CIRCUMFERENTIAL GUIDED WAVES IN ELASTIC AND VISCOELASTIC MULTILAYERED ANNULI

A Dissertation in
Engineering Science and Mechanics
by
Jason Kenneth Van Velsor

© 2009 Jason K. Van Velsor

Submitted in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

August 2009
The dissertation of Jason K. Van Velsor was reviewed and approved* by the following:

Joseph L. Rose  
Paul Morrow Professor of Engineering Science and Mechanics  
Dissertation Advisor  
Chair of Committee

Bernhard R. Tittmann  
Schell Professor of Engineering Science and Mechanics

Clifford J. Lissenden  
Professor of Engineering Science and Mechanics

Ghassan Chehab  
Assistant Professor of Civil Engineering

Albert E. Segall  
Professor of Engineering Science and Mechanics  
Graduate Student Officer  
Department of Engineering Science and Mechanics

*Signatures are on file in the Graduate School
ABSTRACT

People all over the world rely on vast pipeline infrastructures whether it be directly, such as for fuel to cook and heat their homes, or indirectly, such as for delivery to the farmer who must harvest and process the food that will eventually end up on their table. In many cases these pipelines, or at least portions of them, have been in operation for more than half a century. This long-term use, coupled with a lack of integrity assurance programs in the early part of the 20th century, has resulted in the overall degradation and, in many cases, failure of the pipeline. Such failure can be detrimental to the people in the immediate vicinity of the pipeline and the surrounding habitat, and result in significant clean-up, reparation, and litigation costs to the operator. For this reason, most operators have now implemented pipeline integrity and assessment programs, which generally rely quite heavily on non-destructive testing methods.

One of the most common inspection technologies for gas transmission pipeline is the Magnetic Flux Leakage (MFL) In-Line Inspection (ILI) tool. These tools travel through the inside of the pipeline, saturating the pipe wall with a magnetic field, which subsequently “leaks” from the wall in areas of corrosion. While this technology has been found to be a reliable detection tool for most corrosion geometries, it has relatively limited sizing capabilities. For this reason, there is currently significant interest in the use of guided ultrasonic waves for improved inspection capabilities. In a similar fashion to the MFL tools, these ultrasonic tools travel through the inside of the pipe, usually transmitting and receiving ultrasound in the circumferential direction of the pipe.

As wave mechanics can be a very complex subject, many tool developers, and even researchers, equate the propagation of the circumferential wave to that which travels in a plate. While this may be a valid assumption in some cases, it is not in all cases; particularly as radius decreases and wall thickness increases. Furthermore, nowhere in the present body of literature
regarding circumferential waves, has anybody made provisions in their model for the presence of coating layers, even if assuming elastic.

This primary goal of the work presented here is the development of more accurate modeling tools for circumferential guided waves. The generalized characteristic equations for circumferentially propagating shear-horizontal and Lamb [type] waves are derived and the Global Matrix Method is used to extend the solutions to multiple layers. Dispersion curves are presented for several cases of single-layer and multilayered annuli and some physical insights into their meaning are provided. Wave structures are also examined for satisfaction of the appropriate boundary conditions.

The elastic-viscoelastic correspondence principle is employed to allow the use of the elastic solutions for viscoelastic materials, via incorporation of complex material moduli. The Semi-Analytical Finite Element approach is enlisted to calculate the dispersion solutions for elastic/viscoelastic multilayered annuli and to obtain solution in the dispersion spaces for which the fully analytical calculation was found to fail. Excellent agreement is found between the analytical and semi-analytical approaches. The differences between the dispersion curves for elastic multilayered annuli and elastic/viscoelastic annuli are examined.

Two experimental demonstrations are provided to validate the accuracy of the newly development theoretical models and to provide examples of how the theoretical models may be employed to develop new non-destructive testing methodologies. In particular, it is shown how a circumferential shear-horizontal wave is influenced by coating thickness and how these effects may be used to detect disbonds in protective coating layers. Multiple disbond detection features are identified that have resulted directly from consideration of the theoretical models.

Another practical example is included in an appendix and employs a circumferential Lamb [type] wave for the estimation of the remaining wall thickness of a gradually thinned
annulus. This study is completed using finite-element analysis as appropriate test specimens were not available and proper fabrication would be impractical.

It should be noted that there are many other potential applications for the models developed in this work. Other examples include the inspection of gas storage well casings and boiler and heat exchanger tubing. The modeling tools will be especially useful in the development of defect detection, identification, and sizing methods. Future work may involve the modification of the modeling tools to study the influences of soil layers and liquid contents.
# TABLE OF CONTENTS

LIST OF FIGURES ........................................................................................................................................ix

LIST OF TABLES ........................................................................................................................................xiv

ACKNOWLEDGEMENTS ............................................................................................................................... xv

Chapter 1 Introduction ................................................................................................................................. 1

1.1 Motivation ........................................................................................................................................... 1
1.1 Literature Review ................................................................................................................................. 4
  1.1.1 Hallmark Guided Wave Studies ................................................................................................. 4
  1.1.2 Axial Wave Propagation in an Elastic Hollow Cylinder ............................................................ 5
  1.1.3 Circumferential Wave Propagation in an Elastic Hollow Cylinder ........................................... 6
  1.1.4 Wave Propagation in Viscoelastic Materials ........................................................................... 10
  1.1.5 The Semi-Analytical Finite-Element Method ........................................................................ 11
1.2 Overview of Chapters ......................................................................................................................... 12

Chapter 2 An Analytical Solution for Circumferentially Propagating Guided Waves ......................... 15

  2.1 Circumferential Shear Horizontal Waves in a Single-Layer Annulus ............................................ 17
  2.2 Circumferential Lamb [Type] Waves in a Single-Layer Annulus .................................................... 22
  2.3 Extension to Multiple Layer Annuli ................................................................................................. 26

Chapter 3 Numerical Results Following an Analytical Formulation ......................................................... 32

  3.1 Numerical Results for CSH-Waves ................................................................................................. 33
  3.2 Numerical Results for CLT-Waves ................................................................................................. 43
  3.3 Computational Limitations of the Analytical Formulation ............................................................. 55

Chapter 4 Review of Viscoelastic Theory .................................................................................................... 60

  4.1 Constitutive Relations ....................................................................................................................... 60
  4.2 Dynamic Viscoelasticity ................................................................................................................... 62
  4.3 Elastic-Viscoelastic Correspondence Principle ............................................................................ 64
  4.4 Viscoelastic Wave Propagation ...................................................................................................... 66

Chapter 5 A Semi-Analytical Solution for Circumferentially Propagating Guided Waves ................. 69

  5.1 SAFE Formulation for Circumferential Guided Waves ................................................................. 69
  5.2 Considerations for CSH-Waves ....................................................................................................... 78
  5.3 Considerations for CLT-Waves ....................................................................................................... 80

Chapter 6 Numerical Results Following a Semi-Analytical Formulation .............................................. 83

  6.1 Numerical Results for CSH-Waves ................................................................................................. 84
# Contents

6.1.1 Elastic Materials ..................................................... 84
6.1.2 Viscoelastic Materials ................................................. 91
6.2 Numerical Results for CLT-Waves ................................. 93
   6.2.1 Elastic Materials ..................................................... 93
   6.2.2 Viscoelastic Materials ................................................. 95
6.3 Limitations of the SAFE Formulation ............................ 102

Chapter 7 Experimental Results ........................................ 105
   7.1 Effects of Coating Thickness on CSH-Wave Propagation  106
   7.2 Effects of Coating Presence on CSH-Wave Propagation  111

Chapter 8 Summary and Conclusion ................................... 124
   8.1 Review of Work ......................................................... 124
   8.2 Contributions .......................................................... 128
   8.3 Future Directions ....................................................... 130
   8.4 Publications ............................................................ 130

References ............................................................................. 133

Appendix A Circumferential Lamb [Type] Coefficient Matrix Elements ........................................ 140
Appendix B Fundamental Principles of Lorentz Electromagnetic Acoustic Transducers .... 142
Appendix C Approximate Thickness Measurement Using CLT-Waves: A Finite-Element Example ....................................................... 148
   C.1 Introduction to the Finite-Element Method .......... 148
   C.2 A Method for Approximating Wall Thickness Using CLT-Waves .......................... 149

Appendix D Computer Codes .................................................. 156
   D.1 Analytical Formulation: CSH-Waves .......................... 156
      D.1.1 Subfunction: GetInput ........................................ 157
      D.1.2 Subfunction: LayerMatrix .................................... 157
      D.1.3 Subfunction: GlobalMatrix ................................... 158
      D.1.4 Subfunction: Bisection .......................................... 158
      D.1.5 Function: Wavestructure ...................................... 159
      D.1.6 Sample Input File (INPUT.xls) ......................... 160
   D.2 Analytical Formulation: CLT-Waves .......................... 161
      D.2.1 Subfunction: LayerMatrix .................................... 162
      D.2.2 Subfunction: GlobalMatrix ................................... 163
      D.2.3 Subfunction: Bisection .......................................... 163
      D.2.4 Function: Wavestructure ...................................... 164
   D.3 SAFE Formulation: CSH-Waves ............................... 167
      D.3.1 Function: GetInput ............................................. 169
      D.3.2 Subfunction: Integrate_SH .................................. 170
      D.3.3 Function: Wavestructure ...................................... 171
D.3.4 Sample Input File (INPUT.xls) ................................................................. 174
D.4 SAFE Formulation: CLT-Waves ................................................................. 175
  D.4.1 Subfunction: Integrate_LT ................................................................. 177
  D.4.2 Function: Wavestructure ................................................................. 179

Appendix E  Nontechnical Abstract .............................................................. 184
LIST OF FIGURES

Figure 1.1 Failure cause breakdown for natural gas transmission pipeline in the U.S. from 1987 through 2006 (PHMSA 2007) ................................................................. 2

Figure 2.1 Theoretical model used for the development of the governing equations for CSH- and CLT-wave propagation in a single-layer annulus ................................. 16

Figure 2.2 Theoretical model used for the development of the governing equations for CSH- and CLT-wave propagation in the circumferential direction of a multilayer annulus ............................................................................................................................ 28

Figure 3.1 CSH (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.984 aspect ratio. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the OR of the annulus .................................................................................................................. 35

Figure 3.2 CSH (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.8 aspect ratio. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the OR of the annulus .................................................................................................................. 36

Figure 3.3 CSH (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.5 aspect ratio. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the OR of the annulus .................................................................................................................. 37

Figure 3.4 CSH (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.2 aspect ratio. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the OR of the annulus .................................................................................................................. 38

Figure 3.5 CSH (a) Frequency-wavenumber, (b) phase velocity, and (c) group velocity dispersion curves for a multilayered annulus with a 0.8 aspect ratio. Each layer is 4 mm thick. Curves are also shown for the individually isolated layers. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the common interface of the annuli ................................................................................. 40

Figure 3.6 Magnified view of the frequency-wavenumber dispersion curve shown in Figure 3.5(a). Black dots denote points of intersection ................................................. 41

Figure 3.7 Wave structures and stresses for the points identified in Figure 3.5(b) ........................................ 42

Figure 3.8 CLT (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.984 aspect ratio. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the OR of the annulus .................................................................................................................. 44
Figure 3.9 CLT (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.8 aspect ratio. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the OR of the annulus.

Figure 3.10 CLT (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.5 aspect ratio. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the OR of the annulus.

Figure 3.11 CLT (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.2 aspect ratio. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the OR of the annulus.

Figure 3.12 CLT (a) Frequency-wavenumber, (b) phase velocity, and (c) group velocity dispersion curves for a multilayered annulus with a 0.8 aspect ratio. Each layer is 4 $\text{mm}$ thick. Curves are also shown for the isolated Layer 1. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the common interface of the annuli.

Figure 3.13 CLT Frequency-wavenumber dispersion curves for the multilayered annulus described in Table 3.1. Magnified views illustrate the repulsion of CLT-modes in multilayered annuli.

Figure 3.14 Wave structures and stresses for the points $a$, $b$, and $c$ identified in Figure 3.12(b).

Figure 3.15 Wave structures and stresses for the points $d$, $e$, and $f$ identified in Figure 3.12(b).

Figure 3.16 Angular phase velocity dispersion curves for the 3-layered annulus described in Table 3.2. Curves are also shown for the isolated Layer 1 and Layer 3.

Figure 3.17 Wave structures corresponding to points $a$ and $b$ of Figure 3.16.

Figure 3.18 Bessel functions of the first and second kind of order (a) 0, (b) 1, and (c) 100.

Figure 3.19 Frequency-angular wavenumber dispersion curves for CSH-waves propagating in a 2-layer annulus as described in Table 3.3. It is seen that double precision eventually fails to find roots at lower frequencies because of the large angular wavenumber.

Figure 4.1 Graphical illustration of the complex modulus showing the storage modulus, loss modulus, and loss tangent.

Figure 5.1 Model used for the development of the SAFE formulation.
Figure 6.1 CSH (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.984 aspect ratio. Results shown for the analytical formulation (red dots), SAFE formulation (blue crosses), and for a flat plate (black line). Linear wavenumber, phase, and group velocity were calculated at the OR.................................85

Figure 6.2 CSH (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.2 aspect ratio. Results are shown for the analytical formulation (red dots), SAFE formulation (blue crosses), and for an actual plate (black line). ........................................................................................................86

Figure 6.3 Phase velocity dispersion curves for CSH-waves propagating in a 2-layer annulus as described in Table 3.3. Analytical (red dots) and 200 element SAFE (blue lines) results shown. SAFE solution provides an accurate solution in the regions which the analytical method fails, below the black dashed line. Linear phase velocity was calculated at the common interface........................................87

Figure 6.4 Wave structures corresponding to labeled points in Figure 6.3. Calculations performed with SAFE method. The number of elements used is indicated in each plot (# EL).................................................................................................................................88

Figure 6.5 Linear phase velocity dispersion curves for an annulus with an aspect ratio of 0.5 and a shear-wave velocity of 0.9 mm/μs. The phase velocity was calculated at the OR. A small frequency range is shown to easily visualize the error.................................90

Figure 6.6 CSH (a) Frequency-wavenumber, (b) phase velocity, and (c) attenuation dispersion curves for a 12.7 mm thick polyethylene annulus of aspect ratio 0.987 (blue dots) and for a flat plate (black line). The annulus curves were calculated at the OR. .................................................................................................................92

Figure 6.7 CSH (a) Frequency-wavenumber, (b) phase velocity, and (c) attenuation dispersion curves for a 12.7 mm thick polyethylene annulus of aspect ratio 0.5 (blue dots) and for a flat plate (black line). The annulus curves were calculated at the OR....94

Figure 6.8 CLT (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.984 aspect ratio. Results shown for the analytical formulation (red dots), SAFE formulation (blue crosses), and for a flat plate (black line). Annulus curves were calculated at the OR .................................................................96

Figure 6.9 CLT (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.2 aspect ratio. Results shown for the analytical formulation (red dots), SAFE formulation (blue crosses), and for a flat plate (black line). Annulus curves were calculated at the OR ............................................................................................................97

Figure 6.10 Phase velocity dispersion curves for CLT-waves propagating in a 2-layer annulus as described in Table 3.3. Analytical (red dots) and 200 element SAFE (blue
lines) results shown. SAFE solution provides an accurate solution in the regions which the analytical method fails, below the black dashed line. Linear phase velocity was calculated at the common interface.................................98

Figure 6.11 CLT (a) phase velocity and (b) attenuation dispersion curves for a polyethylene annulus of aspect ratio 0.992 obtained using the SAFE method with 300 elements. Plot (c) shows a larger portion of the attenuation axis. Linear phase velocity and attenuation were calculated at the OR..........................................................99

Figure 6.12 Linear (a) phase velocity and (b,c) attenuation dispersion curves for CLT-waves propagating in a 2-layer annulus as described in Table 6.3. The 125-element SAFE solution is shown for both the elastic (red dotted lines) and viscoelastic (blue dots) cases. Linear phase velocity and attenuation were calculated at the common interface. The black solid lines are for a flat steel plate of the same thickness as the steel annulus layer...............................................................101

Figure 6.13 Comparison of wave structure and stress distribution as calculated using the analytical and SAFE techniques for CSH-waves. Plots (b), (d), and (f) correspond to the points b, d, and f shown in Figure 3.5(b).................................................................103

Figure 6.14 Comparison of wave structures as calculated using the analytical and SAFE techniques for CLT-waves. Plots (d), (e), and (f) correspond to the points d, e, and f shown in Figure 3.12(b)...................................................................................104

Figure 7.1 CSH (a) phase velocity, (b) group velocity, and (c) attenuation dispersion curves for a 24 in schedule 10 pipe with various thicknesses of the Bitumastic 50 coating described in Table 7.2. Linear phase and group velocity and attenuation were calculated at the OR. ........................................................................107

Figure 7.2 Experimental setup for coating thickness influence study. ..................................................108

Figure 7.3 Analytic envelopes of the wave packets that have traveled through the 2 ft long circumferential section of pipe with different coating thicknesses, showing increased time-of-flight and decreased amplitude with increasing coating thickness. .....................110

Figure 7.4 Plots showing the (a) time-of-flight and (b) amplitude change as coating thickness is increased. .....................................................................................................110

Figure 7.5 CSH (a) phase velocity, (b) group velocity, and (c) attenuation dispersion curves for tar-glass-felt coated pipe described in Table 7.2. Linear phase and group velocity and attenuation were calculated at the IR. Viscoelastic, elastic, and bare pipe curves shown for comparison...............................................................114

Figure 7.6 Wave structures corresponding to points shown in Figure 7.5.........................................115

Figure 7.7 Magnified view of attenuation dispersion curves shown in Figure 7.5(c). Attenuation units have been converted to dB/m.................................................................116
Figure 7.8  Cross-sectional view of 130 kHz CSH-wave traveling in a 8 in. diameter tar-glass-felt coated pipe with an aspect ratio of 0.94. Note that one traversal around a 24 in. diameter pipe is approximately equivalent to 3 traversals around the 8 in. diameter pipe.

Figure 7.9 RF waveforms showing comparison between 130 kHz CSH-wave traveling in a bare pipe (blue) and in a pipe with a 3 mm tar-glass-felt coating (magenta). Two pulses are seen for each traversal because there are waves traveling in both the CW and CCW directions.

Figure 7.10 Experimental setup showing the unwrapped pipe circumference and a sample waveform with labeled wave packages. The reference pulse may be used to normalize the subsequently received pulses.

Figure 7.11 Ultrasonic waveforms obtained from a 20”-diameter schedule 10 pipe with a tar-glass-felt coating with a (a) 1ft. disbond, (b) 2ft. disbond, and for a (c) bare pipe.

Figure 7.12 STFTs of RF-waveforms for a (a) 1ft disbond, (b) 2ft disbond, and for a (c) bare pipe. Results obtained using a 64-point Hanning window with 32-point overlap.

Figure 7.13 A plot showing the time, frequency, and amplitude disbond detection features for three different disbond sizes in a tar-glass-felt coated pipe.

Figure B.1 Lorentz EMAT configurations for generating Lamb-waves with (a) in-plane displacements and (b) out-of-plane displacements (Thompson 1973b).

Figure B.2 Lorentz EMAT configuration for generating SH-waves (Thompson 1990; Hirao et al. 2003).

Figure B.3 Phase velocity dispersion curves showing potential excitation points for a Lamb-wave EMAT with fixed wavelength $\lambda$.

Figure C.1 (a) Linear phase and (b) group velocity dispersion curves for a 20 in diameter schedule 10 pipe (6.35 mm). Curves also shown for 30%, 50%, and 70% thickness ($t_w$). Linear phase and group velocity were calculated at the IR.

Figure C.2 Finite element model showing original thickness (left) and with gradual wall loss to 70% of the original thickness.

Figure C.3 (a) Theoretically predicted wave structures for the $LT_0$ and $LT_1$ modes in the 100% thickness annulus at a frequency of 175 kHz. (b) Example RF waveform and analytic envelopes for $u_r$ and $u_\theta$ displacement. The transmitted $LT_1$ and $LT_0$ modes are clearly seen.

Figure C.4 (a) Time gated view of the analytic envelopes of the $LT_1$ and $LT_0$ waves for the different gradual wall thickness ($WT$) reduction models. (b) Graphical representation of the change in TOF of the $LT_1$ and $LT_0$ modes.
LIST OF TABLES

Table 3.1 Geometric and material parameters used in the generation of the plots shown in Figure 3.5 through Figure 3.7 and Figure 3.12 through Figure 3.15 ......................................................... 39

Table 3.2 Geometric and material parameters used in the generation of the plots shown in Figure 3.16 ....................................................................................................................... 51

Table 3.3 Geometric and material parameters used in the generation of Figure 3.19 .............. 56

Table 6.1 Error in cutoff frequency for $SH_8$ mode shown in Figure 6.5 ................................. 90

Table 6.2 Dimensions of 0.987 aspect ratio annulus used in the calculation of curves in Figure 6.6 and properties for a polyethylene material (Bartoli et al. 2006) ................. 91

Table 6.3 Dimensions of a 4 in diameter schedule 40 pipe with a 1 mm polyethylene coating, yielding an aspect ratio of 0.879 for the multilayered structure. ......................... 100

Table 7.1 Dimensions and properties of pipe specimen used for coating thickness study. Properties for Bitumastic 50 coating from (Barshinger et al. 2004). ......................... 106

Table 7.2 Dimensions and properties of pipe specimen used for coating disbond detection study .......................................................................................................................... 113

Table 7.3 Amplitude-loss, time-of-flight, and frequency loss for disbonded coating regions ................................................................................................................................. 122
ACKNOWLEDGEMENTS

I truly hope that the quality of this work is representative of the quality of family, friends, and colleagues that have helped me along the way. There is no doubt that this would not have been possible without their support and encouragement. Particularly, I would like to thank the following people:

My wife, Laura, who without hesitation agreed to relocate herself and her career to State College so that I could pursue graduate studies. In seemingly insurmountable times, her keen wit has kept me laughing and her words of encouragement have kept me motivated. This moment would not be as meaningful without her to share it with. Also, she has willingly proofread this document.

My family, especially my mother and sister, whose love knows no bound. Their constant encouragement and unfailing ability to “look at the bright side,” have not only inspired me, but dramatically affected the person I am today. I am also forever indebted to my father, who truly loved his children more than anything else. He would have been the proudest father of them all to have seen my years of diligent work come to fruition.

My advisor, Dr. Joseph Rose, who above all else, has taught me that I am capable of much more than my self-perceptions have previously led me to believe. As his pupil my education has extended far beyond the topic of guided waves and into the realms of business, communication, and philosophy. I am tremendously grateful for the opportunities that I have been afforded as part of his group.
My other dissertation committee members, Drs. Bernhard Tittmann, Cliff Lissenden, and Ghassan Chehab. I have found their classes and discussions incredibly useful and their thoughtful reviews of this work have served only to make it better. For this I am very thankful.

My lab mates, who have helped to make my graduate experience a pleasant one. Whether technical or not, our everyday conversations have always been enjoyable and will be missed. I feel very lucky to have been part of such an exceptional group of people

To all of the employees of FBS, Inc. and especially Steve Owens and Russell Love for their help with mechanical design and hardware-control programming, respectively. The quality of their work has directly enhanced the quality of mine.

And to Bruce Nestleroth and the Battelle Memorial Institute for providing the test specimens used in this work. Bruce has also been an exceptional source of practical information regarding the in-line inspection of pipeline.
For my father, who always emphasized the value of education.
Chapter 1

Introduction

The burgeoning of ultrasonic guided-wave usage in the field of nondestructive evaluation is a testament to its efficacy and practical wide-range application. Inaccessible structures can now be inspected more cost-effectively and in less time than previously possible. More specifically, a significant effort has been put forth on the topic of ultrasonic guided-wave inspection of buried pipeline. A litany of studies regarding wave propagation and scattering in free and multilayer hollow cylinders can be found. The case of the multilayer hollow cylinder is of practical importance as the majority of buried pipeline have some variation of protective coating applied to the outer surface. Coating layers, though often ignored, can have a significant effect on the propagation of ultrasonic guided waves; such as significantly reducing propagation distance and altering propagation velocities and wave structures. The primary goal of the work summarized herein, is to create a physically accurate theoretical model for guided waves traveling in the circumferential direction of both elastic and viscoelastic multilayered annuli and to use this model to develop some practical insights into the non-destructive evaluation of coated pipes and pipe-like structures.

1.1 Motivation

The United States is the planet’s largest consumer of natural gas resources and the planet’s second-largest producer of natural gas resources (CIA 2007). To accommodate the transmission, storage, and distribution of this immense volume of product, the United States has constructed and must maintain a vast infrastructure of pipeline and gas storage wells. To prevent
casualties, injuries, loss of product, destruction of property and environment, and to reduce down-time, operators routinely inspect for defects in their pipeline. Figure 1.1 shows a cause-of-failure breakdown for natural gas transmission line over the last nearly 20 years (PHMSA 2007). From this figure it is seen that approximately one quarter of all natural gas transmission pipeline failures are caused by corrosion. Furthermore, if all non-preventable failures are removed from the chart, such as excavation damage, it is seen that corrosion is the dominant cause of all preventable failures. Preventable failure is defined here to mean a pipeline failure that may have been avoided if preventative measures had been utilized successfully.

At present, the most common NDE tool used to detect corrosion over long distances of pipeline is the Magnetic-Flux Leakage (MFL) In-Line Inspection (ILI) tool. The MFL technique saturates the wall of the pipe with a magnetic field and when a metal-loss defect is encountered, magnetic flux escapes from the wall and is detected by Hall-Effect sensors. While this method is reliable for corrosion detection, it has limited corrosion sizing capabilities. Additionally, the MFL

![Failure Cause Breakdown](image)

**Figure 1.1** Failure cause breakdown for natural gas transmission pipeline in the U.S. from 1987 through 2006 (PHMSA 2007).
technique is generally not capable of detecting closed-face cracks that are oriented parallel to the flux lines. This is a significant drawback as Stress Corrosion Cracking (SCC) is a serious ailment in aging pipeline infrastructures.

Since ILI tools are the only cost-effective option for evaluating long spans of pipeline (>100 miles) in a relatively short amount of time (hours to days), it would be useful to use guided waves excited from an ILI tool-mounted system to improve the defect detection and sizing capabilities of the current MFL tools. Several attempts at this have been made to date with results ranging from poor to decent. In many cases, while the tools used to excite and receive the ultrasound are quite sufficient, it is actually a lack of understanding of wave mechanics that ultimately leads to the selection of non-optimal modes/frequencies. Even in the event that the selected mode/frequency is sufficient for the desired task, a thorough review and understanding of the applicable wave mechanics can often lead to new, and often unobvious, contrast features.

While the detection, classification, and sizing of defects is a critical topic, there has been much interest recently in preventing the onset of corrosion. This is actually done by inspecting the pipe’s protective coating layer for disbonds or other faults. Since the protective layers are non-ferrous in nature, MFL is not capable of detecting such coating defects. In contrast, guided waves are sensitive to coating layers and provide several physically based features that have the potential to separate faulty coating from well-bonded coating. This topic is addressed in detail later in this work.

The primary focus of this work is the development of an accurate model for circumferential guided waves in single and multilayered structures of both elastic and viscoelastic nature. These models can then be used for the development of physically-based NDE methods for defect detection, classification, and sizing and for coating integrity analysis in an ILI environment. Circumferential guided waves are the main focus as full circumferential coverage can be obtained with a minimal number of sending and receiving transducers. If an axial
orientation were to be used, full coverage could only be obtained by increasing the number of sensor pairs until the entire circumference is populated. This approach would be more costly and the tool would become excessively cumbersome and require a much more significant power supply. Additionally, circumferential guided waves are well suited for the detection of axially oriented SCC, which is a major problem for pipeline using industries.

1.1 Literature Review

As the work presented here would not have been possible without the past contributions of others in the field, it is necessary and beneficial to review these works. A review of several of the hallmark papers and works in guided-wave propagation will be presented; followed by a more concentrated review of the recent accomplishments relevant to the work discussed herein. For a thorough review of wave propagation in bounded and unbounded elastic solids, one is pointed to the texts by (Kolsky 1963), (Viktorov 1967), (Auld 1973), (Graff 1991), (Achenbach 1975), and (Rose 1999).

1.1.1 Hallmark Guided Wave Studies

The first comprehensive physical analyses of guided wave propagation emerged in the latter part of the 19th century. At this time, (Pochhammer 1876) and (Chree 1886) were studying wave propagation in infinitely long cylindrical rods. (Lamb 1917), (Rayleigh 1945), and (Love 1944) published work regarding the exact elastic solution for wave propagation in a medium bounded by two parallel surfaces. Rayleigh, who later became a Nobel laureate in physics for his contributions in chemistry, is predominantly known in the field of acoustics for his analysis of the surface wave (Rayleigh 1885), which today is commonly referred to as the Rayleigh Wave.
(Rayleigh 1945) and (Love 1944) also presented approximate solutions for the axisymmetric wave motion in hollow cylinders using shell theory. Though approximate, the shell-theory solution served as a stepping stone for the work discussed in the next section.

1.1.2 Axial Wave Propagation in an Elastic Hollow Cylinder

A majority of the work regarding wave propagation in hollow cylinders that has been completed to date has concentrated on propagation in the axial direction of the cylinder. As stated previously, initial attempts to characterize the axial wave motion in hollow cylinders utilized shell theories. Approximations of this type were limited to axisymmetric propagation.

Improvements to the shell theory approximation were made by (Mirsky et al. 1957) and (Mirsky et al. 1958) who added considerations for thick cylindrical shells and non-axisymmetric motion. (Ghosh 1923/24) was the first to abandon shell theory approximations and develop an exact solution, based on the theory of elasticity, for the axisymmetric propagation of longitudinal vibrations in a hollow cylinder. Expanding on Ghosh’s work, (Gazis 1959a; Gazis 1959b) developed a three-dimensional theory of wave propagation in elastic hollow cylinders, with consideration for both axisymmetric and non-axisymmetric propagation. This is the theory predominantly used today.

As numerical computation ability began to drastically increase in the 1970’s, (Zemanek 1972) revisited the problem of elastic wave propagation in a hollow cylinder and generated real, imaginary, and complex dispersion curves, and radial displacement and stress distributions. Adding another level of complexity, (Ditri et al. 1992; Ditri 1994) provided a theoretical study of the application of surface tractions to hollow cylinders and of the interaction of elastic waves with circumferential defects in hollow cylinders, respectively.
Some of the most recent and practically significant work to be completed recently deals with the concept of guided-wave focusing, that is, forced constructive interference by use of a phased array of transducers. The idea of using non-axisymmetric loading to create constructive interference at some desired point within a pipe arose from source influence and non-axisymmetric wave propagation studies performed by (Ditri et al. 1992) and (Li et al. 2001), respectively. In these studies a Normal Mode Expansion (NME) technique is used to predict the amplitude coefficients for all propagating wave modes. (Li et al. 2002) first demonstrated the ability to achieve energy focusing by using appropriately time-delayed and amplified excitation signals. The focusing theory was expanded to include torsional modes by (Sun et al. 2003; Sun 2004; Sun et al. 2005). (Zhang 2005) provides a comprehensive review of the focusing principle with experimental verification and numerical verification using finite-element methods. The development of guided wave focusing has lead to significantly improved penetration power and defect sizing and location sensitivity.

1.1.3 Circumferential Wave Propagation in an Elastic Hollow Cylinder

As the primary theme of the work being presented here, the topic of circumferential wave propagation in hollow cylinders is extremely relevant. The amount of work published in this area is relatively terse in nature when compared to the body of work relating to wave propagation in the axial direction of hollow cylinders and much of the work addressing practical applications using circumferential guided wave have actually used plate models.

The topic of wave propagation in cylindrical layers is first seen in (Viktorov 1967). In his text, Viktorov identifies the major physical differences between wave propagation in cylindrical structures and planar structures and specifically addresses the topics of Rayleigh waves on concave and convex surfaces and Lamb [type] waves in cylindrical layers. He defines the concept
of angular wavenumber, which is a unique physical phenomenon to cylindrically curved waveguides. In his treatment of Lamb [type] waves in a cylindrical layer, Viktorov forms the characteristic equation for an elastic single layer. Due to the limited computational abilities of the time, Viktorov makes several simplifications in order to arrive at a first approximation. Specifically, he replaces the Bessel and Neumann functions in the characteristic equation with asymptotic representations in terms of semiconvergent Debye series.

Though the surface wave problem was revisited by several authors throughout the years following the publication of Viktorov’s book, the topic of circumferential guided-wave propagation in a hollow cylindrical structure was not addressed again until (Qu et al. 1996; Liu et al. 1998a) who presented the first numerical examples of the dispersion curves and wave structures for time-harmonic Lamb [type] waves propagating in a circular annulus. Shortly thereafter, (Liu et al. 1998b) published a solution of transient wave propagation in a circular annulus using the mode eigenfunction expansion, previously referred to as NME, technique. In their work they identified the ideal modes for detecting radial cracks on the inner wall of an annulus. A similar investigation was completed by (Li et al. 2000) in which they identified the optimal angle-of-incidence in order to detect radial cracks on the inner surface of an annulus.

