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THE ATIYAH-SINGER INDEX FORMULA FOR SUBELLIPTIC OPERATORS ON CONTACT MANIFOLDS

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Abstract

The Atiyah-Singer index theorem gives a topological formula for the index of an elliptic differential operator. The topological index depends on a cohomology class that is constructed from the principal symbol of the operator.

On contact manifolds, the naturally arising geometric operators are not elliptic, but subelliptic. A filtration on the algebra of differential operators that is adapted to these geometric structures, naturally leads to a symbolic calculus that is noncommutative, and a corresponding subelliptic theory can be developed.

For such subelliptic operators we construct a symbol class in the K-theory of a noncommutative C^*-algebra naturally associated to the algebra of symbols. There is a canonical map from this noncommutative K-theory to the ordinary cohomology of the manifold, which gives a class to which the Atiyah-Singer formula can be applied. In this way we define the topological index of a subelliptic operator, and we prove that it is equal to its analytic index.

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Chapter 1

Introduction.

The celebrated Atiyah-Singer Index Theorem gives a topological formula for the analytic index of an elliptic differential operator P on a closed, smooth manifold M,

$$\mathrm{Index}(P) = \int_{T^*M} \mathrm{Ch}([P]) \wedge \mathrm{Td}(M).$$

Developing an approach to index theory for contact manifolds found in the works of Melrose and Epstein [Me], [EM], and Ponge [Po1], [Po2], we prove that the Atiyah-Singer formula can be applied to compute the index of *subelliptic* operators P on contact manifolds. Our method applies in more general cases, and, specifically, leads to a corresponding theorem for subelliptic operators on *foliations*.

Subellipticity, like ellipticity, is a condition that implies *hypoellipticity* (which means that, whenever Pu is smooth in an open set $U \subseteq M$ for some distribution u, then u itself is smooth in U). Hypoellipticity, in turn, implies that P is *Fredholm*, i.e., the analytic index,

$$\mathrm{Index}(P) = \dim \mathrm{Ker}\,(P) - \dim \mathrm{Ker}\,(P^*),$$

is well defined. If P is subelliptic, however, the right hand side of the index formula—the so-called 'topological index'—poses problems. An important and difficult step has been to construct the appropriate cohomology class $\mathrm{Ch}[P]$ in this case.

1.1 Ellipticity and hypoellipticity.

Whereas hypoellipticity is a local property, ellipticity is an infinitesimal condition. Suppose P is a differential operator that, in local coordinates, is given by

$$P = \sum_{|\alpha| \leq d} a_\alpha(x) \partial_x^\alpha.$$

To study the behaviour of P in a small neighborhood of a point $m \in M$, choose coordinates x such that $x = 0$ at m, and then 'blow up' these coordinates $y = t^{-1}x$, letting $t \to 0$. Expressing P in the y coordinates, we have

$$t^d P = P_0 + tP_1 + t^2 P_2 + \cdots,$$

where $P_j = \sum_{|\alpha|=d-j} a_\alpha(ty)\partial_y^\alpha$ is the homogeneous order $(d-j)$ part in P. As $t \to 0$, we have $t^d P$ acting on functions in the tangent space $T_m M$ as the operator,

$$P_m = \sum_{|\alpha|=d} a_\alpha(0)\partial_y^\alpha.$$

Thus, P determines a family $\{P_m\}$ of homogeneous, constant coefficient operators on the fibers of TM that, in some sense, captures the action of P on functions with 'infinitesimal' support. An operator P is elliptic precisely if these approximations P_m are *hypoelliptic*. In this sense, ellipticity is a condition of 'infinitesimal' hypoellipticity that guarantees local (and hence global) hypoellipticity.

1.2 Subelliptic operators.

There are many examples of differential operators that are hypoelliptic, but not elliptic. The best known example is the heat operator $\Delta + \partial_t$ on $\mathbb{R}^{n+1}$. More generally, Hörmander has shown that, if $X_0, \ldots, X_p$ is a family of vector fields on a manifold, such that these vector fields, together with their repeated brackets, span the tangent space at each point of the manifold, then the operator

$$P = X_0 + \sum_{i=1}^{p} X_i^2,$$

is hypoelliptic. Another important example is provided by the Heisenberg group, whose Lie algebra is spanned by invariant vectorfields X, Y, Z, with $[X, Y] = Z$. The operator

$$P = X^2 + Y^2 + i\alpha Z$$

is hypoelliptic, if and only if α is not an odd integer. All these examples show is that the *lower order part* of the operator P plays an importand role in establishing hypoellipticity.

The infinitesimal approximation suited to the analysis of these types of operators was worked out by Stein and his school in the 1970's. In our work, we assume that P is an operator whose characteristic directions form a locally trivial vector bundle $N^* \subseteq T^*M$. If $H \subseteq TM$ denotes the kernel of N^*, we may think of P as an operator that is 'elliptic' only in the H-directions (recall that an operator is elliptic iff it has *no* characteristic directions). In most geometric examples, subelliptic operators have a second order 'elliptic' part, and are of first order in the characteristic directions. Typically, the bundle $H \subseteq TM$ is associated to some geometric structure on M, as in the case of *contact manifolds*, *CR manifolds*, or *foliations*.

Now, by a 'blowing up' procedure analogous to the one described for elliptic operators, but this time blowing up by a factor t^{-1} in the H direction (where P is of order 2, say), and by a factor t^{-2} in the $N = TM/H$ direction (where P is of order 1), we obtain, in the limit, a family of operators P_m on the fibers of $H \oplus N$ that are *homogeneous* in a *graded* sense (where H has degree 1, N degree 2), and that are *invariant* for a certain *nilpotent group structure* on these fibers. In other words, naturally associated to $H \subseteq TM$ is a family of graded nilpotent groups $G_m = H_m \oplus N_m$, and P can be approximated, in an infinitesimal neighborhood of each point m, by an invariant, homogeneous operator P_m on the group G_m.

Rothschild and Stein [RS] (**better reference?**) showed that, just as in the elliptic case, hypoellipticity of all 'model operators' P_m implies hypoellipticity of P. Hypoellipticity of the invariant operators P_m is verified by considering their (noncommutative) Fourier transform $\pi(P_m), \pi \in \hat{G}$. A theorem by Helffer and Nourrigat [HN] states that the invariant homogeneous operator P_m is hypoelliptic iff $\pi(P_m)$ is injective, for each $\pi \neq 0$. If $G_m \cong \mathbb{R}^n$ is abelian, with the trivial grading, this is just the usual condition for ellipticity.

Fredholm operators (Dirac operators, signature operators) have always played a crucial role in noncommutative geometry, but in recent years elliptic theory has, in some instances, been replaced by the more general, and more subtle analysis of subelliptic operators. As important examples we mention the construction by Connes and Moscovici of a spectral triple for a general foliation ([CM]), and the proof by Kasparov and Julg of the Baum-Connes conjecture for subgroups of $SU(n,1)$ ([KJ]).

It is for subelliptic operators associated to a contact structure or foliation that we prove the validity of the Atiyah-Singer formula. The classical theorem for elliptic operators corresponds to the special case where $H = TM$.

It is likely that our appraoch could work for more general 'Heisenberg structures' (arbitary subbundles $H \subseteq TM$), but we have not aimed for maximal generality, but have focused, instead, on the examples of most practical significance. An advantage of our choice was that it allowed us to avoid the use of the *pseudodifferential* calculus for Heisenberg structures, as developed, for example, in [BG]. To deal with more general cases one might have to appeal to this theory.

1.3 A symbol for subelliptic operators.

One of the difficulties in working out an index formula for subelliptic operators is to find the appropriate analog for the cohomology class,

$$\mathrm{ch}([P]) \in H^*(M).$$

In the classical case, the construction of this class relies crucially on the ellipticity of P. Atiyah and Singer construct a class in topological K-theory, $[P] \in K^0(T^*M)$, whose Chern character is the desired cohomology class. However, a different construction, due to Quillen, makes use of the Cayley Transform of the self-adjoint operators

$$\begin{pmatrix} 0 & P_m^* \\ P_m & 0 \end{pmatrix},$$

to obtain an (equivalent) class in *C^*-algebraic* K-theory, $[P] \in K_0(C_0(T^*M))$. We have succeeded in generalizing the latter construction to the case of subelliptic operators P, this time, however, obtaining a class $[P]$ in the K-theory of a *noncommutative* C^*-algebra.

Proving that the constructed object determines a well-defined K-theory element has been one of the most challenging parts of the project. The (relatively straightforward) analysis used for elliptic operators breaks down in this case, and we have made use of techniques involving analysis on nilpotent groups, as can be found, for example, in [HN], [FS], [RS].

1.4 Convolution algebras of groupoids.

Let $T_H M$ denote the bundle of graded nilpotent groups G_m over M described above. As a space, $T_H M$ is diffeomorphic to TM, but, clearly, its fibers carry a different (generally non-abelian) group structure. This object is an example of a *smooth groupoid*.

Expressed most succinctly, a groupoid is a small category with invertible arrows (or: a group with 'partial' composition), and in a smooth groupoid all groupoid operations (composition, inverses, etc.) are smooth. A bundle of groups, like TM or $T_H M$, is a simple example of a groupoid. Another basic example is the pair groupoid $M \times M$, with composition $(x, y) \circ (y, z) = (x, z)$. Groupoids play a central role in noncommutative geometry.

Associated to a smooth groupoid is its *convolution algebra*, which generalizes the group algebra known from representation theory. If G is a Lie group, and $f, g \in C_c^\infty(G)$, then the convolution product of f and g is defined by,

$$(f * g)(\gamma_1) = \int f(\gamma_1 \gamma_2^{-1}) g(\gamma_2)\, d\mu(\gamma_2).$$

Elements $f \in C_c^\infty(G)$ can be represented as bounded operators $\pi(f)$ on the Hilbert space $L^2(G)$, by the formula $\pi(f)g = f * g$. The (reduced) *group C^*-algebra* $C_r^*(G)$ is the completion in the operator norm $\|f\|_{C^*} = \|\pi(f)\|$ of the convolution algebra $C_c^\infty(G)$.

For smooth groupoids, a similar construction exists. The main difference in the definition of the convolution product is that integration does not take place over the entire groupoid $\mathcal{G}$, but only over the set of arrows $\gamma_2 \in \mathcal{G}$ that have the same 'source' as the arrow γ_1. As before, the convolution algebra $C_c^\infty(\mathcal{G})$ is faithfully represented on a particular Hilbert space associated to $\mathcal{G}$. The operator norm completion of the convolution algebra with respect to this representation is the (reduced) groupoid C^*-algebra $C^*(\mathcal{G})$. (For details, see [Co], [Ma].)

The K-theory class associated to a subelliptic operator P, constructed on the basis of the family of invariant operators $\{P_m\}$ on the nilpotent groups G_m, is an element in the K-theory of the (noncommutative) convolution C^*-algebra of the bundle $T_H M$,

$$[P] \in K_0(C^*(T_H M)).$$

However, Connes' Thom Isomorphism for bundles of nilpotent groups (see [Ni2]) gives a natural identification

$$K_0(C^*(T_H M)) \cong K_0(C^*(TM)) \cong K^0(C_0(T^*M)) \cong K^0(T^*M),$$

which lands $[P]$ in $K^0(T^*M)$. We can now take its chern class, and the right hand side of the Atiyah-Singer formula makes sense for subelliptic operators. What remains to prove is equality of the analytic and topological indices thus defined.

1.5 Tangent groupoid and deformation.

Our proof of the index theorem for subelliptic operators follows the approach developed by Alain Connes in [Co], based on his notion of the *tangent groupoid*. The tangent groupoid formalises the notion of 'blowing up' of coordinates that we outlined in our discussion of ellipticity.

The tangent groupoid is obtained by glueing a parametrized family of pair groupoids $M \times M$ to the tangent bundle TM. The tangent groupoid is thus the union of two manifolds,

$$TM \cup M \times M \times (0,1],$$

that are glued together by letting a triple $(a(t), b(t), t)$ converge to a vector $v \in T_m M$, as $t \downarrow 0$, if both $a(t)$ and $b(t)$ converge to m, while

$$t^{-1}\left((a(t) - b(t)\right) \to v$$

for some choice of coordinates. In this sense, the glueing 'blows up' the diagonal in $M \times M$.

The convolution algebra of the tangent groupoid is a field of C^*-algebras over the interval $[0,1]$, where, at $t = 0$, we have the algebra $C^*(TM) \cong C_0(T^*M)$, while at each $t > 0$ we have $C^*(M \times M) \cong \mathcal{K}(L^2(M))$, the algebra of compact operators on $L^2(M)$. This field of algebras corresponds to the well known deformation quantization from $C_0(T^*M)$ to the compact operators, and, by a standard procedure, such a deformation induces a map in K-theory,

$$\mathrm{Ind_t} \ : \ K_0(C_0(T^*M)) \cong K^0(T^*M) \to K_0(\mathcal{K}) \cong \mathbb{Z}.$$

Alain Connes proves that this map is identical to the analytic index map (see [Co], Chapter II.5, [Hi]).

Generalizing this procedure, we construct a groupoid whose $t = 0$ fiber is the convolution algebra $C^*(T_H M)$, from which we obtain a deformation that induces a map

$$\mathrm{Ind_q} \ : \ K_0(C^*(T_H M)) \to \mathbb{Z}.$$

We prove that,

$$\mathrm{Index}\, P = \mathrm{Ind_q}\,([P]),$$

where $[P]$ denotes the class associated to a subelliptic operator P,

Finally, by considering the so-called 'adiabatic' groupoid associated to our tangent groupoid (which is a larger groupoid extending the tangent groupoid to a groupoid over the square $[0,1] \times [0,1]$), we have shown that the natural isomorphism $K_0(C^*(T_H M)) \cong K^0(T^*M)$ commutes with the two index maps $\mathrm{Ind_t}$, $\mathrm{Ind_q}$ obtained from the respective deformations.

$$\begin{tikzcd} K^0(T^*M) \arrow[r, "\mathrm{Ind_t}"] & \mathbb{Z} \\ K_0(C^*(T_H M)) \arrow[u, "\cong"] \arrow[ur, dotted, "\mathrm{Ind_q}"'] & \end{tikzcd}$$

In other words, if we identify $[P] \in K_0(C^*(T_H M))$ with the naturally corresponding class in $K^0(T^*M)$, we see that the Atiyah-Singer formula computes the analytic index.

Chapter 2

Heisenberg structures and osculating groups.

2.1 Introduction.

Central to the analysis of subelliptic operators is a so-called *Heisenberg structure.* A Heisenberg structure on a manifold M is determined by a distribution H, i.e., a locally trivial subbundle H of the tangent space TM. In the literature it is usually assumed that H has codimension one in TM (for example, see [BG]). But this restriction is unnecessary here.

Roughly speaking, given a subelliptic operator P, the distribution H is the bundle of directions in which P is elliptic. In applications, the bundle H is usually associated to the geometry of the manifold M. For example, if M is *foliated* H is the bundle of longitudinal vectors, whereas if M is a *contact manifold* the bundle H consists of vectors in the kernel of the contact form θ. We will study these examples in more detail in section 2.7.

The purpose of this chapter is to construct a bundle of *nilpotent Lie groups* naturally associated to a Heisenberg structure (M, H). These groups were first introduced by Stein as a useful aid in the analysis of the $\bar{\partial}_b$ operator, and have subsequently played a key role in the literature on hypoelliptic operators (see, for example, [RS], [BG]). Typically, one chooses local coordinates $U \to \mathbb{R}^n$ on an open set $U \subseteq M$, and defines a group structure on the coordinate space $\mathbb{R}^n$ by means of an explicit formula. One shows that a change of coordinates on U induces a group isomorphism, so that, up to isomorphism, the groups are well-defined.

There is a simple invariant definition of the associated Lie algebra. The basic equality,

$$[fX, gY] = fg[X,Y] + f(X.g)Y - g(Y.f)X,$$

shows that if X, Y are sections of H then, *modulo* H, the bracket value of the bracket $[X,Y](m)$ at $m \in M$ only depends on the values $X(m)$ and $Y(m)$ at m. In other words, the commutator of vector fields induces a *pointwise* bracket,

$$H_m \otimes H_m \to N_m : \ X \otimes Y \mapsto [X,Y] \bmod H,$$

where $m \in M$, and $N = TM/H$ denotes the quotient bundle. This can be extended to a Lie bracket on $\mathfrak{g}_m = H_m \oplus N_m$, by taking $[\mathfrak{g}_m, N_m] = 0$. Clearly, the Lie algebra $\mathfrak{g}_m$ is 2-step nilpotent. The *osculating group* G_m at m is the simply-connected nilpotent group associated to

$\mathfrak{g}_m$. We will investigate how these invariantly defined groups can be identified with the local groups structures on the coordinate space, as defined in the literature.

We will find it usefull to have a concrete geometric interpretation of the osculating groups, independent of the Lie algebras. We develop an approach which identifies the group elements with explicit geometric objects, which we call *parabolic arrows.* The name suggests an analogy with tangent vectors ('linear arrows'). We show how the group operation arises naturally from the composition of *flows* associated to these arrows.

Notation. Throughout this chapter, M denotes a smooth manifold with Heisenberg structure $H \subseteq TM$. We will write $N = TM/H$ for the quotient bundle, and denote the fiber dimensions by $p = \dim H$, $q = \dim N$, and $n = p + q = \dim M$. We will *not* assume that $q = 1$.

2.2 Parabolic arrows.

When studying a Heisenberg structure (M, H) it is convenient to work with a special type of coordinates.

Definition 1 *Let m be a point on M, and $U \subseteq M$ an open set in M containing m. A coordinate chart $\phi\colon U \to \mathbb{R}^n$, $\phi(m') = (x_1, \ldots, x_n)$ is called an* H-coordinate chart at m, *if $\phi(m) = 0$, and the first p coordinate vectors $\partial/\partial x_i$ $(i = 1, \ldots, p)$ at the point m span the fiber H_m of H at m.*

Tangent vectors can be defined as equivalence classes of smooth curves. By analogy, we introduce an equivalence relation involving second-order derivatives.

Definition 2 *Let $c_1, c_2\colon [-1, 1] \to M$ be two smooth curves that are tangent to H at $t = 0$. For such curves we say that $c_1 \sim_H c_2$ if $c_1(0) = c_2(0)$ and if, choosing H-coordinates centered at $c_1(0) = c_2(0)$, we have*

$$c_1'(0) - c_2'(0) = 0,$$
$$c_1''(0) - c_2''(0) \in H.$$

An equivalence class $[c]_H$ is called a parabolic arrow *at the point $c(0)$. The set of parabolic arrows at $m \in M$ is denoted $T_H M_m$, while*

$$T_H M = \bigcup_{m \in M} T_H M_m.$$

We can give $T_H M$ the topology induced by the C^2-topology on the set of curves, but for the moment we just think of $T_H M$ as a set.

Lemma 3 *The equivalence relation $\sim_H$ is well-defined, i.e., independent of the choice of the H-coordinates.*

Proof. The condition that $c_1'(0) = c_2'(0)$ is clearly invariant. We will show that, assuming $c_1'(0) = c_2'(0)$, the condition $c_1''(0) - c_2''(0) \in H$ on the second derivatives is independent of the choice of H-coordinates.

If ψ is a change of H-coordinates, then:

$$\begin{aligned}\frac{d^2(\psi\circ c)}{dt^2} &= \frac{d}{dt}\left(\sum_j \frac{\partial\psi}{\partial x_j}(c(t))\frac{dc^j}{dt}\right)\\ &= \sum_{j,k}\frac{\partial^2\psi}{\partial x_j\partial x_k}(c(t))\frac{dc^j}{dt}\frac{dc^k}{dt} + \sum_j \frac{\partial\psi}{\partial x_j}(c(t))\frac{d^2c^j}{dt^2}.\end{aligned}$$

At $t=0$ we assumed $dc_1/dt = dc_2/dt$, so that the first term on the right hand side is equal for $\psi\circ c_1$ and $\psi\circ c_2$ (at $t=0$). Therefore:

$$(\psi\circ c_1)''(0) - (\psi\circ c_2)''(0) = \frac{\partial\psi}{\partial x}(m)\ \cdot\ \left(c_1''(0) - c_2''(0)\right).$$

Since ψ is a change of H-coordinates at m, $\partial\psi/\partial x$ preserves H_m, so that $c_1''(0)-c_2''(0)\in H_m$ implies $(\psi\circ c_1)''(0) - (\psi\circ c_2)''(0)\in H_m$.

□

If we fix H-coordinates at $m\in M$, and consider the second-order expansion (in coordinates) of a curve c with $c(0)=m$,

$$c(t) = c'(0)\,t + \frac{1}{2}c''(0)\,t^2 + \mathcal{O}(t^3),$$

we see that any such curve is equivalent, as a parabolic arrow, to a curve $\tilde{c}$ of the form

$$\tilde{c}(t) = (th, t^2n) = (th_1,\ldots,th_p,t^2n_1,\ldots,t^2n_q).$$

This observation forms the basis for the following definition.

Definition 4 *Suppose we are given H-coordinates at $m\in M$. Let $h\in\mathbb{R}^p, n\in\mathbb{R}^q$, and let $c(t)$ be the curve in M defined (in H-coordinates) by*

$$c(t) = (th, t^2n).$$

We call $(h,n) = (h_1,\ldots h_p,n_1,\ldots,n_q)\in\mathbb{R}^{p+q}$ the Taylor coordinates *for the parabolic arrow $[c]_H\in T_HM_m$, induced by the given H-coordinates at m.*

This is analogous to the way in which coordinates on the tangent space T_mM are induced by coordinates on M, with the important difference that Taylor coordinates on T_HM_m are defined for only one fiber (i.e., one point $m\in M$) at a time.

Analogous to the directed line segments that represent tangent vectors, a pictorial representation for the class $[c]_H$ would be a directed segment of a parabola. Hence our name 'parabolic arrow.' Parabolic arrows are what smooth curves look like when you blow up the manifold using the dilations $(h,n)\mapsto(th,t^2n)$, and let $t\to\infty$.

When working with H-coordinates $\phi(m) = x\in\mathbb{R}^n$, we use the notation $x=(x^H,x^N)\in\mathbb{R}^{p+q}$, where

$$x^H = (x_1,\ldots,x_p)\in\mathbb{R}^p,\ x^N = (x_{p+1},\ldots,x_{p+q})\in\mathbb{R}^q.$$

Lemma 5 *If ψ is a change of H-coordinates at m, then the induced change of Taylor coordinates $\psi(h,n) = (h',n')$ for a given parabolic vector in $T_H M_m$ is given by the* quadratic *formula:*

$$h' = D\psi(h),$$
$$n' = [D\psi(n) + D^2\psi(h,h)]^N,$$

where $[v]^N$ denotes the normal component of the vector $v = (v^H, v^N) \in \mathbb{R}^{p+q}$.

Proof. This is just the formula for $(\psi \circ c)''(0)$ from the proof of Lemma 3.

□

Corollary 6 *The smooth structures on the set of parabolic arrows $T_H M_m$ at a point $m \in M$ defined by different Taylor coordinates are compatible, i.e., $T_H M_m$ has a natural structure of a smooth manifold.*

It is clear from Lemma 5 that Taylor coordinates define a structure on $T_H M_m$ that is more than just a smooth structure. This will be fully clarified when we introduce the *group* structure on $T_H M_m$, but part of this extra structure is captured if we consider how parabolic arrow behave when rescaled.

Definition 7 *The family of dilations δ_s, $s > 0$, on the space of parabolic arrows $T_H M_m$ is defined by*

$$\delta_s([c]_H) = [c_s]_H,$$

where $[c]_H$ is a parabolic arrow in $T_H M_m$ represented by the curve $c(t)$, and c_s denotes the reparametrized curve $c_s(t) = c(st)$.

When working in Taylor coordinates $[c]_H = (h,n)$, we simply have

$$\delta_s(h,n) = (sh, s^2 n).$$

Clearly, these dilations are smooth maps and $\delta_{st} = \delta_s \circ \delta_t$.

Considering Taylor coordinates on $T_H M_m$, it is tempting to identify parabolic arrows with vectors in $H \oplus N$. Lemma 5 shows that such an identification is not invariant if we use Taylor coordinates to define it. But we have at least the following result.

Lemma 8 *There is a* natural *identification*

$$T_0(T_H M_m) \cong H_m \oplus N_m$$

of the tangent space $T_0(T_H M_m)$ at the 'origin' (i.e., at the equivalence class $[0]_H$ of the constant curve at m) with the vector space $H_m \oplus N_m$. It is obtained by identifying the coordinates on $T_0(T_H M_m)$ induced by Taylor coordinates on $T_H M_m$, with the natural coordinates on $H_m \oplus N_m$.

Proof. From Lemma 5, we see that Taylor coordinates on $T_0(T_H M_m)$ transform according to the formula

$$h' = D\psi(h),\ n' = D\psi(n)^N,$$

because the quadratic term $D^2\psi(h,h)$ has derivative 0 at 0. This is precisely how the induced coordinates on $H_m \oplus N_m$ behave under coordinate transformation ψ.

□

2.3 Composition of parabolic arrows.

We will now show that the manifold $T_H M_m$ has the structure of a Lie group. Our method is based on composition of local flows of M. By a flow Φ of M we mean a diffeomorphism $\Phi\colon M \times \mathbb{R} \to M$, such that $m \mapsto \Phi_t(m) = \Phi(m,t)$ is a diffeomorphism for each $t \in \mathbb{R}$, while $\Phi_0(m) = m$. A *local flow* is only defined on an open subset $V \subseteq M \times \mathbb{R}$. Two flows Φ, Ψ can be composed:

$$(\Phi \circ \Psi)(m,t) = \Phi(\Psi(m,t),\, t).$$

Using the notation Φ_t for the local diffeomorphism $\Phi_t(m) = \Phi(m,t)$, we can write $(\Phi \circ \Psi)_t = \Phi_t \circ \Psi_t$.

A (local) flow is said to be generated by the vector field $X \in \Gamma(TM)$, if

$$\frac{\partial \Phi}{\partial t}(m,t) = X(m).$$

However, we are specifically interested in flows for which the generating vector field $X_t(m) = \frac{\partial \Phi}{\partial t}(m,t)$ is *not constant*, but depends on t. We will only require that X_0 is a section in H, but we will allow X_t to pick up a component in the N-direction. The reason is that we are not primarily interested in the *tangent vectors* to the flow lines $c_m(t) = \Phi(m,t)$, but in the *parabolic arrows* that they define.

We start with a formula that gives a quadratic approximation (in t) for the composition of two arbitrary flows.

Lemma 9 *Let Φ^X, Φ^Y be two flows in $\mathbb{R}^n$ that are defined near the origin, and let X and Y be their generating vector fields at $t = 0$:*

$$X(x) = (\partial_t \Phi^X)(x,0), \text{ and } Y(x) = (\partial_t \Phi^Y)(x,0).$$

Then the composition of Φ^X and Φ^Y has the following second-order approximation,

$$(\Phi^X_t \circ \Phi^Y_t)(0) = \Phi^X_t(0) + \Phi^Y_t(0) + t^2\,(\nabla_Y X)(0) + \mathcal{O}(t^3),$$

where ∇ denotes the standard connection on $T\mathbb{R}^n$.

Remark. Observe that $X = \partial_t \Phi^X$ is required only at $t = 0$!

Proof. Write $F(r,s) = \Phi^X_r(\Phi^Y_s(0))$. The Taylor series for F gives

$$\begin{aligned} F(t,t) &= t\,\partial_r F(0,0) + t^2 \partial_s \partial_r F(0,0) + t\,\partial_s F(0,0) + \frac{1}{2} t^2 \partial_r^2 F(0,0) \\ &\quad + \frac{1}{2} t^2 \partial_s^2 F(0,0) + \mathcal{O}(t^3) \\ &= F(t,0) + F(0,t) + t^2 \partial_s \partial_r F(0,0) + \mathcal{O}(t^3), \end{aligned}$$

or

$$\Phi^X_t \Phi^Y_t(0) = \Phi^X_t(0) + \Phi^Y_t(0) + t^2\, \partial_s \partial_r \Phi^X_r \Phi^Y_s(0)\big|_{r=s=0} + \mathcal{O}(t^3).$$

At r=0 we have $\partial_r \Phi_r^X = X$, so:

$$\partial_s \partial_r \Phi_r^X(\Phi_s^Y(0))\big|_{r=0} = \partial_s \left(X(\Phi_s^Y(0)) \right).$$

Here $X(\Phi_s^Y(0))$ denotes the vector field X evaluated at the point $\Phi_s^Y(0)$, which can be thought of as a point on the curve $s \mapsto \Phi_s^Y(0)$. The operator ∂_s is applied to the components of this vector, and the chain rule gives

$$\partial_s X(\Phi_s^Y(0))\big|_{s=0} = \sum_i \partial_i X(0) \cdot \partial_s \Phi_s^Y(0)^i\big|_{s=0} = \sum_{i=1}^{p} (\partial_i X)(0)\, Y^i(0) = (\nabla_Y X)(0).$$

□

We are interested in flows Φ for which the *flow lines* $\Phi^m(t) = \Phi(m,t)$ define parabolic arrows. Hence the following definition.

Definition 10 *A* parabolic flow *of (M,H) is a local flow $\Phi\colon V \to M$ (with V an open subset in $M \times \mathbb{R}$) whose generating vector field at $t = 0$,*

$$\frac{\partial \Phi}{\partial t}(m,0)$$

(defined at each point m for which $(m,0) \in V$) is a section of H.

Given a *parabolic* flow Φ, each of the flow lines Φ^m is tangent to H at $t = 0$, and so determines a parabolic arrow $[\Phi^m]_H$ at each $m \in M$ (with $(m,0) \in V$). Once we have defined the smooth structure on $T_H M$ it will become clear that $m \mapsto \Phi^m$ is a smooth section of the bundle $T_H M$. It is an analogue of the notion of a generating vector field, but it is generally only defined at $t = 0$.

We now show how composition of parabolic flows induces a group structure on the fibers of $T_H M$.

Proposition 11 *Let Φ, Ψ be two* parabolic *flows. Then the composition $(\Phi \circ \Psi)_t = \Phi_t \circ \Psi_t$ (defined on an appropriate domain) is also a parabolic flow, and the parabolic vector $[(\Phi \circ \Psi)^m]_H$ at a point $m \in M$ only depends on the parabolic vectors $[\Phi^m]_H$ and $[\Psi^m]_H$ at the same point.*

Proof. Let ∇ denote the standard local connection on TM induced by the H-coordinates at m. Because $\nabla_{fY}(gX) = fg\nabla_Y(X) + f(Y.g)X$, we see that the operation

$$\Gamma^\infty(H) \otimes \Gamma^\infty(H) \to \Gamma^\infty(N) \,:\, (X,Y) \mapsto [\nabla_Y X]^N$$

is $C^\infty(M)$-bilinear. In other words, the N-component of $\nabla_Y X$ at the point $m \in M$ only depends on the values $X(m)$ and $Y(m)$ at m. We denote this N-component by ∇^N:

$$\begin{aligned}
&\nabla^N \,:\, H_m \otimes H_m \to N_m, \\
&\nabla^N(X(m), Y(m)) = [\nabla_Y X]^N(m).
\end{aligned}$$

Lemma 9 implies:

$$\begin{aligned}
&\Phi_t \Psi_t(0)^H = \Phi_t(0)^H + \Psi_t(0)^H + \mathcal{O}(t^2), \\
&\Phi_t \Psi_t(0)^N = \Phi_t(0)^N + \Psi_t(0)^N + t^2 \nabla^N(X(0), Y(0)) + \mathcal{O}(t^3).
\end{aligned}$$

Writing

$$\Phi_t(0)^H = th + \mathcal{O}(t^2),\ \Phi_t(0)^N = t^2 n + \mathcal{O}(t^3),$$
$$\Psi_t(0)^H = th' + \mathcal{O}(t^2),\ \Psi_t(0)^N = t^2 n' + \mathcal{O}(t^3),$$

this becomes

$$\Phi_t\Psi_t(0)^H = t(h + h') + \mathcal{O}(t^2),$$
$$\Phi_t\Psi_t(0)^N = t^2\left(n + n' + \nabla^N(h, h')\right) + \mathcal{O}(t^3).$$

The proposition is a direct corollary of these formulas.

□

It is clear from Proposition 11 that composition of parabolic flows induces a group structure on the set $T_H M_m$, for each $m \in M$, analogous to addition of tangent vectors in $T_m M$. To see that $T_H M_m$ is actually a *Lie group*, we use the explicit formulas obtained in the proof of Proposition 11 .

Proposition 12 *Let Φ, Ψ be two parabolic flows. Given H-coordinates at m, let $X_i \in \Gamma(H)$ $(i = 1, \ldots, p)$ be local sections in H that extend the coordinate tangent vectors ∂_i at m. Let X_i^l $(l = 1, \ldots, n)$ denote the coefficients of the vector field X_i, i.e.,*

$$X_i = \sum X_i^l \partial_l.$$

Let (b_{ij}^k) be the array of constants

$$b_{ij}^k = \partial_j X_i^{p+k}(m)$$

for $i, j = 1, \ldots, p$ and $k = 1 \ldots, q$. It represents a bilinear map $b\colon \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^q$ via

$$b(v, w)^k = \sum_{i,j=1}^{p} b_{ij}^k v^i w^j,$$

with $k = 1, \ldots, q$.

If (h, n) and (h', n') are the Taylor coordinates of $[\Phi^m]_q$ and $[\Psi^m]_q$, respectively, then the Taylor coordinates (h'', n'') of $[(\Phi \circ \Psi)_m]_q$ are given by

$$h'' = h + h',$$
$$n'' = n + n' + b(h, h').$$

Proof. This is a direct corollary of the formulas in the proof of Proposition 11. Simply observe that

$$b(\partial_i, \partial_j)^k = \partial_j X_i^{p+k}(m) = [\nabla_{X_j} X_i]^{p+k}(m) = \nabla(\partial_i, \partial_j)^{p+k},$$

which implies that $b(v, w) = \nabla^N(v, w)$.

□

Corollary 13 *The operation*

$$[\Phi^m]_H * [\Psi^m]_H = [(\Phi \circ \Psi)^m]_H$$

defines the structure of a Lie group on $T_H M_m$.

We call the groups $T_H M_m$ the *osculating groups* associated to the Heisenberg structure (M, H).

Note that, although the value of the array (b_{ij}^k) depends on the choice of coordinates, the group operation on parabolic vectors is defined in a coordinate-independent way. Furthermore, the values of b_{ij}^k do *not* depend on the choice of sections X_i, but only on the choice of coordinates.

Beals and Greiner introduce the group structure by means of the formulas we have derived in Proposition 12 (see [BG], chapter 1). Note that in their treatment $q = 1$, so that the k-index in the array b_{ij}^k is missing. The osculating group itself is simply identified with the coordinate space $\mathbb{R}^n$, and its status as an independent geometric object was left obscure.

Proposition 14 *The natural dilations δ_s of the osculating groups $T_H M_m$, induced by reparametrization of curves, are Lie group automorphisms.*

Proof. That the dilations are group automorphisms follows immediately from the geometric definition of the group operation on $T_H M_m$ in Corollary 13 (by reparametrizing the flows). Alternatively, using Taylor coordinates we have $\delta_s(h, n) = (sh, s^2 n)$, which is clearly a smooth automorphism for the group operation

$$(h, n,) * (h', n') = (h + h', n + n' + b(h, h')).$$

$\square$

The construction of the osculating bundle $T_H M$ is *functorial* for (local) diffeomorphisms. Given a diffeomorphism of manifolds with Heisenberg structure,

$$\phi : \ (M, H) \to (M', H'),$$

such that $D\phi \colon H \to H'$, we could define the *parabolic derivative* $T_H \phi$ of ϕ as the map

$$T_H \phi : \ T_H M \to T_H M' : \ [c]_H \mapsto [\phi \circ c]_H,$$

where $c(t)$ is a curve in M representing a parabolic arrow $[c]_H \in T_H M_m$. A straightforward calculation, similar to the proof of Lemma 3, shows that this is a well-defined map (independent of the choice of the curve c), and functoriality is obvious, i.e.,

$$T_H(\phi \circ \phi') = T_H \phi \circ T_H \phi'.$$

Clearly, if ϕ is a diffeomorphism, then $T_H \phi$ is a group isomorphism in each fiber.

2.4 The Lie algebras of the osculating groups.

According to Lemma 8, we may identify the Lie algebra $\mathrm{Lie}\,(T_H M_m)$ as a vector space with $H_m \oplus N_m$. In the introduction we defined a Lie algebra structure on $H_m \oplus N_m$, and we now show that it is compatible with the group structure on $T_H M_m$. We make use of some general results on two-step nilpotent groups that are discussed in the appendix to this chapter.

Proposition 15 *Let X and Y be two (local) sections of H. Then the value of the normal component $[X,Y]^N(m)$ of the bracket $[X,Y]$ at the point m only depends on the values of X and Y at the point m.*

The Lie algebra structure on $\mathrm{Lie}\, T_H M_m \cong H_m \oplus N_m$ is given by

$$[(h,n),(h',n')] = \left(0,[X,Y]^N(m)\right),$$

where $X, Y \in \Gamma(H)$ are arbitrary vector fields with $X(m) = h, Y(m) = h'$.

Proof. This is a straightforward application of Lemma 31 to the group structure on $T_H M_m$ as described in Proposition 11. We have

$$b(h',h) - b(h,h') = (\nabla_X Y - \nabla_Y X)^N(m) = [X,Y]^N(m).$$

We have already shown that $b(h,h') = (\nabla_Y X)^N(m)$ only depends on $h = X(m)$ and $h' = Y(m)$. □

We are now in a position to define the smooth structure on the total space

$$T_H M = \bigcup T_H M_m.$$

There is a natural bijection

$$\exp : \; H_m \oplus N_m \to T_H M_m,$$

namely the exponential map from the Lie algebra $H_m \oplus N_m$ to the Lie group $T_H M_m$. We give the total space $T_H M$ the smooth structure that it derives from its identification with $H \oplus N$.

Lemma 16 *The smooth structure on $T_H M$, obtained by the fiberwise identification with $H \oplus N$ via exponential maps, is compatible with the Taylor coordinates on each $T_H M_m$, for any choice of H-coordinates at m.*

Proof. Choosing H-coordinates at m, we get linear coordinates on $H_m \oplus N_m$. Taking these coordinates and Taylor coordinates on $T_H M_m$, we have identified $\mathrm{Lie}\, T_H M_m \cong H_m \oplus N_m$. According to Proposition 30, the exponential map $H_m \oplus N_m \to T_H M_m$ is expressed in these coordinates as

$$\exp(h,n) = (h, n + \frac{1}{2} b(h,h)),$$

which is clearly a diffeomorphism. □

A *graded Lie algebra* $\mathfrak{g} = \mathfrak{g}_r \oplus \cdots \oplus \mathfrak{g}_1$ (where elements in $\mathfrak{g}_i$ are of degree i) has a Lie bracket that is compatible with the grading,

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j},$$

with $[\mathfrak{g}_i, \mathfrak{g}_j] = \{0\}$ if $i + j > r$. Dilations δ_t associated to the grading are linear maps defined by

$$\delta_t(X) = t^i X, \text{ for } X \in \mathfrak{g}_i.$$

Clearly, $\delta_t, t > 0$ is a one-parameter group of Lie algebra automorphisms.

