ANALYTICAL AND NUMERICAL
OPTIMIZATION OF AN ELECTRONICALLY
SCANNED CIRCULAR ARRAY

A Thesis in
Electrical Engineering
by
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ABSTRACT

A combined analytical and empirical optimization of an ultra high-frequency (UHF) circular array is presented in this work. This effort can be roughly categorized into three parts. Part 1 is a mathematical and numerical analysis of the general characteristics of circular arrays. The analysis includes a derivation of pattern functions for arrays of isotropic and endfire elements that extends beyond what has been presented in the literature to the limits of manageability (Chapter 2). This includes the derivation of a new closed form expression for the directivity of either isotropic (in the plane of the array) or analytically specified endfire element patterns with a variable elevation beamwidth.

A parameter study of the circular array follows (Chapter 5). An empirical approach was undertaken to answer questions left unresolved from the mathematical analysis, which, because of the inherent complexity, is inconclusive. The resulting parameter study documents the relative performance available from a specific array as a function of the physical size, the number of elements, and the element beamwidth. The study demonstrates that the number of array elements is the primary factor limiting the ability to minimize the beamwidth (in the plane of the array) and sidelobe levels. Other discussions include the unavoidable element-element coupling (Chapter 3), relative energy contained in element patterns of various beamwidths, and the observation of a positive relationship between inter-element coupling and element phase center movement towards the array center.

Utilizing the results from Part 1, Part 2 discusses the optimization strategy of the element and the array (Chapter 6). Two moment method codes—one of which works directly with a quasi-Newton optimizer—were used to complete the physical design. A complete array was fabricated and tested. Two critically important concepts are presented here. The first is that assuming the pre-1983 IEEE definition of gain is adopted, referred to throughout the thesis as system gain, then the voltage excitation that maximizes the system gain for an array of arbitrary geometry is simply proportional to the field contributions at a given
beam angle from the respective elements. The second concept is that despite strong inter-
element coupling, an array with a desirable set of element characteristics can be created
by performing an optimization on an isolated element. This is of significance because the
optimization of the electromagnetic model of the array can be prohibitive, for the sheer
number of unknowns present.

Part 3 develops the appropriate beamforming methods. Several techniques are used. The
first, based upon a linear least-squares method (LLS), is suitable for reception (Chapter
4). Here it is shown that the LLS method can be used to maximize the directivity. Along
this line, the effect that the so-called target null-to-null beamwidth has on the array
efficiency (and consequent system gain) is noted and discussed. Both weighted and
unweighted versions are considered. With weighting, sidelobes of -40 dB are
demonstrated. For transmission, a new means of placing a taper across the aperture while
simultaneously operating all amplifiers at full power is introduced. An eight-port vector
combiner, which forms the basis of this capability, is explained. The sequential quadratic
programming method is employed to permit non-linear array weighting constraints
(Chapter 7). Non-linear constraints are needed to maximize the effectiveness of the
combiner. This approach to a tapered transmit beam affords the full system gain of a
uniform excitation (or more), while reducing peak sidelobes by approximately 11 dB.
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1 INTRODUCTION

The U.S. Navy is the primary means by which the United States projects military force throughout the world on a short-term basis. Approximately one dozen aircraft carriers continually patrol international waters, and are under the direct control of the Commander-in-Chief. Each carrier possesses more destructive force than the entire combined U.S. inventory during World War II. On average, each carrier represents an investment of more than five billion dollars, and carries aircraft assets easily totaling several more.

The E-2C aircraft is a carrier’s first line of defense. As an early warning and control platform, its task is to provide airborne radar coverage around the carrier battlegroup. Practically, this includes everything from detection of enemy air attacks on the battlegroup to warning an inbound strike group of hostile fighter activity. The E-2C serves in a role very similar to that of the USAF E-3 airborne warning and control system (AWACS). Since the mid-1960s, it has been regular a part of the fleet's defensive and offensive forces.

Figure 1.1—The E-2C is catapulted off the carrier deck.
The most distinguishing feature of this aircraft is the 24-foot diameter radome, which rotates at 6 revolutions per minute. Inside the radome is a corporately fed 10-element antenna array [1]. Current upgrade plans include an 18-element active array which in addition to scanning mechanically, can scan electronically up to ± 60° from boresight [2]. This upgrade provides the obvious advantages of electronic scanning, and the ability to implement a two-dimensional (space-time) signal processing architecture, which is estimated to provide a 20 dB enhancement in the radar's sub-clutter visibility [3].

One planned capability is to provide E-2C radar information to other battlegroup participants through the Cooperative Engagement Capability (CEC) [4]. These could include battle cruisers, destroyers, aircraft carriers, other airborne platforms, or ground troops. The CEC network exploits the fact that while the angular position of a radar track has considerable ambiguity, the range ambiguity is very small. Participants at or near orthogonal angles to an unknown target can cooperatively develop high-precision position and velocity information by sharing track data. By providing high-precision radar tracks to net participants, surface-to-air missiles can be fired from an asset like the AEGIS cruiser before it can even detect the threat. With the CEC, participants may request information from another asset at any time. A mechanically scanned antenna cannot simultaneously provide 360° surveillance for the battlegroup while responding instantly to CEC sensor requests. For this reason, the circular array is being looked at seriously as a candidate for the next evolution in airborne surveillance.

The circular electronically scanned array has been a subject of interest for decades. For airborne applications where 360° coverage is essential, the circular array brings at least three benefits: the elimination of mechanical rotation, increased beam agility, and the virtual elimination of scan loss. These advantages have recently led the military to turn to the circular array for surveillance, communications, and other functions. Several other advantages can be discussed from a design perspective. A rotationally symmetric array can be modeled electromagnetically with enormous computational savings, both in terms of speed and memory requirements. From a beamforming perspective, the array elements
can be treated as identical, so that knowledge of one element is sufficient and end effects are nonexistent.

In addition to its advantages, the circular array also presents some significant challenges. The most immediate problem that confronts a designer is how to obtain the gain of an equivalently sized linear array without resorting to an array with hundreds of elements. Another is the beamforming problem. Not only do traditional methods (which depend upon element and array factor separability) fail, but as is shown in Chapter 4, more elements must be used in order to obtain sidelobes equivalent to that of a corresponding linear array. No less challenging is the difficulty of commutating a tapered transmit excitation while fully utilizing the available power and efficiency of the transmitter.

This work is dedicated to the solution of, or at least the minimization of, the foregoing three inherent shortcomings. A number of specific circular arrays that have appeared in the literature are included in the reference section. Few, however, are worthy of serious discussion, since these arrays all use simple elements such as dipoles or horns. Our objective was to obtain the absolute maximum gain possible while maintaining a high degree of pattern control over a specified bandwidth.

1.1 THESIS OBJECTIVES

- Develop and discuss analytical models of the circular array
- Conduct a study of all parameters affecting array performance
- Design a UHF circular array suitable for surveillance radar applications, having the following characteristics:
  1. Operating frequency: 400-450 MHz
  2. Polarization: horizontal
  3. System gain $\geq 20.5$ dBi (see discussion in section 4.4.1)
  4. Azimuth beamwidth $\leq 8.2^\circ$
  5. Elevation beamwidth $\leq 33^\circ$
  6. Spurious radiation $\geq 35$ dB down from peak
• Develop optimal transmit and receive beamforming algorithms

This thesis is organized as follows. In Chapter 2 is an analysis of the pattern function and directivity of arrays with isotropic (in the plane of the ring) or endfire elements. Chapter 3 develops the principles of the embedded element. Chapter 4 discusses a relevant receive-mode beamforming algorithm as well as the capabilities and the "tricks" associated with the linear least-squares method. Chapter 5 builds on the analysis of Chapter 2 by providing additional numerical results not accessible through mathematical means. Chapter 6 provides a complete description of the steps taken to arrive at a final design, as well as the modeling and measurement results. Chapter 7 presents a new method of commutating a tapered transmitter excitation, while making full use of the available transmitter power. Concluding remarks and contributions are provided in Chapter 8.
2 ANALYTICAL PATTERN FUNCTION STUDY

2.1 INTRODUCTION

Circular arrays are a viable solution to a variety of applications. Frequently these applications are defense-related, where either the available space or platform shape dictates the antenna geometry. Civilian applications requiring 360° coverage, such as ground based direction-finding arrays or wireless networks, are also prime candidates for the circular array. This chapter is concerned with configurations where the primary radiation is in the plane of the array. Its purpose is twofold. The first is to decompose the array factor (or pattern function in the case of endfire elements) into a collection of functions that are dependent on changing conditions, i.e., the delineation of non-zero functions when the element pattern, the number of elements, or the ring radius varies. The second is to determine whether a simple rule-of-thumb exists that relates the number of elements and the optimal (in terms of directivity) radius. With the exception of a directivity calculation in section 2.4.1, we deal only with the single ring array. For many applications, it makes sense to use endfire elements to reduce the required number of active elements. Reducing the number of elements has a payoff not only in terms of the necessary hardware required to control the array, but also in terms of the processing load. The computational load on an element-space adaptive radar processor is $O(MN^3)$, where $M$ is the number of active elements, and $N$ is the number of pulses in a coherent processing interval [5].

A survey of the literature on circular arrays reveals that many papers are aimed at achieving some measure of performance where the beam or beams are principally orthogonal to the ring plane [6]-[10]. Other works pertaining to field synthesis in the ring plane include [11], which extends the method of Woodward and Levinson to the circular array to obtain a prescribed pattern. A relationship between a desired sidelobe level and a number of concentric rings is developed in [12]. An iterative method of constrained synthesis is presented in [13]. Quiescent and adaptive beamforming algorithms are investigated in [14]. Additional examples of pattern synthesis can be found in [15]-[19]. Specific examples include a comparison between ring arrays composed of radially and
tangentially-oriented dipoles [20], a high-Q traveling wave structure with a potential for high directivity [21], and a direction-finding Wullenweber array [22]. None of the works listed address our specific objectives. Several works partly do, such as the one of Redlich [23], which states that the number of elements should be greater than $2ka +1$, where $a$ is the ring radius. His technique is based upon a sampling method, and observes that under this condition, spurious radiation arising from higher order terms is small. An optimality condition is not provided. An approximate method of representing the pattern function was given by Lee and Lo [24], but it is limited to isotropic elements and a full active ring. Shortly thereafter, Lo and Lee [25] derived a closed-form expression for the directivity. Their expression is limited to arrays using short dipole elements. Cheng and Tseng [26] extended this formulation to permit the consideration of the element pattern $g(\theta, \phi) = \sin^p \theta$, where $p$ is an integer. Unfortunately, the result is invalid for $p > 1$, yielding a directivity too great for $p = 2, 3$, etc. The expressions obtained in both [25] and [26] can also be found in Ma [27]. Additional discussions of the circular array can be found in [28] and [29].

To provide a sufficient basis for later discussions, we begin by reviewing the array factor for a full ring of isotropic elements in the next section. This is followed by pattern function discussions in section 2.3 for the full-ring, half-ring, and half-ring of endfire elements, respectively. Section 2.4 contains directivity expressions for arrays using isotropic and endfire elements, respectively. Selected examples are presented in section 2.5, and observations and conclusions are recorded in section 2.6.

2.2 ARRAY FACTOR REVIEW

Consider a circular array with $M$ isotropic elements as shown in Fig. 2.1. The electric field pattern can be expressed as

$$E(r, \theta, \phi) = \sum_{m=1}^{M} w_m \frac{e^{-jR_m}}{R_m},$$  \hspace{1cm} (2.1)
where $w_m$ is the complex excitation

$$w_m = I_m e^{j\alpha_m}. \quad (2.2)$$

and $R_m$ is the distance from the $m^{th}$ element to the observation point. In the far-field, this can be rewritten as

$$E(r, \theta, \phi) = \frac{e^{-jkr}}{r} F(\theta, \phi), \quad (2.3)$$

where $\phi_m' = 2\pi m / M$ is the angular position of the $m^{th}$ active element, and the array factor is

$$F(\theta, \phi) = \sum_{m=1}^{M} I_m e^{jk_a \sin \theta \cos (\phi - \phi_m')} e^{j\alpha_m}. \quad (2.4)$$

Figure 2.1—Coordinate system of circular array.
To point a beam in the \((\theta_0, \phi_0)\) direction, a \textit{cophasal} excitation can be selected such that

\[
\alpha_m = -ka \sin \theta_0 \cos (\phi_0 - \phi'_m) .
\]  

(2.5)

After Balanis [29], a new set of variables can be defined such that

\[
\rho = a \left[ (\sin \theta \cos \phi - \sin \theta_0 \cos \phi_0)^2 + (\sin \theta \sin \phi - \sin \theta_0 \sin \phi_0)^2 \right]^{1/2},
\]

(2.6)

\[
\xi = \tan^{-1} \left\{ \frac{\sin \theta \sin \phi - \sin \theta_0 \sin \phi_0}{\sin \theta \cos \phi - \sin \theta_0 \cos \phi_0} \right\} .
\]

(2.7)

Equations (2.6) and (2.7) facilitate a more compact form of (2.4):

\[
F(\theta, \phi) = \sum_{m=1}^{M} I_m e^{ik\rho \cos (\phi'_m - \xi)} .
\]

(2.8)

While not directly applicable to low-sidelobe applications or the endfire elements are used, considerable insight can be gained by considering the case where the elements are isotropic in the plane of the ring and the current magnitudes are equal. For this special case,

\[
F(\theta, \phi) = I_0 \sum_{m=1}^{M} e^{ik\rho \cos (\phi'_m - \xi)} .
\]

(2.9)

The generating function [30]

\[
e^{jz \cos \psi} = \sum_{q=-\infty}^{\infty} j^q e^{jq\psi} J_q (z) ,
\]

(2.10)
where $J_q(z)$ is the Bessel function of the first kind, is used to expand and reorder (2.9):

$$F(\theta, \phi) = I_0 \sum_{q=-\infty}^{\infty} j^q e^{-j\rho \xi} J_q(k \rho) \sum_{m=1}^{M} e^{\frac{j2\pi q m}{M}}.$$  \hspace{1cm} (2.11)

Consider the sum $\sum_{m=1}^{M} e^{\frac{j2\pi q m}{M}}$ and let $p = e^{\frac{j2\pi q}{M}}$; then

$$\sum_{m=1}^{M} p^m = \frac{p-p^{M+1}}{1-p} = \frac{p-p}{1-p} = 0 \text{ for } p \neq 1 \ (q \neq -M, 0, M, \ldots).$$

When $q = 0$ or $\pm kM$ the summation becomes

$$\sum_{m=1}^{M} 1 = M$$

allowing (2.11) to be rewritten as

$$F(\theta, \phi) = M I_0 \sum_{q=-\infty}^{\infty} e^{jqM(\pi/2-\xi)} J_{qM}(k \rho).$$  \hspace{1cm} (2.12)

2.3 ARRAY FACTOR DECOMPOSITION

2.3.1 Full-Ring, Isotropic (in the plane of the ring) Elements

The normalized array factor for a uniformly-spaced linear array (ULA) with equal amplitude excitations is

$$F(\psi) = \frac{\sin(M\psi/2)}{M \sin(\psi/2)}$$  \hspace{1cm} (2.13)
where $\psi = kd + \alpha$, $d$ is the inter-element spacing and $\alpha$ is the phase shift. Numerous antenna books (e.g., [29]) explain a graphical method in which the function $F(\psi)$ is plotted over some angular range, and a circle of radius $kd$ is drawn below it, having an offset of $\alpha$. To compute the array factor at any angle, a line can be drawn straight down from the graph of $F(\psi)$ onto the edge of the circle (visible region). Because $F(\psi)$ is periodic with major lobes at $\psi = 0, \pm 2\pi, \pm 4\pi$, etc., the condition $kd + \alpha \leq 2\pi(1 - 1/M)$ must be maintained in order to prevent any portion of the major lobes from appearing in the visible region. This method provides a clear mental picture about avoiding spurious radiation, and the shape of the function does not change with respect to its argument. The parameters $d$ and $\alpha$ change only the size and offset of the visible region which (2.13) projects onto. Unfortunately, a similar graphical method is not easily constructed for the circular array.

Suppose now we would like to use (2.12) in the same way the array factor in (2.13) was used for a ULA. Consider the argument $k\rho$. Since our primary interest is the antenna’s performance in the $x$-$y$ plane, (2.6) reduces to

$$\rho = a\left[\right. (\cos\phi - \cos\phi_0)^2 + (\sin\phi - \sin\phi_0)^2\left.\right]^{1/2}$$

$$= a\left[2 - 2\cos(\phi - \phi_0)\right]^{1/2} = 2a\left|\sin\left(\frac{\phi - \phi_0}{2}\right)\right| \tag{2.14}$$

Thus, the maximum value that $\rho$ can take on is $2a$. As it turns out, this is the maximum value that $\rho$ can take on for any $\theta$ and $\theta_0$.

Consider Fig. 2.2, where the radius is expressed in terms of the number of elements and inter-element spacing. Assume for now that the inter-element spacing should be $\lambda/2$; then
\[
\sin\left(\frac{\Delta \phi}{2}\right) = \frac{\lambda/4}{a}
\]

so that

\[
a = \frac{\lambda}{4 \sin\left(\frac{\pi}{M}\right)}.
\] (2.15)

We assume that \( M \geq 30 \), so that (2.15) becomes

\[
a \approx \frac{M \lambda}{4\pi},
\] (2.16)

with an error less than 0.2%.

Figure 2.2—Inter-element separation.

Then \( ka \approx M / 2 \), so that for \( \lambda / 2 \) inter-element spacing,

\[
\max[k \rho] \approx M.
\] (2.17)
We now re-examine the array factor in (2.12) in light of the result in (2.17) to determine the number of significant contributors to the series, \( J_{qM}(k\rho) \) is plotted in Fig. 2.3 for \( M = 30 \) and \( q = 0, 1, 2 \) (which shows that the significance of the higher-order terms is dependent not only upon \( M \), but also upon \( k\rho \)). The higher order Bessel functions are practically zero until the argument approaches the order. For large \( M \), the peak occurs at approximately \( k\rho = 1.6 + 1.02M \). Thus, at least the first-order terms \( (q = \pm 1) \) must be included in addition to the principal term \( (q = 0) \). We now inquire as to the range of argument for which the second-order and higher terms may be excluded. From Fig. 2.4(a) we see that for an argument of \( M \), the terms \( J_0(M) \) and \( J_{\pm M}(M) \) are non-zero, but that the higher order terms are zero for \( M \geq 5 \). In Fig. 2.4(b) the inter-element separation is increased to \( 0.9\lambda \), so that the maximum of \( k\rho = 1.8M \). Now the second-order and higher terms are negligible for \( M \geq 30 \).

Once more, the inter-element separation is increased to \( 1.0\lambda \) so that the maximum argument is equal to \( 2M \). Fig. 2.4(c) shows that the second-order term must be included,
which is decreasing slowly and monotonically as $M$ increases. With this in mind, (2.12) can be replaced by (assuming $M$ is even)

$$F(\theta, \phi) = M_0 \sum_{q=-1}^{1} e^{i q M (\pi/2 - \xi)} J_{qM} (k \rho)$$

$$= M_0 \left[ J_0 (k \rho) + 2 J_M (k \rho) \cos \left[ M \left( \frac{\pi}{2} - \xi \right) \right] \right] , \quad k \rho \leq 1.8 M . \quad (2.18)$$

Equation (2.18) can be modified by adding a third term if results for $k \rho > 1.8 M$ are desired. However, as will be demonstrated later, for inter-element spacings approaching a wavelength, spurious radiation becomes a significant problem, so that (2.18) is sufficient for our application.

Figure 2.4—Instead of plotting the Bessel function of fixed order as a function of its argument, the function is plotted such that the order and arguments are related to the number of elements. This permits a determination of the terms that contribute significantly in (2.12) for a given inter-element spacing. Subplots represent inter-element spacings of (a) $0.5 \lambda$, (b) $0.9 \lambda$, and (c) $1.0 \lambda$. 
This result can be further simplified for radiation patterns in the $x$-$y$ plane. From (2.14) we recall that

$$k \rho = 2ka \left| \sin \left( \frac{\phi - \phi_0}{2} \right) \right|$$ for $\theta = \theta_0 = 90^\circ$, and from (2.7),

$$\xi = \tan^{-1} \left( \frac{\sin \phi - \sin \phi_0}{\cos \phi - \cos \phi_0} \right) = \tan^{-1} \left( -\cot \frac{\phi + \phi_0}{2} \right),$$

so that

$$\xi = \frac{\phi + \phi_0}{2} - \frac{\pi}{2}. \quad (2.19)$$

Recalling that

$$\cos M (\pi - \xi) = \cos (M \xi), \quad M \text{ even} \quad (2.20)$$

(2.18) becomes

$$F(\phi) = M I_0 \left[ J_0 \left( 2ka \sin \frac{\phi + \phi_0}{2} \right) + 2J_M \left( 2ka \sin \frac{\phi + \phi_0}{2} \right) \cos \left[ M \left( \frac{\phi + \phi_0}{2} \right) \right] \right] \quad (2.21)$$

for $M$ even.

In order to study the natural "modes" of radiation from the full ring, one further step can be taken to simplify this result by letting the main beam be directed in the $x$-direction ($\phi_0 = 0$). In this case, neglecting constants, the radiation pattern is proportional to
Equation (2.22) lends significant insight into the formation of grating lobes by the ring array. Successive peaks of the \( J_0(k\rho) \) term decrease monotonically, so that there is no chance of grating lobe formation with only this term. The peak of \( J_M(k\rho) \) in Fig. 2.3 occurs at an argument of just slightly greater than \( M \), and for \( k\rho \) less than approximately \( 0.8M \), this term is zero. The argument is maximized at \( \phi = \pi \), meaning that grating lobes first appear at the rear, for \( 2ka \approx M \). Note that there is no contribution in the direction of the main beam arising from this term. The cosine term simply modulates the amplitude. As the radius increases, the \( J_M(k\rho) \) term decreases at \( \phi = \pi \), but is maximized at some point \( \phi < \pi \). Thus, (2.22) reveals the mechanism of grating lobe formation—first at the back, then moving towards the side. As the radius continues to increase, an additional grating lobe forms at the rear, this time smaller, resulting from the second maxima of \( J_M(k\rho) \). It too moves in the direction of the main beam as the radius is increased.

With this in mind, there are only two ways to prevent spurious radiation. The most obvious measure is to simply keep the radius \( a \) small enough to minimize the contribution from the second term in (2.22). As will be shown later, the ring radius can be set such that the directivity reaches a first maximum, which places the first peak of \( J_M(k\rho) \) outside of the visible region. The second measure is to increase \( M \). From Fig. 2.4(a), \( J_M(M) \) is seen to be a monotonically decreasing function; hence, even if grating lobes do appear, their significance is diminished. This makes the circular array fundamentally different from the linear array, whose grating lobe magnitude for a uniform amplitude array is independent of the number of elements.
2.3.2 Half-Ring, Isotropic (in the plane of the ring) Elements

Certain occasions may arise where a smaller portion of an array is utilized. Consider now the case of the half-ring, where $M/2$ elements are active, and $M$ is an integer divisible by 4. From the result in (2.11), we have

$$F(\theta, \phi) = I_0 \sum_{q=-\infty}^{\infty} j^q e^{-jq\xi} J_q(k\rho) \sum_{m=-M/4}^{M/4} e^{j2\pi q m/M}$$

(2.23)

With $M$ active elements, we had the convenience of the summation

$$\sum_{m=1}^{M} e^{j2\pi q m/M} = \begin{cases} M, & q = 0, \pm M, \pm 2M, \ldots \\ 0, & \text{otherwise} \end{cases}$$

However, we now find that

$$\sum_{m=-M/4}^{M/4} e^{j2\pi q m/M} = \begin{cases} \frac{M}{2}, & q = 0, \pm M, \pm 2M, \ldots \\ (-1)^{\frac{1}{2}(q-1)} \left[ j + \cot \left( \frac{\pi q}{M} \right) \right], & q = 1, \pm 3, \pm 5, \ldots \\ 0, & \text{otherwise} \end{cases}$$

(2.24)

The summation limits, selected to permit the simplest answer later, impose the requirement that $M$ must be divisible by 4. Thus, (2.23) becomes, after considering non-zero odd and even-order terms,
\[
F(\theta, \phi) = I_0 \left\{ \frac{M}{2} \sum_{q=0}^{\infty} \epsilon_q J_q M(k \rho) \cos \left[ qM \left( \frac{\pi}{2} - \xi \right) \right] + 2j \sum_{q=0}^{\infty} (-1)^q \cot \left[ \frac{\pi}{M} (2q + 1) \right] J_{2q+1}(k \rho) \sin \left[ (2q + 1) \left( \frac{\pi}{2} - \xi \right) \right] \right\} \tag{2.25}
\]

where \( \epsilon_q \) is the Neumann factor.

For large \( M (> 30) \), the first two lines of (2.25) dominate the function. The last line has a maximum amplitude of unity, since

\[
\cos(z \sin \theta) = 2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(z) \cos[(2k + 1) \theta]. \tag{2.26}
\]

Thus, the array factor can be approximated as

\[
F(\theta, \phi) \approx I_0 \left\{ \frac{M}{2} \sum_{q=0}^{\infty} \epsilon_q J_q M(k \rho) \cos \left[ qM \left( \frac{\pi}{2} - \xi \right) \right] + 2j \sum_{q=0}^{\infty} (-1)^q \cot \left[ \frac{\pi}{M} (2q + 1) \right] J_{2q+1}(k \rho) \sin \left[ (2q + 1) \left( \frac{\pi}{2} - \xi \right) \right] \right\} \tag{2.27}
\]

The above result is a generalization for the 3-D case where \( k \rho \leq 1.8 \). For the case of \( \phi_0 = 0 \), we can follow the argument leading to (2.22) and write that

\[
F(\phi) \propto J_0 \left( 2ka \sin \frac{\phi}{2} \right) + 2J_M \left( 2ka \sin \frac{\phi}{2} \right) \cos \left( \frac{M \phi}{2} \right) + 4j M \sum_{q=0}^{\infty} (-1)^q \cot \left[ \frac{\pi}{M} (2q + 1) \right] J_{2q+1} \left( 2ka \sin \frac{\phi}{2} \right) \sin \left[ \frac{\phi}{2} (2q + 1) \right] \tag{2.28}
\]
The top line is the familiar result containing both desirable (principal term) and undesirable (residual) portions. The imaginary portion of (2.28) contributes entirely to spurious radiation. A close examination reveals that the spurious contribution resulting from the series is not easily predicted, and is present in some form regardless of the number of elements or inter-element spacing. Although the argument of each odd-order Bessel term is a function of the radius, the coefficients, which are determined by the cotangent function, are not. Fig. 2.5 reveals that the largest coefficients occur for $q$ near $0, M/2, M$, etc. It is quite impossible to zero, or come close to zeroing, all components of the series.

Fig. 2.6 shows the radiation arising from each of the three principal components for a specific configuration. In this instance, the inter-element spacing is approximately $0.6 \lambda$. Notice that the radiation arising from line 2 of (2.28) appears in virtually the entire visible region, except near the main beam. To summarize, while the spurious contribution from the $J_M(k\rho)$ term can be largely suppressed by increasing $M$, the contribution resulting from the usage of only one-half of the array cannot.

$$\cot \left( \frac{\pi}{M} (2q+1) \right)$$ for (a) $M = 40$, and (b) $M = 60$. 

Figure 2.5——
Figure 2.6—Contributions to field pattern from zero-order term (solid line), from $M^{th}$ order term (broken line), and series of terms (dotted). The only term contributing to the main lobe is the zero-order term. The contribution from the $M^{th}$ order term can be eliminated by maintaining $k\rho \leq 0.4$, and reduced in general by increasing $M$. The series arises from utilizing half of the array, and is present regardless of the number of elements or inter-element spacing. $M = 40$, $\lambda = 1$, $a = 4$.

2.3.3 Half-Ring, Endfire Elements

The full-ring and half-ring cases with isotropic elements provide considerable insight into the behavior of the circular array. However, no complete investigation of single ring circular arrays would exclude the analysis of the case of the half-ring of endfire elements. In many instances, arrays of endfire elements are used to minimize the number of elements requiring control, and to provide a means of compressing the elevation beamwidth. To avoid unnecessary complication, consider an element pattern that is analytically specified over $360^\circ$. In addition, we avoid selecting an element pattern that
has squared trigonometric terms, or worse yet, raised to an arbitrary exponent. Let the element pattern be

\[ g(\phi) = 1 + c \cos(\phi - \phi') \quad 0 \leq c \leq \infty. \]  

The element pattern is plotted in Fig. 2.7 for several values of \( c \):

![Figure 2.7—The analytical element pattern for c = 1.0 (a), 1.5 (b), 2.0 (c), and 2.5 (d).](image)

Eq. (2.29) provides a "good" element pattern, although the ability to adjust it is limited. For example, with \( c = 1.5 \), the radiation at right angles to the peak is down by approximately 8 dB, while the backlobe is down 14 dB. The half-power beamwidth is roughly 120°, which is well within the range of realistic values available for a circular arrangement of endfire parasitic elements.
The derivation of the pattern function is provided in Appendix A. The result is given below:

\[
F(\phi) = \frac{M}{2} \sum_{q=0}^{\infty} \varepsilon_q J_{qM}(k \rho) \cos \left( \frac{qM \phi}{2} \right) + 2 \sum_{q=0}^{\infty} (-1)^q J_{q+1}(k \rho) \cos \left( \frac{\phi}{2} (2q + 1) \right)
\]

\[
+ 2j \sum_{q=0}^{\infty} (-1)^q \cot \left( \frac{\pi}{M} (2q + 1) \right) J_{2q+1}(k \rho) \sin \left( \frac{\phi}{2} (2q + 1) \right) - c \sum_{q=0}^{\infty} \varepsilon_q (-1)^q J_{2q}(k \rho) \cos(q \phi) \sin \phi
\]

\[
+ \frac{c}{2} \sum_{q=0}^{\infty} \varepsilon_q (-1)^q J_{2q}(k \rho) \cot \left( \frac{\pi}{M} (2q + 1) \right) \cos[\phi(q + 1)] - \cot \left( \frac{\pi}{M} (2q - 1) \right) \cos[\phi(q - 1)] \right)
\]

\[
+ \frac{j c M}{2} \left\{ \sum_{q=0}^{\infty} J_{qM+1}(k \rho) \sin \left( \frac{\phi}{2} (qM - 1) \right) + \sum_{q=1}^{\infty} J_{qM-1}(k \rho) \sin \left( \frac{\phi}{2} (qM + 1) \right) \right\}
\]

As before, the above derivation assumes that \( \phi_0 = 0 \), and that only patterns in the plane of the ring are considered, so that \( k \rho = 2k \sin(\phi/2) \) (we avoid writing this at each instance to save space). Several terms possess the same form as in (2.28), which has been discussed previously. More terms are also introduced which cloud the picture, making an intuitive grasp nearly impossible. However, it should be noted that lines 3 and 4 contain the \( J_0(k \rho) \) term that contributes to the main beam. Although the closed form solution has become unwieldy, a compact closed form expression for the directivity of such an array is developed in the following section.

