ECONOMIC MODELS OF INFORMATION SYSTEMS:
INTERNET AUCTIONS AND INFORMATION GATEKEEPERS

A Thesis in
Business Administration

by
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Abstract

In this thesis, we study two important features of the marketplace based internet economy—the emergence of information gatekeepers and the use of auction-like mechanisms to allocate and price goods sold on the internet.

The first essay “One Auction or Two? Auction with Multiple-units” discusses the application of sequential sale of multi-unit auctions. We analyze the case where both the auctioneer and the bidders are impatient, and find that whether or not to use sequential auctions largely depends on the intensity of market competition. We also explore the problem of whether or not the auctioneer should tell bidders how many items are available for the auction.

The second essay “Paid Placement in Information Gatekeepers” analyzes the practice of paid placement in information gatekeepers, where the gatekeeper biases its outputs to favor certain providers who pay it a placement fee. In addition, to get a better understanding of how different search engines are using paid placement strategies, we model in chapter 4 several paid-placement ranking strategies and compare their revenues via simulation.

But these are only simulations, which can identify which mechanism is "better", without finding out which is the “best”. In the third essay, “Optimal Allocation Mechanisms When the Ranking of Bidders Valuations is Common”, we study the following problem: buyers (content providers) compete for positions offered by the seller (search engine). While each buyer’s valuation for each position is private and independent of other’s valuation, the ranking for these positions is common among all the buyers. I identify the optimal mechanisms to allocate the positions under four different cases, and examine the existence of efficient incentive compatible mechanism (subject to the reserve price).
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Chapter 1

Introduction

The recent advances in information technology have greatly changed our lives. While the boom in e-business of the 1990s might be over, the electronic marketplace is here to stay and the effort to adapt its potentialities to one’s own business is one of the key sources of competitive advantage. In this thesis, we study two important features of this marketplace—the emergence of information gatekeepers and the use of auction-like mechanisms to allocate and price goods sold on the internet. Information gatekeepers use their expertise in a certain field to search and recommend information to facilitate people’s decision making. They become more and more indispensable of our daily life. On the other hand, how information gatekeepers search and present information to users has a great impact not only on consumer’s daily decision making, but also the long term internet structure. Internet auctions create new market places without geographic restrictions and generate a tremendous amount of value; they are also very commonly used in B2B market. But auction mechanisms are not only used in these auction markets. For example, internet search engines also use auctions to sell the top positions to some content providers who are willing to pay. The purpose of this thesis is to identify some interesting features of internet economy.

The first essay “One Auction or Two? Auction with Multiple-units” discusses the application of sequential sale of multi unit auctions. While most standard results suggest that the auctioneer will get the same revenue from selling his items through sequential auctions as through a single auction, why are sequential auctions still observed? We analyze the case where both the auctioneer and the bidders are impatient, and find that whether or not to use sequential auctions largely depends on the intensity of market
competition. We also explore the problem of whether or not the auctioneer should tell bidders how many items are available for the auction. We use backward induction to derive the perfect Bayesian equilibrium strategies for both bidders and auctioneer. We find that it is valuable for the auctioneer with a large number of items to create uncertainty about the number of periods, that is not to reveal the information about numbers of items for sale. The optimal auction in our setting is known; it introduces a reserve price to which the seller is committed not to sell below it. We discuss the case when the auctioneer cannot commit, and find that the effect of the reserve price to raise revenue is reduced, and under some conditions the auction with reserve price generates less expected revenue than the sequential auction discussed above.

The second essay “Paid Placement in Information Gatekeepers” analyzes the practice of paid placement in information gatekeepers, where the gatekeeper biases its outputs to favor certain providers who pay it a placement fee. Information gatekeepers—such as Internet search engines, shopbots, financial advisors, referral sites, and online travel services—are an essential entry point for many information search and decision making tasks. An information gatekeeper uses its repository of relevant information and algorithms to provide users a ranked recommendation list in response to a specific query. We examine a gatekeeper’s bias decisions, how these are affected by the elasticity of demand towards bias, and the interplay between bias and technology aspects of gatekeeper’s quality. A monopoly gatekeeper will choose independence only when the users’ sensitivity to bias is very high. An increase in technological quality will cause the gatekeeper to lower its bias level when the elasticity of marginal demand for quality is lower than the per-user contribution of placement revenue. Competition will cause gatekeepers to reduce bias level. For heterogeneous gatekeepers, the one with higher technology level may or may not choose a lower bias level than opponents, but will capture a larger market share. In choosing the technology level, the gatekeeper with superior cost function will set a higher technology level. Its optimal bias level will be such that its composite quality
measure is better than its opponents. The competitive choices of bias level may in some cases be Pareto inefficient.

In addition, to get a better understanding of how different search engines are using paid placement strategies, we model several paid-placement ranking strategies, including those used by Overture and Google, the two biggest brokers in paid placement advertising, along with some of our own design, and compare their revenues via simulation. All of these mechanisms generate essentially the same revenue when item relevance is very highly correlated with willingness to pay. We find that ranking based on a product of click-thru rate and willingness to pay fares best when relevance is positively but not completely correlated with willingness to pay. Take-it-or-leave-it pricing (where the search engine or broker sets reserve prices for all ranks) also fares well. We propose a new ranking strategy that weights clicks on lower ranked items more than clicks on higher ranked items. This method is shown to converge to the optimal (maximum revenue) ordering faster and more consistently than other methods.

But these are only simulations, which can identify which mechanism is "better", without finding out which is the "best". In the third essay, "Optimal Allocation Mechanisms When the Ranking of Bidders Valuations is Common", we study the following problem: buyers (content providers), each of whom has unit demand, compete for positions offered by the seller (search engine). While each buyer’s valuation for each position is private and independent of other’s valuation, the ranking for these positions is common among all the buyers. We begin with 4 simplified scenarios specifying buyers valuations for different positions: i), each buyer’s valuation declines in the same rate for lower positions; ii), the buyer’s valuation drops faster for lower positions if he has higher value for the higher positions; iii), the opposite of ii); iv), the combination of iii) and iv). We identify the optimal mechanisms to allocate the positions under these four cases, and examine the existence of efficient incentive compatible mechanism (subject to the reserve price), and then discuss the implementation of these mechanisms.
Chapter 2

One Auction or Two: Simultaneous vs. Sequential Sales in Multi-unit Auctions

2.1 Introduction

On April 25, 2003, the BBC Nightly News reported an upcoming auction of 600 Picasso lithographs at Christie’s in New York. There was apparently some concern at Christie’s that auctioning off so many similar items would lead to some of them being sold for only a few hundred dollars apiece. In an article on Christie’s web site[1] Jonathan Rendell, the auction house’s Deputy Chairman, mentions that some estimated prices were as low as five hundred dollars and that the consignor had decided to sell with no reserve price.

This paper seeks to address some of the issues brought up in this recent example. When several similar items (in our model we simplify to identical items) are being sold, should the auctioneer decide to auction off all the items simultaneously or is there a potential benefit from selling the items sequentially in smaller lots? We add another question, more relevant to other contexts to be described later in this section, namely, if there is uncertainty about the total number of items among the bidders, can this benefit the seller?

It is obvious that if potential buyers arrive sequentially over time, there is a reason for holding some items back (as airlines often do) so as to take advantage of high-value buyers who arrive late[2]. However, most of the voluminous literature in economics on

[2]Analogously, a university might have five positions but might wish to fill only some of them in a given year, so as to have access to future entrants into the academic market.
auctions has assumed a fixed, known number of bidders and this clearly corresponds to some real-life auction situations. In the example above, there is no evidence that a substantial number of potential buyers was not present at the time of the proposed auction. In bidding for contracts, the buyer of the services often has a list of qualified bidders and it is a non-trivial task to get into this list from outside.

Internet auction sites provide another illustration of the problem dealt with in this paper. For example, many consumer products like electronics are usually sold in multiple units in B2C auction sites such as eBay.com, Onsale.com, uBid.com, etc; not to mention the transaction of production supplies in B2B procurement market such as Freemarkets.com. (There are many research papers in this field, for example, [8]). Search engines often use some form of auction mechanism to sell off sponsored links (see Feng (2003) [20] for a specific analysis of this problem and Baye and Morgan(2001) for a general analysis of information gatekeepers). In many of these internet auctions, it is not clear if the seller has more items of the same kind that he or she has decided to hold back for a future auction; interested buyers have to take this uncertainty into account in planning their bidding strategies.

Specifically, this paper addresses the two questions: should the auctioneer split his stock of identical items into lots and sell them at different times or should he sell everything simultaneously? What happens if buyers are uncertain about how many items an auctioneer has to offer? In this paper we focus our attention to a standard auction context with a fixed number of bidders. The auction format used here is the simple uniform price (“second price”) auction, with independent private valued bidders. Since each bidder is assumed to want only one item, this allocates items to the highest value bidders in the symmetric equilibrium.

Before giving a synopsis of our results, we should acknowledge that for many people, this problem has been solved. Given that each buyer has a maximum demand of one unit and that the buyer valuations are independent and private and from identical
distributions, the optimal auction is known and is a simple one. Set a reserve price and sell all the items at the highest rejected bid, for example. ([21], [33].) This assumes, like the entire mechanism design literature, that the auctioneer is able to make a binding commitment not to sell the objects below the announced reserve price (see, however, McAfee and Vincent (1997) and Skreta (2001), which are discussed in the next section). For some reason, such perfect commitments are often not characteristic of the real world, as the Picasso example above illustrates. (Note that the consignor there had specified that all the items should be sold—there was no reserve price set.) Also some auctioneers in eBay have been observed to reduce the reserve price at the end of the auction if they fail to sell with the original reserve price.³ They can resell their commodity in eBay for the second time for free if no bids meet the reserve price in the first auction.⁴ Besides the lack of commitment for not selling the unsold stock, the second reason is that reserve prices, announced or secret, appear to be not popular in some sites like eBay. Many bidders generally won’t bid on an item with a reserve price.⁵

In the absence of a reserve price or in the absence of a commonly known perfect commitment not to sell ever below the reserve price, it is by no means evident that a reserve price policy will continue to be optimal. We explore this issue as well in this paper.

As in the Picasso example above, the auctioneer in our model does not control the total number of items he or she has to sell. This distinguishes this problem from that of a monopolistic seller who can choose how much to produce.

Our major results are as follows: First, when the number of items is large and if there is no “free disposal”, that is, either the auctioneer cannot throw away the unsold items, or he cannot commit to do that, the effect of a reserve price to screen out low

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valued bidders is greatly reduced, and the sequential auction may generate more expected revenue for the auctioneer by increasing the intensity of competition in the initial period. We find that if we assume that bidders’ valuations follow a uniform distribution, there exists a cut-off point in the “intensity of competition” (which is expressed as the fraction of number of items for sale relative to number of bidders $\frac{K}{n}$) such that if the intensity of competition is above this cut-off point, a sequential auction performs better in expected revenue than the one-stage auction. Second, in sequential auctions, by keeping the number of auctions a secret, the auctioneer may extract more surplus from the bidders. We study the case where the bidders don’t know the auctioneer’s inventory and has to form some expectation about the auctioneer’s type. We find that in equilibrium, the “high inventory” auctioneer pretends to be a “low inventory” type in the first period and reveals the type in the second period, or reveals the type at the beginning of the first period, depending again on the intensity of competition, as well as bidders’ prior beliefs about the auctioneer’s type. Third, the “optimal auction” without perfect commitment to a reserve price performs less well under some conditions in generating revenue than the sequential auction without reserve price.

The remainder of the paper is organized as follows: in the next section, we discuss some relevant literature, then in section 2.3 we introduce the environment and notation. In section 2.4 we consider the simple case when the auctioneer’s inventory is common knowledge among bidders, and the auctioneer cannot throw away his inventory for free. In subsection 2.4.2 we study the performance of the “optimal auction” without commitment to the reserve price. In section 2.5 we consider the case where the number of items is known only to the seller.

In this paper, we will occasionally refer to “two-period auctions” and sometimes to “two auctions”; these refer to the same object and are both used only for variety.
2.2 Literature Review

The study of sequential VS. simultaneous auction was initiated by Weber (1983) \cite{weber1983}. He studied an independent private values environment with each bidder having unit demand and with the seller having multiple items for sale, just as we do have. However, Weber’s paper has no discounting and ours does. This turns out to make a crucial difference to the results. He finds out that the auctioneer’s expected payoff is the same whether she sells all the items in one period, or one in a period. This applies to both the “second price” auction (uniform price auction) and the first price auction (discriminatory price auction). The expected winning price is constant across periods. Weber’s results appear as a limiting case of our first model as the discount factor goes to 1.

Menezes (1993) \cite{menezes1993} analyzes a 2-period sequential model with fixed cost of delay but does not consider discounting or a comparison of sequential and simultaneous auction.

Both Beam \etal (1996) \cite{beam1996} and Pinker-\etal (2000) \cite{pinker2000} used the actual transaction data on certain internet auction sites to determine how many items should be sold in each round of a sequential auction. However, both the papers assumed naive bidders who do not consider the effects of their current actions on future periods. Bidders in our model are fully rational.

On the question of lack of commitment to a reserve price in “optimal” auctions, McAfee and Vincent (1997) \cite{mcafee1997} study the optimal sequential auction when the auctioneer cannot commit never to sell below the initial reserve price, and Skreta (2002) \cite{skreta2002} studies the optimal mechanism to sell a single item, in a two-period model. However both consider only the sale of a single object, and do not address whether it would be optimal to sell multiple units simultaneously or sequentially.
Another frequently discussed issue in multi-unit auctions, namely, “declining price anomaly”, also exhibited in our model. (This is not our central concern in our paper, though). (Note that the so-called anomaly does not appear in the analysis of Weber (1983)[50].) Among the papers concerned with this anomaly are Ashenfelter(1989)[5], McAfee and Vincent (93) [34], Menezes (1993)[38] and Jeitschko (1999)[18].

2.3 The Model and Notation

We initially consider the situation where the auctioneer receives $K$ items in inventory. There are $n$ symmetric risk neutral bidders. Bidder $i$ has a unit demand if the price is less than or equal to $v_i$, his private value. The $v_i$s are independently drawn from a common, absolutely continuous distribution $F$ on $[0, 1]$. All $K$, $n$ and $F(\cdot)$ are common knowledge. For simplicity we only consider a two-period case, and assume that if the auctioneer decides to use a sequential auction, he or she will sell half the inventory $k = K/2$ in each period (This is common knowledge). The auctioneer’s decision is to choose whether to sell his inventory in one or two auctions.

If $K$ is the auctioneer’s private information, then the number of items put up for sale in the first auction could convey information about $K$ to the bidders. The bidders observe the number of items in the auction at the current period. Depending on the inventory information available to them, they decide whether to bid in the current period. If they bid, they then decide how much to bid. If a bidder wins an item he leaves, and if not, he stays for the second period and decides how much to bid in the second period. As mentioned in the introduction, the auction format is a sealed bid, uniform price auction, where the market clearing price will be determined by the highest rejected price in each period. At the end of the first period, remaining bidders are told how many objects remain for sale for the second period.
The auctioneer is also assumed to be risk neutral and to have a commonly known reserve value of 0 and this is common knowledge, too.

Suppose the bidders and the auctioneer have a common discount factor \(\delta\), \(0 \leq \delta \leq 1\). We also assume that the number of bidders is always greater than the number of items in the auction in each period, that is, \(n > K\). The case where \(n \leq K\) is trivial.

In addition, we use the following notation in the analysis:

1. \(b^t_i\): The bid of bidder \(i\) with valuation \(v_i\) in period \(t\).
2. \(Y_i\): the \(i\)'th highest valuation among \(n-1\) bidders.
3. \(H_i(v)\): the probability of \(Y_i \leq v\), where
   \[
   H_i(v) = \sum_{l=0}^{i-1} \frac{(n-1)!}{(n-1-l)!l!} F(v)^{n-1-l} (1 - F(v))^l,
   \]
   \(i = 1, 2, ... n-1\)
4. \(h_i(v)\): the probability density function of the \(i\)'th highest bid among \(n-1\) bidders,
   where
   \[
   h_i(v) = \frac{(n-1)!}{(n-1-i)!(i-1)!} F(v)^{n-1-i} (1 - F(v))^{i-1} f(v), i = 1, 2, ..., n-1.
   \]
5. \(X_i\): the \(i\)'th highest valuation among \(n\) bidders.
6. \(G_i(v)\): the probability of \(X_i \leq v\), with density function \(g_i(v)\) where
   \[
   G_i(v) = \sum_{l=0}^{i-1} \frac{n!}{(n-i)!!l!} F(v)^{n-l} (1 - F(v))^l, \quad i = 1, 2, ... n
   \]
7. \(g_i(v)\): the probability density function of the \(i\)'th highest bid among \(n\) bidders,
   where
   \[
   g_i(v) = \frac{(n)!}{(n-i)!!l!(i-1)!} F(v)^{n-i} (1 - F(v))^{i-1} f(v), i = 1, 2, ... n.
   \]

2.4 Number of items commonly known

In this section, we consider the case when the auctioneer’s inventory information is commonly known by the bidders. We now study whether or not the auctioneer would
want to use sequential auctions or a single simultaneous auction. First we compare the performance of sequential and simultaneous auctions. In both formats there is no reserve price. Next we compare the sequential auction to an ‘optimal auction” under the assumption that the auctioneer can not commit to the reserve price.

2.4.1 Sequential Auction Vs. Simultaneous Auction

2.4.1.1 Sequential Auction

First consider the sequential auction where the auctioneer sells half of his inventory in each of the two periods. We start with an analysis of the second auction assuming that the $k$ highest bidders win and leave in the first period, and there are only $n - k$ bidders remaining. (This is actually going to be on the equilibrium path, when $k = \frac{K}{2}$. The only out of equilibrium node only occurs when fewer than $K/2$ objects are sold in the first period. This does not affect bids in the second period.)

Using backward induction, we first consider the second-period auction. Since at the beginning of the second period, it is public information to bidders that this is the final period of the auction, standard auction theory tells us that truth telling is the weakly dominant strategy, i.e., bidders bid their true values. Consider a bidder $i$. Suppose the symmetric (increasing) bidding strategy used by the other $n - 1$ players in the first period is given by $\beta(v)$. If bidder $i$ wins, he will pay the amount of the $(k+1)^{th}$ highest bidder’s bidding price among the $n - k$ remaining bidders, or the $2k^{th}$ highest bidder’s value among the other $n - 1$ bidders (except himself) who entered the auction at the beginning of the first period. Here the assumption is being made that the $k$ bidders with the highest values buy and leave at the end of the first period, so that the value of anyone buying in the second period must be less than the realization of $Y_k$ and his first-period bid must be less than the realized price in the first period $\beta(Y_k)$. Note in this paper,
the superscript represents the number of periods, and the subscript represents the order statistics of bidders’ values.

The auction in the second period allocates items efficiently because the bidding function in the second period \( b_i^2(v_i) = v_i \) is increasing in \( v_i \).

Let \( X_i \) be the \( i^{th} \) order statistic of all bidders’ evaluations of the item. The auctioneer’s discounted expected payoff in the second period is

\[
E \left[ R^2 \mid X_k \right] = \delta_k \cdot E[X_{2k+1} \mid X_k = \beta^{-1}(p_1)]
\]

and the discounted unconditional expected second period payoff is \( E \left[ R^2 \right] = \delta_k \cdot E[X_{2k+1}] \).

We then go back to the first period. There are \( n \) bidders at the beginning of the first period. If there is no future auction, every bidder would bid his or her value. But since there will be a second period, bidders have the option to wait and win the item at a possibly lower price, so they shade their first-period bid by their expected payoff from the second period.

Then player \( i \)'s expected payoff is:

\[
\begin{cases} 
  v_i - \beta(Y_k) & \text{if } b^1_i \geq \beta(Y_k) \\
  \delta E \max[(v_i - Y_{2k}, 0) \mid \beta^{-1}(b^1_i) \leq Y_k] & \text{otherwise}
\end{cases}
\]
It can be shown that the optimal bidding function in the first period is:

\[ b_i^1(v_i) = v_i - \frac{\delta}{\text{prob}(Y_{2k} < v_i \leq Y_k)} \cdot \int_0^{v_i} (v_i - y) \left( \begin{array}{c} n - 1 \\ 2k - 1 \end{array} \right) \left( \sum_{i=1}^{k-1} \frac{k - 1 + i}{k - 1} F(y) y_{n-1-2k}^{n-1-2k} (1 - F(y))^{k-i} (1 - F(v_i))^{k+i-1} f(y) dy \right) \]

\[ = v_i - \delta \pi^2(v_i|Y_{2k} < v_i \leq Y_k) \]
\[ = (1 - \delta) v_i + \delta E[Y_{2k}|Y_{2k} < v_i \leq Y_k] \]  

(2.1)

The intuition behind this bidding strategy is that bidders should not feel regret when they win in the first period, expecting that they may win in the second period with a lower price. We can view \( \pi^2(v_i|Y_{2k} < v_i \leq Y_k) \) as \( \pi^2(v_i|\text{winning in the 2nd period}) \). Thus in the first period, a bidder never bids an amount \( x \) more than \( v_i - \pi^2(v_i|Y_{2k} < v_i \leq Y_k) \), because if he does, there is a positive probability that some bid is between \( v_i - \pi^2(v_i|Y_{2k} < v_i \leq Y_k) \) and \( x \). Then if bidder \( i \) wins in the first period, he has to pay more than \( v_i - \pi^2(v_i|Y_{2k} < v_i \leq Y_k) \), which makes his expected payoff less than \( \pi^2(v_i|Y_{2k} < v_i \leq Y_k) \). So he would rather not enter the first period auction, but wait till the second period and bid. On the other hand, he will not bid less than \( v_i - \pi^2(v_i|Y_{2k} < v_i \leq Y_k) \), because an increase of \( \epsilon \) in the bid can increase his probability of winning without affecting the payment, given the bidding strategy above is followed by all the other bidders. A formal proof is in Appendix A.1.

**Proposition 1.** The first period auction efficiently allocates items to the \( k \) bidders with the \( k \) highest valuations.
Since the highest $k$ bids win in the first period, we only need to show that the bidding function in the first period is increasing in $v$. The proof is straightforward so we omit it here.

*Note:* The auction may be *inefficient* in terms of a cost of delay if the auctioneer chooses a two-period auction when $\delta < 1$.

In sequential auctions people are often concerned with the existence of the so-called “price decline anomaly”, that is, the winning prices in the later periods are less than the winning prices for identical objects in the earlier ones. Here, we only need to consider the equilibrium market clearing prices in two periods to show that an expected price decline could take place.

