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Simultaneous estimation of both position and orientation, commonly referred to in the literature as *pose*, is a subject of continuing interest in the aerospace research community and industry. With the advent of large, proliferated constellations in low earth orbit, the ability for a satellite to track and estimate relative pose in autonomous proximity operations is desired. A way to parameterize pose in a compact and singularity-free way that is computationally efficient has given rise to study into an older way to represent kinematics and rigid body motion: dual-quaternions. Dual-quaternions have found their niche in computer graphics applications due to their numerical efficiency, but there are issues involved with using them for real-time pose estimation. The issue with dual-quaternions is that they do not exist within a standard real number space, meaning the usage of more exotic distributions to describe dual-quaternion uncertainty must be used. The traditional solution to this issue is to make a small-angle assumption, meaning an estimation filter will likely break down if the estimate strays too far from the truth. This dissertation addresses these issues with dual-quaternion filtering using two approaches. The first approach is a constrained extended Kalman filter (EKF) and the second is a Bingham-distributed unscented Kalman filter (UKF). The development of both these filters is provided, and they are demonstrated to accurately spacecraft relative pose through simulation.
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Chapter 1
Introduction

1.1 Motivation and Background

Relative motion estimation with six degrees-of-freedom (6-DOF) in three dimensional space is not a trivial problem. In the field of robotics, this problem is typically referred to as *pose* estimation. It is a problem that requires accurate estimation of a rigid body’s 3-D orientation or *attitude* and simultaneous localization of its position in 3-D space.

Classical methods to model this problem divide it in two by solving for attitude and position separately unless the motion of the system is restricted to some fixed frames. Estimates of the position and velocity of a spacecraft typically use the assumption that errors in a state estimate are randomly distributed using n-dimensional Gaussian distributions. The justification for this is, given the first two moments of a distribution, the Gaussian distribution is the highest entropy distribution that can match the first two moments of any probability distribution in $\mathbb{R}^n$. However, when states include angles or unit-vectors, these values cannot be assumed to have Gaussian distributed errors since they live on closed manifolds.

What makes the approach of splitting the 6-DOF estimation problem into two models non-ideal for space applications is in the coupling of orientation and position information in the dynamics and instrumentation that are available aboard a spacecraft. In order for a spacecraft to accurately localize itself, it must first know in which direction it is
facing. What is lost in the sequential estimation of attitude and then position, is the coupling in uncertainty between orientation and position. Uncertainty in attitude is useful information in a filter that is purely focused on localizing a spacecraft’s position. How does one capture the coupling of these uncertainties?

A possible answer to this question is to use unit dual-quaternions to describe 6-DOF transformations. Unit quaternions are sets of four numbers with unit norm that provide the most efficient way to parameterize rotations in three-dimensional space that is singularity free. Dual-quaternions are an extension of quaternions that provide a way to link the dynamics of rigid body rotations and translations into a single algebra that is governed by the rules of dual numbers and quaternions.

Mixed states containing both elements from quaternions and vector spaces have issues in correlating the coupled dynamics and statistics of linear and rotational motion. But the fact that we can adequately parameterize the dynamics with dual-quaternions does not mean that there are no downsides to their usage. The two main road-blocks to using conventional filters for dual-quaternions are that:

1. A unit dual-quaternion has two constraints in order to be valid. The “real” part must be unit norm and the real half must be orthogonal to the dual half.

2. Quaternions exist on the hypersphere $S^3$ and dual-quaternions exist on a dual hypersphere. The classical distribution to characterize noise in vector spaces, the Gaussian normal distribution, does not exist on the hypersphere.

During the 1950’s-1960’s, work was done by mathematicians to define distributions on hyper-surfaces. On the hypersphere, $S^n$, there exist multiple distributions that try to capture the utility of the Gaussian distribution on non-Euclidean surfaces. The most promising distribution, defined by Bingham in 1974 [1], has gained recent attention in the past 25 years due to the following properties: it has antipodal and radial symmetry, it is defined on the sphere, and it is the distribution that maximizes entropy on a hypersphere.
given the second moment of a distribution. The Bingham distribution has been used extensively for the distributions of quaternions on $S^3$, but attention has recently focused on using Bingham distributions for both attitude and linear states.

There have been attempts in the literature of coupling these correlations using conditioned probability distributions by either conditioning a Bingham distribution with a Gaussian or a Gaussian distribution with a Bingham distribution, but these end up causing either discontinuities in the distribution that need to be fixed by using mixture models or by using computationally expensive maximum likelihood parameter optimization problems [2–4].

Dual-quaternions have seen application in a variety of fields besides the aerospace engineering field [5–7]. Dual-quaternions are a combination of the theory of quaternions as defined by Hamilton [8] with the dual-number theory of Clifford and Study [9,10]. The advantage to using dual-quaternions is that they provide a six degree-of-freedom (6-DOF) coupling between rotation and translation naturally by their definition. This allows the natural calculation of statistical cross-covariances between rotation and translation.

The primary motivating factors for using dual-quaternions are that they are a computationally efficient way of representing 6-DOF transformations in a computer. This has led to interest in using dual-quaternion for 6-DOF pose control problems. The use of dual-quaternions in the control of robotic manipulators motivates the development for filters also in the dual-quaternion space.

1.2 Research Overview

The goal of this dissertation is to present a nonlinear filter for dual-quaternions that properly addresses the constraints and statistics of dual-quaternion states. In service to meet this goal, the following sub-objectives were addressed:

1. Define an extended Kalman filter (EKF) that seeks to directly address the two dual-
quaternion constraints: unit norm and orthogonality. This allows an assumption that the noise error is Gaussian, while still forcing the filtered solution to exist on the dual hypersphere.

2. Define a nonlinear “sigma point” filter that avoids the partial derivatives in the EKF. This objective first requires the definition of a new set of points that match the moments for a dual-quaternion state. This is accomplished by modeling the process noise for the quaternion state by a Bingham distribution on the unit hypersphere with the dual part modeled by a constrained Gaussian distribution. The quadrature points for such a parameterization are derived from the unscented transforms of Gilitschenski and Darling [2,11] and the conjugate unscented transform (CUT) [12–16]. Previous attempts at a dual-quaternion Bingham filter have focused on states whose rotations are limited to one or two axes. This dissertation provides the first Bingham Unscented Kalman filter (BUKF) for a dual-quaternion state with six degrees-of-freedom.

3. Compare the performance of these two new filters with existing filters from the literature for dual-quaternion states. This is accomplished by performing Monte-Carlo simulation of all filters for a spacecraft relative motion problem. The relative states of the two spacecraft are estimated provided two different measurement models with varying degrees of measurement noise.

Work that has been done with creating filters using a Bingham distribution for spacecraft pose estimation is extremely limited. The only published papers to the knowledge of the author focus on problems with motion restricted to some fixed plane of rotation or use computationally intensive nonlinear particle filtering techniques or mixture models [4,17]. Current dual-quaternion filters do not accurately capture the statistics of the dual-quaternion state because they make one or more of the following simplifications: errors in attitude are assumed small, meaning that the quaternion error can be modeled
on a three-dimensional subspace tangent to the four-dimensional quaternion; a nonlinear projection is taken from the quaternion space into a three-dimensional space; or the full set of constraints of the dual-quaternion state are not taken into account when updating the covariance and state estimate. A minimum variance update for a dual-quaternion state vector modeled by a Bingham distribution has also not been presented in the literature, and this dissertation will also present a framework to do this while respecting the constraints of a normalized dual-quaternion. A minimum variance framework was chosen because it provides an unbiased estimate, as opposed to a maximum-likelihood filter. A minimum variance nonlinear filter for a dual-quaternion pose will provide utility for many technical disciplines. Any application involving the filtering of 6-DOF motion with large initial estimate error, large process noise, or large measurement noise would benefit from a filter of this design.

1.3 Dissertation Organization

This dissertation is organized in the following way:

Chapter 2: An introductory description of the fundamental theory for how quaternions and dual-quaternions are used to represent rigid body attitudes and pose is given. Following the basics, the kinematics and dynamics for quaternions and dual-quaternion are derived. Finally, the concept of multiplicative error in quaternions is given.

Chapter 3: A brief refresh of probability and statistics is given. The probability distributions used in this dissertation, being the Gaussian and the Bingham, are defined. The distribution for a dual-quaternion assuming Gaussian and Bingham parts is described.

Chapter 4: This chapter focuses on deriving the constrained extended Kalman filter (CEKF) for dual-quaternion states. The chapter starts by giving a brief description of the generalized extended Kalman filter (EKF) and then describes the common multiplicative
EKF (MEKF) that is used for quaternion states. Lastly, the constrained dual-quaternion filter is derived.

Chapter 5: As an alternative to the EKF for filtering nonlinear systems, the UKF is a desirable alternative. Whereas the EKF requires a linearization of the state dynamics about the current estimate, the UKF avoids linearization by way of Gaussian quadrature. The Bingham UKF (BUKF) is the second direct contribution of this dissertation and comes with a description of a new set of dual-quaternion sigma points.

Chapter 6: The example problem, being the relative motion of two satellites in low Earth orbit (LEO), which is used for all simulations is described along with the filter dynamics and measurement models.

Chapter 7: Monte Carlo simulation of the previously defined relative motion problem in LEO is provided with statistical results.

Chapter 8: Conclusions and suggestions for further study and use cases for the filters and techniques discussed in this dissertation.
Chapter 2  
Theory of Dual-Quaternions

2.1 Introduction

Quaternions and dual-quaternions are both products of the 19th century mathematicians R. W. Hamilton, W.K. Clifford, and E. Study. Hamilton discovered his quaternions in the 1830’s-1840’s while searching for a system of imaginary numbers that can represent rotations in three dimensions like complex numbers can for rotations in a two-dimensional plane [8,18]. He is famously attributed to discovering the mechanics for how quaternions behave while on a walk in Dublin and then transcribing their properties onto a nearby bridge.

Following Hamilton’s discovery, Clifford introduced his geometric algebra, which sought to unify the dot product of vectors with the outer product of Grassman to create a unified algebra. Inadvertently in an 1873 conference proceeding, he created a term with peculiar properties that would later be known as the dual operator, $\epsilon$ [9]. Hamilton’s quaternions can be shown to be specific instance of a Clifford Algebra. From the geometric algebra of Clifford, it took nearly twenty years for the dual-quaternion formalism of Study to describe the pose of two frames relative to each other to be realized [10]. A derivation of the Clifford algebra of quaternions and dual-quaternions is given in Appendix A.

Work with quaternions saw a resurgence in the 1960’s-1980’s where their use to
compactly describe, without singularity, the attitude of a coordinate frame in relation to another was leveraged in the early years of satellite attitude estimation [19]. Despite the prevalence of using quaternions to describe rotations in three dimensional space, the use of dual-quaternions to represent pose in 6-DOF systems has only recently in the past 20 years received any attention. Dual-quaternions before were treated as niche mathematical constructs used in the engineering mechanics and theoretical kinematics communities to describe systems such as robotic manipulators [6].

The idea of “pose” has its roots in the computer vision community where it was originally an area focused on identifying the orientations of the human body in static photographs. Any rigid body can have a pose defined to be the position and orientation of that rigid body in 3-D space, and the literature has forked to many different fields in engineering and computer science. For the space domain, the use of dual-quaternions to represent the pose of a satellite are few in number, but interest in their use is growing.

2.2 Pose

The usage of quaternions and dual-quaternions are in the description of orientation and relative position between two right-handed coordinate frames. They can be an efficient parameterization of those quantities, which will be explained in the following sections.

The sections on Rotations and Twists will provide a very brief description on the mathematical tools used to describe the orientations of two frames with respect to each other.

2.2.1 Rotations

Attitude is the relative orientation of two frames in three dimensional space. A rotation is a change in attitude. All rotations in three dimensions are defined as being a part of the Lie group $SO(3)$. Rotations in $SO(3)$ have been classically defined by special
orthonormal matrices called a Direction Cosine Matrices (DCM) that have at least one eigenvalue of +1. An example of a DCM given a rotation about the three principal axes in \( \mathbb{R}^3 \) is given below:

\[
C_3(\psi) = \begin{bmatrix}
\cos(\psi) & \sin(\psi) & 0 \\
-\sin(\psi) & \cos(\psi) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(2.1)

\[
C_2(\theta) = \begin{bmatrix}
\cos(\theta) & 0 & -\sin(\theta) \\
0 & 1 & 0 \\
\sin(\theta) & 0 & \cos(\theta)
\end{bmatrix}
\]

(2.2)

\[
C_1(\phi) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(\phi) & \sin(\phi) \\
0 & -\sin(\phi) & \cos(\phi)
\end{bmatrix}
\]

(2.3)

A vector \( \mathbf{x} \in \mathbb{R}^3 \) can then be represented in a different coordinate frame through simple multiplication with the DCM describing the difference in orientation between the two frames \( A \) and \( B \): \( \mathbf{x}_B = C_{B/A} \mathbf{x}_A \). Since a DCM has nine entries with 6 constraints on the magnitude of its rows and columns, it is impractical to use in applications where attitude is to be integrated in the presence of system noise since a re-normalization of a DCM is a complicated process [19].

Various three parameter parameterizations of rotations such as the Euler angles, Gibbs’ vector whose components are called the Rodrigues parameters (RP) and the modified Rodrigues parameters (MRP) [20]. The Euler angles \((\phi, \theta, \psi)\) were some of the earliest defined. Any orientation in three dimensions can be described as a sequence of one to three Euler angle rotations about the three principle axes. A DCM can be formed from any Euler angle sequence by multiplication of the principle axis DCM, Equations (2.1)-(2.3).
For any rotation, there exists an axis about which there is no change. This unit vector for the axis of rotation $e$, is known as the Euler axis. An Euler rotation is any rotation $\theta$ about a given axis of rotation $e$. The Gibbs’ vector combines these two to form the RP:

$$g = \tan\left(\frac{\theta}{2}\right)e$$

(2.4)

Due to the fact that every pair of Euler angle and axis can give the same rotation as the negative axis and negative rotation, the Gibbs’ vector is said to "double cover" $SO(3)$.

The MRP are similar to the RP, but are a stereographic projection of $SO(3)$ onto $\mathbb{R}^3$ in a similar way that the Mercator projection is a stereographic projection of the globe onto a two dimensional plane. The MRP have the definition:

$$p = \tan\left(\frac{\theta}{4}\right)e$$

(2.5)

MRP, like the RP, also double cover $SO(3)$.

Either in their formulations or kinematics, singularities exist for all of the three parameter orientation descriptions. In order to avoid the issues of enforcing the constraints on a DCM or the singularities of the three parameter descriptions, the quaternion presents itself as the best minimum parameter description of a rotation that is nonsingular. A description of quaternions follows in the next section.

### 2.2.2 Twists

Pose is the combination of rotation and displacement and twist is a change in pose, similar to how a screw rotates as it translates. A twist is a transformation that exists
inside the Lie group $\text{SE}(3)$, and is typically represented by a $4 \times 4$ matrix representation

$$y = Ax + r \quad (2.6)$$

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} A & r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \quad (2.7)$$

where the matrix $A$ is a DCM that describes the rotation of the twist and $r$ is a displacement vector defining the linear translation of the twist. There are two ways of physically describing the twist displacement depending on the order of the rotation and displacement. A displacement $r$ is represented in the initial frame’s coordinates if the displacement is performed before the rotation, and it is represented in the second frame’s coordinates if the rotation is first and then followed by the displacement [6].

### 2.3 Quaternions and Dual Quaternions

#### 2.3.1 Quaternions

A quaternion is a four-dimensional number with the form:

$$q = q_0 + q_1i + q_2j + q_3k \quad (2.8)$$

which consists of a scalar part $q_0$ and a vector part $(q_1, q_2, q_3)$. The set of quaternions is the set $\mathbb{H}$, which can be represented as either a scalar/vector pair

$$q = (q_0, \bar{q}). \quad (2.9)$$
or as a vector in $\mathbb{R}^4$

$$q = [q_1, q_2, q_3, q_4]^T. \quad (2.10)$$

The scalar part of a quaternion, $q_0$, inherits the algebra of scalars in $\mathbb{R}$, and the vector part of a quaternion, $\vec{q}$, has the algebra of a vector space in $\mathbb{R}^3$ [20].

When treated as a vector, the scalar part of the quaternion is sometimes labeled in the literature as $q_4$ instead of $q_0$. They can be considered interchangeable, and both subscripts will be used in this dissertation.

The basis vectors $i$, $j$, and $k$ have the well-known properties described by Hamilton:

$$i^2 = j^2 = k^2 = ijk = -1 \quad (2.11)$$

$$jk = i, \quad kj = -i \quad (2.12)$$

$$ki = j, \quad ik = -j \quad (2.13)$$

$$ij = k, \quad ji = -k. \quad (2.14)$$

A quaternion is defined to be a conjugate of another quaternion if the following relationship is true,

$$q = q_0 + q_1i + q_2j + q_3k \quad (2.15)$$

$$q^* = q_0 - q_1i - q_2j - q_3k. \quad (2.16)$$

The concept is similar to a complex conjugate in number theory.
The quaternion algebra is, for two quaternions $q$ and $r$:

\begin{align*}
q + r &= (q_0 + r_0, \bar{q} + \bar{r}) \quad (2.17) \\
q - r &= (q_0 - r_0, \bar{q} - \bar{r}) \quad (2.18) \\
\lambda q &= (\lambda q_0, \lambda \bar{q}) \quad (2.19)
\end{align*}

with the geometric product defined as

\[ qr = (q_0 r_0 - \bar{q} \cdot \bar{r}, r_0 \bar{q} + q_0 \bar{r} + \bar{q} \times \bar{r}). \quad (2.20) \]

The cross product present in the vector part of the quaternion geometric product means that quaternion multiplication is non-commutative as $\bar{q} \times \bar{r} = -\bar{r} \times \bar{q}$.

In $\mathbb{R}^4$, geometric multiplication of two quaternions can be represented using matrices by,

\[
qr = \begin{bmatrix}
    r_4 & r_3 & -r_2 & r_1 \\
    -r_3 & r_4 & r_1 & r_2 \\
    r_2 & -r_1 & r_4 & r_3 \\
    -r_1 & -r_2 & -r_3 & r_4 \\
\end{bmatrix}
\begin{bmatrix}
    q_1 \\
    q_2 \\
    q_3 \\
    q_4 \\
\end{bmatrix} = [\mathbf{r} \otimes q]. \quad (2.21)
\]

where the operator $[\ldots \otimes]$ has the definition

\[
[q \otimes] = \begin{bmatrix}
    q_4 I_{3 \times 3} - [\bar{q} \times] \bar{q} \\
    -\bar{q}^T & q_4
\end{bmatrix}. \quad (2.22)
\]
and the operator \([\ldots \times]\) denotes the skew symmetric cross product matrix

\[
[x \times] = \begin{bmatrix}
0 & -x_3 & x_2 \\
x_3 & 0 & -x_1 \\
-x_2 & x_1 & 0
\end{bmatrix}
\] (2.23)

Alternatively, multiplication of two quaternions could instead use matrices constructed using the \([\ldots \circ]\) operator defined to be

\[
qr = \begin{bmatrix}
q_4I_{3 \times 3} + [\bar{q} \times] & \bar{q} \\
-\bar{q}^T & q_4
\end{bmatrix}
\begin{bmatrix}
r_1 \\
r_2 \\
r_3 \\
r_4
\end{bmatrix}
= [q \circ] r.
\] (2.24)

A quaternion is said to have a **magnitude** defined as

\[
||q|| = \sqrt{q^*q} = \sqrt{qq^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.
\] (2.25)

With this definition of magnitude, a quaternion that has a magnitude greater than zero has an inverse defined to be:

\[
q^{-1} = \frac{q^*}{||q||^2} = \frac{q^*}{q^*q}.
\] (2.26)

If a quaternion has unit norm, \(q^*q = 1\), its inverse is equal to its complex conjugate. Unit quaternions are important, and the most common quaternion used in engineering due to its ability to describe rotations in three dimensional space.

A quaternion of unit norm, \(q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1\), can describe the rotation of the components of a vector, \(x \in \mathbb{R}^3\), from one coordinate system to another through the
linear operator,

\[ y = q^* x q, \]  \hspace{1cm} (2.27)

A unit quaternion can be used to parameterize the DCM that describes the orientation of a frame \( B \) with respect to frame \( I \) by Equation (2.27),

\[
A(q) = [q^* \odot] [q \otimes] = (q_1^2 - q^T q) I - 2 q_1 [q \times] + 2 q q^T \hspace{2cm} (2.28)
\]

\[
= \begin{bmatrix}
q_1^2 + q_2^2 - q_3^2 & 2(q_1 q_2 + q_3 q_4) & 2(q_1 q_3 - q_2 q_4) \\
2(q_1 q_2 - q_3 q_4) & q_2^2 - q_1^2 - q_3^2 & 2(q_2 q_3 + q_4 q_1) \\
2(q_1 q_3 + q_2 q_4) & 2(q_2 q_3 - q_4 q_1) & q_1^2 - q_1^2 - q_2^2 + q_3^2
\end{bmatrix} \hspace{2cm} (2.29)
\]

The quaternion is the smallest parameterization of \( SO(3) \) that is singularity free in its kinematics and dynamics. Like the RP and MPR, quaternions also doubly cover \( SO(3) \), which can lead to ambiguity since the rotations described by \( q \) and \(-q\) are equivalent.

Multiple rotations in succession can be combined using the quaternion geometric product. If \( q_{O/I} \) is a quaternion that represents the attitude of frame \( O \) with respect to frame \( I \), or rotation necessary to rotate frame \( I \) so that it is coincident with frame \( O \), and \( q_{B/O} \) represents the attitude of frame \( B \) with respect to from \( O \), the quaternion that represents the attitude of frame \( B \) with respect to frame \( I \) is given by

\[
q_{B/I} = q_{O/I} q_{B/O} \hspace{2cm} (2.31)
\]

If we want to define a differential quaternion \( \delta q \) that can be used to represent the attitude error between a reference frame \( A \) with respect to \( I \) and another frame \( B \) with respect to \( I \), it can be done by recognizing that the geometric product of quaternions
must be used,

\[ q_{B/I} = q_{A/I} \delta q \Rightarrow \delta q = q_{A/I}^{r} q_{B/I} \] (2.32)

The vector \( \delta q \) is the multiplicative error between the two attitude quaternions.

### 2.3.2 Dual Quaternions

A dual quaternion is the summation of a quaternion with another quaternion multiplied by the dual-operator, \( \epsilon \).

\[ \hat{q} = q_{r} + \epsilon q_{d} \] (2.33)

The dual-operator \( \epsilon \) has the properties that \( \epsilon^{2} = 0, \ \epsilon \neq 0 \). The dual-quaternion can be represented by the pair of real and dual parts \( \hat{q} = (q_{r}, q_{d}) \) much like the elements of the quaternion. The real and dual elements of the dual quaternion follow the standard quaternion algebra. The algebra of dual quaternions is:

\[
\begin{align*}
\hat{q} + \hat{p} &= (q_{r} + p_{r}, q_{d} + p_{d}) \\
\hat{q} - \hat{p} &= (q_{r} - p_{r}, q_{d} - p_{d}) \\
\lambda \hat{q} &= (\lambda q_{r}, \lambda q_{d})
\end{align*}
\] (2.34-2.36)

with the geometric product of two dual-quaternions being

\[
\begin{align*}
\hat{r} &= \hat{q}\hat{p} = (q_{r} + \epsilon q_{d})(p_{r} + \epsilon p_{d}) \\
&= q_{r}p_{r} + \epsilon(q_{r}p_{d} + q_{d}p_{r})
\end{align*}
\] (2.37-2.38)
This multiplication can be mapped to $\mathbb{R}^8$ for use in a computer via the following linear operation,

$$
\hat{q}\hat{p} = \begin{bmatrix}
[P_r \otimes] & 0_{4 \times 4} \\
[P_d \otimes] & [P_r \otimes]
\end{bmatrix}
\begin{bmatrix}
q_r \\
q_d
\end{bmatrix}.
$$

(2.39)

The conjugate of a dual-quaternion $\hat{q}$ is simply

$$
\hat{q}^* = q_r^* + \epsilon q_d^*.
$$

(2.40)

With this definition of a conjugate dual-quaternion, the magnitude of a dual-quaternion is defined to be

$$
\|\hat{q}\| = \sqrt{\hat{q}^* \hat{q} = \sqrt{q_r^* q_r + \epsilon(q_r^* q_d + q_d^* q_r)}}
$$

(2.41)

This norm shows that a dual-quaternion could have a magnitude that is a dual-number. A dual-quaternion that has a real magnitude must meet a set of specific conditions.