2.4 DIRECTIVITY

2.4.1 Isotropic (in the plane of the ring) Elements

The directivity of an arbitrary antenna is given by [29] to be
A closed-form solution can be obtained for the following element pattern:

\[ g(\theta, \phi) = \sin^p \theta, \quad p = 0, 1, 2 \ldots \]  

(2.32)

For \( p = 1 \), \( g(\theta, \phi) \) takes on the pattern shape of an infinitesimal dipole. For a beam formed on the horizon (\( \theta_o = 90^\circ \)), the numerator is simply

\[ |F(\theta_o, \phi_t)|^2 = \left( \sum_{m=1}^{M} I_m \right)^2. \]  

(2.33)

Consider now the denominator of (2.31). It can be integrated by summing the product of all possible element pairs over all space:

\[ W = \sum_{m=1}^{M} \sum_{n=1}^{M} I_m I_n e^{j(\alpha_n - \alpha_m)} X_{mn}, \]  

(2.34)

where

\[ X_{mn} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi e^{jk \sin \theta [\cos(\phi - \phi'_n) - \cos(\phi - \phi'_m)]} \sin^{2p+1} \theta \, d\theta \, d\phi \]

\[ = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi e^{jk \sin \theta \sin \left\{ \frac{-\phi'_n + \phi'_m}{2} \right\}} \sin^{2p+1} \theta \, d\theta \, d\phi \]  

(2.35)

with
\[ z_{mn} = 2ka \sin \left( \frac{\phi_m - \phi_n}{2} \right). \] (2.36)

The directivity is thus

\[ D = \left( \sum_{m=1}^{M} I_m \right)^2 \frac{W}{W}. \] (2.37)

Using the fact that [30]

\[ \int_0^{2\pi} e^{jz \sin \alpha} d\alpha = 2\pi J_0(z), \]

we can write (2.35) as

\[ X_{mn} = \int_0^{\pi} J_0(z_{mn} \sin \theta) \sin^{2\nu+1} \theta d\theta. \] (2.38)

To evaluate, the following identity from [30] is used:

\[ \int_0^\pi J_\mu(z \sin \theta) \sin^{\mu+1} \theta \cos^{2
\nu+1} \theta \ d\theta = \frac{2^\nu \Gamma(n+1)}{2^{\nu+1}} J_{\mu+n+1}(z). \] (2.39)

The sine term may be converted into a cosine and expanded by the binomial theorem:

\[ \sin^{2\nu} \theta = (1 - \cos^2 \theta)^\nu = \sum_{i=0}^{\nu} \frac{(-1)^i p! \cos^{2i} \theta}{(p-i)! i!} \] (2.40)

which after insertion into (2.38) yields
Finally, letting $\mu = 0$ and $\nu = i - \frac{1}{2}$, we get

$$X_{mn} = \frac{p!}{\sqrt{2z_{mn}}} \sum_{i=0}^{p} \frac{(-1)^i 2^i \Gamma(i + \frac{1}{2})}{i!} J_{i+\frac{1}{2}}(z_{mn}) .$$

(2.42)

With the following relation [31], the gamma function can be eliminated:

$$\Gamma(i + \frac{1}{2}) = \frac{(2i)!\sqrt{\pi}}{i!2^{2i}} .$$

Finally,

$$X_{mn} = \frac{p!\sqrt{\pi}}{\sqrt{2z_{mn}}} \sum_{i=0}^{p} \frac{(-1)^i (2i)!}{(p - i)!2^i (i!)^2 z_{mn}^i} J_{i+\frac{1}{2}}(z_{mn}) .$$

(2.43)

Although the focus of this Chapter is on single ring arrays, the extension to concentric ring arrays is trivial:

$$D = \left( \sum_{q=1}^{Q} \sum_{m=1}^{M} I_{m}^{q} \right)^2$$

(2.44)

where $Q$ is the number of rings, and the superscript $q$ pertains to any quantity associated with the $q^{th}$ ring. Note that neither (2.37) nor (2.44) are restricted to the full array.
2.4.2 Endfire Elements

The endfire element in (2.29) can also be considered; for an element pattern of

\[ g(\theta, \phi) = [1 + c \cos(\phi - \phi')] \sin^p \theta, \]

(2.45)

the integral in the denominator takes the form of

\[ X_{mn} = \int_0^{2\pi} \int_0^\pi [1 + c \cos(\phi - \phi_m')] [1 + c \cos(\phi - \phi_n')] \sin^{2p+1} \theta e^{j2\alpha c \sin \theta \sin(\phi_m - \phi')} \sin(\phi - \phi_m') d\theta d\phi \]

(2.46)

The absolute value presents a difficulty, except when \( 0 \leq c \leq 1 \), where the absolute value may be omitted. For \( c = 1 \), (2.45) represents a realistic element pattern in the azimuth plane (refer to Fig. 2.7(a)). The 40-element array shown in Fig 2.8 has an embedded element pattern as shown to its right. Both patterns have 3 dB beamwidths between 135° and 150°, 6 dB beamwidths of approximately 180°, and negligible radiation from the back. An oblong element pattern having no nulls can be created by setting \( c < 1 \). The integration is performed below for the special case of \( c = 1 \).

![Figure 2.8](image)

(a) (b)

Figure 2.8—(a) 40-element array of endfire elements. (b) Embedded (active) element pattern.
The integration on the right side of (2.46) is performed by multiplying the $\phi$-related terms of $g(\theta, \phi)$ and breaking the integral into 3 parts:

$$X_{mn} = X_{mn}^{a} + X_{mn}^{b} + X_{mn}^{c},$$  \hspace{1cm} (2.47)

where $X_{mn}^{a}$ is the portion of the integral arising from the constant term $1 + \frac{1}{2}\cos(\phi'_{m} - \phi'_{n})$, $X_{mn}^{b}$ is the portion arising from $\cos(\phi - \phi'_{m})$ and $\cos(\phi - \phi'_{n})$, and $X_{mn}^{c}$ is the portion from $\frac{1}{2}\cos(2\phi - \phi'_{m} - \phi'_{n})$. From the developments in the previous section,

$$X_{mn}^{a} = \left[1 + \frac{1}{2}\cos(\phi'_{m} - \phi'_{n})\right] \frac{\sqrt{\pi} p!}{\sqrt{2} z_{mn}} \sum_{i=0}^{p} \frac{(-1)^{i} (2i)!}{(p-i)! (2i)! z_{mn}^{i}} J_{i+\frac{1}{2}}(z_{mn}).$$  \hspace{1cm} (2.48)

Consider now $X_{mn}^{b}$, which takes the form of

$$\frac{1}{4\pi} \int_{0}^{\pi} \int_{0}^{\pi} \cos(\phi - \phi'_{m}) \sin^{2p+1} \theta e^{j2\kappa \sin\theta \sin\left(\frac{\phi - \phi'_{m}}{2}\right)} d\theta d\phi. \hspace{1cm} (2.49)$$

It can be demonstrated that

$$\int_{0}^{\pi} \cos(\phi - \phi_{0}) e^{jz\sin(\phi - \phi_{0})} d\phi = -j2\pi J_{1}(z) \sin(\phi_{0} - \phi_{0}). \hspace{1cm} (2.50)$$

Eq. (2.50) allows us to write for the $\cos(\phi - \phi'_{m})$ and $\cos(\phi - \phi'_{n})$ terms that

$$X_{mn}^{b} = X_{mn}^{b1} + X_{mn}^{b2}$$

$$= -j2\pi J_{1}(z_{mn} \sin \theta) \sin \left(\frac{\phi_{m} + \phi'_{m}}{2} - \phi_{n}'\right) - j2\pi J_{1}(z_{mn} \sin \theta) \sin \left(\frac{\phi_{m} + \phi'_{m}}{2} - \phi'_{n}\right) = 0. \hspace{1cm} (2.51)$$

Finally the integral
\[ X_{mn}^c = \frac{1}{8\pi} \int_0^\pi \int_0^{2\pi} \cos(2\phi - \phi'_m - \phi'_n) \sin^2\theta J_0(2kz\sin\theta\sin(\phi'_m - \phi'_n)) \sin(\phi - \phi'_m) d\theta d\phi \]  \hspace{1cm} (2.52)

is evaluated by using the relation

\[ \int_0^{2\pi} \cos(2\phi - \phi_0) e^{jz\sin(\phi-\phi_0)} d\phi = 2\pi J_1(z) \cos(2\phi_1 - \phi_0) \]  \hspace{1cm} (2.53)

to write (2.52) in a form that is readily equated to (2.39):

\[ X_{mn}^c = \frac{1}{2} \int_0^{\pi/2} J_2(z_{mn} \sin\theta) \sin^3\theta \sin^{2(p-1)}\theta d\theta d\theta . \]  \hspace{1cm} (2.54)

The binomial theorem can be used in a manner akin to that used in computing \( X_{mn}^a \) to write

\[ X_{mn}^c = \frac{\sqrt{\pi} (p-1)!}{\sqrt{2z_{mn}}} \sum_{i=0}^{\pi/2} (-1)^i (2i)! \frac{(2i)!}{(p-1-i)!2^i (i)!^2} J_{i+2}\left( z_{mn} \right) . \]  \hspace{1cm} (2.55)

Thus,

\[ D = \left( \sum_{m=1}^{M} \sum_{n=1}^{M} I_{m} I_{n} e^{j(\alpha_m - \alpha_n)} (X_{mn}^a + X_{mn}^c) \right)^2 . \]  \hspace{1cm} (2.56)

2.5 EXAMPLES

Equations (2.37) and (2.56) are plotted below for a variety of array configurations as a function of the array size. In each case, the directivity reaches a first peak when the inter-element spacing is approximately \( \lambda/2 \). For increasing radius, the directivity falls
because significant energy is contained in spurious radiation. A second peak typically occurs at a radius approximately twice that of the first peak.

Figure 2.9—First directivity maximum occurs at approximately 0.47-0.49 $\lambda$ inter-element spacing. $\lambda = 1$ meter, isotropic elements, full ring active.
Figure 2.10—Directivity of 60-element array of isotropic (in the ring plane) elements. \( \lambda = 1 \) meter, \( p = 2 \) (66° elevation beamwidth).

Figure 2.11—Directivity of 60-element array of isotropic (in the ring plane) elements. \( \lambda = 1 \) meter, \( p = 9 \) (32° elevation beamwidth).
Figure 2.12—Directivity of 40-element endfire array $\lambda = 1$ meter, $p = 2$ (66° elevation beamwidth).

Figure 2.13—Directivity of 40-element endfire array. $\lambda = 1$ meter, $p = 7$ (36° elevation beamwidth).
Figure 2.14—Directivity of 80-element endfire array. $\lambda = 1$ meter, $p = 2$ (66° elevation beamwidth).

Figure 2.15—Directivity of 80-element endfire array. $\lambda = 1$ meter, $p = 7$ (36° elevation beamwidth).
This chapter describes in some detail the fundamental radiating properties of the circular array through its plane. We have shown that the simplest case to analyze is the fully active ring of isotropic elements. By utilizing less than the full ring, additional spurious terms are introduced. In addition, a spurious term is introduced into the pattern function when the inter-element spacing is greater than about 0.4 $\lambda$. This does not imply that the first maximum of directivity occurs here; it appears to occur at very nearly $\lambda/2$ inter-element spacing, regardless of the number of elements or element pattern, or the elevation beamwidth. A second directivity maximum typically occurs at about twice the radius of the first maximum. However, at this point, the spurious radiation can be quite severe. The most complicated case is the half-ring of endfire elements, where the sheer number of terms in the pattern function precludes an intuitive understanding.

Of course, for isotropic elements, the maximum directivity is realized with all elements active. More interesting is the question of the optimal number of active elements for the array of endfire elements. Figs. 2.12-2.15 show that for an element pattern of $g(\phi) = 1 + c \cos(\phi - \phi') \sin^\nu \theta$, at least one-half of the array can be used to realize the maximum directivity. While the "sweet spot" occurs at approximately the same ring radius, the number of active elements required to maximize directivity does not remain constant. Comparing Figs. 2.12 with 2.13 and 2.14 with 2.15 shows that with narrower elevation beamwidths, a smaller active portion of the array maximizes directivity.
3 THE EMBEDDED (ACTIVE) ELEMENT

3.1 INTRODUCTION

In subsequent chapters, numerous references to embedded (or active) elements will be made. Before proceeding, the theory and implications of this critical but overlooked topic will be addressed. We will develop a sufficient foundation to provide a basis for discussions of beamforming methods later. This chapter will focus primarily on three themes:

1) the validity and utility of the linear network concept for array characterization,
2) the invariance of the array pattern to individual source/element mismatch,
3) the impact of load (source impedance) variations on element performance, and the necessity of accurate knowledge of the actual loads used.

3.2 BACKGROUND

As pointed out by Pozar [32], the concept of the active (embedded) element pattern goes back at least 30 years. While ordinary array theory neglects mutual coupling, antenna engineers realized long ago that the pattern characteristics of an antenna element mounted in its array environment are generally different than that of the isolated element. This difference is caused by the adjacent coupling to and re-radiation by its neighbor elements. The embedded element concept is essential for beamforming purposes, as it provides a means of incorporating the effects from neighbor elements and decoupling one from another. Once decoupled, the element patterns can be combined using the principle of linear superposition. Armed with knowledge of the embedded element patterns, the excitations can be computed and the array pattern can be precisely determined without ever requiring use of the electromagnetic simulation again.

There appears to be considerable misunderstanding with respect to this topic not only in the signal processing community, but among antenna engineers as well. This was apparent from a workshop [33] dedicated to the development of space-time adaptive
processing (STAP) algorithms for the circular array. Virtually every presenter stated that if funded, they would develop techniques to counteract the effects of mutual coupling. In addition, a question arose which seemingly no one could answer. The question, in essence, as follows: "Assume that a low-sidelobe excitation is created from a set of embedded element patterns. Given a circular array, where the individual reflection coefficients change as function of the element location, how does one arrive at the excitation correction?" The correct answer to this question is that no correction is required. Given an element pattern that incorporates all mutual coupling throughout the array and the damping effect of the source impedances, the excitations can be computed once, and the patterns predicted accurately, without regard to source-element mismatch. Poor element/source power transfer (high VSWR) manifests itself in a reduction of system gain, but does not affect the shape of the array pattern.

Several references dealing with the embedded element should be discussed briefly. The first textbook to include a discussion was that of Rudge et al., [34] in a chapter contributed by Hansen. He provided the following formula, allowing for the calculation of array gain:

$$G(\theta) = \frac{NR_i g_i(\theta) h^2(\theta)}{R_a(\theta)} \left[1 - |\Gamma(\theta)|^2\right]$$  \hspace{1cm} (3.1)

where \(N\) is the number of array elements, \(R_i\) is the isolated element resistance, \(g_i\) is the isolated element pattern, \(h\) is the normalized array factor, \(R_a\) is the active element resistance, and \(\Gamma\) is the complex reflection coefficient. Also, a relationship between the isolated and active element pattern is drawn:

$$g_a(\theta) = \frac{R_g g_i(\theta)}{R_a(\theta)} \left[1 - |\Gamma(\theta)|^2\right].$$  \hspace{1cm} (3.2)
Both of the foregoing equations are based upon an open-circuited element pattern approach, and assume a linear array with similar reflection coefficients (a large array). These restrictions prohibit the consideration of arrays with arbitrary geometry. Furthermore, this approach does not lend itself readily to measurement, since the active element pattern is not measured directly, but requires the accurate measurement of active and isolated element resistance.

The other derivation of a relationship between isolated and active element patterns was provided by Pozar [32]. In contrast, Pozar represented the array by a scattering matrix, with a resistance placed in series with each generator to simulate the source impedance. Again, the derivation assumes a large uniformly spaced linear array.

Neither of the aforementioned derivations is suitable for arbitrary arrays, and neither directly addresses the question regarding the excitation correction.

Consider the $N$-element array as shown below. No assumptions are made regarding the array, so that each element can be unique, as can the sources or loads. In addition, the array may be arranged in an arbitrary geometry. Various proximity effects may occur, such as pattern broadening, which is often desirable for arrays, or the appearance of nulls, which can cause scan blindness. A framework that can explain the most fundamental properties of the embedded element pattern is desired.

![Figure 3.1—Array with arbitrary voltage sources, loads, and geometry.](image-url)
3.3 NETWORK BASICS AND EQUIVALENCE TO MoM

In order to explain the unique characteristics of the embedded element, established rules of linear network analysis are utilized. A variety of means have been devised to characterize linear networks, including impedance (Z), admittance (Y), hybrid (G and H), and scattering (S) parameters. Regardless of the method used, an $N$-port network is completely specified by an $N \times N$ matrix, whereby the output at any port is determined by multiplying the network matrix by an $N \times 1$ excitation vector. One method may possess one or more advantages for a specific application. S-parameters, for instance, have found great favor with microwave engineers in the last 30 years, as they permit measurement of active components under true load conditions, avoid the need for open or short circuits, etc. [35].

For our application, the Z and Y-parameters provide a convenient mechanism by which to show that no correction to the excitation is required, regardless of the power reflected from one or more elements.

Consider the two-port network shown. Internally, the device may contain any number of components, but the voltage-current relations are determined by the following equations:

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
\]  

(3.3)
where

\[
Z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2 = 0} = \text{input impedance at port 1 with port 2 open-circuited} \tag{3.4a}
\]

\[
Z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1 = 0} = \text{open-circuit voltage produced at port 1 from a current at port 2} \tag{3.4b}
\]

\[
Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2 = 0} = \text{open-circuit voltage produced at port 2 from a current at port 1} \tag{3.4c}
\]

\[
Z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1 = 0} = \text{input impedance at port 2 with port 1 open-circuited} \tag{3.4d}
\]

\[Z_{11}\text{ and } Z_{22}\text{ are input, or }\textit{self}\text{ impedances, and } Z_{12}\text{ and } Z_{21}\text{ are mutual impedances—sometimes referred to as }\textit{transfer}\text{ impedances. A physical interpretation of a transfer impedance is that it represents a voltage developed at one point due to a current induced at another.}\]

The problem can also be formulated in terms of voltage sources and short-circuit currents with the aid of the admittance matrix \(Y\):

\[
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} =
\begin{bmatrix}
Y_{11} & Y_{12} \\
Y_{21} & Y_{22}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
\tag{3.5}
\]

where

\[
Y_{11} = \left. \frac{I_1}{V_1} \right|_{V_2 = 0} = \text{input admittance at port 1 with port 2 short-circuited} \tag{3.6a}
\]
\[ Y_{12} = \frac{I_1}{V_2} \bigg|_{v_1=0} = \text{short-circuit current through port 1 from a voltage at port 2} \quad (3.6b) \]

\[ Y_{21} = \frac{I_2}{V_1} \bigg|_{v_2=0} = \text{short-circuit current through port 2 from a voltage at port 1} \quad (3.6c) \]

\[ Y_{22} = \frac{I_2}{V_2} \bigg|_{v_1=0} = \text{input admittance at port 2 with port 1 short-circuited} \quad (3.6d) \]

The Y matrix is the inverse of the Z matrix.

Consider now the moment method (MoM) for electromagnetics, which provides a means of determining surface currents, impedances, and radiation patterns for metallic and dielectric structures. One example of a simple MoM formulation is the solution of the Electric Field Integral Equation (EFIE) for a thin wire antenna. If the diameter of the wire is small compared to wavelength and also to the length of the wire, then some simplifying physical assumptions can be made. The first is that the contribution from the bases of the wire can be ignored in comparison to that of the lateral surface. The second is that the current is directed along the \( z \)-axis, and that it does not vary with the azimuthal angle \( \phi \). The third assumption is that the current does not flow along the surface of the wire but along its axis; thus, if we introduce cylindrical coordinates \((\rho, \phi, z)\) and let \( a \) be the radius of the wire, we can write for the current density on the lateral surface that

\[ \mathbf{J}(a, \phi, z) = \frac{I(0,0,z)}{2\pi a} \hat{z} = \frac{I(z)}{2\pi a} \hat{z} \quad (3.7) \]

where \( I \) is the current flowing along the axis of the wire. The current is equal to zero at both ends of the wire. From the Stratton-Chu representations of the electromagnetic fields [36], the \( z \)-component of the electric field can be expressed as
\[
\hat{z} \cdot \mathbf{E}(\mathbf{R}') = jkZ \int_{S_a} \hat{z} \cdot \mathbf{J}(\mathbf{R}) g(\mathbf{R}, \mathbf{R}') dS + jZ \frac{\partial}{\partial z'} \int_{S_a} \nabla \cdot \mathbf{J}(\mathbf{R}) g(\mathbf{R}, \mathbf{R}') dS, \ R' \in V \tag{3.8}
\]

where \( S_a \) is the lateral surface of the wire, \( Z = \sqrt{\mu_0 / \varepsilon_0} \), \( k = 2\pi / \lambda \), and \( V \) is the region of space exterior to the surface of the antenna. The observation point \( \mathbf{R}' \) is in space while the integration point \( \mathbf{R} \) is the point \((0,0,z)\). Thus, the Green’s function reads

\[
g(\mathbf{R}, \mathbf{R}') = -\frac{e^{-jk\sqrt{\rho^2 + (z-z')^2}}}{4\pi \sqrt{\rho^2 + (z-z')^2}} \tag{3.9}
\]

By substitute this and (3.7) in (3.8), and integrating in the angular direction, we get

\[
\hat{z} \cdot \mathbf{E}(\mathbf{R}') = -j\frac{kZ}{4\pi} \int_{l} I(z(z)) \frac{e^{-jk\sqrt{\rho^2 + (z-z')^2}}}{\sqrt{\rho^2 + (z-z')^2}} \, dz
\]

\[
- \frac{jZ}{4\pi k} \frac{\partial}{\partial z'} \int_{l} \frac{\partial I(z(z))}{\partial z} \frac{e^{-jk\sqrt{\rho^2 + (z-z')^2}}}{\sqrt{\rho^2 + (z-z')^2}} \, dz, \ R' \in V \tag{3.10}
\]

On the surface of the wire, the tangential component of the electric field is equal to zero except at the gap (origin). There, a harmonic in time \((e^{j\omega t})\) voltage is applied and is denoted by \( V_0 \). It is the result of an impressed uniform electric field given by

\[
\mathbf{E}^I = -\hat{z} V_0 \delta(z) \tag{3.11}
\]

where \( \delta \) is the Dirac \( \delta \)-function. Thus, from (3.10) we can write

\[
V_0 \delta(z') = \frac{jkZ}{4\pi} \int_{l} I(z(z)) \frac{e^{-jk\sqrt{a^2 + (z-z')^2}}}{\sqrt{a^2 + (z-z')^2}} \, dz
\]

\[
+ \frac{jZ}{4\pi k} \frac{\partial}{\partial z'} \int_{l} \frac{\partial I(z(z))}{\partial z} \frac{e^{-jk\sqrt{a^2 + (z-z')^2}}}{\sqrt{a^2 + (z-z')^2}} \, dz, |z'| \leq l \tag{3.12}
\]
This is the integro-differential equation for a thin wire dipole antenna and can be solved numerically using the MoM. The two arms of the wire are subdivided into \( N \) parts each, not necessarily equally, and the set of dividing points is

\[
\{-l = z_{-N}, z_{-N+1}, \ldots, z_{-1}, 0, z_1, z_2, \ldots, z_{N-1}, z_N = l\}
\]  

(3.13)

With respect to these points, triangular basis functions are defined:

\[
f_n(z) = \begin{cases} 
\frac{z - z_{n-1}}{z_n - z_{n-1}}, & z_{n-1} \leq z \leq z_n \\
\frac{z_{n+1} - z}{z_{n+1} - z_n}, & z_n \leq z \leq z_{n+1}, \\
0, & \text{elsewhere}
\end{cases}
\]  

(3.14)

and the current expressed as a linear combination of these

\[
I(z) = \sum_{n=-N+1}^{N-1} I_n f_n(z)
\]  

(3.15)

The same functions are used as testing functions and an inner product is defined according to

\[
(f, g) = \int_{-l}^{l} f(z) g^*(z) \, dz
\]  

(3.16)

Applying this to (3.12) we get

\[
\int_{-l}^{l} V_0 \delta(z') f_n(z') \, dz' = \int_{-l}^{l} A(z') f_n(z') \, dz' + \int_{-l}^{l} \frac{\partial \Phi(z')}{\partial z} f_n(z') \, dz',
\]  

\[
n = -N + 1, -N + 2, \ldots, -1, 0, 1, 2, \ldots, N - 2, N - 1
\]  

(3.17)
where

\[
A(z') = \frac{jkZ}{4\pi} \int_l I(z) e^{-j k \sqrt{a^2 + (z-z')^2}} \, dz, \quad \Phi(z') = \frac{jZ}{4\pi k} \int_l \frac{\partial I(z)}{\partial z} e^{-j k \sqrt{a^2 + (z-z')^2}} \, dz
\]  

(3.18)

We next integrate the second integral in (3.17) by parts to get

\[
\int \frac{\partial \Phi(z')}{\partial z'} f_n(z') \, dz' = \Phi(z') f_n(z') \bigg|_{z_n}^{z_{n+1}} - \int_{z_{n+1}}^{z_n} \Phi(z') \frac{\partial f_n(z')}{\partial z'} \, dz'
\]

\[
= -\frac{1}{z_n - z_{n-1}} \int_{z_{n+1}}^{z_n} \Phi(z') \, dz' + \frac{1}{z_{n+1} - z_n} \int_{z_n}^{z_{n+1}} \Phi(z') \, dz' = \Phi(z_n^+) - \Phi(z_n^-)
\]

(3.19)

where

\[
z_n^+ = (z_n + z_{n+1})/2, \quad z_n^- = (z_{n-1} + z_n)/2
\]

(3.20)

are the midpoints of the respective sub-intervals. For the first integral in (3.11) we write

\[
\int A(z') f_n(z') \, dz' = \int_{z_{n-1}}^{z_n} A(z') f_n(z') \, dz' + \int_{z_n}^{z_{n+1}} A(z') f_n(z') \, dz'
\]

\[
= \int_{z_{n-1}}^{z_n} A(z') \frac{z - z_{n-1}}{z_n - z_{n-1}} \, dz' + \int_{z_n}^{z_{n+1}} A(z') \frac{z_{n+1} - z}{z_{n+1} - z_n} \, dz'
\]

\[
\approx A(z_n^-) \frac{z_n - z_{n-1}}{2} + A(z_n^+) \frac{z_{n+1} - z_n}{2}
\]

(3.21)

Substituting (3.19) and (3.21) in (3.17), we get the system of equations

\[
V_0 = -A(z_0^-) \frac{z_0 - z_{-1}}{2} + A(z_0^+) \frac{z_1 - z_0}{2} + \Phi(z_0^+) - \Phi(z_0^-)
\]

(3.22)
and

\[ 0 = A(z_n^-) \frac{z_n - z_{n-1}}{2} + A(z_n^+) \frac{z_n - z_{n+1}}{2} + \Phi(z_n^+) - \Phi(z_n^-), \quad n = \pm 1, \pm 2, \ldots, \pm (N-1) \]  

(3.23)

Using the definitions (3.18) and substituting in them the current expansion (3.15), we write

\[ A(z_n^+) = \sum_{m=-N+1}^{N-1} I_m A_{nm}^\pm, \quad \Phi(z_n^+) = \sum_{m=-N+1}^{N-1} I_m \Phi_{nm}^\pm \]  

(3.24)

where

\[ A_{nm}^\pm = \frac{jkZ}{4\pi} \int_{z_{nm}}^{z_{n+1}} f_m(z) \frac{e^{-jk\sqrt{a^2 + (z-z_n^\pm)^2}}}{\sqrt{a^2 + (z-z_n^\pm)^2}} \, dz, \quad \Phi_{nm}^\pm = \frac{jZ}{4\pi k} \int_{z_{nm}}^{z_{n+1}} \frac{\partial f_m(z)}{\partial z} \frac{e^{-jk\sqrt{a^2 + (z-z_n^\pm)^2}}}{\sqrt{a^2 + (z-z_n^\pm)^2}} \, dz \]  

(3.25)

From (3.14),

\[ \frac{\partial f_n(z)}{\partial z} = \begin{cases} 
1, & z_{n-1} \leq z \leq z_n \\
\frac{1}{z_n - z_{n-1}}, & z_{n-1} \leq z \leq z_n^-
\end{cases}, \quad z_n \leq z \leq z_{n+1}, \quad n = -N+1, \ldots, -1, 0, 1, 2, \ldots, N-1 \]  

(3.26)

Substituting (3.24) in (3.22) and (3.23) we get

\[ \sum_{m=-N+1}^{N-1} I_m \left( \frac{z_1 A_{0m}^+ - z_{-1} A_{0m}^-}{2} + \Phi_{0m}^+ - \Phi_{0m}^- \right) = V_0 \]  

(3.27)

and
\[
\sum_{m=-N+1}^{N-1} I_m \left( \frac{z_n - z_{n-1}}{2} A_{nm}^- + \frac{z_{n+1} - z_n}{2} A_{nm}^+ + \Phi_{nm}^+ - \Phi_{nm}^- \right) = 0, \quad n = \pm 1, \pm 2, \ldots, \pm (N-1) \tag{3.28}
\]

This is the system of \(2N-1\) equations for determining the coefficients of the current expansion. In matrix notation

\[
ZI = V \tag{3.29}
\]

where \(I\) is the \(2N-1\) column vector of the unknown current coefficients, \(Z\) is the \((2N-1) \times (2N-1)\) matrix of the mutual and self interactions of the different segments of the antenna, and the right-hand side is a \(2N-1\) column vector of applied voltages. The number of non-zero elements of the voltage vector \(V\) are equal to the number of sources. The values of the non-zero elements are equal to the applied voltages and their position in the column vector corresponds to their placement in the wire. The integrals in (3.25) can be computed in a variety of ways. The MoM just described is basically the one of Rao, Wilton, and Glisson [37].

