). Teachers: Beliefs and knowledge. In D. Berliner & R. Calfee (Eds.), *Handbook of educational psychology* (pp. 709–725). New York: Macmillan**.**

Carpenter, T. P., Ansell, E., Franke, M. L., Fennema, E., & Weisbeck, L. (1993). Models of problem solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematics Education, 24,* 428–441.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*, 3–20.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics*. Portsmouth, NH: Heinemann.

Carpenter, T. P., Fennema, E., Peterson, P.L., & Carey, D. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education, 19,* 385–401.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26,* 499–532.

Carpenter, T. P., Hiebert, J., & Moser J. M. (1983). The effect of instruction on children's solutions of addition and subtraction word problems. *Educational Studies in Mathematics, 14,* 55–72.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15*, 179–202.

Carpenter, T. P., Moser, J. M., & Bebout, H.C. (1988). Representation of addition and subtraction word problems. *Journal for Research in Mathematics Education, 19*, 345–357.

Carter, K. (1990). Teachers' knowledge and learning to teach. In W. Houston (Ed.), *Handbook of research on teacher education* (pp. 291–310). New York: Macmillan Publishing Company**.**

Cohen, D. K. (1990). A revolution in one room: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis, 12*, 311–329.

Coleman, J. S., Campbell, E. Q., & Hobson, C. (1966). *Equality of educational opportunity*. Washington, DC: U.S. Government Printing Office.

Collis, K. F., Romberg, T. A., Jurdak, M. E. (1986). A technique for assessing mathematical problem-solving ability. *Journal for Research in Mathematics Education, 17*, 206–221.

Connelly, M., Clandinin, D. J., & He, M.F. (1997). Teachers' personal practical knowledge on the professional knowledge landscape. *Teaching and Teacher Education, 13*, 665–674.

Cramer, K., Post, T. & delMas, R.C. (2002). Initial fraction learning by fourth- and fifth- grade students: A comparison of the effects of using commercial curricula with the effects of using the rational number project curriculum. *Journal for Research in Mathematics Education, 33*, 111–144.

Croninger, R. G., Rice, J. K., Rathbun, A., & Nishio, M. (2007). Teacher qualifications and early learning: Effects of certification, degree, and experience on first-grade student achievement. *Economics of Education Review, 26*, 312–324.

Davis, E. (2006). A model for understanding understanding in mathematics. *Mathematics Teaching in the Middle School, 12*, 190–196.

Doyle, W. (1986). Classroom organization and management. In M. Wittrock (Ed.), *Handbook of research on teaching* (pp. 392–431). New York: Macmillan.

Dunkin, M. J., & Biddle, B. J. (1974). *The study of teaching*. New York: Holt, Rinehart and Winston.

Eisner, E. (2002). From episteme to phronesis to artistry in the study and improvement of teaching. *Teaching and Teacher Education, 18*, 375–385.

Elbaz, F. (1981). The teacher's "practical knowledge": Report of a case study. *Curriculum Inquiry 11*, 43–71.

Eylon, B.S., & Linn, M. (1988). Learning and instruction: An examination of four research perspectives in science education. *Review of Educational Research, 58*, 251–301.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147–164). New York: Macmillan Publishing Company.

Fenstermacher, G. D. (1994). The knower and the known: The nature of knowledge in research on teaching. *Review of Research in Education, 20*, 3–56.

Garrahy, D., & Cothran, D. (2005). Voices from the trenches: An exploration of teachers' management knowledge. *The Journal of Educational Research, 99*, 56–63.

Graeber, A. (1999). Forms of knowing mathematics: What preservice teachers should learn. *Educational Studies in Mathematics, 38*, 189–208.

Grimmett, P. P., & Mackinnon, A. M. (1992). Craft knowledge and the education of teachers. In G. Grant (Ed.), *Review of research in education* (pp. 385–456). Washington, DC: American Educational Research Association.

Grossman, P. (1990). *The making of a teacher: Teacher knowledge and teacher education*. New York: Teachers College Press.

Grossman, P., Wilson, S., & Shulman, L. S. (1989). Teachers of substance: Subject matter knowledge for teaching. In M. Reynolds (Ed.), *Knowledge base for the beginning teacher* (pp. 23–34). Oxford, England: Pergamon Press.

Hiebert, J. & Carpenter, T. P. (1992). Teaching and learning with understanding. In D. Grouws (Ed.), *Handbook or research on mathematics teaching and learning* (pp. 65–97). Reston, VA: National Council of Teachers of Mathematics.

Hiebert, J., Gallimore, R., & Stigler, J. W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Researche,r 31*, 3–15.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*, 371–406.

Hurry, J., Nunes, T., Bryant, P., & Pretzlik, U. (2005). Transforming research on morphology into teacher practice. *Research Papers in Education*, *20*, 187–206.

Jencks, C., & Smith, M. (1972). *Inequality: A reassessment of the effect of family and schooling in America*. New York: Basic Books.

Kendal, M., & Stacey, K. (1999). Varieties of teacher privileging for teaching calculus with computer algebra systems. *International Journal of Computer Algebra in Mathematics Education*, *6*, 233–247.

Kendal, M., Stacey, K. (2001). The impact of teacher privileging on learning differentiation with technology. *International Journal of Computers for Mathematical Learning*, *6*, 143–165.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Leinhardt, G. (1990). Capturing craft knowledge in teaching. *Educational Researcher*, *19*, 18–25.

Leinhardt, G., & Greeno, J. G. (1986). The cognitive skill of teaching. *Journal of Educational Psychology*, *78*, 75–95.

Leinhardt, G., Weidman, C., & Hammond, K. M. (1987). Introduction and integration of classroom routines by expert teachers. *Curriculum Inquiry, 17*, 135–176.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Erlbaum.

McCutchen, D., Abbott, R. D., Green, L. B., Beretvas, S. N., Cox, S., Potter, N. S., et al. (2002). Beginning literacy: Links among teacher knowledge, teacher practice, and student learning. *Journal of Learning Disabilities, 35*, 69–86.

McDonald, F., & Elias, V. (1976). The effects of teacher performance on pupil learning. *Beginning teacher evaluation study*. *Phase II, final report, Vol. I*. Princeton, NJ: Educational Testing Service.

Meijer, P. C. (2002). How can student teachers elicit experienced teachers' practical knowledge? *Journal of Teacher Education*, *53*, 406–419.

Merriam, S. B. (1998). *Qualitative research and case study applications in education.* San Francisco, CA: Jossey-Bass Inc.

Mitzel, H. E. (1960). Teacher effectiveness. In C. Harris (Ed.), *Encyclopedia of educational research* (pp. 1481–1486). New York: Macmillan.