If a dual-quaternion is to be unit norm, it must meet the following two conditions:

$$
q_r^* q_r = 1
$$

(2.42)

$$
q_r^* q_d + q_d^* q_r = 0
$$

(2.43)

These conditions can be translated into $\mathbb{R}^4$ to become

$$
q_r^T q_r = 1
$$

(2.44)

$$
q_r^T q_d = 0.
$$

(2.45)

These conditions imply that the real quaternion, $q_r$, must be a unit quaternion, and the
second condition means that the dual part of the quaternion, $q_d$, must be perpendicular to the real part. There is no norm constraint on the dual part of the dual-quaternion in order for the dual-quaternion to be a unit dual-quaternion.

A function of a dual-number $f(\hat{q})$ is defined to be

$$f(\hat{q}) = f(q_r) + \epsilon q_d \frac{\partial f(\hat{q})}{\partial \hat{q}} (q_r),$$  \hspace{1cm} (2.46)

meaning the magnitude of the dual-quaternion is

$$\|\hat{q}\| = \sqrt{q^*_r q_r + \epsilon (q^*_r q_d + q^*_d q_r) \frac{q^*_r q_r}{q^*_r q_r}}$$  \hspace{1cm} (2.47)

A dual-quaternion inverse exists if and only if the real part of the dual-quaternion has a magnitude greater than zero. If it exists, the inverse dual-quaternion is defined as

$$\hat{q}^{-1} = \frac{q^*_r}{q^*_r q_r} - \epsilon \frac{q^*_r q_d q^*_r}{(q^*_r q_r)^2}.$$  \hspace{1cm} (2.48)

Unlike with the classic quaternion, if a dual-quaternion has unit norm it is not guaranteed to have a conjugate equal to its inverse. The extra condition is that

$$\hat{q}_d^* = -q^*_r q_d q^*_r.$$  \hspace{1cm} (2.49)

The $4 \times 4$ matrix representation for a twist between frames $B$ and $I$ in Equation (2.6) can be parameterized by a dual-quaternion defined to be either,

$$\hat{q}_{B/I} = q_{B/I} + \frac{1}{2} \epsilon q_{B/I} \mathbf{Y}^B,$$  \hspace{1cm} (2.50)
or

\[ \hat{q}_{\mathcal{B}/\mathcal{I}} = q_{\mathcal{B}/\mathcal{I}} + \frac{1}{2} \epsilon t^T q_{\mathcal{B}/\mathcal{I}}. \] (2.51)

depending on the order of the rotation and displacement. This dual-quaternion has unit norm and meets the necessary conditions such that \( \hat{q}^{-1} = \hat{q}^* \).

Much how subsequent rotations can be combined using the geometric product, the geometric product of two dual-quaternions yields the effective twist of both in sequence

\[ \hat{q}_{\mathcal{B}/\mathcal{I}} = \hat{q}_{\mathcal{O}/\mathcal{I}} \hat{q}_{\mathcal{B}/\mathcal{O}}. \] (2.52)

Using this relationship, we can define the differential dual-quaternion twist between two reference frames, letting \( \hat{q} \) be some estimate and \( \hat{q} \) represent the true pose of some coordinate system \( \mathcal{B} \) with respect to coordinate system \( \mathcal{I} \).

\[ \hat{q} = \hat{q} \delta q \quad \Rightarrow \quad \delta q = \hat{q}^* \hat{q} \] (2.53)

### 2.3.3 Kinematics

The kinematics of a quaternion are

\[ \dot{q} = \frac{1}{2} q \omega, \] (2.54)

where \( \omega \) is a pseudovector that represent the rotational velocity of a frame with respect to another using that frame’s basis vectors. These kinematics can be mapped to \( \mathbb{R}^4 \) and
written as

\[ \dot{\mathbf{q}} = \frac{1}{2} [\mathbf{\omega} \otimes \mathbf{q}] = \frac{1}{2} \Omega \mathbf{q} \]  
(2.55)

\[ = \frac{1}{2} \begin{bmatrix}
0 & -\omega_1 & -\omega_2 & -\omega_3 \\
\omega_1 & 0 & \omega_3 & -\omega_2 \\
-\omega_3 & -\omega_2 & 0 & \omega_1 \\
\omega_2 & \omega_3 & -\omega_1 & 0 
\end{bmatrix} \begin{bmatrix}
q_0 \\
q_1 \\
q_2 \\
q_3 
\end{bmatrix}. \]  
(2.56)

One should notice that in Equation (2.55), the angular velocity vector $\mathbf{\omega}$ defined in the rotating coordinates is treated as a quaternion in $\mathbb{R}^4$ with its scalar part equal to zero.

Since these kinematics are linear in $\mathbf{q}$ if the angular velocity $\mathbf{\omega}$ is constant, it can be shown that the solution for $\mathbf{q}(t)$ is equal to

\[ \mathbf{q}(t) = \Phi(t, t_0)\mathbf{q}(t_0), \]  
(2.57)

where the state transition matrix $\Phi(t, t_0)$ has the analytical solution given by Mayo [21]

\[ \Phi(t, t_0) = \cos \left( \frac{\mathbf{\omega}(t - t_0)}{2} \right) I_{4 \times 4} + \left( \frac{\sin \left( \frac{\mathbf{\omega}(t - t_0)}{2} \right)}{\mathbf{\omega}} \right) \Omega, \]  
(2.58)

where $\mathbf{\omega} = \lVert \mathbf{\omega} \rVert$.

This state transition matrix is only valid if the assumption that $\mathbf{\omega}$ is constant throughout the time spanned by $\Delta t = t - t_0$ is true or a justifiable assumption.

The kinematics of a dual-quaternion behave like those of a standard quaternion, where

\[ \dot{\mathbf{\hat{q}}} = \frac{1}{2} \mathbf{\hat{q}} \mathbf{\hat{\omega}}. \]  
(2.59)
The dual-vector velocity vector $\mathbf{\hat{\omega}}$ is then,

$$
\mathbf{\hat{\omega}} = \mathbf{\omega} + \epsilon \mathbf{v},
$$

(2.60)

where $\mathbf{\omega}$ is as previously defined and $\mathbf{v}$ is translational velocity in the translating coordinate system.

These kinematics in matrix-vector form, $\mathbf{\hat{q}} \in \mathbb{R}^8$, are

$$
\dot{\mathbf{\hat{q}}} = \frac{1}{2} \begin{bmatrix}
\mathbf{\Omega} & 0_{4 \times 4} \\
\mathbf{V} & \mathbf{\Omega}
\end{bmatrix} \begin{bmatrix}
\mathbf{q}_r \\
\mathbf{q}_d
\end{bmatrix},
$$

(2.61)

where $\mathbf{V} = [\mathbf{v} \otimes]$. 

Like with the quaternion kinematics, if the assumption that both $\mathbf{\omega}$ and the velocity $\mathbf{v}^I$ in the inertial coordinates can be assumed constant for some time-step $\Delta t = t - t_0$, then there exists an analytic solution given by

$$
\mathbf{\check{q}}(t) = \Phi(t, t_0) \mathbf{q}(t_0),
$$

(2.62)

where

$$
\Phi(\Delta t) = \begin{bmatrix}
\mathbf{\Phi}_{11} & 0_{4 \times 4} \\
\mathbf{\Phi}_{21} & \mathbf{\Phi}_{11}
\end{bmatrix},
$$

(2.63)

$$
\mathbf{\Phi}_{11} = \cos \left( \frac{\theta}{2} \right) I_{4 \times 4} + \frac{\sin \left( \frac{\theta}{2} \right)}{\mathbf{\omega}} \mathbf{\Omega}
$$

(2.64)

$$
\mathbf{\Phi}_{21} = \frac{1}{2} \Delta t \left\{ \cos \left( \frac{\theta}{2} \right) I_{4 \times 4} + \frac{\sin \left( \frac{\theta}{2} \right)}{\mathbf{\omega}} \mathbf{\Omega} \right\} V_0 = \frac{1}{2} \Delta t \mathbf{\Phi}_{11} \mathbf{V}
$$

(2.65)
2.3.4 Dynamics

The dynamics of a dual quaternion are simply an extension of the kinematics of the coordinate frames to a rigid body centered at the rotating frame with mass $m$ and moment of inertia $J$. The angular and linear velocities are no longer considered constant, yielding their dynamic equations:

\[
\dot{\hat{q}} = \frac{1}{2} \begin{bmatrix}
\omega \otimes & 0_{4 \times 4} \\
v \otimes & \omega \otimes
\end{bmatrix}
\begin{bmatrix}
q_r \\
q_d
\end{bmatrix},
\]

\[
\dot{\omega} = J^{-1}(\tau - \omega \times J\omega),
\]

\[
\dot{v} = \frac{f}{m} - \omega \times v,
\]

where $f$ is the total forces acting on the rigid body and $\tau$ is the total torque. Both are represented in the rotating coordinate system.

These dynamics are nonlinear, therefore must be propagated numerically under most circumstances. There exist assumptions that provide a nonlinear solution to these dynamics. If the angular velocity $\omega$ is constant in the rotating frame and the linear acceleration $a = f/m$ is assumed constant in the inertial frame for some time-span $\Delta t = t - t_0$, the solution has the form:

\[
\hat{q}(t) = \Phi(\Delta t)\hat{q}(t_0)
\]

\[
v(t) = \Psi(\Delta t)(v + \Delta t a)
\]
where

\[
\Phi(\Delta t) = \begin{bmatrix} \Phi_{11} & 0_{4\times 4} \\ \Phi_{21} & \Phi_{11} \end{bmatrix},
\]

(2.71)

\[
\Phi_{11} = \cos\left(\frac{\theta}{2}\right) I_{4\times 4} + \frac{\sin\left(\frac{\theta}{2}\right)}{\omega} \Omega
\]

(2.72)

\[
\Phi_{21} = \frac{1}{2} \Delta t \Phi_{11} \left( V(t_0) + \frac{1}{2} \Delta t A \right)
\]

(2.73)

\[
\Psi(\Delta t) = I_{3\times 3} - \frac{\sin(\theta)}{\omega} [\omega \times] + \frac{1 - \cos(\theta)}{\omega^2} ( [\omega \times] )^2
\]

(2.74)

This solution is nonlinear if the state space of the discrete dynamic equations is taken to be \( x = [ q^T \ v ]^T \). Other analytic solutions for the discrete dynamic equations might possibly exist for torque-less systems where some rotational moments of inertia are equivalent.

### 2.4 Quaternion Error

With the definitions for quaternion and dual-quaternion error given in Equations (2.32) and (2.53), assumptions can be used to simplify the kinematics of quaternions and dual-quaternions. The MRP can be used as a way to parameterize this error in three dimensions.

#### 2.4.1 Parameterizations

The quaternion is the smallest number of parameters that it takes to fully parameterize a rotation without singularities, yet there are some drawbacks to using a quaternion in practice. A unit quaternion exists on the unit-sphere \( SO(3) \), therefore, a quaternion sampled from a probability distribution such as the Gaussian distribution in three
dimensions, which has tails that extend out to infinity, can lead to violations of a unit norm constraint [22]. To accommodate this, the three parameter representations, RP and MRP, that have no norm constraints tend to be used in practice when sampling quaternion errors from a Gaussian distribution [23].

A quaternion can be parameterized by a unit vector \( e \) and angle of rotation \( \phi \) to create the Euler Parameters (EP) where

\[
q = \cos \left( \frac{\phi}{2} \right) + \sin \left( \frac{\phi}{2} \right) e,
\]

(2.75)

and \( e \) is the Euler vector, which is the axis of rotation. The Euler parameters

\[
(\cos(\phi/2), \sin(\phi/2)e_1, \sin(\phi/2)e_2, \sin(\phi/2)e_3)
\]

(2.76)

are exactly the coefficients of a unit quaternion \((q_0, q_1, q_2, q_3)\). The three elements (RP) of the Gibbs’ vector are then

\[
g = \frac{\bar{q}}{q_4} = \tan \left( \frac{\phi}{2} \right) e = \tan \left( \frac{\phi}{2} \right) \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}
\]

(2.77)

The downside of using the RP is that as \( q_4 \to 0 \), corresponding to 180 degree rotations, produces singular RP.

This led to the derivation by Wiener [24] of the Modified Rodrigues Parameters (MRP)

\[
p = \frac{\bar{q}}{1 + q_4} = \tan \left( \frac{\phi}{4} \right) e
\]

(2.78)

whose singularity lies at \( 2\pi \). Since one can limit rotations to lie between The MRP are
derived using what is known as the Cayley transform, being

\[ p = (q - 1)(q + 1)^{-1} = (q + 1)^{-1}(q - 1) = \frac{q - 1}{q + 1}. \] (2.79)

Similar to how the unconstrained RP, and MRP are alternative attitude representations derived from a quaternion, there exist analogues with respect to dual-quaternions. These are called the dual Rodrigues parameters (DRP) [25] and the dual modified Rodrigues parameters (DMRP), which are referred to in Reference [26] as a twistor. The DMRP are obtained through the Cayley transformation of a unit dual-quaternion to yield,

\[
\hat{p} = \frac{\hat{q} - 1}{\hat{q} + 1} = p + \frac{1}{2} \epsilon (1 - p) \hat{q}_d^{S/T} (1 - p) = p + \frac{1}{4} \epsilon (1 + p) r^S (1 - p) = p + \frac{1}{4} \epsilon (1 - p) r^T (1 + p),
\] (2.80) (2.81) (2.82) (2.83)

2.5 Summary

In this chapter, the mathematical descriptions for attitude and pose were described. Following this, the mathematical definitions of the quaternion and dual-quaternion algebra were also described. For the purpose of filtering and estimation, the concept of multiplicative error in the quaternion and dual-quaternion algebra was also presented.
Chapter 3  
Overview Probability Distributions

3.1 Introduction

This chapter’s focus is on providing the statistical background information necessary to understand this work in an abridged fashion. It will start with a quick description of what a probability density function is. Then, the basic information theory concept of entropy is introduced. Following this, a description of the distributions used for this work are given: the Gaussian or normal distribution, the Bingham distribution, and the Dual-Bingham distribution.

In filtering, it is important to choose a distribution for a random variable that maximizes the entropy of the space it inhabits. For vectors in Euclidean space, that distribution is the white-noise or Gaussian distribution. The maximum entropy distribution for a random variable that exists on the unit sphere is the Bingham distribution, which is what lends itself well to describing the uncertainty of a quaternion state.

The way to describe a dual-quaternion as a stochastic variable is a joint distribution of a Bingham-distributed quaternion and a Gaussian-distributed vector. The expectation and second moment of such a distribution is described, which can be used in a filter to propagate a description of the distribution using Gaussian quadrature.
3.2 Distribution Functions

Given a continuous random variable $x$, the probability that it has a value less than another given value, $X$ is given by

$$P(x < X) = \int_{-\infty}^{X} p(t)dt$$  \hspace{1cm} (3.1)

This function, $P(x)$, is referred to as the cumulative distribution function (CDF), or in some literature, as the simply the distribution function (DF). The function $p(x)$, is important, as it is known as the probability density function (PDF) for the variable $x$.

3.2.1 Probability Density Functions

Intuition about probability density functions can be made by the analogy between probability and mass in physics. Just as the density of an object is described as the amount of mass per unit volume, the probability density can be thought of as the amount of likelihood a random variable can be a given value in an infinitesimal volume around of that value. If you integrate the density of a rigid body along the whole volume of that rigid body, you receive the total mass of that body. The same is true for probability density, but with probability instead of mass.

A PDF can take any form, however, there is one constraint that all PDFs must have to be a true PDF: an integral of the PDF over the full support of the random variable must be unity.

$$\int_{a}^{b} p(x)dx = 1, \hspace{0.2cm} x \in [a, b]$$ \hspace{1cm} (3.2)

This condition ensures that the probability of any instance of the random variable, $P(x < X)$, is guaranteed to be between zero and one.
### 3.2.2 Joint Distributions

Distributions of more than one variable, \( P(x_1 < X_1, ..., x_n < X_1) \), are defined to be joint distributions of \( n \) random variables. A formal definition of a joint distribution function,

\[
P(x_1 < X_1, ..., x_n < X_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{X_1} p(t_1, ..., t_n) dt_1 ... dt_n \tag{3.3}
\]

This notation for a joint probability can be simplified by representing the random variables as a single random vector \( \mathbf{x} = [x_1 \ ... \ x_n]^{T} \) in \( \mathbb{R}^n \).

\[
P(\mathbf{x} < \mathbf{X}) = \int_{-\infty}^{\mathbf{x}} p(\mathbf{t}) d\mathbf{t} \tag{3.4}
\]

### 3.2.3 Conditional Probability

A conditional probability density function represents the probability of a random variable given knowledge of another random variable, \( p(\mathbf{x}|\mathbf{y}) \). This can be related to a joint density between random variables in a joint density function by

\[
p(\mathbf{x}, \mathbf{y}) = p(\mathbf{x}|\mathbf{y}) p(\mathbf{y}) \tag{3.5}
\]

If knowledge of the random vector \( \mathbf{y} \) does not improve the knowledge of \( \mathbf{x} \), meaning \( p(\mathbf{x}|\mathbf{y}) = p(\mathbf{x}) \), the two random vectors are defined to be independent. Their joint density follows directly from this to be

\[
p(\mathbf{x}, \mathbf{y}) = p(\mathbf{x}) p(\mathbf{y}) \tag{3.6}
\]
A powerful observation of Equation (3.5), is that we can rewrite it as

\[ p(x, y) = p(y|x)p(x) \]  

(3.7)

Equations (3.5) and (3.7) can be equated and manipulated to show that

\[ p(x|y) = \frac{p(y|x)p(x)}{p(y)} \]  

(3.8)

Equation (3.8) is commonly known as Bayes’ rule and is a core principle of Bayesian statistics.

### 3.2.4 Expectation and Moments

Continuing with the mass analogy introduced earlier from physics, the moments of inertia of a rigid body are related to the spread of mass throughout a rigid body. In statistics, the moments of a distribution describe how the probability density is spread throughout the distribution.

The moments are numbered such that the moment \( n \), is given by

\[ E[x^n] = \int_{-\infty}^{\infty} x^n p(x) \, dx \]  

(3.9)

where the \( E[\cdot] \) operator is known as the expectation operator. The zero moment, \( E[1] \), for a distribution, is simply the unity condition for the probability density function.

The first moment, \( \mu = E[x] \), also known as the mean of the distribution, is the value that a random variable is expected to have. This should not be confused with the mode of a distribution, which is the value of \( x \) which has the largest probability density. In most symmetrical distribution, the mean and mode are the same, yet this is not true for distributions that have multiple modes.
A distribution's second moment, $E[x^2]$, does not have much physical significance, however, its second central moment, $E[(x - \mu)^2]$, is commonly referred to as the variance of a distribution. It is a measure on how much the distribution is or is not clustered about the mean. For distributions with zero mean, there is no difference between the second and second central moments.

For multivariate distributions, the second central moment, $E[(\mathbf{x} - \mathbf{\mu})(\mathbf{x} - \mathbf{\mu})^T]$, is often referred to as a covariance matrix. The diagonals of the covariance matrix are the variances of each of the elements of the vector, and the off-diagonal terms are the respective covariances.

Higher order moments do exist, namely the skewness and kurtosis, being the third and fourth moments respectively. However, they do not see much use in practical engineering applications as they are not generally needed to describe entropy maximizing distributions.

### 3.3 Maximum Entropy Distributions

The entropy of a distribution for a continuous random variable, $x$, is a metric on the amount of information needed to describe it, or the amount of uncertainty in knowing the true value of the random variable. Its formal definition is

$$H(x) = -\int_{-\infty}^{\infty} p(x) \ln p(x) \, dx$$  \hspace{1cm} (3.10)

For a distribution on $\mathcal{SO}(n)$ with PDF $p(\mathbf{x})$ to be useful for the purpose of modeling a stochastic process, we should use a distribution to model our system that has maximum entropy. The reason for this is that we want a distribution model that minimizes the effect of our prior knowledge "over-fitting" the estimate as more information is introduced. The Gaussian distribution is widely used for modeling continuous random variables in
\( \mathbb{R}^n \) because it is a maximum entropy distribution in that domain.

### 3.3.1 Gaussian Distribution

The distribution that maximizes entropy in \( \mathbb{R}^n \) given a fixed mean, \( \mu \), and covariance, \( \Sigma \), are known as the multivariate normal or Gaussian distributions. The PDF for a Gaussian distribution is given by

\[
p(x) = \frac{1}{\sqrt{|2\pi \Sigma|}} \exp \left[-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right]
\]

(3.11)

Equation (3.11) is a very famous and well-known formula, and from it one can see the form of many maximum-entropy distributions, the exponential family with form.

\[
p(x) = K \exp \left[f(x)\right]
\]

(3.12)

### 3.3.2 Bingham Distribution

The Bingham distribution is the maximum entropy distribution given a known second moment on the unit \( n \)-sphere [1]. A demonstration of this for the unit circle can be found in Appendix D. The Bingham distribution was initially defined with the three dimensional sphere in mind, but it can be expanded to any \( n \)-sphere with the following PDF:

\[
p(x) = \frac{1}{F(Z)} \exp \left[x^T M Z M^T x\right]
\]

(3.13)

where

\[
F(Z) = |dS|_n \ {}_1F_1 \left(\frac{1}{2};\frac{n}{2};Z\right)
\]

(3.14)
and $|dS|_n$ is the surface area of the unit $n$-sphere and $\mathbf{1}_F \left( \frac{1}{2}; \frac{n}{2}; Z \right)$ is a confluent hypergeometric function of matrix argument. These hyper-geometric functions are computationally intensive to calculate, and their evaluations are usually compiled beforehand and placed into lookup tables for fast reference [4,11] or approximated used saddle-point computations [27].

The matrix $Z$ is a diagonal set of concentration parameters that define the spread of the distribution about the mode. The diagonals of the matrix, $z_i$, must be ordered such that if $i < j \rightarrow z_i < z_j$. A property of confluent hyper-geometric functions of matrix argument is that $F(Z) = F(Z + \lambda I)$, which means that to define a Bingham distribution for the unit $n$-sphere requires $n$ concentration parameters since a $\lambda$ can always be chosen such that the $z_{n+1}$ parameter is fixed to be zero. To see how the concentration matrix affects the probability density of the distribution, see Figure 3.1.

As the diagonals of the Bingham $Z$ concentration matrix become more negative, their probabilities become more focused in the directions of the distribution’s mode. As one of the dimensions approaches zero, all of the probability in the distribution collapses into a smaller subspace. In the case of a Bingham distribution in three dimensions, if a second of the concentration parameters approaches zero, the probabilities in the distribution collapses into a plane.

The first moment of a Bingham distribution is zero due to the antipodal symmetry of the distribution. Antipodal symmetry means that there is a line running through the origin of the distribution where probabilities are mirrored at each end where the rotational line intersects the surface of the unit $n$-sphere. This means that both $\mathbf{x}$ and $-\mathbf{x}$ have the same probability, which motivates the use of the Bingham distribution for use with the quaternions. Since the first moment is zero, it is not considered important when defining a Bingham distribution. However, the mode of the distribution is important, and appears in the $M$ rotation matrix of the distribution.
The second moment of a Bingham distribution $\mathcal{B}(Z, M)$ is

$$ P = M \text{ diag} \left[ \frac{dF(Z)/dZ_1}{F(Z)} \frac{dF(Z)/dZ_2}{F(Z)} \cdots \frac{dF(Z)/dZ_n}{F(Z)} \right] M^T \quad (3.15) $$

Efficient methods for calculating the derivatives of the Bingham normalizing constant have been developed by Kume and Wood in Reference [28].

The Bingham distribution is similar to the Gaussian distribution, and in many sources, it is claimed that there is an underlying normal distribution that is conditioned to lie on the unit hyper-sphere [3, 29]. Unlike a standard Gaussian distribution, the trace of a Bingham distribution is always equal to one.

If quaternions are treated as a random unit vector in $\mathbb{R}^4$, a Bingham distribution describing these unit vectors can be constructed where the mode of the distribution, $q_m$, can be used to define the $M$ matrix to be $M = [q \odot]$. This is due to the fact that a unit quaternion parameterizes a pure rotation in $\mathbb{R}^4$.