The natural decomposition $\text{Lie}\,(T_H M_m) = H_m \oplus N_m$ defines a grading, with $\mathfrak{g}_1 = H_m$ of degree 1 and $\mathfrak{g}_2 = N_m$ of degree 2. The corresponding dilations are $\delta_t(h, n) = (th, t^2 n)$. The dilations of the osculating group $T_H M_m$ induced by reparametrization of curves and the dilations of the graded Lie algebra $H_m \oplus N_m$ are related via the exponential map (see Proposition 30):

$$\exp(\delta_t(h, n)) = \exp(th, t^2 n) = (th, t^2 n + \frac{1}{2} b(th, th)) = \delta_t(h, n + \frac{1}{2} b(h, h)) = \delta_t \exp(h, n).$$

It will be useful to characterize the parabolic arrows whose logarithms are vectors in H.

Proposition 17 *If $c\colon \mathbb{R} \to M$ is a curve such that $c'(t) \in H$ for all $t \in (-\varepsilon, \varepsilon)$, then the parabolic arrow $[c]_H \in T_H M_m$ is the exponential of the tangent vector $c'(0) \in H_m$, where $m = c(0)$.*

Proof. Choose H-coordinates at $m = c(0)$, and let $(h, n) \in \mathbb{R}^{p+q}$ be the corresponding Taylor coordinates of the parabolic arrow $[c]_H$. Because $c'(t) \in H$ for t near 0, we can choose an H-frame $X_1, \ldots, X_n$ in a neighborhood U of m in such a way that $c'(t) = \sum h_i X_i(c(t))$ at every point $c(t) \in U$. With this set up, we compute the second derivative:

$$\frac{d^2 c}{dt^2} = \frac{d}{dt}(\sum_i h_i X_i) \circ c = \sum_j \frac{\partial}{\partial x_j} \left(\sum_i h_i X_i \right) \frac{dc_j}{dt} = \sum_{i,j} h_i \frac{\partial X_j}{\partial x_i} \frac{dc_j}{dt}.$$

Then, at $t = 0$, the normal component of $c''(0)$ is given by,

$$c''(0)^N = \sum_{i,j} \partial_i X_j^N h_i h_j = b(h, h),$$

where $b(h, h)$ is defined as in Proposition 12. It follows that the Taylor coordinates of $[c]_H$ are $(h, \frac{1}{2} b(h, h))$, and therefore, by Proposition 30,

$$\log([c]_H) = (h, 0).$$

□

2.5 Exponential maps.

The osculating Heisenberg structure was originally introduced as a tool for the analysis of subelliptic operators on contact manifolds. In the literature, one typically defines a (local) group structure on an open subset of the manifold itself, via some explicit coordinate expression (see, for example, [BG], [FS1], [RS]). In the preceding sections we developed an infinitesimal and geometric interpretation of the osculating groups, but to establish the relationship with the definitions found in the literature, we must consider the appropriate ways to identify the osculating groups with (open sets in) the manifold. This will be important later, in Chapter 5, where we define the *parabolic tangent groupoid* associated to the bundle $T_H M$.

For a connection ∇ on TM there is the corresponding *exponential map*

$$\exp^\nabla : TM \to M,$$

where $\exp(m, v)$ is defined as the end point $c(1)$ of the unique 'geodesic' $c(t)$ that satisfies

$$c(0) = m,\ c'(0) = v,\ \nabla_{c'(t)} c'(t) = 0.$$

We will use the term 'exponential map' in a generalized sense.

Definition 18 *An* exponential map $\exp\colon TM \to M$ *is a smooth map whose restriction* $\exp_m\colon T_m M \to M$ *to a fiber* $T_H M_m$ *fixes the point* m*, while the derivative* $D\exp_m$ *is the identity map* $T_m(T_m M) = T_m M \to T_m M$ *at the point* m*.*

Determining the ways the osculating groups $T_H M_m$ can be identified with open subsets in the manifold M amounts to developing the appropriate notion of an exponential map for the bundle $T_H M$.

Definition 19 *Let* (M, H) *be a manifold with Heisenberg structure.*
An exponential map *for the Heisenberg structure,*

$$\exp\ :\ T_H M \to M,$$

is a smooth map such that for each $(m, v) \in T_H M$*, the curve* $c(t) = \exp(m, \delta_t v)$ *in* M *defines a parabolic arrow* $[c]_H$ *equal to* $v \in T_H M_m$*.*

Notice that this reduces to Definition 19 if $H = TM$. A convenient way to construct a Heisenberg exponential map is by choosing a smooth system of H-coordinates on M. This is a choice of H-coordinates,

$$E_m\ :\ \mathbb{R}^n \to M,$$

at each point $m \in M$, in such a way that the map $(m, x) \mapsto E_m(x)$ is smooth. Let

$$F_m\ :\ \mathbb{R}^n \to T_H M_m$$

denote the Taylor coordinates induced by the H-coordinates E_m. It is immediately clear from the definition of Taylor coordinates (Definition 4) that the composition

$$\exp = E_m \circ F_m^{-1}\ :\ T_H M_m \to M$$

defines a Heisenberg exponential map.

One can try to obtain a Heisenberg exponential map by means of an ordinary exponential map $\exp\colon TM \to M$. Consider the composition

$$T_H M \xrightarrow{\log} H \oplus N \xrightarrow{j} TM \xrightarrow{\exp} M,$$

where j is an isomorphism induced by a choice of section $N \hookrightarrow TM$. Every exponential map for the Heisenberg structure is induced by an ordinary exponential map in this way. However, it is important to note that *not every* exponential map $TM \to M$ will induce an exponential map for the Heisenberg structure. The condition that every curve $c(t) = \exp(th)$, for $h \in H_m$, represents the parabolic arrow $[c]_H = h \in H_m$ translates into a requirement on the *second derivative* of exp.

One way to deal with this issue is to consider exponential maps arising from connections. Observe that the choice of a Heisenberg structure $H \subseteq TM$ is equivalent to a reduction of the principal frame bundle fTM, to the subbundle fT_HM, whose fiber at $m \in M$ consists of frames $(e_1, \ldots, e_n)$ in T_mM for which $(e_1, \ldots, e_p)$ is a frame in H_m. (A section of fT_HM is what we have called an H-frame.) The bundle fT_HM is a principal bundle with structure group,

$$G_{p,q} = \{\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathrm{Aut}\,(\mathbb{R}^p \oplus \mathbb{R}^q)\} \subseteq GL_n(\mathbb{R}).$$

In other words, a Heisenberg structure on M is equivalent to a $G_{p,q}$-structure on TM, and the natural connections to consider are connections on the principal $G_{p,q}$-bundle fT_HM. For the associated affine connection ∇ on TM, this simply means that if $X \in \Gamma(H)$, then $\nabla_Y X \in \Gamma(H)$, in other words, ∇ restricts to a connection on H. One easily verifies that this last condition implies that the exponential map $\exp^\nabla \colon TM \to M$ satisfies the requirements of Definition 19, when we identify $TM \cong H \oplus N$ with T_HM. We will also use the notation $\exp^\nabla$ for the induced exponential map,

$$\exp^\nabla \;:\; T_HM \to M.$$

Exponential maps induced by connections will play a role in our investigation of the parabolic tangent groupoid.

2.6 Osculating structures in the literature.

In this section we show how our formalism relates to the definitions of the osculating structure found in the literature. One purpose of Definition 19 is to clarify and generalize the various identifications of the osculating groups with coordinate patches in the manifold. The reader who is not burdened by knowledge of this literature may prefer to skip this section.

As a first example, we consider the group structure defined by Beals and Greiner on the coordinate space for given H-coordinates $E_m \colon \mathbb{R}^n \to U$ at a point m (see [BG], section 1.1). We have already seen (see Proposition 12) that the coordinate definition of the osculating group $F_m \colon \mathbb{R}^n \cong T_HM_m$, implicit in the construction of Beals and Greiner, corresponds to what we call the Taylor coordinates F_m on the osculating group induced by H-coordinates E. If we now interpret this identification as a map

$$E_m \circ F_m^{-1} \;:\; T_HM_m \to U,$$

then it satisfies our requirements for an exponential map, as we have seen.

To make things more explicit, suppose for simplicity that M is a subset of $\mathbb{R}^n$, and let $X_1, \ldots, X_n$ be an H-frame with $X_i = \sum X_i^j \partial_j$. At each point $m \in M$, one can define H-coordinates on M by

$$E_m \;:\; \mathbb{R}^n \to M \;:\; v \mapsto m + \sum v_i X_i(m) = m + X(m)v,$$

where $X(m)$ denotes the $n \times n$ invertible matrix $(X_i^j(m))_{ij}$. If $b(m), m \in M$ denotes the array b_{ij}^k for the coordinates E_m as in Proposition 12, then the logarithm $T_HM_m \to H_m \oplus N_m$ is expressed in Taylor coordinates as $\log(h,n) = (h, n - b(h,h))$ (Proposition 30). One can work out an explicit (and rather useless) expression for b^k_{ij}:

$$b^k_{ij}(m) = \sum_l dX^l_i(X_j)(m)\,(X^{-1})^{p+k}_l(m),$$

where $X^{-1}(m)$ denotes the inverse matrix of $X(m)$. (If $X_j(m) = \partial_j$, then $X(m) = 1$, and we retrieve the expression from Proposition 12.) The Taylor coordinates on $T_H M_m$ can then be made explicit as

$$F_m\,:\,\mathbb{R}^{p+q} \to T_H M_m\,:\,v = (h,n) \mapsto \exp(\sum v_i X_i(m) - \sum b^k_{ij}(m) h_i h_j X_{p+k}(m)),$$

where we have a complicated quadratic term

$$\sum b^k_{ij}(m) h_i h_j X_{p+k}(m) = \sum d(h_i X^l_i)(h_j X_j)(m)\,(X^{-1})^{p+k}_l(m) X_{p+k}(m).$$

We obtain an exponential map

$$\exp = E_m \circ F_m^{-1}\,:\,T_H M \to M,$$

defined in a neighborhood of the zero section.

Osculating structures on contact manifolds first appeared in the work of Folland and Stein (see [FS1], sections 13 and 14). Their construction was different from that of Beals and Greiner, and can be descibed as follows (we generalize slightly). On a $(2k+1)$-dimensional contact manifold, with $2k$-dimensional bundle $H \subseteq TM$, choose a (local) H-frame $X_1, \ldots, X_{2k+1}$. It can be shown that this frame can be chosen such that

$$\begin{aligned}[X_i, X_{k+i}] &= X_{2k+1} \bmod H,\ \text{for}\, i = 1, \ldots, p,\\ [X_i, X_j] &= 0 \bmod H,\ \text{for all other values of}\, i \leq j.\end{aligned}$$

Then, for $v \in \mathbb{R}^{2k+1}$, let $E_m(v)$ be the endpoint $c(1)$ of the integral curve $c(t)$ of the vector field $\sum v_i X_i$ with $c(0) = m$, in other words,

$$E_m\,:\,\mathbb{R}^{2k+1} \to M\,:\,v \mapsto \Phi^1_{\sum v_i X_i}(m),$$

where Φ^t_Y denotes the flow generated by a vector field Y. For Folland and Stein, the 'osculating Heisenberg structure' on M is the family of maps E_m, identifying an open subset of the Heisenberg group $H_k = \mathbb{R}^{2k+1}$ (with its standard coordinates), with a neighborhood of $m \in M$.

Translating, we see that the commutator relations for the H-frame allow us to identify the basis $X_i(m) \in H_m \oplus N_m$ of the osculating Lie algebra with the standard basis of the Lie algebra of the Heisenberg group. Accordingly, we have an isomorphism,

$$F_m\,:\,H_k = \mathbb{R}^{2k+1} \to T_H M_m\,:\,v \mapsto \exp(\sum v_i X_i(m)).$$

We recognize F_m as the Taylor coordinates on $T_H M_m$ for the H-coordinates E_m^{-1} at m (we have $b^k_{ij} = b^k_{ji}$), and we see that the osculating structure on M, as defined by Folland and Stein, can be interpreted as an exponential map,

$$\exp = E_m \circ F_m^{-1}\,:\,T_H M_m \to M.$$

Beals and Greiner start their construction with an arbitrary system of H-coordinates E_m, and are therefore required to compensate by a quadratic correction term in the Taylor coordinates F_m. Folland and Stein, on the other hand, choose a system of coordinates E_m that is better suited to the Heisenberg structure, and as a result obtain Taylor coordinates F_m that are simply the linear coordinates on $H_m \oplus N_m$.

Remark. The construction of Folland and Stein corresponds to the exponential map associated to a (local) *flat* $G_{p,q}$-connection ∇ on TM. Locally, such a connection can be represented by the choice of an H-frame $X_1, \ldots, X_n$. For each $v \in \mathbb{R}^n$, the vector field $\sum v_i X_i$ is parallel. Identifying $T_H M$ with $H \oplus N \cong TM$ (the H-frame induces a section $N \subseteq TM$), the exponential map $\exp^\nabla$ is given by,

$$\exp^\nabla \,:\, T_H M \to M \,:\, (m, v) \mapsto E_m \circ F_m^{-1}(v),$$

with coordinate maps E_m, F_m given by,

$$E_m \,:\, \mathbb{R}^n \to M \,:\, v \mapsto \Phi^1_{\sum v_i X_i}(m),$$
$$F_m \,:\, \mathbb{R}^n \to T_H M_m \,:\, v \mapsto \exp(\sum v_i X_i(m)).$$

2.7 Foliations, Heisenberg manifolds, contact manifolds.

In this chapter we discuss some important examples of Heisenberg structures.

Foliations.
The simplest examples are *foliations*. By definition, a foliated manifold is equipped with a distribution $H \subseteq TM$ that is integrable. Integrability of H can be defined in a number of equivalent ways, one of which is to say that $[H, H] \subseteq H$.

Proposition 20 *A distribution $H \subseteq TM$ is integrable, i.e., (M, H) is a foliation, if and only if all the osculating groups $T_H M_m$ are abelian.*

Proof. The expression $[H, H] \subseteq H$ is equivalent to the statement that the normal component $[X, Y]^N$ vanishes for any two sections X, Y in $\Gamma^\infty(H)$, which, according to Proposition 15, is equivalent to the fact that all the Lie algebras $\mathrm{Lie}(T_H M_m)$ are abelian.

□

Remark. An alternative proof uses foliation charts as a special type of H-coordinates. In a foliation chart, the coordinate fields $X_i = \partial_i, i = 1, \ldots, p$, are—by definition—sections in H. Applying Proposition 12, we see that $X_i^{p+k} = 0$ implies $b_{ij}^k = 0$.

Heisenberg manifolds.
A more interesting example is given by the class of *Heisenberg manifolds.* Heisenberg manifolds are manifolds with Heisenberg structure $H \subseteq TM$, where H is of codimension one in TM. (In our notation, $q = 1$, and $p = n - 1$.)

Proposition 21 *If (M, H) is a Heisenberg manifold, then the osculating groups are isomorphic to a direct product $\mathbb{R}^s \times H_r$ of an abelian group $\mathbb{R}^s$ and the $(2r+1)$-dimenisional Heisenberg group H_r. Here $2r + s = p$, but the values of r, s are not necessarily the same for all $m \in M$.*

Proof. The groups $T_H M_m$ that arise in this case are of the type $G_B = \mathbb{R}^{p+1}$ described in section 2.9, where $B\colon \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ is a bilinear form on $\mathbb{R}^p$, and multiplication is given by

$$(h, n) * (h', n') = (h + h', n + n' + B(h, h')).$$

By Lemma 33, we may assume that B is skew-symmetric. From the theory of skew-symmetric forms, we know that a linear coordinate change brings B in canonical form. The form B may be degenerate, and if s is the dimension of the kernel of B, then G_B is isomorphic to the direct product $\mathbb{R}^s \times H_r$, where H_r is the $(2r+1)$-dimensional group associated to the unique non-degenerate skew-symmetric form on $\mathbb{R}^r$ $(p = 2r + s)$. The group operation on H_r may be expressed in coordinates as

$$(x_0, \ldots, x_{2r}) \, * \, (y_0, \ldots, y_{2r}) = (x_0 + y_0 + \frac{1}{2}\sum_{j=1}^{r}(x_j y_{r+j} - x_{r+j} y_j), x_1 + y_1, \ldots, x_{2r} + y_{2r}).$$

Replacing the bilinear form with an equivalent, but non skew-symmetric form (see Lemma 33), we obtain

$$(x_0, \ldots, x_{2r}) \, * \, (y_0, \ldots, y_{2r}) = (x_0 + y_0 + \sum_{j=1}^{r} x_j y_{r+j}, \, x_1 + y_1, \ldots, x_{2r} + y_{2r}).$$

Thus, H_r is isomorphic to the $(2r+1)$-dimensional matrix group:

$$H_r \cong \{ \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_0 \\ 0 & 1 & 0 & \cdots & x_{r+1} \\ 0 & 0 & 1 & \cdots & x_{r+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \; : \; x_i \in \mathbb{R} \}.$$

□

Contact manifolds.

A *contact manifold* is a special kind of Heisenberg manifold. At one extreme, all osculating groups of a Heisenberg manifold could be abelian, in which case we are dealing with a codimension one foliation. Contact manifolds exist at the other extreme, where all osculating groups are Heisenberg groups.

Let M be a manifold of odd dimension $n = 2r + 1$, and $H \subseteq TM$ a codimension one subbundle.

Definition 22 *The distribution $H \subseteq M$ defines a* contact structure *on M if a non-vanishing local one-form $\theta \in \Omega^1(M)$ with $\theta(H) = 0$ is a* contact form*, i.e., if $\theta \wedge (d\theta)^r$ is a nowhere vanishing volume form.*

Observe that the definition does not depend on the choice of θ, because any other choice θ' is related to θ by $\theta' = f\theta$, where f is a non-vanishing smooth function. One easily calculates $\theta'(d\theta')^r = f^{r+1}\theta(d\theta)^r$, i.e., $\theta' = f\theta$ is a contact form if and only if θ is.

Also note that θ is not necessarily defined globally. A global contact form exists if and only if $N = T/H$ is a trivial line bundle.

Proposition 23 *Let (M, H) be a Heisenberg manifold of odd dimension $n = 2r + 1$. If all the osculating groups $T_H M_m$ are isomorphic to the Heisenberg group H_r, then (M, H) is a contact manifold.*

Proof. Let b be the skew-symmetric bilinear form

$$b : H \otimes H \to N : b(X, Y) = [X, Y]^N$$

that defines the Lie algebra structure on $H \otimes N$. If we identify the fibers of the line bundle N with $\mathbb{R}$, than b becomes a two form on H. We will show that $b = -d\theta$.

Choose a local frame $X_1, \ldots, X_{2r} \in \Gamma(H)$ spanning H, and a non-vanishing section $X_0 \in \Gamma(N)$. Every smooth section θ of the bundle $N^* = (TM/H)^*$ canonically identifies with a one form $\theta \in \Omega^1(M)$ (because $N^* \subseteq T^*M$), that satisfies $\theta(H) = 0$. Choose $\theta \in \Gamma(N^*)$ such that $\theta(X_0) = 1$.

For two vector fields $X, Y \in \Gamma(H)$ we have the normal component

$$[X, Y]^N = \theta([X, Y]) \cdot X_0.$$

A basic formula from differential topology relates the de Rham operator d to the Lie bracket of vector fields,

$$d\theta(X, Y) = -\theta([X, Y]) + X\theta(Y) - Y\theta(X).$$

In our case, the two final terms on the right hand side vanish, and we find,

$$b(X, Y) = -d\theta(X, Y) \cdot X_0.$$

If Lie$(T_H M)$ is isomorphic to the Lie algebra of the Heisenberg group H_r, then there exist coordinates for which the two form $b = -d\theta$ takes the canonical form,

$$-d\theta = \sum_{i=1}^{r} dx_i \wedge dx_{r+i},$$

It follows immediately that θ is a contact form.

□

The converse is also true. Before we see why, we consider the example of the Heisenberg group *as* a contact manifold. Let $M = \mathbb{R}^{2r+1}$ be equipped with the structure of the $(2r + 1)$ dimensional Heisenberg group H_r, and let $X_0, X_1, \ldots, X_{2r}$ be the standard basis for its Lie algebra, such that

$$[X_i, X_{i+r}] = X_0, \ i = 1, \ldots, r.$$

Thinking of X_i as right invariant vector fields on M, we define a distribution $H \subseteq TM$ as the span of the fields $X_1, \ldots, X_{2r}$ (excluding X_0). Clearly, for every $m \in M$, Lie$(T_H M_m)$ is isomorphic to the Lie algebra of H_r, so that all osculating groups are isomorphic to H_r. Therefore, by Proposition 23 the Heisenberg group M with distribution H is a contact manifold.

An application of Lemma 34 gives the formulas for right invariant fields on the Heisenberg group. With standard coordinates $(x_0, x_1, \ldots, x_{2r}) \in \mathbb{R}^{2r+1} = M$ we find,

$$\begin{aligned} X_0 &= \frac{\partial}{\partial x_0}, \\ X_i &= \frac{\partial}{\partial x_i} + \frac{1}{2} x_{i+r} \frac{\partial}{\partial x_0},\ i = 1, \ldots, r, \\ X_{i+r} &= \frac{\partial}{\partial x_{i+r}} - \frac{1}{2} x_i \frac{\partial}{\partial x_0},\ i = 1, \ldots, r. \end{aligned}$$

Therefore, the one form

$$\theta = dx_0 + \frac{1}{2} \sum_{i=1}^{r} (x_i dx_{i+r} - x_{i+r} dx_i)$$

satisfies $\theta(X_i) = 0$ for $i = 1, \ldots, 2r$. It is the *canonical contact form* on the Heisenberg group.

Theorem 24 *Every contact manifold (M, H) of dimension $(2r+1)$ is locally isomorphic (as a contact manifold) to an open subset of the Heisenberg group H_r with its canonical contact structure.*

In other words, if θ is a contact form on M, then, around each point in M, there exist local coordinates $(x_0, \ldots, x_{2r})$ for which θ takes the form $\theta = dx_0 + \frac{1}{2} \sum (x_i dx_{i+r} - x_{i+r} dx_i)$.

The proof involves the construction of a *symplectic* structure on $M \times \mathbb{R}^+$, and an application of Darboux's theorem; see [Ar].

As an immediate corollary, we obtain the converse of Proposition 23.

Corollary 25 *Let (M, H) be a contact manifold of dimension $(2r+1)$. Then all osculating groups $T_H M$ are isomorphic to the Heisenberg group H_r.*

Remark. Recall that a distribution $H \subseteq TM$ is integrable if and only if for every one form θ that vanishes on H, the two form $d\theta$ also vanishes when restricted to H. Again, we see that foliations and contact manifolds are in some sense 'opposite' structures.

Homogeneous Heisenberg structures.

Foliations and contact manifolds have an interesting feature in common. In both cases the Heisenberg structure on the manifold is locally isomorhic to the canonical Heisenberg structure of the osculating group. This makes analysis on such manifolds particularly easy.

Definition 26 *A* Heisenberg atlas *for a manifold M is an atlas $\{(U_k, \psi_k)\}$, where the coordinates space $\mathbb{R}^{p+q}$ carries the canonical Heisenberg structure of a (fixed) two-step graded nilpotent group $G \approx \mathbb{R}^{p+q}$, and the coordinate transformations $\psi_k \circ \psi_l^{-1}$ preserve the Heisenberg structure on G.*

A manifold with a Heisenberg atlas is said to have a homogeneous Heisenberg structure*.*

Clearly, the chart $\psi_k \colon U_k \to G$ induces an isomorhism between the osculating groups $T_H M_m$ and the group G, for each $m \in U_k$ (this is obvious at the level of the Lie algebras). Hence, for a manifold with Heisenberg atlas all osculating groups are isomorphic.

Conversely, according to Proposition 20, if all osculating groups are isomorphic to the (graded) abelian group $G = \mathbb{R}^{p+q}$, then the Heisenberg structure is a foliation, and the collection of foliation charts defines a Heisenberg atlas. Likewise, Proposition 23 states that if all osculating groups are isomorphic to a Heisenberg group $G = H_r$, the Heisenberg structure is a contact structure, in which case Darboux coordinates provide the charts for a Heisenberg atlas.

It is an interesting geometric question (to which we don't know the answer) whether isomorphism of all osculating groups (or, perhaps, the stronger condition that the bundle of groups $T_H M$ is locally trivial) implies the existence of a Heisenberg atlas.

2.8 Contact manifolds in complex analysis.

Contact manifolds arise in complex analysis as the boundary of strictly pseudoconvex domains in $\mathbb{C}^k$. Osculating structures were first defined in this context (see [FS1]). A good introductory reference for this material is [Ep].

We briefly recall some basics from complex geometry (see [GH]). As a real manifold, $\mathbb{C}^k = \mathbb{R}^{2k}$ has a tangent space $T_\mathbb{R}\mathbb{C}^k = T\mathbb{R}^{2k}$. Let $T_\mathbb{C}\mathbb{C}^k = T_\mathbb{C}\mathbb{R}^{2k}$ be the complexified bundle. Its complex fiber dimension is $2k$. Standard coordinates are denoted $(z_1, \ldots, z_k) = (x_1 + iy_1, \ldots, x_k + iy_k)$. The *holomorphic tangent bundle* $T^{1,0} \subseteq T_\mathbb{C}\mathbb{C}^k$ consists of *complex* linear combinations of the basis vectors

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right).$$

Likewise, the anti-holomorphic vectors $T^{0,1} = \bar{T}^{1,0}$ are spanned by the vectors $\partial/\partial\bar{z}_j$. Clearly,

$$T_\mathbb{C}\mathbb{C}^k = T^{1,0} \oplus T^{0,1}.$$

There is a natural identification of the holomorphic tangent bundle $T^{1,0}$ with the real tangent space $T_\mathbb{R}\mathbb{C}^k$, via $Z \mapsto \mathrm{Re}Z$, or,

$$T^{1,0} \cong T_\mathbb{R}\mathbb{C}^k \,:\, \frac{\partial}{\partial z_j} \mapsto \frac{1}{2}\frac{\partial}{\partial x_j} \,;\, \sqrt{-1}\frac{\partial}{\partial z_j} \mapsto \frac{1}{2}\frac{\partial}{\partial y_j}.$$

Multiplying vectors in $T^{1,0}$ with $\sqrt{-1}$ induces a linear map J (with $J^2 = -1$) on the fibers of $T_\mathbb{R}\mathbb{C}^k$. The inverse of the isomorphism $T^{1,0} \to T_\mathbb{R}$ can be expressed as

$$v \mapsto v - \sqrt{-1}Jv.$$

At the level of differential forms, one has the ∂ and $\bar{\partial}$ operators,

$$\partial f = df|_{T^{1,0}} = \sum \frac{\partial f}{\partial z_j} dz_j,\ \bar{\partial} f = df|_{T^{0,1}} = \sum \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

Clearly, $df = \partial f + \bar{\partial} f$, while $\bar{\partial} f = 0$ iff f is holomorphic.

Now, let $M = \partial\Omega$ be the smooth boundary of a region $\Omega \subseteq \mathbb{C}^k$. The manifold M has dimension $2k-1$, so it is not a complex manifold. However, the complex structure of $\mathbb{C}^k$ induces a Heisenberg structure on M, as we will see.

Let $T^{1,0}M$ be the intersection of the complexified tangent bundle $T_{\mathbb{C}}M$ with the holomorphic tangent bundle $T^{1,0}\mathbb{C}^k$, and let $T^{0,1}M$ denote the conjugate bundle, i.e., the anti-holomorphic vectors in $T_{\mathbb{C}}M$. Then the complex bundle

$$H_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M \subseteq T_{\mathbb{C}}M$$

is a (complex) codimension 1 subbundle in $T_{\mathbb{C}}M$. Therefore $H = H_{\mathbb{C}} \cap T_{\mathbb{R}}M$ is a *real* codimension 1 subbundle of $T_{\mathbb{R}}M$, and (M, H) is a Heisenberg manifold. Alternatively, we can think of H as the largest subbundle of $TM \subseteq T_{\mathbb{R}}\mathbb{C}^k$ that is closed under the action of J. In this picture, $T^{1,0}M$ and $T^{0,1}M$ are the $\pm\sqrt{-1}$ eigenspaces of J.

Suppose the region Ω is given in the form $\{z|\rho(z) < 0\}$, with ρ a smooth function on $\mathbb{C}^n$ and $d\rho \neq 0$ on the boundary $M = \rho^{-1}(0)$.

Lemma 27 *The (1,0)-form $\theta = -i\partial\rho$ restricts to a real one form on TM. The kernel of θ (as a form on TM) is the bundle H.*

Proof. Because $d\rho(v) = 0$ when $v \in TM$, we have $\partial\rho(v) = -\bar{\partial}\rho(v) = -\overline{\partial\rho(v)}$ (since ρ is real valued), which proves that $\theta(v) = -i\partial\rho(v)$ is real. Also, $\partial\rho(v) = -\bar{\partial}\rho(v)$ implies that θ vanishes on vectors in $T^{1,0}M$ as well as $T^{0,1}M$, so that θ vanishes on all of $H_{\mathbb{C}}$.

We need to prove that θ does not vanish on all of TM. Let $v \in T_{\mathbb{R}}\mathbb{C}^k$ be a normal vector to M. By assumption, $d\rho(v) \neq 0$. Also, $w = Jv \in TM$, which implies $d\rho(w) = 0$. We find,

$$\theta(w) = -i\partial\rho(w) = -id\rho(w - iJw) = d\rho(v) \neq 0.$$

□

To check whether θ is a contact form, one calculates,

$$d\theta = -id\partial\rho = i\partial\bar{\partial}\rho = i\sum_{j,k} \frac{\partial^2\rho}{\partial z_j \partial\bar{z}_k} dz_j \wedge d\bar{z}_k.$$

Notice that $\overline{d\theta} = d\theta$. We have seen that (M, H) is a contact manifold if and only if $d\theta$ is nondegenerate when restricted to H.

Lemma 28 *The form $d\theta$ is nondegenerate on H if and only if it is nondegenerate as a bilinear map*

$$d\theta : T^{1,0}M \otimes T^{0,1}M \to \mathbb{C}.$$

Proof. Nondegeneracy of the real form $d\theta$ on H is equivalent to nondegeneracy of $d\theta$ on $H_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$. The bilinear form

$$d\theta : (T^{1,0}M \oplus T^{0,1}M) \otimes (T^{1,0}M \oplus T^{0,1}M) \to \mathbb{C},$$

vanishes on the summands $T^{1,0} \oplus T^{1,0}$ and $T^{0,1} \oplus T^{0,1}$, so we only need to consider $d\theta$ on $(T^{1,0} \otimes T^{0,1}) \oplus (T^{0,1} \otimes T^{1,0})$. But $d\theta$ is the same on these last two summands, because $d\theta(X, \bar{Y}) = -d\theta(\bar{X}, Y)$ for sections X, Y in $T^{1,0}M$. Therefore $d\theta$ is nondegenerate on $H_{\mathbb{C}}$ if and only if it is nondegenerate on $T^{1,0} \otimes T^{0,1}$.

□

The restriction of $d\theta$ to $T^{1,0} \otimes T^{0,1}$ can be thought of as a *Hermitian form* $\langle\cdot\rangle_L$ on $T^{1,0}M$, called the *Levi form.* It is given by

$$\langle Z_1, Z_2\rangle_L = -id\theta(Z_1, \bar{Z}_2),$$

for sections Z_1, Z_2 in $T^{1,0}M$. We see that the Levi form $\langle\cdot\rangle_L$ is the restriction to $T^{1,0}M$ of the form $\langle\cdot\rangle$ on $T^{1,0}\mathbb{C}^k$ defined by

$$\langle\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\rangle = \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}.$$

(The bundle $T^{1,0}M$ may or may not contain any of the basis vectors $\partial/\partial z_j$.)

The following proposition is a direct corollary of Lemma 28.

Proposition 29 *The natural Heisenberg structure on the smooth boundary $M = \partial\Omega$ of a domain $\Omega \subseteq \mathbb{C}^k$ is a contact structure if and only if the Levi form is nondegenerate.*

As an example, consider the $(2k-1)$ sphere S^{2k-1} as the boundary of the unit ball $B \subseteq \mathbb{C}^k$. The defining function here is

$$\rho(z) = -1 + \sum z_j \bar{z}_j.$$

Thus,

$$\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} = \delta_{jk},$$

which shows that the Levi form is simply the restriction of the standard Hermitian form on $T^{1,0}\mathbb{C}^k$ to $T^{1,0}S^{2k-1}$. Therefore the Levi form is *strictly positive*, and we conclude that the sphere S^{2k-1}, with the Heisenberg structure inherited from its embedding in $\mathbb{C}^k$, is a contact manifold.

If R is the symmetric form on TM defined by the Hessian $\partial^2\rho/\partial x_i \partial x_j$ (for arbitrary coordinates $(x_1, \ldots, x_{2k-1})$), and if $R_{\mathbb{C}}$ denotes the associated *Hermitian* form on $T_{\mathbb{C}}M$, given by $R_{\mathbb{C}}(X, Y) = R(X, \bar{Y})$, then the Levi form is just the restriction of $R_{\mathbb{C}}$ to the bundle $T^{1,0}M$. The region Ω is strictly convex if and only if the real form R is strictly positive. This, in turn, is equivalent to strict positivity of $R_{\mathbb{C}}$, which implies strict positivity of the Levi form. Therefore, the boundary of any strictly convex region in $\mathbb{C}^k$ is a contact manifold. (A domain whose boundary has a strictly positive Levi form is called *strictly pseudoconvex.*)

2.9 Appendix: Two-step nilpotent groups.

We derive some simple facts about two-step nilpotent groups.

Recall that a Lie algebra $\mathfrak{g}$ is called *two-step nilpotent* if $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = 0$. The Campbell–Baker–Hausdorff formula for such Lie algebras has very few non-zero terms:

$$\exp(x) \cdot \exp(y) = \exp(x + y + \frac{1}{2}[x, y]),$$

for $x, y \in \mathfrak{g}$. (For the full CBH-formula, see [Se].) Replacing the bracket $[x, y]$ with an arbitrary (not necessarily skew-symmetric) bilinear map $B : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^q$, we can define a Lie group $G_B = \mathbb{R}^p \times \mathbb{R}^q$ with group operation

$$(h_1, n_1) * (h_2, n_2) = (h_1 + h_2, n_1 + n_2 + B(h_1, h_2)).$$

It is trivial to verify the group axioms (using the bilinearity of B). By Proposition 12, the group structure of parabolic arrows $T_H M_m$ expressed in Taylor coordinates is of this type. Our main goal in this section is to prove the following proposition.

Proposition 30 *Let G_B be the Lie group defined above. With the natural coordinates on $G_B = \mathbb{R}^{p+q}$ and $\operatorname{Lie} G_B = T_0\mathbb{R}^{p+q}$, the exponential map $\exp : \operatorname{Lie} G_B \to G_B$ is expressed as*

$$\exp(h, n) = (h, n + \frac{1}{2}B(h, h)).$$

The proof consists of a string of lemmas.

Lemma 31 *The Lie algebra structure on $\operatorname{Lie} G_B = \mathbb{R}^{p+q}$ is given by the bracket*

$$[(h_1, n_1), (h_2, n_2)] = (0,\ B(h_1, h_2) - B(h_2, h_1))\,.$$

In particular, the Lie algebra structure only depends on the skew-symmetric part $(B - B^T)/2$ of the bilinear map B.

Proof. The neutral element in G_B is $(0, 0)$, and inverses are given by

$$(h, n)^{-1} = (-h, -n + B(h, h)).$$

Commutators in G_B are calculated as follows:

$$\begin{aligned}
&(h_1, n_1) * (h_2, n_2) * (h_1, n_1)^{-1} * (h_2, n_2)^{-1} \\
&= (h_1, n_1) * (h_2, n_2) * (-h_1, -n_1 + B(h_1, h_1)) * (-h_2, -n_2 + B(h_2, h_2)) \\
&= (h_1 + h_2, n_1 + n_2 + B(h_1, h_2)) * \\
&\qquad (h_1 - h_2, -n_1 - n_2 + B(h_1, h_1) + B(h_2, h_2) + B(-h_1, -h_2)) \\
&= (0, B(h_1, h_1) + B(h_2, h_2) + 2B(h_1, h_2) + B(h_1 + h_2, -h_1 - h_2)) \\
&= (0, B(h_1, h_2) - B(h_2, h_1)).
\end{aligned}$$

Replace (h_i, n_i) with (th_i, tn_i) and take the limit as $t \to 0$.

$\square$

We see that the groups G_B are indeed two-step nilpotent, or even abelian in the trivial case where B is symmetric.

Lemma 32 *If $B \colon \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^q$ is a* skew-symmetric *bilinear map, then the exponential map* $\exp \colon \operatorname{Lie}(G_B) \to G_B$ *is the usual identification of $T_0\mathbb{R}^n$ with $\mathbb{R}^n$.*

Proof. For any $(h,n) \in \mathbb{R}^{p+q}$ we have $(th,tn) * (sh,sn) = ((t+s)h,(t+s)n)$. In other words, the map

$$\phi : \mathbb{R} \to G_b : t \mapsto (th,tn)$$

is a group homomorphism. The tangent vector to this one-parameter subgroup at $t = 0$ is $\phi'(0) = (h,n) \in \mathrm{Lie}\,(G_B)$, and by definition $\exp(\phi'(0)) = \phi(1) = (h,n) \in G_B$.

□

Lemma 33 *If $B, C \colon \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^q$ are two bilinear maps that have the same skew-symmetric part, then the quadratic map*

$$\phi : G_C \xrightarrow{\cong} G_B : (h,n) \mapsto (h, n + \tfrac{1}{2}B(h,h) - \tfrac{1}{2}C(h,h)),$$

is a group isomorphism.

Proof. With $S = B - C$:

$$\begin{aligned}
&\phi(h_1,n_1) * \phi(h_2,n_2)\\
&= (h_1, n_1 + \tfrac{1}{2}S(h_1,h_1)) * (h_2, n_2 + \tfrac{1}{2}S(h_2,h_2))\\
&= (h_1 + h_2, n_1 + \tfrac{1}{2}S(h_1,h_1) + n_2 + \tfrac{1}{2}S(h_2,h_2) + C(h_1,h_2))\\
&= (h_1 + h_2, n_1 + n_2 + \tfrac{1}{2}S(h_1,h_1) + \tfrac{1}{2}S(h_2,h_2) + S(h_1,h_2) + B(h_1,h_2))\\
&= (h_1 + h_2, n_1 + n_2 + \tfrac{1}{2}S(h_1+h_2,h_1+h_2) + B(h_1,h_2))\\
&= \phi(h_1 + h_2, n_1 + n_2 + B(h_1,h_2)) = \phi((h_1,n_1) * (h_2,n_2)).
\end{aligned}$$

□

Proof of Proposition 30. Let $C = \frac{1}{2}(B - B^T)$ be the skew-symmetric part of B. The exponential map for G_B is the composite of the following three maps:

$$\mathrm{Lie}(G_B) \xrightarrow{\cong} \mathrm{Lie}(G_C) \xrightarrow{\exp} G_C \xrightarrow{\phi} G_B.$$

The first two of these maps are just the identity map $\mathbb{R}^{p+q} \to \mathbb{R}^{p+q}$ (by Lemmas 31 and 32, respectively). Lemma 33 gives the explicit isomorphism $\phi \colon G_C \cong G_B$, with $C(h,h) = 0$.