Monk, D. H. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review, 13*, 125–145.

Mosteller, F., & Moynihan, D. (1972). *On equality of educational opportunity*. New York: Random House.

Neale, D. C., Smith, D., & Johnson, V. G. (1990). Implementing conceptual change teaching in primary science. *The Elementary School Journal, 91*, 109–131.

Parker, V., & Gerber, B. (2000). Effects of a science intervention program on middle-grade student achievement and attitudes. *School Science and Mathematics, 100,* 236–242.

Polya, G. (1985). *How to solve it: A new aspect of mathematical method*. Princeton, NJ: Princeton University Press.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*, 561–574.

Russell, S. J., Economopoulos, K., Murray, M., & Mokros, J. (2004). *Investigations in number, data, and space.* Glenview, IL: Scott Foresman.

Schoen, H. L., Cebulla, K. J., Finn, K. F., & Fi, C. (2003). Teacher variables that relate to student achievement when using a standards-based curriculum. *Journal for Research in Mathematics Education, 34*, 228–259.

Scott, P. H., Asoko, H. M., & Driver, R. H. (1997). Teaching for conceptual change: A review of strategies. In A. Tiberghien, E. Jossem, & J. Barojas (Eds.), *Connecting research in physics education with research in teacher education.* London, England: The International Commission on Physics Education.

Shulman, L. (1986a). Paradigms and research programs in the study of teaching. In M. Wittrock (Ed.), *Handbook of research on teaching*. New York: Macmillan Publishing Company.

Shulman, L. (1986b). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*, 4–14.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*, 1–22.

Sierpinska, A. (1994). *Understanding in mathematics.* Bristol, PA: The Falmer Press.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114–145.

Simon, M., & Schifter, D. (1991). Towards a constructivist perspective: An intervention study of mathematics teacher development. *Educational Studies in Mathematics, 22*, 309–331.

Smith, E. L., Blakeslee, T. D., & Anderson, C. W. (1993). Teaching strategies associated with conceptual change learning in science. *Journal of Research in Science Teaching, 30*, 111–126.

Smith, S. B., & Baker, S. (2001). Making a difference in the classroom with early literacy instruction. *Teaching Exceptional Children, 33*, 8–14.

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem solving: The case of equation solving. *Learning and Instruction, 18*, 565–579.

Stein, M. K., Baxter, J. A., & Leinhardt, G. (1990). Subject-matter knowledge and elementary instruction: A case from functions and graphing. *American Educational Research Journal, 27*, 639–663.

Stein, M. K., Grover, B. W., & Henningsen, M. A. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*, 455–488.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction*. New York: Teachers College Press.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). New York: Macmillan Publishing Company.

Wiggins, G., & McTighe, J. (1998). *Understanding by design*. Alexandria, VA: Association for Supervision and Curriculum Development.

Winne, P. H., & Marx, R. W. (1977). Reconceptualizing research on teaching. *Journal of Educational Psychology, 69,* 668–678.

Wood, T., Cobb, P., & Yackel, E. (1991). Change in mathematics teaching: A case study. *American Educational Research Journal, 28,* 587–616.

Wright, C., & Nuthall, G. (1970). Relationships between teacher behaviors and pupil achievement in three experimental elementary science lessons. *American Educational Research Journal, 7*, 477–491.

APPENDIX A

DETAILS OF AND TEACHERS' REACTIONS TO THE UNIVERSITY COURSE

In this appendix I describe the common course experiences and reactions of the three teachers. Because they came from the same school district and all participated in my discussion-based course, a portion of their background is similar. The assertions made in the preceding chapters were based on the evidence I collected while observing and interviewing the teachers and students.

Understanding Addition and Subtraction Word Problem Structure

The teachers began the study with an implicit understanding of most of the types of problems outlined by the CGI materials. The first time I asked them to write an addition problem on their papers they all wrote a Join (Result Unknown) problem. I then asked them to write a different kind of addition problem and they were able to think of addition in a different manner. Two teachers wrote a Part–Part–Whole (Whole Unknown) word problem. One teacher explained the difference by saying she was not "adding something to what is there. I have two separate things that I'm looking at versus something joining…to me I would think starting with something and adding to that [is different from] combining two different things" (Pam, Course Day 1, 65-66). As a class we had not labeled any of the categories as "Joining," but *Investigations*, the curriculum they used in their classrooms, utilizes the words "joining" and "separating." One teacher wrote a Join (Change Unknown) problem and suggested it was like the first problem (Join (Result Unknown)) but "had a different kind of unknown." From these additional word problems the teachers were able to identify the fact that the type of action (or lack of action) in the problem and the type of unknown in the problem would make for a differently structured word problem. Slowly, over two days, the teachers were able to identify all of the types of structures outlined by the CGI program.

Once the teachers had identified and explained the structures, they voiced their excitement and desire to make changes in their practice based on their new understanding:

Carmen: This is a new way to think about what you are really asking.

Julie: Just being more aware of these different kinds of problems, right away I'm thinking, wow, I need to be thinking more about it when I'm in my room. I've never thought about it like this. I never knew there were so many kinds.

Pam: I'll never look at addition and subtraction in the same way again. (Julie, Carmen, and Pam, Course Day 3, 3-13)

All of the teachers identified that they needed to change their perspective on how they viewed addition and subtraction word problems.

The teachers explained that it was crucial to have the students "understand what is going on in the problem." They explained in our course how it was very important for teachers to teach in such a way that the students were not memorizing a procedure but understood the structure of the problem. The teachers identified that students should not look for key words to solve a problem, and explained that they wanted to stay away from a solely procedural understanding. They recognized the importance of students being able to picture in their heads the context of the problem, a concept they had used from their work with *Investigations*.

Subdividing the problems based on knowns, unknowns, and action of the problem gave the teachers a different perspective of the concepts of addition and subtraction word problems. They began to realize that a problem they previously would have considered to be “addition” would have been one in which the action of joining was taking place in the problem. They soon identified problems that could be considered addition in which there was no joining at all. Also, the teachers began to identify joining problems that could be solved with subtraction. Similarly, they began to view subtraction as not merely the act of separating. The teachers vocalized a very important aspect of their new understanding, explaining that calling a word problem a subtraction problem does not give it meaning but is only a way to solve a word problem. The teachers agreed with this interpretation and spent the next several weeks developing their understandings of the different structures of addition and subtraction word problems.