Important to the concept of filtering is the idea of zero-mean white noise. While zero-mean white noise might be important for filters in $\mathbb{R}^n$, the analogous noise in $\mathcal{SO}(3)$ would be an identity-mode white noise quaternion. Such a quaternion can be defined by the Bingham distribution $u \sim \mathcal{B}(Z, I)$ whose mode is the identity quaternion. This means that the second moment is then simply given by:

$$ \Sigma = E[uu^T] = \text{diag} \left[ f_1 \ f_2 \ f_3 \ f_4 \right] \quad (3.16) $$

Take the kinematic equation for a quaternion with a process noise $u$ that is modeled as a Bingham distribution with identity mode

$$ q_{k+1} = f(q_k, \omega_k)u_k \quad (3.17) $$
If one was to look at the uncertainty in the state $q$ as $k \to \infty$, it would approach the uniform Bingham distribution, meaning that all orientations are equally likely. This can been seen in one dimension in Figure 3.2. This Figure depicts the values of the PDF for a Bingham distribution on the unit circle. At time $t = 0$, the distribution has probabilities that are concentrated at the modes of the distribution. As over time, with the addition of process noise $u$, the knowledge of the orientation becomes less certain as the probability density is spread out around the unit circle. As $\Delta t$ increases, the PDF for the position around the unit circle approaches that of a uniform distribution. The covariance of a uniform Bingham distribution is one such that

$$P = \frac{1}{4}I$$

(3.18)
Figure 3.2: Bingham PDF in 2D with constant rotation over time.

3.4 The Dual-Bingham Distribution

Now that the distributions of the Bingham distribution for quaternion states and the Gaussian distribution for vector states have been described, a joint distribution for a dual-quaternion state can be derived. Let

\[
q = \begin{bmatrix}
q_r \\
\mathbf{d} = \frac{1}{2} [q \odot \mathbf{r}]
\end{bmatrix}
\] (3.19)

be a vector representation of the dual-quaternion \( \hat{q} = q + \frac{1}{2} \epsilon q r \) in \( \mathbb{R}^8 \) with \( q \sim \mathcal{B}(Z, M) \) and \( r \in \mathcal{N}(\mu, \Sigma) \).

We can construct a PDF for the dual-quaternion state, \( p(q_r, d) \), using the definition for a joint distribution in Equation (3.7).

\[
p(q_r, d) = p(d|q_r)p(q_r)
\] (3.20)

The PDF of the vector \( q_r \) is that of the Bingham distribution \( \mathcal{B}(Z, M) \) and the conditional distribution \( p(d|q_r) \) is a degenerate Gaussian distribution that has been
rotated into four dimensions, making

\[
p(q_r, d) = \frac{1}{F(Z)\sqrt{2\pi\Sigma}} \exp \left[ (r - \mu)^T [q^* \circ] \Sigma [q \circ] (r - \mu) + q_r^T MZM^T q_r \right] \quad (3.21)
\]

While this PDF is of mathematical interest, it is not necessary for use in a minimum-variance filter. A minimum-variance update in a filter only requires a definition of the covariance for the dual-quaternion state. Like the Bingham filter, the expectation of a dual-quaternion state is also zero due to the antipodal symmetry. What is more important for a dual-quaternion, is a definition of the mode

\[
\hat{q}_m = q_m + \frac{1}{2}\epsilon q_m \mu \quad (3.22)
\]

### 3.4.1 Dual-Bingham Covariance

The covariance of this distribution is then

\[
P = E[qq^T] = \begin{bmatrix}
E[q_r q_r^T] & \frac{1}{2}E[q_r ([q \circ] r)^T] \\
\frac{1}{2}E[q_r ([q \circ] r)^T] & \frac{1}{4}E[((q \circ) r)((q \circ) r)^T]
\end{bmatrix} \quad (3.23)
\]

We can evaluate each quadrant of this dual-quaternion covariance separately. The top left quadrant simply evaluates to the covariance of the Bingham distribution for \( q_r \).

\[
\Lambda = M \text{diag} [ f_1 f_2 f_3 f_4 ] M^T = \begin{bmatrix}
\bar{\Lambda} & \lambda \\
\lambda^T & \lambda_1
\end{bmatrix} \quad (3.24)
\]

The cross-covariances in the top-right and bottom-left are provided in the following derivation. For this derivation, it is best to express the dual part of the dual-quaternion
in its vector form \( d = \frac{1}{2} [q \odot] r \).

\[
E[qq^T] = \frac{1}{2} \begin{bmatrix}
\lambda^T \mu + \tilde{\Lambda} [\mu \times] & -\tilde{\Lambda} \mu \\
\mu^T (\lambda_4 I - [\lambda \times]) & -\lambda^T \mu
\end{bmatrix}
\quad (3.25)
\]

An interesting thing about this cross-covariance is that given a covariance of a dual-quaternion state, the underlying mean of the Gaussian distribution of the position can be obtained from the last column of the cross-covariance matrix.

The bottom-right terms of Equation (3.23) are more complicated and will be derived in parts themselves by the following decomposition

\[
E[dd^T] = \frac{1}{4} \begin{bmatrix}
\Delta & \delta \\
\delta^T & \delta_4
\end{bmatrix}
\quad (3.26)
\]

The right column of this expectation matrix is equal to

\[
\begin{bmatrix}
\delta \\
\delta_4
\end{bmatrix} = \begin{bmatrix}
-(\Sigma + U) \lambda - \left( (\Sigma + U) \tilde{\Lambda} - \tilde{\Lambda} (\Sigma + U) \times \right)^{-1} \\
\operatorname{Tr}(\tilde{\Lambda} (\Sigma + U))
\end{bmatrix}
\quad (3.27)
\]

where the outer product of \( \mu \) with itself is introduced, \( U = \mu \mu^T \) and \( \cdot \times \)^{-1} corresponds to the inverse skew symmetric operation.

Lastly, the top-left corner of the covariance follows from the quaternion multiplication to be

\[
\Delta = -(\Sigma + U) [\lambda \times] + (\Sigma + [\lambda \times]) + (1 - 2 \operatorname{Tr}(\tilde{\Lambda})) I (\Sigma + U) \\
+ (\Sigma + U - \operatorname{Tr}(\Sigma + U) I) \tilde{\Lambda} + (\operatorname{Tr}(\Sigma + U) \operatorname{Tr}(\tilde{\Lambda}) - \operatorname{Tr}(\tilde{\Lambda}) I) I
\quad (3.28)
\]

This covariance shows the coupling of both the Bingham and Gaussian distributions on the distribution of the dual-quaternion random variable.
If we assume that the dual-quaternion is an error dual-quaternion

\[
\hat{\delta}q = \hat{q}^* \hat{q} = \delta q + \frac{1}{2} \epsilon \delta q \delta r
\]  

(3.29)

some terms in the following covariance can be simplified by the fact that \( \mu = 0, \lambda = 0, \) and \( M = I. \) This yields a simplified dual-quaternion covariance

\[
\Delta = (\Sigma + (1 - 2\text{Tr}(\bar{\Lambda}))I)\Sigma + (\Sigma - \text{Tr}(\Sigma)I)\bar{\Lambda}
\]

\[+ (\text{Tr}(\Sigma)\text{Tr}(\bar{\Lambda}) - \text{Tr}(\Sigma\bar{\Lambda}))I\]

\[\delta = -\left[\Sigma\bar{\Lambda} - \bar{\Lambda}\Sigma \times\right]^{-1}\]

\[\delta_4 = \text{Tr}(\bar{\Lambda}\Sigma)\]

(3.30)

(3.31)

(3.32)

### 3.5 Summary

Distributions for dual-quaternions are something that are not readily available in the literature, but the preceding chapters are intended to rectify that missing information. This was done by describing the two maximum-entropy distributions that are combined in the dual-quaternion Bingham distribution. A dual-quaternion has both a Bingham-distributed real part and a dual part that is a Gaussian in three dimensions that is rotated by the Bingham-distributed quaternion. The covariance of such a distribution was described for a general case where \( q \sim \mathcal{B}(Z, M) \) and \( r \sim \mathcal{N}(\mu, \Sigma) \) and was then specialized for an error dual-quaternion where \( \delta q \sim \mathcal{B}(Z, I) \) and \( \delta r \sim \mathcal{N}(0, \Sigma). \) With the basic information on the probability distributions used in this work, the following chapters detail how these concepts are applied to the filtering of dual-quaternion states.
Chapter 4

Constrained EKF for Dual-Quaternions

4.1 Introduction

This chapter focuses on describing the various filters that will be compared in this dissertation. What will come first is a brief description of a Kalman Filter (KF) specifically the Extended Kalman Filter (EKF) for nonlinear systems of equations. Secondly, the EKF as it applies to dual-quaternion pose is discussed. Filtering a state that has constraints that need to be met for it to be valid provide a nontrivial engineering problem. Two version of dual-quaternion EKF are discussed.

The first is an multiplicative EKF as defined by Filipe et al. [30]. The multiplicative form of this EKF seeks to replicate the utility of the multiplicative EKF for the normal quaternion attitude state.

The second type a of dual-quaternion EKF is derived in this dissertation as an extension of the constrained filter approach of Zanetti et al. [31] to the dual-quaternion state. This method seeks to find a normalization term that minimizes the a-posteriori state covariance while also enforcing the necessary constraints.
4.2 Nonlinear Filtering Using an EKF

The original KF [32] seeks to estimate the dynamic state of a discrete linear system subject to stochastic process and measurement noise. It does so by calculating the minimum variance estimate of the dynamic state via measurements at a given time-step. Kalman showed that the filter is optimal for linear systems. An EKF, is a sub-optimal filter for nonlinear systems that linearizes the system model about the current estimate.

Let our state $x \in \mathbb{R}^n$ be a discrete random variable with state estimate $\hat{x}_k = E[x_k]$.

The state has nonlinear dynamics defined to be:

$$
x_{k+1} = f(x_k, w_k), \quad w_k \sim \mathcal{N}(0, Q)
$$

$$
y_k = h(x_k) + v_k, \quad v_k \sim \mathcal{N}(0, R)
$$

where $w_k$ as a Gaussian process noise and $v_k$ is a Gaussian measurement noise term with properties

$$
E[w_k] = 0, \quad E[w_k w_k^T] = Q \in \mathbb{R}^{n \times m}
$$

$$
E[v_k] = 0, \quad E[v_k v_k^T] = R \in \mathbb{R}^{p \times p}
$$

$$
E[w_k v_k^T] = 0
$$

The measurement vector, $y_k \in \mathbb{R}^p$, at time-step $k$ is also described by a nonlinear function, $h$. This system of equations can be linearized using a Taylor series expansion truncated to first order about the state estimate $\hat{x}_k$ to yield:

$$
x_{k+1} = f(\hat{x}_k) + \frac{\partial f}{\partial x_k} \bigg|_{x_k=\hat{x}_k} (x_k - \hat{x}_k) + \frac{\partial f}{\partial w_k} \bigg|_{x_k=\hat{x}_k} w_k
$$

$$
y_k = h(\hat{x}_k) + \frac{\partial h}{\partial w_k} \bigg|_{x_k=\hat{x}_k} (x_k - \hat{x}_k) + \frac{\partial h}{\partial v_k} \bigg|_{x_k=\hat{x}_k} v_k
$$
Introducing the error variable $e_k = x_k - \hat{x}_k$, and the following matrix definitions

\begin{align*}
F_k &= \frac{\partial f}{\partial x_k} \Bigg|_{x_k = \hat{x}_k} \\
G_k &= \frac{\partial f}{\partial w_k} \Bigg|_{x_k = \hat{x}_k} \\
H_k &= \frac{\partial h}{\partial w_k} \Bigg|_{x_k = \hat{x}_k}
\end{align*}

(4.6)\hspace{1cm} (4.7)\hspace{1cm} (4.8)

simplifies the discrete linearized dynamics and measurements to be

\begin{align*}
e_{k+1} &= F_k e_k + G_k w_k \\
y_k &= h(\hat{x}_k) + H_k e_k + v_k
\end{align*}

(4.9)\hspace{1cm} (4.10)

The covariance of the state error at step $k$ is then defined as

$$P_k = E[e_k e_k^T].$$

(4.11)

An estimate of the state and covariance at step $k = 0$ are usually known and given by $\hat{x}_0$ and $P_0$ respectively.

All KF have two distinct steps during execution, a prediction step followed by a correction step when a measurement is made available. The prediction step is simply propagation of the state expectation and covariance given by

\begin{align*}
\hat{x}_{k+1}^- &= f(\hat{x}_k^+) \\
P_{k+1}^- &= F_k P_k^+ F_k^T + G_k Q G_k^T.
\end{align*}

(4.12)\hspace{1cm} (4.13)

The measurement update at a step $k$ is similar to a sequential linear-least squares update
given by,

\[
\dot{x}_k^+ = \dot{x}_k^- + K_k(\tilde{y}_k - h(\dot{x}_k^-)) \tag{4.14}
\]

\[
P_k^+ = (I - K_kH_k)P_k^- \tag{4.15}
\]

where \(K_k\) is the Kalman gain matrix defined as

\[
K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1} \tag{4.16}
\]

and the raw measurement at step \(k\) is given by \(\tilde{y}_k\).

### 4.3 DQ-MEKF

There are various sources that have looked at developing an EKF for pose estimation using dual-quaternions [33–35], although the first that incorporated most of the tricks used in attitude filtering that had been developed over many years [36], was developed by Filipe et al [30] as the Dual-Quaternion Multiplicative Extended Kalman Filter (DQ-MEKF).

#### 4.3.1 Multiplicative DQ-MEKF

The multiplicative Kalman filter for quaternion states [19] takes advantage of the fact that a quaternion representing a small rotation will have a real part, \(q_4\), that is approximately unity. With this simplification, only the vector part of the quaternion error state needs to appear in the filter error covariance. This also avoids having to take care of the quaternion norm constraint.

The dual-quaternion MEKF [30] introduces a second simplifying assumption for the dual part of the dual-quaternion. It assumes that, given a small difference in frame
attitudes, the dual-error is simply half of the relative error vector:

\[
\tilde{\delta}q \approx \begin{bmatrix}
\delta \bar{q}_1 \\
\delta r_2 \\
\delta \bar{q}_3 \\
0
\end{bmatrix}
\] (4.17)

The rotational velocity of a rotating frame can be measured directly by rate-integrating gyroscopes. A common model for this is given by Farhrenkopf [37]

\[
\tilde{\omega} = \omega + \beta + \eta
\] (4.18)

where the gyroscope bias, $\beta$, is modeled as having a random drift $\dot{\beta} = \eta_\beta \sim \mathcal{N}(0, \sigma^2_{\eta_\beta})$. A white noise term $\eta \sim \mathcal{N}(0, \sigma^2_{\eta})$ represents a measurement noise for the gyroscope.

Derivations for the linearized system of equations for the dual-quaternion error are given in Appendix B. The standard prediction and update step as given in [20] are preserved in the dual-quaternion version of this filter.

### 4.3.2 Constrained DQ-EKF

In a similar approach to Zanetti et al. [31] for a unit-quaternion state, a minimum variance update given a nonlinear measurement model with additive noise was defined,

\[
\tilde{y}_k = h(x_k) + \eta_k, \quad \eta_k \sim \mathcal{N}(0, R)
\] (4.19)
with a state vector that is a unit dual-quaternion. A unit dual-quaternion \( \mathbf{x} = [\mathbf{x}_1^T \mathbf{x}_2^T]^T \) must have the following two constraints satisfied,

\[
\mathbf{x}_1^T \mathbf{x}_1 - 1 = 0 \tag{4.20}
\]
\[
\mathbf{x}_1^T \mathbf{x}_2 = 0 \tag{4.21}
\]

These constraints (4.20) and (4.21) will be referred to as the norm and orthogonality constraints respectively.

If the state is appended by other unconstrained states, \( \mathbf{x} = [\mathbf{x}_1^T \mathbf{x}_2^T \mathbf{x}_u^T]^T \), the unconstrained states can be ignored in this derived filter since their update step will be the same as in a filter without state constraints.

The \textit{a-priori} state estimate at time-step \( k \) is \( \hat{\mathbf{x}}_k^- \), and the \textit{a-posteriori} estimate after a measurement update is defined to be for an unbiased estimator

\[
\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k(\hat{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-)) = \hat{\mathbf{x}}_k^- + K_k \epsilon \tag{4.22}
\]

Let the error in the estimates be \( \hat{\mathbf{e}}_k^+ = \mathbf{x}_k - \hat{\mathbf{x}}_k^+ \) and \( \hat{\mathbf{e}}_k^- = \mathbf{x}_k - \hat{\mathbf{x}}_k^- \). The estimate measurement at time-step \( k \) is defined to be through a truncated Taylor series

\[
\mathbf{h}(\hat{\mathbf{x}}_k^-) \approx \mathbf{h}(\mathbf{x}) - H_k \hat{\mathbf{e}}_k^- \tag{4.23}
\]

If you substitute (4.23) into (4.22), you get

\[
\hat{\mathbf{e}}_k^+ = (I - K_k H_k) \hat{\mathbf{e}}_k^- - K_k \eta_k \tag{4.24}
\]

Since a measurement update is performed at a given time-step, the \( k \) subscript will be assumed for all future equations. The \textit{a-posteriori} covariance, \( \hat{\mathbf{P}}^+ \), is calculated by
taking the expectation of $\hat{e}^+ (\hat{e}^+)^T$ to get the Joseph form

$$\hat{P}^+ = \hat{P}^- - KH\hat{P}^- - \hat{P}^- H^T K^T + KWK^T$$

(4.25)

where $W = H\hat{P}^- H^T + R$. Both the a-priori and a-posteriori covariances can be subdivided into the following block forms

$$P = \begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

(4.26)

A gain

$$K = \begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \end{bmatrix}$$

(4.27)

would need to be derived so that the update (4.22) minimizes the trace of the a-posteriori covariance (4.25).

For an unconstrained estimate, the updates for $x_1$ and $x_2$ are given by

$$\begin{bmatrix} \tilde{x}_1^+ \\ \tilde{x}_2^+ \end{bmatrix} = \begin{bmatrix} \tilde{x}_1^- \\ \tilde{x}_2^- \end{bmatrix} + \begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \end{bmatrix} \epsilon$$

(4.28)

where $\tilde{K}_1 = P_{11}^T H^T W^{-1}$ and $\tilde{K}_2 = P_{22}^T H^T W^{-1}$ are the unconstrained Kalman gains for states $x_1$ and $x_2$. These are obtained by solving an optimization problem that minimizes the trace of the a-posteriori covariance matrix. This optimization can be augmented
with our constraints in the following way.

\[
\min_{(K_1, K_2)} J = \text{Tr} \left( \hat{P}^+ \right) = \text{Tr} \left( \hat{P}_{11}^+ \right) + \text{Tr} \left( \hat{P}_{22}^+ \right)
\]

\[
\text{s.t. } (\hat{x}_1^+)^T \hat{x}_1^+ - 1 = \epsilon^T K_1^T K_1 \epsilon + 2(\hat{x}_1^-)^T K_1 \epsilon + (\hat{x}_1^-)^T \hat{x}_1^- - 1 = 0
\]

(4.29)

\[
(\hat{x}_1^+)^T \hat{x}_2^+ = \epsilon^T K_2^T K_2 \epsilon + ((\hat{x}_1^-)^T K_2 + (\hat{x}_2^-)^T K_1) \epsilon + (\hat{x}_1^-)^T \hat{x}_2^- = 0
\]

One could numerically solve this optimization at every time-step or solve algebraically for the gain matrices and Lagrange multipliers. This optimization has the Lagrangian,

\[
\mathcal{L} = \text{Tr} \left( \hat{P}_{11}^+ \right) + \text{Tr} \left( \hat{P}_{22}^+ \right) + \lambda_0 \left( (\hat{x}_1^+)^T \hat{x}_1^+ - 1 \right) + \lambda_1 \left( (\hat{x}_1^+)^T \hat{x}_2^+ \right)
\]

(4.30)

with first order optimality conditions

\[
\frac{\partial \mathcal{L}}{\partial K_1} = -(\hat{P}_1^-)^T H^T + K_1 W + \lambda_0 (K_1 \epsilon \epsilon^T + \hat{x}_1^- \epsilon^T) + \frac{\lambda_1}{2} (K_2 \epsilon \epsilon^T + \hat{x}_2^- \epsilon^T) = 0
\]

(4.31)

\[
\frac{\partial \mathcal{L}}{\partial K_2} = -(\hat{P}_2^-)^T H^T + K_2 W + \frac{\lambda_1}{2} (K_1 \epsilon \epsilon^T + \hat{x}_1^- \epsilon^T) = 0
\]

(4.32)

\[
\frac{\partial \mathcal{L}}{\partial \lambda_0} = \epsilon^T K_1^T K_1 \epsilon + 2(\hat{x}_1^-)^T K_1 \epsilon + (\hat{x}_1^-)^T \hat{x}_1^- - 1 = 0
\]

(4.33)

\[
\frac{\partial \mathcal{L}}{\partial \lambda_1} = \epsilon^T K_2^T K_2 \epsilon + ((\hat{x}_1^-)^T K_2 + (\hat{x}_2^-)^T K_1) \epsilon + (\hat{x}_1^-)^T \hat{x}_2^- = 0
\]

(4.34)

For the following derivations, diacritic markers denoting estimates and the superscript denoting \textit{a-posteriori} information is left out for simplicity.

The optimality conditions (4.31)-(4.34) can be reduced by solving for $K_2$ in (4.32) and substituting this result into (4.31) and solving for $K_1$. These results for $K_1$ and $K_2$ can be substituted into (4.33) and (4.34) to obtain the following system of equations for
the Lagrange multipliers.

\[
\lambda_1^2 - \frac{2 (a + c (1 + \lambda_0 r))}{br} \lambda_1 + \frac{4 (1 + \lambda_0 r)}{r^2} = 0
\]  
(4.35)

\[
\lambda_1^2 - \frac{4b}{cr} \lambda_1 + \frac{4a}{cr^2} - \frac{4 \left(1 + \lambda_0 r - \frac{\lambda_1^2}{4}\right)^2}{cr^2} = 0
\]  
(4.36)

where

\[
a = (\tilde{x}_1^+)^T \tilde{x}_1^+
\]  
(4.37)

\[
b = (\tilde{x}_1^+)^T \tilde{x}_2^+
\]  
(4.38)

\[
c = (\tilde{x}_2^+)^T \tilde{x}_2^+
\]  
(4.39)

\[
r = \epsilon^T W^{-1} \epsilon
\]  
(4.40)

are projections of the unconstrained update onto each other and themselves.

It can be seen from inspection of (4.35) that it is linear in \(\lambda_0\), meaning that it can be solved as a function of \(\lambda_1\):

\[
\lambda_0 = \frac{-br^2 \lambda_1^2 + 2(a + c) r \lambda_1 - 4b}{4br - 2cr^2 \lambda_1}
\]  
(4.41)

One can then substitute (4.41) into (4.36) and perform some algebraic manipulation to obtain

\[
\lambda_1^2 - \frac{4b}{cr} \lambda_1 + \frac{4a}{cr^2} = \frac{r^2 \lambda_1^2 \left(\lambda_1^2 - \frac{4b}{cr} \lambda_1 + \frac{4a}{cr^2}\right)^2}{4c \left(\frac{2b}{cr} - \lambda_1\right)^2} = 0
\]  
(4.42)

This is a sextic equation in \(\lambda_1\), which would, in general, by very difficult to solve, especially since two of the roots bracket a singularity at \(\lambda_1 = 2b/(cr)\). However, it can be broken
down by factoring out the leading quadratic terms as such:

\[
\left( \lambda_1^2 - \frac{4b}{cr} \lambda_1 + \frac{4a}{cr^2} \right) \left( 1 - \frac{r^2 \lambda_1^2 \left( \lambda_1^2 - \frac{4b}{cr} \lambda_1 + \frac{4a}{cr^2} \right)}{4c \left( \frac{2b}{cr} - \lambda_1 \right)^2} \right) = 0 \quad (4.43)
\]

This leading quadratic term has two roots defined to be

\[
\lambda_1 = \frac{2b}{cr} \pm \frac{2}{cr} \sqrt{b^2 - ac} \quad (4.44)
\]

For a dual-quaternion state, \( b \) should be nearly close to zero, and \( a \) will be close to unity. These two roots will be imaginary for most dual-quaternion states besides those within a position that is in a small sphere around the origin. The solution for \( \lambda_1 \) must then be one of the four roots obtained from the remaining quartic equation

\[
1 - \frac{r^2 \lambda_1^2 \left( \lambda_1^2 - \frac{4b}{cr} \lambda_1 + \frac{4a}{cr^2} \right)}{4c \left( \frac{2b}{cr} - \lambda_1 \right)^2} = 0 \quad (4.45)
\]

which can be manipulated to obtain

\[
\lambda_1^4 - \frac{4b}{cr} \lambda_1^3 + \frac{4(a - c^2)}{cr^2} \lambda_1^2 + \frac{16b}{r^3} \lambda_1 - \frac{16b^2}{cr^4} = 0 \quad (4.46)
\]

Analytic solutions to quartic equations can be complex, and since this equation has real coefficients, a numerical root finder is sufficient. Once the multipliers \( \lambda_0 \) and \( \lambda_1 \) are obtained using (4.41) and (4.46), it is only a matter of choosing which \( \lambda_1 \) from the four roots that best minimizes the Lagrangian (4.30).