Equation (3.29) takes the same form as Kirchoff’s equations, and demonstrates the equivalence between moment method and Z-parameter network formulations.

### 3.3.1 An Example

The simple two-dipole array of Fig. 3.3 is analyzed using impedance parameters, and is treated as a two-port network. Specifically, the feedpoint of the Antenna 1 serves as Port 1, while Port 2 is the feedpoint of Antenna 2. Using the Numerical Electromagnetics Code [38] as a testbed, \(Z_{11}\) is first determined in precisely the same manner as would be used for a circuit, that is, the ratio of \(V_1/I_1\) is determined with Antenna 2 open-circuited. A mistake reported in several references (e.g., [29]) is that \(Z_{11}\) is measured by removing Antenna 2. While this method may yield acceptable results if coupling is weak, it is inconsistent with the network concept and the definition in (3.4(a)). \(Z_{21}\) is determined by placing a large load (e.g., \(10^6\ \Omega\)) in the feed segment of Antenna 2, so that Port 2 is
practically open-circuited, but $V_{2OC}$ can be determined by multiplying the load resistance times the current through it. $Z_{22}$ and $Z_{12}$ are determined in a like manner. The correct impedance matrix is

$$Z = \begin{bmatrix} 66.5 + j48.8 \Omega & 57.6 - j166.0 \Omega \\ 57.6 - j166.0 \Omega & 514.9 + j139.8 \Omega \end{bmatrix}$$

for the chosen parameters, while the result obtained by measuring $Z_{11}$ and $Z_{22}$ without Antenna 2 present is

$$Z = \begin{bmatrix} 119.8 + j71.6 \Omega & 57.6 - j166.0 \Omega \\ 57.6 - j166.0 \Omega & 889.0 + j152.7 \Omega \end{bmatrix}.$$  

Notice that $Z_{21}$ and $Z_{12}$ are equal, as expected for a reciprocal network.

Two 1-volt (in-phase) excitations are applied to ports 1 and 2, and the currents are recorded in Table 3.1. The currents are computed directly with NEC, and with the use of each of the above $Z$ matrices above.

![Figure 3.3—Method for measuring $Z_{11}/Z_{21}$ for the given example. Antenna 1 length 0.5 meters; Antenna 2 length 0.75 meters; all wire radii 10 mm, separation 0.2 meters, operating frequency 300 MHz.](image-url)
TABLE 3.1
PORT 1/PORT 2 CURRENTS CALCULATED FROM NEC, AND WITH TWO-PORT Z MATRICES

<table>
<thead>
<tr>
<th></th>
<th>Amplitude (mA)</th>
<th>Phase (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direct from NEC</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I₁</td>
<td>7.39</td>
<td>-12.28</td>
</tr>
<tr>
<td>I₂</td>
<td>2.89</td>
<td>41.60</td>
</tr>
<tr>
<td>Z matrix</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I₁</td>
<td>7.39</td>
<td>-12.28</td>
</tr>
<tr>
<td>I₂</td>
<td>2.89</td>
<td>41.60</td>
</tr>
<tr>
<td>Z matrix (Z₁₁ meas. w/ant. 2 removed)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I₁</td>
<td>5.68</td>
<td>-19.18</td>
</tr>
<tr>
<td>I₂</td>
<td>1.57</td>
<td>35.21</td>
</tr>
</tbody>
</table>

The simple array has been treated as a two-port network, despite the fact that 15 segments were used for each dipole. In reality, the electromagnetic code has treated the entire structure as a 30-port problem. We could have placed a source in the first segment and open-circuited every other segment, one at a time, then placed a source in the second segment, etc., until the entire impedance matrix was created. The first purpose of this example is to demonstrate that an antenna system may possess any number of segments, yet can be represented as a reduced-order network representation, in accordance with the number of non-trivial (non-zero) entries in the voltage vector in (3.29). The antennas can be divided in an arbitrary fashion, as long as Port 1 is associated with Antenna 1, and Port 2 is associated with Antenna 2. A given "antenna" is not constrained to having any form of connectivity. This is an important result, as it is often desirable to treat antennas with no physical continuity as one port of a network. One can thus consider the following unusual array of Fig. 3.4, consisting of a one-wavelength loop and a four-element Yagi, also as a two-port system.

Figure 3.4—Two-element array.
The second purpose of this example is to confirm that all portions of the antenna system must remain in place when computing the impedance parameters of the reduced-order network.

3.4 RADIATION PATTERNS

So far the network concept has only been considered for computing driving point impedances. An item of even greater interest is the determination of the embedded element pattern. The embedded element pattern is the radiation pattern resulting from exciting the element (port) in question, while setting all other sources equal to zero. This is, in general, not equal to some linear combination of the isolated element patterns.

Typically, moment method codes use voltage sources to excite an antenna. This fits well with the network model. Once the impedance matrix is inverted, we have

\[ I_m = \sum_{n=1}^{N} Y_{mn} V_n, \quad m = 1, 2, \cdots, N. \] 

(3.30)

Thus, the currents on each segment of the antenna are a linear combination of the voltages applied at each designated port. After the currents are known, the radiation pattern for an arbitrary structure can be determined via several means. One way is by using a more general form of the Stratton-Chu relation:

\[ \mathbf{E}(\mathbf{r}) = jkZ \int_{S} \mathbf{J}(\mathbf{r}) g(\mathbf{r}, \mathbf{r}) dS + \frac{jZ}{k} \int_{S} \mathbf{J}(\mathbf{r}) \cdot \nabla \nabla g(\mathbf{r}, \mathbf{r}) dS, \mathbf{r} \in V, \] 

(3.31)

where \( S \) is the surface of the antenna. This form allows the consideration of arbitrary surface currents, unlike the form in (3.8). In addition, (3.31) may be used with arbitrary geometric primitives, e.g., wires, triangles, quadrilaterals, etc. For materials that can be treated as a perfect conductor, a simple equation as in (3.7) can relate the individual current \( I_m \) to the surface current density \( J_m \) of particular segment type used.
Simply stated, the array radiation pattern is a function of the currents on the structure, which are in turn due to a linear combination of the applied voltage sources. The standing wave ratio at the feedpoint of any given antenna need not be considered when computing the currents or the radiation pattern.

Fig. 3.5 displays the radiation patterns for the "array" of Fig. 3.4. Clearly, the pattern created with both antennas active cannot be obtained from any linear combination of the isolated element patterns. It is not too difficult to imagine, however, that summing the embedded patterns in Fig. 3.5(b) can create the array pattern.
Figure 3.5—Radiation patterns for array of Figure 3.4; in (a), isolated; in (b), embedded; and in (c), with both antennas excited with equal amplitude and in-phase voltage sources. The solid line corresponds to the Yagi, and the broken line to the loop.

3.5 INCLUSION OF LOADS

Array modeling must generally include effects arising from the impedance of the output amplifier or receiver to which each element is connected. If inter-element coupling is weak, accurate modeling of the source impedance may not be of concern. However, for many arrays, large errors in computing element patterns may occur if the source impedances are omitted, particularly if mutual coupling is strong. A case in point is shown in Fig. 3.6, where the set of 50Ω loads act as a convenient damping source and improves pattern characteristics. The linear network model permits the inclusion of such loads:

\[ ZI = (Z_p + Z_L)I = V \]  

(3.32)
where $Z_p$ is the "passive" portion of the impedance matrix from before, and $Z_L$ is the loading matrix. In general, non-zero elements of $Z_L$ will exist only on the diagonal of $Z$.

![Figure 3.6](image)

Figure 3.6—(a) 40-element circular array of Yagis, (b) embedded element patterns with (...) and without (--) 50 ohm loads.

All loads must remain incorporated in the network itself, and not removed when a given source is removed (short-circuited). This is in keeping with physical reality, where some array elements may be turned off or nearly so, yet still present a constant impedance looking into the generator. In the MoM formulation, loads may be added prior to matrix inversion, or afterwards as admittances. In NEC, for example, loads are placed prior to inversion, so that an antenna having $Z$-matrix block symmetry like that of Fig. 3.6 must be symmetrically loaded. Only with a user specified $Y$-parameter network can one place asymmetric loads in a symmetric problem [39].

### 3.6 INVARIANCE TO ELEMENT/SOURCE MISMATCH

One of the natural benefits of large arrays is the ability to achieve very low-sidelobe patterns. Numerical techniques, such as the iterative linear least-squares approach discussed in Chapter 4, utilize the known element pattern to minimize sidelobe energy
over a specified angular region. A natural concern is that a correction to the computed excitation vector may be required, depending on the power reflected from individual elements. However, as stated in the previous section, currents on all segments are completely described by the relation \( \mathbf{I} = \mathbf{YV} \). Since each element pattern is a function only of the surface currents on the structure, then the array pattern is a linear function of the applied voltage sources. This remarkable result is demonstrated using the circular array of Fig. 3.6(a). Using beamforming method discussed in Chapter 4, a low-sidelobe pattern is created. In Fig. 3.7, most sidelobe peaks are approximately 40 dB down from the peak. The sidelobes closest to the main beam are higher because the main beam has been compressed into a small angular region, in an attempt to get undesirable active impedances—recall that the objective is to show pattern invariance to element/source mismatch.
Figure 3.7—(a) Computed array pattern using an external beamforming program, and (b) patterns computed using NEC.

Fig. 3.7(b) clearly shows that the computed excitations, when used in the electromagnetic simulation, yield precisely the same radiation patterns as in Fig. 3.7(a). Fig. 3.8(b) confirms that all elements do not accept power equally.
Figure 3.8—(a) Voltage excitation across aperture, (b) standing wave ratio of individual elements.
3.7 SENSITIVITY TO SOURCE LOAD VARIATION

A real phenomenon that does affect patterns is the variation of the source impedances. Obtaining an analytical formulation that relates the degree of pattern degradation to a known change in a particular source impedance is difficult. This may be an area for future investigation.

A simulation was conducted on the array of Fig. 3.6(a) to roughly assess the effects of load variations. Without changing the excitation derived previously for elements loaded with $50\,\Omega$, the elements were loaded with alternating resistances, i.e., Element 1 is loaded with $50\,\Omega$, Element 2 with $25\,\Omega$, Element 3 with $50\,\Omega$, etc. Patterns are shown in Fig. 3.9 for $50/25\,\Omega$ and $50/45\,\Omega$ alternating schemes.

![Figure 3.9—Patterns demonstrating the effect of variations in source impedance.](image-url)
3.8 SUMMARY

Several conclusions may be drawn from this chapter. Existing analyses of the active element pattern, which assume a large linear array, do not directly apply to the circular array. Fortunately this has minimal impact, since our aim is not to determine the embedded element pattern from the isolated element pattern. With modern electromagnetic simulators, the embedded element pattern can be determined directly by placing a representative load in series with each source, and setting the voltage of all but one source equal to zero. The MoM formulation, which in its ultimate accuracy is representative of actual physical behaviour, is used to demonstrate that an antenna system can be treated as a linear network. The obvious implication is that direct measurements may be performed on a physical antenna and the same principle applies. To obtain an accurate measurement, all loads must reasonably approximate the source impedance that is to be present in the fully operational array. This obviously applies not only to the transmitter modules, but to any receiver modules as well.

The embedded element concept is critical, since it permits a set of highly-coupled elements to be decoupled via the network concept. This provides the assurance that an array pattern will be a linear combination of the individual excitations times their corresponding element patterns. As a result, source/element mismatches and changes in element patterns need not be considered when computing the array excitation. An array may have a number of sources with high VSWR, but this manifests itself only in poor array efficiency, and not in an unpredictable or uncontrollable array pattern. The equivalence between the MoM formulation and the Z-parameter network used for the array model was demonstrated. Regardless of the size of the antenna array or the number of segments in the model, a Z/Y parameter matrix can be used to relate the currents and voltages at the feedpoints of the array, given a set of excitations. This matrix need only be of a size equal to the number of sources or ports.
4 RECEIVE MODE BEAMFORMING

4.1 INTRODUCTION

More than 50 years ago, Dolph published his classic paper on the derivation of a set of array weights that minimizes the beamwidth of a uniformly-spaced linear array (ULA) given a maximum permissible sidelobe level [40]. Numerous other works [41]-[47] have focused on various aspects of pattern synthesis. Each of these papers considered only ULAs of isotropic elements. Other works (such as those of Sureau and Keeping [48] and others previously listed [9]-[17]) extended analytical approaches to handle pattern synthesis for circular or elliptical arrays. While the scope of synthesis techniques was expanded beyond the linear geometry, these methods still lack the ability to minimize spurious energy for element patterns having arbitrary shapes, or worse yet, arrays with dissimilar element patterns.

In recent years, antenna engineers have found relief in gradient search methods, such as proposed in [18], [19], or in matrix-based techniques as used in adaptive processing algorithms [49], [50]. Numerical methods possess a number of advantages, including the ability to handle arbitrary array geometries, as well as arrays whose individual element patterns may differ significantly. Furthermore, these methods provide a means of finding the absolute global minimum of a pattern based cost function.

The linear least-squares (LLS) method is a matrix-based technique that is easily adapted for array beamforming. It is easy to understand, generic, and once implemented, can be applied to virtually any array with the greatest impact being a few minor changes in the beamforming software. Our experience is that the LLS method outperforms derivative-based search methods in terms of computational speed. The only significant limitation is that it cannot place non-linear (e.g., constant power) constraints on individual excitations. Because power constraints are not generally required for reception, the LLS method is ideal for computing receive beam excitations. It can compute the complex excitations for a 60-element array in a fraction of a second using a 400 MHz personal computer.
This chapter will provide a clear explanation the LLS method in the context of array beamforming, and provide practical insights and considerations not addressed in the literature. A primary topic of discussion is that of the array efficiency, both during transmit and receive. Another is the use of the method to maximize directivity. A simple method to modify a required steering vector in order to shape a broad (fan) beam is also presented. Other capabilities of LLS, such as the ability to iteratively shape array patterns and correct for individual element pattern deficiencies, will also be demonstrated. Finally, the amplitude and phase accuracies required to meet the desired objectives will be briefly addressed.

4.2 CALCULATION OF THE EXCITATION VECTOR

Consider an $M$-element array lying in the $x$-$y$ plane. For simplicity, only the pattern performance in the $\theta = 90^\circ$ plane will be treated. The procedure is easily extended to incorporate element data in multiple planes. The far-zone field strength at an angle $\phi$ can be written as

$$F(\phi, w) = \sum_{m=1}^{M} w_m g(m - \phi'_m) \quad (4.1)$$

where $w_m$ is a single complex weight (excitation), the unprimed and primed $\phi$ coordinates represent the observation and element location angles, respectively, and $g(\phi - \phi'_m)$ is the complex electric field contribution at the observation point from the $m^{th}$ element. This can be rewritten, adjusting the notation, as

$$F(\phi_n) = gw \quad (4.2)$$

where

$$g = [g(\phi_1 - \phi'_1), g(\phi_2 - \phi'_2), \ldots, g(\phi_m - \phi'_m)] \quad (4.3)$$
and

\[ \mathbf{w} = [w_1, w_2, \cdots, w_M]^T. \]  \hspace{1cm} (4.4)

A few comments are in order regarding the form of \( \mathbf{g} \). In most references, a complex exponential is associated with each element pattern, the exact form depending on the array geometry and the element position. For MATLAB type simulations, the exponential is generally included, as the element pattern is assumed to be real. For the representation in (4.1), each element pattern has the phase term included. This is generally preferred for physical arrays, where the element patterns are derived either from an electromagnetic model or direct measurements. In either case, the element patterns are already complex, and are referenced to the array center. In addition, element patterns in this form are the only truly accurate representation for a physical array. Strictly speaking, the antenna cannot be represented as a real element pattern and an associated phase center without introducing small errors (see discussion in Section 5.5).

Figure 4.1—Sampling function for LLS derived taper. The unknown far-field pattern is sampled at periodic intervals outside the main beam, where the target function is specified to be zero. Within the main beam, the function is sampled only on boresight, and left undefined elsewhere.
As shown in Fig. 4.1, a desired array pattern is specified. The target function is a vector composed of the $N$ values of $F(\phi_n)$. This function is specified to be zero outside the main beam, unity at the peak, and undefined at other points within a small "corridor". Our intention is to place as much pattern energy as possible in the main beam. We proceed to minimize the squared difference between the target and pattern functions:

$$J = \sum_{n=1}^{N} \left| F(\phi_n) - \hat{F}(\phi_n) \right|^2.$$  \hspace{1cm} (4.5)

As shown by Compton [50], $J$ is minimized by

$$w = R^{-1}v,$$  \hspace{1cm} (4.6)

where

$$R = \begin{bmatrix}
\sum_{n=1}^{N} g_1^*(n) g_1(n) & \sum_{n=1}^{N} g_1^*(n) g_2(n) & \cdots & \sum_{n=1}^{N} g_1^*(n) g_M(n) \\
\sum_{n=1}^{N} g_2^*(n) g_1(n) & \cdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{n=1}^{N} g_M^*(n) g_1(n) & \cdots & \sum_{n=1}^{N} g_M^*(n) g_M(n)
\end{bmatrix},$$  \hspace{1cm} (4.7)

and

$$v = \left[ \sum_{n=1}^{N} g_1^* \hat{F}_n, \sum_{n=1}^{N} g_2^* \hat{F}_n, \ldots, \sum_{n=1}^{N} g_M^* \hat{F}_n \right]^T.$$  \hspace{1cm} (4.8)

In short, $R$ is the covariance matrix of element patterns. In general, the signal processing community treats $R$ as an expected value of signals received by the array elements. The form of $R$ in (4.7) is written explicitly to emphasize that the matrix is filled only over the $N$ samples, and not over 360°. An alternative to the common method of obtaining (4.6) is
provided in Appendix B. Although it results in the same answer, some may find this to be a more intuitive and satisfying means of obtaining the excitation vector.

Equation (4.6) is generally used to provide low-sidelobe patterns, and so it is natural to inquire what happens if the width of the "corridor", or exclusion region, is set to an arbitrarily small value. In fact, as the width of the exclusion region is made smaller and smaller, the directivity approaches its maximum value, until the width is zero, at which point the directivity is absolutely maximized. The equivalence of this simple technique to an analytical directivity maximization technique is demonstrated below.

### 4.3 EQUIVALENCE TO DIRECTIVITY MAXIMIZATION

The general expression for directivity is

\[
D(\phi_h) = \frac{4\pi |F(\theta_h, \phi_h)|^2}{\int_0^{2\pi} \int_0^{\pi} |F(\theta, \phi)|^2 \sin \theta d\theta d\phi}
\]  

(4.9)

To demonstrate the equivalence between the two directivity maximization methods, we represent the beamforming problem in a slightly different form. Let \( G \) be the \( N \times M \) matrix of element patterns, \( \mathbf{f} \) and \( \hat{\mathbf{f}} \) are \( N \times 1 \) actual and desired patterns, respectively, and \( \mathbf{w} \) the complex weight vector as before. Then

\[
\mathbf{f} = G \mathbf{w}.
\]  

(4.10)

From [51], the solution of \( \min_{\mathbf{w}} |G \mathbf{w} - \hat{\mathbf{f}}|^2 \) is

\[
\mathbf{w} = (G^H G)^{-1} G^H \hat{\mathbf{f}}
\]  

(4.11)

where \( H \) denotes the Hermitian transpose. This yields
\[ f = G (G^H G)^{-1} G^H \hat{f}. \]  

(4.12)

In terms of the previous formulation, \( R = G^H G \) and \( v = G^H \hat{f}. \)

To simplify the illustration, and to maintain compatibility with the foregoing two-dimensional beamsteering formulation, only element pattern data in the \( \theta = 90^\circ \) plane will be considered. The results are easily extended to include element data for all \( \phi \) and \( \theta \). In fact, a rigorous maximization of the directivity requires knowledge of the element patterns over \( 4\pi \) steradians. In practice, the following method does an adequate job of maximizing directivity using element data in the principal plane, so long as the element patterns are reasonably well behaved.

By discretizing (4.9) (sampling the \( N \) values of \( F(\phi) \) at \( \theta = \theta_0 \)), we have (neglecting constants)

\[
D_r(\phi_0) = \frac{\left| w^H G^H e_{n_0} \right|^2}{w^H G^H G w} 
\]

where \( e_{n_0} \) is an \( N \times 1 \) unit vector in the \( \phi_0 \) direction. The \( r \) subscript has been added to \( D(\phi_0) \) to denote that this is a relative directivity. The ability to determine the absolute directivity has been lost, but our desire is only to choose the excitation that maximizes \( D \).

A quadratic function in the form of (4.13) is maximized by [51]

\[
w = \left( G^H G \right)^{-1} G^H e_{n_0}. \]

(4.14)

Since \( e_{n_0} = c \hat{f} \), and \( R \) from (4.7) is exactly equal to \( G^H G \) when the exclusion region is set to zero width, the two approaches are numerically equivalent.
4.4 EFFECT UPON ACTIVE IMPEDANCES

4.4.1 Transmission

Before proceeding, the definitions for gain and system gain should be addressed. Prior to 1983, the definition of gain (ANSI STD 145-1973) included "losses" due to source/antenna impedance mismatches. Gain was expressed as the product of efficiency times the directivity:

\[ G = e_r e_a D \quad (4.15) \]

where \( e_r \) is the reflection efficiency, and \( e_a \) is the absorption efficiency. This is the most logical definition to use when describing the performance of antenna arrays, for the following five reasons:

1) Commercial and military antenna ranges measure point-to-point performance. These measurements are invariably called gain, and do not compensate for antenna impedance fluctuations.

2) This definition is the true means by which an antenna's ability (given fixed impedance sources and fixed impedance transmission lines) to couple to another antenna, or to produce a given field strength, is assessed.

3) This definition is the industry standard way to describe array performance—done so in the arrays described in [1]-[3], and also in numerous proposals submitted to the Naval Air Warfare Center and the Office of Naval Research that the author has had the privilege of evaluating.

4) The array gain, according to this definition, as well as the effective radiated power, can be maximized readily according to a simple theorem presented in section 6.10.

5) If this definition is not used, a new definition must be coined that describes the quantity measured on an antenna range.
In 1983, the IEEE adopted a new standard for gain—IEEE STD 145-1983 [52], replacing ANSI 145-1973. The new definition excludes "losses" due to source/antenna impedance mismatch. **In order to clearly discuss the array performance as a whole, a new term—system gain—is introduced. The system gain refers to the quantity described in (4.15)—the pre-1983 definition of gain.**

The reflection efficiency is typically the term of greatest interest (except for electrically small and lossy antennas), as it can be significantly affected by the excitation. Consider first the case of transmission, where each element possesses its own amplifier module. The overall array reflection efficiency is

\[
e_r = \frac{\sum_{m=1}^{M} P_{av}(m) \left(1 - |\Gamma_m|^2 \right)}{\sum_{m=1}^{M} P_{av}(m)} \quad (4.16)
\]

where \(P_{av}(m)\) is the power available at antenna element \(m\), and \(\Gamma_m\) is the complex reflection coefficient. Equation (4.16) holds for the general case, permitting the radiation resistance at any element to be negative (\(|\Gamma_m| > 1\)), in which case power is subtracted from the numerator. This case can occur for a tapered transmitter excitation (demonstrated in Fig. 4.5). It is not necessarily a cause for concern, however, as those elements with negative impedances are generally at a relatively low power, and thus the power absorbed by the source is also small.

To illustrate the effect of the excitation on the array directivity, system gain, and efficiency, consider the following example. In Fig. 4.2(a), a 60-element array of Yagis is arranged in a circular configuration. Each element is tuned for the best possible front/back ratio and VSWR. Each generator is modeled as a Thevenin equivalent, having a 50\(\Omega\) load placed in series with the applied voltage source. This results in the embedded
element pattern of Fig. 4.2(b). The element pattern is well behaved and consequently is easy to work with for array beamforming.

Figure 4.2—(a) Array with 60-fold rotational symmetry. (b) Embedded element pattern.
To determine the array performance, embedded element data is used to compute the excitations according to (4.6). The array performance is computed with a moment method electromagnetic simulator (NEC version 4.1), and the directivity and active impedances are read directly from the output file. System gain is derived from (4.15). Array performance data is provided for several excitations in Table 4.1, each of which is obtained by specifying a different exclusion region width (target beamwidth).

### Table 4.1
**Directivity and System Gain for Transmit Excitations**

<table>
<thead>
<tr>
<th>Target BW (°)</th>
<th>Directivity (dBi)</th>
<th>Efficiency (%)</th>
<th>Sys. Gain (dBi)</th>
<th>3 db BW (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>18.6</td>
<td>96.7</td>
<td>18.5</td>
<td>9.5</td>
</tr>
<tr>
<td>28</td>
<td>18.8</td>
<td>96.6</td>
<td>18.6</td>
<td>9.0</td>
</tr>
<tr>
<td>24</td>
<td>19.2</td>
<td>96.2</td>
<td>19.0</td>
<td>8.6</td>
</tr>
<tr>
<td>20</td>
<td>19.6</td>
<td>94.2</td>
<td>19.3</td>
<td>7.5</td>
</tr>
<tr>
<td>16</td>
<td>20.1</td>
<td>86.7</td>
<td>19.5</td>
<td>6.8</td>
</tr>
<tr>
<td>12</td>
<td>20.7</td>
<td>71.6</td>
<td>19.2</td>
<td>6.2</td>
</tr>
<tr>
<td>0</td>
<td>21.5</td>
<td>47.7</td>
<td>18.3</td>
<td>5.5</td>
</tr>
<tr>
<td>Uniform *</td>
<td>20.3</td>
<td>92.2</td>
<td>19.9</td>
<td>5.9</td>
</tr>
</tbody>
</table>

* 30 elements active

Table 4.1 demonstrates that the array efficiency is a strong function of the excitation. In general, one can trade directivity for sidelobe energy by varying the target beamwidth. For wide target beamwidths, the array efficiency is high, but the directivity is low. As the target beamwidth is decreased, the directivity rises and efficiency drops. Maximum system gain is reached at some point where the efficiency is still reasonable, typically between 80 and 90%. This point depends upon the array geometry and element-element coupling. If the elements of the above array were perfectly decoupled, the array efficiency would not be a function of the excitation, and the peak system gain would occur for a target beamwidth of 0°. For the sample array, the adjacent element-element coupling is approximately 18 dB. Fig. 4.3 shows the array pattern and corresponding excitation resulting from specifying a 24° exclusion region, and in Fig. 4.4 are the results from setting the width equal to zero. A reasonable estimate of the efficiency can be made by looking at the shape of the excitation. As the efficiency drops, the excitation generally becomes increasingly oscillatory. Note also from Table 4.1 that the peak system gain (for
LLS derived excitations) occurs for a 16° exclusion region. For smaller beamwidths, efficiency decreases at a faster rate than the directivity increases.

Figure 4.3—(a) Low-sidelobe pattern created by specifying a 24° exclusion region. (b) Well-behaved excitation leads to high array efficiency.