Join. During the course meetings each teacher was able to explain that the action of two separate groups coming together was integral for a Join problem. As one teacher described Join problems she said, “There has to be some action there. There’s something already there and something comes and joins it.” I observed that Join problems were the easiest for teachers to identify, possibly because as they said, they were the most familiar with them. They were also familiar with Join (Change Unknown) problems, which they at first referred to only as “missing addend” problems. The teachers identified these problems as more difficult and often being the ones given to the students at the end of the year “to throw them off” or “to make them think.” At first when I asked the teachers what the students were supposed to think about when solving Change Unknown problems they could only talk about "how to solve it." By the end of the sessions the teachers interpreted what the students were supposed to think about as "what they know and what they don't know…and how to use what they know to find the missing part" (Carmen, Course Day 5, 78-80).

The teachers quickly identified that Start Unknown problems were the most difficult Join and Separate problems to solve. As one teacher described:

> Start Unknown is harder because it is easy to start with something and then add on. But I had to think for a second. Start unknown is the hardest. Take it one line at a time. What do you have, I feel like that is comforting. Versus, if you start with nothing, what do you have, nothing. (Carmen, Course Day 3, 112-116)

In this course the teachers were able to apply this understanding when directly modeling with cubes a Join (Start Unknown) problem. They explained that when modeling this type of problem they should not first select the number of cubes representing the change, as that is what comes second in the problem. They could directly model the problem by using Trial and Error, as CGI labels it, starting with some quantity of cubes (representing the unknown), adding the known change quantity to it, and then deciding whether it matched the known final quantity.

Separate. It was easy for the teachers to discriminate between Join and Separate problems. The teachers knew these two structures both contained action in the problem, but the action was different. "One you're joining together, one you are taking away (Carmen, Course Day 1, 174-175)". A Separate (Result Unknown) problem was described as the most basic, straightforward, or simple type of subtraction problem. As one teacher said:

> I have seven pieces of candy. I gave four away. How many pieces do I have left? So that, I think, is kind of the basic subtraction problem. It's like the opposite of join. I know both of the start. And I know the change because I gave four away. I don't know the result. (Pam, Course Day 1, 83-86)

Separate (Result Unknown) was the easiest of the three types to model, because "you have both things right away that you need to do. You have them right there" (Julie, Course Day 1, 106-107). The teachers also explained that Change Unknown would be more difficult, with Start Unknown being the most difficult.

The teachers were able to interpret separating as a concept that involved both subtraction and addition, which was difficult for them at first. When discussing a Separate (Start Unknown) problem one teacher explained that she immediately jumped to addition to solve it, even though it is separating. The teachers were able to identify and interpret the action in the problem. They could generalize problems by viewing particular cases of types of problems. They could demonstrate how to model the action and interpret how addition and subtraction can be used as a strategy to solve Join and Separate problems in terms of the action. They acknowledged that identifying the operation they used to solve the problem was not the same as identifying the structure of the problem.

Part–Part–Whole. The teachers discriminated between a Join problem and a Part–Part–Whole problem by realizing that a Part–Part–Whole problem did not have any action in it. Occasionally a teacher would write a Part–Part–Whole problem that would be misconstrued as a Join problem, but then another teacher would correct her. A typical response might be something such as, “I guess it’s not. So they are not really being joined they are just two separate groups that get combined. They do get all counted but they don’t actually physically come together” (Julie, Course Day 1, 231-232). Julie realized that although all the objects get counted in a Part–Part–Whole problem, the objects are not physically combined into a new whole. Upon further reflection the teachers began to interpret Part–Part–Whole problems differently, instead of having parts that were two separate wholes they were two parts of the same whole. For example, one teacher wrote the following problem.

> I have some beads on my bracelet. Three beads are red. Four beads are blue. How many beads are on my bracelet?

To explain how the beads are part of one whole, the teacher says, “So the beads aren’t in two separate places. You’ve got this whole bracelet and it’s made up of two parts.” Furthermore the teacher distinguished this problem from a Join problem by saying, “You aren’t actually threading the beads on. There is no action joining them they are just there.” Another teacher asked, “What if I actually talked about making the bracelet?” and the explanation was, “that would be different because you’d be joining, you’d have three and then adding to it” (Julie and Carmen, Course Day 2, 7-17). Julie was explaining that one of the aspects of Part–Part–Whole problems that make them different from Join problems is that the sum exists at the outset of the problem.

Although it was more of a challenge to visualize parts and wholes than it was to visualize action versus inaction, the teachers were able to apply their understandings of the parts and the whole to identify what types of Part–Part–Whole problems could be made with different quantities unknown. They realized there were only two different types of these problems. They each described how the unknown could either be one of the parts or the whole. The teachers could easily create both types and explained that the Whole Unknown problems were easier to model and solve than the Part Unknown problems.

One teacher wrote the following Part–Part–Whole (Part Unknown) word problem:

> I have seven pencils in my tub. Four of them are red. Some of them are blue. How many are blue?

When asked what makes this type of problem more difficult than a Whole Unknown problem, a teacher summarized the group's conversation by saying, "There's another level of thinking. It's not you take the first two things you hear and give the answer. You have to hear the whole thing and do another step." She demonstrated this by modeling both types of problems. Modeling the Whole Unknown problem the teacher said, "I have three that are red, I have four that are blue, just count them all up." Modeling the Part Unknown problem the teacher said is more difficult. "The student may grab the whole number of pencils first and will have to do some manipulating before making the right combination" (Carmen, Course Day 2, 33-40).

Compare. This last type of problem was not spontaneously generated by any of the teachers. After all of the other types were discussed and the teachers could not think of any other different types of problems, I introduced the Compare problem. The teachers were obviously not unfamiliar with this type of problem, but they were not able to think of problems of this type at the time. However, the teachers were eventually able to explain that a Compare problem, unlike Part–Part–Whole problems, contains two different sets that are compared to each other. They could explain that, like Part–Part–Whole problems, there was no action in the problem. Initially the teachers thought there would only be two types of Compare problems, one with the difference unknown and one with one of the sets unknown, but after writing, modeling, and solving several problems and even teaching the problems they identified that Compare (Referent Unknown) and Compare (Comparison Unknown) were structurally different.

By the end of the study the teachers were able to identify the different kinds of problems when I presented them and they had many experiences writing their own problems when asked for a specific type. The teachers were able to take a problem and change only the type of known or unknown in the problem, generalizing them all as the same structure of problem. The teachers could discriminate among types of problems and they were able compare the difficult level of different problems.