All the previous work concentrated on Lamb [type] wave propagation, but another important class of circumferential wave problem is that of the shear-horizontal (SH-) wave. Although the solution to this type of problem is fundamentally simpler, it is significant as SH-waves have some desirable properties from a practical perspective; such as excellent mode isolation capabilities, insensitivity to viscous materials, and pure excitation with Electromagnetic Acoustic Transducers (EMATs). The problem of time-harmonic SH-wave propagation in a single-layer annulus was first addressed by (Zhao et al. 2004). In their article the authors derived the dispersion equation for circumferential SH-waves and presented several numerical examples of dispersion curves and wave structures. Several different solution cases were compared to the
plate assumption. In a sense, (Zhao et al. 2004) is the SH-wave analog of (Liu et al. 1998a), that addressed Lamb [type] waves.

(Valle et al. 1999) were the first authors to address the propagation of circumferential Lamb [type] waves in a multilayered cylinder, however, their solution was specific to the case of a solid shaft within a cylindrical layer. Furthermore, they were modeling a rotating shaft within a housing and therefore assumed a free-sliding condition at the interface between the shaft and the cylindrical layer, further reducing the generality of their solution. Dispersion curves and wave structures were presented for the sliding shaft problem and it was shown in an appendix that the free-sliding interface is not capable of supporting a Stoneley wave. The authors note the same conclusion for a “frictional” interface. Though it can be considered a multilayer problem, the model developed by Valle et al is so highly specific that its application is very limited. A more general $n$-layer solution is needed that is applicable to hollow structures.

Of particular interest here is the application of circumferential guided waves for the non-destructive testing of gas transmission pipeline and storage well casings. With the exception of (Luo, Rose et al. 2004), who examined the use of circumferential SH-waves for axial-crack detection in pipe, a majority of the work done in this area has used a plate assumption, such as (Hirao et al. 1999) who developed a phased-shift detection and depth estimation technique for axial cracking. (Nestleroth et al. 2002) were the first to study the topic of disbond detection using guided waves implemented with In-Line-Inspection (ILI) tools. In their work, Electromagnetic Acoustic Transducers (EMATs) were employed using the magnetic field present from a Magnetic Flux Leakage (MFL) tool to generate guided Lamb-type waves, which propagated for a short distance along the axial direction of the pipe. Signal amplitude was the single discretionary feature for coating disbond detection. Disbond detection was realized but it was determined that other detection features were needed for a more reliable inspection. Also, the technique did not work as well for thinner coatings such as Fusion Bonded Epoxies (FBEs).
(Aaron et al. 2003) studied the detection and sizing of Stress Corrosion Cracking (SCC) using EMATs deployed from an ILI tool. A Magnetostrictive EMAT technique was employed to generate circumferentially traveling Shear-Horizontal and Shear-Vertical guided waves. A bench scale ILI tool was constructed and used to perform sizing studies on SCC. Coating disbondment was not addressed. (Al-Qahtani et al. 2008) demonstrated a fully operational EMAT tool for the detection of SCC and coating disbondment detection. A circumferential SH-wave technique was employed but, as was the case in the work done by (Nestleroth et al. 2002), signal amplitude was the only contrast mechanism used for coating disbondment detection.

All the works reviewed to this point make the assumption of isotropy as this is the most common case encountered in applications. As composite material technology is rapidly improving and becoming more cost-effective, more structures are being made of this type of material. Drive shafts, aircraft fuselage sections, and high-strength piping are all examples of cylindrical structures that are now being made of anisotropic materials. Wave propagation in these types of materials is much more complex as the governing wave equation cannot be decomposed and the wave fields are therefore coupled. (Towfighi et al. 2002) was the first to develop the dispersion equation for wave propagation in what the paper calls an anisotropic curved plate. The authors introduce a new general solution technique for solving a coupled partial differential equation set. The method is based on Fourier Series expansions of the unknown displacement functions. Verification of the technique is accomplished by comparing the dispersion results to the limiting cases of an anisotropic plate, presented in (Rose 1999), and the isotropic cylinder, presented in (Liu et al. 1998a). Recently, a similar work was completed by (Jiandong et al. 2007) in which Legendre orthogonal polynomial series were used to model circumferential wave propagation in an orthotropic hollow cylinder. Results agreed with those presented by (Towfighi et al. 2002). (Jiandong et al. 2007) also studied mode conversion from the end of an orthotropic curved plate using finite-element simulations.
1.1.4 Wave Propagation in Viscoelastic Materials

The protective bituminous and polymeric coating materials that are applied to the outer surfaces of pipe are normally viscoelastic in nature. For this reason it is diligent to consider wave propagation in viscoelastic materials. Both (Christensen 1981) and (Haddad 1995) provide an excellent review of the theory of viscoelasticity with chapters dedicated to the propagation of stress waves in viscoelastic materials. (Christensen 1981) provides a summary of how a solution to a viscoelastic problem can be developed by solving the corresponding elastic problem with complex viscoelastic material parameters substituted for the elastic counterparts. This is known as the Correspondence Principle.

Because of the recent industrial interest in long-range guided-wave inspections, the problem of axial wave propagation in an elastic hollow cylinder coated with viscoelastic material has received much attention. (Barshinger 2001) presents the analytical framework for axial wave propagation in a hollow cylinder with viscoelastic coating. Additionally, a technique for the acoustic measurement of the viscoelastic material properties is presented for bulk material samples. A concise review of the wave mechanics of the coated pipe system can be found in (Barshinger et al. 2002). Guided-wave scattering, phased-array focusing, and finite-element modeling of wave propagation in hollow cylinders with viscoelastic coatings were studied by (Luo 2005) and (Luo et al. 2005; Luo et al. 2007).

Because the properties of viscoelastic materials vary with time, temperature, exposure to UV radiation, etc., in situ material property measurements are often needed if a specific coated pipe system is to be modeled. To this end, several methods are reviewed here. One is pointed to (Christensen 1981) for a general discussion regarding the measurement of viscoelastic material properties using ultrasonic waves. (Barshinger 2001) introduces a water immersion bulk-wave technique for property measurement. Though it has been found to be time consuming and
impractical, (Luo 2005) was the first to present a truly unobtrusive technique for measuring viscoelastic material properties using ultrasonic waves. (Simonetti 2003) also has developed a technique for the measurement of viscoelastic material properties though the proposed technique requires access to a bulk sample of the coating in liquid form or, if a bulk sample is not available, removal of coating from the pipe under study. Removing the coating from the pipe is not only destructive but it can alter the properties of the coating, essentially invalidating the measurement. Other than the property measurement techniques, (Simonetti 2003) provides an in-depth study of SH- and Lamb wave propagation in plates with viscoelastic bilayers. (Simonetti et al. 2004) provide another presentation of SH-wave propagation in an elastic plate with both elastic and viscoelastic bilayers.

The in situ method used by the present author, to obtain viscoelastic properties, is described in (Van Velsor 2006). The technique uses a reflection method to approximate coating density and a bulk wave pulse-echo technique to measure ultrasonic attenuation. Several other techniques such as the Normalized Amplitude Spectrum, developed by (Guo et al. 1995) based on work by (Haines et al. 1978) are reviewed in (Van Velsor 2006).

1.1.5 The Semi-Analytical Finite-Element Method

When analytical solutions of the phase velocity dispersion curves are not possible or are incomplete, the Semi-Analytical Finite Element (SAFE) method provides an accurate and relatively fast alternative. SAFE methods are most commonly used for structures with arbitrarily shaped cross-sections as closed-form solutions are not available. The SAFE method will be employed in this work as one means of acquiring the roots of the dispersion equations for circumferential guided waves and as such a review of the relevant literature is obligatory.
(Hayashi, Song et al. 2003) provides a concise review of the background of the SAFE technique as well as the governing SAFE equations for wave propagation in an arbitrary cross-section structure. A rod and rail example are used to demonstrate the generality of the method. (Hayashi, Kawashima et al. 2003) utilize the SAFE method to study flexural mode propagation and guided-wave focusing in the axial direction of a hollow cylinder. (Bartoli et al. 2006) provide an approachable review of the SAFE method for arbitrary cross-sections and make considerations for viscoelastic materials. Examples are provided for elastic, viscoelastic, and anisotropic plates and rail. (Mu et al. 2008) utilize the SAFE method to study the propagation of guided waves along the axial direction of pipe coated with viscoelastic material. Flexural modes are studied and a mode differentiation technique based on the orthogonality property is introduced.

1.2 Overview of Chapters

Chapter 2 presents the analytical formulation for the propagation of steady-state time-harmonic waves in an elastic annulus. Both circumferential shear-horizontal and Lamb [type] waves are addressed. The characteristic equations are developed for a single layer and subsequently extended to account for multilayered annuli through the use of the Global Matrix Method. The topic of eigenfunctions is also addressed.

Numerical examples based on the analytical formulation of Chapter 2 are presented in Chapter 3. The effect of decreasing aspect ratio on the dispersion curves is analyzed for the single-layer case for circumferential shear-horizontal and Lamb [type] waves. Dispersion curves are also shown for multilayered annuli and observations regarding the physical nature of circumferential waves are made. Wave structures are shown for several cases and examined for satisfaction of the appropriate boundary and continuity conditions. The chapter concludes with a discussion on the limitations of the computational algorithms based on the analytical formulation.
In anticipation of adding viscoelastic layer considerations to the circumferential wave formulation, Chapter 4 provides a brief review of the linear theory of viscoelasticity as it relates to wave propagation. The constitutive equations are introduced followed by discussions on dynamic viscoelasticity and the elastic-viscoelastic correspondence principle. The chapter concludes with some considerations particularly important to wave propagation in viscoelastic materials.

The Semi-Analytical Finite Element (SAFE) method is applied in Chapter 5 as an efficient means of calculating a complete set of roots of the characteristic equaitons for circumferential SH- and Lamb [type] wave propagation in elastic and viscoelastic multilayered annuli. A general formulation is presented that finds all propagating and non-propagating modes in an annulus. Formulations for calculating the roots associated with circumferential SH-waves only and Lamb [type] waves only are also presented. Expressions are presented for the calculation of the stress fields within the annulus based on the nodal displacement data.

Chapter 6 presents the dispersion results as calculated using the SAFE method introduced in Chapter 5. The results are compared to those in Chapter 3, obtained from the analytical formulation. Phase velocity, group velocity, and attenuation dispersion curves are presented for single-layer viscoelastic annuli and for multilayer elastic/viscoelastic combinations for both circumferential SH- and Lamb [type] waves. A limitation of the SAFE technique for circumferential Lamb [type] waves is noted.

Two experimental demonstrations are presented in Chapter 7. In the first demonstration, a circumferential SH-wave method for coating disbond detection is introduced. A theoretical analysis is completed to determine which wave features will be sensitive to the presence of a coating layer. An experiment is designed in which circumferential waves are propagated through several regions with differently sized coating disbonds.
Chapter 8 summarizes the main findings and contributions of this work. A list of refereed journal publications that have resulted, either directly or indirectly, from this work is provided as well as a list of anticipated publications. Proceedings from several nationally and internationally held conferences are also listed.

Appendix A listed the detailed coefficient matrix components for the case of circumferential Lamb [type] waves. A brief introduction to the fundamental physics and geometries of Lorentz type Electromagnetic Acoustic Transducers (EMATs) is provided in Appendix B. Appendix C gives an introduction to the ABAQUS software theory and introduces the hybrid analytical finite-element philosophy. An example of a hybrid-analytical finite-element approach is provided for circumferential Lamb [type] waves. A method for the estimation of wall thickness based on the analysis of a multi-modal wave signal is introduced. This study is performed using commercial finite-element software, as proper specimens were not readily available.

Appendix D presents the computer codes that were developed for calculating the circumferential wave dispersion curves and wave structures using both the analytical and SAFE methods. Appendix E is a non-technical abstract describing this work.
Chapter 2

An Analytical Solution for Circumferentially Propagating Guided Waves

With modern computing, extremely complex multi-physics problems can easily be solved with numerical approximation methods such as the Finite-Element Method, though despite this, the practice of describing natural phenomenon with analytical expressions continues to be a highly valued one. The primary reason for this is undoubtedly because analytical models provide physical insight that is often lost when numerical approximation methods are used. Following this reasoning, the concentration of this chapter will be the analytical modeling of circumferential guided waves.

In this chapter the governing equations for circumferential SH-waves and circumferential Lamb [type] waves are developed. For brevity, circumferential SH- and Lamb [type] waves will be abbreviated as CSH- and CLT-waves, respectively, from this point onward. Following the development of the single-layer cases, considerations will be made for $n$-layer annuli. The following chapter presents the numerical results following from the analytical derivations presented in this chapter.

The development of the characteristic equations for a single layer annulus is a necessary step toward the development of the multilayer cases. Single-layer considerations for circumferential guided waves have been made by authors such as (Viktorov 1967), (Liu et al. 1998b), (Liu et al. 1998a), and (Zhao et al. 2004). The latter two papers provide detailed derivations of the characteristic equations and wave structures for the single-layer cases of CLT- and CSH-wave propagation, respectively. The development presented here is similar to these two cases but is modified in anticipation of developing the multilayer solutions. Specifically, a generalized boundary value problem will be developed such that the phase term is no longer
associated with the boundary of the annulus, as is the case in (Liu et al. 1998a) and (Zhao et al. 2004). All dimensionless quantities must also be removed from the formulation.

The solution approach used here utilizes the method of displacement potentials and, as a result, applies only to isotropic materials. Additionally, all generalized plane-strain assumptions prevail. Figure 2.1 shows the theoretical model of a single-layer annulus used for the development of the CSH- and CLT characteristic equations.

The development of the characteristic equations begins with the well known Navier’s Equation of Motion, Eq. 2.1 (Graff 1991),

\[(\lambda + 2\mu)\nabla \Delta - 2\mu \nabla \times \omega + \rho f = \rho \ddot{u} ,\]  

where the dilatation of a material, \(\Delta\), is given by

\[\Delta = \nabla \cdot u ,\]  

and the rotation vector, \(\omega\), is given by

\[\omega = \frac{1}{2} \nabla \times u . \]  

\[\text{Figure 2.1} \text{ Theoretical model used for the development of the governing equations for CSH- and CLT- wave propagation in a single-layer annulus.} \]
In Eq. 2.1, \( \rho \) is the material density, \( \lambda \) and \( \mu \) are Lamé Constants, \( \mathbf{f} \) is the body-force vector, and \( \mathbf{u} \) is the displacement vector. Note that the form of Navier’s Equation shown in Eq. 2.1 has the advantage of applicability in any curvilinear coordinate system and the convenience of separated dilatational and rotational components (Graff 1991). From this point forward, it is assumed that there are no body forces present.

For the case of circumferential guided-wave propagation, the displacement vector is given by Eq. 2.4,

\[
\mathbf{u} = u_r(r, \theta, t)\hat{e}_r + u_\theta(r, \theta, t)\hat{e}_\theta + u_z(r, \theta, t)\hat{e}_z. \tag{2.4}
\]

It is seen that, because generalized plane-strain assumptions have been made, the displacement components are functions of \( r \), \( \theta \), and \( t \) only. This does not infer that there cannot be displacement in the \( z \)-direction but, instead, that any \( z \)-displacement must be uniform throughout the entire \( z \)-plane. The specific cases of CSH- and CLT-wave propagation will be discussed next.

### 2.1 Circumferential Shear Horizontal Waves in a Single-Layer Annulus

Shear-horizontal (SH-) waves are waves in which the particle motion is in-plane but orthogonal to the direction of propagation. For the case of CSH-waves, particle displacement would be in the \( z \)-direction with no displacement in the \( r \) or \( \theta \) directions \((u_r = u_\theta = 0)\). Also, from the generalized plane-strain assumption, there must be no variation of any quantity in the \( z \)-direction \((\partial/\partial z = 0)\). Under these conditions, Eq. 2.1 simplifies to

\[
\mu \nabla^2 u_z = \rho \frac{\partial^2 u_z}{\partial t^2} \quad \text{or} \quad \nabla^2 u_z = \frac{1}{c_s^2} \frac{\partial^2 u_z}{\partial t^2}. \tag{2.5}
\]
Note that for the CSH-wave case, a scalar form of the governing displacement equation of motion has been obtained with no need for the use of Helmholtz decomposition. Two forms of the wave equation are shown in Equation 2.5; from which it can be determined that bulk shear waves propagate with a velocity given by

\[ c_s = \sqrt{\frac{\mu}{\rho}}. \]

Assuming time-harmonic motion of the form \( e^{-i\omega t} \) and propagation in the \( \theta \)-direction, one solution of Eq. 2.5 is assumed to be

\[ u_z = \psi(r)e^{i(\rho \theta - \omega t)}, \]

where \( \rho \) is the angular wavenumber (Viktorov 1967; Liu et al. 1998a). It is important to distinguish between the circular wavenumber \( k \) and the angular wavenumber \( \rho \) as the terms circular and angular are often used interchangeably. In this work, the angular wavenumber, \( \rho \), refers specifically to the circular wavenumber, \( k \), multiplied by some radius, \( R \), at which the linear phase velocity is to be determined. It is a dimensionless quantity. Equation 2.8 summarizes this relationship.

\[ \rho = kR. \]

From Eq. 2.8 it is seen that there are two different ways to proceed: either assume a radius and solve for the \( k \)-roots of the characteristic equation at the assumed radius or find the \( \rho \)-roots and then calculate the \( k \)-roots at any radius. For multilayered annuli it will be shown here and in the next section that finding the roots of the characteristic equation in the \( \omega - \rho \) domain provides the most general solution that can subsequently be used to determine the linear phase velocity, \( c_p \), at an arbitrary radius of the multilayered structure by the use of Eqs. 2.9 and 2.10, first introduced by (Viktorov 1967).
From Eq. 2.10 it is seen that there is a linear increase in phase velocity, \( c_p \), from the internal to external surface of the annulus. This increase in phase velocity with radius is necessary to maintain a constant phase front through the thickness of the annulus. This phenomenon is unique to wave propagation along structures that are curved in the direction of wave propagation. Note that this does not include wave propagation in the axial direction of a hollow cylinder as it is typically assumed that the structure is straight in this direction.

Substituting Eq. 2.7 into Eq. 2.5 yields Eq. 2.11, which is a second order ODE, commonly solved using Bessel or Hankel functions (Hayek 2001). The solution of Eq. 2.11 in terms of Bessel functions is of the form

\[
\psi(r) = A_1 J_p(k_s r) + A_2 Y_p(k_s r),
\]

where

\[
k_s = \omega/c_s
\]

is the circular wavenumber of a bulk shear wave.

To solve for the unknown coefficients in Eq. 2.12, the boundary conditions for the problem of interest must be considered. For the case of a single-layer “free” annulus, traction-free boundary conditions are assumed; that is, stress is required to vanish on the inside and outside surfaces of the annulus. For the case of CSH-waves, shear stress is given by
\[ \sigma_{rz} = \mu \frac{\partial u_r}{\partial r} . \]  

Using the recurrence relation for Bessel functions (Abramowitz et al. 1964),

\[ 2 \xi'_r(x) = \xi_{r-1}(x) - \xi_{r+1}(x), \quad \text{where} \quad \xi = J \text{ or } Y, \]

and with the use of Eq. 2.7 and Eq. 2.12, Eq. 2.14 can be written as

\[ \sigma_{rz} = s(r)e^{i\omega t}, \]

where

\[ s(r) = \mu \frac{k}{2} \left\{ A_1 \left( J_{p-1}(k_r) - J_{p+1}(k_r) \right) + A_2 \left( Y_{p-1}(k_r) - Y_{p+1}(k_r) \right) \right\}. \]

The \( e^{i\omega t} \) time dependence is inferred but not explicitly written in Eq 2.16 or Eq. 2.17; as will be the case for the remainder of this work.

In preparation for the case of multiple layers, it is convenient to summarize the relevant stress and displacement eigenfunctions in a single matrix equation, as seen in Eq. 2.18.

\[
\begin{bmatrix}
J_p(k_r) & Y_p(k_r) \\
\mu \frac{k}{2} \left( J_{p-1}(k_r) - J_{p+1}(k_r) \right) & \mu \frac{k}{2} \left( Y_{p-1}(k_r) - Y_{p+1}(k_r) \right)
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
= 
\begin{bmatrix}
\psi(r) \\
s(r)
\end{bmatrix}.
\]

For the single-layer case, only the stress-related component of Eq. 2.18 is needed, whereas for the multilayer case, both the displacement- and stress-related components will be needed.

For the case of the single-layer annulus, the traction-free boundary conditions require that

\[ \sigma_{rz} \bigg|_{r_1, r_2} = 0, \]

resulting in the set of linear homogeneous algebraic equations given by Eq. 2.20 and Eq. 2.21.

\[ \mathbf{D}(p, \omega)\mathbf{A} = \mathbf{0}, \]

where
\[ D(p, \omega) = \begin{bmatrix} J_{p-1}(k_s r_1) - J_{p+1}(k_s r_1) & Y_{p-1}(k_s r_1) - Y_{p+1}(k_s r_1) \\ J_{p-1}(k_s r_2) - J_{p+1}(k_s r_2) & Y_{p-1}(k_s r_2) - Y_{p+1}(k_s r_2) \end{bmatrix}, \quad (2.21) \]

and \( \mathbf{A} = (A_1, A_2)^T \). Nontrivial solutions to this set of equations are found by setting the determinant of the coefficient matrix, \( D(p, \omega) \), to zero;

\[ \det(D(p, \omega)) = 0. \quad (2.22) \]

Equation 2.22 is the characteristic equation for CSH-waves in a single-layer annulus; the eigenvalues of which form the frequency-wavenumber dispersion curves. The angular phase and group velocity dispersion curves can then be calculated using the relations seen in Eq. 2.9 and Eq. 2.23, respectively.

\[ \alpha_g = \frac{\partial \omega}{\partial p}. \quad (2.23) \]

Once the eigenvalues of the characteristic equation have been determined, it is possible to solve for the displacement and stress distribution throughout the wall thickness of the annulus for any given mode and frequency combination. The displacement field is given by Eq. 2.24,

\[ u_z = \left( J_p(k_s r) - \Lambda Y_p(k_s r) \right) e^{i(p \theta - \omega t)}, \quad (2.24) \]

where

\[ \Lambda = \frac{J_{p-1}(k_s r_1) - J_{p+1}(k_s r_1)}{Y_{p-1}(k_s r_1) - Y_{p+1}(k_s r_1)}. \quad (2.25) \]

The nonzero stress components are given by Eq. 2.26 and Eq. 2.27.

\[ \sigma_{rz} = \frac{k_s \mu}{2} \left( J_{p-1}(k_s r) - J_{p+1}(k_s r) - \Lambda \left( Y_{p-1}(k_s r) - Y_{p+1}(k_s r) \right) \right) e^{i(p \theta - \omega t)}. \quad (2.26) \]

\[ \sigma_{\theta z} = \frac{\mu}{r} \frac{\partial u_z}{\partial \theta} = \frac{i p \mu}{r} \left( J_p(k_s r) - \Lambda Y_p(k_s r) \right) e^{i(p \theta - \omega t)}. \quad (2.27) \]
2.2 Circumferential Lamb [Type] Waves in a Single-Layer Annulus

Lamb waves are waves with displacements in two directions: in-plane and out-of-plane. Unlike SH-waves, the in-plane component for the Lamb-wave case is along the line of propagation. Because Lamb waves are technically a type of plate wave, this work refers to circumferential Lamb [type], or CLT-, waves when referring to propagation in an annulus. This distinction is important because of the physical differences in the propagation characteristics, although, as the ratio of inner diameter to outer diameter approaches unity, the annulus solution approaches that of a plate and the two cases become equivalent (Liu et al. 1998a).

By the theorem introduced by Helmholtz, it is possible to dissect the displacement field, \( \mathbf{u} \), into a sum of the gradient of a scalar and the curl of a zero-divergence vector with the use of the scalar and vector potentials, \( \Phi \) and \( \mathbf{H} \) (Morse et al. 1953). This is shown in Eq. 2.28,

\[
\mathbf{u} = \nabla \Phi + \nabla \times \mathbf{H}, \quad \nabla \cdot \mathbf{H} = 0, \tag{2.28}
\]

where \( \nabla \cdot \mathbf{H} = 0 \) is a necessary condition in order to determine a unique solution for the three displacement components from the total of four components of \( \Phi \) and \( \mathbf{H} \). Substituting Eq. 2.28 into Navier’s Equation, Eq. 2.1, Eq. 2.29 is obtained,
\[ \nabla \left( (\lambda + 2\mu)\nabla^2 \Phi - \rho \ddot{\Phi} \right) + \nabla \times \left( -\mu \nabla \times \nabla \times \mathbf{H} - \rho \ddot{\mathbf{H}} \right) = 0. \]  \hspace{1cm} 2.29

For a non-trivial solution of Eq. 2.29, it is required that,

\[ (\lambda + 2\mu)\nabla^2 \Phi = \rho \ddot{\Phi}, \]  \hspace{1cm} 2.30

and

\[ -\mu \nabla \times \nabla \times \mathbf{H} = \rho \ddot{\mathbf{H}}. \]  \hspace{1cm} 2.31

Recalling Eq. 2.6 and also noting that longitudinal waves propagate at a velocity given by,

\[ c_l = \sqrt{\frac{(\lambda + 2\mu)}{\rho}}, \]  \hspace{1cm} 2.32

it is possible to rewrite Eq. 2.30 and Eq. 2.31 as

\[ \nabla^2 \Phi = \frac{1}{c_l^2} \frac{\partial^2 \Phi}{\partial t^2}, \]  \hspace{1cm} 2.33

and

\[ \nabla^2 \mathbf{H} = \frac{1}{c_s^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}, \]  \hspace{1cm} 2.34

respectively. In Eq. 2.34,

\[ \mathbf{H} = H_r \mathbf{e}_r + H_\theta \mathbf{e}_\theta + H_z \mathbf{e}_z. \]  \hspace{1cm} 2.35

Assuming generalized plane-strain and Lamb [type] wave propagation,

\[ H_r = H_\theta = 0, \]  \hspace{1cm} 2.36

since

\[ u_z = \frac{\partial}{\partial z} = 0. \]  \hspace{1cm} 2.37
Applying the conditions shown in Eq. 2.36 and Eq. 2.37, Eq. 2.34 reduces to the scalar equation shown in Eq. 2.38.

\[ \nabla^2 H_z = \frac{1}{c_s^2} \frac{\partial^2 H_z}{\partial t^2}. \quad 2.38 \]

As was done in the CSH-wave case, solutions to Eq. 2.33 and Eq. 2.38 are assumed to be of the form

\[ \Phi = f(r)e^{i(\omega \theta - \omega t)} \quad 2.39 \]

and

\[ H_z = h_z(r)e^{i(\omega \theta - \omega t)}, \quad 2.40 \]

respectively, where \( f \) and \( h_z \) are functions of \( r \), yet to be determined. Substituting Eq. 2.39 into Eq. 2.33 and Eq. 2.40 into Eq. 2.38, two second-order linear ODEs are obtained, as seen in Eq. 2.41 and Eq. 2.42,

\[ f''(r) + \frac{1}{r} f'(r) + \left[ k_i^2 - \left( \frac{p}{r} \right)^2 \right] f(r) = 0, \quad 2.41 \]

\[ h_z''(r) + \frac{1}{r} h_z'(r) + \left[ k_s^2 - \left( \frac{p}{r} \right)^2 \right] h_z(r) = 0, \quad 2.42 \]

where \( k_s \) was defined in Eq. 2.13 and \( k_i \) is the bulk longitudinal wavenumber,

\[ k_i = \omega / c_i. \quad 2.43 \]

The Bessel function solutions to Eq. 2.41 and Eq. 2.42 are shown in Eq. 2.44 and Eq. 2.45, respectively (Hayek 2001).

\[ f(r) = A_s J_p (k_i r) + A_s Y_p (k_i r). \quad 2.44 \]

\[ h_z(r) = A_s J_p (k_i r) + A_s Y_p (k_i r). \quad 2.45 \]
The term $k_r$ was defined in Eq. 2.13; $k_l$ is defined in Eq. 2.43 and is the wavenumber of a bulk longitudinal wave.

In order to solve for the four unknown coefficients in Eq. 2.44 and Eq. 2.45, it is again necessary to apply traction-free boundary conditions. The normal and shear components of stress are required to vanish at the surfaces of the annulus and the expressions for these components are given by Eq. 2.46 and Eq. 2.47, respectively (Chou et al. 1992; Sadd 2005).

\[
\sigma_r = \lambda \left[ \frac{\partial u_r}{\partial r} + \frac{1}{r} \left( u_r + \frac{\partial u_{\theta}}{\partial \theta} \right) \right] + 2\mu \frac{\partial u_r}{\partial r}. \tag{2.46}
\]

\[
\sigma_{r\theta} = \mu \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right). \tag{2.47}
\]

The particle displacement terms, $u_r$ and $u_{\theta}$, are given by Eq. 2.48 and Eq. 2.49, respectively.

\[
u_r \frac{\partial \Phi}{\partial r} + \frac{1}{r} \frac{\partial H_z}{\partial \theta}. \tag{2.48}
\]

\[
u_{\theta} \frac{\partial \Phi}{\partial r} - \frac{\partial H_z}{\partial r}. \tag{2.49}
\]

By inserting Eqs. 2.48 and 2.49 into Eqs. 2.46 and 2.47 and collecting the equations into a single matrix equation, Eq. 2.50 is formed.

\[
\mathbf{D}(p, \omega) \mathbf{A} = \begin{pmatrix} u_r \\ u_{\theta} \\ \sigma_r \\ \sigma_{r\theta} \end{pmatrix},\tag{2.50}
\]

where
and \( A = (A_1, A_2, A_3, A_4)^T \). The individual components of the coefficient matrix, \( \mathbf{D}(p, \omega) \), are stated explicitly in Appendix A. As was the case for CSH-waves, the stress-related components of Eq. 2.51 are needed in the single-layer case and, therefore, the bottom two rows are required to vanish at the boundaries of the annulus. The eigenvalues of the resulting characteristic equation form the frequency-wavenumber dispersion curves for CLT-waves traveling in a single-layer annulus. The angular phase and group velocity dispersion curves can then be calculated using the relations seen in Eq. 2.9 and Eq. 2.23, respectively.

After the eigenvalues of the characteristic equation have been determined, the displacements \( u_r \) and \( u_\theta \) and the stresses \( \sigma_r \) and \( \sigma_{r\theta} \) can be determined by arbitrarily setting one of the four amplitude coefficients to unity and determining the relative amplitude of the other three. Once the relative values of the four coefficients, \( A_1 - A_4 \), are known, Eq. 2.50 and Eq. 2.51 can be used to solve for the displacements and stresses.

This concludes the development of the single-layer CSH and CLT cases. The next section will discuss the extension of the theoretical models to the case of multilayered annuli. Numerical examples are presented for both cases in Chapter 2 and some limitations on computational capability are discussed.

### 2.3 Extension to Multiple Layer Annuli

The extension of the single-layer cases to the multilayer cases is accomplished by one of two fundamental matrix methods: the Transfer Matrix Method (Thomson 1950; Haskell 1953) or...
the Global Matrix Method (Knopoff 1964). Though it may take longer to arrive at a solution, the Global Matrix Method (GMM) is used here as it is more stable and can handle many categories of problems without modification (Lowe 1995). The underlying strategy of the GMM is to develop the displacement and stress equations for each individual layer and then, by applying the boundary and continuity conditions, it is possible to assemble a Global Matrix representing the entire layered system. The characteristic equation of the layered system is then obtained by setting the determinant of the Global Matrix to zero.

Consider the multilayer annulus shown in Figure 2.2. The GMM approach will be illustrated using the case of CSH-wave propagation as this case is simpler and will make the development of the Global Matrix more transparent. The same approach is equally applicable for CLT-wave propagation as the only difference is in the number of boundary and continuity conditions needed to solve the corresponding system of equations. This difference will be addressed later in this section.