Figure 4.3—(a) Low-sidelobe pattern created by specifying a 24° exclusion region. (b) Well-behaved excitation leads to high array efficiency.
Figure 4.4—(a) Pattern resulting in maximum directivity (0° exclusion region). (b) Oscillatory excitation results in low efficiency, and low system gain.
4.4.2 Reception

It is quite tempting to argue that for an active array, the case of reception is fundamentally different from that of transmission. When receiving, each element is in a passive mode, and is irradiated with a plane wave. Unlike the transmission mode, where the dynamic range of excitations can exceed 20 dB, the elements receive a plane wave from the distant source with equal power, according to the element pattern. The active impedance of each element should then be equal to that of the transmit case when exciting each element in the array with equal power, and adjusting the phase of the $m^{th}$ source to equal that of the conjugate of $g_m(\phi_0)$. The overall efficiency is then

$$e_r = \frac{\sum_{m=1}^{M} |w_m|^2 \left(1 - |\Gamma_m^u|\right)}{\sum_{m=1}^{M} |w_m|^2},$$  

(4.17)

where $\Gamma_m^u$ refers to the reflection coefficient for a uniform excitation. Equation (4.17) appears to be the same as (4.16); however, the distinction is that the efficiency is based upon a set of reflection coefficients that is independent of the receiver weights. The individual weight $w_m$ typically exists only in a beamforming computer, and is transparent to the element itself. For this reason, the efficiency should not be significantly affected by the taper on receive, and the efficiency problem associated with the high directivity patterns is averted. To check this theory, we resort to a second means of computing system gain, this time directly from the measured embedded element pattern. The gain of an antenna is given by [53]

$$G(\theta, \phi) = \frac{4\pi|E(\theta, \phi)|^2}{\eta P_{in}},$$  

(4.18)

where $\eta$ is intrinsic impedance of free space and $P_{in}$ is the power input to the antenna. This expression may be converted to system gain by replacing the total power into the
antenna \((P_{in})\) with the power available to the antenna \((P_{av})\). To compute the array system gain from a set of measured patterns, then, the contributions to \(E(\theta, \phi)\) from each element are added, squared, and divided by the total power available to the set of receiver modules (or equivalently, the excitation weights). In a slightly different form,

\[
G_s(\theta, \phi) = \frac{\sum_{m=1}^{M} |w_m g_m(\theta, \phi)|^2}{\sum_{m=1}^{M} |w_m|^2},
\]

(4.19)

where \(G_s\) is the array system gain, and \(g_m(\theta, \phi)\) is the linear system gain (with respect to that of an isotropic element) from the \(m^{th}\) element. The numerator is identically equal to \(w^T R w\), but is in a form convenient for post-processing of measured element data. By setting the two methods equal we obtain

\[
D(\theta, \phi)e_r = G_s(\theta, \phi)
\]

\[
\frac{w^T R(\theta, \phi) w}{(4\pi)^{-1} w^T R_{\text{all} \theta, \phi} w} \sum_{m=1}^{M} |w_m|^2 \left(1 - |\Gamma_m|^2\right) = \frac{w^T R(\theta, \phi) w}{\sum_{m=1}^{M} |w_m|^2}.
\]

(4.20)

The \(R\) matrix in the denominator of the left-hand side of (4.20) must contain a \(\sin \theta\) along with each covariance matrix element since it is a sampled implementation of the directivity expression in (4.9). One additional stipulation about the matrix \(R\) in this special case is that its components must be in the linear system gain (relative to isotropic) format. The denominator of \(D(\theta, \phi)\) should be recognized as the total power radiated if
the array were 100% efficient ($P_{rad}$). By canceling components on both sides, the following expression is obtained:

$$P_{rad} = \sum_{m=1}^{M} |w_m|^2 \left(1 - |\Gamma_m|^2\right),$$

(4.21)

which states the obvious: that the total radiated power is proportional to the sum of power in each individual transmitter module times the percentage of energy transmitted to its respective element. If the proposed efficiency in (4.17) is used in (4.20), then (4.21) states that the set of active (transmitter) reflection coefficients are independent of the excitation, which is in violation of (4.16), and in violation of the well-established transmit/receive reciprocity theorem [54]. Thus, one cannot argue against the principle of reciprocity based upon the nature of the active array architecture. System gain calculations from measured (receive) element patterns in Chapter 6 confirm this assertion.

We can develop a satisfying intuitive argument that explains the reciprocity theorem in the context of an active array. The difference between the directivity and the system gain is easy to understand when transmitting a beam. In this instance, the active impedances (particularly for the case of a high-directivity excitation) deviate from ideal (equal to the selected characteristic impedance), and cause power to reflect back to the amplifier modules. For the case of reception, a bit more imagination must be invoked, since, as noted in the argument surrounding (4.17), the excitation weights do not affect the power induced into a given receiver module; they are transparent to the RF hardware. The mechanism for the difference between directivity and system gain lies in the signal cancellation present in either an analog or digital beamformer. The "trick" to high directivity patterns is that the excitation is selected not so much as to maximize the signal strength, but rather to minimize the denominator in $D(\theta, \phi)$. This is accomplished, simply, by cancellation of element pattern components. Because a reciprocal relationship
exists, then, the amount of raw power cancelled in the beamformer during the reception process is identically equal to the power reflected back to the amplifier modules as a result of the changing (according to the excitation in question) active element impedances during the transmission process.

Each of the reflection curves in Fig. 4.5 has its own unique characteristics. As stated previously, the uniform excitation has the flattest curve, because no elements are swamped by high-powered adjacent sources. The reflection coefficients for the excitation corresponding to a 24° exclusion region are low near beam boresight, where the excitations are well behaved. Only the very large reflections correspond to elements having low power applied to them, and are swamped by adjacent sources. Two of the

![Figure 4.5—Set of reflection coefficients for uniform, low-sidelobe, and high-directivity excitations. Source-element match is best overall for uniform, as no element is "swamped" by neighbors. Low-sidelobe taper has excellent match near boresight. The outlying points correspond to elements having very little power applied to them. High-directivity excitation has many sources that absorb power, despite the lack of low-power sources.](image-url)
sources actually absorb power. Despite this, the overall array efficiency remains very high. The maximum directivity excitation, on the other hand, leads to numerous sources that absorb power (\(|I| > 1\)). In this case, many of the absorbing sources are not low power at all. In addition, the elements having the peak power applied to them (elements 10-20 and 40-50) have reflection coefficients no smaller than about 0.6, which leads to poor array efficiency, and poor system gain.

Figure 4.6—Pattern for uniform excitation.
4.5 LINEAR LEAST-SQUARES BEAMFORMING CAPABILITIES

4.5.1 Perturbation Compensation

One of the deficiencies inherent with all non-numerical beamforming approaches is that they lack the ability to compensate for element patterns having deviations from the desired performance. Such deviations could arise from a variety of factors, like unaccounted-for nearby structures, changes in the element patterns due to mechanical stress/warpage, water seepage into radome structures, or a shift in the impedance of individual sources.

The beauty of the LLS method as defined is that it can compensate for pattern deficiencies, so long as the element perturbations are known. Many applications (e.g., radar) requiring adaptive beamforming use "blind" algorithms that do not depend upon knowledge of element patterns. For our purposes, a precise knowledge of all element patterns is assumed.

To demonstrate the ability to compensate for alterations, we create an element pattern with considerable perturbation from the pattern of Fig. 4.2(b). Placing an obstruction in front of the element creates the pattern shown in Fig. 4.7. The element loses about 5 dB gain on boresight (0°), and radiates considerable energy towards the rear. In short, it could easily represent a failed element. In the simulation experiment the element faces at an angle 42° from the main beam direction. To assess the impact on the pattern performance, the perturbed element is used, but the excitation of Fig. 4.3(b) is not corrected. Next, the perturbed pattern is used in the covariance matrix and steering vector to correct for the deficiencies. Fig. 4.8 demonstrates the difference between the two patterns. The corrected pattern exhibits lower spurious radiation at all angles outside of the main beam.
Figure 4.7—Perturbed element pattern.

Figure 4.8—Both patterns are created with element pattern of Fig. 4.7. Broken line results from uncorrected excitation; solid line results from using perturbed element to compute excitation.
4.5.2 Iteratively Weighted Patterns

One characteristic of the LLS method is that the resulting patterns tend to have spurious energy "piled up" near the main beam, as shown in Figs. 4.3(a). One way to provide an additional degree of pattern control is to iteratively solve for \( \mathbf{w} \), placing additional weight on high-sidelobe areas until the pattern meets the desired specification. Mathematically, a new covariance matrix is specified at each step. For the purpose of implementation, \( \mathbf{R} \) can be expressed as the product of an outer set of matrices composed of element patterns, and a diagonal weight matrix:

\[
\mathbf{R} = \begin{bmatrix}
g_{11}^* & g_{12}^* & \cdots & g_{1N}^* \\
g_{21}^* & g_{22}^* & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
g_{M1}^* & \cdots & g_{MN}^*
\end{bmatrix} \begin{bmatrix}
u_1 & 0 & \cdots & 0 \\
0 & u_2 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & u_N
\end{bmatrix} \begin{bmatrix}
g_{11} & g_{12} & \cdots & g_{1M} \\
g_{21} & g_{22} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
g_{N1} & \cdots & g_{NM}
\end{bmatrix}
\]  

(4.22)
where $g_{mn}$ (or $g_{nm}$) represents the contribution from the $m$th element in the direction of the $n$th measurement point. The only matrix that changes is the center one, which contains the weights. The steering vector $v$ does not change, as there is no contribution to $J$ in the direction of the main beam. The individual weight $u_n$ at step $k$ is determined by

$$u_{n,k} = u_{n,k-1} + c \max\left[0, s - 20 \log \frac{F_{\max}}{F(\phi_n)}\right]$$  \hspace{1cm} (4.23)$$

where $s$ is a threshold sidelobe level in dB relative to the peak, and $c$ is a gain constant. In reality, the method requires a bit of practice, requiring the selection of a target sidelobe level 1-5 dB greater than the level desired, and an appropriate iteration gain that depends upon the problem. Selection of an excessively large iteration gain may result in an ill-conditioned matrix $R$. Iterative weighting can also provide patterns having additional null depth over specified regions to deal with jamming sources. This method was initially proposed by Olen and Compton [55].

Figure 4.10—Weighted pattern has the same 3 dB beamwidth as that of Fig. 4.3(a), but the close-in sidelobe energy has been redistributed.
4.5.3 Spoiled Beams

Ordinarily, the objective is to create a beam with the cleanest possible pattern and an acceptably small beamwidth. On some occasions, it may be desirable to produce a fan beam, so that a broad angular region can be covered. The most obvious course would be to simply set the target beam width to be much wider than before. However, we find that as this width expands beyond a critical point, the main beam splits. To avoid this, a number of points must be specified within the main beam of the target function. Until now, the steering vector was based upon a single non-zero value of the desired beam; that is, \( \hat{F}(n_o) \). Because a single value was specified, the phase of this value was unimportant, as the excitation vector adjusted itself accordingly. Now, however, the problem arises as to how one specifies the phase of the vector \( \hat{f} \) within the bounds of the exclusion region.

In order to observe the effects of using a constant phase target function within the exclusion region, a simulation was conducted in which all elements of the vector \( \hat{f} \) were specified to be \( 1\angle 0^\circ \). The pattern in Fig. 4.11(a) results. The pattern is well behaved, and exhibits a small amount of ripple near the beam edges, as one would expect for such a broad flat-top beam. While this result was somewhat surprising, the NEC output file of every flat-top fan beam tested suggests that the far-field pattern phase can remain stable to within a degree. As such, a purely real target vector results in a nearly ideal flat-top fan beam.

In an attempt to exploit the fact that the array pattern phase can remain very stable over a broad angular region, an additional simulation was conducted to assess the ability to shape the beam edges. A simple window that demonstrates effective main beam shaping is the exponential amplitude taper. The amplitude of the desired pattern samples given by the following relation:

\[
\hat{F}(n) = \text{Re} \left\{ e^{\frac{j2\pi n}{b}|\pi - n_o|} \right\},
\]

\[ (4.24) \]
where $\xi$ is a phase constant, and $b$ is the number of points appearing in the exclusion region. When $\xi = 0$, a flat-top beam results. As $\xi$ increases, the beam becomes more rounded, as shown in Fig. 4.11(b). If the constant becomes too large, nulls appear in the main beam.

### Table 4.2
DIRECTIVITY AND SYSTEM GAIN FOR SEVERAL FAN BEAMS

<table>
<thead>
<tr>
<th>Target BW (°)</th>
<th>Phase Factor $\xi$</th>
<th>Directivity (dBi)</th>
<th>Efficiency (%)</th>
<th>TX Sys. Gain (dBi)</th>
<th>3 db BW (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0</td>
<td>9.1</td>
<td>89.2</td>
<td>8.6</td>
<td>98</td>
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<tr>
<td>100</td>
<td>.45</td>
<td>10.4</td>
<td>96.7</td>
<td>10.3</td>
<td>64</td>
</tr>
<tr>
<td>180</td>
<td>0</td>
<td>6.6</td>
<td>94.4</td>
<td>6.3</td>
<td>178</td>
</tr>
<tr>
<td>180</td>
<td>.45</td>
<td>8.3</td>
<td>97.8</td>
<td>8.2</td>
<td>100</td>
</tr>
</tbody>
</table>

(a)
4.6 SENSITIVITY TO AMPLITUDE AND PHASE ERRORS

The sensitivity of the array pattern to excitation errors should be briefly discussed. A precise relationship that equates the sidelobe levels and errors and is difficult to arrive at, primarily because of the non-linear geometry. Even if one is developed, the degree of pattern degradation becomes in statistical nature. For example, Skolnik [56] showed that the ensemble average power pattern for a uniform array of $M$ by $N$ isotropic elements arranged on a rectangular grid is

$$
|F(\theta, \phi)|^2 = P^2 \epsilon e^{-\bar{\delta}^2} |F_0(\theta, \phi)|^2 + \left[ (1 + \bar{\Delta}^2)P^2 \epsilon e^{-\bar{\delta}^2} \right] \sum_{m=1}^{M} \sum_{n=1}^{N} i_{mn}^2
$$

(4.25)
where $P_e$ is the probability of an element being operational, $\delta$ is the phase error in radians (described by a Gaussian probability density function), $|F_0(\theta,\phi)|^2$ is the no-error pattern, $\Delta$ is the amplitude error, and $i_{mn}$ is the no-error current at the $mn$th element. An equivalent expression for a circular array of endfire elements is difficult to arrive at, because of the inseparability of the element pattern and array factor, as well as the non-isotropic element pattern. Despite this, "rule-of-thumb" formulas that relate the excitation accuracy requirements to intended sidelobe levels are nonetheless useful, such as the following by Miller [57]:

$$\text{rms sidelobe level} = \frac{5}{2^B N}$$  \hspace{1cm} (4.26)

where $B$ is the number of bits in a phase shifter and $N$ is the number of array elements. Again, this formula assumes a linear array of isotropic elements. From examples in this and later chapters, we observe that the "equivalent" number of elements in the circular array is somewhat less than the actual number, based upon an (later) observation that the circular array requires more elements than its linear counterpart to obtain a desired quiescent pattern.

Instead of developing relationships specifically for the circular array, we simply present the results of an ad hoc simulation of random phase and amplitude errors introduced into the 60-element excitation for the array of Fig. 4.2(a). An excitation was created using the iterative weighting method discussed in section 4.5.2 to create a pattern with 45 dB peak sidelobes. Uniformly distributed amplitude and phase errors were introduced, and the best and worst results of 10 trials are recorded in Tables 4.3 and 4.4. For errors no larger than those listed in the tables, the effect on peak system gain is negligible.

A natural question arises regarding the practicality of using elements facing opposite the main beam in order to obtain a cleaner pattern. To examine the feasibility in this, we
TABLE 4.3
EFFECT OF RANDOM PHASE ERRORS ON SIDELOBES

<table>
<thead>
<tr>
<th>+/- Phase Error (°)</th>
<th>Peak Sidelobe</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Best</td>
<td>Worst</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>45</td>
<td>45</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>44</td>
<td>41</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>41</td>
<td>39</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>39</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>37</td>
<td>34</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>35</td>
<td>33</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>34</td>
<td>31</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 4.4
EFFECT OF RANDOM AMPLITUDE ERRORS ON SIDELOBES

<table>
<thead>
<tr>
<th>+/- Amplitude Error (dB)</th>
<th>Peak Sidelobe</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>Worst</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>45</td>
<td>45</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>44</td>
<td>43</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>43</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>40</td>
<td>39</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>40</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>37</td>
<td>35</td>
<td></td>
</tr>
</tbody>
</table>

resort to a classical definition of sensitivity used in control theory [58]. Define the real quantity

$$P = F(\phi)F^*(\phi) = \sum_{m=1}^{M} I_m e^{j\alpha_m} g(\phi - \phi_m') \sum_{m=1}^{M} I_m e^{-j\alpha_m} g^*(\phi - \phi_m')$$ (4.27)

where $I_m$ and $\alpha_m$ are the magnitude and phase of the weight at the $m^{th}$ element, respectively. Consider a sensitivity of the function $P$ to a change with respect to a change in $I_m$:

$$S_{I_m}^P = \frac{\Delta P}{\Delta I_m} = \frac{\Delta P}{I_m} \frac{I_m}{P}$$,
which, for an infinitesimal change, becomes

\[ S_{p_m} = \frac{\partial P}{\partial I_m} \cdot P. \quad (4.28) \]

\[
\frac{\partial P}{\partial I_m} = e^{j\mu_m} g(\phi - \phi_m') \sum_{m=1}^{M} I_m e^{-j\mu_m} g^*(\phi - \phi_m') + e^{-j\mu_m} g^*(\phi - \phi_m') \sum_{m=1}^{M} I_m e^{-j\mu_m} g(\phi - \phi_m')
\]

\[ = 2 \text{Re}\left\{e^{-j\mu_m} g^*(\phi - \phi_m') \sum_{m=1}^{M} I_m e^{-j\mu_m} g(\phi - \phi_m')\right\} = 2 \text{Re}\left\{e^{-j\mu_m} g^*(\phi - \phi_m') F(\phi)\right\} \quad (4.29) \]

Consequently, the sensitivity to a change in magnitude is

\[
S_{p_m} = \frac{\partial P}{\partial I_m} \frac{I_m}{P} = \frac{2 I_m \text{Re}\left\{e^{-j\mu_m} g^*(\phi - \phi_m') F(\phi)\right\}}{F(\phi) F^*(\phi)} \quad (4.30)
\]

A similar definition can be created for the phase sensitivity:

\[ S_{\alpha_m} = \frac{\partial P}{\partial \alpha_m} \frac{\alpha_m}{P} \quad (4.31) \]

In this case, the classical definition causes a problem, since the phase \( \alpha_m \) is periodic. If \( \alpha_m \) is zero for any given element, \( S_{\alpha_m} \) is zero. Clearly (4.31) leads an absurd result. This problem is eliminated if the sensitivity is redefined to be

\[ S_{\alpha_m} = \frac{\partial P}{\partial \alpha_m} \frac{1}{P} \quad (4.32) \]

Using this definition, we find that \( S_{\alpha_m} \) is identical to that of \( S_{p_m} \) in (4.30).

An accurate interpretation of (4.30) should tell us which portion of the pattern is most sensitive to excitation errors, as well as to which elements. If one assumes that an equal-sidelobe pattern is synthesized, then \( |F(\phi)| \) is roughly a constant for all regions outside
the main beam. Thus, the primary factors that determine the pattern sensitivity to errors present at a particular element are the magnitude and the strength of the element pattern in a given direction. That is,

\[ S_{\text{P}}^m, S_{\text{P}}^n \propto I_m \left| g(\phi - \phi_m') \right| \]  

(4.33)

The portion of the pattern most sensitive to errors is not opposite the main beam, but rather the angular space immediately surrounding the main beam. The elements near beam boresight have the largest \( I_m \), as well as a large contribution from the element pattern. This is confirmed in Fig. 4.12, where phase errors of up to \( \pm 3^\circ \) and amplitude errors of up to \( \pm 0.3 \) dB are introduced into the same excitation used to create Tables 4.3 and 4.4. The close-in sidelobes are the ones most affected, as suggested by (4.33).

Figure 4.12—Pattern resulting from the introduction of phase errors of up to \( \pm 3^\circ \) and amplitude errors of up to \( \pm 0.3 \) dB. Maximum sidelobes of no-error quiescent pattern are 45 dB down from peak.
4.7 SUMMARY

A practical look at beamforming with the LLS method was presented using an electromagnetic model of a circular array as the testbed. The LLS method was shown to be effective for forming a variety of low-sidelobe patterns. It also provides a simple method to maximize the directivity of an array with arbitrary geometry. Active impedances and overall reflection efficiency are strongly related to the excitation. Because of inter-element coupling, maximization of the directivity does not generally lead to maximum system gain. By relating two independent methods used for computing array system gain, the principle of transmit/receive system gain reciprocity was confirmed for an active array. The concept of iterative weighting was also demonstrated. Iterative weighting can be used to provide "equal" sidelobes, or to provide increased jammer rejection within a specified angular region. Broad fan beams were synthesized by creating a steering vector that was constant or tapered in amplitude, and constant in phase. The LLS method was used to correct for spurious energy introduced into the array pattern by a failed or severely perturbed element. Finally, a cursory look at the sensitivity to amplitude and phase errors revealed that use of all array elements is indeed a valid concept, and that amplitude and phase accuracies of a few tenths of a dB and a few electrical degrees, respectively, are desirable.
5 A PARAMETER STUDY OF THE CIRCULAR ARRAY

5.1 INTRODUCTION

The circular array whose primary radiation is through the plane of the ring may be implemented in several ways. The array might possess numerous individually controlled elements, each of which has an essentially isotropic radiation pattern in the plane of the array. This type of arrangement would likely consist of a number of monopoles for vertical polarization or turnstile antennas for a horizontally polarized version. A second philosophy might use some number of concentric rings, each of which has a different number of elements. A third implementation would use the Yagi-Uda or another type of endfire element. The endfire approach has the advantage of reducing the number of elements requiring control to a minimum. For military radars, space-time adaptive processing has become a standard, and the endfire approach reduces the necessary computational resources, making an element-space (as opposed to beam-space) processing scheme viable.

The desire to conduct a parameter study of a circular array of endfire elements originated from a difficulty in analytically relating the defining characteristics of the array with its corresponding performance. Numerous papers dealing with the circular array have appeared in the literature over the past fifty years (numerous examples are interspersed in the reference section; also an excellent bibliography appears in Ma [27]), but the number of references pertaining to arrays with endfire elements is limited. The exceptions are concerned only with analytical [17] or numerical [18], [19] beamforming methods for circular or non-specific array configurations. In Section 2.4.2, we derived closed form expressions for the directivity and array patterns for an element pattern of

\[ g(\theta, \phi) = [1 + c \cos(\phi - \phi')] \sin^p \theta \]

where \( p \) is an integer. Unfortunately, this expression does not provide the latitude to significantly adjust the azimuth characteristics of the embedded element pattern. In addition, a decomposition of the array pattern results in an untidy expression that is difficult or impossible to draw practical conclusions from. Clearly, the effect of the embedded element beamwidth upon array performance begs
investigation. In addition to the embedded element beamwidth, we also seek to answer questions regarding the number of array elements and their radial placement. Because of the inability to reach sound conclusions from an analytical approach, an empirical approach was also sought.

This chapter discusses the results of that empirical investigation. To do this, a study of several parameters affecting array performance was conducted—specifically, the phase center diameter, the number of elements, and the element beamwidth. Considered is a circularly symmetric antenna lying in the horizontal plane, consisting of endfire elements arranged in a ring. Each element is assigned a phase center with some radius, and has an associated embedded azimuth beamwidth. Various array configurations are studied, and a comparative summary of performance is presented (section 5.2). Results are given for a variety of excitations for each configuration. Other discussions include the formation of grating lobes in section 5.3, the relative energy contained in element patterns of varying beamwidths (section 5.4), and phase center movement (section 5.5). Section 5.6 provides the summary of results.

5.2 RELATIVE PERFORMANCE STUDY

For the initial study, we examine the performance available from eighteen array configurations resulting from three different element patterns, three array sizes, and arrays of 40 and 60 elements. "Hemispheric" element patterns are defined as

\[
g(\phi - \phi') = \begin{cases} 
\cos^q(\phi - \phi'), & -\frac{\pi}{2} \leq \phi - \phi' < \frac{\pi}{2} \\
0, & \text{otherwise}
\end{cases}
\]  

\text{(5.1)}

where \( q \) is a real number that determines the azimuth beamwidth, and \( \phi' \) is the angular position of the element. Although it appears strange to use element patterns with an infinite front-to-back (F/B) ratio, the back lobes tend to defocus in the array pattern. The pattern of an array with 40 or more elements and a reasonable F/B ratio (> 20 dB) is virtually indiscernible from that of one using hemispheric elements. In addition, using
element patterns with an infinite F/B eliminates confusion about the mechanism of grating lobe formation, discussed in section 5.4. Element patterns with 80°, 120°, and 160° half-power beamwidths were selected for the study. This represents a realistic limiting range of azimuthal beamwidths available from such an arrangement. Because of the geometry and mutual coupling between array elements, embedded element beamwidths tend to be much greater than that of an isolated element. A frequency of 425 MHz was selected for the study, and array sizes of 16, 20, and 24-foot diameters between phase centers were used. This corresponds to an inter-element separation of approximately 0.36 λ for the 16-foot diameter array with 60 elements, and approximately 0.81 λ for the 24-foot diameter array with 40 elements.

Consider a circular array lying in the x-y plane. The array pattern is given by

$$ F(\phi, w) = \sum_{m=1}^{M} w_m g(\phi - \phi'_m) e^{jka \cos(\phi - \phi'_m)} , $$

(5.2)

where $w_m$ is a complex element weight, and $a$ is the radius at which the element is located. A complex exponential is included since the element pattern $g(\phi - \phi'_m)$ is purely real. To relate levels of sidelobes and grating lobes with available beamwidths and directivity, two types of array weights were used for the comparison. The first is the uniform cophasal excitation used in Chapter 2, and the other is the linear least-squares (LLS) method as defined in Chapter 4. For the uniform excitation, we use the number of elements that maximizes the directivity. For the LLS method, we set $M/2 + 1$ elements active. The LLS method is convenient for such a study, since a single parameter can be used to adjust the array pattern characteristics. By varying the angular width of the exclusion region, or target null-null beamwidth, a trade between spurious radiation levels and directivity/beamwidth can be made.

Performance measures include the relative directivity, half-power azimuth beamwidth, maximum sidelobe, and maximum grating lobe, which has been separated from
maximum sidelobe to lend insight into the mechanism of grating lobe formation. The following comments may assist in interpreting the figures:

**Target Null-to-Null BW:** Used as the independent variable, this represents the exclusion region width in degrees. Data was collected for 2° increments. The leftmost data point on each figure is for the uniform excitation, not for a 0° exclusion region.

**Relative Directivity:** Computed directivity with respect to the result for the configuration with 60 elements, 20-foot phase center diameter, and 160° element beamwidth and a uniform amplitude excitation. Since the patterns are two-dimensional (2-D), only energy on the x-y plane is considered in the calculations. Strictly speaking, a volume integral must be computed to obtain the directivity. The relative directivity is nonetheless a useful concept for the study and facilitates rapid collection of data.

**Maximum Sidelobe:** Peak amplitude of closest sidelobe (to main beam) with respect to boresight directivity.

**Maximum Grating Lobe:** Peak amplitude of lobe appearing at least 90° from the main beam.

The data follows in Figs. 5.1-5.18.
Figure 5.1—60 elements, 16-foot phase center diameter, 160° element beamwidth.

Figure 5.2—60 elements, 16-foot phase center diameter, 120° element beamwidth.
Figure 5.3—60 elements, 16-foot phase center diameter, 80° element beamwidth.

Figure 5.4—60 elements, 20-foot phase center diameter, 160° element beamwidth.
Figure 5.5—60 elements, 20-foot phase center diameter, 120° element beamwidth.

Figure 5.6—60 elements, 20-foot phase center diameter, 80° element beamwidth.
Figure 5.7—60 elements, 24-foot phase center diameter, 160° element beamwidth.

Figure 5.8—60 elements, 24-foot phase center diameter, 120° element beamwidth.
Figure 5.9—60 elements, 24-foot phase center diameter, 80° element beamwidth.

Figure 5.10—40 elements, 16-foot phase center diameter, 160° element beamwidth.
Figure 5.11—40 elements, 16-foot phase center diameter, 120° element beamwidth.

Figure 5.12—40 elements, 16-foot phase center diameter, 80° element beamwidth.
Figure 5.13—40 elements, 20-foot phase center diameter, 160° element beamwidth.

Figure 5.14—40 elements, 20-foot phase center diameter, 120° element beamwidth.
Figure 5.15—40 elements, 20-foot phase center diameter, 80° element beamwidth.

Figure 5.16—40 elements, 24-foot phase center diameter, 160° element beamwidth
Figure 5.17—40 elements, 24-foot phase center diameter, 120° element beamwidth

Figure 5.18—40 elements, 24-foot phase center diameter, 80° element beamwidth
5.2.1 Performance vs. Number of Elements

Because the number of elements in the array has such a strong influence over the overall performance and only arrays with 40 and 60 elements had been examined, it was deemed appropriate to include several additional simulations. The array diameter was fixed at 20 feet, the element beamwidth at 120°, and the number of array elements was varied (still using \( M/2 + 1 \) active elements). Included are the results for two cases. For the first case, a maximum sidelobe of -30 dB with respect to the main beam was specified, and the target null-to-null beamwidth was iteratively adjusted until this level was achieved. The number of elements was increased in increments of ten, and the first increment that satisfied the desired sidelobe condition was treated as 0 dB relative directivity. The simulation was again repeated for a -35 dB maximum sidelobe. Data is recorded in Tables 5.1 and 5.2.

<table>
<thead>
<tr>
<th># Elements</th>
<th>BW [Deg]</th>
<th>Grating Lobe [dB]</th>
<th>Rel Directivity [dB]</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>40</td>
<td>11.6</td>
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</tr>
<tr>
<td>50</td>
<td>8.5</td>
<td>31.9</td>
<td>1.5</td>
</tr>
<tr>
<td>60</td>
<td>7.3</td>
<td>38.7</td>
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<td>70</td>
<td>6.5</td>
<td>38.7</td>
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</tr>
<tr>
<td>90</td>
<td>6.2</td>
<td>38.5</td>
<td>3.0</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th># Elements</th>
<th>BW [Deg]</th>
<th>Grating Lobe [dB]</th>
<th>Rel Directivity [dB]</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
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<tr>
<td>40</td>
<td>NA</td>
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<td>0</td>
</tr>
<tr>
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<td>7.9</td>
<td>39.5</td>
<td>0.8</td>
</tr>
<tr>
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<td>7.0</td>
<td>42.5</td>
<td>1.3</td>
</tr>
<tr>
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<td>90</td>
<td>6.6</td>
<td>42.2</td>
<td>1.6</td>
</tr>
</tbody>
</table>
One final simulation was conducted on several array configurations to determine the effect of using less (or more) elements. All results use a fixed target null-to-null beamwidth.

