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FORMAL EXPONENTIAL MAPS AND HOCHSCHILD COHOMOLOGY ASSOCIATED WITH DG MANIFOLDS

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Abstract

In this dissertation, we study two different aspects of dg manifolds.

The first part is devoted to the study of the relation between 'formal exponential maps,' the Atiyah class, and Kapranov $L_\infty[1]$ algebras associated with dg manifolds in the C^∞ context. We prove that, for a dg manifold, a 'formal exponential map' exists if and only if the Atiyah class vanishes. Inspired by Kapranov's construction of a homotopy Lie algebra associated with the holomorphic tangent bundle of a complex manifold, we prove that the space of vector fields on a dg manifold admits an $L_\infty[1]$ algebra structure, unique up to isomorphism, whose unary bracket is the Lie derivative w.r.t. the homological vector field, whose binary bracket is a 1-cocycle representative of the Atiyah class, and whose higher multibrackets can be computed by a recursive formula. For the dg manifold $(T_X^{0,1}[1], \bar{\partial})$ arising from a complex manifold X, we prove that this $L_\infty[1]$ algebra structure is quasi-isomorphic to the standard $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$.

The second part is devoted to the study of Hochschild cohomology of a dg manifold arising from a Lie algebra in terms of Keller admissible triples. We prove that a Keller admissible triple induces an isomorphism of Gerstenhaber algebras between Hochschild cohomologies of the direct-sum type for dg algebras. As an application, we show that the Hochschild cohomology of the dg algebra of smooth functions on a dg manifold arising from a Lie algebra $\mathfrak{g}$ is isomorphic to the Hochschild cohomology of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$. Furthermore, we give a new concrete proof of the Kontsevich–Duflo theorem for finite-dimensional Lie algebras.

Contents

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Chapter 1

Introduction

A differential graded manifold (dg manifold in short) is a geometric object which intertwines various fields including complex geometry and Lie theory. The notion of dg manifolds (a.k.a. Q-manifolds [1, 40, 48]) started to appear in the mathematical physics literature in the study of BRST operators used to understand gauge symmetries, and also appeared in AKSZ formalism which studies topological field theories and topological sigma models [1, 8]. It naturally arose in various fields such as differential geometry, Lie theory, homotopy theory and derived geometry. In particular, strong homotopy Lie algebras (or L_∞ algebras) and Lie algebroids, as well as ordinary smooth manifolds give rise to dg manifolds. A vector bundle E equipped with a smooth section s gives rise to a dg manifold related to the derived intersection of s with the zero section. Moreover, complex manifolds and foliations induce associated dg manifolds.

In this dissertation, we study dg manifold in two different directions. For one direction, we investigate the exponential maps on dg manifolds and obtain higher structures, called Kapranov $L_\infty[1]$ algebras [49]. Our approach to this problem is to use formal exponential maps on graded manifolds, the analogue of the infinite jet of classical exponential maps, introduced in Liao–Stiénon [34]. It turns out that a dg manifold $\mathcal{M}$ admits a formal exponential map compatible with its homological vector field Q if and only if the Atiyah class of the dg manifold $(\mathcal{M}, Q)$ vanishes. Moreover, we find that any formal exponential map on a dg manifold $(\mathcal{M}, Q)$ give rise to an $L_\infty[1]$ algebra structure on the space of vector fields $\mathfrak{X}(\mathcal{M})$. In particular, when the dg manifold arises from a Kähler manifold, we show that our $L_\infty[1]$ algebras coincide with the $L_\infty[1]$ algebras constructed by Kapranov [23].

The other direction is the study of Hochschild cohomology of dg algebras [33]. Since dg algebras are "internally graded," the usual construction of Hochschild cochain complex becomes a double complex. There are two different types of

Hochschild cohomologies for dg algebras corresponding to the two possible choices of totalization of the double complex: one is by taking direct products and the other is by taking direct sums. The direct product Hochschild cohomology is well-studied and natural as it can be described in terms of derived categories and derived functors. However, the Kontsevich formality theorem for dg manifolds [35] (also in [7]) pertains to the direct sum Hochschild cohomology. Following the work of Keller [24] for direct product Hochschild cohomology, we defined a notion of Keller admissible triple which could be understood as a 'Morita equivalence' of dg algebras in terms of direct sum Hochschild cohomologies . Furthermore, combining this notion with the formality theorem for the dg manifold arising from a Lie algebra $\mathfrak{g}$, we give an alternative proof of the Kontsevich–Duflo theorem [16] [28].

In Chapter 2, we recall some basic definitions on dg manifolds and dg coalgebras, and some necessary theorems regarding them. In Chapter 3, we investigate the relation between the Atiyah class and the formal exponential map on dg manifolds. Moreover, we introduce and investigate Kapranov $L_\infty[1]$ algebras on dg manifolds followed by examples when the dg manifolds arise from L_∞ algebras, foliations, and complex manifolds. In Chapter 4, we introduce Keller admissible triples for dg algebras and prove that they induce an isomorphism of Gerstenhaber algebras on direct sum Hochschild cohomologies. As an application, we use this isomorphism to give a new proof of the Kontsevich–Duflo theorem. Note that, except for minor changes, Chapter 2, 3 and 4 are taken from [49] and [33], verbatim.

1.1 Formal exponential map on dg manifolds

The exponential map appears in linearization problems in classical Lie theory and differential geometry.

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. The classical Lie theoretic exponential map is $\exp : \mathfrak{g} \to G$ which is a local diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of $1 \in G$. The local diffeomorphism, in turn, induces an isomorphism on their differential operators evaluated at the points $0 \in \mathfrak{g}$ and $1 \in G$. Recall that the space $D'_0(\mathfrak{g})$ of differential operators on $\mathfrak{g}$ evaluated at $\{0\}$ can be identified with the symmetric tensor algebra $S\mathfrak{g}$, while the space $D'_1(G)$ of differential operators on G evaluated at $\{1\}$ can be identified with the universal enveloping algebra $\mathcal{U}\mathfrak{g}$. Thus, the exponential map $\exp : \mathfrak{g} \to G$ induces an isomorphism of vector spaces $\exp_* : S\mathfrak{g} \to \mathcal{U}\mathfrak{g}$. The induced map $\mathrm{pbw} := \exp_*$ is the well-known symmetrization map and realizes the *Poincaré–Birkhoff–Witt isomorphism* (PBW in short). In fact, both $S\mathfrak{g}$ and $\mathcal{U}\mathfrak{g}$ carry canonical coalgebra structures which pbw preserves. Hence pbw is an isomorphism of coalgebras.

Analogously, upon the choice of an affine connection ∇ on a smooth manifold M, the classical differential geometric exponential map is $\exp^\nabla : T_M \to M \times M$ which is a local diffeomorphism of fiber bundles

$$\begin{tikzcd} T_M \arrow[r, "\exp^\nabla"] \arrow[d, "\pi"'] & M \times M \arrow[d, "\mathrm{pr}_1"] \\ M \arrow[r, "\mathrm{id}"'] & M \end{tikzcd} \tag{1.1}$$

from a neighborhood of the zero section of T_M to a neighborhood of the diagonal Δ in $M \times M$. Similarly to the Lie algebra case, the space of fiberwise differential operators on the vector bundle $\pi : T_M \to M$ evaluated at the zero section can be identified, as a $C^\infty(M)$-coalgebra, to $\Gamma\big(S(T_M)\big)$, while the space of fiberwise differential operators on the fiber bundle $\mathrm{pr}_1 : M \times M \to M$ evaluated at the diagonal Δ can be identified, as a $C^\infty(M)$-coalgebra, to the space $\mathcal{D}(M)$ of differential operators on M. Thus, one gets an isomorphism of $C^\infty(M)$-coalgebras $\mathrm{pbw}^\nabla := \exp^\nabla_* : \Gamma\big(S(T_M)\big) \to \mathcal{D}(M)$, also known as the PBW isomorphism [31]. The explicit formula of pbw^∇ is:

$$\mathrm{pbw}^\nabla(X_0 \odot \cdots \odot X_k)(f) = \left.\frac{d}{dt_0}\right|_0 \left.\frac{d}{dt_1}\right|_0 \cdots \left.\frac{d}{dt_k}\right|_0 f\Big(\exp^\nabla(t_0X_0 + t_1X_1 + \cdots + t_kX_k)\Big), \tag{1.2}$$

for all $X_0, X_1, \cdots, X_k \in \Gamma(T_M)$ and $f \in C^\infty(M)$. Observe that pbw^∇ is the fiberwise ∞-order jet of the exponential map $\exp^\nabla : T_M \to M \times M$ arising from the connection ∇. Thus, pbw^∇ can be understood, in a sense, as a 'formal exponential map' associated with the affine connection ∇.

Recall that a $\mathbb{Z}$-graded manifold $\mathcal{M}$ consists of a smooth manifold M (called the base manifold or body) equipped with its structure sheaf $\mathcal{O}_M$, and a sheaf $\mathcal{A}$ of $\mathbb{Z}$-graded commutative $\mathcal{O}_M$-algebras over M such that there exists a $\mathbb{Z}$-graded vector space V over $\mathbb{K}$ and there exist algebra isomorphisms $\mathcal{A}(U) \cong C^\infty(U;\mathbb{K})\hat{\otimes}_{\mathbb{K}}\widehat{S}(V^\vee)$ for all sufficiently small open sets $U \subset M$. Here, $\mathbb{K}$ denotes the underlying field, either $\mathbb{R}$ or $\mathbb{C}$, and $\widehat{S}(V^\vee) \cong \mathrm{Hom}^\bullet(S(V), \mathbb{K})$ denotes the graded $\mathbb{K}$-algebra of formal power series on V.

It turns out that, despite its geometric origin, the PBW map $\mathrm{pbw}^\nabla : \Gamma\big(S(T_M)\big) \to \mathcal{D}(M)$ can be characterized completely by algebraic recursive formulas involving the connection ∇, without recourse to the associated exponential map [31]. In fact, the algebraic characterization of pbw^∇ is the key to extending it to $\mathbb{Z}$-graded manifolds. Using this recursive algebraic characterization, Liao–Stiénon [34] showed that, for a $\mathbb{Z}$-graded manifold $\mathcal{M}$ and an affine connection ∇ on $\mathcal{M}$, the PBW map $\mathrm{pbw}^\nabla : \Gamma\big(S(T_{\mathcal{M}})\big) \to \mathcal{D}(\mathcal{M})$ is an isomorphism of graded $C^\infty(\mathcal{M})$-coalgebras.

A dg manifold $(\mathcal{M}, Q)$ is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ endowed with a homological vector field, i.e. a degree $+1$ derivation Q of $C^\infty(\mathcal{M})$ satisfying $Q^2 = 0$. The homological vector field Q induces a coderivation L_Q on $\Gamma\big(S(T_\mathcal{M})\big)$ and a coderivation $\mathcal{L}_Q$ on $\mathcal{D}(\mathcal{M})$, which are both Lie derivatives. In other words, there are two dg coalgebras $(\Gamma\big(S(T_\mathcal{M})\big), L_Q)$ and $(\mathcal{D}(\mathcal{M}), \mathcal{L}_Q)$ canonically arising from a dg manifold $(\mathcal{M}, Q)$. It is natural to ask whether $\mathrm{pbw}^\nabla : \big(\Gamma\big(S(T_\mathcal{M})\big), L_Q\big) \to \big(\mathcal{D}(\mathcal{M}), \mathcal{L}_Q\big)$ is an isomorphism of *differential* graded $C^\infty(\mathcal{M})$-coalgebras. In other words, the question is about the existence of 'formal exponential map of dg manifolds'. The answer to this question is captured by the Atiyah class of the dg manifold. Below, we recall the definition of the Atiyah class in terms of affine connections [42].