□

It will be useful to have explicit formulas for the right-invariant vector fields on the Lie group G_B. The bilinear form B is represented by the array of constants,

$$B^k_{ij} = \langle B(e_i,e_j), e_k \rangle, \text{ for } i,j = 1,\ldots,p\ ,\ k = 1,\ldots,q.$$

Lemma 34 *Let $(x_1,\ldots,x_{p+q})$ denote the standard coordinates on $G_B = \mathbb{R}^{p+q}$. The vector space of left-invariant vector fields on G_B is spanned by the vector fields $Y_i = \partial_i$, for $i = p+1,\ldots,p+q$, and the vector fields*

$$Y_i = \partial_i + \sum_{k=1}^{q} \sum_{j=1}^{p} B^k_{ij} x_j \partial_{p+k},$$

for $i = 1,\ldots,p$. In particular,

$$B^k_{ij} = \partial_j Y_i^{p+k}.$$

Proof. Suppose Y is a right-invariant vector field that restricts to some $v \in T_0G_B$ at $x = 0$. Denoting right multiplication by $R_x(y) = y * x$ we have

$$\begin{aligned}
Y = DR_x(v) &= \frac{d}{dt}\bigg|_{t=0} (tv^H, tv^N) * (x^H, x^N) \\
&= \frac{d}{dt}\bigg|_{t=0} (tv^H + x^H, tv^N + x^N + tb(v^H, x^H)) \\
&= (v^H, v^N + b(v^H, x^H)).
\end{aligned}$$

We could write, for short (extending B to $\mathbb{R}^{p+q}$),

$$Y(x) = v + B(v, x).$$

Taking $v = \partial_i$, we obtain the formulas in the lemma. $\square$

Chapter 3

Maximally hypoelliptic operators.

3.1 Introduction.

In this chapter we develop the 'elliptic' theory suitable to the presence of a Heisenberg structure on a compact manifold (M, H). We have restricted our discussion to the case of contact manifolds. However, we believe that many of the results hold in the case of more general Heisenberg structures.

In general, the operators that are of interest in applications act on sections in vector bundles. For simplicity of our exposition, we will only discuss scalar operators. However, there is no difficulty in rephrasing everything in terms of vector bundle operators. We will explicitly mention vector bundles only if the generalization is not trivial.

Recall that a closed Hilbert space operator is *Fredholm* if it has closed range, and its kernel and cokernel are finite dimensional. The *index* of a Fredholm operator P is defined as

$$\operatorname{Index} P = \dim \operatorname{Ker} P - \dim \operatorname{Coker} P = \dim \operatorname{Ker} P - \dim \operatorname{Ker} P^*.$$

It is well-known that the closure of an *elliptic* operator on a compact manifold is Fredholm, and index theory typically deals with elliptic operators.

However, the fact that an elliptic operator is Fredholm is a consequence of their *hypoellipticity*, and the aim of our work is to develop an index theory for a more general class of hypoelliptic operators.

Definition 35 *A differential operator P on a manifold M is* hypoelliptic *if, for any distribution u on M, whenever Pu is C^∞ on an open set $U \subseteq M$, u is C^∞ in U.*

Proposition 36 *If the closure of a symmetric, hypoelliptic operator on a compact manifold M has closed range in $L^2(M)$, it is Fredholm.*

Proof. Let P be hypoelliptic on a compact manifold M. Let $\bar{P}$ be the closure of P as an unbounded operator on $L^2(M)$, with $\bar{P} = P^*$.

For v in the domain of P^*, we have $\langle u, P^*v\rangle = \langle Pu, v\rangle$ for every $u \in C^\infty(M)$. We see that $\bar{P}v = P^*v = Pv$, where Pv should be read in the distribution sense. Hence, hypoellipticity of P implies that the kernel of $\bar{P}$ consists of smooth functions. But the kernel of a closed operator is closed, and a closed subspace of $L^2(M)$ consisting of smooth functions is finite dimensional (see Lemma 37).

Since P has closed range, the cokernel of P is isomorphic to the kernel of $P^* = P$.

□

Lemma 37 *Let M be a compact manifold. A closed linear subspace of $L^2(M)$ that consists entirely of continuous functions is finite dimensional.*

Proof. By Sobolev theory, the inclusion of Banach spaces $C(M) \subseteq L^2(M)$ (with supremum norm on $C(M)$) is compact. But the range of a compact operator cannot contain an infinite dimensional closed linear subspace.

□

Ellipticity of an operator P only depends on the highest order part of the operator, i.e., its principal symbol $\sigma(P)$. In elliptic theory, the requirement that the principal symbol is invertible, leads to the proof of the so-called *a priori estimates*, which state that, if P is an elliptic operator of order d, and A is any differential operator of order less or equal d, then there is a $C > 0$ such that

$$\|Au\| \leq C(\|Pu\| + \|u\|).$$

These estimates are at the heart of elliptic theory. In particular, they imply hypoellipticity of P, and hence Fredholmness. The key technical tool in the proof of the estimates is Fourier theory.

We will show how, by simply changing the notion of 'order' of an operator, in a way which is consistent with the presence of a Heisenberg structure, one is naturally led to a *noncommutative* symbolic calculus. The same ingredients that play a role in the analysis of elliptic operators will appear here, including the key role played by the Fourier transform (in noncommutative harmonic analysis this time). The aim of the chapter is to prove Theorem 53, which states, roughly, that an operator whose (noncommutative) principal symbol is invertible is hypoelliptic.

By taking $H = TM$, this chapter could be read as an introduction to ordinary elliptic theory. The reader familiar with elliptic theory will notice how little has changed. Our approach is essentially a direct translation of the well known results from elliptic theory to a noncommutative setting.

3.2 A noncommutative symbolic calculus.

In this section we define the symbolic calculus for operators on manifolds with Heisenberg structure. A Heisenberg calculus for pseudo-differential operators has been developed independently by Beals and Greiner [BG] and by Taylor [Ta1]. A succinct introduction can be found in [Ep], and [EM]. We have limited the discussion of the calculus to differential operators, which greatly simplifies the discussion. It allows us to give a simple *algebraic* interpretation of the calculus.

Let $\mathfrak{g}^\bullet$ denote the *filtered* Lie algebra of vector fields on M, with filtration induced by $H \subseteq TM$,

$$\mathfrak{g}^1 = \Gamma(H) \subseteq \mathfrak{g}^2 = \Gamma(TM).$$

The associated *graded* Lie algebra is precisely the set of smooth sections in the bundle of osculating Lie algebras $\mathrm{Lie}(T_HM) = \{\mathfrak{g}_m\}$ discussed in Chapter 2. The filtration on the set of vector fields induces a filtration on the algebra of differential operators $\mathcal{P}$ in the obvious way: a vector field $X \in \Gamma(H)$ defines an order 1 operator; any other vector field has order 2; and a product of vector fields ΠX_k has order $\sum d(k)$ or less, where $d(k)$ denotes the order of X_k. Let $\mathcal{P}_H^d$ be the linear span of monomials ΠX_k of order d or less, with $\mathcal{P}_H^0 = C^\infty(M)$. A differential operator P is said to have *Heisenberg order* d if $P \in \mathcal{P}_H^d \setminus \mathcal{P}_H^{d-1}$. We write

$$\mathcal{P}_H = \bigcup_{d=0}^{\infty} \mathcal{P}_H^d$$

for the algebra of differential operators filtered by the Heisenberg order.

We introduce some convenient notation. If $X_1, \cdots, X_n$ is a local H-frame on M, a differential operator P of Heisenberg order d can be locally represented as,

$$P = \sum_{|\alpha| \leq d} a_\alpha X^\alpha.$$

As usual, $\alpha = (\alpha_1, \ldots, \alpha_n)$ denotes a multi-index, while the expression X^α is analogous to the usual notation ∂^α, and is shorthand for,

$$X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}.$$

The Heisenberg degree of the monomial X^α is given by,

$$|\alpha| = \alpha_1 + \cdots + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+q}.$$

The algebra of symbols for the Heisenberg calculus is simply the *graded* algebra $\mathcal{S}_H$ associated to the filtered algebra $\mathcal{P}_H$,

$$\mathcal{S}_H = \bigoplus \mathcal{S}_H^d \ ; \ \mathcal{S}_H^d = \mathcal{P}_H^d / \mathcal{P}_H^{d-1}.$$

Abstractly, the *principal Heisenberg symbol* of a differential operator P of Heisenberg order d is simply the image of P under the quotient map

$$\sigma_H^d \ : \ \mathcal{P}_H^d \to \mathcal{S}_H^d.$$

Clearly, if P has Heisenberg order k, and Q has Heisenberg order l, then

$$\sigma_H^k(P)\sigma_H^l(Q) = \sigma_H^{k+l}(PQ).$$

Because the bracket $[X, Y]$ of two order 1 vector fields may have Heisenberg order 2, the algebra of symbols $\mathcal{S}_H$ is generally *noncommutative*. The algebra $\mathcal{S}_H$ is commutative if and only if all the osculating groups are abelian, i.e., if (M, H) is a foliation.

The set $\mathcal{P}_H^d$ is a finitely generated $C^\infty(M)$-module, and is therefore naturally equivalent to the module of sections in a smooth vector bundle over M. This is not a very meaningful point of view, because multiplication in $\mathcal{P}_H$ is not $C^\infty(M)$-linear and therefore the *algebra structure* of $\mathcal{P}_H^d$ does not correspond to an algebra structure on the fibers of this vector bundle.

The situation is different when we turn to the graded algebra S_H. If $f \in C^\infty(M)$ and $X \in \Gamma(TM)$ then $[X, f] = X.f \in C^\infty(M)$ is again a smooth function. As a result, $S_H^0 = C^\infty(M)$ commutes with all of $\mathcal{S}_H$, or, in other words, multiplication in $\mathcal{S}_H$ is $C^\infty(M)$-linear. It follows that S_H is naturally isomorphic *as a graded algebra* to the module of sections in a bundle of graded algebras. We now describe this bundle of algebras explicitly.

For a point $m \in M$, let I_m denote the ideal

$$I_m = \{f \in C^\infty(M) \mid f(m) = 0\}.$$

Abstractly, the fibers in the vector bundle $\mathcal{U}^d$ corresponding to the module S_H^d are the finite dimensional vector spaces

$$\mathcal{U}_m^d = \mathcal{S}_H^d / I_m \mathcal{S}_H^d.$$

If P is a differential operator in $\mathcal{P}_H^d$, let $[P]^d$ denote the corresponding element in $\mathcal{S}_H^d$, and $[P]_m^d$ the element in $\mathcal{U}_m^d$. Clearly, $[P]_m^d = [Q]_m^d$ in $\mathcal{U}_m^d$ if and only if P and Q agree at the point m up to terms of order less than d.

Multiplication of elements in S_H induces bilinear maps,

$$\mathcal{U}_m^k \otimes \mathcal{U}_H^l \to \mathcal{U}_H^{k+l} \,:\, [P]_m^k \otimes [Q]_m^k \mapsto [PQ]_m^{k+l},$$

thus inducing the structure of a graded algebra on the infinite dimensional vector space

$$\mathcal{U}_m = \bigoplus_{d=0}^{\infty} \mathcal{U}_m^d,$$

where $\mathcal{U}_m^0 = \mathbb{C}$. To see that multiplication in $\mathcal{U}_m$ is well defined, let $[P]_m^k = [P']_m^k$ and $[Q]_m^l = [Q']_m^l$, which means that

$$P = P' + fA + \text{lower order terms}, \; Q = Q' + gB + \text{lower order terms},$$

where A and B are operators of order k, l, respectively, and f and g are smooth functions with $f(m) = g(m) = 0$. Then

$$\begin{aligned} PQ &= P'Q' + P'gB + fAQ' + fAgB + \text{lower order terms} \\ &= P'Q' + gP'B + fAQ' + fgAB + \text{lower order terms}, \end{aligned}$$

and so $[PQ]_m^{k+l} = [P'Q']_m^{k+l}$. We have used the fact that $[P', g]$ and $[A, g]$ are of order one less than P' and A, respectively, which is precisely the reason that multiplication in $\mathcal{S}_H$ is $C^\infty(M)$-linear. One easily checks that the canonical map

$$\mathcal{S}_H \to \bigoplus_{d=0}^{\infty} \Gamma(\mathcal{U}^d) \,:\, [P]^d \mapsto [m \mapsto [P]_m^d]$$

defines an isomorphism of graded algebras. To understand the symbolic calculus, we need to investigate the multiplicative structure on the point-wise algebras $\mathcal{U}_m$. This algebra turns out to be isomorphic to the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_m)$ of the osculating Lie algebra $\mathfrak{g}_m = H_m \oplus N_m$ at m. Note that the grading on $\mathfrak{g}_m$ induces a grading

$$\mathcal{U}(\mathfrak{g}_m) = \bigoplus_{d=0}^{\infty} \mathcal{U}^d(\mathfrak{g}_m).$$

Proposition 38 *Let $X_1, \ldots, X_n$ be a local H-frame in a neighborhood of $m \in M$, and let $Y_i = X_i(m) \in H_m$ ($i = 1, \ldots, p$), and $Y_j = X_j(m)^N \in N_m$ ($j = p+1, \ldots, i = p+q$), so that $Y_1, \ldots, Y_n$ is a basis for the osculating Lie algebra $\mathfrak{g}_m = H_m \oplus N_m$. The map*

$$\chi : \mathcal{U}(\mathfrak{g}_m) \to \mathcal{U}_m = \oplus \mathcal{U}_m^d : \sum_\alpha c_\alpha Y^\alpha \mapsto \sum c_\alpha \left[X^\alpha\right]_m^{|\alpha|}$$

is an isomorphism of graded algebras. It is independent of the choice of H-frame.

Proof. The restriction of χ to the Lie algebra $\mathfrak{g}_m \subseteq \mathcal{U}(\mathfrak{g}_m)$ is compatible with the grading on $\mathfrak{g}_m$ and $\mathcal{U}_m$. It preserves commutators, because by definition of the bracket on $\mathfrak{g}_m$ we have

$$\chi[Y_i, Y_j] = \chi(Y_i)\chi(Y_j) - \chi(Y_j)\chi(Y_i)$$

Therefore, χ extends in a unique way to a graded algebra homomorphism $\mathcal{U}(\mathfrak{g}_m) \to \mathcal{U}_m$. By the Poincaré–Birkhoff–Witt Theorem this homomorphism is identical to the map χ defined above (see [Hu]). Clearly, the map χ is one-to-one and onto, and therefore an isomorphism.

If we choose a different H-frame to define χ, then χ remains the same when restricted to $\mathfrak{g}_m$, and therefore also on $\mathcal{U}(\mathfrak{g}_m)$.

□

Let $\mathcal{U}^d(\mathrm{Lie}(T_H M))$ denote the bundle with fibers $\mathcal{U}^d(\mathfrak{g}_m)$. We have an identification of graded algebras

$$\mathcal{S}_H \cong \bigoplus_{d=0}^{\infty} \Gamma(\mathcal{U}^d(\mathrm{Lie}(T_H M))),$$

and the principal Heisenberg symbol $\sigma_H^d(P)$ of a differential operator of Heisenberg order d can be interpreted as a smooth section in the bundle $\mathcal{U}^d(\mathrm{Lie}(T_H M))$. If we identify $\mathcal{U}(\mathfrak{g}_m)$ with the algebra of right invariant differential operators on the osculating group $G_m = T_H M_m$, then $\mathcal{S}_H$ is realized as the algebra of smooth families of such operators, and the principal symbol $\sigma_H^d(P)$ can be identified with a smooth family of *homogeneous, right-invariant* operators P_m on the osculating groups G_m. We call P_m the *model operator* for P at the point $m \in M$.

With

$$P = \sum_{|\alpha| \leq d} a_\alpha X^\alpha,$$

as above, we have

$$P_m = \sum_{|\alpha| = d} a_\alpha(m) Y^\alpha \in \mathcal{U}(\mathfrak{g}_m),$$

where, as before, $Y_i = X_i(m) \in \mathfrak{g}_m = H_m \oplus N_m$. The algebraic discussion in this section shows that the model operators P_m are invariantly defined, i.e., independent of the choice of H-frame and local representation of P. Once we have constructed the *parabolic tangent groupoid* we will give a geometric proof of this fact (see section 6.2).

Remark. In the special case where $H = TM$, the principal Heisenberg symbol $\sigma_H(P)$ is just the Fourier transform of the usual principal symbol $\sigma(P)$.

3.3 Rockland operators.

We will study the behaviour of the operator P by approximating it near a point $m \in M$ by its model operator P_m, a *homogeneous, right-invariant* operator on the graded nilpotent group $G_m = T_H M_m$, the osculating group at m for the Heisenberg structure (M, H). There is a beautiful criterion that decides whether such a model operator is *hypoelliptic*. Let us first consider the easiest case where the group is just $\mathbb{R}^n$.

Theorem 39 *A homogeneous constant coefficient operator on $\mathbb{R}^n$ is hypoelliptic if and only if it is elliptic.*

(This follows from theorem by Hörmander; see [Fo], p. 216.) This result has been generalized to the case of arbitrary graded nilpotent groups.

Definition 40 *A* Rockland operator *on a graded group G is a differential operator P that is right-invariant, homogeneous, and has the property that $d\pi(P)$ is injective on the space of smooth vectors $\mathcal{S}_\pi$, for every irreducible unitary representation $\pi \in \hat{G}$, except the trivial representation.*

We explain the notation. The unitary dual $\hat{G}$ of G is the set of equivalence classes of irreducible unitary representations π on Hilbert space $\mathcal{H}_\pi$. The space of smooth vectors $\mathcal{S}_\pi \subseteq \mathcal{H}_\pi$ consists of vectors $v \in \mathcal{H}_\pi$ for which the map $g \mapsto \pi(g)v \colon G \to \mathcal{H}_\pi$ is a C^∞ function. In the usual manner, π induces a representation $d\pi$ of elements in the Lie algebra $\mathfrak{g}$ of G as skew-Hermitian (unbounded) operators,

$$d\pi(X)v = \frac{d}{dt}_{(t=0)} \exp(tX)v,$$

for $X \in \mathfrak{g}$, $v \in \mathcal{S}_\pi$. The representation $d\pi$ extends to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, such that for an invariant operator $P \in \mathcal{U}(\mathfrak{g})$, we have an unbounded operator $d\pi(P)$ on $\mathcal{H}_\pi$ with domain $\mathcal{S}_\pi$.

(For the basics of representation theory for nilpotent groups, see, for example, [Ta], [CG].)

Lemma 41 *Let $G = \mathbb{R}^n$ be an abelian group with trivial grading. A homogeneous, constant coefficient operator on G is a Rockland operator if and only if it is elliptic.*

Proof. The unitary dual $\hat{G}$ can be identified with the dual space $(\mathbb{R}^n)^*$. To $\xi \in (\mathbb{R}^n)^*$ is associated the (scalar) unitary representation $\pi_\xi(x) = e^{i\langle \xi, x\rangle}$. This leads to,

$$d\pi_\xi\left(\frac{\partial}{\partial x_j}\right) = \frac{d}{dt}_{(t=0)} \pi_\xi(tx_j) = i\xi_j.$$

In other words, if $P = \sum a_\alpha \partial^\alpha$, then $d\pi_\xi(P) = \sum a_\alpha (i\xi)^\alpha$, which is just the symbol of P. The Rockland condition for P says precisely that the symbol of P is invertible for $\xi \neq 0$.

$\square$

The following elegant theorem was first conjectured by Rockland [Ro], who proved it for the Heisenberg group. Necessity in the general case was proven by Rothschild and Stein [RS]; sufficiency by Helffer and Nourrigat [HN].

Theorem 42 *A right invariant homogeneous operator on a graded group is hypoelliptic if and only if it is a Rockland operator. Moreover, if P is any left invariant operator on a graded group, whose highest order part (in the graded sense) is a Rockland operator, then P is hypoelliptic.*

Sufficiency of the Rockland condition follows from the following *a priori estimates* ([HN], proposition 6.4).

Proposition 43 *Let G be a graded group, with grading of length r. Let d be a common multiple of $1, 2, \ldots, r$ that is greater or equal r^r.*

If P is a Rockland operator of degree d, and A is an invariant differential operator of order $\leq d$, then there exists a constant $C > 0$ such that

$$\|Au\| \leq C\left(\|Pu\| + \|u\|\right),$$

for all $u \in \mathcal{S}(G)$. The norms in the inequality are $L^2(G)$ norms.

These estimates imply hypoellipticity of P. If the degree of P does not satisfy the requirements, then the estimates will still hold for a sufficiently large power P^m of P (if P is Rockland, then so is P^m). Hypoellipticity of P^m then implies hypoellipticity of P. In Appendix A we have verified that the proof of the Rockland theorem given in [HN] still holds for *vector bundle* operators ([HN] only deals with scalar operators).

The following important result was proven by Nelson and Stinespring ([NS], Theorem 2.2) for elliptic operators. Their proof also applies to Rockland operators, because it relies only on the fact that the invertible operator $P^*P + 1$ is hypoelliptic.

Theorem 44 *A formally self-adjoint Rockland operator P with domain on a graded group G is essentially self-adjoint. Moreover, if π is any unitary representation of G, then the closure of the operator $\pi(P)$, with domain $\mathcal{S}_\pi$, is self-adjoint.*

Next, we consider the Sobolev theory which corresponds to the filtration of $\mathcal{U}(\mathfrak{g})$. Let $Y_1, \cdots, Y_n$ be a basis for the Lie algebra $\mathfrak{g}$, compatible with the grading, i.e., $Y_1, \cdots, Y_{r(1)}$ in $\mathfrak{g}^1$, $Y_{r(1)+1}, \cdots, Y_{r(2)}$ in $\mathfrak{g}^2$, etc. Then the *weigthed Sobolev space* $W^k = W^k(G)$ is the completion of $C_c^\infty(G)$ with respect to the norm,

$$\|u\|_{W^k} = \sum_{|\alpha| \leq k} \|Y^\alpha u\|.$$

As usual, these weighted Sobolev spaces are Hilbert spaces. The W^k norms are, up to equivalence, independent of the choice of basis Y_i. Clearly, if A is an arbitrary invariant operator of (weighted) order less or equal d, then A is continuous as an operator $W^{d+k} \to W^k$.

Proposition 45 *Let G be a graded group, with grading of length r. Let d be a common multiple of $1, 2, \ldots, r$ that is greater or equal r^r.*

If P is a Rockland operator on G of order d, then the domain of the closure of P is the weighted Sobolev space $W^d(G)$.

Proof. The expression $\|Pu\| + \|u\|$ is equivalent to the *graph norm* of P. The a priori estimates for a maximally hypoelliptic P can be written as,

$$\|u\|_{W^d} \leq C(\|Pu\| + \|u\|).$$

But, simply because P is of order d, we also have,

$$\|Pu\| + \|u\| \leq C'\|u\|_{W^d}.$$

It follows that the graph norm is equivalent to the Sobolev norm.

□

We will show that the restrictions on the order of P in Proposition 43 can be removed.

Proposition 46 *Let G be a graded nilpotent group. Let P be a formally selfadjoint Rockland operator of order d.*

If $f \in C(\mathbb{R})$ is a continuous function of order $f(x) = \mathcal{O}(|x|^{-k/d})$, as $x \to \infty$, for a positive integer k, then $f(P)$ is bounded as an operator $L^2 \to W^k$.

Proof. We first show that, for any invariant operator A of order less or equal k, there is a constant $C > 0$, such that,

$$\|Au\| \leq C\|(P^2 + 1)^{k/2d}u\|,$$

for all $u \in \mathcal{S}(G)$.

First, observe that $P^2 + 1$ is a bijective map from the domain W^{2d} to L^2. Therefore, the closure of $P^2 + 1$ is selfadjoint, so it makes sense to define the selfadjoint operator $(P^2 + 1)^{k/2d}$.

For suitably large m (depending on the length of the grading of the group), both the selfadjoint operators $(A^*A)^{md}$ and P^{2mk} are of order $2mdk$, and P^{2mk} is a Rockland operator for which the a priori estimates hold,

$$\|(A^*A)^{md}u\| \leq C(\|P^{2mk}u\|^2 + \|u\|^2),$$

for all $u \in W^{2mdk}$. Rewriting these estimates, we derive,

$$0 < \langle (A^*A)^{2md}u, u\rangle \leq C\langle (P^{4mk} + 1)u, u\rangle \leq C\langle (P^2 + 1)^{2mk}u, u\rangle,$$

which holds for all u in the domain of the Rockland operator $(P^2 + 1)^{2mk}$ which is W^{2mdk}. The domain of $(A^*A)^{2md}$ certainly contains W^{2mdk}.

By Lemma **??** (see the Appendix to this chapter) we derive,

$$0 < \langle A^*Au, u\rangle \leq C'\langle (P^2 + 1)^{k/d}u, u\rangle,$$

for all u in the domain of $(P^2 + 1)^{k/d}$, which certainly contains $\mathcal{S}(G)$.

This establishes that $g(P)\colon W^k \to L^2$ is bounded, for $g(x) = (x^2 + 1)^{-k/2d}$. Now, if $f(x) = \mathcal{O}(|x|^{-k/d}$, then $f(x) = h(x)g(x)$ for a bounded function h. Because $h(P)$ is bounded on L^2, $f(P) = h(P)g(P)$ is bounded $W^d \to L^2$.

□

Corollary 47 *The a priori estimates in Proposition 43 hold, regardless of the degree of the Rockland operator P.*

Proof. Take $k = d$ in Proposition 46.

$\square$

The following lemma plays an important technical role.

Lemma 48 *Let G be a graded group, with grading of length r. Let k be a common multiple of $1, 2, \ldots, r$.*

Then for every $\epsilon > 0$, there exists $C > 0$ such that,

$$\|u\|_{W^{k-1}} \leq \epsilon \|u\|_{W^k} + C\|u\|,$$

for all $u \in \mathcal{S}(G)$.

(See Proposition 4.6.1 in [HN]) This lemma is false (in general) for values of k that are not multiples of $1, 2, \ldots, r$. However, for a limited class of groups (that includes the Heisenberg group), we give a simple proof of a stronger version of this lemma. The proof depends on a 'spectral' definition of the Sobolev norms.

Lemma 49 *Let G be a graded, two-step nilpotent group, for which $\mathfrak{g}_1$ generates the Lie algebra $\mathfrak{g}$, i.e., $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$, and let,*

$$\Delta = \sum_{i=1}^{p} -Y_i^2,$$

be the sum of squares of order one vector fields Y_1.

Then the weighted Sobolev norm $\|u\|_{W^k}$ is equivalent to the norm $\|u\|_k = \|(\Delta + 1)^{k/2} u\|$.

Proof. Because the order one vector fields $Y_1, \cdots, Y_p$ generate the Lie algebra $\mathfrak{g}$, the operator Δ is hypoelliptic, and hence a (positive) Rockland operator.

If $k = 2m$ is even, clearly $\|(\Delta + 1)^m u\| \leq C\|u\|_{W^{2m}}$, while, by Proposition 46,

$$\|u\|_{W^{2m}} \leq C\|(\Delta^2 + 1)^{m/2} u\|.$$

Spectral theory gives,

$$\|(\Delta^2 + 1)^{m/2} u\| \leq \|(\Delta + 1)^m u\|.$$

This proves the lemma for even k.

To deal with the odd case $k = 2m + 1$, simply observe that,

$$\|u\|^2_{W^{2m+1}} = \sum_{i=1}^{p} \|Y_i u\|^2_{W^{2m}} + \|u\|^2_{W^{2m}} = \langle (\sum_{i=1}^{p} -Y_i^2 + 1) u, u \rangle_{W^{2m}} = \|(\Delta + 1)^{1/2} u\|^2_{W^{2m}}.$$

$\square$

Corollary 50 *Let G be a graded, two-step nilpotent group, for which $\mathfrak{g}_1$ generates the Lie algebra $\mathfrak{g}$, i.e., $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$.*

Then Lemma 48 holds for arbitrary positive integers k.

Proof. Using Lemma 49, we only need to observe that,

$$\|(\Delta+1)^{(k-1)/2}u\| \le \epsilon\|(\Delta+1)^{k/2}u\| + C\|u\|,$$

which follows by spectral theory.

□

Examples. On the graded abelian group $\mathbb{R}^{p+1}$ the heat operator,

$$P = \sum_{i=1}^{p} -\partial_i^2 + \partial_n,$$

is a Rockland operator. As an exercise the reader should verify that the operator

$$P = \sum_{i=1}^{p} -Y_i^2 + i\alpha Y_n,$$

on the Heisenberg group is a Rockland operator if and only if the complex constant α is not an odd integer. One should make use of the spectral theory of the harmonic oscillator.

3.4 Maximally hypoelliptic operators.

We now turn to the study of hypoelliptic operators on a compact contact manifold (M, H).

Definition 51 *Let (M, H) be a compact contact manifold. A differential operator P on M of H-order d is called* maximally hypoelliptic *if for every differential operator A on M of H-order $\le d$, there is a constant C such that,*

$$\|Au\| \le C(\|Pu\| + \|u\|),$$

for all smooth $u \in C^\infty(M)$. The norms in the inequality are $L^2(M)$ norms.

Remark. The 'a priori' estimates will hold, in particular, if the *ordinary* order of A is less or equal half that of P. Therefore,

$$\|u\|_{d/2} \le C(\|Pu\| + \|u\|),$$

where $\|\cdot\|_{d/2}$ denotes the *ordinary* Sobolev norm. This is expressed by saying that P is *subelliptic*. However, the notion of maximal hypoellipticity is better suited to the Heisenberg structure on M.

Let $\{U_j, \psi_j\}$ be a Heisenberg atlas for M (see Definition 26), with Darboux coordinates $\psi_j \colon U_j \to G$, where G is the Heisenberg group. Choosing an H-basis $Y_1, \cdots, Y_n$ for the right invariant vector fields on G, let $X_1^{(j)}, \cdots, X_n^{(j)}$ denote the pullback of this basis to U_j. Let $\{\varphi_j\}$ be a partition of unity subordinate to $\{U_j\}$. For a positive integer k, we define the *weighted Sobolev space* $W^k = W^k(M, H)$, as the completion of $C^\infty(M)$ with respect to the norm,

$$\|u\|_{W^k}^2 = \sum_j \sum_{|\alpha|\le k} \|\varphi_j X_{(j)}^\alpha u\|^2.$$

As usual, these weighted Sobolev spaces are Hilbert spaces.

Clearly, if A is an arbitrary differential operator of (graded) order less or equal d, it can be written as,

$$A = \sum_j \sum_{|\alpha| \leq k} \varphi_j a_\alpha^{(j)} X_{(j)}^\alpha,$$

with $a_\alpha^{(j)} \in C^\infty(U_j)$. One easily checks that A is continuous as an operator $W^{d+k} \to W^k$. The Sobolev norms are, up to equivalence, independent of the choice of atlas $\{U_j\}$, frame $X^{(j)}$, or partition φ_j.

Lemma 52 *Let (M, H) be a compact contact manifold. For every integer k, and every $\varepsilon > 0$, there exists $C > 0$ such that,*

$$\|u\|_{W^{k-1}} \leq \varepsilon \|u\|_{W^k} + C\|u\|,$$

for any $u \in C^\infty(M)$.

Proof. Corollary 50 gives the local estimates,

$$\sum_{|\alpha| \leq k-1} \|X_{(j)}^\alpha \varphi_j u\| \leq \epsilon \sum_{|\alpha| \leq k} \|X_{(j)}^\alpha \varphi_j u\| + C\|\varphi_j u\|.$$

We must commute $X_{(j)}^\alpha$ and φ_j. First,

$$\|u\|_{W^{k-1}} = \sum_j \sum_{|\alpha| \leq k-1} \|\varphi_j X_{(j)}^\alpha u\| \leq \sum_j \sum_{|\alpha| \leq k-1} \|X_{(j)}^\alpha \varphi_j u\| + \sum_j \sum_{|\alpha| \leq k-1} \| \, [X_{(j)}^\alpha, \varphi_j] u\|$$

The first term on the right hand side gives,

$$\begin{aligned} \sum_j \sum_{|\alpha| \leq k-1} \|X_{(j)}^\alpha \varphi_j u\| &\leq \epsilon \sum_j \sum_{|\alpha| \leq k} \|X_{(j)}^\alpha \varphi_j u\| + C \sum_j \|\varphi_j u\| \\ &\leq \epsilon \|u\|_{W^k} + \epsilon \sum_j \sum_{|\alpha| \leq k} \| \, [X_{(j)}^\alpha, \varphi_j] u\| + C'\|u\|. \end{aligned}$$

Because the commutator $[X_{(j)}^\alpha, \varphi_j]$ is an order $|\alpha| - 1$ operator on M, we obtain,

$$\|u\|_{W^{k-1}} \leq \epsilon \|u\|_{W^k} + \epsilon C'' \|u\|_{W^{k-1}} + C'\|u\| + C'''\|u\|_{W^{k-2}},$$

where C'' and C''' are independent of ϵ. If we assume, by a pocess of induction, that,

$$\|u\|_{W^{k-2}} \leq \epsilon \|u\|_{W^{k-1}} + C\|u\|,$$

the desired result follows.

$\square$

We are now ready to prove the main result of this chapter.

Theorem 53 *Let (M, H) be a compact contact manifold. A differential operator P on M is maximally hypoelliptic if and only if all the model operators $P_m, m \in M$ are Rockland operators.*

Proof. Assume that all model operators are Rockland operators.

By approximating the differential operator P with its model operator P_m in a neighborhood of each $m \in M$, we first show that *for each $m \in M$ there is a neighborhood V of m, and a constant C_V, such that,*

$$\|u\|_{W^d} \leq C_V(\|Pu\| + \|u\|),$$

for functions u with support in V.

Let U be a neigborhood of m, equipped with an isomorphism of Heisenberg structures $\psi\colon U \to G$, where G is the Heisenberg group. Let $Y_1, \cdots, Y_n$ be a basis of invariant vector fields on G, identified with vector fields on U. The Rockland operator P_m now acts on functions $u \in C_c^\infty(U)$, and we have,

$$\|u\|_{W^d} \leq C(\|P_m u\| + \|u\|).$$

All we need to do is compare $\|Pu\|$ and $\|P_m u\|$.

Choosing coordinates $(x_1, \cdots, x_n)$ in U, with $x = 0$ at m, we have,

$$P = P_m + \sum x_j Q_j + S,$$

where Q_j are order d differential operators, and S is of order $d-1$. If u is supported in a ball of small radius $|x| < \epsilon$, we get,

$$\|P_m u\| - \|Pu\| \leq \|(P_m - P)u\| \leq \epsilon C\|u\|_{W^d} + C\|u\|_{W^{d-1}},$$

with C independent of ϵ. Using Lemma 52, we obtain the desired estimate for smooth functions supported in an ϵ neighborhood of m, for ϵ sufficiently small.

To get the global result, choose a finite open cover $\{V_j\}$ such that the above estimates hold locally for each V_j. Choose a smooth partition of unity $\{\varphi_j\}$ subordinate to $\{V_j\}$. Then,

$$\begin{aligned}
\|Au\| \leq \sum \|A\varphi_j u\| &\leq \sum C_j(\|P\varphi_j u\| + \|\varphi_j u\|) \\
&\leq \sum C_j(\|\varphi_j P u\| + \|\,[P, \varphi_j]\,u\| + \|\varphi_j u\|) \\
&\leq C(\|Pu\| + \|u\|) + \sum C_j\|\,[P, \varphi_j]\,u\|.
\end{aligned}$$

Again, Lemma 52 finishes the proof.

Conversely, suppose P is maximally hypoelliptic. A similar argument, reversing the roles of P and P_m, proves that P_m is hypoelliptic. (Note that P_m is invariant, so if it is hypoelliptic in a small open set it is hypoelliptic globally). Theorem 42 implies that P_m is Rockland.

□

3.5 Appendix: inequalities of unbounded operators.

The purpose of this section is to prove the following technical result, which is a version for (unbounded) differential operators of Proposition 1.3.8 in [Pe].

Proposition 54 *Let A, B be two selfadjoint differential operators on a manifold M.*
If $0 \le \langle Au, u\rangle \le \langle Bu, u\rangle$, for all $u \in C_c^\infty(M)$, then for any $0 < r < 1$,

$$\langle A^r u, u\rangle \le \langle B^r u, u\rangle;,\ u \in C_c^\infty(M).$$

Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$, and E the associated resolution of the identity. For a (possibly unbounded) Borel function $f\colon \mathbb{R} \to \mathbb{C}$, the operator $f(A)$ is defined as $f(A) = \int f dE$, where the integral is interpreted as a weak limit. More explicitly, for a pair $u, v \in \mathcal{H}$, one can define the regular, bounded Borel measure $E_{u,v}(\omega) = \langle E(\omega)u, v\rangle$, where $E(\omega)$ is the spectral projection of A associated to the Borel set $\omega \subseteq \mathbb{R}$. Then $f(A)$ is characterized by,

$$\langle f(A)u, v\rangle = \int f(t) dE_{u,v}(t).$$

The domain of $f(A)$ is defined as,

$$\mathcal{D}(f(A)) = \{u \in \mathcal{H} : \int |f|^2 dE_{u,u} < \infty\}.$$

In fact, for $u \in \mathcal{D}(f(A))$,

$$\|f(A)u\|^2 = \int |f|^2 dE_{u,u}.$$

The domain $\mathcal{D}(f(A)) \subseteq \mathcal{H}$ is dense, and $f(A)$ is closed, while $f(A)$ is bounded iff f is E-essentially bounded. (see [Ru], Chapter 13.)

Lemma 55 *Let A be a selfadjoint differential operator on a manifold M. Let f be a Borel function that is bounded by a polynomial, i.e. $|f(t)| \le C(1 + t^{2N})$ for some $N > 0$, and all t in the spectrum of A.*
Then the domain of $f(A)$ contains $C_c^\infty(M)$.

Proof. Let E be the resolution of the identity associated with A. If $u \in C_c^\infty(M)$, then u is in the domain of $1 + A^{2N}$, for any integer $N > 0$. This means that,

$$\int (1 + t^{2N})^2 dE_{u,u}(t) < \infty.$$

Since $E_{u,u}$ is a positive measure, it follows that,

$$\int |f(t)|^2 dE_{u,u}(t) < \infty,$$

and so $u \in \mathcal{D}(f(A))$, by definition.

□

Lemma 56 *Let A be a selfadjoint differential operator on a manifold M. Let f be a Borel function on $\mathbb{R}$, and f_n a sequence of* bounded *Borel functions converging pointwise to f, and such that $|f_n(x)| \leq |f(x)|$, for x in the spectrum of A.*

Then $f_n(A)u$ converges to $f(A)u$, for all u in the domain of $f(A)$.

Proof. Let $u \in \mathcal{D}(f(A))$, i.e., $\int |f|^2 dE_{u,u} < \infty$. Because $|f - f_n|^2 \leq 4|f|^2$, the dominated convergence theorem implies,

$$\|f(A)u - f_n(A)u\|^2 = \int |f - f_n|^2 dE_{u,u} \to 0.$$

□

Lemma 57 *Let A be a strictly positive operator on a Hilbert space $\mathcal{H}$.*

Then the domain $\mathcal{D}(A)$ of A is contained in the domain of $A^{1/2}$, and $A^{1/2}$ maps $\mathcal{D}(A)$ to a dense subset of $\mathcal{H}$.

Proof. We have $\mathcal{D}(A) \subseteq \mathcal{D}(A^{1/2})$, because if $\int |t|^2 dE_{u,u} < \infty$, then clearly $\int |t^{1/2}|^2 dE_{u,u} < \infty$.

To see that $A^{1/2}$ maps $\mathcal{D}(A)$ to a dense subset of $\mathcal{H}$, let $w \in \mathcal{H}$ be such that $\langle A^{1/2}u, w\rangle = 0$ for all $u \in \mathcal{D}(A)$. Then $\langle Au, A^{-1/2}w\rangle = \langle A^{1/2}u, w\rangle = 0$. But because A is invertible, $\{Au \colon u \in \mathcal{D}(A)\} = \mathcal{H}$, so we have $A^{-1/2}w = 0$, and therefore $w = 0$.