**Proposition 2.** The price decline anomaly exists in the multi-unit sequential auction.

**Proof.** The winning price in the second period given the winning price in the first period is: $E\left[ P^2 | X_{k+1} \right] = E \left[ X_{2k+1} | X_{k+1} \right]$. The expected second period auction price in the beginning of the auction should be

$$E \left[ P^2 \right] = E \left[ X_{2k+1} \right]$$

The winning price in the first period is:

$$E \left[ P^1 \right] = E[X_{k+1} - \pi^2_{k+1} (X_{k+1}|X_{k+1} = X_{k+1})] = (1 - \delta)E[X_{k+1}] + \delta E[X_{2k+1}]$$

From this we know that the market clearing price in the first period is always at least as high as that in the second period, and they are equal only when $\delta = 1$, that is, only when there is no time discounting.
Given bidders’ bidding strategies, the revenue that the auctioneer expects in the first period is:

\[
E[R^1] = k \cdot E \left[ X_{k+1} - \pi^2_{k+1} (X_{k+1} | X_{k+1} = X_{k+1}) \right]
\]

\[
= k \cdot E \left[ X_{k+1} - (\delta X_{k+1} - \delta E [Y_{2k} | X_{k+1} = X_{k+1}]) \right]
\]

\[
= k \cdot ((1 - \delta) E [X_{k+1}] + \delta E [X_{2k+1}])
\]

The auctioneer’s expected revenue \( E[R^2] \) when there are two periods is then:

\[
E[R^2] = E[R^1] + E[R^2]
\]

\[
= k \cdot ((1 - \delta) E [X_{k+1}] + \delta E [X_{2k+1}])
\]

\[
= k \cdot \left\{ (1 - \delta) E [X_{k+1}] + 2\delta E [X_{2k+1}] \right\}
\]

\[
= k \cdot \left\{ (1 - \delta) \int_0^1 yg_{k+1}(y)dy + 2\delta \int_0^1 yg_{2k+1}(y)dy \right\}
\]

From this expression we can see that the auctioneer’s expected revenue not only depends on the relative magnitudes of \( n \) and \( k \), but also depends on the value of \( \delta \). The discount factor \( \delta \) has two effects on \( E[R^2] \): the more the bidders discount (the smaller the discount factor), the more aggressively bidders bid in the first period. This can increase the first-period revenue. But at the same time the utility from the second-period auction decreases because of the discounting, this time by the auctioneer. The overall effect of \( \delta \) depends on the trade-off between these two effects.

2.4.1.2 Simultaneous Auction or Sequential Auction?

Suppose now the auctioneer can choose to sell his inventory in either a one-period or two-period auction, and he announces this information so it is common knowledge among bidders at the beginning of the auction. The two-period case has just been discussed. For the single-period auction case, keeping all notation the same, there are \( 2k \) items in the auction \( (2k = K) \). All the bidders will bid their true values, and a bidder
\( i \)'s expected payoff will be:

\[
\pi_i(v) = \int_0^v (v - y_{2k}) h_{2k}(y_{2k}) dy_{2k}
\]

The auctioneer’s expected revenue will be:

\[
E[R1] = 2k \cdot E[X_{2k+1}] = 2k \cdot \int_0^v y_{2k+1}(y) dy
\]

Comparing these two expected payoffs,

\[
E[R2] - E[R1] = k \cdot \{ (1 - \delta) E[X_{k+1}] + 2\delta E[X_{2k+1}] - 2E[X_{2k+1}] \}
\]

\[
= k \cdot (1 - \delta) \cdot (E[X_{k+1}] - 2E[X_{2k+1}])
\]

The sign of this expression is determined by \( n, k \) and \( \delta \). If \( \delta = 1 \), that is, if there is no time discounting, \( E[R2] = E[R1] \), which is just Weber’s standard result. If \( \delta < 1 \), the sign of this difference is determined by \( (E[X_{k+1}] - 2E[X_{2k+1}]) \), again depends upon the distribution of bidders’ values and the relationship of \( n \) and \( k \). Apparently if \( E[X_{k+1}] > 2E[X_{2k+1}] \), then the auctioneer can be better off if he holds a two-period auction, even though his expected payoff in the second period is very small! This is when the market competition for the items is so weak that bidders can get the items at a very low price, it is better to sell only \( k \) items at a higher price by creating some competition.

So even though both the bidders and auctioneer discount future periods, it may not always be better to sell all the items in one period. Which auction performs better is largely determined by the relationship between \( k \) and \( n \), that is, the market competition situation, as well as the discount factor. Discounting encourages people to shade less in the first period, thus to bid more aggressively in the first period, so creating first-period competition. This is beneficial to the auctioneer. Thus discounting has some positive
effect on the auctioneer’s expected revenue. For some combination of $k$ and $n$, this positive effect can overcome the negative effect and make the auctioneer gain more by a sequential auction. We have:

**Proposition 3.** When $\delta = 1$, the one-period auction and two-period auction generate the same results. (See Appendix). When $\delta < 1$, the choice of a one-period or two-period auction will be determined by the relative magnitude of the number of bidders and the number of items for sale, as well as the distribution function of $v_i$. (See Appendix) The two-period auction must be inefficient if any items are sold in the second period because of discounting.

We also find that the auctioneer’s expected revenue from the auction does not necessarily increase with the number of items in the auction. The proof is straightforward, so we omit it here. The intuition is that the price might go down sufficiently to offset the increased sales.

More specifically, if we assume that bidders’ valuations of the item are i.i.d random variables uniformly distributed between $[0, 1]$, we have when $\delta < 1$:

**Proposition 4.** If $k < \frac{1}{3}n$, it is always better to have a one-period auction. If $k > \frac{1}{3}n$, it is always better to hold a two-period auction. When $k = \frac{1}{3}n$, the expected revenue from the simultaneous auction and sequential auction is the same for the auctioneer.

The proof for both proposition 3 and 4 are in the Appendix A.2.

**Remark:** Until now we have only considered the case where the auctioneer sells items in identical lots, if a 2-period auction is used. If we relax this constraint, the question becomes how to optimally allocate items across the two periods. Let $k_1$ be the number of items in the auction in the first period, then in the second period there are $K - k_1$ items remaining. Given the bidders bidding strategy specified in Eq. 2.1 the
auctioneer maximizes:

$$Max \quad k_1 \left\{ E \left[ X_{k_1+1} \right] - \delta \left( E \left[ X_{k_1+1} \right] - E \left[ X_{K+1} \right] \right) \right\} + \delta (K - k_1) E \left[ X_{K+1} \right]$$

Assuming that the bidders’ values follow the uniform distribution on $[0, 1]$, solving for the the first-order condition and rounding to the smallest integer, we get $k_1^* = \min \{ \left\lfloor \frac{1}{2} n \right\rfloor, K \}$. Notice that (1) when $\left\lfloor \frac{1}{2} n \right\rfloor > K$, that is to say, when $K$ is relatively small, the auctioneer will choose a one-period auction. Compare this with our previous result where the auctioneer can only choose identical lots each period. In that environment the auctioneer will choose a one-period auction when $k < \frac{1}{3} n$, or $K < \frac{2}{3} n$. He is more likely to choose a one-period auction there because he can only split his inventory in half. Since $\left\lfloor \frac{1}{2} n \right\rfloor \geq \frac{1}{2} K$, the auctioneer will sell at least half of his inventory in the first period. So he will put more for sale in the first period than in the second period.

2.4.2 Reserve Price Without Commitment

It is well-known that in auction theory, an optimal mechanism for selling multiple items to bidders with independent private values is an auction where all the items are sold at once with an optimal reserve price different from the seller’s valuation. However, in reality, the full commitment assumption may not always hold. Even if the auctioneer wishes to commit to such a reserve price, there may be no credible mechanism allowing him to do so. For example, some auctioneers in eBay are observed to reduce their reserve prices at the end of the auction if they fail to sell with the current reserve price\(^6\). And eBay actually allows the auctioneer to relist for free their items for sale for the second time, if there is no bid that meets the reserve price the first time\(^7\). A similar practice is also observed in retail markets. For example, experienced shoppers know that the


\(^7\)eBay Fee Chart, http://auctionknowhow.com/resource/fees.shtml
current sale is not final unless it is marked “final sale”, while waiting to buying later will incur a waiting cost, plus the risk that the certain commodity may be sold out before it reaches the final sale. The shopper’s decision problem includes the optimal time to participate.

Let’s consider the case that the auctioneer sells all of his ($K$) items with announced reserve price ($r$) in the first period, but bidders know that the auctioneer cannot commit to this reserve price and will resell the unsold items in the next period with reduced reserve price. For simplicity here we only consider a two period model, so the reserve price in the second period is reduced to 0. Suppose bidders don’t know how many other bidders’ valuations are above the reserve price. Anticipating there will be a second auction with positive probability, bidders will adjust their bidding strategy and decide whether or not to participate in the first period auction, and if so, how much to bid.

**Proposition 5.** In equilibrium, there exists a threshold value $\tilde{r} \in [0, 1]$. A bidder with value no less than $\tilde{r}$ will bid his true value in the first period, and bidders whose valuations are lower than $\tilde{r}$ will wait for the second period. That is,

$$
\begin{cases}
(v - r) \text{prob}(n' \leq K) + \sum_{i=K+1}^{n} (v - E[Y_{K,n-1}|n' = i]) \text{prob}(n' = i) \\
\geq \delta \sum_{i=0}^{K} (v - E[Y_{K,n-1}|n' = i]) \text{prob}(n' = i) \quad \text{if } v > \tilde{r} \\
(v - r) \text{prob}(n' \leq K) \leq \delta \sum_{i=0}^{K} (v - E[Y_{K,n-1}|n' = i]) \cdot \text{prob}(n' = i) \quad \text{if } v < \tilde{r}
\end{cases}
$$

(2.2)

and the optimal $\tilde{r}$ is determined by:

$$
(\tilde{r} - r) \cdot \text{prob}(n' \leq K) = \delta \sum_{i=0}^{K} (\tilde{r} - E[Y_{K,n-1}|n' = i]) \cdot \text{prob}(n' = i)
$$

(2.3)

where $E[Y_{K,n-1}|n' = i]$ represents the $K$'th highest bidders valuation among $n - 1$ bidders, given $n'$ bidders’ valuations above $\tilde{r}$. 

Proof is in Appendix A.3.

Given this strategy, the auctioneer’s expected revenue when he or she uses a reserve price \( r \) without commitment is:

\[
E[R(r)] = \sum_{i=K+1}^{n} \tilde{r}^{n-i}(1 - \tilde{r})^{i} \left( \begin{array}{c} n \\ n - i \end{array} \right) K \cdot (E[X_{K+1} | X_{K+1} > \tilde{r} \& n' = i])
\]

\[+ \sum_{i=0}^{K} r^{n-i}(1 - \tilde{r})^{i} \left( \begin{array}{c} n \\ n - i \end{array} \right) (i \cdot \tilde{r} + \delta(K - i) \cdot E[X_{K+1} | X_{K+1} < \tilde{r} \& n' = i]) \]

where \( \tilde{r}(r) \) is determined by Eq. 2.3.

Then the auctioneer’s decision problem is to choose the optimal reserve price \( r \) to announce, that is, \( r^* = \arg \max \{R(r)\} \).

Compare this revenue with the expected revenue from the sequential auction without reserve price. Intuitively, the auction with reserve price works better when \( delta \) is small. When \( \delta \to 0 \) it approaches the theoretical “optimal auction” with full commitment. However when \( delta \) is large, bidders are more willing to wait till the next period for a lower price, which makes the reserve price in the first period less effective in screening out the low-valued bidders. Thus the auction with reserve price may performs worse than the sequential auction. We did not derive a formal sufficient condition for this result, however figure 2.4 shows a numerical example in which for a certain region of \( \delta \) the sequential auction is better than the reserve price auction. In this example, \( n = 7, K = 4, \) and bidders’ valuations follow a uniform distribution in \([0, 1]\).

When \( \delta = 0 \), which is equivalent to the case where the auctioneer can make full commitment, an auction with an optimal reserve price \( \left( \frac{1}{2} \right) \) performs better than an auction without reserve price. With the increase in \( \delta \), the performance of the reserve price auction is reduced. There exists a region of \( \delta \) where the reserve price auction does
Fig. 2.1. Performance of auction with reserve price without commitment Vs. no reserve price with respect to the change of $\delta$. 
not perform as well as the sequential auction without reserve price. When $\delta = 1$, these two auction formats perform equally well.

2.5 Bidders uncertain about number of periods

We now consider the case when the auctioneer does not announce how many periods there will be. Under the setup of our model, this is to say that the auctioneer does not tell bidders his inventory information. So when bidders come to the auction site, they know that there can be at most 2 periods, but they don’t know whether the current auction has one or two periods. This is the case in many of the online auction sites, where items are sold in identical lots but the total amount of inventory on hand is not known to bidders. Milgrom and Weber (1982) [40] indicated that in a single unit case with affiliated values, the auctioneer is always better off by providing as much value-related information as she can to the bidders. The decision to communicate how many periods there will be is a somewhat different choice, but it is interesting to check whether the auctioneer will in any case be better off revealing this information.

We consider the following game: in the beginning Nature determines whether the auctioneer is a High type (H) who has $2k$ items on hand, or a Low type (L) who has only $k$ items on hand. With probability $\rho$ he is a High type, and with $1 - \rho$ he is a Low type. The belief $\rho$ is common knowledge among the bidders and the auctioneer. The auctioneer’s value is 0 and does not reveal how many periods there will be. If he is a Low type, he will sell all $k$ items in one period; if he is a High type, he can choose to sell $2k$ items in one period, or to pretend to be a Low type in the first period by selling $k$ items in the first period, and $k$ items in the second period. There is no way that the Low type auctioneer can signal his type. Only the High type can differentiate himself from the Low type by selling $2k$ items in the first period. Bidders bid based on their beliefs
Fig. 2.2. The game when bidders don’t know the auctioneer’s type
about the total number of items in the auction, or the auctioneer’s type. The game is shown in the graph above:

Again we use backward induction to derive the Perfect Bayesian Equilibrium. When bidders see \( k \) items in the auction, they expect that with probability \( \rho' \) there will be a second period.

Suppose that the High type auctioneer will play \((k,k)\) with probability \( \alpha \), and \((2k,0)\) with probability \( 1 - \alpha \), then \( \rho' = \frac{\alpha \rho}{\alpha \rho + (1 - \rho)} \), where \( \alpha \) satisfies the equation that

\[
\frac{\alpha \rho}{\alpha \rho + (1 - \rho)} = \frac{(3-2\delta)k - (1-\delta)n}{\delta k}.
\]

That is, \( \alpha \) generates a belief \( \rho' \) that makes the bidders play such that the high type auctioneer is indifferent between revealing her type and pooling with the low type player. If we assume that bidders valuations follow a uniform distribution in \([0,1]\), we can specify the equilibrium strategy in the following proposition:

**Proposition 6.** The auctioneer’s equilibrium strategy is:

1. If Low type, always sell \( k \) items in the first period.

2. If High Type,
   1) If \( \rho \leq \frac{(3-2\delta)k - (1-\delta)n}{\delta k} \), pretend to be a Low type by selling \( k \) items in the first period, where \( \alpha = 1 \) and \( \rho' = \rho \).
   2) If \( \rho \geq \frac{(3-2\delta)k - (1-\delta)n}{\delta k} > 0 \), play mixed strategy that with probability \( \alpha \) sells \( k \) items in the first period and with probability \( 1 - \alpha \) sells \( 2k \) items in the first period, where \( \alpha \) satisfies the equation that

\[
\frac{\alpha \rho}{\alpha \rho + (1 - \rho)} = \frac{(3-2\delta)k - (1-\delta)n}{\delta k}.
\]

That is, \( \alpha \) generates a belief \( \rho' \) that makes the bidders play such that the auctioneer is indifferent between revealing her type and pooling with the low type player.

3) If \( \frac{(3-2\delta)k - (1-\delta)n}{\delta k} \leq 0 \), which means \( n \geq \frac{3-2\delta}{1-\delta}k \), always sell \( 2k \) items in the first period. Intuitively, this is the case when there are a large enough number of bidders in the market to generate sufficient competition for the number of items for sale.
The equilibrium strategy for the bidders follows similarly.

Intuitively speaking, when the belief that he is of a High type is small enough, the auctioneer can always benefit from pretending to be a Low type. This condition is satisfied when the market is highly non-competitive, (i.e., $k \geq \frac{(1-\delta)n}{3}$). This is what (2.1) of Proposition 6 says. But if the market is highly competitive, where $n \geq \frac{3-2\delta}{1-\delta}k$ as 2.3) of Proposition 6 shows, the auctioneer is better off by selling all of her items in one period. Otherwise the auctioneer will randomize between revealing and not revealing.

The structure of the equilibrium above is similar to one encountered in many games of incomplete information like this one. However, the result has interesting implications. If the auctioneer is a high type, it might pay for him or her to hide this information, given what the bidders believe about her type. Note that this information or the lack thereof does not affect bidders’ valuations; it does, however, affect bidders’ strategies. Uncertainty about the existence of a second round of auctioning makes a bidder bid more aggressively for the current auction. Since it acts in this way for all bidders, it has an effect similar to an overall rise in bidder valuations.

2.6 Conclusions & Extensions

This paper has considered the simplest version of an important problem in order to gain intuition that can, we hope, be transferred to more complex settings. An auctioneer has $2k$ items for sale and bidders know that he has to sell all of the items. The auctioneer has to decide whether to sell them in two lots or one. There is discounting of future payoffs. We find that when bidders have complete information about how many periods there will be, whether or not to sell the items in one period or two largely depends on the market competition situation. When the market is less competitive, the auctioneer can earn more from selling in two periods. The so-called “optimal auction” with a reserve price sometimes performs worse than the sequential auction without reserve price, when
there is no commitment to the reserve price. We then study the case when the auctioneer does not reveal how many auctions he is going to have and when the bidders believe with probability $\rho$ that he is of High type. In this case we find that when $\rho$ is small enough, the auctioneer can realize the monopoly power by pretending to be a Low type, selling his or her items in two periods. This shows that it is not always better for the auctioneer to offer this piece of information to the bidders. The reason is that even though this is an independent private value auction, the potential for two periods affects the bidders’ strategies, and the auctioneer can take advantage of this common effect to increase market competition among bidders. We then compare this type of auction with an auction with reserve price but without commitment. We find that under some conditions the sequential auction performs better.

Much future work, of course, remains to be done. For example, what will be the equilibrium if there is a capacity limit, so that in each period the auctioneer decides how many items to purchase (hold), instead of purchasing all the items before the auction. Second, it might also be interesting to add uncertainty in the market in different periods, for example of a stochastic number of bidders, so that both the auctioneer and the bidders are not sure about the future when they make decisions in the first period. Third, we have looked at only two periods; the general multi-period case is also of interest. However we feel that the more general case should not be qualitatively different from the one we consider.
Chapter 3

Paid Placement in Information Gatekeepers

3.1 Introduction

Individual and organizational decision makers often turn to advisors, consultants, or counsellors for advice, recommendation, and help in determining which alternatives to consider or how to rank them. While the transportation revolution increased the number of choices available for most decisions (such as those related to purchase of goods and services) by removing barriers related to distance, the information revolution has dramatically increased the capability of consumers to collect information relevant to the decision. For example, the World Wide Web presents repositories of information—from text to multimedia, from amateur opinions to expert thought, from voluntary contributions to commercial interests—on every conceivable topic. The average decision maker, whether searching for information goods or for collateral information related to some other decision or product, is presented with a huge consideration space, much too large for unguided or enumerative search. Consequently, the last few years have witnessed tremendous growth in the area of information gatekeepers. Common examples of information gatekeepers on the Internet include Internet search engines, comparison shopping systems, recommender systems, referral sites, Web portals, online travel services, and other information systems that present, or filter, alternatives using databases and algorithms for information retrieval.

An information gatekeeper is an information system that is able to influence decision-making due to its expertise and knowledge related to the decision topic. Its
vast repository of relevant information and algorithms for matching elements of this repository to consumer requirements allows it to search for, match, and evaluate alternatives, thereby offloading some search and evaluation tasks from the decision maker. These systems are seen as gatekeepers because most decision makers would be unable to conduct search and evaluation without the assistance of such systems. When presented with a query, a gatekeeper system generates a recommendation list. The gatekeeper’s influence may manifest in terms of which alternatives are seriously considered, or perhaps how the decision maker values certain alternatives. The position of an item in the recommendation set, i.e., its rank, affects the likelihood that the item is examined further by the decision-maker. For example, McLuhan (2000) finds that content providers on the Internet experience greater click-through and conversion rates when they are more highly ranked by a search engine. Similarly, Smith et al. (2001) report evidence about the strong correlation between ranking and choice in the case of airline reservation systems.

Given the importance of information gatekeepers in today’s society it is useful to examine the choice and impact of their quality decisions. Two aspects dominate the definition of gatekeeper quality. The first is the technology level (or the intrinsic quality) of the gatekeeper: in the case of an Internet search engine, this includes the information retrieval algorithm, Web crawling algorithm, size of the database index, and user interface. The second element, a managerial decision, is the extent of bias in the gatekeeper’s service i.e., the gatekeeper’s relative inclination to recommend certain candidate alternatives because of incentives provided for making the recommendation. We define bias as the deliberate perturbation of advice (or a search result) in order to derive monetary gain from a third party (a content provider, who might expect this to increase profits in a future transaction with the decision maker). Such bias is referred to as paid placement, sponsored listing or preferential placement. Paid placement usually requires a minor modification of the ranking algorithm or to the display of results, either
Fig. 3.1. Paid placement in a metasearch engine.
of which can be made at very low cost. Such bias is widespread in many information gatekeepers including search engines, information portals, metasearch engines, online travel services, and shopbots. Figure 3.1 provides an illustration.

This paper develops economic models to analyze the occurrence, extent, and impact of bias in information gatekeepers, as well as the interdependence between bias and technology decisions. Internet search engines illustrate the importance of information gatekeepers today as well as our research question in this area. Lawrence and Giles (1999) estimated the publicly indexable Web at 800 million pages, 6 terabytes of text data, on 2.8 million servers, as of February 1999. In the absence of effective search engines, which serve as a gateway to the Web, few people can derive effective value from this vast repository. A recent USA Today (Dec. 11, 2000) article states that 100 million queries are made on U.S. search engines each weekday. A study of Web usage by Media Metrix found that the top 3 search engines were each visited by 61%, 56% and 40% of tracked Internet users during the past month (March 2000). Early Internet search engines were independent, non-commercial, aggregators of content, free of charge to users, and relied on advertising for revenues. The drop in venture capital and advertising revenues forced search engines to look for alternative sources, paid placement is one of the most important. Today, nearly all major search engines and portals employ paid placement. Similarly, shopbots recommend certain vendors not because they provide the lowest price but because it obtains a higher commission from them.