Let $(\mathcal{M}, Q)$ be a dg manifold. Consider the induced cochain complex $\Gamma\big(\mathcal{M}; T^\vee_\mathcal{M} \otimes \mathrm{End}(T_\mathcal{M})\big)$ with the Lie derivative L_Q in the direction of the homological vector field as its coboundary operator. Given an affine connection ∇ on $\mathcal{M}$, consider the $(1,2)$-tensor $\mathrm{At}^\nabla_{(\mathcal{M},Q)} \in \Gamma\big(\mathcal{M}; T^\vee_\mathcal{M} \otimes \mathrm{End}(T_\mathcal{M})\big)$ of degree $+1$ defined by the relation

$$\mathrm{At}^\nabla_{(\mathcal{M},Q)}(X, Y) = [Q, \nabla_X Y] - \nabla_{[Q,X]} Y - (-1)^{|X|} \nabla_X [Q, Y],$$

for all homogeneous vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$. Since $L_Q(\mathrm{At}^\nabla_{(\mathcal{M},Q)}) = 0$, the element $\mathrm{At}^\nabla_{(\mathcal{M},Q)}$ is a 1-cocycle called the **Atiyah cocycle** associated with the affine connection ∇. The cohomology class

$$\alpha_{(\mathcal{M},Q)} := \big[\mathrm{At}^\nabla_{(\mathcal{M},Q)}\big] \in H^1\big(\Gamma\big(\mathcal{M}; T^\vee_\mathcal{M} \otimes \mathrm{End}(T_\mathcal{M})\big)^\bullet, L_Q\big)$$

does not depend on the choice of connection ∇, and therefore is an intrinsic characteristic class called the **Atiyah class** of the dg manifold $(\mathcal{M}, Q)$ [42].

We prove that the Atiyah class captures the obstruction to the existence of formal exponential map compatible with the homological vector field.

Theorem 1.1.1. *Let $(\mathcal{M}, Q)$ be a dg manifold. The Atiyah class $\alpha_{(\mathcal{M},Q)}$ vanishes if and only if there exists an affine connection ∇ on $\mathcal{M}$ such that*

$$\mathrm{pbw}^\nabla : \big(\Gamma\big(S(T_\mathcal{M})\big), L_Q\big) \to \big(\mathcal{D}(\mathcal{M}), \mathcal{L}_Q\big)$$

is an isomorphism of dg coalgebras over the dg ring $(C^\infty(\mathcal{M}), Q)$.

Following the pioneering work of Kapranov [23], it is known that the Atiyah class of a holomorphic vector bundle gives rise to an $L_\infty[1]$ algebra, which plays an important role in derived geometry and the construction of Rozansky–Witten invariants. In a similar fashion, the Atiyah class of a dg manifold induces an $L_\infty[1]$ algebra. Our approach relies on the map pbw^∇.

We give a complete proof of the following theorem, which was announced in [42] without a proof.

Theorem 1.1.2. *Let $(\mathcal{M}, Q)$ be a dg manifold. Each choice of a torsion-free affine connection ∇ on $\mathcal{M}$ determines an $L_\infty[1]$ algebra structure on the space of vector fields $\mathfrak{X}(\mathcal{M})$. While the unary bracket $\lambda_1 : S^1\big(\mathfrak{X}(\mathcal{M})\big) \to \mathfrak{X}(\mathcal{M})$ is the Lie derivative L_Q along the homological vector field, the higher multibrackets $\lambda_k : S^k\big(\mathfrak{X}(\mathcal{M})\big) \to \mathfrak{X}(\mathcal{M})$, with $k \geq 2$, arise as the composition*

$$\lambda_k : S^k\big(\mathfrak{X}(\mathcal{M})\big) \to \Gamma\big(S^k(T_\mathcal{M})\big) \xrightarrow{R_k} \mathfrak{X}(\mathcal{M})$$

induced by a family of sections $\{R_k\}_{k\geq 2}$ of the vector bundles $S^k(T^\vee_\mathcal{M}) \otimes T_\mathcal{M}$ starting with $R_2 = -\operatorname{At}^\nabla_{(\mathcal{M},Q)}$, the Atiyah cocycle corresponding to ∇.

Furthermore, the $L_\infty[1]$ algebra structures on $\mathfrak{X}(\mathcal{M})$ arising from different choices of connections are all isomorphic.

The $L_\infty[1]$ algebras obtained in this way are called the **Kapranov** $L_\infty[1]$ **algebras** of the dg manifold $(\mathcal{M}, Q)$. By construction, each Kapranov $L_\infty[1]$ algebra is completely determined by the Atiyah 1-cocycle and the sections

$$R_k \in \Gamma\big(\mathcal{M}; S^k(T^\vee_\mathcal{M}) \otimes T_\mathcal{M}\big) \cong \Gamma\big(\mathcal{M}; \operatorname{Hom}(S^k(T_\mathcal{M}), T_\mathcal{M})\big)$$

for $k \geq 3$. Thus, it is natural to ask if the R_k's can be described explicitly. The following theorem gives a characterization of the higher multibrackets for the Kapranov $L_\infty[1]$ algebras of a dg manifold. It is worth noting that it is similar to the characterization of higher multibrackets of $L_\infty[1]$ algebra on the Dolbeault complex $(\Omega^{0,\bullet}(T_X^{1,0}), \bar{\partial})$ of a Kähler manifold X described in the original work of Kapranov [23].

Theorem 1.1.3.

1. *The sections $R_n \in \Gamma\big(S^n(T^\vee_\mathcal{M}) \otimes T_\mathcal{M}\big)$, with $n \geq 3$, are completely determined, by way of a recursive formula, by the Atiyah cocycle $\operatorname{At}^\nabla_{(\mathcal{M},Q)}$, the curvature R^∇, and their higher covariant derivatives.*

2. *In particular, if $R^\nabla = 0$, then $R_2 = -\operatorname{At}^\nabla_{(\mathcal{M},Q)}$ and $R_n = \frac{1}{n}\widetilde{d^\nabla} R_{n-1}$, for all $n \geq 3$.*

As examples, the Kapranov $L_\infty[1]$ algebras of two classes of dg manifolds are considered: 1) the dg manifolds associated with finite dimensional $L_\infty[1]$ algebras, and 2) the dg manifolds associated with regular integrable distributions, including the case of the anti-holomorphic tangent bundle in the complexified tangent bundle of a complex manifold. For the dg manifold $(\mathfrak{g}[1], d_{\mathrm{CE}})$ associated with a finite dimensional $L_\infty[1]$ algebra $\mathfrak{g}[1]$, we compute the Atiyah class and prove that the Kapranov $L_\infty[1]$ algebra structure on $\mathfrak{X}(\mathfrak{g}[1])$ can be expressed in terms of the multibrackets of the $L_\infty[1]$ algebra $\mathfrak{g}[1]$. For the dg manifold $(F[1], d_F)$ associated with an integrable distribution $F \subset T_{\mathbb{K}}M$ on a smooth manifold M, we prove that the Kapranov $L_\infty[1]$ algebra structure on $\mathfrak{X}(F[1])$ is quasi-isomorphic to the $L_\infty[1]$ algebra on $\Omega^\bullet_F(T_{\mathbb{K}}M/F)$ arising from the Lie pair $(T_{\mathbb{K}}M, F)$, studied in [31]. In particular, we prove that, for the dg manifold $(T^{0,1}_X[1], \bar\partial)$ associated with a Kähler manifold X, the Kapranov $L_\infty[1]$ algebra structure on $\mathfrak{X}(T^{0,1}_X[1])$ is quasi-isomorphic to the $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega^{0,\bullet}(T^{1,0}_X)$, studied in the original work of Kapranov [23].

1.2 Keller admissible triples and Duflo theorem

The Hochschild cochain complex of an associative algebra A is

$$0 \to A \xrightarrow{d_{\mathcal{H}}} \mathrm{Hom}(A, A) \xrightarrow{d_{\mathcal{H}}} \cdots \xrightarrow{d_{\mathcal{H}}} \mathrm{Hom}(A^{\otimes n}, A) \xrightarrow{d_{\mathcal{H}}} \cdots$$

where $d_{\mathcal{H}} : \mathrm{Hom}(A^{\otimes n}, A) \to \mathrm{Hom}(A^{\otimes n+1}, A)$ is the Hochschild differential induced by the multiplication of A. Alternatively, in terms of derived categories, the Hochschild cochain complex of A is the derived hom $\mathrm{RHom}_{A-A}(A, A)$ in the category of A-A-bimodules.

A similar construction is available for a dg algebra $A = (A, d_A)$. Observe that the differential d_A on A induces an internal differential ∂ on $\mathrm{Hom}(A^{\otimes n}, A)$, for each non-negative integer n, defined by

$$\partial f(a_1 \otimes \cdots \otimes a_n) = d_A \circ f(a_1 \otimes \cdots \otimes a_n) - (-1)^{|f|} \sum_{i=1}^{n} (-1)^{|a_1| + \cdots + |a_{i-1}|} f(\cdots \otimes d_A a_i \otimes \cdots)$$

for $f \in \mathrm{Hom}(A^{\otimes n}, A)$ and $a_i \in A$. Together with the internal differential ∂, one obtains a double complex

$$\begin{tikzcd}
 & \vdots & \vdots & & \vdots & \\
0 \arrow[r] & A^{k+1} \arrow[u, "\partial"] \arrow[r, "d_{\mathcal{H}}"] & \mathrm{Hom}^{k+1}(A, A) \arrow[u, "\partial"] \arrow[r, "d_{\mathcal{H}}"] & \cdots \arrow[r, "d_{\mathcal{H}}"] & \mathrm{Hom}^{k+1}(A^{\otimes n}, A) \arrow[u, "\partial"] \arrow[r, "d_{\mathcal{H}}"] & \cdots \\
0 \arrow[r] & A^{k} \arrow[u, "\partial"] \arrow[r, "d_{\mathcal{H}}"] & \mathrm{Hom}^{k}(A, A) \arrow[u, "\partial"] \arrow[r, "d_{\mathcal{H}}"] & \cdots \arrow[r, "d_{\mathcal{H}}"] & \mathrm{Hom}^{k}(A^{\otimes n}, A) \arrow[u, "\partial"] \arrow[r, "d_{\mathcal{H}}"] & \cdots \\
 & \vdots \arrow[u, "\partial"] & \vdots \arrow[u, "\partial"] & & \vdots \arrow[u, "\partial"] &
\end{tikzcd}$$

where $\mathrm{Hom}^k(A^{\otimes n}, A)$ consists of homogeneous elements of degree k in $\mathrm{Hom}(A^{\otimes n}, A)$ and we write $A^k = \mathrm{Hom}^k(\mathbb{K}, A)$.

Recall that given a double complex, there are two different total complexes: one by taking direct product and the other one by taking direct sum. It turns out that the Hochschild cohomology $\mathrm{HH}_{\prod}(A)$ of a dg algebra A arising from direct product (i.e. direct product Hochschild cohomology) is canonically described in terms of derived category of dg algebras and dg modules. However, the Hochschild cohomology $\mathrm{HH}_{\oplus}(A)$ of a dg algebra A arising from *direct sum* (i.e. direct sum Hochschild cohomology) does not behave well in terms of derived category of dg algebras and dg modules. In particular, quasi-isomorphisms are not respected in direct sum Hochschild cohomologies; it needs an additional condition such as what we call the point-wise nilpotency condition. Despite the defect, direct sum Hochschild cohomology serves an important role in the Kontsevich formality theorem and the Kontsevich–Duflo-type theorem for dg manifolds [35] — see also [7] for comparison.

Acknowledging the importance of direct sum Hochschild cohomology, and inspired by a similar theorem for direct product Hochschild cohomology introduced by Keller [24], we establish an isomorphism of Gerstenhaber algebras between direct sum Hochschild cohomologies of dg algebras via Morita equivalence.