□

Lemma 58 *Let A, B be two (possibly unbounded) strictly positive operators on a Hilbert space $\mathcal{H}$, with dense domains $\mathcal{D}(A), \mathcal{D}(B)$.*

If $0 < \langle Au, u\rangle \leq \langle Bu, u\rangle$ for all $u \in \mathcal{D}(B)$, while $\mathcal{D}(B) \subseteq \mathcal{D}(A)$, then

$$0 < \langle B^{-1}u, u\rangle \leq \langle A^{-1}u, u\rangle,$$

for all $u \in \mathcal{H}$.

Proof. Because A, B are positive, we can take square roots: $C = A^{1/2}$ and $D = B^{1/2}$. By Lemma 57, $\mathcal{D}(B) \subseteq \mathcal{D}(D)$, and also $\mathcal{D}(B) \subseteq \mathcal{D}(A) \subseteq \mathcal{D}(C)$. We may write,

$$\langle Cu, Cu\rangle \leq \langle Du, Du\rangle,$$

for $u \in \mathcal{D}(B)$.

From $\|Cu\| \leq \|Du\|$, we see that $\|CD^{-1}v\| \leq \|v\|$, for $v = Du$, $u \in \mathcal{D}(B)$. Because the set $\{Du \colon u \in \mathcal{D}(B)\}$ is dense in $\mathcal{H}$, we conclude the closure $\overline{CD^{-1}}$ is bounded of norm ≤ 1.

Because $(CD^{-1})^* = (\overline{CD^{-1}})^*$, while $(CD^{-1})^* \supseteq D^{-1}C$, we get,

$$\|D^{-1}Cv\| \leq \|v\|,$$

for all $v \in \mathcal{D}(D^{-1}C) = \mathcal{D}(C)$.

This, in turn, implies $\|D^{-1}u\| \leq \|Cu\|$, for $u = Cv$. But the range of C is $\mathcal{H}$ (because C^{-1} is bounded), so we get the desired result for all $u \in \mathcal{H}$.

□

Proof of Proposition 54. Consider the functions $f_\varepsilon, \varepsilon > 0$,with domain $x \in [0, \infty)$,

$$f_\varepsilon(x) = \int_\varepsilon^\infty \frac{x}{tx+1} t^{-r} dt = x^r \int_{\varepsilon x}^\infty \frac{1}{u+1} u^{-r} du.$$

The functions f_ε are bounded, because,

$$\int_\varepsilon^\infty \frac{x}{tx+1} t^{-r} dt \leq \int_\varepsilon^\infty t^{-r-1} dt = \frac{1}{r} \varepsilon^{-r}.$$

Observe that $c = \int_0^\infty (1+t) t^{-r} dt < \infty$, for $0 < r < 1$. We have pointwise convergence,

$$\lim_{\varepsilon \to 0} f_\varepsilon(x) = c x^r,$$

while $|f_\varepsilon(x)| \leq |cx^r|$. By Lemma 55 and Lemma 56,

$$f_\varepsilon(A) u \to c A^r u \ , \ f_\varepsilon(B) u \to c B^r u,$$

for $u \in C_c^\infty$. Hence, it suffices to show that $f_\varepsilon(A) \leq f_\varepsilon(B)$ for all $\varepsilon > 0$.

Observe that $g_t(x) = x(tx+1)^{-1}$ has supremum $\|g_t\|_\infty = 1/t$, and so,

$$f_\varepsilon = \int_\varepsilon^\infty g_t t^{-r} dt,$$

converges uniformly. Therefore, we have norm convergence,

$$f_\varepsilon(A) = \int_\varepsilon^\infty \frac{A}{tA+1} t^{-r} dt.$$

It therefore suffices to show,

$$\frac{A}{tA+1} \leq \frac{B}{tB+1}.$$

But that follows immediately from $(tA+1)^{-1} \leq (tB+1)^{-1}$, because,

$$\frac{x}{tx+1} = \frac{1}{t}\left(1 - \frac{1}{tx+1}\right).$$

$\square$

Chapter 4

From noncommutative symbol to K-theory class.

4.1 Introduction.

In this chapter we associate a K-theory class to the principal Heisenberg symbol of a maximally hypoelliptic operator on a contact manifold. As we saw in Chapter 3, a filtration on the algebra of differential operators compatible with the Heisenberg structure naturally leads to a noncommutative symbolic calculus. Correspondingly, the K-theory class associated to the principal part of a maximally hypoelliptic operator is a class in the K-theory of a noncommutative C^*-algebra.

We have seen that the algebra of symbols consists of smooth families of differential operators on the oscillating groups $G_m, m \in M$. This algebra can be extended to include smooth families of compactly supported distributions on G_m, with convolution as the product. One can think of this extended algebra as containing the symbols of *parametrices* for maximally hypoelliptic operators. However, we will not develop the symbolic calculus in this direction.

Instead, we consider the subalgebra of 'symbols' consisting of smooth families of functions in the convolution algebras $C_c^\infty(G_m)$. The C^*-algebra $C^*(T_HM)$ is the closure of the latter algebra with respect to a suitable C^*-norm. This C^*-algebra contains the *resolvent* of the Heisenberg symbol $\sigma_H(P)$ of a maximally hypoelliptic operator P. This is the key fact that allows us to construct an element,

$$[\sigma_H(P)] \in K_0(C^*(T_HM)).$$

The C^*-algebra $C^*(T_HM)$ is the reduced C^*-algebra of the smooth *groupoid* T_HM. We review the construction of the convolution C^*-algebra of a general smooth groupoid in sections 4.2 and 4.3. The reader familiar with this construction could skip these sections.

The basic idea for our construction of the K-theory class $[\sigma_H(P)]$ is due to Quillen, who gave a reformulation of the topological K-theory class of an elliptic operator (see [Qu]). The symbol class, as conceived by Atiyah and Singer, was an element in 'compactly supported' K-theory $K^0(T^*M)$. This K-theory class is defined precisely if the principal symbol $\sigma(x,\xi)$ of the operator P is invertible away from $\xi = 0$, i.e., if P is elliptic. There is not much hope of extending this topological construction to operators that are not elliptic. However, C^*-algebraic K-theory is more flexible. First, Quillen developed an algebraic formalism that gives the symbol class of an elliptic operator as a formal difference of projections in $K_0(C_0(T^*M))$. By a slight reformulation, we conceive of Quillen's construction as producing an element in $K_0(C^*(TM))$, i.e., in the K-theory of the convolution C^*-algebra of the groupoid TM. Of course,

$$K^0(T^*M) \cong K_0(C_0(T^*M)) \cong K_0(C^*(TM)).$$

Once conceived in these terms, it is not hard to see what to do in the hypoelliptic case. The important technical step is to prove that the family of resolvents of the model operators P_m defines an element in the C^*-algebra $C^*(T_H M)$.

4.2 Smooth groupoids.

In this section we briefly review the definition of a smooth groupoid, and give some examples that are relevant in what follows. A basic reference for this material is [Ma]. Put most succinctly, a groupoid is a small category with invertible arrows. To fix some terminology and notation, we give the expanded definition.

Definition 59 *A* groupoid *is a structure* $(\mathcal{G}, \mathcal{G}^{(0)}, s, r, m, e, i)$*, where* $\mathcal{G}$ *is a set called the* arrow set*, and* $\mathcal{G}^{(0)}$ *is a set called the* object set*, and* r, s, m, e, i *are various maps defined below. First we have the* source map s*, and the* range map r*:*

$$s : \mathcal{G} \to \mathcal{G}^{(0)},$$
$$r : \mathcal{G} \to \mathcal{G}^{(0)},$$

assigning to each arrow $\gamma \in \mathcal{G}$ *a source object* $s(\gamma)$ *and a range object* $r(\gamma)$*.*

A pair of arrows $(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G}$ *is called* composable*, if* $s(\gamma_1) = r(\gamma_2)$*. The set of pairs of composable arrows is denoted* $\mathcal{G}^{(2)}$*, and we have a* multiplication map m*, defined on composable arrows:*

$$m : \mathcal{G}^{(2)} \to \mathcal{G}.$$

We denote $m(\gamma_1, \gamma_2) = \gamma_1 \cdot \gamma_2$*. Associated to the multiplication map are the* units map e*, and the* inverses map i*:*

$$e : \mathcal{G}^{(0)} \to \mathcal{G},$$
$$i : \mathcal{G} \to \mathcal{G}.$$

We denote $i(\gamma) = \gamma^{-1}$*, and* $e(x) = e_x$*.*

Such a structure is a groupoid, if it satisfies the following six axioms.

Axioms governing the source map in relation to multiplication, units, and inverses:

(1) $s(\gamma_1\gamma_2) = s(\gamma_2),\ \forall \gamma_1, \gamma_2 \in \mathcal{G}$.

(2) $s(e_x) = x,\ \forall x \in \mathcal{G}^{(0)}$.

(3) $s(\gamma^{-1}) = r(\gamma),\ \forall \gamma \in \mathcal{G}$.

The groupoid axioms proper:

(4) *[Associativity]*

$(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3),\ \forall(\gamma_1, \gamma_2), (\gamma_2, \gamma_3) \in \mathcal{G}^{(2)}$.

(5) *[Multiplicative Units]*

$\gamma \cdot e_{s(\gamma)} = e_{r(\gamma)} \cdot \gamma = \gamma,\ \forall\gamma \in \mathcal{G}$.

(6) *[Multiplicative Inverses]*

$\gamma \cdot \gamma^{-1} = e_{r(\gamma)}, \gamma^{-1} \cdot \gamma = e_{s(\gamma)},\ \forall\gamma \in \mathcal{G}$.

Note that axioms 1—3 guarantee that the various multiplications in axioms 4—6 are indeed defined. Some trivial consequences of the axioms should be mentioned. First of all, one easily derives that $(\gamma^{-1})^{-1} = \gamma$. This implies, in particular, that the inverses map i is a bijection. One then deduces the analogs of axioms 1—3 for the range maps:

$$r(\gamma_1\gamma_2) = r(\gamma_1)$$
$$r(e_x) = x,$$
$$r(\gamma^{-1}) = s(\gamma).$$

We see that the range map r and source map s are right inverses for the identity map e. Therefore r and s are surjective, and e is injective.

To clarify axiom (1), we should mention that we think of elements in $\mathcal{G}$ as arrows pointing *from right to left*, with the source of the arrow on the right, and the range of the arrow on the left. With this picture in mind, it is clear that the source of $\gamma_1\gamma_2$ is the same as the source of γ_2.

Definition 59 describes an algebraic groupoid. We now impose topological conditions.

Definition 60 *A* topological groupoid *is a groupoid for which the arrow set $\mathcal{G}$ and object set $\mathcal{G}^{(0)}$ are topological spaces; the set of composable arrows $\mathcal{G}^{(2)}$ is* closed *in $\mathcal{G} \times \mathcal{G}$; the groupoid maps s, r, m, e, i are continuous; and the source and range maps s, r are open maps.*

A smooth groupoid *is a groupoid for which the arrow set $\mathcal{G}$ and object set $\mathcal{G}^{(0)}$ are smooth manifolds; the groupoid maps s, r, m, e, i are smooth; and the source and range maps s, r are submersions.*

Some notation:

$$\mathcal{G}_x = \{\gamma \,|\, s(\gamma) = x\},$$
$$\mathcal{G}^y = \{\gamma \,|\, r(\gamma) = y\},$$
$$\mathcal{G}_x^y = \mathcal{G}_x \cap \mathcal{G}^y.$$

Notice that, in a topological groupoid, the fact that source and range map are open is equivalent to saying that the sets $\mathcal{G}_x$ and $\mathcal{G}^y$ are closed subspaces of $\mathcal{G}$. In the case of a smooth groupoid, the fact that s and r are submersions guarantees that $\mathcal{G}_x$ and $\mathcal{G}^y$ are smooth submanifolds of $\mathcal{G}$. In fact, these sets form the leafs of two foliations of $\mathcal{G}$. Also observe that the set $\mathcal{G}^{(2)}$ of composable arrows is automatically a submanifold of $\mathcal{G} \times \mathcal{G}$, because it is the pre-image of the diagonal in $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ under the submersion $(r, s)\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$.

Example 1. Taking a smooth manifold M as object space, imagine an arrow connecting each pair of points in M. That is to say, we take $\mathcal{G}^{(0)} = M$ with arrow space $\mathcal{G} = M \times M$. The groupoid maps are the obvious ones: thinking of $(x, y) \in \mathcal{G}$ as an arrow from y to x, we get $s(x, y) = y, r(x, y) = x$. Composition of arrows is expressed by the multiplication law

$$(x, y) \cdot (y, z) = (x, z).$$

The units for this operation are the arrows $e(x) = (x, x)$,while $(x, y)^{-1} = (y, x)$.

The groupoid $M \times M$ is called a *pair groupoid*.

Example 2. The tangent space TM of a smooth manifold can be thought of as a smooth groupoid, with addition of vectors as groupoid multiplication. Composable vectors, of course, are those belonging to the same fiber, and the units $e(x)$ are the zero vectors. We may identify the object space with M, and take the units map to be the zero section $e \colon M \to TM$. Clearly, source and range maps are just the base point map, $r = s \colon TM \to M$.

Note that, from a purely algebraic point of view, the groupoid TM is just an uncountable family of disjoint abelian groups. Defining TM as a *smooth* groupoid makes precise the notion that the fibers form a smooth bundle of Lie groups.

Example 3. In the same manner, the bundle of oscillating groups $T_H M$ is a smooth groupoid. Smoothness of the groupoid operations follows from smoothness of the coefficients in the array b^k_{ij}, defined in Proposition 12. Note that, in this case, the group structure on the fibers G_m may vary from point to point.

4.3 The convolution C^*-algebra of a groupoid.

We recall the construction of the *reduced C^*-algebra* $C^*(\mathcal{G})$ associated to a smooth groupoid $\mathcal{G}$. See [Re] for the basics on groupoid C^*-algebras.

Definition 61 *A* Haar system *on a topological groupoid $\mathcal{G}$ is a family of positive measures λ_x, defined on the submanifolds $\mathcal{G}_x = s^{-1}(x)$ for each $x \in \mathcal{G}^{(0)}$, that is (right)* invariant *with respect to groupoid multiplication, and* continuous *in x. To be precise:*

(1) $\lambda_{s(\gamma)}(E \cdot \gamma) = \lambda_{r(\gamma)}(E)$, $\forall \gamma \in \mathcal{G}, \forall E \subseteq \mathcal{G}_{r(\gamma)}$.

(2) The map,

$$x \mapsto \int_{\mathcal{G}_x} f d\lambda_x$$

is continuous on $\mathcal{G}^{(0)}$, for any choice of $f \in C_c(\mathcal{G})$.

For a smooth groupoid, a smooth Haar system *is a Haar system such that each λ_x is a smooth 1-density, and such that $x \mapsto \int f d\lambda_x$ is smooth for any $f \in C^\infty_c(\mathcal{G})$.*

Example 1. According to the definition, a Haar system on the pair groupoid $\mathcal{G} = M \times M$ is a family of measures λ_x on $\mathcal{G}_x = \{(y, x) | y \in M\} = M \times \{x\}$ that is invariant and continuous. Invariance of the family λ_x simply means that all λ_x are equal (as measures on M). Continuity is then automatic. In the smooth case, λ_x should be a fixed smooth 1-density on M.

Example 2. A smooth Haar system on the tangent bundle $\mathcal{G} = TM$ is a family of smooth densities on the fibers $\mathcal{G}_x = T_xM$ for $x \in M$. Invariance is satisfied if the densities λ_x are Haar measures for the Lie group structure on T_xM, i.e., λ_x is a multiple of Lebesgue measure on T_xM. Thinking of Haar measure on T_xM as a linear map,

$$\lambda_x \colon |\Lambda^n| T_xM \to \mathbb{R}_+,$$

we see that the family $\{\lambda_x\}$ can be identified with a single 1-density λ on M. (The bundle $|\Lambda^n|TM$ is the bundle of 1-densities, associated to the frame bundle of TM by means of the representation $GL(n,\mathbb{R}) \to \mathbb{R}_+ \colon A \mapsto |\det(A)|$.) To require that λ_x varies smoothly with x is to say that λ is a smooth 1-density on M.

Example 3. The situation for the groupoid T_HM is similar. If we identify the osculating group T_HM_x with the Lie algebra $H_x \oplus N_x$ via the exponential map, then Haar measure on T_HM_x is just Lebesgue measure on $H_x \oplus N_x$, which, in turn, is naturally identified with Lebesgue measure on T_xM (independent of the choice of section $N \subseteq TM$). So, again, a smooth Haar system on T_HM corresponds to a smooth 1-density on M.

For a smooth groupoid $\mathcal{G}$, the set $C_c^\infty(\mathcal{G})$ of smooth, compactly supported functions on $\mathcal{G}$ is a convolution $*$-algebra. Convolution and star operation are defined as,

$$\begin{aligned}(f * g)(\gamma) &= \int_{\mathcal{G}_x} f(\gamma\nu^{-1}) g(\nu) d\lambda_x(\nu),\\ f^*(\gamma) &= \overline{f(\gamma^{-1})}\end{aligned}$$

where $x = s(\gamma)$. Right invariance of the Haar system garantuees associativity of convolution, and all axioms for a $*$-algebra are easily verified.

For each $x \in \mathcal{G}^{(0)}$ the associated *regular representation* π_x of the $*$-algebra $C_c^\infty(\mathcal{G})$ is the representation on the Hilbert space $L^2(\mathcal{G}_x, \lambda_x)$ obtained by convolution,

$$\pi_x(f)\phi = f * \phi,$$

with $f \in C_c^\infty(\mathcal{G})$ and $\phi \in C_c^\infty(\mathcal{G}_x) \subseteq L^2(\mathcal{G}_x)$.

Definition 62 *Let $\mathcal{G}$ be a smooth groupoid. The* reduced groupoid C^*-algebra $C^*(\mathcal{G})$ *is the completion of the convolution algebra $C_c^\infty(\mathcal{G})$ with respect to the norm,*

$$||f||_{C_r^*(\mathcal{G})} = \sup_{x \in \mathcal{G}^{(0)}} ||\pi_x(f)||.$$

In other words, the reduced C^*-algebra norm is the smallest norm for which all regular representations are continuous.

Example 1. Convolution and star operation for the pair groupoid $M \times M$ are given by the formulas,

$$\begin{aligned}(f * g)(x,y) &= \int_M f(x,z) g(z,y) d\lambda(z),\\ f^*(x,y) &= \overline{f(y,x)},\end{aligned}$$

with $x, y, z \in M$. This corresponds to multiplication and adjoint operation for operator kernels. Indeed, the operator$\pi_y(K)$, $y \in M$, for $K \in C_c^\infty(M \times M)$ is given by,

$$(\pi_y(K)\phi)(x,y) = \int_M K(x,z)\phi(z,y)d\lambda(z).$$

Identifying $\mathcal{G}_y = M$, we see that $\pi_y(K) = \mathrm{Op}(K)$ is independent of $y \in M$. Here $\mathrm{Op}(K)$ denotes the Hilbert-Schmidt operator with Schwartz kernel K, i.e.,

$$\mathrm{Op}(K)\phi(x) = \int_M K(x,z)\phi(z)d\lambda(z).$$

Since $\|\pi_y(K)\| = \|\mathrm{Op}(K)\|$, we have $\|K\|_{C_r^*(\mathcal{G})} = \|\mathrm{Op}(K)\|$, and we get a canonical isomorphism,

$$C^*(M \times M) \cong \mathcal{K}(L^2(M,\lambda)) \,:\, K \mapsto \mathrm{Op}(K),$$

where $\mathcal{K}$ denotes the algebra of compact operators on a Hilbert space.

Example 2. For the tangent space $\mathcal{G} = TM$ we had $\mathcal{G}_x = T_xM$, with Lebesgue measure λ_x on $T_xM = \mathbb{R}^n$. For a smooth function $f \in C_c^\infty(TM)$ the regular representation $\pi_x(f)$ is the operator on $L^2(T_xM, \lambda_x)$ of convolution with f_x, the restriction of f to T_xM. This implies,

$$||\pi_x(f)|| = \sup_{\xi \in T_x^*M} \hat{f}_x(\xi),$$

where $\hat{f}_x$ denotes the Fourier transform of f_x ($\hat{f}_x$ is a Schwartz class function on T^*M), and so,

$$||f||_{C^*(\mathcal{G})} = ||\hat{f}||_\infty.$$

We see that Fourier transform in the fibers of TM gives an isomorphism,

$$C^*(TM) \cong C_0(T^*M).$$

Example 3. The situation for the bundle of oscillating groups T_HM is similar, except that we cannot make use of Fourier transform. Each regular representation $\pi_x, x \in M$ gives a $*$-homomorphism,

$$\pi_x \,:\, C_c^\infty(T_HM) \to C^*(G_x),$$

where $C^*(G_x)$ is the (reduced) group C^*-algebra of the osculating group G_x at $x \in M$. (The nilpotent groups G_m are amenable, so the full group C^*-algebra $C^*(G)$ is naturally isomorphic to the reduced C^*-algebra $C_r^*(G)$.) Then,

$$||f||_{C^*(T_HM)} = \sup_{x \in M} ||\pi_x(f)||_{C^*(G_x)}.$$

Proposition 63 *The function $x \mapsto ||\pi_x(f)||_{C^*(G_x)}$ is continuous on M, for each $f \in C_c(T_HM)$.*

Proof. Suppose for some $x \in M$ we have $\|\pi_x(f)\| = C$. Pick $g_x \in C_c(G_x)$ such that

$$\frac{\langle f_x * g_x, g_x \rangle}{\langle g_x, g_x \rangle} > C - \varepsilon.$$

If we extend g_x to a function $g \in C_c(T_H M)$, then the function

$$y \mapsto \frac{\langle f_y * g_y, g_y \rangle}{\langle g_y, g_y \rangle}$$

is continuous. (Here g_y denotes the function on G_y defined by $g_y(v) = g(y, v)$ for $y \in M, v \in G_y$.) Therefore, for all y in a neighborhood of x we have $\|\pi_y(f)\| > C - 2\varepsilon$. In other words, $x \mapsto \|\pi_x(f)\|$ is *lower semi-continuous*.

□

As a corollary of Proposition 63, we see that $C^*(T_H M)$ is a continuous field of C^*-algebras over M, where the fibers are the group C^*-algebras $C^*(G_x)$. Notice that $C^*(T_H M)$ is commutative if and only if each osculating group G_x is abelian, which means that (M, H) is a foliation.

4.4 An elliptic symbol as a K-theory class.

Let P be an elliptic differential operator on a compact manifold M, acting on the space of smooth sections in a vector bundle E over M, with range in the space of sections in a second bundle F,

$$P : \Gamma(E) \to \Gamma(F).$$

If P is given in local coordinates as,

$$P = \sum_{|\alpha| \le d} a_\alpha(x) \partial^\alpha,$$

with matrix valued coefficients $a_\alpha(x)$, then the principal symbol,

$$\sigma(P)(x, \xi) = \sum_{|\alpha| = d} a_\alpha(x)(i\xi)^\alpha,$$

is invariantly defined as a section,

$$\sigma(P) \in \Gamma(\mathrm{Hom}\,(\pi^* E, \pi^* F)),$$

where $\pi^* E$, $\pi^* F$ denote the pullback of E, F to the cotangent space $T^* M$ via the base point map $\pi \colon T^* M \to M$.

By definition, P is elliptic if its principal symbol $\sigma(P)$ is invertible outside the zero section $M \subset T^* M$. In this case the principal symbol defines a class in compactly supported K-theory,

$$[(\sigma(P), \pi^* E, \pi^* F)] \in K^0(T^* M).$$

There is a nice description of this class as an element in the C^*-algebraic K-theory group $K_0(C_0(T^*M))$. To arrive at this description, and show its equivalence with the topological K-theory class, we use the strong excision property of K-theory as an intermediate step,

$$K^0(T^*M) \cong K^0(\overline{T^*M}, \partial T^*M),$$

where $\overline{T^*M} = T^*M \cup \partial T^*M$ is an arbitrary compactification of T^*M. By choosing a suitable boundary ∂T^*M we can avoid the technical complications presented by the possible nontriviality of the vector bundles E and F. Consider the *graph* Γ_σ of the symbol $\sigma(P)$,

$$\Gamma_\sigma = \{(v, \sigma(v)) \in \pi^*E \oplus \pi^*F\} \subseteq \pi^*E \oplus \pi^*F.$$

As a vector bundle over T^*M, it is isomorphic to π^*E,

$$\Gamma_\sigma \cong \pi^*E \,:\; (v, \sigma(v)) \mapsto v.$$

But, because σ is homogeneous (i.e. $\sigma(x, t\xi) = t^d\sigma(x, \xi)$) and *elliptic*, there is a sense in which Γ_σ converges *at infinity* (as $\xi \to \infty$) to the bundle π^*F. Let $\overline{T^*M}$ be the compactification obtained by adding a point at infinity in each fiber of T^*M (or any other compactification that covers this one). Then π^*E, π^*F, and Γ_σ extend to bundles over $\overline{T^*M}$ in the obvious way, without the need for trivialization, and the compactly supported class $[(\sigma, \pi^*E, \pi^*F)]$ is equivalent to the relative element,

$$[\Gamma_\sigma] - [\pi^*F] \in K^0(\overline{T^*M}, \partial T^*M).$$

This translates easily into algebraic language. Choose a hermitian metric in the fibers of the bundles E and F. Let e_σ denote the *graph projection* of the symbol σ, i.e., the orthogonal projection of $\pi^*E \oplus \pi^*F$ onto Γ_σ,

$$e_\sigma \in C_b(\mathrm{End}\,(\pi^*E \oplus \pi^*F)).$$

Let e_F denote the projection of $\pi^*E \oplus \pi^*F$ onto π^*F. Then $e_sigma - e_F$ is an element in $C_0(\mathrm{End}\,(\pi^*E \oplus \pi^*F))$, and we therefore have a well-defined K-theory class

$$[e_\sigma] - [e_F] \in K_0(C_0(\mathrm{End}\,(\pi^*E \oplus \pi^*F))).$$

Here $C_0(\mathrm{End}\,(\pi^*E \oplus \pi^*F))$ is the C^*-algebra of continuous sections in $\mathrm{End}\,(\pi^*E \oplus \pi^*F)$ that vanish at infinity. In terms of relative K-theory, we actually have an element in the group,

$$K_0(C_b(\mathrm{End}\,(\pi^*E \oplus \pi^*F)), C_0(\mathrm{End}\,(\pi^*E \oplus \pi^*F))).$$

While $\mathrm{End}(\pi^*E \oplus \pi^*F)$ may not be a trivial $\mathcal{M}_{2N}(\mathbb{C})$ bundle over T^*M, we still have *Morita equivalence*,

$$C_0(\mathrm{End}\,(\pi^*E \oplus \pi^*F)) \sim_{\mathrm{Morita}} C_0(T^*M),$$

and hence,

$$K_0(C_0(\mathrm{End}\,(\pi^*E \oplus \pi^*F))) \cong K_0(C_0(T^*M)).$$

To get the equivalent class in $K_0(C_0(T^*M))$, choose a bundle G such that $E \oplus F \oplus G$ is trivial. Then e_σ and e_F are projections in,

$$C_b(\mathrm{End}\,(\pi^*E \oplus \pi^*F)) \subseteq C_b(\mathrm{End}\,(\pi^*E \oplus \pi^*F \oplus \pi^*G)) \cong C_b(T^*M) \otimes \mathcal{M}_k(\mathbb{C}),$$

where k denotes the fiber dimension of $E \oplus F \oplus G$. (In fact, e_σ and e_F will be elements in $C(T^*M)^+ \otimes \mathcal{M}_k$.)

It is clear that the element $[e_\sigma] - [e_F] \in K_0(C_0(T^*M))$ is equivalent to the class $[\Gamma_\sigma] - [\pi^*F] \in K^0(\overline{T^*M}, \partial T^*M)$ (to get a perfect correspondence, we should take the Cech-Stone compactification for $\overline{T^*M}$), and hence to the compactly supported class $[(\sigma, \pi^*E, \pi^*F)] \in K^0(T^*M)$.

4.5 The analytic index as K-theory class.

The beauty of the algebraic construction just described is that it can also be applied directly to P itself. Choosing a measure on M, and hermitian structures in the bundles E and F, we can think of P as an unbounded Hilbert space operator,

$$P: \ L^2(M, E) \to L^2(M, F).$$

Since P is a differential operator, it is closable. The *graph projection* e_P of P is the projection of the Hilbert space $\mathcal{H} = L^2(M, E) \oplus L^2(M, F)$ onto the graph of the closure $\bar{P}$ of P, which is a closed subspace of $\mathcal{H}$. If e_{L^2F} denotes the projection of $\mathcal{H}$ onto $L^2(M, F)$, we can construct the K-theory class,

$$[e_P] - [e_{L^2F}] \in K_0(\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H})) \cong K_0(\mathcal{K}(\mathcal{H})).$$

The following lemma states that this is a well-defined element.

Lemma 64 *For a closed operator $P\colon L^2(M, E) \to L^2(M, F)$, the difference of projections $e_P - e_{L^2F}$ is a compact operator on $L^2(M, E) \oplus L^2(M, F)$ if and only if $(1 + P^*P)^{-1}$ is compact.*

Proof. An elementary exercise in linear algebra shows that if $A \in \mathcal{M}_n(\mathbb{C})$ is an $n \times n$ matrix, then the graph projection of A is the $2n \times 2n$ matrix,

$$\begin{pmatrix} (1 + A^*A)^{-1} & (1 + A^*A)^{-1}A^* \\ A(1 + A^*A)^{-1} & A(1 + A^*A)^{-1}A^* \end{pmatrix}.$$

Likewise, we can describe the graph projection of P as a 2×2 matrix of operators on $\mathcal{H}$,

$$e_P = \begin{pmatrix} (1 + P^*P)^{-1} & (1 + P^*P)^{-1}P^* \\ P(1 + P^*P)^{-1} & P(1 + P^*P)^{-1}P^* \end{pmatrix}.$$

(When we write P here, we mean the closure $\bar{P}$.) The entries in the matrix should be interpreted according to the direct sum decomposition $\mathcal{H} = L^2(M, E) \oplus L^2(M, F)$. Observe that from $P(1+P^*P) = (1+PP^*)P$ we get $(1+PP^*)^{-1}P = P(1+P^*P)^{-1}$, and then $P(1+P^*P)^{-1}P^* = (1 + PP^*)^{-1}(1 + PP^* - 1) = 1 - (1 + PP^*)^{-1}$. Therefore,

$$e_P - e_{L^2F} = e_P - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (1 + P^*P)^{-1} & (1 + P^*P)^{-1}P^* \\ P(1 + P^*P)^{-1} & -(1 + PP^*)^{-1} \end{pmatrix}.$$

If this is compact, then so is the matrix entry $(1+P^*P)^{-1}$. Conversely, if $(1+P^*P)^{-1}$ is compact then so are all matrix entries in $e_P - e_{L^2F}$. For example, using a polar decomposition $P = \sqrt{P^*P}U$, we get,

$$(1+P^*P)^{-1}P^* = (1+P^*P)^{-1}\sqrt{P^*P}U = f(P^*P)U,$$

with $f(x) = \sqrt{x}/(1+x)$. Because $f \in C_0(\mathbb{R})$ and P^*P has compact resolvent, the operator $f(P^*P)$ is compact.

□

Observe that for a maximally hypoelliptic operator, the a priori estimates imply that $(1+P^*P)^{-1}$ is bounded as an operator $L^2 \to W^{2d}$, and is therefore compact. Hence, the lemma applies.

There is an nice description of the graph projection, which is sometimes more convenient. Consider the self-adjoint operator on $\mathcal{H}$,

$$D = \begin{pmatrix} 0 & -iP \\ iP^* & 0 \end{pmatrix},$$

and let u be the Cayley transform of D, i.e., the unitary,

$$u = (D+i)(D-i)^{-1}.$$

The decomposition of $\mathcal{H}$ is encoded in the grading operator,

$$\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

that satisfies $\epsilon^2 = 1$, and $\epsilon D = -D\epsilon$. It follows that $\epsilon u = u^*\epsilon$, and then $(\epsilon u)^2 = 1$. This last equality implies that $\frac{1}{2}(\epsilon u + 1)$ is a projection.

Lemma 65 *With the above notation,*

$$e_P = \frac{1}{2}(\epsilon u + 1)\ ,\ e_{L^2F} = \frac{1}{2}(\epsilon + 1).$$

Proof. We obtain an expression for u as a 2×2 matrix as follows,

$$\begin{aligned} u &= (D+i)^2(D^2+1)^{-1} \\ &= \begin{pmatrix} -1+P^*P & -2P^* \\ 2P & -1+PP^* \end{pmatrix} \begin{pmatrix} (1+P^*P)^{-1} & 0 \\ 0 & (1+PP^*)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1-2(1+P^*P)^{-1} & -2(1+P^*P)^{-1}P^* \\ 2(1+PP^*)^{-1}P & 1-2(1+PP^*)^{-1} \end{pmatrix}, \end{aligned}$$

and therefore,

$$\frac{1}{2}(\epsilon u + 1) = \begin{pmatrix} (1+P^*P)^{-1} & (1+P^*P)^{-1}P^* \\ (1+PP^*)^{-1}P & 1-(1+PP^*)^{-1} \end{pmatrix} = e_P.$$

□

Observe that, $u = (D+i)(D-i)^{-1} = 1 + 2i(D-i)^{-1}$, and therefore,

$$e_P - e_{L^2F} = \frac{1}{2}(\epsilon u + 1) - \frac{1}{2}(\epsilon + 1) = i\epsilon(D-i)^{-1},$$

which shows, again, that $e_P - e_{L^2F}$ is compact if and only if the resolvent $(D-i)^{-1}$ is compact, i.e., if D is Fredholm.

Proposition 66 *With the notation used above, we have,*

$$[e_P] - [e_{L^2F}] = [\mathrm{Ker}\, P] - [\mathrm{Ker}\, P^*] \in K_0(\mathcal{K}(\mathcal{H})),$$

where $[\mathrm{Ker}\, P], [\mathrm{Ker}\, P^*]$ *denote the projections of* $\mathcal{H}$ *onto the kernels of* P *and* P^**.*

Proof. Consider the family of projections $e_t = \frac{1}{2}(\epsilon u_t + 1)$, with $u_t = (tD+i)(tD-i)^{-1}$. By functional calculus, the family u_t is norm continuous. Observe that the spectral function $z \mapsto (z+i)(z-i)^{-1}$ maps 0 to -1, and ∞ to $+1$. Now, D having compact resolvent, its spectrum is discrete. Therefore, as $t \to \infty$, the family u_t converges in norm to the operator,

$$u_\infty = -[\mathrm{Ker}\, D] + (1 - [\mathrm{Ker}\, D]) = 1 - 2[\mathrm{Ker}\, D].$$

By definition of D, we have,

$$[\mathrm{Ker}\, D] = \begin{pmatrix} [\mathrm{Ker}\, P] & 0 \\ 0 & [\mathrm{Ker}\, P^*] \end{pmatrix},$$

and we see that the homotopy of projections e_t converges in norm to the projection,

$$e_\infty = \frac{1}{2}(\epsilon u_\infty + 1) = \frac{1}{2}(\epsilon - 2\epsilon[\mathrm{Ker}\, D] + 1) = \begin{pmatrix} [\mathrm{Ker}\, P] & 0 \\ 0 & 1 - [\mathrm{Ker}\, P^*] \end{pmatrix}.$$

Now use,

$$[e_{L^2F}] = [1 - [\mathrm{Ker}\, P^*]] + [\mathrm{Ker}\, P^*] \in K_0(\mathcal{K}(\mathcal{H})).$$

□

Hence, the same algebraic formula that defines the K-theory class associated to an elliptic symbol $\sigma(P)$, defines the analytic index of P as a K-theory class. Of course, the symbol class could have been constructed starting from the self adjoint element,

$$\sigma(D) = \begin{pmatrix} 0 & -i\overline{\sigma(P)} \\ i\sigma(P) & 0 \end{pmatrix}.$$

We point out that the K-theory element in $K_0(C_0(\mathrm{End}(\pi^*E \oplus \pi^*F))$ is well-defined precisely because the resolvent of $\sigma(D)$ is an element in the C^*-algebra $C_0(\mathrm{End}(\pi^*E \oplus \pi^*F)$, which occurs if and only if the symbol is *elliptic*. Likewise, the analytic index of P in $K_0(\mathcal{K}(L^2(E) \oplus L^2(F)))$ is well-defined if and only if the resolvent of D is a compact operator. Since ellipticity of $\sigma(D)$ implies that D has compact resolvent, the two resolvents (of $\sigma(D)$ and of D itself) and the corresponding K-theories should be related. This relationship will be clarified by the construction of the *tangent groupoid* and its associated C^*-algebra and K-theory, described in a later chapter.

4.6 The Heisenberg symbol as K-theory class.

In this section we adapt Quillen's construction to the noncommutative, hypoelliptic case. The appropriate K-theory group for the class associated to the Heisenberg symbol of a maximally hypoelliptic operator is $K_0(C^*(T_H M))$, where $C^*(T_H M)$ denotes the convolution C^*-algebra of the smooth groupoid $T_H M$, the bundle of osculating groups for the Heisenberg structure (M, H). As is to be expected from the above discussion, the key technical issue is to show that, for a maximally hypoelliptic operator, the *resolvent* of its symbol defines an element in a C^*-algebra that is Morita equivalent to $C^*(T_H M)$.

Starting from a maximally hypoelliptic operator P, we have the Heisenberg symbol $\sigma_H(P)$, which is a smooth family of Rockland operators P_m on the osculating groups $G_m, m \in M$. As before, we define the self-adjoint operators,

$$D_m = \begin{pmatrix} 0 & -iP_m \\ iP_m^* & 0 \end{pmatrix},$$

and Cayley transforms,

$$u_m = (D_m + i)(D_m - i)^{-1}.$$

Note that the family $\{D_m\}$ is just the Heisenberg symbol $\sigma_H(D)$ of the self-adjoint operator D associated to P. All operators D_m are Rockland operators.

If P acts on sections in bundles E and F, then D_m acts on sections in the (always trivial) bundle $\pi^*E \oplus \pi^*F$ over G_m. (Here π is the base point map $\pi\colon T_H M \to M$.) As before, ϵ denotes the grading operator associated to the decomposition $\pi^*E \oplus \pi^*F$. Following Quillen's formalism, we want to define a K-theory class,

$$[\sigma_H(P)] = [e_\sigma] - [e_F] = [\frac{1}{2}(\epsilon u + 1)] - [\frac{1}{2}(\epsilon + 1)] \in K_0(C^*(T_H M)).$$

We must interpret this with a little care. First of all, the graph projection,

$$e_\sigma = \frac{1}{2}(\epsilon u + 1),$$

can be thought of as a *family* of projections parametrized by $m \in M$,

$$e_m = \frac{1}{2}(\epsilon u_m + 1) \in \mathrm{End}\,(E_m \oplus F_m),$$

and we will need to show that

$$e_m \in C^*(G)^+ \otimes \mathrm{End}\,(E_m \oplus F_m) \cong \mathcal{M}_{2N}(C^*(G_m)^+),$$

where $2N$ denotes the fiber dimension of $E \oplus F$. Since $u_m = 1 + 2i(D_m - i)^{-1}$, it suffices to prove that,

$$(D_m - i)^{-1} \in \mathcal{M}_{2N}(C^*(G_m)).$$

Secondly, the family $\{e_m\}$ should define a projection,

$$e_\sigma \in C^*(T_H M) \otimes_M C(\mathrm{End}(E \oplus F)).$$

Here $C^*(T_HM) \otimes_M C(\mathrm{End}(E \oplus F))$ denotes the C^*-algebra of continuous sections in the field $\{C^*(G_m)^+ \otimes \mathrm{End}\,(E_m \oplus F_m)\}$. By choosing a bundle G such that $E \oplus F \oplus G$ is trivial, we obtain,

$$C^*(T_HM) \otimes_M C(\mathrm{End}(E \oplus F)) \subseteq \mathcal{M}_k(C^*(T_HM)^+).$$

Again, it suffices to show that the family of resolvents $\{(D_m - i)^{-1}\}$ defines a continuous section in $\{\mathcal{M}_{2N}(C^*(G_m))\}$ (here we can simply choose a *local* trivialization of $E \oplus F$).