Preferential placement in Internet-based information gatekeepers has evolved in a similar way as in more traditional gatekeepers. For example, the transformation of search engines from independent to commercial sites finds a parallel in the radio stations industry: while there were only independent radio stations in the early 1900s, today there

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1 http://www.usatoday.com/life/cyber/tech/cti895.htm
2 http://searchenginewatch.com/reports/mediametrix.html
3 See http://www.searchenginewatch.com/sereport/01/05-metasearch.html
are about 10,000 commercial radio stations in the U.S. and only about 2,500 independent stations. Commercial radio stations employ a pay for play policy, accepting money from record companies and “pushing” their music by playing these songs more frequently. The music played on radio stations influences sales of recorded music since listeners tend to rely on the expertise of radio deejays, just as consumers turn to shopbots in making purchase decisions. Pay-for-play caused huge controversy (known as the payola scandal) in the middle of the last century. Today, however, it has evolved into a legal scheme so long as the radio station announces the play with a sponsorship line [23]. The similarity between paid placement and pay-for-play underlines the need for increased managerial and policy attention on bias in information gatekeepers.

First focusing on bias as a managerial decision regarding quality, this paper addresses the following questions: When should a gatekeeper employ (or not employ) bias in its recommendations? How does a gatekeeper set its bias level, and how do parameters such as technology level and advertising rates affect this choice? How does bias affect market outcomes, including gatekeeper profits and market coverage? How does competition affect these outcomes, and what can we predict about quality levels in the market for information gatekeepers? Sections 3.3 and 3.4 set up a single period model answering these questions. Then in Section 3.5 we endogenize the gatekeeper’s technology level, and extend the analysis to a two-period game where each gatekeeper sets the technology level in the first period and bias level in the next period. We describe the subgame perfect equilibrium and discuss its implications. In the end (Section ??) we study the socially optimal bias choice and compare the result with the market outcome outcome. We conclude with a summary of our results and possible application of our work to other forms of Internet-based information intermediaries.
3.2 Literature Review

Bay and Morgan (2001) [10] argue that modern markets for information tend to be dominated by “information gatekeepers” that specialize in collating, aggregating, and searching massive amounts of information available on the Web, and can often charge consumers, advertisers, and information providers, for their ability to acquire and transmit information. Wise and Morrison (2001) [51] emphasize the increasing role of information gatekeepers in today’s economy, noting that in business-to-business markets, “value has shifted from the product itself to information about the product.” Given the broad relevance of information gatekeepers to modern society, and the analogies between the Internet and more traditional forms of communication, the issue of bias has received attention in the media communications, telecommunications policy and social sciences literatures (see e.g., [36] and [91]). Introna and Nissenbaum (2000) [27], in a socio-political qualitative analysis, argue that search engines raise political questions, due to their systematic exclusion of certain sites and systematic favoring of others. They argue that paid placement, as a commercial practice, goes against the idea of the Internet as a democratizing force. Introna and Nissenbaum (2000) [27] wonder whether the market mechanism could serve as an acceptable corrective, and conclude that it is unlikely to do so.

Preferential placement is also discussed in many other business settings such as the airline industry. Airlines began developing computerized reservation systems in the 1970s, and such systems quickly became indispensable for both travel providers (airlines) and travelers (via their travel agents). Display bias soon became a big issue, Smith et al. (2001) [49] write: “Flights for the carrier operating the [system] were sorted higher in the list of candidate itineraries.” Copeland et al. (1995) [14] provide specific evidence that placement bias had significant impact on user decisions and provider profits, noting that “An average of 70% of all bookings were made from the first screen display (the first
options) presented to the travel agent.” In the 1980s, the Department of Transportation regulated the design of such systems, however the airlines that owned the leading systems found other ways to maintain some preferential placement. With the advent of Internet-based travel search engines in the 1990s, placement bias has become an issue in this setting as well. Recently, Northwest Airlines complained about placement bias in a leading Internet travel search engine, Expedia, arguing that “Expedia favors airlines that pay higher fees” [52]. Paid placement bias also occurs in the financial advice industry. Howard and Underwriter (2002) [47] cited “statistically significant evidence of advisors recommending one provider’s offering over another because it paid a higher commission”, and propose “independent financial advisors ... need to become genuinely independent”, where the “independent advisor” means “an advisor who is not paid by a provider”.

In devising a paid placement strategy, gatekeepers must therefore balance placement revenues against the loss of paid placement. Bhargava and Choudhary (2001b) [12] and Corbett and Karmarkar (1999) [15] study the case where an information intermediary has the option to charge subscription fees for customers and listing fees for suppliers. However, Bhargava and Choudhary (2001b) [12] consider only a one-sided network benefit, and their model does not incorporate advertising revenue. Corbett and Karmarkar (1999) [15] model two-sided network benefits, but assume homogeneous content providers and do not incorporate advertising revenue. Baye and Morgan (2001) [10] applied a game theoretic model to study a similar question. None of these papers considers the possibility that the gatekeeper may bias its outputs due to payments from content providers. In a model of an advertising-based website, Dewan et al. (2001) [17] studied the “tradeoff between revenue per customer and the number of customers who choose to visit the site.” They model the advertising policy as the website’s lifetime decision and model the problem as an infinite horizon planning model with discounting, with the objective to maximize the market value of the firm. Gabszewics et al. (1999) [21] analyze the case of two TV channels competing in both the audience market and the
advertising market. But in both these models, advertising is the only revenue source, hence no trade-off between different resources is considered.

Besides the tension between placement revenues and costs, information gatekeepers must also consider the effect of placement bias under competition. In their classical work on quality and competition Shaked and Sutton (1982) [45, 46] discuss the impact of vertical differentiation on competition, when consumers are differentiated in their income levels. They find that under certain range of consumers valuation, no more than 2 firms can earn positive profit, and one of the firm will set the highest possible quality levels in order to minimize the competition in qualities. Banker et al (1998) [9] model consumer demand as a linear function of the price and quality levels, and analyze what is the link between quality and competitive intensity, under three different interpretations of competition intensity (namely, firms differ in “intrinsic demand potentials,” firms can “not allowed to cooperate to set quality levels,” and “number of firms entering the market”).

3.3 Optimal Bias Level for a Monopolist Gatekeeper

In developing an economic model of a gatekeeper, we consider three types of entities: users of the gatekeeper, content providers, and third-parties such as advertisers and licensing firms. The gatekeeper has a database index of content providers, developed using its technologies for information gathering and indexing. The information gathering and indexing technologies, ranking and retrieval algorithms, and user interface, together comprise the technological quality $\xi$ of the gatekeeper, and determine its intrinsic demand. When presented with a query, it applies retrieval and ranking algorithms to generate a recommendation list, and presents this list to the user within some interface that contains features for traversing and managing this list. The recommendation list may include certain items for which content providers paid for placement. The gatekeeper’s degree of paid placement is public information. This section treats technology
level as given and analyzes the case of a single gatekeeper. Section 3.4 extends the analysis to duopolistic competition while Section 3.5 considers the choice of technology level for competing gatekeepers.

3.3.1 Model Setup

How does paid placement affect the gatekeeper’s demand function? Our reasoning is as follows. Users visit the gatekeeper because of its expertise and knowledge, expecting a high correlation between the gatekeeper’s ranking of an item and their own expected utility from pursuing that item. When the ranking list is biased, however, this expected correlation is reduced: even when the paid items are clearly identified, users are not sure that a high-ranked item is highly relevant, or that a paid item is irrelevant. The user incurs a cost in trying to make judgements about items on the paid vs. unpaid lists. This analysis has anecdotal support in the literature, as indicated in Sections 3.1 and 3.2. For example, consumer organizations have argued that paid placement bias in Internet-based travel search services hurts travelers [52]. Similarly, Goodman (2000) [24] argues that search engines must act as “referees—fair arbiters of relevance” or they will lose market share. Hence, paid placement bias reduces the gatekeeper’s perceived quality and leads to a fall in demand. This finding is confirmed and extended in our own survey about search engine usage, covering 200 users of an academic search service (NEC Research Institute ResearchIndex) in July 2002 (please see Appendix ??). The survey indicated that most users do not favor paid listings. However, a higher bias does not necessarily cause a user to switch to another search engine. Users consider both the technological quality and the bias level in evaluating their choices, and most users stated that they have a “favored search engine” indicating that a gatekeeper with greater quality and lower bias will capture a greater, but not necessarily entire, market share.

We write $M(\xi, x)$ to represent the market demand function for a gatekeeper with technological quality $\xi$ and bias level $x$. There are different measures of bias level in an
information gatekeeper. For example, it can be defined as the difference between the results with or without paid placement. For simplicity, in this paper, we ignore the fact that paid links can be presented in different ranks and it might create different bias levels (we discuss this question explicitly in Chapter 3). Thus we define bias level $x$ as the absolute number of paid placements relative to the total number of results presented.

The gatekeeper’s intrinsic demand is $\mathcal{M}(\xi, 0)$ i.e., the demand when it employs no bias. The function $\mathcal{M}(\xi, x)$ is increasing in $\xi$ and decreasing in $x$, so that $\mathcal{M}_\xi(\xi, x) > 0$ and $\mathcal{M}_x(\xi, x) < 0$. $\mathcal{M}_\xi$ is the marginal demand for technological quality, and $\mathcal{M}_x$ is the marginal reduction in demand due to bias. We define the sensitivity of the gatekeeper’s demand function to changes in bias as

**Definition 1 (Bias Elasticity of Demand).** The elasticity of demand towards bias is $\epsilon(x) = \frac{-\mathcal{M}_x(\xi,x)x}{\mathcal{M}(\xi,x)}$.

and the sensitivity to $\mathcal{M}_\xi(\xi,x)$, the marginal demand due to increase in quality, as

**Definition 2 (Elasticity of Marginal Demand).** The elasticity of marginal demand for quality is $\kappa(\xi, x) = \frac{-\mathcal{M}_\xi(\xi,x)x}{\mathcal{M}_\xi(\xi,x)}$.

The relative decline in demand due to a small amount of bias is $\frac{-\mathcal{M}_x(\xi,x)}{\mathcal{M}(\xi,x)}$. To facilitate the analysis, we make the following assumption.

**Assumption 1.** The relative decline in demand is non-decreasing in $x$, i.e., $\frac{\partial}{\partial x}\left(\frac{-\mathcal{M}_x(\xi,x)}{\mathcal{M}(\xi,x)}\right) \geq 0$.

Assumption 1 is trivially satisfied when $\mathcal{M}_{xx} < 0$, is equivalent to the condition $\frac{-\mathcal{M}_{xx}}{\mathcal{M}_x} < \frac{-\mathcal{M}_x}{\mathcal{M}}$, and implies that the elasticity of demand towards bias is non-decreasing in bias ($\epsilon'(x) > 0$).

The gatekeeper offers preferential placement to content providers who pay a placement fee $\gamma$. Since we ignore the ranking of the paid links, for simplicity we assume that
all content providers pay the same $\gamma$ if they are listed in the paid portion. Providers that receive preferential treatment expect additional profits due to increased transactions with users. For example, Copeland et al. (1995) report that American Airlines earned over $100 million per year due to favorable display bias in the SABRE reservation system. This extra revenue potential would obviously be lower either if fewer travel agents used SABRE or if SABRE provided preferential treatment to many airlines. Thus, a content provider’s expected benefit from paid placement increases in the gatekeeper’s number of users but decreases in the number of providers receiving such treatment. Noting that the gatekeeper’s market coverage $M$ is its quality measure from the content providers’ perspective, we use a quality-adjusted linear demand function (as in Banker et al. (1998)) to represent the demand for preferential placement as

$$\gamma = bM(\xi, x) - e x$$

where $b$ (the content provider’s per-user expected increase in profits due to preferential placement) and $e$ are suitable non-negative constants. The gatekeeper’s paid placement revenue equals $\pi_p = \gamma \cdot x$.

Apart from its paid placement revenues, the gatekeeper generates revenues due to its user base, including revenues from third-party firms such as advertisers, fees for licensing its information retrieval technologies and usage fee. For simplicity in exposition, we refer to all user-based revenues as advertising revenues. Let $s$ represent the (advertising) value per user, then the gatekeeper’s revenues from its user base equal $\pi_u = s \cdot M(\xi, x)$.

To summarize, the impact of paid placement is twofold. Users of the gatekeeper expect a neutral and fair marketplace, hence paid placement negatively impacts the gatekeeper’s perceived quality and demand. Since loss of market share causes a fall in
advertising revenue, the gatekeeper must trade-off potential revenues from paid placement with those from advertising, and choose a paid placement level $x$ to maximize its total profit. Thus, the gatekeeper’s decision problem is to choose the optimal bias level $x^*(\xi)$, given its technological quality $\xi$, in order to maximize profit

$$\pi^*(\xi) = \max_x \pi(\xi, x) = \max_x \left( sM(\xi, x) + \gamma x = (s + bx)M(\xi, x) - ex^2 \right)$$

(3.1)

The profit function is concave when $-\frac{M_{xx}}{M_x} < \frac{2b}{s+bx}$ for all $x$.

### 3.3.2 Optimal Placement Strategy

From first-order conditions, a candidate optimal bias level is given by

$$sM_x(\xi, x) + bxM_x(\xi, x) + bM(\xi, x) - 2ex = 0$$

(3.2)

To interpret the first-order condition, rewrite Eq. (3.2) as $(bM(\xi, x) - ex)\Delta x = -sM_x(\xi, x)\Delta x - \gamma x x \Delta x$. The left hand side represents the increase in placement revenue for a small change in the paid placement level: this is the marginal benefit from an increase in bias. The first term on the right is the fall in user-based revenue from lower demand, and the second term is the fall in placement revenue due to the lower fee; the sum of the two is the marginal cost of bias. At the optimal bias level, marginal revenue equals marginal cost. Examining second-order conditions (see Appendix), we explain how the gatekeeper’s decision to employ paid placement depends on the decline in demand due to paid placement.

**Proposition 7.** The gatekeeper chooses to be independent (i.e., $x^* = 0$) when the relative loss in intrinsic demand is high, i.e., $-\frac{M_x(\xi, 0)}{M(\xi, 0)} \geq \frac{b}{s}$. When $-\frac{M_x(\xi, 0)}{M(\xi, 0)} < \frac{b}{s}$ and the profit function is concave, the gatekeeper employs paid placement, and the optimal bias level $x^* > 0$ is given by Eq. (3.2).
Proposition 7 states the gatekeeper’s paid placement strategy in terms of its intrinsic demand function \( M(\xi, 0) \). Intuitively, the gatekeeper chooses to be independent when it suffers a sufficiently big relative drop in demand on adopting paid placement. As the value per user \( s \) increases (or \( b \), the content provider’s per-user value of paid placement, decreases), a greater absolute loss in demand is needed to prevent the gatekeeper from adopting paid placement. A smaller value per user, \( s \) (or a larger \( b \)) increases the likelihood that the gatekeeper implements paid placement, since it is willing to sacrifice user-based revenues for placement revenues. The proofs of Proposition 7 and other results are given in the technical appendix.

We can extend this proposition to understand how a gatekeeper’s technological quality level affects its bias. Higher quality gatekeepers face, ceteris paribus, greater demand. Are they more likely to be independent? Interestingly, we find that this is not always the case. From Proposition 7, we derive

**Corollary 3.1.** Given the same market situation \((b, e, s)\), higher-quality gatekeepers are more likely to choose independence when the relative decline in intrinsic demand is increasing in quality, i.e., when \( \frac{\partial}{\partial \xi} \left( \frac{-M_x(\xi, 0)}{M(\xi, 0)} \right) > 0 \). When \( \frac{\partial}{\partial \xi} \left( \frac{-M_x(\xi, 0)}{M(\xi, 0)} \right) < 0 \), higher quality gatekeepers are less likely to choose independence.

More generally, consider the sensitivity of the placement strategy to its quality of service. How will a gatekeeper’s placement strategy change as its technological quality \( \xi \) improves? An increase in \( \xi \) increases the gatekeeper’s demand, increasing the potential advertising revenue. An increase in bias level will increase placement revenues but reduces the demand and therefore also the potential advertising revenue. The net effect on demand combines the conflicting effects due to increased quality and increased bias. The net effect on profits combines the effect on placement and advertising revenues, hence it must depend on the elasticity of marginal demand \( \kappa(\xi, x) \) and the fractional contribution of paid placement in the gatekeeper’s revenue function. Proposition 8 formalizes...
this result. The term \( \frac{-M_{\xi x}x}{M_\xi} \) is the elasticity of marginal demand for quality \((\kappa(\xi, x))\), while the term \( \frac{bx}{s+bx} \) is the placement revenue per user relative to the total revenue per user.

**Proposition 8.** If \( \frac{-M_{\xi x}x}{M_\xi} < \frac{bx}{s+bx} \) for all \( x \), the gatekeeper increases its bias level with an increase in its technology level \( \xi \). If \( \frac{-M_{\xi x}x}{M_\xi} > \frac{bx}{s+bx} \) for all \( x \), the gatekeeper reduces its bias level. In either case, the gatekeeper’s profit increases with \( \xi \).

When \( M_{\xi x} > 0 \), then Proposition 8 trivially yields that \( x^*(\xi) \) increases with \( \xi \). Intuitively, when the demand function falls less steeply for higher quality \((M_{\xi x} > 0)\), then the gatekeeper—in making its optimal tradeoff between placement and user-based revenues—finds it profitable to increase the weight of placement, hence it increases its bias level as it improves quality. When the demand falls significantly more steeply for higher quality, then the gatekeeper prefers a higher level of independence as its quality improves.

Next, we consider the impact of per user profit. How do changes in \( s \) affect the search engine’s paid placement strategy?

**Proposition 9.** An increase in per user profit, \( s \), decreases the number of paid listings \( x^* \), hence the gatekeeper increases its market coverage \( M \). The search engine also improves its total profits \( \pi \). Correspondingly, the gatekeeper increases its bias level with an increase in \( b \) and decrease its bias level with an increase in \( e \).

To understand this result, consider the gatekeeper’s tradeoff between its two revenue sources. As \( s \) increases, the potential for advertising revenue increases, hence a partial sacrifice of revenues brought by users imposes a greater cost on the gatekeeper. Therefore, it reduces its level of paid placement in order to provide greater utility to users, and captures a greater percentage of potential advertising revenues. Similarly,
when the negative externality of multiple paid placements among the content providers
\((e)\) is higher, for example, when consumers’ attention is more limited, the gatekeeper
will set a lower bias level.

### 3.4 Competition between Gatekeepers

This section examines paid placement bias under competition, and we analyze two
settings: one, when there are two symmetric gatekeepers with identical technology level;
and two, when the gatekeepers have different technology levels. We discuss the impact
of competition on the gatekeeper’s choice of bias level, and the impact of optimal bias
decisions on the competitive intensity achieved in the market. A monopoly gatekeeper
will choose bias level in a way that causes some users to depart the market; increase
in competition, or increase in (technological) quality, will cause the number of users to
increase. In general, we define an increase in the competitive level of the industry as an
increase in market coverage.

#### 3.4.1 Gatekeepers with the Same Technology

Suppose the market contains two gatekeepers who are endowed with the same
technology level \(\xi\), hence have the same intrinsic demand. Let \(x_1\) and \(x_2\) represent the
bias levels of the two gatekeepers. The total demand for gatekeeper services in this
setting is \(M(\xi, \min\{x_1, x_2\})\). How do the gatekeepers split this demand when users
are heterogeneous in their tolerance toward bias? Clearly, users with tolerance between
\(\min\{x_1, x_2\}\) and \(\max\{x_1, x_2\}\) will prefer the less biased gatekeeper. Users who have
tolerance higher than \(\max\{x_1, x_2\}\) find both gatekeepers attractive, and some fraction
\(\alpha (\geq \frac{1}{2})\) will choose the one with lower bias. For simplicity, we set \(\alpha = \frac{1}{2}\), but our results
hold even when the gatekeepers do not split this segment of the market equally. Since
the two gatekeepers have the same technology level, we omit \(\xi\) in the demand function.
and write it as $M(x_i; x_j)$, which represents the demand for the gatekeeper with bias level $x_i$ when the competing gatekeeper has bias level $x_j$. Hence if $M(x)$ denotes the monopoly demand at bias level $x$, then

$$M(x_i; x_j) = \begin{cases} 
M(x_i) - \frac{1}{2}M(x_j) & \text{if } x_i < x_j \\
\frac{1}{2}M(x_i) & \text{if } x_i \geq x_j
\end{cases}$$

Consider the effect of bias on demand for gatekeeper 1.

1. $(x_1 \geq x_2)$ If it employs a greater bias, gatekeeper 1 captures only half its monopoly market, so $M(x_1; x_2) = \frac{1}{2}M(x_1)$. It charges a fee $\gamma_1 = \frac{1}{2}M(x_1) - ex_1$ for paid placement. The gatekeeper’s profit function is:

$$\pi_1 = \frac{s}{2}M(x_1) + \frac{b}{2}M(x_1)x_1 - ex_1^2 \quad (3.3)$$

2. $(x_1 < x_2)$ If it has the lower bias, then gatekeeper 1 has a larger market share $M(x_1; x_2) = M(x_1) - \frac{1}{2}M(x_2)$. It charges placement fee $\gamma_1 = bM(x_1) - \frac{1}{2}bM(x_2) - ex_1$. Given the competitor’s bias level $x_2$, the gatekeeper’s profit function is

$$\pi_1 = sM(x_1) - \frac{s}{2}M(x_2) + bM(x_1)x_1 - \frac{b}{2}M(x_2)x_1 - ex_1^2 \quad (3.4)$$

To solve for gatekeeper 1’s best response to gatekeeper 2’s bias level, $\Omega_1(x_2)$, we compute first order conditions for Eq. (3.3) and Eq. (3.4). Since the two gatekeepers are symmetric, we determine gatekeeper 2’s best response $\Omega_2(x_1)$ in a similar way. Figure 3.2 displays these response functions. The unique pure strategy Nash equilibrium is

$$x_1^* = x_2^* = \hat{x}^* = \arg \max_x s \frac{1}{2}M(x) + \frac{b}{2}M(x)x - ex^2 \quad (3.5)$$
Fig. 3.2. Best response functions $\Omega_i$ and optimal bias $x_i^*$ for gatekeepers with identical technology level.