Theorem 1.2.1. *Let A and B be dg algebras and let X be a dg A-B-bimodule. We say that (A, X, B) is a Keller admissible triple if 'partial' Hochschild complexes*

$$0 \to \mathrm{Hom}_A(X, X) \hookrightarrow \mathrm{Hom}(X, X) \xrightarrow{d^A_{\mathcal{H}}} \cdots \xrightarrow{d^A_{\mathcal{H}}} \mathrm{Hom}(A^{\otimes n} \otimes X, X) \xrightarrow{d^A_{\mathcal{H}}} \cdots$$
$$0 \to \mathrm{Hom}_{B^{\mathrm{op}}}(X, X) \hookrightarrow \mathrm{Hom}(X, X) \xrightarrow{d^B_{\mathcal{H}}} \cdots \xrightarrow{d^B_{\mathcal{H}}} \mathrm{Hom}(X \otimes B^{\otimes n}, X) \xrightarrow{d^B_{\mathcal{H}}} \cdots$$

are exact and the action maps

$$\rho_A : (A, d_A) \to \big(\mathrm{Hom}_{B^{\mathrm{op}}}(X, X), \partial_X\big),$$
$$\rho_B : (B^{\mathrm{op}}, d_B) \to \big(\mathrm{Hom}_A(X, X), \partial_X\big)$$

are quasi-isomorphisms satisfying the weak cone-nilpotency condition defined in Section 4.3.2, then there is an isomorphism of Gerstenhaber algebras

$$\mathrm{HH}_{\oplus}(A) \cong \mathrm{HH}_{\oplus}(B)$$

between the direct sum *Hochschild cohomology of the dg algebras A and B.*

An example of Keller admissible triple is $(A, d_A) = (\mathcal{U}\mathfrak{g}, 0)$, $(X, d_X) = (\mathcal{U}\mathfrak{g} \otimes \Lambda^\bullet \mathfrak{g}, d_X)$, $(B, d_B) = (\Lambda^\bullet \mathfrak{g}^\vee, d_{\mathrm{CE}})$, where both d_X and d_{CE} are referred as Chevalley-Eilenberg differentials on each graded vector space.

The above theorem provides a way to construct an explicit isomorphism between the Hochschild cohomologies of, in a sense, "Morita equivalent" dg algebras. As an application, we obtain a new concrete proof of the Kontsevich–Duflo theorem for Lie algebras.

Given a finite dimensional Lie algebra $\mathfrak{g}$, the PBW isomorphism $\mathrm{pbw} : S\mathfrak{g} \to \mathcal{U}\mathfrak{g}$ induces an isomorphism $\mathrm{pbw} : (S\mathfrak{g})^{\mathfrak{g}} \to Z(\mathcal{U}\mathfrak{g})$ from the $\mathfrak{g}$-invariant subspace of $S\mathfrak{g}$ to the center of $\mathcal{U}\mathfrak{g}$. This isomorphism does not preserve the natural associative algebra structures carried by $(S\mathfrak{g})^{\mathfrak{g}}$ and $Z(\mathcal{U}\mathfrak{g})$ but it can be modified by the square root of the Duflo element so as to become an isomorphism of associative algebras. The Duflo element $J \in \widehat{S}(\mathfrak{g}^\vee)$ is the formal power series on $\mathfrak{g}$, defined by $J(x) = \det\left(\frac{1-e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}\right)$ for $x \in \mathfrak{g}$. The Duflo element J acts on $(S\mathfrak{g})^{\mathfrak{g}}$ by a formal differential operator and so does its square root $J^{1/2}$. A remarkable theorem due to Duflo [15] asserts that the composition $\mathrm{pbw} \circ J^{1/2} : (S\mathfrak{g})^{\mathfrak{g}} \to Z(\mathcal{U}\mathfrak{g})$ is an isomorphism of associative algebras. Duflo's theorem is a generalization of the Harish–Chandra theorem, which is about the center of the universal enveloping algebra of a semi-simple Lie algebra, to any finite dimensional Lie algebra.

Kontsevich [28] proposed a new proof of the Duflo theorem as an application of his formality theorem. More precisely, he considered the tangent map of the formality morphism at the Lie–Poisson structure on $\mathfrak{g}^\vee$ which is regarded as a Maurer–Cartan element in the differential graded Lie algebra of polyvector fields $\mathcal{T}^\bullet_{\mathrm{poly}}(\mathfrak{g}^\vee)$ on $\mathfrak{g}^\vee$. This approach led to the Kontsevich–Duflo theorem: the map

$$\mathrm{pbw} \circ J^{1/2} : H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S\mathfrak{g}) \to H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}\mathfrak{g}) \tag{1.3}$$

is an isomorphism of graded algebras. The classical Duflo theorem is an isomorphism of these cohomologies in degree 0.

Following a conjecture formulated by Shoikhet [50], Liao–Stiénon–Xu [35, Theorem 4.3] proved that a Kontsevich–Duflo-type theorem holds for all finite dimensional dg manifolds $(\mathcal{M}, Q)$: the map

$$\mathrm{hkr} \circ \mathrm{Td}^{1/2}_{(\mathcal{M},Q)} : \mathbb{H}^\bullet\big({}_{\oplus}\mathcal{T}_{\mathrm{poly}}(\mathcal{M}), [Q, -]\big) \to \mathbb{H}^\bullet\big({}_{\oplus}\mathcal{D}_{\mathrm{poly}}(\mathcal{M}), d_{\mathcal{H}} + [\![Q, -]\!]\big) \tag{1.4}$$

is an isomorphism of graded algebras. Here hkr is the Hochschild–Kostant–Rosenberg map, and $\mathrm{Td}^{1/2}_{(\mathcal{M},Q)}$ is the square root of the Todd class acting on ${}_{\oplus}\mathcal{T}_{\mathrm{poly}}(\mathcal{M})$ by contraction.

Shoikhet also suggested that the Kontsevich–Duflo theorem (1.3) could be recovered by applying (1.4) to the dg manifold $(\mathfrak{g}[1], d_{\mathfrak{g}})$. Indeed, in doing so, we obtain the isomorphism of graded algebras

$$\mathrm{hkr} \circ \mathrm{Td}^{1/2}_{\mathfrak{g}[1]} : \mathbb{H}^\bullet\big({}_{\oplus}\mathcal{T}_{\mathrm{poly}}(\mathfrak{g}[1]), [d_{\mathfrak{g}}, -]\big) \to \mathbb{H}^\bullet\big({}_{\oplus}\mathcal{D}_{\mathrm{poly}}(\mathfrak{g}[1]), d_{\mathcal{H}} + [\![d_{\mathfrak{g}}, -]\!]\big)$$

from the cohomology of polyvector fields to the cohomology of polydifferential operators on $\mathfrak{g}[1]$. The cohomology $\mathbb{H}^\bullet\big({}_\oplus\mathcal{T}_{\mathrm{poly}}(\mathfrak{g}[1]), [d_\mathfrak{g}, -]\big)$ is naturally identified with the Chevalley–Eilenberg cohomology $H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S\mathfrak{g})$, while the cohomology $\mathbb{H}^\bullet\big({}_\oplus\mathcal{D}_{\mathrm{poly}}(\mathfrak{g}[1]), d_\mathcal{H}+[\![d_\mathfrak{g}, -]\!]\big)$ coincides with the direct sum Hochschild cohomology $\mathrm{HH}^\bullet_\oplus(\Lambda\mathfrak{g}^\vee, d_{\mathrm{CE}})$.

By applying Theorem 1.2.1 to the Keller admissible triple $(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g}\otimes\Lambda^\bullet\mathfrak{g}, \Lambda^\bullet\mathfrak{g}^\vee)$, one gets the isomorphism of Gerstenhaber algebras $\mathrm{HH}^\bullet_\oplus(\Lambda\mathfrak{g}^\vee, d_{\mathrm{CE}}) \cong \mathrm{HH}^\bullet_\oplus(\mathcal{U}\mathfrak{g})$. Composing it with the standard Cartan–Eilenberg isomorphism [5] $\mathrm{HH}_\oplus(\mathcal{U}\mathfrak{g}) \xrightarrow{\cong} H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}\mathfrak{g})$ yields the desired isomorphism of graded algebras

$$\Phi_D : \mathbb{H}^\bullet\big({}_\oplus\mathcal{D}_{\mathrm{poly}}(\mathfrak{g}[1]), d_\mathcal{H} + [\![d_\mathfrak{g}, -]\!]\big) \cong \mathrm{HH}^\bullet_\oplus(\Lambda\mathfrak{g}^\vee, d_{\mathrm{CE}}) \xrightarrow{\cong} H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}\mathfrak{g}).$$

Also, one can easily see that the isomorphism $\prod_k \Gamma\big(\Lambda^k T^\vee(\mathfrak{g}[1])\big) \xrightarrow{\cong} \widehat{S}(\mathfrak{g}^\vee)$ identifies the Todd class $\mathrm{Td}_{\mathfrak{g}[1]} \in \prod_k \Gamma\big(\Lambda^k T^\vee \mathfrak{g}[1]\big)$ with the Duflo element $J \in \widehat{S}(\mathfrak{g}^\vee)$. Hence, we obtain the following theorem:

Theorem 1.2.2. *Given a finite-dimensional Lie algebra $\mathfrak{g}$, the diagram*

$$\begin{tikzcd}
\mathbb{H}^\bullet\big({}_\oplus\mathcal{T}_{\mathrm{poly}}(\mathfrak{g}[1]), [d_\mathfrak{g}, -]\big) \arrow[r, "\mathrm{hkr}\circ\mathrm{Td}^{1/2}_{\mathfrak{g}[1]}"] \arrow[d, "\Phi_T"', "\cong"] & \mathbb{H}^\bullet\big({}_\oplus\mathcal{D}_{\mathrm{poly}}(\mathfrak{g}[1]), d_\mathcal{H} + [\![d_\mathfrak{g}, -]\!]\big) \arrow[d, "\cong"', "\Phi_D"] \\
H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S\mathfrak{g}) \arrow[r, "\mathrm{pbw}\circ J^{1/2}"] & H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}\mathfrak{g})
\end{tikzcd}$$

commutes.

As a consequence, we obtain an alternative proof of the Kontsevich–Duflo theorem.

Notations and conventions. The symbol $\mathbb{K}$ denotes either of the fields $\mathbb{R}$ or $\mathbb{C}$. Unless otherwise stated, all grading in this paper are $\mathbb{Z}$-gradings.

For a smooth manifolds M over $\mathbb{K}$, the sheaf of germs of smooth $\mathbb{K}$-valued functions on M is denoted $\mathcal{O}_M = \mathcal{O}_M^\mathbb{K}$. The algebra of globally defined smooth functions on M is $C^\infty(M) = \mathcal{O}_M(M)$.

We reserve the symbol $\mathcal{M}$ to denote dg manifold and 'dg' means 'differential graded.' All dg manifolds in this dissertation will be finite dimensional.

A (p, q)-shuffle is a permutation σ of the set $\{1, 2, \cdots, p+q\}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. The set of (p, q)-shuffles will be denoted by $\mathfrak{S}_p^q$.

We use Sweedler's (sumless) notation for the comultiplication Δ in any coalgebra C:

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)}, \quad \forall c \in C.$$

Let A be a graded ring, and let V, W be two left (respectively, right) graded A-modules. The space $\mathrm{Hom}_A(V,W) = \mathrm{Hom}^\bullet_A(V,W)$ (respectively, $\mathrm{Hom}_{A^{\mathrm{op}}}(V,W) = \mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(V,W)$) of morphisms of left (respectively, right) A-modules from V to W is naturally $\mathbb{Z}$-graded: the symbol $\mathrm{Hom}^r_A(V,W)$ (respectively, $\mathrm{Hom}^r_{A^{\mathrm{op}}}(V,W)$) denotes the space of morphisms of left (respectively, right) A-modules of degree r from V to W.

Given a graded vector space V, the suspension of V is denoted by $V[1]$. Hence, we have $V[1]^n = V^{n+1}$. We denote by $\mathrm{s} : V[1] \to V$ the degree-shifting map of degree $+1$ and by $\mathfrak{s} : V \to V[1]$ the degree-shifting map of degree -1.

Given a homogeneous element x in a graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V^k$, we write $|x|$ to denote the degree of x. Thus $|x| = d$ means that $x \in V^d$.

For $x \in V$, $\xi \in V^\vee$, we denote $\langle\, \xi \mid x \,\rangle := \xi(x)$ and $\langle\, x \mid \xi \,\rangle := (-1)^{|x||\xi|}\langle\, \xi \mid x \,\rangle$. The pairing is extended to a pairing of tensor algebras $\langle - \mid - \rangle : TV \times TV^\vee \to \mathbb{K}$ by

$$\langle\, x_1\otimes\cdots\otimes x_p \mid \xi_1\otimes\cdots\otimes\xi_q \,\rangle := \begin{cases} (-1)^{\sum\limits_{i=1}^{p}\sum\limits_{j=i+1}^{p}|\xi_i||x_j|} \langle\, x_1 \mid \xi_1 \,\rangle\cdots\langle\, x_p \mid \xi_p \,\rangle, & \text{if } p = q,\\ 0, & \text{if } p \neq q; \end{cases}$$

and to a pairing of symmetric algebras $\langle - \mid - \rangle : SV \times SV^\vee \to \mathbb{K}$ by

$$\langle\, x_1 \odot \cdots \odot x_p \mid \xi_1 \odot \cdots \odot \xi_q \,\rangle := \sum_{\sigma\in S_q} \varepsilon \cdot \langle\, x_1 \otimes \cdots \otimes x_p \mid \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(q)} \,\rangle,$$

where $\varepsilon = \pm 1$ is the number such that $\xi_1 \odot \cdots \odot \xi_q = \varepsilon \cdot \xi_{\sigma(1)} \odot \cdots \odot \xi_{\sigma(q)}$ in $SV^\vee$. Similarly, we also have the pairings $\langle - \mid - \rangle : TV^\vee \times TV \to \mathbb{K}$ and $\langle - \mid - \rangle : SV^\vee \times SV \to \mathbb{K}$.

