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# TRAVELING WAVES ARISING IN THE APPROXIMATION AND CONTROL OF PDES

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# Abstract

This dissertation is in the area of partial differential equations. It contains two parts: (1). The backward Euler approximation to conservation laws. (2). The traveling profiles for invasive species Model.

In the first part, we study a hyperbolic system of conservation laws and construct an approximation by a backward Euler scheme, where time is discretized while space is still described by a continuous variable $x \in \mathbb{R}$. We prove the global existence and uniqueness of these approximate solutions, and the invariance of suitable subdomains. Furthermore, given a left and a right state $u_l$, $u_r$ connected by an entropy-admissible shock, we construct a traveling wave profile for the backward Euler scheme connecting these two asymptotic states in two main cases. Namely: (i) a scalar conservation law, where the jump $u_l$–$u_r$ can be arbitrarily large, and (ii) a strictly hyperbolic system, assuming that the jump $u_l$–$u_r$ occurs in a genuinely nonlinear family and is sufficiently small.

In the second part, we study a parabolic system of PDEs describing the spreading of an invasive species. The two models are worked on: (i) all insects carry the infection and contaminate the trees, and (ii) only the infected insects contaminate the tree. We try to find the traveling front for these two models with some proper speed. In the first model (i), we can get the optimal control added to evolution equations for the given cost function. We discover that the more we slow down the traveling front speed, the more cost we will pay. In second model (ii), the smallest traveling speed for the existence of the traveling front could be found, and in this case, no matter what controls are added to the system, the smallest traveling speed will not decrease unless the density of all insects decreases at the uninfected location, or the density of all insects decreases to zero at some places.

# Table of Contents

# List of Figures

# List of Symbols

$\mathbb{R}$ The 1-dimensional real number space.

$\mathbb{R}^m$ The $m$-dimensional real number space.

$\mathcal{C}(\Omega)$ The set of the continuous functions on $\Omega$, the subset of some number field.

$\mathcal{C}_n(\Omega)$ The set of the $m$th continuously differentiable functions on $\Omega$, the subset of some number field.

$\mathcal{C}_{n,m}(\Omega_1 \times \Omega_2)$ The set of the functions which are $n$th continuously differentiable on $\Omega_1$ and $m$th continuously differentiable on $\Omega_2$. $\Omega_1, \Omega_2$ are the subsets of some number fields respectively.

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# Chapter 1 Introduction

In this dissertation, we study the traveling wave solutions for some types of partial differential equations. Many phenomena in physics, chemistry, biology can be described by the partial differential equations, so this area is deeply studied. The problems of the partial differential equations usually come with some additional conditions, such as initial conditions, boundary conditions, asymptotic conditions and so on. Some special equations can be solved explicitly such as heat equations, wave equations and some other linear differential equations. However, for more general nonlinear parabolic and elliptic equations, the solutions cannot be found in the explicit form.

There are many ways to solve the PDEs, such as Fourier series, Laplace transform, fixed point of the contractive maps, etc. Finding the traveling wave solutions is one of them. This method will give some possible solutions for some special kinds of PDEs and these wave-form solutions could help us to analyze the properties of other non-wave solutions.

In the rest of this chapter, we will discuss the reasons we do research on the wave solutions and analyze the basic properties of the wave solutions. We will try to use the phase portrait method to prove the existence of the wave solutions to the general scalar diffusion equation, which is defined on $]0,T] \times \Omega \subset I\!R^m$ and takes the form as

$$\frac{\partial u}{\partial t} = \sum_{i,j} a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_i b_i(t,x)\frac{\partial u}{\partial x_i} + F(t,x,u). \tag{0.1}$$

In Chapter 2, we will review some basic facts about hyperbolic systems of conservation laws. We will analyze the conditions for the generations of three structures which will be used to construct the solutions for the Riemann problems: the rarefaction waves, shocks and contact discontinuities.

In Chapter 3, we will study the backward Euler method as a way to construct approximate solutions. The existence and uniqueness of backwards Euler approximations for systems of conservation laws will be proved.

In Chapter 4, we will introduce the general model of biology problems, including the assumptions and the generation process of the diffusion equations in this model from these problems.

In Chapter 5, we will study the existence and optimal control of traveling profiles to two models simplified from the general model in Chapter 4: the first model will think that all individuals are infected; the second model will consider the infection rate is not always 100%. We discovered that the infected speed could be reduced if we added the control to the environments. However, the infected speed in the second model cannot be controlled unless the total amount of the individuals decreases.

# 1 Introduction to the application of traveling profiles

The traveling wave solutions are usually considered when PDEs have both time and spatial variables, for example the heat equations (a kind of parabolic equations) and the wave equations (a kind of hyperbolic equations). We will take the diffusion equation (0.1) (a kind of parabolic equations) as an example because the biology model we will study in **Chapter 5** contains a special case of diffusion equations.

The main idea to find a wave solution is to construct a solution for a PDE such that the solution travels along a direction at a constant speed and the shape of the solution does not change with the wave traveling. To realize this assumption for the solutions, we suppose the solution takes the form as

$$u(t,x) = w(x_1 - ct), \tag{1.2}$$

for some speed $c \in I\!R$ and $w(\xi)$ is defined on $I\!R$, without loss of generality. This form of solution considers that $u$ only depend on the linear combination of $t$, $x_1$. Then, for other spatial variable, $\partial u/\partial x_i = 0$, $i = 2, \ldots, m$. In this way, $u$ is constant with respect to each plane $\{(t, x_1, \ldots, x_m) | x_1 - ct = \mathit{constant}\}$. Hence, for any fixed time $t$, $u$ will keep the form as $w$ but shift right $ct$ units.

In order to find the solution like (1.2), the main method is to transfer the partial differential equation into an ordinary differential equation with only variable $\xi$. We should suppose that $b_i$, $a_{ij}$ and $F(t,x,u)$ could be represented as a wave as $b_i(x_1 - ct)$, $a_{ij}(x_1 - ct)$ and $F(x_1 - ct, u)$. With this additional assumption for $b_i$, $a_{ij}$, $F$, after substituting (1.2) into the equation (0.1), we could get

$$-cw'(\xi) = a_{11}(\xi)w''(\xi) + b_1(\xi)w'(\xi) + F(\xi, w(\xi)). \tag{1.3}$$

Instead of considering the wave solution traveling in one variable direction as (1.2), the planar wave solution which takes the form as

$$u(t, x) = w(y \cdot x - t),$$

where $y$ is a $m$-dimensional constant vector, will consider the wave solution traveling in the direction $y$.

## 1.1 Technical tools of wave solutions

A wave solution contributes to a profile which moves with a fixed speed but doesn't change the its shape. Because of special properties of wave solutions, there are many advantages to do research on the wave solutions.

The most important advantage is that it will help us to find a solution for the PDEs with both time and space variables, because after substituting the wave form (1.2) into the PDEs, they will turn into ODEs. Generally, ODEs are easier to solve: three main methods can be applied to solve ODEs transformed from the corresponding PDEs:

**Leray-Schauder method** Consider the speed as a constant unknown. Define a Leray-Schauder degree satisfying certain properties such that we can deform the initial system with the constant unknown speed into a suitable model system for which the degree is different from zero on the boundary of a sufficiently large-radius ball. Then, the existence of the solution for the initial system could be proved by finding the solution on this model system.

**Phase portrait method** Transfer the second order differential equation system into the first order ODE system by introducing the new variable equal to the derivatives.

For example, (1.3). If set $p = w'$. $w$, $p$ will satisfy

$$w' = p, \qquad a_{11}p' = -cp - b_1 p - f(\xi, w).$$

We could find the wave solution by finding the trajectories satisfying the 1-order ODEs above. If the asymptotic conditions for $w$ such as $\lim_{\xi\to\pm\infty} = w_\pm$ exists, the trajectories should be restricted to the ones connecting the points $(w_-, 0)$ and $(w_+, 0)$ in the phase space $(w, p)$.

**Theory of bifurcations** This method will consider the speed as a new variable together with $w$ in (1.3). This method is typically used to solve periodic wave problems and multi-dimensional wave problems.

We will use **theory of bifurcations** to construct the backward Euler approximation for the systems of conservation laws in **Chapter 3** and **phase portrait method** to solve the invasive model in **Chapter 5**.

Another advantage to find wave solutions is that we could use them to analyze the properties of general solutions of the PDEs. For example, if we could find a wave solution for a parabolic equation, the maximum principle and the comparison principle of parabolic equations could help us to determine the upper or lower bounds for the possible non-wave solutions for the parabolic equation, which will guarantee the existence of the non-wave solutions and provide the estimation of the bounds of these non-wave solutions.

## 1.2 Maximum principle when $f$ independent of $t$, $x$

In this part, we will analyze the the strong maximum principle of the parabolic equations to show the usage of the wave solution method. We consider a special diffusion equation (0.1) where $F$ only depends on $u$ and rewrite the equation into

$$u_t + Lu - f(u) = 0, \qquad Lu = -\sum_{ij} a_{ij} u_{x_i x_j} + \sum_i b_i u_{x_i} + cu \tag{1.4}$$

defined on $Q_T \doteq ]0, T] \times \Omega \subset I\!R^m$. In the following discussion, we will suppose that $Q_T$ is connected. $a_{ij}$, $b_i$, $i, j = 1, \ldots, m$ and $c$ are continuous functions on $\overline{Q}_T$ and $a_{ij} = a_{ji}$. $f \in \mathcal{C}_1(I\!R, I\!R)$.

To make the parabolic equation satisfy the maximum principle, we need that the operator $L$ is non-divergence form, which means that the coefficients of the second-order derivatives satisfies

$$\sum_{ij} a_{ij} \zeta_i \zeta_j \geq \theta |\zeta|^2, \qquad \text{for some constant } \theta > 0.$$

By the analysis for the second-order parabolic equations in [25, 26], the strong maximum principle holds for some special parabolic equations.

**Theorem 1.1.** *Assume that $d(t,x)$ is bounded in $\overline{Q}_T$ and $v(t,x) \in \mathcal{C}_{2,1}(Q_T) \cap \mathcal{C}(\overline{Q}_T)$ and satisfies*

$$v_t + Lv - d(t,x)v \geq 0 \text{ in } Q_T, \qquad v(0,x) \geq 0. \tag{1.5}$$

*with $L$ is non-divergence form and $c$ in $L$ is bounded. Then, either* **(1)** *$v > 0$ in $Q_T$ or* **(2)** *$v \equiv 0$ in $\bar{Q}_T$.*

**Proof.** See Appendix A. The more standard and general proof of this theorem could be found in **Corollary 9.14** of [46]. □

Then, use the **Theorem 1.1**, the following theorems could be gotten

**Theorem 1.2.** *If $\underline{u}$ and $\overline{u}$ are sub- and super-solution of (1.4) and $\underline{u}(0,x) \leq \overline{u}(0,x)$, then either* **(1)** *$\underline{u} < \overline{u}$ in $Q_T$ or* **(2)** *$\underline{u} \equiv \overline{u}$ in $\overline{Q}_T$.*

**Theorem 1.3.** *If $u$ is the solution of (1.4) with the initial condition $u(0,x) = u_0(x)$ and $u_0$ is the time-independent solution to (1.4), then either* **(1)** *$u$ is strictly increasing in $t$ in $Q_T$ with fixed $x$ or* **(2)** *$u \equiv u_0$ in $\overline{Q}_T$.*

**Theorem 1.4.** *Assume that $\underline{u}$ and $\overline{u}$ are sub- and supersolution of (1.4) in $Q_T$ and satisfy*

**1).** $\underline{u}(0,x) \leq \overline{u}(0,x)$ *for $x \in \Omega$;*

**2).** *if $\Omega \neq I\!R^m$, $D\underline{u}(t,x) \cdot \vec{n}(x) + \beta(t,x)\underline{u}(t,x) \leq D\overline{u}(t,x) \cdot \vec{n}(t,x) + \beta(t,x)\overline{u}(t,x)$ for $(t,x) \in \partial\Omega \times ]0,T[$ where $\beta(t,x) \geq 0$ on $\partial\Omega \times ]0,T[$ and $\vec{n}(x)$ is the outward normal at $x \in \partial\Omega$.*

*If $u_0(x)$ is a continuous function in $\Omega$ and satisfies $\underline{u}(0,x) \leq u_0(x) \leq \overline{u}(0,x)$, then there exists a unique solution $u(t,x)$ of (1.4) such that $u(0,x) = u_0(x)$ and $\underline{u} \leq u \leq \overline{u}$ in $\overline{Q}_T$.*

The proofs of these three theorems could be found **Section 4.1** in [26] and **Theorem 10.1** in [46]. The **Theorem 10.1** in [46] is the comparison theorems, which is an important result when we do the qualitative analysis for the diffusion equations and we will greatly use it in **Chapter 5**.

# Chapter 2 Introduction to Conservation Laws

In this section, we will review some basic theories about the systems of conservation laws.

The concept of the conservation laws comes from the physics systems where the integrals of some physical quantities will always keep constant, for example, the energy and momentum. If we write these physical relationships into the mathematical expression, we could get a system of differential equations.

A first-order system of $n$ conservation laws takes the form:

$$u_t + f(u)_x = 0, \tag{0.1}$$

where $u = (u_1, \ldots, u_n)$ is the vector of conserved quantity and $f(u) : \Omega \subset I\!R^n \to I\!R^n$ is a smooth function, called the flux.

Integrating (0.1) over any interval $[a, b]$, we obtains

$$\frac{d}{dt}\int_a^b u(t,x)dx = -\int_a^b f(u)_x dx = f(u(a)) - f(u(b)).$$

This means that the total amount of the quantity $u$ inside $[a, b]$ can change only due to the difference of the flow, $f(u)$, across the two endpoints. Hence, the integral of the dependent variables over the space variable will never change with respect to time if there is no out-source added into or in-source moved out of the system.

To simplify the problems related to conservation laws, we will mainly focus on strictly hyperbolic systems, a special kind of systems of conservation laws whose eigenvalues will keep distinct from each other.

# 1 Strictly hyperbolic systems of conservation laws

We can write the equations (0.1) into the quasilinear form due to the smoothness of $f(u)$:

$$u_t + A(u)u_x = 0, \qquad A(u) \doteq Df(u) : I\!R^n \to I\!R^{n\times n}. \tag{1.2}$$

The property of $A(u)$ will greatly affect the properties of $u$. Strictly hyperbolic system is a special kind of systems of conservation laws with a special $A(u)$.

**Definition 1.1.** *We say* (1.2) *is a strictly hyperbolic system if* $A(u)$ *has* $n$ *distinct real eigenvalues*

$$\lambda_1(u) < \cdots < \lambda_n(u). \tag{1.3}$$

Since $A(u)$ has $n$ distinct eigenvalues, it will also have $n$ distinct right eigenvecters $\{r_1(u), \ldots, r_n(u)\}$ and $n$ corresponding left eigenvectors $\{l_1(u), \ldots, l_n(u)\}$. These two bases of eigenvectors could be normalized, so that

$$|r_i| \equiv 1, \qquad r_i \cdot l_j = \delta_{ij}.$$

## 1.1 Weak solution

The most common problem related to the conservation laws is the Cauchy problem:

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x). \tag{1.4}$$

**Definition 1.2.** *A function* $u : [0,T] \times I\!R \to I\!R^n$ *is a weak solution for* (1.4) *if*

*(1).* $u,\ f(u) \in \mathbf{L}^1_{loc}([0,T] \times I\!R), I\!R^n)$ *and*

$$\int_0^T \int_{-\infty}^{+\infty} u\phi_t + f(u)\phi_x dx\, dt + \int_{-\infty}^{+\infty} \bar{u}\phi(0,x)dx = 0 \tag{1.5}$$

*for all* $\phi \in \mathcal{C}^1_c(]-\infty, T[\times I\!R, I\!R^n)$.

*(2).* $u(0,x) = \bar{u}(x)$.

## 1.2 Rankine-Hugoniot conditions

Rankine-Hugoniot conditions will generate when we consider the Cauchy problem (1.4) with

$$\bar{u}(x) = \begin{cases} u^- & x < 0 \\ u^+ & x > 0 \end{cases}. \tag{1.6}$$

**Definition 1.3.** *The Rakine-Hugoniot conditions are that for any pair of constants* $u^-$, $u^+$*, there exists a constant number* $\lambda$ *to make*

$$\lambda(u^+ - u^-) = f(u^+) - f(u^-) \tag{1.7}$$

With this condition, we can give the result for the Cauchy problem with the initial data (1.6)

**Theorem 1.1.** *The Cauchy problem* (1.4) *with the initial data* (1.6) *will have the solution taking the form as:*

$$u(t,x) = \begin{cases} u^- & x < \lambda t \\ u^+ & x > \lambda t \end{cases}, \tag{1.8}$$

*if and only if* $u^+$, $u^-$ *satisfy the Rakine-Hugoniot conditions for some real* $\lambda$*.*

From the theorem above, we can discover that the scalar case of the Cauchy problem for a class of hyperbolic systems of conservation can always be solved because the value of $\lambda$ can be calculated from (1.7) if $u^+$, $u^-$, $f(u^+)$, $f(u^-)$ are scalar values. While the multidimensional case could only be solved under some special requirements.

The discontinuity along the line $\{x = \lambda t\}$ is the shock and $\lambda$ is the speed of this shock. In this way, the Rankine-Hugoniot conditions could be interpreted as

$$[\text{speed of the shock}] \times [\text{jump in the state}] = [\text{jump in the flux}].$$

We can rewrite (1.7) as following,

$$\lambda(u^+ - u^-) = \int_0^1 Df(\theta u^+ + (1-\theta)u^-) \cdot (u^+ - u^-) d\theta \doteq A(u^+, u^-) \cdot (u^+ - u^-).$$

$A(u^+, u^-)$ is the average of the differential matrices $Df$. The Rankine-Hugoniot conditions hold if and only if $u^+ - u^-$ is exactly an eigenvector of the average matrix $A(u^+, u^-)$. Then, $\lambda$, the speed of the shock, will be the corresponding eigenvalue.

## 1.3 Admissibility conditions

In fact, the definition for weak solution in **Definition 1.2** cannot restrict the possible weak solution to a unique one. We need some additional conditions to choose the admissible weak solution from several possible ones.

There are three kinds of admissibility conditions:

**Vanishing viscosity.** For a weak solution $u$, there exists a sequence of smooth solution $u^\epsilon$ to

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon, \tag{1.9}$$

which converge to $u$ in $\mathbf{L}^1_{loc}$ as $\epsilon \to 0^+$.

**Entropy inequality.** For a weak solution $u$, for every pair of function $(\eta, q)$ where $\eta : I\!R^n \to I\!R$ is a convex and continuously differentiable function and $q$ satisfies

$$Dq(v) = D\eta(v) \cdot Df(v), \qquad v \in I\!R^n, \tag{1.10}$$

the following inequality,

$$\iint \eta(u)\phi_t + q(u)\phi_x \, dx \, dt \geq 0,$$

holds for every smooth function $\phi \geq 0$ with compact support on $[0, T] \times I\!R$.

*Remark:* In the pair of function $(\eta, q)$ satisfying (1.10), $\eta$ will be called an *entropy* and $q$ will be the corresponding *entropy flux*.

**Lax condition.** For a weak solution $u$, at every point $(t, x)$ of the jump, the left and right states of this point $u^-$, $u^+$ satisfy

$$\lambda_i(u^-) \geq \lambda_i(u^+, u^-) \geq \lambda_i(u^+), \tag{1.11}$$

where $\lambda_i(u^+)$ and $\lambda_i(u^-)$ are the $i$th eigenvalues of the differential matrix $A(u^+)$ and $A(u^-)$ respectively. $\lambda_i(u^+, u^-)$ is equal to the speed of the jump at the point $(t, x)$ and hence is the $i$th eigenvalue of the average of differential matrices, $A(u^+, u^-)$.

The weak solution is called admissible if it satisfies any one of these three admissibility conditions above. For example, (1.8) is an admissible solution to the Cauchy problem (1.4) with (1.6) if (1.7) holds

## 1.4 Riemann problem

The Riemann problem for the system of conservation laws is to solve the system (0.1) with the piecewise constant initial data (1.6).

To make it simpler, we should add additional hypothesis for the system (0.1):

- The coefficients of $f(u)$ is smooth.
- Each characteristic field is either genuinely non-linear or linearly degenerate.

The explanations of “genuinely non-linear” and “linearly degenerate” are given below

**Definition 1.4.** *For an $n \times n$ matrix like $A(u)$, it has eigenvalues as $\lambda_i(u)$ and the corresponding right eigenvectors as $r_i(u)$. If*

$$D\lambda_i(u) \cdot r_i(u) \neq 0, \qquad \textit{for all } u \in \Omega,$$

*then we say that the $i$th charactersitic field is genuinely non-linear.*

*If*

$$D\lambda_i(u) \cdot r_i(u) = 0, \qquad \textit{for all } u \in \Omega,$$

*then we say that $i$th characteristic field is linearly degenerate.*

By adjusting the sign of the eigenvector, we can say that the genuinely non-linear characteristic field, for example, the $i$th characteristic field, satisfies

$$D\lambda_i(u) \cdot r_i(u) > 0, \qquad \text{for all } u \in \Omega. \tag{1.12}$$

The solution of the Riemann problem will only consist of three kinds of structures: rarefaction wave, shock and contact discontinuity, which we will look into one by one.

### 1.4.1 rarefaction waves

We define the rarefaction curve firstly.

**Definition 1.5.** *The $i$th rarefaction curve through $u_0$ is the solution to the Cauchy problem*

$$\frac{du}{d\sigma} = r_i(u(\sigma)), \quad \sigma \in \mathbb{R}, \qquad u(0) = u_0,$$

*where $r_i(u(\sigma))$ is the $i$th right eigenvector of the matrix $A(u(\sigma))$. This solution is unique and can be denoted as $R_i(\sigma)(u_0)$ satisfying*

$$R_i(0)(u_0) = u_0; \qquad R_i(\sigma')(R(\sigma)(u_0)) = R_i(\sigma' + \sigma)(u_0).$$

Now, for the given $u^-$, $u^+$ in (1.6), if there exists a $\sigma^+ > 0$ such that $u^+ = R_i(\sigma^+)(u^-)$, the function below

$$u(t,x) = \begin{cases} u^-, & x < \lambda_i(u^-)t \\ u^+, & x > \lambda_i(u^+)t \\ R_i(\sigma)(u^-), & x/t = \lambda_i(R_i(\sigma)(u^-)) \end{cases} \tag{1.13}$$

is the weak solution of (1.4) with (1.6).

**Definition 1.6.** *The solution of the Riemann problem taking the form of* (1.13) *is called a centred rarefaction wave.*

If the $i$th characteristic field is genuinely non-linear and (1.12) holds for suitable $\lambda_i$ and $r_i$, then the map $\sigma \mapsto \lambda_i(R_i(\sigma)(u_0))$ is strictly increasing because

$$\frac{d\lambda_i(R_i(\sigma)(u_0))}{d\sigma} = D\lambda_i(R_i(\sigma)(u_0)) \cdot \frac{dR_i(\sigma)(u_0)}{d\sigma} = D\lambda_i(R_i(\sigma)(u_0)) \cdot r_i(R_i(\sigma)(u_0)) > 0.$$

From the analysis above, it's easy to see that for a given $u^-$, if the Riemann problem could be solved by the solution as (1.13), $u^+$ must be on the positive part of the rarefaction curve $R_i(u^-)(\sigma)$. This is because from the form of (1.13), we require $\lambda_i(u^-) < \lambda_i(u^+)$. Since $d\lambda_i(R_i(\sigma)(u_0))/d\sigma > 0$ and $u^+ = R_i(\sigma)(u^-)$, we see that $\sigma > 0$.

This implies that only the positive part of the rarefaction curve $\{R_i(\sigma)(u^-), \sigma > 0\}$ is important for a given left side $u^-$.

### 1.4.2 shocks

As we discussed in **Theorem 1.1**, if the given $u^-$, $u^+$ in (1.6) satisfy (1.7), then the function (1.8) is the weak solution of (1.4) with (1.6).

**Definition 1.7.** *The solution of the Riemann problem taking the form of* (1.8) *is called the shock.*

Next, we try to discover the rule to find the suitable $u^+$ and $\lambda$ to satisfy (1.7) for a given $u^-$. In fact, we see that (1.7) is a system of $n$ equations with $n+1$ variables $u^+$, $\lambda$. So, we should be able to find a 1-dimensional curve of points $u^+$'s with corresponding $\lambda$'s satisfying (1.7).

**Theorem 1.2.** *For every $u_0 \in \Omega$, there exist $\sigma_0 > 0$ and one smooth curve $S_i : [-\sigma_0, \sigma_0] \mapsto \Omega$, together with scalar function $\lambda_i : [-\sigma_0, \sigma_0] \mapsto \Omega$, for each characteristic field $i = 1 \ldots, n$, such that*

$$f(S_i(\sigma)(u_0)) - f(u_0) = \lambda_i(\sigma)(u_0)(S_i(\sigma)(u_0) - u_0), \qquad \textit{for } \sigma \in [-\sigma_0, \sigma_0].$$

*The curves $S_i(\sigma)(u_0)$, $i = 1, \ldots, n$ are called the shock curves and satisfy the conditions that*

$$S_i(0)(u_0) = u_0, \qquad \lambda_i(0)(u_0) = \lambda_i(u_0)$$

*Properly choose $\sigma$ such that $|dS_i/d\sigma| \equiv 1$. Then, $S_i(\sigma)(u_0)$ will satisfy*

$$\frac{d}{d\sigma} S_i(\sigma)(u_0)\bigg|_{\sigma=0} = r_i(u_0), \qquad \frac{d}{d\sigma}\lambda_i(\sigma)(u_0)\bigg|_{\sigma=0} = \frac{1}{2} D\lambda_i(u_0) \cdot r_i(u_0)$$

**Proof.** See **Theorem 5.1** in [11]. □

With the theorem above, we know that for a given $u^-$, we can always find some values $\sigma^+$ close to 0 such that the combination of $u^+ = S_i(\sigma^+)(u^-)$ and $\lambda = \lambda_i(\sigma^+)(u^-)$, satisfies (1.7). If the $i$th characteristic field is genuinely nonlinear, by our additional hypothesis for the system (0.1), $D\lambda_i(u) \cdot r_i(u) > 0$ for all $u \in \Omega$ and hence $\frac{d\lambda_i(\sigma)(u^-)}{d\sigma}\Big|_{\sigma=0} > 0$. Besides, since the shock contains a jump, the admissible condition (1.11) should be satisfied, which requires that $\lambda \leq \lambda_i(u^-)$. That means

$$\sigma^+ \cdot \frac{d}{d\sigma}\lambda_i(\sigma)(u^-)\bigg|_{\sigma=0} \approx \int_0^{\sigma^+} \frac{d}{d\sigma}\lambda_i(\sigma)(u^-)\, d\sigma = \lambda_i(\sigma^+)(u^-) - \lambda_i(u^-) \leq 0.$$

So, $\sigma^+ < 0$. Thus, for a given $u^-$, $u^+$ must be on the negative part of the shock curve originating from $u^-$ if the Riemann problem could be solved as (1.8).

### 1.4.3 contact discontinuity

The contact discontinuity happens only when the characteristic field, for example $i$th characteristic field, is linearly degenerate. In this case, the $i$th shock and rarefaction curves coincide. That is

$$R_i(\sigma)(u_0) = S_i(\sigma)(u_0), \qquad \sigma \in [-\sigma_0, \sigma_0].$$

In this case, $\lambda_i(R_i(\sigma)(u_0)) \equiv \lambda_i(u_0)$ for all $\sigma$ because

$$\frac{d\lambda_i(R_i(\sigma)(u_0))}{d\sigma} = D\lambda_i(R_i(\sigma)(u_0)) \cdot \frac{dR_i(\sigma)(u_0)}{d\sigma} = D\lambda_i(R_i(\sigma)(u_0)) \cdot r_i(R_i(\sigma)(u_0)) \equiv 0.$$

That means that any value between the jumps $u^-$, $u^+$ will travel at the speed $\lambda_i(u_0)$.

In this case, if the given $u^-$, $u^+$ in (1.6) satisfy that $u^+ = S_i(\sigma^+)(u^-) = R_i(\sigma^+)(u^-)$, the function taking the form (1.8) with $\lambda = \lambda_i(u^-) = \lambda_i(u^+)$ will be the solution to the Riemann problem (0.1) with (1.6).

### 1.4.4 general solution for Riemann problem

We define the characteristic curve $\Psi(\sigma)(u_0)$ for each characteristic field.

We need to reparametrize the rarefaction curves and shock curves: for the $i$th characteristic field,

- If this characteristic field is genuinely nonlinear, scale the right eigenvector such that $D\lambda_i(u) \cdot r_i(u) \equiv 1$ for all $u \in \Omega$. In this way,

  $$\frac{d}{d\sigma}\lambda_i(R_i(\sigma)(u_0)) \equiv 1, \qquad \frac{d}{d\sigma}\lambda_i(S_i(\sigma)(u_0)) \equiv 1,$$

  and hence

  $$\lambda_i(R_i(\sigma)(u_0)) = \lambda_i(S_i(\sigma)(u_0)) = \lambda_i(u_0) + \sigma.$$

- If this characteristic field is linearly degenerate, the rarefaction curve and the shock curve coincide and the eigenvalue along the curve will keep constant. So, set $|r_i(u)| \equiv 1$ for all $u \in \Omega$ and parametrize this curve by the arc-length.

With the reparametrized curves, we can discover some properties of the characteristic curves

**Theorem 1.3.** *The characteristic curve for the $i$th characteristic field originating at $u_0$ is the composition function as following*

$$\Psi_i(\sigma)(u_0) = \begin{cases} R_i(\sigma)(u_0), & \sigma \geq 0 \\ S_i(\sigma)(u_0), & \sigma < 0 \end{cases}. \tag{1.14}$$

*$\Psi(\sigma)(u_0)$ is twice continuously differentiable at $\sigma = 0$ and smooth at any other places.*

**Proof.** See the analysis in Page 98 of [11]. □

For the general case where $u^+$ doesn't lie on the $i$th shock or rarefaction curve originating from $u^-$ for some $i$, we need to decompose the value between $u^-$, $u^+$.

**Theorem 1.4.** *Define the composite map as following*

$$\Lambda(\sigma_1, \dots, \sigma_n)(u^-) = \Psi_n(\sigma_n) \circ \cdots \circ \Psi_1(\sigma_1)(u^-) \tag{1.15}$$

*For every compact set $K \subset \Omega$, there exist $\delta > 0$ such that for any $u^- \in K$ and $|u^+ - u^-| < \delta$, there exist a group of values $(\sigma_1, \dots, \sigma_n)$ to make $u^+ = \Lambda(\sigma_1, \dots, \sigma_n)(u^-)$.*

**Proof.** See **Theorem 5.3** in [11]. □

With the theorem above, we can find intermediate values between $u^-$, $u^+$, which are $\omega_i = \Psi_i(\sigma_i) \circ \cdots \circ \Psi_1(\sigma_1)(u^-)$ for $i = 0, \cdots, n$. Notice that $\omega_n = u^+$, $\omega_0 = u^-$. We can see that for any $i = 1, \dots, n$, $\omega_i = \Psi_i(\sigma_i)(\omega_{i-1})$, so the Riemann problem with the initial data as

$$\bar{u}(x) = \begin{cases} \omega_{i-1} & x < 0 \\ \omega_i & x > 0 \end{cases}$$

could be solved by a shock or rarefaction wave.

Denote that $\lambda_i^- = \lambda_i(\omega_{i-1})$, $\lambda_i^+ = \lambda_i(\omega_i)$ if $\omega_{i-1}$, $\omega_i$ correspond to a rarefaction wave solution and $\lambda_i^- = \lambda_i^+ = \lambda_i(\omega_{i-1}, \omega_i)$ if $\omega_{i-1}$, $\omega_i$ correspond to a shock wave or contact discontinuity solution for a Riemann problem. Because the system is strictly hyperbolic, we can get

$$\lambda_1^- \le \lambda_1^+ < \lambda_2^- \le \lambda_2^+ < \cdots < \lambda_n^- \le \lambda_n^+.$$

In sum, the general solution of the Riemann problem is

$$u(t,x) = \begin{cases} u^-, & x < \lambda_1^- t \\ u^+, & x > \lambda_n^+ t \\ \omega_i, & \lambda_i^+ t < x < \lambda_{i+1}^- t,\ i = 1, \dots, n-1 \\ R_i(\sigma)(\omega_{i-1}), & \lambda_i^- t < x < \lambda_i^+ t,\ \sigma = x/t - \lambda_i(\omega_{i-1}),\ i = 1, \dots, n \end{cases} . \tag{1.16}$$

# 2 Various approximation methods

In fact, it is hard to directly write out the exact analytical solutions for all Cauchy problems defined in (1.4). Hence, some approximate methods have been developed.