Using these values for the Lagrange multipliers, the Kalman gains are:

\[
K_1 = \bar{K}_1 - \frac{\lambda_0 - \frac{\lambda_1^2}{1 + 2 \lambda_0 - \frac{\lambda_1^2}{4} \lambda_1^2}}{1 + 2 \lambda_0 - \frac{\lambda_1^2}{4} \lambda_1^2} \left( \bar{K}_1 \epsilon \epsilon^T + \bar{x}_1 \epsilon^T \right) W^{-1} - \frac{\lambda_1}{2 \left( 1 + 2 \lambda_0 - \frac{\lambda_1^2}{4} \lambda_1^2 \right)} \left( \bar{K}_2 \epsilon \epsilon^T + \bar{x}_2 \epsilon^T \right) W^{-1} \quad (4.47)
\]
\[ K_2 = \hat{K}_2 - \frac{\lambda_1}{2(1 + \lambda_0 r - \frac{\lambda_1 r^2}{4})} (\hat{K}_1 \epsilon \epsilon^T + \hat{\chi}_1 \epsilon^T) W^{-1} + \frac{\lambda_1 r}{4(1 + \lambda_0 r - \frac{\lambda_1 r^2}{4})} (\hat{K}_2 \epsilon \epsilon^T + \hat{\chi}_2 \epsilon^T) W^{-1} \] (4.48)

If the values are substituted for the gain matrices into update equation (4.22), the following update is achieved

\[ \hat{x}^+ = G(\epsilon, \lambda_0, \lambda_1) \hat{x}^- + \hat{K}(\epsilon, \lambda_0, \lambda_1) \epsilon \] (4.49)

\[
\begin{bmatrix}
\hat{x}_1^+ \\
\hat{x}_2^+
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{1 + \lambda_0 r - \frac{\lambda_1 r^2}{4}} I & -\frac{\lambda_1 r}{2(1 + \lambda_0 r - \frac{\lambda_1 r^2}{4})} I \\
-\frac{\lambda_1 r}{2(1 + \lambda_0 r - \frac{\lambda_1 r^2}{4})} I & \frac{1 + \lambda_0 r - \frac{\lambda_1 r^2}{4}}{1 + \lambda_0 r - \frac{\lambda_1 r^2}{4}} I
\end{bmatrix}
\begin{bmatrix}
\hat{x}_1^- \\
\hat{x}_2^-
\end{bmatrix} +
\begin{bmatrix}
\frac{1}{1 + \lambda_0 r - \frac{\lambda_1 r^2}{4}} \hat{K}_1 - \frac{\lambda_1 r}{2(1 + \lambda_0 r - \frac{\lambda_1 r^2}{4})} \hat{K}_2 \\
-\frac{\lambda_1 r}{2(1 + \lambda_0 r - \frac{\lambda_1 r^2}{4})} \hat{K}_1 + \frac{1 + \lambda_0 r - \frac{\lambda_1 r^2}{4}}{1 + \lambda_0 r - \frac{\lambda_1 r^2}{4}} \hat{K}_2
\end{bmatrix} \epsilon
\] (4.50)

In a second approach to this constrained update, the optimal quaternion update that minimizes the a-posteriori covariance is simply a brute force normalization as per Zanetti et al. [31].

\[ \hat{x}_1^+ = \frac{\hat{x}_1^- + K_1 \epsilon}{\|\hat{x}_1^- + K_1 \epsilon\|} \] (4.51)

If this result is assumed, then if the following optimization is performed will handle the second dual-quaternion constraint:

\[
\min_{(K_2)} J = \text{Tr}(\hat{P}_+^+) = \text{Tr}(\hat{P}_{22}^+)
\] (4.52)

\[ \text{s.t. } (\hat{x}_1^+)^T \hat{x}_2^+ \epsilon^T = \epsilon^T K_1^T K_2 \epsilon + ((\hat{x}_1^+)^T K_2 + (\hat{x}_2^+)^T K_1) \epsilon + (\hat{x}_1^+)^T \hat{x}_2^- = 0
\]

Solving this optimization problem yields the simple result that the dual state update is just a projection of the \( x_2 \) state to be perpendicular to \( x_1 \)

\[ \hat{x}_2^+ = \left( I - (\hat{x}_1^+)^T \hat{x}_1^+ \right) \left( \hat{x}_2^- + K_2 \epsilon \right) \] (4.53)

The issue with both of these approaches is that the optimal Kalman gains \( K_1 \) and
are functions of the innovations \( \epsilon \), meaning the could not have been pulled out of the expectation integrals in solving for the Jordan form of the \textit{a-posteriori} covariance. This means that the calculated covariance for these constrained updates are not the actual covariance of the state error.

A third approach to try and alleviate this issue involves using Gaussian quadrature for the constrained update. For the following derivations, let the Taylor series expansion of the innovations be forgone, keeping them nonlinear.

If in the Joseph form of our \textit{a-posteriori} covariance (4.25), one could not pull out the gain matrix \( K \), resulting in the following equation

\[
\dot{\hat{P}}^+ = \dot{\hat{P}}^- - E[K \epsilon (\hat{\epsilon}^-)^T] - E[\hat{\epsilon}^- \epsilon^T K^T] + E[K \epsilon \epsilon^T K^T]
\] (4.54)

We can solve for the expectations in (4.54) by using Gaussian quadrature. An unscented transform should be sufficient here. Let

\[
E[K \epsilon (\hat{\epsilon}^-)^T] = \sum_{i=1}^{N} \omega_i K_i \epsilon_i (\hat{\epsilon}^-)^T
\] (4.55)

\[
E[\hat{\epsilon}^- \epsilon^T K^T] = \sum_{i=1}^{N} \omega_i \hat{\epsilon}_i \epsilon_i^T K_i^T
\] (4.56)

\[
E[K \epsilon \epsilon^T K^T] = \sum_{i=1}^{N} \omega_i K_i \epsilon_i \epsilon_i^T K_i^T
\] (4.57)

The minimization problem to reduce the \textit{a-posteriori} trace of the covariance would
then be

\[
\min_{K_i} \quad J = \text{Tr} \left( \hat{P}^- + \sum_{i=1}^{N} \omega_i \left( K_i \epsilon_i \epsilon_i^T K_i^T - K_i \epsilon_i (\hat{e}_i^-)^T - \hat{e}_i^- \epsilon_i^T K_i^T \right) \right)
\]

\[\text{s.t.} \quad (\hat{x}_{i,1}^+)^T \hat{x}_{i,1}^+ - 1 = 0 \]
\[
(\hat{x}_{i,1}^+)^T \hat{x}_{i,2}^+ = 0, \quad \forall i = \{1, ..., N\}
\]

The solution for each of these \(K_i\) follow the same numerical solution as in the first constrained update.

### 4.4 Summary

In this chapter, the two main methods for Kalman filtering a dual-quaternion state using an EKF were given. The multiplicative version of the EKF runs into the issue that the constraints are applied post-hoc to the estimate at each time step. A new constrained CEKF was also developed that used an additive error assumption. The full algorithm for this new filter can be found in Table 4.1. The additive constrained versions of the EKF introduce errors by assuming additive errors on a state that is not a normal vector space. In addition, as will be demonstrated by simulation, the constrained Kalman filters for a dual-quaternion state do not produce an unbiased estimate.
### Table 4.1: CEKF Algorithm

| Model | \( \dot{x} = f(t_k, x_k, u_k), \ u_k \sim \mathcal{N}(\ 0, \ Q \) ) \n
| \( y = h(x_k) + v_k, \ v_k \sim \mathcal{N}(\ 0, \ R \) ) | \n
| Initialize | \( \hat{x}(t_0) = \hat{x}_0 \) \n| \( \hat{P}(t_0) = \hat{P}_0 \) | \n
| Solve for \( \lambda \) | \n| Gain | \( \lambda_1^4 - \frac{4b}{cr} \lambda_1^3 + \frac{4(a - c^2)}{cr^2} \lambda_1^2 + \frac{16b}{r^3} \lambda_1 - \frac{16b^2}{cr^4} = 0 \) \n| \( G = \begin{bmatrix} \frac{1}{1+\lambda_0 r - \frac{\lambda_1 r}{4}} I & -\frac{\lambda_1 r}{2(1+\lambda_0 r - \frac{\lambda_1 r}{4})} I \\ -\frac{\lambda_1 r}{2(1+\lambda_0 r - \frac{\lambda_1 r}{4})} I & \frac{1+\lambda_0 r}{1+\lambda_0 r - \frac{\lambda_1 r}{4}} I \end{bmatrix} \) \n| \( \bar{K} = \begin{bmatrix} \frac{1}{1+\lambda_0 r - \frac{\lambda_1 r}{4}} \bar{K}_1 - \frac{\lambda_1 r}{2(1+\lambda_0 r - \frac{\lambda_1 r}{4})} \bar{K}_2 \\ -\frac{\lambda_1 r}{2(1+\lambda_0 r - \frac{\lambda_1 r}{4})} \bar{K}_1 + \frac{1+\lambda_0 r}{1+\lambda_0 r - \frac{\lambda_1 r}{4}} \bar{K}_2 \end{bmatrix} \) \n
| Update | \( \dot{x}^+ = G(\epsilon, \lambda_0, \lambda_1) \dot{x}^- + \bar{K}(\epsilon, \lambda_0, \lambda_1) \epsilon \) \n| \( \dot{P}^+ = \dot{P}^- - \bar{K} H \dot{P}^- - \dot{P}^- H^T \bar{K}^T + \bar{K} W \bar{K}^T \) | \n
| Propagate | \( \Phi(t_{k+1}, t_k) \dot{P}_k^+ \Phi(t_{k+1}, t_k)^T + Q_k \) | \n
| \n
52 |
Chapter 5  
UKF for Dual-Quaternions

5.1 Introduction

As an alternative to the EKF for filtering nonlinear systems, the UKF is a desirable alternative. Whereas the EKF requires a linearization of the state dynamics about the current estimate, the UKF avoids linearization by way of Gaussian quadrature.

Beginning this chapter, a section describing the practice of using quadrature to calculate moment integrals precedes a definition of the *Unscented Transform* (UT). The UT is the essential building block for the UKF, which allows it to avoid the linearization that is used in the EKF for moment propagation.

This chapter will describe to the reader a standard UKF for vector states, an applied standard UKF in the dual-quaternion space using dual MRP, and lastly, a UKF assuming a Bingham distributed quaternion part of the dual-quaternion state.

The Bingham UKF (BUKF) is a direct contribution of this dissertation, and is an extension of the previous filter for attitude quaternion states by Gilitschenski et. al [11]. To begin the description of the Bingham UKF, a derivation of the necessary sigma pts needed to propagate the dual-quaternion state is presented.
5.2 Propagating Moments using Quadrature

Gaussian quadrature is a mathematical tool for computing integrals numerically using a grid of discrete points. Since the moments of a probability distribution for continuous random variables are defined using integrals over the domain of the random variable, they can be used to reconstruct the central moments of a probability distribution given a finite number of points that fall within that domain. Mathematically this concept can be represented by

\[
E[x^n] = \int_{-\infty}^{\infty} (x - \mu)^n p(x) dx = \sum_{i=0}^{N} \omega_i (\sigma_i - \mu)^n
\]  (5.1)

where \( \sigma_i \) and \( \omega_i \) are the respective sigma points and weights. A set of sigma points and weights are accurate to \( n \)-th order if they correctly match the \( n \)-th moment of the distribution.

This concept can be explained by the following example. Let \( x \sim \mathcal{N}(0, I) \in \mathbb{R}^3 \) be a Gaussian distributed random variable in with zero mean and identity variance. We can choose a symmetric set of six points and weights such that the following conditions are met:

\[
E[1] = \int_{\mathbb{R}^3} p(x) dx = \sum_{i=0}^{5} \omega_i = 1
\]  (5.2)

\[
E[x] = \int_{\mathbb{R}^3} xp(x) dx = \sum_{i=0}^{5} \omega_i \sigma_i = 0
\]  (5.3)

\[
E[xx^T] = \int_{\mathbb{R}^3} xx^T p(x) dx = \sum_{i=0}^{5} \omega_i \sigma_i \sigma_i^T = I
\]  (5.4)

Condition (5.3), along with any other odd order moment, is automatically met by choosing
the symmetric points.

\[ \sigma_{0,1} = [ \pm \delta \ 0 \ 0 ]^T \]
\[ \sigma_{2,3} = [ 0 \ \pm \delta \ 0 ]^T \quad , \quad \omega_i = \omega, \ \forall i \] (5.5)
\[ \sigma_{4,5} = [ 0 \ 0 \ \pm \delta ]^T \]

If we also assume that all weights \( \omega_i \) are equivalent for each sigma point, we can use conditions (5.4) and (5.2) to solve for our weights and spread of the points, \( \delta \).

\[ \sum_{i=0}^{5} \omega = 6 \omega = 1 \Rightarrow \omega = \frac{1}{6} \] (5.6)
\[ 2 \omega \delta^2 = \frac{1}{3} \delta^2 = 1 \Rightarrow \delta = \sqrt{3} \] (5.7)

The weights and sigma points for this normal distribution are then:

\[ \sigma_{0,1} = [ \pm \sqrt{3} \ 0 \ 0 ]^T \]
\[ \sigma_{2,3} = [ 0 \ \pm \sqrt{3} \ 0 ]^T \quad , \quad \omega = \frac{1}{6} \] (5.8)
\[ \sigma_{4,5} = [ 0 \ 0 \ \pm \sqrt{3} ]^T \]

This example has sigma points that are accurate to up to 2nd order. In order to match higher-order moments, more sigma points that fully capture the domain would be required.

These points can linearly transformed to work for any continuous random variable, \( y \sim N( \mu, \Sigma ) \in \mathbb{R}^3 \), using

\[ y = \sqrt{\Sigma} x + \mu \] (5.9)

Every sigma point in the quadrature method for \( x \) can be transformed using Equation (5.9) in order to estimate moments for the random variable \( y \).
The example in this section’s use of a normal distribution can be considered arbitrary as quadrature methods will work for any probability distribution over the domain. However, if a Gaussian distribution is assumed, the previous example can be generalized to any continuous random variable \( x \sim \mathcal{N}(\mu, \Sigma) \in \mathbb{R}^n \) using the Unscented Transform (UT).

5.2.1 The Unscented Transform

The UT by Julier et al. [38] is a generalized way to calculate the moments of a normally-distributed random variable that is accurate to third order. A tuning parameter \( \kappa \) is chosen such that \( n + \kappa = 3 \). The points, \( X_i \), and weights, \( \omega_i \), are then defined to be

\[
X_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \end{bmatrix}^T
\]
\[
\omega_0 = \frac{\kappa}{n + \kappa}
\]
\[
X_i = \pm \sqrt{(n + \kappa)} I_i
\]
\[
\omega_i = \frac{1}{2(n + \kappa)},
\]

where \( I_i \) is the \( i \)th column of the identity matrix \( I \in \mathbb{R}^{n \times n} \). One can see that our previous example is in fact a specific instance of the UT for a continuous random variable with \( n = 3 \), meaning \( \kappa = 0 \). Once the dimension of the random variable is higher than three, the weights for the central sigma point \( \omega_0 \) will be negative. This can be seen as non-ideal, yet it works in practice.

If the distribution is not symmetric, the UT will not be sufficient in reconstructing the full shape of the underlying PDF. Points for the UT are placed along each of the principle axes, but in order to capture higher order moments that represent how asymmetric a PDF might be, points along other axes of symmetry are required. This is the motivation for the Conjugate Unscented Transform (CUT) [12,13,15,16]. A full derivation for the CUT points will not be provided in this dissertation, but the idea of using the conjugate points that are not on the primary axes will be used in the sigma points for the dual-quaternion
state.

5.3 The Unscented Kalman Filter

This section is organized into two parts. Firstly, a generalized UKF for vector states is defined. After the general form of the UKF is defined, a specialization for the UKF to a dual-quaternion state is presented.

5.3.1 General UKF

Unscented Kalman filters were first described by Julier and Uhlmann in Reference [39,40]. An EKF works best when the linearized approximation of the nonlinear dynamics or measurement model is sufficient due to the error being appropriately small. In cases where this is most likely not true, such as is common with initial estimates, the EKF estimate will diverge. The UKF works on the assumption that it is easier to estimate the Gaussian distribution of the estimate error than it is to estimate a nonlinear dynamic system. With this in mind, the state for a UKF is sometimes augmented to include the process and measurement noise in order to better treat them in the nonlinear dynamics and measurement equations.

The UKF brings with it several advantages over the EKF:

1. Error expectations are lower.
2. Non-differentiable functions can be estimated.
3. Derivations of Jacobian matrices are unnecessary.
4. The filter can be more accurate [40].

The greatest disadvantage to the UKF as compared the EKF is that it can, in general, be more computationally costly. This is due to the necessity of propagating all sigma
points through the nonlinear dynamics. However, it cannot be said that this is more costly numerically in terms of computer cycles without knowing the complexity of the Jacobian and state transition matrix calculations that are in the EKF linearizations.

The sigma points for the UKF are UT points described in the previous section (5.10). This necessarily means that the UKF assumes a Gaussian distributed error state uncertainty. The Gaussian distribution is known to be the maximum entropy distribution in $\mathbb{R}^3$, which makes this assumption not detrimental in most applications. Where this assumption might be most detrimental are when the state is constrained to lie on the surface of a geometry that does not extend to infinity, which is the support of the Gaussian PDF.

The sigma points are centered about the current estimate, the mean, with scattering based on the current estimate of the state error covariance. If the state $x$ has length $n$, then the sigma points are computed to be

$$\sigma_k \leftarrow \pm \sqrt{(n + \lambda)}I_k$$  \hfill (5.12)

$$\chi_k^{(0)} = \hat{x}_k$$  \hfill (5.13)

$$\chi_k^{(i)} = \sqrt{P_k} \sigma_k^{(i)} + \hat{x}_k, \text{ for } i = \{1, \ldots, 2n\}$$  \hfill (5.14)

where $\sqrt{P_k}$ is matrix square root of the a-prior error covariance. The matrix square root can be implemented in various ways, with the Cholesky transform

$$P = LL^T$$  \hfill (5.15)

where $L$ is a lower left triangular matrix "square root", however, square root free algorithms do exist in the literature [41].

The generalized UKF introduces a new weighting factor, $\lambda = \alpha^2(n + \kappa) - n$ which uses a tuning constant $\alpha$ to allow control of the sigma point spread about the mean.
Typical values of $\alpha$ are in the range $1 \times 10^{-4} \leq \alpha \leq 1$.

5.3.1.1 Prediction

The prediction step of the UKF starts with each sigma point being propagated forward in time using the nonlinear dynamic equations. At the next time-step, a Gaussian quadrature calculation of the mean and error covariance using $2n+1$ sigma points is performed,

$$\hat{x}_{k+1}^- = \sum_{i=0}^{2n} \pi_{\text{mean}}^{(i)} \chi_{k+1}^{(i)} \quad (5.16)$$

$$\hat{P}_{k+1}^- = \sum_{i=0}^{2n} \pi_{\text{cov}}^{(i)} (\chi_{k+1}^{(i)} - \hat{x}_{k+1}^-)(\chi_{k+1}^{(i)} - \hat{x}_{k+1}^-)^T + Q \quad (5.17)$$

where $Q$ is the measurement noise covariance and the weights for the mean and covariance are given by

$$\pi_{\text{mean}}^{(0)} = \frac{\lambda}{n + \lambda} \quad (5.18)$$

$$\pi_{\text{cov}}^{(0)} = \frac{\lambda}{n + \lambda} + (1 - \alpha^2 + \beta) \quad (5.19)$$

$$\pi_{\text{mean}}^{(i)} = \pi_{\text{cov}}^{(i)} = \frac{1}{2(n + \lambda)}, \quad \forall i \in \{1, 2, ..., 2n + 1\} \quad (5.20)$$

The additional tuning parameter $\beta$ incorporates prior knowledge, if it exists, of the distribution.

5.3.1.2 Update

For the measurement update, the form for an unbiased estimator is used just as in the KF and EKF:

$$\hat{x}_k^+ = \hat{x}_k^- + K_k(\tilde{y}_k - h(\hat{x}_k^-)) \quad (5.21)$$
The Kalman gain is by definition
\[ K_k = P_{xy}^k (P_{yy}^k)^{-1}. \] (5.22)

where the the cross-covariance \( P_{xy}^k \) and measurement covariance \( P_{yy}^k \) are calculated from the sigma points as the a-prior covariance was in the prediction step.

\[
P_{yy}^k = \sum_{i=0}^{2n} \pi^{(i)} \text{cov} ((\tilde{y} - h(\chi_k^{(i)}))(\tilde{y} - h(\chi_k^{(i)}))^T) \quad (5.23)
\]

\[
P_{xy}^k = \sum_{i=0}^{2n} \pi^{(i)} \text{cov} ((\chi_k^{(i)} - \hat{x}_k^-)(\tilde{y} - h(\chi_k^{(i)}))^T \quad (5.24)
\]

The a-posteriori covariance is then
\[
\hat{P}_k^+ = \hat{P}_k^- - K_k P_{yy}^k K_k^T \quad (5.25)
\]

At every step, the sigma points are resampled from the distribution of the error estimate and propagated as described.

A version of the UKF for dual-quaternion pose estimation was derived by Deng et al [42]. This derivation uses the DMRP to represent the dual-quaternion error to avoid the norm constraints during the propagation step. This error parameterization has been included into other more recent versions of the dual-quaternion EKF [43].
5.3.2 Dual-Quaternion UKF

The dual-quaternion UKF has a state $\mathbf{x}$ with the form

$$\mathbf{x} = \begin{bmatrix} \hat{q} \\ x_1 \\ \vdots \\ x_n \end{bmatrix}$$

(5.26)

where the dual-quaternion is treated as a vector in eight dimensions and the state is augmented with any number of additional vector states. These augmented states could be measurement biases, velocities, or process and measurement noise terms.

The issue with this mixed state is then how to generate sigma points for the dual-quaternion part of the state with its constraints and the more normal vector part of the state. This is accomplished by parameterizing the error state of the dual-quaternion using the MRP. The MRP can be treated as any other Euclidean vector space, meaning the standard UT can be used. By representing the error state in the MRP, the singularity at $2\pi$ can be largely avoided assuming small errors.

The error state is then

$$\delta \hat{\mathbf{x}}_k = \mathbf{x} - \hat{\mathbf{x}}_k = \begin{bmatrix} \hat{p} \\ x_1 \\ \vdots \\ x_n \end{bmatrix}$$

(5.27)

Since this error state is a simple vector space, the normal prediction and update equations for the generalized UKF can be used.
5.4 Unscented Filtering for Bingham States

In this section, sets of sigma points are presented for a Bingham distributed random variable and for a state with mixed Bingham and Gaussian distributed random variables. With these sigma points defined, a UKF for Bingham states is described.

5.4.1 Bingham Quadrature

A method for generating sigma points for a Bingham distributed random variable, \( q \sim \mathcal{B}(I, Z) \in SO(3) \), is given by Gilitshenski in [11]. These points take advantage of the fact that both \( q \) and \(-q\) represent the same orientation in order to reduce the amount of sigma points that are used in the calculations. These 14 sigma points are produced for a canonical Bingham distribution where the second moment is given by \( P = \text{diag}[f_1 \ f_2 \ f_3 \ f_4]^T \). A reminder for the reader, a canonical Bingham distribution is a quaternion Bingham distribution where the mode is the identity quaternion.

\[
\begin{align*}
\omega_m : & \quad \sigma^{(0)} = \pm [0 \ 0 \ 0 \ 1]^T \\
\omega_{\alpha_1} : & \quad \sigma^{(\pm\alpha_1)} = \pm [\pm \sin(\alpha_1) \ 0 \ 0 \ \cos(\alpha_1)]^T \quad (5.28) \\
\omega_{\alpha_2} : & \quad \sigma^{(\pm\alpha_2)} = \pm [0 \ \pm \sin(\alpha_2) \ 0 \ \cos(\alpha_2)]^T \\
\omega_{\alpha_3} : & \quad \sigma^{(\pm\alpha_3)} = \pm [0 \ 0 \ \pm \sin(\alpha_3) \ \cos(\alpha_3)]^T
\end{align*}
\]

These sigma points produce the following constraints that can be used to solve for the angles \( \alpha_i \) given our moment equations

\[
2 \left( \omega_m + 2 \sum_{i=1}^{3} \omega_{\alpha_i} \right) = 1 \quad (5.29)
\]

\[
4\omega_{\alpha_i} \sin^2(\alpha_i) = f_i, \quad i \in \{1, 2, 3\} \quad (5.30)
\]

\[
2 \left( \omega_m + 2 \sum_{i=1}^{3} \omega_{\alpha_i} \cos^2(\alpha_i) \right) = f_4 \quad (5.31)
\]
The property that a Bingham distributed random variable has a covariance with trace equal to one, \(\text{tr}(P) = 1\), means that conditions (5.29) and (5.31) are redundant. Solving Equation (5.30) for \(\alpha_i\) yields
\[
\alpha_i = \sin^{-1}\left(\sqrt{\frac{f_i}{4\omega_{\alpha_i}}}\right), \quad i = \{1, 2, 3\}
\] (5.32)

For \(\alpha_i\) to be real, \(4\omega_{\alpha_i}\) must be greater than or equal to \(f_i\).