To begin, each layer is treated as a single-layer, and a time-harmonic steady-state solution to Navier’s Equation is assumed. As seen in Eq. 2.52, the assumed solution is valid for layer $m$.

$$u_z^{(m)} = \psi^{(m)}(r)e^{i(\rho_0-\omega t)}, \quad r_m \leq r \leq r_{m+1}.$$  \hspace{1cm} (2.52)

Using a slightly modified version of Eq. 2.12, the expression for $\psi^{(m)}(r)$, in terms of Bessel functions, is

$$\psi^{(m)}(r) = A_1^{(m)} J_p (k_1^{(m)} r) + A_2^{(m)} Y_p (k_2^{(m)} r),$$  \hspace{1cm} (2.53)

where there are two unknown coefficients, $A_{1,2}^{(m)}$, for each layer, $m$. The relevant stress equation for layer $m$ is found from using Eq. 2.53 in Eq. 2.14.
For the CSH case, there is one displacement-related and one stress-related equation for each layer, yielding a total of $2N$ equations ($4N$ for CLT-waves). Therefore, $2N$ boundary/continuity conditions will be required in order to solve the system. As was done in the single-layer case, the innermost and outermost boundaries are assumed to be traction-free. For the case of a multilayered system, some description of the interaction at the interface between two layers must be made. This is done through continuity conditions in which displacement and stress are required to be continuous at the interface. The boundary and interfacial continuity conditions for CSH-wave propagation are summarized in Eq. 2.54. For completeness, the boundary and continuity conditions for the CLT case are also summarized in Eq. 2.55.

$$\left\{ \sigma_{rz} \right\}_{\text{free surface}} = 0, \quad \left\{ \frac{u_z}{\sigma_{rz}} \right\}_{\text{layer } m \text{ interface } m+1} = \left\{ \frac{u_z}{\sigma_{rz}} \right\}_{\text{layer } m+1 \text{ interface } m+1}.$$  \hspace{1cm} 2.54
Next, using Eq. 2.18, the layer matrices are formed for the bottom, $D_B^{(m)}$, and top, $D_T^{(m)}$, of each layer. For CSH-waves, the bottom and top layer matrices for layer $m$ are shown in Eq. 2.56 and Eq. 2.57, respectively.

$$D_B^{(m)} = \begin{bmatrix}
J_p(k_m^m r_m) \\
\mu m k_s^m J_p(k_m^m r_m) - J_{p+1}(k_s^m r_m)
\end{bmatrix} \begin{bmatrix}
Y_p(k_m^m r_m)
\mu m k_s^m Y_p(k_m^m r_m)
\end{bmatrix}$$  \hspace{1cm} 2.56

$$D_T^{(m)} = \begin{bmatrix}
J_p(k_m^{m+1} r_m) \\
\mu m k_s^m J_p(k_m^{m+1} r_m) - J_{p+1}(k_s^m r_m)
\end{bmatrix} \begin{bmatrix}
Y_p(k_m^{m+1} r_m)
\mu m k_s^m Y_p(k_m^{m+1} r_m)
\end{bmatrix}$$  \hspace{1cm} 2.57

For the free-surface boundary conditions only the second row of the above two matrices are needed, which yields the following for the innermost and outermost surfaces (interfaces):

$$\begin{bmatrix}
D_B^{(1)} \\
D_T^{(N)}
\end{bmatrix} \begin{bmatrix}
A_1^{(1)} \\
A_2^{(1)}
\end{bmatrix} = \begin{bmatrix}
D_B^{(N)} \\
D_T^{(N)}
\end{bmatrix} \begin{bmatrix}
A_1^{(N)} \\
A_2^{(N)}
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}$$  \hspace{1cm} 2.58

where the subscript “2” after the $D_{B,T}^{(1,N)}$ matrices indicates that only the second row of the corresponding matrix is being considered as this is the row relating to stress. The two equations shown in Eq. 2.58 will constitute the first and last rows of the Global Matrix. All rows in between result from the interfacial continuity conditions.

Using the notation introduced in Eq. 2.56 and Eq. 2.57, the interfacial continuity conditions can be written as follows:
or, rearranging to obtain a homogeneous form,

\[
\begin{bmatrix}
D_{r}^{(m)} & -D_{g}^{(m+1)} \\
D_{g}^{(m)} & -D_{r}^{(m+1)} \\
A_{1}^{(m)} & A_{2}^{(m+1)} \\
A_{2}^{(m)} & A_{1}^{(m+1)}
\end{bmatrix}
\begin{bmatrix}
A_{1}^{(m)} \\
A_{2}^{(m)} \\
A_{1}^{(m+1)} \\
A_{2}^{(m+1)}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

With both the boundary and continuity conditions expressed in homogeneous form, it is now possible to construct the Global Matrix. This is done by assembling the layer matrices into a single matrix in which the individual matrices are matched according to interface. That is, the top coefficient matrix of layer \( m \) should fall into the same rows of the global matrix as the bottom coefficient matrix of layer \( m+1 \). The global matrix for an \( N \)-layered system is shown on the left in Eq. 2.61.

\[
\begin{bmatrix}
D_{r}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
D_{r}^{(2)} & -D_{g}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & D_{r}^{(m-1)} & -D_{g}^{(m-1)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D_{r}^{(m)} & -D_{g}^{(m+1)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{g}^{(N)}
\end{bmatrix}
\begin{bmatrix}
A_{1}^{(1)} \\
A_{2}^{(1)} \\
\vdots \\
A_{1}^{(m-1)} \\
A_{2}^{(m-1)} \\
\vdots \\
A_{1}^{(N-1)} \\
A_{2}^{(N-1)}
\end{bmatrix} =
\begin{bmatrix}
A_{1}^{(0)} \\
A_{2}^{(0)} \\
\vdots \\
A_{1}^{(N)} \\
A_{2}^{(N)} \\
\vdots \\
A_{1}^{(N)} \\
A_{2}^{(N)}
\end{bmatrix}.
\]

where,

\[
A^{(m)} = \begin{bmatrix} A_{1}^{(m)} \\ A_{2}^{(m)} \end{bmatrix}, \quad \text{and} \quad 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

It is seen from Eq. 2.61 that for CSH-waves traveling in a multilayered system, the result will be a \( 2N \) square Global Matrix. For the case of CLT wave propagation in a multilayered system, the Global Matrix will be a \( 4N \) square matrix since there are two stress components.
which must satisfy the boundary conditions and two stress and two displacement components which must satisfy the interfacial continuity conditions.

After the formation of the Global Matrix, the remainder of the solution process is identical to the single-layer cases; the determinant of the Global Matrix yields the characteristic equation whose roots are the eigenvalues for the structure under study. Once the eigenvalues of the characteristic equation are identified, the unknown constants, \( A_{1,2}^{(1)} \) through \( A_{1,2}^{N} \), can be determined by arbitrarily setting one to unity and the solving for the relative value of the others.

Again, by performing all necessary calculations in \( \omega-p \) space, the angular phase and group velocities can be determined; from which the linear phase and group velocities can be determined at any arbitrarily chosen radius. Alternatively, one may initially choose some radius, \( R \), for which a solution is desired and then solve in the \( \omega-k \) space to directly determine the linear phase and group velocities.

This brings the theoretical development of circumferential guided-wave propagation to an end. Both the single-layer CSH-wave and CLT-wave cases have been presented and the extension to the multilayer case was illustrated using the CSH-wave case. Though not shown explicitly, the extension to multiple layers for CLT-waves follows the exact same process as that of CSH-waves. The next chapter will provide numerical verification of the analytical results presented in this chapter.
Chapter 3

Numerical Results Following an Analytical Formulation

While the derivations of the characteristic equations do provide some physical insight, of more practical interest is the resultant propagating wave modes and shapes for a particular structure. These are determined by numerical solution of the characteristic equations as closed-form solutions do not exist. All numerical computations and corresponding results presented in this chapter were performed using MATLAB R2007a (The MathWorks 2007) unless otherwise noted. The computational codes developed as part of this work can be seen in Appendix D.

Following the development of the characteristic equations for CSH- and CLT-waves for single and multilayered annuli, presented in Chapter 2, it is necessary to verify the validity of the presented equations. In this work, verification will be achieved by one or more methods, including: direct comparison of numerical results with published results, by comparison of numerical results with the limiting case of a plate, and by analysis of the eigenfunctions for satisfaction of the appropriate boundary and continuity conditions.

All materials are assumed to be elastic in this chapter and as such, only real angular wavenumbers, \( p \), will be considered. Situations in which \( p \) would be complex, such as for evanescent modes or viscoelastic materials, will be considered in the chapters to come. The problem of determining the propagating wave modes at some circular frequency, \( \omega \), reduces to determining the values of \( p \) for which non-trivial solutions, \( i.e. \) non-zero roots, of the characteristic equations exist. Any number of root-seeking algorithms may be employed to find the roots of the characteristic equations. Some common algorithms include the Bisection method, the Newton-Rhapson method, the Secant method, and the \textit{regula falsi} method.
For real \( p \), the Bisection method is regarded as a more robust routine as it guarantees convergence, though only at the sacrifice of computational speed (Jaluria 1996). For this reason, as well as for its simplistic nature and ease of coding, the Bisection routine is employed in this work. The basic premise of the Bisection routine is to locate regions where \( f(x) \) and \( f(x+\delta) \) result in oppositely signed values, where \( f(x) \) is the function for which the roots are to be determined. The increment, \( \delta \), is then continually decreased until \( f(x) \) and \( f(x+\delta) \) both approach zero to within some specified precision.

The process of generating dispersion curves involves determining the roots of the characteristic equation over some frequency range that is of interest, typically determined by the application at hand. For the elastic materials considered here, the angular wavenumber roots, \( p \), are determined at fixed increments of \( \omega \), resulting in the frequency-wavenumber \( (\omega-p) \) dispersion curves from which it is possible to determine the angular or linear phase and group velocity dispersion curves using the relations introduced in Chapter 2.

Sample numerical results will now be presented for both CSH- and CLT-waves in single and multilayered elastic annuli with the primary goal of proving the validity of the analytical treatment presented in Chapter 2. Some defining characteristics of circumferential guided waves will be discussed and the chapter will conclude with a discussion on the computational limitations associated with the analytical formulation of the circumferential guided wave problem.

### 3.1 Numerical Results for CSH-Waves

As a first check of the validity of the analytical derivation presented in Section 2.1 for CSH-waves, a plate approximation is assumed and the dispersion curves are compared to those obtained using a formulation for SH-waves in a flat plate. As the inner-to-outer radius ratio (or aspect ratio) approaches unity the CSH-wave solution should approach the solution for the flat
plate. This is intuitive from a physical standpoint and is mathematically proven in (Zhao et al. 2004).

Figure 3.1 shows a comparison of the SH-wave dispersion curves for a steel plate (black solid lines) and the CSH-wave dispersion curves for a steel annulus with an aspect ratio of 0.984 (red dotted lines). A bulk shear wave velocity of 3.23 mm/μs was used in the computation and a closed-form solution for SH-waves in a plate, as presented in (Graff 1991), was used to generate the plate solution shown in the figure. As expected, it is seen that the plate approximation is a sufficient model for an annulus of this aspect ratio, though small differences in the curves begin to manifest as the frequency and wavenumber increase.

Figure 3.2, Figure 3.3, and Figure 3.4 show dispersion curve comparisons of a steel plate and annuli with aspect ratios of 0.8, 0.5, and 0.2, respectively. It is evident from these figures that as the aspect ratio of the annulus decreases, the plate-like assumption is no longer a valid approximation for CSH-waves in an annulus. Several interesting phenomena arise as the aspect ratio decreases; firstly, the fundamental mode, labeled $SH0$ in each figure, becomes increasingly dispersive. Secondly, the group velocities of the higher order modes can exceed that of the fundamental mode as the aspect ratio decreases. These two phenomena are not true for wave propagation in plate and along the axis of a pipe and, therefore, are defining characteristics of CSH-waves in an annulus. Also worth noting is the fact that the cutoff frequencies are very close for CSH-waves in an annulus and for SH-waves in a plate, even for very small aspect ratios.

The parameters used for the generation of the curves seen in Figure 3.1 through Figure 3.4 were purposely chosen to coincide with those published in (Zhao et al. 2004) so that the curves depicted here could be compared with the published results. Both results were found to be in agreement, further validating the analytical derivation and computational code used herein. The next step is to go beyond the published work and present several numerical results for CSH-waves in a multilayered annulus.
Figure 3.1 CSH (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.984 aspect ratio. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the OR of the annulus.
Figure 3.2 CSH (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.8 aspect ratio. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the OR of the annulus.
Figure 3.3 CSH (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.5 aspect ratio. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the OR of the annulus.
Figure 3.4 CSH (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.2 aspect ratio. The circular wavenumber, \( k \), and the linear phase and group velocity are calculated at the OR of the annulus.
Figure 3.5 shows the dispersion curves for a multilayer annulus with properties as summarized in Table 3.1. In each of the plots, the dispersion curves are shown for the multilayer annulus (red solid lines), Layer 1 only (black dashed line), and for Layer 2 only (black dashed-dotted line). The wavenumber and linear phase and group velocities are calculated at the mid-plane of the multilayer structure and at the top of Layer 1 and the bottom of Layer 2.

It is seen from plots (a) and (b) of Figure 3.5 that any intersection in the solution space for the individual layers, also belongs to the solution space of the multilayer system. This is seen more clearly in the magnified view of Figure 3.5(a), shown in Figure 3.6. Physically, what these points signify are locations in the dispersion space where the boundary and continuity conditions are satisfied for all three annular structures. As seen in the wave structure (or eigenfunction) plots (d), (e), and (f) of Figure 3.7, the intersection points are characterized by zero shear stress on the surfaces of the single-layer annuli and at the surfaces and interface of the multilayer annulus. Similar observations have been made by (Simonetti et al. 2004) for multilayered plates. The wave structures shown in Figure 3.7 correspond to the points labeled in Figure 3.5(b). Furthermore, these intersections represent “resonance” points of Layer 1. Resonance, in this context, is taken to mean a mode/frequency combination in which the largest possible displacement is achieved in the layer with the higher acoustic impedance. From Figure 3.7 (a) – (c) and (g) – (i) it is seen that points not located near the resonances show dominant particle displacement in Layer 2. From Figure 3.7(c) it is also seen that said resonance points coincide with the peaks of the group velocity dispersion curves.

| Table 3.1 | Geometric and material parameters used in the generation of the plots shown in Figure 3.5 through Figure 3.7 and Figure 3.12 through Figure 3.15 |
|---|---|---|---|---|
| Layer 1 | $r_{\text{inner}}$ (m) | $r_{\text{outer}}$ (m) | $\rho$ (kg/m$^3$) | $c_l$ (m/s) | $c_s$ (m/s) |
| | 0.032 | 0.036 | 7850 | 5850 | 3230 |
| Layer 2 | 0.036 | 0.04 | 1500 | 1700 | 900 |
Figure 3.5 CSH (a) Frequency-wavenumber, (b) phase velocity, and (c) group velocity
dispersion curves for a multilayered annulus with a 0.8 aspect ratio. Each layer is 4 mm thick.
Curves are also shown for the individually isolated layers. The circular wavenumber, \( k \), and the
linear phase and group velocity are calculated at the common interface of the annuli.
As observed in Figure 3.5(b), where \( r_{N+1} \) and \( m \) are as seen in Figure 2.2 and \( r \) is the arbitrary radius for which the linear phase velocity dispersion curves have been calculated. From Eq. 3.1 it is seen that in the limiting case, the phase velocity of all modes approaches the bulk shear-wave velocity of the slowest layer with the maximum potential phase velocity occurring at the outer radius of the annulus; all other radii having proportionally slower phase velocities. As previously stated, the variation in linear phase velocity with radius is a necessary requirement to maintain a constant phase front through the thickness of the annulus.

One last interesting observation that will be made is regarding the linear group velocity dispersion curves seen in Figure 3.5(c). For the multilayered annulus, it is seen that the peaks of the linear group velocity dispersion curves approximately trace, or approach, the linear group velocity curves of the higher-impedance layer; in this case Layer 1. The actual peak values,

**Figure 3.6** Magnified view of the frequency-wavenumber dispersion curve shown in Figure 3.5(a). Black dots denote points of intersection.

\[
\lim_{f \to \infty} c_s(r) = r \frac{r}{r_{N+1}} \min(c_s^m),
\]

where \( r_{N+1} \) and \( c_s^m \) are as seen in Figure 2.2 and \( r \) is the arbitrary radius for which the linear phase velocity dispersion curves have been calculated. From Eq. 3.1 it is seen that in the limiting case, the phase velocity of all modes approaches the bulk shear-wave velocity of the slowest layer with the maximum potential phase velocity occurring at the outer radius of the annulus; all other radii having proportionally slower phase velocities. As previously stated, the variation in linear phase velocity with radius is a necessary requirement to maintain a constant phase front through the thickness of the annulus.
Figure 3.7 Wave structures and stresses for the points identified in Figure 3.5(b).
though, are always lower than the curves for the isolated high-impedance layer. A practical exploitation of this phenomenon will be discussed and demonstrated in Chapter 7.

The next section will present some numerical examples for CLT-waves in single and multilayered annuli. The chapter will end with a discussion of the limitations of the numerical methods based on the analytical formulation of the circumferential wave problem.

3.2 Numerical Results for CLT-Waves

As was done for the case of CSH-waves, a first check of the analytical solution for CLT-waves will be completed by comparing a circumferential wave plate approximation with the results obtained for a flat plate. Figure 3.8 shows a comparison of the Lamb-wave dispersion curves for a steel plate (black solid lines) and the CLT-wave dispersion curves for a steel \((c_s = 3.23 \text{ mm}/\mu\text{s}, c_l = 5.96 \text{ mm}/\mu\text{s}, \text{ and } \rho = 7850 \text{ kg/m}^3)\) annulus with an aspect ratio of 0.984 (red dotted lines). The Lamb-wave dispersion curves were obtained using an available code based on the formulation for plates presented in (Rose 1999). As was also seen in the CSH-wave case, the plate approximation provides an accurate representation of a flat plate with small discrepancies beginning to manifest at high frequencies and large wavenumbers. The first two modes, \(LT0\) and \(LT1\), are labeled in each plot. For the plate approximation, these modes correspond to the fundamental antisymmetric and symmetric modes of the flat plate, respectively. When referring to waves propagating in an annulus, mode shapes are no longer perfectly symmetric or antisymmetric and are therefore referred to as \(LT0\) through \(LTm\) in this work, where \(m\) represents any positive integer extending to infinity.
Figure 3.8 CLT (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.984 aspect ratio. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the OR of the annulus.
Figure 3.9 CLT (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.8 aspect ratio. The circular wavenumber, \( k \), and the linear phase and group velocity are calculated at the OR of the annulus.
Figure 3.10 CLT (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.5 aspect ratio. The circular wavenumber, \( k \), and the linear phase and group velocity are calculated at the OR of the annulus.
Figure 3.11 CLT (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.2 aspect ratio. The circular wavenumber, \( k \), and the linear phase and group velocity are calculated at the OR of the annulus.
Several defining characteristics of CLT-waves are seen in Figure 3.8 through Figure 3.11. To start, the mode corresponding to the $S0$ plate mode no longer intercepts the vertical axis of the phase velocity dispersion plot at the plate velocity. It instead asymptotically tends toward infinite phase velocity. In the group velocity curves, the same mode tends toward zero as frequency tends toward zero. Physically this means that at low frequencies, or large wavelengths, a standing wave is formed in the annulus; a direct result of the ability of the wave to propagate around the annulus and constructively interfere with itself. Since the energy in an assumed infinite plate cannot propagate back onto itself, even very low frequencies will propagate at some finite velocity commonly referred to as the plate velocity.

Another defining characteristic of CLT-waves is clearly seen for annuli with smaller aspect ratios (0.5 and below for the plots shown). A new mode is seen extending from the origin of the frequency-wavenumber dispersion curves, which quickly peaks and then intercepts the $kd$-axis at the point $(kd=1-\eta)$, where $\eta$ represents the aspect ratio of the cylinder. (Liu et al. 1998a) provides a nice discussion on this mode and concludes that it is a result of the inability of the inner surface of the annulus to support a surface wave. As pointed out by (Viktorov 1967), convex surfaces may support surface waves while concave surfaces may not. Therefore, any surface wave that forms on the inner surface of the annulus will quickly leak back into the annulus, where it is then reflected off of the outer surface, subsequently forming the new mode. This mode is not seen in plates as they can support surfaces waves on both sides. The reason that the mode intercepts the $kd$-axis at $(kd=1-\eta)$ is because the wavelength is equal to the outer circumference at this point and no propagation is possible.

Based on the discussion in the last paragraph, it is possible to conclude that the phase velocity of the CLT-mode that is analogous to the $A0$ mode in a plate, will converge to the Rayleigh surface wave velocity as frequency tends toward infinity. The Rayleigh wave propagates on the outer surface of the annulus. It is also seen from Figure 3.9 through Figure 3.11.
that as the aspect ratio of the annulus begins to decrease, the CLT-mode once analogous to the $S0$ plate mode, becomes quite dissimilar to the $S0$ plate mode. This phenomenon is a manifestation of the inability of small aspect ratio annuli to support symmetric wave displacement profiles.

Figure 3.12 presents the CLT-wave dispersion curves for a multilayered annulus with the properties and geometry previously specified in Table 3.1. In the plots, the curves corresponding to the multilayered annulus are shown in red and the curves for just Layer 1 are shown by the black dashed lines. The individual curves for Layer 2 are not shown so to prevent the plots from becoming cluttered. Figure 3.13 shows two magnified regions of plot (a) in Figure 3.12 where there appear to be intersecting modes. Examination of the magnified views reveals that the CLT-modes in the multilayered annulus do not actually intersect. Similar observations were made by (Überall et al. 1994) and (Simonetti 2004) for Lamb-waves in multilayered plates.

Figure 3.14 and Figure 3.15 show the wave structures and stress distributions for points $a$, $b$, $c$ and $d$, $e$, $f$ in Figure 3.12(b), respectively. It is observed from Figure 3.14 that for regions of the multilayer dispersion curves that do not trace the curves for the high impedance layer, Layer 1, wave propagation occurs primarily in the lower impedance layer, Layer 2. As was shown in the case of CSH-waves and for now CLT-wave in Figure 3.15, the points at which the multilayer curves intersect the curves for the isolated high-impedance layer constitute resonance points of that layer. Furthermore, from Figure 3.12(c) it is seen that the peaks of the group velocity dispersion curves for the multilayer coincide with the resonance points, though the peak values only approach the group velocity of the modes in the isolated high impedance layer and are always slightly slower. As an example, these resonance points are of significant practical importance for the non-destructive evaluation of coated metallic annuli.
Figure 3.12 CLT (a) Frequency-wavenumber, (b) phase velocity, and (c) group velocity dispersion curves for a multilayered annulus with a 0.8 aspect ratio. Each layer is 4 mm thick. Curves are also shown for the isolated Layer 1. The circular wavenumber, $k$, and the linear phase and group velocity are calculated at the common interface of the annuli.
The computational algorithms developed for the CLT-wave case can be viewed in Appendix D. To demonstrate the applicability of the developed code to \( n \)-Layered annuli, the angular phase velocity dispersion curves for a 3-layered annulus, with the properties listed in Table 3.2, are shown in Figure 3.16. Angular phase velocity is plotted here as there is no interface that is common to all three layers. The angular phase velocity dispersion curves are also shown for the isolated high impedance layers, Layer 1 and Layer 3. The wave structures for the points labeled \( a \) and \( b \) are shown in Figure 3.17.

**Table 3.2** Geometric and material parameters used in the generation of the plots shown in Figure 3.16

<table>
<thead>
<tr>
<th></th>
<th>( r_{\text{inner}} ) (m)</th>
<th>( r_{\text{outer}} ) (m)</th>
<th>( \rho ) (kg/m(^3))</th>
<th>( c_1 ) (m/s)</th>
<th>( c_s ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer 1</td>
<td>0.028</td>
<td>0.031</td>
<td>7850</td>
<td>5850</td>
<td>3230</td>
</tr>
<tr>
<td>Layer 2</td>
<td>0.031</td>
<td>0.032</td>
<td>1500</td>
<td>1700</td>
<td>900</td>
</tr>
<tr>
<td>Layer 3</td>
<td>0.032</td>
<td>0.035</td>
<td>2750</td>
<td>6300</td>
<td>3100</td>
</tr>
</tbody>
</table>

**Figure 3.13** CLT Frequency-wavenumber dispersion curves for the multilayered annulus described in Table 3.1. Magnified views illustrate the repulsion of CLT-modes in multilayered annuli.
Figure 3.14 Wave structures and stresses for the points a, b, and c identified in Figure 3.12(b).
Figure 3.15 Wave structures and stresses for the points \(d\), \(e\), and \(f\) identified in Figure 3.12(b).
Figure 3.16 Angular phase velocity dispersion curves for the 3-layered annulus described in Table 3.2. Curves are also shown for the isolated Layer 1 and Layer 3.

Figure 3.17 Wave structures corresponding to points \( a \) and \( b \) of Figure 3.16.
From Figure 3.17 it is seen that the 3-layered annulus has resonances for both of the high impedance layers, Layer 1 and Layer 3, where the dispersion curves for the isolated layers intersect that of the multilayered annulus. Furthermore, in the region for which the wave structures are shown, the $LT2$ and $LT3$ modes approximate the $S0$ plate mode and thus dominant in-plane displacement is seen in each of the resonating layers. This demonstrates that it is possible to selectively excite the layers of a multilayer annulus, which again has significant implications for non-destructive testing using ultrasonic waves.

This concludes the presentation of the numerical results as obtained from the analytical formulation. The next section will discuss some of the limitations of the developed code and present some methods for addressing said limitations.

3.3 Computational Limitations of the Analytical Formulation

This section is devoted to identifying the computational limitations of the computer code based on the analytical formulation of the circumferential wave problem. In fact, there is one primary limitation associated with the numerical computation of the circumferential guided-wave dispersion curves and it has to do with the calculations of Bessel Functions. For cylindrical geometries Bessel Functions are inevitably encountered. (Barshinger 2001) solves the problem of wave propagation in the axial direction of a hollow cylinder with a viscoelastic coating. In the axial formulation, the characteristic equation is composed of $0^{th}$ and $1^{st}$ order Bessel (or Hankel) functions of the first and second kind (i.e. $J_0(x)$, $J_1(x)$, $Y_0(x)$, and $Y_1(x)$). The expressions for these common functions can be found in any advanced mathematics text or handbook, such as (Hayek 2001) or (Abramowitz et al. 1964).

The necessity of only four Bessel functions is a convenience that is not encountered in the case of circumferential wave propagation. As was shown in Eq. 2.12 for CSH-waves, and in
Eqs. 2.44 and 2.45 for CLT-waves, the order of the Bessel functions is not necessarily an integer and is dependent on wavenumber and radius since \( p = kr \). Thus for large radius annuli, the Bessel functions will be of very high order, which will naturally present a problem for small arguments of the Bessel function of the second kind, \( Y_p(x) \). To illustrate this, consider the plots of the Bessel functions shown in Figure 3.18. Plots (a) and (b) show the 0th and 1st order Bessel functions, respectively, that are used in the calculation of the dispersion curves for wave propagation along the axial direction of a pipe. The Bessel function of the first kind intersects the \( y \)-axis for an argument of zero whereas the Bessel function of the second kind asymptotically approaches negative infinity. This typically does not present a problem in computations as values for \( Y_0(x) \) and \( Y_1(x) \) can typically be calculated for arguments approaching zero.

Now consider plot (c) in Figure 3.18, which has an arbitrarily selected order of 100. It is immediately seen that computing any Bessel function of the second kind will be very difficult for small arguments. Furthermore because the argument of the Bessel function involves a product with frequency for circumferential waves, the computation for small arguments is a necessity.

Figure 3.19 shows the frequency-angular wavenumber dispersion curves for the multilayered annulus described in Table 3.3. In this plot two sets of curves are shown; one set (red dotted lines) calculated using double precision in MATLAB R2007a and one set (black solid lines) calculated using arbitrary precision arithmetic in MATEMATICA 4.0.1.0 (Wolfram Research 1999). A close review of the curves reveals the computation using double precision first fails to find a root near the point \((p, \omega) = (670, 2.4)\) and then continuously fails to find roots for the

| Table 3.3 Geometric and material parameters used in the generation of Figure 3.19 |
|-------------------------------|------------------|------------------|------------------|------------------|------------------|
| \( r_{\text{inner}} \) (m)   | \( r_{\text{outer}} \) (m) | \( \rho \) (kg/m\(^3\)) | \( c_1 \) (m/s) | \( c_s \) (m/s) |
| Layer 1                       | 0.24765          | 0.254            | 7850            | 5960            | 3260            |
| Layer 2                       | 0.254            | 0.257            | 1500            | 1400            | 900             |
Figure 3.18 Bessel functions of the first and second kind of order (a) 0, (b) 1, and (c) 100.
$SH_0$ and $SH_1$ modes. Only frequencies up to 500 kHz are shown in Figure 3.19 and it has been observed that even more roots are missed at higher frequencies, especially when considering large radius annuli with low impedance layers, such as polymers or other viscoelastic materials. Similar problems are encountered in the calculation of the CLT-wave dispersion curves for large radius annuli.

As seen in Figure 3.19, the use of arbitrary precision arithmetic is able to provide the complete solution for the dispersion space analyzed, albeit with a significantly reduced computational efficiency. For the case of CLT-waves, it was also found that a larger region of the dispersion space could be computed though the use of arbitrary precision arithmetic, though the algorithm did also eventually fail to find roots. Other problems arising from the numerical computation based on the analytical formulation involved the calculation of wave structures. In general they could not be computed for very low phase velocities, such as those encountered with low impedance materials.
Because the arbitrary precision arithmetic approach still has root finding limitations and because the calculation time increased by an approximate factor of 10 for use of said approach, it is desirable to investigate alternative methods, even if approximate. One such approximate approach was introduced by (Gridin et al. 2003) for CSH- and CLT-waves and involves the asymptotic reduction of the exact dispersion relations. The method is tedious, involving from five to nine different sets of equations whose use is dependent on the values of the angular wavenumber. Additionally, the method is only applicable to single-layer annuli and thus not of general interest to this study. Viktorov also used a similar approximation method when he first addressed the topic of circumferential waves and Rayleigh waves on concave and convex surfaces (Viktorov 1967). To arrive at a first approximation, he replaced the Bessel functions in the characteristic equation with asymptotic representations in terms of semiconvergent Debye series (Watson 1966).

With the anticipation of adding considerations for viscoelastic layered annuli, it was decided that an approximate method based on the Semi-Analytical Finite Element method would be most appropriate for obtaining complete solutions of the dispersion space. In this manner it is not necessary to calculate the complex arbitrary order Bessel functions that would result from viscoelastic considerations. Additionally, this method has been shown to be sufficiently accurate by several other authors (Hayashi, Kawashima et al. 2003; Hayashi, Song et al. 2003; Bartoli et al. 2006; Mu et al. 2008).

This concludes the fully analytical treatment of circumferential guided waves. The next chapter will introduce the basic concepts of viscoelastic theory that will be needed to add considerations for viscoelastic annulus layers. Chapter 5 will then introduce the SAFE method as applied to annuli with viscoelastic layers.
Chapter 4
Review of Viscoelastic Theory

The purpose of this chapter is to review some of the basic concepts of viscoelastic theory as it applies to wave propagation. The displacement fields of ultrasonic waves certainly satisfy the small strain assumption of linear viscoelasticity and thus this theory applied here. Further assumptions are that all viscoelastic materials under consideration are free of micro-cracking and are at the constant temperature at which the corresponding viscoelastic material constants were calculated. First the viscoelastic constitutive equations will be presented, followed by discussions on the dynamic loading of viscoelastic materials and the elastic-viscoelastic correspondence principle. The chapter concludes with a discussion of the treatment of wave propagation in viscoelastic materials. For a in depth review of viscoelastic theory, one is pointed to the texts of (Christensen 1981) and (Haddad 1995). (Schapery 1967) also provides an in-depth review of the linear viscoelastic theory and has published many other articles on the non-linear theory.

4.1 Constitutive Relations

Under the assumptions of elasticity theory, stress is a single-valued continuous function of strain. In other words, when a load is applied or removed, the material reacts instantaneously and all deformation is recoverable. The instantaneous nature of elastic materials is evident from the lack of any temporal quantities in the governing equations.

Viscoelastic materials exhibit some of the properties of elastic materials but also exhibit properties of viscous materials. As a result, stress in a viscoelastic body is a function of both the current and past strain and strain rates. Unlike the elastic materials, the reaction of the material is
now dependent on the history of the material and the rate at which the load is applied. Like elastic materials, all deformation is recoverable in linear viscoelastic materials, given enough time. From these observations, one would expect the governing equations of viscoelasticity to be time-dependent; and they are.

Some of the governing equations of viscoelasticity will now be presented. As this discussion is provided as an overview, many of the details of the viscoelasticity formulation cannot be presented. The reader is pointed to (Christensen 1981) for the detailed formulation. For cross-referencing convenience, the discussion and notation used here follows from that of Christensen.