## 2.1 Vanishing viscosity approximations

The method has been considered when we talked about vanishing viscosity admissibility condition. We add small viscosity to the end of the original system of equations (0.1), just like (1.9). Then, the system of conservation laws becomes a parabolic system. The parabolic system will guarantee a continuous solution $u^\epsilon$. As Figure 2.1 shows, when we take the coefficient of the viscosity term, $\epsilon$, to 0, $u^\epsilon$ is expected to converge to the weak solution of (0.1) in $\mathbf{L}^1_{loc}$ norm.

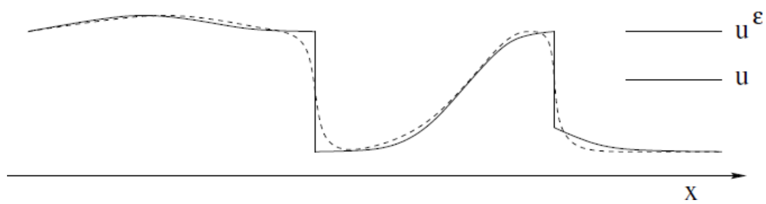


**Figure 2.1.** Vanishing viscosity approximation.

However, the problem of this method is that the solution for the parabolic system like (1.9) is still hard to estimate. So, we need other approximations.

## 2.2 Glimm approximations

Before starting the Glimm approximations, we make an assumption that all eigenvalues of the $Df(u)$ is between $[0, 1]$. This is possible if we make a proper affine transformation of the coordinates $x$ and $t$.

The Glimm approximations try to approximate the analytical solution for (0.1) on each cell of the grid. It will be operated in five steps:

1. Define a step size $\Delta t = \Delta x$. Then, the grid points will be like $(t_j, x_i) = (j\Delta t, i\Delta x)$
2. For each time $t_j$, choose a random $\theta_j$ uniformly distributed on $[0, 1]$.
3. Build up the piecewise constant approximation, denoted as $u(0, \cdot)$, for the initial data $\bar{u}$. $u(0, \cdot)$ is constant between any two grid points for $t = 0$. $u(0, \cdot)$ only jumps at the grid points such as $(0, x_i)$.
4. By induction, assume that a piecewise constant approximation $u(t_{j-1}, \cdot)$ has been built. At each grid point, solve the Riemann problems for $t \in [t_{j-1}, t_j[$ as Figure 2.2 shows.

5. At the time $t_j$, construct the piecewise constant approximation $u(t_j,\cdot)$ by defining

$$u(t_j,x)=u(t_j-,x_i+\theta_j\Delta x), \qquad \text{for } x\in]x_i,x_{i+1}[.$$

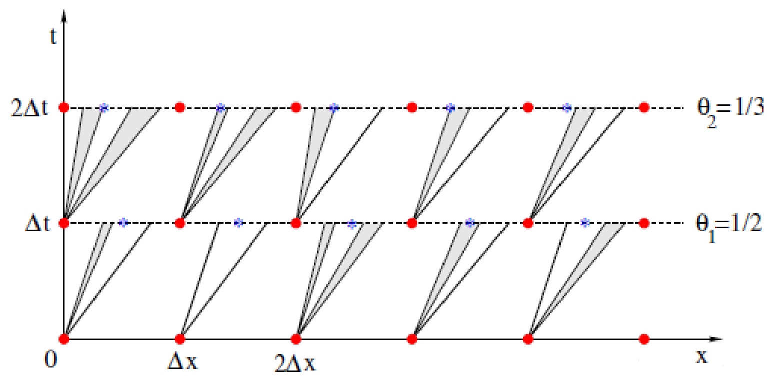


**Figure 2.2.** Glimm approximation.

In the step 4, when we solve the Riemann problem at one grid point such as $(t_{j-1},x_i)$, because of the assumption that all eigenvalues of $Df(u)$ is between $[0,1]$, the max distance that the shocks or the rarefactions travel within $t$ time, is $t$. So, any rarefactions, shocks and contact discontinuities generated from this grid point, will be contained in the interval $]x_i,x_{i+1}[$ for any time $t\in]t_{j-1},t_j[$. Hence, when we try to set the constant value, $u(t_j-,x_i+\theta_j\Delta x)$, for $u(t_j,x)$ on $]x_i,x_{i+1}[$, $u(t_j-,x_i+\theta_j\Delta x)$ only depend on the solution of the Riemann problem originating at $(t_{j-1},x_i)$.

In fact, use the Glimm approximations, the following fundamental theorem for the global existence of the entropy solutions for the Cauchy problem (1.4) could be proved:

**Theorem 2.1.** *Suppose that the system of conservation laws defined in* (1.4) *is strictly hyperbolic,* $f(u)$ *is smooth with respect to* $u$ *and each characteristic fields of the system is either genuinely non-linear or linearly degenerate. Then, there exists a constant* $\delta_0>0$ *such that if*

$$\text{Tol. Var. } \{\bar{u}\}\leq\delta_0,$$

*the Cauchy problem* (1.4) *will have a weak solution. And this weak solution is admissible if the pair of* $(\eta,q)$ *satisfying* (1.9) *could be found.*

**Proof.** The proof could be found in [28, 39].

## 2.3 Front tracking approximations

The Glimm approximations are based on the random restarting procedure. The front tracking approximations are quite similar to the Glimm approximation and could also be used to prove **Theorem 2.1**

For front tracking approximations, we construct it in the three steps:

1. Construct the piecewise constant approximation, denoted as $u(0,\cdot)$, for the initial data $\bar{u}$, satisfying

$$\text{Tol. Var. } \{u(0,\cdot)\} \leq \text{Tol. Var. } \{\bar{u}\}, \qquad \|u(0,\cdot) - \bar{u}\|_{\mathbf{L}^1} < \epsilon,$$

   and that $u(0,\cdot)$ has only finite jumps at $x_1, \ldots, x_N$.

2. Solve the Riemannn problem at each jump point $x_\alpha$ for $\alpha = 1, \ldots, N$. If the solution of the Riemann problem contains the rarefaction, approximate this rarefaction by a centred rarefaction fan containing several small jumps traveling at a speed close to the characteristic speed and keep constant between two adjascent jumps.

3. Resolve the Riemann problem when the wave-fronts interact and approximate the new rarefactions if necessary, as in Figure 2.3.

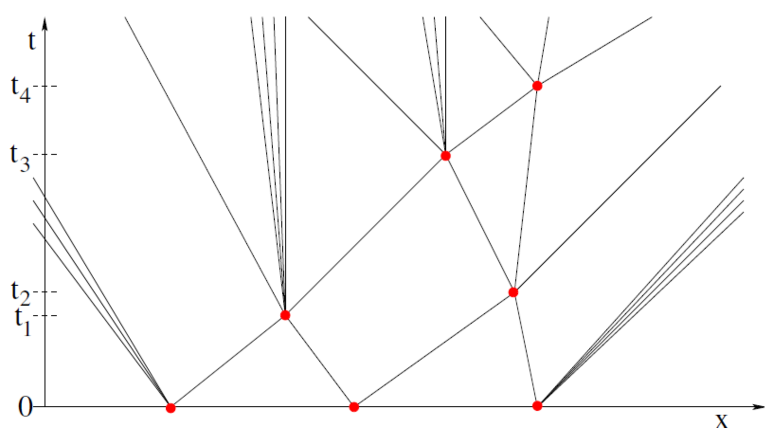


**Figure 2.3.** Front Tracking.

The exact method to discretize the rarefaction could be found in Section 7.2 in [11], in which the simplified solver for the Riemann problem is also introduced. The proof of **Theorem 2.1** can be found in the Section 7 in [11] as well.

## 2.4 The method of lines

This method was firstly introduced in [6]. The key of the method of lines is to only discretize the space and keep the time continuous. Denote the space step $\Delta x = \epsilon$ and set $U_k(t) \doteq u(t, k\epsilon)$ for $k \in \mathcal{Z}$. Then, we can find that

$$\frac{d}{dt}U_k(t) = -\frac{f(U_k(t)) - f(U_{k-1}(t))}{\epsilon}.$$

In the same paper [6] was proved that $\sum_k |U_k(t) - U_{k-1}(t)|$ is uniformly bounded for any $t \geq 0$.

Additionally, if define

$$u^\epsilon(t, x) = U_k(t), \qquad (k-1)\epsilon < x \leq k\epsilon,$$

$u^\epsilon(t, x)$ could be converged to a unique limit $u(t, x)$ in $\mathbf{L}_1$ for any $t$ as $\epsilon \to 0$.

# Chapter 3

# On Backward Euler Approximations for Systems of Conservation Laws

## 1 Introduction

Consider the hyperbolic system of conservation laws

$$u_t + f(u)_x = 0, \tag{1.1}$$

where $f : I\!R^n \mapsto I\!R^n$ is smooth function with Jacobian matrix $Df(u) = A(u)$. Given an initial datum

$$u(0, x) = \bar{u}(x), \tag{1.2}$$

with small total variation, it is well know that a unique entropy weak solution exists, globally in time [14]. Several approximations methods have been studied in the literature [11,24,33]. For some of them, rigorous convergence results are known [8,10]. In particular, semi-discrete schemes, where space is discretized while time remains a continuous variable, have been studied in [5, 7]

On the other hand, solutions generated by backward Euler approximations have been relatively less explored. These are also “semidiscrete" approximations, but now it is the time variable that is discretized, while space remains continuous. In connection with a scalar conservation law, backward Euler approximations provide a basic tool for the construction of a contractive semigroup [23, 40]. For hyperbolic systems, however, little is known. Aim of the present paper is to establish some results in this direction.

To construct backward Euler approximations, we fix a time step $\varepsilon > 0$, and set $t_k = k\varepsilon$. Then, if $u(t_{k-1}, \cdot)$ is given, an approximate value for $u(t_k, \cdot)$ is computed by solving

$$u(t_k, x) \;=\; u(t_{k-1}, x) - \varepsilon f(u(t_k, \cdot))_x \,. \tag{1.3}$$

Equivalently,

$$\Big[I + \varepsilon Df(u(t_k, x))\Big] u_x(t_k, x) \;=\; u(t_{k-1}, x). \tag{1.4}$$

If the matrix $I + \varepsilon Df(u)$ has a uniformly bounded inverse on the domain under consideration, then for each $k \geq 1$ the profile $u_k(\cdot) = u(t_k, \cdot)$ is obtained by solving an ODE, with suitable asymptotic conditions at $x \to \pm\infty$.

Throughout the following, we shall assume that the all matrices $A(u)$ have uniformly bounded norm, say

$$\|A(u)\| \;\leq\; M \qquad \text{for all } u \in I\!R^n. \tag{1.5}$$

Starting from the quasilinear system $u_t + A(u)u_x = 0$ and performing the affine change of coordinates

$$\tau \;=\; Mt, \qquad y \;=\; x + 2Mt,$$

we obtain the system

$$u_\tau + 2u_y + \frac{1}{M}\, A(u) u_y \;=\; 0.$$

We shall thus work with a system of the form

$$u_t + [2u + f(u)]_x \;=\; 0, \tag{1.6}$$

assuming that the matrix $A(u) \;=\; Df(u)$ satisfies

$$\|A(u)\| \;\leq\; 1, \qquad\qquad \|A(u) - A(v)\| \;\leq\; L|u - v| \qquad \text{for all } \; u, v \in I\!R^n, \tag{1.7}$$

for some Lipschitz constant $L$.

**Lemma 1.1.** *Under the assumptions (1.7), the matrix $2I + A(u)$ has a uniformly bounded, Lipschitz continuous inverse:*

$$\big\|(2I{+}A(u))^{-1}\big\| \;\leq\; 1, \qquad\qquad \big\|(2I{+}A(u))^{-1}{-}(2I{+}A(v))^{-1}\big\| \;\leq\; L|u{-}v| \qquad \textit{for all } \; u, v \in I\!R^n. \tag{1.8}$$

**Proof.** For every $w \in I\!R^n$ we have

$$\big|(2I + A(u))w\big| \;\geq\; 2|w| - |A(u)||w| \;\geq\; |w|.$$

Hence the inverse matrix $(2I - A(u))^{-1}$ satisfies the first inequality in (1.8).

Next, for every $w \in I\!R^n$ we have

$$\begin{aligned}\big|(2I + A(u))^{-1}w - (2I + A(v))^{-1}w\big| \;&=\; \left|\int_0^1 \frac{d}{ds}(2I + A(su + (1-s)v)^{-1}\, w\, ds\right|\\ &\leq\; \int_0^1 \frac{1}{\big\|(2I + A(su + (1-s)v)^{-1}\big\|^2} \left|\frac{d}{ds}(2I + A(su + (1-s)v))\right| |w|\, ds\\ &\leq\; L\,|u - v|\,|w|.\end{aligned}$$

This proves the second inequality in (1.8). □

Aim of our analysis is to establish four main results. For a fixed time step $\varepsilon > 0$, we shall prove

- Global existence, uniqueness of backward Euler approximations.
- Positive invariance of suitable domains $S \subset I\!R^n$.
- Existence of traveling profiles, corresponding to large entropy admissible shocks, for scalar conservation laws
- Existence of traveling profiles, for small, entropy admissible shocks, in the case of genuinely nonlinear hyperbolic systems.

Existence of traveling profiles for scalar conservation laws with nonlocal flux has been studied in [29, 42, 45]. For semidiscrete approximations of genuinely nonlinear systems, traveling profiles connecting the left and right states of a small shock were obtained in [5], by constructing a center manifold on a suitable functional space. Our proof relies on similar techniques.

## 2 Solving the backward Euler step

Setting $w = u(t_{k-1})$, $u = u(t_k)$, the Backward Euler step for (1.6) amounts to solving the ODE

$$u + \varepsilon(2I + A(u))u_x \;=\; w. \tag{2.1}$$

Equivalently

$$u'(x) \;=\; g(x,u) \;\doteq\; (2I + A(u))^{-1}\frac{w(x)-u}{\varepsilon}\,. \tag{2.2}$$

As usual, by a Carathéodory solution to (2.2) we mean an absolutely continuous function $u$ which satisfies

$$u(b)-u(a) \;=\; \int_a^b g(x,u(x))\,dx, \tag{2.3}$$

for every $a < b$.

We now prove the existence and uniqueness of the solution to the Backward Euler step.

**Theorem 2.1.** *Consider the system of conservation laws (1.6), where the matrix* $A(u) = Df(u)$ *satisfies (1.7). Given a step size* $\varepsilon > 0$, *for every* $w \in \mathbf{L}^1(I\!\!R; I\!\!R^n)$ *the ODE (2.2) has a unique solution* $u = E(w) \in \mathbf{L}^1(I\!\!R; I\!\!R^n)$. *Moreover, one has*

$$\|u\|_{\mathbf{L}^1} \;\leq\; 9\,\|w\|_{\mathbf{L}^1}\,, \tag{2.4}$$

$$\|E(w_1)-E(w_2)\|_{\mathbf{L}^1} \;\leq\; \exp\left\{\frac{10L}{\varepsilon}\|w_2\|_{\mathbf{L}^1}\right\}\cdot 9\|w_1-w_2\|_{\mathbf{L}^1}, \tag{2.5}$$

*for every* $w_1, w_2 \in \mathbf{L}^1$.

**Proof. 1.** We first consider the case where $w \in \mathcal{C}^1_c$ is continuously differentiable with compact support. In this case, there exists $x_0 \in I\!\!R$ such that $w(x) = 0$ for $|x| \leq x_0$. We can define $u(x) = 0$ for $x \leq x_0$, and solve the Cauchy problem (2.2) on the half line $[x_0, +\infty[$ with initial data $u(x_0) = 0$. We observe that this Cauchy problem has unique local solution, because the right hand side is locally Lipschitz continuous w.r.t. both variables $x$ and $u$. The fact that the solution is globally defined follows from the sublinear growth of $g$. Indeed, the first bound in (1.7) implies

$$|g(x,u)| \;\leq\; \frac{|w(x)|+|u(x)|}{\varepsilon}\,.$$

**2.** By (1.7), for every $v \in I\!\!R^n$ we have

$$|v| \;\leq\; 2|v| - |A(u)v| \;\leq\; \big|(2I + A(u))v\big| \;\leq\; |2v| + |A(u)v| \;\leq\; 3|v|.$$

Hence, for every $w \in I\!R^n$, we have

$$\frac{1}{3}|w| \;\leq\; |(2I + A(u))^{-1}w| \;\leq\; |w|. \tag{2.6}$$

Furthermore, from the inequalities

$$|v|^2 \;\leq\; \big\langle v, (2I + A(u))v \big\rangle \;\leq\; 3|v|^2,$$

taking $v = (2I + A(u))^{-1}w$ and using (2.6) we obtain

$$\frac{1}{9}|w|^2 \;\leq\; |(2I + A(u))^{-1}w|^2 \;\leq\; \big\langle (2I + A(u))^{-1}w, w \big\rangle \;\leq\; 3|(2I + A(u))^{-1}w|^2 \;\leq\; |w|^2.$$

**3.** To get an a priori estimate on the size of the solution, we observe that

$$\begin{aligned} \frac{d}{dx}|u(x)| \;&=\; \frac{\big\langle u\;,\;(2I + A(u))^{-1}w \big\rangle}{\varepsilon|u|} - \frac{\big\langle u\;,\;(2I + A(u))^{-1}u \big\rangle}{\varepsilon|u|} \\ &\leq\; \frac{1}{\varepsilon}|w(x)| - \frac{1}{9\varepsilon}|u(x)|, \end{aligned} \tag{2.7}$$

This yields

$$\begin{aligned} \int_{x_0}^{+\infty} |u(x)|\,dx \;&\leq\; \int_{x_0}^{+\infty} \left( \int_{x_0}^{x} e^{(y-x)/9\varepsilon} \cdot \frac{|w(y)|}{\varepsilon}\,dy \right) dx \\ &=\; \int_{x_0}^{+\infty} |w(y)| \left( \int_{y}^{+\infty} \frac{1}{\varepsilon} e^{(y-x)/9\varepsilon} dx \right) dy \\ &=\; 9 \int_{x_0}^{+\infty} |w(y)|\,dy. \end{aligned} \tag{2.8}$$

This proves that, for $w \in \mathcal{C}^1_c$, the solution $u = E(w)$ of (2.2) satisfies (2.4).

**4.** Next, let two functions $w_1, w_2 \in \mathcal{C}^1_c$ be given. Choose $x_0 \in I\!R$ so that $w_1(x) = w_2(x) = 0$ for all $x \leq x_0$. Let $u_1, u_2$ be two corresponding solutions to (2.2).

We now estimate

$$
\begin{aligned}
&\frac{d}{dx}|u_1(x)-u_2(x)| \\
&\quad = \frac{\Big\langle u_1-u_2\ ,\ (2I+A(u_1))^{-1}(w_1-u_1)-(2I+A(u_2))^{-1}(w_2-u_2)\Big\rangle}{\varepsilon|u_1-u_2|} \\
&\quad \leq -\frac{\Big\langle u_1-u_2\ ,\ (2I+A(u_1))^{-1}(u_1-u_2)\Big\rangle}{\varepsilon|u_1-u_2|}+\frac{1}{\varepsilon}\,\|(2I+A(u_1))^{-1}\|\,|w_1-w_2| \\
&\qquad +\frac{1}{\varepsilon}\left\|(2I+A(u_1))^{-1}-(2I+A(u_2))^{-1}\right\|\big(|u_2|+|w_2|\big) \\
&\quad \leq -\frac{|u_1-u_2|}{9\varepsilon}+\frac{|w_1-w_2|}{\varepsilon}+\frac{L}{\varepsilon}\big(|u_2|+|w_2|\big)\,|u_1-u_2|.
\end{aligned}
\tag{2.9}
$$

As in (2.8) we thus obtain

$$
\begin{aligned}
&\int_{x_0}^{+\infty}|u_1(x)-u_2(x)|\,dx \\
&\quad\leq \int_{x_0}^{+\infty}\left(\int_{x_0}^{x}\exp\left\{\int_y^x\frac{-(1/9)+L|u_2(z)|+L|w_2(z)|}{\varepsilon}\,dz\right\}\cdot\frac{|w_1(y)-w_2(y)|}{\varepsilon}\,dy\right)dx \\
&\quad\leq \exp\left\{\frac{L\|u_2\|_{\mathbf{L}^1}+L\|w_2\|_{\mathbf{L}^1}}{\varepsilon}\right\}\cdot\int_{x_0}^{+\infty}|w_1(y)-w_2(y)|\left(\int_y^{+\infty}\frac{1}{\varepsilon}e^{(y-x)/9\varepsilon}dx\right)dy \\
&\quad\leq \exp\left\{\frac{10L}{\varepsilon}\|w_2\|_{\mathbf{L}^1}\right\}\cdot 9\int_{x_0}^{+\infty}|w_1(y)-w_2(y)|\,dy.
\end{aligned}
\tag{2.10}
$$

This establishes the Lipschitz estimate (2.5), for functions $w_1,w_2\in\mathcal{C}^1_c$.

**5.** It now remains to extend the map $w\mapsto u=E(w)$ by continuity, for all $w\in\mathbf{L}^1$. Given $w\in\mathbf{L}^1$, take any sequence $w_n\in\mathcal{C}^1_c$ such that $\|w_n-w\|_{\mathbf{L}^1}\to 0$. Let $u_n\in\mathbf{L}^1$ be the corresponding solutions to (2.2).

By (2.5) it follows

$$
\|u_m-u_n\|_{\mathbf{L}^1}\ \leq\ \exp\left\{\frac{10L}{\varepsilon}\|w_n\|_{\mathbf{L}^1}\right\}\cdot 9\|w_m-w_n\|_{\mathbf{L}^1}\ \leq\ C\,\|w_m-w_n\|_{\mathbf{L}^1}, \tag{2.11}
$$

for every $m,n$. This implies that the sequence $(u_n)_{n\geq1}$ is a Cauchy sequence in $\mathbf{L}^1$. Hence it converges to a unique limit $u\in\mathbf{L}^1(I\!R;\,I\!R^n)$. By possibly selecting a subsequence, we can assume the pointwise convergence $u_n(x)\to u(x)$ for a.e. $x\in I\!R$.

**6.** We claim that the limit $u$ is absolutely continuous (after possibly modifying its values on a set of measure zero), and satisfies (2.2). Using (2.2) and (1.8), for every $m,n\geq 1$ one obtains

$$\|u'_m-u'_n\|_{\mathbf{L}^1}\;\leq\;\frac{1}{\varepsilon}\|w_m-w_n\|_{\mathbf{L}^1}+\frac{1}{\varepsilon}\big(1+L\|w_n\|_{\mathbf{L}^1}+L\|u_n\|_{\mathbf{L}^1}\big)\|u_m-u_n\|_{\mathbf{L}^1}.$$

Therefore, the sequence of derivatives $(u'_n)_{n\geq 1}$ is a Cauchy sequence in $\mathbf{L}^1$ as well, and converges to some limit is $v\in\mathbf{L}^1(I\!R;\,I\!R^n)$. Consider the integral

$$\hat{u}(x)\;=\;\int_{-\infty}^{x}v(y)dy.$$

The definition of $v,\hat{u}$ implies that, for any $x\in I\!R$,

$$|u_n(x)-\hat{u}(x)|\;\leq\;\int_{-\infty}^{x}|u'_n(y)-v(y)|dy\;\leq\;\|u'_n-v\|_{\mathbf{L}^1}\to 0,\ \text{as }n\to\infty.$$

This proves the convergence $u_n(x)\to\hat{u}(x)$, uniformly for $x\in I\!R$. In particular, this implies $\hat{u}(x)=u(x)$ for a.e. $x\in I\!R$. This shows that the function $u=E(w)$ constructed in step **5** is absolutely continuous (up to redefining its values on a set of measure zero), and provides a solution to (2.2).

**7.** In this last step, we prove that the equation (2.2) has a unique Carathéodory solution $u\in\mathbf{L}^1$.

For a given $w\in\mathbf{L}^1(\mathbb{R})$, suppose that there exists two solutions: $u_1$ and $u_2$.

Given $\epsilon_0>0$, there exists $x_0$ such that

$$\int_{-\infty}^{x_0}|w(x)|\,dx\;<\;\epsilon_0,\qquad\int_{-\infty}^{x_0}|u_1(x)|\,dx\;<\;\epsilon_0,\qquad\int_{-\infty}^{x_0}|u_2(x)|\,dx\;<\;\epsilon_0,\tag{2.12}$$

$$|u_1(x_0)|\;<\;\epsilon_0,\qquad\qquad|u_2(x_0)|\;<\;\epsilon_0.$$

since $u\in\mathbf{L}^1(I\!R)$, which implies $\liminf_{|x|\to\infty}|u(x)|=0$.

We observe that $u_1,u_2$ satisfy the same equation (2.2) on $[x_0,+\infty[$ and

$$|u_1(x_0)-u_2(x_0)|\;<\;2\epsilon_0\,.$$

By (2.9), $u_1,u_2$ satisfy

$$\frac{d}{dx}\big|u_1(x)-u_2(x)\big|\;\leq\;\Big(-\frac{1}{9\varepsilon}+\frac{L}{\varepsilon}(|u_2|+|w|)\Big)\big|u_1(x)-u_2(x)\big|.$$

This yields

$$\begin{aligned}\big|u_1(x)-u_2(x)\big| &\leq \exp\left\{-\frac{1}{9\varepsilon}(x-x_0)+\int_{x_0}^{x}\frac{L}{\varepsilon}\big(|u_2|+|w|\big)dx\right\}\big|u_1(x_0)-u_2(x_0)\big| \\ &\leq \exp\left\{-\frac{1}{9\varepsilon}(x-x_0)+\frac{L}{\varepsilon}\big(\|u_2\|_{\mathbf{L}^1}+\|w\|_{\mathbf{L}^1}\big)\right\}2\epsilon_0\,,\end{aligned}$$

$$\begin{aligned}\int_{x_0}^{+\infty}\big|u_1(x)-u_2(x)\big|\,dx &\leq \int_{x_0}^{+\infty}\exp\left\{-\frac{1}{9\varepsilon}(x-x_0)+\frac{L}{\varepsilon}\big(\|u_2\|_{\mathbf{L}^1}+\|w\|_{\mathbf{L}^1}\big)\right\}2\epsilon_0\,dx \\ &\leq 9\varepsilon\,\exp\left\{\frac{L}{\varepsilon}\big(\|u_2\|_{\mathbf{L}^1}+\|w\|_{\mathbf{L}^1}\big)\right\}2\epsilon_0.\end{aligned}$$

Combining the above estimate with (2.12), we obtain

$$\|u_1-u_2\|_{\mathbf{L}^1} = \left(\int_{-\infty}^{x_0}+\int_{x_0}^{+\infty}\right)\big|u_1(x)-u_2(x)\big|\,dx \leq 2\epsilon_0+9\varepsilon\,\exp\left\{\frac{L}{\varepsilon}\big(\|u_2\|_{\mathbf{L}^1}+\|w\|_{\mathbf{L}^1}\big)\right\}2\epsilon_0.$$

Since the above inequality holds for every $\epsilon_0>0$, we conclude that $u_1=u_2$, hence the Carathéodory solution $u\in\mathbf{L}^1$ is unique. □

# 3 An invariance property

In the remainder of this paper we focus on the case where the system (1.1) is strictly hyperbolic [11, 24, 33]. More precisely, we shall assume

**(A)** *The system (1.1) is strictly hyperbolic, with $\mathcal{C}^2$ coefficients. For every $u\in\mathbb{R}^n$ the Jacobian matrix $Df(u)$ has positive and distinct eigenvalues:*

$$0 < \lambda_1(u) < \cdots < \lambda_n(u). \tag{3.1}$$

As customary, we shall denote by

$$\big\{r_i(u),\ldots,r_n(u)\big\},\qquad \big\{\ell_i(u),\ldots,\ell_n(u)\big\}. \tag{3.2}$$

dual bases of right and left eigenvectors of the matrix $Df(u)$.

Given $w\in\mathbf{L}^1(\mathbb{R};\,\mathbb{R}^n)$, the backward Euler step amounts to finding $u\in\mathbf{L}^1(\mathbb{R};\,\mathbb{R}^n)$ such that

$$u'(x) = g(x,u) \doteq \big(Df(u)\big)^{-1}\frac{w(x)-u}{\varepsilon}\,. \tag{3.3}$$

Notice that here the matrix $Df(u)$ is invertible, because all of its eigenvalues are strictly positive.

In this section we prove an invariance property of Backward Euler approximations, in the same spirit as [32]. Let $S \subset I\!R^n$ be a closed domain. Assuming that the initial datum $\bar{u}$ takes values inside $S$, we seek conditions on $S$ ensuring that all approximations constructed by the backward Euler scheme still take values inside $S$.

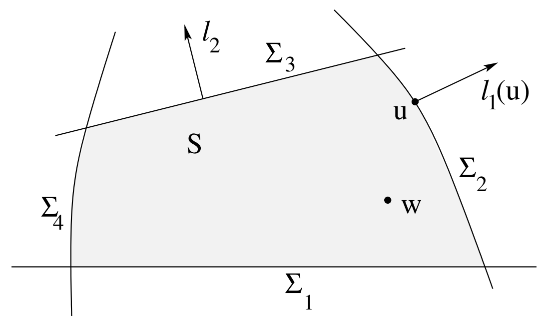


**Figure 3.1.** A positively invariant domain. Each boundary $\Sigma_k$ is perpendicular to one of the left eigenvectors $\ell_i$ of the Jacobian matrix $Df(u)$.

**Theorem 3.1.** *Let the assumptions* **(A)** *hold. Let $S \subset I\!R^n$ be a closed, convex set. Assume that the boundary $\partial S$ is contained in the union of finitely many $\mathcal{C}^1$ hypersurfaces*

$$\Sigma_k \;=\; \big\{u \in I\!R^n\,;\quad \varphi_k(u) = 0\big\}, \qquad k = 1, \ldots, N,$$

*such that, for each $u \in \Sigma_k$, the gradient $\nabla\varphi_k(u)$ is a left eigenvector of $Df(u)$. Then $S$ is positively invariant for the backward Euler scheme.*

**Proof. 1.** Given $w \in \mathbf{L}^1(I\!R; I\!R^n)$ with $w(x) \in S$ for all $x$, we need to show that the solution to (2.2) remains inside $S$ as well.

As a first step, we show that this is true when $w \in \mathcal{C}^1_c$. As remarked in the proof of Theorem 2.1, in this case there exists $x_0 \in I\!R$ such that $w(x) = 0$ for $|x| \le x_0$. We can thus define $u(x) = 0$ for $x \le x_0$, and solve the ODE (3.3) on the half line $[x_0, +\infty[$ with initial data $u(x_0) = 0$.

By a well known invariance property for solutions to ODEs (see for example [41]), it suffices to prove that

$$u(x) \in \partial S \quad \Longrightarrow \quad u'(x) = g(x, u) \in T_{u(x)}(S) \tag{3.4}$$

where $T_u(S)$ denotes the tangent cone to the set $S$ at the point $u$.