Strategies Used to Solve Addition and Subtraction Word Problems

Direct modeling. The teachers came into the study with a better grasp of the strategies children use to solve problems than they did of the structure of addition and subtraction word problems. When asked how most children would directly model a Join (Result Unknown) problem they all identified that children would use *Joining All* and even used that terminology without my input, possibly from their experience with *Investigations*. The teachers explained that when using the manipulatives on a Join (Result Unknown) problem the students would place the manipulatives in two distinct groups and then physically join them together. Similarly, before instruction the teachers all identified that a common strategy for solving a Compare (Difference Unknown) problem would be using *Matching*. Both of these modeling strategies were ones with which the teachers were familiar by name and had seen used in their classrooms often.

Joining To, Separating From, and *Separating To* were also strategies the teachers had seen extensively, although they did not have a common language for describing them. All the teachers could easily explain how children might solve problems using these strategies and could identify them as common strategies for the appropriate types of problems. The teachers were able to use the manipulatives without much instruction and were able to interpret how their students and the students in the videos shown in class used the manipulatives. When one teacher was asked whether a student would just subtract the two numbers given in a Join (Change Unknown) problem the teacher responded, “They might, but not at the beginning. They would probably [begin with the starting amount and then] take a few out and go, one, two, three, four, five, six, oh I need more and then go back” (Pam, Course Day 3, 3-7). The teacher explained the process of *Joining To* the original amount until the Result amount is obtained. The teachers were able to describe the similarities between *Joining To* and *Separating To* as, “They are not sure how many to either separate or join so they are just adding or taking away until they find the right number.” The teachers also identified that a very common student mistake is to forget that the answer to the problem is not the number counted up to or down to, but the number of steps that were counted. In our course and in their practice they often explained that many students would struggle to decide how many needed to be added on but then forget what their final answer represented in the problem, either with direct modeling or with a counting strategy.

The teachers had not worked with very many Start Unknown problems in their classrooms so they were less sure that the students were capable of using their own strategies to solve these problems. On the second day of the course, one teacher described how she might assist her students with a Join (Start Unknown) word problem.

> If I were helping them, which I would probably have to do, I would say, if she had one and she got three more how many would she have? If she had two and she got three more how many would she have? That is how I would talk them through it. (Julie, Course Day 3, 40-43)

As a group we discussed how this strategy could be helpful to students and gave it the label of *Trial and Error*. By the end of the course all the teachers identified this as a valuable strategy, especially for Start Unknown problems, and even explained that they might not have to help all their students with this type of problem and that some students could potentially discover this strategy themselves. By the end of the study the teachers identified Start Unknown problems as, although challenging, viable options even for first-grade students. They interpreted these problems as ones children had the potential to reason about and for which *Trial and Error* could be useful when directly modeling.

Counting strategies. The teachers were even more familiar with counting strategies than direct modeling strategies, especially the second-grade teachers, whose students tended to use direct modeling less often. They were able to explain the transition from direct modeling to using counting strategies. For example, when talking about a Join (Result Unknown) problem the teachers identified and accurately demonstrated that the first step is being able to model using *Joining All*. The next step they described is using their fingers to represent the objects and then counting them all. The teachers explained that this is also a form of direct modeling, because the fingers represent the objects, but it transitions the students to *Counting On From First* and then *Counting On From Larger* (or *Counting Down*), in which they are no longer representing the objects but only using a counting strategy.

Each teacher, in her own way, was able describe the increased difficulty level of *Counting On To* and *Counting Down To.* To the first-grade teacher's amazement she said, "I have actually seen kids [solve Join (Change Unknown) problems] in their heads where they will go… (The teacher bobs her head while counting to herself)." She added, "Those are usually my sharper kids, they can actually keep track in their head of how many times they are bopping their head or whatever. But I've seen them actually moving their head to count" (Julie, Course

Day 3, 52-56). Another teacher added to a conversation a few days later her understanding about the different levels at which children solving Change Unknown problems.

> …Or the difference between the kids who needed their fingers and then I had some kids who could mentally count up six, seven, eight, nine, and keep track that that was one, two, three, four. So they are keeping track of two different numbers in their head at the same time. Those were the kids who were just a little bit below my kids who just automatically knew it. (Carmen, Course Day 5, 155-160)

This teacher explained that some students will need their fingers to keep track of how many they are *Counting On To* or *Counting Down To* while others can do it without the aide, and still other students will be able to solve these problems by just recalling facts.

Making Changes to Lesson Plans

Given their newfound understandings of addition and subtraction word problems the teachers were dissatisfied with their prior teaching methods regarding this subject. They quickly identified that they had been teaching almost exclusively Join and Separate (Result Unknown) problems. The teachers explained that they were doing the students a disservice by focusing on these two types of problems. They interpreted this type of teaching as training the students to do only those types of problems and without even fully developing their understandings of them. They also believed they were making it more difficult for students to understand any other type of addition and subtraction word problem in the future. When the students were given several Change Unknown problems at the end of the year, the teachers believed they were "thrown off" because they had been overexposed to Result Unknown problems. The teachers explained that by the end of the year the students were "trained" to just find the two numbers and add them or subtract them based on procedural cues given in the problem.

The teachers found this to be contradictory to the message *Investigations* gives the teachers and students by telling them to picture what is going on in the problem and imagine the story in their heads. The teachers thought this was a good idea, but in the past were always unhappy with the types of problems the students had to solve; the teachers interpreted these problems as ones the students did not need to think deeply about. They were happy to have more complex types of problems to give the students so that imagining the story was a necessary step in solving the problems.

The following excerpt from the transcript describes how the teachers understood teaching addition and subtraction word problems before the study and indicates changes they intended to make in the future:

Pam: So the first *Investigations* [unit] to introduce combining is…I read a story problem that they picture in their mind. 'There are twelve children playing tag on the playground. Then ten more children join the game. How many children are playing tag now?' And then they explain to me what is happening. They say they will picture our playground and 'these kids are out and then more are there.' 'Well should the answer to this problem be more than twelve or less than twelve? Explain your thinking.' Nobody is like 'hmmm'; they all know it's more.

Carmen: I was just thinking as we were talking about this, story problems with *Investigations*, when you introduce joining problems you do one with the whole class and you do that, 'think about what's happening, picture it,' and everybody talks. But they are always these easy ones and the kids always know how to do it, and it's almost like, why am I bothering with this lesson? Everyone knows how to do it. I almost think that it might be worthwhile to switch that up and do some of these more challenging ones as the whole class lesson, so the kids who struggle with these would hear some strategies from the kids who are getting it.

Julie: See, personally I am feeling like I am going to be interested. I always hated this lesson.

Jana: Which one?

Pam: The first day of combining. Picture twelve plus ten, is it more or less than twelve. How do you know?