The Cartesian tensor form the stress-strain relations are given in Eq. 4.1 and Eq. 4.2.

\[
\sigma_{ij}(t) = \int_0^t G_{ijkl}(t - \tau) \frac{d\epsilon_{kl}(\tau)}{d\tau} d\tau,
\]

\[
\epsilon_{ij}(t) = \int_{-\infty}^t J_{ijkl}(t - \tau) \frac{d\sigma_{kl}(\tau)}{d\tau} d\tau,
\]

where \( \tau \) is the time variable and \( t \) is the current time. In Eq. 4.1 and Eq. 4.2, \( G_{ijkl}(t) \) and \( J_{ijkl}(t) \) are tensors of what are termed relaxation and creep functions, respectively. They are mechanical properties of the material and must be determined by experiment. Since the present work is primarily concerned with isotropic materials, the viscoelastic stress-strain relations can be simplified as follows:

\[
s_{ij}(t) = \int_{-\infty}^t G_{ijkl}(t - \tau) \frac{d\epsilon_{kl}(\tau)}{d\tau} d\tau.
\]

\[
\sigma_{kk}(t) = \int_{-\infty}^t G_{ijkl}(t - \tau) \frac{d\epsilon_{kl}(\tau)}{d\tau} d\tau.
\]
In the above equations $G_1(t)$ and $G_2(t)$ are independent relaxation functions and $J_1(t)$ and $J_2(t)$ are independent creep functions. The term $\delta_{ij}$ represents the Kronecker delta function. It can be seen from the equations that $G_1(t)$ and $J_1(t)$ are associated with deviatoric (shape changing) quantities and $G_2(t)$ and $J_2(t)$ are associated with dilatational (volume changing) quantities.

### 4.2 Dynamic Viscoelasticity

As mentioned previously, the relaxation and creep functions are determined experimentally and the resulting relationships between stress and strain are often expressed using mechanical models consisting of combinations of springs and dashpots. Some examples include the Maxwell and Kelvin-Voigt models, and the generalized Maxwell and Kelvin models. For wave propagation problems, an entirely different representation of the viscoelastic mechanical properties is used; the complex moduli.

Allow the isotropic stress-strain condition to be expressed as follows:

\[
\hat{\sigma} = \int_{-\infty}^{t} G_\alpha(t - \tau) \frac{d\hat{\varepsilon}(\tau)}{dt} \, d\tau,
\]  

4.8
where \( \alpha = 1 \) for deviatoric stresses and \( \alpha = 2 \) for dilatational stresses. As shown in Eq. 4.9, the relaxation function \( G_\alpha(t) \) is expressed as the sum of a constant and a time-dependent function.

\[
G_\alpha(t) = \hat{G}_\alpha + \tilde{G}_\alpha(t) \quad \text{where} \quad \tilde{G}_\alpha(t) \to 0 \quad \text{as} \quad t \to \infty.
\]  

(4.9)

Under steady state oscillatory conditions, the strain history is expressed as a time-harmonic function,

\[
\tilde{\varepsilon}(t) = \tilde{\varepsilon}_0 e^{i\omega t},
\]

(4.10)

where \( \tilde{\varepsilon}_0 \) is the amplitude and \( \omega \) is the frequency of oscillation. Stress is taken to have the same steady state harmonic form as strain and thus its general form will be,

\[
\tilde{\sigma}(t) = G_\alpha^*(i\omega)\tilde{\varepsilon}_0 e^{i\omega t},
\]

(4.11)

where \( G_\alpha^*(i\omega) \) is a complex frequency-dependent function known as the complex modulus. By substituting Eq. 4.9 into Eq. 4.8 and then Eq. 4.10 and Eq. 4.11 into the result, the following two equations result:

\[
G_\alpha'(\omega) = \hat{G}_\alpha + \omega \int_0^\infty \hat{G}_\alpha(\eta) \sin \omega \eta \, d\eta,
\]

(4.12)

\[
G_\alpha^*(\omega) = \omega \int_0^\infty \hat{G}_\alpha(\eta) \cos \omega \eta \, d\eta,
\]

(4.13)

where \( \eta = t - \tau \) and

\[
G_\alpha^*(i\omega) = G_\alpha'(\omega) + iG_\alpha^*(\omega).
\]

(4.14)

The real and imaginary parts of the complex modulus are referred to as the storage and loss modulus, respectively.

Another important representation of the stress-strain relation is given in Eq. 4.15,
Comparing Eq. 4.15 to Eq. 4.10, it is seen that for steady state oscillatory problems, there is a $\alpha \varphi$ phase lag between stress and strain. The quantity $\tan^{-1}\left(\frac{G''(\omega)}{G'(\omega)}\right)$ is referred to as the loss tangent and provides a measure of the relative viscoelasticity of a material. Figure 4.1 provides a graphical representation of the complex modulus. As seen in the figure, $0^\circ \leq \varphi_a \leq 90^\circ$, where a loss tangent of $0^\circ$ represents a completely elastic material and a loss tangent of $90^\circ$ represents a completely viscous material. Therefore, any loss tangent between these two values will represent a viscoelastic material.

4.3 Elastic-Viscoelastic Correspondence Principle

It follows from the discussion in the preceding paragraph and from Figure 4.1 that elasticity and viscosity are actually special cases of viscoelasticity. As a result one would expect
to be able to obtain the governing equations of elasticity from the governing equations of viscoelasticity. To illustrate this, consider the Fourier and Inverse Fourier transforms shown in Eq. 4.17 and Eq. 4.18, respectively.

\[ f(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt. \quad 4.17 \]

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega t} d\omega. \quad 4.18 \]

Using \( G_1^*(i\omega) \) and Eq. 4.12 through Eq. 4.14 in the Fourier transformed version of Eq. 4.3 yields the following deviatoric stress-strain relation:

\[ \bar{\sigma}_{ij}(\omega) = G_1^*(i\omega)\bar{e}_{ij}(\omega). \quad 4.19 \]

A similar process applied to Eq. 4.4, using \( G_2^*(i\omega) \), yields Eq. 4.20 as a transformed version of the dilatational stress-strain relation.

\[ \bar{\sigma}_{kk}(\omega) = G_2^*(i\omega)\bar{\varepsilon}_{kk}(\omega), \quad 4.20 \]

Eq. 4.19 and Eq. 4.20 are seen to be of the same form as the generalized Hooke’s Law for elastic materials if the elastic variables were to be replaced by the corresponding transformed viscoelastic variables and if the elastic material properties are replaced with complex viscoelastic material properties; and, indeed, if the loss tangent were to be 0°, the imaginary component of the complex moduli would vanish and the standard elasticity relations would follow.

Though not shown here, the remaining governing equations of linear viscoelasticity also take on the same form as the governing equations of linear elasticity when integral transform methods are used. This analogy is often referred to as the elastic-viscoelastic correspondence principle and it has significant implications. Because the time-dependence of the stress-strain relations can be removed using integral transform methods, it is possible to obtain the
transformed viscoelastic solution by solving the corresponding elastic problem with complex material properties. (Read 1950) was the first to utilize the Fourier integral transform for such purposes and (Schapery 1967) and (Christensen 1981) provide detailed presentations of the correspondence principle using Laplace integral transforms.

4.4 Viscoelastic Wave Propagation

According to the correspondence principle introduced above, the problem of wave propagation in the circumferential direction of a linear viscoelastic cylinder begins with Navier’s equation with complex moduli, as seen in Eq. 4.21.

\[
\left[ \lambda^*(i\omega) + 2\mu^*(i\omega) \right] \nabla \Delta - 2\mu^*(i\omega) \nabla \times \omega = \rho \ddot{\mathbf{u}},
\]

where body forces are assumed zero and \( \Delta \) and \( \omega \) have been defined in Eq. 2.2 and Eq. 2.3, respectively. The complex shear modulus, \( \mu^*(i\omega) \), is equivalent to \( G_i^*(i\omega) \) and the complex Lamé constant, \( \lambda^*(i\omega) \), is given by Eq. 4.22

\[
\lambda^*(i\omega) = G_i^*(i\omega) - \frac{2}{3} G_i^*(i\omega).
\]

Another useful result of the correspondence principle is the ability to use all of the viscoelastic counterparts in relations among elastic constants. This was done in Eq. 4.22.

Since the entire solution process for circumferential wave propagation was outlined in Chapter 2, there is no need to repeat it here. The primary difference between the elastic and viscoelastic cases will be that for the viscoelastic case, the roots of the dispersion equation will be complex, as seen in Eq. 4.23.

\[
p^*(i\omega) = p'(\omega) + ip''(\omega).
\]
The physical significance of the terms in Eq. 4.23 are as follows: \( p'(\omega) \) is the angular wavenumber of the propagating modes and \( p^*(\omega) \) is the angular attenuation with which the corresponding wave propagates. This is intuitive as it was shown that the real part of the complex moduli is responsible for energy storage and the imaginary part responsible for energy dissipation.

It has been shown that, though viscoelastic theory is more complicated than elastic theory, the inclusion of viscoelastic layers in the multilayered annulus requires no change in the formulation of the problem, with the exception of using complex material properties in place of the elastic ones. (Barshinger et al. 2004) provides the following expressions for the complex moduli:

\[
\mu^*(\omega) = \rho c_s^2, \tag{4.24}
\]

\[
\lambda^*(\omega) = \rho (c_l^2 - 2c_s^2), \tag{4.25}
\]

where

\[
c_{s,l}^* = \left( \frac{1}{c_{s,l}(\omega)} + i \frac{\alpha_{s,l}(\omega)}{\omega} \right)^{-1}, \tag{4.26}
\]

and \( c_s \) and \( c_l \) are the bulk shear and longitudinal wave velocities, respectively, and \( \alpha_s \) and \( \alpha_l \) are the attenuation constants of the bulk shear and longitudinal waves, respectively. The units of the attenuation term in Eq. 4.26 are \( Np/mm \). Other authors have formulated a similar expression to Eq. 4.26 where the attenuation is specified per unit wavelength (Bernard et al. 2001). In this work, a hysteretic material model is assumed. That is, the bulk wave attenuation is assumed to increase linearly with frequency and the term \( \alpha/\omega \) is constant. The Kelvin-Voigt model is another material model in which attenuation is a quadratic function of frequency, though this model is less commonly used in the ultrasonic frequency ranges of interest here.
It is seen from Eqs. 4.24 through 4.26 that the complex moduli necessary for characterizing isotropic materials can be determined by experimentally measuring the bulk shear and longitudinal velocity and attenuation. Though the bulk longitudinal and shear wave velocities are functions of frequency for viscoelastic materials, it is usually assumed that they are constant over the typical frequency ranges used in ultrasonics (Bernard et al. 2001; Barshinger et al. 2004; Luo 2005; Bartoli et al. 2006). Techniques for measuring these properties have been the topic of many articles (Guo et al. 1995; Barshinger 2001; Luo 2005; Van Velsor 2006).

Chapter 5 will introduce the SAFE method as it applies to circumferential guided waves. The viscoelastic principles discussed in the present chapter will be incorporated into the formulation via complex moduli. Examples of the resultant attenuation dispersion curves will be presented in Chapter 6.
Chapter 5

A Semi-Analytical Solution for Circumferentially Propagating Guided Waves

Other authors have presented SAFE solutions for wave propagation in plate and arbitrarily shaped cross-sections such as rod and rail. The SAFE method has also been applied to pipe, but only for wave propagation in the axial direction. This work addresses for the first time, a SAFE solution for guided waves propagating in the circumferential direction of an annulus consisting of elastic and/or viscoelastic layers.

Of particular interest are the regions in which the analytical approach fails, as was shown in Chapter 3, though the SAFE solution will be presented for the entire dispersion space of interest. The increased computational efficiency of the SAFE method will also have added benefits for the computation of the attenuation dispersion curves, which will be addressed in the next chapter. This section provides an overview of the SAFE method as it applies to circumferential guided waves in multilayered annuli. Both CSH- and CLT-waves will be addressed.

5.1 SAFE Formulation for Circumferential Guided Waves

A SAFE formulation for circumferentially propagating waves will be presented here. The formulation will generally follow that presented by (Hayashi, Song et al. 2003; Bartoli et al. 2006) for propagation along the straight axis of a structure with arbitrary cross section. To account for the curvature of an annular waveguide, a polar coordinate system is adopted and the treatment of circumferentially propagating waves becomes analogous to that of a plate in the cartesian coordinate system. The differences arising from the switching of coordinate systems
will be explicitly noted as they arise. Initial considerations will be made for both CSH- and CLT-waves simultaneously and a method for separating the solutions will be presented subsequently.

Consider the annular cross section shown in Figure 5.1. Under the generalized plane-strain assumption, the faces in the \( z-\theta \) plane extend to \(+/-\) infinity, as illustrated. Propagation is assumed in the \( \theta \)-direction and is described by the function \( e^{ip\theta} \), where \( p \) is the angular wavenumber as previously discussed in the analytical formulation presented in Chapter 3. Again, this is a fundamental difference between wave propagation in a curved waveguide and waves propagating in a plate, for both the analytical and SAFE formulations.

Also shown in Figure 5.1 is a sample linear element used to discretize the annulus. The degrees of freedom (DOF) are shown for nodes \( i \) and \( j \). Note that because the primary interest here is the propagation of waves in an annulus with sides extending toward infinity, it is only necessary to discretize the domain in the radial direction. If wave propagation in the circumferential direction of an arbitrarily shaped toroid were of interest, two-dimensional discretization of the \( r-z \) plane would be necessary and the formulation presented here would be applicable.

**Figure 5.1** Model used for the development of the SAFE formulation.
The displacement, strain, and stress components for any point in the domain of interest are as follows:

\[
\mathbf{u} = \begin{bmatrix} u_r, u_\theta, u_z \end{bmatrix}^T, \quad 5.1
\]

\[
\mathbf{e} = \begin{bmatrix} e_r, e_\theta, e_z, e_{r\theta}, e_{\theta z}, e_{rz} \end{bmatrix}^T, \quad 5.2
\]

\[
\mathbf{\sigma} = \begin{bmatrix} \sigma_r, \sigma_\theta, \sigma_z, \sigma_{r\theta}, \sigma_{\theta z}, \sigma_{rz} \end{bmatrix}^T, \quad 5.3
\]

where \( T \) denotes the transpose of a vector or matrix. Under the generalized plane-strain assumption, the displacement vector for a propagating wave can be expressed as

\[
\mathbf{u}(r, \theta, t) = \begin{bmatrix} u_r(r, \theta, t) \\ u_\theta(r, \theta, t) \\ u_z(r, \theta, t) \end{bmatrix} = \begin{bmatrix} U_r(r) \\ U_\theta(r) \\ U_z(r) \end{bmatrix} e^{i(p\theta - \omega t)}, \quad 5.4
\]

where the \( \theta \) and time dependence are accounted for in the analytic periodic exponential function. As expected from the generalized plane-strain assumption, there is no variation of any component in the \( z \)-direction and determining the displacement is reduced to solving for three one-dimensional functions of \( r \). Seeing the discretized version of Eq. 5.4 in Eq. 5.5, it becomes apparent why the method is called the Semi-Analytical Finite Element method. The displacement of a single element is a product of a standard finite-element approximation with an entirely analytic wave propagation term.

\[
\mathbf{u}^{(e)}(r, \theta, t) = \sum_k N_k \mathbf{u}_k^{(e)} e^{i(p\theta - \omega t)} = \begin{bmatrix} \frac{R_r - r}{\ell} & 0 & 0 & \frac{r - R_r}{\ell} & 0 & 0 \\ 0 & \frac{R_\theta - r}{\ell} & 0 & 0 & \frac{r - R_\theta}{\ell} & 0 \\ 0 & 0 & \frac{R_z - r}{\ell} & 0 & 0 & \frac{r - R_z}{\ell} \end{bmatrix} \begin{bmatrix} U_{ri}^{(e)} \\ U_{\theta i}^{(e)} \\ U_{zi}^{(e)} \\ U_{rj}^{(e)} \\ U_{\theta j}^{(e)} \\ U_{zj}^{(e)} \end{bmatrix} e^{i(p\theta - \omega t)}, \quad 5.5
\]

or,
\[
\mathbf{u}^{(e)}(r, \theta, t) = \sum_k N_k \mathbf{U}_k^{(e)} e^{i(p\theta-q t)} = \left[ N_i \mathbf{I}, N_j \mathbf{I} \right] \mathbf{U}_i^{(e)} e^{i(p\theta-q t)}, \]

where \( \mathbf{I} \) is the identity matrix. For linear one-dimensional elements, \( N_i \) and \( N_j \) are the element shape functions given by

\[
N_i = \frac{R_j - r}{\ell}, \quad 5.7
\]

\[
N_j = \frac{r - R_i}{\ell}. \quad 5.8
\]

Now consider strain as a function of displacement as seen in Eq. 5.9,

\[
\mathbf{\epsilon} = \left[ \mathbf{L}_r \frac{\partial}{\partial r} + \mathbf{L}_\theta \frac{\partial}{\partial \theta} + \mathbf{L}_z \frac{\partial}{\partial z} + \mathbf{K}_r + \mathbf{K}_\theta + \mathbf{K}_z \right] \mathbf{u}. \quad 5.9
\]

where,

\[
\mathbf{L}_r = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \mathbf{L}_\theta = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1/r & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \mathbf{L}_z = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad 5.10
\]

and

\[
\mathbf{K}_r = \begin{bmatrix}
0 & 0 & 0 \\
1/r & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \mathbf{K}_\theta = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \mathbf{K}_z = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}. \quad 5.11
\]

Since there are no derivative operators associated with the \( \mathbf{K} \) matrices, they may be combined into a single matrix as shown in Eq. 5.12.
The discretized version of Eq. 5.9 may then be written as

$$\textbf{K}_{\text{equiv}} = \begin{bmatrix} 0 & 0 & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad 5.12$$

The discretized version of Eq. 5.9 may then be written as

$$\varepsilon^{(e)}(r, \theta, t) = \left[ L_r \frac{\partial}{\partial r} + L_\theta \frac{\partial}{\partial \theta} + L_z \frac{\partial}{\partial z} + K_r + K_\theta + K_z \right] \mathbf{N}(r) \mathbf{U}^{(e)} e^{i(\rho \theta - \omega t)}, \quad 5.13$$

where

$$\mathbf{N}(r) = \begin{bmatrix} N_1, N_2, N_3 \end{bmatrix}. \quad 5.14$$

Note that the $\partial / \partial z$ term in Eqs. 5.9 and 5.13 vanish under the generalized plane-strain assumption but are maintained for generality. For arbitrary two-dimensional cross sections, the generalized plane-strain assumption is removed and the corresponding two-dimensional shape functions must be employed. Standard two-dimensional linear and quadratic shape functions can be found in any book on the topic of finite-element analysis, such as (Segerlind 1976).

Following the notation used by (Hayashi, Song et al. 2003; Bartoli et al. 2006), Eq. 5.13 can be rewritten as

$$\varepsilon^{(e)} = (\mathbf{B}_1 + i \rho \mathbf{B}_2) \mathbf{U}^{(e)} e^{i(\rho \theta - \omega t)}, \quad 5.15$$

where

$$\mathbf{B}_1 = L_r \frac{\partial \mathbf{N}}{\partial r} + L_z \frac{\partial \mathbf{N}}{\partial z} + \mathbf{K}_{\text{equiv}} \mathbf{N}, \quad 5.16$$

and

$$\mathbf{B}_2 = L_\theta \mathbf{N}. \quad 5.17$$
Note the use of the angular wavenumber $p$ in Eq.
5.15 in place of the circular wavenumber $k$ that is used to describe propagation along straight structures.

For linear one-dimensional elements $B_1$ and $B_2$ are given by

$$B_1 = \begin{bmatrix}
-1/\ell & 0 & 0 & 1/\ell & 0 & 0 \\
\frac{(r_i-r_{j})}{rt} & 0 & 0 & \frac{(r_{j}-r_i)}{rt} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{(r_{j}-r_i)}{rt} & -1/\ell & 0 & 0 & \frac{(r_{j}-r_i)}{rt} + 1/\ell & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/\ell & 0 & 0 & 1/\ell & 0
\end{bmatrix}, \quad 5.18$$

and

$$B_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{(r_i-r_{j})}{rt} & 0 & 0 & \frac{(r_{j}-r_i)}{rt} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{(r_{j}-r_i)}{rt} & 0 & 0 & \frac{(r_{j}-r_i)}{rt} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \quad 5.19$$

Now that the discretized form of strain has been expressed as a function of displacement, the governing equations may be considered. Instead of considering the equations of motion, the variational principle of mechanics known as Hamilton’s Principle is used in the SAFE formulation. As seen in Eq.
5.20 for conservative systems, Hamilton’s principle mathematically states that the actual path followed by some particle or system of particles is the one that renders the integral in Eq.
5.20 stationary with respect to all possible neighboring paths that the system could be imagined to take between two instants $t_1$ and $t_2$, provided the initial and final configurations of the system are prescribed (Meirovitch 1967).

$$\delta \int_{t_1}^{t_2} (T - V) dt = 0, \quad 5.20$$
where $T$ is the kinetic energy present in the system,

$$ T = \frac{1}{2} \int_{\Omega} \mathbf{u}^T \rho \mathbf{u} \, d\Omega , \quad 5.21 $$

and $\mathbf{u}$ is the first derivative of displacement with respect to time. The potential, or strain, energy is given by $V$,

$$ V = \frac{1}{2} \int_{\Omega} \mathbf{\varepsilon}^T \mathbf{C} \mathbf{\varepsilon} \, d\Omega , \quad 5.22 $$

and $\Omega$ represents the volume. By using the conservative form of Hamilton’s Principle, it is assumed that the displacement field across the thickness is not affected by damping. This has been shown to be a reasonable assumption by (Shorter 2004) and has also been adopted by (Bartoli et al. 2006) for the treatment of viscoelastic plate and by (Mu et al. 2008) for the treatment of axial wave propagation in hollow cylinders with viscoelastic coatings.

In Eq. 5.22, $\mathbf{C}$ is the material stiffness matrix given by

$$ \mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{33} & C_{34} & C_{35} & C_{36} \\ C_{44} & C_{45} & C_{46} \\ \text{symmetric} & & & \\ C_{55} & C_{56} \\ C_{66} & & & \end{bmatrix} , \quad 5.23 $$

where for viscoelastic materials

$$ C_{mn} = C_{mn}^* = C_{mn}' + iC_{mn}'' . \quad 5.24 $$

As discussed in Chapter 4, the real part of $\mathbf{C}^*$ is associated with the storage modulus while the imaginary part is associated with the loss modulus. For isotropic materials
Equation 5.26 is obtained by integrating the kinetic energy term, Eq. 5.21, by parts.

\[
T = \frac{1}{2} \left[ \mathbf{u}^T \rho \frac{\partial \mathbf{u}}{\partial t} \bigg|_{\Omega} - \oint_{\Omega} \mathbf{u}^T \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} d\Omega \right]. \tag{5.26}
\]

Once Eq. 5.26 is substituted into Eq. 5.20 the first term will vanish as \( \delta \mathbf{u} \) must be zero at the boundaries of the layer by the definition of virtual displacements. Discarding the first term from Eq. 5.26, Hamilton’s Principle can be rewritten as

\[
\int_{t_i}^{t_f} \left[ \int_{\Omega} \delta(\mathbf{e}^v) \mathbf{C} \mathbf{e} \ d\Omega + \int_{\Omega} \delta(\mathbf{u}^v) \rho \mathbf{u} \ d\Omega \right] dt = 0, \tag{5.27}
\]

or in discrete form,

\[
\int_{t_i}^{t_f} \left\{ \sum_{e=1}^{n_e} \left[ \int_{\Omega_e} \delta(\mathbf{e}^{(e)v}) \mathbf{C}_e \mathbf{e}^{(e)} d\Omega_e + \int_{\Omega_e} \delta(\mathbf{u}^{(e)v}) \rho \mathbf{u}^{(e)} d\Omega_e \right] \right\} dt = 0. \tag{5.28}
\]

Substituting the expressions for displacement and strain, Eqs. 5.5 and 5.15, respectively, into the discretized form of Hamilton’s Principle, Eq. 5.29 is obtained (Bartoli et al. 2006).

\[
\int_{t_i}^{t_f} \left\{ \delta \mathbf{U}^T \left[ \mathbf{K}_1 + ip \mathbf{K}_2 + p^2 \mathbf{K}_3 - \omega^2 \mathbf{M} \right] \mathbf{U} \right\} dt = 0, \tag{5.29}
\]

where

\[
\mathbf{K}_1 = \bigcup_{e=1}^{n_e} \mathbf{k}_1^{(e)}, \quad \mathbf{K}_2 = \bigcup_{e=1}^{n_e} \mathbf{k}_2^{(e)}, \quad \mathbf{K}_3 = \bigcup_{e=1}^{n_e} \mathbf{k}_3^{(e)}, \quad \mathbf{M} = \bigcup_{e=1}^{n_e} \mathbf{m}^{(e)} \tag{5.30}
\]

and
which may be calculated using numerical integration techniques such as Gaussian Quadrature.

The symbol $\mathcal{U}$ represents the assembly of the element matrices into the global stiffness matrix.

Because $\delta \mathbf{U}^T$ must vanish at $t_1$ and $t_2$, non-trivial solutions of Eq. 5.29 will result when

$$[\mathbf{K}_1 + i\rho \mathbf{K}_2 + p^2 \mathbf{K}_3 - \omega^2 \mathbf{M}] M \mathbf{U} = 0,$$

where $M$ is the number of degrees of freedom of the system, which for the formulation presented here, will be equal to 3 times the number of nodes in the discretized region. In the next sections, it will be shown how the dimensionality of Eq. 5.35 can be reduced by considering the CSH- and CLT-wave cases separately. This is desirable from both a computational efficiency perspective and from a practical perspective as realistic sources do not generate both types of waves.

As a final step, it will prove beneficial to transform Eq. 5.35 into a first-order eigensystem of $p$, as shown in Eq 5.36.

$$[\mathbf{A} - p \mathbf{B}]_{2M} \mathbf{Q} = \mathbf{0},$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{K}_1 - \omega^2 \mathbf{M} \\ \mathbf{K}_1 - \omega^2 \mathbf{M} & i\mathbf{K}_2 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{K}_1 - \omega^2 \mathbf{M} & 0 \\ 0 & -\mathbf{K}_3 \end{bmatrix}.$$
The values of $\omega$ and $p$ that satisfy Eq. 5.36 are the eigenvalues of the system and result in the $\omega$-$p$ dispersion curves. As indicated by the subscript $2M$ in Eq. 5.36, the total number of roots found will be twice the number of degrees of freedom; $M$ roots corresponding to forward propagating waves and $M$ roots corresponding to backward propagating waves. In the case of elastic materials, complex roots correspond to evanescent, or non-propagating, modes and in the case of viscoelastic materials, complex roots correspond to propagating waves which attenuate with distance.

The next two sections will take the general formulation just presented and make specific considerations for CSH- and CLT-waves. As was the condition for using Helmholtz decomposition in the analytical formulation, the CSH- and CLT-wave solutions arising from the SAFE method can only be seperated for isotropic materials.

### 5.2 Considerations for CSH-Waves

For the case of CSH-waves, displacement is assumed in the $z$-direction only and the discretized displacement equation shown in Eq. 5.5 immediately reduces to

$$\mathbf{u} = \mathbf{Q} \mathbf{U}.$$

By

$$5.39$$

Next, Eq. 5.15 is considered. Noting that the only non-zero strain components are $e_{rz}$ and $e_{\theta z}$, any row in the $\mathbf{B}_1$ and $\mathbf{B}_2$ matrices corresponding to a zero-strain component is removed and any column that is multiplied by a nodal displacement in the $r$- or $\theta$-direction is removed, resulting in

$$u^{(c)}(r, \theta, t) = \left[ \frac{R_j - r}{L} \begin{array}{c} U_{rj} \\ U_{zj} \end{array} \right] e^{i(r \rho - \omega t)}.$$
Again, considering the non-zero strain components, the material stiffness matrix for CSH-wave can be reduced to

\[ B_1 = \begin{bmatrix} 0 & 0 \\ -1/\ell & 1/\ell \end{bmatrix}, \]

and

\[ B_2 = \begin{bmatrix} R_j - r & r - R_i \\ r\ell & r\ell \\ 0 & 0 \end{bmatrix} \]

Again, considering the non-zero strain components, the material stiffness matrix for CSH-wave can be reduced to

\[ C = \begin{bmatrix} C_{55} & C_{56} \\ C_{65} & C_{66} \end{bmatrix}, \]

which, for isotropic materials gives

\[ C = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}, \]

Note that Eq. 5.44 will be complex if viscoelastic materials are being considered.

So it can be seen that by considering only the strain and displacement components relevant to CSH-wave propagation, the number of degrees of freedom has been reduced by a factor of 3, thereby reducing the size of the matrix that must be solved by the same factor. Not only will this reduce the computation time significantly, but it will result in a dispersion plot showing only CSH-modes instead of all possible propagating modes.

Once the nodal displacement values have been determined, the non-zero stress components can be calculated using the following relations:

\[ \sigma_{rz}^{(e)} = \frac{\mu}{\ell} (U_{zj} - U_{zi}) , \]

\[ \sigma_{\theta z}^{(e)} = \frac{ip\mu}{r\ell} \left[ (R_j - r)U_{zi} + (r - R_i)U_{zi} \right] \text{ for } R_i \leq r \leq R_j. \]
Note that if $\sigma_{\theta z}$ is calculated at one of the two element nodes, Eq. 5.46 reduces to

$$\sigma_{\theta z}^{(e)} = \frac{ip\mu}{R_j \ell} U_{zi} = \frac{ip\mu}{R_j \ell} U_{zf}.$$  \hspace{1cm} \text{(5.47)}$$

It is assumed in Eqs. 5.45 through 5.47 that the node locations and material properties are those associated with element $e$. This finishes the simplification of the generalized SAFE formulation for CSH-waves. The next section will present a similar treatment for CLT-waves.

### 5.3 Considerations for CLT-Waves

For the case of CLT-waves, displacement is assumed in the $r$- and $\theta$-direction only and the discretized displacement equation shown in Eq. 5.5 immediately reduces to

$$\mathbf{u}^{(e)}(r, \theta, t) = \begin{bmatrix} \frac{R_j - r}{\ell} & 0 & \frac{r - R_j}{\ell} & 0 \\ 0 & \frac{R_j - r}{\ell} & 0 & \frac{r - R_j}{\ell} \end{bmatrix} \begin{bmatrix} U_{ri} \\ U_{th} \\ U_{ij} \\ U_{\theta j} \end{bmatrix} e^{i(p\theta - \omega t)}. \hspace{1cm} \text{(5.48)}$$

Next, Eq. 5.15 is again considered. Noting that the zero-strain components are $\varepsilon_z$ and $\varepsilon_{rz}$, any row in the $\mathbf{B}_1$ and $\mathbf{B}_2$ matrices associated with these components is removed and any column that is multiplied by a nodal displacement in the $z$-direction is removed, resulting in

$$\mathbf{B}_1 = \begin{bmatrix} -1/\ell & 0 & 1/\ell & 0 \\ \frac{R_j - r}{r\ell} & 0 & \frac{r - R_j}{r\ell} & 0 \\ 0 & \frac{r - R_j}{r\ell} - 1/\ell & 0 & \frac{R_j - r}{r\ell} + 1/\ell \end{bmatrix}. \hspace{1cm} \text{(5.49)}$$

and
Again, considering the non-zero strain components, the material stiffness matrix for CLT-waves can be reduced to

\[
B_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{R_j - r}{r \ell} & 0 & \frac{r - R_i}{r \ell} \\
\frac{R_j - r}{r \ell} & 0 & \frac{r - R_i}{r \ell} & 0
\end{bmatrix}.
\]

Note that Eq. 5.52 will be complex if viscoelastic materials are being considered.

Because there are still two displacement components per node, the number of degrees of freedom is only reduced by 1/3 when considering only CLT-wave propagation. Although, this still reduces the total size of the matrix to be solved, again reducing the computation time needed to solve for the CLT-wave dispersion curves.

Once the nodal displacement values have been determined, the non-zero stress components can be calculated using the following relations:

\[
\sigma_r^{(e)} = \frac{\lambda}{r \ell} \left( R_j (U_{ri} + i p U_{\theta i}) - R_i (U_{rj} + i p U_{\theta j}) \right) + \frac{2(\lambda + \mu)}{\ell} \left( U_{rj} - U_{ri} \right),
\]

\[
\sigma_\theta^{(e)} = \frac{\lambda}{r \ell} \left( R_j (U_{\theta j} + i p U_{ri}) - R_i (U_{\theta i} + i p U_{ri}) \right) + \frac{\mu}{\ell} \left( U_{\theta j} - U_{\theta i} \right).
\]
Again, it is assumed that any node locations or material properties in Eqs. 5.53 through 5.55 are those associated with element \( e \). Furthermore, \( r \) must be within the region \( R_i \leq r \leq R_j \).