We now prove these two key results about the family of resolvents $(D_m - i)^{-1}$ of the model operators D_m of the self-adjoint, maximally hypoelliptic operator D.

Proposition 67 *Let G be a graded nilpotent group, with grading of arbitrary length. If L is a formally self-adjoint Rockland operator on G, then*

$$f(L) \in C^*(G)$$

for all functions $f \in C_0(\mathbb{R})$.

proof. A theorem of Folland and Stein states that, for the *positive* Rockland operator L^2, the distribution kernel K of $\exp(-L^2)$,

$$e^{-L^2}\phi = K * \phi,$$

is a Schwartz-class function on the group G ([FS], chapter 4.B). Hence, certainly $K \in L^1(G)$, and therefore,

$$e^{-L^2} \in C^*(G).$$

(Here, we identify $C^*(G)$ with its image under the left regular representation on $L^2(G)$.)

Next, the distribution kernel of the operator Le^{-L^2} is $(L\delta) * K = LK$. The function LK, again, is of Schwartz class, because any left invariant homogeneous operator $L = \sum a_\alpha(x)\partial^\alpha$ has polynomial coefficients $a_\alpha(x)$. We conclude that also,

$$Le^{-L^2} \in C^*(G).$$

Because the functions e^{-x^2} and xe^{-x^2} separate points on $\mathbb{R}$, these two functions generate $C_0(\mathbb{R})$ as a C^*-algebra. This implies the general result.

$\square$

Remark. Folland and Stein prove the theorem quoted in our proof only for scalar operators L. However, their proof relies only on the existence of a priori estimates for a sufficiently large power of L, i.e., on the theorem by Helffer and Nourrigat we referred to in the previous chapter (see Theorem 43). In Appendix A we verify that the a priori estimates hold if L is a matrix of operators, and therefore Proposition 67 also holds for such operators.

Proposition 67 establishes our first requirement on the resolvent of the model operators,

$$(D_m - i)^{-1} \in \mathcal{M}_{2N}(C^*(G_m)).$$

This result holds in absolute generality. Our second requirement is that the family $\{(D_m - i)^{-1}\}$ defines a *continuous section* in the field of C^*-algebras $\{\mathcal{M}_{2N}(C^*(G_m))\}$ over M. We will only prove this for the special case where the bundle of osculating groups is *locally trivial.* In this case, the field $\{\mathcal{M}_{2N}(C^*(G_m))\}$ is also locally trivial, and we can choose local isomorphisms,

$$C^*(G_m) \cong C^*(G),$$

where G is a fixed graded group. With these identifications, the model operators D_m form a family of Rockland operators on a single group G, with smoothly varying coefficients.

In the case of a contact manifold, for example, Darboux coordinates induce Taylor coordinates on the osculating groups, by means of which the groups are identified with the Heisenberg group, with its standard coordinates. In general, a *Heisenberg atlas* will accomplish the same for any trivial Heisenberg structure.

Having chosen a local trivialization of the bundle $T_H M = \{G_m\}$ on an open set $U \subseteq M$, a continuous section in the field $\{\mathcal{M}_{2N}(C^*(G_m))\}$ is just a continuous map,

$$U \to \mathcal{M}_{2N}(C^*(G)).$$

In other words, having identified the resolvents $(D_m - i)^{-1}$ as elements in the same C^*-algebra, we must establish that they form a norm-continuous family.

Definition 68 *Let G be a graded nilpotent group, and let $X_1, \cdots X_n$ be a basis for the Lie algebra of G compatible with the grading. Let $U \subseteq \mathbb{R}^n$ be an open set.*

A family of right-invariant operators $\{D_m, m \in U\}$ on a graded group G,

$$D_m = \sum_{|\alpha| \leq d} a_\alpha(m) X^\alpha,$$

is called a continuous family*, if the coefficients a_α are continuous functions on U.*

Lemma 69 *A continuous family of right invariant differential operators of order d on a graded group G is norm continuous as a family of bounded operators $W^d(G) \to L^2(G)$.*

Proof. This follows immediately from the definitions. Clearly, if $|\alpha| \leq d$, then $\|X^\alpha\|_{W^d \to L^2} \leq 1$. Therefore,

$$\|D_{m_1} - D_{m_0}\|_{W^d \to L^2} \leq C \sum_\alpha \|a_\alpha(m_1) - a_\alpha(m_0)\|_\infty.$$

□

Proposition 70 *Let G be a graded nilpotent group. If $\{D_m, m \in U\}$ is a continuous family of self-adjoint Rockland operators on G, then the map,*

$$U \to \mathcal{B}(L^2(G)) : \ m \mapsto f(D_m),$$

is norm continuous for any $f \in C_0(\mathbb{R})$.

Proof. Consider the equality,

$$(D_{m_0} - i)^{-1} - (D_{m_1} - i)^{-1} = (D_{m_1} - i)^{-1}(D_{m_1} - D_{m_0})(D_{m_0} - i)^{-1}.$$

By Proposition 46, we have,

$$\|(D_{m_0} - i)^{-1}\|_{L^2 \to W^d} \leq \infty,$$

while,

$$\|(D_{m_1} - i)^{-1}\|_{L^2 \to L^2} \leq 1.$$

and, finally, by Lemma 69,

$$\|D_{m_1} - D_{m_0}\|_{W^d \to L^2} \to 0,$$

as $m_1 \to m_0$. Combining these facts we obtain,

$$\|(D_{m_1} - i)^{-1}(D_{m_1} - D_{m_0})(D_{m_0} - i)^{-1}\|_{L^2 \to L^2} \to 0$$

as $m_1 \to m_0$ (keeping m_0 fixed). Therefore the family of resolvents $(D_m - i)^{-1}$ is norm continuous.

The result for general $f \in C_0(\mathbb{R})$ follows by spectral theory.

□

Again, the same is true when D_m is a matrix of Rockland operators, by the same proof. We summarize:

Theorem 71 *Let P be a maximally hypoelliptic differential operator on a manifold with Heisenberg structure (M, H), acting on the space of sections in a vector bundle E, with range in the space of sections in a bundle F. Let $\sigma_H(P) = \{P_m\}$ denote the Heisenberg symbol of P. The graph projection,*

$$e_m \in \mathcal{B}(L^2(G_m, E_m \oplus F_m)),$$

of the Rockland operator P_m defines an element in the C^*-algebra,

$$e_m \in C^*(G_m)^+ \otimes \mathrm{End}\,(E_m \oplus F_m) \cong \mathcal{M}_{2N}(C^*(G_m)^+).$$

Assuming that the bundle $T_H M$ of osculating groups is locally trivial, the family of graph projections $\{e_m,\, m \in M\}$ defines a projection,

$$e_\sigma \in C^*(T_H M)^+ \otimes_M C(\mathrm{End}\,(E \oplus F)) \subseteq \mathcal{M}_k(C^*(T_H M)^+).$$

Finally, let

$$e_F \in C(\mathrm{End}\,(E \oplus F))$$

denote the family of projections $E_m \oplus F_m \to F_m$. We identify e_F with the element,

$$1 \otimes e_F \in C^*(T_H M)^+ \otimes_M C(\mathrm{End}\,(E \oplus F)) \subseteq \mathcal{M}_k(C^*(T_H M)^+).$$

Then the K-theory class,

$$[\sigma_H(P)] = [e_\sigma] - [e_F] \in K_0(C^*(T_H M)),$$

is well defined.

If $H = TM$, then P is elliptic, and $[\sigma_H(P)]$ is equivalent to the topological K-theory class associated to the principal symbol $\sigma(P)$ of P,

$$[(\sigma(P), \pi^* E, \pi^* F)] \in K^0(T^* M).$$

It seems reasonable to conjecture that the assumption of triviality of the bundle $T_H M$ can be dropped. However, since this assumption holds for the most important examples (contact manifolds, foliations), it is not a serious restriction for us.

Chapter 5

The parabolic tangent groupoid.

5.1 Introduction.

Quillen's construction associates to the principal symbol $\sigma(P)$ of an elliptic operator on a compact manifold a K-theory class in $K_0(C_0(T^*M))$, which corresponds to the topological K-theory class $[\sigma(P)] \in K^0(T^*M)$ defined by Atiyah and Singer. The same algebraic formula associates to the operator P itself a class in $K_0(\mathcal{K}(L^2(M)))$, the K-theory group of the C^*-algebra of compact operators on $L^2(M)$. Under the isomorphism $K_0(\mathcal{K}) \cong \mathbb{Z}$, induced by the trace map, the latter class corresponds to the analytic index of P.

In [Co], Connes introduces a smooth groupoid whose convolution C^*-algebra combines in a single continuous field both of these C^*-algebras $C_0(T^*M)$ and $\mathcal{K}(L^2(M))$. Based on the idea that the tangent bundle TM can be identified with the normal bundle of the diagonal $\Delta = M \subset M \times M$ in the Cartesian square $M \times M$, Connes glues the groupoid TM to the parametrized family $M \times M \times (0, 1]$ by blowing up the diagonal in $M \times M$ as $t \in (0, 1]$ converges to 0. The resulting groupoid is called the *tangent groupoid* of the manifold M. A simple construction involving the K-theory of the convolution algebra of the tangent groupoid leads to a natural map,

$$K_0(C_0(T^*M)) \to K_0(\mathcal{K}(L^2(M))) \cong \mathbb{Z},$$

which turns out to be equal to the analytic index map. Connes sketches a proof of the Atiyah-Singer index theorem based on these ideas (see [Co], section II.5).

Our proof of the index theorem for subelliptic operators is based on a generalization of Connes' tangent groupoid to the case where (M, H) is a manifold with Heisenberg structure. The basic idea is to glue the groupoid $M \times M \times (0, 1]$ to the bundle of osculating groups $T_H M$ by blowing up the diagonal in $M \times M$ using *graded dilations*, corresponding to the grading of the osculating groups. Our groupoid will give rise to a map,

$$K_0(C^*(T_H M)) \to K_0(\mathcal{K}(L^2(M))) \cong \mathbb{Z},$$

which leads to the index theorem that we will formulate and prove in the next chapter.

5.2 The groupoid $\mathbb{T}_H M$.

As a generalization of Connes' tangent groupoid, which relates the total space of the tangent bundle TM to the pair groupoid $M \times M$, we define a similar groupoid in which the bundle TM is replaced by $T_H M$. We shall refer to this groupoid as the *parabolic tangent groupoid* of a manifold with Heisenberg structure, and denote it by $\mathbb{T}_H M$.

As an algebraic groupoid, $\mathbb{T}_H M$ is the disjoint union,

$$\mathbb{T}_H M = (\bigcup_{t \in (0,1]} \mathcal{G}_t) \cup (\bigcup_{m \in M} \mathcal{G}_m),$$

of a parametrized family of pair groupoids with the collection of osculating groups,

$$\begin{aligned} \mathcal{G}_t &= M \times M, \ t \in (0,1], \\ \mathcal{G}_m &= T_H M_m, \ m \in M. \end{aligned}$$

Clearly, the union $\cup \mathcal{G}_t = M \times M \times (0,1]$ by itself is a smooth groupoid, and the same is true, as we have seen, for the bundle of osculating groups $\cup \mathcal{G}_m = T_H M$. We write $\mathcal{G}_0 = T_H M$, and $\mathcal{G}_{(0,1]} = M \times M \times (0,1]$. Each groupoid $\mathcal{G}_t, t \in [0,1]$ has object space M, and the object space for the total groupoid $\mathcal{G} = \mathbb{T}_H M$ is the manifold,

$$\mathcal{G}^{(0)} = M \times [0,1].$$

We will endow $\mathbb{T}_H M$ with the structure of a manifold with boundary, by glueing $\mathcal{G}_0$ as the $t = 0$ boundary to $\mathcal{G}_{(0,1]}$. The topology of $\mathbb{T}_H M$ will be such that $\mathcal{G}_{(0,1]}$ is an open subset of $\mathbb{T}_H M$, and a curve $(a(t), b(t), t)$ in $\mathcal{G}_{(0,1]} = M \times M \times (0,1]$ converges, as $t \to 0$, to a parabolic arrow $(m, v) \in T_H M$ if,

$$\begin{aligned} M \ni m &= \lim_{t \to 0} a(t) = \lim_{t \to 0} b(t), \\ T_H M_m \ni v &= [a]_H * [b]_H^{-1}, \end{aligned}$$

where we assume that $a'(0), b'(0) \in H$. Recall that $[a]_H, [b]_H$ denote the *parabolic arrows* defined by the curves a, b, while the expression $[a]_H * [b]_H^{-1}$ denotes the product of $[a]_H$ with the inverse of $[b]_H$ in the osculating group $T_H M_m$ (see Chapter 2). If $H = TM$, the osculating groups are abelian, and the last condition simplifies to,

$$T_m M \ni v = a'(0) - b'(0) = \lim_{t \to 0} \frac{a(t) - b(t)}{t},$$

which is the topology of Connes' tangent groupoid.

With this defintion of the topology, it is easy to see that the groupoid operations for $\mathbb{T}_H M$ are continuous. For example, in $\mathcal{G}_{(0,1]}$ we have,

$$(a(t), b(t), t) \ \cdot \ (b(t), c(t), t) = (a(t), c(t), t),$$

while in $\mathcal{G}_0$,

$$([a]_H * [b]_H^{-1}) \ * \ ([b]_H * [c]_H^{-1}) = [a]_H * [c]_H^{-1},$$

assuming that $a(0) = b(0) = c(0)$ and $a'(0), b'(0), c'(0) \in H_m$. However, we will not rigorously develop this point of view. Instead, we glue $\mathcal{G}_0$ to $\mathcal{G}_{(0,1]}$ in an alternative way, more convenient for practical use, by defining a smooth structure on $\mathbb{T}_H M$. For that purpose we make use of an exponential map for the Heisenberg structure,

$$\exp : T_H M \to M.$$

(See section 2.5; it might be useful for the reader to quickly review the content of that section before continuing.) We define a map,

$$\psi : T_H M \times [0, 1) \to \mathbb{T}_H M,$$

by,

$$\psi(m, v, t) = (\exp_m(\delta_t v), m, t), \text{ for } t > 0,$$
$$\psi(m, v, 0) = (m, v) \in T_H M_m.$$

Here δ_t denotes the Heisenberg dilation in the osculating group $T_H M_m$.

The smooth structure on $T_H M \times [0, 1)$ induces a smooth structure in an open neigborhood of $\mathcal{G}_0 = T_H M$ in $\mathbb{T}_H M$. (Compare the definition of the smooth structure of the tangent groupoid in [Co], p. 103.)

Theorem 72 *With the above structure, $\mathbb{T}_H M$ is a smooth groupoid with boundary.*

As we will see in the next sections, this is an immediate corollary of Lie's Third Theorem for *Lie algebroids.*

We make a few remarks. First of all, recall that, if we identify $T_H M$ with TM via a map,

$$T_H M \xrightarrow{\log} H \oplus N \xrightarrow{j} TM,$$

induced by the choice of a section $N \subseteq TM$, then an exponential map $T_H M \to M$ determines an ordinary exponential map $TM \to M$. But, as we have seen, not every exponential map for TM corresponds to a good exponential map for $T_H M$. If we allowed arbitrary exponential maps $TM \to M$ in the definition of the parabolic tangent groupoid, the different charts would fail to be smoothly compatible. This is a subtle but important issue, and the concepts outlined in chapter 2 were developed largely to assist the construction of the groupoid $\mathbb{T}_H M$.

To be more explicit, recall that an exponential map for $T_H M$ can be constructed locally, in an open set $U \subseteq M$, by choosing a system of H-coordinates,

$$E_m : \mathbb{R}^n \to U.$$

Recall that H-coordinates are such that, for each $m \in U$, the coordinate vectors $dE_m(\partial_i)$, for $i = 1, \ldots, p$, are vectors in H_m at the point m. A system of H-coordinates E_m is such that $(m, v) \mapsto E_m(v)$ is a smooth map $U \times \mathbb{R}^n \to U$. Then let,

$$F_m : \mathbb{R}^{p+q} \to T_H M_m,$$

denote the Taylor coordinates on the osculating group $T_H M_m$, induced by the H-coordinates E_m (Definition 4). With this set-up, the composition,

$$\exp(m, v) = E_m(F_m^{-1}(v)),$$

is a (local) exponential map that is compatible with the Heisenberg structure, and every Heisenberg exponential map is locally of this kind (see Section 2.5). From this perspective, the equivalent of the map ψ that was defined above is the chart,

$$\psi' : U \times \mathbb{R}^p \times \mathbb{R}^q \times [0, 1) \to \mathbb{T}_H M,$$

defined by,

$$\begin{aligned}
\psi'(m, h, n, t) &= (E_m(th, t^2 n), m, t), \text{ for } t > 0, \\
\psi'(m, h, n, 0) &= F_m(m, h, n) \in T_H M_m.
\end{aligned}$$

The effect of this construction is clear: it 'blows up' the diagonal in $M \times M$ (as t gets smaller) by a factor t^{-1} in the direction of H, and by a factor t^{-2} in the direction that is transversal to H. The reason we need to choose different coordinates E_m for each point $m \in M$ is easy to understand: the bundle H may not be integrable.

The choice of coordinates at $t = 0$ is perhaps less transparant, but the success of the construction crucially depends on it. Recall that, if we make the *canonical* identification of $T_H M$ with the bundle $H \oplus N$ (by means of the Lie exponential map in the fibers), then the Taylor coordinates F_m are explicitly given by,

$$\log F_m(m, h, n) = (h, n - \tfrac{1}{2} b_m(h, h)) \in H_m \oplus N_m,$$

where $b_m(h, h)$ is a quadratic form that depends on the coordinates E_m (see Propositions 12 and 30). If the coordinates E_m are chosen in such a way that the corresponding bilinear form b_m is skew-symmetric (for example, as in the construction of Folland and Stein discussed in Section 2.6), then this quadratic correction term vanishes, and we can work simply with the natural coordinates on $H_m \oplus N_m$ at $t = 0$. Correspondingly, one solution to the construction of the parabolic tangent groupoid would be to work with 'preferred' coordinate systems E_m, i.e., H-coordinates for which b_m is skew-symmetric. But if one allows *arbitrary* H-coordinates E_m to 'blow up' the diagonal in $M \times M$, then the coordinates on $T_H M \cong H \oplus N$ must be adjusted by a quadratic correction term, as indicated. It is precisely to clarify this situation that we studied Taylor coordinates on the osculating groups in Chapter 2, and carefully distinguished between the bundle of groups $T_H M$ and the bundle of Lie algebras $H \oplus N$.

We now show that, with the above choices, the manifold structure on $\mathbb{T}_H M$ is well defined. Later we will see that this follows directly from Lie's Third Theorem for Lie algebroids, and the reader may choose to skip the rest of this section. Propositions 73 and 75 are not necessary for what follows, but they do illustrate how a *direct* proof of Theorem 72 could be developed, without reference to Lie algebroids. More importantly, the proofs of Propositions 73 and 75 show the relevance of the corrected groupoid coordinates at $t = 0$, if arbitrary H-coordinates E_m are allowed. The theory of integration of Lie algebroids only applies to charts ψ that are obtained by means of exponential maps induced by *connections* on $H \oplus N$ (see Section 2.5), and for such charts the correction term $b_m(h, h)$ vanishes automatically. The proofs in this section establish the correctness of our definition in the general case.

Proposition 73 *For different choices of the exponential map* $\exp\colon T_HM \to M$, *the maps* $\psi\colon T_HM\times[0,1]\to \mathbb{T}_HM$, *defined in the manner described above, have smooth transition functions. In other words,* $\mathbb{T}_HM$ *has a well-defined structure of smooth manifold, independent of the choice of exponential map.*

The basic ingredient of the proof is the following technical lemma.

Lemma 74 *Let* $\phi\colon T_HM\to T_HM$ *be a diffeomorphism that preserves the fibers; fixes the zero section* $M\subset T_HM$*; and at the point* m *has derivative* $D\phi_m = \mathrm{id}$*, and a second derivative that satisfies* $D^2\phi_m(h,h)\in H_m$*, for* $h\in H_m$*. Then the map,*

$$\widetilde{\phi}\ :\ T_HM\times\mathbb{R}\ \to T_HM\times\mathbb{R},$$

defined by,

$$\begin{aligned}&\widetilde{\phi}(m,v,t) = (\delta_t^{-1}\phi(m,\delta_t v),t),\\ &\widetilde{\phi}(m,v,0) = (m,v,0),\end{aligned}$$

is a diffeomorphism.

Proof. Clearly, $\widetilde{\phi}$ is smooth on the open subset where $t\neq 0$. We must prove that $\widetilde{\phi}$ is smooth in a neighborhood of the $t=0$ fiber.

For convenience of notation, we identify T_HM with $H\oplus N$ via the logarithm. The proof is based on a simple Taylor expansion near $t=0$. For a choice of coordinates on $H\oplus N$ we have,

$$\phi(m,v) = \phi(m,0) + D\phi_m(v) + \frac{1}{2}D^2\phi_m(v,v) + R(m,v).$$

The remainder term $R=R(m,v)$ satisfies a bound $|R|<C|v|^3$, for $|v|<1$. Now write $v=h+n$ with $h\in H_m, n\in N_m$. Then,

$$\begin{aligned}\phi(m,th+t^2n) &= \phi(m,0) + tD\phi_m(h) + t^2D\phi_m(n)\\ &\quad + \frac{1}{2}t^2D^2\phi_m(h,h) + t^3D^2\phi_m(h,n) + \frac{1}{2}t^4D^2\phi_m(n,n) + R(m,th+t^2n)\\ &= \phi(m,0) + tD\phi_m(h) + t^2D\phi_m(n) + \frac{1}{2}t^2D^2\phi_m(h,h) + t^3R'.\end{aligned}$$

The error term $R'=r'(m,h,n,t)$, satisfies a bound $|R'|\leq C$ for $|h|<\epsilon|t|^{-1}$, $|n|<\epsilon|t|^{-2}$. Observe that these inequalities hold in an open neighborhood of the $t=0$ fiber in $T_HM\times\mathbb{R}$.

The assumptions on ϕ allow the simplification,

$$\phi(m,\delta_t v) = (m,\, th + t^2n + \frac{1}{2}t^2D^2\phi_m(h,h) + t^3R'),$$

where $D^2\phi_m(h,h)\in H_m$. We find,

$$\delta_t^{-1}\phi(m,\delta_t v) = (m,\, v + tR''),$$

where, again, the coefficient of the remainder R'' is uniformly bounded in a neighborhood of the $t=0$ fiber. This implies continuity of $\widetilde{\phi}$.

By the same reasoning, expanding ϕ in a higher order Taylor series, one obtains,

$$\widetilde{\phi}(m,v,t) = (m,\, v + \sum_{k=1}^{r} a_k t^k + R_r t^r, t),$$

where the coefficients $a_k = a_k(m,v)$ are smooth functions, independent of t, arising from the derivatives of ϕ, while the coefficient R_r of the remainder is uniformly bounded in a neighborhood of $t = 0$. This implies smoothness of $\widetilde{\phi}$.

□

Proof of Proposition 73 . Let ψ and ψ' be the two maps,

$$\psi, \psi' \,:\, T_H M \times [0,1] \to \mathbb{T}_H M,$$

constructed in the manner explained above, for two different exponential maps $E, E' \,:\, T_H M \to M$. We must prove that the transition function $\widetilde{\phi} = \psi^{-1} \circ \psi'$ is smooth. We have,

$$\widetilde{\phi}(m,v,t) = (\delta_t^{-1} E_m^{-1}(E'_m(\delta_t v)), m, t), \text{ for } t \neq 0,$$
$$\widetilde{\phi}(m,v,0) = (m,v,0).$$

The defining property of exponential maps for $T_H M$ (Definition 19) immediately implies that the composition $\phi_m = E_m^{-1} \circ E'_m$ satisfies the assumptions of Lemma 74. Hence, $\widetilde{\phi}$ is smooth.

□

The next proposition shows that the manifold structure on $\mathbb{T}_H M$ is compatible with our earlier, informal definition of the topology on $\mathbb{T}_H M$ by means of parabolic arrows.

Proposition 75 *Suppose $a(t), b(t)$ are smooth curves in M with $a(0) = b(0) = m$, such that $a'(0)$ and $b'(0)$ are in H_m. Then in $\mathbb{T}_H M$, endowed with the manifold structure defined above,*

$$\lim_{t \to 0} (a(t), b(t), t) = [a]_H * [b]_H^{-1} \in T_H M_m.$$

First proof. If we *assume* that the curve $(a(t), b(t), t)$ in $\mathcal{G}_{(0,1]}$ extends to a smooth curve in $\mathbb{T}_H M$, then there is a nice proof that makes use of parabolic flows. Let $v_0 \in T_H M_m$ be the point in $\mathcal{G}_0$ to which the curve in $\mathcal{G}_{(0,1]}$ converges, and let v_t be the parabolic arrow defined by, $\psi(b(t), v_t, t) = (a(t), b(t), t)$, i.e.,

$$a(t) = \exp_{b(t)}(\delta_t v_t).$$

By definition of the manifold structure on $\mathbb{T}_H M$, we have $(b(t), v_t) \to (m, v_0)$. We see that the section $v_t, t \in [0,1]$ is smooth along $b(t)$, and can be extended to a section V in a neigborhood of $m = b(0)$. Now define a flow,

$$\Phi_v^t(m') = \exp_{m'}(\delta_t V(m')).$$

By definition of Heisenberg exponential maps, the curve,

$$t \mapsto \exp_{m'}(\delta_t V(m'))$$

has parabolic arrow $V(m')$. In other words, Φ_v^t is a parabolic flow, and in particular,

$$[\Phi_v^t(m)]_H = V(m) = v_0 \in T_H M_m.$$

Clearly $a(t) = \Phi_v^t(b(t))$. Extend $b(t)$ to a parabolic flow Φ_b^t, such that $b(t) = \Phi_b^t(m)$. Then we see that,

$$[a]_H = [\Phi_v^t \circ \Phi_b^t(m)]_H = [\Phi_v^t(m)]_H * [\Phi_b^t(m)]_H = v_0 * [b]_H,$$

which means that $v_0 = [a]_H * [b]_H^{-1}$.

□

Second proof. To prove the proposition without the extra assumption of convergence, and to illustrate a different technique, we give a second proof.

We use the map ψ' defined above to describe the manifold structure on $\mathbb{T}_H M$. We need a system of H-coordinates E_m, and the corresponding Taylor coordinates F_m. Let us identify an open set $U \subseteq M$ with $\mathbb{R}^n$ (via a coordinate map that we suppress in the notation). Given an H-frame X_i on U, we have a system of coordinates,

$$E_m \,:\, \mathbb{R}^n \to U \,:\, v \mapsto m + \sum v_i X_i(m) = m + Xv.$$

Here $X = (X_i^j)$ denotes the $n \times n$ matrix whose columns are the vector-values functions $X_i \colon U \to \mathbb{R}^n$.

Expand a and b in the coordinates on $U \cong \mathbb{R}^n$ as,

$$\begin{aligned} a(t) &= th + t^2 k + \mathcal{O}(t^3), \\ b(t) &= th' + t^2 k' + \mathcal{O}(t^3), \end{aligned}$$

assuming that $a(0) = b(0) = 0$. We have Taylor coordinates,

$$F_m(h, n) = [a]_H, \; F_m(h', n') = [b]_H,$$

where $n = k^N$, $n' = k'^N$ are the normal components of k, k'. With the notation of Proposition 12, we compute,

$$\begin{aligned} F_m^{-1}([a]_H * [b]_H^{-1}) &= (h, n) * (h', n')^{-1} \\ &= (h, n) * (-h', -n' + b(h', h')) \\ &= (h - h', n - n' - b(h, h') + b(h', h')). \end{aligned}$$

Now let $(a(t), b(t), t) = \psi'(b(t), x(t), y(t), t)$, i.e.,

$$a(t) = E_{b(t)}(tx(t), t^2 y(t)),$$

where the coordinates $(x(t), y(t)) \in \mathbb{R}^{p+q}$ depend on t. We must show that,

$$\lim_{t \to 0} (x(t), y(t)) = (-h + h', -n + n' - b(h, h') + b(h, h)).$$

We approximate the coordinates $(x(t), y(t)) \in \mathbb{R}^{p+q}$ by a Taylor expansion of $E_{b(t)}^{-1}(a(t))$, using the explicit form of E_m, as follows,

$$
\begin{aligned}
(tx(t), t^2 y(t)) &= X_{b(t)}^{-1} (a(t) - b(t)) \\
&= a(t) - b(t) + t\, D(X^{-1})_0 \big(h', a(t) - b(t)\big) + \mathcal{O}(t^3) \\
&= t(h - h') + t^2(k - k') + t^2\, D(X^{-1})_0(h', h - h') + \mathcal{O}(t^3),
\end{aligned}
$$

Let us explain the calculation. In the first step we expanded $X^{-1}(b(t))$. Because $a(t) - b(t) = \mathcal{O}(t)$, it sufficed to consider only the first derivative,

$$
\frac{\partial}{\partial t}\, X^{-1}(b(t))\big|_{t=0} = D(X^{-1})_0.h'.
$$

In the second step we expanded $a(t) - b(t)$, again ignoring terms of order $\mathcal{O}(t^3)$.

Reversing the dilation, we find,

$$
(x(t), y(t)) = \big(h - h', n - n' + D(X^{-1})_0^N(h', h - h')\big)\ + \mathcal{O}(t).
$$

Because $X_0 = 1$, we have $D(X^{-1})_0 = -DX_0$, while the normal component $DX_0^N(h', h - h')$ is equal to $b(h', h - h')$, by definition of the bilinear form b. This gives the desired result.

□

5.3 Lie algebroids.

It is not too difficult to give a direct proof of smoothness of the groupoid operations in $\mathbb{T}_H M$. However, one can bypass all the tedious Taylor expansions by appealing to a general integrability result for Lie algebroids. In fact, Propositions 73 and 75 above were included mainly for illustration of some of the techniques involved in their proofs, and are not essential in what follows.

In this section we review the basic facts about Lie algebroids. The material is taken from [Ma].

Definition 76 *Let B be a smooth manifold. A* Lie algebroid *$\mathcal{A}$ with base B is a smooth vector bundle $p\colon \mathcal{A} \to B$, together with a Lie bracket on its sections,*

$$
[\ ,\ ] : \ \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A}),
$$

that is $\mathbb{R}$-bilinear, skew-symmetric, and satisfies the Jacobi identity, and a vector bundle map

$$
q : \ \mathcal{A} \to TB,
$$

called the anchor *of $\mathcal{A}$, such that*

$$
\begin{aligned}
q([X, Y]) &= [q(X), q(Y)], \\
[X, fY] &= f[X, Y] + (q(X).f)Y,
\end{aligned}
$$

for $X, Y \in \Gamma(\mathcal{A})$, $f \in C^\infty(B)$.

The notion was introduced by J. Pradines in 1967 (see [Pa]). The English term 'anchor' was coined by K. MacKenzie. The anchor map relates the bracket and module structures of $\Gamma(\mathcal{A})$.

Lemma 77 *The Lie bracket of a Lie groupoid is a* local *operation, i.e., if sections X and X', Y and Y' agree on some open subset of B, then so do $[X,Y]$ and $[Y,Y']$.*

Proof. Let $f \in \Gamma(B)$ have support in the open set $U \subseteq B$ where Y and Y' agree. Comparing $[X,fY] = f[X,Y] + (q(X).f)Y$ and $[X,fY'] = f[X,Y'] + (q(X).f)Y'$, we derive that $f[X,Y] = f[X,Y']$. It follows that $[X,Y] = [X,Y']$ in U. Likewise $[X,Y'] = [X',Y']$.

□

Associated to every smooth groupoid $\mathcal{G}$ is a Lie algebroid, denoted by $A\mathcal{G}$. This notion generalizes the construction of the Lie algebra associated to a Lie group, which consists of right invariant vector fields on the group, and which, as a vector space, can be identified with the tangent space at the unit element of the group. We first consider the notion of right invariant vector fields on a groupoid $\mathcal{G}$. Right multiplication by an element $\gamma \in \mathcal{G}$ induces a diffeomorphism between s-fibers,

$$R_\gamma \,:\, \mathcal{G}_{s(\gamma')} \to \mathcal{G}_{s(\gamma)} \,:\, \gamma' \mapsto \gamma'\gamma.$$

Clearly, the notion of 'right invariance' only makes sense for vector fields on $\mathcal{G}$ that are tangent to the fibers $\mathcal{G}_x = s^{-1}(x)$ of the source map $s\colon \mathcal{G} \to \mathcal{G}^{(0)}$. These are called *vertical vector fields.* We denote the bundle of vertical vectors of $\mathcal{G}$ by $T^s\mathcal{G}$,

$$T^s\mathcal{G} = \bigcup_{x\in\mathcal{G}^{(0)}} T\mathcal{G}_x = \{v \in T\mathcal{G} \mid Ds(v) = 0\} \ \subseteq T\mathcal{G}.$$

Here Ds denotes the derivative $Ds : T\mathcal{G} \to T\mathcal{G}^{(0)}$ of the source map s. Because s is a submersion, $T^s\mathcal{G}$ is a locally trivial vector bundle.

Definition 78 *A* right invariant vector field *is a vertical vector field $X \in \Gamma^\infty(T^s\mathcal{G})$ that is invariant under right multiplication, i.e.,*

$$DR_\gamma \cdot X(\gamma') = X(\gamma'\gamma),$$

for all $\gamma, \gamma' \in \mathcal{G}$.

Observe that the commutator of two right invariant vector fields is again a right invariant vector field, because right multiplication commutes with the bracket operation on $\Gamma(T^s\mathcal{G})$

Now, let $A\mathcal{G}$ be the restriction of the $T^s\mathcal{G}$ to the space of units $B = \mathcal{G}^{(0)}$, i.e.,

$$A\mathcal{G} = \bigcup_{x\in\mathcal{G}^{(0)}} T_{e(x)}\mathcal{G}_x.$$

The vector bundle $A\mathcal{G} \to B$ is locally trivial, since it is the restriction of the locally trivial bundle $T^s\mathcal{G}$. Because a right invariant vector field is determined by its values on the unit space $\mathcal{G}^{(0)}$ (the value in $T_{e(x)}\mathcal{G}_x$ determines the value in all of $T\mathcal{G}_x$), restriction to $\mathcal{G}^{(0)}$ induces a one-to-one correspondence between right invariant vector fields on $\mathcal{G}$ and arbitrary sections in $\Gamma(A\mathcal{G})$. With this correspondence, the commutator of right invariant vectorfields induces a Lie bracket on $\Gamma(A\mathcal{G})$.

There is now only one possible choice for the anchor map $q\colon A\mathcal{G} \to TB$. To see this, let $\widetilde{X}, \widetilde{Y}$ denote the right invariant fields on $\mathcal{G}$ associated to $X, Y \in \Gamma(A\mathcal{G})$. For a given function $f \in C^\infty(B)$, the right invariant function $\tilde{f} \in C^\infty(\mathcal{G})$ that extends f is given by

$$\tilde{f}(\gamma) = f(r(\gamma)),$$

where $r\colon \mathcal{G} \to B$ denotes the range map of $\mathcal{G}$. Clearly, $\tilde{f}(\gamma') = \tilde{f}(\gamma'\gamma)$. Identifying $B = e[B] \subseteq \mathcal{G}$, we find,

$$\begin{aligned}
[X, fY](x) = [\widetilde{X}, \tilde{f}\widetilde{Y}](e_x) &= \tilde{f}[\widetilde{X}, \widetilde{Y}](e_x) + (\widetilde{X}.\tilde{f})\tilde{Y}(e_x) \\
&= f[X, Y](x) + (\widetilde{X}.(f \circ r))(e_x)Y(x).
\end{aligned}$$

We conclude that the anchor map satisfies the second axiom iff,

$$\begin{aligned}
(q(X).f)(x) = (\widetilde{X}.(f \circ r))(e_x) &= (d(f \circ r)(e_x) \cdot \widetilde{X}(e_x) \\
&= df(x) \cdot Dr(e_x) \cdot X(x) = (Dr(X).f)(x),
\end{aligned}$$

or simply,

$$q(X) = Dr(X).$$

In other words, the anchor $q\colon A\mathcal{G} \to TB$ is the projection of vertical vectors along the range map r. It then follows trivially that $q([X, Y]) = [q(X), q(Y)]$.

We summarize.

Definition 79 *The Lie algebroid $A\mathcal{G}$ associated to a smooth groupoid $\mathcal{G}$ is the bundle of vertical vectors $T^s\mathcal{G}$ restricted to the base space $B = \mathcal{G}^{(0)}$, where the Lie bracket $[X, Y]$ of two sections $X, Y \in \Gamma(A\mathcal{G})$ is the restriction to B of the commutator $[\widetilde{X}, \widetilde{Y}]$ of the unique right invariant vector fields $\widetilde{X}, \widetilde{Y}$ that extend X, Y, while the anchor $q\colon A\mathcal{G} \to B$ is the derivative $q = Dr$ of the range map $r\colon \mathcal{G} \to B$, restricted to $A\mathcal{G}$.*

Example 1. The Lie algebroid of the pair groupoid $\mathcal{G} = M \times M$ is obtained as follows. Because $\mathcal{G}_m = s^{-1}(m) = M \times \{m\}$, we have $T\mathcal{G}_m = TM \times \{m\}$, and $T^s\mathcal{G} = TM \times M$. A right invariant vector field is a section in $TM \oplus TM$ that is constant in the first component, and zero in the second. So right invariant vectorfields on $M \times M$ identify naturally with sections in TM, and this identification preserves the commutator structure.

The space of units is embedded as the diagonal in $M \times M$, and so

$$A\mathcal{G} = \bigcup_{m \in M} T_m M \times \{m\} \approx TM.$$

Of course, the base of $A\mathcal{G}$ is just $B = M$. The bracket on $A\mathcal{G}$ corresponds to the usual bracket operation of vector fields on M. The anchor is the identity map $q\colon A\mathcal{G} \to TB = TM$. A Lie algebroid where the anchor is a submersion is called *totally transitive*.

Example 2. The tangent space TM and parabolic tangent space $T_H M$ are examples of 'bundles of Lie groups' (the Lie group structure is not necessarily locally trivial).

If the groupoid $\mathcal{G}$ is a bundle of Lie groups, the Lie algebroid $A\mathcal{G}$ is the family of Lie algebras naturally associated to $\mathcal{G}$, with the obvious bracket. Since source and range maps coincide, we have $Dr = Ds$. Therefore, since vertical vectors are in the kernel of Ds, we see that the anchor map $q = Dr = 0$ is trivial. This is equivalent to $[X, fY] = f[X,Y]$, which means that the Lie bracket is a pointwise operation. Lie algebroids with trivial anchor, $q = 0$, are called *totally intransitive*.

Note. Mackenzie [Ma] reserves the term 'bundle' of Lie algebras for the locally trivial case, i.e., where the family is locally isomorphic (as a Lie algebroid) to a product $U \times \mathfrak{g}$ of an open set U and a *fixed* Lie algebra $\mathfrak{g}$.