**Proposition 10.** When the two gatekeepers have an identical technology level $\xi$, they choose the same bias level $\hat{x}^*$, obtain equal market share, and make the same profits. Whether the gatekeepers set zero or positive bias is determined by the same conditions as in the monopoly analysis (Proposition 7). The change in bias level with a change in technology level follows the same logic as in Proposition 8.

How does competition affect the users’ and content providers’ welfare, and how does the bias level under competition ($\tilde{x}^*$) compare with the bias level $x^*$ chosen by a single gatekeeper who has demand function $\mathcal{M}$? Were the gatekeepers to choose the same bias level $x^*$, they would then split the total market $\mathcal{M}(x^*)$, each attracting only half the original users. The lower user base reduces the content providers’s willingness-to-pay for paid placement as well. Each gatekeeper’s user-based revenues reduce by half, but their placement revenues fall more than half. A decrease in bias, at this level, will increase both user-based and placement revenues, making it profitable to reduce bias level. An alternative explanation may be derived by considering the demand elasticity.
toward bias at the optimal bias level. The competition between the gatekeepers requires setting the bias at a level where the elasticity is lower than at the monopoly bias level. However, if the gatekeepers choose the same $x^*$ and get half the demand, the elasticity at this point is higher than the optimal level. We summarize the results below.

**Proposition 11.** The optimal bias level $\hat{x}^*$ of two gatekeepers with identical technology level $\xi$ is less than the corresponding bias level for a monopoly gatekeeper. This competition between the gatekeepers increases users welfare, while the total gatekeeper surplus and the total surplus of content providers are both reduced.

### 3.4.2 Gatekeepers with Different Technology

Now consider the case where the two gatekeepers have different technology levels, hence differ in their intrinsic demand. How do differences in technological quality affect the choice of bias levels under competition? Specifically, compared to the case of two gatekeepers with identical quality, how does the choice of bias levels change when one gatekeeper improves its technology? Does this improvement in technology level, and the corresponding change in competitive intensity, cause a further increase—or a decrease—in user welfare?

Let $\xi_i$ represent the technology level of gatekeeper $i$ where, without loss of generality, $\xi_1 > \xi_2$. For ease of comparison with the case of competition between identical gatekeepers, we assume that $\xi_2 = \xi$ as before, and gatekeeper 1 has a better technology $\xi_1$. Following a similar argument as in the previous section, the demand function for gatekeeper $i$ is

$$M(\xi_i, x_i; \xi_j, x_j) = \begin{cases} M(\xi_i, x_i) - \frac{\xi_i}{\xi_i+\xi_j}M(\xi_j, x_j) & \text{if } M(\xi_i, x_i) > M(\xi_j, x_j) \\ \frac{\xi_i}{\xi_i+\xi_j}M(q_i, x_i) & \text{otherwise} \end{cases}$$

(3.6)
where \( M(\xi, x) \) represents the demand when there is only one gatekeeper with technology level \( \xi \) and bias level \( x \). In order to determine choice of bias levels of gatekeepers with different technologies, it is useful to view \( M(\xi, x) \) as a measure of the composite quality level of the gatekeeper. So, the gatekeeper with higher composite quality measure will gain a larger market share.

Following the same logic as before, if \( M(\xi_1, x_1) \leq M(\xi_2, x_2) \), then gatekeeper 1 has \( \frac{\xi_1}{\xi_1 + \xi_2} \) of its original market \( M_1(\xi_1, x_1) \). Its fee for paid placement is \( \gamma_1 = \frac{\xi_1}{\xi_1 + \xi_2} b M_1(\xi_1, x_1) - e x_1 \). We write the profit function

\[
\pi_1 = \frac{\xi_1}{\xi_1 + \xi_2} (s M(\xi_1, x_1) + b M(\xi_1, x_1) x_1) - e x_1^2 \tag{3.7}
\]

For the other case, if \( M(\xi_1, x_1) > M(\xi_2, x_2) \), then gatekeeper 1 has a larger market share: \( M(\xi_1, x_1) - \frac{\xi_1}{\xi_1 + \xi_2} M(\xi_2, x_2) \). Then \( \gamma_1 = b M(\xi_1, x_1) - \frac{\xi_1}{\xi_1 + \xi_2} b M(\xi_2, x_2) - e x_1 \). Thus the profit function is

\[
\pi_1 = s M(\xi_1, x_1) - \frac{\xi_2}{\xi_1 + \xi_2} s M(\xi_2, x_2) + b M(\xi_1, x_1) x_1 - \frac{\xi_2}{\xi_1 + \xi_2} b M(\xi_2, x_2) x_1 - e x_1^2 \tag{3.8}
\]

Taking \( \xi_1 \) and \( \xi_2 \) as given, we can compute the best response functions of each gatekeeper, depicted as \( \tilde{\Omega}_1 \) and \( \tilde{\Omega}_2 \) in Figure 3.3. Let \((\tilde{x}_1^*, \tilde{x}_2^*)\) denote the equilibrium bias levels. How do the technology differences impact the bias decisions of each of the gatekeepers? Intuitively, when the two gatekeepers have different technology levels, the gatekeeper with higher technology level has advantage in the competition. It can either further differentiate itself from the lower technology gatekeeper by choosing \( x_1 < x_2 \), or take advantage of the users’ tolerance towards bias because of its higher technology by setting \( x_1 \geq x_2 \). The lower technology gatekeeper’s choices are to set a very high bias level to differentiate itself from superior technology gatekeeper, or to reduce its bias level and try to capture as much market share as possible. Proposition 12 gives the
Fig. 3.3. Best response functions (thick curves) and optimal bias for gatekeepers with different technology levels. The thin curves represent the best response functions for identical gatekeepers.

equilibrium bias levels $\tilde{x}^*_i$ for each gatekeeper, and they are compared to $\hat{x}^*$, the bias level for competing but identical gatekeepers.

**Proposition 12.** The gatekeeper with lower technology $\xi_2$ will set a bias level $\tilde{x}^*_2$ below $\hat{x}^*$. Gatekeeper 1 may increase or decrease its bias level from $\hat{x}^*$. The sufficient condition for gatekeeper 1 to set bias below $\hat{x}^*$ is

$$\left( \frac{\partial(-\mathcal{M}_x/M)}{\partial \xi} \geq 0 \right) \text{ and } \left( \kappa(\xi, x) > \frac{bx}{s+bx} \right) \text{ at } \hat{x}^*$$

In either case, the gatekeeper with better technology delivers better composite quality, hence achieves greater market coverage in equilibrium. The total market covered by both gatekeepers is $\mathcal{M}(\xi_1, \tilde{x}^*_1)$.

One interesting observation due to Proposition 12 is that the equilibrium market coverage of gatekeeper 1 is always greater than that of gatekeeper 2, even when gatekeeper 2 decreases its bias level and gatekeeper 1 increases its bias level. Second, the
gatekeeper (2) with lower technology always decreases its bias level below $\hat{x}^*$. Third, whether gatekeeper 1 reduces its bias level below $\hat{x}^*$ depends on a) the elasticity of marginal demand with respect to bias, $\kappa(\xi, x)$, and b) the fraction of placement revenues per user, $\frac{bx}{s+bx}$. The intuition behind these results is that when $\kappa(\xi, x)$ is high, then the gatekeeper with better technology will choose lower bias in order to maintain a better composite quality; this is more likely to happen when the technology levels are not significantly differentiated. If $\kappa(\xi, x)$ is not sufficiently large, then gatekeeper 1 can exploit its technology superiority to set higher bias; this is more possible when the technology levels are greatly differentiated, because it will cost a lot more for the gatekeeper with lower technology to set up a higher composite quality measure. Finally, we note that when the demand function falls less steeply for higher $\xi$ ($M_{\xi x} > 0$), then gatekeeper 1 will always increase its bias level.

We have discussed the equilibrium bias strategy when the two gatekeepers have different technology levels. How will this difference affects users’ welfare?

**Proposition 13.** When there exists a gatekeeper with lower technology level, the gatekeeper with better technology ($\xi_1$) will set a bias level ($\hat{x}^*(\xi_1)$) below its monopoly bias level ($x^*(\xi_1)$); Under competition, the total market coverage when gatekeeper 1 has better technology ($M(\xi_1, \hat{x}^*(\xi_1))$) is larger than that when the two gatekeepers have the same technology levels ($M(\xi, x^*(\xi_1))$).

So the entrance of a second gatekeeper, even though it has a lower technology level, will force the existing gatekeeper to reduce its bias level, and this increases the total market coverage, as well as users’ welfare. In the same way, if one gatekeeper improves its technology level, users welfare is larger than when the two gatekeeper has the identical but lower technology level.
3.5 Setting Quality

To this point, our analysis of gatekeepers’ bias decisions has treated technology level as given. Now we consider the case where gatekeepers choose both their technology and bias levels. We formulate the following game: There are two gatekeepers in the market endowed with different cost structures \( c_1(\xi) \) and \( c_1(\xi) \). The cost function \( c(\xi) \) is increasing and convex in \( \xi \), and after \( \bar{\xi} \) it approaches to \( \infty \). In the first period the two gatekeepers choose their technology levels simultaneously. In the beginning of the second period, they observe each other’s technology levels and choose their bias levels simultaneously. What will be the equilibrium behavior of these two gatekeepers?

We use backward induction to identify the subgame perfect equilibrium in technology and bias. The second period bias decision given technology level has already been worked out in the last section. Now in the first period the two gatekeepers have to decide how to determine their technology levels. Will both of the firms set quality as high as possible? Or will they differentiate themselves by setting different technology levels? Intuitively the gatekeeper with the superior cost function has competition advantage, so will it set a higher technology level?

**Proposition 14.** When the cost of improving technological quality is not very steep, then both gatekeepers will set their technology levels at \( \bar{\xi} \), and they will share the market for information. When the cost of technology is sufficiently steep then the gatekeeper with superior cost function (gatekeeper 1) will set a higher quality level, which is increasing in \( \xi_2 \).

This is to say that if the improvement in technology is not too costly (please see Appendix B.3 for the precise statement), both firms expect that the increased revenue from advertising and paid placement will cover the cost of technology improvement, hence they will set their technology level as high as possible. Otherwise, there exists a cutoff point in the technology level, after that the benefit of raising technology level
does not compensate its cost. In this case, the gatekeeper with superior cost function will take advantage, set a higher technology level, capturing a higher market share.

3.6 Conclusions

This article considers information gatekeepers as a gateway connecting users and content providers, and analyzes the optimal paid placement (bias) strategy. This work applies to a large variety of information gatekeepers. Apart from Internet search engines (e.g. Google.com), other categories of information gatekeepers to which our work applies include recommender systems (e.g., at Amazon.com), comparison shopping services (e.g., mySimon.com), e-marketplaces and exchanges (e.g., FreeMarkets), and more traditional information gatekeepers such as investment advisors and television networks. Like search engines, many information gatekeepers generate user-based revenues, but also seek to obtain revenues from their provider-base by offering some form of preferential placement. For example, some Internet booksellers are influenced by advertising fees in determining their bestseller lists. Similarly, certain Internet exchanges provide preferential service (such as real time notification or favorable recommendation to buyers) to some clients in return for higher fees.

On the one hand, paid placement appears to be a financial necessity; on the other, bias level is one component of the gatekeeper’s composite quality measure, thus paid placement can hurt the gatekeeper’s perceived quality, in turn market share and its potential for revenues brought by users. We have developed a mathematical model for optimal design of paid placement. We find that under certain conditions, (namely, when the users responsiveness toward bias is not large enough), the gatekeeper will set a non-zero bias level. We determine sensitivity of the placement strategy to the user market characteristics, the gatekeeper’s technology level, and the advertising rate. Analyzing the duopoly case, we showed that competition reduces the gatekeepers’ bias
level. Two competing gatekeepers who have identical technology will choose identical bias level. When the gatekeepers are endowed with different technology levels, the one with better technology captures larger market share. We consider further the game where the gatekeepers set technology level in the first period and bias level in the second. The gatekeeper with superior cost function has a competitive advantage in this game: it will set a higher technology level, and capture a larger market share, even though it might set higher bias.

We are interested in extending our analysis to explore the impact of bias strategy on the gatekeeper industry’s long term structure, as well as on society’s total welfare. One task is to allow gatekeepers to improve their technology over time, thus the technology decision is not one time. Under this setting, it may be beneficial for a gatekeeper to set a higher bias level in the first stage, and use placement revenue to improve its technology in the next stage. Our models can also be extended to examine conditions under which the information gatekeeper will begin to charge users, and more specifically the case where the gatekeeper differentiates between users by offering two versions: a fee-based premium service with no bias in the query results, and a free basic version with paid placement bias. The fee-based premium version will bring additional user revenues to the search engine, however it may reduce placement revenues because paid placement becomes less attractive to content providers. In addition, the search engine’s market coverage and placement fee may change as well. This method has already been noticed in certain markets. For example, Levene and Guardian (2002)\textsuperscript{[32]} pointed out “the Financial Service Authority is understood to be considering compelling independent financial advisors to offer customers a clear choice between paying fees or forking out for recommendations through commission.” Our model can be extended to determine if it is optimal for the gatekeeper to offer differentiated service. Similar models can be developed to examine the impact of differentiation based on advertising. For example, some search engines have already began to offer fee-based premium search services that
contain no advertising. If this is the trend, it may eventually change people’s view of
Internet search engines as a free resource for fair information.
Chapter 4

Implementing Paid Placement in Search Engines: Computational Evaluation of Alternative Mechanisms

4.1 Introduction

Many web sites such as Internet search engines, web portals, and comparison shopping services, aim to provide information or recommendations to users who might be searching for information or trying to make a purchase decision. Paid placement advertising has established itself as an important revenue resource for such information-oriented web sites, especially for Internet search engines. Paid placement is a mechanism where the search engine deliberately biases its recommendations (or sequence of results) in return for a fee from providers who wish to get preferential placement on the results page. Paid placement has begun to dominate standard advertising, for both the payor and receiver of placement fees, since such links offer a greater probability of attracting interested buyers to the provider’s web site. A recent article in The Economist (April 18, 2002, “The Internet sells its soul”) attests to the growing important of paid placement (“...Internet search-terms are up for sale, as advertisers bid to push their sites up to the top of search-engine listings”) and notes that provider firms are willing to pay substantial amounts to search engines to direct traffic to their sites through paid placement links (e.g., firms selling digital cameras to Internet shoppers pay 60 cents per-click in return for a prominent position on results pages for the term “digital camera”). However, Such gatekeepers attract consumers because of their content and attempt to get advertising revenue from firms that wish to sell products to these consumers, thus creating a tradeoff between content and advertising (see e.g., [17] and [21]).
The tradeoff between placement and user-based revenues implied by the disutility that paid placement creates for users, makes the featured listings section a scarce resource for the search engine. When the search engine faces significant demand for this resource from content providers, how should it determine which providers get the opportunity to be included in the featured section? An obvious solution seems to be that preferential placement should be given to providers who have highest willingness to pay (WTP), and that the ones with higher WTP should be given a higher ranking. Each provider would pay a price corresponding to their ranking, regardless of the actual performance, i.e., the number of user click-throughs achieved. The market clearing price is easy to determine in this case. However, the uncertainty about the increased click-throughs due to paid placement causes a downward shift in the provider’s overall WTP for paid placement. This would result in a lower clearing price and lower placement revenues for the search engine. Recent research by [?] has shown that a contingency pricing scheme (where providers’ payment is contingent on their receiving click-throughs) is likely to improve placement revenues in face of such uncertainty. Indeed, the industry has largely adopted a model of paid placement where the price for placement is conditional on the number of click-throughs. Under this model, therefore, allocating the paid listings slots to those with the highest WTP is not an optimal strategy for the search engine: if the provider with highest WTP happens to be one with low relevance, this may result in very low placement fees. Alternately, should the search engine allot the highest rank to the provider who earns (or is likely to earn) the most click-throughs? How should the search engine determine the price for each rank in this case? Perhaps one should use a mechanism that combines WTP and the potential for click-throughs?

In practice, we observe many different approaches for allocating and pricing paid placement slots. Overture operates real-time auctions where providers bid how much they are willing to pay (per click) for appropriate words or phrases. Google’s AdWords paid placement program, however, ranks listings according to the product of willingness
to pay and the actual click-through rate of the listings. The motivation is to promote listings that receive a higher click-through rate, even if they make a lower bid, since click-through can be thought of as a proxy for relevance to the user. These paid listings appear to the right of standard search results on Google’s own site, and appear as sponsored links on their partner sites (including AOL.com and Yahoo!). Other paid placement brokers (including FindWhat.com and LookSmart.com) use other strategies for identifying, ranking, distributing, pricing, and displaying their paid listings.

In the following sections, we study four mechanisms for allocating and pricing the paid placement slots to content providers. The current section describes the mechanisms that are in use by two leading Internet search engines, Google and Overture, as well as two alternative mechanisms that are based on fixed payment and clickthrough-based payment respectively. In each case, we describe how the search engine chooses providers for paid placement, and develop a model for computing the search engine’s revenues from paid placement based on the relevance, willingness to pay, and the click-through rate of listings. Specifically, we describe the following four ranking protocols: (1) rank according to willingness to pay, (2) rank according to willingness to pay times click-through rate, (3) rank according to click-through rate, and (4) take-it-or-leave-it pricing for each rank, where reserve prices are set by the search engine.

4.2 Revenue Model

Suppose that $s$ listing companies, each with a single listing to advertise, compete for placement on a search engine for a particular search query. Each listing company $j$ is willing to pay $v_j$ per click for their listing. The listing has a “true” relevance score $\alpha_j$, encoding how likely the link is useful to users. We assume that the click-through for any listing is a function of both its relevance score and its position on the search engine’s results page. We use an exponentially decaying attention model, where the probability
that a user clicks on a link decreases by a factor of $\delta > 1$ for every position below the top position in the list of search results. We also assume that click-through increases linearly with relevance. So the average click-through for item $j$ listed in the $i$th position is $\frac{\alpha_j}{\delta^i-1}$.

We conceptualize the content provider’s true relevance and willingness to pay for a featured listing as random variables $\alpha_j$ and $v_j$, respectively. Let $f(\alpha_j, v_j)$ denote the joint density function. Define $r : I \rightarrow J$ to be the allocation function which allocates position $i$ to listing company $j$, where $J$ also contains a fictitious null provider, since the search engine may prefer to leave certain slots unfilled. Let $P_i$ represent the payment for position $i$. For the click-through based mechanisms, let $p_i$ represent the payment per click-through, so that in equilibrium $P_i = p_i \frac{\alpha_j}{\delta^i-1}$ where $j = r(i)$. Then the search engine’s placement revenue is:

$$E \left[ \sum_{i}^k P_i \right] = \sum_{i}^k \int_{0}^{1} \int_{0}^{1} P_i f(\alpha_{r(i)}, v_{r(i)}) d\alpha_{r(i)} dv_{r(i)}$$

where $k$ is the number of paid listings that the search engine decides to display out of the $s$ total bidders.

4.3 Mechanisms

We consider the following mechanisms:

1. **Rank by willingness to pay** (Overture). Every listing company bids their willingness to pay (per click). The highest $k$ listings are displayed, ranked according to their bid amount. Companies pay what they bid (first price auction). Let $r_i$ be the $i$’th element of the sorted list of values $v_j$. Since, in equilibrium, a bidder will bid the expected second-highest value below his own value, the amount paid is $p_i = v_{i+1}$ for this mechanism.
2. **Rank by willingness to pay times click-through** (Google). Every listing company bids their willingness to pay (per click). We compute the product of their willingness to pay and the click-through that they generate. They highest $k$ listings are displayed, ranked according to this product. Let $r(i)$ be the $i$th element of the sorted list of $\alpha r(i) v_r(i)$. In equilibrium, a bidder will pay the least amount to remain in their position. So if the bidder’s willingness to pay is greater than that of the company one position below (i.e., $v_r(i) \geq v_r(i+1)$), then the bidder will pay $p_i = v_r(i+1)$. Otherwise, the bidder will pay $p_i = \frac{v_r(i+1)\alpha r(i+1)}{\alpha r(i)}$.

3. **Rank by click-through.** Every listing company bids their willingness to pay (per click). The $k$ listings which generate the highest click-through rate are displayed, ranked by their click popularity. Every listed item pays the amount of the next highest bidders’ willingness to pay. Let $r(i)$ denote the $i$th element of the sorted list of $\alpha_j$. Then the owner of the $i$th ranked listing pays $p_i = v_r(i+1)$ if $v_r(i) \geq v_r(i+1)$. Otherwise, the listing owner pays $p_i = \frac{v_r(i+1)\alpha r(i+1)}{\alpha r(i)}$.

4. **Posted price for each position.** In this mechanism, the search engine secretly fixes a reserve price for each of the positions it has for a certain period. Each listing company bids a certain amount for that period to be listed. The highest $k$ bids are compared to the search engine’s reserve price for each position. If the $i$th highest bid is higher than the reserve price for the $i$th position, it is admitted for the $i$th position and pays the reserve price; otherwise, it is compared to the $(i+1)^{th}$ reserve price. If it is higher than the $(i + 1)^{th}$ reserve price, it is admitted and pays the $(i+1)^{th}$ reserve price, but the total available positions for paid placement will also be reduced by 1, and so on.
4.4 Revenue Performance for Different Ranking Mechanisms

Our computational simulations comparing the revenue performance of the different mechanisms were based on the following setting. The search engine has $k = 5$ paid positions to fill. There are $s$ different listing companies that desire paid placement, whose relevance scores ($\alpha_j$) and willingness to pay ($v_j$) are correlated according to a truncated bivariate normal distribution between 0 and 1. The means are 0.5 and the standard deviations are 0.167. By construction, the covariance thus can vary between -0.167 and 0.167. We believe that this modeling choice captures the intuition that relevance and willingness to pay are usually correlated (since presumably content providers want their listings to appear when users are likely to click on them, i.e., when the listings appear relevant to the users), but are not necessarily completely correlated, since many other factors (including budget constraints, mistakes, etc.) play a role.

4.4.1 Impact of Correlation between WTP and Relevance

First we compute the performance of each mechanism and its sensitivity to the correlation between a content provider’s true relevance and willingness to pay for paid placement. The following result is derived with $s = 10$ providers who desire paid placement and an attention decay factor $\delta = 2$. For each value of correlation in $[-1, 1]$, we computed the average revenue over 200 simulations (each simulation consists of a draw for each of the content provider’s relevance and WTP).