Many equations throughout the paper have the following general shape:

$$A(X_1, X_2, \ldots, X_n) = (-1)^{\sum_{(i,j)\in\mathscr{K}} |X_{\sigma(i)}||X_{\sigma(j)}|} B(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}), \quad (1.5)$$

where $X_1, X_2, \ldots, X_n$ is a finite collection of $\mathbb{Z}$-graded objects; σ is a permutation of the set of indices $\{1, 2, \ldots, n\}$; $\mathscr{K}$ is the set of couples (i,j) of elements of $\{1, 2 \ldots, n\}$ such that $i < j$ and $\sigma(i) > \sigma(j)$; and A and B are n-ary operations on the $\mathbb{Z}$-graded objects $X_1, X_2, \ldots, X_n$ whose output is an object of degree $|X_1| + |X_2| + \cdots + |X_n|$. The factor $(-1)^{\sum_{(i,j)\in\mathscr{K}} |X_{\sigma(i)}||X_{\sigma(j)}|}$ appearing in the right hand side of (1.5) is called the *Koszul sign* of the permutation σ of the graded objects $X_1, X_2, \ldots, X_n$. It will customarily be abbreviated as ε since its actual value — either $+1$ or -1 — can be recovered from a careful inspection of both sides of the equation. We will also use the more explicit abbreviation $\varepsilon(X_1, X_2, \cdots, X_n)$ if the collection of $\mathbb{Z}$-graded objects being permuted is not immediately clear. As explained by Boardman in [3], this sign is mostly inconsequential and it is not necessary to devote much attention or thought to it. In fact, the right hand side of (1.5) can be a sum of several terms so it would be more correct to say that the general shape of the equations is

$$A(X_1, X_2, \ldots, X_n) = \sum_k (-1)^{\sum_{(i,j)\in\mathscr{K}_k} |X_{\sigma_k(i)}||X_{\sigma_k(j)}|} B_k(X_{\sigma_k(1)}, X_{\sigma_k(2)}, \ldots, X_{\sigma_k(n)}).$$

Chapter 2

Preliminaries

This chapter is to provide basic definitions and facts related to differential graded manifolds and differential graded coalgebras.

2.1 Graded manifolds

Let M be a smooth manifold over $\mathbb{K}$, and $\mathcal{O}_M$ be the sheaf of $\mathbb{K}$-valued smooth functions over M. A **graded manifold** $\mathcal{M}$ with support M consists of a sheaf $\mathcal{A}$ of graded commutative $\mathcal{O}_M$-algebra on M such that there is a $\mathbb{Z}$-graded vector space V satisfying

$$\mathcal{A}(U) \cong \mathcal{O}_M(U) \hat{\otimes}_{\mathbb{K}} \widehat{S}(V^\vee) = \prod_{k=0}^{\infty} \mathcal{O}_M(U) \otimes S^k(V^\vee)$$

for sufficiently small open set $U \subset M$. Here, $\mathcal{O}_M(U) \hat{\otimes}_{\mathbb{K}} \widehat{S}(V^\vee)$ means the graded algebra of formal power series on V with coefficients in $\mathcal{O}_M(U)$. The global section of the sheaf $\mathcal{A}$ will be denoted by $C^\infty(\mathcal{M}) = \mathcal{A}(M)$. We say a graded manifold $\mathcal{M}$ is finite dimensional if $\dim M < \infty$ and $\dim V < \infty$. By default, graded manifold $\mathcal{M}$ will always be finite dimensional.

Remark 2.1.1. Strictly speaking, the sheaf $\mathcal{A}$ is *not* a sheaf of graded algebra. The algebra of formal power series $\widehat{S}(V^\vee)$ over a graded vector space V is decomposed as $\widehat{S}(V^\vee) = \prod_{n \in \mathbb{Z}} \left(\widehat{S}(V^\vee)\right)^n$ where $\left(\widehat{S}(V^\vee)\right)^n$ consists of homogeneous formal power series of degree n. That is, $\widehat{S}(V^\vee)$ is *not* a graded algebra but a projective limit of graded algebras. However, we abuse the terminology "graded algebra" and use it for $\widehat{S}(V^\vee)$ and consequently for the sheaf $\mathcal{A}$.

Remark 2.1.2. In the literature, the definition of graded manifolds varies. One variation is that the sheaf of functions $\mathcal{A}$ over a graded manifold is defined by

$\mathcal{A}(U) \cong \mathcal{O}_M(U) \otimes S(V^\vee)$ for sufficiently small open subset $U \subset M$. However, we allow for formal power series rather than polynomials. Another variation is that the graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V^n$ in $\mathcal{A}(U) \cong \mathcal{O}_M(U)\hat{\otimes}\widehat{S}(V^\vee)$ is assumed to satisfy $V^0 = \{0\}$. By doing so, all coordinate functions in degree 0 are smooth variables. However, we allow that coordinate functions in degree 0 can be formal variables.

By $\mathcal{I}_\mathcal{A}$, we denote the sheaf of ideal of $\mathcal{A}$ consisting of functions vanishing at the support M of $\mathcal{M}$. That is, for sufficiently small $U \subset M$,

$$\mathcal{I}_\mathcal{A}(U) \cong \mathcal{O}_M(U)\hat{\otimes}_\mathbb{K}\widehat{S}^{\geq 1}(V^\vee).$$

Given graded manifolds $\mathcal{M} = (M, \mathcal{A})$ and $\mathcal{N} = (N, \mathcal{B})$, a **morphism** $\mathcal{M} \to \mathcal{N}$ of graded manifolds consists of a pair (f, ψ), where $f : M \to N$ is a morphism of smooth manifolds and $\psi : f^*\mathcal{B} \to \mathcal{A}$ is a morphism of sheaves of graded commutative $\mathcal{O}_M$-algebras such that $\psi(f^*\mathcal{I}_\mathcal{B}) \subset \mathcal{I}_\mathcal{A}$. We often use the notation $\phi : \mathcal{M} \to \mathcal{N}$ to denote such a morphism. Then $\psi = \phi^*$. Also, we write $\phi^* : C^\infty(\mathcal{N}) \to C^\infty(\mathcal{M})$ to denote the morphism on global sections. Note that the condition $\psi(f^*\mathcal{I}_\mathcal{B}) \subset \mathcal{I}_\mathcal{A}$ is equivalent to ψ being continuous w.r.t the $\mathcal{I}$-adic topology.

Example 2.1.3. Let $\mathbb{E} = \bigoplus_{i\in\mathbb{Z}} E_i$ be a graded vector bundle over a smooth manifold M. A typical example of a graded manifold is $\mathcal{M} = \mathbb{E}$ where its graded algebra of smooth functions is $\Gamma\big(M; \widehat{S}(\mathbb{E}^\vee)\big)$. Since we are interested in *finite dimensional* graded manifolds, we assume that $E_i = 0$ except for finitely many indices i and each E_i is of finite rank. As a special case, when M is a point a graded vector space V is an example of a graded manifold.

Vector bundles in the category of graded manifolds are called **graded vector bundles**. More precisely, let $\mathcal{E} = (E, \mathcal{B})$ and $\mathcal{M} = (M, \mathcal{A})$ be graded manifolds. A graded vector bundle is a map $\Phi = (\pi, \Psi) : \mathcal{E} \to \mathcal{M}$ such that $\pi : E \to M$ is a smooth vector bundle and $\Psi : \mathcal{A} \to \pi_*\mathcal{B}$ is an inclusion locally characterized by the canonical inclusion

$$\mathcal{O}_M(U)\hat{\otimes}_\mathbb{K}\widehat{S}(V^\vee) \cong \mathcal{A}(U) \xrightarrow{\Psi(U)} \mathcal{B}(\pi^{-1}(U)) \cong \mathcal{O}_E(\pi^{-1}(U))\hat{\otimes}_\mathbb{K}\widehat{S}((V \oplus W)^\vee).$$

Note that $\Phi = (\pi, \Psi) : \mathcal{E} \to \mathcal{M}$ is a morphism of graded manifolds under the identification $\Psi \in \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{A}, \pi_*\mathcal{B}) \cong \mathrm{Hom}_{\mathcal{O}_E}(\pi^*\mathcal{A}, \mathcal{B})$.

Given a graded vector bundle $\Phi : \mathcal{E} \to \mathcal{M}$, a **section** $s : \mathcal{M} \to \mathcal{E}$ of $\mathcal{E}$ over $\mathcal{M}$ is a morphism of graded manifolds such that $\Phi \circ s = \mathrm{id}_\mathcal{M}$. In terms of smooth functions, s induces a morphism of graded algebra $s^* : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{M})$ such that $s^* \circ \Phi^* = \mathrm{id}_{C^\infty(\mathcal{M})}$. We write the $C^\infty(\mathcal{M})$-module of sections of $\mathcal{E}$ over $\mathcal{M}$ by the usual notation $\Gamma(\mathcal{E}) = \Gamma(\mathcal{M}; \mathcal{E})$. It can be checked that a morphism of graded manifolds $\mathcal{E} \to \mathcal{M}$ forms a graded vector bundle over $\mathcal{M} = (M, \mathcal{A})$ is equivalent to saying that the sheaf

$$\Gamma(\mathcal{E}) : U \mapsto \Gamma(\mathcal{M}|_U; \mathcal{E}|_U)$$

of $\mathcal{A}$-modules is locally free.

For a graded manifold $\mathcal{M}$ with support M, its tangent bundle $T_{\mathcal{M}}$ is a graded manifold with support T_M and is a graded vector bundle over $\mathcal{M}$. Its sections are called **vector fields** on $\mathcal{M}$ and the space of vector fields $\Gamma(\mathcal{M}; T_{\mathcal{M}}) = \Gamma(T_{\mathcal{M}})$ can be identified with that of graded derivations of $C^\infty(\mathcal{M})$. We also write $\Gamma(\mathcal{M}; T_{\mathcal{M}}) = \mathfrak{X}(\mathcal{M})$. Observe that $\mathfrak{X}(\mathcal{M})$ admits a Lie algebra structure, whose Lie bracket coincides with the graded commutator

$$[X, Y] = X \circ Y - (-1)^{|X| \cdot |Y|} Y \circ X$$

for homogeneous elements $X, Y \in \mathfrak{X}(\mathcal{M})$ regarded as derivations of $C^\infty(\mathcal{M})$. Indeed $T_{\mathcal{M}}$ is a graded Lie algebroid [41].

Example 2.1.4. As a special case of Example 2.1.3, a $\mathbb{Z}$-graded vector space V is an example of a graded manifold $\mathcal{M} = V$. Let W be a $\mathbb{Z}$-graded vector space and W_0 be a (ungraded) vector space. Then as in Example 2.1.3, the $\mathbb{Z}$-graded vector bundle $W_0 \times V \times W \to W_0$ forms a graded manifold $\mathcal{E} := W_0 \times V \times W$. Then $\mathcal{E} \to V$ is a graded vector bundle over a graded manifold V. This can be visualized by a diagram

$$\begin{tikzcd}
W_0 \times V \times W \arrow[r] \arrow[d] & W_0 \arrow[d] \\
V \arrow[r] & \{\mathrm{pt}\}
\end{tikzcd}$$

where each arrow is a projection. In particular, the tangent bundle $\mathcal{E} = T_{\mathcal{M}}$ of $\mathcal{M} = V$ is when $W_0 = \{pt\}$ and $W = V$.