This concludes the simplification of the generalized SAFE formulation for CLT-waves and of the SAFE formulation in general. The next chapter will discuss the numerical implementation of the SAFE method and present some examples. Convergence will be examined and results will be compared to those obtained from the analytical formulation.
Chapter 6

Numerical Results Following a Semi-Analytical Formulation

Following the mathematical formulation of the SAFE method for circumferential guided waves presented in the previous chapter, it is necessary to confirm the validity of the resulting solutions. For elastic materials, this validation will be completed through direct comparison with the results presented in Chapter 3 for the analytical formulation. For the case of viscoelastic materials, results will be validated by utilizing the plate approximation to compare the circumferential wave results with the results for the viscoelastic plate considered by (Bernard et al. 2001; Bartoli et al. 2006). Wave structures will also be examined for satisfaction of the appropriate boundary and continuity conditions as a further means of validation.

All SAFE computations presented here were completed in MATLAB R2007a (The MathWorks 2007). The codes developed for this purpose are included in Appendix D for reference. For the case of elastic layers only, there are generally two ways to proceed following the assembly of the finite-element matrices: search for values of $\omega$ that satisfy the characteristic equation at fixed increments of $p$, or search for values of $p$ at fixed increments of $\omega$. Either approach will yield the same result. Furthermore, a simple Bisection routine may be employed for the location of the roots. If this approach is taken, it is recommended that the form of the characteristic equation shown in Eq. 5.35 be employed to minimize the size of the resulting matrices which must be solved.

For cases involving one or more viscoelastic layers, the bisection routine is no longer applicable and a two-dimensional root seeking algorithm is needed to locate the complex roots of the characteristic equation. Several methods for accomplishing this task are discussed by (Lowe
1995). Alternatively, one may use the generalized eigenvalue problem form shown in Eq. 5.36 and then employ MATLAB’s native eigenvalue solver,

\[ [x, \Lambda] = \text{eig}(A, B), \]

which essentially solves the problem

\[ (A - \Lambda B)x = 0, \]

where \( \Lambda \) are the generalized eigenvalues corresponding to the generalized right eigenvectors given by \( x \) (The MathWorks 2007). The convenience of this function should be apparent as it is capable of solving for both the eigenvalues and eigenvectors for both real and complex input.

The next two sections will present several numerical examples for CSH- and CLT-waves in elastic and viscoelastic annuli. Convergence studies will also be completed to determine the minimum number of elements needed to provide an accurate solution for each case.

### 6.1 Numerical Results for CSH-Waves

#### 6.1.1 Elastic Materials

The CSH dispersion results for the steel annuli presented in Figure 3.1 and Figure 3.4 are again plotted in Figure 6.1 and Figure 6.2 with the results obtained from the SAFE calculation superimposed. Figure 6.1 corresponds to the near-plate aspect ratio of 0.984 and Figure 6.2 corresponds to an annulus with a very small aspect ratio of 0.2. All circumferential results were plotted for the outer radius of the annulus. The results from the SAFE calculation, shown as blue crosses, were calculated using 50 linear finite elements and it is evident from the plots that excellent agreement with the analytical result is achieved, indicating that the SAFE approach is an accurate alternative for the cases shown. It is noted that even for the plate approximation case, the analytical and SAFE results show small but perceivable differences from the actual plate,
Figure 6.1 CSH (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.984 aspect ratio. Results shown for the analytical formulation (red dots), SAFE formulation (blue crosses), and for a flat plate (black line). Linear wavenumber, phase, and group velocity were calculated at the OR.
Figure 6.2 CSH (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.2 aspect ratio. Results are shown for the analytical formulation (red dots), SAFE formulation (blue crosses), and for an actual plate (black line).
especially evident in the group velocity plots. This illustrates the importance of having a correct physical model for circumferential waves as most practical structures encountered have smaller aspect ratios than the case plotted in Figure 6.1. Furthermore, when dealing with ultrasound, even very small changes in velocity will manifest as measurable shifts in the time-of-flight of a wave.

One of the primary motivations for the development of the SAFE solution resulted from the tendency of the analytical method to miss roots at high frequencies for large radius annuli with low-impedance layers. It was shown in Figure 3.19 that the use of arbitrary precision arithmetic was able to provide a complete solution space for CSH-waves in the region examined, though the method will also eventually fail as frequency increases. Figure 6.3 shows the linear phase velocity dispersion curves for the same multilayered annulus corresponding to Figure 3.19 and described in Table 3.3. In Figure 6.3 the frequency axis has been extended to 1 MHz so to clearly see where the analytical solution fails. The red dots correspond to the double precision

![Figure 6.3](image-url)

**Figure 6.3** Phase velocity dispersion curves for CSH-waves propagating in a 2-layer annulus as described in Table 3.3. Analytical (red dots) and 200 element SAFE (blue lines) results shown. SAFE solution provides an accurate solution in the regions which the analytical method fails, below the black dashed line. Linear phase velocity was calculated at the common interface.
analytical results and the blue solid line corresponds to the 200-element SAFE solution. It is seen that, below the failure threshold of the analytical solution (black dashed line), the SAFE solution provides a complete and accurate result. Note the very small differences seen at high frequency and phase velocity would be eliminated with the addition of more elements.

Another reason for pursuing the SAFE method was because there were regions of the dispersion space where, though roots were found, the computation of wave structures was not possible. Furthermore, it is desirable to calculate wave structures in the regions where the analytical computation completely failed. Figure 6.4 shows the wave structures corresponding to the points labeled in Figure 6.3 as calculated using the SAFE method. It is seen that all boundary and continuity conditions are satisfied. Of particular interest is the wave structure corresponding to point \( c \) as this area could not be investigated with the analytical approach. As would be expected, it is seen that the wave propagates only in the low impedance layer.

![Figure 6.4](image)

**Figure 6.4** Wave structures corresponding to labeled points in Figure 6.3. Calculations performed with SAFE method. The number of elements used is indicated in each plot (# EL).
The number of elements used in each case is indicated in each plot of Figure 6.4 by a number followed by “EL”. Note that for the points investigated, convergence was achieved with fewer elements than were needed in the computation of this dispersion curves. This is because the total number of elements used in the calculation of the dispersion curves is determined by the smallest possible wavelength in the dispersion space of interest. The total number of elements needed to calculate the dispersion curves can be determined from the relation:

\[ NOE = \frac{\eta f_{\text{max}} \sum_{m=1}^{V} d_m}{c_{p,\text{min}}} \]  \hspace{1cm} (6.3)

where \( d_m \) is the thickness of layer \( m \), and \( f_{\text{max}} \) and \( c_{p,\text{min}} \) are the maximum frequency and minimum estimated phase velocity in the dispersion space of interest, respectively. The minimum estimated phase velocity is generally the bulk shear velocity of the lowest impedance layer for CSH-waves and the Rayleigh surface wave velocity of lowest impedance layer for CLT-waves. The term \( \eta \) in Eq. 6.3 represents the number of elements per minimum wavelength and will generally be determined by the type of element used (e.g. linear or quadratic) and the machine precision.

Consider a single-layer annulus with a 0.5 aspect ratio and a bulk shear wave velocity of 0.9 mm/\( \mu \)s. Figure 6.5 shows the linear phase velocity dispersion curves for the OR as calculated for increasing \( \eta \). Only the portion from 0.85 MHz to 1 MHz is shown so that the error in the SAFE calculation is easily visualized. As a relative measure of error, the value of the \( SH8 \) mode cutoff frequency has been calculated and is shown in Table 6.1. It is seen that for the linear elements used, \( \eta = 50 \) provides a nearly identical solution as the analytical computation and \( \eta = 25 \) provides a solution with an acceptable amount of error and half the number of elements. Depending on the specific dispersion region of interest, \( \eta \) may be reduced even further. It is assumed that the same general trends apply for the case of CLT-waves, though, the Rayleigh wave velocity should be used in the computation of \( NOE \).
Figure 6.5 Linear phase velocity dispersion curves for an annulus with an aspect ratio of 0.5 and a shear-wave velocity of 0.9 mm/μs. The phase velocity was calculated at the OR. A small frequency range is shown to easily visualize the error.

Table 6.1 Error in cutoff frequency for $SH8$ mode shown in Figure 6.5

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$f_{cutoff}$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical</td>
<td>0.9003</td>
<td>0.00%</td>
</tr>
<tr>
<td>50</td>
<td>0.9008</td>
<td>0.06%</td>
</tr>
<tr>
<td>25</td>
<td>0.9022</td>
<td>0.21%</td>
</tr>
<tr>
<td>15</td>
<td>0.9056</td>
<td>0.59%</td>
</tr>
<tr>
<td>10</td>
<td>0.912</td>
<td>1.30%</td>
</tr>
<tr>
<td>5</td>
<td>0.9496</td>
<td>5.48%</td>
</tr>
</tbody>
</table>
6.1.2 Viscoelastic Materials

With the elastic CSH-wave SAFE results validated, it is now appropriate to consider viscoelastic materials. The differences between elastic and viscoelastic materials were discussed in Chapter 4. To validate the code for viscoelastic materials, a single-layer viscoelastic annulus with a near-plate aspect ratio of 0.987 is explored. For comparison reasons, the annulus is assumed to be made of a polyethylene material as described in (Bartoli et al. 2006) and summarized in Table 6.2. Note that (Bartoli et al. 2006) specifies the bulk-wave attenuation constants per unit wavelength. To transform these constants to the form used in Eq. 4.26, they are divided by the factor $2\pi c_s\lambda$. Figure 6.6 shows the frequency-wavenumber and linear phase and attenuation dispersion curves for both the annulus (blue dotted lines) and for a flat plate of the same material (black solid lines). The results for the flat plate were generated using the closed-form solution presented in (Graff 1991). It is seen that the results from the SAFE computation using viscoelastic material properties matches very well with the closed-form solution for the plate. Excellent agreement is also found with Figure 3 in (Bartoli et al. 2006), who essentially arrived at the closed-form solution shown in Figure 6.6 by way of a SAFE computation for plate.

From Figure 6.6(a) it is seen that the wave modes in the viscoelastic annulus, and plate, no longer intercept the $\omega$-axis. This is a result of there being no non-propagating modes for the viscoelastic case. Instead, all modes are propagating, albeit with some associated attenuation, as shown in Figure 6.6(c). It is observed that the $SH0$ mode propagates with the least amount of attenuation of any mode and that said attenuation increases linearly with frequency.

Table 6.2 Dimensions of 0.987 aspect ratio annulus used in the calculation of curves in Figure 6.6 and properties for a polyethylene material (Bartoli et al. 2006)

<table>
<thead>
<tr>
<th>$r_{inner}$ (m)</th>
<th>$r_{outer}$ (m)</th>
<th>$\rho$ (kg/m$^3$)</th>
<th>$c_1$ (m/s)</th>
<th>$\alpha_1/\omega$</th>
<th>$c_s$ (m/s)</th>
<th>$\alpha_s/\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0127</td>
<td>953</td>
<td>2344</td>
<td>0.0117</td>
<td>953</td>
<td>0.0478</td>
</tr>
</tbody>
</table>
Figure 6.6 CSH (a) Frequency-wavenumber, (b) phase velocity, and (c) attenuation dispersion curves for a 12.7 mm thick polyethylene annulus of aspect ratio 0.987 (blue dots) and for a flat plate (black line). The annulus curves were calculated at the OR.
Figure 6.7 shows the dispersion curves for a 0.5 aspect ratio single-layer annulus with the same thickness and material properties listed in Table 6.2. Note that the inner and outer radius values in the table do not apply to this case. A very interesting phenomenon is seen in Figure 6.7(c) that has never been observed before; the higher-order CSH wave modes propagate with less attenuation than the fundamental $SH_0$ mode at frequencies above their corresponding “cutoff” frequencies. The term “cutoff” is used in quotations as there are not actual cutoff frequencies in the case of viscoelastic annuli. It is interpreted here to mean the frequency region above which dispersion is minimized. It is observed that the attenuation curves of the higher-order modes cross that of the $SH_0$ mode, reach a minima, subsequently converging to the same attenuation curve of the $SH_0$ mode. Furthermore, the minimum attenuation reached by each mode is increasing at a decreasing rate. From a practical perspective, this suggests that if it is desirable to excite a high-frequency CSH-wave in a viscoelastic annulus, higher-order modes have the potential to propagate longer distances than the fundamental mode.

This concludes this chapter’s presentation of the numerical results for CSH-waves as obtained using the SAFE method. Two more examples of elastic annuli coated with a viscoelastic material will presented in Chapter 7. The remaining sections of this chapter are devoted to the SAFE results for CLT-waves in elastic and viscoelastic annuli.

6.2 Numerical Results for CLT-Waves

6.2.1 Elastic Materials

Figure 6.8 and Figure 6.9 show the CLT-wave dispersion curves for steel ($c_s = 3.23 \text{ mm/μs}$, $c_l = 5.96 \text{ mm/μs}$, and $\rho = 7850 \text{ kg/m}^3$) annuli with aspect ratios of 0.984 and 0.2, respectively, as calculated using the SAFE method. The analytical results that were shown in
Figure 6.7 CSH (a) Frequency-wavenumber, (b) phase velocity, and (c) attenuation dispersion curves for a 12.7 mm thick polyethylene annulus of aspect ratio 0.5 (blue dots) and for a flat plate (black line). The annulus curves were calculated at the OR.
Figure 3.8 and Figure 3.11 are also shown (red dots) for reference, as are the results for a flat plate (black lines). The results from the SAFE computation (blue crosses) were obtained using 50 linear elements and are seen to be in excellent agreement with the analytical results. The wavenumber-frequency (thickness) and linear phase and group velocity annulus curves were calculated at the OR. This serves as a first validation of the SAFE method for CLT-waves.

As was seen for the CSH-wave case, the analytical computation is prone to missing roots when large radius annuli and low impedance materials are being studied. Figure 6.10 shows the CLT-wave dispersion curves for the annulus described in Table 3.3. The linear phase velocity was calculated at the common interface of the two layers. It is seen that the analytical results (red dots) are missing roots for the $LT0$ mode above approximately 200 kHz and for the $LT1$ mode above approximately 375 kHz. The failure threshold shown in the figure is included as a visual aid so to easily see where the analytical method fails and is not to be interpreted literally. It is seen that the 200-element SAFE solution (blue lines) provides excellent agreement with the analytical results (red dots) and is capable of finding roots in the regions where the analytical solution failed.

### 6.2.2 Viscoelastic Materials

To validate the SAFE method for calculating the CLT-wave dispersion curves for viscoelastic materials, the results obtained for the polyethylene annulus described in Table 6.2 are again compared to the SAFE results obtained by (Bartoli et al. 2006) for a plate made of the same material. Additionally, (Bernard et al. 2001) presents an analytical solution for a plate of the same material and geometry to which the plate-approximation result presented here can be compared.
Figure 6.8 CLT (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.984 aspect ratio. Results shown for the analytical formulation (red dots), SAFE formulation (blue crosses), and for a flat plate (black line). Annulus curves were calculated at the OR.
Figure 6.9 CLT (a) Frequency-wavenumber (thickness), (b) phase velocity, and (c) group velocity dispersion curves for a steel plate and an annulus with a 0.2 aspect ratio. Results shown for the analytical formulation (red dots), SAFE formulation (blue crosses), and for a flat plate (black line). Annulus curves were calculated at the OR.
A direct comparison of Figure 6.11 with Figure 2 of (Bartoli et al. 2006) and Figure 6 of (Bernard et al. 2001) has shown that the SAFE results presented here, obtained using an annular-approximation of a plate, match very well with the published results. The only significant differences noted are in the number of modes displayed in each author’s figure. In the case of (Bartoli et al. 2006), this is because a high-attenuation filter is used to remove all the highly attenuated modes, which essentially makes the phase velocity plot more visually decipherable. While they have not published their attenuation filtering thresholds, it has been found that plotting only the modes with attenuation less than 3.6 $Np/mm$ yielded an almost identical plot. Because (Bernard et al. 2001) have generated the plot analytically, they have the convenience of plotting finite number of modes and, for the modes they plotted, excellent agreement is seen with that presented in Figure 6.11.

Figure 6.10 Phase velocity dispersion curves for CLT-waves propagating in a 2-layer annulus as described in Table 3.3. Analytical (red dots) and 200 element SAFE (blue lines) results shown. SAFE solution provides an accurate solution in the regions which the analytical method fails, below the black dashed line. Linear phase velocity was calculated at the common interface.
Figure 6.11 CLT (a) phase velocity and (b) attenuation dispersion curves for a polyethylene annulus of aspect ratio 0.992 obtained using the SAFE method with 300 elements. Plot (c) shows a larger portion of the attenuation axis. Linear phase velocity and attenuation were calculated at the OR.
Now consider an annulus with the geometry and properties summarized in Table 6.3 that correspond to a 4 in diameter schedule 40 pipe with a 1 mm thick polyethylene coating. This particular geometry yields an aspect ratio of 0.879 for the multilayered annulus. The linear phase velocity and attenuation dispersion curves, calculated at the common interface, for this annulus are shown in Figure 6.12. To obtain the plots shown, all modes with attenuation greater than $0.7 \text{ Np/mm} \ (\approx 6080 \text{ dB/m})$ were removed.

Some noteworthy phenomena are observed in the phase velocity dispersion curves of Figure 6.12(a). Most notably, it is seen that for the case of the viscoelastic coated annulus (VE) shown here, the modes appear to cross each other in the regions where the curves for the equivalent elastic (E) case veer away from one another. However, the modes do not share a common root at these intersections as they do not cross at the same points in the attenuation dispersion curves. It is also seen from the attenuation dispersion curves that these phase velocity intersections correspond to attenuation minima of the mode traveling dominantly in the low-impedance layer and maxima of the mode traveling dominantly in the high-impedance layer. For example, consider the point just below 300 kHz where the $LT0$ mode intersects the $LT2$ mode. The attenuation maxima and minima for this point are clearly seen in Figure 6.12(b).

Another very obvious difference between the viscoelastic multilayered annulus and the equivalent elastic one is the presence of a mode extending linearly from the origin to approximately 400 kHz, where it veers upward and eventually joins (not shown) with the mode in the top right corner of the phase velocity plot. Similar modes have been observed by other authors.

### Table 6.3

<table>
<thead>
<tr>
<th></th>
<th>$r_{\text{inner}}$ (m)</th>
<th>$r_{\text{outer}}$ (m)</th>
<th>$\rho$ (kg/m$^3$)</th>
<th>$c_1$ (m/s)</th>
<th>$\alpha_1/\omega$</th>
<th>$c_s$ (m/s)</th>
<th>$\alpha_s/\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer 1</td>
<td>0.05113</td>
<td>0.05715</td>
<td>7850</td>
<td>5850</td>
<td>0</td>
<td>3230</td>
<td>0</td>
</tr>
<tr>
<td>Layer 2</td>
<td>0.05715</td>
<td>0.05818</td>
<td>953</td>
<td>2344</td>
<td>0.0117</td>
<td>953</td>
<td>0.0478</td>
</tr>
</tbody>
</table>
Figure 6.12  Linear (a) phase velocity and (b,c) attenuation dispersion curves for CLT-waves propagating in a 2-layer annulus as described in Table 6.3. The 125-element SAFE solution is shown for both the elastic (red dotted lines) and viscoelastic (blue dots) cases. Linear phase velocity and attenuation were calculated at the common interface. The black solid lines are for a flat steel plate of the same thickness as the steel annulus layer.
in viscoelastic plates (Bernard et al. 2001) and, in particular, (Simonetti 2004) who attributes this behavior to the existence of non-propagating modes with non-purely-imaginary roots in the dispersion curves of the elastic structure.

Perhaps the most important conclusion that can be drawn from Figure 6.12 is that if interest is only in the modes that will propagate for reasonably long distances, the dispersion curves for the equivalent elastic multilayered annulus can be used and any region in which two modes veer from one another should be avoided due to the high attenuation associated with the viscoelastic case. However if this is done, it should be realized that the resulting group velocity dispersion curves will only be valid in the regions where the curves tend to those of the isolated high-impedance layer, as these are the regions of minimum attenuation. It was shown by (Bernard et al. 2001) that for attenuation minima, the group velocity and energy velocity associated with the viscoelastic case are similar.

It has been shown that the SAFE numerical approximation method is capable of accurately solving for the roots of the respective characteristic equations for both the elastic and viscoelastic cases for CSH-waves and CLT-waves. The next section will discuss a limitation of the SAFE method.

6.3 Limitations of the SAFE Formulation

Though the SAFE method has been shown to be able to accurately calculate the dispersion curves, it is not without its limitations. Figure 6.13 and Figure 6.14 show a comparison of wave structures as calculated using the analytical and SAFE methods. It is seen that the SAFE method is able to accurately calculate the correct wave structures. In the case of CSH-waves, it is seen the SAFE method is also able to accurately calculate the stress distribution profiles. However, the stress distributions are not shown for the CLT-wave case as they were found to not
agree well with the analytical calculation. This is a result of there being no prescribed boundary or continuity conditions for the SAFE formulation and has been observed by other authors as well (Gao 2007). The SAFE calculation is accurate for the CSH-wave stresses as they are both single-variable functions of displacement, whereas in the CLT-wave case, the stresses are functions of two displacements and thus discontinuities arise at the layer interfaces and the appropriate stresses do not necessarily vanish at the boundaries.

It has been shown the SAFE method can produce accurate displacement wave structures for both the CSH-wave and CLT-wave cases and accurate stress distributions for the CSH-wave case. If stress distributions are required for the CLT-wave case, the analytical expressions should be used.

This concludes the presentation of the SAFE numerical results. Complete dispersion solutions are now available for the dispersion space of interest. The next chapter will present some experimental results that verify the trends observed in the theoretical models.

Figure 6.13 Comparison of wave structure and stress distribution as calculated using the analytical and SAFE techniques for CSH-waves. Plots (b), (d), and (f) correspond to the points b, d, and f shown in Figure 3.5(b).
Figure 6.14 Comparison of wave structures as calculated using the analytical and SAFE techniques for CLT-waves. Plots (d), (e), and (f) correspond to the points d, e, and f shown in Figure 3.12(b).
Chapter 7
Experimental Results

The previous chapters have dealt with the development and validation of the tools necessary to accurately model circumferential guided waves in single and multilayered elastic and viscoelastic annuli. Such tools have significant potential for applications in nondestructive testing using ultrasonic guided waves. There are many practical situations in which a structure can be accurately represented by the annulus model. For example, the practical analog to the generalized plane-strain assumption used in the development of the circumferential wave models is a hollow cylinder whose boundaries in the axial direction are far removed from the wave field, relative to wavelength. Because ultrasonic wavelengths are typically very short, most all piping systems fit this criteria. Examples of applications include the nondestructive testing of gas transmission line, gas storage well casing, boiler and heat-exchanger tubing, and many others.

The goal of this chapter is to provide experimental validation of the theoretically based modeling tools that have been developed in the previous chapters. Two experiments are discussed. The first experiment is designed to study the effect of increasing coating thickness on the propagation of CSH-waves. Data is collected for several different coating thicknesses and the results are compared to the theoretical predictions. The second experiment studies the effect of partially coated regions on CSH-waves. Several differently sized sections of coating are removed and several theoretically predicted wave features that are sensitive to the presence of the coating layers are explored.

Note that Appendix C presents a method for the approximation of wall thickness using CLT-waves. The theoretical models are used to develop a multi-mode time-based technique. The theoretical results are validated using ABAQUS/Explicit finite-element modeling software. A
brief introduction to the software and method are also included in Appendix C. The results are not presented here as appropriate specimens were not available and the method could not be demonstrated experimentally.

7.1 Effects of Coating Thickness on CSH-Wave Propagation

According to the CSH-wave dispersion examples provided in the previous chapters, one effect of the addition of a lower-impedance layer to a single-layer annulus is a decrease in the maximum achievable group velocity for any specific wave mode. Furthermore, if the lower-impedance layer is viscoelastic in nature, some wave attenuation will be introduced. Figure 7.5 summarizes these effects for a 24 in diameter schedule 10 pipe with various thicknesses of a Bitumastic 50 coal-tar mastic coating with the properties published by (Barshinger et al. 2004) and summarized in Table 7.1. The curves were calculated for the outer radius of the pipe.

As seen from Figure 7.1, as the thickness of the coating layer increases, there is a corresponding decrease in linear phase and group velocity and an increase in attenuation. It should be noted that when dealing with viscoelastic materials, the velocity of the wave group is no longer necessarily the group velocity, but instead the energy velocity. However, it has been shown by (Bernard et al. 2001) that for regions where attenuation reaches a minimum, there is generally good agreement between the group and energy velocity. Furthermore, since it is seen in Figure 7.1(b) that there is excellent correspondence between the viscoelastic phase velocity

**Table 7.1** Dimensions and properties of pipe specimen used for coating thickness study. Properties for Bitumastic 50 coating from (Barshinger et al. 2004).

<table>
<thead>
<tr>
<th></th>
<th>$r_{\text{inner}}$ (m)</th>
<th>$r_{\text{outer}}$ (m)</th>
<th>$\rho$ (kg/m$^3$)</th>
<th>$c_1$ (m/s)</th>
<th>$\alpha_1/\omega$</th>
<th>$c_s$ (m/s)</th>
<th>$\alpha_s/\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mild Steel</strong></td>
<td>0.29845</td>
<td>0.3048</td>
<td>7850</td>
<td>5850</td>
<td>0</td>
<td>3230</td>
<td>0</td>
</tr>
<tr>
<td><strong>Bitum. 50</strong></td>
<td>0.3048</td>
<td>varied</td>
<td>1500</td>
<td>1860</td>
<td>0.023</td>
<td>750</td>
<td>0.24</td>
</tr>
</tbody>
</table>
Figure 7.1 CSH (a) phase velocity, (b) group velocity, and (c) attenuation dispersion curves for a 24 in schedule 10 pipe with various thicknesses of the Bitumastic 50 coating described in Table 7.2. Linear phase and group velocity and attenuation were calculated at the OR.
dispersion curves and the equivalent elastic curves, there will be good correspondence between
the group velocity curves for the two cases. Thus, the group velocity curves shown in
Figure 7.1(b) are that of the equivalent elastic solution.

To verify the trends seen in Figure 7.1, an experiment was designed in which two
sensors, a transmitter and a receiver, were placed around the circumference of a 24 in schedule 10
pipe as shown in Figure 7.2. To generate the CSH-waves in the pipe wall, Lorentz-type
Electromagnetic Acoustic Transducers (EMATs) were used. Appendix B provides a review of the
fundamental physics and geometries of EMATs for guided wave excitation.

It should be explicitly noted here that EMATs are not centered at a particular frequency,
but rather fixed at a specific wavelength. For this study, an EMAT that generated the $SH0$
mode at 130 kHz was used. For this sensor, $\lambda/2$ (and subsequently the magnet width) is approximately
equal to 0.5 in, as determined for a phase velocity of 3.23 mm/$\mu$s.

Figure 7.2 Experimental setup for coating thickness influence study.
The EMATs were placed on either side of a 2 ft long circumferential section of the pipe, as seen in the photograph in Figure 7.2. Data was collected for the no-coating case and subsequently for coating thicknesses of 3 mils, 7 mils, and 13 mils. The sensors were not moved at any time during the experiment so that any variation in the time-of-flight of the wave could be attributed to the coating presence. The coating was left to cure for 24 hours after each application, according to the manufacturer’s recommendations, before data collection and recoating. The average coating thickness was determined from a minimum of 32 randomly acquired measurements using a calibrated Linear-Variable Differential Transducer (LVDT).

Figure 7.3 shows the analytic envelopes of the wave packets that have propagated through the bare pipe (black) and pipe with 3 mils (blue), 7 mils (green), and 13 mils (red) of Bitumastic 50 coating. The envelope amplitudes and changes in time-of-flight, as compared to the bare pipe case, are summarized in the figure. It is seen that as the coating thickness increases, there is a corresponding increase in time-of-flight and decrease in amplitude. Therefore, the experimentally measured trends agree well with the theoretical predictions. Furthermore, it is seen that the attenuation is more severe for the case of the 13 mil coating, which agrees with the prediction from the attenuation dispersion curves in Figure 7.1(c). The time and amplitude shifts are summarized in Figure 7.4.

According to the theoretical models, the change in time-of-flight between the bare pipe and 13 mil coating cases should be 1.1 $\mu$s and the change in amplitude should be approximately 0.24 dB. The actual measured values are larger at 1.6 $\mu$s and 2.58 dB, respectively. This is an indication that the actual material properties are slightly different than the published quantities. More importantly, though, is the fact that the general trends match as the measurement of acoustic properties of coating materials are inherently difficult and can be highly dependent on temperature.
Figure 7.3 Analytic envelopes of the wave packets that have traveled through the 2 ft long circumferential section of pipe with different coating thicknesses, showing increased time-of-flight and decreased amplitude with increasing coating thickness.

Figure 7.4 Plots showing the (a) time-of-flight and (b) amplitude change as coating thickness is increased.
The general trends of the theoretical models have been verified by experimentation. It has been verified that increasing coating thickness causes a reduction in the propagation velocity of a group of waves and that attenuation becomes increasingly severe with coating thickness. As seen from the dispersion curves in Figure 7.1, the increase in time-of-flight and attenuation could be maximized by moving to a higher frequency, essentially increasing sensitivity to the coating presence. The next section presents another experimental demonstration and practical application for a pipe with a much thicker coating. In addition to the time and amplitude features just examined, a frequency-based feature is studied.

7.2 Effects of Coating Presence on CSH-Wave Propagation

The specific goal of this section is to use the theoretical models and the experimental results presented in the last section to design an experiment capable of detection missing or disbonded sections of coating. To aid in the selection of guided wave modes and frequencies that would be sensitive to coating presence, or lack thereof, several desired wave properties are defined. An acceptable mode/frequency for coating disbond detection should adhere to the following characteristics:

1. The selected modes/frequencies should be affected by the presence, or lack of presence, of coating in more than one way (e.g. attenuation, velocity, frequency, etc.).
   - Modes exhibiting velocity variation due to coating presence will provide a time-based characterization feature.
   - While all modes are attenuated by coating, modes with moderate attenuation must be selected over modes exhibiting excessive attenuation or too little attenuation. This will provide an amplitude-based characterization feature.
• Modes whose attenuation varies moderately with frequency will illustrate a
  frequency filtering effect and will therefore provide a frequency-based
  characterization feature.

2. The selected modes/frequencies should not be sensitive to variations in pipe wall
   thickness.
  • By choosing a mode/frequency whose velocity varies with coating presence,
    but not with wall thickness, it will be possible to design a sensor that can be
    used on multiple pipe sizes.
  • Selecting this type of mode will also prevent the confusion of corrosion areas
    with areas of disbonded coating.

3. The selected modes/frequencies should not mistake a wet interface for well
   bonded coating.
  • To avoid confusing water with well-bonded coating, a mode with completely
    in-plane or dominantly in-plane displacement must be selected.

4. The final dimensions and specifications of the sensors needed to generate the
   selected modes/frequencies must conform to the limitations imposed by the ILI
   tool and environment.
  • EMATs must work reliably, even with small fluctuations in sensor liftoff.
    This is accomplished by using strong magnets, high excitation voltages, and
    as low a frequency as possible, although, very low frequency EMATs may
    have impractically large footprints.

CSH-waves satisfy many of these general criteria. For example, because liquids cannot
generally support shear motion, CSH-waves will not leak energy at wet interfaces. Also, with
fewer modes in the dispersion space of interest, mode conversion can be avoided making mode
differentiation much simpler. Of particular interest are modes in the coated annulus which nearly
converge to the $SH0$ bare-annulus mode. These modes will be nearly non-dispersive and will nearly converge to the bulk-shear velocity of the steel independently of thickness. Thus it will not be sensitive to variations in wall thickness.

Case studies have shown that coal-tar-felt coatings are a commonly used type of coating prone to disbonding (Nestleroth et al. 2002). A sample was obtained and the longitudinal properties of the coating were measured experimentally and the shear properties were estimated using the methods described in (Van Velsor 2006) using an assumed Poisson ratio typical of tar-based materials ($\nu = 0.35$). All dimensions and material properties are summarized in Table 7.2.

The linear phase, group, and attenuation dispersion curves were calculated for the inner radius (IR) of the pipe and are shown in Figure 7.5. Three sets of curves are shown for comparison: the viscoelastic case (blue dots), the elastic case (red lines) and the bare pipe case (black dashed line). It is seen that the resonant points of the pipe layer are characterized by maxima in the group velocity curves and minima in the attenuation dispersion curves. Since the regions of interest in this study are the resonant points of the pipe layer, where attenuation approaches zero, it is assumed, as was legitimized in the previous section, that the group velocity provides a reasonable enough approximation of the energy velocity. Furthermore, as the phase velocity curves of the viscoelastic model trace those of the elastic model in the resonance regions, it is acceptable to use the group velocity curves for the elastic case, but only in these regions.