**TABLE 5.3**
PERFORMANCE VS. # ACTIVE ELEMENTS (60 ELEMENTS, 16 FT. ARRAY, 160° BW)

<table>
<thead>
<tr>
<th># Active Elements</th>
<th>Rel Directivity [dB]</th>
<th>Beamwidth [Deg]</th>
<th>Max SL</th>
<th>Max GL</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>0</td>
<td>8.3</td>
<td>-32.3</td>
<td>-33.5</td>
</tr>
<tr>
<td>15</td>
<td>-1.5</td>
<td>11.5</td>
<td>-20.3</td>
<td>-27.1</td>
</tr>
<tr>
<td>21</td>
<td>-0.7</td>
<td>9.6</td>
<td>-25.2</td>
<td>-30.5</td>
</tr>
<tr>
<td>60</td>
<td>0.1</td>
<td>7.9</td>
<td>-35.3</td>
<td>-35.5</td>
</tr>
</tbody>
</table>

**TABLE 5.4**
PERFORMANCE VS. # ACTIVE ELEMENTS (60 ELEMENTS, 20 FT. ARRAY, 120° BW)

<table>
<thead>
<tr>
<th># Active Elements</th>
<th>Rel Directivity [dB]</th>
<th>Beamwidth [Deg]</th>
<th>Max SL</th>
<th>Max GL</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>0</td>
<td>7.5</td>
<td>-32.0</td>
<td>-39.1</td>
</tr>
<tr>
<td>15</td>
<td>-1.1</td>
<td>9.7</td>
<td>-22.1</td>
<td>-31.7</td>
</tr>
<tr>
<td>21</td>
<td>-0.5</td>
<td>8.3</td>
<td>-27.3</td>
<td>-36.4</td>
</tr>
<tr>
<td>60</td>
<td>0</td>
<td>7.5</td>
<td>-32.5</td>
<td>-39.9</td>
</tr>
</tbody>
</table>

5.3 GRATING LOBE FORMATION

In Chapter 2, we presented a mathematical explanation for grating lobe formation and their movement as the inter-element separation increases. Fig. 5.19 provides an illustration of this movement. All patterns were created with the LLS method using a fixed target null-to-null beamwidth.

The figure below demonstrates the mechanism by which grating lobes form first at the rear for an inter-element separation of approximately 0.5λ, and move from the rear to the side for increasing separation. In Fig. 5.20(a) we depict a circular array consisting of some number of elements having a phase center to phase center distance of 0.5λ. By utilizing one-half of the array, the natural phase difference between two elements at the edge of the active portion (circled) is approximately 180°. The radiation pattern for a pair
Figure 5.19—40 elements, 120° element beamwidth, with inter-element separation of (a) 0.44λ, (b) 0.56λ, (c) 0.67λ, and (d) 0.78λ.

of isotropic elements with this combination of spacing and phasing takes the form of a figure-eight. One lobe radiates in the general direction of the main beam, while the other radiates rearward. While the element patterns used in this study are of course not isotropic, they are relatively broad, and tend to support this form of radiation. Assuming
that a taper is applied to the array, the power applied to these edge elements is small, so that grating lobe energy remains small (as in Fig. 5.19(a)).

As the inter-element separation increases, a more serious condition occurs. For the situation in Fig. 5.20(b) where the separation is $0.6\lambda$, the natural phase difference for the edge elements is now $216^\circ$. As one considers each successive pair of elements moving in the direction of the main beam, the phase difference is observed to be decreasing. We find that some pair of elements will have a relative phase difference of approximately $180^\circ$, but this new pair will be closer to the beam boresight angle. This undesirable condition is set up between a pair of elements having more power applied to them. As the inter-element separation grows, this condition occurs closer and closer to the azimuth angle of the main beam, where the power is at a maximum. This describes both the angular movement and growth of grating lobes as shown by the sequence in Fig. 5.19.

![Figure 5.20—Physical mechanism of grating lobe formation. Inter-element separation of (a) 0.5\(\lambda\) (b) 0.6\(\lambda\).](image)

**5.4 RELATIVE ENERGY IN ELEMENT PATTERNS**

Although the study considers only 2-D patterns, an additional observation should be noted regarding the energy contained in the $x$-$y$ plane for element patterns. Various array configurations were modeled with the Numerical Electromagnetic Code (NEC). These
configurations included arrays of 40 to 60 elements, using the Yagi-Uda, Log-Periodic, and active endfire (dipole array) elements. The embedded element beamwidth was adjusted by varying the radial placement of each element. Frequently the element required some adjustments (i.e., wire lengths and spacings) to compensate for changes in the F/B ratio resulting from mutual coupling. Once a desirable element pattern was obtained, an integration of the energy through the x-y plane was performed.

In general, the energy contained the x-y plane is positively correlated to the embedded element beamwidth. A concrete relationship is difficult to arrive at, because the difference between narrow and broad element patterns depends upon the individual array. It should be noted that in every case considered, the difference in energy ranged from +0.2 to +0.7 dB when comparing the broadest possible element pattern for a given configuration (140° to 150°) to one of 80° to 90°. Fortunately, there is a one-to-one correspondence between the differences in the element pattern energy and the overall array directivity. As such, any differences in the pattern energy can be treated independently, and added later for comparison.

5.5 PHASE CENTER MOVEMENT

An interesting phenomenon associated with the circular array is the tendency for the phase centers to move inward as the element-to-element coupling increases. Consider a single element of the circular arc shown below ($f_0 = 425$ MHz). A simulation was run with the NEC, where the Yagi was placed along the x-axis. Define a far-field phase measurement $\psi(\phi)$ of the element, which is taken over the entire azimuth space ($-\pi \leq \phi < \pi$). The Yagi was initially placed such that the center of the reflector lay at $x = 0, \ y = 0$. The Yagi was then moved rearward (towards the reflector end) by an amount $\Delta x$, and a new phase measurement was taken. This process was repeated until sufficient measurements were obtained. In reality, one simulation is sufficient, as a new far-field phase measurement $\psi'(\phi)$ can be determined by
\[ \psi'(\phi) = \psi(\phi) + k\Delta x \cos(\phi) \]  \hspace{1cm} (5.3)

Figure 5.21—Circular arc used for isolated and embedded element phase center determination. Shown with 127-inch translation distance from array center to reflector. Individual element length = 32 inches.

This leads to a family of phase curves—each associated with a particular location along the Yagi’s symmetric axis. The phase curves from the output file are plotted in Fig. 5.22.

Figure 5.22—Far-field phase (sampled at 1x10^6 meters from antenna) of isolated element (\(\phi\)-plane component), placing various positions of test antenna on the coordinate center. Top curve corresponds to \(x=0\) inches, bottom curve corresponds to \(x=31\) inches (1-inch increments).

The majority of references pertaining to phase center determination are analytical methods that depend on the knowledge of structure currents on or aperture fields across a specific geometric shape [59], [60]. A method proposed by Best [61] provides an
accurate albeit aspect dependent determination of a phase center location from a set of numerical data. We prefer a numerical method whereby an aspect independent phase center of an antenna can be determined from a similar set of modeled or measured data. Suppose $N$ measurements are taken; an expected value of the phase can be determined by

$$\psi_{\text{mean}} = \frac{1}{N} \sum_{n=1}^{N} \psi(\phi_n)$$

(5.4)

The effective phase center of the radiating object can be determined by minimizing the sum of the differences between the individual phase measurements $\psi(\phi_n)$ and $\psi_{\text{mean}}$. Equation (5.4) is sufficient for the case of elements that are approximately isotropic in the $x$-$y$ plane. A more appropriate method would weight the far-zone phase contributions by the energy radiated in the respective direction. For example, in the pattern below, we weight the phase measured at $\phi = 0^\circ$ more heavily than the phase measured at $\phi = 180^\circ$, where the pattern strength is down approximately 20-25 dB.

Figure 5.23—Element patterns for isolated (...) and embedded cases with 127-inch translation distance from array center to reflector (- - -), and 175-inch translation distance (—).
Define a new quantity, the *weighted* expected value:

\[
\psi_{\text{weighted mean}} = \frac{1}{\sum_{n=1}^{N} |F(\phi_n)|^2} \sum_{n=1}^{N} |F(\phi_n)|^2 \psi(\phi_n),
\]

(5.5)

where \(F(\phi)\) is a complex electric field value. We also need to know the absolute difference between an individual phase measurement, \(\psi(\phi)\), and the weighted expected value:

\[
U = \left| \psi(\phi_n) - \frac{1}{\sum_{n=1}^{N} |F(\phi_n)|^2} \sum_{n=1}^{N} |F(\phi_n)|^2 \psi(\phi_n) \right|,
\]

(5.6)

\(U\) represents a single weighted phase error. We now consider \(U\) for all values of \(\phi\), once more applying a weighting scheme so that the emphasis placed upon the difference is proportional to the radiated energy:

\[
\gamma = \frac{1}{|F(\phi_n)|^2} \sum_{n=1}^{N} |F(\phi_n)|^2 U.
\]

(5.7)

\(\gamma\) represents a mean weighted phase error. Equation (5.7) returns a single value for each curve in Fig. 5.22 (which shows only \(0^\circ \leq \phi \leq 90^\circ\)). If these points are plotted for all possible phase curves, a minimum error is reached at some point, which is the effective phase center. Fig. 5.24 shows the phase center movement rearward as the coupling increases. For the most tightly packed array, phase centers moved rearward by more than the length of the element itself. The minimum phase error of the embedded patterns was somewhat higher than that of the isolated pattern, since the phase of the embedded
element patterns has higher ripple. The curves below correspond to the element patterns in Fig. 5.23.

![Graph showing mean weighted phase error for isolated and embedded elements.]

**Figure 5.24**—Mean weighted phase error plotted for isolated and embedded elements.

### 5.6 SUMMARY OF RESULTS

A considerable amount of data has been presented in this chapter. The following conclusions can be drawn:

1) Embedded element beamwidth is not critical. Figs. 5.1-5.18 shows that the element beamwidth barely affects the directivity or the array beamwidth. A small advantage appears to belong to the narrower element patterns, since they do not support the grating lobe condition as readily as the broader patterns.
2) Although some endfire effect does come from the depth of the array for elements with very large beamwidths, the increased element-to-element coupling associated with broad element patterns tends to result in lower array efficiency \(i.e.,\) the reflection efficiency, as discussed in Chapter 4. These two effects (increased energy for broad elements vs. decreased array efficiency) tend to offset each other, further validating point #1.

3) The arrays appear to work best at an inter-element spacing of approximately \(0.5\lambda\), as observed in Chapter 2. As the spacing increases, the directivity falls as more energy is contributed to spurious energy. Clearly, the 40-element arrays worked best with a 16-foot phase center diameter, while the 60-element arrays returned the optimum performance with a 20-foot diameter array.

4) The number of elements has a marked effect on the array's ability to compress the azimuth beamwidth and spurious radiation levels. As demonstrated in Tables 5.1 and 5.2, a point of diminishing returns comes for a fixed radius array. Once this point is reached, additional elements do not result in additional performance.

5) Utilization of less than one-half of the array typically results in decreased directivity, at least for element beamwidths equal to or greater than \(120^\circ\).

6) Grating lobes form nearly opposite to the main beam direction for inter-element spacings of approximately \(0.5\lambda\). As the spacing increases, the grating lobes increase in size, and move in the direction of the main beam.

7) Increased element-element coupling causes broader element patterns, and also causes the element phase centers to move towards the center of the array. When adding or subtracting elements to a physical array, one should not assume that phase centers remain at the same radius. As such, performance can be sensitive to changes in the number of array elements.
8) The number of elements that maximized directivity for the uniform cophasal excitation was typically about $M/2$, and slightly fewer for the $80^\circ$ element patterns. The peak directivity is not highly sensitive to the addition or deletion of a few active elements.
6 ARRAY DESIGN, FABRICATION, AND MEASUREMENT

6.1 INTRODUCTION

This chapter describes the optimization, fabrication, modeling, and measurement results of the element and the array. A significant portion pertains to the thought processes, methodologies, and steps undertaken to arrive at a final design. The design process involved numerous decisions, and included movement along paths that did not result in success. To avoid discussing these would leave many unanswered questions. On the other hand, addressing each issue in minute detail with sufficient supporting analysis would result in an excessively long chapter. To maintain a reasonable balance, each topic is discussed briefly. Measurements of a complete 54-element ¼ scale model array are compared with the electromagnetic models for the element and the array.

6.2 SELECTION OF PROTOTYPE ARRAY PARAMETERS

The first step was to select a prototype set of array characteristics based upon the program requirements and results from the array studies in Chapters 2 and 5. As they appeared in the thesis introduction, the array requirements are as follows:

- Operating frequency: 400-450 MHz
- Polarization: horizontal
- System gain: ≥ 20.5 dBi
- Azimuth beamwidth: ≤ 8.2°
- Elevation beamwidth: ≤ 33°
- Spurious Radiation ≤ 35 dB down from peak
To recap the results of the array study, we recall that the embedded element beamwidth did not play a significant role in the overall array pattern characteristics. The phase center location did have some effect on the directivity, beamwidth, and spurious radiation, but it too had a limited effect when compared to the number of elements in the array. The array studies of Chapters 2 and 5 do not provide information about meeting the system gain and elevation beamwidth requirements. They do, however, indicate that a good starting point would be to choose a number of elements that meets the specified azimuth beamwidth and spurious radiation levels. The number of elements is also a convenient parameter to fix, since it is not a function of the element design, coupling, etc., as are the element beamwidth and phase center location. For convenience, Table 5.2 is copied below:

Table 6.1 shows the relationship between the number of array elements (with approximately ½ active), the azimuth plane beamwidth, and the grating lobes, given a -35 dB maximum sidelobe. The simulation was run for the case of a 20-foot array diameter and a 120° element beamwidth. Based upon these results, arrays having less than 50 elements were eliminated from consideration. While the data in Table 5.3 shows that additional performance can be obtained by using elements in the "rear", we prefer to not base any decisions on this information. The use of all array elements, while reducing spurious radiation levels, results in a small (if any) increase in array directivity, and adds to the required beamforming processor load. From Table 6.1 we see that the smallest number of elements (at least in increments of 10) that can satisfy the requirement is 60.

<table>
<thead>
<tr>
<th># Elements</th>
<th>BW [Deg]</th>
<th>Grating Lobe [dB]</th>
<th>Rel Directivity [dB]</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>40</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>50</td>
<td>9.4</td>
<td>33.6</td>
<td>0</td>
</tr>
<tr>
<td>60</td>
<td>7.9</td>
<td>39.5</td>
<td>0.8</td>
</tr>
<tr>
<td>70</td>
<td>7.0</td>
<td>42.5</td>
<td>1.3</td>
</tr>
<tr>
<td>80</td>
<td>6.7</td>
<td>43.5</td>
<td>1.5</td>
</tr>
<tr>
<td>90</td>
<td>6.6</td>
<td>42.2</td>
<td>1.6</td>
</tr>
</tbody>
</table>
6.3 ELEMENT SELECTION

We had thus far considered only ideal element patterns, with no regard as to how the assumed set of desirable characteristics (e.g., pattern smoothness, good F/B ratio, etc.) would be obtained. Forced with an element selection and design, we next considered all physical limitations and constraints imposed by the enclosure, power handling requirements, other co-existing antennas, and electromagnetic code limitations. These are discussed below.

**Allotted Space:** The entire UHF Electronically Scanned Array (UESA) must fit into an aerodynamic radome whose exterior surface is 24 feet in diameter and 30 inches in height at the center. The exterior surface is specified by the equation of an ellipsoid:

\[
\frac{x^2}{144^2} + \frac{y^2}{144^2} + \frac{z^2}{15^2} = 1, \quad (6.1)
\]

where all distances are measured in inches from the center of the radome. The radome material is composed of a lightweight honeycomb material of 1-inch thickness sandwiched between two high-strength epoxy skins of 0.030-inch thickness each.

**Power Handling:** Each element must be capable of handling up to 60 kilowatts peak power at 35,000 feet with a 6% duty cycle.

**IFF Antenna Coexistence:** The UHF array must work in conjunction with a vertically polarized Identification: Friend or Foe (IFF) system, whose frequency of operation is 1.03-1.09 GHz. The UHF array must not interfere with the IFF array, which also takes the form of a circular array, but with a smaller radius. Discussion of the IFF system will be limited in this document, due to classification issues.
Because of these restrictions, several types of antennas were eliminated immediately. One type is a dense array of vertically oriented monopoles, which is fed against a groundplane that runs horizontally through the radome. The monopoles would be mounted on the top and bottom of the groundplane so that the take-off angle of the beam would be on the $\theta = 90^\circ$ plane. The "groundplane" is actually a pair of metal sheets separated by several inches to permit transmit/receive (T/R) modules or coaxial lines to be placed between them. This type of arrangement has some attractive qualities. For one, its construction is straightforward and simple. More importantly, this approach provides an easy way to utilize the entire aperture simultaneously, and electromagnetic models of the monopole array suggest that higher gain than any other approach is likely [62].

Unfortunately, the monopole array also has its difficulties (aside from the likely interaction with the IFF array). The first problem is that of feeding such an array. A densely packed array would require 200-300 element pairs (top and bottom). Current upgrade plans for the E-2C include a substantial increase in the surveillance radar transmitter power. The projected (in year 2005) weight estimate for the upgraded transmitter is 2000 pounds. The overall permissible weight of the complete radome assembly is only 2650 pounds, 800 of which is reserved for the radome shell. This eliminates the possibility of placing the T/R modules inside the radome. The other option is to run coaxial lines from the fuselage based transmitter to the radome, but the sheer weight of the coax in this case would also eliminate the monopole array from consideration. The monopole array has one more drawback. It has far too many channels to process with a space-time adaptive processor, which is necessary for the reduction of ground clutter.

Another possible array configuration is based upon a turnstile antenna. The turnstile antenna is, in its simplest form, a pair of crossed dipoles. The advantage of a turnstile array is that it can provide horizontal polarization. Unfortunately, it suffers the same weight and processing related shortcomings as the monopole array above. Because the
IFF array must radiate through the UHF array, other types of "solid" antennas, like patches, horns, or trough structures [63] were also not considered.

Endfire arrays of dipoles were considered. Ordinary endfire arrays do not exhibit the directivity of a Yagi-Uda array. The Hansen-Woodyard type array exhibits greater directivity than the ordinary endfire array—approximately [64]

\[ D \approx 7.28 \frac{L}{\lambda} \]  

where \( L \) is the overall length of the antenna in wavelengths. Even these, however, have not unilaterally demonstrated a realistic potential for greater directivity than the Yagi-Uda. In addition, fully active endfire arrays are in general more difficult to feed, requiring an \( N \)-way splitter, and coaxial lines that add weight and increase loss. Another problem experienced with this type of array is that the front elements tend to exhibit negative resistances [65]. For these reasons, active endfire arrays have been largely abandoned over the past several decades.

Conversely, the Yagi-Uda array has the potential for substantial gain, has a single feed, and with modern computers, its characteristics can be tailored for a specific application. For these reasons, the Yagi has increased tremendously in popularity, and represented the logical choice to serve as the UESA array element.

### 6.4 ELEMENT DESIGN

Given that a Yagi-Uda was the selected starting point, the next task was to maximize the effective aperture given the physical size and power handling constraints. The first question was to determine which electromagnetic code or codes to use. The choice came down to several packages available for use at the Naval Air Warfare Center-Aircraft Division (NAWCAD) in Patuxent River, Maryland. The first possibility was a finite difference time domain (FDTD) code—the \( XFDTD \) [66] software—written by Remcom, Inc. This package was ultimately not used, because the Cartesian gridding scheme does
not lend itself well to thin wires lying at odd angles to the grid. The unwritten "industry standard" for modeling (or measurement) of an embedded element is that the mutual coupling of seven to nine consecutive elements should be accounted for (four on either side of the active element). Another software package considered for the design was *EIGER* [67], a new hybrid MoM/high frequency code written by Lawrence Livermore Laboratories. Unfortunately, the EIGER beta version was not released until September 1999. The remaining choice was the Numerical Electromagnetic Code (NEC). NEC has several characteristics which make it well suited to the job. Being a MoM (integral) code, it has no geometrical limitations other than those imposed by the computational considerations. As such, elements can be located at arbitrary positions and angles. The frequency domain approach is also suited to a UESA array design, since it is a relatively narrow band application. Perhaps the most powerful capability of the NEC is that it can exploit the rotational symmetry of the problem. With no symmetry to exploit, the NEC has a matrix storage requirement that is proportional to $N^2$, a matrix fill time proportional to $N^2$, and matrix factor time proportional to $N^3$, where $N$ is the number of wire segments. With rotational symmetry, the storage and fill time requirements are reduced by a factor of $M$, and the factor time is reduced by $M^2$, where $M$ is the number of symmetric sections. For an array of approximately 60 elements, this amounts to an enormous payoff in the execution speed and memory requirement. In fact, using the rotational symmetry allows the entire array to be modeled faster than if a "adequate" portion (e.g., a 60° arc) were modeled with no symmetry.

NEC does impose certain geometric limitations. It is essentially a wire code, so that the element design must be based upon wires. Although it can model plates, the plates are implemented with a magnetic field integral equation (MFIE) formulation. A consequence of this approach is that any structures "built" with MFIE plates must be enclosed, and must have some non-zero thickness [39]. Volumes that approach zero are not permitted. As such, the surface patches in NEC are of limited use in modeling antennas.
**Note:** For further discussion here and to avoid confusion of array elements with elements of the Yagi antenna (*e.g.*, a 6-element Yagi), we shall refer to the Yagi elements as *wires*.

It should be obvious that aside from what has already been determined about the array configuration, that the best way to maximize the effective aperture of the array is to compress the elevation beamwidth of the embedded element to the smallest possible value. Common knowledge of the Yagi-Uda antenna tells us that we should make the overall length as long as possible. The difficulty with this is that the longest wire, the reflector, is in the rear where the least space is available. The typical length for a reflector in term of wavelengths is approximately $\frac{\lambda}{2}$ or slightly longer. At the center frequency (425 MHz), 60 full-length reflectors do not fit even when placed at a 10-foot radius. If the reflectors are placed 10 feet from the center of the radome, then the maximum length of a given element is 2 feet. Such an arrangement would lead to poor volumetric utilization. Any means of lengthening the element would result in increased gain not only because they are longer, but because the available vertical space increases towards the center of the radome.

One variation of the Yagi is the so-called *cubical quad*, or quad for short [68]. The quad is essentially a Yagi that uses loops instead of dipole elements. The length of the loops is approximately one wavelength long. Like the Yagi, the reflector is slightly longer than one wavelength, and the directors are slightly shorter. The quad was considered to be an excellent candidate for the circular array, since the longest side of a (square) loop is approximately $\frac{\lambda}{4}$. This allows the element length to be much greater than for a traditional Yagi.

Unfortunately, none of the attempts to construct a circular array model using quads were successful. The main problem was that the vertical wires of each loop are parallel to those of the adjacent element, making an efficient form of backward wave coupler and causing large deviations of the active impedances. The other problem is that the vertical wires would likely be a source of interference for the vertically polarized IFF.
One other non-successful attempt was made to use some very long (14-wire) Yagis. The philosophy was to use an element whose length was nearly 2/3 the radius of the radome, and stagger the layers so that 60 elements would fit. Numerous NEC models were generated. While inter-element coupling was not excessive, the element patterns were poorly behaved, having ripple as high as 9 dB.

One way to shorten the length of a traditional Yagi is to place "disks" on the end of the wires in a manner akin to a top-hat monopole [69]. This is perhaps the best way, but such an arrangement is not supported directly by NEC. An alternative is to bend some of the rear wires, which ultimately became the solution to the element length problem.

The next question to address was what diameter tubing to use. Three possibilities were initially considered, largely because of their availability. These included 0.5-inch, 0.75-inch, and 1.0-inch thicknesses. Several competing factors drove the selection of the tubing thickness. The first is the power handling ability. Corona is typically the limiting factor for an antenna of this sort. Corona is most likely to occur on the ends of driven wire, where the electric fields are many times higher than at the feedpoint. A simulation was conducted to gauge the order of the field strengths that could appear. NEC has the ability to determine the electric and magnetic fields immediately surrounded a structure. The simulation was conducted using a resonant dipole, tuned to 425 MHz. For a dipole lying along the \( y \)-axis and fed with a 1-volt source, the electric fields on the \( y \)-axis are as shown in Fig. 6.1. The peak fields occur at the end, directly on the axis. According to NEC, a field strength of nearly 220 volts/meter per volt of input appears at the end of a dipole constructed of 0.5-inch tubing. From a report published by Hayt et al., [70], breakdown at 450 MHz at an altitude of 35,000 (standard temperature and pressure) occurs at 7.5 x \( 10^5 \) volts/meter. If we assume that the antenna will be designed to have a 50 \( \Omega \) impedance, and that the maximum VSWR ever experienced will be 3:1, then the maximum input voltage (for a given power level) occurs when the input impedance is 150 \( \Omega \). For a 60 kilowatt per element power level, this corresponds to a peak of approximately 3000 volts at the input (we further assume that a two-layer Yagi will
result, so that there are two feeds per element). Multiplying times the peak value from Fig. 6.1, we arrive at peak of nearly $6.5 \times 10^5$ volts/meter for the 0.5-inch tubing. If the 0.75-inch inch diameter tubing is selected, the peak fields are reduced to about $3.9 \times 10^5$ volts/meter, which provides a safety factor of nearly 2.

Figure 6.1—Electric field on the axis of a UHF dipole (1-volt input). The dipole end is at $y = 0.159$ meters.

The idea of using larger diameter tubing was abandoned for several reasons. Ultimately, both are related to the thin-wire nature of NEC. Although the code can handle wires with significant radii, it assumes that the currents are axial only and uniformly distributed around the wire circumference. In three parts of the antenna, this can be assumed not to be the case. The first example is in the corners required to shorten the overall length of the rearmost wires. Currents are known to flow non-uniformly through a corner of a wire structure with substantial radius [71]. The second case is for the driven wire and first director pair, which are typically in close proximity (and couple strongly) for broadband Yagis. The third case is the feed itself. The transmission line will attach to one side of the
driven wire, and non-uniformities of the current are assured. Using excessively large tubing would only exacerbate the computational inaccuracies.

6.5 OPTIMIZATION WITH NEC-OPT

Numerous analytical formulations [72] and experimental investigations [73] pertaining to the design of the Yagi have appeared in the literature over the past seventy years. In recent years, more attention has been focused on numerical optimization techniques [74]. A common method of optimizing the Yagi (or other antennas) is to utilize an existing electromagnetic code in conjunction with an optimizer in an attempt to minimize a user defined error function. A block diagram of such a framework is shown below.

![Antenna design suite architecture.](image)

A prototype optimizer developed by Paragon Technology called NEC-OPT [75] was obtained and used for the first stage of the element optimization. Based upon the NEC 4.0 engine, it has an architecture as in Fig. 6.2. NEC-OPT uses a quasi-Newton optimization algorithm [76], which finds the solution of \( \min_{x \in \mathbb{R}^n} f(x) \) using only function values. Given a starting point \( x_c \), the search direction is computed according to the formula

\[
d = -H^{-1}g_c, \quad (6.3)
\]

where \( H \) is a positive definite approximation of the Hessian matrix, and \( g_c \) is the gradient evaluated at \( x_c \). A line search is then used to find a new point.
\[ x_n = x_c + \lambda d, \quad \lambda > 0 \]  \hspace{1cm} (6.4)

such that

\[ f_n \leq f_c + \alpha g^T d, \quad \alpha \in (0,0.5). \]  \hspace{1cm} (6.5)

The procedure is iterated until \( \|g(x)\| \leq \varepsilon \), where \( \varepsilon \) is a gradient tolerance. So long as the optimality condition is not achieved, the Hessian approximation can be updated according to the following formula, for example [76]

\[ H_{k+1} = H_k - \frac{H_k s_k s_k^T H_k}{s_k^T H_k s_k} + \frac{y_k y_k^T}{y_k^T s_k^T}, \]  \hspace{1cm} (6.6)

where \( s = x_{k+1} - x_k \) and \( y_k = \nabla f(x_{k+1}) - \nabla f(x_k) \).

The quasi-Newton optimization algorithm is appropriate for this application, since analytical derivatives are not available. The prototype NEC-OPT permits 20 variables to be optimized, and allows a user-defined error function to be minimized. The error function can be composed of any combination of NEC outputs, such as directivity, VSWR (with arbitrary \( Z_0 \)), F/B ratio, etc. The individual components of the error function may be weighted to bring about the desired characteristics. In addition to the optimization variables, NEC-OPT provides function variables, which are related to the optimization variables through a mathematical expression. The function variables provide additional flexibility to the optimization process.

NEC-OPT also permits global optimization to be performed. It can step through an \( N \)-dimensional problem space, either systematically or in a random fashion. Each optimization variable can be bounded to limit the amount of search space. The
optimization variables associated with each "run" are recorded in a file along with the composite error. The ability to optimize globally is critical—even the simplest antennas usually result in an ill-defined error function, as opposed to the bowl-shaped quadratic function represented by the pattern function, where the error minimum can be found on the first try.

6.6 ELEMENT EVOLUTION AND MODELS

In order to maximize the effective aperture of the element, a two (vertical) layer element framework was assumed. A three-layer arrangement was also considered, but was abandoned because 1) use of three layers eliminates the possibility of using z-axis symmetry during the optimization process, and 2) it was anticipated that since the vertical separation is limited, the greatest gain would be achieved with a two-level arrangement. Three levels would likely result in vertical "overpopulation", since at a radius of 8 feet from the radome center, less than one wavelength of available vertical space exists. One additional payoff of limiting the element to two layers is computational savings. Despite the advantage realized from the use of rotational symmetry, the computational and memory resources required to model the complete array are still significant.

Because the array performance is dictated by the embedded element patterns, the logical solution would be to optimize the embedded element pattern. The difficulty with this philosophy is the computation time required. The prototype element had approximately 250 segments, and with 60 elements in the array, this corresponds to a computational time of about 9 minutes. For the initial attempt to optimize a Yagi element, 6 frequencies were used, with 400 MHz as the lower limit and 450 MHz as the upper limit. After observing significant deviation of the VSWR from the specified goal between steps, the number of steps was increased to 11 to ensure consistent behavior across the frequency band.