Julie: Right, because all the kids are bored because they already know how to do it. I mean you just see them like… [imitates students making a bad face]. You know what it almost seemed to me, when we give them this Join problem Result Unknown. And you know sometimes I feel like the kids are looking at me like, 'no duh,' you know, when you say 'do we have more than?' It's like, yeah, of course we have more. Like it seems like that is not the type of problem we want to give them to visualize. I mean I could give them that problem. They do not need to visualize that problem to know that there is more.

Pam: The other ones let's visualize.

Julie: That would be better in my mind.

Carmen: So then when we do this lesson which starts with this 'lame-o' problem that only three kids are challenged by, it will be better to introduce several different combining stories. (Carmen, Julie, Pam, Course Day 3, 529-580)

The three teachers all believed the diversity of the new problem types they had learned would be useful in teaching the students to understand the problems.

Although the teachers were dissatisfied with their lessons in the past, they did not know how to change them. In examining past lesson plans one teacher realized that instead of focusing on "harder addition and subtraction word problem concepts" the lessons just began to use larger numbers as the year went on. The teachers identified that the students moved into larger numbers too quickly, but did not previously know other ways to change the focus of the lessons. The teachers expressed excitement over being able to challenge the students' thinking with lesson plans that were meant to expand their knowledge of addition and subtraction besides just increasing the magnitude of the numbers with which the students could work.

They also explained that some of the problems they had given in the past were challenging for more "superficial reasons" that had less to do with understanding addition and subtraction word problems. For example, one teacher explained:

> My higher kids, they can do these [Result Unknown problems] with their eyes closed, and challenging them isn't giving them higher numbers. They are supposed to get harder, but the difference between the harder ones is that the numbers get bigger. This is the only one that's rough, here's the tricky problem. Steve and Rosa pick cherries. They ate eighteen cherries for lunch. Rosa ate six cherries after lunch. How many cherries did they eat? This throws them because of this first line, they picked cherries. They are trying to figure out 'what's that got to do with it?' In light of what we've just done this seems less important. (Julie, Course Day 3, 76-84)

The teachers agreed that although it is important for students to learn how to identify which information is relevant in a word problem, there are many more important, interesting, and mathematical ways to challenge the students.

As a group, and without my input, the teachers began deciding what kinds of problems they would give their students. When occasionally they turned to me for advice, I made it clear that they were the teaching experts and needed to decide for themselves how they would apply the research I had shown them. They did not want to omit Result Unknown problems altogether, of course, but they wanted to "mix them up" with other kinds. At first the teachers tried using the problems from *Investigations* by just changing the relationships. The group wrote the problems for the first relevant unit in the first-grade class in this manner. They did not keep track of how many of each type of word problem they were writing but claimed they were trying to "give them some of everything." When they finished writing this first packet of problems I noticed they had not written any Compare (Difference Unknown), Compare (Referent Unknown), Join (Start Unknown) or Separate (Start Unknown) problems (See Appendix B for a full list of problems). I did not question them about this at the time, because I did not want to influence their decision making. When I asked them about this during our next meeting, they had not realized the Compare problems had not been included. They also admitted that they were worried that Start Unknown problems would be too challenging for the first-grade class and so they resisted including them.

The teachers found the process of changing the old problems to be challenging. They were able to explain that it did not make much sense to change a Join or Separate (Result Unknown) problem that has action into a problem without action such as a Compare (Difference Unknown) problem. This reinforced how structurally different the types of problems were. For the next round of word-problem writing they chose to utilize the problems in the curriculum simply as a guide to decide which numbers to use in the problems. The teachers would consider a textbook problem and sometimes use the context as well.

When the teachers got together to write problems for the beginning unit of the second-grade class the next week they were more organized and clear about their goals. They kept track of which kinds of problems they were writing and made sure they wrote two of each of the eleven types of addition and subtraction word problems. They decided, because of the way they wanted to spend time in class, that this was too many problems to be included in the packet and so they narrowed it down to 16 problems, eliminating problems they felt were more awkwardly worded but without focusing on any structures in particular (see Appendix B).

After the teachers wrote the new packets for their lesson plans they decided to give their students a "pretest." In our course meetings we had discussed that the CGI group claimed students had the potential to be able to solve these addition and subtraction word problems through their own modeling and thinking, and the teachers wanted to observe this for themselves prior to any instruction. They were interested in identifying what their students would do completely on their own and then comparing the results. So they decided to give all the students the same Join (Change Unknown) problem.

> I had five apples. My friend gave me some more. Now I have nine apples. How many apples did my friend give me?

The teachers chose a Join (Change Unknown) problem because they were considered to be the "tough" problems they had been using in the *Investigations* units, the problems "thrown in at the end to stump the students." The teachers were curious as to how the students would react when given this problem before any Result Unknown problems. The teachers also discussed the size of the numbers used in the pretest question. They explained that they wanted to make sure to keep the numbers relatively small, so they could test the students' ability to understand the structure of the problem without getting weighed down by the size of the numbers. All three teachers gave the problem to the students and then watched to identify which strategies the students would use to solve the problem. During the next course meeting they came together to discuss the results.

I believe this was a significant turning point for the three teachers. All three teachers were very excited that the students had appropriate strategies to approach the problem. In each of the classes most of the students solved the problem correctly. In the first-grade class, all but four of the students did the problem correctly. Furthermore, the first-grade teacher was pleasantly surprised that her most advanced group of students used recalled facts, the slightly less advanced group of students used a counting strategy (almost all used *Counting On To*), and the group that she considered the least advanced group of students used direct modeling to solve the problem, just as we had discussed in class. All but one or two of the students answered the problem correctly in the second-grade classrooms. Almost all of the students in the second-grade classes recalled a fact to solve the problem and did not need to use another strategy. Three or four students in the second-grade classes used the *Counting On To* strategy.

After the pretest the three teachers were significantly more confident in the abilities of their students. They always seemed excited about learning more mathematics themselves, but now they were excited to apply their understandings with their students. I also believe that after the pretest they felt more connected to the research and viewed their students' work as being a part of the body of research instead of being a separate case. As one teacher said, "It was fun to try it out. Now I can see it with my own students instead of just reading or thinking about it. I thought it sounded good before, but now I really see it better." And another teacher added to it saying, "This certainly shows that [we] can put in problems like this, and from the beginning, because they certainly are capable of doing it" (Julie, Course Day 7, 442-445)

After the first round of data collection, in January of 2009, the teachers talked to each other outside of our course and decided that as a group they preferred to also rewrite the word problems in the other addition and subtraction word-problem unit that was being taught in April and May. They contacted me and asked whether I would like to attend the group session during which they wrote the addition and subtraction word problems. When we met as a group the first-grade teacher decided she wanted her students to do problems of all varieties so the teachers wrote problems for the first-grade class and included either one or two problems of all structures (see Appendix B). The second-grade teachers decided they would make their packet consist of one problem of each type so the group wrote those 11 problems (see Appendix B).