**2.** To fix ideas, consider a point $u \in \partial S$, say with

$$\begin{cases} \varphi_k(u) &= 0 \qquad k \in \mathcal{I}, \\ \varphi_k(u) &> 0 \qquad k \notin \mathcal{I}, \end{cases} \tag{3.5}$$

for some subset $\mathcal{I} \subseteq \{1, \ldots, N\}$. By assumption, for every $k \in \{1, \ldots, N\}$ there exists an index $i = i(k) \in \{1, \ldots n\}$ such that the gradient $\nabla\varphi_k(u)$ is parallel to the left eigenvector $\ell_i(u)$. As shown in Fig. 3.1, we assume that the vector $\ell_i$ points outward from the set $S$. The tangent cone to the set $S$ at the point $u$ is now given by

$$T_u(S) = \Big\{ \mathbf{v} \in I\!R^n\,;\ \ \big\langle \ell_i(u), \mathbf{v} \big\rangle \leq 0 \quad \text{for all } i \in I \Big\}, \tag{3.6}$$

where

$$I = \big\{ i(k)\,;\ \ k \in \mathcal{I} \big\}. \tag{3.7}$$

**3.** Now let $u = u(x)$ be the solution of (3.3), with $w \in \mathcal{C}^1_c$. Consider any point $x$ where $u(x) \in \partial S$, and let $I \subset \{1, \ldots n\}$ be the corresponding set of indices constructed as in (3.5)–(3.7). According to (3.6) we need to show that

$$\Big\langle \ell_i(u(x)), u'(x) \Big\rangle \leq 0 \qquad\qquad \text{for all } i \in I. \tag{3.8}$$

The convexity of $S$ implies

$$S \subseteq \Big\{ w \in I\!R^n\,;\ \ \Big\langle \ell_i\big(u(x)\big)\,,\ w - u(x) \Big\rangle \leq 0 \Big\}. \tag{3.9}$$

Using (3.3), (3.9), and the fact that $\ell_i(u)$ is an eigenvector of $Df(u)$ with eigenvalue $\lambda_i(u) > 0$, we obtain

$$\begin{aligned} \Big\langle \ell_i(u(x)), u'(x) \Big\rangle &= \Big\langle \ell_i(u(x)),\ \big(Df(u(x))\big)^{-1} \frac{w(x) - u(x)}{\epsilon} \Big\rangle \\ &= \Big\langle \ell_i(u(x)),\ \frac{1}{\lambda_i(u(x))} \frac{w(x) - u(x)}{\epsilon} \Big\rangle \leq 0. \end{aligned}$$

By the invariance principle for solutions of ODEs, this implies that $u(x) \in S$ for all $x \in I\!R$.

**4.** The previous analysis shows that the backward Euler step $u$ takes values inside $S$

for every $w \in \mathcal{C}_c^1$. We claim that the conclusion remains valid also if $w \in \mathbf{L}^1$. Indeed, consider a sequence of functions $w_m \in \mathcal{C}_c^1$, $m \geq 1$, converging to $w$ in $\mathbf{L}^1$. By the previous steps, the corresponding solutions $u_m$ of (3.3) satisfy $u_m(x) \in S$ for all $x \in I\!R$. As in the proof of Theorem 2.1, by (2.11) the sequence $(u_m)_{m\geq 1}$ is Cauchy, and converges to a unique limit $u \in \mathbf{L}^1(I\!R; I\!R^n)$, which satisfies (3.3). Since $S$ is closed, this implies $u(x) \in S$ for all $x \in I\!R$, completing the proof. □

# 4 Traveling wave profiles

In the remainder of the paper, always under the assumptions **(A)**, we study solutions to the backward Euler scheme in the form of traveling waves, so that

$$u(t,x) \;=\; w(x - c\,t). \tag{4.1}$$

By (1.3), it follows that $w$ should satisfy

$$f(w(x))_x \;=\; \frac{w(x + c\,\varepsilon) - w(x)}{\varepsilon}, \tag{4.2}$$

which leads to the ODE with non-local argument

$$w'(x) \;=\; (Df(w(x))^{-1} \frac{w(x + c\,\varepsilon) - w(x)}{\varepsilon}\,. \tag{4.3}$$

It is of interest to construct solutions such that

$$\lim_{x\to -\infty} w(x) \;=\; w^-, \qquad\qquad \lim_{x\to +\infty} w(x) \;=\; w^+, \tag{4.4}$$

for some constant states $w^-, w^+ \in I\!R^n$.

The equation (4.2) can be written as

$$\frac{d}{dx}\left( f(w(x)) - \frac{1}{\varepsilon}\int_x^{x+c\varepsilon} w(y)\,dy \right) \;=\; 0. \tag{4.5}$$

Integrating (4.5) we obtain

$$f(w(x)) - \frac{1}{\varepsilon}\int_x^{x+c\varepsilon} w(y)\,dy \;=\; \mathbf{v}, \tag{4.6}$$

for some constant vector $\mathbf{v} \in I\!R^n$. Letting $x \to \pm\infty$ and assuming (4.4) we obtain

$$f(w(x)) - \frac{1}{\varepsilon}\int_x^{x+c\varepsilon} w(y)\,dy \;=\; f(w^+) - cw^+ \;=\; f(w^-) - cw^-. \tag{4.7}$$

In particular, (4.7) yields the Rankine-Hugoniot jump conditions

$$f(w^+) - f(w^-) \;=\; c(w^+ - w^-). \tag{4.8}$$

**Remark 4.1.** A second order Taylor approximation of the right hand side of (4.2) yields

$$\varepsilon f\big(w(x)\big)_x \;=\; c\varepsilon w_x(x) + \frac{(c\varepsilon)^2}{2} w_{xx}(x).$$

Hence

$$\frac{c^2\varepsilon}{2}\, w'' \;=\; \big[f - cw\big]_x\,.$$

Notice that this is the same equation satisfied by a viscous traveling wave, with viscosity coefficient $c^2\varepsilon/2$. In first approximation, we thus expect that the solution to (4.2) will satisfy

$$w'(x) \;=\; \frac{2}{\varepsilon c^2}\big[f(w(x)) - cw(x) - C\big],$$

for some integration constant $C$.

**Remark 4.2.** In (4.2), it is not restrictive to assume $\varepsilon = 1$. Indeed, if $w$ satisfies

$$f(w(x))_x \;=\; w(x+c) - w(x), \tag{4.9}$$

then $w_\varepsilon(x) = w(x/\varepsilon)$ provides a solution to (4.2).

# 5 Traveling profiles for a scalar conservation law

In this section we consider a scalar conservation law

$$u_t + f(u)_x \;=\; 0, \tag{5.1}$$

and assume that the left and right states $u^- > u^+$ are connected by an entropy admissible shock with Rankine-Hugoniot speed

$$c \;=\; \frac{f(u^+) - f(u^-)}{u^+ - u^-}\,. \tag{5.2}$$

More precisely, we shall assume the stability conditions

$$f'(u^-) \ > \ c \ > \ f'(u^+), \tag{5.3}$$

$$f\big(\theta u^+ + (1-\theta)u^-\big) \ < \ \theta f(u^+) + (1-\theta)f(u^-) \qquad \text{for all } 0 < \theta < 1, \tag{5.4}$$

together with

$$M \ \geq \ f'(u) \ \geq \ c_0 \ > \ 0 \qquad \text{for all } u \in I\!R. \tag{5.5}$$

By (5.5) the characteristic speed remains uniformly positive. Taking $\varepsilon = 1$, the delay differential equation (4.3) describing a traveling wave profile takes the form

$$z'(x) \ = \ \frac{1}{f'\big(z(x)\big)}\big[z(x+c) - z(x)\big]\,. \tag{5.6}$$

The goal of this section is to prove the existence of a traveling wave profile for the Backward Euler scheme, connecting the states $u^-, u^+$.

**Theorem 5.1.** *Let $f : I\!R \mapsto I\!R$ be a $\mathcal{C}^1$ flux function satisfying (5.2)–(5.5), for some states $u^+ < u^-$. Then there exists a solution to the equation (5.6), with asymptotic conditions*

$$\lim_{x\to-\infty} z(x) \ = \ u^-, \qquad\qquad \lim_{x\to+\infty} z(x) \ = \ u^+.$$

**Proof. 1.** We construct a sequence of approximate traveling profiles

$$z_n : \ ]-\infty, n+c] \ \mapsto \ [u^+, u^-],$$

defined as follows. We start by setting

$$z_n(x) \ = \ u^+ + e^{-x} \qquad \text{for } x \in [n, n+c]. \tag{5.7}$$

Then we solve the delay differential equation (5.6) backwards, and construct the values of $z_n(x)$ for $x \in ]-\infty, n]$.

**2.** We claim that $z_n$ is monotonically decreasing on each interval $[n-kc, n-(k-1)c]$ for $k \geq 1$. Indeed, when $u^+ \geq 0$ and $z_n(x) = u^+ + e^{-x}$, $z_n(x)$ is monotonically decreasing on the interval $[n, n+c]$. Since $f'(z) \geq c_0 > 0$, by (5.6) the derivative $z'(x) < 0$ for all $x \in ]n-c, n]$. By induction on $k$, we conclude that this derivative remains negative, and $z_n$ is monotone decreasing on the entire domain $]-\infty, n+c]$.

**3.** This step will establish an upper bound for $z_n$. By (4.8) we can introduce the constant

$$C \doteq f(u^+) - cu^+ = f(u^-) - cu^-. \tag{5.8}$$

Moreover, the same argument used at (4.6) now yields

$$f\big(z_n(x)\big) - \int_x^{x+c} z_n(y)\,dy = f(z_n(n)) - \int_n^{n+c} z_n(y)\,dy \doteq C_n\,. \tag{5.9}$$

Letting $n \to +\infty$ we obtain

$$\lim_{n\to+\infty} C_n = \lim_{n\to+\infty} f\big(u^+ + e^{-n}\big) - \int_n^{n+c} \big(u^+ + e^{-y}\big)\,dy = f(u^+) - cu^+ = C. \tag{5.10}$$

Next, since $z_n(x)$ is decreasing w.r.t. $x$, we have

$$f\big(z_n(x)\big) - cz_n(x) \leq f\big(z_n(x)\big) - \int_x^{x+c} z_n(y)\,dy = C_n. \tag{5.11}$$

Thanks to (5.3), we can find a sequence of points $\widetilde{u}_n > u^-$, with

$$\lim_{n\to\infty} \widetilde{u}_n = u^-$$

and such that

$$f(\widetilde{u}_n) - c\widetilde{u}_n > C_n \tag{5.12}$$

for all $n$ sufficiently large. Combining (5.11) with (5.12), by continuity we conclude that

$$z_n(x) \leq \widetilde{u}_n \tag{5.13}$$

for all $x \leq n$ and $n \geq 1$ sufficiently large.

**4.** By (5.13) it follows that every function $z_n$ is decreasing and takes values within an interval $[u^+, \widetilde{u}_n]$, where $\widetilde{u}_n \to u^-$. As a consequence, the limit

$$z_n^- \doteq \lim_{x\to-\infty} z_n(x) \tag{5.14}$$

is well defined. We claim that

$$\lim_{n\to\infty} z_n^- = u^-. \tag{5.15}$$

Indeed, by possibly taking a subsequence, we can assume

$$z_n^- \;\to\; z^- \;\in\; [u^+, u^-]. \tag{5.16}$$

Since

$$f(z_n^-) - cz_n^- \;=\; C_n \;\to\; C \qquad \text{as} \quad n \to \infty, \tag{5.17}$$

this already implies

$$z^- \;=\; u^- \qquad \text{or} \qquad z^- \;=\; u^+.$$

To rule out the second alternative, we argue as follows. By (5.3), there exist $\varepsilon, \delta > 0$ such that

$$\frac{d}{du}\big[f(u) - cu\big] \;\leq\; -\varepsilon \qquad\qquad \text{for} \;\; u \in [u^+,\, u^+ + \delta].$$

However, as long as $z_n(x) \in [u^+, u^+ + \delta]$,

$$\begin{aligned} f\big(z_n(x)\big) - cz_n(x) \;&=\; f(u^+ + e^{-n}) - c(u^+ + e^{-n}) + \int_{u^+ + e^n}^{z_n(x)} \big[f'(u) - c\big]\, du \\ &\leq\; f(u^+ + e^{-n}) - \int_n^{n+c} (u^+ + e^{-x})\, dx - \varepsilon\big[z_n(x) - u^+ - e^{-n}\big] \\ &=\; C_n - \varepsilon\big[z_n(x) - u^+ - e^{-n}\big]. \end{aligned} \tag{5.18}$$

We now observe that the right hand side of (5.18) remains strictly smaller that $C_n$ as long as

$$u^+ - e^{-n} \;<\; z_n(x) \;\leq\; u^+ + \delta.$$

By (5.17) this implies $z_n^- \geq u^+ + \delta$, for every $n$ large enough, ruling out the possibility that $z_n^- \to u^-$. Hence (5.15) must hold.

**5.** Next, we claim that, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$z_n(x) \in [u^+ + \varepsilon, u^- - \varepsilon] \qquad \Longrightarrow \qquad z_n'(x) \;\leq\; -\delta, \tag{5.19}$$

for all $n$ sufficiently large.

If not, in view of (5.6) and the fact that $f'$ is uniformly positive, we could find a sequence of points $x_n$ such that

$$z_n(x_n) \;\in\; [u^+ + \varepsilon, u^- - \varepsilon], \qquad\qquad z_n(x_n + c) - z_n(x_n) \;\to\; 0.$$

Since $z_n(\cdot)$ is monotone decreasing, we have $z_n(x) - z_n(y) \to 0$ for any $x, y \in [x_n, x_n + c]$ and hence

$$\int_{x_n}^{x_n+c} z_n(y)dy \;\to\; c z_n(x_n).$$

Taking a subsequence, we can assume

$$z_n(x_n) \to \overline{z} \;\in\; [u^+ + \varepsilon, u^- - \varepsilon],$$

By (5.1) this implies

$$f(\overline{z}) - c\,\overline{z} \;=\; C.$$

However, this equation does not have solutions within the interval $[u^+ + \varepsilon, u^- - \varepsilon]$. This contradiction shows that (5.19) must hold.

**6.** By possibly performing a horizontal shift, and consider the functions

$$u_n(x) \;=\; z_n(x - a_n),$$

where $a_n$ is chosen so that

$$u_n(0) \;=\; \frac{u^+ + u^-}{2}\,.$$

For any $\varepsilon > 0$, let $\delta > 0$ be as in (5.19). Setting

$$M_\varepsilon \;\doteq\; \frac{u^- - u^+}{\delta}\,,$$

for every $n \geq 1$ sufficiently large we achieve

$$\begin{cases} u_n(x) - u^+ < \varepsilon \quad \text{for} \;\; x > M_\varepsilon\,, \\ \\ u_n(x) - u^- > -\varepsilon \quad \text{for} \;\; x < -M_\varepsilon\,. \end{cases} \tag{5.20}$$

By the Ascoli-Arzela theorem, by possibly taking a subsequence we obtain the uniform convergence $u_n(x) \to u(x)$ on the interval $[-M_\varepsilon, M_\varepsilon]$. Outside this interval, thanks to (5.20) we have

$$\limsup_{m,n\to+\infty} \sup_{x\in I\!R} \big|u_m(x) - u_n(x)\big| \;\leq\; 2\varepsilon. \tag{5.21}$$

To complete the construction, we take a sequence $\varepsilon_\nu \downarrow 0$. Repeating the previous construction, we obtain a subsequence $(u_n^{(\nu)})_{n\geq 1}$ which satisfies (5.21) with $\varepsilon$ replaced by $\varepsilon_\nu$. By a standard diagonal procedure, this yields a subsequence uniformly converging on the whole real line.

**7.** It remains to prove that the limit $u(x) = \lim_{n\to\infty} u_n(x)$ provides a traveling wave solution. From the identity (5.1), letting $n \to \infty$ we obtain

$$f\big(z(x)\big) - \int_x^{x+c} z(y)\,dy \;=\; C \qquad\qquad \text{for all } x \in I\!R. \tag{5.22}$$

Since $z$ is continuous and $f \in \mathcal{C}^1$, this integral equation yields (5.6). □

# 6 Traveling profiles for hyperbolic systems

Let $\Omega \subset I\!R^n$ be an open set, and let $f : \Omega \mapsto I\!R^n$ be a smooth flux function such that, for every $u \in \Omega$, the Jacobian $Df(u)$ has $n$ real distinct, positive eigenvalues $0 < \lambda_1(u) < \cdots < \lambda_n(u)$. If the $k$-th characteristic field is genuinely nonlinear, for every right state $u_r \in \Omega$ there exists a 1-parameter family of left states $u_l$ which are joined to $u_r$ by an entropy-admissible shock [11, 36, 46].

At a fixed state $u_r$, let $\{r_1, \ldots, r_n\}$ and $\{\ell_1, \ldots, \ell_n\}$ be bases of right and left eigenvectors for the Jacobian matrix $A = Df(u_r)$, normalized so that

$$|r_k| \;=\; 1, \qquad \nabla\lambda_k(u_r)\cdot r_k \;>\; 0, \qquad \ell_i \cdot r_j \;=\; \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The main result of this section is the existence of traveling profiles for the backward Euler approximation scheme.

**Theorem 6.1.** *In the above setting, every $u_r \in \Omega$ has a neighborhood $\mathcal{V}$ with the following property. If $u_l \in \mathcal{V}$ is a left state connected to $u_r$ by an admissible shock of the genuinely nonlinear $k$-th family, then there exists a traveling profile for the backward Euler scheme having $u_l, u_r$ as asymptotic limits.*

*Namely, there exists a smooth function $u : I\!R \mapsto \Omega$ and a speed $c$ such that*

$$u'(x) \;=\; \big(Df(u(x)\big)^{-1} \frac{u(x + c\,\varepsilon) - u(x)}{\varepsilon}\,, \tag{6.1}$$

$$\lim_{x\to-\infty} u(x) \;=\; u_l, \qquad\qquad \lim_{x\to+\infty} u(x) \;=\; u_r\,. \tag{6.2}$$

**Remark 6.1.** A similar result was proved in [5], in the case of semidiscrete approximations, where space is discretized but time remains continuous. In such case, (6.1) is replaced by

$$u'(x) \;=\; \mu\Big(f\big(u(x)\big) - f\big(u(x-1)\big)\Big)\,. \tag{6.3}$$

We shall follow the same steps of the proof in [5], with the appropriate modifications.

Most of the theory of delay differential equations is stated for positive delays (see [30]), hence for convenience we apply the rescaling $x \to -\frac{x}{c\varepsilon}$ which leads to the following equation

$$u' \;=\; c\big(Df(u(x))\big)^{-1}\big(u(x) - u(x-1)\big), \tag{6.4}$$

to be solved in $I\!R^n$, with the asymptotic conditions

$$\lim_{x\to-\infty} u(x) = u_r, \qquad \lim_{x\to+\infty} u(x) = u_l. \tag{6.5}$$

Here $u$ should be considered as a $\frac{1}{c\varepsilon}$-stretching and reflection of the original solution for (4.3).

We write (6.4) as a system of $n+1$ delay differential equations

$$\begin{cases} u'(x) = c(x)\big(Df(u(x))\big)^{-1}\big(u(x) - u(x-1)\big), \\ c'(x) = 0, \end{cases} \tag{6.6}$$

with the additional asymptotic condition $\lim_{x\to\pm\infty} c(x) = c$, where the shock speed $c$ is determined by the Rankine-Hugoniot equations.

It will be convenient to introduce the space $\mathcal{C} \doteq \mathcal{C}\big([-1,0]\,;\ \mathbb{R}^{n+1}\big)$ endowed with norm

$$\left\| \begin{pmatrix} \phi \\ e \end{pmatrix} \right\|_{\mathcal{C}} \;\doteq\; \sup_{-1\le\theta\le 0} \left| \begin{pmatrix} \phi(\theta) \\ e(\theta) \end{pmatrix} \right|.$$

The system (6.6) can now be rewritten as a functional differential equation on $\mathcal{C}$, namely

$$\frac{d}{dx} D\bigg( \begin{pmatrix} u \\ c \end{pmatrix}_x \bigg) = F\bigg( \begin{pmatrix} u \\ c \end{pmatrix}_x \bigg) \tag{6.7}$$

where

$$D\bigg( \begin{pmatrix} \phi \\ e \end{pmatrix} \bigg) = \begin{pmatrix} \phi(0) \\ e(0) \end{pmatrix} \quad \text{and} \quad F\bigg( \begin{pmatrix} \phi \\ e \end{pmatrix} \bigg) = \begin{pmatrix} e(0)\big(Df(\phi(0))\big)^{-1}\big(\phi(0) - \phi(-1)\big) \\ 0 \end{pmatrix}.$$

Notice that $F$ is bounded and Lipschitz continuous operator on $\mathcal{C}$. To find the profile of (6.7), we need to define the solution operator $T(x)$ of (6.7). According to Lemma 7.1 in [30], we have the followig result.

**Theorem 6.2.** *For any initial data* $\begin{pmatrix} u \\ c \end{pmatrix}_0 (\theta) = \begin{pmatrix} \phi \\ e \end{pmatrix}(\theta)$, *with* $\begin{pmatrix} \phi \\ e \end{pmatrix} \in \mathcal{C}$ *and* $\theta \in [-1, 0]$, *there exists a unique solution of the functional differential equation* (6.7) *and the associated strongly continuous solution operator* $T(x) : \mathcal{C} \to \mathcal{C}$, *with* $x > 0$, *satisfies*

$$T(x) \begin{pmatrix} \phi \\ e \end{pmatrix}(\theta) = \begin{pmatrix} u \\ c \end{pmatrix}_x (\theta) = \begin{pmatrix} u \\ c \end{pmatrix}(x + \theta). \tag{6.8}$$

*The infinitesimal generator* $\mathcal{A}$ *of* $T(x)$, *defined as*

$$\mathcal{A} \begin{pmatrix} u \\ c \end{pmatrix} = \lim_{x \to 0} \frac{1}{x} \left[ T(x) \begin{pmatrix} u \\ c \end{pmatrix} - \begin{pmatrix} u \\ c \end{pmatrix} \right],$$

*is given by*

$$\mathcal{A} \begin{pmatrix} \phi \\ e \end{pmatrix} = \begin{pmatrix} \phi \\ e \end{pmatrix}'$$

*on the domain*

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} \phi \\ e \end{pmatrix} \in \mathcal{C} : \begin{pmatrix} \phi \\ e \end{pmatrix}'(0) = \begin{pmatrix} e(0)\big(Df(\phi(0))\big)^{-1}\big(\phi(0) - \phi(-1)\big) \\ 0 \end{pmatrix} \right\}. \tag{6.9}$$

In order to construct the traveling profile for (6.7), we first need to construct the center manifold of the linearized system. We thus linearize this system around the constant solution $\begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix} \in \mathcal{C}$ and obtain

$$\begin{pmatrix} u \\ c \end{pmatrix}'(x) = \begin{pmatrix} u \\ c \end{pmatrix}_x'(0) = \int_{-1}^{0} d\rho(\theta) \begin{pmatrix} u \\ c \end{pmatrix}_x (\theta), \tag{6.10}$$

$$\rho(\theta) = \begin{pmatrix} \lambda_k(u_r)\big(Df(u_r)\big)^{-1}\big(H(\theta) - H(\theta+1)\big) & 0 \\ 0 & 0 \end{pmatrix},$$

where $H$ is the Heaviside function. Following the definition in Section 7.1 in [30], we can introduce the transposed of the linear system (6.10) on the space $\mathcal{C}^* \doteq \mathcal{C}([0,1],(\mathbb{R}^{n+1})^*)$, namely

$$(\alpha,\upsilon)'(0) \;=\; -\int_{-1}^{0}(\alpha,\upsilon)(-\theta)d\rho(\theta) \;=\; \lambda_k(u_r)\big(Df(u_r)\big)^{-1}\cdot\big(\alpha(1)-\alpha(0),0\big). \quad (6.11)$$

Here the dual product is defined as

$$\left\langle (\alpha,\upsilon),\begin{pmatrix}\phi\\ e\end{pmatrix}\right\rangle \;\doteq\; \alpha(0)\phi(0)+\upsilon(0)e(0)-\int_{-1}^{0}\lambda_k(u_r)\alpha(\theta+1)(Df(u_r))^{-1}\phi(\theta)d\theta. \; (6.12)$$

The characteristic equation for the linear system is given by

$$\det\Big(zI-\int_{-1}^{0}e^{z\theta}d\rho(\theta)\Big) \;=\; z\prod_{i=1}^{n}\big(z+\lambda_k(u_r)\lambda_i(u_r)^{-1}(e^{-z}-1)\big) \;=\; 0.$$

Since $\frac{1-e^{-z}}{z}$ is a decreasing function with a removable singularity at $z=0$, the characteristic equation has a zero of order $n+2$ at $z=0$, and $n-1$ additional zeros of order one.

The center manifold of (6.10) is the eigenspace of the eigenvalue 0. Using $^t$ to denote a transposition, the normalized basis for the center manifold and its adjoint basis can be written as

$$\Phi_0=\begin{pmatrix}0\\1\end{pmatrix} \qquad \Psi_0=\begin{pmatrix}0\\1\end{pmatrix}^t$$

$$\Phi_i=\begin{pmatrix}r_i(u_r)\\0\end{pmatrix},\; i\in\{1,\dots,n\}\backslash\{k\} \qquad \Psi_i=\frac{\lambda_i(u_r)}{\lambda_i(u_r)-\lambda_k(u_r)}\begin{pmatrix}l_i(u_r)\\0\end{pmatrix}^t,\; i\in\{1,\dots,n\}\backslash\{k\}$$

$$\Phi_k=\begin{pmatrix}r_k(u_r)\\0\end{pmatrix} \qquad \Psi_k=\tfrac{2}{3}\begin{pmatrix}l_k(u_r)\\0\end{pmatrix}^t-2\begin{pmatrix}l_k(u_r)\theta\\0\end{pmatrix}^t$$

$$\Phi_{n+1}=2\begin{pmatrix}r_k(u_r)\theta\\0\end{pmatrix} \qquad \Psi_{n+1}=\begin{pmatrix}l_k(u_r)\\0\end{pmatrix}^t$$

We now consider the center manifold for the nonlinear system (6.7) and its solution operator $T(x)$ defined in (6.8):

**Theorem 6.3.** *There exists a neighborhood* $\mathcal{U}$ *of* $\begin{pmatrix}u_r\\ \lambda_k(u_r)\end{pmatrix}$ *in* $\mathcal{C}$ *and a center manifold*

$\mathcal{M}$ *such that the following holds.*

1. *Every* $T(x)$*-orbit starting in* $\mathcal{M}$ *remains in* $\mathcal{M}$ *as long as it stays in* $\mathcal{U}$*.*

2. *Invariant sets under* $T(x)$ *in* $\mathcal{U}$ *are also in* $\mathcal{M}$*.*

3. $\mathcal{M} \cap \mathcal{U} = \left\{ \begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix} + (\Phi_0, \dots, \Phi_{n+1})y + h(y); y \in \mathcal{Z} \right\}$*, where* $\mathcal{Z}$ *is a neighborhood of 0 in* $\mathbb{R}^{n+2}$ *and* $h : \mathcal{Z} \to \mathcal{Q}$ *is a smooth function with* $h(0) = 0, dh(0) = 0$*. The codomain* $\mathcal{Q}$ *is the complement space of the generalized eigenspace for the eigenvalue* $0$*, which is spanned by* $(\Phi_0, \dots, \Phi_{n+1})$*, in* $\mathcal{C}$*.*

**Remark 6.2.** *We observe here few facts relevant in the next steps of the proof of Theorem 6.1.*

i. *The basis* $\{\Phi_0, \dots, \Phi_{n+1}\}$ *consists of functions contained in the domain* $D(\mathcal{A})$*.*

ii. *There exists an* $(n+2) \times (n+2)$ *matrix* $B$ *such that*

$$\mathcal{A}(\Phi_0, \dots, \Phi_{n+1}) = (\Phi_0, \dots, \Phi_{n+1})B\,.$$

*It is immediate to check that such a matrix has the rows* $(b_i)_{i \in \{0, \dots, n+1\}}$ *all equal to* $0$ *except for* $b_k = (0, \dots, 0, 2)$*.*

iii. *The product* $\left\langle \begin{pmatrix} \Psi_0 \\ \vdots \\ \Psi_{n+1} \end{pmatrix}, h(y) \right\rangle$ *is equal to* $0$ *for any* $y \in \mathcal{Z}$*. Therefore we can define a diffeomorphism* $L : \mathcal{M} \cap \mathcal{U} \to \mathcal{Z}$ *by setting*

$$L\left( \begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix} + (\Phi_0, \dots, \Phi_{n+1})y + h(y) \right) = \left\langle \begin{pmatrix} \Psi_0 \\ \vdots \\ \Psi_{n+1} \end{pmatrix}, (\Phi_0, \dots, \Phi_{n+1})y + h(y) \right\rangle = y.$$

## 6.1 The flow on the center manifold $\mathcal{M}$

In the next step, we construct a flow $\left\{T(x)\begin{pmatrix}\phi\\ e\end{pmatrix}, x\geq 0\right\}$ on $\mathcal{M}\subset\mathcal{C}$ such that

$$T(x)\begin{pmatrix}\phi\\ e\end{pmatrix}(\theta) \;=\; \begin{pmatrix}u\\ c\end{pmatrix}(x+\theta)$$

with the initial data $\begin{pmatrix}u\\ c\end{pmatrix}_0 = \begin{pmatrix}\phi\\ e\end{pmatrix}$. By Theorem 6.3, we can turn this problem into a Cauchy problem on $\mathbb{R}^{n+2}$.

**Lemma 6.1.** *Let* $\begin{pmatrix}u_r\\ \lambda_k(u_r)\end{pmatrix}+\begin{pmatrix}\phi\\ e\end{pmatrix}\in\mathcal{M}\cap\mathcal{U}$ *and* $x\geq 0$ *be fixed. If* $T(x)\Big(\begin{pmatrix}u_r\\ \lambda_k(u_r)\end{pmatrix}+\begin{pmatrix}\phi\\ e\end{pmatrix}\Big)\in\mathcal{M}\cap\mathcal{U}$*, then*

$$T(x)\Big(\begin{pmatrix}u_r\\ \lambda_k(u_r)\end{pmatrix}+\begin{pmatrix}\phi\\ e\end{pmatrix}\Big)=\begin{pmatrix}u_r\\ \lambda_k(u_r)\end{pmatrix}+(\Phi_0,\ldots,\Phi_{n+1})y(x)+h\big(y(x)\big)\,,$$

*where* $y(x)$ *is the unique solution of the following Cauchy problem:*

$$\begin{cases} y'(x) = By + (\Psi_0^t,\ldots,\Psi_{n+1}^t)^t(0)F(y)\,,\\[2mm] y(0) \;=\; \left\langle (\Psi_0^t,\ldots,\Psi_{n+1}^t)^t\,, \begin{pmatrix}\phi\\ e\end{pmatrix}\right\rangle, \end{cases} \tag{6.13}$$

*with* $B$ *as defined in ii. of Remark 6.2,* $F(y):=\begin{pmatrix}0\\ \widehat{F}\big((\Phi_0,\ldots,\Phi_{n+1})y+h(y)\big)\\ 0\end{pmatrix}$ *and*

$$\widehat{F}\Big(\begin{pmatrix}\phi\\ e\end{pmatrix}\Big)=\big((\lambda_k(u_r)+e(0))(Df(u_r+\phi(0)))^{-1}-\lambda_k(u_r)(Df(u_r))^{-1}\big)(\phi(0)-\phi(-1))\in\mathbb{R}^n$$

*for* $\begin{pmatrix} \phi \\ e \end{pmatrix} \in \mathcal{C}$.

*Proof:* We can find a similar proof in Section 10.2 of [30]. □

## 6.2 A smaller submanifold $\widetilde{\mathcal{M}}$ of the center manifold $\mathcal{M}$

The flow we want to find on $\mathcal{M}$ should satisfy the following conditions:

(a) It has start from $\begin{pmatrix} u_r \\ \sigma \end{pmatrix}$ and end at $\begin{pmatrix} u_l \\ \sigma \end{pmatrix}$ for some constant $\sigma$,

(b) For every point $\begin{pmatrix} \phi \\ e \end{pmatrix}$ on this flow, the flow afterwards has be given by

$$\left\{ T(x) \begin{pmatrix} \phi \\ e \end{pmatrix}; \quad x \geq 0 \right\}.$$

The flow satisfying these two conditions can be the solution of (6.7) and hence of (6.6) and (6.4). Hence, if we can find this flow on $\mathcal{M}$, we can get one solution to (6.4).

We firstly claim that the flow satisfying these two conditions is on $\mathcal{M}$. We can see that $\begin{pmatrix} u_r \\ \sigma \end{pmatrix}$ and $\begin{pmatrix} u_l \\ \sigma \end{pmatrix}$ belong to $\mathcal{M}$. Hence, by Theorem (6.3), when $u_l$ is close to $u_r$ and $\sigma$ is close to $\lambda_k(u_r)$, the flow satisfying conditions (a) and (b) should be contained in $\mathcal{M}$.