**Table 7.2** Dimensions and properties of pipe specimen used for coating disbond detection study

<table>
<thead>
<tr>
<th></th>
<th>$r_{\text{inner}}$ (m)</th>
<th>$r_{\text{outer}}$ (m)</th>
<th>$\rho$ (kg/m$^3$)</th>
<th>$c_1$ (m/s)</th>
<th>$c_s$ (m/s)</th>
<th>$\alpha_1/\omega$</th>
<th>$\alpha_s/\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Layer 1</strong></td>
<td>0.29845</td>
<td>0.3048</td>
<td>7850</td>
<td>5850</td>
<td>3230</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>Layer 2</strong></td>
<td>0.3048</td>
<td>0.3079</td>
<td>1500</td>
<td>1400</td>
<td>680</td>
<td>0.007</td>
<td>0.035</td>
</tr>
</tbody>
</table>
Figure 7.5  CSH (a) phase velocity, (b) group velocity, and (c) attenuation dispersion curves for tar-glass-felt coated pipe described in Table 7.2. Linear phase and group velocity and attenuation were calculated at the IR. Viscoelastic, elastic, and bare pipe curves shown for comparison.
The wave structures for the points shown in Figure 7.5(a) can be seen in Figure 7.6. As expected, it is seen that resonant, or near resonant, points provide maximum displacement profiles in the pipe layer. Additionally, the resonant points corresponding to the fundamental mode of the bare pipe, $SH0^p$, are nearly non-dispersive and will not show significant phase velocity shifting for thinning pipe walls. It is also seen from Figure 7.6(a) that there is large displacement at the pipe/coating interfaces, which is desirable for coating disbond detection.

With a theoretical model to work with, it is possible to now consider wave features which may be sensitive to coating disbonding. The first, and perhaps most significant, feature follows directly from the consideration of the group velocity dispersion curve shown in Figure 7.5(b) and was discussed in the previous section. It is seen that the peak group velocity of the resonance points are slower for the case of the coated pipe. This implies that the presence and size of a coating disbond will affect the time-of-flight of a circumferential guided wave. This is a non-obvious and significant finding, as time-based features tend to be more reliable than others.
Two more disbond detection features are related to the attenuation of the circumferential guided wave caused by the presence of well-bonded coating. The first, and most obvious and commonly used feature, which was also addressed in the previous section, is a reduction in the amplitude of the traversed wave. A disbond will show as an increase in received wave amplitude as compared to a region of perfectly bonded coating. The second attenuation related feature is the frequency-dependent attenuation of the propagating wave, as shown in Figure 7.7. For the region shown, there is an approximately linear decrease in the attenuation minima and therefore one would expect the higher frequency content of an ultrasonic pulse to attenuate more quickly than lower frequencies. It other words, a well bonded coating acts as a frequency filter and if a disbond is present, more higher frequency content will be received.

As a side note, the topic of material property measurement often arises when studying wave propagation in coated structures and it is interesting to note that the plot seen in Figure 7.7 may be of use in determining the bulk shear-wave attenuation constant if it is unknown. By experimentally locating the attenuation minima over a range of frequencies, the attenuation

![Figure 7.7](image)

**Figure 7.7** Magnified view of attenuation dispersion curves shown in Figure 7.5(c). Attenuation units have been converted to dB/m.
constant may be determined via the inverse problem. Similarly, the group, or energy, velocity maxima may be of use in determining the bulk shear-wave velocity of the coating material.

Prior to experimental verification, it is useful to perform some numerical studies to verify some of the predictions of the analytical modeling. Finite-element modeling is a quick and cost effective virtual experiment that has the added benefit of being free of experimental error and is therefore useful for verifying the trends just discussed. A finite-element model was created using ABAQUS/Explicit. A brief introduction to the finite element method is included in Appendix C. Due to a prohibitively large number of elements, it was not possible to model a 24 in schedule 10 pipe with a coating layer. In order to overcome this problem, a pipe section with an outer diameter of 8 in and an aspect ratio of 0.94 was created. For a bare pipe, this is numerically equivalent to the 24 in diameter pipe. In the model, the dimensions and properties of the coating layer remained the same as in the case of the 24 in pipe. The excitation source was modeled to imitate the loading achieved using an SH-wave EMAT with a wavelength of 25.4 mm. This loading was chosen so to generate the \( SH1 \) mode when excited at approximately 130 kHz.

A cross section of the pipe at several different times can be seen in Figure 7.8. From these pictures it can be seen that the wave is severely attenuated after just two traversals of the 8 in diameter pipe. Note that, based on circumferential length, one traversal around a 24 in diameter pipe is approximately equivalent to 3 traversals around the 8 in model, suggesting that the wave would be severely attenuated in one traversal around a 24 in diameter pipe with the perfectly bonded coating. Accordingly, it may be necessary to use multiple sensor pairs to achieve disbond detection in large diameter pipe.
The resulting RF waveforms of a CSH-wave traveling in the circumferential direction of the bare and coated pipe models are shown in Figure 7.9. It is seen that the time period for one traversal in the coated pipe model is longer than that for the bare pipe. The time-of-flight in the 8 in diameter coated pipe model is 222 μs, corresponding to a velocity of approximately 2.9 mm/μs, compared to approximately 3.23 mm/μs for a bare pipe.

Based on the analysis of the finite-element model presented in Figure 7.8 and Figure 7.9, the velocity and attenuation trends obtained from the model agree with the trends obtained from

Figure 7.8 Cross-sectional view of 130 kHz CSH-wave traveling in a 8 in. diameter tar-glass-felt coated pipe with an aspect ratio of 0.94. Note that one traversal around a 24 in. diameter pipe is approximately equivalent to 3 traversals around the 8 in. diameter pipe.
theoretical analysis and experimental results from the previous section. With the numerical and analytical models in agreement, a proof-of-concept experiment was designed.

Once again, Lorentz–type EMAT sensors are employed. A sensor configuration is adopted, which will allow for the normalization of all received pulses to an initial reference pulse. An illustration of the sensor configuration is seen in Figure 7.10. The transmitter (T) and receiver (R) are separated by some arc length (ΔS). The reference pulse is the pulse that travels directly from the transmitter to the receiver and its point-of-reception is indicated by the orange dot in the figure. The reference pulse can be used to normalize subsequently received signals in the amplitude, time, and frequency domains. Note that in Figure 7.10, the “green” signal is the first signal to completely traverse the circumference of the pipe, whereas the “red” signal never traverses the area in between the transmitter and receiver prior to first reception. For this reason, the received “green” pulse is the one used to extract information regarding coating integrity.

**RF Waveforms**

![RF Waveforms](image)

**Figure 7.9** RF waveforms showing comparison between 130 kHz CSH-wave traveling in a bare pipe (blue) and in a pipe with a 3 mm tar-glass-felt coating (magenta). Two pulses are seen for each traversal because there are waves traveling in both the CW and CCW directions.
Figure 7.11 shows the results from a coating disbond detection experiment obtained using the sensor configuration described by Figure 7.10 on a 20 in schedule 10 pipe with the same coating thickness and properties described in Table 7.2. The CSH-wave dispersion curves for the 20 in schedule 10 pipe are nearly identical to the ones shown in Figure 7.5 for a 24 in schedule 10 pipe and, for this reason, not shown again here. A full-circumference, 1 ft x 1 ft, and 1 ft x 2 ft disbond were created. The acquired signals from each simulated disbond area can be seen in Figure 7.11. The peaks of interest are marked with blue arrows and include the reference pulse and the first complete traversal of the pipe. Table 7.3 summarizes the time-of-flight and amplitude-loss data for all three disbond states. There is more than a 20 μs difference between the bare pipe case and the 1 ft disbond case and nearly a 10 μs difference between the 1 ft and 2 ft disbond cases. Accounting for the noise level of the system and experimental error, time-of-flight differences in excess of 2 μs can be reliably measured. It is clearly seen that as the disbond size...
increases, the time-of-flight and amplitude loss both decrease. Thus two of the three proposed disbond detection features have been verified experimentally.

As previously discussed, the tendency of attenuation to increase with frequency can be used as a coating disbond detection feature. The absence of coating will result in a frequency spectrum with higher frequency content. As the amount of well-bonded coating increases, higher frequency content will be filtered out by absorption. An experimental demonstration of this concept is shown in Figure 7.12, in which the Short-Time Fourier Transforms (STFTs) of the data sets displayed in Figure 7.11 are plotted. The difference between the two white lines, marking the maximum frequency content of the reference pulse and the first CCW traversal, represents the amount of lost frequency content after one complete circumferential traversal of the pipe. The results are summarized in Table 7.3, along with the time and amplitude data, and it is seen that as the disbond size increases, the amount of lost frequency content decreases since there is less coating to absorb it. Therefore, it has been experimentally demonstrated that by monitoring the frequency content of the received signal, it is possible to determine if a coating disbond is present.

Figure 7.13 summarizes the results of the coating disbond proof-of-concept study. All three features are plotted on the same scale. The sensitivity of the technique can be increased by combining the three features into a single feature, such as through addition or multiplication.

It is interesting to note that the coating disbond detection method introduced here has subsequently undergone several rounds of field testing and has been found to work reliably. This concludes the experimental verification portion of this work. Note that a simple method for estimating the remaining wall thickness of a pipe using CLT-waves, which is validated by finite-element modeling, is introduced in Appendix C.
Figure 7.11 Ultrasonic waveforms obtained from a 20”-diameter schedule 10 pipe with a tarring-glass-felt coating with a (a) 1ft. disbond, (b) 2ft. disbond, and for a (c) bare pipe.

Table 7.3 Amplitude-loss, time-of-flight, and frequency loss for disbonded coating regions

<table>
<thead>
<tr>
<th></th>
<th>1ft x 1ft</th>
<th>1ft x 2ft</th>
<th>Bare</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude Loss (dB)</td>
<td>35.14</td>
<td>30.87</td>
<td>8.19</td>
</tr>
<tr>
<td>Time-of-Flight</td>
<td>522.4</td>
<td>513.1</td>
<td>499.3</td>
</tr>
<tr>
<td>Lost Frequency Content</td>
<td>69%</td>
<td>50%</td>
<td>10%</td>
</tr>
</tbody>
</table>
Figure 7.12 STFTs of RF-waveforms for a (a) 1ft disbond, (b) 2ft disbond, and for a (c) bare pipe. Results obtained using a 64-point Hanning window with 32-point overlap.

Figure 7.13 A plot showing the time, frequency, and amplitude disbond detection features for three different disbond sizes in a tar-glass-felt coated pipe.
Chapter 8
Summary and Conclusion

8.1 Review of Work

The primary motivation behind this work was to develop the tools necessary to accurately model wave propagation in the circumferential direction of elastic and viscoelastic multilayered annuli to aid in the development of new methods for non-destructive testing using ultrasonic guided waves and to improve those methods already existing. It has been common among publishing researchers to make simplifying assumptions when studying circumferential guided waves. Most commonly, the annulus is approximated as a plate without mention of the relevance of such an assumption. Among those who have used a proper annular model, none have approached the problem of multiple layers or viscoelastic media, which are, more commonly than not, encountered in actual applications. This work has removed these assumptions and has directly approached the topic of wave mechanics in multilayered elastic and viscoelastic annuli for both circumferential shear-horizontal and Lamb [type] waves.

This study began with a generalized re-formulation of the boundary value problem for circumferential waves in a single-layer annulus as all other formulations to date have been constructed in such a manner that limits the applicability to only one layer. A boundary-independent phase constant was utilized and no non-dimensionalized quantities were employed. Generalized plane-strain and isotropic material assumptions were made, thus enabling the use of Helmholtz decomposition. The coefficient matrices were developed for both CSH- and CLT-waves by direct solution of the wave equation and by the method of displacement potentials,
respectively. A multilayer formulation was presented based on the Global Matrix Method and the newly developed generalized formulation.

Numerical solutions to the characteristic equations for CSH- and CLT-waves were presented in the third chapter. It was shown that the results from the generalized formulation matched well with the previously published results for a single-layer annulus. Results were also checked by comparing results of a near-unity aspect ratio cylinder with that of a flat plate. Wave structure plots were generated and analyzed for satisfaction of the appropriate boundary and continuity conditions. The effect of decreasing aspect ratio on the dispersion space was analyzed. Several solutions were presented for CSH- and CLT-waves in multilayered annuli and some physical insights into the interpretation of the dispersion curves were presented. Most notably, it was shown that the resonant excitation of a specific layer could be achieved by exciting the regions of the dispersion space where the curves of the multilayered structure intercept the curves for the isolated layer in which resonance is desired. The term “resonance” was taken to mean a point at which maximum displacement is achieved. Finally, some computational limitations were discussed where it was concluded that the computation based on the analytical solution failed to find roots for large radius annuli with low acoustic impedance layers because of the inability to calculate small arguments of high-order Bessel functions. This limitation served as an argument for the development of the Semi-Analytical Finite Element solution for circumferential waves. Anticipation of adding viscoelastic layer capabilities added additional motivation for this development.

Following the analytical development of the dispersion equations, a concise review of the foundations of linear isothermal viscoelasticity was presented. It was shown that for the steady-state oscillatory loading of viscoelastic materials, stress lags strain by a phase constant, the tangent of which provides a relative measure of the elastic and dissipative properties of the material. Furthermore, the elastic-viscoelastic correspondence principle was used to show that the
governing equations of elasticity could be used to describe the behavior of viscoelastic materials provided complex material moduli were employed. The meaning of the roots of the characteristic equation was discussed. Finally, it was shown how the complex moduli could be calculated by experimentally measuring the bulk-wave velocity and attenuation of a material.

The Semi-Analytical Finite-Element formulation for circumferentially traveling waves was introduced. Detailed derivations were shown for linear elements and considerations for other element types were made. Hamilton’s variational principle of mechanics was used to develop the characteristic SAFE equation for circumferential waves and a hysteretic model was introduced for viscoelastic materials. Finally, provisions were made for decoupling the CSH- and CLT-wave solutions for isotropic materials and the discretized stress equations were developed.

Numerical results following the SAFE development were provided in Chapter 6. It was shown that nearly perfect agreement is achieved between the analytical and SAFE solutions for elastic media and that the SAFE method provided accurate solutions in the same regions of the dispersion space that the analytically-based computation failed. Viscoelastic solutions were introduced for single-layer annuli. In the case of CSH-waves, a unique phenomenon related to the annular geometry was witnessed. Specifically, it was seen that the attenuation of the higher-order CSH-modes can propagate with less attenuation than the fundamental mode. Linear phase, group, and attenuation dispersion curves were presented for elastic/viscoelastic multilayered structures and it was shown that the resonant points of the elastic layer corresponded to minima in the attenuation dispersion curves. It was found the stress distributions arising from the SAFE calculation for CLT-waves was not accurate and therefore the analytical expressions should be used in these cases.

Two experiments were presented. The first studied the effect of coating thickness on the propagation of the fundamental $SH0$ mode. Transmitting and receiving EMATs were placed in a fixed position around the circumference of a 20 in schedule 10 pipe and the area between the
sensors was coated with a coal-tar mastic coating. Data was collected for several different coating thicknesses and the experimental results were shown to agree well with the theoretically predicted trends. It was shown that measurable changes in time-of-flight and attenuation occurred for coatings as thin as 0.003 in.

In the second experiment, a method was developed for the detection of disbonds in the protective coating layer of gas transmission pipeline using CSH-waves. Several disbond detection features were identified through an analysis of the dispersion curves. Most notably, a time-based disbond detection feature was identified as the presence of the coating results in a decrease in the maximum group velocity of the propagating wave. Accordingly, disbonds would manifest as decreased in the time-of-flight of a CSH-wave. Another feature identified was related to the frequency dependent attenuation of the wave. The time-based feature was verified through finite-element modeling and both features were subsequently verified through experimental demonstration. A self-normalizing sensor arrangement was also introduced so to provide a more reliable relative amplitude measurement of the received waves.

Several appendices of practical interest were also included in this work. Appendix B provided a concise review of the physics and geometric configurations of Lorentz EMATs for generating Lamb- and SH-waves. Basic design considerations were introduced. Two different sensor configurations for generating Lamb-waves were discussed and detailed drawings were provided. The periodic permanent magnet configuration for generating SH-waves was also discussed and a detailed drawing was provided.

Appendix C included an example of an application involving CLT-waves. The study was not included in Chapter 7 as it was performed using finite-element analysis and experimental results were not presented. A multi-mode thickness estimation method was introduced based on the time-of-flight of the $LT0$ and $LT1$ modes. A dispersion analysis for several annuli of different thicknesses was completed and it was noted that there were drastic changes in the group
velocities of the two modes at frequencies near 175 kHz. More importantly, the velocity of the LT0 modes was seen to decrease with decreasing thickness and the velocity of the LT1 modes was seen to increase with decreasing thickness, suggesting that by monitoring the relative arrival times between the two modes, an estimation of the average wall thickness could be realized. The proposed method was explored using a finite-element model in which wall thickness was gradually decreased to a minimum so to not have any reflecting surfaces. The modeling results were shown to agree well with the theoretical predictions and changes in relative arrival time as large as 30 μs were achieved for wall thicknesses which were 30% of the original thickness at the thinnest point. It was found that the arrival time of the LT0 mode changed more drastically with thickness, which also agreed with the theoretical predictions.

8.2 Contributions

This work has immediate relevance to the non-destructive testing of cylindrical structures such as gas transmission pipeline, gas storage wells, and boiler and heat exchanger tubing. The importance of this work increases with decreasing annulus diameter and increasing wall thickness as it has been shown that plate assumptions are not valid in these cases. Furthermore, through the consideration of multilayered elastic/viscoelastic annuli, it has been shown that there exist optimal excitation points that can minimize attenuation, which is generally desirable for the inspection of coated structures. Correspondingly, there are regions of very high attenuation which should be avoided. In short, the primary contribution resulting from this work is the development of a set of tools for modeling circumferential wave propagation that will aid in the development of new and more accurate non-destructive testing methods. Particular emphasis is placed on the enhanced mode/frequency selection capabilities provided by the new more accurate models, such
as those that have been demonstrated in Chapter 7 for the case of coating disbond detection.

Some of the detailed contributions that have resulted from this work are as follows:

1. Presentation of a generalized formulation for circumferential wave propagation in annular structures, a necessary step toward the development of the $n$-layer solutions.


5. Development of Semi-Analytical Finite Element method for calculating wave structures of circumferential shear-horizontal and Lamb [type] guided waves in annuli composed of an arbitrary number of elastic or viscoelastic layers.

6. Showed for the first time that the higher-order CSH wave modes propagate with less attenuation than the fundamental $SH0$ mode above certain frequencies in a single-layer viscoelastic annulus. A phenomenon not observed for wave propagation in a plate.


8.3 Future Directions

There are several directions in which this work could potentially be expanded. With the basic tools for the theoretical modeling of circumferential waves in multilayered annuli now available, defect identification, localization, and sizing studies would be a logical next step for research efforts. While an example problem of thickness estimation was introduced here, there are many other defect geometries to be examined. Efforts should concentrate on those that cannot be reliably detected using MFL technology and those that have the potential to enhance MFL sizing predictions. These include metal-loss defects that are long in the circumferential direction and narrow in the axial direction, pitting defects, and axially oriented cracks.

Another area for expansion would be the modification of the theory and tools presented here to account for an infinite soil layer on the outside of the annulus and for liquid layers on the inside or outside of the annulus. These conditions typically exist for buried pipeline, pipeline carrying petroleum products, and submersed pipeline. Modified theories of this type would be of substantial practical importance.

8.4 Publications

The following refereed journal publications have arisen from aspects of this work and other work completed throughout the tenure of this work:


Also, several articles are in preparation for publication and are as follows:


Much of this work has also been presented at research conferences in the United States and abroad. The proceedings are as follows:


References


The MathWorks, I., MATLAB 7.4.0.287 (R2007a), 2007.


Wolfram Research, I.,MATHEMATICA 4 4.0.1.0, 1999.


Appendix A

Circumferential Lamb [Type] Coefficient Matrix Elements

\[ d_{11} = \frac{k_L}{2} J_{p-1}(k_L r) - \frac{k_L}{2} J_{p+1}(k_L r) \]

\[ d_{12} = \frac{k_L}{2} Y_{p-1}(k_L r) - \frac{k_L}{2} Y_{p+1}(k_L r) \]

\[ d_{13} = i \frac{p}{r} J_p(k_S r) \]

\[ d_{14} = i \frac{p}{r} Y_p(k_S r) \]

\[ d_{21} = i \frac{p}{r} J_p(k_L r) \]

\[ d_{22} = i \frac{p}{r} Y_p(k_L r) \]

\[ d_{23} = \frac{k_S}{2} J_{p+1}(k_S r) - \frac{k_S}{2} J_{p-1}(k_S r) \]

\[ d_{24} = \frac{k_S}{2} Y_{p+1}(k_S r) - \frac{k_S}{2} Y_{p-1}(k_S r) \]

\[ d_{31} = \frac{\mu}{r^2} \left( \frac{r^2 \kappa^2 k_L^2}{4} \left( J_{p-3}(k_L r) - 2 J_p(k_L r) + J_{p+3}(k_L r) \right) \right) \]

\[ + \frac{r k_L (\kappa^2 - 2)}{2} \left( J_{p-1}(k_L r) - J_{p+1}(k_L r) \right) - p^2 (\kappa^2 - 2) J_p(k_L r) \]

where \( \kappa = c_t / c_s \)
\[ d_{32} = \frac{\mu}{r^2} \left( \frac{r^2k^2}{4} \left( Y_{p-2}(k_zr) - 2Y_{p}(k_zr) + Y_{p+2}(k_zr) \right) \right. \\
\quad \quad \quad \quad \quad + \left. \frac{rk_z(k^2 - 2)}{2} \left( Y_{p-1}(k_zr) - Y_{p+1}(k_zr) \right) - p^2(k^2 - 2)Y_p(k_zr) \right) \]

\[ d_{33} = \frac{\mu}{r^2} \left( irpk_s \left( J_{p-1}(k_sr) - J_{p+1}(k_sr) \right) - i2 J_p(k_sr) \right) \]

\[ d_{34} = \frac{\mu}{r^2} \left( irpk_s \left( Y_{p-1}(k_sr) - Y_{p+1}(k_sr) \right) - i2 pY_p(k_sr) \right) \]

\[ d_{41} = \frac{\mu}{r^2} \left( irpk_L \left( J_{p-1}(k_Lr) - J_{p+1}(k_Lr) \right) - i2 J_p(k_Lr) \right) \]

\[ d_{42} = \frac{\mu}{r^2} \left( irpk_L \left( Y_{p-1}(k_Lr) - Y_{p+1}(k_Lr) \right) - i2 pY_p(k_Lr) \right) \]

\[ d_{43} = \frac{\mu}{r^2} \left( -\frac{r^2k^2}{4} \left( J_{p-2}(k_sr) - 2J_p(k_sr) + J_{p+2}(k_sr) \right) \right. \\
\quad \quad \quad \quad \quad \quad \quad \quad + \left. \frac{rk_s}{2} \left( J_{p-1}(k_sr) - J_{p+1}(k_sr) \right) - p^2J_p(k_sr) \right) \]

\[ d_{44} = \frac{\mu}{r^2} \left( -\frac{r^2k^2}{4} \left( Y_{p-2}(k_sr) - 2Y_p(k_sr) + Y_{p+2}(k_sr) \right) \right. \\
\quad \quad \quad \quad \quad \quad \quad \quad + \left. \frac{rk_s}{2} \left( Y_{p-1}(k_sr) - Y_{p+1}(k_sr) \right) - p^2Y_p(k_sr) \right) \]
Electromagnetic Acoustic Transducers (EMATs) offer a noncontact method of generating ultrasonic waves in a structure that is particularly useful if the sensor or structure is in motion. Additionally, they are capable of operating at elevated temperatures and can maintain consistent coupling on rough or oxidized surfaces (Alers 2004). They are of particular interest in this work as circumferential guided waves are used for the in-line inspection of pipe, in which an inspection tool travels through the pipe at a near constant velocity and EMATs are used to generate and receive ultrasound.

The two primary components of a typical EMAT consist of a current-carrying coil and a magnet or group of magnets. When the coil is placed within close proximity to the surface of a metallic structure, the dynamic current $I e^{i\omega t}$ passing through the coil traces induces an eddy current, of frequency $\omega$ and opposite polarity, within the surface layer of the structure (Thompson 1973b). If aligned properly, the interaction of the eddy current and magnetic field $B_0$, which for the present purpose is assumed to be static, results in a dynamic Lorentz force $f_\omega$ on the lattice of the material (Thompson 1973a). Mathematically this is state as,

$$f_\omega = \mu_0 J_\omega \times B_0,$$

where $J_\omega$ is the dynamic eddy current density and $\mu_0$ is the permeability of free space. If the material under investigation is ferromagnetic, there are also magnetic and magnetostrictive forces present, though the magnetic forces are small compared to the magnetostrictive ones and are normally neglected (Thompson 1978). Furthermore, when high magnetic fields are applied, such as those produced by modern Neodymium Rare-Earth (NdFeB) magnets, the magnetostrictive
forces are much smaller than the Lorentz forces and are therefore not considered in this review (Thompson 1977).

Depending on the relative orientation of the magnetic field to the induced eddy currents, both Lamb and shear horizontal (SH) guided waves can be generated and received within a structure. Figure B.1 and Figure B.2 show the Lorentz EMAT configurations for generating Lamb- and SH-waves, respectively. In the Lamb-wave case, the magnetic field can be oriented at any angle in the plane of propagation (Thompson 1973b), though from a practical perspective, it is usually oriented out-of-plane or in-plane, as shown in Figure B.1 (a) and (b), respectively. An out-of-plane magnetic field will result in dominant in-plane Lorentz forces and thus dominant in-plane displacement. Likewise, an in-plane magnetic field will result in dominant out-of-plane Lorentz forces and will therefore generate dominant out-of-plane displacement. In the case of Figure B.1(a), a permanent magnet is placed directly above the coil. For dominant out-of-plane displacement, Figure B.1(b), two permanent magnets are placed on the sides of the coil corresponding to the wave propagation direction. A ferromagnetic backing plate can be placed on top of the magnets to enhance the field within the plate being inspected (Nestleroth et al. 2002). Despite the direction of the Lorentz forces, either type of Lamb-wave EMAT can be used to generate symmetric and antisymmetric modes.

Figure B.2 shows the Lorentz EMAT configuration for generating SH-waves. In this arrangement, a group of periodically polarized magnets (PPM) are placed above the coil (Thompson 1990; Hirao et al. 2003). The result is an in-plane Lorentz force that is oriented orthogonally to the wave propagation direction and alternates direction with each column of magnets (as viewed from the top). In addition to being noncontact, SH-wave EMATs are particularly appealing as their efficiency can exceed that of traditional piezoelectric devices for generating guided waves.
Figure B.1 Lorentz EMAT configurations for generating Lamb-waves with (a) in-plane displacements and (b) out-of-plane displacements (Thompson 1973b).
As seen in Figure B.1 and Figure B.2, the wavelength of the generated wave is a fixed parameter that is determined by the coil trace spacing for Lamb-waves and by the magnet width for SH-waves. Therefore, an EMAT must be designed to excite a particular mode at a particular frequency according to the relation

\[ \lambda = \frac{c_p}{f} \]  

\text{B.2}

where \( c_p \) is the phase velocity of the mode to be excited at frequency \( f \). As shown in Figure B.3, a fixed wavelength sensor has a diagonal activation line in the phase velocity dispersion space. Any

\textbf{Figure B.2} Lorentz EMAT configuration for generating SH-waves (Thompson 1990; Hirao et al. 2003).
mode that intersects the activation line can be excited by driving the current in the coil at a frequency corresponding to the intersection point.

(Thompson 1973b) has shown, theoretically and experimentally, that the amplitude of a wave excited by an EMAT decreases exponentially as the coil is lifted away from the surface of the structure being inspected. This is because the induced eddy currents from each coil trace begin to overlap and cancel as the coil is lifted away. Based on this observation, it is noted that as the wavelength becomes smaller, the amount acceptable coil liftoff will decrease due to the closer proximity of the traces. It is therefore advantageous to keep the coil as close to the surface as practically possible.

One last comment should be made regarding EMAT efficiency. The sensitivity of a pair of sending and receiving EMATs is usually described by a transfer impedance (Alers 2004), which is a ratio of the voltage output at the receiver terminals to the current input at the transmitter terminals. To optimize the transfer impedance, an electronic tuning network consisting of transformers, resistors, inductors, and capacitors is used to match the electronic

Figure B.3 Phase velocity dispersion curves showing potential excitation points for a Lamb-wave EMAT with fixed wavelength $\lambda$. 
impedance of the source electronics with the transmitting coil. It is also common to attach a preamplifier to the output of the receiving coil prior to digitization of the signal. The design of said networks and preamplifiers is beyond the scope of this review and is therefore not included. Additionally, there has also been a significant amount work done regarding the optimized design of the current-carrying coils used for EMATs. Things such as the trace cross-sectional area and coil width-to-length ratio will affect the performance of the EMAT. For an in-depth review of the electronic aspects of EMAT tuning and impedance matching, one is referred to (Maxfield 2004).
Appendix C

Approximate Thickness Measurement Using CLT-Waves: A Finite-Element Example

This appendix serves as a brief introduction to the finite-element method as implemented by ABAQUS/Explicit. An introduction to the finite-element method is provided and the hybrid analytical/numerical finite-element philosophy is discussed. An example application of CLT-waves is introduced and the method is verified using

C.1 Introduction to the Finite-Element Method

When complex structural geometries are to be considered, analytical solutions are often not available and approximation methods are employed. Several of these approximation methods include the boundary-element method, the finite-difference method, and the finite-element method. ABAQUS utilizes the finite element method for nearly all analyses, though dynamic analyses in ABAQUS use a finite difference method to propagate solutions in the time domain.

The ABAQUS finite-element strategy is based on the variational principles of mechanics, such as Hamilton’s Principle that was introduced in Chapter 5 where it was employed as part of the Semi-Analytical Finite-Element method. It essentially collects the potential energy, strain energy, and external work present in a system into a scalar integral, which is evaluated between two instants in time. Given some prescribed initial and final conditions, the condition which minimizes this integral, as seen in Eq. C.1,

\[ \int_{t_1}^{t_2} (\delta T + \delta W) dt = 0, \quad C.1 \]
yields the equations of motion for the system (Meirovitch 1967). In Eq. C.1, \( T \) is the kinetic energy present in the system and \( W \) is the work done on the system, which is equivalent to the strain energy for conservative systems.

In the present work, the viscous damping caused by a viscoelastic coating, is incorporated in the ABAQUS model as a stiffness-proportional damping factor \( \beta_R \) per (Luo 2005), who shows that if the complex elastic modulus \( E^* \) and the complex shear modulus \( G^* \) of a coating are known, the stiffness-proportional damping factor can be calculated using Eq. C.2.

\[
\beta_R = \frac{E^*}{\omega E'} - \frac{\alpha_R}{\omega^2}. \tag{C.2}
\]

Because the mass-proportional Rayleigh damping factor \( \alpha_R \) only affects the damping of modes with frequencies below the ultrasonic range, the term is typically dropped from Eq. C.2.

In the hybrid analytical finite-element approach, the analytical models are used to select the mode and frequency of excitation and subsequently to design the proper loading conditions and geometries for the finite-element model. This analytically based input is a crucial aspect of finite element modeling as the finite-element solution is highly dependent on the initial conditions and loading. An example of the hybrid analytical finite-element approach is illustrated next using CLT-waves.

**C.2 A Method for Approximating Wall Thickness Using CLT-Waves**

When using circumferential waves there is a unique opportunity to analyze a signal that has propagated around the entire circumference, finally arriving again at the excitation point. This is the through-transmission method and it is an ideal method for the inspection of pipe using ILI tools. A method of estimating the remaining wall thickness, or amount of wall loss, using a CLT multi-mode through transmission signal is introduced here. Other guided wave thickness
estimation techniques, based on frequency-shifting phenomena, have been introduced by (Luo et al. 2003; Luo and Rose 2004).

In some cases a corrosion region may not have distinct edges but instead may have gradual thinning over a large area. In these situations, the MFL tools are typically used for in-line inspection can estimate the axial length relatively well but have difficulty estimating the circumferential extent. Though guided waves may also have difficulty estimating the circumferential extent, as there may be no edge reflections, this issue can be circumvented by performing a direct thickness estimate in the circumferential direction using the through transmission mode of inspection. Either the entire circumference can be investigated using a single set of sensors or it can be segmented with multiple sensor pairs if localization is critical.