Other Changes in Practice

The most obvious changes in the teachers' practice were the kinds of problems given to the students and the amount of time spent on addition and subtraction word problems. The teachers all claimed that during the year of this study they spent significantly more time on the subject during both units it was taught, and with a greater variety of problems. As one teacher summarized:

> I had kids that were really more confident about what they were doing than in the past and I was trying to think about what had changed from last year. I think that this is way more in-depth than

> I've ever done it. I mean I probably spent the three days [before] and that was okay, we're moving on. This time I am devoting an entire week [to both units] and that is more than I've done. I feel like [before] when it comes around in the spring we only touched on it a little bit. (Pam, Course Day 10, 315-325)

The teachers in the past were so bored with these lessons that they devoted little time to them, about three days, twice per year. They were unable to improve the lessons because they did not have the mathematical understanding to do so. They were all very excited to be able to finally make changes to lessons they had found unfulfilling in the past. The changes to their practice were to include more variety of problems and to spend a greater amount of time on the units. The manner in which each teacher included a variety of problems was slightly different in each classroom as described in Chapters Four, Five and Six.

There were other changes in the teachers' practice that were less deliberate and seemed to evolve over time. For example, all three teachers began to examine more closely the words they spoke in class, identifying that their language choices could influence the way their students understood the word problems. In the past the teachers were mostly using Result Unknown problems, so they were accustomed to the answer to the problems being the "total" or "what was left." The teachers noticed immediately that they had to be careful when using the word *total*, which they had a tendency to use for representing the largest number in the problem, even if they were not joining anything together. One teacher used the word *total* in reference to the starting quantity in a Separate (Start Unknown) problem. Explaining the problem she said, “You take the change and the result and your answer is the *total*.” Another teacher agreed with her methods but seemed concerned about her interpretation of the word “total,” that it was not fully representing the structure of the problem. She said, “I am not saying you are wrong, but maybe you could use other words that make more sense in the problem, like ‘what you start with’” (Pam, Course Day 1, 785-788). The teachers learned that just obtaining the answer is not the same as accurately describing the process.

The teachers were also cautious about how they used the word, or overused the word, *answer*. They explained that always referring to the sum of or difference between two numbers as the *answer* is not wrong but it is not very descriptive. For example, when a Part–Part–Whole (Whole Unknown) problem is asking students to combine the two parts and find the whole, the teachers discussed how instead of just asking the students for the “answer,” it would be important to talk about what the “whole” is, especially within the context of the problem. The teachers began to discriminate between words such as *whole* and *answer* and recognized that the more specific they were in the words they chose, the more meaning they likely gave to the word problems.

For a period of time the teachers seemed to be quite cautious about their word choice, hesitating to state problems because they did not want to say the wrong thing. In the beginning they had increased their vocabulary, trying to use words other than *answer* and *total* to describe the structure of the problems, but they realized that sometimes they used the words inaccurately. For example, when talking about a Compare (Difference Unknown) problem one teacher described one of the sets being compared as the "whole." Another teacher corrected her saying, "Not necessarily. There is no whole. There are two separate groups. There isn't a total, it's how many you have and one just has a bit more" (Carmen, Course Day 2, 56-58) But they tried to support each other and identified this struggle as positive and necessary to the growth of their own understandings.

The teachers identified the importance of language in writing word problems. They explained that they did not want their students using key words to solve the problems and that those key words could be misleading or could be used incorrectly to solve problems. As one teacher explained:

> It doesn't help the students understand the problem, you are just looking for this word, and then I add…It's kind of like teaching them, 'cross the number out and then make it one lower'…it's more of a procedural thing than a true understanding. The language itself can be confusing if they are only thinking about language and not trying to figure out what's going on. (Julie, Course Day 1, 115-120)

When writing the new addition and subtraction word problems they tried not to use words that were obvious key words.

For example, when writing a Join (Result Unknown) problem the teachers at first asked, "How many brushes are there in all?" They then talked about removing the words "in all" and agreed it was better. When I asked them why, one teacher responded, "It doesn't sound the way our kids talk." Another added, "They might say how many brushes altogether? That might be more natural" (Julie and Pam, Course Day 11, 999-1022). Another teacher added that in the past she noticed students searching for “in all” in problems and that was something she wanted to avoid.

The teachers explained that some words were too obvious of an indication of which operation could be used to solve the problem. They identified that they wanted the students to have to think about the problem, understand the structure of the problem, and then apply this understanding to solve the problem.

Pam: Ted had $.75. He went to the store and spent $.29 on the candy bar. How much money did he have left?

Julie: Could you say…I mean this sort of indicates…I mean it would be the same anyway but what if you said ‘he went to the store and spent $.29 on gum. He spent the rest on a candy bar. How much did the candy bar cost?’ Still the same problem but it doesn’t…

Carmen: That’s good Julie.

Julie: It makes it like they have to think a little bit more. As soon as you say ‘how much money did he have left,’ it’s like, okay, we subtract. But if we…this way they do subtract but it’s not, again like we were saying, putting the big minus sign on the top of the page. (Carmen, Julie, Pam, Course Day 11, 3514-3542)

This conversation came up many times in the other problem-writing sessions when the teachers did not want to make the problems so mundane and typical of “math book problems.” Teachers would say things such as, “Yeah, we definitely want to change that. Yeah, there’s a big subtraction sign on it.” The teachers understood that the problems should be written how “normal people talk” without using key words and “giving away” what operation could be used to solve the problem.

In the course, all three teachers began immediately questioning their practice of how they teach the different ability levels in their classroom. Their initial reaction was to use the variety of problems to challenge their higher achieving students. On the first day of learning about different problem structures one teacher commented:

> So when we do these word problems with our class and we are trying to differentiate instruction, if we have kids who are really good in addition, we don’t have to go and get the bigger numbers to work with, we could just change the structure, keep the problem but change the structure of the problem, change the unknown, and it would make it more challenging for our kids who are really good with these number concepts to start with. That would save me a whole lot of time. I totally have a new perspective on what I can do to extend my students! (Julie, Course Day 1, 644-650)

For the rest of that class this seemed to be the general perspective of the teachers on how their understandings were going to change their practice. They were very excited about applying their new mathematical understandings to challenge the understanding of their high-level students.