If we introduce a tuning parameter \(0 \leq \kappa \leq 1\), we can design the weights \(\omega_m\) and \(\omega_{\alpha_i}\) to meet condition (5.29) and have \(\alpha_i\) be real.

\[
\omega_m = \frac{\kappa}{2} f_4
\] (5.33)

\[
\omega_{\alpha_i} = \frac{1}{4} \left( f_i + (1 - \kappa) \frac{f_4}{3} \right)
\] (5.34)

These sigma points were demonstrated by Gilitschenski to fully capture the moments of a Bingham distributed quaternion state up to second order.

### 5.4.2 A Mixed Quaternion State

A set of points that can be used for a mixed quaternion state, \(z = [q^T, \, x^T]^T\), where \(q \in \mathbb{H}\) and \(x \in \mathbb{R}^n\) was developed by Darling [2, 3] as an extension of the points given by Gilitschenski.
These points and weights are

\[
\sigma^{(0)} = \pm \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}^T
\]

\[
\sigma^{(\alpha_1)} = \pm \begin{bmatrix} \pm \sin(\alpha_1) & 0 & 0 & \cos(\alpha_1) & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}^T
\]

\[
\sigma^{(\alpha_2)} = \pm \begin{bmatrix} 0 & \pm \sin(\alpha_2) & 0 & \cos(\alpha_2) & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}^T
\]

\[
\sigma^{(\alpha_3)} = \pm \begin{bmatrix} 0 & 0 & \pm \sin(\alpha_3) & \cos(\alpha_3) & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}^T
\]

\[
\sigma^{(\delta_1)} = \pm \begin{bmatrix} 0 & 0 & 0 & 1 & \pm \delta & 0 & \cdots & 0 & 0 \end{bmatrix}^T
\]

\[
\sigma^{(\delta_\ell)} = \pm \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & \pm \delta \end{bmatrix}^T
\]

The conditions to solve for \(\delta\) and \(\alpha_i\) are

\[
2 \left( \omega_m + 2n\omega_\delta + 2 \sum_{i=1}^{3} \omega_{\alpha_i} \right) = 1
\]

\[
4\omega_{\alpha_i} \sin^2(\alpha_i) = f_i, \quad i \in \{1, 2, 3\}
\]

\[
4\omega_\delta \delta^2 = 1
\]

\[
2 \left( \omega_m + 2 \sum_{i=1}^{3} \omega_{\alpha_i} \cos^2(\alpha_i) \right) = f_4
\]

which can be solved to yield

\[
\delta = \sqrt{\frac{1}{4\omega_\delta}}, \quad \alpha_i = \sin^{-1} \left( \sqrt{\frac{f_i}{4\omega_{\alpha_i}}} \right) \quad i = \{1, 2, 3\}
\]

The choice of weights is accomplished by introduction of two tuning parameters, \(\lambda\)
and $\kappa$, with $\lambda + \kappa < 1$. The weights for these points are defined to be

$$\omega_m = \frac{\lambda}{2} f_4$$  \hspace{1cm} (5.45)  \\
$$\omega_\delta = \frac{\kappa}{4n} f_4$$  \hspace{1cm} (5.46)  \\
$$\omega_{\alpha_i} = \frac{1}{4} \left( f_i + (1 - \lambda - \kappa) \frac{f_4}{3} \right)$$  \hspace{1cm} (5.47)

Like the points of Gilitschenski, the points of Darling are accurate up to second order by matching the covariances. However, unlike points from the UT, they are not capable of capturing the full shape of the PDF for this mixed state. The cross-covariances between the Bingham and Euclidean states must be approximated numerically. Darling provides a maximum-likelihood estimation method for approximating this PDF in Reference [2]. Also, the ability to transform these points with $q \sim \mathcal{B}(0, Z)$ and $x \sim \mathcal{N}(0, I)$ to a state with $p \sim \mathcal{B}(M, Z)$ and $y \sim \mathcal{N}(\mu, \Sigma)$ is not as straightforward as in Equation (5.9).

### 5.4.3 The UKF for a Bingham state

Gilitschenski proposed the UKF for a Bingham state in Reference [11]. This UKF uses a lot of the earlier work by Glover et al. [17] for performing the prediction and update steps.

First it is assumed that the discrete nonlinear dynamics in quaternion space are to be be

$$q_{k+1} = f(q_k)u_k$$  \hspace{1cm} (5.48)  \\
y = q_kv_k  \hspace{1cm} (5.49)$$

where $f$ is some nonlinear process and the process noise $u_k$ is a Bingham distributed quaternion $u_k \sim \mathcal{B}(I, Q)$. The measurement is the quaternion state with some
measurement noise quaternion \( v_k \sim B(I, R) \), which rotates the true state by some error quaternion.

Let the second moment of the quaternion state for a Bingham Distribution be as previously defined:

\[
P = E[qq^T] = M \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} M^T \tag{5.50}
\]

where \( M \) is the a 4x4 rotation matrix representing the mode of the distribution. For unit quaternions, \( M = [q_m \otimes \] where \( \pm q_m \) is the mode of the Bingham distribution.

### 5.4.3.1 Prediction

Sigma points for the state are constructed using either the Gilitschenski points for a purely quaternion state or with the Darling points for a mixed quaternion and vector state. Each of these sigma points is then propagated forward using the nonlinear dynamics \( f \). For simplicity of the following definition and to avoid the calculation of the cross-covariances of Darling’s method, the UKF of Gilitschenski will be described.

Once all sigma points are propagated forward, the covariance of the state, \( q \), can be calculated using the sigma points at time \( k + 1 \),

\[
\hat{P}_{k+1} = E[q_{k+1}q_{k+1}^T] = 2(\omega_0\sigma_0^T\sigma_0^T + 2 \sum_{i=1}^{3} \omega_{\alpha_i}\sigma_{\alpha_i}\sigma_{\alpha_i}^T) \otimes Q \tag{5.51}
\]

The composition, \( \otimes \), of two Bingham covariance matrices is defined to be, given two
Bingham covariances of the following structure:

\[ P = \begin{bmatrix} \tilde{P} & \mathbf{p} \\ \mathbf{p}^T & p_4 \end{bmatrix}, \quad Q = \begin{bmatrix} \tilde{Q} & \mathbf{q} \\ \mathbf{q}^T & q_4 \end{bmatrix} \]  

(5.52)

as

\[ S = P \otimes Q = \begin{bmatrix} \tilde{S} & \mathbf{s} \\ \mathbf{s}^T & s_4 \end{bmatrix} \]  

(5.53)

\[ \tilde{S} = \mathbf{p}\mathbf{q}^T + \mathbf{q}\mathbf{p}^T + \tilde{P}[\mathbf{q} \times] - \tilde{Q}[\mathbf{p} \times] \]

\[ + (\tilde{P} + [\mathbf{p} \times] + (1 - 2\text{Tr}(\tilde{P}))I)\tilde{Q} \]

\[ + (\tilde{Q} - [\mathbf{q} \times] + (1 - 2\text{Tr}(\tilde{Q}))I)\tilde{P} \]

\[ + (\text{Tr}(\tilde{P})\text{Tr}(\tilde{Q}) - \text{Tr}(\tilde{P}\tilde{Q}))I \]  

(5.54)

\[ \mathbf{s} = p_4\mathbf{q} + q_4\mathbf{p} + [\mathbf{p} \times] \mathbf{q} - \tilde{Q}\mathbf{p} - \tilde{P}\mathbf{q} - [\tilde{Q}\tilde{P} - \tilde{P}\tilde{Q} \times]^{-1} \]

(5.55)

\[ s_4 = \text{Tr}(\tilde{P}\tilde{Q}) + p_4 q_4 - 2\mathbf{p}^T\mathbf{q} \]  

(5.56)

The form of this covariance was first given by Glover et. al [17].

### 5.4.3.2 Update

The first step in the measurement update, given a Bingham covariance matrix, \( P_k \), is to calculate the concentration parameter, \( Z^- \), of the underlying Bingham distribution. The first step to accomplish this is to perform a simple diagonalization of the covariance by calculating the eigenvalues and eigenvectors.

\[ P = M\text{diag}[f_1 \ f_2 \ f_3 \ f_4]M^T \]  

(5.57)
Once the diagonal elements (eigenvalues) of this covariance can be found, a variety of methods are shown in the literature to calculate what the \textit{a-priori} concentration matrix, $Z^-$, would be [44]. The method that is most prevalent in the recent literature is to use a fast saddlepoint method developed by Kume and Wood [27].

Given a measurement $\hat{y}$ at time $k$, the measurement update can be derived from examination of the product of two Bingham distributed variables. Glover et. al showed that the product of two random Bingham variables is another Bingham variable with PDF given by

$$f(y) = f(q)f(v) = K_q \exp[q^T M_q M_q^T q] K_v \exp[v^T M_v R M_v^T v]$$

$$= K_{qv} \exp[q^T M_q Z_q M_q^T q + v^T M_v R M_v^T v] = K_{qv} \exp[q^T C q]$$

(5.59)

We can use the relationship that $v = q^* \tilde{y}$ to simplify the exponent of Equation (5.59) into a single matrix, $C$, which can be diagonalized to get

$$C = M_{q_q^+} \text{diag}Z^+ M_{q_q^+}^T$$

(5.60)

which has an \textit{a-posteriori} Bingham distribution $q \sim B(M_{q_q^+}, Z^+)$.

### 5.5 A Bingham UKF for Dual-Quaternions

This section presents the sigma points derived for this dissertation in order to propagate a dual-quaternion state while respecting the two necessary constraints. These points are then used inside a Bingham UKF that is similar in design to the quaternion UKF of Gilitschenski.
5.5.1 Dual-Quaternion Sigma Points

The points of Darling for a mixed state with a Bingham distributed quaternion with Gaussian parts cannot properly capture the PDF of a dual-quaternion for two reasons: the points do not conform to the constraints of a dual-quaternion where the dual part must be tangent to the real part of the quaternion and correlation between the real and dual-parts is not matched.

To capture this correlation between the real and dual parts of the dual-quaternion, a set of points that comply with all constraints and that also are not completely tangent to each other are developed. These points are similar to the conjugate Unscented points (CUT) points of Adurthi [12]. In total, there are 98 points, but the symmetry of the dual-quaternion space means that only 49 sigma points need to be propagated.

The first set of the sigma points are the pure quaternion points of Gilitschenski.

\[
\omega_m : \quad \boldsymbol{\sigma}^{(0)} = \pm\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T
\]
\[
\omega_{\alpha_1} : \quad \boldsymbol{\sigma}^{(\pm\alpha_1)} = \pm\begin{bmatrix} \pm \sin(\alpha_1) & 0 & 0 & \cos(\alpha_1) & 0 & 0 & 0 & 0 \end{bmatrix}^T
\]
\[
\omega_{\alpha_2} : \quad \boldsymbol{\sigma}^{(\pm\alpha_2)} = \pm\begin{bmatrix} 0 & \pm \sin(\alpha_2) & 0 & \cos(\alpha_2) & 0 & 0 & 0 & 0 \end{bmatrix}^T
\]
\[
\omega_{\alpha_3} : \quad \boldsymbol{\sigma}^{(\pm\alpha_3)} = \pm\begin{bmatrix} 0 & 0 & \pm \sin(\alpha_3) & \cos(\alpha_3) & 0 & 0 & 0 & 0 \end{bmatrix}^T
\]

These fourteen points are then joined by the following twelve points for a pure displacement that are seen in the Darling points

\[
\omega_{\delta_1} : \quad \boldsymbol{\sigma}^{(\pm\delta_1)} = \pm\begin{bmatrix} 0 & 0 & 0 & 1 & \pm \delta & 0 & 0 & 0 \end{bmatrix}^T
\]
\[
\omega_{\delta_2} : \quad \boldsymbol{\sigma}^{(\pm\delta_2)} = \pm\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & \pm \delta & 0 & 0 \end{bmatrix}^T
\]
\[
\omega_{\delta_3} : \quad \boldsymbol{\sigma}^{(\pm\delta_3)} = \pm\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & \pm \delta & 0 \end{bmatrix}^T
\]

These 26 points would not be able to fully capture the cross-covariances between the real and dual parts of dual-quaternion state. The following points are defined to be
dual-quaternion twists composed of a single axis rotation as in the Gilitschenski points and a single axis displacement defined in $\mathbb{R}^3$ with a normally distributed random variable $z \sim \mathcal{N}(0, I)$.

The first set of 24 conjugate points are a composition of the Gilitschenski points with a displacement along that same axis of rotation.

\[
\begin{align*}
\omega_{\alpha_{i,i}} : & \quad \sigma^{(\pm i,i)} = \pm \begin{bmatrix}
\pm \sin(\alpha_1) & 0 & 0 & \frac{\pm \delta \cos(\alpha_1)}{2} & 0 & 0 & \frac{\pm \delta \sin(\alpha_1)}{2}
\end{bmatrix}^T \\
\omega_{\alpha_{j,j}} : & \quad \sigma^{(\pm j,j)} = \pm \begin{bmatrix}
0 & \pm \sin(\alpha_2) & 0 & \frac{\pm \delta \cos(\alpha_2)}{2} & 0 & \frac{\pm \delta \sin(\alpha_2)}{2}
\end{bmatrix}^T \\
\omega_{\alpha_{k,k}} : & \quad \sigma^{(\pm k,k)} = \pm \begin{bmatrix}
0 & 0 & \pm \sin(\alpha_3) & \cos(\alpha_3) & 0 & \frac{\pm \delta \cos(\alpha_3)}{2} & 0 & \frac{\pm \delta \sin(\alpha_3)}{2}
\end{bmatrix}^T
\end{align*}
\]

(5.63)

The next 48 points follow the same logic as the previous 24 conjugate points.

\[
\begin{align*}
\omega_{\alpha_{i,j}} : & \quad \sigma^{(\pm i,j)} = \pm \begin{bmatrix}
\pm \sin(\alpha_1) & 0 & 0 & \frac{\pm \delta \cos(\alpha_1)}{2} & 0 & 0 & \frac{\pm \delta \sin(\alpha_1)}{2}
\end{bmatrix}^T \\
\omega_{\alpha_{i,k}} : & \quad \sigma^{(\pm i,k)} = \pm \begin{bmatrix}
\pm \sin(\alpha_1) & 0 & 0 & \frac{\pm \delta \cos(\alpha_1)}{2} & 0 & \frac{\pm \delta \sin(\alpha_1)}{2}
\end{bmatrix}^T \\
\omega_{\alpha_{j,i}} : & \quad \sigma^{(\pm j,i)} = \pm \begin{bmatrix}
0 & \pm \sin(\alpha_2) & 0 & \frac{\pm \delta \cos(\alpha_2)}{2} & 0 & \frac{\pm \delta \sin(\alpha_2)}{2}
\end{bmatrix}^T \\
\omega_{\alpha_{j,k}} : & \quad \sigma^{(\pm j,k)} = \pm \begin{bmatrix}
0 & \pm \sin(\alpha_2) & 0 & \frac{\pm \delta \cos(\alpha_2)}{2} & 0 & \frac{\pm \delta \sin(\alpha_2)}{2}
\end{bmatrix}^T \\
\omega_{\alpha_{k,i}} : & \quad \sigma^{(\pm k,i)} = \pm \begin{bmatrix}
0 & 0 & \pm \sin(\alpha_3) & \cos(\alpha_3) & 0 & \frac{\pm \delta \cos(\alpha_3)}{2} & \frac{\pm \delta \sin(\alpha_3)}{2}
\end{bmatrix}^T \\
\omega_{\alpha_{k,j}} : & \quad \sigma^{(\pm k,j)} = \pm \begin{bmatrix}
0 & 0 & \pm \sin(\alpha_3) & \cos(\alpha_3) & 0 & \frac{\pm \delta \cos(\alpha_3)}{2} & \frac{\pm \delta \sin(\alpha_3)}{2}
\end{bmatrix}^T
\end{align*}
\]

(5.64)

Like the points by Gilitschenski [11], these points have a symmetry. This comes from the fact that orientations $q$ and $-q$ are equivalent, which means that there are 4 possible dual-quaternion combinations that are equivalent. The first moment of a Bingham distributed random variable is zero, so it is sufficient to match the zeroth and second moments.

If we assume the standard parameterization for a dual-quaternion as in Equation (2.50), we can see that the dual part of the dual-quaternion will have statistics of a normally distributed random variable that is rotated by a Bingham distributed quaternion.
The covariance for such a distribution is given by:

\[ \Sigma = \text{diag} \left[ f_1 \quad f_2 \quad f_3 \quad f_4 \quad \frac{1 - f_1}{4} \quad \frac{1 - f_2}{4} \quad \frac{1 - f_3}{4} \quad \frac{1 - f_4}{4} \right] \]  

(5.65)

The conditions to solve for \( \delta_i \) and \( \alpha_i \) are

\[ 2 \left( \omega_m + 6 \omega_\delta + 14 \sum_{i=1}^{3} \omega_{\alpha_i} \right) = 1 \]  

(5.66)

\[ 28 \omega_{\alpha_i} \sin^2(\alpha_i) = f_i, \quad i = \{1, 2, 3\} \]  

(5.67)

\[ \left( \omega_\delta + 2 \sum_{i=1}^{3} \omega_{\alpha_i} - \frac{f_i}{14} \right) \delta^2 = \frac{1 - f_i}{4}, \quad i = \{1, 2, 3\} \]  

(5.68)

\[ 2 \left( \omega_m + 6 \omega_\delta + 14 \sum_{i=1}^{3} \omega_{\alpha_i} \cos^2(\alpha_i) \right) = f_4 \]  

(5.69)

\[ 2 \sum_{i=1}^{3} \omega_{\alpha_i} \delta^2 \sin^2(\alpha_i) = \frac{1 - f_4}{4} \]  

(5.70)

Equation (5.67) can be first solved for \( \sin^2(\alpha_i) \) to get

\[ \sin^2(\alpha_i) = \frac{f_i}{28 \omega_{\alpha_i}} \]  

(5.71)

which that be substituted into (5.70) to produce the equation

\[ \frac{2}{7} (f_1 + f_2 + f_3) \delta^2 = 1 - f_4 \]  

(5.72)

Simple manipulation of (5.71) and (5.72) provides the relationships

\[ \delta = \sqrt{\frac{7}{2}}, \quad \alpha_i = \sin^{-1} \left( \sqrt{\frac{f_i}{28 \omega_{\alpha_i}}} \right) \quad i = \{1, 2, 3\} \]  

(5.73)

Like with the pure Bingham distribution sigma points, we can introduce the parameter,
$\kappa \in [0, 1)$, to define our weights to be

$$
\omega_m = \omega_\delta = \frac{\kappa}{14} f_4
$$

(5.74)

$$
\omega_{\alpha_i} = \frac{1}{28} \left( f_i + (1 - \kappa) \frac{f_4}{3} \right)
$$

(5.75)

### 5.5.1.1 Dual-Quaternion Covariance Analysis

These sigma points can be demonstrated to fully capture the second moment of a dual-quaternion state, $\hat{q} = q + \epsilon d$ whose covariance is given by

$$
P = \mathbb{E}[\hat{q}\hat{q}^T] = \begin{bmatrix}
P_B & \mathbb{E}[qd^T] \\
\mathbb{E}[dq^T] & \mathbb{E}[dd^T]
\end{bmatrix}
$$

(5.76)

where $P_B$ is a Bingham covariance of the real part of the dual-quaternion, $q$ with the structure

$$
P_B = \begin{bmatrix}
\Lambda & \lambda \\
\lambda^T & \lambda_4
\end{bmatrix}
$$

(5.77)

Given a displacement vector $r \sim \mathcal{N}(\mu, \Sigma)$ to describe the dual-part of the state as $d = \frac{1}{2} q r$, the cross covariance of the dual-quaternion covariance can be shown to be

$$
\mathbb{E}[qd^T] = \frac{1}{2} \begin{bmatrix}
\lambda \mu^T + \Lambda [\mu \times] & -\Lambda \mu \\
\mu^T (\lambda_4 I - [\lambda \times]) & -\lambda^T \mu
\end{bmatrix}
$$

(5.78)

A useful observation from this cross covariance is, the mode (and mean) for the distribution of the displacement vector $r$ can be found from the first three entries of the last column by taking an inverse of $\Lambda$.

The purely dual part of the dual-quaternion covariance is defined using the covariance
composition used in the previous section for the process noise in the quaternion UKF.

\[
E[dd^T] = \frac{1}{4} \begin{bmatrix} \Delta & \delta \\ \delta & \delta_4 \end{bmatrix}
\]  
(5.79)

\[
\Delta = -(\Sigma + M)[\lambda \times] + (\Lambda + [\lambda \times] + (1 - 2 \text{Tr}(\Lambda))I)(\Sigma + M)
+ (\Sigma + M - (\text{Tr}(\Sigma) + \text{Tr}(M))I)\Lambda
+ ((\text{Tr}(\Sigma) + \text{Tr}(M))\text{Tr}(\Lambda) - (\text{Tr}(\Sigma\Lambda) + \text{Tr}(M\Lambda)))I
\]  
(5.80)

\[
\delta = -(\Sigma + M)\lambda - [(\Sigma + M)\Lambda - \Lambda(\Sigma + M) \times]^{-1}
\]  
(5.81)

\[
\delta_4 = \text{Tr}(\Lambda \Sigma) + \text{Tr}(\Lambda M)
\]  
(5.82)

where the matrix \( M = \mu \mu^T \) is not to be confused with \( M \) from the Bingham distribution and the vector \( \delta \) is not to be confused with \( \delta = \sqrt{7/2} \) as given in the sigma point definitions.

Mapping the sigma points for the canonical dual-quaternion distribution with \( q \sim \mathcal{B}(I, Z) \) and \( z \sim \mathcal{N}(0, I) \) to one where \( q \sim \mathcal{B}(M, Z) \) and \( r \sim \mathcal{N}(\mu, \Sigma) \) is not so straight-forward. If \( \Sigma = I \), then these points can be easily transformed by multiplying them by the dual-quaternion mode given by \( \hat{q}_m = q_m + \frac{1}{2} e q_m \mu \). This is unlikely in practice, since most normal distributions will not have a covariance equal to identity.