Example 2.1.5. More generally, let $\mathcal{M} = \mathbb{E}$ be a graded manifold associated with a $\mathbb{Z}$-graded vector bundle $\mathbb{E}$ over M, as in Example 2.1.3. A typical example of graded vector bundle $\mathcal{E}$ over $\mathcal{M} = \mathbb{E}$ is a $\mathbb{Z}$-graded vector bundle object over a $\mathbb{Z}$-graded vector bundle $\mathbb{E}$. That is, it satisfies the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] & E \arrow[d] \\
\mathbb{E} \arrow[r] & M
\end{tikzcd}$$

where E is a vector bundle over M, and by forgetting the $\mathbb{Z}$-graded structure on $\mathbb{E}$, both $\mathcal{E} \to E$ and $\mathcal{E} \to \mathbb{E}$ are $\mathbb{Z}$-graded vector bundles over smooth manifolds.

2.2 Differential graded manifolds

A **differential graded manifold** (dg manifold in short) is a graded manifold $\mathcal{M}$ together with a homological vector field, i.e. a vector field $Q \in \mathfrak{X}(\mathcal{M})$ of degree $+1$ satisfying $[Q, Q] = Q \circ Q + Q \circ Q = 0$.

A morphism $\Phi : (\mathcal{M}, Q) \to (\mathcal{N}, R)$ of dg manifolds is a morphism of graded manifolds compatible with the homological vector fields.

Given two dg manifolds $(\mathcal{E}, \mathcal{Q})$ and $(\mathcal{M}, Q)$, we say a morphism of dg manifolds $\Phi : (\mathcal{E}, \mathcal{Q}) \to (\mathcal{M}, Q)$ is a dg vector bundle if $\Phi : \mathcal{E} \to \mathcal{M}$ is a graded vector bundle such that the homological vector field $\mathcal{Q} : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ is stable in the space of fiberwise linear functions $C^\infty_{\mathrm{lin}}(\mathcal{E}) \cong \Gamma(\mathcal{M}; \mathcal{E}^\vee)$ on $\mathcal{E}$ over $\mathcal{M}$. Given a graded vector bundle $\Phi : \mathcal{E} \to \mathcal{M}$, it can be checked that $\Phi : (\mathcal{E}, \mathcal{Q}) \to (\mathcal{M}, Q)$ is a dg vector bundle if and only if $\Gamma(\mathcal{M}; \mathcal{E})$ is equipped with a dg $(C^\infty(\mathcal{M}), Q)$-module structure.

For a dg manifold $(\mathcal{M}, Q)$, its tangent bundle $T_\mathcal{M}$ is naturally a dg manifold, with the homological vector field being the complete lift[1] of Q. In fact $T_\mathcal{M}$ is a dg Lie algebroid over $\mathcal{M}$ [41, 42].

Example 2.2.1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $(\mathfrak{g}[1], d_{\mathrm{CE}})$ is a dg manifold — its algebra of functions is $C^\infty(\mathfrak{g}[1]) \cong \Lambda^\bullet \mathfrak{g}^\vee$ and its homological vector field Q is the Chevalley–Eilenberg differential d_{CE}.

This construction admits an 'up to homotopy' version: Given a $\mathbb{Z}$-graded finite dimensional vector space $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, the graded manifold $\mathfrak{g}[1]$ is a dg manifold, i.e. admits a homological vector field, if and only if $\mathfrak{g}$ admits a structure of curved L_∞ algebra.

Example 2.2.2. Let M be a smooth manifold. Then $(T_M[1], d_{\mathrm{dR}})$ is a dg manifold — its algebra of functions is $C^\infty(T_M[1]) \cong \Omega^\bullet(M)$ and its homological vector field Q is the de Rham differential d_{dR}. Likewise, a complex manifold X gives rise to a dg manifold $(T_X^{0,1}[1], \bar\partial)$ whose algebra of functions $C^\infty(T_X^{0,1}[1])$ is $\Omega^{0,\bullet}(X)$ and whose homological vector field Q is the Dolbeault operator $\bar\partial$.

Example 2.2.3. Let s be a smooth section of a vector bundle $E \to M$. Then $(E[-1], \iota_s)$ is a dg manifold — its algebra of functions is $C^\infty(E[-1]) \cong \Gamma\big(\Lambda^{-\bullet} E^\vee\big)$ and its homological vector field is $Q = \iota_s$, the interior product with s. This dg manifold can be thought of as a smooth model for the (possibly singular) intersection of s with the zero section of the vector bundle E, and is often called a 'derived intersection', or a *quasi-smooth derived manifold* [2].

Both situations in Example 2.2.2 are special instances of Lie algebroids, while Example 2.2.3 is a special case of derived manifolds [2].

[1] It is also called tangent lift in the literature [42, 35].

2.2.1 Atiyah class

Let $\mathcal{M}$ be a graded manifold and $\mathcal{E}$ be a graded vector bundle over $\mathcal{M}$. We say a $\mathbb{K}$-linear map

$$\nabla : \mathfrak{X}(\mathcal{M}) \otimes_{\mathbb{K}} \Gamma\big(\mathcal{E}\big) \to \Gamma\big(\mathcal{E}\big)$$

of degree 0 is a **linear connection** on $\mathcal{E}$ over $\mathcal{M}$ if the following axioms are satisfied:

1. $C^\infty(\mathcal{M})$-linearity w.r.t. the first argument: $\nabla_{fX} s = f \nabla_X s$;
2. ∇_X is a derivative: $\nabla_X(fs) = X(f)s + (-1)^{|f|\cdot|X|} f \nabla_X s$,

where $f \in C^\infty(\mathcal{M})$ and $X \in \mathfrak{X}(\mathcal{M})$ are homogeneous elements, and $s \in \Gamma\big(\mathcal{E}\big)$.

The **covariant derivative** associated with a linear connection ∇ is the $\mathbb{K}$-linear map

$$d^\nabla : \Gamma\big(\Lambda^p T^\vee_{\mathcal{M}} \otimes \mathcal{E}\big) \to \Gamma\big(\Lambda^{p+1} T^\vee_{\mathcal{M}} \otimes \mathcal{E}\big)$$

of (internal) degree 0, defined by

$$\begin{aligned}
&\big(d^\nabla \omega\big)(X_1 \wedge \cdots \wedge X_{p+1}) \\
&\qquad = \sum_{i=1}^{p+1} (-1)^{i+1} \varepsilon \cdot \nabla_{X_i} \big(\omega(X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_{p+1})\big) \\
&\qquad\quad + \sum_{i<j} (-1)^{i+j} \varepsilon \cdot \omega([X_i, X_j] \wedge X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge \widehat{X}_j \wedge \cdots \wedge X_{p+1}),
\end{aligned}$$

for all homogeneous $\omega \in \Gamma\big(\Lambda^p T^\vee_{\mathcal{M}} \otimes \mathcal{E}\big)$ and $X_1, \cdots, X_{p+1} \in \mathfrak{X}(\mathcal{M})$. The symbol $\varepsilon = \varepsilon(\omega, X_1, \cdots, X_{p+1})$ denotes the Koszul signs arising from the reordering of the homogeneous objects $\omega, X_1, \cdots, X_{p+1}$ in each term of the right hand side.

We say ∇ is an **affine connection** on $\mathcal{M}$ if it is a linear connection on $T_{\mathcal{M}}$ over $\mathcal{M}$. Given an affine connection ∇ on $\mathcal{M}$, the $(1,2)$-tensor $T^\nabla \in \Gamma\big(T^\vee_{\mathcal{M}} \otimes T^\vee_{\mathcal{M}} \otimes T_{\mathcal{M}}\big)$ of degree 0, defined by

$$T^\nabla(X, Y) = \nabla_X Y - (-1)^{|X|\cdot|Y|} \nabla_Y X - [X, Y]$$

for any homogeneous vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, is called the **torsion** of ∇. We say an affine connection ∇ is **torsion-free** if $T^\nabla = 0$. It is well known that affine torsion-free connections always exist [34].

The **curvature** of an affine connection ∇ is the degree 0 element of $(1,3)$-tensor $R^\nabla \in \Omega^2\left(\mathcal{M}, \mathrm{End}(T_{\mathcal{M}})\right)$, defined by

$$R^\nabla(X, Y)Z = \nabla_X \nabla_Y Z - (-1)^{|X|\cdot|Y|} \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for any homogeneous vector fields $X, Y, Z \in \mathfrak{X}(\mathcal{M})$. If the curvature R^∇ vanishes identically, the affine connection ∇ is called **flat**.

Let $(\mathcal{M}, Q)$ be a dg manifold. We define an operator $\mathcal{Q}$ of degree $+1$ on the graded $C^\infty(\mathcal{M})$-module $\Gamma\big(\mathcal{M}; T_\mathcal{M}^\vee \otimes \operatorname{End}(T_\mathcal{M})\big)$:

$$\mathcal{Q} : \Gamma\big(\mathcal{M}; T_\mathcal{M}^\vee \otimes \operatorname{End}(T_\mathcal{M})\big)^\bullet \to \Gamma\big(\mathcal{M}; T_\mathcal{M}^\vee \otimes \operatorname{End}(T_\mathcal{M})\big)^{\bullet+1} \tag{2.1}$$

by the Lie derivative along the homological vector field Q:

$$(\mathcal{Q}F)(X, Y) = [Q, F(X, Y)] - (-1)^k F([Q, X], Y) - (-1)^{k+|X|} F(X, [Q, Y])$$

for any section $F \in \Gamma\big(\mathcal{M}; T_\mathcal{M}^\vee \otimes \operatorname{End}(T_\mathcal{M})\big)^k$ of degree k and homogeneous vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$. One can easily check that $\mathcal{Q}^2 = 0$. Therefore

$$\Big(\Gamma\big(\mathcal{M}; T_\mathcal{M}^\vee \otimes \operatorname{End}(T_\mathcal{M})\big)^\bullet, \mathcal{Q}\Big)$$

is a cochain complex.

Now given an affine connection ∇, consider the $(1, 2)$-tensor $\operatorname{At}^\nabla_{(\mathcal{M},Q)} \in \Gamma\big(\mathcal{M}; T_\mathcal{M}^\vee \otimes \operatorname{End}(T_\mathcal{M})\big)$ of degree $+1$, defined by

$$\operatorname{At}^\nabla_{(\mathcal{M},Q)}(X, Y) = [Q, \nabla_X Y] - \nabla_{[Q,X]} Y - (-1)^{|X|} \nabla_X [Q, Y]$$

for any homogeneous vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$.

Proposition 2.2.4 ([42]). *In the above setting, the following statements hold.*

1. *If the affine connection ∇ on $\mathcal{M}$ is torsion-free, then we have $\operatorname{At}^\nabla_{(\mathcal{M},Q)} \in \Gamma\big(\mathcal{M}; S^2(T_\mathcal{M}^\vee) \otimes T_\mathcal{M}\big)$. In other words,*

$$\operatorname{At}^\nabla_{(\mathcal{M},Q)}(X, Y) = (-1)^{|X|\cdot|Y|} \operatorname{At}^\nabla_{(\mathcal{M},Q)}(Y, X).$$

2. *The element $\operatorname{At}^\nabla_{(\mathcal{M},Q)} \in \Gamma\big(\mathcal{M}; T_\mathcal{M}^\vee \otimes \operatorname{End}(T_\mathcal{M})\big)^1$ is a 1-cocycle.*

3. *The cohomology class $[\operatorname{At}^\nabla_{(\mathcal{M},Q)}]$ does not depend on the choice of connection.*

The element $\operatorname{At}^\nabla_{(\mathcal{M},Q)}$ is called the **Atiyah cocycle** associated with the affine connection ∇. The cohomology class

$$\alpha_{(\mathcal{M},Q)} := [\operatorname{At}^\nabla_{(\mathcal{M},Q)}] \in H^1\Big(\Gamma\big(\mathcal{M}; T_\mathcal{M}^\vee \otimes \operatorname{End}(T_\mathcal{M})\big)^\bullet, \mathcal{Q}\Big)$$

is called the **Atiyah class** of the dg manifold $(\mathcal{M}, Q)$ [42]. See also [50] and [37, Footnote 6].

2.3 Fedosov construction on graded manifolds

This section is to give a brief description of Fedosov construction of graded manifolds. We refer readers to [13, 17, 34] for more details.