Finding 1. For every mechanism, the revenue earned is increasing in the covariance between the content provider’s relevance score and its willingness to pay.

Figure 4.1 displays this increasing trend of revenues with respect to covariance. It is easy to see that the ranking mechanism used by Google and Overture perform the best. Google’s approach of ranking by the product of WTP and Relevance performs significantly better when relevance and WTP are negatively correlated, and the two
Fig. 4.1. Average revenue versus covariance for the four tested ranking strategies.
mechanisms yield nearly equivalent revenues when relevance and WTP are positively correlated. These results are summarized below. Table ?? in the Appendix shows the t-Test using Two-Sample Assuming Unequal Variances, for the revenues generated by the two mechanisms. Table ?? shows the same t-Test for the case where the covariance is positive.

**Finding 2.** *Ranking by WTP*Relevance generates more revenues than Ranking purely by WTP, except in the area where the correlation is large. When the correlation is positive, the difference between these two mechanisms is not significant with significance level 0.05.

Figure 4.1 indicates that the posted price mechanism works quite well when the relevance and WTP are negatively correlated. When the correlation is negative, the posted price mechanism yields significantly greater revenues than Overture’s mechanism, but when the correlation is positive, it yields significantly lower revenues than Overture’s (and Google’s) mechanism. It is interesting to note that Overture’s mechanism does worse than the posted price mechanism when the relevance and WTP are negatively correlated. This happens because when there is negative correlation, Overture’s mechanism systemically picks the providers who will achieve lower click-throughs, hence earns lower revenues, while the posted price mechanism yields greater revenues since the payment is not contingent on click-throughs.

**Finding 3.** *Google’s Rank by WTP*Relevance dominates all other mechanisms except when WTP and Relevance are strongly correlated. The posted price mechanism performs better than Overture’s under negative correlation but is worse when the correlation is positive. The Rank by popularity (click-through) mechanism is always dominated by the other mechanisms.
4.4.2 Impact of Attention Decay Factor

Now we examine how the performance of the different mechanisms is influenced by $\delta$, the difference in the attention that a certain listing item can get in different positions. The following figures show the performance of each mechanism with respect to $\delta$, when the correlation between paid listing firms relevance score and willingness to pay is, respectively, strongly negative (left subfigure) and strongly positive (right subfigure).

Fig. 4.2. Revenue versus the attention decay parameter $\delta$, under negative and positive correlation between relevance and willingness to pay.

**Finding 4.** The revenue generated from paid placement decreases as $\delta$ increases.

This result is intuitive. As $\delta$ increases, the lower rank positions become less attractive thus will generate lower revenues. We note that the search engine can control
the decay factor, for example, by designing a user-friendly interface which allows users to maintain attention over a larger subset of paid listings.

### 4.4.3 Impact of Demand for Paid Placement

The search engine’s potential for placement revenues depends on the overall demand for preferential placement. Intuitively, when more providers compete for paid placement, this will increase the market clearing price and the search engine’s revenues. Our simulation extends this intuition by revealing that this relationship is also affected by the covariance between the paid listings’ relevance score and willingness to pay. Figure 4.3 demonstrates that placement revenues increase with the level of demand when WTP and relevance are positively correlated, but that an increase in demand does not yield greater revenues (except in the case of Google’s mechanism, where revenue increases modestly with demand) when the correlation is negative.

![Figure 4.3](image-url)

**Fig. 4.3.** Revenue versus the demand for paid listings $s$, under negative and positive correlation between relevance and willingness to pay.
Finding 5. When the correlation between the paid listings’ relevance and WTP is highly positive (cov=0.15 in this case), the increase in the demand of paid placement (number of potential paid links) increases the search engine’s revenue. While when the correlation between the paid listings’ relevance and WTP is very negatively correlated (cov=-0.15 in this case), there is no obvious increasing trend when increasing the demand.

This result seems counter-intuitive at the first sight, observing competition does not necessarily increase revenues. However, this can be explained by the correlation between the paid links’ relevance and WTP. For example, ranking by click through seems generate less revenue when the competition increases. This is because if ranked by click through, the higher the relevance, the lower the willingness to pay when these two are extremely negatively correlated, while the effect of the click through is reduced with the lower ranks.

4.5 Ranking Dynamics

The previous section analyzed the steady state performance of various ranking mechanisms. For the three mechanisms where the ranking is based partly on the relevance of the listing, knowledge of the listing’s relevance is required to determine the steady state ranking. Since the search engine does not know the relevance for each listing, it approximates the relevance by observing the click-through rate for each listing. The approximation itself should improve over time as the search engine gathers more data, hence the ranking produced by each mechanism needs to be revised over time until the steady state solution is obtained. This section studies the ranking dynamics for Google’s Rank by WTP*Relevance mechanism, and compares two different approaches for dynamic ranking. We ignore the posted price mechanism because it generates a ranking that remains unchanged for a period of time, and we ignore the rank by click-through mechanism because it is always dominated by the other mechanisms. We ignore
Overture’s ranking by willingness to pay mechanism because it is the most stable one: in equilibrium the ranking is based purely on the content providers’ willingness to pay, and in equilibrium the content provides will bid their true values.

Google’s ranking mechanism provides an interesting test bed to examine ranking dynamics. Since this mechanism ranks listings based on the product of WTP and click-through rate, the ranking will change over time as the observed click-through rate itself changes. In Google’s implementation of this mechanism, each click-through earned by a listing equally impacts the approximation of the listing’s relevance, regardless of whether the click was earned at a high rank or a low rank. Intuitively, lower-ranked items are less likely, ceteris paribus, to be selected by a user. Therefore, if a low-ranked listing does get selected by the user, it should result in a greater increase in its relevance score. Therefore, in order to study the dynamics of the rank by WTP*Relevance mechanism, we propose a new dynamic ranking rule where the relevance score is computed by weighting each click-through by the listing items position when it received a click through. Intuitively, a lower-ranked item is more likely to move up the list under this weighted ranking mechanism than under Google’s unweighted ranking mechanism.

We performed computational simulations to examine the dynamics of each ranking mechanism, with the following setting. There are 5 paid placement slots available and there are 5 content providers. In each simulation run, each provider’s relevance score and willingness to pay are i.i.d. random variables which follow a truncated normal distribution in [0, 1]. The decay factor $\delta$ is 2. In each period of a simulation, a provider’s probability $q$ of getting a click-through is $\alpha / \delta^i$, a function of its true relevance score $\alpha$ and its position $i$ in the result page. Since there is no information about click-through rates in the beginning, each simulation run consists of a trial period where every item is given a chance to be on the top as well as every other position: the click-through they earned is recorded, and the weighted and unweighed products (scores) are calculated. At the end of the trial period, the items are presented and ranked according to their scores.
the remaining periods, the ranks are revised according to the weighted and unweighted ranking rules. Once a lower ranked item achieves a larger score, it can be promoted to a higher position. Each run consists of a maximum of 1000 periods (the simulation stops earlier if it converges), and we executed 200 runs for each mechanism. For each run, we record the number of periods to converge to the equilibrium allocation, and terminate it after 1001 periods if there is no convergence. In each period, the average distance between the equilibrium allocation and the current allocation is also calculated as the sum of the absolute value of the difference between the current rank and equilibrium rank of each listing.

Our objective in this section is to examine how the weighted and unweighted dynamic ranking mechanisms behave over time, which one approaches to steady state faster, and to what extent the non-steady state allocations are different from the (optimal) equilibrium allocation for each mechanism. Specifically, for each mechanism we compute how many periods are required to reach equilibrium, and the distance between the current allocation and the equilibrium allocation.

4.5.1 Impact of Trial Period

The trial period, in which each candidate provider is given a chance to be placed in each paid placement slot, is a way to offer each link a fair chance to be tested for their true relevance. The length of the trial period has a significant affect on the performance of each mechanism, since trials are costly (they involve allocations that generate poor placement revenues) but generate useful information for approximating relevance. Thus a longer trial period is likely to produce a better approximation of relevance (thereby resulting in greater placement revenues in future periods) at the expense of lower placement revenues during the trial period.

Table 4.2 shows the average number of periods to converge for each of the two mechanisms. Table 4.1 shows the number of runs that converged to the equilibrium when
<table>
<thead>
<tr>
<th>trial period</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
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<tr>
<td>weighted</td>
<td>65</td>
<td>96</td>
<td>112</td>
<td>116</td>
<td>125</td>
<td>125</td>
<td>133</td>
<td>130</td>
</tr>
<tr>
<td>unweighted</td>
<td>42</td>
<td>73</td>
<td>73</td>
<td>78</td>
<td>92</td>
<td>92</td>
<td>96</td>
<td>99</td>
</tr>
</tbody>
</table>

Table 4.1. Number of runs which converged within 1000 periods.

<table>
<thead>
<tr>
<th>trial period</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>weighted</td>
<td>595.87</td>
<td>558.71</td>
<td>476.83</td>
<td>470.89</td>
<td>417.12</td>
<td>423.06</td>
<td>382.62</td>
<td>380.32</td>
</tr>
<tr>
<td>unweighted</td>
<td>804.87</td>
<td>650.25</td>
<td>643.76</td>
<td>616.17</td>
<td>560.69</td>
<td>558.59</td>
<td>533.80</td>
<td>519.01</td>
</tr>
</tbody>
</table>

Table 4.2. Number of periods at which the mechanism converges.

the paid listings’ relevance and WTP were independent (COV=0). The data indicate
the following results.

**Finding 6.** An increase in the length of the trial period increases the speed of convergence, but at a decreasing rate.

**Finding 7.** The weighted mechanism converges faster to the equilibrium state than the unweighted mechanism, both in terms of the number of runs which converge, and average time of convergence.

### 4.5.2 Impact of Correlation

Now we examine the impact of correlation between WTP and Relevance on the convergence properties of the two mechanisms. Figure 4.4 shows the average distance between the current run and the equilibrium state each period, when the covariance between the paid links relevance and willingness to pay is -0.15 and 0.15. The simulations are based on experiments where the trial period consisted of 5 periods.
Fig. 4.4. Distance to equilibrium versus time for the weighted and unweighted ranking mechanisms, under negative (left) positive (right) correlation between relevance and willingness to pay.
Finding 8. Both mechanisms converge closer to the equilibrium as the correlation between WTP and Relevance increases. The weighted mechanism always converges faster than the unweighted mechanism. The difference between the two mechanisms increases with increase in covariance.

The difference between the two mechanisms’ performance is statistically significant at the 0.05 significance level. Tables ?? and ?? in the Appendix display the t-test statistics for COV=-0.15 and COV=0.15, respectively.

4.6 Conclusions

This essay discusses the ranking mechanism of the paid links. Assuming the paid links’ true relevance and willingness to pay are random variables, we modelled several ranking mechanisms, either used by the leading search services (ranking by WTP as in Overture, and ranking by WTP*Relevance as in Google), or by common sense (ranking by click-through, and by fixed payment with posted price). We tested the steady state performance of such mechanisms, with respect the correlation between the content provider’s relevance and WTP, attention decay factor, as well as the number of competing content providers. We found that Google’s mechanism performs best in almost all the cases, while Overture’s mechanism is also pretty good under some conditions. We then go ahead to test the dynamic behavior of Overture and Google’s mechanism, and propose a weighted mechanism. We found that these mechanisms converges faster and more stable if we increase the length of the trial period. We also found that our proposed mechanism converges faster and more stable than Google’s ranking mechanism.
Chapter 5

Optimal Allocation Mechanisms
When the Ranking of Bidders Valuations is Common

5.1 Introduction

In the last two years Internet search engines began to sell top slots in their result pages to those content providers who were willing to pay. Soon the revenue from these so-called "featured listing" or "sponsored links" became vital as search engines' banner ads business went down. Initiated by Goto.com (now Overture.com), now nearly all major search engines (eg. Google.com) and portals (eg. Yahoo.com) employ paid placement, and Overture.com has become the biggest and most successful paid placement broker on the Internet.\footnote{At Search Engine Watch \url{http://searchenginewatch.com/webmasters/paid.html}} Different search engines use different mechanisms to sell those top positions, though most of them use auctions. For example, Overture used to use real time pay-your-bid auctions to sell their paid positions, where bidders bid for the positions they were interested in and won if their bids exceed the current winning bid, and paid what they bid. Goto.com even used to show the current winning price in their result pages next to the paid link, so it is also an open auction. Google is using some variant of a second price auction to sell their positions through their so called “AdWord Program”. Such auctions have common features; namely that a bidder’s valuation is higher for a higher position than for a lower position, because a higher placement attracts more attention from the search engine users; it is also reasonable to suppose that the second ranked position cannot be allocated until the highest one has been, though we shall question this assumption later on. This kind of auction can also be applied in other
contexts. An example could be of several tasks are waiting in a line for processing, each with a certain deadline approaching and tasks bid for the position in the waiting queue. In this way each task has a different valuation for a certain position, but their preference toward the positions are ranked in the same way. We are interested in the seller’s revenue maximization problem, namely, how to design a mechanism to allocate the positions to the buyers and receive the maximum amount of payment from the buyers.

This kind of problem is studied in the matching / assignment problem literature, though, unlike here, usually in complete information environments. Consider the matching problem with a finite set of indivisible objects and a finite set of indivisible agents. For one sided matching problem, where the objects being allocated have no preference towards the agents, such as assigning dormitory rooms to students, or legislators to committees, etc, randomization ([29], [2]), which is Pareto efficient and strategy-proof, is often used. Bogomolnaia and Moulin(2002) [13] show when the agents rank the objects in the same way, the “Random Serial Dictatorship” is Pareto inferior to their proposed “Probabilistic Serial” mechanism. For two-sided matching problems such as the problem of college admission, or allocation of medical interns to hospitals, the Gale-Shapley algorithm [22] is often used. This, too, is stable and strategy-proof. (Abdulkadiroglu and Sonmez [1]). But in all of these mechanisms, there is no monetary transfer involved, and the preference is known. Kamecke(2001) [29] studied the conditions when the rational agents should use the equivalent bidding strategy in two English multiple object auction as in the Gale-Shapley algorithm. However, the main focus of the matching literature is on efficiency rather than optimality.

The closest literature related to this paper is optimal (multiple unit) auction design. Optimal auction design studies how the designer (the auctioneer) designs a mechanism to maximize his expected revenue (the sum of the expected payments of the buyers) which is incentive compatible and individually rational. We follow very closely Myerson(81) [41], where he studied the optimal mechanism to sell a single object, and
found that it is optimal to sell the object to the bidder with the highest valuation, given his virtual value is positive. As a generalization, Maskin and Riley (1990) studied the optimal auction of selling multiple identical objects, using a similar approach. For optimal auctions with heterogeneous objects, many literatures consider multi-unit demand. Then the question comes with demand reduction: when the objects are complements or substitutes, or when the number of buyers is small or large, whether or not to sell the items separately or in bundles, [43], [7], [4]. Menezes 1998 studied a pooled auction in the environment of identical or perfectly correlated objects where every bidder submits a single bid, and bidders with higher bids have the rights to choose their ideal objects earlier. While our paper categorize the case of correlated objects into 4 different cases, the environment in [39] is similar to one of the cases we studied. For efficiency in multi-unit auction, Dasgupta and Maskin (2000) and Perry and Reny (1999) show how the Vickrey mechanism can be generalized to varieties of multi-unit environment to achieve efficiency, as long as each bidder’s signal is one dimensional. Jeihiel and Moldovanu (2001) shows that when bidders have multi-dimensional signals, efficiency is usually not obtained.

Different from the above two streams, the problem we address in this paper considers the optimal allocation of non-identical objects, where buyers’ values for these objects are ranked in the same order, and each buyer only needs one object (unit demand). We categorize the environment into four cases based on how bidders’ valuation drops with the rank of the positions, relative to other bidders, namely, parallel, convergent, divergent, and convergent then divergent. In each case, the optimal allocation rule and payment scheme is characterized. We find the optimal allocation rule is quite different under different cases. Thus to understand the buyers preference characteristics is vital in determining the optimal mechanism. And as long as the buyers’ valuations for lower positions decrease at different rates, the seller can extract much more surplus out of the buyers, where the optimal expected payment is at least as high as the highest rejected
buyer’s valuation for that particular position; under some conditions these optimal allocation rules are even efficient in terms of maximizing the total welfare of the seller and the buyers. We also find that this optimal mechanism cannot usually be implemented by simple sequential “highest rejected bid” auctions.

This paper is organized as following. In section 5.2 we introduce the model setup and notation. We then discuss the optimal mechanism under four different specification of buyers preferences in Section 5.3. Some issues related to the implementation of the optimal mechanism are discussed in Section 5.4, and we conclude in Section 5.5.

5.2 Model

Assume that the set of risk neutral buyers $N = \{1, 2, \ldots, n\}$ compete for $K$ positions. Buyers have independent private types. Buyer $i$’s type $t_i$ is distributed over the interval $T_i = [a, b]$ ($a \geq 0$) according to the distribution function $F_i$ with associated density function $f_i$. Let $T = \times_{j=1}^n T_j$ denote the product of the sets of buyers’ types, and for all $i$, let $T_{-i} = \times_{j \neq i} T_j$. Define $f(t)$ to be the joint density of $t = (t_1, t_2, \ldots, t_n)$. Similarly, define $f_{-i}(t_{-i})$ to be the joint density of $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$. Since the types are independent, $f(t) = f_1(x_1) \times f_2(x_2) \times \cdots \times f_n(x_n)$, and $f_{-i}(t_{-i}) = f_1(x_1) \times \cdots \times f_{i-1}(x_{i-1}) \times f_{i+1}(x_{i+1}) \times \cdots \times f_n(x_n)$.

Let $v^k_i(t_i)$ represent buyer $i$’s valuation for $k$’th position, which is non-increasing in $k$. For simplicity we write $v^k_i(t_i)$ and $v_i^k$ interchangeably. Let $t_0$ represent the seller’s type, so his valuation of the $K$ positions is given by $v^k_0$. Assume that all buyers’ valuation functions are either parallel, or there exists a position $\mu \in (-\infty, \infty)$ for which all bidders’ valuations are the same and this is common knowledge. Assume we can separate the expression of types from the expression of the difference between any two values for a certain position, more specifically, assume $v^k_i(t_i) - v^k_j(t_j) = (t_i - t_j)S(k)$, $i, j = 0, 1, \ldots, n$, ...
where $S(k)$ is independent of $t$. For simplicity, in this paper we assume buyers’ valuations are linear in the rankings.

By the “Revelation Principle” ([3], [25], [42]), without loss of generality, we restrict our attention to direct mechanisms. Let $P : T \rightarrow \triangle$ represent the allocation rule, where $\triangle$ is the set of probability distributions over the set of buyers and the set of positions, and $X : T \rightarrow \mathbb{R}^N$ represents the payment rule. Our goal is to identify the optimal mechanism $(P, X)$ which is incentive compatible and individually rational. Following Myerson, let $p_i(t)$ represent the probability for buyer $i$ to win one position and $x_i(t)$ be the buyer $i$’s expected payment for his winning position. More specifically, let $p_{ik}(t)$ represent the probability that the buyer $i$ wins the $k$th position, $x_{ik}(t)$ be buyer $i$’s expected payment for $k$th position. Then we have $p_i(t) = \sum_k p_{ik}(t)$, and $x_i(t) = \sum_k x_{ik}(t)$.

Suppose the seller uses the direct mechanism $(P, X)$, then buyer $i$’s expected utility is:

$$U_i(p, x, t_i) = \sum_k \int_{T_i} \left[ v_{ik}(t_i)p_{ik}(t_i, t_{-i}) - x_{ik}(t_i, t_{-i}) \right] f_{-i}(t_{-i}) dt_{-i}$$

The seller’s expected utility is:

$$U_0(p, x) = \sum_k \int_T \left[ v_{0k}(t) \left( 1 - \sum_N p_{ik}(t) \right) + \sum_N x_{ik}(t) \right] f(t) dt$$  \hspace{1cm} (5.1)$$

where

$$p_{ik}(t) \geq 0 \ \forall i, \ \forall k, \ \forall t \in T$$  \hspace{1cm} (5.2)$$

$$\sum_N p_{ik}(t) \leq 1 \ \forall k, \ \forall t \in T$$  \hspace{1cm} (5.3)
\[
\sum_{K} p_{i}^k(t) \leq 1 \quad \forall i, \forall t \in T
\] (5.4)

For the buyers, the “Individual Rationality” condition ensures that by not participating, a buyer can guarantee himself a payment of zero:

\[
U_i(p, x, t_i) \geq 0 \quad \forall i, \forall t_i
\] (5.5)

The “Incentive Compatibility” condition ensures that every buyer report his true type. This is written as:

\[
U_i(p, x, t_i; t_i) \leq U_i(p, x, s_i; t_i) = \sum_{K} \int_{i-1}^{T} v_i^k(t_i)p_i^k(t_i, t_{-i}) - x_i^k(t_{-i}, s_i)f_{-i}(t_{-i})dt_{-i} \forall i, \forall t_i, \forall s_i \neq t_i
\] (5.6)

Thus our goal is to identify the optimal \(p_i^k(t)\) and \(x_i^k(t)\) to maximize the expected payoff of the seller. That is,

\[
\max (5.1) \quad \text{s.t. } (5.2), (5.3), (5.4), (5.5), \text{and } (5.6)
\]

Consider the IC constraint.