We can calculate the necessary covariance in \( r \) given a dual-quaternion covariance \( P \), by first defining the nominal covariance, \( P_N \), where the displacement covariance is identity, and subtracting that from \( P \). If we look at only the vector part of the dual covariance, if the underlying Gaussian distribution has an identity covariance, it would
make the upper left corner of the dual covariance equal to

\[
\Delta_N = (\Lambda + [\lambda \times] + (1 - 2\text{Tr}(\Lambda))I)M + M(\Lambda - [\lambda \times])
\]

\[
- (1 + \text{Tr}(M))\Lambda + (1 + \text{Tr}(M)\text{Tr}(\Lambda) - \text{Tr}(M\Lambda))I
\]

(5.83)

The vector part of the difference is given by

\[
E = \Delta - \Delta_N = -\Sigma [\lambda \times] + (\Lambda + [\lambda \times] + (1 - 2\text{Tr}(\Lambda))I)\Sigma + (\Sigma + (1 - \text{Tr}(\Sigma))I)\Lambda
\]

\[
+ (\text{Tr}(\Sigma)\text{Tr}(\Lambda) - \text{Tr}(\Sigma\Lambda) - 1)I
\]

(5.84)

The equation

\[
E + \Sigma [\lambda \times] - (\Lambda + [\lambda \times] + (1 - 2\text{Tr}(\Lambda))I)\Sigma - (\Sigma + (1 - \text{Tr}(\Sigma))I)\Lambda
\]

\[
- (\text{Tr}(\Sigma)\text{Tr}(\Lambda) - \text{Tr}(\Sigma\Lambda) - 1)I = 0
\]

(5.85)

can be solved for \(\Sigma\) numerically. With \(\Sigma\), the sigma points can be transformed to the new distribution by the dual-quaternion mode

\[
\hat{q} = q_m + \frac{1}{2}e_{q_m}(\sqrt{\Sigma}z + \mu)
\]

(5.86)

The Gaussian covariance of position can also be extracted from the dual-quaternion sigma points for both the rotating body frame and the reference frame using the following formulae:

\[
\Sigma_A = 2\sum_{i=1}^{N}\omega_i(r(\sigma^i) - r(\sigma^0))(r(\sigma^i) - r(\sigma^0))^T
\]

(5.87)
\[ \Sigma_B = 2 \sum_{i=1}^{N} \omega_i (d(\sigma^i) - d(\sigma^0))(d(\sigma^i) - d(\sigma^0))^T \]  \hspace{1cm} (5.88)

where

\[ d(\tilde{q}) = 2q_dq_r^* \]  \hspace{1cm} (5.89)

and

\[ r(\tilde{q}) = 2q_r^*q_d \]  \hspace{1cm} (5.90)

The following numerical example will demonstrate the ability of the dual-quaternion sigma points to capture the distribution of a dual-quaternion as it changes over time. Let two frames \( A \) and \( B \) start at time \( t_0 = 0 \) with the location of frame \( B \) with respect to frame \( A \) be defined by the Gaussian variable \( d \sim \mathcal{N}(0, \Sigma_B) \) where \( d \) is the position vector of frame \( B \) relative to frame \( A \) in \( B \) frame coordinates and \( \Sigma_B = 0.2I_{3 \times 3} \). Let the frame \( B \) be oriented with respect to frame \( A \) by the Bingham distributed quaternion \( q \sim \mathcal{B}(Z, M) \) with \( Z = \text{diag}(-100, -30, -30, 0) \), \( M = [q_m \otimes] \), \( q_m(t_0) = 0.5 + 0.5i + 0.5j + 0.5k \).

Let the frames have constant velocities with respect to each other in the rotating frame \( B \) with relative angular velocity \( \omega = [1 1 1]^T \) and Cartesian velocity \( v = [1 0 0]^T \). Propagating the sigma points forward in time for ten seconds, \( t_f = 10 \), and converting each dual-quaternion point to positions in the \( A \) and \( B \) frames, the evolution of the distributions over time can be observed.

Figure 5.1 shows how the Gaussian distribution of the position vector in the \( B \) frame is preserved during the transformation. However, relative motion states are most commonly defined in the non-rotating or 'pseudo-inertial' frames, which do rotate, but are treated as non-rotating for model simplification. In Figure 5.2 the non-Gaussian distribution of the sigma points in frame \( A \) after propagation can be observed.

Analysis of the tuning parameter \( \kappa \) shows that it effects the spread of the sigma points in the inertial frame after propagation. The sigma points in frame \( A \) can be seen for
κ = 0.45, κ = 0.1 and κ = 0.9 in Figures 5.2 and 5.3 respectively. A κ value in the range of κ ∈ [0, 0.45] can be shown through Monte Carlo analysis to be effective at accurately capturing the distribution for position in the non-rotating reference frame.

![Position Sigma Points (B Frame), κ = 0.45](image1)

(a) X-Y projection

![Position Sigma Points (B Frame), κ = 0.45](image2)

(b) Isometric view

Figure 5.1: Position sigma points in the \( \mathcal{B} \) frame at times \( t_0 \) and \( t_f \).

![Position Sigma Points (A Frame), κ = 0.45](image3)

(a) X-Y projection

![Position Sigma Points (A Frame), κ = 0.45](image4)

(b) Isometric view

Figure 5.2: Position sigma points in the \( \mathcal{A} \) frame at times \( t_0 \) and \( t_f \).

A small Monte Carlo simulation using one thousand points was run to demonstrate how the proposed dual-quaternion sigma points can accurately capture the dual-quaternion statistics after propagation. Figure 5.4 shows the results of this Monte Carlo. Shown in
Figure 5.3: Position sigma points in the $\mathcal{A}$ frame with varying tuning parameter $\kappa$.

This figure are each individual point sampled from the dual-quaternion distribution at times $t_0 = 0$ and $t_f = 10s$ along with the $3\sigma$ bounds of a Gaussian distributed variable with the same mean and covariance of the actual non-Gaussian distribution. It can be seen that many points fall outside of these bounds as the distribution of points curves in an arc about the original position at the initial time. Distributions such as these have been documented frequently in the literature as arising from mixed states that contain both Gaussian distributed and variables from circular distribution such as the Von Mises or Bingham distributions. Another interesting observation is that the sigma points in this test case with $\kappa = 0.45$ have points lying outside of the bounds for a Gaussian distribution with the same first and second moment as the one depicted in these results. The sample covariance for the Monte Carlo points also strongly agrees with one calculated using Equation (5.87).

### 5.5.2 A Bingham Dual-Quaternion UKF

Using much the same procedure as the UKF of Gilitschenski, the dual-quaternion form of this UKF handles the real part of the dual-quaternion in the same way.
5.5.2.1 Prediction

The sigma points of the dual-quaternion are each propagated using the dual-quaternion dynamics

\[ \hat{q}_{k+1} = f(\hat{q}_k)\hat{u}_k \] (5.91)

\[ \hat{y} = \hat{q}_k\hat{v}_k \] (5.92)

The second moment of the dual-quaternion is then constructed via the sigma points to be

\[ \hat{P}_{k+1} = E[\hat{q}_{k+1}\hat{q}_{k+1}^T] = 2(\sum_{i=1}^{N} \omega_i \sigma_i^i \sigma_{k+1}^i)^T \otimes Q \] (5.93)

5.5.2.2 Update

The quaternion part of the state is then updated using the same method of decomposing the covariance by its eigenvalues and then fusing the dual-quaternion measurement. Measurements for the position are then done using the standard UKF update equations.
using the transformations from the dual-quaternion sigma points in Equations (5.90) or (5.89)

5.6 Summary

Two UKF for estimation of a dual-quaternion state were described in this chapter. The first filter by Deng et al. [26] uses a standard UKF where normally distributed Gaussian uncertainty in the state is assumed by definition. It does this by modeling the error as a stereographic projection of the error onto the real numbers in three dimensions by the dual MRP. The second filter, algorithm in Table 5.1, treats the quaternion part of the dual-quaternion state as a unit vector in $\mathcal{SO}(3)$. The dual part of the dual-quaternion is then a fusion of the Bingham distributed quaternion state and a normally distributed distance in $\mathbb{R}^3$. This was accomplished by first defining the covariance of a dual-quaternion using these assumptions and deriving a method to generate sigma points given a dual-quaternion covariance. Through simulation, these sigma points were then shown to better capture the non-Gaussian distribution of the position as measured in the non-rotating reference frame.
Table 5.1: BUKF Algorithm

| Model | \[ \hat{q}_{k+1} = f(\hat{q}_k)\hat{u}_k \]  
|       | \[ y_q = q_kv_k, v_k \sim B(I, A_R) \]  
|       | \[ y_r = h(\hat{δ}q) + v_k, v_k \sim N(0, R) \]  
|       | \[ \hat{u}_k = u_{k,r} + \epsilon \frac{1}{2} u_{k,r} \delta_{k,u}, u_{k,r} \sim B(I, Z_Q), \delta_u \sim N(0, Q) \]  
| Initialize | \[ \hat{q}(t_0) = \hat{q}_0 \]  
|           | \[ \hat{P}_b(t_0) - \hat{P}_{b0}\hat{P}_r(t_0) = \hat{P}_{r0} \]  
| Gain | \[ K_k = \frac{P^{xy}(P^{yy})^{-1}}{P^{yy}} \]  
| Update | \[ C = M_q Z_q M_q^T + [y_q \otimes] M_v R M_v^T [y_q \otimes]^T \]  
|         | \[ = M_{q_{in}} \text{diag} Z^+ M_{q_{in}}^T \]  
|         | \[ \hat{\delta}_k^+ = \hat{\delta}_k^- + K \epsilon \]  
|         | \[ \hat{P}_{k,r}^- = 2 \sum_{i=1}^{N} \omega_i (r(\sigma_i) - r(\sigma^0))(r(\sigma_i) - r(\sigma^0))^T \]  
|         | \[ \hat{P}_{k,r}^+ = \hat{P}_{k,r}^- - K_k P^{yy} K_k^T \]  
| Propagate | \[ \hat{P}_{k+1}^- = E[q_{k+1}q_{k+1}^T] = 2(\sum_{i=1}^{N} \omega_i \sigma_i \sigma_i^T) \otimes Q \]  

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Chapter 6  
Simulation Models

6.1 Introduction

In this chapter, an example problem involving the relative motion of two satellites in a low Earth orbit (LEO) is defined. This example problem is used to showcase the performance of the two filters derived in this work, the CEKF and BUKF, in the presence of varying degrees of measurement noise.

Both satellites are modeled using only relative Newtonian gravitation around the Earth. In common astrodynamics jargon for spacecraft relative motion, one satellite, the "chief", is considered to be located at the center of a rotating frame, Figure 6.1, that has a variety of names in the literature: the Hill’s frame, the LVLH frame, the RIC frame, and many others. The second satellite, named the "deputy", has a relative position defined in the rotating Hill’s frame. In the scenario presented in this chapter, the chief satellite is passive, and the deputy is actively sensing and attempting to orientate and locate itself in relation to the chief satellite.
Figure 6.1: The RIC frame is the local horizontal frame defined with the $R$ axis in the direction of the orbit radius from the center of the Earth and the $C$ axis to be tangent to the orbital plane. The $I$ unit vector then completes the right-handed coordinate set.

### 6.2 Dynamic Model Description

Let the relative position between two satellites as defined in the rotating RIC frame be $\mathbf{p} = [x \ y \ z]^T$. Let $r$ and $\theta$ be the radius and *true anomaly* of the chief satellite orbit. True anomaly is a classical orbital parameter defined to be the angle inside the orbital plane between the radius of perigee and the current orbit position vector. The
Newtonian gravitational dynamics of the relative state $\rho$ are then

\[
\ddot{x} - 2\dot{\theta}\dot{y} - \dot{\theta}^2x + \frac{\mu(r + x)}{((r + x)^2 + y^2 + z^2)^{3/2}} - \frac{\mu}{r^2} = u_x \quad (6.1)
\]

\[
\ddot{y} + 2\dot{\theta}\dot{x} + \dot{\theta}^2y + \frac{\mu y}{((r + x)^2 + y^2 + z^2)^{3/2}} = u_y \quad (6.2)
\]

\[
\ddot{z} + \frac{\mu z}{((r + x)^2 + y^2 + z^2)^{3/2}} = u_z \quad (6.3)
\]

\[
\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = u_r \quad (6.4)
\]

\[
\ddot{\theta} + \frac{2i\dot{\theta}}{r} = u_\theta \quad (6.5)
\]

These equations are considered idealized, in that they are only considering the accelerations due to point-mass Newtonian gravity. In reality, a planet, moon, or asteroid has an unevenly distributed mass that will cause perturbations to this idealized relative motion. On top of the spherical harmonics of the gravitational potential, non-conservative forces such as drag, solar radiation pressure, and third body effects also lead to these idealized equations as being incomplete. For the purpose of this numerical example, all perturbing effects of relative satellite motion are not modeled explicitly. These perturbations can be approximated for in a filter by modeling them as Gaussian white noise $u_i \sim \mathcal{N}(0, Q_i)$.

The initial conditions for this example were chosen to match those used in Kim et al. [45] in order to show a one-to-one comparison with the dual-quaternion filters and the traditional more quaternion-vector filters given in the literature. The deputy satellite will start in a state near a natural motion ellipse (NMC) around the chief with initial state $x = [\rho \; \dot{\rho}]^T$. An NMC is a trajectory that exists only in the Newtonian dynamics and only if the chief is in a perfectly circular orbit. If there are any perturbations present or if the chief orbit is eccentric, the relative motion ellipse can become unstable.
Table 6.1: Chief orbital elements at time $t_0$. These orbital elements are considered constant over the length of the simulation. In reality, these will vary over an orbit due to orbital perturbations.

<table>
<thead>
<tr>
<th>Element</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>semi-major axis ($a$)</td>
<td>6,998.455 km</td>
</tr>
<tr>
<td>eccentricity ($e$)</td>
<td>0.00172</td>
</tr>
<tr>
<td>inclination ($i$)</td>
<td>28.5 deg</td>
</tr>
<tr>
<td>right-ascension ($\Omega$)</td>
<td>125 deg</td>
</tr>
<tr>
<td>argument of perigee ($\omega$)</td>
<td>5 deg</td>
</tr>
<tr>
<td>true anomaly at ($\theta_0$)</td>
<td>0 deg</td>
</tr>
</tbody>
</table>

It is assumed by the filters that the deputy is at an initial relative state given by

$$\hat{x} = \begin{bmatrix} 200 [m] & 200 [m] & 100 [m] & 0.01 [m/s] & -0.4325 [m/s] & 0.01 [m/s] \end{bmatrix}^T \quad (6.6)$$

This state is given the rotating RIC frame, $\mathcal{H}$. The true state of the deputy will be randomly chosen at each simulation based on the initial state estimate $\hat{x}$ and an initial estimate of the error covariance given by $P_0 = E[\delta x \delta x^T]$.

The chief orbit will be known and have eccentricity $e = 0.00172$ and semi-major axis $a = 6,998.455$ km. The full set of classical orbital elements describing the chief orbit can be found in Table 6.1 and their definitions can be seen in Figure 6.2.

For these simulations, the body-fixed frame of the chief satellite and the rotating Hill’s frame are assumed to be coincident. Another way to consider this assumption is that the chief’s body frame is aligned such that its $x$-axis is pointing outward from the Earth, and that it is rotating about its $z$-axis with an angular velocity equal to that of the RIC frame.

### 6.2.1 Filter Internal Models

Using only relative measurements, it is not possible to pin the inertial state of either satellite. Due to this, the chief’s orbit is not included in the filter state. For all simulations present in this dissertation, the relative position and orientation of the deputy satellite
Figure 6.2: The classical orbital elements as they are defined. Eccentricity is a unitless parameter that defines the shape of the orbit ellipse, with $e = 0$ being a perfect circle, and $e = 1$ being a parabola.

in relation to the chief will be represented by a dual-quaternion. For all filters with the exception of the BUKF, this dual-quaternion state is augmented by additional states.

6.2.1.1 Dual-Quaternion Relative Pose

In contrast to Kim et al., the relative pose of the deputy and chief will be represented by a dual-quaternion instead of a vector and quaternion pair. Let the chief spacecraft have a fixed attitude in the rotating RIC frame with body-fixed coordinate system $A$ and the deputy spacecraft have body-fixed coordinate axes given by $B$. The dual-quaternion that describes the relative pose of these two frames is given by the dual-quaternion $\hat{q}_{B/A} = q_{B/A} + \epsilon d_{B/A}$. The relative position of frame $B$ with respect to $A$ appears in the dual part of the dual-quaternion

$$\hat{q}_{B/A} = q_{B/A} + \frac{1}{2} \epsilon \rho_A q_{B/A}$$  \hspace{1cm} (6.7)

For this example, the rotational motion of the spacecraft is assumed to have a constant
angular velocity
\[ \omega = \begin{bmatrix} -0.0020 & 0.0000 & 0.0011 \end{bmatrix}^T \text{[rad/s]} \] (6.8)

and the position velocity, \( \mathbf{v}_A \), is assumed to be governed by the relative motion Equations (6.1)-(6.3).

For the relative motion problem, the full internal state for all of the filters, with the exception of the BUKF, is the \( 14 \times 1 \) state vector
\[
\mathbf{x} = \begin{bmatrix} \hat{q}_{B/A} & \beta & \mathbf{v}_A \end{bmatrix}^T
\] (6.9)

where \( \beta \) is a gyroscopic bias term. In the BUKF it is an assumption that a direct measurement of the velocity in the body-fixed coordinates of the deputy spacecraft is known. This could be accomplished through integration of an onboard inertial measurement unit (IMU) to provide a velocity measurement with Gaussian white noise \( \eta_v \sim \mathcal{N}(0, \sigma_v^2 I_{3 \times 3}) \).

Bias terms that exist inside IMU accelerometers is not estimated inside the filter.

6.2.1.2 Rate-Integrating Gyroscope Modeling

The model for a kinematic velocity of attitude given measurements by rate-integrating gyroscopes is,
\[
\tilde{\omega} = \omega + \beta + \eta_\omega
\] (6.10)

where a tilde represents a measured velocity, \( \beta \) is the steady-state bias term, and \( \eta_\omega \sim \mathcal{N}(0, \sigma_\omega^2 I_{3 \times 3}) \) is a zero-mean Gaussian white noise. This model was first defined by Farrenkopf [37], and has become the standard way to model angular velocities as measured by rate-integrating gyroscopes.
The time varying nature of the gyro bias is modeled by another random process where

\[
\dot{\beta}_\omega = \xi_\omega \tag{6.11}
\]

and \( \xi_\omega \sim \mathcal{N}(0, \sigma^2_\xi I_{3 \times 3}) \).

In all filters except for the BUKF, where the measured angular velocity is used directly, the gyro measurement bias is appended to the relative state and estimated inside the filter.

6.2.1.3 Error State

Inherent to the descriptions of the standard MEKF and UKF for dual-quaternion states is the small-angle assumption for attitude error. Let the dual-quaternion error be defined as

\[
\delta \hat{q}_{B/A} = \hat{q}_{B/A}^* \hat{q}_{B/A}
\]

\[
= \delta q_{B/A} + \frac{1}{2} \epsilon (\delta q_{B/A} \rho_B - \dot{\rho}_B \delta q_{B/A})
\]

(6.12)

If the small angle assumption is taken, the relative error between frame \( B \) and \( A \) can be described by a first-order approximation

\[
\tilde{\delta q} \approx \begin{bmatrix}
\frac{\delta \theta}{2} \\
1 \\
\frac{\delta \phi}{2} \\
0
\end{bmatrix}
\]

(6.13)

where \( \delta \theta \) is twice of the vector part of the error quaternion, and the dual error is given by \( \delta \phi \approx \delta r_B \). The error state for these filters is then the \( 12 \times 1 \) state vector

\[
\delta x = \begin{bmatrix}
\delta \theta_{B/A} & \delta \phi_{B/A} & \delta \beta & \delta v_A
\end{bmatrix}^T
\]

(6.14)
A first order approximation with a small angle assumption for the kinematics of this state vector for use in both EKFs to propagate the covariance is given by

\[ \delta \dot{x} = F \delta x + G \nu \] (6.15)

\[ F = \begin{bmatrix} -[\hat{\omega} \times] & 0 & -I & 0 \\ -[\hat{\omega} \times] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \] (6.16)

\[ G = \begin{bmatrix} -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \] (6.17)

The state transition matrix for the error vector can then be approximated to be

\[ \Phi = I - \Delta t F \] (6.18)

The nonlinear error dynamics are used in both UKF versions,

\[ \dot{\delta q} = \frac{1}{2} (\delta \hat{q} \hat{\omega} - \hat{\omega} \delta \hat{q}) + \frac{1}{2} \delta \hat{q} \hat{\omega} \] (6.19)

6.2.1.4 Vision-Based Navigation

The vision based navigation sensor system (VISNAV) produces image space coordinates \((X_i, Y_i)\) for a given focal length \(f\) that can be transformed into position coordinates given object space locations attached to the chief spacecraft and an estimate of the current...
Table 6.2: Beacon locations on the chief spacecraft. Since the chief spacecraft is fixed in the rotating RIC frame, these beacons are also defined in the relative motion frame

<table>
<thead>
<tr>
<th>$x$ (m)</th>
<th>$y$ (m)</th>
<th>$z$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>-0.5</td>
<td>-0.5</td>
<td>0</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.5</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5</td>
<td>0.1</td>
</tr>
<tr>
<td>0.0</td>
<td>0.2</td>
<td>-0.1</td>
</tr>
</tbody>
</table>

relative attitude [46]. The measurement model is then

$$b_i = \frac{1}{\sqrt{f^2 + X_i^2 + Y_i^2}} \begin{bmatrix} -X_i \\ -Y_i \\ f \end{bmatrix} = C(q_{B/A})r_i + \nu_i$$  \hspace{1cm} (6.20)$$

$$r_i = \frac{1}{\sqrt{(X_i - x_A)^2 + (Y_i - y_A)^2 + (Z_i - z_A)^2}} \begin{bmatrix} X_i - x_A \\ Y_i - y_A \\ Z_i - z_A \end{bmatrix} = \frac{p_i}{\|p_i\|}$$  \hspace{1cm} (6.21)$$

where $C(q_{B/A})$ is the DCM for frame $B$ with respect to frame $A$ given the quaternion $q_{B/A}$. The vector $\rho = [x_A \ y_A \ z_A]^T$ is the position of the deputy spacecraft in the RIC frame and $x = [X_i \ Y_i \ Z_i]^T$ is the position of beacon $i$ in the RIC frame.

These unit vectors have a the Jacobian with respect to the filter state given by

$$H_i = \begin{bmatrix} C(q_{B/A})r_i \times & C(q_{B/A})[\frac{p_i}{\|p_i\|}] \times & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (6.22)$$

for usage inside the EKF. For these simulations, the chief spacecraft is assumed to have fixed beacons located on it at the locations in Table 6.2.

While there are six beacons affixed to the chief spacecraft, at each measurement time, the number of beacons visible is not constant due to occlusion from the body of the chief spacecraft. Since neither the chief or deputy spacecraft have a defined geometry,
occlusion is simulated by constructing a plane perpendicular to the relative position vector from the chief to the deputy. All beacons that are on the side of this plane that is opposite of the deputy spacecraft are considered not visible to the deputy spacecraft’s sensors.

Since the beacon locations are known to the deputy spacecraft in the body-fixed space of the chief, both relative position and orientation can be estimated given sufficient unit vectors to the beacons.

The optimal attitude can be obtained with at least two unit vectors by solution of Wahba’s Problem [47]. Wahba’s Problem seeks to find the DCM, $C$, that minimizes the loss function

$$L(C) = \frac{1}{2} \sum_{i=1}^{N} a_i \| b_i - C r_i \|^2$$

(6.23)

where $a_i$ are weights and $b_i$ and $r_i$ are as previously defined. The weights $a_i$ are optimally chosen to be inverse variances of the measurement uncertainty. Most solutions to Wahba’s problem involve quaternion parameterizations of $C$. Davenport’s q-method [48] was the first to provide a useful way of solving Wahba’s problem and was used inside the BUKF to pre-process the VISNAV unit vectors to produce a quaternion measurement for the Bingham filter.

Let

$$K(B) = \sum_{i=1}^{N} a_i [ b_i \otimes ]^T [ r_i \oslash ]$$

(6.24)

Davenport’s q-method states that the largest eigenvalue of $K(B)$ has a corresponding eigenvector $q$ that is the optimal estimate of the attitude quaternion.

Another, more computationally efficient method to solve Wahba’s problem that would be used in practice is the QUEST algorithm [49]. However, the simplicity of solving Davenport’s q-method with the help of an eigenvalue solver did not warrant using the more complicated iterative solution of QUEST for these simulations on modern hardware.

For the test cases where the attitude measurement is a noisy quaternion, the relative
position of the satellite is measured using a standard active range-finding laser. These type of sensors provide a range $\rho$, azimuth $\phi$ and elevation $\theta$ to describe a vector to the target. The relative position measured in the body-fixed frame is then

$$
\mathbf{p}_B = \begin{bmatrix}
\rho \cos(\phi) \cos(\theta) \\
\rho \sin(\phi) \cos(\theta) \\
\rho \sin(\theta)
\end{bmatrix} 
$$

(6.25)

### 6.3 Summary

The general scenario, dynamics models and measurement models used for simulation in the following chapter have been defined. An example working filter was built for each of the dual-quaternion filters described in Chapter 4 and Chapter 5. These filters were then tasked to process measurements in the example scenario with varying degrees of measurement noise and compared against more conventional quaternion filters.
Chapter 7  
Numerical Examples

7.1 Introduction

The conventional filters for satellite pose, the UKF and MEKF, are compared against the CEKF and BUKF filters defined in chapters 4 and 5 respectively. The filters are applied to two different autonomous navigation problems:

1. The spacecraft orientation is determined by unit vectors pointing to reflective 'beacons' located on the body of the chief satellite in known locations inside the chief spacecraft’s body-fixed frame. The VISNAV laser ranging system developed by Texas A&M is an example of this case [46].

2. A quaternion measurement is produced through some image-processing technique. The exact method used to obtain this quaternion is outside the scope of this dissertation, but modern techniques involve the training of convolutional neural networks (CNN) on a training set of spacecraft images taken from a variety of angles and lighting conditions. It will be assumed that the quaternion output from this technique is not guaranteed to favor \( +q \) or \( -q \) since both represent the same relative orientations.