However, the $(n+2)$-dim manifold $\mathcal{M}$ is too large. We need to restrict the flow to a smaller manifold by reconsidering the exact form of (6.7).

**Lemma 6.2.** *Let* $\widetilde{\mathcal{M}}$ *be defined as*

$$\widetilde{\mathcal{M}} := \left\{ \begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix} + \begin{pmatrix} \phi \\ e \end{pmatrix} \in \mathcal{M};\ f(u_r+\phi(0)) - f(u_r) = (\lambda_k(u_r)+e(0)) \int_{-1}^{0} \phi(\theta) d\theta \right\}.$$

*There exists a neighborhood* $\widetilde{\mathcal{U}} \subset \mathcal{U}$ *of* $\begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix}$ *such that* $\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{U}}$ *is a 2-dimensional* *invariant manifold under $T$. The tangent space of $\widetilde{\mathcal{M}}$ at* $\begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix}$ *is spanned by* $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} r_k(u_r) \\ 0 \end{pmatrix}$.

The proof is similar to the proof of Lemma 7 in [5].

**Lemma 6.3.** *If $\sigma$ is close to $\lambda_k(u_r)$ then* $\begin{pmatrix} u_r \\ \sigma \end{pmatrix}$ *belongs to $\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{U}}$ and the flow starting from this point will be contained in $\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{U}}$ if it is contained in $\widetilde{\mathcal{U}}$. Hence, the flow solving* (6.7) *should be contained in $\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{U}}$ for a suitable $\sigma$.*

*Proof:* Multiply both sides of (6.4) for $\big(Df(u)\big)^{-1}$ and integrate from $-\infty$ to $x$. The result is exactly the restriction in the definition of $\widetilde{\mathcal{M}}$. Thus, if the solution to (6.4) exists and its values are contained in $\widetilde{\mathcal{U}}$, they belong also to $\widetilde{\mathcal{M}}$, hence to $\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{U}}$.

## 6.3 The flow from $u_r$ to $u_l$ with speed $\sigma$

For $\varepsilon_0 > 0$ small enough, consider the segment

$$R := \left\{ \begin{pmatrix} u_r \\ \sigma \end{pmatrix}; \quad \big|\sigma - \lambda_k(u_r)\big| \le \varepsilon_0 \right\} \tag{6.14}$$

and the curve

$$H := \left\{ \begin{pmatrix} u_l \\ \sigma \end{pmatrix}; \sigma(u_l - u_r) = f(u_l) - f(u_r), \quad \big|\sigma - \lambda_k(u_r)\big| \le \varepsilon_0 \right\} \tag{6.15}$$

on $\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{U}}$. The same argument as in Lemma 6 in [5] now yields

**Lemma 6.4.** *The point* $\begin{pmatrix} \phi \\ e \end{pmatrix} \in \mathcal{M} \cap \mathcal{U}$ *is a fixed point if and only if $y_{n+1} = 0$, where $y_{n+1}$ is the last component of* $y = L\Big(\begin{pmatrix} \phi \\ e \end{pmatrix}\Big)$.

Next, we have the following result.

**Lemma 6.5.** *Let $R$ and $H$ as defined respectively in* (6.14) *and* (6.15). *Then*

**i.** *$L(R)$ and $L(H)$ are transverse curves in $\mathcal{Z} \subset \mathbb{R}^{n+2}$, therefore $R$ and $H$ are transverse to each other in $\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{U}}$.*

**ii.** *$R, H$ are both transverse to any flow in $\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{U}}$.*

**iii.** *The set of fixed points of (6.13) on $\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{U}}$ is given by $R \cup H$. A point in $R \setminus \begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix}$ is a repelling equilibrium. Moreover, a point in $H \setminus \begin{pmatrix} u_r \\ \lambda_k(u_r) \end{pmatrix}$ is an attracting equilibrium if and only if $\lambda_k(u_r) \leq \sigma \leq \lambda_k(u_l)$.*

*Proof:* The proof is similar to the one in Section 5 of [5], but we shall give a more detailed analysis of the equilibrium points.

**1.** For any point $\begin{pmatrix} u_r \\ \sigma \end{pmatrix} \in R$, $y_0 = \sigma - \lambda_k(u_r)$ and $y_k = 0$ where $k$ denotes the class which the shock connecting $u_r$ and $u_l$ belongs to. For any point $\begin{pmatrix} u_l \\ \sigma \end{pmatrix} \in H$, $y_0 = \sigma - \lambda_k(u_r)$, $y_k = l_k(u_r) \cdot (u_l - u_r)$ for the same $k$ and hence $y_0 \approx \frac{y_k}{2}$. Hence, $L(R)$ and $L(H)$ are transverse to each other.

**2.** The component $y_0$ is constant along the flow satisfying the two condition in Section **6.2** while $y_0$ will always change along $R$, $H$.

**3.** For $\begin{pmatrix} u_r \\ \sigma \end{pmatrix} \in R$, $L\Big(\begin{pmatrix} u_r \\ \sigma \end{pmatrix}\Big) = (\sigma - \lambda_k(u_r), 0, \dots, 0)^t$ and $(\Phi_0, \dots, \Phi_{n+1})y + h(y) \approx \big(\sigma - \lambda_k(u_r)\big)\Phi_0$.

Therefore around this point, (6.13) yields

$$y_k' \approx \Big(2 + \frac{4\big(\sigma - \lambda_k(u_r)\big)}{3\lambda_k(u_r)}\Big) y_{n+1}, \quad y_{n+1}' \approx \frac{2\big(\sigma - \lambda_k(u_r)\big)}{\lambda_k(u_r)} y_{n+1}, \quad y_i' \approx 0.$$

The point is a repelling equilibrium if and only if $y_{n+1} \to 0$ as $x \to -\infty \Leftrightarrow \sigma \geq \lambda_k(u_r)$.

**4.** For $\begin{pmatrix} u_l \\ \sigma \end{pmatrix} \in H$, $L\Big(\begin{pmatrix} u_l \\ \sigma \end{pmatrix}\Big) = (\sigma - \lambda_k(u_r), l_1(u_r) \cdot (u_l - u_r), \dots, l_n(u_r) \cdot (u_l - u_r), 0)^t$ and $(\Phi_0, \dots, \Phi_{n+1})y + h(y) \approx \begin{pmatrix} u_l - u_r \\ \sigma - \lambda_k(u_r) \end{pmatrix}$.

Therefore around this point, (6.13) yields

$$\begin{aligned} y_0' &= 0 \\ y_i' &\approx \frac{2\lambda_i(u_r)}{\lambda_i(u_r) - \lambda_k(u_r)}\sigma l_i(u_r)\big(Df(u_l)\big)^{-1} r_k(u_r) y_{n+1} \\ y_k' &\approx 2y_{n+1} + \frac{4}{3}\Big(\sigma l_k(u_r)\big(Df(u_l)\big)^{-1} r_k(u_r) - 1\Big) y_{n+1} \\ y_{n+1}' &\approx 2\Big(\sigma l_k(u_r)\big(Df(u_l)\big)^{-1} r_k(u_r) - 1\Big) y_{n+1} \end{aligned}$$

Take $u_l = S_k(z), \sigma = \lambda_k(z)$ with $S_k(0) = u_r, \lambda_k(0) = \lambda_k(u_r)$ as Theorem 5.1 in [11]. By analyzing $z$, $y_{n+1} \to 0$ as $x \to +\infty$ if and only if $\lambda_k(u_r) \le \sigma \le \lambda_k(u_l)$ and $\begin{pmatrix} u_l \\ \sigma \end{pmatrix}$ is an attracting equilibrium point if and only if $y_{n+1} \to 0$ as $x \to +\infty$.

## 6.4 Proof of Theorem 6.1

By the Lemma 6.5, we can prove that if $u_l, u_r$ are connected by a $k$-shock with the speed $\sigma$, then there exist a profile connecting these two points, $\begin{pmatrix} u_r \\ \sigma \end{pmatrix}$ and $\begin{pmatrix} u_l \\ \sigma \end{pmatrix}$.

This is because if $u_l, u_r$ are connected by a $k$-shock with speed $c$, $u_l$, $u_r$ and $c$ will satisfy (4.8) and hence $\lambda_k(u_r) < c < \lambda_k(u_l)$. Then, by Lemma 6.5, $\begin{pmatrix} u_r \\ c \end{pmatrix}$ is a repelling equilibrium point and $\begin{pmatrix} u_l \\ c \end{pmatrix}$ is an attracting equilibrium point. So, the flow starting from $\begin{pmatrix} u_r \\ c \end{pmatrix}$ will end at $\begin{pmatrix} u_l \\ c \end{pmatrix}$.

By Lemma 6.1, this flow corresponds to a solution of (6.7) and hence of (6.4). By applying a reflection an dilatation we obtain the solution of the original problem (6.1) and (6.2).

# Chapter 4
# Background of Diffusion Equations and Its Traveling Wave Solutions

The problem of the population of species in the environment or the chemical reactions can be usually described by the diffusion equations as

$$\frac{\partial \rho_i}{\partial t} = \frac{\partial}{\partial x}\left(D_i \frac{\partial \rho_i}{\partial x} + C_i \rho_i\right) + F_i(t, x, 0, [\rho(t, \cdot)]), \quad \text{for } i = 1, \ldots, n, \tag{0.1}$$

$\rho_i$ represents the density of the species $i$ or the chemical material $i$ and we denote $\rho = (\rho_1, \ldots, \rho_n)$ as the composition of all species or chemical materials in the environment. $D_i(t, x)$ and $C_i(t, x)$ are $\mathcal{C}_1$ function with $F_i$ is a continuous function for $x, t$ and a bound functional for $\rho$.

In this chapter, we will simply introduce the biology background of diffusion equations. Then, we will analyze the properties of the diffusion equations and the wave solutions of the diffusion equations.

## 1 The diffusion equation from biology problems

We introduce some notations firstly. For each species $i$, we have

- the density function $\rho_i(t, x)$. Hence $\int_a^b \rho_i(t, x)dx$ is the total number of the individuals of the species $i$ in the interval $[a, b]$ at the time $t$. Besides, $\rho_i(t, x)$ is twice continuously differentiable in $x$ and once in $t$ and defined on $(t, x) \in [0, +\infty[\times] - \infty, +\infty[$.

- the redistribution kernel function $k_i(h, t, x, \xi)$, $h > 0$. It defines the coefficient for the density at $(t, x)$ passing to the density at $(t + h, x + \xi)$. If there is no new individual added into the species $i$, then the density of the species $i$ at the location

$y$ and the time $s > t$ could be calculated as

$$\rho_i(s, y) = \int_{-\infty}^{+\infty} k_i(s - t, t, x, y - x)\rho_i(t, x)\, dx.$$

- the new added individual functional $F_i(t, x, h, [\rho(t, \cdot)])$, $h > 0$. It defines the amount of new added individuals at the location $x$ between the time $t$ and $t + h$. $\rho(t, \cdot)$ is defined as $(\rho_1, \ldots, \rho_n)(t, \cdot)$ where $\rho_i$, $i = 1, \ldots n$ are the density function of the species $i$. The density of the species $i$ at the location $y$ and the time $s > t$ with new individual can be written as

$$\rho_i(s, y) = \int_{-\infty}^{+\infty} k_i(s - t, t, x, y - x)\rho_i(t, x)\, dx + (s - t)F_i(t, y, s - t, [\rho(t, \cdot)]). \quad (1.2)$$

Besides, there are some additional requirements for $k_i$ and $F_i$, for all $i = 1, \ldots, n$,

**Condition 1** $k_i(h, t, x, \xi)$ is always positive and $\int_{-\infty}^{+\infty} k_i(h, t, x, \xi) d\xi < C$. $C$ is independent of $h$.

**Condition 2** $\lim_{h \to 0} \int_{-\infty}^{+\infty} \xi^2 k_i(h, t, x, \xi) d\xi = 0$.

**Condition 3** $F_i$ is a continuous functional on $\mathbf{L}^1$ with the kernel $k_i$, uniformly with respect to $h$. More specifically, for every $(x, y)$, there exists a number $\delta(t, x, \epsilon)$ such that for any two densities of species $\rho^{(1)}$ and $\rho^{(2)}$, if

$$\int_{-\infty}^{+\infty} k_i(h, t, y, x - y)\big|\rho_i^{(1)}(t, y) - \rho_i^{(2)}(t, y)\big| dy < \delta(t, x, \epsilon)$$

for all $i = 1, \ldots, n$, then

$$\big|F_i(t, x, h, [\rho^{(1)}(t, \cdot)] - F_i(t, x, h, [\rho^{(2)}(t, \cdot)]\big| < \epsilon$$

for all $i$.

Now, the following theorem could be proved

**Theorem 1.1.** *If there exist a series of the kernel functions and the new individual functionals such that* (1.2) *hold for all* $i$ *and the three conditions above are satisfied, then there exist a series of functions* $D_i(t, x) \geq 0$, $C_i(t, x)$ *such that for each* $i$, $\rho_i(t, x)$ *is twice continuously differentiable in* $x$ *and once in* $t$ *as well as* $\rho_i$, $i = 1, \ldots, n$ *satisfy* (0.1).

**Proof:** See **Theorem 1.3** in [26].

# 2 Qualitative analysis for diffusion equation

The general diffusion equation system is defined on the $(m+1)$-dimensional space: $Q_t \doteq (t,x) \in ]0,T] \times \Omega \subset I\!R^m$. It has the form of

$$\frac{\partial u}{\partial t} + Lu = F(t,x,u). \tag{2.3}$$

Here, $F$ is an $n$-dimensional function. $L$ is an $n$-dim differentiation operator for $n$-dimensional $u \in \mathcal{C}_{1,2}([0,T]\times\Omega, I\!R^n)$. The formula of each component of $L$ is as following

$$L_k u = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} H_{ki}, \quad -H_{ki} = \sum_{j=1}^{m} D_{kij}(t,x)\frac{\partial u}{\partial x_j} + C_{ki}(t,x)u, \qquad k = 1, \dots n, \tag{2.4}$$

where $D_{kij}$, $C_{ki}$ are all the $n$-dim vectors.

If $\Omega$ is not the whole space $I\!R^n$, we assume that the boundary of it, denoted as $\partial\Omega$, has a unit normal, denoted as $\vec{n}(x)$, which is smooth on each point on $\partial\Omega$. Then we will have three kinds of boundary conditions on $]0,T[\times\partial\Omega$:

**Dirichlet condition:** $u(t,x) = c(t,x)$;

**Neumann condition:** $\vec{n}(x)\cdot(H_{k1},\cdots,H_{km}) = a_k(t,x)$ for $k = 1,\dots,n$;

**Robin condition:** $\vec{n}(x)\cdot(H_{k1},\cdots,H_{km}) = b_k(t,x)u + a_k(t,x)$ for $k = 1,\dots,n$;

where $a_i \in I\!R$ and $b_i \in I\!R^n$, $b_i > 0$.

With these definitions, we can introduce several basic problems related to diffusion equation system:

The **initial boundary value problem** is the most common problem. This problem requires to find a $\mathcal{C}_{1,2}$ function $u : \overline{Q}_T \to I\!R^n$ satisfying the following conditions:

- $u$ satisfies (2.3);
- $u$ satisfies the initial condition $u(0,x) = u_0(x) \in I\!R^n$ on $\overline{\Omega}$;
- $u$ satisfy one of the three boundary conditions.

The **initial value problem** is similar to the initial boundary problem but only the first two conditions above need to be satisfied. In fact, the diffusion system (2.3) could be defined on $]0,T]\times I\!R^m$. Besides, the boundary conditions could be replaced by the asymptotic conditions.

## 2.1 Comparison theorems

In **Chapter 1**, we have discussed the usage of the max principles and comparison theorems to prove the existence of the solutions of (1.4). In this section, we will explain this theorem in details and use them to prove the existence of the subsolution of the specific diffusion equations in **Chapter 5**.

The comparison theorem could be proven in the certain case (♠) where

- (2.3) is a scalar case meaning $n = 1$;
- $Lu$ takes the form as in (1.4) and $L$ is non-divergence, $c$ is bounded;
- $F(t, x, u)$ is $\mathcal{C}_1$ in $u$.

Then, the comparison theorem will take the form as

**Theorem 2.1.** *Under the case described above, for any two $C_{2,1}(Q_T) \cap C(\overline{Q}_T)$ functions $u$ and $v$ satisfying*

- $u_t + Lu - f(t, x, u) \geq v_t + Lv - f(t, x, v)$ *for* $(t, x) \in Q_T$*;*
- $u(0, x) \geq v(0, x)$ *for* $x \in \Omega$*;*
- *if* $\Omega \neq I\!R^m$, $D_x u \cdot \vec{n}(t, x) + \beta(t, x)u \geq D_x v \cdot \vec{n}(t, x) + \beta(t, x)v$ *for* $(t, x) \in ]0, T[\times \partial\Omega$ *where* $\vec{n}$ *is the outward normal of* $\Omega$ *and* $\beta \geq 0$*;*

*then* $u(t, x) \geq v(t, x)$ *for* $(t, x) \in \overline{Q}_T$.

*Moreover, if* $u(0, x) > v(0, x)$ *for* $x \in \Omega_1 \subset \Omega$*, then* $u(t, x) > v(t, x)$ *for* $(t, x) \in [0, T] \times \overline{\Omega}_1$.

**Proof.** The basic proof could be found in **Theorem 10.1** in [46]. The only thing to pay attention to is that when we change the unknown variable $w = u - v$ to $z = e^{kt}w$, we should be careful to choose $k$ to make $k + c - \max_u |f_u(t, x, \cdot)| \geq 0$. □

## 2.2 Existence and uniqueness for initial value problems of diffusion equations

If $F(t, x, u)$ is independent of $u$, (2.3) is a heat equation. Hence, the general solution to the initial value problem for (2.3) and the initial value $u(0, x) = u_0(x)$, takes the form as

$$u(t, x) = \int_{I\!R^m} \Phi(t, x - y)u_0(y)\, dy + \int_0^t \int_{I\!R^m} \Phi(t - s, x - y)f(s, y)\, dyds,$$

where

$$\Phi(t,x) = \begin{cases} (4\pi t)^{-m/2} e^{-\frac{|x|^2}{4t}}, & t > 0 \\ 0, & t < 0 \end{cases}, \qquad \text{with } |x| = \sqrt{x_1^2 + \cdots + x_m^2}. \tag{2.5}$$

If $F(t,x,u)$ is not independent of $u$, the strong solution for the initial value problem of (2.3) haven't been proven in the general case. So, we will look into the special case where

$$Lu = -D\Delta u, \qquad D > 0$$

because the case (♠) specifies that $L$ is non-divergence. In fact, $L$ could be simplified into the operator that

$$Lu = -\Delta u$$

by scaling the spatial variable $x \to x/\sqrt{D}$. Then, we could have the theorem that

**Theorem 2.2.** *For the initial value problem*

$$\begin{cases} \frac{du}{dt} - \Delta u &= F(t,x,u), \\ u(0,x) &= u_0(x), \end{cases} \tag{2.6}$$

*if $F(t,x,u)$ is continuously differentiable with respect to $x$, $u$ and $u_0$ is bounded and has only a finite points of discontinuity, then the solution for* (2.6)*, denoted as $v$, will exist and be bounded for the value of $t < \infty$. $v \in \mathcal{C}_{1,2}(]0,T] \times \Omega, I\!R^n)$ and $v(0,x) = u_0(x)$ at the each point where $u_0$ is continuous. Moreover, $v$ is the unique solution.*

**Proof.** The standard proof could be found in **Theorem 1** in [35]. The only thing to pay attention to is that the kernel used in the original proof is (2.5) with $m = 1$. In our case, we should utilize the general kernel without the restriction $m = 1$. □

# 3 Wave solution for diffusion equation

From the analysis in the **Chapter 2**, we see the wave solution method could efficiently simplify the diffusion equations when this kind of equations contain both spatial and time variables. While the method of wave solutions cannot be applied to the general diffusion equations, it could still be used to solve the diffusion equations with the special form

$$\frac{\partial u}{\partial t} = A\Delta u + f(u), \tag{3.7}$$

where the system is defined on $Q_T$ for some time $T$, $u \in I\!R^n$, $A$ is an $n \times n$ real matrix.

We consider the planar wave $u(t,x) = w(x_1 - ct)$ for the equations above and get

$$Aw'' + cw' + f(w) = 0. \tag{3.8}$$

The wave solutions usually have asymptotic conditions such as

$$\lim_{\xi \to \pm\infty} w(\xi) = w_\pm.$$

If $w'(\xi)$ is bounded when $\xi \to \pm\infty$, it is easy to see that $w_\pm$ are both the zeros of $f(w)$.

In conclusion, to find the wave solution for the system (3.7), we need to find a suitable pair of $c$ and $w$ to satisfy (3.8) and the asymptotic conditions.

## 3.1 Scalar case

In the following discussion, we will look into the scalar diffusion equation (0.1) defined on $Q_T \doteq [0,T] \times \Omega$ with $t, x$-independent $f$ and $n = 1$, which takes the form as

$$u_t - u_{xx} - f(u) = 0. \tag{3.9}$$

We could consider the travelling wave solution of it. The wave solution has the form

$$u(t,x) = U(x - ct). \tag{3.10}$$

Under this case, $U$ will satisfy that

$$U'' + cU' + f(U) = 0. \tag{3.11}$$

The $c$ is the speed of the wave.

To make it easier, we only focus on the function $U \in [0,1]$ and make the following assumption about $f(u)$:

$$f(u) \in \mathcal{C}_1([0,1]), \qquad f(0) = 0,\ f(1) = 0. \tag{3.12}$$

If we hope $u(t,x)$ is a stable solution, we need $U(z)$ satisfying that

$$U'(z) \to 0, \qquad \text{as } z \to \pm\infty,$$

because $u_t, u_x, u_{xx}$ will vanish as $x \to \pm\infty$. By this way, the limits of $U(z)$ as $z \to \pm\infty$ also exist and should be the zeros of $f(u)$. In this case, we know that $f(u)$ has zeros 0, 1. Hence, we require that

$$U(-\infty) = 0, \qquad U(+\infty) = 1. \tag{3.13}$$

In the following sections, we will try to analyze the solution of (3.11) with the asymptotic conditions (3.13) and the assumptions (3.12), by using the **phase portrait method**.

## 3.2 General properties of wave solutions

In order to use phase portrait method for the second order differential equation (3.11), we introduce the second variable $P$ and rewrite (3.11) into the 1-order differential equation system.

$$\begin{cases} U' &= P, \\ P' &= -cP - f(U). \end{cases} \tag{3.14}$$

The asymptotic condition (3.13) could be tranformed into the conditions that

$$(U,P)(-\infty) = (0,0), \qquad (U,P)(+\infty) = (1,0). \tag{3.15}$$

**Theorem 3.1.** *If $U \in [0,1]$ is the solution to (3.11) and satisfies $U(-\infty) = 0$, $U(+\infty) = 1$, $U$ will be an increase function.*

**Proof.** See Appendix A. □

In fact, for the system (3.14), we can rewrite it into the scalar case by eliminating the independent variable $z$ as the following

$$\frac{dP}{dU} = -c - \frac{f(U)}{P}, \tag{3.16}$$

defined on $U \in [0,1]$ with the boundary conditions

$$P(0) = P(1) = 0. \tag{3.17}$$

(3.16) with (3.17) is a new kind of question and we will prove that this question will 1-1 correspond to the problem (3.14) with its asymptotic conditions (3.15).

**Theorem 3.2.** *With the assumption about* (3.12)*, there is a 1-1 correspondence between the positive solution to* (3.16) *with the boundary conditions* (3.17) *and the flow of* (3.14) *satisfying the asymptotic conditions* (3.15) *except for the shift in* $z$*.*

**Proof.** See Appendix A. □

By the **Theorem 3.2** we can see that the three problems: the problem (3.11) with the asymptotic conditions (3.13), the problem (3.14) with (3.15) and the problem (3.16) with the boundary conditions (3.17), are equivalent to each other.

Multiple (3.16) by $P$ and integrate from 0 to 1 in $U$, getting

$$0 = -c \int_0^1 P\, dU - \int_0^1 f(U)\, dU.$$

By **Theorem 3.1**, $P \geq 0$ for $U \in [0, 1]$. So, we can conclude that the sign of $c$ is opposite to the sign of $\int_0^1 f(U)\, dU$.

## 3.3 Critical points

From the above, we know that the existence of traveling wave front for the equation (3.11) depends on the value of $c$. To find a wave front with a suitable speed $c$, we will look into the system (3.14). Because (3.14) is a homogeneous system, we could easily calculate out the critical points for this system, which are $(U, P) = (1, 0)$, $(0, 0)$. The types of these two critical points will greatly effect the existence of the wave front. We only look into the critical point $(0, 0)$. The situation of $(1, 0)$ is similar to $(0, 0)$.

We linearize (3.14) at $(0, 0)$ and get

$$\frac{d}{dz}\begin{pmatrix} U \\ P \end{pmatrix} = A \begin{pmatrix} U \\ P \end{pmatrix}, \qquad A \doteq \begin{pmatrix} 0 & 1 \\ -f'(0) & -c \end{pmatrix}.$$

The eigenvalues of $A$ is

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4f'(0)}}{2}.$$

Then, we consider five cases.

CASE 1: $f'(0) < 0$. $\lambda_{1,2}$ are two different real numbers with the opposite signs and the eigenvectors will take the form of

$$r_{1,2} = \begin{pmatrix} 1 \\ \lambda_{1,2} \end{pmatrix}. \tag{3.18}$$

The point $(0,0)$ will be a saddle. The flow leaving this point will increase in both $U$ and $P$ directions while the flow entering this point will increase in $U$ but decrease in $P$.

CASE 2: $f'(0) = 0$. $\lambda_{1,2} = 0, -c$. The eigenvectors will take the form of (3.18) as well. Since one eigenvalue is 0, one eigenspace will span a center manifold. The situation of the other eigenspace depends on the sign of $c$.

- If $c < 0$, the non-zero eigenvalue is positive and its corresponding flowing will leave $(0,0)$ and increase in $U$ and $P$.
- If $c = 0$, the center manifold will be 2-dimensional and the flow passing $(0,0)$ cannot be analyzed by just linearizing (3.14) around $(0,0)$.
- If $c > 0$, the eigenvalue is negative and its corresponding flow will enter $(0,0)$ and increase in $U$ but decrease in $P$.

CASE 3: $0 < f'(0) < c^2/4$. $\lambda_{1,2}$ are two different real numbers but their signs are the same. The eigenvectors will take the form as (3.18) as well.

- If $c < 0$, the two eigenvalues will be positive. Then, the flow along any linear combination of these two vectors will leave the point $(0,0)$. Hence, $(0,0)$ is a source (node). The flows going along the directions in the first quadrant will have $U$ and $P$ increase.
- If $c > 0$, the two eigenvalues will be negative. Then, the flow along any linear combination of these two vectors will enter the point $(0,0)$. Hence, $(0,0)$ is a sink (node). The flows going along the directions in the forth quadrant will have $U$ decrease and $P$ increase.

CASE 4: $f'(0) = c^2/4$ ($c \neq 0$). $\lambda_{1,2} = -c/2$. The eigenvectors corresponding to the eigenvalue $-c/2$ are

$$r_1 = \begin{pmatrix} 1 \\ -\frac{c}{2} \end{pmatrix}, \qquad r_2 = \frac{4}{c^2+4} \begin{pmatrix} \frac{c}{2} \\ 1 \end{pmatrix}.$$

The flow from $(0,0)$ could be parametrized as following:

$$\begin{pmatrix} U \\ P \end{pmatrix}(z) = C_1 e^{\lambda_1 z} r_1 + C_2 e^{\lambda_1 z}(r_2 + z r_1).$$

Hence, $(0,0)$ is still a node. If $c < 0$, $(0,0)$ is a source. If $c > 0$, $(0,0)$ is a sink.

CASE 5: $f'(0) > c^2/4$. $\lambda_{1,2}$ are two conjugate complex numbers. So, $(0,0)$ is a spiral. The flow from $(0,0)$ could be parametrized as:

$$\begin{pmatrix} U \\ P \end{pmatrix}(z) = C_1 e^{az}\left[\cos bz \begin{pmatrix} 1 \\ a \end{pmatrix} - \sin bz \begin{pmatrix} 0 \\ b \end{pmatrix}\right] + C_2 e^{az}\left[\cos bz \begin{pmatrix} 0 \\ b \end{pmatrix} + \sin bz \begin{pmatrix} 1 \\ a \end{pmatrix}\right]$$

with $a = -c/2$ and $b = \sqrt{4f'(0) - c^2}/2$. If $c < 0$, the spiral will leave $(0,0)$. If $c = 0$, there will be several ellipse-form flows going around the $(0,0)$ and never passing $(0,0)$. If $c > 0$, the spiral will enter $(0,0)$.

## 3.4 Orbits

Our goal is to find an orbit in $(U,P)$-space connecting $(0,0)$ and $(1,0)$ with $P \geq 0$. Hence, the system (3.14) should at least satisfy the following necessary conditions:

**(1).** $(0,0)$ should have a flow leaving with $U$, $P$ both increasing,

**(2).** $(1,0)$ should have a flow entering with $U$ increasing but $P$ decreasing.

So, we need $f$ to satisfy some conditions.

By the previous analysis about the critical points, to satisfy the first condition, the only possible cases are

- $f'(0) < 0$.
- $f'(0) = 0$ and $c < 0$.
- $0 < f'(0) \leq c^2/4$ and $c < 0$.

To satisfy the second conditios, the only possible cases are

- $f'(1) < 0$.
- $f'(1) = 0$ and $c > 0$.

- $0 < f'(1) \leq c^2/4$ and $c > 0$.

Since the sign of $c$ cannot be changed, among all combinations of the conditions for $f(0)$ and $f(1)$, only five possible combinations satisfy the necessary conditions of the existence of orbits.

- $f'(0) < 0$ and $f'(1) < 0$ and $c$ can be any real value.
- $f'(0) = 0$ and $f'(1) < 0$ and $c < 0$.
- $0 < f'(0) \leq c^2/4$ and $f'(1) < 0$ and $c < 0$.
- $f'(0) < 0$ and $f'(1) = 0$ and $c > 0$.
- $f'(0) < 0$ and $0 < f'(1) \leq c^2/4$ and $c > 0$.

We will look into three kinds of orbits. The first kind of orbits will satisfy the first conditions. The second kind of orbits will satisfy the second or forth conditions. The last kind of orbits will satisfy the third or fifth conditions.

### 3.4.1 saddle-saddle orbits

This kind of orbits corresponds to the first condition above. The solution for (3.14) should be an increase function because of its asymptotic conditions and **Theorem 3.1**.

**Theorem 3.3.** *Suppose that $f(u) \leq 0$ when $u \searrow 0$. Denote $P_{c_i}(U)$, $i = 1, 2$ as two solutions satisfying* (3.16) *with $P(0) = 0$ and entering the first quadrant at the origin, for $c = c_i$, $i = 1, 2$ respectively. Then, if*

**(1).** $c_1 = c_2$, $P_{c_1}(U) \equiv P_{c_2}(U)$;

**(2).** $c_1 > c_2$, $P_{c_1}(U) < P_{c_2}(U)$;

*when $U > 0$ and $P_{c_1} > 0$ for all value between $0$ and $U$.*

*More advancedly, the map $c \mapsto P_c(U)$ is continuous on $c$.*

**Proof.** See the **Lemma 4.14** in [26] and the **Proposition 4.5** in [4] □

We want to understand the further behavior of the solution of (3.16) after it enters the first quadrant of the origin. So, we consider the the half strip

$$Q \doteq \big\{(U,P)\,|\,0<U<1,\,P>0\big\}. \tag{3.19}$$

We firstly claim that

**Lemma 3.1.** *When $c<0$ and $|c|$ is considerably large. The orbit $(U,P(U))$ entering $Q$ from the origin, where $P(U)$ is the solution of* (3.16) *with $P(0)=0$, will leave $Q$ from the half vertical line $\{(1,P)\,|\,P>0\}$.*

**Proof.** See Appendix A. □

In fact, the existence of the solution to (3.16) with (3.17) depends greatly on the number of zeros of $f(U)$ inside $]0,1[$. Basically, we have the following conclusion:

**Theorem 3.4.** *If $f(u)$ satisfies $f'(0)$, $f'(1)<0$ and has only one zero inside $]0,1[$, there will be a unique pair of the solution $P_{c_0}(U)$ and the speed $c_0$ to make $P_{c_0}(U)$ be the solution of* (3.16) *with* (3.17).