Let \(Q_i(p, t_i) = \sum_{K} \int_{T_{-i}} S(k)p_i^k(t_i, t_{-i})f_{-i}(t_{-i})dt_{-i}.\)

**Proposition 15.** When \(S(k) \geq 0\), an allocation mechanism is feasible if and only if:

\[
\text{if } s_i \leq t_i, \quad \text{then } Q(p, x, s_i) \leq Q(p, x, t_i)
\] (5.7)

\[
U_i(p, x, t_i) = U_i(p, x, a) + \int_{a}^{t_i} Q_i(p, s_i)ds_i
\] (5.8)
Proof. To show the “only if” part,

\[ U(p, x, a) \geq 0 \quad (5.9) \]

Using (5.2), (??), and (??)

\[ U(p, x, s_i; t_i) \]
\[ = \sum K \int_{T_{-i}} \left[ v^k_i(t_i)p^k_i(t_{-i}, s_i) - x^k_i(t_{-i}, s_i) \right] f_{-i}(t_{-i}) dt_{-i} \]
\[ = \sum K \int_{T_{-i}} \left[ (s_i + (t_i - s_i)S(k)) p^k_i(t_{-i}, s_i) - x^k_i(t_{-i}, s_i) \right] f_{-i}(t_{-i}) dt_{-i} \]
\[ = U_i(p, x, s_i) + \sum K \int_{T_{-i}} ((t_i - s_i)S(k)) p^k_i(t_{-i}, s_i)f_{-i}(t_{-i}) dt_{-i} \]
\[ = U_i(p, x, s_i) + (t_i - s_i)Q_i(p, s_i) \]

The incentive compatibility constraint implies that:

\[ U_i(p, x, t_i; t_i) \geq U_i(p, x, s_i; t_i) + (t_i - s_i)Q_i(p, t_i) \quad \forall s_i \quad (5.10) \]

Use (5.10) twice we get:

\[ (t_i - s_i)Q_i(p, s_i) \leq U_i(p, x, t_i) - U_i(p, x, s_i) \leq (t_i - s_i)Q_i(p, t_i) \quad (5.11) \]

So

\[ Q_i(p, x_i, s_i) \leq Q_i(p, x_i, t_i) \quad (5.12) \]

when \( s_i \leq t_i \)

Let \( t_i - s_i = \delta \), then (5.11) can also be written as:

\[ \delta Q_i(p, s_i) \leq U_i(p, x, s_i + \delta) - U_i(p, x, s_i) \leq \delta Q_i(p, s_i + \delta) \quad (5.13) \]
If we assume that the utility function is continuous and differentiable everywhere, this equation is integrable and can be written as: 
\[ \int_a^{t_i} Q_i(p, s_i)ds_i = U_i(p, x, t_i) - U_i(p, x, a), \]
so
\[ U_i(p, x, t_i) = U_i(p, x, a) + \int_a^{t_i} Q_i(p, s_i)ds_i \]
(5.14)

From the other direction (the “if” part), to show (5.10), assume \( s_i \leq t_i \), then using (5.7) and (5.8) we get:
\[
U_i(p, x, t_i) = U_i(p, x, s_i) + \int_{s_i}^{t_i} Q_i(p, r_i)dr_i \\
\geq U_i(p, x, s_i) + \int_{s_i}^{t_i} Q_i(p, s_i)dr_i \\
= U_i(p, x, s_i) + (t_i - s_i)Q_i(p, s_i)
\]

If \( s_i > t_i \), then
\[
U_i(p, x, t_i) = U_i(p, x, s_i) - \int_{t_i}^{s_i} Q_i(p, r_i)dr_i \\
\geq U_i(p, x, s_i) - \int_{t_i}^{s_i} Q_i(p, s_i)dr_i \\
= U_i(p, x, s_i) + (t_i - s_i)Q_i(p, s_i)
\]

So when \( S(k) \geq 0 \), \((p_i, x_i)\) is an optimal mechanism if it satisfies (5.7), (5.8), (5.9), (5.2), (5.3), and (5.4) and maximizes (5.1).

When \( S(k) < 0 \), everything follows through except (5.8) now becomes:
\[ U_i(p, x, t_i) = U_i(p, x, b) - \int_{t_i}^{b} Q_i(p, s_i)ds_i \]
(5.15)

and (5.9) becomes
\[ U_i(p, x, b) \geq 0 \]
(5.16)
We will discuss this further in section 5.3.3.

We can re-arrange the objective function (5.1):

\[
U_0(p, x) = \sum_K \int_T v_0^k(t) \left( 1 - \sum_N p_i^k(t) \right) + \sum_N x_i^k(t) f(t) dt \\
= \sum_K \int_T v_0^k(t) f(t) dt + \sum_K \int_T \sum_N p_i^k(t) \left( v_i^k(t) - v_0^k(t) \right) f(t) dt \\
+ \sum_K \int_T \sum_N \left( x_i^k(t) - p_i^k(t) v_j^k(t) \right) f(t) dt
\]

For the last term of the rearranged objective function,

\[
\sum_K \int_T \sum_N \left( x_j^k(t) - v_i^k(t) v_j^k(t) \right) f(t) dt
\]

Plug this back to the objective function:

\[
U(p_i, x_i, t_0) = \sum_K \int_T v_0^k f(t) dt - NU_i(p, x, a) + \sum_K \int_T \sum_N \left[ (v_i^k(t_i) - v_0^k) - S(k) \frac{1 - F(t_i)}{F(t_i)} \right] p_i^k(t_i, t_{i-1}) f(t) dt
\]

Maximizing (5.17) is equivalent to:

\[
\max \sum_K \int_T \sum_N \left( v_i^k(t_i) - v_0^k \right) - S(k) \frac{1 - F(t_i)}{F(t_i)} p_i^k(t_i, t_{i-1}) f(t) dt - N \cdot U_i(p, x, a)
\]
such that

$$U_i(p, x, a) \geq 0$$

$$Q_i(p, x, s_i) \leq Q_i(p, x, t_i), \text{ if } s_i \leq t_i$$

$$p_i^k(t) \geq 0 \ \forall i, \ \forall k, \ \forall t \in T$$

$$\sum_N p_i^k(t) \leq 1 \ \forall k, \ \forall t \in T$$

$$\sum_K p_i^k(t) \leq 1 \ \forall i, \ \forall t \in T$$

Since buyers’ valuations drop for a lower ranked position, based on how their valuations drop relative to other’s (they may drop in the same rate, or some may drop faster than the others), we categorize the situations into four cases, namely, parallel, convergent, divergent, and convergent then divergent. We are going to discuss the optimal mechanism for these different cases in Section 2. Figure 5.1 shows these 4 cases, where $\mu$ represent the position for which each buyer has the same valuation.

### 5.3 Four Different Cases

#### 5.3.1 The Parallel Case

First consider the case where every buyer’s valuation for a lower position drops at the same rate. In the example of paid placement practice of search engines, if the competing content providers for a certain keyword have relatively the same taste or budget, the change of their valuation for positions maybe relatively stable, thus we may approximate their valuation function by assuming that their valuations drop at the same
rate. Let \( v^k_i(t_i) = t_i - \alpha k \), where \( \alpha > 0 \) is a constant. Thus the type of the buyers are characterized by the intersections of their utility functions with the value axis (vertical axis in the figure). From now on assume that the seller’s valuations for the items are 0. Then \( v^k_i(t_i) - v^k_0 = t_i - \alpha k (S(k) = 1) \). From now on we assume the seller’s valuation for the positions are 0. So Eq. (5.18) becomes:

\[
\max \sum_K \int_T \left( \sum_N \left( t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} - \alpha k \right) p^k_i(t) \right) f(t) dt - N \cdot U_i(p, x, a) \tag{5.19}
\]

This is very similar to Myerson’s optimal auction design (1981) problem. Let \( c(t_i) = t_i - \frac{1 - F_i(t)}{f_i(t)} \) represent the virtual value of buyer \( i \). If the regularity condition is satisfied that \( t_i - \frac{1 - F_i(t)}{f_i(t)} \) is strictly increasing in \( t_i \), and we can make that \( U_i(p, x, a) = 0 \). then Eq. (5.19) is maximized when the objects are assigned to the \( K \) highest buyers given that their virtual values are higher than \( \alpha K \). More importantly, as long as the winners are determined, it doesn’t matter which buyer gets which objects, as long as they win. This is because in the objective function, we can fully separate the part containing \( t_i \) and the part containing \( k \). Notice that even though it seems that the reserve price seems to be different in the ranking of positions, it is actually a constant \( \left( r(k) = \text{solves}\left\{ c(t_i) - \alpha K = 0 \right\} \right) \) according to this allocation rule. That is to say, as long as a buyer is eligible to win the last position, he is eligible to win every other position.

To formally state the allocation rule, let \( A_i(t_i, k) \) represent the ranked list (from the highest to the lowest) of the term in the objective function after the \( \sum_N \) and before \( p^k_i \), in this parallel example, it is \( t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} - \alpha k \) in the objective function (5.19). Define \( A_{-i}(t_{-i}, k) \) as the same list but without \( i \)’th entry, and \( B_{j}(t_j, k) \), \( j = 1, 2, \ldots n - 1 \) as the \( j \)'th highest element in the list of \( A_{-i}s \). Then define

\[
z_k(t_{-i}) = \inf \left\{ s_i | A(s_i, k) \geq 0 \text{ and } A(s_i, k) \geq B_k(t_j, k), \forall j \neq i \right\} \tag{5.20}
\]
Proposition 16. In the parallel case, the optimal incentive compatible allocation rule is to allocate one position to each of the $K$ bidders who have the highest virtual values, given that their types satisfy that $c(t_i) - \alpha K \geq 0$ (or $A(t_i, k) \geq 0$). But once the winners are determined, it doesn’t matter how to allocate the positions among the winners. In other words,

$$p_i(t_{-i}, s_i) = \begin{cases} 1 & \text{if } s_i \geq zK(t_{-i}) \\ 0 & \text{if } s_i < zK(t_{-i}) \end{cases} \quad (5.21)$$

Proof. The optimality of this allocation rule is obvious because (5.19) will be maximized if we pick the buyers with the highest $K$ virtual values, given $t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$ is non-negative.

To check for the incentive compatibility constraint that $Q_i(p, x, s_i) \leq Q_i(p, x, t_i)$ when $s_i \leq t_i$, notice that if $c(t_i) \geq \max\{C_K(t_j), \alpha k\}, j \in n$, where $C_K(t_j)$ is the $K$'th highest virtual value among all the other buyers, then he wins. Since $S(k) = 1$, this equation purely means the conditional probability of winning 1 item given type $t_i$ is higher than type $s_i$ ($s_i \leq t_i$). Since the highest $K$ buyers win, and $c(t_i)$ is increasing, we know that the probability that the probability that $A(s_i, k) \geq B_K(t_j, k), \forall j \neq i$ is increasing in $t_i$. So whenever buyer $i$ could win by submitting $s_i$, he could also win by submitting $t_i$ where $t_i > s_i$. So $Q(p_i, x_i, t_i)$ is indeed increasing in $t_i$.

Now consider the payment scheme $x$. According to (5.8), $U_i(p, x, t_i) = U_i(p, x, a) + \int_a^{t_i} Q_i(p, s_i)ds_i$, so we have the optimal expected payment function determined by:

$$\sum_K x_i^k(t) = \sum_K v_i^k p_i^k(t) - \sum_K \int_a^{t_i} S(k)p_i^k(s)f(s)ds$$

$$= \sum_K v_i^k p_i^k(t) - S(k) \int_a^{t_i} S(k)p_i^k(s)f(s)ds$$

$$= \sum_K v_i^k p_i^k(t) - S(k)(t_i - zK(t_{-i})) \quad (5.22)$$

So if buyer $i$ wins position $k$, his payment will be $x_i^k = v_i^k - v_i^k + z_i^k$, where $z_i^k$ is defined as the $K$ th highest type (other $i$) buyer’s valuation for the $k$ th position, or
$z_K(t_{i-1})$'s valuation for position $k$. This means, it doesn't matter for a winning buyer which position he is allocated, as long as he is paying the $K+1$'s buyers valuation for that particular position, his utility is the same for each position he consumes.

This mechanism can be implemented as a “psudo-second price auction”, where every bidder bid their type, and the highest $K$ bidders win. The higher their types, the higher the positions they are allocated. And each one pays the highest rejected bidder’s valuation for his winning position. To better understand this mechanism, assume that buyers types follow a uniform distribution between $[0,1]$. Then the reserve price for each position is the same, that is, $\frac{1+\alpha K}{2}$. If there are 3 positions for sale and $\alpha$ is 0.05, the reserve price is 0.575 for each position. If the realized types are $t_1 = 0.9$, $t_2 = 0.8$, $t_3 = 0.75$, $t_4 = 0.7$, then the positions 1, 2, 3 will be allocated to buyer 1, 2, 3, respectively, with the expected payment for those position $v^1(0.7)$, $v^2(0.7)$, $v^3(0.7)$, which are, 0.65, 0.6, 0.55. On the other hand, if the realized types are: $t_1 = 0.9$, $t_2 = 0.7$, $t_3 = 0.5$, $t_4 = 0.2$. Then $t_1$ is allocated to position 1, $t_2$ is allocated to position 2, while position 3 is not allocated.

5.3.2 The Convergent Case

This describes the case that the higher type buyer’s valuation drops faster for a lower position than a lower type buyer’s. In other words, the lower position means less to a high type buyer than to a low type buyer. For example, if a relatively unknown company has a big marketing budget to attract customer attention, it may have a strong incentive to win a higher position, but its valuations for a lower position may drop much quicker than its competitors, because the reduced attention from those lower positions does not serve the company’s strategic goal.

Let $v^K_i = \beta - t_i(k - \mu)$, where $\mu > K$ is the position for which each buyer has the same valuation. In this example, $\mu$ is the horizontal value of the point where each utility function crosses. This function guarantees that the higher type buyer (larger $t_i$)
has a higher valuation for each position than a lower valued buyer. Thus \( v_k^i - v_j^k = (t_i - t_j)(\mu - k) \), and \( S(k) = (\mu - k) > 0 \).

So Eq. (5.18) becomes:

\[
\max \sum_K \int_T \left( \sum_N \left( \beta + \left( t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \right)(\mu - k) \right) p_i^k(t) \right) f(t) dt - N \cdot U_i(p, x, a) \tag{5.23}
\]

To maximize this expression, if the distribution function satisfies the regularity condition, again the positions should be allocated to those buyers who have larger virtual values, that is, the highest \( K \) buyers whose values satisfy \( \beta + t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \geq 0 \). Notice that the reserve price \( r(k) = \text{solve} \{ c \cdot (\mu - k) + \beta = 0 \} \) is decreasing in \( k \). And the winning virtual value can be negative because of the presence of a positive constant \( \beta \), different from Myerson’s case.

Then how to allocate the positions among the buyers?

**Proposition 17. The optimal incentive compatible allocation rule when the virtual value \( t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \) is increasing is to allocate the higher positions to the buyers with higher virtual values, as long as \( \beta + \left( t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \right)(\mu - k) \geq 0 \). In another word,**

\[
p_i^k(t_i, s_{-i}) = \begin{cases} 
1 & \text{if } z_k(t_{-i}) \leq s_i \leq z_{k-1}(t_{-i}) \forall k \\
0 & \text{otherwise}
\end{cases} \tag{5.24}
\]

Where \( z_k(t_{-i}) \) is defined in the same way as in Section 5.3.1. The only difference is that here \( A_k \) is the \( k'th \) highest element in the list of \( \beta + \left( t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \right)(\mu - k)s \). We also define \( z_0(t_{-i}) \) equal to \( b \), the upper bound of the buyers’ value distribution.

**Proof.** Since \( S(k) = \mu - k \) is positive, the objective function will be maximized if the seller allocates the higher position (larger \( \mu - k \)) to the buyers of higher type (larger...
\[ t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \]. To see this, let \( y_j \) be the \( j \)th highest value of \( t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \), comparing \( y_j(\mu - k) + y_{j+1}(\mu - k - 1) \) and \( y_{j+1}(\mu - k) + y_j(\mu - k - 1) \). Note that the difference between these two expressions is that: \( y_j - y_{j+1} > 0 \). This can be generalized to the case where there are more than 2 values.

Thus, according to this allocation rule, each position has a reserve price and the reserve prices are decreasing in the ranking of the positions; at the same time, the types which receive these positions are also decreasing in the ranking of positions. That means, for a certain position \( k \), if there is no \( t_i \) that satisfies the reserve price condition, then that particular position will not be allocated, but a position lower than that may still be allocated. This unallocated position can occur in the top, the middle, or the bottom of the ranking. There are some measures that the seller can take to make a specific position unavailable. For example, if there \( K \) top paid links in a search engine and the \( k' \)th position is not sold, then the search engine can insert one of its own ads (an ad about the search engine itself) into that slot or insert a fake web link there; if these \( K \) positions represent the order in a queue where all the jobs are waiting for processing, and the unallocated slot is the \( k' \)th one, then the seller can deliberately delay the processing for all the jobs after the \( k' \)th.

Now we are going to determine the payment function according to Eq.(5.8):

\[ \sum_K x_{ki} = \sum_K v_{ki} p_{ki}^k - \sum_K \int_a^{t_i} S(k)p_{ki}^k f(s)ds \]

Notice that if buyer \( i \) is allocated the first object, then Eq.(5.22) becomes that:

\[ x_{ki}^1 = v_{ki}^1 - \int_a^{t_i} S(1)p_{ki}^1 f(s)ds \]

2The case when there is a constraint that no lower position can be allocated before a higher ranked one is filled (that is, \( \sum_{i \in N} p_i^k \geq \sum_{i \in N} p_i^{k+1}, \forall k = 1, 2, ..., K - 1 \)) is not considered here. However, the results are expected to be qualitatively similar
Thus we get \( v_i^1 - Z_1^1(t_i|z_1 \leq t_i) \). Thus the optimal payment for the first object is \( x_i^1 = Z_1^1(t_i|z_1 \leq t_i) \).

Now consider the second object. The buyer \( i \) can win the second object only if his type is between the first highest and second highest buyers' type other than his own. Thus we get \( \int_a^t \mathcal{S}(2)p_i^2 f(s)ds = S(2)(t_i - z_2(t_i|z_2 \leq t_i \leq z_1|z_2 \leq t_i) = (v_i^2 - Z_2^2(t_i|z_2 \leq t_i \leq z_1)) \cdot prob(z_2 \leq t_i \leq z_1|z_2 \leq t_i) \). Thus the optimal payment for the second object is:

\[
\begin{align*}
x_i^2 &= v_i^2(1 - \text{prob}(z_2 \leq t_i \leq z_1|z_2 \leq t_i) + Z_2^2(t_i|z_2 \leq t_i \leq z_1) \cdot \text{prob}(z_2 \leq t_i \leq z_1|z_2 \leq t_i) \\
&= v_i^2 \left( 1 - \frac{\int_0^{t_i} ((n-1)!)^{-1} F(y)^{n-1-k}(1-F(t_i))f(y)dy}{\int_0^{t_i} ((n-1)!)^{-1} F(y)^{n-1-k}(1-F(y))f(y)dy} \right) \\
&\quad + \frac{\int_0^{t_i} ((n-1)!)^{-1} F(y)^{n-1-k}(1-F(t_i))f(y)dy}{\int_0^{t_i} ((n-1)!)^{-1} F(y)^{n-1-k}(1-F(y))f(y)dy} \\
&\quad + \frac{\int_0^{t_i} ((n-1)!)^{-1} F(y)^{n-1-k}(1-F(t_i))f(y)dy}{\int_0^{t_i} ((n-1)!)^{-1} F(y)^{n-1-k}(1-F(y))f(y)dy}
\end{align*}
\]

The optimal payment for the rest of the positions can be obtained in the same way.

In general,

**Proposition 18.** The optimal payment for the first position is

\[ x_i^1 = Z_1^1(t_i|z_1 \leq t_i) \]

and for the \( k' \)th position (\( k > 1 \)) is:

\[
\begin{align*}
x_i^k(t_i) &= v_i^k(1 - \text{prob}(z_k \leq t_i \leq z_{k-1}|z_k \leq t_i) + Z_k^k(t_i|z_k \leq t_i \leq z_{k-1}) \cdot \text{prob}(z_k \leq t_i \leq z_{k-1}|z_k \leq t_i) \\
&= v_i^k \left( 1 - \frac{\int_0^{t_i} ((n-1)!)^{-1} F(y)^{n-1-k}(1-F(t_i))^{k-1}f(y)dy}{\int_0^{t_i} ((n-1)!)^{-1} F(y)^{n-1-k}(1-F(y))^{k-1}f(y)dy} \right) \\
&\quad + \frac{\int_0^{t_i} ((n-1)!)^{-1} F(y)^{n-1-k}(1-F(t_i))^{k-1}f(y)dy}{\int_0^{t_i} ((n-1)!)^{-1} F(y)^{n-1-k}(1-F(y))^{k-1}f(y)dy}
\end{align*}
\]

(5.25)
Lemma 5.1. In the convergent case, in expectation, if a buyer wins a position, he will pay at least as much as the next highest type bidder’s valuation for that position.

**Proof.** We only need to show for a specific position $k$, $v^k_i - Z^k_k(t_i | z_k \leq t_i) \leq v^k_i - Z^k_k(t_i | z_k \leq z_{k-1}) \cdot \text{prob}(z_k \leq t_i \leq z_{k-1} | z_k \leq t_i)$. This is obvious because:

$$v^k_i - Z^k_k(t_i | z_k \leq t_i) = (v^k_i - Z^k_k(t_i | z_k \leq z_{k-1})) \cdot \text{prob}(z_k \leq t_i \leq z_{k-1} | z_k \leq t_i) + (v^k_i - Z^k_k(t_i | z_k \leq z_{k-1} \leq t_i)) \cdot \text{prob}(z_k \leq t_i \& z_{k-1} \leq t_i | z_k \leq t_i)$$

and both of the terms to the right of “=” are non-negative.

Thus it is obvious that the seller can extract a lot more surplus from the buyers than in the parallel case, where every winner pays the highest rejected buyer’s valuation for his winning position. It is also true that this mechanism performs better than the simple second-price sequential auction, where the buyers pay the highest rejected bid for the winning position. Intuitively, when different buyers’ valuations for lower positions fall at different rates, the seller has an incentive to match the position to the buyers optimally to maximize his expected payoff. To charge a higher price for a lower position can prevent a higher type buyer from shading his bid to win a lower position. This increases the seller’s expected payoff, comparing to the sequential second price auction.

To better understand this mechanism, let’s assume that buyers’ types follow a uniform distribution between $[0, 1]$. Then for the $k$th position, the reserve price is $\frac{1}{2} - \frac{\beta}{2(\mu - k)}$, which is decreasing in $k$. For example, if there are 3 positions available, and $\mu = 6$, $\beta = 1$, then the reserve price for positions 1, 2, 3 are $\frac{2}{5}$, $\frac{3}{8}$, $\frac{1}{3}$, respectively. If there are 4 buyers with realized types $t_1 = 0.8$, $t_2 = 0.6$, $t_3 = 0.4$, $t_4 = 0.2$, then the position 1, 2, 3 will be allocated to buyer 1, 2, 3, respectively, with the expected payment for those position $v^1_1(0.6)$, $v^2_2(0.467)$, $v^3_3(0.333)$. On the other hand, if the realized types
are: \( t_1 = 0.8, t_2 = 0.35, t_3 = 0.32, t_4 = 0.2 \). Then \( t_1 \) is allocated to position 1, \( t_2 \) is allocated to position 3, while position 2 is not allocated.