Consider the linear phase and group velocity dispersion curves for a 20 in schedule 10 pipe (red lines) shown in Figure C.1. The linear phase velocity is shown in Figure C.1(a) and the group velocity in Figure C.1(b). Again, the curves are calculated at the inside surface of the annulus as this is where excitation and reception take place using ILI tools. Dispersion curves are also shown for pipe of the same nominal diameter but with 30%, 50%, and 70% thickness reductions. It is seen that reduced thickness results in decreased $LT_0$ and increased $LT_1$ phase and group velocities. Accordingly, one would expect the time-of-flight of a wave propagating through a thinned region to decrease for $LT_1$ modes and increase for $LT_0$ modes, thus providing a time-based measurement which can be related to the thickness of the annulus in the region between the transmitting and receiving sensors.
To explore the potential for thickness estimation using this phenomenon, a two-dimensional finite-element model was created in ABAQUS. As seen in Figure C.2, the model consisted of a pipe cross section with gradual wall loss over a region slightly less than 180°. Four models were generated; a full thickness model and models with 30%, 50%, and 70% thickness reduction at the thinnest point (9 o’clock). Loading was modeled to imitate that of a Lamb-wave EMAT and is seen as a segmented region near the 12 o’clock position. For EMAT excitation (fixed wavelength), the activation line is a diagonal line in the phase velocity dispersion space as Figure C.1 (a) Linear phase and (b) group velocity dispersion curves for a 20 in diameter schedule 10 pipe (6.35mm). Curves also shown for 30%, 50%, and 70% thickness ($t_w$). Linear phase and group velocity were calculated at the IR.

To explore the potential for thickness estimation using this phenomenon, a two-dimensional finite-element model was created in ABAQUS. As seen in Figure C.2, the model consisted of a pipe cross section with gradual wall loss over a region slightly less than 180°. Four models were generated; a full thickness model and models with 30%, 50%, and 70% thickness reduction at the thinnest point (9 o’clock). Loading was modeled to imitate that of a Lamb-wave EMAT and is seen as a segmented region near the 12 o’clock position. For EMAT excitation (fixed wavelength), the activation line is a diagonal line in the phase velocity dispersion space as Figure C.1 (a) Linear phase and (b) group velocity dispersion curves for a 20 in diameter schedule 10 pipe (6.35mm). Curves also shown for 30%, 50%, and 70% thickness ($t_w$). Linear phase and group velocity were calculated at the IR.
shown in Figure C.1(a). The wave length of the excitation region was chosen to generate a dominant $LT1$ mode at a frequency of 175 kHz. This excitation region was chosen as there is a noticeable difference in the group velocity of both the $LT0$ and $LT1$ modes.

Figure C.3(a) shows the theoretically predicted wave structures for the $LT0$ and $LT1$ modes, at a frequency of 175 kHz, in the full wall thickness annulus. Figure C.3(b) shows the $r$- and $\theta$-displacement waveforms and combined analytic envelope as received at the receiving node shown in Figure C.2. Because the receiving node was placed on the inside radius of the pipe, of particular interest in Figure C.3(a) is the relative amplitude of the $u_r$ and $u_\theta$ displacements at the very bottom of the wave structure plots. Excellent agreement is seen between the theoretical prediction and the finite-element model. The $u_r$ displacement is slightly larger than the $u_\theta$ displacement for the $LT0$ mode and the $u_\theta$ displacement is more than twice that of the $u_r$ displacement for the $LT1$ mode. Note that the waves shown have propagated through the thinned region in the counter-clockwise direction. A break was put in the circumferential direction and modeled with infinite elements so to be able to clearly resolve the waveforms traveling through

![Figure C.2](image)

**Figure C.2** Finite element model showing original thickness (left) and with gradual wall loss to 70% of the original thickness.
the thinned region. As expected from the designed loading, the faster \textit{LT1} mode is the dominant mode with a large in-plane displacement component. The \textit{LT0} mode at 175 kHz is also generated, though less efficiently. As will be shown, the fact that both modes are generated is advantageous for this application.

Figure C.4(a) shows the analytic envelopes for the full-thickness model as well as the 30\%, 50\%, and 70\% gradual wall loss models. An obvious shift in the time-of-flight of both the \textit{LT0} and \textit{LT1} modes is seen, the values of which are summarized in Figure C.4(b). As the wall thickness reduces the velocity of the \textit{LT1} mode increases monotonically and the velocity of the \textit{LT0} mode decreases monotonically. The decrease in velocity of the \textit{LT0} mode is more significant. This agrees with the trends seen in the group velocity dispersion curves shown in Figure C.1(b).

A simple way to estimate the remaining thickness would be to monitor the time-of-flight between the peaks of the \textit{LT1} and \textit{LT0} modes. An increase in this quantity would be indicative of wall thinning and the estimated wall thickness would be comparable to an average over the region between the two sensors.

This method is immediately applicable to bare pipe and pipe with thin, low-attenuation coatings as the time shifts caused by the coating presence will be much smaller than that caused by the wall thinning. For pipe with a heavier, more attenuative coating, it would be necessary to first verify that the coating is intact and well bonded, using the method introduced in Section 7.2 for example, before extracting information regarding wall thickness. Provided the coating is intact, the relative time-of-flight between the two modes should still have good correlation with wall thickness.
Figure C.3  (a) Theoretically predicted wave structures for the LT0 and LT1 modes in the 100% thickness annulus at a frequency of 175 kHz. (b) Example RF waveform and analytic envelopes for \( u_r \) and \( u_\theta \) displacement. The transmitted LT1 and LT0 modes are clearly seen.
Figure C.4 (a) Time gated view of the analytic envelopes of the LT1 and LT0 waves for the different gradual wall thickness (WT) reduction models. (b) Graphical representation of the change in TOF of the LT1 and LT0 modes.
Appendix D

Computer Codes

D.1 Analytical Formulation: CSH-Waves

% Jason K. Van Velsor (c)2009
% Written in MATLAB R2007a

% This code calculates the circumferential shear horizontal wave
% dispersion curves for a multilayered elastic annulus. The circular
% frequency (w) roots are calculated at fixed increments of angular
% wavenumber (p). In some cases, typically low shear wave velocities
% ( <1 mm/us ) and high frequencies, the Bessel functions cannot be
% calculated. If this occurs and some roots are not found, the SAFE
% code may be used.
% clc, clear all;

[NoL, a, b, rho, CL, CS] = GetInput; % Reads input from Excel file
fstop = 1.1;   % Ending frequency in MHz
dp = 1;

wstart = 0.001;
dw = 0.005;
jj = 0;

kstop = 2*pi*fstop/min(CS)*1.2;
pstart = 0.01*a(1);
pstop = kstop*b(NoL);

for p = pstart:dp:pstop
    jj = jj+1;
    ii = 0;
    for w = wstart:dw:2*pi*fstop;
        for LL = 1:1:NoL
            DB1{LL} = LayerMatrix(w,p,CL(LL),CS(LL),rho(LL),a(LL));
            DT1{LL} = LayerMatrix(w,p,CL(LL),CS(LL),rho(LL),b(LL));
            DB2{LL} = LayerMatrix(w+dw,p,CL(LL),CS(LL),rho(LL),a(LL));
            DT2{LL} = LayerMatrix(w+dw,p,CL(LL),CS(LL),rho(LL),b(LL));
        end
    DG1 = GlobalMatrix(NoL, DB1, DT1);
DG2 = GlobalMatrix(NoL,DB2,DT2);
f1 = det(DG1);
f2 = det(DG2);

min = w; max = w+dw;
if f1*f2 <= 0
   ii = ii+1;
   root(jj,ii) = Bisection(f1,f2,min,max,p,CL,CS,a,b,rho,NoL);
end

end

D.1.1 Subfunction: GetInput

function [NumberOfLayers, InnerRadius, OuterRadius, LayerDensity,...
   LayerCL, LayerCT] = GetInput

[filename1,pathname] = uigetfile('*.xls','Select Input File');
homedirectory = pwd;
        cd(pathname);
infile = xlsread(filename1);
        cd(homedirectory);

[m,n] = size(infile);
infile(:,n) = [];
reshape = isnan(infile);
for ii=1:1:n-1;
    if reshape(1,ii) == 0
        Input(:,ii) = infile(:,ii);
    end
end
[x,NumberOfLayers] = size(Input);
InnerRadius = Input(1,:).*1000;
OuterRadius = Input(2,:).*1000;
LayerDensity = Input(3,:)./1000;
LayerCL = Input(4,:)./1000;
LayerCT = Input(5,:)./1000;

D.1.2 Subfunction: LayerMatrix

% Calculates the matrix elements for Guided Circumferential SH Waves
function [D] = LayerMatrix(w,p,CL,CS,rho,r)
ks = w/CS;
mu = rho*CS^2;

D(1,1) = besselj(p,ks*r);
D(1,2) = bessely(p,ks*r);
D(2,1) = mu*ks/2*( besselj(p-1,ks*r) - besselj(p+1,ks*r) );
D(2,2) = mu*ks/2*( bessely(p-1,ks*r) - bessely(p+1,ks*r) );

D.1.3 Subfunction: GlobalMatrix

function DG = GlobalMatrix(NoL,DB,DT)
% Assembles global matrix from layer matrices
DG1 = zeros([2*NoL,2*NoL]);
DG2 = zeros([2*NoL,2*NoL]);
DG(1,1:1:2)=DB{1}(2,:);
DG(2*NoL,(2*NoL-1):1:2*NoL)=DT{NoL}(2,:);
for ii = 2:1:NoL
    DG((2*ii-2):1:(2*ii-1),(2*ii-3):1:(2*ii-2)) = DT{ii-1}(:,:);
    DG((2*ii-2):1:(2*ii-1),(2*ii-1):1:(2*ii)) = -DB{ii}(:,:);
end

D.1.4 Subfunction: Bisection

function [max] = Bisection(f1,f2,min,max,p,CL,CS,a,b,rho,NoL)
% A simple bisection routine to locate the roots of a function
eps = 1;
while eps >= 1e-7
    mid = (min+max)/2;
    fmin = f1; fmax = f2;
    for LL = 1:1:NoL
        DB(LL) = LayerMatrix(mid,p,CL(LL),CS(LL),rho(LL),a(LL));
        DT(LL) = LayerMatrix(mid,p,CL(LL),CS(LL),rho(LL),b(LL));
    end
    DG = GlobalMatrix(NoL,DB,DT);
    fmid = det(DG);
    if fmin*fmid <= 0;
        max = mid;
    end
end
else fmin*fmid > 0;
    % 'Flag 2'
    min = mid;
end
eps = abs(max/max-min/max);
end

D.1.5 Function: Wavestructure

clear all; clc;

[NoL, a, b, rho, CL, CS] = GetInput;

%----- Input root info for desired wave structure ---------------- %
f = 0.1163; % frequency
cp = 3.421; % linear phase velocity
R = b(1); % radius at which linear phase velocity was calculated
%----------------------------------------------------------------%

w = 2*pi*f;
p = R*w/cp;

for LL = 1:1:NoL
    DB{LL} = LayerMatrix(w,p,CL(LL),CS(LL),rho(LL),a(LL));
    DT{LL} = LayerMatrix(w,p,CL(LL),CS(LL),rho(LL),b(LL));
end

DG = GlobalMatrix(NoL,DB,DT);
X = zeros(2*NoL,1); X(1,1) = 1;
DG = [DG X];
Coef = rref(DG); Coef = Coef(:,2*NoL+1); Coef = Coef./Coef(1,1);

for ii = 1:1:NoL
    % ---------- Displacement Uz ---------------------
    r(ii,:) = a(ii):(b(ii)-a(ii))/500:b(ii);
    ks = w/CS(ii);
    Uz(ii,:) = Coef(2*ii-1)*besselj(p,ks*r(ii,:)) + ...
               Coef(2*ii)*bessely(p,ks*r(ii,:));

    % ---------- Stress SIGMARz ---------------------
    mu = rho(ii)*CS(ii)^2;
    Srz(ii,:) = mu*ks/2*( Coef(2*ii-1)*besselj(p-1,ks*r(ii,:)) - ... 
                         besselj(p+1,ks*r(ii,:))) + Coef(2*ii)*bessely(p-1,ks*r(ii,:)));
    Stz(ii,:) = imag( i*mu*p./r(ii,:) .* ...
                     ( Coef(2*ii-1)*besselj(p,ks*r(ii,:)) + ... 
                       Coef(2*ii)*bessely(p,ks*r(ii,:)) ));
end

% ------------ Normalize -----------------------------
Mxu = max(max(abs(Uz)));
Mxs = max([max(abs(Srz)) max(abs(Stz))]);

figure('Position',[100,100,350,600],'Color','white');
for jj=1:1:NoL
    hold on;
    plot(Uz(jj,:)/Mxu,r(jj,:),'-k',Srz(jj,:)/Mxs,r(jj,:),'--k',...
    Stz(jj,:)/Mxs,r(jj,:),'-.k');
end
box on; xlim([-1.1 1.1]); ylim([a(1) b(NoL)]);
set(gca,'Layer','top'); xlabel('u_z , \sigma_r_z , \sigma_{\theta_z}');
ylabel('r (mm)'); legend('u_z','\sigma_r_z','\sigma_{\theta_z}',0);
if NoL > 1
    for hh=2:1:NoL
        plot((-1.1:0.1:1.1), a(hh)*ones(1,23),':k');
    end
end
set(gca, 'FontSize', 18);

D.1.6 Sample Input File (INPUT.xls)
D.2 Analytical Formulation: CLT-Waves

% Jason K. Van Velsor (c)2009
% Written in MATLAB R2007a
%
% This code calculates the circumferential Lamb-type wave
% dispersion curves for a multilayered elastic annulus. The circular
% frequency (w) roots are calculated at fixed increments of angular
% wavenumber (p). In some cases, typically low shear wave velocities
% ( <1 mm/us ) and high frequencies, the Bessel functions cannot be
% calculated. If this occurs and some roots are not found, the SAFE
% code may be used.
%
clc, clear all;

[NoL, a, b, rho, CL, CS] = GetInput; % Reads input from Excel file
fstop = 1;   % Ending frequency in MHz
dp = 0.5;

wstart = 0.001;
dw = 0.01;
kstop = 2*pi*fstop/min(CS)*1.2;
pstart = 0.001*a(1);
pstop = kstop*b(NoL);

jj = 0;

for p = pstart:dp:pstop
jj = jj+1;
ii = 0;
for w = wstart:dw:2*pi*fstop;
   for LL = 1:1:NoL
      DB1(LL) = LayerMatrix_Beta(w,p,CL(LL),...
                  CS(LL),rho(LL),a(LL));
      DT1(LL) = LayerMatrix_Beta(w,p,CL(LL),...
                  CS(LL),rho(LL),b(LL));
      DB2(LL) = LayerMatrix_Beta(w+dw,p,CL(LL),...
                  CS(LL),rho(LL),a(LL));
      DT2(LL) = LayerMatrix_Beta(w+dw,p,CL(LL),...
                  CS(LL),rho(LL),b(LL));
   end

DG1 = GlobalMatrix(NoL,DB1,DT1);
DG2 = GlobalMatrix(NoL,DB2,DT2);
f1 = det(DG1);
f2 = det(DG2);

min = w; max = w+dw;
if f1*f2 <= 0
   ii = ii+1;
end
D.2.1 Subfunction: \texttt{LayerMatrix}

\begin{verbatim}
function [D] = LayerMatrix_Beta(w,p,CL,CS,rho,r)

K = CL/CS;
kL = w/CL;
kS = w/CS;
mu = rho*CS^2;

D(1,1) = kL/2 * besselj(p-1,kL*r) - kL/2 * besselj(p+1,kL*r);
D(1,2) = kL/2 * bessely(p-1,kL*r) - kL/2 * bessely(p+1,kL*r);
D(1,3) = i*p/r * besselj(p,kS*r);
D(1,4) = i*p/r * bessely(p,kS*r);
D(2,1) = i*p/r * besselj(p,kL*r);
D(2,2) = i*p/r * bessely(p,kL*r);
D(2,3) = kS/2 * besselj(p+1,kS*r) - kS/2 * besselj(p-1,kS*r);
D(2,4) = kS/2 * bessely(p+1,kS*r) - kS/2 * bessely(p-1,kS*r);

D(3,1) = mu/r^2 * ( (r*K*kL)^2/4 * ( besselj(p-2,kL*r) - 2*besselj(p,kL*r) + besselj(p+2,kL*r) ) + ... 
* r*K*(K^2-2)/2 * ( besselj(p-1,kL*r) - besselj(p+1,kL*r) + p^2*(K^2-2) * besselj(p,kL*r) )
* mu/r^2 * ( (r*K*kL)^2/4 * ( bessely(p-2,kL*r) - 2*bessely(p,kL*r) + bessely(p+2,kL*r) ) + ... 
* r*K*(K^2-2)/2 * ( bessely(p-1,kL*r) - bessely(p+1,kL*r) + p^2*(K^2-2) * bessely(p,kL*r) )

D(3,3) = mu/r^2 * ( i*r*p*kS * ( besselj(p-1,kS*r) - ... 
* besselj(p+1,kS*r) ) - i*2*p * besselj(p,kS*r) );
D(3,4) = mu/r^2 * ( i*r*p*kS * ( bessely(p-1,kS*r) - ... 
* bessely(p+1,kS*r) ) - i*2*p * bessely(p,kS*r) );
end
\end{verbatim}
\begin{align*}
D(4,1) &= \mu/r^2 \left( i*r*p*kL \cdot (\text{besselj}(p-1,kL*r) - \text{besselj}(p+1,kL*r)) - i*2*p \cdot \text{besselj}(p,kL*r) \right); \\
D(4,2) &= \mu/r^2 \left( i*r*p*kL \cdot (\text{bessely}(p-1,kL*r) - \text{bessely}(p+1,kL*r)) - i*2*p \cdot \text{bessely}(p,kL*r) \right); \\
D(4,3) &= \mu/r^2 \left( -(r*kS)^2/4 \cdot (\text{besselj}(p-2,kS*r) - 2*\text{besselj}(p,kS*r) + \text{besselj}(p+2,kS*r)) + r*kS/2 \cdot (\text{besselj}(p-1,kS*r) - \text{besselj}(p+1,kS*r)) - p^2 \cdot \text{besselj}(p,kS*r) \right); \\
D(4,4) &= \mu/r^2 \left( -(r*kS)^2/4 \cdot (\text{bessely}(p-2,kS*r) - 2*\text{bessely}(p,kS*r) + \text{bessely}(p+2,kS*r)) + r*kS/2 \cdot (\text{bessely}(p-1,kS*r) - \text{bessely}(p+1,kS*r)) - p^2 \cdot \text{bessely}(p,kS*r) \right); \\
\end{align*}

**D.2.2 Subfunction: GlobalMatrix**

```matlab
function DG = GlobalMatrix(NoL,DB,DT)
DG(1,1:1:4)=DB{1}(3,:);
DG(2,1:1:4)=DB{1}(4,:);
DG(4*NoL-1,(4*NoL-3):1:4*NoL)=DT{NoL}(3,:);
DG(4*NoL,(4*NoL-3):1:4*NoL)=DT{NoL}(4,:);
for ii = 2:1:NoL
    DG((4*ii-5):1:(4*ii-2),(4*ii-7):1:(4*ii-4)) = DT{ii-1}(:,:);
    DG((4*ii-5):1:(4*ii-2),(4*ii-3):1:(4*ii)) = -DB{ii}(:,:);
end
```

**D.2.3 Subfunction: Bisection**

```matlab
function [max] = Bisection_Beta(f1,f2,min,max,p,CL,CS,a,b,rho,NoL)
% A simple bisection routine to locate the roots of a function

eps = 1;
while eps >= 0.0000001
    mid = (min+max)/2;
    fmin = f1; fmax = f2;
    for LL = 1:1:NoL
        DB(LL) = LayerMatrix_Beta(mid,p,CL(LL),CS(LL),rho(LL),a(LL));
        DT(LL) = LayerMatrix_Beta(mid,p,CL(LL),CS(LL),rho(LL),b(LL));
    end
```
DG = GlobalMatrix(NoL,DB,DT);
fmid = det(DG);

if fmin*fmid <= 0;
  % 'Flag 1'
  max = mid;
else fmin*fmid > 0;
  % 'Flag 2'
  min = mid;
end
eps = abs(max/max-min/max);
end

D.2.4 Function: Wavestructure

clear all; clc;

%----- Input root info for desired wave structure -----------------%
f = 0.2877;  % frequency
cp = 5.933;  % linear phase velocity
R = b(1);   % radius at which linear phase velocity was calculated
%-----------------------------------------------------------------

[NoL, a, b, rho, CL, CS] = GetInput;

w = 2*pi*f;
p = R*w/cp;
for LL = 1:1:NoL
  DB{LL} = LayerMatrix_Beta(w,p,CL(LL),CS(LL),rho(LL),a(LL));
  DT{LL} = LayerMatrix_Beta(w,p,CL(LL),CS(LL),rho(LL),b(LL));
end
DG = GlobalMatrix(NoL,DB,DT);
X = zeros(4*NoL,1); X(1,1) = 1;
DG = [DG X];
Coef = rref(DG); Coef = Coef(:,4*NoL+1); Coef = Coef./Coef(1,1);

%---------- Displacement Ur -----------------
for ii = 1:1:NoL
  r(ii,:) = a(ii):(b(ii)-a(ii))/500:b(ii);
  kl = w/CL(ii);
  ks = w/CS(ii);
  mu = rho(ii)*CS(ii)*2;
  K = CL(ii)/CS(ii);
  Ur(ii,:) = Coef(4*ii-3)*kl/2.*(besselj(p-1,kl*r(ii,:))) ...
             - kl/2.*(besselj(p+1,kl*r(ii,:))))...
+ Coef(4*ii-2)*(kl/2.*(bessely(p-1,kl*r(ii,:)))) ... 
- kl/2.*(bessely(p+1,kl*r(ii,:))))... 
+ Coef(4*ii-1)*i*p./r(ii,:).*bessely(p,ks*r(ii,:))... 
+ Coef(4*ii)*i*p./r(ii,:).*bessely(p,ks*r(ii,:));

Ut(ii,:) = imag( Coef(4*ii-3)*i*p./r(ii,:).*bessely(p,kl*r(ii,:))... 
+ Coef(4*ii-2)*i*p./r(ii,:).*bessely(p,kl*r(ii,:))... 
+ Coef(4*ii-1)*(ks/2.*(besselj(p+1,ks*r(ii,:))) ... 
- ks/2.*(bessely(p-1,ks*r(ii,:))))... 
- ks/2.*(bessely(p-1,ks*r(ii,:)))) )

Sr(ii,:) = mu./r(ii,:).^2 .* ( ... 
Coef(4*ii-3)*((r(ii,:)*K*kl).^2/4 .* ... 
( besselj(p-2,kl*r(ii,:)) - 2*besselj(p,kl*r(ii,:)) + ... 
besselj(p+2,kl*r(ii,:)) ) + r(ii,:)*kl*(K^2-2)/2 .* ... 
( besselj(p-1,kl*r(ii,:)) - besselj(p+1,kl*r(ii,:)) ) - ... 
p^2*(K^2-2) * besselj(p,kl*r(ii,:)) ) ... 
+ Coef(4*ii-2)*((r(ii,:)*K*kl).^2/4 .* ... 
( besselj(p-2,kl*r(ii,:)) - 2*besselj(p,kl*r(ii,:)) ... 
+ besselj(p+2,kl*r(ii,:)) ) + r(ii,:)*kl*(K^2-2)/2 .* ... 
( besselj(p-1,kl*r(ii,:)) - besselj(p+1,kl*r(ii,:)) ) - ... 
p^2*(K^2-2) * besselj(p,kl*r(ii,:)) ) ... 
+ Coef(4*ii-1)*((i*r(ii,:))*p*ks .* ... 
( besselj(p-1,ks*r(ii,:)) - besselj(p+1,ks*r(ii,:)) ) ... 
- i*2*p *besselj(p,ks*r(ii,:)) ) + ... 
Coef(4*ii)*((i*r(ii,:))*p*ks .* ... 
( besselj(p-1,ks*r(ii,:)) - besselj(p+1,ks*r(ii,:)) ) - ... 
i*2*p * besselj(p,ks*r(ii,:)) ) ... 
); 

Srt(ii,:) = imag( mu./r(ii,:).^2 .* ( ... 
Coef(4*ii-3)*((i*r(ii,:))*p*kl .* ( besselj(p-1,kl*r(ii,:)) ... 
- besselj(p+1,kl*r(ii,:)) ) - i*2*p * ... 
besselj(p,kl*r(ii,:)) ) + Coef(4*ii-2)*((i*r(ii,:))*p*kl .* ... 
( besselj(p-1,kl*r(ii,:)) - besselj(p+1,kl*r(ii,:)) ) - ... 
i*2*p * besselj(p,kl*r(ii,:)) ) ... 
+ Coef(4*ii-1)*(-(r(ii,:)*ks).^2/4 .* ... 
( besselj(p-2,ks*r(ii,:)) - 2*besselj(p,ks*r(ii,:)) + ... 
besselj(p+2,ks*r(ii,:)) ) + r(ii,:)*ks/2 .* ... 
( besselj(p-1,ks*r(ii,:)) - besselj(p+1,ks*r(ii,:)) ) ... 
p^2 * besselj(p,ks*r(ii,:)) ) ... 
+ Coef(4*ii)*(-(r(ii,:)*ks).^2/4 .* ... 
( besselj(p-2,ks*r(ii,:)) - 2*besselj(p,ks*r(ii,:)) + ... 
besselj(p+2,ks*r(ii,:)) ) + r(ii,:)*ks/2 .* ... 
( besselj(p-1,ks*r(ii,:)) - besselj(p+1,ks*r(ii,:)) ) - ... 
p^2 * besselj(p,ks*r(ii,:)) ) ... 
) );
\begin{verbatim}
St(ii,:) = mu./r(ii,:).^2 .* ( ...
    Coef(4*ii-3)*( (r(ii,:)*kl).^2*(K^2-2)/4 .* ...
        ( besselj(p-2,kl*r(ii,:)) - 2*besselj(p,kl*r(ii,:)) +...
    besselj(p+2,kl*r(ii,:)) ) - r(ii,:)*kl*K^2/2 .* ...
        ( besselj(p+1,kl*r(ii,:)) - besselj(p-1,kl*r(ii,:)) ) ...  
    + Coef(4*ii-2)*( (r(ii,:)*kl).^2*(K^2-2)/4 .* ...
        ( bessely(p-2,kl*r(ii,:)) - 2*bessely(p,kl*r(ii,:)) +...
    bessely(p+2,kl*r(ii,:)) ) - r(ii,:)*kl*K^2/2 .* ...
        ( bessely(p+1,kl*r(ii,:)) - bessely(p-1,kl*r(ii,:)) ) ...  
    + Coef(4*ii-1)*( i*r(ii,:)*p*ks .* ( besselj(p-1,ks*r(ii,:))...
    - besselj(p+1,ks*r(ii,:)) ) - i*2*p * ...
    besselj(p,ks*r(ii,:)) ) + Coef(4*ii)*( i*r(ii,:)*p*ks .*...
        ( bessely(p-1,ks*r(ii,:)) - bessely(p+1,ks*r(ii,:)) ) ...  
    - i*2*p * bessely(p,ks*r(ii,:)) ) ...  
    )
end
%
% % ---------- Normalize --------------------------
Mxu = max([max(abs(Ur)) max(abs(Ut))]);
Mxs = max([max(abs(Sr)) max(abs(Srt)) max(abs(St))]);

figure('Position',[100,100,350,600],'Color','white');
for jj=1:1:NoL
    hold on;
    plot(Ur(jj,:)/Mxu,r(jj,:),'--k');
    plot(Ut(jj,:)/Mxu,r(jj,:),'-.k');
    % plot(Srt(jj,:)/Mxs,r(jj,:),'--k');
    % plot(Sr(jj,:)/Mxs,r(jj,:),'-.k');
    % plot(St(jj,:)/Mxs,r(jj,:),':k');
end
box on; xlim([-1.1 1.1]); ylim([a(1) b(NoL)]);
set(gca,'Layer','top'); ylabel('radius (mm)');
legend('u_r','u_\theta',0);
%legend('\sigma_r \_\thetaeta','\sigma_r','\sigma_\theta',0);
set(gca, 'FontSize', 18);
if NoL > 1
    for hh=2:1:NoL
        plot((-1.1:0.1:1.1), a(hh)*ones(1,23),':k');
    end
end
set(gca, 'FontSize', 18);
\end{verbatim}
D.3 SAFE Formulation: CSH-Waves

% Jason K. Van Velsor (c)2009
% Written in MATLAB R2007a

% -------------------------------------------------------------------
% This code calculates the circumferential shear horizontal wave
% dispersion curves for a multilayered elastic/viscoelastic annulus
% using the Semi-Analytical Finite Element (SAFE) method. The angular
% wavenumber (p) roots are calculated at fixed increments of
% circular frequency (w).
% -------------------------------------------------------------------
clc, clear all;

% --------------------- User Inputs -----------------------%
NoE = 200; % Number of Elements
fstop = 0.5; % Ending frequency in MHz
dw = 0.007;
% Load Excel File with Layer Properties
[NoL, a, b, rho, CL, AL, CS, AS] = GetInput;
% ---------------------------------------------------------%

% ----------------- Discretization of Domain ----------------- %
NoN = NoE + 1; % Number of Nodes

% ----- Compute relative layer thicknesses ----- %
% Individual layer thickness
for aa=1:1:NoL
    thk(aa) = b(aa) - a(aa);
end
% Layer thickness relative to total thickness
rthk = thk / (b(NoL) - a(1));

% ----- Discretize each layer based on relative thickness ----- %
for bb=1:1:NoL
    % Calculates number of elements per layer
    rNoE(bb) = round(rthk(bb)*NoE);
    % Calculates the element size for each layer
    for cc=1:1:rNoE(bb)
        L{bb}(cc)=thk(bb)/rNoE(bb);
    end
end

% Corrects for rounding errors which can potentially lead to
% the presence of an extra element
if sum(rNoE) ~= NoE
    L{NoL} = [];
    rNoE(NoL) = rNoE(NoL) - 1;
    for dd=1:1:rNoE(NoL)
        L{NoL}(dd)=thk(NoL)/rNoE(NoL);
    end
end
L = [L(:, :)]; % Assembles element sizes into a single array

% ----- Calculate node locations ----- %
N(1) = a(1);
for ee=1:1:NoE
    N(ee+1) = N(1) + sum(L(1:1:ee));
end

% -------------------------------------------------------------- %
% ---------------- Compute Elemental Matrices ------------------ %
start = 1; % Sets node 1 as the starting point
for ff = 1:1:NoL

    % CSi(ff) = 1 / ( 1/CS(ff) - i*AS(ff) ); %Barshinger
    CSi(ff) = CS(ff) / ( 1 - i*AS(ff)/2/pi ); %Bartoli
    mu = rho(ff)*CSi(ff)^2; % Shear Modulus (Lame' Constant)

    C = [mu 0; 0 mu]; % Reduced Stiffness Matrix

    for gg = start:1:sum(rNoE(1:1:ff))
        [k1{gg}, k2{gg}, k3{gg}, mass{gg}] = ...
            Integrate_SH(N(gg), N(gg+1), L(gg), C, rho(ff));
    end
    start = sum(rNoE(1:1:ff))+1;
end

% -------------------------------------------------------------- %
% ------------------- Compute Global Matrices ------------------- %
K1 = zeros(NoN,NoN);
K2 = zeros(NoN,NoN);
K3 = zeros(NoN,NoN);
M = zeros(NoN,NoN);
for hh = 1:1:NoE
    start = hh;
    stop = hh+1;
    K1(start:1:stop,start:1:stop) = K1(start:1:stop,start:1:stop) + k1{hh}(1:1:2,1:1:2);
    K2(start:1:stop,start:1:stop) = K2(start:1:stop,start:1:stop) + k2{hh}(1:1:2,1:1:2);
    K3(start:1:stop,start:1:stop) = K3(start:1:stop,start:1:stop) + k3{hh}(1:1:2,1:1:2);
    M(start:1:stop,start:1:stop) = M(start:1:stop,start:1:stop) + mass{hh}(1:1:2,1:1:2);
end

% -------------------------------------------------------------- %
% --------------------- Solve Eigenvalue Problem ------------------- %
wstart = 0;
jj = 0;
for w = wstart:dw:2*pi*fstop
    jj = jj+1;
    ws(jj) = w;
    A = [zeros(size(K1)), K1-w^2*M; K1-w^2*M, K2];
    B = [K1-w^2*M, zeros(size(K1)); zeros(size(K1)), -K3];
    p(:,jj) = eig(A,B);
    PC = w/2/pi/fstop*100;
    clc;
    [num2str(PC), '%', ' Complete']
end

% Creates a new roots matrix with only low-attenuation roots
max_attenuation = -4; %(Np/mm)
pfilt = p;
nn = 1;
for ii = 1:1:jj
    mm = 1;
    for kk = 1:1:2*NoN
        if (imag(pfilt(kk,ii)) > 0) || (imag(pfilt(kk,ii)/b(1))...
            < max_attenuation)
            pfilt(kk,ii) = real(pfilt(kk,ii)); pfilt(kk,ii) = NaN;
            mm = mm+1;
        end
    end
    nn = nn+1;
end