**Proof.** See Appendix A. □

The condition that $f(u)$ has only one zero between $]0,1[$ is a sufficient condition that guarantees the existence of the solution to the problem (3.16) with (3.17). In fact, by the **Theorem 2.4** in [27], any saddle-saddle situation will contribute to a unique positive solution with a unique speed $c^*$ for (3.16) with (3.17).

### 3.4.2 saddle to degenerate point orbits

This kind of orbits corresponds to the second and forth conditions above. In these two conditions, one of critical points will be degenerate.

Degenerate point refers to the critical point where one eigenspace around this point corresponds to a center manifold. The other eigenspace corresponds to an unstable/stable manifold iff $c<0/c>0$. The structure of the unstable/stable manifold of this kind of critical point will be similar to the structure of the unstable/stable manifold of saddles.

In fact, by the same theorem **Theorem 2.4** in [27], we can conclude that these two cases

- $f'(0)=0$ and $f'(1)<0$ and $c<0$.
- $f'(0)<0$ and $f'(1)=0$ and $c>0$.

will contribute to a unique positive solution with a unique speed $c^*$ for (3.16) with (3.17).

### 3.4.3 saddle-node orbits

This kind of orbits corresponds to the third and fifth conditions above. We only consider the third case:

$$0 < f'(0) \leq c^2/4 \text{ and } f'(1) < 0 \text{ and } c < 0.$$

The analysis of the fifth condition we concluded above is similar to the analysis for the third one.

For this case, we will make another assumption that $f(u) > 0$ for $u \in ]0,1[$. With this assumption, we have the conclusion as below:

**Theorem 3.5.** *Suppose that $f(u)$ satisfies $0 < f'(0) \leq c^2/4$, $f'(1) > 0$, $f(u)$ is positive on $]0,1[$ and $f(0) = f(1) = 0$. Then, for any speed $c \leq -2\sqrt{\nu}$, where $\nu$ is defined as*

$$\nu \doteq \sup_{U_0 \in ]0,1]} f(U)/U, \tag{3.20}$$

*there will be a solution to* (3.16) *with* (3.17)*.*

**Proof.** See Appendix A. □

The fact is that $-2\sqrt{\nu}$ is not the maximal speed to make the problem (3.16) with (3.17) solvable. The maximal speed $c^*$ should be some value satisfying $-2\sqrt{\nu} \leq c^* \leq -2\sqrt{f'(0)}$. The speed $c^*$ could be approximated in the special case where $f(u) > 0$ on $]0,1[$ and $f'(u) < f'(0)$ on $]0,1]$, by constructing the family of the solutions to (3.9) and converging the solutions to a wave front as in **Section 3** of [35].

# Chapter 5 Controlled Traveling Profiles for Models of Invasive Species

## 1 Introduction

Consider a reaction-diffusion equation of the form

$$u_t \;=\; \sigma\Delta u + f(u,\alpha). \tag{1.1}$$

Here $t \geq 0$ is time, $x \in I\!\!R^n$ is the spatial variable, while $u = u(t,x)$ denotes the density of an invasive biological species, such as mosquitoes. We assume that, by implementing a control $\alpha = \alpha(t,x) \geq 0$, the population can be partly removed. This will slow down, or even reverse, its spatial propagation.

By a rescaling of the dependent variable we shall always assume that, when $\alpha = 0$, i.e. in absence of control, one has

$$f(0,0) \;=\; f(1,0) \;=\; 0.$$

In other words, the maximum population density sustained by the environment (i.e., the carrying capacity) is normalized so that $u^{max} = 1$.

Given an initial density

$$u(0,x) \;=\; \bar u(x) \tag{1.2}$$

and a time interval $[0,T]$, a natural objective can be stated as

$$\text{minimize:} \quad \mathcal{J} \;\doteq\; \int_0^T \left( \int_{R^n} \big[u(t,x) + \alpha(t,x)\big]\, dx \right) dt\,. \tag{1.3}$$

The right hand side of (1.3) accounts for the population size, plus the cost of the control, integrated over time.

Thanks to the fact that $u = 0$ and $u = 1$ are equilibrium states, in many cases the solution to (1.1) can be approximately described in terms of the set $\Omega(t)$ where $u(t,x) \approx 1$. Namely, if the diffusion coefficient $\sigma > 0$ is small, we expect that the difference $\big\|\chi_{\Omega(t)} - u(t,\cdot)\big\|_{\mathbf{L}^1}$ will also be small. The characteristic function $\chi_{\Omega(t)}$ of the set $\Omega(t)$ thus provides a good approximation to the density $u(t,\cdot)$ itself. Based on this observation, in [12] it was proposed to replace the problem (1.3) by an optimization problem for the moving set $\Omega(t)$. More precisely, let $c(t,x)$ be the speed at which the boundary of the set $\Omega(t)$ moves, in the direction of the interior normal, at a point $x \in \partial\Omega(t)$. The new optimization problem then takes the form

$$\text{minimize:} \quad \mathcal{J} \;=\; \int_0^T \left( \text{meas}\big(\Omega(t)\big) + \int_{\partial\Omega(t)} E\big(c(t,x)\big)\, dx \right) dt. \tag{1.4}$$

The cost function $E(c)$, which is integrated over the boundary of the set $\Omega(t)$, measures the effort needed to push the boundary inward with speed $c$. As shown in [12], it is this particular function that provides the link between the two problems (1.4) and (1.3). A rigorous justification of this approximation procedure can be achieved via a sharp interface limit.

With this motivation in mind, in the present paper we study the function $E(\cdot)$, for various nonlinear parabolic equations, or systems. In our basic setting, $E(c)$ is defined as the minimum cost of a control $\alpha(\cdot)$ which yields a traveling wave solution to (1.1) with speed $c$. This leads to the problem

$$\text{minimize:} \quad \|\alpha\|_{\mathbf{L}^1} \tag{1.5}$$

among all integrable functions $\alpha \geq 0$ such that there exists a solution to the ODE

$$\sigma U'' + c\,U' + f(U,\alpha) \;=\; 0, \tag{1.6}$$

with asymptotic conditions

$$U(-\infty) \;=\; 0, \qquad\qquad U(+\infty) \;=\; 1. \tag{1.7}$$

In the models considered in [12], the function $f(u,\alpha)$ has linear dependence on the control variable $\alpha$. Namely, the two main cases

$$f(u,\alpha) \;=\; F(u)+\alpha, \qquad \text{or} \qquad f(u,\alpha) \;=\; F(u)+\alpha u,$$

were studied. Thanks to this assumption, the difference in cost between any two admissible controls can be directly computed by Stokes' formula [9,31]. This yields a straightforward way to identify the optimal solution.

In the present paper, our first goal is to extend the analysis of controlled traveling waves to a more general class of functions $f$, possibly nonlinear also w.r.t. the control variable $\alpha$. In this case the techniques from [31] cannot be implemented, and the construction of optimal profiles requires a more careful analysis.

In the second part of the paper, we focus our study on two systems of PDEs, describing the interaction between disease-carrying insects and infected trees. A relevant example is provided by *Xylella fastidiosa*, which is a plant pathogenic bacterium that attacks olive trees. It is transmitted by a meadow spittlebug, the *Philaenus spumarius*, a sap-feeding insect. In [3] a detailed model for spatial propagation of a Xylella was introduced. This is described by a system of four equations for the densities of (i) healthy and infected insects, and (ii) healthy and infected trees. Here we consider two simplified models, that will allow a more detailed mathematical analysis.

**Model 1.** Assume that:

- The insect population spreads by diffusion and reproductive growth.
- By spraying pesticides, some of the insects can be removed. This slows down, or even reverses, their spatial propagation.
- All insects carry the infection, and contaminate the trees.

Calling

- $u = u(t,x) \in [0,1]$ the density of insects,
- $\theta = \theta(t,x) \in [0,1]$ the fraction of trees that are infected,
- $\alpha = \alpha(t,x) \geq 0$ the control function,

the evolution of these variables can be described by

$$\begin{cases} u_t \;=\; \Delta u + f(u,\alpha)\,, \\ \theta_t \;=\; \kappa_1 u(1-\theta)\,. \end{cases} \tag{1.8}$$

Here the constant $\kappa_1$ is an infection rate. The function $f = f(u, \alpha)$, modeling the controlled population growth, can take different forms. For example:

(i) Logistic growth + insect removal by pesticides or mosquito nets. This leads to

$$f(u, \alpha) \;=\; \kappa_3(1-u)u - \alpha u. \tag{1.9}$$

(ii) Weed removal, reducing the carrying capacity of the ecosystem. A possible model is

$$f(u, \alpha) \;=\; u(u-u^*)\left[\frac{1}{1+\alpha} - u\right], \tag{1.10}$$

where $u^* \in [0, 1/2]$. Notice that in this case the maximum population supported by the environment shrinks to $(1+\alpha)^{-1} < 1$ as the control $\alpha$ increases. This is another way to reduce the density of insects.

For the above model, a natural goal is to minimize

$$\begin{aligned} \mathcal{J} \;&=\; \int_0^T \left( \int_{\mathbb{R}^2} \big[\alpha(t,x) + \theta(t,x)\big]\, dx \right) dt \\ &=\; [\text{cost of the control}] + [\text{fruit production loss}]. \end{aligned} \tag{1.11}$$

for given initial data.

**Model 2.** We here assume that

- Newly born insects are healthy. Only later in life they can be infected, by the presence of contaminated trees.
- Infected insects contaminate the trees, and contaminated trees infect the new insects.
- By spraying pesticides, some of the insects can be removed.

In addition to the previous variables, calling

- $I = I(t,x) \in [0,1]$ the fraction of insects which are infected,
- $v = Iu$ the density of infected insects,

we thus consider the system of evolution equations

$$\begin{cases} u_t &= \Delta u + f(u) - \alpha u\,, \\ (Iu)_t &= \Delta(Iu) + \kappa_2(1-I)u\theta - \alpha Iu - d\,Iu, \\ \theta_t &= \kappa_1 Iu(1-\theta)\,. \end{cases} \tag{1.12}$$

The constants $\kappa_1, \kappa_2$ are infection rates, while $d$ is a death rate.

Motivated by [12], for the three models (1.1), (1.8), (1.12), we are interested in (i) the existence of controlled traveling profiles having a given speed $c$, and (ii) control functions $\alpha(\cdot)$ which achieve these traveling profiles and have minimum cost.

We now summarize the main results, proved in the remainder of the paper. In Section 2 we study the scalar equation (1.1). By a rescaling of the spatial variable, it is not restrictive to assume $\sigma = 1$. In absence of control, by the standard theory in [26, 47] it is known that the equation admits a traveling wave solution with a suitable speed $c^* < 0$. Here we prove that, given any speed $c > c^*$, there exists a control function $\alpha(\cdot)$ with finite cost which yields a traveling profile with speed $c$. More precisely (see Fig. 5.1), setting

$$u = U(x-ct), \qquad \alpha = \alpha(x-ct),$$

we construct a solution to

$$U'' + cU' + f(U,\alpha) \ = 0, \tag{1.13}$$

with asymptotic conditions (1.7). In Section 3 we prove that a suitable control function $\alpha^*(\cdot)$ can be chosen, having minimum cost. Necessary conditions for optimality are then derived in Section 4. In turn, these can be used in a shooting method, to numerically compute optimal solutions. Plots of an optimal traveling profile, and of the minimum cost $E(c)$ as a function of the speed $c$, are shown in Fig. 5.7 and Fig. 5.8, respectively.

In Section 5 we study Model 1. Here the main result shows that, for every wave speed $c \in [c^*, 0[$, the system (1.8) admits a controlled traveling wave with speed $c$. In other words, by removing part of the pest population, the speed at which the contamination advances can be slowed down to almost zero.

The last two sections are concerned with Model 2. Looking for traveling wave solutions of (1.12) of the form

$$u(t,x) \ = \ U(x-ct), \qquad I(t,x) \ = \ I(x-ct), \qquad \theta(t,x) \ = \ \Theta(x-ct),$$

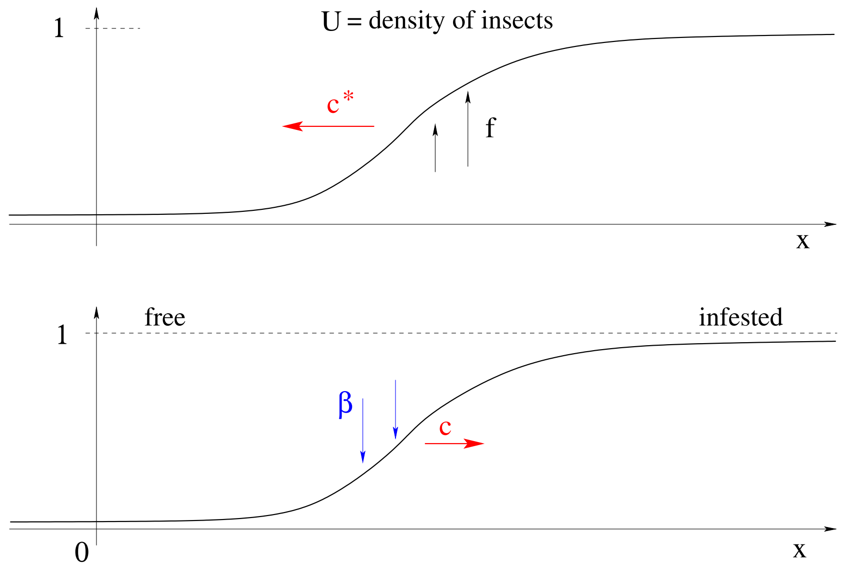


**Figure 5.1.** Traveling profiles for Model 1. Above: without any control, the insect population spreads toward the left, with a speed $c^* < 0$. Below: applying a control, part of the population is removed. This yields a new traveling wave profile, with speed $c > c^*$.

we are led to the system of three ODEs:

$$\begin{cases} U'' + cU' + f(U) - \alpha U &= 0, \\ (IU)'' + c(I'U + IU') + \kappa_2(1-I)U\Theta - \alpha IU - d\,IU &= 0, \\ c\Theta' + \kappa_1(1-\Theta)IU &= 0. \end{cases} \tag{1.14}$$

Two scenarios can be considered. In Section 6 we study (1.14) with asymptotic conditions

$$\begin{cases} U(-\infty) &= 0, \\ I(-\infty) &= 0, \\ \Theta(-\infty) &= 0, \end{cases} \qquad \begin{cases} U(+\infty) &= 1, \\ I(+\infty) &= I^*, \\ \Theta(+\infty) &= 1, \end{cases} \tag{1.15}$$

where $I^* = \kappa_2/(\kappa_2 + d)$. In other words, the density of insects is vanishingly small as $x \to -\infty$, but large for $x \to +\infty$. All trees are healthy in the limit as $x \to -\infty$, while they are increasingly infected as $x \to +\infty$. In this case, controlling the contamination essentially amounts to slowing down the spreading of the insect population (see Fig. 5.1). Observing that the density of infected insects trivially satisfies $IU \leq U$, by a comparison argument we prove that, if the control $\alpha = \alpha(x - ct)$ yields a traveling profile with speed $c < 0$ for the first equation in (1.14), then the same control yields a traveling profile for the entire system (1.14), with the same speed.

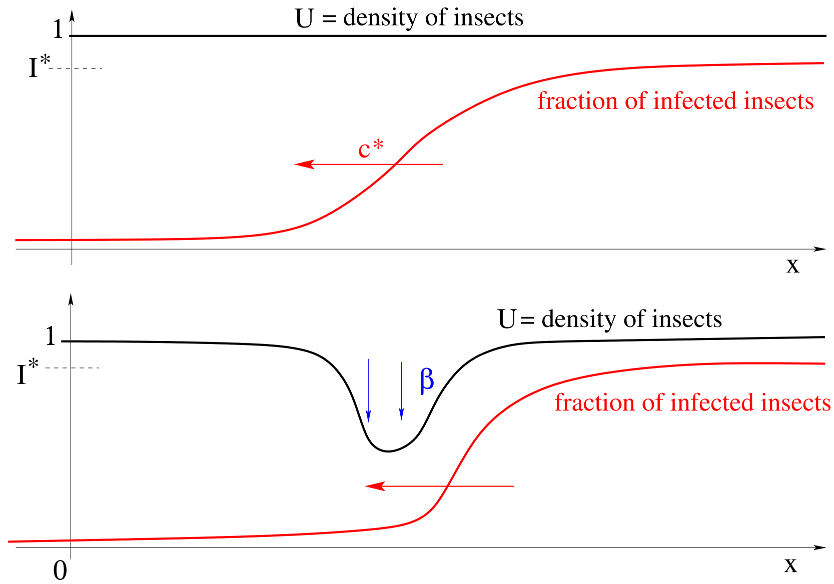


**Figure 5.2.** Traveling profiles for Model 2. Above: without any control, the insect population reaches everywhere its maximum value $U = 1$, while the fraction of infected insects keeps increasing, propagating to the left with speed $c^* < 0$. Below: applying a control, part of the population is removed, in a neighborhood of the interface between healthy and infected individuals. This yields a different traveling wave profile. However, our analysis shows that the propagation speed cannot be affected.

Finally, in Section 7 we consider again the system (1.14), but with asymptotic conditions

$$\begin{cases} U(-\infty) &= 1, \\ I(-\infty) &= 0, \\ \Theta(-\infty) &= 0, \end{cases} \qquad \begin{cases} U(+\infty) &= 1, \\ I(+\infty) &= I^*, \\ \Theta(+\infty) &= 1. \end{cases} \tag{1.16}$$

Notice that here the density of insects is large for $x \to +\infty$ as well as for $x \to -\infty$. Insects and trees are all healthy in the limit as $x \to -\infty$, while they are increasingly infected as $x \to +\infty$.

In the uncontrolled case where $\alpha = 0$, one would have a traveling wave profile where the insect population is everywhere constant: $U(x) = 1$. On the other hand, as shown at the top of Fig. 5.2, the fraction of infected trees and insects keeps increasing. Indeed, the contamination advances toward the left, with speed $c^* < 0$.

An interesting question now arises. Assume that, by applying a control, we locally reduce the population density $U$. As shown at the bottom of Fig. 5.2, this will create a buffer between a region (to the right) where most of the trees and insects are infected, and a region (to the left) where trees and insects are still largely healthy. Can this strategy effectively reduce the speed at which the contamination advances?

Our analysis shows that the answer is negative. Indeed, the speed of a traveling wave must satisfy a constraint stemming from the linearization of the system (1.14) at the asymptotic state $(U, I, \Theta) = (1, 0, 0)$. We now observe that any control $\alpha(\cdot)$ with finite cost must be integrable, hence vanishingly small as $x \to -\infty$. As a consequence, the presence of this additional control cannot remove the above constraint on the wave speed. A precise statement of the result is given in Theorem 7.1.

Traveling profiles for systems of parabolic equations is a classical subject, with an extensive literature. See for example [26, 34, 38, 47] and references therein. Control problems for nonlinear parabolic equations, such as optimal harvesting problems, were studied in [18, 19, 37, 43]. For more accurate models of the spreading and control of invasive populations we refer to [1–3, 22, 44]. Given an effort function $E(c)$, optimization problems for a moving set of the form (1.4) have been recently studied in [13], proving the existence of optimal strategies and establishing necessary conditions for optimality. Control problems for a moving set, describing the support of a population, have also been considered in [16, 20, 21].

# 2 Controlling a traveling front

Given $c \in I\!R$, as in (1.5)–(1.7) we seek a control $\alpha(\cdot)$ with minimum $\mathbf{L}^1$ norm, that produces a traveling wave with speed $c$. Assuming for simplicity that $\sigma = 1$, and using the notation

$$\beta \;=\; f(u,0) - f(u,\alpha), \qquad\qquad f(u) \;=\; f(u,0), \tag{2.1}$$

we can write (1.1) in the form

$$u_t \;=\; \Delta u + f(u) - \beta. \tag{2.2}$$

In addition, we introduce the cost function $L$ implicitly defined by

$$L(u,\beta) \;=\; \alpha. \tag{2.3}$$

The optimization problem for traveling wave profiles can now be stated as follows.

**(OTW)** *Given functions $f(u)$ and $L(u,\beta)$, and a speed $c \in I\!R$, find a nondecreasing profile*

*$U : I\!R \mapsto [0,1]$ and a control function $\beta : I\!R \mapsto I\!R_+$ which minimize the cost*

$$J(U,\beta) \doteq \int_{-\infty}^{+\infty} L\big(U(x),\beta(x)\big)\,dx\,, \tag{2.4}$$

*subject to*

$$U'' + cU' + f(U) - \beta = 0, \qquad U(-\infty) = 0, \quad U(+\infty) = 1. \tag{2.5}$$

**Example 2.1.** When $f(u,\alpha)$ is the function in (1.9), with the notation introduced at (2.1), (2.3) we obtain

$$f(u) = \kappa_3(1-u)u, \qquad \beta = \alpha u, \qquad L(u,\beta) = \alpha = \frac{\beta}{u}\,. \tag{2.6}$$

On the other hand, when $f(u,\alpha)$ is the nonlinear function in (1.10), one obtains

$$f(u) = u(u-u^*)(1-u), \qquad \beta = \left(1 - \frac{1}{1+\alpha}\right) u(u-u^*). \tag{2.7}$$

Notice that in this case the control $\alpha$ will be effective only in the region where $u \in [u^*,1]$, because for $u < u^*$ this control will actually increase the population growth. As Lagrangian function, one should take

$$L(u,\beta) = \begin{cases} 0 & \text{if } \beta = 0, \\ \dfrac{\beta}{(u-u^*)u - \beta} & \text{if } 0 \le \beta < (u-u^*)u\,, \\ +\infty & \text{in all other cases.} \end{cases} \tag{2.8}$$

The optimization problem **(OTW)** will be studied under the following assumptions on the source function $f$ and the cost function $L$.

**(A1)** *$f \in \mathcal{C}^2$, and moreover*

$$f(0) = f(1) = 0, \qquad f'(0) < 0, \qquad f'(1) < 0. \tag{2.9}$$

*In addition, $f$ vanishes at only one intermediate point $u^* \in ]0,1[$, where $f'(u^*) > 0$.*

**(A2)** *For every $u \in ]0,1[$ the map $\beta \mapsto L(u,\beta) \in I\!R_+ \cup \{+\infty\}$ is strictly convex and has* *superlinear growth. More precisely, there exist constants $C_1 > 0$ and $p > 1$ such that*

$$L(u,0) = 0, \qquad L(u,\beta) \geq C_1\,\beta^p \qquad \textit{for all } \beta \geq 0 \textit{ and } u \in [0,1]. \tag{2.10}$$

As a preliminary, we review some basic facts on traveling waves for reaction-diffusion equations of the form

$$u_t = f(u) + u_{xx}\,. \tag{2.11}$$

By definition, a traveling profile for (2.11) with speed $c$ is a solution of the form

$$u(t,x) = U(x - ct). \tag{2.12}$$

This can be found by solving

$$U'' + cU' + f(U) = 0. \tag{2.13}$$

Assuming that $f(0) = f(1) = 0$, we seek a solution $U : I\!R \mapsto [0,1]$ of (2.13) with asymptotic conditions (1.7). Setting $P = U'$, we thus need to find a heteroclinic orbit of the system

$$\begin{cases} U' = P, \\ P' = -\,cP - f(U), \end{cases} \tag{2.14}$$

connecting the equilibrium points $(0,0)$ with $(0,1)$. A phase plane analysis of the system (2.14) yields

**Theorem 2.1.** *Consider the problem (2.13), (1.7), where $f$ satisfies* **(A1)***. Then, there exists a unique $c^* \in I\!R$ and a unique (up to a translation) traveling profile $U$ with speed $c^*$.*

For a detailed proof, see Theorem 4.15 in [26]. It can be shown that the traveling profile $U$ is monotone increasing. A phase portrait of the system (2.14) for various values of $c$ is sketched in Fig. 5.3.

For any given speed $c > c^*$, we seek a control in feedback form $\beta = \beta(u) \geq 0$, with finite cost, that yields a traveling wave with speed $c$. The main result of this section is

**Theorem 2.2.** *Let $f$ satisfy the assumptions* **(A1)** *and let $c^*$ be as in Theorem 2.1. Then, for every $c > c^*$, there exist a bounded function $\beta : ]0,1[ \mapsto I\!R_+$ with compact*

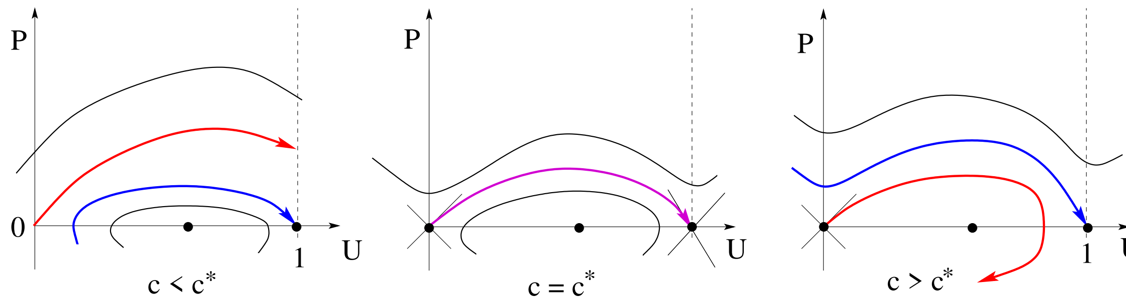


**Figure 5.3.** A traveling profile for (2.11) corresponds to a heteroclinic orbit for the system (2.14), connecting the points $(0,0)$ and $(1,0)$. Under the assumptions **(A2)**, such an orbit exists for one specific value $c = c^*$.

*support, such that the equation*

$$U'' + cU' + f(U) - \beta(U) \;=\; 0, \qquad U(-\infty) = 0, \quad U(+\infty) = 1. \tag{2.15}$$

*admits a solution.*

**Proof. 1.** We will construct a solution of the first order system

$$\begin{cases} U' &= P, \\ P' &= -cP - f(U) + \beta(U), \end{cases} \tag{2.16}$$

with asymptotic conditions

$$(U,P)(-\infty) \;=\; (0,0), \qquad (U,P)(+\infty) \;=\; (1,0), \tag{2.17}$$

for some function $\beta(\cdot)$ of the form

$$\beta(U) \;=\; \begin{cases} \gamma & \text{if } \; u_0 < U < u^*, \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

Here $u^*$ is the zero of $f$ considered in **(A1)**, while $u_0 \in ]0, u^*[$ and $\gamma > 0$ are suitable constants.

**2.** If $\beta \equiv 0$, computing the Jacobian matrix at a point $(U, P)$ one finds

$$A(U,P) \;=\; \begin{pmatrix} 0 & 1 \\ -f'(U) & -c \end{pmatrix}. \tag{2.19}$$

Solving

$$\lambda^2 + c\lambda + f'(U) \;=\; 0,$$

one obtains

$$\lambda \;=\; \frac{-c \pm \sqrt{c^2 - 4f'(U)}}{2}\,. \tag{2.20}$$

We observe that the assumptions (2.9) imply that both $(0,0)$ and $(1,0)$ are saddle points. In particular, the ODE

$$\frac{d}{dU}P(U) \;=\; -\,c - \frac{f(U)}{P} \tag{2.21}$$

has a solution $U \mapsto P^\flat(U)$ through $(0,0)$ with slope

$$\frac{dP^\flat}{dU}(0) \;=\; \frac{-c + \sqrt{c^2 - 4f'(0)}}{2} \;>\; 0\,.$$

It also has a second solution $P^\sharp$ through the point $(1,0)$, with slope

$$\frac{dP^\sharp}{dU}(1) \;=\; \frac{-c - \sqrt{c^2 - 4f'(1)}}{2} \;<\; 0\,.$$

In the special case where $c = c^*$, these solutions exactly match, as in Fig. 5.3, center. On the other hand, when $c > c^*$, as shown in Fig. 5.4 these two solutions satisfy

$$P^\flat(U) \;<\; P^\sharp(U) \qquad\qquad \text{for all } U \in [0, u^*].$$

Now consider the backward Cauchy problem

$$\frac{d}{dU}P(U) \;=\; -\,c - \frac{f(U)}{P} + \gamma, \qquad\qquad U \in [0, u^*], \tag{2.22}$$

with terminal data

$$P(u^*) \;=\; P^\sharp(u^*). \tag{2.23}$$

By choosing $\gamma > 0$ suitably large, the solution to (2.22)-(2.23) will satisfy

$$P(U) < P^\flat(U)$$

at some point $0 < U < u^*$. Calling $u_0 \in [0, u^*]$ the point where $P(u_0) = P^\flat(u_0)$, and defining $\beta(\cdot)$ as in (2.18), we achieve the desired conclusion. □

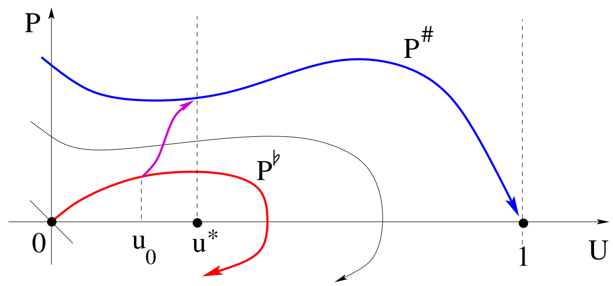


**Figure 5.4.** Trajectories of (2.16) in the case $c > c^*$, $\beta(U) \equiv 0$. Here $P^\flat$ and $P^\sharp$ are the trajectories through $(0,0)$ and through $(1,0)$, respectively.

## 2.1 Existence of a control with finite cost.

According to Theorem 2.2, for every speed $c \geq c^*$ one can find a control $\beta = \beta(U)$ which yields a traveling wave with speed $c$. However, in some cases such as (2.8), one has

$$\begin{cases} L(U, \beta) \ < \ +\infty & \text{if } \beta < \widehat{\beta}(U), \\ L(U, \beta) \ = \ +\infty & \text{if } \beta \geq \widehat{\beta}(U), \end{cases} \tag{2.24}$$

for some function $\widehat{\beta}$. Therefore, some of the traveling waves considered in the above theorem may have infinite cost.

To understand in which cases a traveling wave exists with finite cost, consider any function $\widehat{f}$ that satisfies the assumptions on $f$ stated in **(A1)**, together with

$$f(u) - \widehat{\beta}(u) \ \leq \ \widehat{f}(u) \ \leq \ f(u) \qquad \text{for all } u \in [0,1]. \tag{2.25}$$

Call $\widehat{c}$ the speed of a traveling wave for the corresponding equation

$$u_t \ = \ u_{xx} + \widehat{f}(u).$$

**Theorem 2.3.** *In the above setting, for every speed $c \in [c^*, \widehat{c}[$ there exists a control $\beta = \beta(u)$ with finite cost, such that the equation (2.15) has a solution.*

**Proof. 1.** We can assume $\widehat{c} > c^*$, since otherwise there is nothing to prove. By

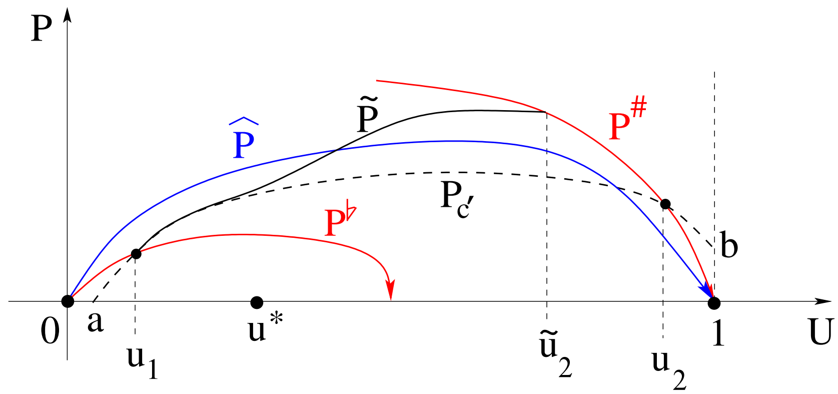


**Figure 5.5.** The trajectories considered in the proof of Theorem 2.3.

assumption, the system

$$\begin{cases} U' &= P, \\ P' &= -\widehat{c}P - \widehat{f}(U) \end{cases} \tag{2.26}$$

has a heteroclinic orbit joining (0,0) with (1,0). With reference to Fig. 5.5, we call $P = \widehat{P}(U)$ the corresponding solution to

$$\frac{dP}{dU} = -\widehat{c} - \frac{\widehat{f}(U)}{P}.$$

In addition, we denote by $P = P^\flat(U)$ and $P = P^\sharp(U)$ the solutions to

$$\frac{dP}{dU} = -c - \frac{f(U)}{P},$$

with boundary data

$$P^\flat(0) = 0 \qquad \text{and} \qquad P^\sharp(1) = 0,$$

respectively.