NEC-OPT runs the model, computes the error function, and perturbs each optimization variable (e.g., a wire length) a small amount to obtain a numerical derivative. As
implemented, NEC-OPT runs the model twice for each optimization variable to compute a gradient. The prototype element employed fifteen optimization variables. Thus, the time required by NEC-OPT to make a single adjustment to the embedded element would be 9 minutes × 11 frequencies × 2 steps/derivative × 15 variables = approximately 50 hours. Typically the optimization process requires between 30 and 40 adjustments for a local optimization, and 20 to 30 runs must usually be made to ensure a satisfactorily low error function. In other words, with the available computer resources, optimization of the embedded element was not feasible. Instead, an isolated element optimization approach was chosen. It was possible, in the end, to associate desirable and undesirable pattern and impedance characteristics of the embedded element with the corresponding characteristics of the isolated element.

The error function used to design the final element consisted of 1) average gain over a ±40° sector (from element boresight) on the θ = 90° plane, and 2) element VSWR, with a weight of 10 and a threshold of 1.3. The heavy weighting was required to get the desired impedance match, and the threshold of 1.3 made the optimizer work on other portions of the band when the VSWR at a given frequency reached 1.3 or lower. The 1.3 value was chosen as a result of numerous trials. If the VSWR of the element rose higher than 1.5, the array efficiency was typically poor (<85%). No emphasis was placed on elevation sidelobes or F/B ratio. The average gain goal by itself was sufficient to provide well-behaved element patterns. Specification of average gain over a ±40° sector was employed to keep the azimuth pattern of the isolated element pattern from becoming too narrow, which led to excessive ripple in the embedded element pattern.

While analysis and computer optimization were used whenever possible, the "art" can never be removed from the design of an antenna such as the UESA. No analysis, for example, can provide information regarding the maximum permissible length of the element. This and other decisions are based upon little more than intuition, observation, and iteration. The NEC-OPT optimizer can select the proper length of a wire, the space
between two wires, etc. It cannot decide what the basic element should look like, how many wires should be used, or which direction the shortening stubs should point. The first element optimized with NEC-OPT is shown in Fig. 6.3. It used eight wires on each of two layers, and had shortening stubs placed on the four rearmost wires. The length of each shortening stub was fixed, and the length of each horizontal wire and all inter-wire spacings were optimized. By using function variables, the vertical separation between the upper and lower layers was specified to be a function of the radial position of each individual wire, so that each wire would just clear the inside of the radome surface. The characteristic impedance of the Yagi was selected to be 37 $\Omega$, as this yielded an error function slightly better than obtained with a 50 $\Omega$ characteristic impedance. The design was iterated until 60 elements could be placed in the array with at least 1-inch clearance from any portion of one element to the next. This Yagi resulted in well-behaved embedded element patterns, but the shortening stubs of adjacent elements coupled strongly, causing the active impedances to swing wildly from their nominal value. Overall array efficiencies as defined by (4.16) were poor, typically around 60%.

A second element was optimized in which the shortening stubs were placed at odd angles to minimize element-element coupling (shown below). The intent was to make the left and right side stubs at 90° relative to each other, so that any stub is orthogonal to the closest stub from an adjacent element. This holds true for the driven wire and the second director. The reflector stubs are bent at 180° to each other since they are the longest stubs,
and this limits the vertical intrusion towards the radome center. This desire was driven primarily from concerns about interference with the IFF array. The first director stubs are bent 45° relative to each other. Originally this wire had stubs very similar to the second director, but it was impossible to obtain the desired VSWR bandwidth without placing at least one stub parallel to the driven wire. A bandwidth of nearly 12% is very aggressive for a Yagi antenna. The key to obtaining this kind of bandwidth is in strong coupling between the driven wire/first director pair. This concept is borrowed from *the Optimized Wideband Antenna* reported by Breakall [77].

The element of Fig. 6.4 became the basis for the first promising array model. The new stubs made a marked improvement in the overall array efficiency. With 60 elements, computed array efficiencies were on the order of 85%. An objective was to obtain efficiencies in excess of 90%, so the number of array elements was reduced by 6. With 54 elements, array efficiencies as high as 95% were observed with the NEC model.

A single full-size prototype element was fabricated at a nearby machine shop (Fig. 6.5). The wires were supported in Rohacell foam (dielectric constant ~ 1.1), and adjustable plungers were installed in the end of each shortening stub, since the ability of NEC to accurately model corners in conjunction with large radius wires was not known. The element VSWR, measured on a Hewlett-Packard 8510C, was a disappointment.
Regardless of the position of the plungers, the element exhibited poor VSWR above 430 MHz (> 3:1).

A new MoM code called WIPL [78]) was purchased shortly before the design had to be finalized (January '99). WIPL has the ability to model plates (using the electric field integral equation), wires, and dielectrics. A "brute force" model of the element was constructed, where the four rearmost wires were modeled as eight-sided structures using thin plates, as shown in Fig. 6.6. By doing so, all effects pertaining to the large diameter wires—currents around the corners, feed non-uniformities, and non-uniformities resulting from strong mutual coupling—would be accounted for. The front four wires were modeled as "thin wires" as in NEC. The front wires are straight and have greater wire-wire separation, so that currents could be expected to be uniformly distributed. This model required more than 1000 unknowns. WIPL immediately showed why the element exhibited poor VSWR in the upper half of the band. The radiation resistance dropped rapidly above 430 MHz.

WIPL was used to design a new element with acceptable VSWR. The necessary modifications included slight changes to the lengths of the driven wire, first and second
directors, and replacement of the fourth director by two separate directors. The input impedance of both elements is shown in Figs. 6.7 and 6.8. In addition to obtaining a radiation resistance that was more constant, the average value increased such that the element could be driven without the need for a 37-50Ω transformer. As an added bonus, the difference between the wire WIPL and NEC models of the corrected element was sufficiently small such that the array modeling could be performed in NEC using the circular symmetry command.

Figure 6.6—Upper half of geometry for (a) initial element, and (b) corrected element.
Figure 6.7—Radiation resistance of initial and corrected isolated elements.

Figure 6.8—Reactance of initial and corrected isolated elements.
Figure 6.9—VSWR of corrected element.

Figure 6.10—Input Impedance of isolated and embedded elements (from NEC).
The standard method of computing the directivity $D(\theta, \phi)$ is to relate the intensity of the electric or magnetic field at a far-zone point in space and the power input to the antenna. While the radiation patterns result from an integration of the currents over the entire structure, the impedance is the ratio of the applied electric field divided by the current through a single wire segment. As such, an accurate determination of the directivity depends upon an equally accurate determination of the input impedance.

One way to remove this dependency is to integrate the radiation pattern over all space. If the integral of the directivity taken over all space is other than $4\pi$, a correction factor is applied. For example, at 400 MHz, the NEC model exhibited 11.8% excess energy radiated, requiring a $-0.48$ dB correction to all points.

In general, the "health" of an electromagnetic model whose excess (or deficit) radiated energy is greater than 15-20% should be called into question [79]. Interestingly, the NEC model required significant corrections at all frequencies. This is attributed at least in part to the optimization process. Since the optimizer seeks to minimize an error function that is based upon the code outputs, the optimizer will attempt to find the "Achilles heel" of the code. Optimistic results are expected from such an automated process, subject to the accuracy of the code and the particular model in question.

Table 6.2 shows the computed directivities of the element on its boresight angle, correction factors, and corrected directivities for both the NEC and WIPL models. Once corrected, both models agree at all points within the band to an error of less than 0.1 dB.

<table>
<thead>
<tr>
<th>Freq (MHz)</th>
<th>Computed Dir. (dBi)</th>
<th>Correction (dB)</th>
<th>Corrected Dir. (dBi)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>NEC</td>
<td>WIPL</td>
<td>NEC</td>
</tr>
<tr>
<td>400</td>
<td>13.06</td>
<td>12.49</td>
<td>-0.48</td>
</tr>
<tr>
<td>425</td>
<td>13.70</td>
<td>13.02</td>
<td>-0.53</td>
</tr>
<tr>
<td>450</td>
<td>14.30</td>
<td>13.48</td>
<td>-0.67</td>
</tr>
</tbody>
</table>
Since WIPL does not yet have the rotational symmetry capability, all array modeling was performed with NEC. In Fig. 6.11 and 6.12 are the isolated and embedded element patterns.

Figure 6.11—Principal plane cuts of (a) azimuth ($\theta = 90^\circ$) and (b) elevation patterns of isolated element at 400 (…), 425 (- -), and 450 (–) MHz. Peak value of the graph is 13.6 dBi.
Figure 6.12—Principal plane cuts of (a) azimuth ($\theta = 90^\circ$) and (b) elevation patterns of embedded element at 400 (…), 425 (- -), and 450 (–) MHz. Peak value of the graph is 9.7 dBi.
6.7 ARRAY MODELS

The possible amount of array modeling data is enormous, given multiple frequencies, excitations, and number of elements used. In this section we provide results at the three principal frequencies of interest—400, 425, and 450 MHz. For each frequency, pattern results are given for three excitations:

1) Uniform excitation with 27 active elements.

2) Iteratively weighted linear least-squares derived excitation, with 27 active elements.

3) Iteratively weighted linear least-squares derived excitation, with 54 active elements.

Using techniques discussed in sections 4.2 and 4.5.2, Excitations 2 and 3 were adjusted to provide the lowest possible sidelobes while maintaining an 8.2° azimuth beamwidth. In addition, Excitation 3 was also adjusted to provide the smallest possible azimuth beamwidth while providing sidelobes 35 dB down from the pattern peak. Because of the similarity in the pattern shapes for each frequency, only selected pattern plots at 425 MHz are shown. Fig. 6.13 shows the azimuth and elevation patterns resulting from a 27 element uniform excitation, while the patterns of Fig. 6.14 are the result of an iteratively weighted linear least-squares excitation with all elements active. All array efficiencies are based upon the reflection coefficients obtained as a result of applying the excitation in question, as discussed in Section 4.4. The directivity of all patterns was determined by averaging and normalizing the pattern over 4π steradians, as in the previous section. Table 6.3 provides a performance summary of each possible combination.
Figure 6.13—NEC (a) azimuth and (b) elevation array patterns for uniform distribution at 425 MHz (27 active elements). The "gain probe" in (a) reads the peak value before applying the directivity correction.
Figure 6.14—NEC azimuth array patterns for (a) 40 dB and (b) 35 dB sidelobes at 425 MHz (54 active elements).
6.8 ARRAY FABRICATION

A complete 1/4\textsuperscript{th} scale model of the UESA array was built at the Pennsylvania State University by a team of electrical engineering students. The 1/4\textsuperscript{th} scale was chosen to provide an array that could be conveniently transported to NAWCAD for testing, and because the required parts (3/16-inch rod, 1.6-1.8 GHz splitters, etc.) were commonly available. Scale models have their advantages, such as reducing the array size, weight, and the size of the antenna range required, but also present a number of challenges. One such challenge is the construction tolerances. To determine the accuracy needed, a MATLAB program was written which generated a NEC file of the isolated element. Each wire length and wire-wire spacing was treated as a variable, to which a uniformly distributed random error was introduced. By trial and error, the required accuracy was determined to be approximately 0.04 inches for the full size element, or 0.01 inches for the scale model. Errors less than or equal to this amount had a negligible effect on the modeled radiation patterns or impedances.

The conductivity was also an issue of concern. A well known paper on frequency scaling by Sinclair [80] shows that to model an antenna in a new coordinate system, the

<table>
<thead>
<tr>
<th>Freq (MHz)</th>
<th>Excitation</th>
<th>Directivity (dBi)</th>
<th>Efficiency (%)</th>
<th>Sys. Gain (dBi)</th>
<th>Az BW (\degree)</th>
<th>El BW (\degree)</th>
<th>Peak Sidelobe</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>Uniform</td>
<td>20.43</td>
<td>95</td>
<td>20.22</td>
<td>7.5</td>
<td>33.9</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>27 LS</td>
<td>20.24</td>
<td>95</td>
<td>20.02</td>
<td>8.2</td>
<td>35</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>54 LS</td>
<td>20.45</td>
<td>95</td>
<td>19.99</td>
<td>7.7</td>
<td>30.2</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>54 LS</td>
<td>20.2</td>
<td>95</td>
<td>19.98</td>
<td>8.2</td>
<td>31</td>
<td>41</td>
</tr>
<tr>
<td>425</td>
<td>Uniform</td>
<td>21.05</td>
<td>94</td>
<td>20.79</td>
<td>6.8</td>
<td>31</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>27 LS</td>
<td>20.65</td>
<td>96</td>
<td>20.47</td>
<td>8.2</td>
<td>32.1</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>54 LS</td>
<td>20.5</td>
<td>95</td>
<td>20.28</td>
<td>7.7</td>
<td>30.6</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>54 LS</td>
<td>20.3</td>
<td>96</td>
<td>20.12</td>
<td>8.2</td>
<td>31.8</td>
<td>41</td>
</tr>
<tr>
<td>450</td>
<td>Uniform</td>
<td>21.25</td>
<td>90</td>
<td>20.79</td>
<td>6.5</td>
<td>30.9</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>27 LS</td>
<td>20.49</td>
<td>93</td>
<td>20.17</td>
<td>8.2</td>
<td>32.4</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>54 LS</td>
<td>20.75</td>
<td>91</td>
<td>20.34</td>
<td>7.7</td>
<td>30.2</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>54 LS</td>
<td>20.49</td>
<td>93</td>
<td>20.17</td>
<td>8.2</td>
<td>31</td>
<td>40</td>
</tr>
</tbody>
</table>
permittivity and permeability of any materials remain the same, and the frequency and the required conductivity (for metallic structures) are inversely proportional to the scale. Brass was selected as the desired material for the Yagi wires, since it is soft and easy to work with. Although its conductively is poor compared to copper (1.57 x 10^7 S/m vs. 5.8 x 10^7 S/m), the effect of the conductivity can be modeled with NEC. NEC showed that the impact on gain resulting from conductor loss at 1.8 GHz was less than 0.02 dB.

The material used to support the wires was also a cause for concern. A metal boom was considered, but was abandoned because of the difficulty in machining. Rohacell foam is appropriate for scale models of single elements, but lacks the structural strength required to support an array like the UESA. A variety of non-conductive materials were examined, but ultimately a nylon-like substance called Delrin was selected for its machinability, strength, and affordability. Concerns were raised about the dielectric constant of Delrin ($\varepsilon_r \approx 4$). In particular, it was feared that a trapped wave might appear in the Delrin rib and disrupt the operation of the Yagi. The WIPL code provided the ability to assess the impact of the dielectric on performance. Models of various traditional Yagis showed that a thin Delrin boom had a small effect on the input impedance, and negligible effect on radiation patterns. In general, the radiation resistance was nominally 3 Ω less, and the reactance became "flatter" across the frequency range of interest. A dielectric rib was not added to the WIPL model of the isolated element because of limitations in the number of unknowns that could be run with the prototype WIPL code. A 3-wire WIPL Yagi model is shown in Fig. 6.15.

![Figure 6.15—WIPL model of 3-wire Yagi with Delrin boom. "Wires" (white plates) and boom (shaded plates) are modeled with plates. Only 1/4th of the antenna is shown, as two symmetry planes were used.](image-url)
The balun and feed assembly was the final and most problematic construction detail. The graduate student team spent several months building and testing numerous elements of various configurations. Although the best measured patterns resulted from a split-wire balun as in Fig. 6.5(b), the difficulty in producing this type of balun at the scale model size and its extremely fragile nature made this option unfeasible. Ferrite baluns were also considered, but the inavailability of ferrite sleeves with a sufficiently high permeability at the scaled frequency caused this option to be abandoned as well. The final configuration included a parallel plate transition section, which provided a convenient connection from the driven element to the RG-141 coaxial transmission line (refer to Fig. 6.16(a)). The parallel plate was constructed from dual-sided circuit board with a Duroid substrate sandwiched between the metallic layers. The width was chosen to provide a characteristic impedance of approximately $50\,\Omega$. The plate section was glued into a channel cut into the Delrin rib. A bazooka balun as in Fig. 6.16(b), made from $\frac{1}{4}$-inch outer diameter (OD) refrigerant tubing, was placed 5 inches from the parallel plate to RG-141 interface. A 5-inch distance resulted in the least pattern perturbation in the NEC model. As discussed in the next section, this balun caused pattern irregularities at 1.8 GHz, which was the only deviation from an otherwise excellent set of measurement results.

Figure 6.16—(a) Mechanical configuration of feed. (b) Bazooka balun model—center conductor modeled with wires, shield and sleeve are modeled with plates.
6.9 MEASUREMENTS

The UESA scale model was transported to Patuxent River Maryland for testing in the NAWCAD large anechoic chamber (Fig. 6.17). The chamber dimensions are 140 feet x 40 feet x 40 feet, which satisfies the "rule of thumb" far zone distance of \(\frac{2D^2}{\lambda}\), where \(D\) is the largest dimension of the array. This amounts to approximately 120 feet at 1700 MHz. Due to the model's construction, elevation patterns were not taken for the embedded elements. In order to obtain an elevation cut, the array would have to be oriented vertically, which it was not designed for. The measurement process took three full days. Azimuth patterns of approximately one dozen embedded element patterns were taken, as well as azimuth and elevation cuts of five isolated elements. Measurements were taken at 1.6, 1.7, and 1.8 GHz. Calibration of the transmit horn was performed prior to and after the completion of testing, and losses from all splitters and feedlines were removed from the results that follow. With the exception of measured element VSWR, all plots are treated as if taken at 400-450 MHz for ease of comparison with modeled data.

Figure 6.17—Entire array inside large anechoic chamber.
Close-ups of the scale model are shown below. Measured vs. computed isolated element patterns are provided in Figs. 6.19(a) – 6.19(g). The measured VSWR plots of a sample of isolated elements (Figs. 6.20) show increasing mismatch at 1.8 GHz, indicating a balun problem.

Figure 6.18—(a) Single UESA scale model element, (b) complete array, and (c) view from array "hub".
Figure 6.19—400 MHz modeled vs. measured results of isolated element: (a) azimuth ($\theta = 90^\circ$) and (b) elevation. Peak value of the graph is 12.6 dBi.
Figure 6.19—425 MHz modeled vs. measured results of isolated element: (c) azimuth (θ = 90°) and (d) elevation. Peak value of the graph is 13.1 dBi.
Figure 6.19—450 MHz modeled vs. measured results of isolated element: (e) azimuth ($\theta = 90^\circ$) and (f) elevation. Peak value of the graph is 13.6 dBi.
Figure 6.19—(g) Typical 450 MHz azimuth and elevation measurements of isolated element. Peak value of the graph is 7.3 dBi. Balun ineffectiveness and uncontrolled feedline radiation was the cause of poor performance for many elements at 450 MHz.

Figure 6.20—VSWR of a sample of isolated elements.
Table 6.4 provides a comparison between the modeled and measured beamwidths and system gains of the isolated element.

**TABLE 6.4**  
**COMPARISON OF NEC MODELED AND MEASURED RESULTS (ISOLATED ELEMENT)**

<table>
<thead>
<tr>
<th>Freq (MHz)</th>
<th>NEC Azimuth (°)</th>
<th>NEC Elevation (°)</th>
<th>NEC Sys. Gain (dBi)</th>
<th>Measured Azimuth (°)</th>
<th>Measured Elevation (°)</th>
<th>Measured Sys. Gain (dBi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>48.9</td>
<td>41.0</td>
<td>12.6</td>
<td>48.0</td>
<td>41.3</td>
<td>12.5</td>
</tr>
<tr>
<td>425</td>
<td>45.6</td>
<td>38.6</td>
<td>13.1</td>
<td>44.2</td>
<td>39.4</td>
<td>13.1</td>
</tr>
<tr>
<td>450*</td>
<td>41.4</td>
<td>36.4</td>
<td>13.6</td>
<td>43.4</td>
<td>38.8</td>
<td>12.6</td>
</tr>
</tbody>
</table>

* Atypical—see conclusion (section 6.10).

The measured and computed embedded elements are plotted in Fig. 6.21, while in Fig. 6.22 are shown the measured VSWR. Fig. 6.23 shows the relationship between the system gain and azimuth beamwidth for synthesized receive beams, and a sample very low-sidelobe pattern synthesized from measured data is given in Fig. 6.24.

![Figure 6.21](image_url)  
(a)  
Figure 6.21—(a) 400 MHz modeled vs. measured azimuth (θ = 90°) embedded element results. Peak value of the graph is 8.7 dBi.
Figure 6.21—Modeled vs. measured azimuth ($\theta = 90^\circ$) embedded element results at (b) 425 MHz and (c) 450 MHz. Peak value of the graphs are 9.3 and 9.8 dBi, respectively.
Figure 6.22—VSWR of a sample of embedded elements.

Figure 6.23—Relationship of system gain and azimuth beamwidth for patterns synthesized from measured embedded element data. The curves hold (to within a few tenths of a dB) for both the 27 and 54 active element cases. Sidelobes are not plotted against a second axis, since they are a strong function of the number of active elements. Sidelobes of specific patterns are provided in Table 6.5.
6.10 SYSTEM GAIN MAXIMIZATION

An interesting consequence of the reciprocity relation is that the system gain can be maximized from embedded element data without knowledge of active impedances. Taking a slightly different approach than the one put forth for the maximization of the directivity in section 4.3, we express the array system gain as in section 4.4.2:

\[
G(\theta, \phi) = \frac{4\pi w^T R(\theta, \phi) w}{\sum_{m=1}^{M} |w_m|^2} = \frac{4\pi w^T R(\theta, \phi) w}{w^T I w}.
\] (6.7)

The following matrix identity states that for a quadratic of the form

\[
f = \frac{x^T A x}{x^T B x},
\] (6.8)
the vector $\mathbf{x}$ required to maximize $f$ is the eigenvalue associated with maximum eigenvalue of $B^{-1}A$ [81]. Thus, the maximum system gain is given by

$$G_{\text{max}}(\theta, \phi) = 4\pi \left[ \lambda_{\text{max}} \{ \mathbf{R}(\theta, \phi) \} \right]$$

(6.9)

and the excitation vector $\mathbf{w}$ is the eigenvector associated with it. A sample maximized system gain pattern is shown in Fig. 6.25, and Table 6.5 confirms that the system gain obtained with this method is indeed the highest available. While interesting, the patterns associated with the maximum system gain excitation typically have very high sidelobes.

![Figure 6.25—Maximized system gain pattern synthesized from 425 MHz measured embedded element data (54 active elements).](image)

An interesting observation is that the envelope of the excitation obtained through this method is proportional to the element pattern from which it is derived. A sample element/excitation pair is shown in Fig. 6.26. Demonstrating this for the general case is trivial; express the $m$th excitation as $w_m = I_m e^{j\alpha_m}$, and the contribution from the $m$th
Figure 6.26—Magnitude of element pattern and excitation derived from (6.9). An element pattern with considerable ripple was chosen to illustrate the excellent correlation.

<table>
<thead>
<tr>
<th>Freq (MHz)</th>
<th>Excitation</th>
<th>Sys. Gain (dBi)</th>
<th>Az BW (°)</th>
<th>Peak Sidelobe (dB rel Max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>Maximum</td>
<td>20.87</td>
<td>6.5</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Uniform</td>
<td>20.42</td>
<td>7.3</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>27 LS</td>
<td>19.83</td>
<td>8.2</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>54 LS</td>
<td>19.23</td>
<td>7.8</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>54 LS</td>
<td>19.65</td>
<td>8.2</td>
<td>40</td>
</tr>
<tr>
<td>425</td>
<td>Maximum</td>
<td>21.42</td>
<td>6.1</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>Uniform</td>
<td>21.04</td>
<td>6.7</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>27 LS</td>
<td>20.22</td>
<td>8.1</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>54 LS</td>
<td>19.50</td>
<td>7.7</td>
<td>35</td>
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<tr>
<td></td>
<td>54 LS</td>
<td>20.15</td>
<td>8.2</td>
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<td>19.33</td>
<td>5.8</td>
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<td></td>
<td>Uniform</td>
<td>18.60</td>
<td>7.0</td>
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<tr>
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<td></td>
<td>54 LS</td>
<td>NA</td>
<td>NA</td>
<td>40</td>
</tr>
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</table>
element pattern in the direction of the main beam as $g_m e^{j\beta_m}$. The system gain can be expressed as

$$G = \frac{\sum_{m=1}^{M} I_m e^{j\alpha_m} g_m e^{j\beta_m} \sum_{m=1}^{M} I_m e^{-j\alpha_m} g_m e^{-j\beta_m}}{\sum_{m=1}^{M} I_m^2}. \quad (6.10)$$

Taking derivatives with respect to $I_m$ and $\alpha_m$, and substituting in the maximum system gain excitation, i.e., $I_m e^{j\alpha_m} = g_m e^{j\beta_m}$, the derivatives $\partial G/I_m$ and $\partial G/\alpha_m$ are zero for all $m$.

6.11 COMPARATIVE CRITIQUE OF NEC AND WIPL

The Numerical Electromagnetic Code has been a modeling tool widely used by antenna engineers for decades. To summarize briefly, structures modeled with NEC must be done so with one of two basic geometrical primitives: wires and/or magnetic surface patches. As stated in section 6.4, the magnetic surfaces patches are implemented with the magnetic field integral equation. The most pressing implication of this is that a structure modeled with magnetic surface patches must be completely closed, and its volume must not approach zero with respect to its surface area. The surface patches can be either triangular or quadrilateral, in which case the patches have well-defined vertices, or "arbitrary", in which case a patch is completely described by its area, position, and orientation. Wires can be attached to the surface patches, either at specified vertices, or at an arbitrary location on a given patch, in which case the patch is sub-divided into smaller patches. Because of the requirement of the structure to be enclosed, the lack of a viewer for the surface patches, and a personal (and undocumented) observation of relative poor performance in predicting radiation and impedances for various known antennas, patches are not frequently used.

Virtually all structures modeled with NEC are done so with wires. Non-enclosed solid structures are not directly supported; however, solid walls may in general be simulated satisfactorily with a wire mesh, as long as the holes in the mesh are small in comparison...
to the wavelength. The thickness of the wires used can and does affect the electrical properties of the solid mesh. Suggestions have been made regarding the proper selection of a proper radius [82]-[84]; however, there does not appear to be a unanimous consensus among users regarding this.

Other shapes such as helices, catenary wires (wires with droop, as in a power line), and arcs can be created in NEC with high-level commands. These commands still use individual wire segments as a basis. Wire segments are specified by the positions of each end, and the wire radius. For best results, 10 (or more) segments should be used per wavelength, and as reported by Burke in [39], the radius should not exceed the segment length. Best results are obtained if the length is two times (or more) greater than the radius. These two guidelines can place a user in a bind for the case of very thick wires, as are often desired for broadband or high-power applications.

NEC has been, since its beginnings, a frequency domain moment method (MoM) code. The most recent version uses a formulation whereby the matching (imposition of boundary conditions) is done on the axis of the wire, and the current flows axially along the surface of the wire. This is a fairly recent change, common to versions of NEC-4.x, and represents a reversal of older formulations. As noted by Burke in [39], this results in greater accuracy, particularly for thick wires. All versions of NEC use "local domain" basis functions, existing independently over individual wire segments [85].

Because NEC has a long history and has had considerable investments, it possesses several features that are difficult to find in other codes. One of the most powerful of these features is its ability to exploit the symmetry of a rotationally symmetric problem. As stated in section 6.4, the matrix storage and fill-time savings are proportional to $M$, and the factoring time is reduced by nearly $M^2$, where $M$ is the number of symmetric sections. These savings are sufficient to reduce an intractable problem into one that can be handled with a modest personal computer. This is a true discrete body-of-revolution (DBOR) capability, where each section can have a different source. Some codes, like EIGER [67], possess the ability to exploit rotational symmetry, but do not permit asymmetrically
excited sections, and are thus not the most efficient means for modeling an array like the UESA.

NEC's possesses several other unique features that are seldom seen in other codes. One such feature is the high-accuracy Sommerfeld/Norton reflection method [86], [87], which provides greater radiation pattern accuracy for antennas mounted at low heights (less than a wavelength) over lossy earth. Another unique capability is the numerical Green's function. The numerical Green's function allows the user to solve and store a large structure in the form of a special file format, which is essentially a pre-factored solution to the impedance matrix for the structure itself. This file becomes the basis of a model file, to which other new structures may be added. The code computes the currents on the entire structure based upon the Green's function file (the original structure), the new structure, and an interaction matrix. This capability is very useful in tuning the placement of an antenna on a ship, for example. Another unique capability is the ability to model thin dielectric coatings on the surface of the wires. NEC can also compute radiation patterns and impedances for wires that are partially buried in ground. It also has the ability to model end-caps for wires, and can embed two-port networks, lumped loads, and transmission lines in models. With the exception of lumped loads, none of the capabilities listed in this paragraph were required to model the UESA array.

Like NEC, WIPL-D (a contraction for Wires/Plates with Dielectrics) is a new frequency domain MoM code. Because it is still early in its development cycle, WIPL-D (or WIPL for short) does not possess many of the more powerful features of NEC—no rotational symmetry (with asymmetric sources), no Sommerfeld/Norton reflection, no dielectric coatings on wires, and no numerical Green's function—as of yet. While WIPL is currently weak in terms of these "add-on" capabilities, it excels in its ability to handle a variety of geometrical primitives. WIPL has wires similar to NEC, although the wires in WIPL are actually conic sections (wires with independent radii at either endpoint). In addition to wires, WIPL provides the ability to model quadrilateral plates. These plates may be non-planar, as long as the quadrilaterals are concave; that is, they do not have
interior corners. They are also implemented with an electric field integral equation, so that there is no requirement for solid structures to be enclosed, as with the NEC magnetic surface patches. Wires can be joined to the plates at arbitrary locations, and the built-in mesh generator remeshes the plates such that a junction is formed at the new attachment point [88]. This is a powerful feature of WIPL—the end of a wire may be moved about on the surface of a solid structure without having to remesh the structure (so that a node is present at the attachment point). Moreover, the newly generated attachment nodes are positioned at the exterior surface of the wire (see Fig. 6.27). This ensures that the flow of current from the wire to the attaching plate is representative of physical reality. Solid dielectric regions may be modeled in WIPL, and are done so with the concept of domains. A region consists of a volume completely enclosed by plates with at least one side having a common domain. Domains (aside from the default free-space domain) are defined by the user, and can have real and imaginary parts for both the relative permittivity and relative permeability, as well as a conductivity in Siemens per meter. Anisotropic materials are not permitted.