It was not until the following week that one of the teachers made the following observation:

> I would think that why shouldn’t [lower achieving students] have something harder that they are starting to work on solving, that they

> might not have it mastered but we come back around to it. But why shouldn't they be stretched anyhow now. Why wait until after they have only done joining. And then we are going to say, now we want to stretch you, but we've already set up a kind of a template.
>
> (Carmen, Course Day 3, 638-643)

The teachers discussed that they would be setting the lower achieving students up for failure if they only gave them Join (Result Unknown) problems until they had them mastered and then gave them problems with another structure. Once this idea was brought up by one teacher they all agreed with it, wanting to give their students of all abilities a chance to think mathematically about the problems. They wanted all of their students to have to interpret the action of the problem and the types of knowns and unknown and use that interpretation to the solve problem.

They also realized that they needed to identify what mathematics the struggling students found to be difficult; for example, were they struggling with understanding the structure or were they struggling with the size of the numbers in the problem?

> It's kind of like in the olden days where we didn't let kids do fun things until they had the basics and the kids who were struggling never got the fun things. It's like we don't want them not to be exposed to it just because they are having a hard time with numbers…it's almost like, are the kids who are struggling, are they stuck on this problem because of the numbers involved? Wouldn't you want to give them some other problems but with lower numbers

> to help them develop different strategies, but at the same time you don't want to blow their mind. Because those are two different goals, in a way. I want them to be more efficient at solving problems before we make them do it with larger numbers. (Julie, Course Day 8, 749-759)

By the time the teachers began teaching the students, they agreed to give all their students the same problems. The problems began with what teachers considered "lower numbers" for their students, and as time progressed, the teachers wanted the size of the numbers to increase. They wanted the students to begin with lower numbers so they could concentrate on understanding the structure of problems without also having to learn how to manipulate larger numbers. The teachers allowed the students to progress through the packets on their own timeline so the highest achieving students got to work with larger numbers more quickly.

The teachers also identified that even though the students of all abilities were working on the same types of problems, they would solve them using different strategies. They explained that students who were struggling with understanding the structure of problems might need to model the problem. The teachers considered students who still needed to model problems as students who were not ready to move on to working with larger numbers.

APPENDIX B

GROUP WRITTEN WORD PROBLEMS

First-Grade Word Problems Part 1

1. Rosa had 6 toy cars. Her mom gave her 6 more. How many toy cars does Rosa have now?
2. Steve has 5 marbles in his pocket. He has 9 marbles in a bag. How many marbles does he have?
3. A squirrel ate 8 nuts. He ate 7 seeds. How many things did he eat?
4. Rosa had 11 library books. 4 of them were funny. How many books were not funny?
5. Steve and Rosa had 12 apples. They ate 6 of them. How many apples are left?
6. Rosa had 15 pennies. She spent 6 pennies. How many pennies did she have then?
7. Steve has 3 cookies. Rosa has 2 more than Steve. How many cookies does Rosa have?
8. 10 children were playing at the park. Then 8 more came to play. How many children were at the park?
9. Kim and Tito picked 11 red flowers. They picked 5 white flowers. How many flowers did they pick?
10. Kim had 16 pennies in her pocket. The pocket had a hole. 8 pennies fell out. Now how many pennies are in her pocket?
11. Tim has 14 fish in the tank. 9 fish are red. How many fish are not red?

12. Kim saw 20 ducks on the pond. Then 9 ducks flew away. How many ducks were still on the pond?
13. Kim saw some cats on her walk. Tim saw 2 cats on his walk. Kim and Tim saw 5 cats altogether. How many cats did Kim see?
14. Steve and Rosa picked cherries. They ate 18 cherries for lunch. Rosa ate 6 cherries after lunch. How many cherries did they eat?
15. Rosa had 6 stamps. Her father gave her some more. Now she has 9 stamps. How many stamps did her father give her?
16. Steve had 10 candies. He ate some candies. Now he has 3 candies. Now many did he eat?
17. Rosa had 21 balloons. She gave 3 of them away. Now how many balloons does she have?
18. Rosa’s mom had 23 hens. She sold 8 of them. Then how many hens did she have?
19. Steve rode 3 miles on his bike in the morning. He rode 9 miles altogether. How many miles did he ride in the afternoon?
20. Kim made 7 paper hats. Tito made 6 paper hats. How many paper hats did they make?
21. Tito had 15 cookies. He gave 9 to his friends. How many cookies did Tito have then?

First Grade Word Problems Part 2

1. Ken found 12 white shells at the beach. He found 6 brown shells. How many shells did he find?

2. Two goats had 15 carrots. Then they ate 8 of them. How many carrots were left?
3. Tim had 13 erasers. Then his friend gave him some more. Now he has 20. How many erasers did his friend give him?
4. Sam has 9 stickers. Chris has 8 more than Sam. How many stickers does Chris have?
5. There are some blue beads on a string. There are 15 red beads on the string. There are 25 beads altogether. How many blue beads are there?
6. Jen washed 19 paint brushes. Alex washed 8 brushes. How many brushes did they wash?
7. Tom has 9 red balloons. He has 3 more red balloons than blue balloons. How many blue balloons does Tom have?
8. Alex made some tacos. Sue made 12 more tacos. Now they have 22 tacos. How many tacos did Alex make?
9. Jane had 17 crayons. Her father gave her some more. Now she has 25. How many did her father give her?
10. Joe has 8 marbles. Ken has 12 marbles. How many more marbles does Ken have than Joe?
11. Eleven people were on a bus. Some more people got on. Now there are 15 people on the bus. How many people got on?
12. Kim and Joe collected 16 cans. The next day they collected 14 more cans. How many cans did they collect?

13. 30 children were playing ball. Some went home. 11 children were still playing. How many children went home?
14. There were some dogs playing at the dog park. 7 of them went home. Now there are 14 dogs left. How many were at the park to start?
15. Peg caught some fish. Jack caught 19 fish. Jack caught 13 less fish than Peg. How many fish did Peg catch?
16. Jill baked some cookies. Liz baked 29 cookies. Jill baked 13 more cookies than Liz. How many cookies did Jill bake?
17. John has 18 pennies. He spent some pennies. Now he has 7 pennies. How many pennies did he spend?
18. Liz drew 8 big stars. Then she drew some little stars. Now she has 14 stars on her paper. How many little stars did she draw?
19. A frog ate some flies. Then it ate 12 crickets. It ate 24 bugs altogether. How many flies did the frog eat?
20. Jake baked 30 muffins. Some were blueberry. 12 were strawberry. How many muffins were blueberry?

Second-Grade Word Problems Part 1

1. Sammy has 46 Pokemon cards. Tyson gave him 38 more. How many cards does Sammy have altogether?
2. Helena is planting a garden. She planted 32 tomato seeds. She wants to plant 50 seeds altogether. How many more seeds does she need?