D.3.1 Function: GetInput

function [NumberOfLayers, InnerRadius, OuterRadius,...
    LayerDensity, LayerCL, LayerAL, LayerCT, LayerAT] = GetInput

    [filename1,pathname] = uigetfile('*.xls','Select Input File');
    homedirectory = pwd;
    cd(pathname);
    infile = xlsread(filename1);
    cd(homedirectory);

    [m,n] = size(infile);
    infile(:,n) = [];
    reshape = isnan(infile);
    for ii=1:1:n-1;
        if reshape(1,ii) == 0
            Input(:,ii) = infile(:,ii);
        end
    end
\[ x, \text{NumberOfLayers} \] = size(Input); 
InnerRadius = Input(1,:).*1000; 
OuterRadius = Input(2,:).*1000; 
LayerDensity = Input(3,:)./1000; 
LayerCL = Input(4,:)./1000; 
LayerAL = Input(5,:); 
LayerCT = Input(6,:)./1000; 
LayerAT = Input(7,:); 

D.3.2 Subfunction: Integrate_SH

\textbf{function} \ [k1,k2,k3,mass] = Integrate_SH(Ri, Rj, el, C, rho) 
\textbf{syms} \ r 
B1 = [0,0; -1/el, 1/el]; 
B2 = [(Rj-r)/(r*el), (r-Ri)/(r*el); 0, 0]; 
Nm = [(Rj-r)/el, (r-Ri)/el]; 
prek1 = r*B1'*C*B1; 
prek2 = r*(B1'*C*B2 - B2'*C*B1); 
prek3 = r*B2'*C*B2; 
pre_mass = r*Nm'*rho*Nm; 

% ----------------- Computing elemental k1 matrix -----------------%
k1(1,1) = quad(inline(prek1(1,1)), Ri, Rj); 
k1(1,2) = quad(inline(prek1(1,2)), Ri, Rj); 
k1(2,1) = quad(inline(prek1(2,1)), Ri, Rj); 
k1(2,2) = quad(inline(prek1(2,2)), Ri, Rj); 

% ----------------- Computing elemental k2 matrix -----------------%
k2(1,1) = 0; 
k2(1,2) = 0; 
k2(2,1) = 0; 
k2(2,2) = 0; 

% ----------------- Computing elemental k3 matrix -----------------%
k3(1,1) = quad(inline(prek3(1,1)), Ri, Rj); 
k3(1,2) = quad(inline(prek3(1,2)), Ri, Rj); 
k3(2,1) = quad(inline(prek3(2,1)), Ri, Rj); 
k3(2,2) = quad(inline(prek3(2,2)), Ri, Rj); 

% ----------------- Computing elemental mass matrix -----------------%
mass(1,1) = quad(inline(pre_mass(1,1)), Ri, Rj); 
mass(1,2) = quad(inline(pre_mass(1,2)), Ri, Rj);
mass(2,1) = quad(inline(pre_mass(2,1)), Ri, Rj);
mass(2,2) = quad(inline(pre_mass(2,2)), Ri, Rj);

% ------------------------------- D.3.3 Function: Wavestructure ------------------------------- %
clc, clear all;

f = 0.1671;  % Frequency of desired wave structure
w = 2*pi*f;
[NoL, a, b, rho, CL, AL, CS, AS] = GetInput;

% ------------------------------- Discretization of Domain ------------------------------- %
NoE = 200;  % Number of Elements
fstop = 0.5;  % Ending frequency in MHz
NoN = NoE + 1;  % Number of Nodes

% ---- Compute relative layer thicknesses ---- %
for aa=1:1:NoL
    thk(aa) = b(aa) - a(aa);
end
% Layer thickness relative to total thickness
rthk = thk / (b(NoL) - a(1));

% ---- Discretize each layer based on relative thickness ---- %
for bb=1:1:NoL
    rNoE(bb) = round(rthk(bb)*NoE);
    for cc=1:1:rNoE(bb)
        L{bb}(cc)=thk(bb)/rNoE(bb);
    end
end

% Corrects for rounding errors which can potentially lead to the presence of an extra element
if sum(rNoE) ~= NoE
    L{NoL} = [];
    rNoE(NoL) = rNoE(NoL) - 1;
    for dd=1:1:rNoE(NoL)
        L{NoL}(dd)=thk(NoL)/rNoE(NoL);
    end
end

L = [L{:,:}];  % Assembles element sizes into a single array

% ---- Calculate node locations ---- %
N(1) = a(1);
for ee=1:1:NoE
    N(ee+1) = N(1) + sum(L(1:1:ee));
end
% ------------------------------------------------------------- %
% ---------------- Compute Elemental Matrices ----------------- %
start = 1;   % Sets node 1 as the starting point
for ff = 1:1:NoL
    CSi(ff) = 1 / ( 1/CS(ff) - i*AS(ff) );
    mu(ff) = rho(ff)*CSi(ff)^2;   % Shear Modulus (Lame' Constant)
    C = mu(ff);           % Stiffness Matrix
    for gg = start:1:sum(rNoE(1:1:ff))
        [k1{gg}, k2{gg}, k3{gg},mass{gg}] = ...
            Integrate_SH(N(gg), N(gg+1), L(gg), C, rho(ff));
    end
    % Sets the first node of layer ff+1 as the starting
    % point for the next round of computations
    start = sum(rNoE(1:1:ff))+1;
end
% ------------------------------------------------------------------ %
% ------------------- Compute Global Matrices ------------------- %
K1 = zeros(NoN,NoN);
K2 = zeros(NoN,NoN);
K3 = zeros(NoN,NoN);
M = zeros(NoN,NoN);
for hh = 1:1:NoE
    start = hh;
    stop = hh+1;
    K1(start:1:stop,start:1:stop) = ...
        K1(start:1:stop,start:1:stop) + k1{hh}(1:1:2,1:1:2);
    K2(start:1:stop,start:1:stop) = ...
        K2(start:1:stop,start:1:stop) + k2{hh}(1:1:2,1:1:2);
    K3(start:1:stop,start:1:stop) = ...
        K3(start:1:stop,start:1:stop) + k3{hh}(1:1:2,1:1:2);
    M(start:1:stop,start:1:stop) = ...
        M(start:1:stop,start:1:stop) + mass{hh}(1:1:2,1:1:2);
end
% ---------------------------------------------------------------- %
% ----------- Find Roots of Characteristic Equaiton -------------- %
A = [zeros(size(K1)), K1-w^2*M; K1-w^2*M, K2];
B = [K1-w^2*M, zeros(size(K1)); zeros(size(K1)), -K3];
[V,D] = eig(A,B);
% --------------------------------------------------------------- %
% removes highly attenuated modes
Dfilt = D;
nn = 1;
for ii = 1:1:2*NoN
    mm = 1;
    for kk = 1:1:2*NoN
        if (imag(Dfilt(kk,ii)) > 0) || (imag(Dfilt(kk,ii)/b(1)) < -1)
            Dfilt(kk,ii) = real(Dfilt(kk,ii)); Dfilt(kk,ii) = NaN;
            mm = mm+1;
        end
    end
    nn = nn+1;
end

% ------------ Plot Wave Structure -------------- %
% Row and Column of root corresponding to desired wave structure
rac = 402;
Uz = real(V(NoN+1:1:2*NoN,rac)) / max(abs(real((V(NoN+1:1:2*NoN,rac)))));
start = 1;
for ii = 1:1:NoL
    for jj = start:1:sum(rNoE(1:1:ii))
        trz(jj) = real(mu(ii)/L(jj) * (Uz(jj+1)-Uz(jj)));
        toz(jj) = imag( i*mu(ii)*D(rac,rac)/N(jj)*Uz(jj) );
    end
    start = sum(rNoE(1:1:ii))+1;
end
Mxs = max([abs(trz) abs(toz)]);
figure('Position',[100,100,350,600],'Color','white');
plot(Uz, N','--k'); hold on;
plot(trz/Mxs, N(1:1:NoE)','.k'); hold on;
plot(toz/Mxs, N(1:1:NoE)','.k'); hold on;
box on; xlim([-1.1 1.1]); ylim([a(1) b(NoL)]);
set(gca,'Layer','top'); xlabel('u_z , \sigma_r_z , \sigma_\theta_z');
ylabel('r (mm)'); legend('u_z', '\sigma_r_z', '\sigma_\theta_z',0);
if NoL > 1
    for hh=2:1:NoL
        plot((-1.1:0.1:1.1), a(hh)*ones(1,23),':k');
    end
end

set(gca, 'FontSize', 18);
### D.3.4 Sample Input File (INPUT.xls)

<table>
<thead>
<tr>
<th></th>
<th>Layer 1</th>
<th>Layer 2</th>
<th>Layer 3</th>
<th>Layer 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Inner Radius (m)</td>
<td>0.0127</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Outer Radius (m)</td>
<td>0.0254</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Density (kg/m³)</td>
<td></td>
<td>953</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>CL (m/s)</td>
<td></td>
<td></td>
<td>2344</td>
</tr>
<tr>
<td>5</td>
<td>α₁ (Np/mm) /Lo</td>
<td>0.055</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>CS (m/s)</td>
<td></td>
<td>953</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>α₂ (Np/mm) /Lo</td>
<td>0.286</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

![Excel Spreadsheet Screenshot](image-url)
D.4 SAFE Formulation: CLT-Waves

% Jason K. Van Velsor (c)2009
% Written in MATLAB R2007a

% -------------------------------------------------------------------
% This code calculates the circumferential Lamb [type] wave
% dispersion curves for a multilayered elastic/viscoelastic annulus
% using the Semi-Analytical Finite Element (SAFE) method. The angular
% wavenumber (p) roots are calculated at fixed increments of
% circular frequency (w).
% -------------------------------------------------------------------
clc, clear all;

[NoL, a, b, rho, CL, AL, CS, AS] = GetInput;

% ------------------ Discretization of Domain -------------------- %
NoE = 200; % Number of Elements
fstop = 1.1; % Ending frequency in MHz
NoN = NoE + 1; % Number of Nodes

% ----- Compute relative layer thicknesses ----- %
for aa=1:1:NoL
    % Individual layer thickness
    thk(aa) = b(aa) - a(aa);
end
% Layer thickness relative to total thickness
rthk = thk / (b(NoL) - a(1));

% ----- Discretize each layer based on relative thickness ---- %
for bb=1:1:NoL
    % Calculates number of elements per layer
    rNoE(bb) = round(rthk(bb)*NoE);
    for cc=1:1:rNoE(bb)
        % Calculates the element size for each layer
        L{bb}(cc)=thk(bb)/rNoE(bb);
    end
end

% Corrects for rounding errors which can potentially
% lead to the presence of an extra element
if sum(rNoE) ~= NoE
    L{NoL} = [];
    rNoE(NoL) = rNoE(NoL) - 1;
    for dd=1:1:rNoE(NoL)
        L{NoL}(dd)=thk(NoL)/rNoE(NoL);
    end
end

L = [L(:,:)]; % Assembles element sizes into a single array

% ----- Calculate node locations ----- %
\[
\text{N}(1) = a(1);
\text{for} \ ee = 1:1:\text{NoE} \\
\quad \text{N}(ee+1) = \text{N}(1) + \text{sum}(L(1:1:ee));  \\
\text{end}
\]

\% --------------------------------------------------------------------------------------
\% ------------------ Compute Elemental Matrices ------------------ 
\% start = 1; \% Sets node 1 as the starting point
\text{for} \ ff = 1:1:\text{NoL} \\
\quad \% \ CSi(\text{ff}) = 1 / ( 1/\text{CS}(\text{ff}) - i*\text{AS}(\text{ff}) ) ; \% \text{Barshinger}
\quad \text{CSi}(\text{ff}) = \text{CS}(\text{ff}) / ( 1 - i*\text{AS}(\text{ff})/2/\pi ) ; \% \text{Bartoli}
\quad \% \text{mu} = \rho(\text{ff})*\text{CSi}(\text{ff})^2 ; \% \text{Shear Modulus}
\quad \% \text{CLi}(\text{ff}) = 1 / ( 1/\text{CL}(\text{ff}) - i*\text{AL}(\text{ff}) ) ; \% \text{Barshinger}
\quad \text{CLi}(\text{ff}) = \text{CL}(\text{ff}) / ( 1 - i/\text{AL}(\text{ff})/2/\pi ) ; \% \text{Bartoli}
\quad \% \text{lam} = \rho(\text{ff})*\text{CLi}(\text{ff})^2 - 2*\text{CSi}(\text{ff})^2 ; \% \text{Lame'} \text{ Constant}
\quad \text{E} = \rho(\text{ff})*\text{CSi}(\text{ff})^2 * ((3*\text{CLi}(\text{ff})^2 - 4*\text{CSi}(\text{ff})^2) \\
\quad \quad / (\text{CLi}(\text{ff})^2 - \text{CSi}(\text{ff})^2)) ; \% \text{Young's Modulus}
\quad \% \text{v} = 1/2 * ( (\text{CLi}(\text{ff})^2 - 2*\text{CSi}(\text{ff})^2) ... \\
\quad \quad / (\text{CLi}(\text{ff})^2 - \text{CSi}(\text{ff})^2)) ; \% \text{Poisson's Ratio}
\quad \% \text{lam} = \text{E} * \text{v} / ( (1+\text{v}) * (1-2*\text{v} )) ; \% \text{Lame'} \text{ Constant}
\quad \text{mu} = \text{E} / ( 2*(1+\text{v}) ) ; \% \text{Shear Modulus}
\% Reduced Stiffness Matrix
\text{C} = [ (\text{lam}+2*\text{mu}), \text{lam}, 0;... \\
\quad \text{lam}, (\text{lam}+2*\text{mu}), 0;... \\
\quad \text{0}, \text{0}, \mu ];
\text{for} \ gg = \text{start}:1:\text{sum}(\text{rNoE}(1:1:\text{ff})) \\
\quad [\text{k1}(gg), \text{k2}(gg), \text{k3}(gg), \text{mass}(gg)] = ...
\quad \text{Integrate_LT}(\text{N}(gg), \text{N}(gg+1), \text{L}(gg), \text{C}, \rho(\text{ff}));
\text{end}
\text{start} = \text{sum}(\text{rNoE}(1:1:\text{ff}))+1;
\text{end}
\% --------------------------------------------------------------------------------------
\% ------------------- Compute Global Matrices ------------------- 
\text{K1} = \text{zeros}(2*\text{NoN},2*\text{NoN});
\text{K2} = \text{zeros}(2*\text{NoN},2*\text{NoN});
\text{K3} = \text{zeros}(2*\text{NoN},2*\text{NoN});
\text{M} = \text{zeros}(2*\text{NoN},2*\text{NoN});
\text{for} \ hh = 1:1:\text{NoE} \\
\quad \text{start} = 2*\text{hh}; - 1;
\quad \text{stop} = \text{start} + 3;
\quad \text{K1}(\text{start}:1:\text{stop},\text{start}:1:\text{stop}) = ...
\quad \quad \text{K1}(\text{start}:1:\text{stop},\text{start}:1:\text{stop}) + \text{k1}(\text{hh})(1:1:4,1:1:4);
\quad \text{K2}(\text{start}:1:\text{stop},\text{start}:1:\text{stop}) = ...
\quad \quad \text{K2}(\text{start}:1:\text{stop},\text{start}:1:\text{stop}) + \text{k2}(\text{hh})(1:1:4,1:1:4);
\quad \text{K3}(\text{start}:1:\text{stop},\text{start}:1:\text{stop}) = ...
\quad \quad \text{K3}(\text{start}:1:\text{stop},\text{start}:1:\text{stop}) + \text{k3}(\text{hh})(1:1:4,1:1:4);
\quad \text{M}(\text{start}:1:\text{stop},\text{start}:1:\text{stop}) = ...
\quad \quad \text{M}(\text{start}:1:\text{stop},\text{start}:1:\text{stop}) + \text{mass}(\text{hh})(1:1:4,1:1:4);
\text{end}
% Create Matrix Multiplier
T = eye(2*NoN,2*NoN);
for ii=2:2:2*NoN
    T(ii,ii) = i;
end
K1 = T'*K1*T;
K2 = (T'*K2*T)./( -i);
K3 = T'*K3*T;
M = T'*M*T;

% Find Roots of Characteristic Equation
wstart = 0;
dw = 0.005;
jj = 0;
for w = wstart:dw:2*pi*fstop
    jj = jj+1;
    ws(jj) = w;
    A = [zeros(size(K1)), K1-w^2*M; K1-w^2*M, K2];
    B = [K1-w^2*M, zeros(size(K1)); zeros(size(K1)), -K3];
    d(:,jj) = eig(A,B);
end
pfilt = d;
nn = 1;
for ii = 1:1:jj
    mm = 1;
    for kk = 1:1:4*NoN
        if (imag(pfilt(kk,ii)) > 0) || (imag(pfilt(kk,ii)/b(1)) < -2)
            pfilt(kk,ii) = real(pfilt(kk,ii)); pfilt(kk,ii) = NaN;
            mm = mm+1;
        end
    end
    nn = nn+1;
end

D.4.1 Subfunction: Integrate_LT

function [k1,k2,k3,mass] = Integrate_LT(Ri, Rj, el, C, rho)
    syms r
    B1 = [ -1/el, 0, 1/el, 0;...
          (Rj-r)/(r*el), 0, (r-Ri)/(r*el), 0;...
          0, (r-Rj)/(r*el)-1/el, 0, (Ri-r)/(r*el)+1/el ];
    B2 = [ 0, 0, 0, 0;...
\[ 0, \frac{(R_j-r)}{(r^*el)}, 0, \frac{(r-R_i)}{(r^*el)}; \ldots \]
\[ \frac{(R_j-r)}{(r^*el)}, 0, \frac{(r-R_i)}{(r^*el)}, 0 \]

\[
Nm = [(R_j-r)/el, 0, (r-R_i)/el, 0; \ldots \\
0, (R_j-r)/el, 0, (r-R_i)/el ]
\]

\[
prek1 = r*B1'*C*B1;
prek2 = r*(B1'*C*B2 - B2'*C*B1);
prek3 = r*B2'*C*B2;
pre_mass = r*Nm'*rho*Nm;
\]

% -------------- Computing elemental k1 matrix ---------------%
\[
k1(1,1) = \text{quad}(@\text{prek1}(1,1), Ri, Rj);
k1(1,2) = 0;
k1(1,3) = \text{quad}(@\text{prek1}(1,3), Ri, Rj);
k1(1,4) = 0;
\]
\[
k1(2,1) = 0;
k1(2,2) = \text{quad}(@\text{prek1}(2,2), Ri, Rj);
k1(2,3) = 0;
k1(2,4) = \text{quad}(@\text{prek1}(2,4), Ri, Rj);
\]
\[
k1(3,1) = \text{quad}(@\text{prek1}(3,1), Ri, Rj);
k1(3,2) = 0;
k1(3,3) = \text{quad}(@\text{prek1}(3,3), Ri, Rj);
k1(3,4) = 0;
\]
\[
k1(4,1) = 0;
k1(4,2) = \text{quad}(@\text{prek1}(4,2), Ri, Rj);
k1(4,3) = 0;
k1(4,4) = \text{quad}(@\text{prek1}(4,4), Ri, Rj);
\]

% -------------- Computing elemental k2 matrix ---------------%
\[
k2(1,1) = 0;
k2(1,2) = \text{quad}(@\text{prek2}(1,2), Ri, Rj);
k2(1,3) = 0;
k2(1,4) = \text{quad}(@\text{prek2}(1,4), Ri, Rj);
\]
\[
k2(2,1) = \text{quad}(@\text{prek2}(2,1), Ri, Rj);
k2(2,2) = 0;
k2(2,3) = \text{quad}(@\text{prek2}(2,3), Ri, Rj);
k2(2,4) = 0;
\]
\[
k2(3,1) = 0;
k2(3,2) = \text{quad}(@\text{prek2}(3,2), Ri, Rj);
k2(3,3) = 0;
k2(3,4) = \text{quad}(@\text{prek2}(3,4), Ri, Rj);
\]
\[
k2(4,1) = \text{quad}(@\text{prek2}(4,1), Ri, Rj);
k2(4,2) = 0;
k2(4,3) = \text{quad}(@\text{prek2}(4,3), Ri, Rj);
k2(4,4) = 0;\]
%  % -------------- Computing elemental k3 matrix ---------------%  
k3(1,1) = quad(inline(prek3(1,1)), Ri, Rj);
k3(1,2) = 0;
k3(1,3) = quad(inline(prek3(1,3)), Ri, Rj);
k3(1,4) = 0;

k3(2,1) = 0;
k3(2,2) = quad(inline(prek3(2,2)), Ri, Rj);
k3(2,3) = 0;
k3(2,4) = quad(inline(prek3(2,4)), Ri, Rj);

k3(3,1) = quad(inline(prek3(3,1)), Ri, Rj);
k3(3,2) = 0;
k3(3,3) = quad(inline(prek3(3,3)), Ri, Rj);
k3(3,4) = 0;

k3(4,1) = 0;
k3(4,2) = quad(inline(prek3(4,2)), Ri, Rj);
k3(4,3) = 0;
k3(4,4) = quad(inline(prek3(4,4)), Ri, Rj);
%  % --------------------------------------------------------------%

% -------------- Computing elemental mass matrix ---------------%  
mass(1,1) = quad(inline(pre_mass(1,1)), Ri, Rj);
mass(1,2) = 0;
mass(1,3) = quad(inline(pre_mass(1,3)), Ri, Rj);
mass(1,4) = 0;

mass(2,1) = 0;
mass(2,2) = quad(inline(pre_mass(2,2)), Ri, Rj);
mass(2,3) = 0;
mass(2,4) = quad(inline(pre_mass(2,4)), Ri, Rj);

mass(3,1) = quad(inline(pre_mass(3,1)), Ri, Rj);
mass(3,2) = 0;
mass(3,3) = quad(inline(pre_mass(3,3)), Ri, Rj);
mass(3,4) = 0;

mass(4,1) = 0;
mass(4,2) = quad(inline(pre_mass(4,2)), Ri, Rj);
mass(4,3) = 0;
mass(4,4) = quad(inline(pre_mass(4,4)), Ri, Rj);
%  % --------------------------------------------------------------%

D.4.2 Function: Wavestructure
%  % Jason Van Velsor
clc, clear all;
\[ f = 0.3062; \]
\[ w = 2\pi f; \]
\[ \text{[NoL, a, b, rho, CL, AL, CS, AS]} = \text{GetInput}; \]

% ---------------------- Discretization of Domain ---------------------- %
NoE = 100; % Number of Elements
fstop = 0.5; % Ending frequency in MHz
NoN = NoE + 1; % Number of Nodes

% ----- Compute relative layer thicknesses ----- %
for aa = 1:1:NoL
    % Individual layer thickness
    thk(aa) = b(aa) - a(aa);
end
% Layer thickness relative to total thickness
rthk = thk / (b(NoL) - a(1));

% ----- Discretize each layer based on relative thickness ----- %
for bb = 1:1:NoL
    % Calculates number of elements per layer
    rNoE(bb) = round(rthk(bb)*NoE);
    for cc = 1:1:rNoE(bb)
        % Calculates the element size for each layer
        L{bb}(cc) = thk(bb) / rNoE(bb);
    end
end

% Corrects for rounding errors which can potentially
% lead to the presence of an extra element
if sum(rNoE) ~= NoE
    L{NoL} = [];
    rNoE(NoL) = rNoE(NoL) - 1;
    for dd = 1:1:rNoE(NoL)
        L{NoL}(dd) = thk(NoL) / rNoE(NoL);
    end
end

L = [L{:,:,:}]; % Assembles element sizes into a single array

% ----- Calculate node locations ----- %
N(1) = a(1);
for ee = 1:1:NoE
    N(ee+1) = N(1) + sum(L(1:1:ee));
end

% -------------------------- Compute Elemental Matrices -------------------------- %

% CSi(ff) = 1 / ( 1/CS(ff) - i*AS(ff) );
CSi(ff) = CS(ff) / ( 1 - i*AS(ff)/2/pi );
% mu = rho(ff)*CSi(ff)^2;   % Shear Modulus (Lame' Constant)
% CLi(ff) = 1 / ( 1/CL(ff) - i*AL(ff) );
CLi(ff) = CL(ff) / ( 1 - i*AL(ff)/2/pi );
% lam = rho(ff)*(CLi(ff)^2 - 2*CSi(ff)^2);   % Lame' Constant
E = rho(ff)*CSi(ff)^2 * ( (3*CLi(ff)^2 - 4*CSi(ff)^2)... 
/ (CLi(ff)^2 - CSi(ff)^2) );   % Young's Modulus
v = 1/2 * ( (CLi(ff)^2 - 2*CSi(ff)^2) / ... 
( CLi(ff)^2 - CSi(ff)^2 ) );   % Poisson's Ratio
lam(ff) = E*v / ( (1+v) * (1-2*v) );   % Lame' Constant
mu(ff) = E / ( 2*(1+v) );   % Lame' Constant

% Stiffness Matrix
C = [ (lam(ff)+2*mu(ff)),          lam(ff),          0;...
lam(ff),   (lam(ff)+2*mu(ff)),          0;...
0,            0,         mu(ff) ];

% Computes k and m matrices for each element in layer 'ff'
for gg = start:1:sum(rNoE(1:1:ff))
    [k1{gg}, k2{gg}, k3{gg},mass{gg}] = ... 
    Integrate_LT(N(gg), N(gg+1), L(gg), C, rho(ff));
end

% Sets the first node of layer ff+1 as the starting point
% for the next round of computations
start = sum(rNoE(1:1:ff))+1;

% ------------------------------------------------------------------ %
% ------------------- Compute Global Matrices ------------------- %
K1 = zeros(2*NoN,2*NoN);
K2 = zeros(2*NoN,2*NoN);
K3 = zeros(2*NoN,2*NoN);
M = zeros(2*NoN,2*NoN);
for hh = 1:1:NoE
    start = 2*hh - 1;
    stop = start + 3;
    K1(start:1:stop,start:1:stop) = ... 
    K1(start:1:stop,start:1:stop) + k1{hh}(1:1:4,1:1:4);
    K2(start:1:stop,start:1:stop) = ... 
    K2(start:1:stop,start:1:stop) + k2{hh}(1:1:4,1:1:4);
    K3(start:1:stop,start:1:stop) = ... 
    K3(start:1:stop,start:1:stop) + k3{hh}(1:1:4,1:1:4);
    M(start:1:stop,start:1:stop) = ... 
    M(start:1:stop,start:1:stop) + mass{hh}(1:1:4,1:1:4);
end

% ------------------------------------------------------------------ %
%

% ------------------------------- Create Matrix Multiplier ------------------------------- %
T = eye(2*NoN,2*NoN);
for ii=2:2:2*NoN
    T(ii,ii) = i;   % Identity matrix with i's in cells associated with U Theta displacements
end
K1 = T'*K1*T;           % Global matrices multiplied by T to remove imaginary unit from characteristic equation
K2 = (T'*K2*T)./-(-i);   % (K1 + i*p*K2 + p^2*K3 - w^2*M)*U = 0 -- > (K1hat + p*K2hat + p^2*K3hat - w^2*Mhat)*Uhat = 0
K3 = T'*K3*T;           % (K1 + i*p*K2 + p^2*K3 - w^2*M)*U = 0   -- > (K1hat + p*K2hat + p^2*K3hat - w^2*Mhat)*Uhat = 0

% ----------- Find Roots of Characteristic Equation ------------- %

A = [zeros(size(K1)), K1-w^2*M; K1-w^2*M, K2];
B = [K1-w^2*M, zeros(size(K1)); zeros(size(K1)), -K3];
[V,D] = eig(A,B);

% ---------------------------------------- %
% filters out highly attenuated roots
Dfilt = D;
nn = 1;
for ii = 1:1:4*NoN
    mm = 1;
    for kk = 1:1:4*NoN
        if (imag(Dfilt(kk,ii)) > 0) || (imag(Dfilt(kk,ii)/b(1)) < -0.5)
            Dfilt(kk,ii) = real(Dfilt(kk,ii)); Dfilt(kk,ii) = NaN;
            mm = mm+1;
        end
    end
    nn = nn+1;
end

% ------------ Plot Wave Structure -------------- %
%row and column of root corresponding to desired wave structure
rac = 398;
Ur = real(V(2*NoN+1:2:4*NoN,rac));
Uo = real(V(2*NoN+2:2:4*NoN,rac));
Mxu = max( [max(abs(Ur)) max(abs(Uo)) ] );

start = 1;
for ii = 1:1:NoL
    for jj = start:1:sum(rNoE(1:1:ii))
        trr(jj) = lam(ii)* ( (Ur(jj+1)-Ur(jj))/L(jj) + ... 
            1/N(jj)*(N(jj+1)-N(jj))*Ur(jj)/L(jj) + ... 
            i*D(rac,rac)/N(jj)*N(jj+1)-N(jj)*Uo(jj)/L(jj) ) ... 
            + 2*mu(ii)/L(jj)*(Ur(jj+1)-Ur(jj));
    end
    start = sum(rNoE(1:1:ii))+1;
end
Mxs = max( [abs(trr)] )

figure(2); %figure('Position', [100,100,350,600], 'Color', 'white'); plot(-Ur/Mxu, N', '.r'); hold on;
plot(-Uo/Mxu, N', 'or'); hold on;
plot(trr/Mxs, N(1:1:NoE)', '.k'); hold on;
%plot(trz/Mxs, N(1:1:NoE)', '-'k'); hold on;
%plot(toz/Mxs, N(1:1:NoE)', '.k'); hold on;
box on; xlim([-1.1 1.1]); ylim([a(1) b(NoL)]);
set(gca, 'Layer', 'top');
ylabel('r (mm)'); legend('u_r', 'u_\theta', 0);

if NoL > 1
    for hh=2:1:NoL
        plot((-1.1:0.1:1.1), a(hh)*ones(1,23), ':k');
    end
end

set(gca, 'FontSize', 18)
Appendix E

Nontechnical Abstract

An ultrasonic wave is a propagating wave with a frequency above the audible range (i.e. \(>20\ kHz\)). In air, ultrasound is a pressure wave and in solids, ultrasound is a stress wave. If an ultrasonic wave travels uninterrupted in a solid, it is commonly referred to as a bulk wave. When the nearest boundaries of a structure are comparable to the wavelength of the ultrasonic wave, complex interference patterns are created and, when the interference is constructive, guided waves are formed and can propagate for long distances. Several examples of the thousands of applications involving guided waves include defect detection, location, and sizing, material characterization, ice detection and removal, and quantitative measurement of structural dimensions, fluid viscosity, and flow rate.

Differently shaped structural boundaries, thicknesses, and material properties result in different wave interference patterns and, therefore, not all structures can support the same types of waves. For this reason, one refers to a set of curves, commonly referred to as dispersion curves, which are specific to the structure being inspected. These curves provide a visual representation of the modes that have potential to propagate in said structure and how their velocity varies with frequency. Most importantly, they serve as a starting point for the design of experiments and non-destructive testing methods. It is therefore necessary for the models to be as accurate as possible to achieve the best experimental and inspection results as possible.

The work introduced here, specifically addresses the mechanics of guided wave propagation in the circumferential direction of annular shaped structures, such as pipes and tubes. These structures are of interest as gas transmission pipeline, gas storage-well casings, boiler and heat exchanger tubing, and many other annular shaped structures are prone to corrosion and
cracking and ultrasonic guided waves offer a time-efficient and cost-effective means for their inspection. Specific considerations are made for multilayered annuli in this work, which may consist of an arbitrary number of elastic or viscoelastic materials. Elastic materials are those which, when deformed, instantaneously return to their original shapes, such as steel. Viscoelastic materials however, would take some time to return to their original shape. This is an important incorporation into the theoretical model as annular structures, like pipeline, are often coated with protective polymeric or tar-based materials.

After initial verification of the theoretical models, two experiments are introduced. The first experiment is designed to study the effect of coating thickness on wave propagation. Data is collected for several different coating thicknesses and the results are compared with the predictions from the theoretical models. Agreement is found. The second experiment explores the potential of detecting disbonds in the coating layer. Theoretical modeling is used to predict wave features that are sensitive to disbonding and said features are verified experimentally.

The overall contribution of this work is the development of theoretical modeling tools for circumferential guided waves which are more physically representative of the real inspection environments that are encountered. Such models will allow for the development of new advanced non-destructive testing methods as well as for the improvement of current methods. The modeling tools will be especially useful in the development of defect detection, identification, and sizing methods. Future work may involve the modification of the modeling tools to study the influences of soil layers and liquid contents.
VITA

Jason K. Van Velsor

EDUCATION

       The Pennsylvania State University, University Park, PA

       The Pennsylvania State University, University Park, PA

B.S.  Engineering Science, 1999-2003
       The Pennsylvania State University, University Park, PA

SELECT PUBLICATIONS


PROFESSIONAL AFFILIATIONS

ASNT, Student Member

Society of Engineering Science, Student Member