Figure 6.27—(a) Input and (b) output geometry of a simple monopole antenna.

The wire formulation is similar to NEC in the sense that the current flows axially along the surface of the wire. It differs in the sense that WIPL-D is an "entire domain" code, where novel polynomial basis functions (for both the wires and the plates) are used to solve for the electric (and in the case of dielectric materials, magnetic also) currents [89]. Polynomials for each individual wire or quadrilateral may range from degree 1 to 9 in order to accurately represent the currents. The entire domain characteristic permits wires
or plates that are much larger in comparison to that of NEC or other local domain MoM codes, which typically require at least 10 to 15 segments (per linear dimension) per wavelength to ensure accuracy. The polynomial order may be selected manually or automatically as a function of the wire or plate size in terms of wavelength. Higher-order polynomials typically yield more accurate results, but require more unknowns and thus result in longer run times. This concept appears to work well, as long as one does not try to place a pair of electrically large plates or wires in close proximity to each other, where they are separated by an electrically small distance and parallel (broadside coupled) to each other. Under these circumstances, the entire domain concept appears to break down, yielding questionable or clearly erroneous results (one can always revert to traditional meshing guidelines and avert this effect if necessary). In general, however, the means by which WIPL-D computes impedance and radiation patterns appears to both accurate and fast in informal comparisons with other MoM codes like NEC, PATCH [90], and EIGER. These statements are based upon informal observations, and as of yet, unsupported by formal documentation.

Since both NEC and WIPL-D have similar wire primitives, and also because wires form the basis of the UESA antenna array, the performance of wires in both codes is of great interest. In general, a physical antenna that is constructed of wires or tubular structures can be modeled most efficiently (rapidly) by using the wire primitive. For very thin wires (wires with length to diameter ratios of 100:1 to 1000:1), virtually any moment method formulation will result in very accurate and converged results. Unfortunately, many antennas require wires that are electrically thick, having a length to diameter ratio of perhaps 3-10. As discussed in several papers by Werner [91], [92], accurately computing the currents on such a wire (and which may be in close proximity to other thick wires) is difficult.

Two simulations were conducted in order to roughly compare the wire performance of NEC and WIPL. For the first test, a narrowband Yagi, shown in Fig. 6.28, was designed with NEC-OPT. The antenna was optimized for forward directivity given a maximum
VSWR of 1.3 (based upon a 50 Ω input impedance) from 430 to 434 MHz. As can be seen in Figs. 6.29 and 6.30, the impedance and directivity results are virtually identical.

Figure 6.28—10-wire Yagi used for "thin-wire" test. Wires are accurately represented graphically in terms of length-to-diameter perspective.

Figure 6.29—Impedances for 10-wire Yagi.
The second test was designed to assess the thick-wire performance of each code. The "initial" element discussed in section 6.6 was selected as a near ideal test object for this purpose, because considerable differences between measured and modeled impedances were observed. Exact impedance measurements are difficult to come by, since the physical balun (not modeled) moves the actual feedpoint rearward from the driven wire. However, the wire/plate WIPL model accurately identified the impedance problem associated with the initial element, and was in good agreement with the corrected element. Thus, the wire/plate WIPL model is regarded as truth in Fig. 6.31, and is compared to the wire WIPL and NEC models. Remarkably, despite the considerable investments and history of NEC, and the vast attention focused on its thick-wire performance, the wire WIPL model appears to yield results that are closer to the truth data, although substantial differences are still present between the wire/plate and wire WIPL models.
6.12 SUMMARY OF RESULTS

In this chapter, we discussed the various attempts to arrive at successful array design. The most successful approach is a traditional circular arrangement of Yagis. The final array configuration is based upon the analytical and empirical studies of Chapters 2 and 5. The elements were designed specifically for placement in a tightly packed circular arrangement. An isolated element design approach was selected primarily because of available computer resources. While an embedded element design approach is preferable, the computer resources required to implement it are prohibitive. It appears possible, however, to correlate isolated element behavior with embedded element behavior, and choose the error function weighting in the optimization process accordingly. Perhaps the most challenging task is to fit enough elements in the array to permit the desired pattern control, while decoupling them enough to provide good array efficiency (> 90%). The natural broadening associated with the embedded element beamwidth allows for the
utilization of approximately ½ of the array, and permits several convenient transmitter module switching architectures. One possibility is discussed in the following chapter.

Agreement between the two electromagnetic element models is excellent. After averaging and correcting the directivity, peak isolated element system gains agreed to within 0.1 dB. Excellent correlation of modeled and measured element system gains and beamwidths was also demonstrated. From Table 6.4, azimuth and elevation beamwidths are in agreement to approximately 1°, except at 450 MHz. Comparison of Tables 6.3 and 6.5 shows that the synthesized array results are, in general, equally encouraging. Synthesized array pattern azimuth beamwidths and associated sidelobe levels were as expected. The greatest difference was in the modeled vs. synthesized array system gains. At 400/425 MHz, the measured system gain exceeded that of the modeled by several tenths of a dB for the uniform excitation, while for tapered excitations, the measured system gain was a bit under projected levels. The measured system gain dropped significantly below its expected value when the azimuth beamwidth was compressed from 8.2° to 7.7° (~ 0.8 dB). In chapter 4, we showed that compression of the azimuth beamwidth causes the array excitation to become more oscillatory. This suggests that the true active impedances become increasingly difficult for NEC to compute under such conditions.

The extreme discrepancy of the measured 450 MHz results is attributed to the balun design. It was known before the design finalization date that the best performance was realized with the split-wire balun. Despite this, the bazooka balun was selected because it was simple to fabricate and mechanically robust. Occasionally, reasonable results were obtained with individual bazooka balun elements, as shown in Figs. 6.19(e) and 6.19(f). Even this particular element exhibited about a 1 dB shortfall in gain, and the rest of the elements had severely perturbed patterns, as the element of Fig. 6.19(g). The resulting embedded element patterns have deep nulls in the forward regions of the pattern. Clearly the pattern of Fig. 6.21(c) (one of the better ones) suggests that energy is coupled into the elevation plane by feedline radiation, since a significant amount of energy is missing
from the $\theta = 90^\circ$ plane. The effect on the array performance is that the measured system gain is down by *at least* 2 dB, and that the quality of possible array patterns is limited. From Table 6.4, the 40 dB pattern is "not applicable". In addition, for certain low sidelobe patterns, the difference between modeled and measured system gains exceeded 5 dB. This is attributed to the oscillatory excitation resulting from attempting to control sidelobes with a number of severely perturbed patterns, leading to very poor array efficiency.

Despite the disappointing performance at the upper end of the band, the project can be viewed as an overall success. With the exception of the measured 450 MHz results, the goals laid out at the beginning of the chapter were met or exceeded, although not simultaneously. The performance goals set forth were very aggressive in accordance with the desired radar system requirements. Although it appears that this particular circular array does not quite deliver the performance available from a linear array of equivalent size, we must remember that it possesses several distinct advantages, such as the virtual lack of scan loss, and high beam agility.

One of the more important conclusions is that of the confirmation of transmit/receive reciprocity as discussed in section 4.4. If this fundamental principle did not hold for the case of the active array, then the receive mode efficiency should remain high for the patterns with the highest directivity. However, the curves in Fig. 6.23 show that the system gain is poor for the patterns with compressed beamwidth. This suggests that the array efficiency is poor on receive when directivity is highest, just as it would be during transmit. However, the reciprocity has several important implications. It means that the absolute system gain can be determined for any array excitation exclusively from a set of measured embedded element patterns, without knowledge of the set of active impedances. Furthermore, the system gain can be absolutely maximized, again without knowledge of the active impedances or an impedance matrix. Finally, the excitation required to maximize array system gain is equal to the conjugate of the set of complex field contributions from the individual element patterns.
7 TRANSMITTER POWER MANAGEMENT

7.1 INTRODUCTION

So far we only two types of array excitations have been discussed. The first is the uniform amplitude cophasal variety as presented in Chapter 2. This is the trivial excitation, where the power fed to each element is equal, and the phase is set equal to the conjugate of the far-field phase at the beam angle. This results in high gain, but also high (~11 dB) sidelobes. In chapter 4, we covered the linear least-squares (LLS) method, which generally results in somewhat reduced gain but low sidelobes.

The primary objective of airborne surveillance radar is to detect airborne targets at the longest possible range. Unlike ground-based radar, returns from the sea surface, ground, and buildings cannot easily be removed with doppler filtering. In order to reduce clutter, radar transmitters virtually always have tapered transmitter (as well as receiver) excitations.

For a linear array, implementation of a tapered transmitter excitation is straightforward. Frequently, a fixed-size transmitter power module serves as a building block, and modules are combined as required to approximate a particular excitation. For the circular array, such a "block tapered" scheme is not viable, as no simple means of commutation exists. A tapered excitation can be implemented with linear power modules. In this architecture, each element has a dedicated module, and the taper is implemented by applying the appropriate input power to each. Unfortunately, most tapered excitations require a power ratio of approximately 20 dB between the center and edge elements. The typical DC-RF efficiency of a linear power amplifier operating at a level –20 dB from its peak is usually less than 1% [93]. As a result, the composite transmitter efficiency is poor, perhaps as little as 10-20% of the maximum efficiency obtainable. One is faced with the (inevitably poor) choice of either high average radiated powered with high sidelobes, or low average radiated power with low sidelobes.
A variety of architectures have been suggested for commutating a particular excitation about a circular aperture. These include a mechanically rotated waveguide assembly [94], which in addition to the size imposed by UHF waveguide suffers the obvious beam agility limitations, a switch network [95], which is appropriate for a tapered receive excitation (requires high power phase shifters for transmitting), and "lenses", including the $R-2R$ [96] and geodesic [97] varieties. Both of the lens schemes require the inputs to be tapered in order to provide a tapered output, as does the Butler matrix [98]. The Butler matrix, which was extremely popular in the sixties and seventies, is a beamforming architecture using hybrid couplers and phase shifters arranged exactly analogous to an FFT structure. As stated in previously in Chapter 6, the transmitter weight allowance places a considerable constraint on the transmitter architecture. If a tapered transmitter is implemented with linear power amplifiers, the modules must be sized according to the power required at the center element. Consequently, the total module resources must greater than that of the linear array, and module utilization is poor. What is desired is a transmitter architecture that satisfies two criteria:

1) Allows all power amplifier modules to be operated at or near full power to obtain maximum efficiency,

2) Permits power from edge modules to be "gathered up" and redistributed to center elements.

Such a device is presented in the following section. Referred to hereafter as the vector combiner, or simply the combiner, this device is based upon an ordinary branchline coupler that can be found in any microwave design text [99]. By definition, the device is equivalent to a $4 \times 4$ Butler matrix, but its use is quite different from that of the typical Butler matrix. The emphasis of this chapter is not so much on the combiner itself, but on the special beamforming methods that must be used in conjunction with the combiner, and the required module amplitude and phase accuracies required.
7.2 VECTOR COMBINER THEORY

The branch line coupler of Fig. 7.1 is a form of directional coupler commonly used for splitting or combining signals. A signal input at Port 1 is isolated from Port 4 (which may also be an input), and splits equally in power between Ports 2 and 3 with a 90° phase shift. The length of each leg is adjusted so that it introduces a 90° phase delay. The attractiveness of this device, as compared to a three-port circuit such as the Wilkinson combiner, is that the input ports are isolated without the need for an isolation resistor. This permits two signals of arbitrary phase and amplitude to be summed together without dissipating power in the device. The S-parameter matrix is

\[
[S] = \frac{-1}{\sqrt{2}} \begin{bmatrix}
0 & j & 1 & 0 \\
j & 0 & 0 & 1 \\
1 & 0 & 0 & j \\
0 & 1 & j & 0
\end{bmatrix} \tag{7.1}
\]

![Figure 7.1—(a) 90° hybrid (branch line) coupler, (b) commonly used symbol.](image)

In addition to thinking of the device as a splitter or combiner, it can also be thought of as an amplitude to phase converter and vice-versa. Consider the following two scenarios: in the first a 1-volt signal is applied at Port 1, and a signal at Port 4 of variable amplitude and in-phase with the signal at Port 1. As shown in Fig. 7.2, the output amplitude at Ports
2 and 3 are always equal, and the relative phase varies from 0° when the inputs are equal in amplitude, to 90° when the amplitude of the signal at Port 4 is 0.

In the second scenario, the amplitude of the signals at Ports 1 and 4 is held equal, and the phase at Port 4 is varied from 0°-90° relative to Port 1. In this instance, the relative output phase is 0°, and the output power is distributed between Port 2 and 3 as shown in Fig. 7.3(b). This mode of operation is of considerable interest, since the coupler can be used to switch power between the two output ports by setting the relative phase at Ports 1 and 4 to ±90°. Since there are no moving parts to fail, a single hybrid coupler fabricated in air dielectric stripline is capable of switching enormous power levels. Because the hybrid can function as a switch, it allows one transmit module to feed one of two elements on either side of the array. Its primary utility is that it reduces the number of modules, but it cannot support a tapered excitation.

Figure 7.2—(a) Output phase, and (b) amplitude relationships for in-phase signals applied at Ports 1 and 4. The input at Port 1 is held at 1 volt.
Consider now the butterfly arrangement of hybrids in Fig. 7.4, where Ports 1-4 are inputs, and Ports 5-8 serve as outputs. Four transmitter modules work together as a group. Only one output signal is desired at Port 5 or 6, and one at Port 7 or 8. The module phases can be chosen to split the power arriving at the output hybrids H₃ and H₄. By setting the relative phase of module pairs 3 and 4 to ± 90° with respect to modules 1 and 2, the signal at one port of each output hybrid can be cancelled. A unique set of module (relative) phases permits the output power to be divided arbitrarily.

Figure 7.3—(a) Output phase, and (b) amplitude relationships for equal amplitude signals applied at Ports 1 and 4.
The drawback of this device is that the two non-zero outputs, one from each of \( H_3 \) and \( H_4 \), cannot take on arbitrary phases. A specific module phase relationship was required to cancel the signal at one of Ports 5/6 and one of Ports 7/8. In addition, it has a problem with the physical layout, since a crossover is not practical for a stripline implementation. The structure in Fig. 7.5 solves both problems. The upper hybrids have been reversed, and phase shifters have been added in two paths to enable arbitrary phase relationships at the outputs. As we will discuss later, the phase shifters need not provide continuous phase adjustment, but need only consist of a limited number of differential delay lines. This is important, since UHF phase shifters capable of handling tens of kilowatts are in general not available.

If the outputs are connected to elements spatially separated by 90°, and staggered such that the outputs from either \( H_3 \) or \( H_4 \) are spatially separated by 180°, the vector combiner permits a cosine taper excitation to be placed across \( \frac{1}{2} \) of the array, since

\[
\cos^2(\phi) + \cos^2(\phi + 90°) = 1.
\] (7.2)
Figure 7.5—Block diagram of vector combiner complete with phase shifters to permit arbitrary output phase, and with layout that avoids the need for crossover.

The combiner provides a simple means of exploiting the tendency of the circular array to develop its maximum gain with approximately ½ of the array active. Just as importantly, all modules are operated at full power and maximum efficiency. Fig. 7.6 shows the pattern (based upon the NEC model at 425 MHz) resulting from a strict cosine pattern. Although some improvement is present, the reduction of the peak sidelobe level is only about 4 dB compared to the uniform excitation (-11 dB sidelobes).

Figure 7.6—Pattern resulting from "cosine taper".
One option for obtaining the array excitation is the LLS method discussed in Chapter 4. A sample pattern is created below. To provide for an equitable comparison, the target null-null beamwidth was adjusted such that the half-power azimuth beamwidth was the same as that of the "cosine pattern" (7.9°). From Fig. 7.7, we see that peak sidelobes are approximately 10 dB lower than when the cosine taper was used.

The problem with the LLS method is that the resulting excitation is unconstrained to any form. As mentioned above, four transmitter modules work together to deliver two outputs, which are spatially separated by 90°. Consider the excitation of Fig. 7.8, which gives rise to the pattern of Fig. 7.7. Since 27 elements are active, any two outputs separated by 14 positions are from the same combiner. Modules 1 and 15 collectively deliver the highest power from any combiner—approximately 1 unit of power. Modules 8 and 22 deliver only $0.5^2 + 0.41^2 = .42$ units of power. Thus, the collective power of this pair is approximately 3.8 dB lower than that provided by the highest power combiner. In order to maintain the shape of the excitation, the power from this pair of outputs must be reduced by 3.8 dB from its maximum possible power. The result is that the total transmitter utilization is poor—down 2.45 dB overall from the maximum possible power.

![Figure 7.7—Pattern obtained using LLS method.](image)
7.3 EXCITATION OPTIMIZATION

Clearly, the combiner power dilemma calls for a beamforming algorithm that permits some form of constraint on the excitation. The LLS method allows one to impose linear constraints with the following relation:

\[ Cw = f \]  \hspace{1cm} (7.3)

where \( C \) is a constraint matrix, and \( f \) is a vector derived from a prescribed set of conditions as described by Compton [50]. For the problem of imposing power constraints, the constraint relation of (7.3) is of little value. For a non-linear array, the phases of the individual excitations are not constant. In order to obtain the excitation vector with a particular optimization algorithm, the real and imaginary parts are specified separately. Simple bounds or linear equality constraints are insufficient to limit the power level applied to a particular element within a specified range. Other algorithms besides LLS which can impose linear equality or inequality constraints are the so-called standard...
and revised "simplex" methods [100], linear and quadratic programming [101], and non-linear least-squares [102].

An optimization algorithm that has seen increasing use is the sequential quadratic programming (SQP) method [103]. SQP is reported to outperform all other derivative based methods in terms of efficiency, accuracy, and percentage of successful solutions over a large number of test problems [104]. To summarize briefly, the SQP algorithm takes a general problem and approximates it with a quadratic of the form

$$\nabla f(x_k)^T d + \frac{1}{2} d^T H d,$$

where \( d \) is the current step direction and \( H \) is the Hessian approximation to the Lagrangian function:

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$  \hspace{1cm} (7.4)

The quadratic problem is solved subject to the constraints

$$\nabla g_i(x_k)^T d + g_i(x_k) = 0 \hspace{0.5cm} i = 1, \cdots m_e$$

$$\nabla g_i(x_k)^T d + g_i(x_k) \leq 0 \hspace{0.5cm} i = m_e + 1, \cdots m$$  \hspace{1cm} (7.5)

where \( g_i(x_k) \) is a constraint, \( m \) is the total number of constraints, and \( m_e \) is the number of equality constraints.

The general problem is converted into a quadratic sub-problem and the constraints are linearized at each major iteration. The process is repeated until the general problem is minimized.

For the transmitter problem, the power of SQP lies in the ability to bound the range of total power output of any single combiner. For 27 active elements, we specify that

$$p_{min} \leq \text{Re}\{w^2(i)\} + \text{Im}\{w^2(i)\} + \text{Re}\{w^2(i+14)\} + \text{Im}\{w^2(i+14)\} \leq p_{max}$$  \hspace{1cm} (7.6)
where $p$ is a unit of power, and $w(i)$ is a complex voltage level applied to the $i^{th}$ element. By setting $p_{\min}$ and $p_{\max}$ to allow a narrow range of possible values, the overall shape of the excitation is permitted to vary somewhat, but not so much as to significantly reduce the overall transmitter output power.

As an example, we consider a possible transmit excitation for the UESA array. By specifying $p_{\min} = 0.9$ and $p_{\max} = 1.1$, the maximum variation in power from any pair of combiners (two sets of four transmitter modules) is $10 \log_{10}(1.1/0.9) = 0.87$ dB. The resulting excitation, shown in Fig. 7.9, strongly resembles but deviates slightly from the shape of a cosine taper. The pattern, shown in Fig. 7.10, has peak sidelobes of about -22 dB from the mainlobe peak—only about 3 dB worse than that obtained with the unconstrained taper. The composite reduction in transmitter power is only 0.4 dB, and the azimuth beamwidth is 7.9°, equal to the rigid cosine pattern. The system gain (based upon measured patterns) as computed with (4.19) is 0.3 greater at 400 and 425 MHz, than that obtained with the uniform distribution (see Table 6.5). This brings the system gains at 400 and 425 MHz to 20.72 and 21.34 dB, respectively. The SQP excitation optimization was not performed at 450 MHz, since a suitable element pattern was not available.

The concept of "taper gain" should not be foreign, given the discussion of system gain maximization in Section 6.10. For the UESA array, it so happens that the natural cosine taper roughly matches the field contributions from an element pattern. It should be noted that the system gain achieved with this taper approaches, but does not exceed that of, the maximum system gain excitations, also presented in Section 6.10. Only the LLS derived taper, which ordinarily returns array weights that decrease more rapidly in amplitude (measured from the beam boresight angle) than the element pattern contributions, results in decreased system gain.
Figure 7.9—Constrained taper. The power reduction of any module set is limited to 0.87 dB.

Figure 7.10—Pattern associated with constrained taper.
7.4 REQUIRED MODULE ACCURACY

Since the distribution of transmitter power depends upon the vector summation and cancellation of multiple signals, the feasibility of the combiner approach depends upon the accuracy and repeatability of the individual modules. If the transmitter exhibits excessive amplitude and phase variations between modules, the excitation may be adversely affected, leading to higher than expected spurious energy. To determine the required module-module consistency needed, we resort to contour plots of the S-parameter matrix of a single hybrid coupler. We begin by examining several cases, treating Ports 1 and 4 as inputs:

1) All power being directed out Port 2,
   a) The degree of cancellation at Port 3 for the above,

2) A signal channeled out Port 2, –18 dB from the maximum available power.

A –18 dB signal level is chosen as the lower limit, since the amplitude (voltage) of the outputs for the taper in Fig. 7.9 spans an 18 dB range. In each instance, the signal applied to Port 1 is 1∠0° volts. For the idealized S-parameter matrix of (7.1), a 1∠90° signal is required at Port 4 to obtain a signal from Port 2 of 29 ∠−90°. Fig. 7.11 reveals that the greatest challenge to Case 1 is obtaining a sufficient level of cancellation at Port 3. Since an 18 dB range for the excitations is assumed, we desire an ability to cancel signals at the inactive ports by 25-30 dB with respect to the highest power output. The error ellipses of Fig. 7.11(a) shows that for 30 dB cancellation, the module-to-module phase difference must never exceed 3.7°, and the amplitude imbalance must never exceed 0.6 dB. Fig. 7.11(b), which maps the Port 2 amplitude error as a function of Port 4 errors, shows that there is little amplitude sensitivity to phase errors, and that Port 2 amplitude errors are roughly half those at Port 4. Fig. 7.11(c) shows that the Port 2 phase errors are insensitive to amplitude errors, and that the input-output phase errors are also decreased roughly by a factor of two. Fig 7.12 repeats the input-output error mapping, but for the case –18 dB desired signal from Port 2.
(a)

(b)
Figure 7.11—(a) Cancellation at Port 3 [dB], when all power is directed out Port 2, (b) Port 2 amplitude errors [dB], and (c) Port 2 phase errors [degrees] as a function of Port 4 amplitude and phase errors.
The combiner sensitivity to module errors can also be analyzed with a mathematical approach. Using phasor notation, we apply a signal $V_1 = \sqrt{2}e^{j\phi}$ at Port 1, and $V_4 = \sqrt{2}ae^{j\phi}$ at Port 4 to a single hybrid coupler. As before, we let the amplitude and phase of the input to Port 4 vary. The output at Port 2 is then $V_2 = e^{-j\phi} - ae^{j\phi}$. We can consider a total differential:

$$dV_2 = \frac{\partial V_2}{\partial \phi} d\phi + \frac{\partial V_2}{\partial a} da$$

(7.7)

The absolute value of (7.7) is $|dV_2| = \sqrt{(ad\phi)^2 + (da)^2}$, which indicates that the change in voltage of an individual excitation is not a function of its level. A relative change may also be defined:
\[
\left| \frac{dV_2}{V_2} \right| = \frac{(a\phi)^2 + (da)^2}{\sqrt{1 - 2a \sin \phi + a^2}}
\]  
(7.8)

If we compare the relative change in voltage for the full power output \((\phi = 90^\circ)\) and the \(-18\) dB output \((\phi = -104.5^\circ)\), we see that the sensitivity is 7.92 times greater (18 dB) for the lower power output. A phase sensitivity can also be defined. The phase at Port 2 can be described by

\[
\theta = \tan^{-1} \left( \frac{1 + a \sin \phi}{a \cos \phi} \right)
\]  
(7.9)

A total differential can be taken as in (7.7):

\[
d\theta = \frac{a(a + \sin \phi)}{1 + 2a \sin \phi + a^2} d\phi - \frac{\cos \phi}{1 + 2a \sin \phi + a^2} da
\]  
(7.10)

Although \(|d\theta|\) becomes a more complicated function of \(\phi\) than the relative change in voltage, the denominator takes the same form as (7.8), and as expected, the greatest sensitivity occurs at \(\phi = -90^\circ\).

The most difficult situation to handle analytically is for the case of complete cancellation, as in Fig. 7.11(a). Here the quiescent operating point changes rapidly, and the most convenient means of determining the degree of cancellation is to simply plot the resulting amplitude or cancellation ratio as a function of all possible amplitude and phase errors. A high degree of cancellation is important to minimize the amount of power radiated from the "backside" of the array.

If Figs. 7.11(a), 7.11(b), and 7.12(a) are plotted in terms of output voltage instead of dB relative to the desired voltage, the observation regarding the constancy of \(|dV|\) for all
outputs is confirmed. This property makes the simulation of random errors straightforward. In Fig. 7.13, a Monte Carlo simulation is performed on the excitation of Fig. 7.9. To each element (including those in the rear) is added a voltage that is 25 dB lower in power compared to the center element, and with random phase. A typical pattern is plotted in Fig. 7.13. The distortion is minimal, and the pattern is comparable to that of Fig. 7.10.

![Figure 7.13](image)

Figure 7.13—Pattern resulting from the addition of a –25 dB (relative to peak excitation) signal with random phase to each element.

A natural question arises as to why the error analysis was applied only to a single hybrid coupler, and not to the entire device. For one, the analysis applied above shows that for the eight-port device, the differential change at any output port is not a function of \( \phi \), only of the module errors, as for the single hybrid. For the device of Fig. 7.4, if the input is defined as

\[
V_{in} = V_1 + V_2 + V_3 + V_4 = 2e^{i\phi} + 2ae^{i\phi} + 2be^{i\phi} + 2ce^{i\phi},
\]

the differential output at Port 5 is
Furthermore, a close examination of the vector combiner reveals that the maximum errors are present at the output of hybrids $H_1$ and $H_2$. For each of the output hybrids ($H_3$ and $H_4$), two signals of equal amplitude are combined to form a single output. This is the same situation where the output amplitude or phase errors of a single hybrid were observed to be approximately $\frac{1}{2}$ that of the module errors when all power was directed to a single output, as shown in Figs. 7.11(b) and 7.11(c). Consequently, additional errors are not introduced by the output hybrids; in fact, it can be rightly argued that the error sensitivity is decreased through $H_3$ and $H_4$.

Until now, the phase shifters have been neglected. Without them, the phase of the active port from each output hybrid cannot be chosen arbitrarily. Otherwise, reducing the power of two modules is the only way to obtain the desired phase at both ports. As demonstrated in Fig. 7.2(a), the relative output phase (of a single hybrid device) can be adjusted with the input power of one module. This is an undesirable means of phase shifting, though, and has the same consequence of using an unconstrained taper—loss of transmitter power.

Consider the combiner in Fig. 7.5. Assuming the lines connecting the input and output hybrids are $\frac{1}{4}$ wavelength, the S-parameter matrix is

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & j & 1 & 1 & -j \\
0 & 0 & 0 & 0 & 1 & -j & j & 1 \\
0 & 0 & 0 & 0 & e^{-j\phi} & e^{j(\frac{\pi}{4}-\phi)} & e^{-j(\frac{\pi}{4}+\phi)} & e^{-j\phi} \\
0 & 0 & 0 & 0 & e^{-j(\frac{\pi}{4}+\phi)} & e^{-j\phi} & e^{-j\phi} & e^{j(\frac{\pi}{4}-\phi)} \\
e^{-j\phi} & e^{-j(\frac{\pi}{4}+\phi)} & e^{j(\frac{\pi}{4}-\phi)} & e^{-j\phi} & 0 & 0 & 0 & 0 \\
e^{-j\phi} & e^{-j(\frac{\pi}{4}+\phi)} & e^{j(\frac{\pi}{4}-\phi)} & e^{-j\phi} & 0 & 0 & 0 & 0 \\
1 & j & -j & 1 & 0 & 0 & 0 & 0 \\
-j & 1 & 1 & j & 0 & 0 & 0 & 0
\end{bmatrix}
$$

(7.12)
A unique input can be determined for every possible output by multiplying the inverted matrix times the desired output vector. To illustrate the need for phase shifters, consider the following two examples. To simulate the absence of phase shifters, let $\phi = 0$. We desire two outputs of equal phase and amplitude from Ports 5 and 7, such as $V_5 = V_7 = \sqrt{2}e^{j0}$. The required inputs are $V_1 = 0$, $V_2 = e^{-j\frac{\pi}{4}}$, and $V_3 = V_4 = e^{j\frac{\pi}{4}}$, and all modules are operated at equal power. Consider now the case where the relative output phase is $\pm 90^o$, such as $V_5 = \sqrt{2}e^{j\frac{\pi}{4}}$ and $V_7 = \sqrt{2}e^{j0}$. The required inputs are now $V_1 = 0$, $V_2 = \sqrt{2}e^{-j\frac{\pi}{4}}$, $V_3 = 0$, and $V_4 = \sqrt{2}e^{j0}$. In order to obtain the desired output phase relationship, modules 1 and 3 shut down completely, and Modules 2 and 4 increase power by a factor of 2. By setting the phase shifters such that $\phi = 90^o$, this condition is averted. This example was selected because the worst case of "module shut down" occurs when the outputs are equal in power.