Throughout this section, $\mathcal{M}$ is a finite dimensional graded manifold and ∇ is a torsion-free affine connection on $\mathcal{M}$. There is an induced linear connection on $\widehat{S}(T^\vee_{\mathcal{M}})$, which is denoted by the same symbol ∇ by abuse of notation.

Consider the map $\nabla^{\natural} : \mathfrak{X}(\mathcal{M}) \times \Gamma\big(S(T_{\mathcal{M}})\big) \to \Gamma\big(S(T_{\mathcal{M}})\big)$

$$\nabla^{\natural}_Y \boldsymbol{X} = (\mathrm{pbw}^{\nabla})^{-1}\big(Y \cdot \mathrm{pbw}^{\nabla}(\boldsymbol{X})\big)$$

for any $Y \in \mathfrak{X}(\mathcal{M})$ and $\boldsymbol{X} \in \Gamma\big(S(T_{\mathcal{M}})\big)$.

Lemma 2.3.1. *The above map $\nabla^{\natural}$ defines a flat connection on $S(T_{\mathcal{M}})$.*

Abusing notation, we write the same symbol $\nabla^{\natural}$ to denote the induced flat connection on $\widehat{S}(T^\vee_{\mathcal{M}})$. Then the associated covariant derivative $d^{\nabla^{\natural}}$ satisfies $(d^{\nabla^{\natural}})^2 = 0$.

In the following, we use the identification

$$\Omega^p(\widehat{S}(T^\vee_{\mathcal{M}})) \cong \Gamma\big(\Lambda^p(T^\vee_{\mathcal{M}}) \otimes \widehat{S}(T^\vee_{\mathcal{M}})\big) \cong \Gamma\big(\operatorname{Hom}(\Lambda^p(T_{\mathcal{M}}) \otimes S(T_{\mathcal{M}}), \mathbb{K})\big)$$

and the total degree of $\omega \in \Omega^p(\widehat{S}(T^\vee_{\mathcal{M}}))$ is $p + |\omega|$, where p is the cohomological degree and $|\omega|$ is the internal degree.

Define two operators

$$\delta : \Omega^p(\widehat{S}(T^\vee_{\mathcal{M}})) \to \Omega^{p+1}(\widehat{S}(T^\vee_{\mathcal{M}}))$$

and

$$\mathfrak{h} : \Omega^p(\widehat{S}(T^\vee_{\mathcal{M}})) \to \Omega^{p-1}(\widehat{S}(T^\vee_{\mathcal{M}}))$$

by

$$\begin{aligned}(\delta\omega)\,&(X_1 \wedge \cdots \wedge X_{p+1}; Y_1 \odot \cdots \odot Y_{q-1}) \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} \varepsilon \cdot \omega(X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_{p+1}; X_i \odot Y_1 \odot \cdots \odot Y_{q-1})\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{h}\omega)\,&(X_1 \wedge \cdots \wedge X_{p-1}; Y_1 \odot \cdots \odot Y_{q+1}) \\ &= \frac{1}{p+q} \sum_{j=1}^{q+1} \varepsilon \cdot \omega(Y_j \wedge X_1 \wedge \cdots \wedge X_{p-1}; Y_1 \odot \cdots \odot \widehat{Y}_j \odot \cdots \odot Y_{q+1}),\end{aligned}$$

for all $\omega \in \Omega^p(\widehat{S}(T^\vee_{\mathcal{M}}))$ and all homogeneous $X_1, \cdots, X_{p+1}, Y_1, \cdots, Y_{q+1} \in \mathfrak{X}(\mathcal{M})$. The symbol ε denotes the Koszul signs: either $\varepsilon(X_1, \cdots, X_{p+1}, Y_1, \cdots, Y_{q-1})$ or $\varepsilon(X_1, \cdots, X_{p-1}, Y_1, \cdots, Y_{q+1})$, as appropriate.

Both δ and $\mathfrak{h}$ are $C^\infty(\mathcal{M})$-linear, and δ is the Koszul operator. Observe that δ has total degree $+1$ and $\mathfrak{h}$ has total degree -1. However neither δ nor ω change the internal degree: $|\delta\omega| = |\omega|$ and $|\mathfrak{h}\omega| = |\omega|$ for $\omega \in \Omega^p(\widehat{S}(T^\vee_{\mathcal{M}}))$.

Remark 2.3.2. In [13, 17, 34], the operator $\mathfrak{h}$ is written as δ^{-1}. We avoid this notation because $\mathfrak{h}$ is not an inverse map of δ, and it is rather a homotopy operator.

Lemma 2.3.3. *The operator δ satisfies $\delta^2 = 0$. That is,*

$$0 \to \Omega^0(\widehat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{\delta} \Omega^1(\widehat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{\delta} \Omega^2(\widehat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{\delta} \cdots$$

forms a cochain complex. Moreover, it satisfies

$$\delta \circ \mathfrak{h} + \mathfrak{h} \circ \delta = \mathrm{id} - \pi_0$$

where $\pi_0 : \Omega^\bullet(\widehat{S}(T^\vee_{\mathcal{M}})) \to C^\infty(\mathcal{M})$ is the natural projection.

We have the following theorem

Theorem 2.3.4 ([34, Theorem 5.6])**.** *Let $\mathcal{M}$ be a finite dimensional graded manifold and ∇ be a torsion-free affine connection on $\mathcal{M}$. Then the covariant derivative $d^{\nabla^\natural}$ decomposes as*

$$d^{\nabla^\natural} = d^\nabla - \delta + \widetilde{A^\nabla},$$

where the operator $\widetilde{A^\nabla} : \Omega^\bullet\big(\widehat{S}(T^\vee_{\mathcal{M}})\big) \to \Omega^{\bullet+1}\big(\widehat{S}(T^\vee_{\mathcal{M}})\big)$, is a (total) degree $+1$ derivation determined by $A^\nabla \in \Omega^1\big(\mathcal{M}, \widehat{S}^{\geq 2}(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}}\big)$, satisfying

$$\mathfrak{h} \circ A^\nabla = 0.$$

Remark 2.3.5. The operator $\widetilde{A^\nabla}$ increases the cohomological degree by $+1$ while it preserves the internal degree. That is, although the total degree of $\widetilde{A^\nabla}$ is $+1$, we have the internal degree $\left|\widetilde{A^\nabla}\right| = 0$.

Write

$$A^\nabla = \sum_{n\geq 2} A^\nabla_n, \qquad A^\nabla_n \in \Omega^1(\mathcal{M}, S^n(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}}).$$

Let $R^\nabla \in \Omega^2\big(\mathcal{M}; \mathrm{End}(T_{\mathcal{M}})\big)$ denote the curvature of ∇.

Proposition 2.3.6. *We have the following recursive formula for A_n^∇:*

$$A_2^\nabla = \mathfrak{h} \circ R^\nabla ,$$
$$A_{n+1}^\nabla = \mathfrak{h} \circ \left(d^\nabla A_n^\nabla + \sum_{p+q=n} \frac{1}{2}[A_p^\nabla, A_q^\nabla] \right), \quad \forall n \geq 2.$$

Proof. By Theorem 2.3.4, the covariant derivative $d^{\nabla^\natural} = d^\nabla - \delta + A^\nabla$ and satisfies $(d^{\nabla^\natural})^2 = 0$.

By Lemma 2.3.3, we know $\delta^2 = 0$ and $\delta \circ \mathfrak{h} + \mathfrak{h} \circ \delta = \mathrm{id} - \pi_0$. Also, $(d^\nabla)^2 = R^\nabla$. Since ∇ is torsion-free, we have

$$[\delta, d^\nabla] = \delta \circ d^\nabla + d^\nabla \circ \delta = 0.$$

As a result, $(d^{\nabla^\natural})^2 = 0$ implies that

$$\delta \circ A^\nabla + A^\nabla \circ \delta = R^\nabla + d^\nabla A^\nabla + \frac{1}{2}[A^\nabla, A^\nabla]$$

By applying the operator $\mathfrak{h}$, we get

$$A^\nabla = \mathfrak{h} \circ \delta \circ A^\nabla = \mathfrak{h} \circ \left(R^\nabla + d^\nabla A^\nabla + \frac{1}{2}[A^\nabla, A^\nabla] \right)$$

because $\mathfrak{h} \circ A^\nabla = 0$ and $\pi_0 \circ A^\nabla = 0$.

Since $\mathfrak{h}\big(\Omega^2(\widehat{S}^q(T_\mathcal{M}^\vee)) \subset \Omega^1(\widehat{S}^{q+1}(T_\mathcal{M}^\vee))$, applying the canonical projections

$$\Omega^1(\mathcal{M}, \hat{S}(T_\mathcal{M}^\vee) \otimes T_\mathcal{M}) \to \Omega^1(\mathcal{M}, S^n(T_\mathcal{M}^\vee) \otimes T_\mathcal{M})$$

(for each $n \geq 2$) to the equality

$$A^\nabla = \mathfrak{h} \circ \left(R^\nabla + d^\nabla A^\nabla + \frac{1}{2}[A^\nabla, A^\nabla] \right) \in \Omega^1(\mathcal{M}, \hat{S}(T_\mathcal{M}^\vee) \otimes T_\mathcal{M})$$

yields the relations

$$A_2^\nabla = \mathfrak{h} \circ R^\nabla,$$
$$A_{n+1}^\nabla = \mathfrak{h} \circ \left(d^\nabla A_n^\nabla + \sum_{p+q=n} \frac{1}{2}[A_p^\nabla, A_q^\nabla] \right), \quad \forall n \geq 2. \tag{2.2}$$

This completes the proof. □

Corollary 2.3.7. *Under the same hypothesis as in Theorem 2.3.4, the element $A_n^\nabla \in \Omega^1(\mathcal{M}, S^n(T_\mathcal{M}^\vee) \otimes T_\mathcal{M})$, with $n \geq 2$, is completely determined by the curvature R^∇ and its higher covariant derivatives. In fact, A_n^∇ satisfies the recursive formula* (2.2) *involving A_k^∇, with $k \leq n-1$.*

2.4 Differential graded coalgebras and comodules

In this section, we summarize the necessary facts about dg coalgebras and dg comodules. We refer the reader to [14, 36] for a general introduction to coalgebras.

2.4.1 dg coalgebras

Let $\mathcal{R}$ be a graded commutative ring. A **graded coalgebra** C over $\mathcal{R}$ is a graded $\mathcal{R}$-module equipped with an $\mathcal{R}$-linear map $\Delta : C \to C \otimes_{\mathcal{R}} C$ of degree 0 called comultiplication satisfying the following conditions:

1. (Coassociativity)

$$(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta : C \to C \otimes_{\mathcal{R}} C \otimes_{\mathcal{R}} C.$$

2. (Counit) There is an $\mathcal{R}$-linear map $\epsilon : C \to \mathcal{R}$ of degree 0 such that

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}_C .$$

Let $\mathrm{tw} : C \otimes_{\mathcal{R}} C \to C \otimes_{\mathcal{R}} C$ be the map defined by

$$\mathrm{tw}(c_1 \otimes c_2) = (-1)^{|c_1|\cdot|c_2|} c_2 \otimes c_1,$$

for homogeneous elements $c_1,\ c_2 \in C$. A graded coalgebra C is called **cocommutative** if it satisfies $\Delta = \mathrm{tw} \circ \Delta$.

An $\mathcal{R}$-linear map $\phi : C \to C$ satisfying

$$\Delta \circ \phi = (\mathrm{id}_C \otimes \phi + \phi \otimes \mathrm{id}_C) \circ \Delta$$

is called an $\mathcal{R}$**-coderivation** of the graded $\mathcal{R}$-coalgebra C. We denote the collection of all $\mathcal{R}$-coderivations of C by $\mathrm{coDer}_{\mathcal{R}}(C)$.

Let $(\mathcal{R}, d_{\mathcal{R}})$ be a dg commutative ring, and (C, d_C) be a dg $(\mathcal{R}, d_{\mathcal{R}})$-module. Then the map

$$d_{C^{\otimes 2}} : C \otimes_{\mathcal{R}} C \to C \otimes_{\mathcal{R}} C$$

defined by

$$d_{C^{\otimes 2}}(c_1 \otimes c_2) = d_C(c_1) \otimes c_2 + (-1)^{|c_1|} c_1 \otimes d_C(c_2)$$

for homogeneous elements $c_1, c_2 \in C$, is a well-defined degree $+1$ differential. Such a differential is called the induced differential on $C \otimes_{\mathcal{R}} C$.