**2.** Next, choose any speed $c'$ with

$$c^* < c < c' < \widehat{c}.$$

The strict inequality $c' < \widehat{c}$ implies that the system

$$\begin{cases} U' &= P, \\ P' &= -c'P - \widehat{f}(U), \end{cases} \tag{2.27}$$

has an orbit joining a point $(a,0)$ on the positive $U$-axis with a point $(1,b)$, with $a,b>0$. We call $P=P_{c'}(U)$ this profile.

**3.** Still referring to Fig. 5.5, consider the intersection points $0<u_1<u_2<1$, defined by

$$P^\flat(u_1) \;=\; P_{c'}(u_1), \qquad\qquad P^\sharp(u_2) \;=\; P_{c'}(u_2).$$

Define the control

$$\widetilde\beta(U) \;\doteq\; \max\Big\{\widehat\beta(U)-(c'-c)P_{c'}(U)\,,\;0\Big\}. \tag{2.28}$$

Notice that this implies

$$-c'-\frac{f(U)-\widehat\beta(U)}{P_{c'}(U)} \;\leq\; -c-\frac{f(U)-\widetilde\beta(U)}{P_{c'}(U)} \qquad \text{for all } U\in[u_1,u_2].$$

Calling $P=\widetilde P(U)$ the solution to

$$\frac{dP}{dU} \;=\; -c-\frac{f(U)-\widetilde\beta(U)}{P}\,, \qquad\qquad \widetilde P(u_1) \;=\; P_{c'}(u_1),$$

a comparison argument yields

$$\widetilde P(U)\geq P_{c'}(U) \qquad \text{for all } U>u_1\,. \tag{2.29}$$

Therefore, the curve $P=\widetilde P(U)$ will intersect the trajectory $P^\sharp$ at some point $\widetilde u_2\leq u_2$.

**4.** We claim that the concatenation of trajectories

$$P(U) \;=\; \begin{cases} P^\flat(U) & \text{if} \quad U\in[0,u_1],\\ \widetilde P(U) & \text{if} \quad U\in[u_1,\widetilde u_2],\\ P^\sharp(U) & \text{if} \quad U\in[\widetilde u_2,1], \end{cases} \tag{2.30}$$

provides a solution to (2.14) with finite cost.

Indeed, for $U\in[0,u_1]\cup[\widetilde u_2,1]$ the above solution corresponds to a control $\beta=0$, with zero cost.

Furthermore, for $U \in [u_1, \widetilde{u}_2]$, in view of (2.4) the cost is

$$\int_{u_1}^{\widetilde{u}_2} \frac{L\big(U, \widetilde{\beta}(U)\big)}{\widetilde{P}(U)}\, dU. \tag{2.31}$$

By (2.28) we have

$$\widehat{\beta}(U) - \widetilde{\beta}(U) \; > \; \delta \; > \; 0 \qquad\qquad \text{for all } U \in [u_1, \widetilde{u}_2].$$

Hence the numerator $L\big(U, \widetilde{\beta}(U)\big)$ remains uniformly bounded for $U \in [u_1, \widetilde{u}_2]$. Finally, the denominator $\widetilde{P}(U)$ is uniformly positive, because of (2.29). □

# 3 Existence of an optimal strategy

Extending one of the results in [12] to this more general nonlinear setting, we now prove

**Theorem 3.1.** *Let $f, L$ satisfy the assumptions* **(A1)** *and* **(A2)**. *Then, for any wave speed $c > c^*$, if (2.5) has a solution with finite cost $J(U, \beta) < \infty$, then the problem* **(OTW)** *has an optimal solution.*

**Proof. 1.** Following the direct method in the Calculus of Variations, we consider a minimizing sequence $(u_n, \beta_n)_{n \geq 1}$. That is, a sequence of solutions to (2.5) such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} L(u_n, \beta_n)\, dx \; = \; \inf_{(u,\beta)} \int_{-\infty}^{\infty} L(u, \beta)\, dx. \tag{3.1}$$

Here the infimum is taken over all solutions $(u, \beta)$ of (2.5). By a translation in the $x$-variable, we can assume that

$$u_n(0) \; = \; u^* \qquad \text{for all } n \geq 1. \tag{3.2}$$

**2.** By the growth condition (2.10), it follows that the norms $\|\beta_n\|_{\mathbf{L}^p}$ are uniformly bounded.

**3.** In this step we prove that the functions $u_n$ are uniformly Lipschitz continuous. Since all these functions are nondecreasing, it suffices to show that their derivative $p_n(x) = u'(x)$ is bounded above, uniformly for all $x \in I\!R$. Calling $M$ the maximum value of the function $f$ on $[0,1]$, from (2.16) it follows

$$p_n'(x) \;=\; -c\cdot p_n(x) - f(u_n(x)) + \beta_n \;\geq\; -cp_n(x) - M. \tag{3.3}$$

In turn, for any $x_0 \in I\!R$ this yields the lower bound

$$p_n(x) = e^{-c(x-x_0)}p_n(x_0) - \int_{x_0}^{x} e^{-c(x-\xi)}M\,d\xi.$$

Integrating the above equation above from $x_0$ to $x_0+1$, and observing that $u_n(x) \in [0,1]$, in the case $c \neq 0$ we obtain

$$\begin{aligned}
1 \;\geq\; u_n(x_0+1) - u_n(x_0) \;&=\; \int_{x_0}^{x_0+1} p_n(x)\,dx \\
&\geq\; \int_{x_0}^{x_0+1} e^{-c(x-x_0)}p_n(x_0)\,dx - \int_{x_0}^{x_0+1}\int_{x_0}^{x} M\,e^{-c(x-\xi)}d\xi\,dx \\
&=\; \frac{1-e^{-c}}{c}p_n(x_0) - M\cdot\Big(\frac{1}{c} + \frac{e^{-c}-1}{c^2}\Big).
\end{aligned}$$

This yields the bound

$$p_n(x_0) \;\leq\; \frac{c}{1-e^{-c}} + M\Big(\frac{1}{1-e^{-c}} - \frac{1}{c}\Big).$$

Notice that this bound is uniformly valid for every $x_0 \in I\!R$ and $n \geq 1$. We thus conclude that all functions $u_n$ have uniformly bounded derivatives, hence are uniformly Lipschitz continuous.

In the case $c = 0$, the above computation is simply replaced by $1 \geq p_n(x_0) - \frac{M}{2}$, leading to the same conclusion.

**4.** Since all functions $u_n$ are uniformly Lipschitz continuous, by possibly taking a subsequence, we can assume the convergence

$$u_n(x) \;\to\; u(x) \tag{3.4}$$

uniformly for $x$ in bounded sets. Moreover, since the $\mathbf{L}^p$ norms of the functions $\beta_n$ are uniformly bounded, we have the weak convergence $\beta_n \rightharpoonup \beta$ for some function $\beta \in \mathbf{L}^p(I\!R)$.

We can write the differential equation satisfied by $u_n$ in integral form:

$$u_n'(x_2) - u_n'(x_1) + cu_n(x_2) - cu_n(x_1) = \int_{x_1}^{x_2} \big[ - f(u_n) + \beta_n \big]\, dx, \tag{3.5}$$

which is valid for every $x_1 < x_2$. Taking the limit as $n \to \infty$ and recalling the uniform convergence $u_n \to u$ and the weak convergence $\beta_n \rightharpoonup \beta$ in $\mathbf{L}^p$, we obtain

$$u'(x_2) - u'(x_1) + cu(x_2) - cu(x_1) = \int_{x_1}^{x_2} \big[ - f(u) + \beta \big]\, dx. \tag{3.6}$$

Therefore, $u \in W^{2,p}_{loc}(I\!R)$, and the ODE in (2.5) is satisfied.

**5.** In the next two steps, using the assumptions **(A2)** on $L$, we prove the lower semicontinuity relation:

$$\int_{-\infty}^{+\infty} L\big(u(x), \beta(x)\big)\, dx \ \leq\ \liminf_{n\to\infty} \int_{-\infty}^{+\infty} L\big(u_n(x), \beta_n(x)\big)\, dx. \tag{3.7}$$

Since $L \geq 0$, we have

$$\int_{-\infty}^{+\infty} L\big(u(x), \beta(x)\big)\, dx \ =\ \lim_{R\to+\infty} \int_{-R}^{R} L\big(u(x), \beta(x)\big)\, dx\,. \tag{3.8}$$

Next, for every $m \geq 1$ consider the function

$$L^{(m)}(u, \beta) \ \doteq\ \min_{z\in[0,\beta]} \Big\{ L(u, z) + m(\beta - z) \Big\}. \tag{3.9}$$

In view of **(A2)**, $L^{(m)}$ is continuous w.r.t. both variables $u, \beta$, and Lipschitz continuous with constant $m$ in the variable $\beta$, uniformly for every $u$. Indeed, $L^{(m)}$ is the largest function $\leq L$ with these properties. Since

$$L(u, \beta) \ =\ \lim_{m\to\infty} L^{(m)}(u, \beta),$$

we have

$$\int_{-R}^{R} L(u(x), \beta(x)\big)\, dx \ =\ \lim_{m\to\infty} \int_{-R}^{R} L^{(m)}\big(u(x), \beta(x)\big)\, dx. \tag{3.10}$$

To prove (3.7), it thus suffices to show that

$$\int_{-R}^{R} L^{(m)}\big(u(x), \beta(x)\big)\, dx \ \leq\ \liminf_{n\to\infty} \int_{-R}^{R} L^{(m)}\big(u_n(x), \beta_n(x)\big)\, dx, \tag{3.11}$$

for any given $R, m \geq 1$.

**6.** By the convexity of the maps $\beta \mapsto L^{(m)}\big(u(x),\beta\big)$ it follows

$$\begin{aligned}\int_{-R}^{R} & L^{(m)}\big(u_n(x),\beta_n(x)\big)\,dx \\ & \geq \int_{-R}^{R} L^{(m)}\big(u_n(x),\beta(x)\big)\,dx + \int_{-R}^{R} L^{(m)}_\beta\big(u_n(x),\beta(x)\big)\cdot\big(\beta_n(x)-\beta(x)\big)\,dx.\end{aligned} \tag{3.12}$$

Thanks to the uniform bound

$$\big|L^{(m)}(u,\beta)\big| \;\leq\; m\beta \qquad\qquad \text{for all } u\in[0,1],\quad \beta\geq 0,$$

and the uniform convergence $u_n \to u$, the first integral on the right hand side of (3.12) satisfies

$$\lim_{n\to\infty}\int_{-R}^{R} L^{(m)}\big(u_n(x),\beta(x)\big)\,dx \;=\; \int_{-R}^{R} L^{(m)}\big(u(x),\beta(x)\big)\,dx\,. \tag{3.13}$$

Using the uniform convergence $u_n \to u$, the weak convergence $\beta_n \rightharpoonup \beta$, and observing that $\|L_\beta\|_{\mathbf{L}^\infty} \leq m$, we conclude that the second integral on the right hand side of (3.12) satisfies

$$\lim_{n\to\infty}\int_{-R}^{R} L^{(m)}_\beta\big(u_n(x),\beta(x)\big)\cdot\big(\beta_n(x)-\beta(x)\big)\,dx \;=\; 0. \tag{3.14}$$

Together, (3.13)-(3.14) imply (3.11), and hence (3.7).

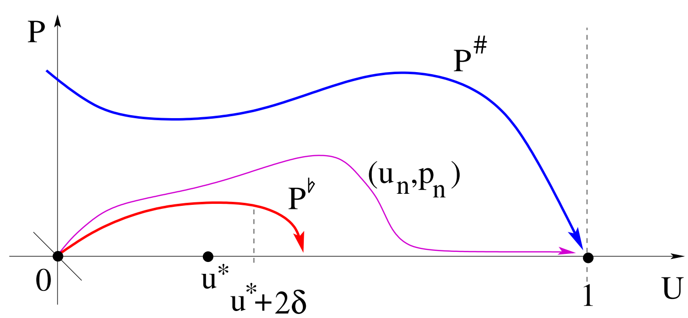


**Figure 5.6.** All trajectories $x \mapsto \big(u_n(x), p_n(x)\big)$ take values in the region between $P^\flat$ and $P^\sharp$.

**7.** In this step we complete the proof by establishing the limits

$$u^- \;\doteq\; \lim_{x\to-\infty} u(x) \;=\; 0, \qquad\qquad u^+ \;\doteq\; \lim_{x\to+\infty} u(x) \;=\; 1. \tag{3.15}$$

Notice that the fact that every $u_n$ satisfies the above limits, together with the convergence

$u_n \to u$ uniformly on bounded sets, does not suffice to conclude (3.15). From the monotonicity of $u$ it only follows that the limits $u^-, u^+$ exists, with $0 \leq u^- \leq u^+ \leq 1$. To achieve (3.15), a more careful argument is needed.

With reference to Fig. 5.6, we observe that any trajectory $x \mapsto \big(u_n(x), p_n(x)\big)$ is contained in the region between $P^\flat$ and $P^\sharp$, i.e. the unstable manifold through $(0,0)$ and the stable manifold through $(1,0)$, respectively. In particular, for some constants $C, \delta > 0$ independent of $n$, we have the implication

$$u_n(x) \in [0, u^* + 2\delta] \quad \Longrightarrow \quad u_n'(x) \;=\; p_n(x) \;\geq\; C u_n(x). \tag{3.16}$$

This immediately implies

$$u^- \;=\; 0, \qquad\qquad u^+ \;\geq\; u^* + 2\delta. \tag{3.17}$$

It remains to prove that $u^+ = 1$. We prove this claim by contradiction. By (3.17) we can find $R > 0$ large enough so that $u(x) > u^* + \delta$ for all $x > R$. Integrating the differential equation in (2.5) on the interval $[R, 2R]$, we obtain

$$u'(2R) - u'(R) + c\big(u(2R) - u(R)\big) + \int_R^{2R} f(u(x))dx \;=\; \int_R^{2R} \beta(u(x))dx. \tag{3.18}$$

Since $u$ is Lipschitz continuous, the quantities

$$u'(2R), \quad u'(R), \quad u(2R), \quad u(R),$$

are all uniformly bounded. On the other hand,

$$\int_R^{2R} f(u(x))dx \;\geq\; R \cdot \min_{u \in [u^*+\delta,\, u^+]} f(u). \tag{3.19}$$

If $u^+ < 1$, then the minimum in (3.19) is strictly positive. Hence the right hand side of (3.19) approaches infinity as $R \to +\infty$. In particular, choosing $R$ large enough, from (3.18) we obtain

$$\int_R^{2R} \beta(u(x))dx \;\geq\; 1.$$

Repeating the same argument on the intervals $\big[kR, (k+1)R\big]$, we again obtain

$$\int_{kR}^{(k+1)R} \beta(u(x))dx \;\geq\; 1, \qquad\qquad k = 1, 2, 3, \ldots \tag{3.20}$$

By the assumption (2.10), in view of (3.20), this implies

$$
\begin{aligned}
J(u,\beta) \;&\geq\; \int_R^{+\infty} L\big(u(x),\beta(x)\big)\,dx \;=\; \sum_{k\geq 1}\int_{kR}^{(k+1)R} L\big(u(x),\beta(x)\big)\,dx \\
&\geq\; \sum_{k\geq 1}\int_{kR}^{(k+1)R} C_1\beta^p(x)\,dx \;\geq\; \sum_{k\geq 1}\int_{kR}^{(k+1)R} C_1\left(\frac{1}{R}\right)^p dx \;=\; \sum_{k\geq 1} C_1 R^{1-p} \;=\; +\infty.
\end{aligned}
\tag{3.21}
$$

We thus obtain a contradiction with the previous step, where we proved that the cost $J(u,\beta)$ is finite. This completes the proof. □

# 4 Necessary conditions for optimality

Given a speed $c > c^*$, assume that $(U,\beta)$ yield an optimally controlled traveling wave profile, as in Theorem 3.1. We seek necessary conditions to determine this profile.

In terms of the $U$-$P$ coordinates, as in (2.16), this means that the control $\beta = \beta(U) \geq 0$ minimizes the cost functional

$$
J(\beta) \;=\; \int_0^1 \frac{L\big(U,\beta(U)\big)}{P(U)}\,dU, \tag{4.1}
$$

subject to

$$
\frac{dP}{dU} \;=\; -\,c + \frac{\beta - f(U)}{P}, \qquad P(0) = P(1) = 0. \tag{4.2}
$$

To apply the Pontryagin Maximum Principle [15, 17], we first compute

$$
\frac{\partial}{\partial P}\left(\frac{L(U,\beta)}{P}\right) \;=\; -\,\frac{L(U,\beta)}{P^2}, \qquad\qquad \frac{\partial}{\partial P}\left(-c + \frac{\beta - f(U)}{P}\right) \;=\; -\,\frac{\beta - f(U)}{P^2}.
$$

The PMP now yields the existence of an adjoint variable $Y(\cdot)$ satisfying the linear equation

$$
\frac{dY}{dU} \;=\; \frac{\beta(U) - f(U)}{P^2(U)}\,Y + \frac{L\big(U,\beta(U)\big)}{P^2(U)}\,, \tag{4.3}
$$

such that, at a.e. $U \in [0,1]$, the following optimality condition holds:

$$
\beta(U) \;=\; \operatorname*{argmin}_{\beta\geq 0}\left\{\left(-c + \frac{\beta - f(U)}{P(U)}\right) Y(U) + \frac{L\big(U,\beta\big)}{P(U)}\right\}. \tag{4.4}
$$

Equivalently,

$$\beta(U) \;=\; \operatorname*{argmin}_{\beta\geq 0}\Big\{\beta Y(U)+L(U,\beta)\Big\}. \tag{4.5}$$

Note that, in the region where $\beta(U)>0$, by (4.5) we must have

$$Y(U)+L_\beta\big(U,\beta(U)\big) \;=\; 0. \tag{4.6}$$

Differentiating w.r.t. $U$ and using (4.3)-(4.6), we obtain

$$\begin{aligned}\frac{d}{dU}L_\beta\big(U,\beta(U)\big)+\frac{d}{dU}Y(U) \;&=\; \frac{d}{dU}L_\beta\big(U,\beta(U)\big)+\frac{\beta(U)-f(U)}{P^2(U)}\,Y+\frac{L\big(U,\beta(U)\big)}{P^2(U)}\\ &=\; \frac{d}{dU}L_\beta\big(U,\beta(U)\big)-\frac{\beta(U)-f(U)}{P^2(U)}\,L_\beta\big(U,\beta(U)\big)+\frac{L\big(U,\beta(U)\big)}{P^2(U)} \;=\; 0.\end{aligned} \tag{4.7}$$

In most cases, the control $\beta$ will be active only on some interval $]u_1,u_2[$ , so that

$$\left\{\begin{array}{ll}\beta(U)>0 & \text{if}\;\; U\in\,]u_1,u_2[\,,\\[2mm] \beta(U)=0 & \text{if}\;\; U\in[0,u_1]\cup[u_2,1]\,.\end{array}\right. \tag{4.8}$$

From (4.5) and the strict convexity of $L(u,\cdot)$ it now follows

$$\lim_{U\to u_1+}\beta(U) \;=\; \lim_{U\to u_2-}\beta(U) \;=\; 0.$$

The optimal solution can thus be obtained by solving the ODE in (4.7) over an interval $[u_1,u_2]$, whose endpoints are determined by the two additional boundary conditions

$$Y(u_1)+L_\beta(u_1,0) \;=\; Y(u_2)+L_\beta(u_2,0) \;=\; 0. \tag{4.9}$$

## 4.1 Numerical computation of optimally controlled traveling profiles.

In a typical application, the optimal traveling profile with a given speed $c>c^*$ can be computed as follows.

STEP 1: Observing that both $(0,0)$ and $(1,0)$ are both saddle points for the system (2.14), compute the unstable manifold $P=P^\flat(U)$ through $(0,0)$, and the stable manifold $P=P^\sharp(U)$ through $(1,0)$, as shown in Fig. 5.7.

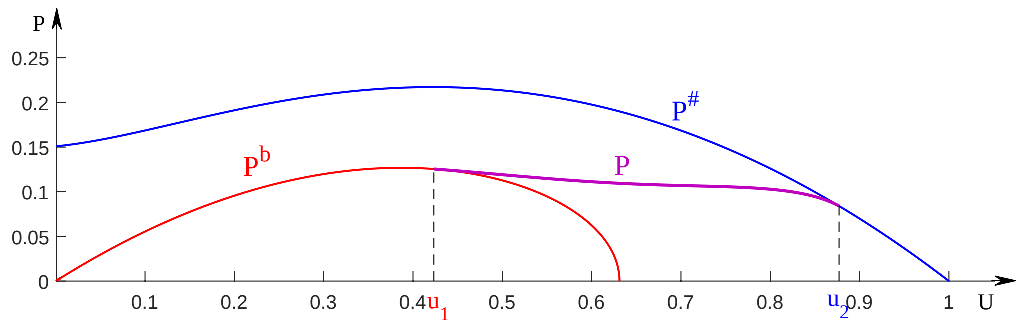


**Figure 5.7.** The optimal traveling profile for the given speed $c = -0.1$, in the $U, P$ coordinates.

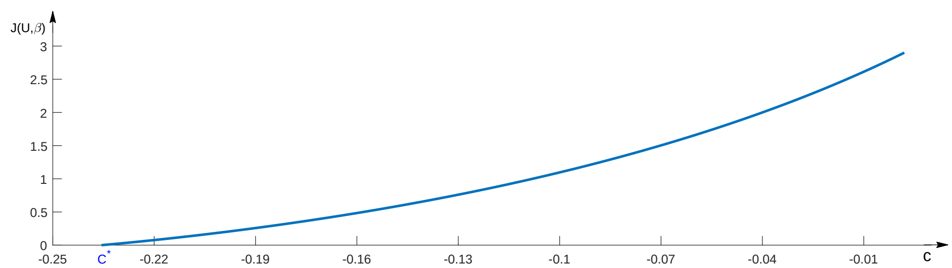


**Figure 5.8.** The minimum cost $E(c)$, depending on the wave speed $c \geq c^*$.

STEP 2: Determine the interval $[u_1, u_2]$ and the portion of the optimal trajectory $U \mapsto P(U)$ for $U \in [u_1, u_2]$ by solving the system of two equations

$$\begin{cases} \dfrac{dP}{dU} & = & -c + \dfrac{\beta - f(U)}{P}\,, \\ \dfrac{d\beta}{dU} & = & \dfrac{1}{L_{\beta\beta}(U,\beta)} \cdot \left[ \dfrac{\beta - f(U)}{P^2(U)}\, L_\beta(U,\beta) - \dfrac{L(U,\beta)}{P^2(U)} - L_{U\beta}(U,\beta) \right], \end{cases} \tag{4.10}$$

with the four boundary conditions

$$\begin{cases} P(u_1) & = & P^\flat(u_1), \\ P(u_2) & = & P^\sharp(u_2), \end{cases} \qquad \begin{cases} \beta(u_1) & = & 0, \\ \beta(u_2) & = & 0. \end{cases} \tag{4.11}$$

Note that the solution to a system of two first order ODEs is determined by two boundary conditions. Here the two additional conditions $\beta(u_1) = \beta(u_2) = 0$ are needed to determine the endpoints of the interval $[u_1, u_2]$.

**Example 4.1.** For sake of illustration, we consider here the optimization problem for a traveling profile, choosing $f(u)$ and $L(u, \beta)$ as in (2.7)-(2.8), with $u^* = 1/3$. If no control is present, a numerical simulation shows that the speed of the traveling profile is $c^* \approx -0.2356$.

For various speeds $c > c^*$, we seek the minimum cost of a control that produces a traveling profile with speed $c$. This is achieved following the above steps 1 and 2. To achieve Step 2, for a given $u_1$, consider the solution $U \mapsto \big(P(U), \beta(U)\big)$ of (4.10) with initial data

$$P(u_1) \;=\; P^\flat(u_1), \qquad\qquad \beta(u_1) \;=\; 0. \tag{4.12}$$

This solution is prolonged until $P(U) = P^\sharp(U)$. More precisely, let $u_2 > u_1$ be the first point such that

$$P(u_2) \;=\; P^\sharp(u_2).$$

The above construction yields a map

$$u_1 \;\mapsto\; \phi(u_1) \;\doteq\; \beta(u_2).$$

By a shooting method, we determine $u_1 \in [u^*, 1]$ such that $\phi(u_1) = 0$. This yields the desired solution.

In the case where the wave speed is $c = -0.1$, a numerical simulation of the optimal traveling profile, is shown in Fig. 5.7. The minimum cost, for increasing values of the wave speed $c \geq c^*$, is shown in Fig. 5.8.

# 5 Controlled traveling profiles for Model 1

In this section we consider controlled traveling profiles for the system (1.8), say with

$$u(t,x) \;=\; U(x-ct), \qquad \theta(t,x) \;=\; \Theta(x-ct), \qquad \alpha = \alpha(x-ct).$$

Since (1.8) is in triangular form, for any speed $c > c^*$ the existence of an optimal traveling profile $U$ for the first equation has already been proved in Theorem 3.1. The next result shows that, if $c < 0$, then the second equation in (1.8) also admits a traveling profile with speed $c$.

We recall that the functions $x \mapsto \big(U(x), \Theta(x)\big)$ should satisfy

$$U'' + cU' + f(U, \alpha(U)) \;=\; 0, \qquad\qquad U(-\infty) = 0, \quad U(+\infty) = 1, \tag{5.1}$$

$$c\Theta' + \kappa_1 U(1-\Theta) \;=\; 0, \qquad\qquad \Theta(-\infty) = 0, \qquad \Theta(+\infty) = 1. \tag{5.2}$$

A solution $\Theta$ of (5.2) will be constructed assuming the integrability condition

$$\int_{-\infty}^{0} U(x)\,dx \;<\; +\infty. \tag{5.3}$$

**Theorem 5.1.** *Let $U : \mathbb{R} \mapsto [0,1]$ be an increasing solution to (5.1), such that (5.3) holds. Then a solution to (5.2) exists if and only if $c < 0$.*

**Proof.** To construct the function $\Theta$ in (5.2), we begin by solving

$$\frac{-\Theta'}{1-\Theta} \;=\; \frac{\kappa_1}{c}U, \qquad\qquad \Theta(-\infty) = 0.$$

An integration yields

$$\ln\big(1-\Theta(x)\big) \;=\; \frac{\kappa_1}{c}\int_{-\infty}^{x} U(y)\,dy,$$

$$\Theta(x) \;=\; 1 - \exp\left\{\frac{\kappa_1}{c}\int_{-\infty}^{x} U(y)\,dy\right\}.$$

Since $\kappa_1 > 0$, if $c < 0$ then

$$\lim_{x\to-\infty} \Theta(x) \;=\; 0, \qquad\qquad \lim_{x\to+\infty} \Theta(x) \;=\; 1.$$

On the other hand, if $c > 0$ then

$$\lim_{x\to+\infty} \ln\big(1-\Theta(x)\big) \;=\; \lim_{x\to+\infty} \frac{\kappa_1}{c}\int_{-\infty}^{x} U(y)\,dy \;=\; +\infty.$$

This contradicts the condition $\Theta(x) \in [0,1]$. Hence, no such traveling profile exists. □

A key assumption of the previous theorem was the boundedness of the integral in (5.3). We now show that this is always satisfied in the setting considered in Theorems 2.2 and 3.1.

**Lemma 5.1.** *Assume that $f : [0,1] \mapsto \mathbb{R}$ satisfies the assumptions* **(A1)**. *Then for any $c > c^*$ and any solution $U$ of (2.15) with $\beta(U) \geq 0$, the integrability condition (5.3) holds.*

**Proof.** As remarked in (3.16), under the assumptions **(A1)** any traveling wave solution must satisfy

$$U'(x) \;\geq\; C\,U(x) \qquad \text{whenever} \qquad U(x) \in [0, u^* + 2\delta], \tag{5.4}$$

for some positive constants $C, \delta > 0$. Calling $x^* \in I\!R$ the point where $U(x^*) = u^*$, from the differential inequality (5.4) we deduce

$$U(x) \;\leq\; e^{-C(x^*-x)}u^* \qquad\qquad \text{for all } x \in ]-\infty, x^*]. \tag{5.5}$$

This implies that, as $x \to -\infty$, the function $U(x)$ converges to zero exponentially fast. Hence the integrability condition (5.3) holds. □

# 6 Traveling profiles for Model 2

In this section we begin a study of the system (1.12), assuming that the function $f$ satisfies the assumptions in **(A1)** together with

$$f(u) \;\geq\; -\,du \qquad\qquad u \in [0,1]. \tag{6.1}$$

Introducing the variable $v = Iu$ = density of infected insects, we thus consider the system

$$\begin{cases} u_t \;=\; u_{xx} + f(u) - \alpha u\,, \\[2mm] v_t \;=\; v_{xx} + \kappa_2(u-v)\theta - \alpha v - d\,v, \\[2mm] \theta_t \;=\; \kappa_1(1-\theta)\,v\,. \end{cases} \tag{6.2}$$

For future use, we recall a basic definition [26, 47].

**Definition 6.1.** *A $\mathcal{C}^1$ function $F : I\!R^m \mapsto I\!R^m$, say $F(w) = \big(F_1(w), \ldots, F_m(w)\big)$ is* **quasi-monotone** *on a convex domain $\mathcal{D} \subseteq I\!R^m$ if*

$$\frac{\partial F_i}{\partial w_j}(w) \;\geq\; 0 \qquad\qquad \textit{for all } i \neq j, \quad w = (w_1, \ldots, w_m) \in \mathcal{D}.$$

Motivated by (6.2), we observe that the map $F : I\!R^3 \mapsto I\!R^3$ defined by

$$F(u,v,\theta) \;=\; \Big(f(u) - \alpha(x)u\,,\; \kappa_2(u-v)\theta - \alpha(x)v - d\,v\,,\; \kappa_1(1-\theta)v\Big), \tag{6.3}$$

is quasi-monotone on the domain

$$\mathcal{D} \;\doteq\; \Big\{(u,v,\theta)\,; \qquad 0 \leq v \leq u \leq 1, \quad \theta \in [0,1]\Big\}. \tag{6.4}$$

By a comparison argument we obtain

**Lemma 6.1.** *Let $f$ satisfy the assumptions* **(A1)** *together with the inequality (6.1). Then the domain $\mathcal{D}$ is positively invariant for the system (6.2). Namely, for any control function $\alpha = \alpha(t,x) \geq 0$, let $(u,v,\theta)$ be a solution to (6.2) such that, at time $t = 0$, $(u,v,\theta)(0,x) \in \mathcal{D}$ for all $x \in I\!R$. Then $(u,v,\theta)(t,x) \in \mathcal{D}$ for all $x \in I\!R$ and $t \geq 0$.*

**Proof.** We first observe that the triples

$$(u^-, v^-, \theta^-)(t,x) \;=\; (0,0,0), \qquad\qquad (u^+, v^+, \theta^+)(t,x) \;=\; (1,1,1),$$

provide a subsolution and a supersolution to the system (6.2), respectively. This implies that the three functions $u, v, \theta$ all take values within the interval $[0,1]$.