### 5.3.3 The Divergent Case

This describes the case that the higher type buyer’s valuation drops slower for a lower position than a lower type buyer’s. For example, Amazon.com is a big player in the paid placement market. It spends a big amount of marketing money in attracting customers in every search engine, but it probably doesn’t care which positions it wins. While a small company’s valuation for a lower position may drop much faster.

Let \( v^k_i = \beta - t_i(k - \mu) \), where \( \mu < 1 \). Then \( v^k_i - v^k_j = (t_i - t_j)(\mu - k) \), and \( S(k) = (\mu - k) < 0 \). We can repeat the analysis of the last section, except that because \( S(k) < 0 \), some of the incentive compatibility conditions \((\text{5.8})\) and \((\text{5.9})\) should be rewritten as \((\text{5.15})\) and \((\text{5.16})\):

\[
U_i(p, x, t_i) = U_i(p, x, b) - \int_{t_i}^{b} Q_i(p, s_i) \, ds_i
\]

\[
U_i(p, x, b) \geq 0
\]
The objective function becomes:

\[
\sum_{K} \int_{T} \left( x_{j}^{k}(t) - p_{i}^{k} v_{j}^{k}(t) \right) f(t) dt
\]

\[=
- \sum_{K} \int_{a}^{b} U_{i}^{k}(p_{i}, x_{i}^{k}, t_i) f(t_i) dt_i
\]

\[=
- \sum_{K} \int_{a}^{b} \left( U_{i}^{k}(p_{i}, x_{i}^{k}, b) - \int_{t_i}^{b} Q_{i}^{k}(p_{i}, s_i) ds_i \right) f(t_i) dt_i
\]

\[=
- U_{i}(p, x, b) + \int_{a}^{b} \left( \sum_{K} \int_{T_{-i}} S(k) p_{i}^{k}(t) f_{-i}(t_i - t) dt_i \right) dt_i
\]

\[=
- U_{i}(p, x, b) + \int_{T} F_{i}(t_i) \sum_{K} \int_{T_{-i}} S(k) p_{i}^{k}(t) f_{-i}(t_i - t) dt_i
\]

Thus the objective function becomes:

\[
\max \sum_{K} \int_{T} \left( \sum_{N} \left( \beta + \left( t_i + \frac{F_{i}(t_i)}{f_{i}(t_i)} \right) S(k) \right) p_{i}^{k}(t) \right) f(t) dt - N \cdot U_{i}(p, x, b)
\]

(5.26)

Define \( \tilde{c}(t_i) = t_i + \frac{F_{i}(t_i)}{f_{i}(t_i)} \) as the modified virtual value. Since \( S(k) \) is negative, if the modified virtual value is non-decreasing in \( t_i \), (for example, uniform distribution, exponential distribution satisfy this condition), this objective function will be maximized if the \( K \) lowest types have been selected, given that \( \beta + \left( t_i + \frac{F_{i}(t_i)}{f_{i}(t_i)} \right) S(k) \) is non-negative. Notice that the reserve price \( r(k) = \text{solve}\{\tilde{c}(t_i) \cdot (\mu - k) + \beta = 0\} \) is again decreasing in \( k \), thus the lower the position, the lower the reserve price. Notice that according to this allocation rule, the lower \( t_i \)s are allocated the lower positions with tighter reserve price conditions. Thus we can allocate the positions from the bottom to the top. If for a certain position \( k \) we can not find any buyer’s type lower than the reserve price, then this mechanism just automatically shifts all the allocated positions up \( k \) ranks. in other words, after the buyers’ type are realized, we can identify the number of positions available \( (k^*) \) by calculating how many buyer’s types are lower than the
modified virtual value. Then identifies the lowest \( k^* \) positions, and allocate the highest position to the highest \( t_i \), and so on. Thus all the unavailable slots (if any) occurs nor in the top, neither in the middle, but in the bottom.

One thing needs to be noted is, if \( t_i + \frac{F_i(t_i)}{f_i(t_i)} \) is increasing, when \( \beta \) is large enough, more specifically, when \( \beta > -\left(b + \frac{1}{f(b)}\right)S(K) \), which means if every buyers valuation for the last position is high enough, since the reserve price condition is the tightest for the last position, then each buyer’s type satisfies the reserve price condition for the rest of the positions (in other words, the reserve price condition becomes \( t_i \leq 1 \)). Thus this mechanism automatically satisfies efficiency and maximize the total payoff of the buyers and the seller.

How to allocate the items among these winners? Different from in section 5.3.1 and 5.1(b), let’s define \( A_i(t_i, k) \) as the ranked list (from the lowest to the highest) of the term \( \left(t_i + \frac{F_i(t_i)}{f_i(t_i)}\right)(\mu - k) \), and \( B_j(t_j, k) \), \( j = 1, 2, ... n - 1 \) as the \( j' \)th lowest element in the list of \( A_i \)s. Then

**Proposition 19.** The optimal incentive compatible allocation mechanism in the diverging case is to allocate the lower position to the buyers with lower modified virtual values, given that buyer’s type satisfies the reserve price condition. in other words, if we define \( d_k = \sup \{ s_i | \beta + A(s_i, k) \geq 0 \text{ and } A(s_i) \leq B_k(t_j), \forall j \neq i \} \), we can write down the allocation rule as:

\[
p^k_i(t_{-i}, s_i) = \begin{cases} 
1 & \text{if } d_{K-k}(t_{-i}) \leq s_i \leq d_{K-k+1}(t_{-i}) \ \forall k \\
0 & \text{otherwise}
\end{cases}
\] (5.27)

where \( d_0(t_{-i}) \) is defined to be \( a \), the lower bound of the value distribution.

**Proof.** First we want to show that to allocate a lower position to a lower modified virtual value (lower type) is optimal. Let \( 0 < A_1 < A_2 < A_3 \) and \( 0 < B_1 < B_2 < B_3 \). The
objective is to minimize $\sum_{i,j} A_iB_j$. And $A_1B_3 + A_2B_2 + A_3B_1 < A_1B_2 + A_2B_3 + A_3B_1$ because it is equivalent to $A_1(B_3 - B_2) + A_2(B_2 - B_3) < 0$; $A_1B_3 + A_2B_2 + A_3B_1 < A_1B_1 + A_2B_2 + A_3B_3$ because it is equivalent to $A_1(B_3 - B_1) + A_3(B_1 - B_3) = (A_1 - A_1)(B_3 - B_1) < 0$. This result can be generalized to the case where $i \geq 3$.

To check whether this allocation rule is incentive compatible, revisit the constraint (5.7) that $Q(p_i, x_i, s_i) \leq Q(p_i, x_i, t_i)$, if $s_i \leq t_i$. Notice that here $Q(p_i, x_i, t_i) = \sum_K \int_{T_i} S(k)p_i^k \cdot f_i(t_i) dt_i \leq 0$. Thus the higher $t_i$, the less possibility that the buyer is going to win, and the less negative $S(k)$, thus constraint (5.7) is indeed satisfied.

The payment scheme can be worked out accordingly as in section 5.3.2. That is, the optimal payment for the lowest position ($K$) is $D^K_1(t_i|t_i \leq d_1)$; and the optimal payment for the position $1 \leq l < K$ is:

$$x^k_i(t_i) = v^k_i(1 - \text{prob}(d_{K-k} \leq t_i \leq d_{K-k+1}|d_{K-k} \leq t_i) + D^k_i(t_i|d_{K-k} \leq t_i \leq d_{K-k+1})$$

$$= v^k_i \left(1 - \int_{t_i}^{1} \frac{(n-1)!}{(n-1-k)!} \cdot F(t_i)^{k-1} \cdot (1-F(y))^{n-1-k} \cdot f(y)dy \right)$$

$$+ \int_{t_i}^{1} \frac{(n-1)!}{(n-1-k)!} \cdot F(t_i)^{k-1} \cdot (1-F(y))^{n-1-k} \cdot f(y)dy$$

$$+ \int_{t_i}^{1} \frac{(n-1)!}{(n-1-k)!} \cdot F(y)^{k-1} \cdot (1-F(y))^{n-1-k} \cdot f(y)dy$$

(5.28)

Thus we know that other than the buyer which has the lowest modified virtual value (the buyer with the least steep slope), all the other winners are paying higher than the next lowest type buyer’s valuation for that winning position.

On the other hand, if $t_i + \frac{F_i(t_i)}{f_i(t_i)}$ is decreasing in $t_i$, then to allocate the lower position to the buyers with lower modified virtual value actually means the higher $t_i$, the greater probability to win an item. This violates the incentive compatible constraint (5.7) that $Q(p_i, x_i, s_i) \leq Q(p_i, x_i, t_i)$, if $s_i \leq t_i$. In this case, randomization is one way
to allocate the position. This is incentive compatible, but we do not have any result for optimality.

Again, if we assume that buyers’ types follow uniform distribution between \([0, 1]\), then the reserve price condition for position \(k\) is: \(t_i \leq \beta \frac{2}{(k-\mu)}\), where the reserve price is decreasing in \(k\) again, and the reserve price condition is more and more tight with the increase in the ranking. Thus the unallocated position will only be in the bottom. But if \(\beta\) is large enough such that \(\beta \geq 2(K - \mu)\), then every type satisfies the reserve price condition and this mechanism is automatically efficient.

For example, if there are 3 positions available, and \(\mu = 0, \beta = 3\), then the reserve price for positions 1, 2, 3 are 1, 0.75, 0.6, respectively. If there are 4 buyers with realized types \(t_1 = 0.8, t_2 = 0.6, t_3 = 0.4, t_4 = 0.2\), then the position 1, 2, 3 will be allocated to buyer 2, 3, 4, respectively, with the expected payment for those position \(v_2^1(0.7102), v_3^2(0.5333), v_4^3(0.4)\), which are 0.8694, 0.8668, 1, respectively, while their values for their winning positions are 1.2, 1.4, 2, respectively. On the other hand, if the realized types are: \(t_1 = 0.9, t_2 = 0.85, t_3 = 0.8, t_4 = 0.2\). Then \(t_3\) is allocated to position 1, and \(t(4)\) is allocated to position 2, while position 3 is not allocated. In this case, the unallocated position will always be in the bottom.

5.3.4 Convergent then Divergent

This is the extreme case of the convergent case, where if a buyer has a higher valuation for a top position, his valuation for a lower position may be lower than his competitors. Again in the example of paid placement practice in search engine industry, a small company’s willingness to pay for a top position maybe higher than a established big company like Amazon.com, but because a small company often has a tight budget, its valuation for a bottom position may be much lower than Amazon. Write the utility function as \(v^k_i = \beta - t_i(k - \mu)\), where \((k - \mu) > 0\) before a certain \(\tilde{k}\) between \([1, K]\) and after that \(\tilde{k}, (k - \mu) < 0\) is the point where each utility function crosses. In this
setting, the higher valuation a buyer has for a higher position, the quicker that his valuation drops for the lower positions. Analyzing the utility function in the same way, 
\[ v^k_i - v^k_j = (t_i - t_j)(\mu - k), \]  
and \( S(k) = (\mu - k) > 0 \) when \( k \leq \bar{k} \) and \( S(k) < 0 \) when \( k > \bar{k} \).

Again one of the incentive compatibility conditions (5.8) should be rechecked because the sign of \( S(k) \) changes before and after \( k = \bar{k} \).

If we change the expression of Eq. (5.8) into

\[ U_i(p, x, t_i) = U_i(p, x, w) + \int_{t_i}^{t_i} Q_i(p, s_i) ds_i \quad \text{if} \quad t_i \geq w \]

and

\[ U_i(p, x, t_i) = U_i(p, x, w) - \int_{t_i}^{t_i} Q_i(p, s_i) ds_i \quad \text{if} \quad t_i < w \]

where \( w \in (a, b) \), and we have

\[ U_i(p, x, w) \geq 0 \]

The objective function becomes:

\[
\sum K \int_T \left( x^k_j(t) - p^k_i(t)v^k_j(t) \right) f(t) dt \\
= -\sum K \int_a^b U^k_i(p_i^k, x_i^k, t_i) f(t_i) dt_i - \sum K \int_a^w U^k_i(p_i^k, x_i^k, t_i) f(t_i) dt_i \\
= -\sum K \int_a^w \left( U^k_i(p_i^k, x_i^k, w) - \int_a^w Q^k_i(p_i^k, s_i) ds_i \right) f(t_i) dt_i \\
- \sum K \int_a^b \left( U^k_i(p_i^k, x_i^k, w) + \int_a^w Q^k_i(p_i^k, s_i) ds_i \right) f(t_i) dt_i \\
= -U_i(p, x, w) + \int_a^w Q^k_i(p_i^k, s_i) ds_i f(t_i) dt_i - \int_a^b Q^k_i(p_i^k, s_i) ds_i f(t_i) dt_i \\
= -U_i(p, x, w) + \int_a^w \left( \int_a^{s_i} Q_i(p, s_i) f(t_i) dt_i ds_i - \int_a^{s_i} Q_i(p, s_i) f(t_i) dt_i ds_i \right) f(t_i) dt_i ds_i \\
= -U_i(p, x, w) + \int_a^w (F(s_i)) Q_i(p, s_i) ds_i - \int_a^w (1 - F(s_i)) Q_i(p, s_i) ds_i \\
= -U_i(p, x, w) + \int_a^w \left( F(t_i) \sum K \int_{T_{-i}} S(k)p^k_i(t_i, t_{-i}) f_{-i}(t_{-i}) dt_{-i} \right) dt_i \\
- \int_a^w \left( 1 - F(t_i) \sum K \int_{T_{-i}} S(k)p^k_i(t_i, t_{-i}) f_{-i}(t_{-i}) dt_{-i} \right) dt_i 
\]
where \( \sum K \int_a^w U^k_i(p^k_i, x^k_i, t_i) f(t_i) dt_i \) can be written as

\[
\sum K \int_a^w \left( U^k_i(p^k_i, x^k_i, w) - \int_a^w Q^k_i(p^k_i, s_i) ds_i \right) f(t_i) dt_i
\]

which implies that \( \int_a^w Q^k_i(p^k_i, s_i) ds_i f(t_i) dt_i \) is negative.

Thus the objective function becomes:

\[
\max \sum K \int_a^w \left( \sum N \left( \beta + \left( t_i + \frac{F_i(t_i)}{f_i(t_i)} \right) (\mu - k) \right) p^k_i(t) \right) f(t) dt + \int_b^w \left( \beta + \left( t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \right) (\mu - k) \right) p^k_i(t) f(t) dt - N \cdot U_i(p, x, w)
\]

Assume that both the virtual value \( t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \) and the modified virtual value \( t_i + \frac{F_i(t_i)}{f_i(t_i)} \) are non-decreasing (for example, the uniform distribution satisfies these two conditions). To maximize this objective function, if we make \( U_i(p, x, w) \) equal to 0, notice that when \( t_i \geq w, \mu - k \geq 0 \), it is optimal to allocate the higher \( \lceil \bar{k} \rceil - k \) positions to the buyers with the highest virtual values \( t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \); when \( t_i < w, \mu - k < 0 \), it is optimal to allocate the lower \( K - \lceil \bar{k} \rceil \) positions to the buyers with the lowest modified virtual value of \( t_i + \frac{F_i(t_i)}{f_i(t_i)} \) when \( \mu - k \leq 0 \).

More specifically, proposition 20 describes this allocation rule.

**Proposition 20.** If the distribution of \( t_i \) satisfies the regularity conditions and \( t_i + \frac{F_i(t_i)}{f_i(t_i)} \) is non-decreasing, the optimal allocation mechanism is: for each \( k < \mu \), allocated the highest remaining position to the buyers with the highest remaining \( t_i \), if \( t_i \geq w \) and \( \beta + \left( t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \right) (\mu - k) \geq 0 \), until \( k = \lfloor \mu \rfloor \), otherwise leave that particular position unassigned; for each \( k > \mu \), allocate the remaining lowest position to the buyers with the remaining lowest \( t_i \), if \( t_i < w \) and \( \beta + \left( t_i + \frac{F_i(t_i)}{f_i(t_i)} \right) (\mu - k) \geq 0 \); otherwise shift the allocation up for 1 rank.
Proof. Follow the cases when \( \mu - k \geq 0 \) and \( \mu - k < 0 \), which follow the proof in sections 5.3.2 and 5.3.3.

The above gives the optimal mechanism given a specific \( w \in (a, b) \). Let this value be \( V(w) \). Now the problem is how to identify that optimal \( w \)? Our objective function now becomes:

\[
\max_w V(w)
\]

(5.31)

Obviously \( w^* \) is a function of \( \mu, n, \) and \( K \). For example, the ideal \( w^* \) should have the property that there are at least \( [K - \mu] \) buyers whose types are below \( w \), and at least \( [\mu] \) buyers whose types are above \( w \). And this indicates that, given \( K \), the optimal \( w \) should be non-increasing in \( \mu \). But to complete this mechanism, \( w \) should be preannounced. So there is positive probability that the above condition can not be satisfied, thus this mechanism is not efficient in addition to the existence of reserve price for each position, because there is positive probability that a certain buyer whose type satisfies the reserve price, will give the seller higher profit if he wins, but can not win because its type falls on the a “wrong” side of \( w \). Figure 5.2 shows one example of how the optimal \( w \) changes with \( \mu \), with the assumption that bidders’ types are uniformly distributed between \([0,1]\), and \( \beta \) is large enough so the reserve price condition is always satisfied. (In this example \( n = 7, k = 4, \) and \( \mu \) can be anywhere between 1 and 4.)

From Figure 5.2 we can see that when the reserve price condition is satisfied, \( w \) is decreasing with \( \mu \). Intuitively, the larger the \( \mu \), the more types should be above \( w \), thus the smaller the \( w \). But one thing need to be noted is that once \( k \) and \( n \) is fixed, the only determinant of \( w \) is between which two positions \( \mu \) is located, while the exact position of \( \mu \) between those two positions does not matter.

Also in the example of uniform distribution between \([0,1]\), the reserve price condition for position \( k < \mu \) is \( t_i \geq \frac{1}{2} - \frac{\beta}{2(\mu - k)} \); for position \( k \geq \mu \) is \( t_i \leq \frac{\beta}{2(k - \mu)} \). Combining with the choice of \( w \), then the necessary condition to allocate position \( k \) before \( k \) is
\[
t_i \geq \max\{w, \frac{1}{2} - \frac{\beta}{2(\mu-k)}\} \quad \text{and for position after } k \text{ is } t_i < \min\{w, \frac{\beta}{2(k-\mu)}\}. \]

For example, let \( K = 3, \mu = 1.5, \beta = 0.5, w = 0.5 \). Then we have the actual reserve price for position 1 before \( \hat{k} \) is \( \max\{0.5,0\} = 0.5 \), and the actual reserve price for position 2 and 3 (after \( \hat{k} \)) are: \( \min\{0.5,1\} = 0.5 \), and \( \min\{0.5,0.333\} = 0.333 \). If the realized types are: \( t_1 = 0.8, t_2 = 0.6, t_3 = 0.4, t_4 = 0.2 \), then the position 1, 2, 3 will be allocated to buyer 1, 3, 4. On the other hand, if the realized types are: \( t_1 = 0.45, t_2 = 0.4, t_3 = 0.3, t_4 = 0.2 \), then the first position is not allocated, while the second position is allocated to buyer 3, and the third position is allocated to buyer 4.

### 5.4 Implementation

Here we present the optimal allocation and payment mechanism under the four different cases in figure 5.3.

Notice that in the mechanisms we discussed above, we assumed that the seller can commit to leave a position unfilled if no buyer’s type satisfies the reserve price condition. In the search engine advertising case, this can be done by inserting either a fake ad or the ad of the search engine itself. In a waiting queue, this can be done by credibly delaying the job processing time or prolonging the waiting time. On the other hand, if the seller cannot commit to it, that is, the \( k' \) th position has to be filled first in order to fill the \( k+1' \) th position, then we have an extra constraint,

\[
\sum_{i \in N} p_i^k \leq \sum_{i \in N} p_i^{k-1} \quad k = 2, ..., K
\]  

(5.32)

Following the same maximization process, we find that this constraint is binding only in the convergent case, and the converging portion of the last case, because only in these two cases will a higher \( t_i \) be assigned to a higher position, with a higher reserve price.

This practice of leaving a position unfilled is commonly observed in reality. For example, in the airline industry, passengers in the coach class are not allowed to sit in the
first class without paying extra, even when the first class is not full. It’s also observed in some competitions, sometimes the highest award given is the second prize, while the first prize remains un-assigned. This guarantees that those who can pay for first class don’t understate their values, or high competition standard. In our model, this makes sure that buyers do not reduce their bids, hoping to win a more desirable position when there is lack of competition.

5.5 Conclusion

In this paper we show how the earlier work about optimal auctions ([41] and [33]) can be extended and applied to the allocation of non-identical objects where every buyer only has unit demand, and their preferences for these objects are ranked in the same order. We find that the optimal way to sell these non-identical objects is quite different when buyers preferences for different objects changes in different way. Thus to understand the buyers preference characteristics is vital in determining the optimal mechanism. We find that when buyers’ valuations for a lower position drop at different rates, the seller can extract more surplus from the buyers than when they drop at the same rate. Compare to the single unit or multiple identical unit case, where the inefficiency is created by the reserve price (under the assumption of symmetric buyers), this optimal allocation mechanism can be inefficient because of the heterogeneity of the items, here because of the choice of the “pivot” type in the fourth case.

We use linear utility function in this paper, and assume that there is some position for which all utility functions give the identical value. Actually the linear function assumption can be relaxed as long as the second assumption is satisfied. This assumption may seem special but it only says that there exists a position (probably very far away) such that every buyer’s valuation for that position is the same (like a position in the very bottom of the result page). Many commonly used utility functions when dealing
with heterogeneous consumers have this property (for example, $U(\theta) = \theta q$ where $q$ is the quality of a product). In future research, we hope to study more general settings than the one discussed in this paper.
Fig. 5.1. Different cases of buyers utility with respect to the ranking of the positions
Fig. 5.2. How the optimal $w$ changes with $\mu$
Fig. 5.3. Optimal allocation and payment schemes under the four different cases
A.1 First Period Bidding Strategy in a Sequential Auction

Bidders’ equilibrium bidding strategy in the first period, (Eq. 2.1). We get buyer i’s first-period expected payoff as

$$\pi(v_i) = \pi^1(v_i | \beta^{-1}(b^1_i) > Y_k) P \left( \beta^{-1}(b^1_i) > Y_k \right) + \pi^2(v_i | \beta^{-1}(b^1_i) \leq Y_k, v_i \geq Y_{2k}) P \left( \beta^{-1}(b^1_i) \leq Y_k, v_i \geq Y_{2k} \right)$$

(Here we are assuming that the first-period price is not revealed to the bidders. Cases where the price is revealed will follow similarly).