These numerical examples demonstrate the ability of the CEKF and BUKF to handle
Table 7.1: Initial conditions for the simulations. Values for the continuous time process noise matrices is also given. For use inside the EKF, these continuous time process noise values are discretized. The Bingham process noise matrix is also given for the BUKF simulations. The BUKF process noise contains both a Bingham and dual Gaussian-Bingham covariance element.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Simulation Truth Values and Initial Filter estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}_0$</td>
<td>[0.7001, 0.0221, -0.0190, 0.7134, 69.4390, 105.3021, -38.3830, -72.4292]</td>
</tr>
<tr>
<td>$\hat{\theta}'_0$</td>
<td>[0.7071, 0.0000, 0.0000, 0.7071, 70.7107, 106.0660, -35.3553, -70.7107]</td>
</tr>
<tr>
<td>$\hat{v}_A$</td>
<td>[0.0101, -0.4325, 0.0100] (m/s)</td>
</tr>
<tr>
<td>$\hat{v}'_A$</td>
<td>[0.0100, -0.4325, 0.0100] (m/s)</td>
</tr>
<tr>
<td>$Q$</td>
<td>diag([1e-9$I_{3\times3}$, 1e-19$I_{6\times6}$, 1e-21$I_{3\times3}$])</td>
</tr>
<tr>
<td>$Q_{Bingham}$</td>
<td>diag([1e-7, 1e-7, 1e-7, (1 - 3e-7), 0.1, 0.1, 0.1, 0.1])</td>
</tr>
</tbody>
</table>

noisy attitude measurements with large errors, and without sensitivity to positive and negative quaternions.

Additionally, each filter was simulated for two minutes in a Monte Carlo analysis of the measurement residuals with $N = 1000$. Root mean squares were taken of the measurement residuals for the first thirty seconds of each simulation and at the end of each simulation to compare both short term and steady-state convergence of each filter in all scenarios.

For all simulations, the same initial conditions for the filters, “truth” states, and process noise were used. These are given in Table 7.1. Filters were run with a measurement update rate of 100Hz, or 0.01 seconds between filter updates. The measurements were received at a rate of 10Hz, or every 0.1 seconds. Each filter was run individually for a one minute simulation. During this simulation, each filter will perform a total of 600 measurement updates. Since this filter will run in a LEO orbit, it is necessary for it to run at this speed since orbits in LEO are exposed to much more orbital perturbations from two-body motion than at higher altitudes, where a slower filter would be sufficient. The internal error state with three-sigma filter covariances are shown alongside the raw attitude error in Euler angles with trajectory tracks.

Section 7.2.1 provides an analysis of the filter performances given the VISNAV
measurements for each filter, which is subsequently followed by Section 7.2.2 which
provides an analysis of each filter given noisy quaternion measurements that have error
sampled from a Bingham distribution. Each subsection also provides the measurement
noise characteristics that were used in each simulation. Each measurement model has
two sets of noise statistics: one of “low” and one of “high” intensity.

Results are shown by figures that depict the attitude quaternion estimate as compared
to truth over the duration of the simulation. The attitude error is shown in the form
of Euler angles due to their more intuitive conception. The trajectory of the deputy
spacecraft in the relative motion frame is also shown with the filter estimated trajectory.

To show the accuracy of the filters, each filter was run 1000 times with the same
initial conditions with different measurements. The root mean square (RMS),

\[ x_{RMS} = \sqrt{\frac{\sum_{i=1}^{N} x_i^2}{N}} \]  

of the angular attitude errors, the position error, and filter residuals before each mea-
surement update was calculated after 30 seconds and two minutes simulation.

7.2 Numerical Results

Firstly, single run feature-point tracking results are presented for all filters, which is
followed by statistical results from the Monte Carlo runs.

7.2.1 Feature-Point Tracking Results

For the VISNAV filter simulations, a low noise intensity measurement with signal-to-noise
ratio (SNR) of 166 dB and a high noise intensity with an SNR of 126 dB was used.
Values for the measurement covariances of each of these cases can be seen in Table 7.2.
Table 7.2: VISNAV noise characteristics

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Low Noise</th>
<th>High Noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_v$</td>
<td>7e-4 (rad)</td>
<td>7e-5 (rad)</td>
</tr>
</tbody>
</table>

The measurement model for the VISNAV unit vectors is such that

$$\tilde{b}_i = C\tilde{r}_i + v_i, \quad v_i \sim \mathcal{N}(0, \, \sigma_v^2(I - [C\tilde{r}_i \times] [C\tilde{r}_i \times]^T))$$  \hspace{1cm} (7.2)

The noise vector $v_i$ is sufficiently small such that it lies on the tangent plane to the 3-sphere of the attitude DCM. Its covariance matrix

$$R = \sigma_v^2(I - [C\tilde{r}_i \times] [C\tilde{r}_i \times]^T)$$  \hspace{1cm} (7.3)

is singular and can be approximated in the filter as $R = \sigma_v^2I$ without any loss of precision.

7.2.1.1 MEKF and UKF Results

The conventional filters perform incredibly well with measurements of this type. These filters assume a small angle assumption, and the high SNR in both noise profiles for this case do not violate this. Figures 7.1-7.8 show the performance of these filters at tracking the reference attitude and trajectory.

All figures of the quaternion state show the filter estimate, the best quaternion estimate using Davenport’s q-method labeled as 'measurement', and both the positive and negative truth quaternion. While the quaternion estimate using the q-method is shown in all plots, only the BUKF uses this estimate internally. The Davenport q-method is shown on all plots to demonstrate that the quaternion estimated from the VISNAV system will always be "+q" rather than "−q".

Figures 7.1-7.4 show the single run results for the dual-quaternion MEKF of Tsiotras et al. [50] in the two noise cases with the VISNAV measurements. In situations with
low measurement noise, the MEKF can accurately track the relative quaternion. With increased measurement noise, the MEKF is more slow to adjust both position and attitude estimates. Errors do not reach steady state until approximately 50 seconds.

Figure 7.1: MEKF deputy quaternion. Low measurement noise VISNAV case.

Figures 7.5-7.8 show single results for the twistor based UKF of Deng et al [42]. The conventional UKF also has no trouble tracking the relative attitude given small measurement noise with attitude error below 0.02 degrees. With increased measurement noise, the UKF quaternion and position estimates suffer from increased jitter.
Figure 7.2: MEKF trajectory. Low measurement noise VISNAV case.

Figure 7.3: MEKF deputy quaternion. High measurement noise VISNAV case.
Figure 7.4: MEKF trajectory. High measurement noise VISNAV case.

Figure 7.5: UKF deputy quaternion. Low measurement noise VISNAV case.
Figure 7.6: UKF Trajectory. Low measurement noise VISNAV case.

Figure 7.7: UKF deputy quaternion. High measurement noise VISNAV case.
7.2.1.2 CEKF and BUKF Results

With the constrained filter, the results are clearly biased. This bias is inherent to filters that are derived in this way. In a minimum variance estimator, the state estimate is updated using the previous estimate of the state and the measurement residuals, or innovations. This update has the form

\[
\hat{x}_k^+ = G\hat{x}_k^- + K\epsilon
\]  

(7.4)

and for an unbiased estimate, the matrix \( G \) needs to be an identity matrix so that the update is

\[
\hat{x}_k^+ = \hat{x}_k^- + K\epsilon
\]  

(7.5)

With the CEKF, both \( G \) and \( K \) are functions of the innovations and the Lagrange multipliers.

\[
\hat{x}^+ = G(\epsilon, \lambda_0, \lambda_1)\hat{x}^- + K(\epsilon, \lambda_0, \lambda_1)\epsilon
\]  

(7.6)
\[
\begin{bmatrix}
\hat{x}_1^+ \\
\hat{x}_2^+
\end{bmatrix} = \begin{bmatrix}
\frac{1}{1+\lambda_0 r} I & -\frac{\lambda_1 r}{2(1+\lambda_0 r)} I \\
-\frac{\lambda_1 r}{2(1+\lambda_0 r)} I & \frac{1+\lambda_0 r}{1+\lambda_0 r} I 
\end{bmatrix} \begin{bmatrix}
\hat{x}_1 \\
\hat{x}_2
\end{bmatrix} + \begin{bmatrix}
\frac{1}{1+\lambda_0 r} \tilde{K}_1 - \frac{\lambda_1 r}{2(1+\lambda_0 r)} \tilde{K}_2 \\
-\frac{\lambda_1 r}{2(1+\lambda_0 r)} \tilde{K}_1 + \frac{1+\lambda_0 r}{1+\lambda_0 r} \tilde{K}_2
\end{bmatrix} \epsilon
\] (7.7)

This update shows that unless both \(\lambda_0\) and \(\lambda_1\) are zero, meaning both constraints are already satisfied, then the \(G\) matrix will not be identity. These biases are clearly visible in all of Figures 7.9-7.12. With the high noise case, the biases in the attitude lead to a pronounced bias in the position estimate as well.

In fact, this Kalman gain has a fundamental flaw in its derivation. The optimization function to derive it assumed that it was a constant that could be pulled outside of any expectation integrals. Since it is a function of innovations, this assumption was violated in the solution of this covariance. This was recognized in the original paper, by citing that the covariance obtained with this constrained update is not the true covariance, but a "pseudo-covariance" [31].

Figures 7.9-7.12 show single case results for the CEKF with the low and high measurement noise VISNAV measurements. For the low noise case, the constrained EKF formulation has similar performance to the MEKF, but despite the similar performance, the biases in attitude and position that appear in this formulation can be seen even with low measurement noise cases. With increased measurement noise, the biases in this formulation become more obvious.

Figures 7.13-7.16 show single test case results for the BUKF with the VISNAV measurements. Even with measurements that are restricting the quaternion states to one hemisphere, the BUKF still has no preference for \(+q\) or \(-q\). Both states are likely for the filter to predict. The BUKF has similar performance to the UKF at following the reference attitude in these tests, however, there is much more noise present in the position estimates. In this higher noise case, the BUKF begins to provide a much smoother estimate to the reference quaternion than the standard UKF. As with the lower noise test-case, the noisy position estimate of the BUKF is still visible.
Figure 7.9: CEKF deputy quaternion. Low measurement noise VISNAV case.

Figure 7.10: CEKF Trajectory. Low measurement noise VISNAV case.
Figure 7.11: CEKF deputy quaternion. High measurement noise VISNAV case.

Figure 7.12: CEKF Trajectory. High measurement noise VISNAV case.
Figure 7.13: BUKF deputy quaternion. Low measurement noise VISNAV case.

Figure 7.14: BUKF Trajectory. Low measurement noise VISNAV case.
Figure 7.15: BUKF deputy quaternion. High measurement noise VISNAV case.

Figure 7.16: BUKF Trajectory. High measurement noise VISNAV case.
7.2.1.3 Filter Accuracy Comparisons

Monte Carlo simulation of 1000 runs demonstrate that after 30 seconds, the BUKF has the most consistency when estimating for both position and attitude. Error RMS results for both noise profiles can be seen in Figures 7.17 and 7.18.
Figure 7.17: Monte Carlo results for all filters with the less noisy VISNAV measurements.
Figure 7.18: Monte Carlo results for all filters with the noisy VISNAV measurements.
7.2.2 Extreme Noise Quaternion Measurements

For cases where the measurements are an extremely noisy and sampled from a Bingham distribution, the SNR of the measurements are 0 dB for the lower noise case and -23 dB for the higher noise case. It is a reminder to the reader that an SNR value of 0 dB means that the magnitude of the noise is equal to the magnitude of the signal and that a negative SNR corresponds to a measurement where the noise has a greater magnitude than that of the signal. Values for the measurement covariances for each of these cases can be seen in Table 7.3. The measurement model for the noisy quaternion measurements is such that

\[
\tilde{q} = qv, \quad v \sim B(Z_R, I)
\]  

Since the measurement noise is a Bingham distributed quaternion \(v\), there is equal probability that it could be \(+v\) or \(-v\). Both represent the same pointing error in \(SO(3)\).

The position measurements for these cases model an active laser-range finder with measurements being range, azimuth, and elevation from the deputy to the target.

7.2.2.1 MEKF and UKF

The MEKF and UKF were not intended for this type of noise, and their performances in these simulations demonstrate this fact. Both with the 0 dB and -23 dB noise, they diverge within the first few measurement updates. The UKF is able to more ably track the truth quaternion, but the bimodal nature of the noise distribution leads to an extremely

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Low Noise</th>
<th>High Noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_{v_A})</td>
<td>1e-4 (m/s)</td>
<td>1e-4 (m/s)</td>
</tr>
<tr>
<td>(Z_R)</td>
<td>diag([-720, -720, -720, 0])</td>
<td>diag([-50, -50, -50, 0])</td>
</tr>
<tr>
<td>(\sigma_\rho)</td>
<td>1 (mm)</td>
<td>20 (cm)</td>
</tr>
<tr>
<td>(\sigma_\theta) and (\sigma_\phi)</td>
<td>1e-6 rad</td>
<td>1e-6 rad</td>
</tr>
</tbody>
</table>
noisy attitude estimate and biased position estimate.

Figures 7.19-7.22 show single test case results for the conventional MEKF. In the extreme noise cases, the MEKF is not capable of handling measurements that could switch sign. Increased noise in the measurements leads to much larger errors. Position errors for the MEKF in this most extreme noise scenario are in the hundreds of meters.

![Deputy Quaternion Estimate over time](image)

**Figure 7.19: MEKF deputy quaternion. High SNR**

Figures 7.23-7.26 show single test case results for the conventional UKF. Like the MEKF, the UKF model includes a small angle assumption for the error that is broken in this test case. Despite difficulty, the conventional UKF is better at tracking the relative attitude than the MEKF. In similar fashion as with the attitude estimate, the UKF also has increased accuracy in the position estimate in the most extreme noise case.
Figure 7.20: MEKF trajectory. High SNR

Figure 7.21: MEKF deputy quaternion. Low SNR
Figure 7.22: MEKF trajectory. Low SNR

Figure 7.23: UKF deputy quaternion. High SNR
Figure 7.24: UKF trajectory. High SNR

Figure 7.25: UKF deputy quaternion. Low SNR
7.2.2.2 CEKF and BUKF

This type of high noise case is where the CEKF and BUKF perform best. The CEKF continues to display the biasing of the state estimate that is inherent to its formulation, and the BUKF shows how easily capable of handling the Bingham distributed measurement noise.

Figures 7.27-7.30 show the single test case for the CEKF with the extremely noise quaternion measurements. Since the constrained CEKF does not have a small angle assumption in its definition, it performs much more competently with higher measurement noise. The bias is, however, more pronounced, yet despite biases in the attitude measurements, the position estimate is converging slowly to the true trajectory. In the highest noise case, the CEKF continues to outperform both the MEKF and UKF. Despite obvious biases, the error in the attitude estimate is below 10 degrees for the CEKF in the highest noise test case.

Figures 7.31-7.34 show results for a single test case with the BUKF. The BUKF is...
Figure 7.27: CEKF deputy quaternion. High SNR

Figure 7.28: CEKF Trajectory. High SNR
Figure 7.29: CEKF deputy quaternion. Low SNR

Figure 7.30: CEKF Trajectory. Low SNR
able to track the relative attitude the most ably in the higher noise cases, and despite obvious jitter in the position estimate, the BUKF is able to follow the reference trajectory faithfully. Despite taking time to converge, the BUKF is able to recover and track the reference trajectory in the highest noise case.

Figure 7.31: BUKF deputy quaternion. High SNR
Figure 7.32: BUKF Trajectory. High SNR

Figure 7.33: BUKF deputy quaternion. Low SNR
Monte Carlo simulation for the high measurement noise cases clearly show the superior statistics for the BUKF and CEKF, which are both able to converge to small error estimates faster than both the MEKF and UKF, which diverge during the simulations due to their small angle error assumptions.

For the lower level test case with extreme bimodal noise, the BUKF is able to outperform all of the other filters in terms of attitude estimate error. The CMEKF comes in second place due to it not being defined with a small angle assumption. Both the MEKF and UKF however, are able to more accurately estimate the spacecraft range in this lower noise test case. Error RMS results for the low noise case for thirty second and two minute simulations can be seen in Figures 7.35 and 7.36.

In the highest noise test case, both the UKF and EKF fall off even further in both attitude and position error. The BUKF is the best in both attitude and position error, with the CEKF coming in a close second place, despite its biases. Error RMS results for
the high noise case for thirty second and two minute simulations can be seen in Figures 7.37 and 7.38.
Figure 7.35: Monte Carlo 30 sec, low SNR

RMS Error after 30 sec with low noise
Figure 7.36: Monte Carlo 2 mins, low SNR
Figure 7.37: Monte Carlo 30 secs, high SNR

RMS Error after 30 sec with high noise
Figure 7.38: Monte Carlo 2 min, high SNR

RMS Error after 2 min with high noise
7.3 Summary

From the filter comparisons, the conventional filters work best in the VISNAV case were the measurement noise is very small due to the small angle assumptions present in these filters. The CEKF and the BUKF do not have these assumptions, yet perform worse in the low noise VISNAV cases for a variety of reasons.

The CEKF does not make the same small angle assumption by treating the dual-quaternion state as a vector in $\mathbb{R}^8$. The downside to this treatment is that in order to enforce the two constraints of a dual-quaternion, the unbiased estimate of a minimum variance update is violated. However, despite this inherent bias, this filter is able to much better function in situations where the SNR is low. This increased performance of this filter in such situations are due to it not needing the small angle assumption of the filter error that the MEKF and UKF require.

The BUKF performs less optimally in situations were the measurement noise characteristics are not that of a Bingham distributed rotation. In addition, the BUKF requires an extra step when provided with vector measurements in order to convert these into a quaternion measurement. However, in circumstances where signal to noise ratios are low, the BUKF performs incredibly well as compared to all of the other filter formulations. In a high noise environment, it converges quickly to its steady state error.
Chapter 8  |  Conclusions and Future Work

This dissertation has presented derivations and simulation results for two new estimators that utilize dual-quaternions for their pose representations. This involved presenting the motivations for using dual-quaternions as being their numerical efficiency and linear kinematics. For use in the nonlinear Bingham distributed UKF, a new set of dual-quaternion sigma points was created as an extension of previous attempts and shown to accurately calculate dual-quaternion moments. This Chapter will summarize the results from this dissertation and provide suggestions for future areas of research that follow from its conclusions.

8.1 Conclusions

Preceding the derivations for both filters, the necessary background and math information was discussed. Firstly, the rules for quaternion and dual-quaternion algebras were presented along with how to utilize these algebras to provide compact descriptions of 6-DOF motion. Lastly, the base statistical concepts and distributions used within this work were described.

Before each of the new filters were derived, a more conventional filter was presented from the literature to serve as a counterpoint to both the CEFK and BUKF respectively.
These filters were then compared against each other in numerical Monte Carlo simulation using a spacecraft relative motion model and camera-based localization measurements.

The Monte Carlo simulation results provided in Chapter 7 demonstrate the capability of the CEKF and BUKF to accurately track spacecraft attitude and position in relative motion space when provided with very noisy measurements that is superior to the small-angle assumption based filters. These more conventional filters broke down in the presence of measurements with very low signal-to-noise ratios, where the new filters did not.

In the presence of very large measurement noise both the CEKF and BUKF outperformed the MEKF and UKF presented in the literature, however, in the simulations with lower measurement noise, the biases that arise in the derivation of the CEKF are more obvious. Due to this, unless more work is done to minimize or negate the biased estimate of the CEKF, the BUKF is preferred for any practical application.

8.2 Future Work

While this dissertation puts forward two completely derived dual-quaternion filters, there is room for future study

1. The BUKF defined in this work only estimates the dual-quaternion state. It is assumed that rate information such as velocity and angular velocity are measured on-board using strap-down accelerometer integration and rate-integrating gyroscopes respectively. Extending these filters to also estimate velocities by incorporating the attitude and accelerometer dynamics models would also be worthwhile research.

2. Biases in the CEKF could theoretically be eliminated by resolving an issue of the constrained optimization by removing the assumption that the Kalman gain is constant. In the constrained solution, the Kalman gain is a function of measurement
innovations meaning that the original optimization step of moving the Kalman gain out of the expectation integrals is not correct. Estimation of these expectations using Gaussian quadrature and performing the same optimization steps could solve the bias issue with the CEKF.

3. In addition to just improving the CEKF gain to fix bias issues, the technique involved for developing this filter can be expanded into other element sets with constraints similar to those in a dual-quaternion state. A full study of which orbital element sets can be used with this CEKF is a good area for future study.

4. Relative motion states might often include model or measurement parameters such as ballistic coefficients for drag or clock drift parameters if an on-board clock is being modeled for one-way pseudorange measurements. An analysis for how the covariance matrix and cross-diagonal terms for these extra parameters and the Bingham distributed quaternion states would need to be done in order to include these extra filter states.
Appendix A

Quaternion Clifford algebra

A.1 Quaternions

A quaternion can be described in the language of geometric algebra as an even multivector of a three-dimensional vector space, $\mathcal{G}_3$. We begin with three orthonormal vectors $e_1$, $e_2$, and $e_3$. This derivation follows the “right-handed” one favored by Hestenes and Sobczyk [51], which means that each of the vectors have a square defined by

$$e_1^2 = e_2^2 = e_3^2 = 1.$$  \hfill (A.1)

The basis vectors of this three-dimensional space are then the linear combinations of these vectors:

$$1, \ e_1, \ e_2, \ e_3, \ e_1e_2, \ e_1e_3, \ e_2e_3, \ e_1e_2e_3,$$  \hfill (A.2)

where geometric multiplication of two perpendicular vectors is equivalent to the Grassman outer-product of two vectors,

$$B = e_1 \wedge e_2,$$  \hfill (A.3)
which is termed to be a bivector. A bivector is a directed area that physically represents the area of a parallelogram created within the interior of two vectors. This concept is shown in Figure A.1

![Diagram of bivector](image)

Figure A.1: The outer product of vectors \( \mathbf{A} \) and \( \mathbf{B} \) is a bivector \( \mathbf{A} \wedge \mathbf{B} \) with directed area given in blue.

The outer product of two orthonormal vectors produces a directed unit area. The multiplication of all the vectors in the three-dimensional space, \( \sigma = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \), is a trivector, or directed unit volume. It is sometimes referred to in three-dimensional space as a pseudoscalar due to the property that it is commutative in multiplication.

\[
\sigma \mathbf{e}_1 = \mathbf{e}_1 \sigma
\]  

(A.4)

With bivectors, there is no commutative multiplication.

\[
\mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_i.
\]  

(A.5)

Using this relationship, we can reduce the six available combinations of the vector set to three distinct bivectors.
These three principle bivectors can be defined as:

\[ i = e_3 e_2 \]  \hspace{1cm} (A.6)  
\[ j = e_1 e_3 \]  \hspace{1cm} (A.7)  
\[ k = e_2 e_1 \]  \hspace{1cm} (A.8)  

which have the well-known properties described by Hamilton:

\[ i^2 = j^2 = k^2 = ijk = -1 \]  \hspace{1cm} (A.9)  
\[ jk = i, \  kj = -i \]  \hspace{1cm} (A.10)  
\[ ki = j, \  ik = -j \]  \hspace{1cm} (A.11)  
\[ ij = k, \  ji = -k. \]  \hspace{1cm} (A.12)  

The basis vectors \( e_i \) are related to the bivectors \( i, j, k \) through the pseudoscalar relationships:

\[ i = -e_1 \sigma \]  \hspace{1cm} (A.13)  
\[ j = -e_2 \sigma \]  \hspace{1cm} (A.14)  
\[ k = -e_3 \sigma. \]  \hspace{1cm} (A.15)  

This relationship is known as a duality. This duality leads some sources to denote the bivectors in \( \mathcal{G}_3 \) to be pseudovectors, as they behave exactly like vectors under rotations. This concept of duality can be used to define the cross product operation of two vectors to be the dual operation of the Grassman outer product,

\[ e_i \times e_j = -\sigma(e_i \wedge e_j) = \sigma(e_j \wedge e_i) = e_k \]  \hspace{1cm} (A.16)
A generalized multivector in $\mathcal{G}_3$ can then be given as

$$\mathbf{M} = q_0 + x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} + \alpha \sigma. \quad (A.17)$$

The even sub-algebra, $\mathcal{G}_3^+$, is simply the scalar and bivector parts of a generalized multivector. The even part of a multivector in three-dimensions is a quaternion given by:

$$\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \quad (A.18)$$

A “left-handed” derivation where $\mathbf{e}_i = -1$ is also possible, and left to the reader to verify.