Next, let $(u,v,\theta)$ be any solution. Then the function $w = u - v$ satisfies

$$w_t \;=\; u_t - v_t \;=\; \Delta w + \big[f(u) + dv\big] - \kappa_2 \theta w - \alpha w \;\geq\; \Delta w - \big[\kappa_2\theta + \alpha + d\big] w. \tag{6.5}$$

Indeed, by (6.1) it follows

$$f(u) + d\,v \;=\; f(u) + du - [du - dv] \;\geq\; -\,d(u-v).$$

From (6.5) we conclude that, if $w(0,x) \geq 0$ for all $x \in I\!R$, then also $w(t,x) \geq 0$ for all $t \geq 0, x \in I\!R$. □

In this section we focus the analysis on

**CASE 1:** *The density of insects is large for $x \to +\infty$, but vanishingly small as $x \to -\infty$. All trees and insects are healthy in the limit as $x \to -\infty$, while they are increasingly infected as $x \to +\infty$.*

We seek traveling wave solutions of (6.2), having the form

$$u(t,x) \;=\; U(x-ct), \qquad v(t,x) \;=\; V(x-ct), \qquad \theta(t,x) \;=\; \Theta(x-ct), \qquad \alpha = \alpha(x-ct). \tag{6.6}$$

This leads to the system

$$\left\{\begin{array}{rcl} U'' + cU' + f(U) - \alpha(x)\,U &=& 0\,,\\[2mm] V'' + cV' + \kappa_2(U-V)\Theta - d\,V - \alpha(x)\,V &=& 0,\\[2mm] c\,\Theta' + \kappa_1 V(1-\Theta) &=& 0\,,\end{array}\right. \tag{6.7}$$

with asymptotic conditions

$$\begin{cases} U(-\infty) &= 0\,, \\ V(-\infty) &= 0, \\ \Theta(-\infty) &= 0\,. \end{cases} \qquad\qquad \begin{cases} U(+\infty) &= 1\,, \\ V(+\infty) &= V^*, \\ \Theta(+\infty) &= 1\,. \end{cases} \tag{6.8}$$

Here $V^* = \kappa_2/(\kappa_2 + d)$.

Assuming that the function $f$ satisfies **(A1)**, there exists a unique speed $c^* < 0$ such that the uncontrolled scalar equation

$$u_t \;=\; u_{xx} + f(u)$$

admits a traveling wave solution with speed $c^*$. Moreover, by the analysis in [12], for every $c > c^*$, there exists a non-negative control function $\alpha(\cdot)$ with minimum $\mathbf{L}^1$ norm, such that the first equation in (6.2) admits a traveling profile with speed $c$.

**Definition 6.2.** *Let $\mathcal{D}$ be the domain at (6.4). Given an integrable function $\alpha \in \mathbf{L}^1(I\!R)$ and a constant $c < 0$, we say that the triple of functions $(U, V, \Theta) : I\!R \mapsto \mathcal{D}$ is a supersolution (respectively, a subsolution) of the system (6.7) if*

*(i) The functions $U, V$ are in $W^{2,1}(I\!R)$, i.e., they have integrable second derivatives.*

*(ii) The function $\Theta$ is absolutely continuous.*

*(iii) The left hand sides of (6.7) are $\leq 0$ (respectively: $\geq 0$) at a.e. point $x \in I\!R$.*

Starting with a solution to the first equation, constructing a supersolution to the whole system (6.7) is an easy matter.

**Lemma 6.2.** *Let $u = U(x)$ be a stationary solution for the first equation in (6.7), for some control $\alpha \in \mathbf{L}^1(I\!R)$ and some speed $c < 0$. Define*

$$\overline{\Theta}(x) \;=\; 1 - \exp\left\{\frac{\kappa_1}{c}\int_{-\infty}^{x} U(y)\,dy\right\}.$$

*Then the triple of functions*

$$(u^+, v^+, \theta^+)(t,x) \;\doteq\; \Big(U(x),\, \min\big\{U(x), V^*\big\},\, \overline{\Theta}(x)\Big) \tag{6.9}$$

*provides an upper solution to the system (6.7).*

**Proof.** We need to show that, by inserting the functions $u^+, v^+, \theta^+$ in (6.7), the left hand sides are all $\leq 0$. A direct computation yields

$$c\theta^+_x + \kappa_1(1-\theta^+)v^+ \;=\; \kappa_1(1-\overline{\Theta})(v^+ - U) \;\leq\; 0.$$

Moreover, at points where $v^+(x) = U(x)$, by (6.1), one has

$$(v^+)''+c(v^+)'+\kappa_2(U-v^+)\theta^+-\alpha(x)v^+-dv \;=\; U''+cU'-\alpha(x)U-dU \;=\; -f(U)-dU \;\leq\; 0.$$

Finally, at points where $U(x) \geq V^*$ and hence $v^+(x) = V^*$ one has

$$\begin{aligned}(v^+)'' + c(v^+)' + \kappa_2(U-v^+)\theta^+ - \alpha(x)v^+ - dv^+ \;&=\; \kappa_2(U-V^*)\overline{\Theta} - \alpha(x)V^* - dV^* \\ \leq\; \kappa_2(1-V^*) - dV^* \;&=\; 0,\end{aligned}$$

completing the proof. □

In the remainder of this section we will show that the same control $\alpha(\cdot)$ yields a traveling profile for the system (6.2), i.e. a stationary solution to (6.7) with asymptotic conditions (6.8) as $x \to \pm\infty$. In view of Lemma 6.2, relying on the monotonicity property of the system (6.7), to prove the result it remains to construct a subsolution $(u^-, v^-, \theta^-)$, with the same asymptotic conditions (6.8).

Introducing the variable $W = V'$, the last two equations in (6.7) are equivalent to the first order system

$$\begin{cases} V' \;=\; W, \\ W' \;=\; -\,cW - \kappa_2(U-V)\Theta + \alpha V + d\,V, \\ \Theta' \;=\; -\dfrac{\kappa_1}{c}\,V(1-\Theta)\,, \end{cases} \tag{6.10}$$

Linearizing (6.10) at the point $(U, V, W, \Theta) = (1, 0, 0, 0)$ we obtain

$$\begin{pmatrix} V' \\ W' \\ \Theta' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ d & -c & -\kappa_2 \\ -\kappa_1/c & 0 & 0 \end{pmatrix} \begin{pmatrix} V \\ W \\ \Theta \end{pmatrix} + G(V,\Theta) + H(V,\Theta,x), \tag{6.11}$$

where

$$G(V,\Theta) \;=\; \begin{pmatrix} 0 \\ \kappa_2 V\Theta \\ \kappa_1 V\Theta/c \end{pmatrix}, \qquad H(V,\Theta,x) \;=\; \begin{pmatrix} 0 \\ \kappa_2\Theta\big(1-U(x)\big)+V\alpha(x) \\ 0 \end{pmatrix}. \tag{6.12}$$

The eigenvalues of the $3\times 3$ matrix in (6.11) are the roots of the characteristic polynomial

$$p(\lambda) \;=\; \det\begin{pmatrix} \lambda & -1 & 0 \\ -d & c+\lambda & \kappa_2 \\ \kappa_1/c & 0 & \lambda \end{pmatrix} \;=\; \lambda^3 + c\lambda^2 - d\lambda - \frac{\kappa_1\kappa_2}{c}. \tag{6.13}$$

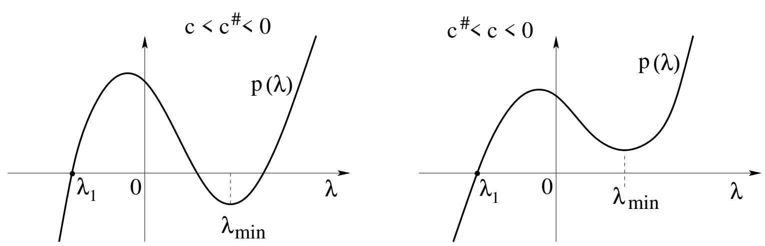


**Figure 5.9.** Left: the characteristic polynomial (6.13) in the case $c < c^\sharp$. Right: the case $c^\sharp < c < 0$.

Since

$$p(0) \;=\; -\frac{\kappa_1\kappa_2}{c} \;>\; 0,$$

as shown in Fig. 5.9 the polynomial $p(\lambda)$ will have two positive real roots if and only if $p(\lambda_{\min}) \le 0$, where

$$\lambda_{\min} \;=\; \frac{-c+\sqrt{c^2+3d}}{3}$$

is the positive zero of $p'(\lambda)$. That is

$$p(\lambda_{\min}) \;=\; -\frac{\kappa_1\kappa_2}{c} + \frac{cd}{3} + \frac{2}{27}c^3 - \frac{2}{27}(c^2+3d)^{\frac{3}{2}} \;\le\; 0. \tag{6.14}$$

Differentiating the left hand side of (6.14) w.r.t. $c$, we obtain

$$\frac{\kappa_1\kappa_2}{c^2} \;+\; \frac{d}{3} \;+\; \frac{2}{9}c^2 \;-\; \frac{2}{9}(c^2+3d)^{\frac{1}{2}}c \;>\; 0 \qquad \text{for all} \quad c \;<\; 0.$$

Therefore, if $c^\sharp < 0$ is a value for which (6.14) is satisfied as an equality, than any value

$c \leq c^\sharp$ will satisfy the inequality (6.14).

To explicitly determine the value $c = c^\sharp$ for which the expression in (6.14) vanishes, we move the last term to the right side, square both sides and simplify the equation to get

$$\frac{c^2 d^2}{27} + \frac{4d^3}{27} + \frac{4}{27}\kappa_1\,\kappa_2\,c^2 + \frac{2\,\kappa_1\,\kappa_2\,d}{3} - \frac{\kappa_1^2\kappa_2^2}{c^2} \;=\; 0.$$

We solve the above equation for $c^2$, and take the negative square root. This yields

$$c^\sharp \;\doteq\; -\left(\frac{-2d^3 - 9\kappa_1\kappa_2 d + 2\big(d^2 + 3\kappa_1\kappa_2\big)^{3/2}}{d^2 + 4\kappa_1\kappa_2}\right)^{1/2}. \tag{6.15}$$

From the above analysis it follows

**Lemma 6.3.** *For $c^\sharp < c < 0$, the $3 \times 3$ Jacobian matrix at (6.11) has one negative eigenvalue and two complex conjugate eigenvalues, with positive real part.*

Calling

$$\lambda_1 \,<\, 0, \qquad \lambda_2 \,=\, a + ib, \qquad \lambda_3 \,=\, a - ib, \tag{6.16}$$

the three eigenvalues, with $a, b > 0$, we obtain three corresponding eigenvectors:

$$\mathbf{v}_i \;=\; \begin{pmatrix} 1 \\ \lambda_i \\ -\kappa_1/c\lambda_i \end{pmatrix}, \qquad\qquad i = 1, 2, 3. \tag{6.17}$$

Notice that $\mathbf{v}_1$ has real entries, while $\mathbf{v}_2, \mathbf{v}_3$ are complex valued. Taking the real and imaginary parts, we obtain the two vectors

$$\mathbf{w}_2 \;=\; \begin{pmatrix} 1 \\ a \\ -\dfrac{\kappa_1 a}{c(a^2+b^2)} \end{pmatrix}, \qquad\qquad \mathbf{w}_3 \;=\; \begin{pmatrix} 0 \\ b \\ \dfrac{\kappa_1 b}{c(a^2+b^2)} \end{pmatrix}, \tag{6.18}$$

which satisfy

$$\Sigma \;\doteq\; \mathrm{span}\{\mathbf{w}_2, \mathbf{w}_3\} \;=\; \mathrm{span}\{\mathbf{v}_2, \mathbf{v}_3\}. \tag{6.19}$$

In particular, a direct computation shows that the linear system

$$\begin{pmatrix} V' \\ W' \\ \Theta' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ d & -c & -\kappa_2 \\ -\kappa_1/c & 0 & 0 \end{pmatrix} \begin{pmatrix} V \\ W \\ \Theta \end{pmatrix} \tag{6.20}$$

admits the solution

$$\begin{pmatrix} V \\ W \\ \Theta \end{pmatrix}(x) = Ae^{ax} \begin{pmatrix} \sin bx \\ b\cos bx + a\sin bx \\ \frac{\kappa_1}{c(a^2+b^2)}\big(b\cos bx - a\sin bx\big) \end{pmatrix}. \tag{6.21}$$

We can now prove the main result of this section, on the existence of controlled traveling waves for Model 2.

**Theorem 6.1.** *Let $f$ satisfy the assumptions* **(A1)** *together with (6.1). Let $c^\sharp < c < 0$ and let $U : I\!R \mapsto [0,1]$ be an increasing solution to the first equation in (6.7), with asymptotic conditions as in (6.8), for some nonnegative control function $\alpha \in \mathbf{L}^1(I\!R)$ with bounded support. Then there exist solutions $V, \Theta$ of the remaining two equations in (6.7), with asymptotic conditions (6.8).*

**Proof.** By Lemma 6.2 we already have an upper solution of (6.7) satisfying the asymptotic conditions (6.8). It remains to construct a lower solution.

**1.** Let $\varphi_0 \in ]0, \pi/2[$ be the angle such that

$$\cos\varphi_0 = \frac{a}{\sqrt{a^2+b^2}}, \qquad \sin\varphi_0 = \frac{b}{\sqrt{a^2+b^2}}.$$

Then by (6.21) the functions

$$\widehat{V}(x) = e^{ax}\sin bx, \qquad \widehat{\Theta}(x) = -\frac{\kappa_1}{c\sqrt{a^2+b^2}}e^{ax}\sin\big[bx - \varphi_0\big] \tag{6.22}$$

provide one particular solution to the linear system (6.20), as shown in Fig. 5.10.

**2.** Given $x_0 \in I\!R$, for any $\varepsilon > 0$, call $(V_\varepsilon, W_\varepsilon, \Theta_\varepsilon)$ the solution to the system (6.7) with initial data

$$V_\varepsilon(x_0) = 0, \qquad W_\varepsilon(x_0) = V'_\varepsilon(x_0) = \varepsilon\widehat{V}'(0), \qquad \Theta_\varepsilon(x_0) = \varepsilon\widehat{\Theta}(0), \tag{6.23}$$

in the special case where $\alpha(x) = 0$ and $U(x) = 1$ for all $x$. By standard ODE theory, as $\varepsilon \to 0$ we have the convergence

$$\varepsilon^{-1}V_\varepsilon(x+x_0) \to \widehat{V}(x), \qquad \varepsilon^{-1}V'_\varepsilon(x+x_0) \to \widehat{V}'(x), \qquad \varepsilon^{-1}\Theta_\varepsilon(x+x_0) \to \widehat{\Theta}(x), \tag{6.24}$$

uniformly for $x$ in bounded intervals.

Since we are assuming that the control $\alpha(\cdot)$ has bounded support, for any $\epsilon_0 > 0$ we can choose $x_0 > 0$ large enough so that

$$\alpha(x) = 0, \qquad 1 - \epsilon_0 \le U(x) \le 1, \qquad \text{for all } x \ge x_0\,. \tag{6.25}$$

By choosing $\varepsilon, \epsilon_0 > 0$ small enough, we obtain an exact solution of (6.7) on an interval $[x_0, x_1]$, with $x_1 - x_0 \le 4\pi/b$ and $\alpha, U$ as in (6.25), such that

$$\begin{cases} V_\varepsilon(x_0) = 0, \\ V_\varepsilon(x_1) = 0, \end{cases} \qquad \begin{cases} \Theta_\varepsilon(x_0) < 0, \\ \Theta_\varepsilon(x_1) > 0, \end{cases} \qquad \begin{cases} V_\varepsilon(x) > 0 \quad \text{for} \quad x_0 < x < x_1\,, \\ V'_\varepsilon(x_1) < 0\,. \end{cases} \tag{6.26}$$

Restricted to the half line $]-\infty, x_1]$, our lower solution is then defined as

$$v^-(x) = \begin{cases} 0 & \text{if } x < x_0\,, \\ V_\varepsilon(x) & \text{if } x \in [x_0, x_1]\,, \end{cases} \qquad \theta^-(x) = \begin{cases} 0 & \text{if } x < x_0\,, \\ \max\big\{\Theta_\varepsilon(x), 0\big\} & \text{if } x \in [x_0, x_1]\,. \end{cases} \tag{6.27}$$

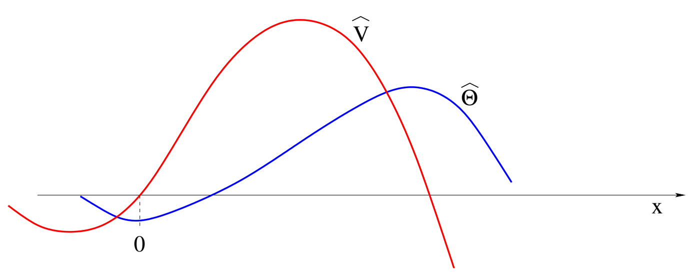


**Figure 5.10.** A particular solution (6.22) to the linear system (6.20).

**3.** Next, we extend this subsolution to the remaining half line $[x_1, +\infty[$. As a first step, we define the constant function

$$\widetilde{\theta}(x) \doteq \Theta_\varepsilon(x_1) > 0,$$

and let $\widetilde{v}$ be the solution to

$$v'' \;=\; -\,cv' - \kappa_2(1-\epsilon_0 - v)\tilde{\theta}_1 + dv, \tag{6.28}$$

on the domain $x \in [x_1, +\infty[$ , with boundary conditions

$$v(x_1) \;=\; 0, \qquad\qquad v(+\infty) \;=\; V^\dagger \;\doteq\; \frac{\kappa_2(1-\epsilon_0)\Theta_\varepsilon(x_1)}{\kappa_2\Theta_\varepsilon(x_1)+d}\,. \tag{6.29}$$

An explicit computation yields

$$\widetilde{v}(x) \;=\; V^\dagger\big(1-e^{\lambda_0(x-x_1)}\big), \qquad\qquad \lambda_0 \;=\; \frac{-c-\sqrt{c^2+4(\kappa_2\Theta_\varepsilon(x_1)+d)}}{2}.$$

Notice that, for $x > x_1$, the couple $(\widetilde{v}, \widetilde{\theta})$ provides a lower solution to the last two equations in (6.7). However, this subsolution does not yet satisfy the asymptotic conditions in (6.8). One more step is thus needed.

**4.** For $x > x_1$ we let $\theta^-$ be the solution to

$$\theta' \;=\; \frac{-\kappa_1}{c}\widetilde{v}(x)(1-\theta), \qquad\qquad \theta(x_1) = \Theta_\varepsilon(x_1), \tag{6.30}$$

where $\widetilde{v}$ is the function constructed in the previous step. More explicitly, this means

$$\theta^-(x) \;=\; 1 - \big(1-\Theta_\varepsilon(x_1)\big)\exp\left\{\int_{x_1}^{x}\frac{\kappa_1}{c}\widetilde{v}(z)\,dz\right\}.$$

Observe that, since $c < 0$ and $\widetilde{v}_1(x) \to V^\dagger > 0$ as $x \to +\infty$, the above solution $\theta^-$ is monotone increasing and satisfies $\theta^-(x) \geq \widetilde{\Theta}_\varepsilon(x_1)$ as $x \in [x_1, +\infty)$ and $\theta^-(x) \to 1$ as $x \to +\infty$.

We then define $v^-$ to be the solution of

$$v'' \;=\; -\,cv' - \kappa_2(U(x)-v)\theta^-(x) + dv, \tag{6.31}$$

on the domain $x \in [x_1, +\infty[$ , with boundary conditions

$$v(x_1) \;=\; 0, \qquad\qquad v(+\infty) \;=\; V^* \;\doteq\; \frac{\kappa_2}{\kappa_2+d}\,. \tag{6.32}$$

Observing that

$$U(x) \geq 1 - \epsilon_0\,, \qquad \theta^-(x) \geq \widetilde{\theta} \qquad \text{for all } x \geq x_1\,,$$

by a comparison argument we conclude

$$v^-(x) \geq \widetilde{v}(x) \qquad \text{for all } x \geq x_1\,.$$

It is now clear that the couple $(v^-, \theta^-)$ provides a subsolution, restricted to the half line $[x_1, +\infty[$. Since $v^-(x) \geq 0$ for all $x \in I\!R$ while $v^-(x_1) = 0$, it follows that at the junction point $x_1$ the left and right derivatives of $v^-$ satisfy

$$(v^-)'(x_1-) \leq 0 \leq (v^-)'(x_1+). \tag{6.33}$$

Hence $(v^-, \theta^-)$, shown in Figure 5.11, is a subsolution defined on the whole real line, which satisfies all the asymptotic conditions in (6.8).

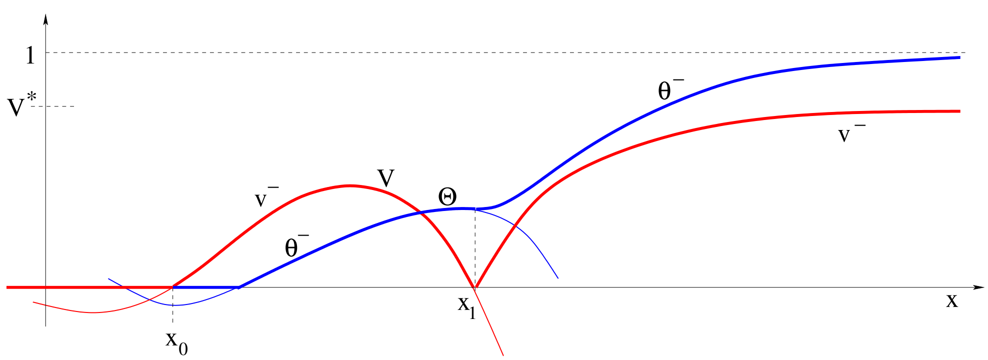


**Figure 5.11.** The lower solution constructed at (6.27) and at (6.30)-(6.32), separately on the half lines where $x \leq x_1$ and $x \geq x_1$.

**5.** Having constructed a supersolution and a subsolution of (6.7) with

$$v^-(x) \leq v^+(x), \qquad \theta^-(x) \leq \theta^+(x) \qquad \text{for all } x \in I\!R, \tag{6.34}$$

the existence of an exact solution follows by a standard monotonicity argument. Namely, since the (component-wise) supremum of two subsolutions is also a subsolution, we can define

$$\big(V(x), \Theta(x)\big) = \sup_{(v^-,\theta^-)\in\mathcal{S}} \big(v^-(x), \theta^-(x)\big),$$

where the supremum is taken over the set $\mathcal{S}$ of all subsolutions which satisfy (6.34). More precisely:

$$v^-(x) \;\leq\; \min\bigl\{U(x), V^*\bigr\}, \qquad\qquad \theta^-(x) \;\leq\; \overline{\Theta}(x).$$

By construction, our subsolutions and supersolutions all satisfy the same asymptotic conditions at (6.8). Hence the same holds for the exact solution. □

# 7 Nonexistence of controlled traveling profiles with slow speed

In this section we continue the analysis of the system (6.2), focusing on

**CASE 2:** The density of insects is large for $x \to +\infty$ as well as for $x \to -\infty$. Insects and trees are all healthy in the limit as $x \to -\infty$, while they are increasingly infected as $x \to +\infty$.

We consider the possibility of using a control $\alpha(\cdot)$ to reduce the density of insects in the intermediate region between the healthy and contaminated zone. In principle, this should provide a "buffer zone", separating the healthy population from the sick one, thus slowing down the spread of the contamination. Our analysis, however, will show that this strategy is not effective. Namely, it cannot yield any traveling wave profile with slower propagation speed.

To state a precise result in this direction, we first study the asymptotic behavior of a traveling wave as $x \to -\infty$. To fix ideas, let a speed $c < 0$ be given. We seek a control $\alpha \in \mathbf{L}^1(I\!\!R)$ and a solution of (6.2) in the form of a traveling wave (6.6). This leads again to the system (6.7). However, the asymptotic conditions (6.8) are now replaced by

$$\left\{\begin{array}{rcl} U(-\infty) &=& 1\,,\\ V(-\infty) &=& 0,\\ \Theta(-\infty) &=& 0\,, \end{array}\right. \qquad\qquad \left\{\begin{array}{rcl} U(+\infty) &=& 1\,,\\ V(+\infty) &=& V^*,\\ \Theta(+\infty) &=& 1\,. \end{array}\right. \tag{7.1}$$

**Theorem 7.1.** *Let $c^\sharp$ be the constant in (6.15), and consider any speed $c$ with $c^\sharp < c < 0$. Then the system (6.7) does not admit any solution $x \mapsto \bigl(U(x), V(x), \Theta(x)\bigr) \in \mathcal{D}$ with asymptotic conditions (7.1), for any control $\alpha \in \mathbf{L}^1(I\!\!R)$.*

**Proof.** The proof will be achieved by showing that, even by adding a control $\alpha \in \mathbf{L}^1(I\!\!R)$ in the equations (6.10), one cannot achieve solutions such that $V(x), \Theta(x)$ converge to zero as $x \to -\infty$, and satisfy the constraint $V(x), \Theta(x) \in [0,1]$ for all $x \in I\!\!R$.

The argument will be given in several steps.

**1.** Assume that, on the contrary, a traveling wave solution $(U, V, \Theta)$, exists, with the prescribed asymptotic behavior as $x \to -\infty$. A contradiction will be obtained by showing that the control $\alpha(\cdot)$ cannot be integrable.

As a preliminary, we observe that the assumption $\alpha \in \mathbf{L}^1(R)$ implies that the traveling wave profile $U(\cdot)$, i.e. the solution to

$$U'' + cU' + f(U) - \alpha(x)U \; = \; 0, \qquad\qquad U(-\infty) \; = \; U(+\infty) \; = \; 1, \tag{7.2}$$

satisfies

$$\int_{-\infty}^{0} \big(1 - U(x)\big)\, dx \; < \; +\infty. \tag{7.3}$$

On the space $I\!R^3$, it will be convenient to use a new system of coordinates $y = (y_1, y_2, y_3)$ corresponding to the basis $\{\mathbf{v}_1, \mathbf{w}_2, \mathbf{w}_3\}$ defined in (6.17), (6.18). Let $x \mapsto Y(x) = \big(Y_1(x), Y_2(x), Y_3(x)\big)$ be the coordinates of the traveling profile $(V, W, \Theta)$ w.r.t. this new basis. By construction, the system (6.10) can be written as

$$\begin{pmatrix} Y_1' \\ Y_2' \\ Y_3' \end{pmatrix} \; = \; \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} + \widetilde{G}(Y) + \widetilde{H}(Y, x), \tag{7.4}$$

where, in view of (6.12) and (7.3), the nonlinear perturbations $\widetilde{G}, \widetilde{H}$ satisfy the bounds

$$\big|\widetilde{G}(Y)\big| \; \leq \; C_0\, |Y|^2, \qquad\qquad \big|\widetilde{H}(Y, x)\big| \; \leq \; C_0\, |Y|\, \beta(x), \tag{7.5}$$

for some constant $C_0$ and some integrable function $\beta \in \mathbf{L}^1\big(] - \infty, 0]\big)$.

**2.** By (7.4)-(7.5) there exists a constant $C_1$ such that

$$\left| \frac{d}{dx} \big|Y(x)\big| \right| \; \leq \; C_1 \Big(1 + \big|\beta(x)\big|\Big) \big|Y(x)\big|.$$

Since $\beta \in \mathbf{L}^1$, we conclude that the vector $Y(x)$ cannot vanish at any point $-\infty < x \leq 0$.

**3.** Introducing the radius $r(x) = \big|Y(x)\big|$, we now consider the normalized vector $\xi$, such that

$$\xi(x) \;=\; (\xi_1,\xi_2,\xi_3)(x) \;=\; \frac{Y(x)}{\big|Y(x)\big|}\,, \qquad\qquad Y(x) \;=\; r(x)\xi(x)\,.$$

By (7.4), denoted the $3\times 3$ matrix in (7.4) as $A$, this vector $\xi$ satisfies

$$\xi'(x) \;=\; A\xi + \tilde g(r,\xi) + \tilde h(r,\xi,x) - \Big\langle A\xi + \tilde g(r,\xi) + \tilde h(r,\xi,x)\;,\;\xi\Big\rangle\xi\,, \tag{7.6}$$

where

$$\tilde g(r,\xi) \;=\; r^{-1}\widetilde G(r\xi), \qquad\qquad \tilde h(r,\xi,x) \;=\; r^{-1}\widetilde H(r\xi,x). \tag{7.7}$$

Since $r(x)\to 0$ as $x\to-\infty$, by (7.5) and (7.7) we have

$$\big|\tilde g(r,\xi)\big| \le C_0\big|r(x)\big|, \qquad \big|h(r,\xi,x)\big| \;\le\; C_0\big|\beta(x)\big| \qquad\qquad \lim_{x\to-\infty}\Big|\tilde g\big(r(x),\xi\big)\Big| \;=\; 0,$$

uniformly for all $|\xi|=1$.

**4.** Based on the previous step, we observe that, as $x\to-\infty$, the evolution of the normalized vector $\xi=(\xi_1,\xi_2,\xi_3)$ satisfies an equation of the form

$$\xi' \;=\; A\xi - \big\langle A\xi\,,\;\xi\big\rangle\xi + g(x) + h(x), \tag{7.8}$$

where $h\in\mathbf{L}^1$ while $\lim_{x\to-\infty} g(x)=0$. We claim that, as $x\to-\infty$, two cases are possible

**Case 1:** $\xi_1(x)\to\pm1$.

**Case 2:** $\xi_1(x)\to 0$.

Indeed, by (7.6), the first component of the vector $\xi$ satisfies the ODE

$$\xi_1'(x) \;=\; (\lambda_1-a)\xi_1(1-\xi_1^2) + g_1(x) + h_1(x), \tag{7.9}$$

where $h_1\in\mathbf{L}^1$ while $\lim_{x\to-\infty} g_1(x)=0$.

For any $\delta\in\,]0,1/2]$, consider the set

$$\begin{aligned} I_\delta \;=\; \Bigg\{\bar x\le 0\,; \quad \big|\xi_1(\bar x)\big|\ge\delta, \quad \int_{-\infty}^{\bar x}\big|h_1(y)\big|\,dy < \frac{\delta}{2}\,,\\ \big|g_1(x)\big| < (a-\lambda_1)\frac{\delta(1-\delta^2)}{2} \qquad \text{for all } x\le\bar x\Bigg\}. \end{aligned} \tag{7.10}$$

Assume that one of these sets $I_\delta$ is nonempty, say $\bar{x} \in I_\delta$. We claim that

$$\big|\xi_1(x)\big| \;>\; \frac{\delta}{2} \qquad \text{for all } x \leq \bar{x}. \tag{7.11}$$

Indeed, consider the function

$$\phi(x) \;\doteq\; \big|\xi_1(x)\big| - \int_{-\infty}^{x} \big|h_1(y)\big|\,dy. \tag{7.12}$$

Recalling that $\lambda_1 < 0 < a$, for $x \leq \bar{x}$ by (7.10) we have the implication

$$|\xi_1| \in \left[\frac{\delta}{2}, \delta\right] \quad\Longrightarrow\quad \phi'(x) \;\leq\; (\lambda_1 - a)|\xi_1|(1-\xi_1^2) + \big|g_1(x)\big| \;<\; 0.$$

If there exist $x_1 < x_2 \leq \bar{x}$ such that

$$\frac{\delta}{2} \;=\; \big|\xi_1(x_1)\big| \;<\; \big|\xi_1(x_2)\big| \;=\; \delta, \tag{7.13}$$

then

$$\frac{\delta}{2} \;=\; \big|\xi_1(x_2)\big| - \big|\xi_1(x_1)\big| \;=\; \phi(x_2) - \phi(x_1) + \int_{x_1}^{x_2} \big|h_1(y)\big|\,dy \;<\; \int_{-\infty}^{\bar{x}} \big|h_1(y)\big|\,dy \;<\; \frac{\delta}{2}\,,$$

reaching a contradiction.

Using (7.11), we now show that

$$\lim_{x\to-\infty} \big|\xi_1(x)\big| \;=\; \lim_{x\to-\infty} \phi(x) \;=\; 1. \tag{7.14}$$

Indeed, the first identity is an immediate consequence of (7.12). To prove the second equality, let any $\varepsilon \in \,]0, \delta/4]$ be given. Choose $x^* < \bar{x}$ such that

$$\int_{-\infty}^{x^*} \big|h_1(y)\big|\,dy < \varepsilon, \qquad\qquad \big|g_1(x)\big| < \varepsilon \qquad \text{for all } x \leq x^*.$$

Observing that

$$\frac{\delta}{4} \;\leq\; \big|\xi_1(x)\big| - \varepsilon \;\leq\; \phi(x) \;\leq\; \big|\xi_1(x)\big| \qquad \text{for all } x \leq x^*,$$

from (7.9) we obtain

$$\phi'(x) \;\leq\; (\lambda_1 - a)\frac{\delta}{4}\big(1-\phi^2(x)\big) + \varepsilon \;<\; 0,$$

where the last inequality holds as long as

$$1-\phi^2(x) \;\geq\; \frac{4\varepsilon}{(a-\lambda_1)\delta}\,.$$

We thus conclude

$$\liminf_{x\to-\infty}\big(1-\phi^2(x)\big) \;\leq\; \frac{4\varepsilon}{(a-\lambda_1)\delta}\,.$$

Since $\varepsilon > 0$ can be arbitrarily small, this yields the second identity in (7.14).