Definition 2.4.1. Let $(\mathcal{R}, d_{\mathcal{R}})$ be a dg commutative ring. A **dg coalgebra** (C, d_C) over $(\mathcal{R}, d_{\mathcal{R}})$ is a dg $(\mathcal{R}, d_{\mathcal{R}})$-module (C, d_C), equipped with a graded coalgebra structure on C over $\mathcal{R}$ where the comultiplication and the counit map respect the differentials. That is,

$$\Delta \circ d_C = d_{C^{\otimes 2}} \circ \Delta,$$
$$\epsilon \circ d_C = d_{\mathcal{R}} \circ \epsilon$$

where $\Delta : C \to C \otimes_{\mathcal{R}} C$ is the comultiplication and $\epsilon : C \to \mathcal{R}$ is the counit map.

2.4.2 Convolution algebras

Let (A, d_A) be a dg algebra with unit 1_A, and (C, d_C) be a dg coalgebra with counit ϵ. The **convolution product** $\star : \mathrm{Hom}(C, A) \times \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, A)$ is the multiplication on the space $\mathrm{Hom}(C, A)$ of $\mathbb{K}$-linear maps defined by

$$f \star g := \mu \circ (f \otimes g) \circ \Delta, \tag{2.3}$$

where $f, g \in \mathrm{Hom}(C, A)$, μ is the multiplication on A, and Δ is the comultiplication on C. One can check that $\big(\mathrm{Hom}(C, A), \star\big)$ is a graded algebra with the unit $1_{\mathrm{Hom}(C,A)} \in \mathrm{Hom}(C, A)$, where

$$1_{\mathrm{Hom}(C,A)}(x) := \epsilon(x) \cdot 1_A.$$

Furthermore, one can show that the linear map $d_{\mathrm{Hom}(C,A)} : \mathrm{Hom}^{\bullet}(C, A) \to \mathrm{Hom}^{\bullet+1}(C, A)$,

$$d_{\mathrm{Hom}(C,A)}(f) := d_A \circ f - (-1)^{|f|} f \circ d_C,$$

is a derivation of degree one, and thus the triple $\big(\mathrm{Hom}(C, A), \star, d_{\mathrm{Hom}(C,A)}\big)$ is a dg algebra which is referred as the **convolution dg algebra**.

Definition 2.4.2. Let V be a graded vector space. The **symmetric coalgebra** over V is the graded vector space $SV := \bigoplus_{n=0}^{\infty} V^{\odot n}$ whose counit is the projection $\epsilon_S : SV \twoheadrightarrow S^0 V \cong \mathbb{K}$, and whose coproduct Δ_S is determined by

$$\Delta_S(1) = 1 \otimes 1, \qquad \Delta_S(v) = 1 \otimes v + v \otimes 1,$$
$$\Delta_S(\mathbf{x} \odot \mathbf{y}) = \Delta_S(\mathbf{x}) \odot \Delta_S(\mathbf{y}),$$

for any $v \in V$, $\mathbf{x}, \mathbf{y} \in SV$.

More precisely, the coproduct Δ_S can be computed by the formula

$$\Delta_S(v_1 \odot \cdots \odot v_n) = \sum_{i=0}^{n} \sum_{\sigma \in S_{i,n-i}} \varepsilon \cdot \big(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}\big) \otimes \big(v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(n)}\big),$$

where $\varepsilon = \pm 1$ is determined by the Koszul sign convention. Also note that the coalgebra $(SV, \Delta_S, \epsilon_S)$ is cocommutative.

Example 2.4.3. Let $\mathfrak{g}$ be a Lie algebra. We have the graded coalgebra $S(\mathfrak{g}[1])$. The Lie bracket $[-,-]_{\mathfrak{g}}$ induces a (graded) symmetric operation

$$\mathrm{s}^{-1} \circ [-,-]_{\mathfrak{g}} \circ (\mathrm{s} \otimes \mathrm{s}) : S^2(\mathfrak{g}[1]) \to \mathfrak{g}[1] : x \odot y \mapsto -\mathrm{s}^{-1}[\mathrm{s}x, \mathrm{s}y]_{\mathfrak{g}}$$

of degree one, where $\mathrm{s} : \mathfrak{g}[1] \to \mathfrak{g}$ is the degree-shifting map. This operation induces a degree-one coderivation $\partial_{\mathfrak{g}} : S(\mathfrak{g}[1]) \to S(\mathfrak{g}[1])$ defined by

$$\partial_{\mathfrak{g}}(x_1 \odot \cdots \odot x_n) = \sum_{i<j} (-1)^{i+j}\, \mathrm{s}^{-1}[\mathrm{s}x_i, \mathrm{s}x_j]_{\mathfrak{g}} \odot x_1 \odot \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \odot x_n\,,$$

for $x_1, \cdots, x_n \in \mathfrak{g}[1]$, and one obtain the dg coalgebra $(S(\mathfrak{g}[1]), \partial_{\mathfrak{g}})$.

Let $A = \mathbb{K}$ be the trivial dg algebra with zero differential. The associated convolution dg algebra $\mathrm{Hom}(S(\mathfrak{g}[1]), \mathbb{K}) \cong S(\mathfrak{g}[1])^\vee$ is equipped with the differential

$$d_{\mathfrak{g}}(f) = -(-1)^{|f|} f \circ \partial_{\mathfrak{g}}\,. \tag{2.4}$$

It is straightforward to show that the convolution product $\star$ coincides with the canonical multiplication on $S(\mathfrak{g}[1])^\vee$ in the sense that

$$(f \star g)(\mathbf{x}) = \langle\, f \odot g \mid \mathbf{x} \,\rangle$$

for $f, g \in \mathrm{Hom}(S(\mathfrak{g}[1]), \mathbb{K}) \cong S(\mathfrak{g}[1])^\vee$ and $\mathbf{x} \in S(\mathfrak{g}[1])$.

2.4.3 Twisting cochains

Let (C, d_C) be a dg coalgebra, and (A, d_A) be a dg algebra. An element $\tau \in \mathrm{Hom}^1(C, A)$ of degree $+1$ is called a **twisting cochain** if it satisfies the Maurer–Cartan equation

$$d_{\mathrm{Hom}(C,A)}(\tau) + \tau \star \tau = 0$$

in the convolution algebra $\big(\mathrm{Hom}(C, A), \star, d_{\mathrm{Hom}(C,A)}\big)$. The **twisted tensor product** $A \otimes_\tau C$ is a dg vector space whose underlying space is the tensor product $A \otimes C$, and the differential d_τ is defined by

$$d_\tau = d_A \otimes \mathrm{id}_C + \mathrm{id}_A \otimes d_C - (\mu \otimes \mathrm{id}_C)(\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_C)(\mathrm{id}_A \otimes \Delta).$$

Note that the twisted tensor product $A \otimes_\tau C$ is a left dg (A, d_A)-module. Let (M, d_M) be a left dg (A, d_A)-module. The space $\mathrm{Hom}_A(A \otimes_\tau C, M)$ is equipped with a canonical differential d_τ induced by the dg structures. This dg vector space $(\mathrm{Hom}_A(A \otimes_\tau C, M), d_\tau)$ will be denoted by $\mathrm{Hom}^\tau(C, M)$ and will be referred as the **twisted Hom space**. See, for example, [53, 36, 32].

Let $\mathfrak{g}$ be a Lie algebra. We have the dg coalgebra $(S(\mathfrak{g}[1]), \partial_{\mathfrak{g}})$ and the dg algebra $(\mathcal{U}\mathfrak{g}, 0)$. One can show that the map $\tau : S(\mathfrak{g}[1]) \to \mathcal{U}\mathfrak{g}$ defined by the composition

$$\begin{tikzcd} S(\mathfrak{g}[1]) \arrow[r, "\mathrm{pr}_{\mathfrak{g}[1]}"'] \arrow[rrr, bend left, "\tau"] & \mathfrak{g}[1] \arrow[r, "-\mathrm{s}"'] & \mathfrak{g} \arrow[r, hook] & \mathcal{U}\mathfrak{g} \end{tikzcd}$$

is a twisting cochain. Here, $\mathrm{s} : \mathfrak{g}[1] \to \mathfrak{g}$ is the degree shifting map, and $\mathfrak{g} \hookrightarrow \mathcal{U}\mathfrak{g}$ is the natural embedding. The twisted tensor product $\mathcal{U}\mathfrak{g} \otimes_\tau S(\mathfrak{g}[1])$ coincides with the Chevalley–Eilenberg chain complex $(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), d_X)$ described in Section 4.4.

Let (B, d_B) be a dg algebra. We say (B, d_B) is a **dg $\mathfrak{g}$-algebra** if it is endowed with an infinitesimal action of $\mathfrak{g}$ (i.e. a Lie algebra morphism $\mathfrak{g} \to \mathrm{Der}^0(B)$) such that

$$d_B(\mathrm{s}x \cdot b) = \mathrm{s}x \cdot d_A(b),$$

for $\mathrm{s}x \in \mathfrak{g}$, $b \in B$. Since a dg $\mathfrak{g}$-algebra (B, d_B) is also a dg $\mathcal{U}\mathfrak{g}$-module, the graded vector space $\mathrm{Hom}(S(\mathfrak{g}[1]), B) \cong \mathrm{Hom}_{\mathcal{U}\mathfrak{g}}(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), B)$ is endowed with the differential d_τ and the convolution product $\star$.

Proposition 2.4.4. *Let (B, d_B) be a dg $\mathfrak{g}$-algebra. The graded vector space $\mathrm{Hom}(S(\mathfrak{g}[1]), B)$, equipped with the differential d_τ and the convolution product $\star$, is a dg algebra.*

Proof. Since $\big(\mathrm{Hom}(S(\mathfrak{g}[1]), B), \star, d_{\mathrm{Hom}(S(\mathfrak{g}[1]),B)}\big)$ is a dg algebra, it suffices to show the compatibility of the convolution product $\star$ and $d_\tau - d_{\mathrm{Hom}(S(\mathfrak{g}[1]),B)}$. This is a consequence of the assumption $\mathfrak{g}$ acts on B by derivations. □

In the setting of Proposition 2.4.4, the differential d^τ is also written as d^B_{CE}, called the **Chevalley–Eilenberg differential**. More explicitly,

$$\begin{aligned} &d^B_{\mathrm{CE}}(f)(x_1 \odot \cdots \odot x_n) \\ &\quad = (d_B \circ f)(x_1 \odot \cdots \odot x_n) + \sum_i (-1)^{i+|f|}\, \mathrm{s}x_i \cdot f(x_1 \odot \cdots \widehat{x_i} \cdots \odot x_n) \\ &\qquad - (-1)^{|f|} \sum_{i<j} (-1)^{i+j} f(\mathrm{s}^{-1}[\mathrm{s}x_i, \mathrm{s}x_j]_{\mathfrak{g}} \odot x_1 \odot \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \odot x_n), \end{aligned} \tag{2.5}$$

for $f \in \mathrm{Hom}(S(\mathfrak{g}[1]), B)$ and $x_1, \cdots, x_n \in \mathfrak{g}[1]$.

Remark 2.4.5. Let $(B, d_B) = (B, 0)$ be a dg $\mathfrak{g}$-algebra with zero differential. In the literature, there is a more common formulation of Chevalley–Eilenberg differential $\tilde{d}_{\mathrm{CE}} : \mathrm{Hom}(\Lambda^\bullet \mathfrak{g}, B) \to \mathrm{Hom}(\Lambda^{\bullet+1}\mathfrak{g}, B)$ given by the formula

$$\begin{aligned} \tilde{d}_{\mathrm{CE}}(\tilde{f})(\mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}x_n) := &\sum_{i=1}^n (-1)^{i+1} \mathrm{s}x_i \cdot \tilde{f}(\mathrm{s}x_1 \wedge \cdots \widehat{\mathrm{s}x_i} \cdots \wedge \mathrm{s}x_n) \\ &+ \sum_{i<j} (-1)^{i+j} \tilde{f}([\mathrm{s}x_i, \mathrm{s}x_j]_{\mathfrak{g}} \wedge \mathrm{s}x_1 \wedge \cdots \widehat{\mathrm{s}x_i} \cdots \widehat{\mathrm{s}x_j} \cdots \wedge \mathrm{s}x_n) \,. \end{aligned}$$

The space $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, B)$ is equipped the product $\tilde{\star} : \mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, B)\times\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, B) \to \mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, B)$,

$$(\tilde{f} \mathbin{\tilde{\star}} \tilde{g})(\mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}x_{n+m}) := \sum_{\sigma\in S_{n,m}} (-1)^\sigma\ \tilde{f}(\mathrm{s}x_{\sigma(1)} \wedge \cdots \wedge \mathrm{s}x_{\sigma(n)})\tilde{g}(\mathrm{s}x_{\sigma(n+1)} \wedge \cdots \wedge \mathrm{s}x_{\sigma(n+m)})\,.$$

The triple $\big(\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, B),\ \tilde{\star}\ , \tilde{d}_{\mathrm{CE}}\big)$ is a dg algebra.