### A.2 Dual-Quaternions

Dual-quaternions have a similar derivation as quaternions, except they are a degenerate case of the even algebra in four dimensions, $\mathcal{G}_4^+$. They are a degenerate case because they are a projection of the 4th order space onto three-dimensional space through the following vector norm identities,

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1$$

$$\mathbf{e}_4^2 = 0. \quad (A.20)$$

Using this relationship, the pseudoscalar in this degenerate algebra has the relationship that,

$$\epsilon^2 = (\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4)^2 = 0. \quad (A.21)$$

This pseudoscalar, $\epsilon$, is referred to in literature on dual-numbers as the dual variable. Its formal definition in the theory of dual-numbers is $\epsilon \mid (\epsilon^2 = 0) \land (\epsilon \neq 0)$. The use of the
designation dual should not be confused with its use in the case of vector and bivector duality in the geometric algebra of three dimensions.

The geometric algebra of four-dimensional space has six unique bivector combinations:

\[ \mathbf{i} = \mathbf{e}_3 \mathbf{e}_2 \]  \hspace{1cm} (A.22)
\[ \mathbf{j} = \mathbf{e}_1 \mathbf{e}_3 \]  \hspace{1cm} (A.23)
\[ \mathbf{k} = \mathbf{e}_2 \mathbf{e}_1 \]  \hspace{1cm} (A.24)
\[ \epsilon \mathbf{i} = \mathbf{e}_1 \mathbf{e}_4 \]  \hspace{1cm} (A.25)
\[ \epsilon \mathbf{j} = \mathbf{e}_2 \mathbf{e}_4 \]  \hspace{1cm} (A.26)
\[ \epsilon \mathbf{k} = \mathbf{e}_3 \mathbf{e}_4 \]  \hspace{1cm} (A.27)

In four dimensions, a bivector dual is itself another bivector.

Much like the quaternion, a dual-quaternion is then an even subset of these degenerate four-dimensional multivectors that can be written as,

\[
\hat{\mathbf{q}} = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k} + \epsilon(q_0' + q_1' \mathbf{i} + q_2' \mathbf{j} + q_3' \mathbf{k})
\]  \hspace{1cm} (A.28)
\[
\hat{\mathbf{q}} = q_r + \epsilon q_d,
\]  \hspace{1cm} (A.29)

where \( q_r \) and \( q_d \) are both quaternions and are the real and dual parts of \( \hat{\mathbf{q}} \) respectively. The dual and real parts of a dual-quaternion can be isolated with the following operators,

\[ \langle \hat{\mathbf{q}} \rangle_r = q_r \]  \hspace{1cm} (A.30)
\[ \langle \hat{\mathbf{q}} \rangle_d = q_d \]  \hspace{1cm} (A.31)
Appendix B

Dual-Quaternion STM Derivation

Extended Kalman filters for quaternion and dual-quaternion estimation use an error state that is free from the unit norm constraint on the quaternion itself. For quaternion filters, this can be accomplished by having the error state be the vector part of an error quaternion, RP, or MRP.

The dual-quaternion filter derived in this work uses the error state given in Filipe et al. [30] which uses the vector part of the dual-quaternion error together with the gyroscope and velocity measurement biases as the EKF state vector.

\[
x = \begin{bmatrix} \overline{\delta q} \\ \beta_\omega \\ \beta_v \end{bmatrix} \in \mathcal{R}^{12} \quad \text{(B.1)}
\]

An error dual-quaternion is defined as a quaternion that will rotate from the estimate state to the true state such that,

\[
\hat{\delta q} = \hat{q}^* \hat{q}. \quad \text{(B.2)}
\]
The derivation of the dual-quaternion error kinematics are then

\[ \dot{\hat{\delta}q} = \dot{\hat{\delta}q}^* \hat{q} + \hat{q}^* \dot{\hat{\delta}q} \]
\[ = \frac{1}{2} (\dot{\hat{q}}^* \dot{\hat{\omega}} - \dot{\hat{\omega}} \dot{\hat{q}}^*) = \frac{1}{2} (\delta q \dot{\omega} - \dot{\omega} \delta q) \] (B.3)

The dual-velocity vector can be defined as the addition of some estimate dual-velocity vector with some error dual-vector denoted by

\[ \hat{\omega} = \hat{\omega} + \hat{\delta} \omega \] (B.5)

which now make the dual-quaternion error kinematics

\[ \dot{\hat{\delta}q} = \frac{1}{2} (\delta q \dot{\omega} - \dot{\omega} \delta q) + \frac{1}{2} \hat{\delta} q \hat{\delta} \omega \] (B.6)

A helpful way find analytic solution to this problem is to look at the real and dual parts separately. They are, to first-order, respectively,

\[ \dot{\delta} q_r = -\omega \times \delta q_r + \frac{1}{2} \delta \omega \] (B.7)
\[ \dot{\delta} q_d = -v \times \delta q_r - \omega \times \delta q_d + \frac{1}{2} (\delta v + \delta \omega) \] (B.8)

If we take the angular and linear velocity errors to be equal to \( \delta \omega = - (\beta \omega + \eta \omega) \) and \( \delta v = - (\beta v + \eta v) \) and substitute these into the error kinematics in matrix-vector form, this yields,

\[ \dot{\delta} q_r = \Omega \delta q_r - \frac{1}{2} (\beta \omega + \eta \omega) \] (B.9)
\[ \dot{\delta} q_d = V \delta q_r + \Omega \delta q_d - \frac{1}{2} (\beta \omega + \beta v + \eta \omega + \eta v) \] (B.10)

Taking the expectation the real part of the dual-quaternion’s error kinematics and solving
yields,
\[
\delta \mathbf{q}_r(t) = \Phi_{11}(t, t_0) \delta \mathbf{q}_r(t_0) + \Phi_{13}(t, t_0) \beta \omega \tag{B.11}
\]

where, using \( \Delta t = t - t_0 \),
\[
\Phi_{11}(\Delta t) = I + \frac{\sin(\omega \Delta t)}{\omega} \Omega + \frac{1 - \cos(\omega \Delta t)}{\omega^2} \Omega^2 \tag{B.12}
\]
\[
\Phi_{13}(\Delta t) = -\frac{1}{2} \left( \Delta t I + \frac{1 - \cos(\omega \Delta t)}{\omega} \Omega + \frac{\omega \Delta t - \sin(\omega \Delta t)}{\omega^3} \Omega^2 \right) \tag{B.13}
\]

These matrices are well known and are found in most texts for the error dynamics of a quaternion with a zero-order-hold placed on the bias term [20].

The derivation for the solution for the dual part of the quaternion error takes the solution for the real part to yield
\[
\delta \mathbf{q}_d(t) = \Phi_{11}(t, t_0) \delta \mathbf{q}_d(t_0) + \Phi_{21}(t, t_0) \delta \mathbf{q}_r(t_0) + \Phi_{23}(t, t_0) \beta \omega + \Phi_{13}(t, t_0) \beta \nu \tag{B.14}
\]

where
\[
\Phi_{21}(t, t_0) = \Phi_{11}(t, t_0) \int_{t_0}^{t} \Phi_{11}(t_0, \tau) V \Phi_{11}(\tau, t_0) d\tau \tag{B.15}
\]
\[
\Phi_{23}(t, t_0) = \Phi_{11}(t, t_0) \int_{t_0}^{t} \Phi_{11}(t_0, \tau) V \Phi_{13}(\tau, t_0) d\tau + \Phi_{13}(t, t_0). \tag{B.16}
\]
These integrals were evaluated by hand to yield,

\[
\int_{t_0}^{t} \Phi_{11}(t_0, \tau)V\Phi_{11}(\tau, t_0)d\tau = \Delta t V + \frac{1 - \cos(\omega\Delta t)}{\omega^2}(V\Omega - \Omega V)
\]
\[
+ \frac{\omega\Delta t - \sin(\omega\Delta t)}{\omega^3}(V\Omega^2 + \Omega^2 V)
\]
\[
+ \frac{\sin(2\omega\Delta t) - 2\omega\Delta t}{4\omega^5}\Omega V\Omega 
\]
\[
+ \frac{6\omega\Delta t - 8\sin(\omega\Delta t) + \sin(2\omega\Delta t)}{4\omega^5}(\Omega^2 V\Omega^2) 
\] 

(B.17)

\[
\int_{t_0}^{t} \Phi_{11}(t_0, \tau)V\Phi_{13}(\tau, t_0)d\tau = -\frac{1}{2}\left(\frac{1}{2}\Delta t^2 V + \frac{\omega\Delta t - \sin(\omega\Delta t)}{\omega^3}V\Omega \right)
\]
\[
+ \frac{\Delta t \cos(\omega\Delta t)}{\omega^2}V\Omega + \frac{\omega^2\Delta t^2 + 2\cos(\omega\Delta t) - 2}{2\omega^4}V\Omega^2 
\]
\[
+ \frac{\omega\Delta t(\omega\Delta t - 2\sin(\omega\Delta t)) - 2\cos(\omega\Delta t) + 2}{2\omega^4}\Omega^2 V 
\]
\[
- \frac{\cos(\omega\Delta t)(\cos(\omega\Delta t) - 2) + 1}{2\omega^4}\Omega V\Omega 
\]
\[
+ \frac{2\omega\Delta t - 4\sin(\omega\Delta t) - \sin(2\omega\Delta t) + 4\omega\Delta t \cos(\omega\Delta t)}{4\omega^5}\Omega V\Omega^2 
\]
\[
+ \frac{6\Delta t - 8\sin(\omega\Delta t) + \sin(2\omega\Delta t)}{4\omega^5}\Omega^2 V\Omega 
\]
\[
+ \frac{(\omega\Delta t - \sin(\omega\Delta t) - 1)(\omega\Delta t - \sin(\omega\Delta t) + 1) + 1}{2\omega^6}\Omega^2 V\Omega^2 
\].

(B.18)

With these integrals, the state transition matrix for the dual-quaternion error is given by

\[
\Phi(\Delta t) = \\
\begin{bmatrix}
\Phi_{11} & 0_{3\times3} & \Phi_{13} & 0_{3\times3} \\
\Phi_{21} & \Phi_{11} & \Phi_{23} & \Phi_{13} \\
0_{3\times3} & 0_{3\times3} & I_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_{3\times3}
\end{bmatrix}
\] 

(B.19)
Appendix C

Dual-Quaternion Noise Derivation

This section seeks to perform the same static analysis to calculate the dual-quaternion discrete process noise covariance matrix, $Q_k$, that Farrenkopf had shown for the standard quaternion [37].

A brief summary of that analysis yields the approximate discrete covariance matrix

$$Q_k = \int_0^{\Delta t} \Phi(t)G(t)Q(t)G^T(t)\Phi^T(t)dt \approx \begin{bmatrix} (\sigma_\omega^2 \Delta t + \frac{1}{3} \sigma_\xi^2 \Delta t^3)I_{3\times3} & -\left(\frac{1}{2} \sigma_\xi^2 \Delta t^2\right)I_{3\times3} \\ -\left(\frac{1}{2} \sigma_\xi^2 \Delta t^2\right)I_{3\times3} & \left(\sigma_\xi^2 \Delta t\right)I_{3\times3} \end{bmatrix}$$

(C.1)

The approximate matrix is created with the assumption that $\|\omega\|\Delta t < \pi/10$ [40]. This formulation also has the Rodrigues parameters as the error quaternion representation.

The derivation of this matrix for the dual-quaternion case makes the same assumption, with one more additional assumption that $\|v\|\Delta t^2 < < \sigma_\omega^2 \Delta t$. The $G$ matrix used in this
calculation is given by

\[
G = \begin{bmatrix}
-\frac{1}{2}I_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
-\frac{1}{2}I_{3\times3} & -\frac{1}{2}I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & I_{3\times3} & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_{3\times3}
\end{bmatrix}.
\] (C.2)

With these assumptions, the integral for the dual-quaternion process noise covariance is

\[
Q_k = \begin{bmatrix}
\left(\frac{1}{4}\sigma^2_\omega \Delta t + \frac{1}{4}\sigma^2_\xi \Delta t^3\right)I & \frac{1}{4}\sigma^2_\omega \Delta tI & -\frac{1}{4}\sigma^2_\xi \Delta t^2I & 0 \\
\frac{1}{4}\sigma^2_\omega \Delta tI & (\frac{1}{4}\sigma^2_\omega + \frac{1}{4}\sigma^2_\xi) \Delta t + \frac{1}{3}\sigma^2_\xi \Delta t^3I & -\frac{1}{4}\sigma^2_\xi \Delta t^2I & -\frac{1}{4}\sigma^2_\xi \Delta t^2I \\
-\frac{1}{4}\sigma^2_\xi \Delta t^2I & -\frac{1}{4}\sigma^2_\xi \Delta t^2I & \sigma^2_\xi \Delta tI & 0 \\
0 & -\frac{1}{4}\sigma^2_\xi \Delta t^2I & 0 & \sigma^2_\xi \Delta tI
\end{bmatrix}.
\] (C.3)
Appendix D

Maximum Entropy Distributions on $SO(1)$

For the unit $n$-sphere, $SO(n)$, distributions are difficult to define. Unlike in $\mathbb{R}^n$, distributions on closed manifolds they have a constrained unit-norm support.

$$x \in S^n \quad x^T x = 1 \quad (D.1)$$

Distributions on these spherical surfaces have given rise to a discipline known as *directional statistics*. Directional statistics seek to provide probability distributions for support variables that wrap, such as the angles of a polar coordinate systems. For example, a distribution in $SO(1)$ (the unit circle) can be described using the single polar variable $\theta$. In this distribution, for a given value of the state, $\theta \pm 2n\pi$, should all have the same probability.
D.1 Fixed Circular Mean

The circular mean of a unit vector $\mathbf{x} = [\cos(\theta) \ \sin(\theta)]^T$ is given by definition to be

$$\bar{\mathbf{x}} \triangleq \frac{I_1(\kappa)}{I_0(\kappa)} \left[ \begin{array}{c} \cos(\mu) \\ \sin(\mu) \end{array} \right]$$

(D.2)

where $I_i$ are the modified Bessel functions of the first kind and $\kappa$ is a measure of concentration for the distribution about mean. The parameter $\kappa$ will be discussed further later.

We can find a maximum entropy distribution for the unit circle with a given circular mean by solving the following maximization problem:

$$\min_{p(\theta)} \ J = -\int_0^{2\pi} p(\theta) \ln(p(\theta)) d\theta$$

s.t. $\int_0^{2\pi} p(\theta) d\theta - 1 = 0$

(D.3)

$$\int_0^{2\pi} \mathbf{x} p(\theta) d\theta - \frac{I_1(\kappa)}{I_0(\kappa)} [\cos(\mu) \ \sin(\mu)]^T = 0$$

The Lagrangian for this minimization problem is then

$$\mathcal{L} = -\int_0^{2\pi} p(\theta) \ln(p(\theta)) d\theta + \lambda_0 \left( \int_0^{2\pi} p(\theta) d\theta - 1 \right)$$

$$+ \lambda^T \left( \int_0^{2\pi} \mathbf{x} p(\theta) d\theta - \frac{I_1(\kappa)}{I_0(\kappa)} [\cos(\mu) \ \sin(\mu)]^T \right)$$

(D.4)

(D.5)

where $\lambda_0$ and $\lambda = [\lambda_1 \ \lambda_2]^T$ are Lagrange multipliers for the minimization. The necessary conditions for a minimum are given by the partial derivatives of the Lagrangian, such
that

\[
\frac{\partial L}{\partial p(\theta)} = -\ln(p(\theta)) - 1 + \lambda_0 + \mathbf{x}^T \mathbf{\lambda} = 0 \quad (D.6)
\]
\[
\frac{\partial L}{\partial \lambda_0} = \int_0^{2\pi} p(\theta) d\theta - 1 = 0 \quad (D.7)
\]
\[
\frac{\partial L}{\partial \mathbf{x}} = \int_0^{2\pi} \mathbf{x}^T p(\theta) d\theta - \frac{I_1(\kappa)}{I_0(\kappa)} \begin{bmatrix} \cos(\mu) \\ \sin(\mu) \end{bmatrix}^T = 0 \quad (D.8)
\]

Solving Equation (D.6) for \( p(\theta) \), yields

\[
p(\theta) = k \exp \left[ \lambda_1 \cos(\theta) + \lambda_2 \sin(\theta) \right] \quad (D.9)
\]

Substitution of (D.9) into (D.7) gives the integral

\[
e^{\lambda_0 - 1} \int_0^{2\pi} e^{\lambda_1 \cos(\theta) + \lambda_2 \sin(\theta)} d\theta = 1 \quad (D.10)
\]

The solution of this integral involves the modified Bessel functions of the first kind, and is given by

\[
k \left( 2\pi I_0 \left( \sqrt{\lambda_1^2 + \lambda_2^2} \right) \right) = 1 \quad \text{(D.11)}
\]

\[
k = \frac{1}{2\pi I_0 \left( \sqrt{\lambda_1^2 + \lambda_2^2} \right)} \quad \text{(D.12)}
\]

which brings the PDF of our distribution to be

\[
p(\theta) = \frac{1}{2\pi I_0 \left( \sqrt{\lambda_1^2 + \lambda_2^2} \right)} \exp \left[ \lambda_1 \cos(\theta) + \lambda_2 \sin(\theta) \right] \quad (D.13)
\]
Substitution of (D.13) into (D.8) gives us the integral

\[
\frac{1}{2\pi I_0 \left( \sqrt{\lambda_1^2 + \lambda_2^2} \right)} \int_0^{2\pi} \left[ \cos \theta \begin{bmatrix} \cos \theta + \lambda_1 \cos \theta + \lambda_2 \sin \theta \end{bmatrix} e^{\lambda_1 \cos \theta + \lambda_2 \sin \theta} d\theta = \frac{I_1(\kappa)}{I_0(\kappa)} \begin{bmatrix} \cos(\mu) \\ \sin(\mu) \end{bmatrix} \right]
\]

\begin{equation}
(\text{D.14})
\end{equation}

\[
\frac{1}{2\pi I_0 \left( \sqrt{\lambda_1^2 + \lambda_2^2} \right)} \int_0^{2\pi} \left[ \cos(\theta) \sinh(\lambda_1 \cos \theta) \cosh(\lambda_2 \sin \theta) \right] \sin(\theta) \cosh(\lambda_1 \cos \theta) \sinh(\lambda_2 \sin \theta) d\theta = \frac{I_1(\kappa)}{I_0(\kappa)} \begin{bmatrix} \cos(\mu) \\ \sin(\mu) \end{bmatrix}
\]

\begin{equation}
(\text{D.15})
\end{equation}

\[
\frac{1}{2\pi I_0 \left( \sqrt{\lambda_1^2 + \lambda_2^2} \right)} \left[ \frac{2\pi I_1 \left( \sqrt{\lambda_1^2 + \lambda_2^2} \right)}{\sqrt{\lambda_1^2 + \lambda_2^2}} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \right] = \frac{I_1(\kappa)}{I_0(\kappa)} \begin{bmatrix} \cos(\mu) \\ \sin(\mu) \end{bmatrix}
\]

\begin{equation}
(\text{D.16})
\end{equation}

\[
\frac{I_1(\kappa)}{\kappa I_0(\kappa)} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{I_1(\kappa)}{I_0(\kappa)} \begin{bmatrix} \cos(\mu) \\ \sin(\mu) \end{bmatrix}
\]

\begin{equation}
(\text{D.17})
\end{equation}

You can see from (D.17) that \( \lambda_1 = \kappa \cos(\mu) \) and \( \lambda_2 = \kappa \sin(\mu) \). Substitution into (D.13) gives the pdf

\[
p(\theta) = \frac{1}{2\pi I_0(\kappa)} \exp \left[ \kappa(\cos(\mu) \cos(\theta) + \sin(\mu) \sin(\theta)) \right]
\]

\begin{equation}
(\text{D.18})
\end{equation}

\[
= \frac{1}{2\pi I_0(\kappa)} \exp \left[ \kappa \cos(\theta - \mu) \right]
\]

\begin{equation}
(\text{D.19})
\end{equation}

This distribution is known within literature as the Von Mises distribution.
D.2 Fixed Variance

If instead, we wanted to fix a variance $\Sigma$ and not the circular mean of the distribution, we have a new maximization problem given by:

$$\min_{p(x)} J = -\int_{x \in \Sigma} p(x) \ln(p(x))dx$$

subject to:

$$\int_{x \in \Sigma} p(x)dx - 1 = 0 \quad \text{(D.20)}$$

$$\int_{x \in \Sigma} xx^T p(x)dx - \Sigma = 0$$

This gives a Lagrangian

$$\mathcal{L} = -\int_{x \in \Sigma} p(x) \ln(p(x))dx + \lambda_0 \left( \int_{x \in \Sigma} p(x)dx - 1 \right) + \text{Tr} \left( \Lambda \left( \int_{x \in \Sigma} xx^T p(x)dx - \Sigma \right) \right) \quad \text{(D.21)}$$

with first order optimality conditions,

$$\frac{\partial \mathcal{L}}{\partial p(x)} = -\ln(p(x)) - 1 + \lambda_0 + x^T \Lambda x = 0 \quad \text{(D.23)}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_0} = \int_{x \in \Sigma} p(x)dx - 1 = 0 \quad \text{(D.24)}$$

$$\frac{\partial \mathcal{L}}{\partial \Lambda} = \int_{x \in \Sigma} xx^T p(x)dx - \Sigma = 0 \quad \text{(D.25)}$$

Solving for $p(x)$ from (D.23) and substituting in our parameterization from $x$ gives

$$p(x) = k \exp \left[ x^T \Lambda x \right] \quad \text{(D.26)}$$
Substitution of (D.26) into (D.24) provides

\[ k \left\{ \int \exp \left[ \mathbf{x}^T \Lambda \mathbf{x} \right] dS \right\} = 1 \]  \hspace{1cm} (D.27)

This integral has a well known solution in terms of confluent hyper-geometric functions of matrix argument,

\[ k \left\{ |dS| {}_1 F_1 \left( \frac{1}{2} : \frac{n}{2} ; \Lambda \right) \right\} = 1 \]  \hspace{1cm} (D.28)

where $|dS|$ is the circumference of the unit circle. Our integration constant $k$ is solved to be

\[ k = \frac{1}{|dS| {}_1 F_1 \left( \frac{1}{2} : \frac{1}{2} ; \Lambda \right)} = \frac{1}{F(\Lambda)} \]  \hspace{1cm} (D.29)

For any square, symmetric matrix, we can perform an eigenvalue decomposition such that

\[ \Lambda = M Z M^T = M \text{ diag}(\{z_1, \ldots, z_n\}) M^T \]  \hspace{1cm} (D.30)

where $M$ is an orthogonal matrix of eigenvectors and $Z$ is a diagonal matrix of eigenvalues.

Substitution of (D.26), (D.29) and D.30 into D.25 with $y = M^T x$ gives

\[ \frac{1}{F(M Z M^T)} M \left\{ \int y y^T \exp \left[ y^T Z y \right] dS \right\} M^T = M \tilde{\Sigma} M^T \]  \hspace{1cm} (D.31)

\[ \frac{1}{F(Z)} \left\{ \int y y^T \exp \left[ y^T Z y \right] dS \right\} = \tilde{\Sigma} \]  \hspace{1cm} (D.32)

where $\tilde{\Sigma}$ is the variance of $y$ and the relationship that $F(\Lambda) = F(Z)$ is due to the properties of the definition for confluent hyper-geometric functions of matrix argument. They are functions of the eigenvalues for a given matrix $\Lambda$, and since $\Lambda$ and $Z$ have
the same eigenvalues, they produce the same results for a given matrix and a diagonal matrix of its eigenvalues.

With the following useful identity,

\[
\frac{d}{dZ} \left[ \exp \left[ y^T Z y \right] \right] = y y^T \exp \left[ y^T Z y \right]
\]  
(D.33)

we can rewrite D.32 to get

\[
\frac{1}{F(Z)} \left\{ \int \frac{d}{dZ} \left[ \exp \left[ y^T Z y \right] \right] dS \right\} = \frac{1}{F(Z)} \left\{ \frac{d}{dZ} \left[ \int \exp \left[ y^T Z y \right] dS \right] \right\}
\]  
(D.34)

\[
\Sigma = \frac{1}{F(Z)} \left\{ \frac{d}{dZ} \left[ F(Z) \right] \right\}
\]
(D.35)

The original covariance is then simply

\[
\Sigma = M \left[ \frac{dF(Z)/dZ}{F(Z)} \right] M^T
\]  
(D.36)

Finally, the PDF for a maximum entropy distribution on the unit-circle with a fixed second moment is then given by

\[
p(x) = \frac{1}{F(Z)} \exp \left[ x^T M Z M^T x \right]
\]  
(D.37)
References


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