The previous analysis has shown that, if one of the sets $I_\delta$ is nonempty, then (7.14) holds, hence Case 1 occurs.

The remaining possibility is that all sets $I_\delta$ are empty. In this case, for every $\delta > 0$ we can find $x^* < 0$ such that

$$\int_{-\infty}^{x^*} \big|h_1(y)\big|\,dy < \frac{\delta}{2}\,, \qquad \big|g_1(x)\big| < (a-\lambda_1)\frac{\delta(1-\delta^2)}{2} \qquad \text{for all } x \leq x^*.$$

This implies $\big|\xi_1(\bar x)\big| < \delta$ for all $\bar x \leq x^*$, otherwise $\bar x \in I_\delta$ against the assumption. We thus conclude that Case 2 holds true.

**5.** We show that Case 1 leads to a contradiction. Indeed, given $\varepsilon > 0$, by choosing $x_0 << 0$ we achieve

$$|Y(x)| \leq \varepsilon, \qquad \int_{-\infty}^{x} \big|\beta(x)\big|\,dx \;\leq\; \varepsilon, \qquad |Y(x)| \;\leq\; 2|Y_1(x)|, \qquad \text{for all } x < x_0\,. \tag{7.15}$$

In this case, for any $x_1 < x < x_0$ we have

$$\big|Y_1(x)\big| \;\leq\; e^{\lambda_1(x-x_1)}\big|Y_1(x_1)\big| + C\int_{x_1}^{x} e^{\lambda_1(x-y)}\Big(\big|Y_1(y)\big| + \big|\beta(y)\big|\Big)\big|Y_1(y)\big|\,dy.$$

Letting $x_1 \to -\infty$ we obtain

$$
\begin{aligned}
\big|Y_1(x)\big| &\leq C \int_{-\infty}^{x} e^{\lambda_1(x-y)} \Big(\big|Y_1(y)\big| + \big|\beta(y)\big|\Big) \big|Y_1(y)\big|\,dy \\
&\leq C\varepsilon \int_{-\infty}^{x} e^{\lambda_1(x-y)} \Big(\varepsilon + \big|\beta(y)\big|\Big)\,dy \\
&\leq C\frac{\varepsilon^2}{|\lambda_1|} e^{\lambda_1 x} + C\varepsilon \int_{-\infty}^{x} \lambda_1 e^{\lambda_1(x-y)} \left(\int_y^x |\beta(z)|\,dz\right) dy \\
&\leq C\varepsilon\Big(C_1\varepsilon + C_2 \int_{-\infty}^{x} |\beta(z)|\,dz\Big) \;\leq\; \frac{\varepsilon}{4}\,,
\end{aligned}
\tag{7.16}
$$

provided that $\varepsilon > 0$ is chosen sufficiently small. By the third inequality on (7.15) we conclude $\big|Y(x)\big| \leq \varepsilon/2$ for all $x < x_0$.

Iterating this argument, we obtain $\big|Y(x)\big| \leq 2^{-k}\varepsilon$ for every $k \geq 1$, hence $Y(x) = 0$ for all $x \in ]-\infty, x_0]$, reaching a contradiction.

**6.** We now show that Case 2 also leads to a contradiction. By step **3**, the last two components satisfy an ODE of the form

$$
\begin{pmatrix} \xi_2' \\ \xi_3' \end{pmatrix} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \xi_3 \end{pmatrix} + \phi_1(x) + \phi_2(x),
\tag{7.17}
$$

with

$$
\lim_{x\to-\infty} \phi_1(x) = 0, \qquad \int_{-\infty}^{0} \phi_2(x)\,dx < +\infty.
$$

On the plane $\Sigma$ at (6.19), it will be convenient to use polar coordinates $(r, \vartheta)$. More precisely, by (7.17) the evolution of the angle variable has the form

$$
\frac{d}{dx}\vartheta(x) = b + \widetilde{\phi}_1(x) + \widetilde{\phi}_2(x).
\tag{7.18}
$$

where

$$
\widetilde{\phi}_1(x) \to 0, \qquad \int_{-\infty}^{x_0} \big|\widetilde{\phi}_2(x)\big|\,dx < +\infty.
$$

As shown in Fig. 5.12, this implies that the trajectory makes infinitely many loops around the origin, close to the plane $\Sigma$. But this is impossible, because in this case, for some values of $x$, one of the components $V(x)$, $\Theta(x)$ must be negative. This concludes the proof of the theorem. □

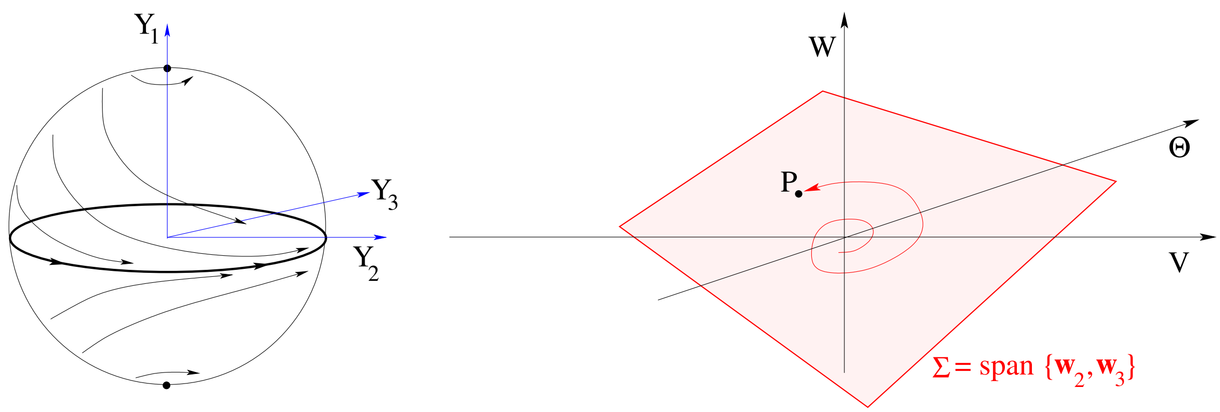


**Figure 5.12.** Left: the dynamics of the unit vector $\xi = Y/|Y|$, on the surface of the unit ball in $I\!R^3$. Right: on the plane $\Sigma = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, for certain values of the angular component $\vartheta$, the point $P$ with polar coordinates $(r, \vartheta)$ lies outside the admissible set where $V \geq 0$ and $\Theta \geq 0$.

# Appendix
# Proof of the Previous Theorems and Lemmas

## 1 Proof of Theorem 1.1 in Chapter 1

**Proof:** 1. We firstly suppose the differential inequality in (1.5) is strict. Then,

$$v_t + Lv - d(t,x)v > 0 \text{ in } Q_T.$$

2. We could prove that if $t > 0$ is close enough to 0, $u(t,x) > 0$. There are two situations needed to be discussed: for some $x$, **(1)** $v(0,x) = 0$; **(2)** $v(0,x) > 0$.

For the case **(1)**, we can get that $v(0+,x) = 0$, $v_{x_i x_j}(0+,x) \geq 0$ and $v_{x_i}(0+,x) = 0$ because $v$ is $\mathcal{C}_2(I\!R^m)$ for $t > 0$ and is continuous on $I\!R^m$ for $t \geq 0$. Since $L$ is non-divergence, $Lv(0+,x) \leq 0$. For the differential inequality in (1.5), we get

$$v_t(0+,x) + Lv(0+,x) - d(0+,x)v(0+,x) > 0.$$

$$v_t(t_0+,x) > -Lv(0+,x) \geq 0.$$

Since $v$ is $\mathcal{C}_1(]0,T])$ for the fixed $x$, we could deduce that $v(t,x) > 0$ when $t$ is close to 0.

For the case **(2)**, for any fixed $x \in \Omega$, since $v \in \mathcal{C}(\overline{Q}_T)$, we could conclude that if $v(0,x) > 0$, we could make $t$ small enough to make $v(t,x) \geq 0$ from the integration inequality.

3. With 2, we only need to prove that $\max_{\overline{Q}_T} v = \max_{\Gamma_T} v$, where $\Gamma_T \doteq \overline{Q}_T - Q_T$ is satisfied with $v(0,x) > 0$. We prove it by contradiction.

Suppose that there exists a time $t_0 > 0$ when $v$ firstly reach 0 at some points, for example $x_0$. Then, at $(t_0,x_0)$, $v(t_0,x_0) = 0$, $v_t(t_0,x_0) \leq 0$, $v_{x_i}(t_0,x_0) = 0$ and

$v_{x_ix_j}(t_0, x_0) \geq 0$. In this way, $v_t(t_0, x_0) + Lv(t_0, x_0) - d(t_0, x_0)v(t_0, x_0) \leq 0$, which is against the strict inequality in 1.

5. Now, we could consider the equal case for (1.5) without the assumption in 1. Denote $M \doteq \sup_{(t,x)\in Q_T} |d(t, x)| + 1$ and define the function $v^\epsilon \doteq v + \epsilon e^{-Mt}$. Then, $v^\epsilon$ satisfies

$$v_t^\epsilon + Lv^\epsilon - d(t, x)v^\epsilon = v_t + Lv - d(t, x)v + \epsilon M e^{-Mt} - d(t, x)\epsilon e^{-Mt} > 0.$$

Hence, by 2 and 3, we could conclude that $\max_{\overline{Q}_T} v^\epsilon = \max_{\Gamma_T} v^\epsilon$. Take $\epsilon \to 0$, $\max_{\overline{Q}_T} v = \max_{\Gamma_T} v$.

6. In conclusion, we prove that if the conditions in **Theorem 1.1** hold, we could prove the weak maximum principle that $\max_{\overline{Q}_T} v = \max_{\Gamma_T} v$. Since $v(0, x) \geq 0$, we could deduce that $v(t, x) \geq 0$ on $Q_T$.

The strong maximum principle (**Theorem 1.1**) could be gotten from the weak one. The differential inequality in (1.5) could be written as

$$v_t + Lv + Mv \geq (M + d(t, x))v \geq 0. \tag{1.1}$$

Utilize the theorems in Section 7.2 in [25], we could get the result in **Theorem 1.1**. □

# 2 Proof of Theorem 3.1 in Chapter 4

**Proof.** By the definition in (3.14), we try to prove $P > 0$ except for the two end points by contradiction. When the flow of (3.14) starting from $(0, 0)$, $P$ will be positive, otherwise $U$ will turn to negative.

If $P = 0$ somewhere where $z_0 \neq \pm\infty$, we can suppose that the flow $(U, P)$ defined in (3.14) firstly passes the point like $(U_0, 0)$ at $z = z_0$ with $0 < U_0 < 1$. There are two situations:

If $P'(z_0) = 0$, then $f(U_0) = 0$. Hence, the flow $(U, P) = (U_0, 0)$ will be the solution to (3.14) with the initial condition $(U, P)(z_0) = (U_0, 0)$. This is against the requirement for the solution.

If $P'(z_0) \neq 0$, since $z_0$ is the first location where $P$ becomes 0, $P'(z_0) < 0$. When $P < 0$, $U$ will become smaller. Since we need the solution finally go to $(1, 0)$, so $(U, P)$ will finally go up and cross the horizontal line $P = 0$ at some point $z_1 > z_0$. Since $U \in [0, 1]$ and 0 is a critical point, $0 < U(z_1) < U(z_0)$. Since the flow $(U, P)$ will finally go right to $(1, 0)$, the part of $(U, P)$ defined on $]-\infty, z_0[$ will finally intersect with the part of $(U,P)$ defined on $]z_1,+\infty[$ at some point $(U_i,P_i)$. Because the value of $(U',P')$ are only related to the value of $(U,P)$, these two parts of the flow will coincide with each other once them meet at one point. This means that the part of $(U,P)$ defined on $]z_1,+\infty[$ after $(U_i,P_i)$ will also pass $(U_0,0)$ and repeat the previous path and hence will never go to $(1,0)$. □

# 3 Proof of Theorem 3.2 in Chapter 4

**Proof.** If we can find a flow satisfying (3.14) and the asymptotic conditions $(U,P)(-\infty)=0$ and $(U,P)(+\infty)=1$, from the **Theorem 3.1**, $U(z)$ will increase from $-\infty$ to $+\infty$, which means that $U$ could be treated as an independent variable. Then,

$$\frac{dP}{dU}=\frac{dP}{dz}\Big(\frac{dU}{dz}\Big)^{-1}=-c-\frac{f(U)}{P}.$$

Since $(U,P)$ pass the line $U=0$ only when $P=0$ and the line $U=1$ only when $P=0$ due to the monotonicity, the boundary conditions, $P(0)=P(1)=0$, satisfy.

The converse direction is difficult. Supposing that we can find a flow that satisfying (3.16) with the boundary conditions, we generate the flow of $(U,P)$ by integrating

$$\frac{dU}{P(U)}=dz, \qquad (U,z)\text{ pass }\Big(\frac{1}{2},0\Big).$$

Then,

$$\begin{cases} U(z) &= F^{-1}(z) \\ P(z) &= (F^{-1})'(z)=P(U(z)) \end{cases}, \qquad F(U)\doteq\int_{1/2}^{U}\frac{dU}{P(U)}\text{ is increasing.} \tag{3.2}$$

It is easy to check the flow $(U(z),P(z))$ defined in (3.2) satisfies (3.14). The rest is to check the asymptotic conditions. Supposing $\lim_{z\to z^-}U(z)=0$, $\lim_{z\to z^+}U(z)=1$, the rest thing is to prove that $z^-=-\infty$ and $z^+=+\infty$.

We hope to prove that there exists a positive $\alpha$ to make $P<\alpha U$. Since $f(0)=0$, we can find a $\beta>0$ such that $|f(U)|<\beta U$. For a given $\beta$, we can find $\alpha$ to make $\beta/\alpha-c<\alpha$. We can check $P\le\alpha U$. Otherwise, supposing $U_0$ is the first value from 0 to make $P(U_0)>\alpha U_0$, then $(dP/dU)(U_0)\le -c+\beta/\alpha\le\alpha$. In this way,

$$\alpha\ge\frac{dP}{dU}(U_0)=\lim_{\delta\to 0}\frac{P(U_0)-P(U_0-\delta)}{\delta}>\frac{\alpha U_0-\alpha(U_0-\delta)}{\delta}=\alpha,$$

which is impossible. Hence, $P \leq \alpha U$ is always true.

By the definition of $F(U)$ in (3.2),

$$z_0 = F(U(z_0)) = \int_{1/2}^{0} \frac{dU}{P(U)} \geq \int_{1/2}^{0} \frac{dU}{\alpha U} = -\infty.$$

Similarly, we can prove $z_1 = +\infty$ by finding constant $\gamma$ to make $P(U) < \gamma(1-U)$. □

# 4 Proof of Lemma 3.1 in Chapter 4

**Proof.** Since $f'(0) < 0$, we can suppose $U_0$ is the largest number to make $f(u) < 0$ for $0 < u < U_0$. Because $f'(1) > 0$ and $f(1) = 0$, we can deduce $U_0 < 1$ and hence $U_0$ will be the first zero of $f(u)$ on $]0,1[$.

Let $c < -\sqrt{\max|f|/U_0}$, then we can prove that $P' > 0$ on $]0,1[$. Hence $P(U)$ will keep positive on $]0,1[$ and finally leave $Q$ from the line $\{(1,P)\,|\,P > 0\}$ as $U$ increases.

Now, we prove that if $c < -\sqrt{\max|f|/U_0}$, $P' > 0$ on $]0,1[$. In fact, since $f(u) < 0$ on $]0,U_0[$, for any $U \in ]0,U_0]$, we can prove by contradict that $P(U) > 0$ and estimate that

$$P'(U) = -c - \frac{f(U)}{P(U)} > -c > 0, \qquad P(U) > -cU.$$

For $U \in ]U_0,1[$, prove $P' > 0$ by contradict. If there exists the first number $U_m \in ]U_0,1[$ such that $P'(U_m) = 0$. In this way, $-c - f(U_m)/P(U_m) = P'(U_m) = 0$ and $P(U_m) = P(U_0) + \int_{U_0}^{U_m} P'(u)\,du > P(U_0) > -cU_0$. However,

$$-c - \frac{f(U_m)}{P(U_m)} > \sqrt{\frac{\max|f|}{U_0}} - \frac{f(U_m)}{U_0}\sqrt{\frac{U_0}{\max|f|}} > 0.$$

This lead to the contradict and hence $P' > 0$ on $]U_0,1[$. Therefore, $P' > 0$ holds on $]0,1[$. □

# 5 Proof of Theorem 3.4 in Chapter 4

**Proof.** By **Theorem 3.3**, $P_c(U)$ will decreases for all $U > 0$ as $c$ increases, until $P_c(U) < 0$. And the decrease of $P_c(U)$ for a fixed $U$ will be continuous with respect to $c$. Thus, when we increase the value of $c$, the solution $P_c(U)$ will become lower and lower as Figure .1. Eventually, we can get the special solution $P_{c_0}(U)$ for a special $c_0$ such

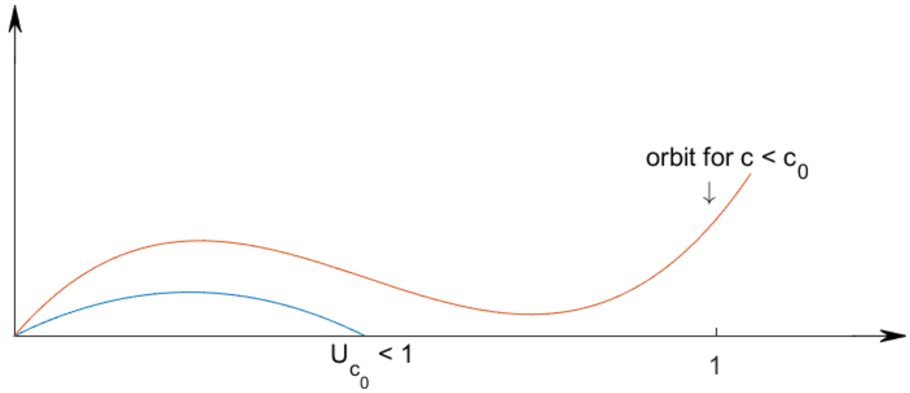


**Figure .1.** The orbits of (3.14) starting from the point $(0,0)$ for different speed $c$.

that $P_{c_0}(U)$ firstly leaves $Q$ from the horizontal segment $\{(U,0)\,|\,0 \le U \le 1\}$ rather than $\{(1,P)\,|\,P \ge 0\}$. We denote the leaving point as $U_{c_0} \in ]0,1]$ and so $P_{c_0}(U_{c_0}) = 0$.

We claim that $U_{c_0} = 1$. We prove it by contradiction. If $U_{c_0} < 1$, we can deduce that $P'_{c_0}(U_{c_0}) = 0$. Otherwise, there should be $U_r$, a little larger than $U_{c_0}$, to make $P_{c_0}(U_r) < 0$. Then, we can find a smaller $c_r$ to make $P_{c_r}(U_r) = 0$, which is against the fact that $P_{c_0}$ is the first solution leaving $Q$ from $\{(U,0)\,|\,0 \le U \le 1\}$ with $c$ increasing. Also, since

$$\lim_{u \to U_{c_0}} -c - \frac{f(u)}{P_{c_0}(u)} = \lim_{u \to U_{c_0}} P'_{c_0}(u) = 0,$$

we can get that $f(U_{c_0}) = 0$.

We could also claim that $U_{c_0}$ is not the only zero for $f(u)$. Since we know that $P_{c_0}(0) = 0$, there should be another point $U_m \in ]0, U_{c_0}[$ to make $P'_{c_0}(U_m) = 0$ and $P_{c_0}(U_m) > 0$. Then, $f(U_m) = -(P'_{c_0}(U_m) + c) \cdot P_{c_0}(U_m) = -cP_{c_0}(U_m) > 0$. Because $f(u) < 0$ when $u$ is quite small, there should be another zero for $f(u)$ between 0 and $U_m$. This is against the condition that there are only one zero inside $]0.1[$. Hence, $U_{c_0} < 1$ is impossible. Thus, $U_{c_0} = 1$. □

# 6 Proof of Theorem 3.5 in Chapter 4

**Proof.** Similar to the saddle-saddle orbits, we try to analyze the behaviors of the solution to (3.16) by analyzing the orbit ending at $P(1) = 0$ in a triangle region $\triangle \doteq \{(U,P)\,|\,0 < P < \mu U,\ 0 < U < 1\}$. The three vertices are denotes as $O$, $A$, $B$ as Figure .2. $\mu$ is a positive number we will determine later.

Since $f'(1) < 0$ guarantees that $(1,0)$ is a saddle, there should be a flow $(U, P(U))$

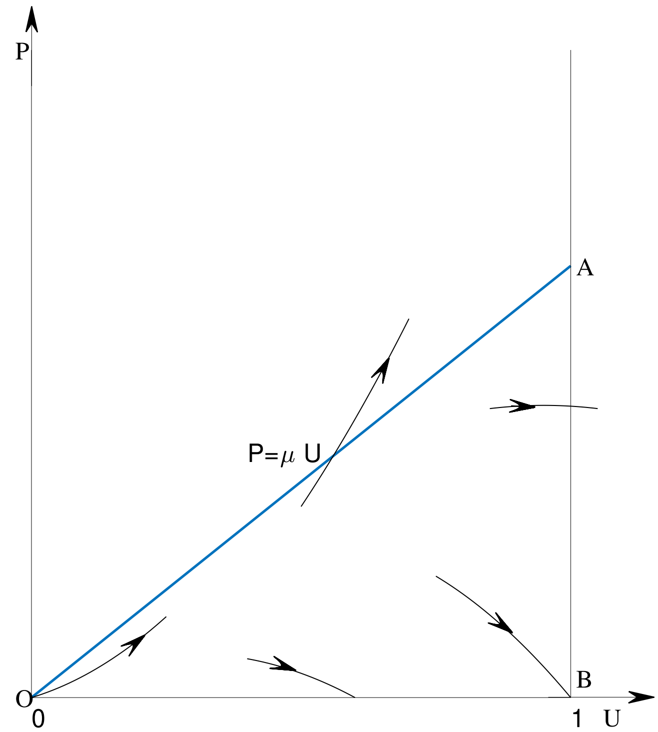


**Figure .2.** The triangle region $\triangle$ and the flows near the edges.

entering $(1,0)$ from $\triangle$. We hope to discover the place where the flow $(U, P(U))$ enter $\triangle$.

The flows passing the points on the edge $AB$ will leave $\triangle$ because the flow will satisfy (3.14) and $U' = P > 0$. So the flow will cross $AB$ from left to right.

The flows passing the points on the edge $OB$ will also leave $\triangle$ because $P' = -cP - f(U) = -f(U) < 0$. So the flow will cross $OB$ from top to bottom.

We hope that the flows passing the points on the edge $OA$ will leave $\triangle$ as well. Because, in this way, the only place where the flows can enter $\triangle$ is the critical point $(0,0)$. Hence, there will be the flow connecting $(0,0)$ and $(1,0)$ and thus corresponding to the solution to (3.16) with (3.17).

The edge $OA$ is parametrized by $(U, \mu U)$, $0 \le U \le 1$. We attempt to adjust the value $\mu$ to make all flows passing $OA$ leave $\triangle$. In fact, the direction of the flow is $(U', P')(U_0, \mu U_0) = (\mu U_0, -c\mu U_0 - f(U_0))$ where $U_0 \in ]0,1]$. If the flow leaves $\triangle$, for all possible $U_0$, we need that

$$\mu \le \inf_{U_0 \in ]0,1]} \frac{P'}{U'}(U_0, \mu U_0) = \inf_{U_0 \in ]0,1]} \frac{-c\mu U_0 - f(U_0)}{\mu U_0} = -c - \sup_{U_0 \in ]0,1]} \frac{f(U_0)}{\mu U_0} = -c - \frac{\nu}{\mu}, \quad (6.3)$$

where

$$\nu \doteq \sup_{U_0 \in ]0,1]} f(U)/U. \tag{6.4}$$

We can easily see that: **(1).** $\nu < +\infty$ and **(2).** $\nu > f'(0)$ because $\lim_{U\to 0} f(U)/U = f'(0)$ and $f(u)$ is bounded on $[0,1]$.

The inequality (6.3) is equivalent to the two conditions: **(i).** $c^2 - 4\nu \geq 0$. **(ii).** $\big(-c-\sqrt{c^2-4\nu}\big)/2 \leq \mu \leq \big(-c+\sqrt{c^2-4\nu}\big)/2$. From the condition **(i)**, we see that the speed $c$ should satisfy that $c < -2\sqrt{\nu}$. If **(i)** holds, it is possible to find a $\mu > 0$ satisfying **(ii)**. $\square$

# Bibliography


[1] Anita, S., Capasso, V., and Dimitriu, G. Region control for a spatially structured malaria model. *Math. Meth. Appl. Sci. 42* (2019), 2909–2933.

[2] Anita, S., Capasso, V., and Mosneagu, A. M. Global eradication for spatially structured populations by regional control. *Discr. Cont. Dyn. Syst. Series B, 24* (2019), 2511–2533.

[3] Anita, S., Capasso, V., and Scacchi, S. Controlling the spatial spread of a xylella epidemic. *Bull. Math. Biology 83:32* (2021).

[4] Aronson, D. G., and Weinberger, H. F. Multidimensional nonlinear diffusion arising in population genetics. *Advances in Mathematics 30* (1978), 33–76.

[5] Benzoni-Gavage, S. Semi-discrete shock profiles for hyperbolic systems of conservation laws. *Phys. D115* (1998), 109–123.

[6] Bianchini, S. Bv solutions of the semidiscrete upwind scheme. *Arch. Rational Mech. Anal. 167* (2003), 1–87.

[7] Bianchini, S. Bv solutions of the semidiscrete upwind scheme. *Arch. Rational Mech. Anal. 167* (2003), 1–81.

[8] Bianchini, S., and Bressan, A. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Annals of Mathematics 161* (2005), 223–342.

[9] Boscain, U., and Piccoli, B. *Optimal Syntheses for control Systems on 2-D manifolds.* Springer, New York, 2004.

[10] Bressan, A. The unique limit of the glimm scheme. *Arch. Rational Mech. Anal. 130* (1995), 205–30.

[11] Bressan, A. *Hyperbolic Systems of Conservation Law.* The One Dimensional Cauchy Problem. Oxford University Press, Oxford, 2000., 2005.

[12] Bressan, A., Chiri, M., and Salehi, N. On the optimal control of propagation fronts. *Math. Models & Methods Appl. Sci. 32* (2022), 1109–1140.

[13] Bressan, A., Chiri, M., and Salehi, N. Optimal control of moving sets. *J. Differential Equations* (to appear).

[14] Bressan, A., Liu, T. P., and Yang, T. $l^1$ stability estimates for $n \times n$ conservation laws. *Arch. Rational Mech. Anal. 149* (1999), 1–22.

[15] Bressan, A., and Piccoli, B. Introduction to the mathematical theory of control. *AIMS Series in Applied Mathematics, Springfield Mo.* (2007).

[16] Bressan, A., and Zhang, D. Control problems for a class of set valued evolutions. *Set-Values Var. Anal. 20* (2012), 581–601.

[17] Cesari, L. *Optimization Theory and Applications.* Springer-Verlag, 1983.

[18] Coclite, G. M., and Garavello, M. A time dependent optimal harvesting problem with measure valued solutions. *SIAM J. Control Optim. 55* (2017), 913–935.

[19] Coclite, G. M., Garavello, M., and Spinolo, L. V. Optimal strategies for a time-dependent harvesting problem. *Discrete Contin. Dyn. Syst. Ser. S 11* (2018), 865–900.

[20] Colombo, R. M., Lorenz, T., and Pogodaev, N. On the modeling of moving populations through set evolution equations. *Discrete Contin. Dyn. Syst. 35* (2015), 73–98.

[21] Colombo, R. M., and Pogodaev, N. On the control of moving sets: Positive and negative confinement results. *SIAM J. Control Optim. 51* (2013), 380–401.

[22] Colombo, R. M., and Rossi, E. A modeling framework for biological pest control. *Math. Biosci. Eng. 17* (2020), 1413–1427.

[23] Crandall, M. G. The semigroup approach to first order quasilinear equations in several space variables. *Israel J. Math. 12* (1972), 108–132.

[24] Dafermos, C. *Hyperbolic Conservation Laws in Continuum Physics, Fourth edition.* Springer-Verlag, Berlin, 2016.

[25] Evans, L. C. *Partial Differential Equations, Second Edition.* American Mathematical Society, Providence, R.I., 2010.

[26] Fife, P. C. *Mathematical Aspects of Reacting and Diffusing Systems.* Lecture Notes in Biomathematics. Springer, 1979.

[27] Fife, P. C., and McLeod, J. B. The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Rat. Mech. Anal. 65* (1977), 335–361.

[28] Glimm, J. Solutions in the large for nonlinear hyperbolic systems of equations. *Pure Appl. Math 18* (1965).

[29] Guerra, G., and Shen, W. Vanishing viscosity and backward euler approximations for conservation laws with discontinuous flux. *SIAM J. Math. Anal. 51* (2019), 3112–3144.

[30] Hale, J. K., and Lunel, S. M. V. *Introduction to Functional Differential Equations*. Springer-Verlag, New York, 1993.

[31] Hermes, H., and Haynes, G. On the nonlinear control problem with control appearing linearly. *SIAM J. Control 1* (1963), 85–108.

[32] Hoff, D. Invariant regions for systems of conservation laws. *Trans. Amer. Math. Soc. 289* (1985), 591–610.

[33] Holden, H., and Risebro, N. *Front Tracking for Hyperbolic Conservation Laws*. Springer-Verlag, Berlin, 2002.

[34] Hosono, Y., and Ilyas, B. Traveling waves for a simple diffusive epidemic model. *Math. Models Methods Appl. Sci. 5* (1995), 935–966.

[35] Kolmogorov, A. N., Petrovskit, I. G., and Piskunov, N. S. A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem. *Selected Works of A. N. Kolmogorove 38* (1937), 242–270.

[36] Lax, P. Hyperbolic systems of conservation laws ii. *Comm. Pure Appl. Math. 10* (1957), 537–566.

[37] Lenhart, S. M., and Montero, J. A. Optimal control of harvesting in a parabolic system modeling two subpopulations. *Math. Models Methods Appl. Sci. 11* (2001), 1129–1141.

[38] Liang, X., and Zhao, X. Q. Spreading speeds and traveling waves for abstract monostable evolution systems. *J. Funct. Anal. 259* (2010), 857–903.

[39] Liu, T. P. The deterministic version of the glimm scheme. *Commun. Math. Phys. 57* (1977).

[40] M. G. Crandall, T. M. L. Generation of semigroups of nonlinear transformations on general banach spaces. *Amer. J. Math. 93* (1971), 265–298.

[41] Martin, R. H. *Nonlinear Operators and Differential Equations in Banach Spaces*. Wiley-Interscience, New York, 1976.

[42] Ridder, J., and Shen, W. Traveling waves for nonlocal models of traffic flow. *Discrete Contin. Dyn. Syst. 39* (2019), 4001–4040.

[43] Ruiz-Balet, D., and Zuazua, E. Control under constraints for multi-dimensional reaction-diffusion monostable and bistable equations. *J. Math. Pures Appl. 143* (2020), 345–375.

[44] Seirin, L., Baker, R., Gaffney, E., and White, S. Optimal barrier zones for stopping the invasion of aedes aegypti mosquitoes via transgenic or sterile insect techniques. *Theoretical Ecology 6* (2013), 427–442.

[45] Shen, W. Traveling waves for conservation laws with nonlocal flux for traffic flow on rough roads. *Netw. Heterog. Media 14* (2019), 709–732.

[46] Smoller, J. *Shock Waves and Reaction-Diffusion Equations.* Springer-Verlag, New York, 1983.

[47] Volpert, A. I., Volpert, V. A., and Volpert, V. A. *Traveling Waves Solutions of Parabolic Systems*, vol. 140 of *Transl. Math. Monographs.* Amer. Math. Society, 1994.

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