5.4 Lie's Third Theorem for Lie algebroids.

Lie's Third Theorem states that every Lie algebra can be integrated by a unique simply connected Lie group. The result depends on an identification, via the *exponential map*, of a neighborhood of the unit element in the Lie group with a neighborhood of the origin in the Lie algebra. The corresponding theorem for Lie algebroids was proven, in full generality, by M. Crainic and R. L. Fernandes in [CF]. Here we describe an integrability theorem due to V. Nistor in [Ni1], which does not treat the most general case, but is better suited to our purposes.

If $\mathcal{G}$ is a smooth groupoid with Lie algebroid $\mathcal{A} = A\mathcal{G}$, then an *exponential map* $\mathcal{A} \to \mathcal{G}$ is defined for a connection ∇ on the vector bundle $\mathcal{A}$, generalizing the notion of an exponential map from a Lie algebra to a Lie group. We now describe this construction. Right multiplication establishes an isomorphism between the fiber $A_{r(\gamma)}$ at the range of an arrow $\gamma \in \mathcal{G}$ and the vertical tangent space $T_\gamma \mathcal{G}_{s(\gamma)}$,

$$DR_\gamma \,:\, \mathcal{A}_{r(\gamma)} \xrightarrow{\cong} T_\gamma \mathcal{G}_{s(\gamma)}.$$

Thus, the bundle of vertical vectors $T^s\mathcal{G}$ is identified with the pull-back via the range map $r^*\mathcal{A}$ of the bundle $\mathcal{A}$. The connection on $\mathcal{A}$ induces a connection on $r^*\mathcal{A} \cong T^s\mathcal{G}$, which, in turn, restricts to a connection ∇^x on the tangent space $T\mathcal{G}_x$ of each s-fiber $\mathcal{G}_x$. Let $\exp^{\nabla^x}$ be the corresponding exponential map,

$$\exp^{\nabla^x} \,:\, \mathcal{A}_x = T_{e(x)}\mathcal{G}_x \to \mathcal{G}_x.$$

To give a different description, consider a path $g\colon [0,1] \to \mathcal{G}_x$. The derivative $g'(t)$ defines a path $a\colon [0,1] \to \mathcal{A}$ by,

$$a(t) = DR_{g(t)^{-1}}\, g'(t).$$

Let $\pi\colon \mathcal{A} \to B$ denote the base point map, and consider the base path $\pi(a(t)) = r(g(t)) \in B$. It satisfies,

$$\frac{d}{dt}(\pi(a(t)) = \frac{d}{dt}(r(g(t)) = Dr(g'(t)) = Dr(DR_{g(t)^{-1}}\, g'(t)) = q(a(t)).$$

We used the fact that $q = Dr$, and $r \circ R_\gamma = r$. Given $v \in A_x$, the exponential $\exp^{\nabla^x}$ is the end point $g(1)$ of a path g in $\mathcal{G}_x$ that starts at the unit $g(0) = e_x$, in direction $g'(0) = v$, such that the corresponding path $a(t)$ is parallel in $\mathcal{A}$, i.e.,

$$\nabla_{q(a(t))}a(t) = 0.$$

This uniquely determines the path $a(t)$ in $\mathcal{A}$, and hence $g(t)$ in $\mathcal{G}$. (If $q(a(t)) = 0$, then $a(t)$ remains constant.) The maps $\exp^{\nabla^x}$ fit together into a global exponential map,

$$\exp^{\nabla} : \mathcal{A} \to \mathcal{G},$$

which is a diffeomorphism in a neighborhood of the zero section $B \subseteq \mathcal{A}$ (see [NWP]).

Now assume that $\mathcal{A}$ is a Lie algebroid, whose base space has an $\mathcal{A}$-invariant stratification in the following sense. The base space B is a *stratified manifold* if it is the union $B = \cup S$ of disjoint submanifolds S, called *strata*. If B is a manifold with corners, then the strata must be submanifolds without corners, while the closure (in B) of each stratum S is a submanifold, possibly with corners, that is contained entirely in a unique open face of B. (The reason we allow B to have corners is that in the next chapter we will construct a groupoid with base space $B = M \times M \times [0,1]^2$.) A stratification of the base space of a Lie algebroid $\mathcal{A}$ is called *$\mathcal{A}$-invariant*, if for each $x \in B$, the range of $q\colon \mathcal{A}_x \to T_x B$ is contained in $T_x S$,

$$q[\mathcal{A}_x] \subseteq T_x S,$$

where S is the stratum that contains x. This is equivalent to the condition that each local diffeomorphism of the form $\exp(q(X))$, for $X \in \Gamma(\mathcal{A})$, preserves the strata of M, or to the condition that the restriction $\mathcal{A}_S$ of $\mathcal{A}$ to S is a Lie algebroid on S, for each S. Algebraically, $\mathcal{A}$ is the disjoint union of the Lie algebroids $\cup \mathcal{A}_S$.

Now also suppose that each Lie algebroid $\mathcal{A}_S$ is *integrable*, which means that there exists a smooth groupoid $\mathcal{G}_S$ whose Lie algebroid is $A\mathcal{G}_S = \mathcal{A}_S$. One can always choose *s-simply connected* groupoids $\mathcal{G}_S$, which means that its s-fibers $\mathcal{G}_x = s^{-1}(x)$ are simply connected. (For every smooth groupoid $\mathcal{G}$, there exists a unique s-simply connected groupoid with the same Lie algebroid as $\mathcal{G}$.) Such s-simply connected groupoids $\mathcal{G}_S$ can be glued together to a groupoid $\mathcal{G}$ that integrates $\mathcal{A}$.

Given a connection ∇ on $\mathcal{A}$, let ∇^S be its restriction to $\mathcal{A}_S$. We obtain exponential maps,

$$\exp^{\nabla^S} : \mathcal{A}_S \to \mathcal{G}_S.$$

The disjoint union $\mathcal{G} = \cup \mathcal{G}_S$ does not have a global smooth structure, but we can still define,

$$\exp^{\nabla} : \mathcal{A} = \cup \mathcal{A}_S \to \mathcal{G} = \cup \mathcal{G}_S,$$

by fitting together the maps $\exp^{\nabla^S}$.

Via this exponential map, the smooth structure on $\mathcal{A}$ induces charts on $\mathcal{G} = \cup \mathcal{G}_S$ in a neighborhood of the base space B, as long as $\exp^{\nabla}$ is injective in a neighborhood of B. According to Nistor's integrability result, if the groupoids $\mathcal{G}_S$ are s-simply connected, these charts can be extended to an atlas, for which $\mathcal{G}$ becomes a smooth groupoid.

Theorem 80 *Let $\mathcal{A}$ be a Lie algebroid with an $\mathcal{A}$-invariant stratification of the base space $B = \cup S$. Suppose that each of the restrictions $\mathcal{A}_S$ is integrable by an s-simply connected smooth groupoid $\mathcal{G}_S$, and that the exponential map $\mathcal{A} \to \mathcal{G} = \cup \mathcal{G}_S$ is injective in a neighborhood of the base space B (for some connection on $\mathcal{A}$).*

Then there exists a unique manifold structure on the disjoint union $\mathcal{G} = \cup \mathcal{G}_S$, for which $\mathcal{G}$ is a smooth groupoid with $\mathcal{A} = A\mathcal{G}$.

(Theorem 2 and 3 in [Ni1]. The condition on the exponential map is missing in [Ni1], and the correction is published in [BN].)

5.5 The Lie algebroid of $\mathbb{T}_H M$.

We now consider the Lie algebroid of the parabolic tangent groupoid $\mathbb{T}_H M$. It is much easier to define than the groupoid itself, and it can be constructed independently of the groupoid $\mathbb{T}_H M$. Once the Lie algebroid is constructed, Theorem 80 will imply smoothness of the groupoid $\mathbb{T}_H M$.

The base space of the groupoid $\mathbb{T}_H M$ can be stratified,

$$B = M \times [0,1] = \cup S,$$

with strata S_t, S_m defined as follows,

$$\begin{aligned} S_t &= M \times \{t\}\ , \text{for}\, t \in (0,1] \\ S_m &= \{(m,0)\}\ \text{for}\, m \in M. \end{aligned}$$

Observe that source $s(\gamma)$ and range $r(\gamma)$ of an arrow $\gamma \in \mathcal{G} = \mathbb{T}_H M$ always belong to the same stratum. The restriction of $\mathcal{G}$ to a stratum $S \subseteq B$ is denoted,

$$\mathcal{G}_S = s^{-1}(S) = r^{-1}(S),$$

and we have a disjoint union of (algebraically connected) groupoids,

$$\mathbb{T}_H M = \cup \mathcal{G}_S.$$

Here,

$$\begin{aligned} \mathcal{G}_{S_t} &= \mathcal{G}_t \cong M \times M,\ t \in (0,1] \\ \mathcal{G}_{S_m} &= \mathcal{G}_m \cong T_H M_m,\ m \in M. \end{aligned}$$

Each subgroupoid $\mathcal{G}_S$ has the structure of a smooth groupoid (independent of our definition of the smooth structure on $\mathbb{T}_H M$), and has Lie algebroid $A_S = A\mathcal{G}_S$,

$$\begin{aligned} A_{S_t} &= A_t \cong TM,\ t \in (0,1] \\ A_{S_m} &= A_m \cong H_m \oplus N_m,\ m \in M \end{aligned}$$

where $H_m \oplus N_m$ is the Lie algebra of $T_H M_m$. We define the total space of the vector bundle $\mathcal{A} = \cup A_S$ over $M \times [0,1]$ by an identification,

$$\phi :\ H \oplus N \times [0,1] \to \cup A_S,$$

given by,

$$\begin{aligned} \phi(m,h,n,t) &= th + t^2 j(n) \in T_m M \subset A_t\ t > 0 \\ \phi(m,h,n,0) &= (h,n) \in H_m \oplus N_m = A_m,\ m \in M. \end{aligned}$$

Here we have chosen a section $j\colon N \hookrightarrow TM$, but one easily verifies that the resulting smooth structure on $\mathcal{A}$ is independent of this choice.

Of course, the basic idea (the use of the dilations for the Heisenberg structure) is the same as for the construction of the smooth structure of the groupoid $\mathbb{T}_H M$, but it is much easier to carry out for the Lie algebroid.

Corresponding to the stratification of the base space, we denote a section in $\Gamma(\mathcal{A})$ as a family of sections in A_S. Explicitly, $\langle X_m, X_t\rangle \in \cup A_S$ denotes a section in $\Gamma(\mathcal{A})$ with $X_m \in \Gamma(H_m \oplus N_m)$ for $m \in M$, and $X_t \in \Gamma(TM)$ for $t \in (0,1]$. We summarize the structure of $\mathcal{A}$ in a definition.

Definition 81 *With the above notation, the smooth structure on $\mathcal{A} = \cup A_S$ is characterized by,*

1. *If $X \in \Gamma(H)$, then $\langle X(m), tX\rangle \in \Gamma(\mathcal{A})$.*
2. *If $Y \in \Gamma(N)$, then $\langle Y(m), t^2 X\rangle \in \Gamma(\mathcal{A})$, where $X \in \Gamma(TM)$ is an arbitrary lift of Y.*

The Lie bracket and anchor map for $\mathcal{A}$ are obtained from those of A_S,

$$
\begin{aligned}
[\,\langle X_m, X_t\rangle, \langle Y_m, Y_t\rangle\,] &= \langle\, [X_m, Y_m], [X_t, Y_t]\,\rangle,\\
q(\,\langle X_m, X_t\rangle\,) &= \langle\, q(X_m), q(X_t)\,\rangle = \langle 0, X_t\rangle.
\end{aligned}
$$

Proposition 82 *With the above definitions, $\mathcal{A}$ is a Lie algebroid.*

Proof. We only need to verify that the bracket of two smooth sections is a smooth section. That $q(X) \in \Gamma(TM)$ for $X \in \Gamma(A)$ is trivial.

Let $X, Y \in \Gamma(H)$, and $f, g \in \Gamma(B)$, and denote $f_t(m) = f(t,m), g_t(m) = g(t,m)$. Then we have sections in $\Gamma(\mathcal{A})$,

$$
\begin{aligned}
f\langle X(m), tX\rangle &= \langle f_0(m)X(m),\ tf_tX\rangle,\\
g\langle Y(m), tY\rangle &= \langle g_0(m)Y(m),\ tg_tY\rangle,
\end{aligned}
$$

with bracket,

$$
\begin{aligned}
&\langle\, [f_0(m)X(m), g_0(m)Y(m)],\ t^2[f_tX, g_tY]\,\rangle\\
&= fg\langle\, [X(m), Y(m)],\ t^2[X,Y]\,\rangle + \langle\, 0,\ t^2(f_t(X.g_t)Y - g_t(Y.f_t)X)\,\rangle.
\end{aligned}
$$

The first term on the right hand side is in $\Gamma(\mathcal{A})$, because, by definition of the Lie algebra structure on $H_m \oplus N_m$,

$$[X,Y](m) \bmod H_m = [X(m), Y(m)] \in N_m.$$

The second term is a section in $\Gamma(H)$ for $t \in (0,1]$. Since it is of order t^2, and has no $\Gamma(N)$ part, it is also in $\Gamma(\mathcal{A})$.

The cases where X and/or Y are sections in $\Gamma(N)$ are similar.

□

5.6 Integration of the Lie algebroid of $\mathbb{T}_H M$.

In this section we show that Theorem 72 is a corollary of Lie's third theorem for Lie algebroids. We first describe the s-simply connected smooth groupoid that integrates the Lie algebroid $\mathcal{A}$ constructed in the previous section. We will see that it agrees with $\mathbb{T}_H M$ in a neighborhood of the $t = 0$ boundary $\mathcal{G}_0 = T_H M$, which is sufficient to prove Theorem 72.

Let $\mathcal{A} = \cup A_S$ be the Lie algebroid constructed in the previous section. Each Lie algebroid A_S, for the stratification $M \times [0,1] = \cup S$ described above, can be integrated to an s-simply connected groupoid $\widetilde{\mathcal{G}}_S$. For the Lie algebra $A_m = H_m \oplus N_m$, the osculating group $\mathcal{G}_m = T_H M_m$ is simply connected, so we take $\widetilde{\mathcal{G}}_m = T_H M_m$. However, the s-simply connected groupoid that integrates $A_t = TM$ is not the pair groupoid $\mathcal{G}_t = M \times M$, but the *homotopy groupoid* $\widetilde{\mathcal{G}}_t = \Pi_M$

of M. While $M \times M$ contains an arrow connecting each pair of points in M, arrows in Π_M are homotopy classes of paths in M with fixed end points. Source and range of an arrow $\gamma = [c]$ represented by a path $c\colon [0,1] \to M$ are the end points of the path: $s(\gamma) = c(0)$, $r(\gamma) = c(1)$. Composition of paths gives the groupoid multiplication.

Proof of Theorem 72 According to Theorem 80 there is a unique smooth groupoid $\widetilde{\mathcal{G}} = \cup \widetilde{\mathcal{G}}_S$ that integrates $\mathcal{A}$. We investigate how it is related to $\mathbb{T}_H M$. We first establish a relationship between the homotopy groupoid Π_M and the pair groupoid $M \times M$. Let us denote the source fiber $s^{-1}(m)$ in Π_M by $\widetilde{M}_m$. It is the universal cover of the component M_m of M that contains m. The range map $r\colon \widetilde{M}_m \to M_m$ is the corresponding covering map. The map,

$$\Pi_M \to M \times M \,:\; \gamma \to (r(\gamma), s(\gamma)),$$

is a morphism of smooth groupoids, and it is a diffeomorphism in a neigborhood of the unit space M. Thus, Π_M and $M \times M$ can be identified as 'local groupoids' near the unit space. Choosing such a local identification, we obtain an identification of $\mathbb{T}_H M$ and $\widetilde{\mathcal{G}}$ as local groupoids in a neighborhood of the base space $M \times [0,1]$ that contains the entire $t = 0$ boundary.

To obtain a chart on $\widetilde{\mathcal{G}}$ in this neighborhood, we need a connection on $\mathcal{A}$, and construct the associated exponential map $\mathcal{A} \to \widetilde{\mathcal{G}}$. We will see that the chart we obtain agrees with the smooth structure on $\mathbb{T}_H M$.

To begin with, choose a connection ∇ on TM that is compatible with the Heisenberg structure (i.e., a G-connection, see section 2.5). Such a connection restricts to a connection $\nabla^H = \nabla|_H$ on H, and if we choose a section $j\colon N \hookrightarrow TM$, then ∇ also induces a connection $j^{-1}\nabla j$ on N. Thinking of j as an identification, we will denote the induced connection on $H \oplus N$ simply by ∇. It is important to observe that ∇ commutes with the dilations, i.e., $\nabla \delta_t X = \delta_t \nabla X$, for $X \in \Gamma(H \oplus N)$.

Let $\overline{\nabla}$ be the connection on $H \oplus N \times [0,1]$ given by,

$$\overline{\nabla} = \nabla + \frac{\partial}{\partial t}.$$

Pushing forward via the identification $\phi\colon H \oplus N \times [0,1] \to \mathcal{A} = \cup A_S$, we obtain a connection on the Lie algebroid $\mathcal{A}$, which we also denote by $\overline{\nabla}$. Restricted to $t \in (0,1]$, the map ϕ is the dilation δ_t in the fibers of $H \oplus N \cong TM$. Because ∇ commutes with these dilations, the restriction $\overline{\nabla}^{S_t}$ of $\overline{\nabla}$ to the stratum $S_t = M \times \{t\}$ is identical to the connection ∇ on $A_t = TM$ that we started with. (The action of $\overline{\nabla}$ in the direction of $\partial/\partial t$ is irrelevant.)

We obtain exponential maps,

$$\exp^{\nabla^x}\colon\; A_x \to \widetilde{\mathcal{G}}_x,$$

that are given by,

$$\begin{aligned}
&\exp^{\overline{\nabla}^x} \,:\; T_m M \to \widetilde{M}_m \,:\; v \mapsto \exp^{\nabla}(v),\ \text{for}\, x = (m,t), t \in (0,1],\\
&\exp^{\overline{\nabla}^x} \,:\; H_m \oplus N_m \to T_H M_m \,:\; (h,n) \mapsto \exp(h,n),\ \text{for}\, x = (m,0).
\end{aligned}$$

Here $\exp^{\nabla}$ denotes the exponential map $T\widetilde{M}_m \to \widetilde{M}_m$ induced by the connection ∇ pulled back from M_m to the cover $\widetilde{M}_m$. Identifying $\mathbb{T}_H M$ with $\widetilde{\mathcal{G}}$ in a neigborhood of the $t = 0$ boundary, we see that we have a commutative diagram,

$$\begin{tikzcd}
H \oplus N \times [0,1] \arrow[r, "\phi"'] \arrow[d, "\exp"'] & \mathcal{A} \arrow[d, "\exp^{\overline{\nabla}}"] \\
T_H M \times [0,1] \arrow[r, "\psi"] & \mathbb{T}_H M
\end{tikzcd}$$

where ψ and ϕ are the maps defined in sections 5.2 and 5.5 above. Since the maps $\exp, \phi, \psi$ are diffeomorphisms, it follows that $\exp^{\overline{\nabla}}$ defines a smooth chart on $\mathbb{T}_H M$. Therefore, the identification of $\widetilde{\mathcal{G}}$ and $\mathbb{T}_H M$ in a neighborhood of the $t = 0$ boundary is a diffeomorphism. It follows that $\mathbb{T}_H M$ satisfies the axioms of a smooth groupoid in a neighborhood of the $t = 0$ boundary. For $t \in (0, 1]$, smoothness of $\mathbb{T}_H M$ clear.

□

5.7 The parabolic tangent groupoid of a contact manifold.

In this section we describe a nice local model for the parabolic tangent groupoid $\mathbb{T}_H M$ when (M, H) is a contact manifold. Recall that Darboux coordinates on a contact manifold M identify an open subset $U \subseteq M$ with the $(2r+1)$-dimensional Heisenberg group $G = H_r$. The Heisenberg group H_r has a standard contact structure, and U is identified with H_r *as a contact manifold.*

If $U \subseteq M$ is any open subset, then the parabolic tangent groupoid $\mathbb{T}_H U = T_H U \cup U \times U \times (0, 1]$ of U is the restriction of the groupoid $\mathbb{T}_H M$ to the subset $U \times [0, 1]$ of the space of units $M \times [0, 1]$. To understand $\mathbb{T}_H M$ it suffices to describe the smooth structure of the groupoid $\mathbb{T}_H U$ near $t = 0$, for a cover of M by open sets U. Therefore, to understand the parabolic tangent groupoid of a contact manifold it suffices to study the parabolic tangent groupoid $\mathbb{T}_H G$ of the Heisenberg group $G = H_r$.

The tangent bundle TG of a group G is trivialized by right translation of the fiber $T_0 G$ at the identity element. This identifies each fiber $T_x G, x \in G$ with the Lie algebra $\mathfrak{g}$ of G,i.e.,

$$TG \cong G \times \mathfrak{g}.$$

For the Heisenberg group $G = H_r$, the Lie algebra is spanned by $2r + 1$ vectors $X_i, Y_i, Z, i = 1, \ldots, r$, with commutation relation $[X_i, Y_i] = Z$. The standard contact hyperplane bundle $H \subseteq TG$ is the subbundle of $TG = G \times \mathfrak{g}$ whose fiber at each point is spanned by the vectors $X_i, Y_i \in \mathfrak{g}$. We obtain a section $N = T/H \hookrightarrow TG$ by identifying the fiber of the quotient bundle N with the span of the vector $Z \in \mathfrak{g}$. From here on we will make that identification, so that $H \oplus N = G \times \mathfrak{g}$. It is important to notice that this is an identification, not just of vector bundles, but of bundles of Lie algebras. As usual, this leads to an identification $TG = T_H G$,

$$TG = G \times \mathfrak{g} \to T_H G = G \times G \,:\, (x, \xi) \mapsto (x, \exp(\xi)).$$

The right invariant trivialization of TG gives rise to a (flat) connection ∇ on G, which restricts to a connection on the contact bundle $H \subseteq TG$ (since H is right invariant). Thus, ∇ is compatible with the contact structure, and we obtain an exponential map,

$$\exp^{\nabla} \,:\, T_H G = G \times G \to G \,:\, \exp^{\nabla}_x(y) = yx.$$

Note that $x \in G$ denotes the base point, while $y \in G \cong T_H G_x$ symbolizes the parabolic arrow at x. The reason we get yx and not xy as the image of the exponential map is that we have chosen a *right* invariant trivialization of TG.

Consider the defining chart for the parabolic tangent groupoid,

$$\psi : \; T_H G \times [0,1] \to \mathbb{T}_H G = T_H G \cup G \times G \times (0,1]$$

that was introduced above. Identifying $T_H G = G \times G$, we see that the map

$$\psi : \; G \times G \times [0,1] \to \mathbb{T}_H G$$

is expressed by the formulas

$$\begin{aligned} &\psi(x,y,t) = (\delta_t(y)x, x, t) \in G \times G \times (0,1], \text{ for } t > 0, \\ &\psi(x,y,0) = y \in T_H G_x. \end{aligned}$$

Here, the map ψ is a bijection, and we can describe the groupoid operations of $\mathbb{T}_H G$ directly in the (x,y,t) coordinates. At $t = 0$ it is clear that simply,

$$\psi(x,y,0)\psi(x,v,0) = \psi(x,yv,0).$$

If $t > 0$, we have

$$\psi(x,y,t)\psi(u,v,t) = (\delta_t(y)x, x, t)\,(\delta_t(v)u, u, t) = (\delta_t(y)x, u, t),$$

where the two elements are composable if and only if $\delta_t(v)u = x$. This implies $\delta_t(y)x = \delta_t(yv)u$, and we obtain the simple expression,

$$\psi(x,y,t)\psi(u,v,t) = \psi(u,yv,t).$$

If $t = 0$, the requirement that $x = \delta_t(v)u$ becomes $x = u$. We therefore have the same expression for the groupoid operation at $t = 0$ and $t > 0$. In fact, the operation as expressed in the (x,y,t) coordinates corresponds precisely to the composition law for elements in the *action groupoid* $(G \times [0,1]) \rtimes_\alpha G$, with $y \in G$ acting on $(x,t) \in G \times [0,1]$ by

$$\alpha(y)(x,t) = (\delta_t(y)x, t).$$

Indeed, the composition rule for elements in this action groupoid is

$$((x,t),y)\,((u,t),v) = ((u,t),yv),$$

where it is required that the source (x,t) of the first element is equal to the range $\alpha(v)(u,t) = (\delta_t(v)u, t)$ of the second element. We can summarize as follows.

Proposition 83 *The parabolic tangent groupoid of the Heisenberg group $G = H_r$ is isomorphic, as a smooth groupoid, to the action groupoid $B \rtimes_\alpha G$, where G acts on the base space $B = G \times [0,1]$ by*

$$\alpha(y)(x,t) = (\delta_t(y)x, t).$$

An explicit isomorphism $B \rtimes_\alpha G \to \mathbb{T}_H G$ is given by,

$$((x,t),y) \mapsto (\delta_t(y)x, x, t) \in G \times G \times (0,1], \text{ for } t > 0$$
$$((x,0),y) \mapsto y \in G \cong T_H G_x,$$

where the isomorphism $G \cong T_H G_x$ is obtained by the right invariant trivialization of the bundle of osculating groups $T_H G$, plus the choice of an identification $T_H G_0 = G$.

Observe that this isomorphism preserves the base space $B = G \times [0,1]$ of the groupoids. Also note that the same construction works for *any* graded nilpotent group.

5.8 A characterization of elements in $C^*(\mathbb{T}_H M)$.

In this section we consider the convolution C^*-algebra $C^*(\mathbb{T}_H M)$ of the parabolic tangent groupoid $\mathbb{T}_H M$, and we derive a concrete characterization of the elements in this C^*-algebra in the case of a contact manifold M. This explicit characterization is one of the key technical steps in the proof of our index theorem.

To define $C^*(\mathbb{T}_H M)$ we choose a smooth Haar system on $\mathbb{T}_H M$. In Examples 1 and 3 of section 4.3 we described Haar systems on the groupoids $T_H M$ and $M \times M$. Fixing a smooth 1-density λ on M, we saw how λ induces Lebesgue measure λ_m on $T_H M_m$, for $m \in M$. This extends to a Haar system on $\mathbb{T}_H M$ by taking the dilated densities $t^{-p-2q}\lambda$ as measures on the s-fibers $\mathcal{G}_{(m,t)} = M$, for $(m,t) \in M \times (0,1]$. Here $p + 2q$ is the homogeneous dimension of the osculating groups $T_H M_m$.

To understand the structure of $C^*(\mathbb{T}_H M)$ it is useful to consider the various restriction maps associated to the stratification of the groupoid. Restriction at $t = 0$ of functions in the convolution algebra $C_c(\mathbb{T}_H M)$ determines a $*$-homomorphism to the convolution algebra $C_c(T_H M)$. This maps extends by continuity to a surjective $*$-homomorphism of the corresponding C^*-algebras,

$$\pi_0 : C^*(\mathbb{T}_H M) \to C^*(T_H M).$$

Further restriction to the individual osculating groups gives morphisms

$$\pi_m : C^*(\mathbb{T}_H M) \to C^*(T_H M_m).$$

Likewise, for $t \in (0,1]$ we obtain surjective $*$-homomorphisms,

$$\pi_t : C^*(\mathbb{T}_H M) \to C^*(M \times M) \cong \mathcal{K}(L^2(M, t^{-p-2q}\lambda)).$$

Observe that the C^*-algebras $\mathcal{K}(L^2(M, t^{-p-2q}\lambda))$ are all naturally isomorphic to $\mathcal{K}(L^2(M,\lambda))$. If the action of a (compact) operator Q on functions $C^\infty(M)$ is known, it determines a well-defined element in this algebra, independent of the value of t. We can therefore ignore this scaling factor, and will simply write $\mathcal{K}(L^2(M))$. (Note: the situation is different when we are given an operator *kernel* $k(x,y)$; the operator it defines *does* depend on the scaling factor of the measure.)

Thus, every element $Q \in C^*(\mathbb{T}_H M)$ gives rise to a family of elements

$$\langle Q_m, Q_t \rangle = \langle \pi_m(Q), \pi_t(Q_t) \rangle,$$

with

$$Q_m \in C^*(T_H M_m),\ Q_t \in \mathcal{K}(L^2(M)).$$

Our aim in this section is to determine precisely which families $\langle Q_m, Q_t \rangle$ represent elements in $C^*(\mathbb{T}_H M)$. An obvious requirement is that the elements $\{Q_m\}$ represent an element in the C^*-algebra $C^*(T_H M)$. A similarly obvious property for the family $\{Q_t\}$ follows from the following lemma.

Lemma 84 *Let M, T be smooth manifolds, and let $\mathcal{G} = M \times M \times T$ be the smooth groupoid that, algebraically, is a family of copies of the pair groupoid $M \times M$, with parameter space T.*

Then $C^*(\mathcal{G})$ is isomorphic to the C^*-algebra $C_0(T, \mathcal{K})$.

Proof. The structure of the convolution C^*-algebra does not depend on the choice of Haar system. So for simplicity, we can take a Haar system on $C^*(M \times M \times T)$ that is the same in each copy of $M \times M$.

With $k \in C_c^\infty(M \times M \times T)$, let $k_t \in C_c^\infty(M \times M)$ denote the restriction of k to $t \in T$, i.e., $k_t(x, y) = k(x, y, t)$. With this notation, the norm on $C^*(\mathcal{G})$ is, by definition,

$$||k||_{C_r^*} = \sup_{t \in T} ||\mathrm{Op}(k_t)||,$$

where $\mathrm{Op}(k_t)$ denotes the operator on $L^2(M)$ with Schwartz kernel k_t. Consider the map,

$$T \to \mathcal{K}(L^2(M)) :\ t \mapsto \mathrm{Op}(k_t)$$

We will show it is continuous. Because k is continuous with compact support, $t \mapsto k_t$ is continuous in the sup norm for k_t. Also, for $f \in C_c^\infty(M \times M)$, such that the support of f is contained in a fixed compact set $V \subseteq M \times M$, we have the uniform estimate,

$$||\mathrm{Op}(f)|| \leq C\, ||f||_\infty,$$

with $C = \mathrm{Vol}(V)^{1/2}$. In other words, $f \mapsto \mathrm{Op}(f)$ is continuous, if we take the sup norm for f. this proves continuity of $t \mapsto \mathrm{Op}(k_t)$.

We see that,

$$C_c^\infty(\mathcal{G}) \to C_0(T, \mathcal{K}) :\ k \to [t \mapsto \mathrm{Op}(k_t)].$$

is an isometric $*$-homomorphism. The range is dense, and we conclude that this map completes to an isomorphism of C^*-algebras.

□

We easily derive that if $\langle Q_m, Q_t \rangle$ represents an element in $C^*(\mathbb{T}_H M)$, then $t \mapsto Q_t$ is a norm-continuous family of operators that is uniformly bounded. This represents the second property that $\langle Q_m, Q_t \rangle$ must satisfy.

Another easy property of elements in $C^*(\mathbb{T}_H M)$ is that the family $\{Q_t\}$ is *asymptotically local.*

Proposition 85 *If $\langle Q_m, Q_t \rangle \in C^*(\mathbb{T}_H M)$, then the family Q_t is* asymptotically local*, i.e., given two continuous functions $\phi, \psi \in C(M)$ with disjoint supports, we have*

$$\lim_{t \to 0} \|\phi\, Q_t\, \psi\| = 0.$$

Proof. Let us first consider $k \in C_c(\mathbb{T}_H M)$. Then $Q_t = \pi_t(k)$ is the compact operator on $L^2(M)$ with continuous Schwartz kernel $k_t(x, y) = k(x, y, t)$, where $(x, y) \in M \times M$. The manifold structure on $\mathbb{T}_H M$ was defined by blowing up the diagonal in $M \times M$ as $t \to 0$. Therefore the compact support of the kernel $k_t(x, y)$ will concentrate near the diagonal of $M \times M$ as $t \to 0$, and for sufficiently small t we have

$$\phi\, \pi_t(k)\, \psi = 0.$$

The result for arbitrary $Q \in C^*(\mathbb{T}_H M)$ follows by approximation. For every $\varepsilon > 0$ there exists an element $k \in C_c(\mathbb{T}_H M)$ such that $\|Q_t - \pi_t(k)\| < \varepsilon$ for every $t \in (0, 1]$ (by definition of the norm on $C^*(\mathbb{T}_H M)$). We see that for sufficiently small t,

$$\|\phi\, Q_t\, \psi\| = \|\phi\, Q_t\, \psi - \phi\, \pi_t(k)\, \psi\| \leq \|\phi\|_\infty \|\psi\|_\infty \varepsilon.$$

□

The next property is the crucial one, and it is less obvious. It establishes the relation between the family Q_t and the family Q_m. If we pick an appropriate exponential map for the Heisenberg structure,

$$\exp\, :\; T_H M \to M,$$

then the osculating group $T_H M_m$ is identified with an open neighborhood $U \subseteq M$ of $m \in M$. In this way the operator Q_m, which acts on $L^2(T_H M_m)$, can be identified with an operator on $L^2(U) \subseteq L^2(M)$. If δ_t are the natural dilations of the osculating group, then

$$\delta_t^* f(x) = f(\delta_t x)$$

denotes the induced action on $L^2(T_H M_m)$, and we consider the family of operators

$$\delta_{t^{-1}}^* Q_m \delta_t^*.$$

These operators are thought of as acting on $L^2(M)$, and we compare their aympptotic behaviour as $t \to 0$ with that of the operators Q_t. Of course, this comparison is only meaningful locally, in a neighborhood of the point m.

Proposition 86 *Let M be a contact manifold.*

If $\langle Q_m, Q_t\rangle \in C^*(\mathbb{T}_H M)$, then for every $m \in M$ and every $\varepsilon > 0$ there exists a neighborhood V of m such that if h denotes the characteristic function of V, then

$$\limsup_{t \to 0}\, \|\, hQ_t h - h\, \delta_{t^{-1}}^* Q_m \delta_t^*\, h\| < \varepsilon.$$

Here the operators $\delta_{t^{-1}}^* Q_m \delta_t^*$, defined on the osculating group, are identified with operators on M with the help of Darboux coordinates near the point m. The norm $\|\cdot\|$ appearing in the estimate is the operator norm of bounded operators on $L^2(M)$.

We will say that Q_t is locally asymptotically equivalent *to $\delta_{t^{-1}}^* Q_m \delta_t^*$.*

Proof. It suffices to consider what happens in the case where $M = G$ is the Heisenberg group. We identify $\mathbb{T}_H G \cong (G \times [0,1]) \rtimes G$, as in Proposition 83.

To begin, we assume $k \in C_c((G \times [0,1]) \rtimes G)$. We think of the operators $\delta^*_{t^{-1}} k_m \delta^*_t$ as acting on the source map fibers of the tangent groupoid at the appropriate value of t. The isomorphism of the tangent groupoid with the action groupoid is effected by δ_t-dilations of the fibers (each a copy of $M = G$). Therefore, when representing the operators $\delta^*_{t^{-1}} k_m \delta^*_t$ on the fibers of the *action* groupoid, we simply get k_m for each $t > 0$.

The neutral element $0 \in G$ corresponds to $m \in M$. According to the composition law in the action groupoid we have,

$$((0,t),x) = ((\delta_t y, t), xy^{-1})\,((0,t),y).$$

It follows that the regular representation $k_t = \pi_t(k)$, associated to the fixed base point $(0,t) \in G \times [0,1]$, is given by

$$\pi_t(k) f(x) = \int_G k(\delta_t y, t, xy^{-1})\, f(y) dy.$$

At $t = 0$ this gives the convolution operator $k_m = \pi_m(k)$,

$$\pi_m(k) f(x) = \int_G k(0,0,xy^{-1})\, f(y) dy.$$

Let f be a characteristic function of some neighborhood of 0 in the fibers $G = \delta_t M$ of the action groupoid, which is fixed for all values of t. Then $h(x) = f_t(x) = f(\delta_t x)$ corresponds to the characteristic function of a fixed neighborhood V of $m \in M$. Let $a_t(x,y)$ denote the Schartz kernel of $f_t k_t f_t$ and $b_t(x,y)$ that of $f_t k_m f_t$. These kernels are given by

$$\begin{aligned}
a_t(x,y) &= f(\delta_t x)\; k(\delta_t y, t, xy^{-1})\; f(\delta_t y),\\
b_t(x,y) &= f(\delta_t x)\; k(0,0,xy^{-1})\; f(\delta_t y).
\end{aligned}$$

By taking the support U of f sufficiently small, and letting $t < r$ for sufficiently small $r > 0$, we can obtain a uniform estimate

$$|a_t(x,y) - b_t(x,y)| < \varepsilon.$$

To see this, consider the values (x,y,t) for which $a_t(x,y) - b_t(x,y) \neq 0$. We must have $\delta_t y \in U$, but also $xy^{-1} \in K$, where $K \subseteq G$ is the compact set

$$K = \{v \in G \mid ((u,t),v) \in \mathrm{supp}(k),\ u \in U, t \in [0,r]\}.$$

This set is compact because k has compact support. Then, because k is continuous, the difference

$$|k(\delta_t y, t, xy^{-1}) - k(0,0,xy^{-1})|$$

can be made arbitrarily small for all $\delta_t y \in U, t < r$, by taking U and $r > 0$ small enough. This estimate will be *uniform* in $xy^{-1} \in K$ (because K is compact), which gives the required estimate on $\|a_t - b_t\|_\infty$.

Now let us denote $c_t(x,y) = a_t(x,y) - b_t(x,y)$. The fact that $xy^{-1} \in K$ if $c_t(x,y) \neq 0$ implies uniform estimates

$$\int |c_t(z,y)| dy < \mathrm{Vol}(K)\varepsilon, \ \int |c_t(x,z)| dx < \mathrm{Vol}(K)\varepsilon,$$

that hold for each $z \in G$. Hence,

$$\|f_t k_t f_t - f_t k_m f_t\| < \mathrm{Vol}(K)\varepsilon,$$

for all $t < r$. This gives the required estimate.

The result for general $\langle Q_m, Q_t \rangle \in C^*(\mathbb{T}_H M)$ follows immediately by approximation, by choosing $k \in C_c(\mathbb{T}_H M)$ such that $\|Q_m - k_m\| < \varepsilon$ and $\|Q_t - k_t\| < \varepsilon$ for all $m \in M$, $t \in (0,1]$. □

The properties derived so far are sufficient to characterize the elements in $C^*(\mathbb{T}_H M)$.

Proposition 87 *Let M be a contact manifold.*

A family of operators $\langle Q_m, Q_t \rangle$ with $Q_m \in C^*(T_H M_m)$ and $Q_t \in \mathcal{K}(L^2(M))$ represents an element in $C^*(\mathbb{T}_H M)$ if and only if it has all of the following properties:

(1) the family Q_m defines an element in $C^*(T_H M)$;

(2) the family Q_t is norm continuous and uniformly bounded;

(3) the family Q_t is asymptotically local (as defined in Proposition 85);

(4) the family Q_t is locally aymptotically equivalent to $\delta^*_{t^{-1}} Q_m \delta^*_t$ for each $m \in M$ (as defined in Proposition 86).