The range required of the phase shifters to prevent module shut down is 0-180°. Commercially available devices that can handle tens of kilowatts of power are not available. One method of providing the necessary phase shift is by installing a set of switched lines between the input and output hybrids at each interconnection, as in Fig. 7.14. Each set possesses four lines of varying lengths. If the phase lengths of the lines connected to H3 (denoted by $\phi_1$) are 168.75°, 123.75°, 78.75°, and 33.75°, and the lengths of the lines connected to H4 (denoted by $\phi_2$) are 33.75°, 22.5°, 11.25°, and 0°, then the required range can be covered in steps of 11.25° with a maximum error of less than 6°. Fine phase shifting can be accomplished by slight reduction in power of two modules, as demonstrated in Fig. 7.2. The desired lines can be selected via series PIN diodes, which have demonstrated the ability to switch 3000 volts @ 25 amps rms up to 500 MHz [105].
Figure 7.14—Vector combiner architecture with switched line sections to provide for arbitrary output phasing without significant "module shutdown".

7.5 SUMMARY

In this chapter, we presented a means of simultaneously allowing a tapered excitation to be applied to the circular array while obtaining maximum power and efficiency from a set of transmitter modules. This method is based upon an eight-port vector combiner consisting of four branch line couplers arranged in a manner analogous to a two-stage FFT. This arrangement permits a tapered excitation to be applied to ½ of the array. Four transmitter modules are phased such that two combiner outputs are activated, the sum of which is approximately constant in power. The two active outputs are physically separated by 90°.

Because the combiner's natural taper is proportional to the cosine function over the range $-\pi/2 \leq \phi \leq \pi/2$, the resulting pattern does not exhibit the best possible sidelobes. To avoid the power reduction associated with an unconstrained excitation, an SQP algorithm is used to optimize the excitation while imposing non-linear constraints on the output power levels. The resulting excitation provides a reasonable balance between the peak sidelobes and the reduction in overall transmitter power. The peak sidelobes were shown to be within 3 dB of those obtained using an unconstrained taper, while at the
same time limiting the reduction in overall transmitter power to approximately 0.4 dB. The system gain actually increases by 0.3 dB over the uniform distribution.

The accuracy of the vector combiner is limited by the module-module amplitude and phase accuracy. The magnitude of the error voltage associated with each possible output was shown to be a constant. The module accuracy required to obtain sufficient cancellation at inactive output ports is the case limiting useful operation. The combiner appears to operate successfully with the 54-element array so long as the cancellation is greater than 25 dB relative to the maximum individual excitation. From Fig. 7.11(a), this corresponds to peak module-module amplitude and phase errors not greater than 1 dB and 6°, respectively.

It should be noted that the combiner concept was developed after the array design was finalized. Had the process occurred in reverse order, the final array would likely have used 52 or 56 elements. Although the odd two elements could be fed with a complete combiner (two ports of which would be terminated), the natural course would be to select the number of array elements to be divisible by four.
8 CONCLUSION

To avoid duplication of the chapter conclusions, we will briefly recap the developments and describe the contributions of this thesis.

The theoretical and empirical studies presented here provide a more comprehensive fundamental treatment of the circular array than is currently available in the literature. Previously, discussions of the pattern function of circular arrays were limited to uniformly excited arrays of isotropic elements with all elements active [24], [27]-[29]. We demonstrated this to be the trivial case, and also showed that the pattern function can be accurately represented with the first two terms of an infinite series of Bessel functions, so long as the inter-element separation (IS) is less than 0.9 $\lambda$ (in opposition to that claimed in [29]). We further showed that while some spurious radiation is always present as a result of the secondary maxima of the $J_0(k\rho)$ term, any grating lobes are formed as a result of the $J_{\pm m}(k\rho)$ terms (up to 0.9 $\lambda$ IS). These terms are essentially zero for IS = 0.4 $\lambda$, and small for IS = 0.5 $\lambda$. Beyond this, grating lobes become a problem, not only in terms of the pattern, but in terms of directivity as well. This observation appears to hold for isotropic or endfire elements, and for tapered excitations.

If one-half of the elements of the same array are active, an additional series was introduced into the pattern function. This series cannot be truncated at a convenient point, and is composed of odd-order Bessel function ranging from $-\infty$ to $\infty$. The functions with order close to zero ($J_{-3}(k\rho), J_{-1}(k\rho), J_1(k\rho), \ldots$) have maxima close to the origin, and consequently cannot be eliminated by making the IS reasonably small. These terms contribute entirely to spurious radiation, since the value of the function is zero at the origin (beam boresight). The obvious conclusion is that there is no benefit whatever to be derived from using a portion of an array of isotropic elements.

The pattern function of an array of endfire elements was decomposed. The result is that a number of infinite series are required to express the pattern function, making any simple
understanding of this configuration impossible. Because the array of endfire elements is of great interest, a clear case was made for an additional non-analytical study. Pattern function discussions of circular arrays consisting of any element type where less than all elements are active have not appeared in the literature. The closed-form directivity expressions for element patterns of \( \sin^p \theta \) and \((1+\cos\phi)\sin^p \theta\) are a first. The closest to this is the derivation by Lo and Lee [25] for an array of short dipoles \( (g(\theta) = \sin \theta) \). Cheng and Tseng [26] attempted to extend this for \( p > 1 \), apparently by attempting to extrapolate the previous formulation. No derivation is provided, and the equation provided yields the erroneous result of more than 5 dBi for a single element when \( p = 2 \), and more than 8 dBi when \( p = 3 \).

A parameter study like that of Chapter 5 has also not appeared in open literature. It addresses a number of unanswered questions raised from the theoretical analysis of Chapter 2. The most important results of this survey is that the element beamwidth is almost inconsequential, that the IS should be very nearly \( \lambda/2 \), and that the number of array elements is the overwhelming factor limiting the ability to control sidelobes and compress the azimuth beamwidth. Other observations are that the peak directivity for an endfire array is reached when approximately one-half the array is active, regardless of the excitation, (for an element beamwidth \( \geq 120^\circ \)), that a point of diminishing returns for additional array elements is eventually reached, and that an element phase center can undergo significant movement along the radial dimension as a function of mutual coupling.

With respect to the receive mode beamforming, the linear least-squares (LLS) method has been utilized for many years by those involved with adaptive array processing, and no contribution can be claimed. In addition, Olen and Compton [55] have previously showed that LLS could be combined with an iterative weighting scheme to provide equal sidelobes akin to that of the Dolph-Chebyshev excitation for linear arrays, or to provide additional suppression against jamming sources. However, a number of issues associated with LLS have been overlooked or ignored. To our knowledge, no one has addressed the
effect of the exclusion region width upon array efficiency. No one has demonstrated that LLS can be used to maximize directivity. Neither has anyone documented an example of creating a low-sidelobe pattern with an array having one or more perturbed elements, and no one to our knowledge has addressed the fan beam problem. All these issues are discussed in Chapter 4.

The transmitter commutation concept proposed in Chapter 7 is clearly a contribution to the circular array. The vector combiner is an eight-port device consisting of a butterfly arrangement of hybrid couplers. Working in conjunction with an excitation optimized with the sequential quadratic programming method, it represents a realistic means of scanning a transmit beam while simultaneously using a tapered excitation and fully utilizing the transmitter's available power and efficiency. With it we demonstrated that the peak sidelobes could be reduced by approximately 11 dB with respect to a uniform illumination.

The array optimization answers a number of questions regarding a physical design. One can simulate array patterns ad infinitum, but creating an electromagnetic array model with good F/B ratio, high efficiency, and low pattern ripple is not a trivial task. The results presented in Chapter 6 demonstrate that these objectives can be achieved. Array efficiencies of more than 90% are remarkable considering the extremely close spacing of the final array. The design goals were purposely set very high. Perhaps one of the most remarkable observations is that a sensitive parasitic element can be optimized in an isolated sense, placed in the dense array environment, and expected to maintain its set of desirable characteristics. This is of critical importance, since the time needed to optimize an isolated element is orders of magnitude less than the time required to optimize an embedded element.

Several avenues for future research present themselves. In Chapter 4 we performed an ad-hoc simulation of pattern sensitivity to amplitude and phase errors, and developed a simple mathematical argument to show that a low sidelobe pattern is degraded rapidly in
the neighborhood of the main beam. An exact expression relating the pattern degradation
to excitation errors was not, however, arrived at. In addition, a better vector combiner
may exist, which instead of having a 4 x 4 configuration (inputs x outputs), has perhaps a
6 x 6, 8 x 8, 9 x 9, or some other yet unknown architecture. The current vector combiner
takes the form of an FFT, so that a third stage could be added to create a version with
eight inputs and outputs. This does complicate matters somewhat. For one, the number of
array elements should then be divisible by eight. Another issue is loss, and yet another is
that for each additional stage, the switching diodes must handle twice the power. Finally,
a conventional 8 x 8 version would require that stripline crossovers be used in the final
stage, an undesirable feature for lines carrying tens of kilowatts.

The most interesting area for future research lies in conquering the problem of wasted
space without significantly increasing the number of elements requiring control. Many
schemes were envisioned and tried that would take advantage of the volume present in
the center of the radome. None were successful. Many of these are briefly mentioned in
Chapter 6; still others are not. A quick calculation shows, particularly with the ellipsoidal
shape of the E-2C radome, that the array designed herein uses less than one-half of the
available space. If one were able to utilize this space, the gain could perhaps be doubled.
This area affords the greatest challenge, as well as the greatest reward.
APPENDIX A

DERIVATION OF $F(\phi)$ FOR THE HALF CIRCLE WITH ENDFIRE ELEMENTS

Let the element pattern $g(\phi) = 1 + c \cos(\phi - \phi')$; also let the beam-pointing angle $\phi_0$ be 0. We wish to compute the radiation from a half circle such that $\phi'_m$ ranges from $-M/4 + 1$ to $M/4$. Then

$$F(\phi) = \sum_{m=-M/4+1}^{M/4} \left[ 1 + c \cos(\phi - \phi'_m) \right] e^{jk\rho\cos(\phi'_m - \xi)}.$$  

(A-1)

We are interested in the radiation pattern in the $x$-$y$ plane, where for the special case of $\phi_0 = 0$, we have that $\xi = (\phi - \pi)/2$ and $\rho = 2a\sin(\phi/2)$. Equation (A-1) can be rewritten as

$$F(\phi) = \sum_{m=-M/4+1}^{M/4} \left[ 1 + c \cos(\phi - \phi'_m) \right] e^{-jk\rho\sin(\phi'_m - \phi/2)}.$$  

(A-2)

We retain $k\rho$ for the course of the derivation for the sake of brevity.

$$F(\phi) = \sum_{m=-M/4+1}^{M/4} \left[ 1 + \frac{c}{2} \left( e^{ij\phi'_m} + e^{-ij\phi'_m} \right) \right] \sum_{q=-\infty}^{\infty} \frac{e^{-jq(\phi'_m - \phi/2)}}{J_q(k\rho)}.$$  

(A-3)

We split the summation into 3 parts, recalling that $\phi'_m = \frac{2\pi m}{M}$:
\[
F(\phi) = \sum_{q=-\infty}^{\infty} e^{j q \phi} J_q(k \rho) \sum_{m=-M/4+1}^{M/4} e^{-j 2\pi q m M} + \frac{ce^{j \phi}}{2} \sum_{q=-\infty}^{\infty} e^{j q \phi} J_q(k \rho) \sum_{m=-M/4+1}^{M/4} e^{-j \frac{2\pi (q+1)m}{M}} \\
+ \frac{ce^{-j \phi}}{2} \sum_{q=-\infty}^{\infty} e^{j q \phi} J_q(k \rho) \sum_{m=-M/4+1}^{M/4} e^{-j \frac{2\pi (q-1)m}{M}} 
\]
\[(A-4)\]

We will consider these terms separately, and denote the first, second, and third parts as (A-4a), (A-4b), and (A-4c) respectively.

From (A-4a),

\[
\sum_{q=-\infty}^{\infty} e^{j q \phi} J_q(k \rho) \sum_{m=-M/4+1}^{M/4} e^{-j 2\pi q m M} = M \frac{J_0(k \rho)}{2} + M \sum_{q=1}^{\infty} J_{qM}(k \rho) \cos \left( \frac{qM \phi}{2} \right) \\
+ \sum_{q \text{ odd}} e^{j q \phi} J_q(k \rho) \left\{ (-1)^{\lfloor\frac{q}{2}\rfloor} \left[ \cot \left( \frac{q\pi}{M} \right) - j \right] \right\} \\
= M \sum_{q=-\infty}^{\infty} \varepsilon_q J_{qM}(k \rho) \cos \left( \frac{qM \phi}{2} \right) + 2 \sum_{q=0}^{\infty} (-1)^{q} J_{2q+1}(k \rho) \sin \left[ \frac{\phi}{2} (2q+1) \right] \\
+ 2j \sum_{q=0}^{\infty} (-1)^{q} J_{2q+1}(k \rho) \sin \left[ \frac{\phi}{2} (2q+1) \right] \cot \left[ \frac{\pi}{M} (2q+1) \right] 
\]
\[(A-5)\]

From (A-4b),

\[
\frac{ce^{j \phi}}{2} \sum_{q=-\infty}^{\infty} e^{j q \phi} J_q(k \rho) \sum_{m=-M/4+1}^{M/4} e^{-j \frac{2\pi (q+1)m}{M}} = c \frac{e^{j \phi}}{2} \sum_{q \text{ even}} e^{j q \phi} J_q(k \rho)(-1)^{q/2} \left\{ -j + \cot \left[ \frac{\pi}{M} (q+1) \right] \right\} \\
+ \frac{c Me^{j \phi}}{4} \sum_{q=-\infty}^{\infty} e^{j q \phi} J_q(k \rho), \quad q = -M - 1, -1, M - 1, \ldots 
\]
\[
\sum_{q=0}^{\infty} \varepsilon_q (-1)^q J_{2q}(k\rho) \cos(q\phi)
\]

\[
+ \frac{c}{2} \sum_{q=0}^{\infty} \varepsilon_q (-1)^q \cos[q(\phi + 1)] J_{2q}(k\rho) \cot \left[ \frac{\pi}{M} (2q + 1) \right] + \frac{cM}{4} \sum_{q=-\infty}^{\infty} e^{j(qM+1)^{\phi}/2} J_{qM-1}(k\rho)
\]  

(A-6)

From (A-4c),

\[
\frac{ce^{-j\phi}}{2} \sum_{q=-\infty}^{\infty} \sum_{m=-M/4}^{M/4} e^{-2\pi(q-1)m/M} e^{-j\phi/2} J_q(k\rho) = \frac{ce^{-j\phi}}{2} \sum_{q \text{ even}} e^{-j\phi/2} J_q(k\rho)(-1)^{q/2} \left[ j - \cot \left[ \frac{\pi}{M} (q-1) \right] \right]
\]

\[
+ \frac{cMe^{-j\phi}}{4} \sum_{q=-\infty}^{\infty} e^{-j\phi/2} J_q(k\rho), \quad q = -M + 1, 1, M + 1, \ldots
\]

\[
= \frac{je^{-j\phi}}{2} \sum_{q=-\infty}^{\infty} \varepsilon_q (-1)^q J_{2q}(k\rho) \cos(q\phi)
\]

\[
- \frac{c}{2} \sum_{q=-\infty}^{\infty} \varepsilon_q (-1)^q \cos[q(\phi - 1)] J_{2q}(k\rho) \cot \left[ \frac{\pi}{M} (2q - 1) \right] + \frac{cM}{4} \sum_{q=-\infty}^{\infty} e^{-j(qM-1)^{\phi}/2} J_{qM+1}(k\rho)
\]  

(A-7)

Combining and simplifying equations (A-5), (A-6), and (A-7), we have that

\[
F(\phi) = \frac{M}{2} \sum_{q=0}^{M/2} \varepsilon_q J_{qM}(k\rho) \cos \left[ \frac{qM\phi}{2} \right] + 2 \sum_{q=0}^{\infty} (-1)^q J_{2q+1}(k\rho) \cos \left[ \frac{\phi}{2} (2q + 1) \right]
\]

\[
+ 2j \sum_{q=0}^{\infty} (-1)^q \cot \left[ \frac{\pi}{M} (2q + 1) \right] J_{2q+1}(k\rho) \sin \left[ \frac{\phi}{2} (2q + 1) \right] - c \sum_{q=0}^{\infty} \varepsilon_q (-1)^q J_{2q}(k\rho) \cos(q\phi) \sin \phi
\]

\[
+ \frac{c}{2} \sum_{q=0}^{\infty} \varepsilon_q (-1)^q J_{2q}(k\rho) \left\{ \cot \left[ \frac{\pi}{M} (2q + 1) \right] \cos[\phi(q+1)] - \cot \left[ \frac{\pi}{M} (2q - 1) \right] \cos[\phi(q-1)] \right\}
\]

\[
+ \frac{jcM}{2} \left\{ \sum_{q=0}^{\infty} J_{qM+1}(k\rho) \sin \left[ \frac{\phi}{2} (qM - 1) \right] + \sum_{q=1}^{\infty} J_{qM-1}(k\rho) \sin \left[ \frac{\phi}{2} (qM + 1) \right] \right\}
\]

(A-8)
APPENDIX B

DERIVATION OF RECEIVE-MODE EXCITATION VECTOR

(Continued from equation 4.5)

We seek to choose the vector \( \mathbf{w} \) that minimizes \( J \). Since \( w_m \) is complex we write

\[
 w_m = x_m + jy_m, \quad m = 1, 2, \ldots, M. \tag{B-1}
\]

A necessary condition for an extremum is

\[
 \frac{\partial J}{\partial x_l} = 0, \quad \frac{\partial J}{\partial y_l} = 0, \quad l = 1, 2, \ldots, M. \tag{B-2}
\]

For notational simplicity, redefine \( \mathbf{g} \):

\[
 \mathbf{g}(\phi_n) = \mathbf{g}(n) = [g_1(n), g_2(n), \ldots, g_M(n)], \quad n = 1, 2, \ldots, N. \tag{B-3}
\]

Also rewrite \( F_n = F(\phi_n) \) and \( \hat{F}_n = \hat{F}(\phi_n) \), and apply (B-2) to (4.5):

\[
 \frac{\partial J}{\partial x_l} = \frac{\partial}{\partial x_l} \sum_{n=1}^{N} (F_n - \hat{F}_n)(F_n^* - \hat{F}_n^*) = \sum_{n=1}^{N} \left\{ \frac{\partial F_n}{\partial x_l} (F_n^* - \hat{F}_n^*) + \frac{\partial F_n^*}{\partial x_l} (F_n - \hat{F}_n) \right\}
\]

\[
 = 2 \text{Re} \sum_{n=1}^{N} \frac{\partial F_n}{\partial x_l} (F_n^* - \hat{F}_n^*). \tag{B-4}
\]

Similarly,

\[
 \frac{\partial J}{\partial y_l} = 2 \text{Re} \sum_{n=1}^{N} \frac{\partial F_n}{\partial y_l} (F_n^* - \hat{F}_n^*). \tag{B-5}
\]

From (4.1) and (B-3),

\[
 \frac{\partial F_n}{\partial x_l} = \frac{\partial}{\partial x_l} \sum_{n=1}^{N} w_m g_m(n) = g_l(n), \tag{B-6}
\]

\[
 \frac{\partial F_n}{\partial y_l} = \frac{\partial}{\partial y_l} \sum_{n=1}^{N} w_m g_m(n) = jg_l(n). \tag{B-7}
\]
Substitute (B-6) and (B-7) into (B-4) and (B-5) to get

\[
\frac{\partial J}{\partial x_j} = 2 \text{Re} \sum_{n=1}^{N} g_i(n)(F_n^* - \hat{F}_n^*)
\]  

(B-8)

\[
\frac{\partial J}{\partial y_j} = 2 \text{Re} \left\{ \sum_{n=1}^{N} g_i(n)(F_n^* - \hat{F}_n^*) \right\}.
\]  

(B-9)

In general, if \( s = u + jv \), then

\[
\text{Re}(js) = \text{Re}(ju - v) = -\text{Im}(s)
\]  

(B-10)

From (B-2), (B-8), and (B-9),

\[
\text{Re} \sum_{n=1}^{N} g_i(n)(F_n^* - \hat{F}_n^*) = 0, \quad \text{Im} \sum_{n=1}^{N} g_i(n)(F_n^* - \hat{F}_n^*) = 0,
\]

or simply

\[
\sum_{n=1}^{N} g_i(n)(F_n^* - \hat{F}_n^*) = 0.
\]  

(B-11)

Because the sum is zero, we can also write this as

\[
\sum_{n=1}^{N} g_i^*(n)(F_n^* - \hat{F}_n^*) = 0, \quad l = 1,2,\ldots, M.
\]  

(B-12)

Expanding now the expression for the field \( F_n \), we have that

\[
\sum_{n=1}^{N} g_i^*(n) \left( \sum_{m=1}^{M} w_m g_{m}(n) - \hat{F}_n^* \right) = 0.
\]  

(B-13)

Considering now all rows \((l's)\):

\[
\sum_{n=1}^{N} \left\{ \sum_{j=1}^{M} \sum_{m=1}^{M} g_{l}^* (n) g_{m}(n) w_{m} \right\} = \sum_{n=1}^{N} \left\{ \sum_{j=1}^{M} g_{l}^* (n) \hat{F}_n \right\}.
\]  

(B-14)

The excitation vector \( \mathbf{w} \) can be determined by expressing (B-14) as

\[
\mathbf{Rw} = \mathbf{v},
\]  

(B-15)
where

\[
\mathbf{R} = \begin{bmatrix}
\sum_{n=1}^{N} g_1^*(n) g_1(n) & \sum_{n=1}^{N} g_1^*(n) g_2(n) & \cdots & \sum_{n=1}^{N} g_1^*(n) g_M(n) \\
\sum_{n=1}^{N} g_2^*(n) g_1(n) & \sum_{n=1}^{N} g_2^*(n) g_2(n) & \cdots & \sum_{n=1}^{N} g_2^*(n) g_M(n) \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{n=1}^{N} g_M^*(n) g_1(n) & \sum_{n=1}^{N} g_M^*(n) g_2(n) & \cdots & \sum_{n=1}^{N} g_M^*(n) g_M(n)
\end{bmatrix}
\]  

(B-16)

and \( \mathbf{v} = \begin{bmatrix}
\sum_{n=1}^{N} \hat{g}_1 \hat{F}_n & \sum_{n=1}^{N} \hat{g}_2 \hat{F}_n & \cdots & \sum_{n=1}^{N} \hat{g}_M \hat{F}_n
\end{bmatrix}^T \).  

(B-17)
APPENDIX C

NEC GEOMETRY FILES FOR SELECTED FIGURES

The following files are provided as a compact form of detailed geometry information, so that simulation results may be easily reproduced. See reference [38] for a program users guide.

Figure 2.8

CE
GW 1 12 0 -6.6 0 0 6.6 0 .25
GW 10 11 0 -6.3 0 0 6.3 0 .25
GM 0 0 0 0 0 5.3 0 0 10 1 10 11
GW 20 12 7.7 -5.7 0 7.7 5.7 0 .25
GW 30,12, 13.0,-5.1,0, 13.0,5.1,0, .25
GM 0 0 90 0 0 0 0 0
GM 0 0 0 0 0 60 0 0
GR 50 40
GS0 0 .0254
GE
PT -1
EX 0 10 6 0 1 0
LD 0 10 6 6 50 0 0
FR 0 1 0 0 435 2
RP 0 1 361 1010 90 0 0 1
EN

Figure 3.4

CE
GW 1 12 0 -6.6 0 0 6.6 0 .25
GW 10 11 0 -6.3 0 0 6.3 0 .25
GM 0 0 0 0 0 5.3 0 0 10 1 10 11
GW 20 12 7.7 -5.7 0 7.7 5.7 0 .25
GW 30 12 13.0 -5.1 0 13.0 5.1 0 .25
GA 111 26 5 0 359.9999 .25
GM 0 0 0 45 0 0 0 111 1 111 26
GM 0 0 0 45 45 20 0 0 111 1 111 26
GS0 0 .0254
GE
PT -1
LD 0 10 6 6 50 0 0
LD 0 111 7 7 160 0 0
EX 0 111 6 0 1 0
Figure 3.6

(same as figure 3.4 with load cards removed)

Figure 3.9

CE
GW 1 12 0 -6.6 0 0 6.6 0 .25
GW 10 11 0 -6.3 0 0 6.3 0 .25
GM 0 0 0 0 5.3 0 0 10 1 10 11
GW 20 12 7.7 -5.7 0 7.7 5.7 0 .25
GW 30 12 13.0 -5.1 0 13.0 5.1 0 .25
GM 0 0 90 0 0 0 0 0
GM 0 0 0 0 60 0 0
GM 100 1 0 0 9 0 0 0
GR 200 20
GS0 0 .0254
GE
PT -1
EX 0 10 6 0 1.165951825 3.034131220
EX 0 110 6 0 -2.207811700 3.968483100
EX 0 210 6 0 -0.378665139 2.837039454
EX 0 310 6 0 -4.460672973 2.571244016
EX 0 410 6 0 -0.872683498 1.844076189
EX 0 510 6 0 -4.287654054 -0.974065071
EX 0 610 6 0 0.288907481 -1.677673259
EX 0 710 6 0 -2.163980853 -3.034388352
EX 0 810 6 0 4.527941804 -0.690793682
EX 0 910 6 0 -4.661475691 -0.31540018
EX 0 1010 6 0 5.801091435 6.209002005
EX 0 1110 6 0 -5.824695585 -9.078075615
EX 0 1210 6 0 0.445221516 12.366571605
EX 0 1310 6 0 10.289228805 -12.415096365
EX 0 1410 6 0 -20.078225565 0.967073903
EX 0 1510 6 0 8.808751875 28.359249585
EX 0 1610 6 0 42.847788210 -9.248286345
EX 0 1710 6 0 4.865833573 -56.358918000
EX 0 1810 6 0 -44.873691945 -44.098547865
EX 0 1910 6 0 -69.930388500 -18.323102445

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Figure 4.2

CE
GW 1 12 0 -6.6 0 0 6.6 0 .25
GW 10 11 0 -6.3 0 0 6.3 0 .25
GM 0 0 0 0 0 5.3 0 0 10 1 10 11
GW 20 12 7.7 -5.7 0 7.7 5.7 0 .25
GW 30 12 13.0 -5.1 0 13.0 5.1 0 .25
GM 0 0 90 0 0 0 0 0
GM 0 0 0 0 90 0 0
GR 50 60
GS0 0 .0254
GE
PT -1
EX 0 10 6 0 1.00000 0.00000
LD 0 10 6 6 40 0 0
FR 0 1 0 0 435 2
RP 0 1 181 1010 90 0 0 1
EN
Figure 5.21

CE
GW 1 12 0 -6.6 0 0 6.6 0 .25
GW 10 11 5.51 -6.09 0 5.51 6.09 0 .25
GW 20 11 10 -5.7 0 10 5.7 0 .25
GW 30 11 16.7 -5.61 0 16.7 5.61 0 .25
GW 32 11 24.2 -5.65 0 24.2 5.65 0 .25
GW 34 11 32.4 -5.45 0 32.4 5.45 0 .25
GM 0 0 0 0 0 0 0 127 0 0
GM 0 0 0 0 0 0 0 8.9 1 1 3 4
GM 100 10 0 0 6.66 0 0 0
GS 2 0 0
GE 0 -1 0
EX 0 510 6 0 1 0
LD 0 10 6 6 25 0 0
LD 0 110 6 6 25 0 0
LD 0 210 6 6 25 0 0
LD 0 310 6 6 25 0 0
LD 0 410 6 6 25 0 0
LD 0 510 6 6 25 0 0
LD 0 610 6 6 25 0 0
LD 0 710 6 6 25 0 0
LD 0 810 6 6 25 0 0
LD 0 910 6 6 25 0 0
LD 0 1010 6 6 25 0 0
FR 0 1 0 0 425 5
RP 0 1 121 1000 90 0 0 3
EN

Figure 6.4

GW 1 9 0 -3.79 0 0 3.79 0 .375
GW 2 4 0 3.79 0 -3.6 3.79 0 .375
GW 3 4 0 -3.79 0 3.6 -3.79 0 .375
GM 0 0 0 0 0 0 0 8.9 1 1 3 4
GW 10 11 0 -4.68 0 0 4.68 0 .375
GW 11 3 0 4.68 0 -2.75 4.68 0 .375
GW 12 3 0 -4.68 0 0 -4.68 -2.75 .375
GM 0 0 0 0 0 6.07 0 8.7 10 1 12 3
GW 20 10 0 -4.32 0 0 4.32 0 .375
GW 21 2 0 4.32 0 -1.414 4.32 -1.414 .375
GW 22 2 0 -4.32 0 0 -4.32 -2 .375
GM 0 0 0 0 0 7.82 0 8.5 20 1 22 2
GW 30 10 0 -4.19 0 0 4.19 0 .375
GW 31 2 0 4.19 0 -1.414 4.19 -1.414 .375
Figure 6.18(b)
Figure 6.28
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VITA

James Matthew Stamm was born in Danville, PA, on January 11, 1962. Upon graduating from high school, he entered active duty military service in the U.S. Air Force. In 1989, he graduated from Temple University with a BSEE degree. He worked at the Naval Air Warfare Center—Aircraft Division (NAWCAD), Patuxent River, MD, from 1989 until 1995 as a test and evaluation engineer with the Communication, Navigation and Identification Laboratory. During this period, he pursued graduate studies at Florida Institute of Technology and John Hopkins University, concentrating in communications and microwave circuit design. He entered the PhD program at the Pennsylvania State University in Fall of 1995. Since December 1996, he has been with the RF Sensors Branch, also at NAWCAD, where he has worked on high-gain radar and communications antennas. His current research interests are in the area of antennas and microwave circuit design, with an emphasis on computational methods and optimization.