3. Michaela was playing war. She began with a stack of cards. In the first round, she won 18 more cards. Now she has 39 cards. How many cards were in her stack to begin the game?
4. On one November morning, there were 72 leaves on a tree. The wind blew 36 of them off the tree. How many leaves are left on the tree?
5. Sattui had a basket with 46 toys in it. Gretzky came and took some of them. When Sattui went back to his basket, he only had 22 toys left. How many toys did Gretzky take?
6. Erikka had some horses in a show. The judges came and took 16 of them because they won a prize. She had 48 horses left. How many horses did she bring to the show?
7. Noah built a Lego house using 89 Legos. He used 42 black Legos and the rest were blue. How many blue Legos did he use?
8. Addison and Tyson were playing football. At halftime, Addison scored 28 points and Tyson scored 42 points. How many points does Addison need to score to tie the game?
9. Brenden and Garrett went on a hike to go bird watching. Brenden saw 57 birds. Garrett saw 22 more birds than Brenden. How many birds did Garrett see?
10. Ryan went on some rides. Ben went on 33 rides. Ben went on 13 more rides than Ryan. How many did Ryan go on?
11. Anna and Kelly went on a trip to Paris, France. They decided to go to the Eiffel Tower. 38 people were waiting in line to ride the elevator. When

they got to the top, they saw 44 more people. How many people did they see at the Eiffel Tower?

12. While Lindsey was waiting for the bus to take her to school, she counted 52 flowers blooming. When she got to school, she saw 34 more flowers blooming. How many flowers did she see blooming?
13. Tyler has 42 Legos. Carter gave him 19 more. How many Legos does Tyler have altogether?
14. Katie has a stamp collection. She has 24 stamps. She wants to collect 60 stamps altogether. How many more stamps does she need?
15. Gabi was raking leaves. She began with a pile of leaves. After 10 minutes she raked 17 more leaves. Now she has 36 leaves. How many leaves were in her pile originally?
16. At the football game, there were 95 fans in the stands. At halftime, 29 of them went to the snack stand. How many fans were left in the stands?

Second-Grade Word Problems Part 2

1. Kim and Jay counted animals in the park. They counted some robins and 37 squirrels. There were 60 animals altogether. How many robins did they see?
2. Sally collected some seashells on Monday. The next day she collected 49 shells. Now she has 73 shells. How many shells did she collect on Monday?

3. Kim had 46 daisies growing in her garden. Spot came and stepped on some daisies. Now there are only 27 daisies standing. How many daisies did Spot trample?
4. One second-grade class flew 28 kites. A third-grade class flew 53 kites. How many less kites did the second-graders fly?
5. There were 39 girls in a bicycle race. There were 52 boys in the race. How many children were in the race?
6. There were 19 dogs at the park. Some more dogs came to play. Now there are 34 dogs at the park. How many dogs came to play?
7. Nate checked out some books from the library on Monday. He returned 17 books on Tuesday. He still has 23 books checked out. How many books did he check out on Monday?
8. Julie has 67 M&Ms. Her friend has 25 M&Ms more than Julie. How many M&Ms does her friend have?
9. Pete caught some fish. Sally caught 13 more fish than Pete. Sally caught 51 fish. How many fish did Pete catch?
10. Ben had $.75. He went to the store and spent $.29 on gum. She spent the rest on a candy bar. How much did the candy bar cost?
11. Anna earned $.35 cleaning the yard. She earned $.47 washing the dishes. How much money did she earn?

APPENDIX C

INDIVIDUALLY WRITTEN WORD PROBLEMS

Pam's Supplemental Problems

1. There were 38 leaves on the tree in my backyard. After a big gust of wind, some of the leaves blew away leaving only 17 leaves on the tree. How many leaves blew away?
2. Sally had some pretzels on her desk. During snack, Ted came up to her desk and took 22 of them. Now Sally only has 19 pretzels. How many pretzels did she have before Ted took some?
3. Last week the temperature outside was 54 degrees. This week it has been 29 degrees outside! How much colder has it been this week than last week?

Carmen's Supplemental Problems: Numbers to 100

1. Jake has 73 pennies. When he gets to 100 pennies, he can trade them for a dollar. How many more pennies does he need?
2. Sue has 4 quarters. She bought an apple. Now she has 49 cents. How much did the apple cost?
3. Kay and Sam were playing *Get to 100*. Sam's marker was on 65. How far is Sam from 100?
4. Pat brought in 100 marbles to school. He made two piles. In one pile he had 71 marbles. How many were in the other pile?
5. Judy had 100 jelly bellies. She took a handful from the bowl. Now there are 82 jelly bellies in the bowl. How many did Judy take from the bowl?

6. Jerry is playing *Roll a Square*. He has some cubes. He needs 38 more to get to 100. How many cubes does Jerry have?

Jana Lunt
The Pennsylvania State University

EDUCATION

May 2010 **The Pennsylvania State University**
Ph.D., Curriculum & Instruction, Emphasis: Mathematics Education

PRESENTATIONS

April 2010 **State College Area School District**: *Understanding Students' Solving Strategies for Addition and Subtraction Word Problems.*

Jan/Feb 2010 **State College Area School District**: Three Seminars: *Understanding Teaching Addition and Subtraction Word Problems.*

Aug 2008 **Mid-Atlantic Center for Mathematics Teaching and Learning**: Student Research Conference: *Effects of Teacher Understanding of Addition and Subtraction Word Problems on Student Understanding*.

Nov 2006 **Psychology of Mathematics Education–North American Chapter** Merida, Yucatan, Mexico. *Ways in Which Prospective Secondary Mathematics Teachers Deal With Mathematical Complexity.*

Oct 2005 **Psychology of Mathematics Education–North American Chapter** Roanoke, VA. *Prospective Secondary Mathematics Teachers Unravel the Complexity of Covariation Through Structural and Operational Perspectives.*

GRANTS AND SCHOLARSHIPS

2007-2008 **Diefenderfer Scholarship** for $21,800. Graduate Student Endowed Scholarship, College of Education, Penn State University

2004-2007 **Mid-Atlantic Center for Mathematics Teaching and Learning**, Graduate Fellowship, National Science Foundation

2006 **Graduate Student Travel Grant**, Penn State University

2005 **Graduate Student Travel Grant**, Penn State University

PUBLICATIONS

Heid, M. K., Lunt, J., Portnoy, N., Zembat, I. O. (2006). Ways in which prospective secondary teachers deal with mathematical complexity. In S. Alatorre, J. L. Cortina, M. Saiz, A. Mendez (Eds.), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Merida, Yucatan, Mexico: PME-NA.