Proof. We have proven necessity of the four properties listed, and must now show that they are sufficient. Let $\mathcal{D}$ be the set of elements $\langle Q_m, Q_t \rangle$ that satisfy all four properties. Because of properties (1) and (2), we can think of $\mathcal{D}$ as a subset of the C^*-algebra

$$C^*(T_H M) \oplus C_b((0,1], \mathcal{K})).$$

We first verify that $\mathcal{D}$ is a norm-closed $*$-subalgebra, which will prove that $\mathcal{D}$ is a C^*-algebra. It is easy to see that the set $\mathcal{D}$ is norm closed. Let $\langle Q_m, Q_t \rangle$ and $\langle R_m, R_t \rangle$ be two families in $\mathcal{D}$. One easily verifies that $\langle Q_m + R_m, Q_t + R_t \rangle$, as well as $\langle Q^*_m, Q^*_t \rangle$ are contained in $\mathcal{D}$. For the product $\langle Q_m R_m, Q_t R_t \rangle$, proporties (1) and (2) clearly hold. To verify property (3), let $\phi_1, \phi_2 \in C(M)$ be two functions with disjoint support. Choose two other function $\psi_1, \psi_2 \in C(M)$, such that ψ_1 and ψ_2 have disjoint supports as well, and such that $\psi_j(x) = 1$ whenever $x \in \mathrm{supp}(\phi_j)$. Then $(1 - \psi_j)$ and ϕ_j have disjoint supports as well, and therefore

$$\lim_{t\to 0} \|\phi_1 Q_t - \phi_1 Q_t \psi_1\| = \lim_{t\to 0} \|\phi_1 Q_t (1 - \psi_1)\| = 0,$$
$$\lim_{t\to 0} \|R_t \phi_2 - \psi_2 R_t \phi_2\| = \lim_{t\to 0} \|(1 - \psi_2) R_t \phi_2\| = 0.$$

Because both Q_t and R_t are uniformly bounded in norm, it follows that

$$\lim_{t\to 0} \|\phi_1 Q_t R_t \phi_2\| = \lim_{t\to 0} \|\phi_1 Q_t \psi_1 \psi_2 R_t \phi_2\| = 0.$$

This establishes property (3). Property (4) is proven in a similar way, this time using asymptotic equivalence of $Q_t h$, hQ_t, and $hQ_t h$, etc.

We see that $\mathcal{D}$ is indeed a C^*-algebra, and by the previous two propositions we know that $C^*(\mathbb{T}_H M) \subseteq \mathcal{D}$. To see that the two are isomorphic, consider the restriction map

$$\mathcal{D} \to C^*(T_H M) : \langle Q_m, Q_t \rangle \mapsto \{Q_m\}.$$

We only need to show that it has the same kernel as the corresponding map,

$$C^*(\mathbb{T}_H M) \to C^*(T_H M) : \langle Q_m, Q_t \rangle \mapsto \{Q_m\}.$$

(both maps are surjective), which is $C_0((0,1], \mathcal{K})$ (this follows from Lemma 84).

Suppose therefore that $\langle Q_m, Q_t \rangle \in \mathcal{D}$ and that $Q_m = 0$ for all $m \in M$. Choose $\varepsilon > 0$. By property (3) there exists a neigborhood V_m of each point $m \in M$, such that the characteristic function h_m of V_m satisfies

$$\limsup_{t \to 0} \|h_m Q_t h_m\| < \epsilon.$$

But since Q_t is asymptotically local, this actually implies,

$$\limsup_{t \to 0} \|Q_t\| < \epsilon,$$

and therefore certainly $\|Q_t\| \to 0$, which proves the claim.

$\square$

Chapter 6

The index theorem for maximally hypoelliptic operators.

6.1 Introduction.

In this chapter we prove our main theorem (Theorem 99), which states that the analytic index of a maximally hypoelliptic operator on a compact contact manifold is computed by the topological index formula of Atiyah and Singer. Our proof is a development of the tangent groupoid proof of the classical index theorem for elliptic operators, proposed by Connes (see [Co], Section II.5). For a detailed exposition of this proof we refer to [Hi]. (Higson's paper is phrased in the language of E-theory, and is easily translated into terms involving the tangent groupoid.)

Restriction to the $t = 0$ and $t = 1$ fibers in the tangent groupoid $\mathbb{T}M$ gives rise to two maps in K-theory, denoted π_0 and π_1, respectively,

$$\begin{array}{ccc} K_0(C^*(\mathbb{T}M)) & \xrightarrow{\pi_1} & K_0(C^*(M \times M)) \cong \mathbb{Z} \\ \downarrow{\scriptstyle \pi_0} & & \\ K^0(T^*M) & & \end{array}$$

Because π_0 is an isomorphism, Connes obtains a map,

$$\mathrm{Ind_a} = \pi_1 \circ \pi_0^{-1} \; : \; K^0(T^*M) \to \mathbb{Z}.$$

The topological index of Atiyah and Singer also defines a map in K-theory,

$$\mathrm{Ind_t} \; : \; K^0(T^*M) \to \mathbb{Z} \; : \; x \mapsto \int_{T^*M} \mathrm{Ch}(x) \wedge \mathrm{Td}(M).$$

First Connes proves that $\mathrm{Ind_a}$ is the analytic index map for elliptic operators P,

$$\text{Index } P = \mathrm{Ind_a}([\sigma(P)]),$$

and then shows that $\mathrm{Ind_a} = \mathrm{Ind_t}$, thus proving the Atiyah-Singer index theorem. (See [Co], [Hi].)

In the same manner, the parabolic tangent groupoid gives rise to a map,

$$\mathrm{Ind_q} : K_0(C^*(T_H M)) \to \mathbb{Z}.$$

The first step in the proof of our index theorem is to show that for a maximally hypoelliptic operator P we have

$$\text{Index } P = \mathrm{Ind_q}([\sigma_H(P)]).$$

Any differential operator P on M gives rise to a smooth invariant family $\langle P_m, t^d P\rangle$ of operators on the parabolic tangent groupoid $\mathbb{T}_H M$. The fact that $\mathrm{Ind_q}$ is equal to the analytic index will follow immediately once it is established that, in the case of a subelliptic operator P, the resolvents of the individual operators in the invariant family $\langle P_m, t^d P\rangle$ define an element in the C^*-algebra $C^*(\mathbb{T}_H M)\otimes\mathcal{M}_k$. Section 6.3 is devoted to proving this key result. Our proof is based on the characterization of elements in $C^*(\mathbb{T}_H M)$ derived in Section 5.8. The corresponding fact for elliptic operators is much easier to prove, because there the osculating groups are abelian, and one can use Fourier Theory.

Next, there is a natural isomorphism

$$\Psi : K_0(C^*(T_H M)) \to K^0(T^*M),$$

which allows us to identify the symbol class $[\sigma_H(P)]$ of a subelliptic operator with an element in $K^0(T^*M)$. Thus, we can apply the topological index formula to the symbol of a subelliptic operator P. We show that, under the natural isomorphism Ψ, the index map that is induced by the parabolic tangent groupoid agrees with the index map that is induced by the ordinary tangent groupoid, i.e.,

$$\mathrm{Ind_q} = \mathrm{Ind_a} \circ \Psi.$$

The proof of this fact relies on the construction of a larger groupoid that interpolates, as it where, between the parabolic tangent groupoid $\mathbb{T}_H M$ and the ordinary tangent groupoid $\mathbb{T}M$. Using the fact proven by Connes that $\mathrm{Ind_a} = \mathrm{Ind_t}$, we obtain

$$\text{Index } P = \int_{T^*M} \mathrm{Ch}(\Psi([\sigma_H(P)])) \wedge \mathrm{Td}(M).$$

6.2 Model operators and the tangent groupoid.

In this section we see how a differential operator P and the model operators P_m can be assembled into a single operator $\mathbb{P}$ on the parabolic tangent groupoid. This clarifies the relationship between a differential operator and its principal symbol. In fact, the tangent groupoid provides a convenient context for an alternative definition of the principal symbol, which has the advantage that it is coordinate independent. (Put differently, the effect of coordinate transformations has been taken care of in the construction of the groupoid itself.)

Let P be a differential operator on a manifold M with Heisenberg structure $H \subseteq TM$. Choosing an H-frame $X_1, \cdots, X_n$, we can present P locally as,

$$P = \sum_{|\alpha|\leq d} a_\alpha X^\alpha.$$

The model operator for P at $m \in M$ was defined as,

$$P_m = \sum_{|\alpha|=d} a_\alpha(m) Y^\alpha,$$

where Y_i denotes the invariant vector field on $T_H M_m$ that correspond to $X_i(m) \in H_m \oplus N_m$. The operator P_m is invariantly defined as a right invariant operator on the osculating group $T_H M_m$. We give an alternative proof of this fact by showing how the collection of model operators can be interpreted invariantly as the asymptotic 'limit' of the family $t^d \delta_t^* P \delta_{t^{-1}}^*$ as $t \to 0$ (here, as elsewhere, δ_t denotes the Heisenberg dilations in the osculating groups). This limit receives a precise meaning if we think of the family $t^d P$ as a right invariant family of operators on the parabolic tangent groupoid.

Proposition 88 *Let P be a differential operator on (M, H) of Heisenberg degree d. Then the model operators $P_m, m \in M$ are well-defined as right invariant operators on the osculating groups $T_H M_m$, independent of the way P is presented locally by a choice of H-frame.*

The principal Heisenberg symbol $\sigma_H(P) = \{P_m\}$, regarded as a single operator on $T_H M$, has smooth coefficients. Moreover, the family,

$$\mathbb{P} = \langle P_m, t^d P \rangle,$$

is a right invariant operator with smooth coefficients on the parabolic tangent groupoid $\mathbb{T}_H M$.

Proof. Let X_i be an H-frame on M. For simplicity, we assume that the frame is defined *globally*. The argument is not essentially different if X_i are only defined locally.

According to Definition 81 each of the families,

$$\mathbb{X}_i = \langle Y_i(m), t^{\nu_i} X_i \rangle,$$

corresponds to a local smooth section in the Lie algebroid of $\mathbb{T}_H M$, and hence to a right invariant vector field on the groupoid. Here $\nu_i \in \{1, 2\}$ is, of course, the degree of X_i, depending on whether X_i is a section in H or not.

Thinking of $\mathbb{X}_i$ as a single operator on $\mathbb{T}_H M$, we can form the product,

$$\mathbb{X}^\alpha = \mathbb{X}_1^{\alpha_1} \cdots \mathbb{X}_n^{\alpha_n} = \langle Y^\alpha(m), t^k X^\alpha \rangle,$$

where $k = |\alpha| = \sum \alpha_i \nu_i$ is the weighted degree of the monomial.

Now consider the function $a_\alpha \in C^\infty(M)$. It can be identified with a function on $\mathcal{G}^{(0)} = M \times [0, 1]$ that is independent of $t \in [0, 1]$, and then induces a right invariant function $\tilde{a}_\alpha = a_\alpha \circ r$ on $\mathbb{T}_H M$. Observe that, when restricted to the group $T_H M_m$ at $t = 0$, $\tilde{a}_\alpha$ is just the scalar $a_\alpha(m)$. For $t \in (0, 1]$, the function $\tilde{a}_\alpha$ corresponds to a_α evaluated on the first component in $M \times M$. We obtain a right invariant operator on $\mathbb{T}_H M$,

$$\tilde{a}_\alpha \mathbb{X}^\alpha = \langle a_\alpha(m) Y^\alpha(m),\, t^k a_\alpha X^\alpha \rangle = \langle \sigma_H(a_\alpha X^\alpha),\, t^k a_\alpha X^\alpha \rangle.$$

To analyse the situation for a general operator P, we expand it as a sum of homogeneous terms,

$$P = P_0 + P_{-1} + P_{-2} + \cdots + P_{-d},$$

where $P_{-k} = \sum_{|\alpha|=d-k} a_\alpha X^\alpha$ is of degree $d-k$. We have right invariant operators,

$$\mathbb{P}_{-k} = \sum_{|\alpha|=d-k} \tilde{a}_\alpha \mathbb{X}^\alpha = \langle \sigma_H(P_{-k}), t^{d-k} P_{-k} \rangle.$$

Note that $\sigma_H(P_{-k})$ denotes the *order* $d-k$ symbol of P_{-k}, not the order d symbol! Finally,

$$\begin{aligned}\mathbb{P} &= \langle P_m, t^d P \rangle = \langle \sigma_H(P_0), t^d P_0 \rangle + \langle 0, t^d P_{-1} \rangle + \langle 0, t^d P_{-2} \rangle + \cdots \\ &= \mathbb{P}_0 + t\mathbb{P}_{-1} + t^2 \mathbb{P}_{-2} + \cdots .\end{aligned}$$

This proves that $\mathbb{P}$ is an invariant differential operator on $\mathbb{T}_H M$ with smooth coefficients.

The fact that $\sigma_H(P)$ is independent of the choice of H-frame X_i is now a trivial corollary. Clearly, at $t \in (0,1]$, the definition of $\mathbb{P}$ is independent of the choice of H-frame and the corresponding local presentation of P, and by continuity the value of $\mathbb{P}$ at $t=0$ (i.e., the family of model operators) is entirely determined by $t^d P$.

□

Remark. It is a routine matter to adapt our discussion to the case of operators acting on sections in vector bundles $E, F \to M$,

$$P : \Gamma(E) \to \Gamma(F).$$

By trivializing E and F in some open set in M, the operator P can be presented as a matrix $P = (P_{ij})$ of scalar differential operators. The symbol of P is simply the matrix of symbols $\sigma_H(P) = (\sigma_H(P_{ij}))$. At each point $m \in M$ the symbol should be thought of as an invariant operator acting on sections in the bundles π^*E, π^*F on $T_H M_m$, obtained by pull back of E, F by the base point map $\pi \colon T_H M \to M$.

In fact, the bundles E, F on M induce *right invariant* bundles $\widetilde{E}, \widetilde{F}$ on $\mathbb{T}_H M$ by pull back via the composition $\mathbb{T}_H M \xrightarrow{r} M \times [0,1] \to M$ (r denotes the range map in the groupoid). Because at $t=0$ the range map r in $\mathbb{T}_H M$ is equal to the base point map π for $T_H M$, the bundles $\widetilde{E}, \widetilde{F}$ restrict to π^*E, π^*F on $T_H M$ at $t=0$. At $t>0$ the bundles $\widetilde{E}, \widetilde{F}$ simply restrict to the bundles E, F in each s-fiber M. The family $\langle P_m, t^d P \rangle$ then defines an invariant operator,

$$\mathbb{P} : \Gamma(\widetilde{E}) \to \Gamma(\widetilde{F}).$$

The proof of Proposition 88 applies without fundamental changes.

6.3 Subelliptic operators and the C^*-algebra of $\mathbb{T}_H M$.

Let (M, H) be a compact contact manifold, and $P \colon \Gamma(E) \to \Gamma(F)$ a maximally hypoelliptic operator on M, acting on sections in smooth vector bundles E, F. Let D be the self-adjoint operator,

$$D = \begin{pmatrix} 0 & -iP \\ iP^* & 0 \end{pmatrix},$$

acting on sections in $\Gamma(E \oplus F)$, with associated self-adjoint model operators,

$$D_m = \begin{pmatrix} 0 & -iP_m \\ iP_m^* & 0 \end{pmatrix},$$

acting on sections in $\Gamma(\pi^* E_m \oplus \pi^* F_m)$ (pulling back E, F via $\pi\colon T_H M \to M$). In Chapter 4 we described two K-theory classes associated to the operator D. The first one, constructed from the graph projection of D itself, corresponded to the analytic index of P as an element in $K_0(\mathcal{K}(L^2(M))) \cong \mathbb{Z}$. An analogous construction involving the graph projections of the model operators D_m gave us the symbol class

$$[\sigma_H(P)] \in K_0(C^*(T_H M)).$$

Now consider the right invariant operatoron the parabolic tangent groupoid $\mathbb{P} = \langle P_m, t^d P\rangle$, and the corresponding self-adjoint operator $\mathbb{D} = \langle D_m, t^d D\rangle$. Our aim is to construct an element in $K_0(C^*(\mathbb{T}_H M))$ that restricts at $t = 0$ to the Heisenberg symbol of P, and at $t = 1$ to its analytic index. We will construct this K-theory class starting from the operator $\mathbb{P}$ by the same procedure that produced the two K-theory classes associated to P and the family $\{P_m\}$, respectively. The crucial technical step in this construction is to show that the family of resolvents

$$(\mathbb{D} - 1)^{-1} = \langle (D_m - i)^{-1},\, (t^d D - i)^{-1}\rangle$$

defines an element in $C^*(\mathbb{T}_H M)$. In Section 5.8, Proposition 87 we derived a characterization of elements in $C^*(\mathbb{T}_H M)$ as families $\langle Q_m, Q_t\rangle$ with certain properties. We will now show that the family $f(\mathbb{D})$ satisfies these properties.

Proposition 89 *Let (M, H) be a compact contact manifold, and $D\colon \Gamma(E) \to \Gamma(F)$ a maximally hypoelliptic, self-adjoint operator on M of Heisenberg order d.*

If $\mathbb{D} = \langle D_m, t^d D\rangle$ denotes the invariant family of operators on $\mathbb{T}_H M$ associated to D, then

$$f(\mathbb{D}) \in \mathcal{M}_k(C^*(\mathbb{T}_H M))$$

for any $f \in C_0(\mathbb{R})$.

Remark. As in Chapter 4, the integer k denotes the dimension of a trivial vector bundle over M that contains $E \oplus F$ as a direct summand.

Proof. According to Proposition 87, we must verify four properties for the family $\langle f(D_m), f(t^d D)\rangle$. By functional calculus it suffices to prove this for the function $f(x) = (x - i)^{-1}$. We know that the family $\{f(D_m)\}$ defines an element in $\mathcal{M}_k(C^*(T_H M))$ (Proposition 67). By functional calculus, the map $t \mapsto f(t^d D)$ is norm-continous, while $\|f(t^d D)\|$ is uniformly bounded. Also, $f(t^d D)$ is compact because D is Fredholm. This proves the first two properties listed in Proposition 87.

Next, we must show that if $\phi, \psi \in C^\infty(M)$ are two functions with disjoint supports, then

$$\lim_{t\to 0} \|\phi f(t^d D)\psi\| = 0.$$

This follows immediately from Lemma 94 below. Finally, we must verify that for each $m \in M$ there exists a neighborhood V of m for each $\varepsilon > 0$ such that, if h denotes the characteristic function of V, then

$$\limsup_{t\to 0} \|h f(\delta^*_{t^{-1}} D_m \delta^*_t) h - h f(t^d D) h\| < \varepsilon.$$

Here the osculating group $T_H M_m$ is identified with a neighborhood of m by means of a choice of Darboux coordinates centered at m, so that we can think of D_m as an operator on M. (See section 5.8 for details.) Because D_m is homogeneous of order d with respect to the dilations δ_t of the osculating group, we have simply

$$\delta^*_{t^{-1}} D_m \delta^*_t = t^d D_m,$$

so we must show

$$\limsup_{t\to 0} \|hf(t^d D_m)h - hf(t^d D)h\| < \varepsilon.$$

The standard trick for comparison of two commutators gives us

$$(t^d D_m - i)^{-1} - (t^d D - i)^{-1} = t^d\,(t^d D_m - i)^{-1}(D - D_m)(t^d D - i)^{-1}.$$

Because $\|[f(t^d D), h]\| \to 0$ as $t \to 0$, we can write

$$\limsup_{t\to 0} \|\, hf(t^d D_m)h - hf(t^d D)h\,\| = \limsup_{t\to 0}\; t^d\,\|\, hf(t^d D_m)(D - D_m)hf(t^d D)\,\|.$$

We decompose the product of operators as follows,

$$\begin{aligned}
&t^d\,\|hf(t^d D_m)(D - D_m)hf(t^d D)\| \\
&\quad = t^d\,\|hf(t^d D_m)\|_{L^2\to L^2}\,\|(D - D_m)h\|_{W^d\to L^2}\,\|f(t^d D)\|_{L^2\to W^d} \\
&\quad \le t^d\,\|(D - D_m)h\|_{W^d\to L^2}\,\|f(t^d D)\|_{L^2\to W^d}.
\end{aligned}$$

Here W^d denotes the order d Sobolev space for the Heisenberg group G. By Lemma 93 below, the norm $\|f(t^d D)\|_{L^2\to W^d}$ is of order $\mathcal{O}(t^{-d})$, which means that

$$t^d\,\|f(t^d D)\|_{L^2\to W^d} < C.$$

For the remaining factor, observe that because D_m is the highest order part of D at m the norm $\|(D - D_m)h\|_{W^d\to L^2}$ can be made arbitrarily small by choosing a sufficiently small neighborhood of m as the support of h. This proves the last property in Proposition 87, and completes the proof.

□

In the remainder of this section we derive some technical results that were used in the proof of Proposition 89.

Lemma 90 *Let (M, H) be a compact contact manifold. Let D be a formally selfadjoint, maximally hypoelliptic operator of order d. Let $\Delta = (D^2 + 1)^{1/2d}$.*

The operators Δ^r $(r \in \mathbb{R})$ map $C^\infty(M)$ bijectively to $C^\infty(M)$, and if k is a positive integer then Δ^{-k} is bounded as an operator $L^2 \to W^k$.

Proof. The a priori estimates imply that the domain of the closure of the maximally hypoelliptic operator $D^2 + 1$ is the weighted Sobolev space W^{2d}. Then the closure of $D^2 + 1$ is a bijective map from the domain W^{2d} to L^2, and is therefore selfadjoint. So it makes sense to define the selfadjoint operator $\Delta = (D^2 + 1)^{1/2d}$.

To prove the first statement, observe that $C^\infty(M)$ is the intersection of the Sobolev spaces W^k, which is equal to the intersection of the domains of the maximally hypoelliptic operators Δ^{4dk} ($k = 1, 2, 3, \ldots$). Let E is the projection valued spectral measure of the invertible, selfadjoint operator Δ (See [Ru], or section 3.5). Then $u \in C^\infty(M)$ if and only if

$$\int |\lambda|^N dE_{u,u} < \infty$$

for all integers $N > 0$. Here $E_{u,u}$ denotes the positive measure

$$E_{u,u}(\omega) = \langle E(\omega)u, u\rangle,$$

where $\omega \subseteq \mathbb{R}$ is a Borel set. If $v = \Delta^r u$ then

$$\langle E(\omega)v, v\rangle = \langle E(\omega)\Delta^{2r}u, u\rangle = \int_\omega \lambda^{2r} dE_{u,u},$$

which shows that $dE_{v,v} = \lambda^{2r} dE_{u,u}$ (see [Ru]). It follows that $u \in C^\infty$ if and only if $v \in C^\infty$.

To prove the second claim, let A be a differential operator of Heisenberg order k. The same argument used in the proof of Proposition 46 shows that

$$\|Au\| \leq C\|\Delta^k u\|,$$

for all $u \in C^\infty$. By the first result we may substitute $u = \Delta^{-k}v$ with $v \in C^\infty$, and we get

$$\|A\Delta^{-k}v\| \leq C\|v\|.$$

Therefore $\|\Delta^{-k}v\|_{W^k} \leq C'\|v\|$ holds for all $v \in C^\infty$, and hence for all $v \in L^2$.

□

Corollary 91 *If $f \in C_0(\mathbb{R})$ has rapid decay, i.e., $f(x) \leq C_k(1+x^2)^{-k}$ for all $k > 0$, then $f(D)$ is a smoothing operator, i.e., it maps $L^2(M)$ to $C^\infty(M)$.*

Lemma 92 *Let r be a fixed real number, $0 < r \leq 1$. The supremum of the family of bounded functions $f_t \in C_b(\mathbb{R})$ with $t > 0$,*

$$f_t(x) = \frac{(x+i)^r}{tx+i},$$

behaves asymptotically as $\|f_t\|_\infty \sim Ct^{-r}$ as $t \downarrow 0$. Here $C > 0$ is a constant that depends on r.

Proof. To find the supremum of f_t, consider the square of the absolute value

$$h(x) = |f_t(x)|^2 = \frac{(x^2+1)^r}{t^2x^2+1}$$

and its derivative

$$h'(x) = \frac{x\left((r-1)t^2x^2 + (r-t^2)\right)}{(t^2x^2+1)^2},$$

First, let $r = 1$, and assume that $0 < t < 1$. In this case f_t reaches its supremum as $x \to \pm\infty$, and we find $\|f_t\|_\infty = t^{-1}$, as desired.

Now suppose $r < 1$, and assume $0 < t^2 < r$ so that $r - t^2 > 0$. Then $h'(x) = 0$ if either $x = 0$, or

$$x^2 = \frac{r - t^2}{(1-r)t^2}.$$

At $x = 0$ the function h has a local minimum $h(0) = 1$. As $x \to \pm\infty$ the value of $h(x)$ tends to zero. The supremum for h is attained at $x^2 = (r - t^2)/(1 - r)t^2$. This means that $x^2 \sim Ct^{-2}$, and therefore $h(x) \sim C't^{-2}$.

□

Lemma 93 *Let D be a selfadjoint, maximally hypoelliptic operator on (M, H) of order d. If $f \in C_0(\mathbb{R})$ satisfies $f(x) < C(1 + x^2)^{-1/2}$, then for any $k \leq d$*

$$\| f(t^d P) \|_{W^k} = \mathcal{O}(t^{-k}).$$

Proof. Let A be a differential operator of Heisenberg order $k \leq d$. We write,

$$Af(t^d P) = A(P + i)^{-k/d} \ \cdot \ (P + i)^{k/d}(t^d P + i)^{-1} \ \cdot \ (t^d P + i)f(t^d P).$$

According to Lemma 92, we have the asymptotic behaviour,

$$\|(P + i)^{k/d}(t^d P + i)\| = \mathcal{O}(t^{-k}).$$

Lemma 90 shows that $A(P + i)^{-k/d}$ is bounded, while the assumption on f implies that there is a bound $(t^d x + i)f(t^d x) < C$.

□

Lemma 94 *Let D be a selfadjoint, maximally hypoelliptic operator on (M, H) of order d. Let $f(x) = (x + \lambda)^{-1}$, $\lambda \notin \mathbb{R}$, and $\varphi \in C^\infty(M)$. Then*

$$\| [f(t^d D), \varphi] \| = \mathcal{O}(t).$$

Proof. We have,

$$[f(t^d D), \varphi] = -t^d f(t^d D)[D, \varphi]f(t^d D).$$

The commutator $[D, \varphi]$ is of Heisenberg order $(d - 1)$, so $\| [D, \varphi]f(t^d D) \| = \mathcal{O}(t^{d-1})$ by Lemma 93. Also, $\|f(t^d D)\| \leq 1$.

□

6.4 The analytic index map for subelliptic operators.

In this section we construct the analytic index map in K-theory

$$K_0(C^*(T_H M)) \to \mathbb{Z}$$

that sends the Heisenberg symbol $[\sigma_H(P)]$ to the analytic index of the subelliptic operator P. The idea is the same as that used by Connes for elliptic operators (in [Co]). The technical result that makes the idea work for subelliptic operators is Proposition 89, proven in the previous section.

The restriction map π_0 at $t = 0$ gives rise to a short exact sequence,

$$0 \;\to\; C_0(\,(0,1],\,\mathcal{K}) \;\to\; C^*(\mathbb{T}_H M) \;\xrightarrow{\pi_0}\; C^*(T_H M) \;\to\; 0.$$

(See section 5.8.)

Lemma 95 *The C^*-algebra $C^*((0,1],\mathcal{K})$ is contractible (to zero).*

Proof. Recall that a contraction of a C^*-algebra A is a $*$-homomorphism,

$$A \to C([0,1], A) \colon a \mapsto [s \mapsto a_s],$$

such that $a_0 = a, a_1 = 0$ for all $a \in A$. For $f \in A = C_0((0,1],\mathcal{K})$ define,

$$f_s(t) = f(t - s)$$

for $s \in [0,1]$, $t \in (s,1]$. If $t[0,s]$ we let $f_s(t) = 0$.

This is a contraction of $C_0((0,1],\mathcal{K})$. We only need to show that $s \mapsto f_s$ is continuous. Any continuous function $f \colon [0,1] \to \mathcal{K}$ with $f(0) = 0$ is *uniformly* continuous. Therefore, for $\varepsilon > 0$, there is $\delta > 0$, such that

$$||f_s - f_{s'}||_{C_0((0,1],\mathcal{K})} = \sup_t ||f(t-s) - f(t-s')||_{\mathcal{K}} < \varepsilon,$$

whenever $|s - s'| < \delta$.

$\square$

Corollary 96 *The restriction map $\pi_0 \colon C^*(\mathbb{T}_H M) \to C^*(T_H M)$ induces an isomorphism in K-theory.*

Proof. This follows immediately from the exact sequence in K-theory for the quotient map π_0.

$\square$

All the restriction maps $\pi_t \colon C^*(\mathbb{T}_H M) \to \mathcal{K}(L^2(M))$, for $t \neq 0$, are homotopic in the sense that $t \mapsto \pi_t(a)$ is continuous for each $a \in C^*(\mathbb{T}_H M)$ (by Lemma 84). Therefore, each π_t induces the same map in K-theory,

$$\pi_t \;:\; K_0(C^*(\mathbb{T}_H M)) \to K_0(\mathcal{K}) \cong \mathbb{Z}.$$

Definition 97 *The composition* $\mathrm{Ind}_H = \pi_1 \circ \pi_0^{-1}$,

$$\begin{tikzcd}
K_0(C^*(\mathbb{T}_H M)) \arrow[d, "\pi_0", "\cong"'] \arrow[dr, "\pi_1"] & \\
K_0(C^*(T_H M)) \arrow[r, "\mathrm{Ind_H}"'] & \mathbb{Z}
\end{tikzcd}$$

is called the deformation index *for* (M, H).

Proposition 89 immediately implies the following theorem.

Theorem 98 *Let (M, H) be a compact contact manifold, and P a maximally hypoelliptic operator on M. Then,*

$$\operatorname{Index} P = \operatorname{Ind}_H\left([\sigma_H(P)]\right).$$

In other words, Ind_H *is the analytic index map for maximally hypoelliptic operators.*

Proof. Proposition 89 implies that there is a well-defined unitary,

$$U = (\mathbb{D} + i)(\mathbb{D} - i)^{-1} = 1 - 2i(\mathbb{D} - i)^{-1} \in \mathcal{M}_k(C^*(\mathbb{T}_H M)^+),$$

and a projection,

$$\frac{1}{2}(\epsilon U + 1) \in \mathcal{M}_k(C^*(\mathbb{T}_H M)^+).$$

We thus obtain a K-theoretic 'index' for the invariant family $\mathbb{P} = \langle P_m, t^d P \rangle$,

$$[\mathbb{P}] = [\frac{1}{2}(\epsilon U + 1)] - [\frac{1}{2}(\epsilon + 1)] \in K_0(C^*(\mathbb{T}_H M)).$$

Its restrictions to $t = 0$ and $t = 1$ are precisely the two K-theory classes constructed in Chapter 4,

$$\begin{aligned} \pi_0([\mathbb{P}]) &= [\sigma_H(P)] \in K_0(C^*(T_H M)), \\ \pi_1([\mathbb{P}]) &= \operatorname{Index} P \in K_0(\mathcal{K}(L^2(M))) \cong \mathbb{Z}, \end{aligned}$$

which proves,

$$\operatorname{Index} P = \pi_1 \pi_0^{-1}([\sigma_H(P)]).$$

□

6.5 The topological index of a subelliptic operator.

We can now prove our index theorem for subelliptic operators.

Theorem 99 *Let (M, H) be a compact contact manifold, and $P\colon \Gamma(E) \to \Gamma(F)$ a maximally hypoelliptic differential operator on M. Then*

$$\operatorname{Index} P = \int_{T^*M} \operatorname{Ch}(\Psi([\sigma_H P])) \wedge \operatorname{Td}(M).$$

In other words, the analytic index of P is computed by the topological index formula of Atiyah and Singer. Here

$$\Psi : \ K_0(C^*(T_H M)) \xrightarrow{\cong} K^0(T^*M)$$

denotes a *canonical* isomorphism in K-theory that we will define below.

Our proof of the index theorem relies on an enlargement of the parabolic tangent groupoid by introduction of a second parameter $s \in [0,1]$. The larger groupoid is called the *adiabatic groupoid* of $\mathbb{T}_H M$ (see, for example, [Ni2]). Recall that we can think of $\mathbb{T}_H M$ as a family of groupoids over the unit interval $[0,1]$, where at $t = 0$ we have the bundle of osculating groups $T_H M$, while at each $t > 0$ we have a copy of $M \times M$. Algebraically, the adiabatic groupoid $\mathbb{T}_H M^{ad}$ is the union of a family of groupoids $\mathcal{G}_{(t,s)}$ parametrized by $(t,s) \in [0,1]^2$, and defined as follows:

$$
\begin{aligned}
\mathcal{G}_{(t,s)} &= M \times M \text{ for } t > 0, s > 0,\\
\mathcal{G}_{(0,s)} &= T_H M, \text{ for } s > 0\\
\mathcal{G}_{(t,0)} &= TM, \text{ for } t > 0\\
\mathcal{G}_{(0,0)} &= H \oplus N.
\end{aligned}
$$

Since each groupoid $\mathcal{G}_{(t,s)}$ has unit space M, the unit space of the adiabatic groupoid $\mathbb{T}_H M^{ad}$ is the manifold with corners $M \times [0,1] \times [0,1]$. We will define the smooth structure on $\mathbb{T}_H M^{ad}$ indirectly by constructing its Lie algebroid, but before we turn to the algebroid let us indicate the essential features of the groupoid structure. Along each horizontal or vertical segment in the square $[0,1]^2$ we have a family of copies of the same groupoid, with a single exceptional groupoid at $t = 0$ or $s = 0$. In the horizontal segments the groupoids along $(0,1] \times \{s\}$ are blown up using Heisenberg dilations $\delta_t^{-1}, t \in (0,1]$ and glued to the exceptional groupoid at $(0,s)$. In the vertical direction we simply scale by a factor $s^{-1}, s \in (0,1]$ to blow up the groupoids along $\{t\} \times (0,1]$ and glue them to the $(t,0)$ groupoid. At each horizontal segment $[0,1] \times \{s\}$ for positive $s > 0$ we get a copy of the parabolic tangent groupoid $\mathbb{T}_H M$. At each vertical segment $\{t\} \times [0,1]$ with positive $t > 0$ we have the usual tangent groupoid $\mathbb{T}M$. We will see below what happens at the exceptional edges $[0,1] \times \{0\}$ and $\{0\} \times [0,1]$.

Schematically:

$$
\begin{tikzcd}
(t,s) = (0,0) \arrow[r] & H \oplus N \arrow[r, dotted, dash, "\delta_t^{-1}"] \arrow[d, dotted, dash, "s^{-1}"'] & TM \arrow[d, dotted, dash, "s^{-1}"', "\mathbb{T}M"] & \\
& T_H M \arrow[r, dotted, dash, "\mathbb{T}_H M", "\delta_t^{-1}"'] & M \times M & (t,s) = (1,1) \arrow[l]
\end{tikzcd}
$$

We verify that $\mathbb{T}_H M^{ad}$ is a smooth groupoid, by considering its Lie algebroid. We start with the family of Lie algebroids $\mathcal{A}\mathcal{G}_{(t,s)}$ associated to the groupoids $\mathcal{G}_{(t,s)}$ described above:

$$
\begin{aligned}
A\mathcal{G}_{(t,s)} &= TM, \text{for } t > 0,\\
A\mathcal{G}_{(0,s)} &= \mathrm{Lie}(T_H M) = H \oplus N \text{ as the bundle of osculating Lie algebras for } s > 0,\\
A\mathcal{G}_{(0,0)} &= H \oplus N \text{ as a bundle of } \textit{abelian} \text{ Lie algebras for } s = 0.
\end{aligned}
$$

At points with $t > 0$ we have the Lie algebroid TM, whose bracket is the commutator of vector fields. At $t = 0$ we have the Lie algebroid $\mathrm{Lie}(T_H M) = H \oplus N$ of $T_H M$, except at the point $(t,s) = (0,0)$ where we have the vector bundle $H \oplus N$ as the Lie algebroid of the *groupoid* $H \oplus N$, which is just a bundle of abelian groups.

To describe the smooth structure of $\mathcal{A} = \cup A\mathcal{G}_{(t,s)}$ we choose a section $N \subseteq TM$ and identify $H \oplus N \cong TM$. As a smooth vector bundle over $M \times [0,1]^2$, the Lie algebroid $\mathcal{A}$ is isomorphic to the bundle $TM \times [0,1]^2$ by the identification,

$$\phi\,:\,TM \times [0,1]^2 \to \cup A\mathcal{G}_{(t,s)},$$

given by a combination of parabolic dilations δ_t and scaling by s:

$$\begin{aligned}
&\phi(m,v;t,s) = (m, s\delta_t v) \in T_mM,\ t>0, s>0,\\
&\phi(m,v;t,0) = (m, \delta_t v) \in T_mM,\ t>0,\\
&\phi(m,v;0,s) = (m, sv) \in H \oplus N,\ s>0,\\
&\phi(m,v;0,0) = (m, v) \in H \oplus N.
\end{aligned}$$

One easily verifies that the bracket operation and anchor map on $\mathcal{A} = \cup A\mathcal{G}_{(t,s)}$ are smooth. Lie's Third Theorem for stratified Lie algebroids implies that the Lie groupoid $\mathbb{T}_H M^{ad}$ has a unique smooth structure such that $\mathcal{A}$ is its Lie algebroid.

The groupoid $\mathbb{T}_H M^{ad}$ gives rise to a commutative diagram in K-theory, induced by restriction of functions on $\mathbb{T}_H M^{ad}$ to each of the four corners of the square $[0,1]^2$. We proceed step-by-step.

Lemma 100 *Restriction of elements in $C^*(\mathbb{T}_H M^{ad})$ to the $(t,s) = (0,0)$ corner,*

$$e_{(0,0)}\,:\,C^*(\mathbb{T}_H M^{ad}) \to C^*(H \oplus N) \cong C_0(H^* \oplus N^*),$$

induces an isomorphism in K-theory.

Proof. Let $\mathcal{G}$ denote the groupoid that is the union of the $t=0$ and $s=0$ edges in $\mathbb{T}_H M^{ad}$. The restriction map $C^*(\mathbb{T}_H M^{ad}) \to \mathcal{G}$ induces an isomorphism in K-theory, because the kernel of this map is the contractible ideal $C_0((0,1]^2, \mathcal{K})$. The kernel of the map that further restricts $\mathcal{G}$ to the corner $(t,s) = (0,0)$ is again a contractible ideal.

□

Now let $e_{(t,1)}$ denote restriction to the edge $s = 1$,

$$e_{(t,1)}\,:\,C^*(\mathbb{T}_H M^{ad}) \to C^*(\mathbb{T}_H M),$$

and $e_{(1,s)}$ restriction to the edge $t=1$,

$$e_{(1,s)}\,:\,C^*(\mathbb{T}_H M^{ad}) \to C^*(\mathbb{T}M).$$

Further restriction to the corner $(t,s) = (1,1)$ gives two $*$-homomorphisms,

$$\begin{aligned}
&e_{(1,1)}\,:\,C^*(\mathbb{T}_H M) \to C^*(M \times M)\\
&e'_{(1,1)}\,:\,C^*(\mathbb{T}M) \to C^*(M \times M).
\end{aligned}$$

We obtain a commutative diagram,

$$\begin{tikzcd}
K_0(C^*(\mathbb{T}_H M^{ad})) \arrow[r, "e_{(1,s)}"] \arrow[d, "e_{(t,1)}"] & K_0(C^*(\mathbb{T}M)) \arrow[d, "e'_{(1,1)}"] \\
K_0(C^*(\mathbb{T}_H M)) \arrow[r, "e_{(1,1)}"'] & K_0(C^*(M \times M))
\end{tikzcd}$$

But $e_{(1,1)}$ is just our deformation index Ind_H, while $e'_{(1,1)}$ is the deformation index associated to the usual tangent groupoid. As is known from the tangent groupoid proof of the Atiyah-Singer index theorem for elliptic operators, this map $e'_{(1,1)}$ is equal to the topological index $\mathrm{Ind_t}$ (see [Co], [Hi]). Therefore, the diagram above can be read as follows,

$$\begin{array}{ccc} K^0(H^* \oplus N^*) & \xrightarrow{e_{(1,s)}} & K^0(T^*M) \\ \Big\downarrow e_{(t,1)} & & \Big\downarrow \mathrm{Ind_t} \\ K_0(C^*(T_HM)) & \xrightarrow[\mathrm{Ind_H}]{} & \mathbb{Z} \end{array}$$

We will see that the remaining maps $e_{(1,s)}$ and $e_{(t,1)}$ are natural isomorphisms. The map $e_{(1,s)}$ is trivial.