First suppose $r = \beta^{-1}(b^1_i) > v_i$, then Eq. A.1 can be written as:

$$\pi(v_i) = \pi^1(v_i | \beta^{-1}(b^1_i) \geq Y_k) P \left( \beta^{-1}(b^1_i) \geq Y_k \right) + \pi^2(v_i | \beta^{-1}(b^1_i) \leq Y_k, v_i \geq Y_{2k}) P \left( \beta^{-1}(b^1_i) \leq Y_k, v_i \geq Y_{2k} \right)$$

$$= \int_r^\infty (v_i - \beta(y)) \frac{(n-1-k)!}{(n-k)!} F(y)^{n-1-k} (1 - F(y))^{k-1} f(y) dy + \delta (1 - F(r))^k \int_0^r (v_i - y) \frac{(n-1)!}{(n-1-k)!} f(y) dy$$

$$\sum_{j=1}^k \left( \frac{n-1}{k-1+j} \right) F(y)^{n-1-2k} (1 - F(y))^{k-j} (1 - F(r))^{j-1} f(y) dy$$

(A.1)

The case when $\beta^{-1}(b^1_i) < v_i$ is similar.
Take the first derivative of either case, for example Eq. A.1 with respect to \( r \), we have:

\[
\pi_r(v_i) = (v_i - \beta(y)) \frac{(n-1)!}{(n-1-k)!k!} F(v_i)^{n-1-k} (1 - F(v_i))^{k-1} f(v)
\]

\[
-\delta k (1 - F(r))^{k-1} f(r) \cdot \left[ \sum_{j=2}^{k-1} \binom{n-1}{k-1+j} \frac{n-1}{j-1} \right] (1 - F(v_i))^{j-2}
\]

\[
\cdot \int_0^r (v_i - y) \frac{(n-1)!}{(n-1-2k)!((2k-1)!} F(y)^{n-1-2k} (1 - F(y))^{k-j} f(y)dy
\]

Set \( \pi_r(v_i) = 0 \) and \( r = v_i \), multiply both sides by \( 1 - F(v_i) \), and take out \( (j-1)(1 - F(v)) \) from the summation form on the right hand side, we have

\[
(v_i - \beta(v_i)) \sum_{i=k}^{2k-1} \frac{(n-1)!}{(n-1-i)!} F(v_i)^{n-1-i} (1 - F(v_i))^i
\]

\[
= \delta (1 - F(v_i))^k \int_0^r (v_i - y) \frac{(n-1)!}{(n-1-2k)!((2k-1)!} f(y)dy
\]

which is equivalent to:
\[(v_i - \beta(v_i))\text{prob}(Y_{2k} < v_i \leq Y_k)\]
\[= \delta \left(1 - F(y)\right)^k \int_0^{v_i} \frac{(n-1)!}{(n-1-k)!} F(y)^{n-1-2k} (1 - F(y))^{k-j} (1 - F(v))^{j-1} \right) f(y) dy \]
\[
\sum_{j=1}^{k-1} \left\{ \sum_{j=1}^{k-1} \frac{n-1}{k-1+j} F(y)^{n-1-2k} (1 - F(y))^{k-j} (1 - F(v))^{j-1} \right\} f(y) dy \]
\[= \delta \pi^2(v_i|Y_{2k} < v_i \leq Y_k) \]
\[= (1 - \delta)v_i + \delta E[Y_{2k}|Y_{2k} < v_i \leq Y_k] \]

Notice that when \(\delta = 1\) and \(k = 1\), we have

\[\beta(v) = v - \pi^2(v|Y_2 < v \leq Y_1)\]
\[= v - \pi^2(v|v = Y_1)\]
\[= v - (v - E[Y_2|v = Y_1]) = E[Y_2|v = Y_1]\]

which is the same as the Weber’s standard results (83), where \(\beta_1(v) = E[Y_2|v = Y_1].\)
A.2 Auctioneer’s choice of either one period or two-period auction

proposition 4. The difference of revenue between holding a two-period auction and a one-period auction is:

\[ E[R_2] - E[R_1] = k \cdot ((1 - \delta) E[X_{k+1}] - 2(1 - \delta) E[X_{2k+1}]) \]  \hspace{1cm} (A.2)

When \( \delta = 1 \),

\[ E[R_2] - E[R_1] = k \cdot ((1 - 1) E[X_{k+1}] - 2(1 - 1) E[X_{2k+1}]) = 0 \]  \hspace{1cm} (A.3)

so the auctioneer gets the same results no matter whether he sells the items in one period or two periods, which is the standard result of Weber(83). More specifically if we assume that bidders’ values follow a uniform distribution, then

\[ E[R_2] = k \cdot \left\{ (1 - \delta) E[X_{k+1}] + 2\delta E[X_{2k+1}] \right\} \]
\[ = k \cdot \left\{ (1 - \delta) \frac{n-k}{n+1} + 2\delta \frac{n-2k}{n+1} \right\} \]
\[ = \frac{k}{n+1} \left\{ (1 + \delta)n - (1 + 3\delta)k \right\} \]

When \( n \geq 3k \), \( E[R_2] \) is increasing in \( \delta \); when \( n < 3k \), \( E[R_2] \) is decreasing in \( \delta \). So given the distribution of the bidders values, the effect of the discounting factor on the sequential auction depends on the market competition. ■

proposition 3. The difference of revenue between holding a two-period auction and a one-period auction is:

\[ E[R_2] - E[R_1] = E[R_2] - E[R_1] \]
\[ = k \cdot ((1 - \delta) E[X_{k+1}] - 2(1 - \delta) E[X_{2k+1}]) \]
\[ = k \cdot (1 - \delta) (E[X_{k+1}] - 2E[X_{2k+1}]) \]
\[ = \frac{k(1-\delta)}{n+1} (3k - n) \]
so when \( k < \frac{1}{3} \cdot n \), \( E[R2] > E[R1] \), the auctioneer is better off by offering two-period auction.

\[ E[R2] > E[R1] \]

A.3 Reserve Price Without Commitment

Proposition 5. First we show that there exists such a \( \tilde{r} \). We can reinterpret Eq. 2.2 as,

\[
(v - r) \cdot \sum_{i=0}^{K} \text{prob}(n' = i) \quad \text{Vs.} \quad \delta \sum_{i=0}^{K} (v - E[Y_{K,n-1}|n' = i]) \cdot \text{prob}(n' = i)
\]

and compare the LHS and RHS term by term. For each term, given \( n' = i \), \( \frac{\partial LHS}{\partial v} = 1 \).

The \( RHS = \delta(v - \int_{0}^{v} \frac{y h_{K}(y)dy}{H_{K}(v)}) \), and we can show that

\[
\frac{\partial LHS}{\partial v} = \delta \cdot \left( 1 - \left( 1 - \frac{H_{2}(v) - h(v) \int_{0}^{v} H(y)dy}{H_{2}(v)} \right) \right) < 1
\]

and is non-negative. This shows that both the LHS and RHS are increasing, while the LHS is increasing faster than RHS. Plus when \( v = r \), \( LHS = 0 \) while \( RHS \geq 0 \). So there exists a non-negative \( v^* \) which makes \( LHS = RHS \). Let \( \tilde{r} = v^* \) if \( v^* \in [0, 1] \), otherwise \( \tilde{r} = 1 \). This is Eq. 2.3.

Next we will show it is Nash Equilibrium to follow the strategy in proposition 5. Suppose that all the bidders except \( i \) follows this strategy. Now for bidder \( i \), if his valuation is less than \( \tilde{r} \), then he has no incentive to deviate by bidding in the first period because his only chance of winning is when \( n' < K \); then he pays at least \( r \) for the winning item. But then according to Eq. 2.2 his expected payoff is lower than winning in the second period and paying a lower price. When \( n' \geq K \), he cannot win bidding in either period. So bidding in the first period won’t help him.

If bidder \( i' \)'s valuation is higher than \( \tilde{r} \), then if he bid in the first period, he has no incentive to deviating from bidding his true value because by bidding lower there
is positive probability that he will not be among the winners, and by bidding higher there is positive probability that he is going to pay more than his true value if he wins. According to Eq. 2.3, given that he will bid his true value, he has no incentive to deviate to wait till the second period. This concludes our proof.

A.4 Bidders uncertain about number of periods

**proposition 6.** Assuming bidders’ value follows a uniform distribution in $[0, 1]$,

$$E[R_H] - E[R'_H] = k \cdot \left[ (1 - \rho' \delta) E[X_{k+1}] - (\delta + \rho' \delta - 2) E[X_{2k+1}] \right]$$

$$= \frac{k}{n+1} \left[ (1 - \rho' \delta) (n-k) + (\delta + \rho' \delta - 2) (n-2k) \right]$$

The critical value of $\rho'$ is determined by the equation

$$
(1 - \rho' \delta) (n-k) + (\delta + \rho' \delta - 2) (n-2k) = 0 \quad (A.4)
$$

This is satisfied when $\rho' = \frac{(3-2\delta)k-(1-\delta)n}{\delta k}$. We now focus on $\frac{(3-2\delta)k-(1-\delta)n}{\delta k}$.

Notice that $\rho' = \frac{\alpha \rho}{\alpha \rho + 1 - \rho} < \rho$ and is increasing in $\alpha$.

1). If $\rho < \frac{(3-2\delta)k-(1-\delta)n}{\delta k}$, then $\rho' < \frac{(3-2\delta)k-(1-\delta)n}{\delta k}$, so

$$
(1 - \rho' \delta) (n-k) + (\delta + \rho' \delta - 2) (n-2k) > 0 \quad (A.5)
$$

which means $E[R_H] - E[R'_H] > 0$. Thus the auctioneer prefers to pretend to be a Low type in the first period and sell $k$ items in each period. So $\alpha = 1$ and $\rho' = \rho$.

2). If $\rho > \frac{(3-2\delta)k-(1-\delta)n}{\delta k}$, then

a) If $0 \leq \frac{(3-2\delta)k-(1-\delta)n}{\delta k} \leq 1$, there exists one solution $\alpha^*$ such that

$$
\frac{\alpha^* \rho}{\alpha^* \rho + 1 - \rho} = 1 - \frac{1-\delta^2}{\delta} \cdot \frac{n-2k}{k},
$$

and with the belief generated by this probability, the auctioneer will be indifferent between selling in one period or two. So the auctioneer will choose to pretend to be a Low type with probability $\alpha^*$. 
b) If \( \frac{(3-2\delta)k-(1-\delta)n}{\delta k} < 0 \), then \( \rho' > \frac{(3-2\delta)k-(1-\delta)n}{\delta k} \), then \( E[R_H] - E[R'_H] < 0 \) and the auctioneer is better off by selling all of her items in one period. \( \blacksquare \)
Appendix B

B.1 Monopoly Case

Proposition 7. Consider the first-order condition Eq. 3.2, \((s + bx)M_x(\xi, x) + bM(\xi, x) - 2ex = 0\). Let \(-\frac{M_x}{M} \geq \frac{b}{s}\) at \(x = 0\). Then, due to Assumption 1, we know that \(-\frac{M_x}{M} \geq \frac{b}{s}\) for all \(x\), hence the first derivative is always negative, and the optimal bias level is at the boundary \(x^* = 0\). For the other case where \(-\frac{M_x}{M} < \frac{b}{s}\) at \(x = 0\), then the FOC yields the optimal solution when the profit function is concave, and \(x^* > 0\).

Proposition 8. By envelop theorem, \(\frac{\partial \pi^*}{\partial \xi} = sM(\xi, x) + bM(\xi, x)\), which is again positive. This implies that the gatekeeper’s total profit is increasing in its quality.

Taking the first derivative of the LHS of Eq. 3.2 with respect to \(\xi\), we get \(sM_x(\xi, x) + bxM_{xx}(\xi, x) + bM_x(\xi, x)\). If this term is positive, or equivalently \(-\frac{M_x}{M} < \frac{b}{s}\), then an increase in \(\xi\) will increase the LHS, thus \(x\) in the RHS should also increase, which means that an increase in the gatekeeper’s quality allows it to increase its fraction of paid placement. Conversely, if the term is negative, then it is optimal to reduce the bias level when \(\xi\) increases.

Proposition 9. From Eq. 3.2, \(sM_x(\xi, x) + bxM_x(\xi, x) + bM(\xi, x) = 2ex\), since \(M_x(\xi, x)\) is negative, so an increase in \(s\) reduces the left hand side, in turn the right hand side. Hence \(x^*\) decreases (and \(\gamma\) increases) when \(s\) increases. That \(M\) increases is obvious. By envelop theorem, \(\frac{\partial \pi^*}{\partial s} = M(\xi, x)\), which is at least non-negative because the market coverage \(M\) is non-negative. The argument is similar for the case where \(b\) increases.
B.2 Duopoly Case

**Proposition 11.** Consider the first derivative of Eq. 3.5: \( \frac{s + bx}{2} \mathcal{M}_x(x) + \frac{b}{2} \mathcal{M}(x) - 2ex \). This derivative is always lower than the corresponding term for the monopoly case (Eq. 3.2). Hence, when the profit function is concave, then \( \hat{x}^* \) is strictly less than \( x^* \) when \( x^* > 0 \), and equals 0 when \( x^* = 0 \).

Since \( x^* < x^* \), the bias level of both search engine is lower than that in the monopoly case, hence this increases users welfare. Since \( \gamma_i = b\mathcal{M}(x_i) - ex_i \), a decrease in the bias level increases the fee for paid placement, and this reduces the content providers payoff.

The total profits of the search engines are smaller than the monopoly case, i.e., \( \pi_1^* + \pi_2^* < \pi^* \). To confirm this, notice that the new profit function is identical to the monopolist’s profit function, except for two extra negative term \( -\frac{s}{2} \mathcal{M}(x_2) \) and \( -\frac{b}{2} \mathcal{M}(x_2)x_1 \). This implies that the increase in the total advertisement revenue and higher placement fee (caused by the lower bias) to the gatekeeper does not compensate for the decrease in placement revenue by reducing the bias level.

**Proposition 12.** Consider the case where gatekeeper 1 increases its technology level to \( \xi_1 \) and gatekeeper 2 keeps \( \xi_2 \) unchanged, hence \( \frac{\xi_1}{\xi_1 + \xi_2} > \frac{1}{2} \). Now consider two choices for gatekeeper 1’s bias level: \( x_1^1 \) and \( x_1^2 \), which gives him a higher or lower composite quality measure, respectively. In the case of higher composite quality measure, it tries to maximize

\[
\frac{s\mathcal{M}_1^1 + b\mathcal{M}_1^1 - e(x_1^1)^2}{\xi_1 + \xi_2} - \frac{\xi_2}{\xi_1 + \xi_2} s\mathcal{M}_2 - \frac{\xi_2}{\xi_1 + \xi_2} b\mathcal{M}_2 x_1^1 \tag{B.1}
\]

otherwise he will have smaller market share and maximize

\[
\frac{\xi_1}{\xi_1 + \xi_2} s\mathcal{M}_1^2 + \frac{\xi_1}{\xi_1 + \xi_2} b\mathcal{M}_1^2 x_1^1 - e(x_1^2)^2 \tag{B.2}
\]
Since \( M^2_1 \leq M_2 \leq M^1_1 \), then Eq. B.1 holds:
\[
\frac{\xi_1}{\xi_1 + \xi_2} sM_2 + \frac{\xi_1}{\xi_1 + \xi_2} bM_2 x_1^1 - e(x_1^1)^2.
\]
Compare the RHS with Eq. B.2. They have the same functional form
\[
\frac{\xi_1}{\xi_1 + \xi_2} sM(\xi_1, x_1) + \frac{\xi_1}{\xi_1 + \xi_2} bM(\xi_1, x_1) y - ey^2.
\]
The only difference is that Eq. B.2 has one constraint \( y = x_1 \), and the RHS has one corresponding constraint that \( y \leq x_1 \). So the maximization of the RHS will perform at least as good as Eq. B.2 because it is less constrained. This in turn shows that Eq. B.1 performs at least as good as Eq. B.2. That means, if gatekeeper 1 has higher technology level than gatekeeper 2, then he will generate more revenue if he has a higher composite quality measure than the case with a really high bias thus a lower composite quality measure.

To understand gatekeeper 1’s bias choice compared with the identical technology case \((\hat{x}^*)\), consider the two FOCs—when it has the same technology level as gatekeeper 2:
\[
\frac{\xi_1}{\xi_1 + \xi_2} sM x_1 (\xi_1, x_1) + \frac{\xi_1}{\xi_1 + \xi_2} bM x_1 (\xi_1, x_1) = 2ex_1
\]
and when it has higher technology level:
\[
\frac{\xi_1}{\xi_1 + \xi_2} sM x_1 (\xi_1, x_1) + \frac{\xi_1}{\xi_1 + \xi_2} bM x_1 (\xi_1, x_1) = 2ex_1.
\]
Whether or not the LHS of the first FOC is smaller than that of the latter, depends on the change rate with respect to \( \xi_1 \), which is
\[
\frac{\xi_2}{\xi_1 + \xi_2} [sM x_1 + bM x_1 x_1 + bM] + \frac{\xi_1}{\xi_1 + \xi_2} [sM x_1 + bM x_1 x_1 + bM x_1].
\]
The sufficient condition to make this term non-negative is:
\[
\frac{\partial}{\partial x} \left( \frac{\xi_2}{\xi_1 + \xi_2} sM x_1 + \frac{\xi_1}{\xi_1 + \xi_2} bM x_1 x_1 + \frac{\xi_1}{\xi_1 + \xi_2} bM x_1 \right).
\]
Proposition 13. When gatekeeper 1 is a monopolist with technology level \( \xi_1 \), the FOC for its profit function is:
\[
sM x (\xi, x) + bx M x (\xi_1, x_1) + bM (\xi_1, x_1) = 2ex_1.
\]
If there exists a gatekeeper 2 with technology level \( \xi_2 < \xi_1 \), then according to Proposition 12, gatekeeper 1 will have a larger market share, thus the FOC for its profit function becomes:
\[
sM x_1 (\xi_1, x_1) + bx_1 M x_1 (\xi_1, x_1) + bM (\xi_1, x_1) - \frac{\xi_2}{\xi_1 + \xi_2} bM x_2 = 2ex_1,
\]
whose LHS is less than the LHS when it is a monopolist. Thus the gatekeeper with better technology level will choose a bias level lower than its monopoly level.

Where there are two gatekeepers, gatekeeper 1 has higher composite quality measure than gatekeeper 2, thus
\[
\mathcal{M}(\xi_1, \hat{x}_1^* (\xi_1)) > \mathcal{M}(\xi_2, \hat{x}_2^* (\xi_2)),
\]
which is in turn greater
than gatekeeper 2’s composite quality measure when it has the same quality level as gatekeeper 1, $M(\xi_2, x_2^*(\xi_2))$, which is the same as $M(\xi, x_1^*(\xi))$.

\section*{B.3 Setting quality}

\textbf{Proposition 14.} When gatekeeper $i$ sets a lower technology level, it will have a smaller market share. Thus its profit is: $\xi_i \xi_i + \xi_j \left[ sM(\xi_i, x_i) + bM(\xi_i, x_i) \right] - e x_i^2 - c(\xi_i)$.

The FOC of this function, taking the optimal bias level choice into consideration, is: $\frac{\xi_j}{(\xi_i + \xi_j)^2} (sM(\xi_i, x_i) + bM(\xi_i, x_i)) + \frac{\xi_i}{\xi_i + \xi_j} \left( sM_{xi}(\xi_i, x_i) + bM_{xi}(\xi_i, x_i) \right) - c'(\xi_i)$. If this function is convex, that is when $\frac{d}{d\xi} \left[ \frac{\xi_j}{(\xi_i + \xi_j)^2} (sM(\xi_i, x_i) + bM(\xi_i, x_i)) + \frac{\xi_i}{\xi_i + \xi_j} \left( sM_{xi}(\xi_i, x_i) + bM_{xi}(\xi_i, x_i) \right) - c'(\xi_i) \right] > c''(\xi_i)$ for all $\xi \in [0, \bar{\xi}]$, both of the firms will choose boundary solution where $\xi = \bar{\xi}$.

Otherwise, the gatekeeper with superior cost function (gatekeeper 2) will set a higher technology level, so he will choose $\xi$ based on the objective function: \( \max x_i s[M(\xi_1, x_1) + bM(\xi_1, x_1)] - \frac{\xi_2}{\xi_1 + \xi_2} (sM(\xi_2, x_2) + bM(\xi_2, x_2)) x_1 - e x_i^2 - c(\xi_i) \). The solution to this equation ($\xi_1^*$) is greater than $\xi_2^*$, which is in equilibrium path.

\section*{B.4 Social Choice}

\textbf{Proposition ??}. The first derivative of the social welfare function with respect to $x_1$ and $x_2$ is:

\begin{align*}
\frac{\partial SW(2)}{\partial x_1} &= f'M_{x1} + sM_{x1} + bM(x_1) - \frac{1}{2} bM(x_2) - e x_1 = 0 \quad (B.3) \\
\frac{\partial SW(2)}{\partial x_2} &= \frac{1}{2} bM(x_2) - e x_2 = 0 \quad (B.4)
\end{align*}

Comparing with the socially optimal bias level when there is only one gatekeeper ($y^*$), gatekeeper 1 should have a lower bias level ($y_1$) since its FOC has one extra negative term $-\frac{1}{2} bM(x_2)$; similarly gatekeeper 2’s bias level ($y_2$) is higher than $y^*$. Therefore
$y_2 > y_1$. This generates higher revenue than the case where the two gatekeepers have the same bias level, because it is the same maximization problem without the constraint $M(x_1) = M(x_2)$.

Now we compare the socially optimal bias levels with the competitive bias level $\hat{x}^*$. It is clear that $y_2 > \hat{x}^*$ since $x^* > \hat{x}^*$. For gatekeeper 1, the condition for $y_1 < \hat{x}^*$ is identical to the condition for $y^* < x^*$ as given in Proposition ?? . To verify this, note that the FOC for the duopoly case can also be written as: $sM_{x_1} + bM(x_1) + bM_{x_1}x_1 - \frac{1}{2}bM(x_2) - 2ex_1 = 0$, given that $x_1 = x_2$, then the difference between the different FOCs is just $f'M_x - bM_xx + ex$, which is just the condition we considered in the last section.
References


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