Lemma 101 *The map*

$$e_{(1,s)} \,:\, K^0(H^* \oplus N^*) \to K^0(T^*M)$$

in the diagram above is the isomorphism induced by an arbitrary identification $H^* \oplus N^* = T^*M$, arising from the choice of a section $N \subseteq TM$.

Proof. Let the groupoid $\mathcal{G}$ denote the restriction of $\mathbb{T}_H M^{ad}$ to the $s = 0$ edge. It is the union of $H \oplus N$ at $t = 0$ with the groupoid $TM \times (0, 1]$. Even though $TM \times (0, 1]$ is glued to $H \oplus N$ by means of Heisenberg dilations, the resulting groupoid $\mathcal{G}$ is isomorphic to $TM \times [0, 1]$.

The map $e_{(1,s)}$ is induced by restriction to $t = 1$ in $\mathcal{G} \cong TM \times [0, 1]$, composed with the inverse (in K-theory) of restriction at $t = 0$. By homotopy invariance of K-theory, this is the identity map.

□

The last step is far from trivial.

Proposition 102 *The map*

$$e_{(t,1)} \,:\, K^0(H^* \oplus N^*) \to K_0(C^*(T_HM))$$

is an isomorphism in K-theory.

This is a special case of Lemma 3 in [Ni2]. We give a brief sketch of the idea of the proof. For every nilpotent (or even solvable) group G there exists a (non-unique) split exact sequence,

$$0 \to G' \to G \to \mathbb{R} \to 0.$$

Therefore G is a semi-direct product of groups $G \cong G' \rtimes \mathbb{R}$, and we get a crossed product

$$C^*(G) \cong C^*(G') \rtimes \mathbb{R}.$$

Thus, by the Connes–Thom isomorphism for crossed products with $\mathbb{R}$ we have

$$K_0(G) \cong K_0(C^*(G') \rtimes \mathbb{R}) \cong K(C^*(G') \otimes C_0(\mathbb{R})).$$

(See [Co1] for the Connes–Thom isomorphism.) By induction we obtain

$$K_0(C^*(G)) \cong K^0(\mathbb{R}^n),$$

where n is the (ordinary) dimension of G. It is not hard to show that the map $e_{(t,1)}$ restricts to the Connes–Thom isomorphism for the osculating group $T_H M_m$ in each of the fibers of $T_H M$, where $\mathbb{R}^n$ is naturally identified with the Lie algebra $H_m \oplus N_m$ of the group. Once this is established, the proof of Proposition 102 is completed by a Mayer–Vietoris argument. (See [Ni2] for details.)

Lemma 101 and Proposition 102 combined gives us a natural isomorphism,

$$\Psi = e_{(1,s)} \circ e_{(t,1)}^{-1} \; : \; K_0(C^*(T_H M)) \xrightarrow{\cong} K^0(T^*M),$$

and our commutative diagram reduces to

$$\begin{tikzcd} & K^0(T^*M) \arrow[d, "\mathrm{Ind_t}"] \\ K_0(C^*(T_H M)) \arrow[ur, bend left, "\Psi"] \arrow[r, "\mathrm{Ind_H}"'] & \mathbb{Z} \end{tikzcd}$$

In particular,

$$\mathrm{Ind}_H([\sigma_H P]) = \mathrm{Ind_t}(\Psi([\sigma_H P])).$$

In combination with Theorem 98 this proves our main Theorem 99.

Appendices

Appendix A

Subelliptic estimates for Rockland operators.

A.1 Introduction.

In this appendix we outline the proof of a theorem of Helffer and Nourrigat [HN] concerning subelliptic estimates for Rockland operators. Our aim is to convince the reader that the proof of their theorem, which applies to *scalar* Rockland operators, generalizes to operators that act on sections in a trivial vector bundle. We will frequently refer to parts of the paper by Helffer and Nourrigat [HN], and assume that the reader has a copy of it at hand. Section A.3 below corresponds to the content of section 6 in [HN], which contains the proof of the main theorem, while sections A.4 and A.5 below contain the crucial technical results from sections 2 and 4 in [HN] respectively, indicating what needs to be changed to make the proofs work for vector bundle operators. None of the changes is fundamental.

In what follows, G denotes a graded nilpotent group, with graded Lie-algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$, and L denotes a left-invariant differential operator G. In [HN] the operator L is a scalar operator, while we assume here that L acts on smooth sections in a trivial complex vector bundle $G \times \mathbb{C}^N$. One can think of L as an $N \times N$ matrix (L_{ij}), where each L_{ij} is a left-invariant scalar operator.

Recall that if π is a unitary representation of G on Hilbert space $\mathcal{H}_\pi$, then π induces an (unbounded) representation of the Lie-algebra $\mathfrak{g}$, by

$$\pi(X)u = \frac{d}{dt}\bigg|_{(t=0)} \pi(\exp(tX))u,$$

for $X \in \mathfrak{g}$. The domain of $\pi(X)$ consists of vectors $u \in \mathcal{H}_\pi$ for which the limit converges. This extends to a representation (still denoted π) of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. An element in $\mathcal{H}_\pi$ that is in the domain of all the operators $\pi(A), A \in \mathcal{U}(\mathfrak{g})$ is called a *smooth vector* of π. The space of smooth vectors is denoted by $\mathcal{S}_\pi$. (In standard presentations of *irreducible* representations π, $\mathcal{H}_\pi = L^2(\mathbb{R}^k)$ for some k, and $\mathcal{S}_\pi$ is precisely of the Schwartz class on $\mathbb{R}^k$.)

If L is a left-invariant operator on G with matrix valued coefficients, we may think of L as an element in the algebra $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{M}_N(\mathbb{C})$, while $\pi(L)$ is an unbounded operator on the Hilbert space $\mathcal{H}_\pi \otimes \mathbb{R}^N = \mathcal{H}_\pi^{\oplus N}$ with domain $\mathcal{S}_\pi \otimes \mathbb{R}^N = \mathcal{S}_\pi^{\oplus N}$. If L is the matrix (L_{ij}), then simply $\pi(L) = (\pi(L_{ij}))$. L is homogeneous of order m if all L_{ij} are homogeneous of order m.

Definition 103 *A differential operator L acting on sections in a trivial bundle $G \times \mathbb{C}^N$ is called a* Rockland operator*, if it is left-invariant, homogeneous, and for every non-trivial irreducible unitary representation $\pi \in \hat{G}$ the operator $\pi(L)$ is injective on $\mathcal{S}_\pi^{\oplus N}$.*

The theorem that concerns us here is the following ([HN], proposition 6.4).

Proposition 104 *Let m be a common multiple of $1, 2, \ldots, r$ which is larger than r^r. If L is a Rockland operator of degree m, then there is a constant $C > 0$ such that,*

$$\|u\|_{H^m} \leq C \left(\|Lu\|_{L^2} + \|u\|_{L^2}\right),$$

for all sections $u \in \mathcal{S}(G, \mathbb{C}^N)$.

This result implies hypoellipticity of *any* Rockland operator L, because the estimates imply hypoellipticity for L^k, for a suitable k.

Remark. The converse is also true. For left invariant homogeneous operators on graded groups, the Rockland condition is *equivalent* to hypoellipticity. The theorem, in this form, was first conjectured by Rockland ([Ro]), who proved it for the case of the Heisenberg group. Necessity of the Rockland condition was proven by Beals ([Be]). (We have not verified necessity of the Rockland condition for hypoelliptic vector bundle operators, since it is not relevant for our purposes. However, we strongly suspect that there is no problem there.)

We outline the proof found in [HN], in order to show that it applies, with minimal changes, to vector bundle operators. In broad outlines, the proof proceeds as follows. We assume, inductively, that the estimates hold for groups of length $r-1$. In particular, it is assumed to be true for the group G/G_r, where $G_r = \exp(\mathfrak{g}_r)$. Throughout the proof, various other inductions are carried out. Starting from the Rockland condition, i.e., injectivity of $\pi(L)$ for irreducible π, and the induction hypothesis, an inequality of the general type

$$\sum \|\pi(A_i)u\| \leq C \sum \|\pi(L)u\|$$

is derived, where, in this case, A_i ranges over a basis of the order m part of $\mathcal{U}(\mathfrak{g})$. One introduces a more general class of representations $\pi_{(\xi,V)}$, that includes, at one extreme, the irreducible representations (where V is a 'large' subalgebra of $\mathfrak{g}$), and at the other extreme the regular representation (where $V = 0$). It is shown (inductively) that the same inequality holds for all such representations, as long as V contains $\mathfrak{g}_r$. From this result one arrives in one more step at the desired inequality for the regular representation (corresponding to the case $V = 0$).

The main technique used in $[HN]$ is the *explicit* presentation of the representations of G on spaces $L^2(\mathbb{R}^k)$, where each $X \in \mathfrak{g}$ is presented as a first order differential operator with polynomial coefficients. These explicit computations reveal precise relations between various different representations, that provide the main inductive technique ('reduction of the number of variables') in the paper. We will not describe these explicit computations here, but will accept the main technical lemmas that are derived from it.

A.2 The representations $\pi_{(\xi,V)}$.

In Kirillov theory, the irreducibel representations of the nilpotent group G are constructed from elements in the dual $\mathfrak{g}^*$ of the Lie algebra. To every $\xi \in \mathfrak{g}^*$ is associated a bilinear form,

$$B_\xi(X, Y) = \langle \xi, [X, Y] \rangle,$$

for $X, Y \in \mathfrak{g}$. It is shown that one can always find a subalgebra $V \in \mathfrak{g}$, which is *maximally isotropic* for B_ξ, i.e., $V = V^\perp$. Restricting ξ to a homomorphism $V \to \mathbb{R}$, there is a corresponding character of the subgroup $G_0 = \exp(V) \subseteq G$,

$$G_0 \to U(1) : \ \exp(v) \to e^{i\langle v,\xi\rangle}.$$

The representation of G induced by this character of G_0 is irreducible, and all irreducible unitary representations are obtained in this way. (Furthermore, the irreducible representations of G associated to two different functionals in $\mathfrak{g}^*$ are unitarily equivalent if and only if these functionals are in the same co-adjoint orbit.)

To carry out various inductive processes, Helffer and Nourrigat introduce a larger class of representations, by starting with a pair (ξ, V), where $V \in \mathfrak{g}$ is a now an arbitrary subalgebra, and $\xi \in V^*$. The representation of G induced by the character $G_0 \to U(1)$ that corresponds to ξ is denoted by $\pi_{(\xi,V)}$. To make these representations more explicit, one can always choose a 'good' transversal S of V in $\mathfrak{g}$, such that the diffeomorphism,

$$\phi : \ V \times S \to G \ ; \ (v, x) \mapsto \exp(v)\exp(x),$$

identifies the coset space $G_0 \backslash G$ with S, and in such a way that Lebesgue measure on $S \subseteq \mathfrak{g}$ corresponds to the G-invariant measure on $G_0\backslash G$. With this setup, the induced representation on $L^2(S)$ is expressed by

$$\pi_{(\xi,V)}(g)f(x) = e^{i\langle v,\xi\rangle} f(\sigma),$$

where $v = v(x, g) \in V$ and $\sigma = \sigma(x, g) \in S$ are uniquely determined by

$$\exp(x)g = \exp(v)\exp(\sigma),$$

for given $x \in S, g \in G$. Observe that v and σ are polynomials in $(x, g) \in S \times G$. It easily follows that each $\pi(X)$, for $X \in \mathfrak{g}$, is a first order differential operator on S with polynomial coefficients. Many of the results in [HN] rely on an explicit working out of what the operators $\pi(X)$ look like, by strategically choosing convenient subalgebras V and transversals S.

Associated to a representation π of G there are Sobolev type subspaces of the representation space $\mathcal{H}_\pi$. For a non-negative integer m, the space H^m_π is defined as the space of vectors in $u \in \mathcal{H}_\pi$ for which $\pi(L)u \in \mathcal{H}_\pi$, for all $L \in \mathcal{U}(\mathfrak{g})$ of homogeneous order less or equal m. Choosing a basis $\{Y_i\}$ for $\mathfrak{g}$, a Hilbert space norm on H^m_π is given by,

$$\|u\|^2_{H^m_\pi} = \sum_{d(\alpha)\leq m} \|\pi(Y^\alpha)u\|^2.$$

If we take for π the right regular representation on $L^2(G)$, then H^m_π is just the weighted Sobolev space $H^m(G)$.

Remark. In [HN], the spaces H^m_π are denoted by $H^m_{(\xi,V)}$ for the representations $\pi = \pi_{(\xi,V)}$. The notation $H^m_{(0)}$ in [HN] refers to the spaces associated to the right regular representation (which is 'induced' by the trivial representation of the trivial subgroup, hence $V = \{0\}$).

A.3 Proof of the subelliptic estimates.

We must prove the existence of an inequality

$$\|u\|_{H^m} \leq C\left(\|Lu\| + \|u\|\right),$$

for all $u \in \mathcal{S}(G)^{\oplus N}$. The proof, given in section 6 of [HN], proceeds by induction on the length r of the grading of $\mathfrak{g}$, assuming that the result is true for graded groups with length of grading $r-1$.

The first step involves an application of the following result.

Proposition 105 *Let G be a graded group, with grading of length r. If m is an integer multiple of $1, 2, \ldots, r$, then, for each $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$, such that*

$$\|u\|_{H^{m-1}} \leq \varepsilon\|u\|_{H^m} + C_\varepsilon\|u\|,$$

for all $u \in \mathcal{S}(G, \mathbb{C}^N)$.

Proof. For scalar functions u, this is Proposition 4.6.1 in [HN]. If

$$u = (u_1, \ldots, u_N) \in \mathcal{S}(G, \mathbb{C}^N) = \mathcal{S}(G)^{\oplus N}$$

is a section in the bundle $G \times \mathbb{C}^N$, then the obvious equality $\|u\|^2_{H^m} = \sum_i \|u_i\|^2_{H^m}$ immediately implies the generalized result.

□

By definition,

$$\|u\|^2_{H^m} = \sum_{d(\alpha)=m} \|Y^\alpha u\|^2 + \|u\|^2_{H^{m-1}}.$$

Proposition 105 implies that the H^m norm is equivalent to

$$\sum_{d(\alpha)=m} \|Y^\alpha u\|^2 + \|u\|^2.$$

Therefore, in order to prove Proposition 104, it suffices to prove the inequality

$$\|Au\| \leq C\|Lu\|,$$

for any $A \in \mathcal{U}(\mathfrak{g})$ homogeneous of order m. The constant $C > 0$ may depend on A. For the next step, we invoke the following technical result.

Proposition 106 *Let V_1, V_2 be two subalgebras of $\mathfrak{g}$ such that*

$$[V_2, V_2] \subseteq V_1 \subseteq V_2,$$

and let $\xi_1 \in V_1^*$ be a functional that vanishes on commutators $[V_2, V_2]$.

Given two finite families $(A_i)_{i\in I}$, $(B_j)_{j\in J}$ of elements in $\mathcal{U}(\mathfrak{g})$, and a constant $C > 0$. Then, if the inequality

$$\sum_{i\in J} \|\pi(A_i)u\| \leq C \sum_{j\in J} \|\pi(B_j)u\|,$$

with $\pi = \pi_{(\xi_1, V_1)}$ holds for all $u \in \mathcal{S}_\pi$, then the same inequality holds for $\pi = \pi_{(\xi_2, V_2)}$, where $\xi_2 \in V_2^*$ is an arbitrary extension of ξ_1. Conversely, if the inequality holds for all extensions $\xi_2 \in V_2^*$ of $\xi_1 \in V_1^*$, with the same constant $C > 0$, then it also holds for ξ_1.

The proof of this proposition is outlined in section A.4.

Taking $V_1 = \{0\} \subseteq V_2 = \mathfrak{g}_r$, the proposition shows that the inequality $\|Au\| \leq C\|Lu\|$ follows if we can show that

$$\|\pi(A)u\| \leq C\|\pi(L)u\|,$$

for all representations $\pi = \pi_{(\xi,\mathfrak{g}_r)}$, $\xi \in \mathfrak{g}_r^*$, for a constant $C > 0$ that is independent of ξ.

The case $\xi = 0$ is easily dealt with. The representation $\pi_{(0,\mathfrak{g}_r)}$ of G is induced by the trivial representation of the normal subgroup $G_r = \exp(\mathfrak{g}_r)$. The space $\mathcal{H}_\pi$ identifies with the space of functions on G/G_r, and $\pi_{(0,\mathfrak{g}_r)}$ factors through the regular representation of the quotient group G/G_r. This group has a grading of length (at most) $r-1$, and the inequality is true by the induction hypothesis.

So it suffices to show the inequality for $\xi \neq 0$. Because A, L are both homogeneous of degree m, is suffices to show *(why ?!)* the inequality for all $\xi \in \mathfrak{g}_r^*$ with $\|\xi\| = 1$. This, in turn, would follow from

$$\|u\|_{H_\pi^m} \leq C\|\pi(L)u\|,$$

for $\pi = \pi_{(\xi,\mathfrak{g}_r)}$, $\|\xi\| = 1$.

The constant C appearing in this inequality must be independent of ξ. Next we show that it suffices to find a constant $C > 0$ for each ξ separately. This is achieved by showing that, if the inequality holds for some $\xi_0 \in \mathfrak{g}_r^*$, then it holds (perhaps for a larger constant $C > 0$) in a neighborhood of ξ_0. Compactness of the unit sphere in $\mathfrak{g}_r^*$ then gives a uniform estimate for all ξ. What allows us to make this step is the following result. We introduce here the convenient notation,

$$\mathfrak{g}^p = \mathfrak{g}_p \oplus \cdots \oplus \mathfrak{g}_r \ ; \ V_p = V \cap \mathfrak{g}^p,$$

for a subalgebra $V \subseteq \mathfrak{g}$. If $V \subseteq \mathfrak{g}^p$, we get a chain of subalgebras

$$V = V_p \supseteq V_{p+1} \supseteq \cdots \supseteq V_r.$$

Proposition 107 *Let $V \subseteq \mathfrak{g}$ be a subalgebra that is contained in $\mathfrak{g}^p$, and let $\xi_0 \in V^*$. Suppose that for two finite families $(A_i)_{(i\in I)}, (B_j)_{(j\in J)}$ of elements in $\mathcal{U}(\mathfrak{g})$ we have the inequality*

$$\sum_{i\in I} \|\pi_{(\xi_0,V)}(A_i)u\| \leq C \sum_{j\in J} \|\pi_{(\xi_0,V)}(B_j)u\|.$$

Suppose that all A_i, B_j are of order less or equal m. Then there is a constant $C' > 0$ such that, for every $0 < \varepsilon \leq 1$ there is a neighborhood $U \subseteq V^*$ of ξ_0, so that every $\xi \in U$ for which $\xi = \xi_0$ on V_{p+1}, satisfies

$$\sum_{i\in I} \|\pi_{(\xi,V)}(A_i)u\|^2 \leq C' \left(\sum_{j\in J} \|\pi_{(\xi,V)}(B_j)u\|^2 + \varepsilon\|u\|^2_{H_\pi^{m-1}} \right).$$

The proof of the proposition is dicussed in section A.5.

This proposition shows that if we have the inequality

$$\|u\|_{H^m_\pi} \leq C\|\pi(L)u\|,$$

for $\pi = \pi_{(\xi_0,\mathfrak{g}_r)}$, then, for $\pi = \pi_{(\xi,\mathfrak{g}_r)}$ with ξ in some neighborhood of ξ_0 we have the inequality

$$\|u\|_{H^m_\pi} \leq C'\left(\|\pi(L)u\| + \|u\|_{H^{m-1}_\pi}\right).$$

Proposition 105 then implies the existence of a constant C'' such that $\|u\|_{H^m_\pi} \leq C''\|\pi(L)u\|$ for ξ near ξ_0.

To sum up what has been achieved so far. In order to prove Proposition 104, it suffices to show that, for every representation $\pi_{(\xi,V)}$, $V = \mathfrak{g}_r$, with $\|\xi\| = 1$, there is a constant $C > 0$ (dependent on $\xi \in V^*$), such that

$$\|u\|_{H^m_\pi} \leq C\|\pi(L)u\|,$$

for all $u \in \mathcal{S}_\pi$.

The proof of these estimates will be achieved inductively. We first prove that these inequalities hold for arbitrary V for which $\pi_{\xi,V)}$ is irreducible. An inductive process will then derive the desired inequalities for $V = \mathfrak{g}_r$, which finishes the proof of Proposition 104.

Proposition 108 *Let $\pi_{(\xi,V)}$ be irreducible (which implies $\mathfrak{g}_r \subseteq V$). Suppose that ξ, when restricted to $\mathfrak{g}_r$, is of norm 1. Then, if L is a Rockland operator, there exists a constant $C > 0$, such that*

$$\|u\|_{H^m_\pi} \leq C\|\pi_{(\xi,V)}(L)u\|,$$

for every $u \in H^m_\pi$.

Proof. As a first step, we claim that there exists a $C > 0$ such that,

$$\|u\|_{H^m_\pi} \leq C(\|\pi_{(\xi,V)}(L)u\| + \|u\|),$$

for $u \in \mathcal{S}_\pi$. In fact, this is true for every subalgebra $V \subseteq \mathfrak{g}$ that contains $\mathfrak{g}_r$,and $\xi \in V^*$ for which the restriction to $\mathfrak{g}_r$ is of norm less or equal 1.

First, consider $V = \mathfrak{g}_r$, and $\xi = 0$. As we have seen before, $\pi_{(0,\mathfrak{g}_r)}$ factors through the regular representation of the group G/G_r, and $\pi(L)$ can be identified with a Rockland operator of order m on G/G_r. The desired inequalities are assumed as part of the induction hypothesis of Proposition 104.

Next, consider $V = \mathfrak{g}_r$ with arbitrary $\xi \in V^*$. Proposition 107 with $\xi_0 = 0$ implies that there is a constant $C > 0$ for which

$$\|u\|_{H^m_\pi} \leq C(\|\pi(L)u\| + \|u\|_{H^{m-1}_\pi}),$$

for all $\xi \in V^*$ in some neighborhood of the origin. We may assume that $\|\xi\| \leq 1$. Now applying Proposition 105 we get the desired estimates.

Finally, let $V \subseteq \mathfrak{g}$ be an arbitrary subalgebra that contains $\mathfrak{g}_r$, and $\xi \in V^*$ such that ξ vansihes on commutators $[V,V]$. (This is certainly the case if V is maximally isotropic for the bilinear form $\langle \xi, [X,Y] \rangle$.) One successively applies Proposition 106 to the chain of subalgebras

$$\mathfrak{g}_r = V_r \subseteq V_{r-1} \subseteq \cdots \subseteq V_1 = V.$$

This proves the claim.

Now, pick a 'good' transversal S for V in $\mathfrak{g}$. If $V = \mathfrak{g}$, the representation $\pi_{(\xi,V)}$ is scalar, and the inequality is trivial. We assume therefore that $V \neq \mathfrak{g}$, and thus that S is larger than a point. In this case proposition 1.5.1 in [HN] states that the space of smooth vectors $\mathcal{S}_\pi$ is dense in H^m_π. The inequality we have just proven therefore extends to $u \in H^m_\pi$.

A classical lemma by Peetre states that, if E and F are reflexive Banach spaces, such that $E \subseteq F$ is a compact inclusion, and $L\colon E \to F$ is a bounded linear operator, then the image of L is closed and its kernel is finite dimensional if and only if there exists a constant $C > 0$ such that

$$\|u\|_E \leq C(\|Lu\|_F + \|u\|_F),$$

for $u \in E$ ([LM], p.171). Now, $\pi(L)\colon H^m_\pi \to \mathcal{H}_\pi$ is bounded, and, by proposition 5.5.2 in [HM], the inclusion $H^m_\pi \subseteq \mathcal{H}_\pi$ is compact if $m \geq r^r$. The inequalities we have proven therefore imply that the operator

$$\pi_{(\xi,V)}(L) \,:\, H^m_\pi \to \mathcal{H}_\pi$$

has closed image in $\mathcal{H}_\pi$. Moreover, by proposition 5.7.1 in [HN1], these same inequalities imply that, if $u \in H^m_\pi$, and $\pi(L)u \in \mathcal{S}_\pi$, then $u \in \mathcal{S}_\pi$. In particular, injectivity of $\pi(L)$ on $\mathcal{S}_\pi$ implies injectivity on H^m_π. (In [HN1] this fact is proven for scalar operators L, but the proof given there generalizes trivially to the case of vector bundle operators.) The open mapping theorem for $\pi(L)$ now gives us the desired estimate.

□

To complete the proof of Proposition 104 one proves that the inequality of Proposition 108 holds, not only for irreducible $\pi_{(\xi,V)}$, but for a more general class of representations that Helffer and Nourrigat call *maximal of order p*. The case $p = r$ will correspond to the condition $V = \mathfrak{g}_r$, which is what we desire. The case $p = 1$ is the irreducible case, proven in Proposition 108. The induction process proceeds successively from $p = 1$ to $p = r$.

Let $V \subseteq \mathfrak{g}^p$, with $p = 1, \ldots, r$, with subalgebras $V_j = V \cap \mathfrak{g}^j$, as before. Given $\xi \in V$, we call the representation $\pi_{(\xi,V)}$ *maximal of order p* if, for an (arbitrary) extension $\xi_1 \in \mathfrak{g}^*$ of ξ, each subalgebra V_j is maximally isotropic in $\mathfrak{g}^j$ for the bilinear form $B(X,Y) = \langle \xi_1, [X,Y] \rangle$. This implies in particular that $\mathfrak{g}_r \subseteq V$. (Note that for $p = 1$ we have V maximally isotropic in $\mathfrak{g}$, and therefore an irreducible representation $\pi_(\xi, V)$. Of course, 'maximal of order 1' is a stronger condition, but every irreducible representation is equivalent to one that is maximal of order 1.) The induction is carried out in lemma 6.5 of [HN], and is a straightforward application of Propositions 106 and 105, like the one we have already seen. Because nothing changes in the case of vector bundle operators, we refer the reader to [HN] for details.

A.4 Proof of Proposition 106.

In this section we verify that Proposition 106 holds for vector bundle operators. The case for scalar operators is given in [HN] as Proposition 2.1 and Remark 2.2. We follow the argument given in section 2 of [HN].

Starting with subalgebras $V_1 \subseteq V_2 \subseteq \mathfrak{g}$, such that $[V_2, V_2] \subseteq V_1$, one constructs a good transversal S for V_2, and an arbitrary complement T of V_1 in V_2. Then $T \times S$ is a good transversal for V_1.

Let $\mathcal{F}$ denote the Fourier transform in the T variables on $\mathcal{S}(T \times S)$ which induces an isometry $L^2(T \times S) \cong L^2(T^* \times S)$. We have $\xi_1 \in V_1^*$ and an extension $\xi_2 \in V_2^*$. Denote the difference by $\xi' = \xi_2 - \xi_1 \in T^*$. An explicit calculation shows that with these choices,

$$\mathcal{F}\pi_{(\xi_1,V_1)}(A)\mathcal{F}^{-1}u(\xi', s) = \pi_{(\xi_1+\xi',V_2)}(A)u_{\xi'}(s),$$

for $u \in \mathcal{S}(T^* \times S)$, $A \in \mathcal{U}(\mathfrak{g})$, where $u_{\xi'}(s) = u(\xi', s)$. More explicitly, both operators $\mathcal{F}\pi_{(\xi_1,V_1)}(A)\mathcal{F}^{-1}$ and $\pi_{(\xi_1+\xi',V_2)}(A)$ are given by the same expression,

$$a(\xi', s, \partial_s) = \sum_\alpha a_\alpha(\xi', s)\partial_s^\alpha,$$

in which the derivatives $\partial/\partial\xi'$ do not appear. Moreover, the coefficients $a_\alpha(\xi', s)$ are polynomial in (ξ', s). This is a consequence of the explicit way in which the transversals are chosen, together with the assumption that $[V_2, V_2] \subseteq V_1$.

Given this fact, it follows trivially that the same equality holds if $u = (u_i)$ is a section in a trivial bundle, and $A = (A_{ij})$ is a matrix of operators $A_{ij} \in \mathcal{U}(\mathfrak{g})$ acting on such sections, because for any representation π of G, $\pi(A)$ is simply given by the matrix $(\pi(A_{ij}))$.

Proposition 106 is an almost immediate consequence of this equality, and the proof, as given in [HN], applies without change to the case of vector bundle operators.

A.5 Proof of Proposition 107.

In this section we verify that Proposition 107 holds for vector bundle operators. The proof for scalar operators ([HN] Proposition 4.1.1 and Remark 4.2.2) is given in section 4 of [HN], specifically in section 4.5. We indicate the changes that need to be made to the proof given in [HN].

First we show that the vector bundle version of lemma 4.5.1 in [HN] holds. We need to explain some of the setup and notation developed in sections 3 and 4 of [HN].

Let V be a subalgebra of $\mathfrak{g}$. Denoting $G_0 = \exp(V) \subseteq G$, consider the conjugate subgroup $G_h = e^{-h}G_0e^h$, for $h \in \mathfrak{g}$. There is a natural correspondence between the representations induced by the characters of G_0 and G_h. Let V_h denote the Lie algebra of G_h. If $\xi_h \in V_h^*$ is the element that corresponds to $\xi \in V^*$ under the conjugacy $V \cong V_h$, then $\pi(\xi, V)$ and $\pi(\xi_h, V_h)$ are unitarily equivalent.

In order to do explicit calculations, [HN] construct a subspace $S \subseteq \mathfrak{g}$, which can serve as a 'good' transversal for all V_h simultaneously, where V is fixed, and $h \in \mathfrak{g}$ varies. In this way, all the represenations $\pi(\xi_h, V_h)$ can be made explicit on the same space $L^2(S)$. The construction is as follows. Let $I_k \subseteq \mathfrak{g}_k$ be the projection of V on $\mathfrak{g}_k$, i.e.,

$$I_k = \{x \in \mathfrak{g}_k \mid \exists v \in V : \, x - v \in \mathfrak{g}^{k+1}\}.$$

One can then choose arbitrary transversals $S_k \oplus I_k = \mathfrak{g}_k$, and set $S = S_r \times \cdots \times S_1$. This S is then a 'good' transversal for all subalgebras V_h. To describe the induced representations $\pi_{(\xi_h, V_h)}$ on $L^2(S)$, the space S is identified with $G_h \backslash G$ via the diffeomorphism

$$S \to G_0 \backslash G : \ (s_r, \ldots, s_1) \mapsto e^{s_r} \cdots e^{s_1}.$$

The unitary equivalence between $\pi(\xi_h, V_h)$ and $\pi(\xi, V)$ is implemented by a diffeomorphism

$$\tau_h \colon S \to S,$$

that preserves Lebesgue measure on S. This diffeomorphism also preserves the subspaces $S_1 \times \cdots \times S_j$, for $j = 1, \ldots, r$.

We now suppose that $V \subseteq \mathfrak{g}^p = \mathfrak{g}_p \oplus \cdots \oplus \mathfrak{g}_r$, as in Proposition 107. In this case $S_j = \mathfrak{g}_j$ for $j = r, \ldots, p-1$. The proof of Proposition 107 will involve the construction of a particular smooth partition of unity on $\tilde{S} = \mathfrak{g}_r \times \cdots \times \mathfrak{g}_{p-1} \subseteq S$. The functions $\psi \in C_c^\infty(\tilde{S})$ figuring in this partition are thought of as functions on S that only depend on the variables $(x_r, \ldots, x_{p-1})$. Because the diffeomorphisms τ_h preserve $\tilde{S}$, we may define,

$$T_h \psi = \psi \circ \tau_h,$$

for $\psi \in C_c^\infty(\tilde{S})$. We have $T_h^{-1} = T_{-h}$. In [HN] Lemma 4.5.2, a special partition of unity is constructed, that will be important in the proof of the Propostion 107.

One can choose $\psi \in C_c^\infty(\tilde{S})$, together with a discrete set of points $Z \in \tilde{S}$, such that

$$\sum_{h \in Z} T_{-h} \psi(x)^2 = 1,$$

and moreover, for each $B \in \mathcal{U}(\mathfrak{g})$, there is a constant $C > 0$ such that for all $x \in \hat{S}$,

$$\sum_{h \in Z} |T_{-h}\, \pi_{(0,V)}(B)\, \psi(x)|^2 \leq C.$$

Observe that the transversal S is stable under dilations δ_t of $\mathfrak{g}$, so that we have a notion of homogeneity for functions on S. These dilations are used in a crucial way, to 'shrink' the partition of unity, if necessary. Some final notation. For $0 < t \leq 1$, we write $h_t = \delta_{t^{-1}} h$, and write $\pi_{\xi(h,t)}$ instead of the cumbersome $\pi_{(\xi_{h_t}, V_{h_t})}$. Also $\psi_t = \psi \circ \delta_t$.

We are now ready to guide the reader to the changes that need to be made to the lemmas and constructions of section 4 in [HN]. First we must address lemma 4.5.1, [HN] p.926. It reads as follows.

Under the hypotheses of Proposition 107, there exists a constant $C > 0$, and a finite number of elements $A_j^k, B_j^k \in \mathcal{U}(\mathfrak{g})$ (here j ranges over a finite index set, and $k = 1, \ldots, N$), where all A_j^k are of order $m-1$ or less, and B_j^k of order m or less, such that for each $h \in \mathfrak{g}$, $0 < t \leq 1$ and $\xi \in \Omega_t$,

$$\begin{aligned} &\|\psi_t\, \pi_{\xi(h,t)}(A) v\|^2 \\ &\quad \leq C \left(\|\psi_t\, \pi_{\xi(h,t)}(P) v\|^2 \, + \, t^2 \sum_{k=1}^{N} \sum_j \| \left(\pi_{(0,V)}(B_j^k) \psi \right)_t \pi_{\xi(h,t)}(A_j^k)\, v_k \|^2 \right), \end{aligned}$$

for all sections $v = (v_1, \ldots, v_N) \in \mathcal{S}(S, \mathbb{R}^N)$.

Here $\Omega_t \subseteq V^*$ is a specific neighborhood of ξ, defined in section 4.1, which depends on the parameter t. The only difference with lemma 4.5.1 as stated in [HN] is the additional summation over the k index. Notice that, even though A, P are operators acting on sections $\mathcal{S}(S, \mathbb{R}^N)$, the A_j^k, B_j^k are simply scalar operators.

The proof of this lemma is exactly as in [HN] (see p.927–929), with two small amendmends. First, for $B = A$ or $B = P$, lemma 4.4.2 in [HN] is invoked to obtain inequality (4.5.5),

$$\| \left[\psi_t, \pi_{\xi(h,t)}(B)\right] v\|^2 \leq \sum_j \| \left(\pi_{(0,V_{h,t})}(B_j)\psi_t\right) \, \pi_{\xi(h,t)}(A_j)v\|^2,$$

with $A_j, B_j \in \mathcal{U}(\mathfrak{g})$ as above. In our case, where B is a matrix (B_{ik}), we have

$$\| \left[B, \psi\right] v\|^2 = \sum_{i=1}^N \| \sum_{k=1}^N [B_{ik}, \psi] v_k\|^2 \leq \sum_{i,k=1}^N \|[B_{ik}, \psi] v_k\|^2.$$

We can therefore replace the proven inequality for the scalar case by the following version for matrices,

$$\| \left[\psi_t, \pi_{\xi(h,t)}(B)\right] v\|^2 \leq \sum_{k=1}^N \sum_j \| \left(\pi_{(0,V_{h,t})}(B_j^k)\psi_t\right) \, \pi_{\xi(h,t)}(A_j^k)v_k\|^2.$$

Likewise, inequality (4.5.7) obtained from lemma 4.4.3 in [HN],

$$\begin{aligned} &\| \left(\pi_{\xi(h,t)}(B) \, - \, \pi_{\xi_0(h,t)}(B)\right) \psi_t v\|^2 \\ &\leq C \sum_{j+k+l<m,\, j\geq 1} \; \sum_{n\in N} \|g_j^n(x,\eta) \, \left(\pi_{(0,V)}(B_k^n)\psi_t\right) \, \pi_{\xi(h,t)}(A_l^n)\, v\|^2, \end{aligned}$$

for scalar $B = A, P$, can be generalized by considering,

$$\| \left(\pi_\xi(B) - \pi_{\xi_0}(B)\right) \psi_t v\|^2 \leq \sum_{i,k=1}^N \| \left(\pi_\xi(B_{ik}) - \pi_{\xi_0}(B_{ik})\right) \psi_t v_k\|^2.$$

Again, an extra summation over $k = 1, \ldots, N$ is added on the right hand side of inequality (4.5.6), while v is there replaced by its components v_k. The rest of the proof is devoted to deriving the estimate (4.5.11),

$$|g_j^n(x,\eta)| \leq Ct^j,$$

which, together with the results obtained so far, proves the lemma.

With this adapted form of lemma 4.5.1 in [HN], we can now show how to prove Proposition 107 (generalizing Proposition 4.1.1 in [HN]). We indicate how the proof in [HN] needs to be adapted.

The first part of the proof can be copied as it is (see [HN], p.931–932), until the point where lemma 4.5.1 is invoked to prove the inequality

$$\begin{aligned} &\|(T_{-h}\psi)_t \, \pi_{(\xi,V)}(A)\, u\|^2 \leq \\ &C \left(\|(T_{-h}\psi)_t \, \pi_{(\xi,V)}(A)\, u\|^2 \, + \, t^2 \sum_j \| \left(T_{-h}\pi_{(0,V)}(B_j)\psi\right)_t \, \pi(\xi,V)(A_j)\, u\|^2 \right). \end{aligned}$$

With the adapted version of this lemma, the final term $t^2 \sum_j \|(\ldots)u\|^2$ will be replaced by $t^2 \sum_{k=1}^{N} \sum_j \|(\ldots)u_k\|^2$. With this change, the final inequality that is derived on p.932 becomes,

$$\|\pi_{(\xi,V)}(A)u\|^2 \leq C \left(\|\pi_{(\xi,V)}(P)u\|^2 + t^2 \sum_{k=1}^{N} \sum_j \|\pi_{(\xi,V)}(A_j^k)u_k\|^2 \right).$$

Because each $A_j^k \in \mathcal{U}(\mathfrak{g})$ is of order less or equal $m-1$, for sufficiently small t we have,

$$t^2 \sum_{k=1}^{N} \sum_j \|\pi_{(\xi,V)}(A_j^k)u_k\|^2 \leq \varepsilon \|u\|_{H^{m-1}_{(\xi,V)}},$$

which proves Proposition 107.

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Vita of Johannes van Erp

Johannes van Erp was born in The Netherlands in 1962. He studied Architectural Design at the Academy for the Visual Arts at Arnhem, but ended up working for ten years as a free-lance systems analyst during the era of mainframe computers.

In 1995, while still working part-time consulting jobs, he decided to follow his passion for mathematics, and went back to college. In 2000 he finished a Master's Degree in Mathematics at the University of Amsterdam as a student of Klaas Landsman. While working on his Master's thesis, he had studied some of the papers on index theory by Nigel Higson, and was attracted by the approach to mathematics that he found in those papers. He decided to continue his graduate work at Penn State. The result of five inspiring years of study and research under the guidance of Professor Higson is presented in this thesis.