Let $\eta : \mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, B) \to \mathrm{Hom}(S^\bullet(\mathfrak{g}[1]), B)$ be the map

$$\eta(\tilde{f})(x_1 \odot \cdots \odot x_n) := (-1)^{1+\cdots+n}\ \tilde{f}(\mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}x_n)$$

for $\tilde{f} \in \mathrm{Hom}(\Lambda^n\mathfrak{g}, B)$, $x_1, \cdots, x_n \in \mathfrak{g}[1]$. One can check that

$$\eta : \big(\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, B),\ \tilde{\star}\ , \tilde{d}_{\mathrm{CE}}\big) \to \big(\mathrm{Hom}^\tau(S^\bullet(\mathfrak{g}[1]), B), \star\big)$$

is an isomorphism of dg algebras.

Remark 2.4.6. Since the reduced Chevalley–Eilenberg chain complex $\mathcal{U}\mathfrak{g} \otimes_\tau S(\mathfrak{g}[1])/\mathbb{K}$ is acyclic, by a theorem [36, Theorem 2.3.1] of twisting cochains, one has a quasi-isomorphism from the cobar complex of $S(\mathfrak{g}[1])$ to $\mathcal{U}\mathfrak{g}$. This quasi-isomorphism induces a map $\mathrm{Hoch}^\bullet_\oplus(S(\mathfrak{g}[1])^\vee, d_\mathfrak{g}) \to \mathrm{Hom}^\bullet(S(\mathfrak{g}[1]),\mathcal{U}\mathfrak{g})$ which coincides with the map in the proof of [4, Theorem 4.10]. See also [26].

2.4.4 Differential graded comodules and cogenerators

Let (C, Δ, ϵ) be a graded coalgebra. A **(right) graded comodule** (M, ϕ_M) over C is a graded vector space M, equipped with a linear map $\phi_M : M \to M \otimes C$ of degree zero such that

(i) $(\phi_M \otimes \mathrm{id}_C) \circ \phi_M = (\mathrm{id}_M \otimes\Delta) \circ \phi_M$;

(ii) $\mu_{M,\mathbb{K}} \circ (\mathrm{id}_M \otimes \epsilon) \circ \phi_M = \mathrm{id}_M$,

where $\mu_{M,\mathbb{K}} : M \otimes \mathbb{K} \to M$ is the scalar multiplication. Let (M, ϕ_M), (N, ϕ_N) be comodules over C. A **morphism of comodules** is a linear map $\Psi \in \mathrm{Hom}^\bullet_\mathbb{K}(M, N)$ such that

$$(\Psi \otimes \mathrm{id}_C) \circ \phi_M = \phi_N \circ \Psi. \tag{2.6}$$

We denote by $\mathrm{coHom}_C(M, N)$ the space of morphisms of comodules from (M, ρ_M) to (N, ρ_N).

A **(right) dg comodule** (M, ϕ_M, d_M) over a dg coalgebra $(C, \Delta, \epsilon, d_C)$ is a (right) graded comodule over C, together with a linear map $d_M : M \to M$ of degree $+1$ such that $d_M \circ d_M = 0$ and

$$\phi_M \circ d_M = (d_M \otimes \mathrm{id}_C + \mathrm{id}_M \otimes d_C) \circ \phi_M.$$

Example 2.4.7. Let (C, Δ, d_C) be a dg coalgebra and (A, μ, d_A) be a dg algebra. If $\tau : C \to A$ is a twisting cochain, the twisted tensor product $A \otimes_\tau C$ equipped with $\mathrm{id}_A \otimes \Delta : A \otimes C \to A \otimes C \otimes C$ is a right dg comodule over C. In particular, the Chevalley–Eilenberg chain complex $(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), d_X)$ is a right dg $S(\mathfrak{g}[1])$-comodule.

Lemma 2.4.8. *Let (M, ϕ_M, d_M), (N, ϕ_N, d_N) be dg comodules. The space $\mathrm{coHom}_C(M, N)$ is a dg vector subspace of $\mathrm{Hom}_{\mathbb{K}}(M, N)$ whose differential ∂ is defined by*

$$\partial(\Psi) := d_N \circ \Psi - (-1)^{|\Psi|}\Psi \circ d_M.$$

Let $(C, \Delta, \epsilon, d_C)$ be a dg coalgebra, and let

$$(A, \mu, 1_A, d_A) = \big(\mathrm{Hom}(C, \mathbb{K}), \star, 1_{\mathrm{Hom}(C,\mathbb{K})}, d_{\mathrm{Hom}(C,\mathbb{K})}\big)$$

be the convolution dg algebra. Let (M, ϕ_M, d_M) be a right dg comodule over C. We define the action map $\rho_M : A \otimes M \to M$,

$$\rho_M(f \otimes m) := \mu_{M,\mathbb{K}} \circ (\mathrm{id}_M \otimes f) \circ \phi_M(m),$$

where $m \in M$, $f \in A = \mathrm{Hom}(C, \mathbb{K})$.

Proposition 2.4.9. *Let (M, ϕ_M) be a right dg comodule over a dg coalgebra C. The triple (M, ρ_M, d_M) is a left dg module over the convolution dg algebra $A = \mathrm{Hom}(C, \mathbb{K})$.*

Furthermore, let (N, ϕ_N, d_N) be another right dg comodule over C, and (N, ρ_N, d_N) be the associated dg module over A. A linear map $\Psi : M \to N$ is a morphism of right comodules over C if and only if Ψ is a morphism of left modules over A, i.e. $\mathrm{coHom}_C(M, N) = \mathrm{Hom}_A(M, N)$.

Proof. The first part of the proposition can be shown by a direct computation. See [14, Proposition 2.2.1]. We only prove the second part here. Let (N, ϕ_N, d_N) be another right dg comodule over C, and $\Psi : M \to N$ be a $\mathbb{K}$-linear map. If Ψ is a comodule morphism over C, then

$$\begin{aligned}
\Psi \circ \rho_M(f \otimes m) &= \Psi \circ \mu_{M,\mathbb{K}} \circ (\mathrm{id}_M \otimes f) \circ \phi_M(m) \\
&= (-1)^{|f||\Psi|}\mu_{N,\mathbb{K}} \circ (\mathrm{id}_N \otimes f) \circ (\Psi \otimes \mathrm{id}_C) \circ \phi_M(m) \\
&= (-1)^{|f||\Psi|}\mu_{N,\mathbb{K}} \circ (\mathrm{id}_N \otimes f) \circ \phi_N \circ \Psi(m) \\
&= (-1)^{|f||\Psi|}\rho_N\big(f \otimes \Psi(m)\big),
\end{aligned}$$

i.e. $\Psi : (M, \rho_M) \to (N, \rho_N)$ is a module morphism over A. Conversely, if Ψ is a module morphism over A, then

$$\begin{aligned}
\mu_{N,\mathbb{K}} \circ (\mathrm{id}_N \otimes f) \circ \Big(\phi_N \circ \Psi\Big)(m) &= \rho_N(f \otimes \Psi(m)) \\
&= (-1)^{|f||\Psi|}\Psi(\rho_M(f \otimes m)) \\
&= (-1)^{|f||\Psi|}\Psi \circ \mu_{M,\mathbb{K}} \circ (\mathrm{id}_M \otimes f) \circ \phi_M(m) \\
&= \mu_{N,\mathbb{K}} \circ (\mathrm{id}_N \otimes f) \circ \Big((\Psi \otimes \mathrm{id}_C) \circ \phi_M\Big)(m)
\end{aligned}$$

for any $f \in \mathrm{Hom}(C, \mathbb{K})$, any $m \in M$. Thus, we have

$$(\Psi \otimes \mathrm{id}_C) \circ \phi_M = \phi_N \circ \Psi,$$

i.e. Ψ is a comodule morphism over C. $\square$

Example 2.4.10. Since $M = \mathcal{U}\mathfrak{g} \otimes_\tau S(\mathfrak{g}[1]) = (\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), d_X)$ is a right dg comodule over $(S(\mathfrak{g}[1]), \partial_\mathfrak{g})$, it is also a left dg module over $(S(\mathfrak{g}[1])^\vee, d_\mathfrak{g})$ by Proposition 2.4.9. More explicitly, the module structure $\rho : S(\mathfrak{g}[1])^\vee \otimes \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]) \to \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])$ is characterized by

$$\begin{aligned}
\rho\big(\xi \otimes (u \otimes x_1 \odot \cdots \odot x_n)\big) &= \mu_{M,\mathbb{K}} \circ (\mathrm{id}_M \otimes \langle\, \xi \mid - \,\rangle) \circ (\mathrm{id}_{\mathcal{U}\mathfrak{g}} \otimes \Delta_S)(u \otimes x_1 \odot \cdots \odot x_n) \\
&= \sum_{i=1}^{n} (-1)^{i+1} \langle\, \xi \mid x_i \,\rangle\; u \otimes x_1 \odot \cdots \widehat{x_i} \cdots \odot x_n \,,
\end{aligned}$$

for $\xi \in S^1(\mathfrak{g}[1])^\vee$, $u \in \mathcal{U}\mathfrak{g}$, $x_1, \cdots, x_n \in \mathfrak{g}[1]$. Furthermore, since $S(\mathfrak{g}[1])^\vee$ is graded commutative, the left action ρ induce a right action $\tilde{\rho}$ on M:

$$\begin{aligned}
\tilde{\rho}\big((u \otimes x_1 \odot \cdots \odot x_n) \otimes \xi\big) &= (-1)^n \rho\big(\xi \otimes (u \otimes x_1 \odot \cdots \odot x_n)\big) \\
&= \sum_{i=1}^{n} (-1)^{n-i} \langle\, x_i \mid \xi \,\rangle\; u \otimes x_1 \odot \cdots \widehat{x_i} \cdots \odot x_n \,,
\end{aligned}$$

which coincides with the contraction action $\llcorner$ defined in Section 4.4.

Cogenerators of graded comodules

Let A be a $\mathbb{K}$-algebra. A generator of a A-module (N, ρ_N) can be considered as a vector space W, together with a linear map $\iota_W : W \to N$ such that the map

$$A \otimes W \xrightarrow{\mathrm{id}_A \otimes \iota_W} A \otimes N \xrightarrow{\rho_N} N$$

is surjective. Inspired by this point of view, we define cogenerator as follows.

Definition 2.4.11. Let (C, Δ, ϵ) be a graded coalgebra, and (M, ρ_M) be a graded comodule over C. A **cogenerator** of M is a graded vector space V, together with a degree-preserving linear map $\pi_V : M \to V$, such that the composition

$$p_V := (\pi_V \otimes \mathrm{id}_C) \circ \phi_M : M \xrightarrow{\phi_M} M \otimes C \xrightarrow{\pi_V \otimes \mathrm{id}_C} V \otimes C$$

is injective.

Since the diagram

$$\begin{tikzcd}
M \arrow[r, "\phi_M"'] \arrow[rr, bend left, "p_V"] \arrow[d, "\phi_M"'] & M \otimes C \arrow[r, "\pi_V \otimes \mathrm{id}_C"'] \arrow[d, "\mathrm{id}_M \otimes \Delta"] & V \otimes C \arrow[d, "\mathrm{id}_V \otimes \Delta"] \\
M \otimes C \arrow[r, "\phi_M \otimes \mathrm{id}_C"] \arrow[rr, bend right, "p_V \otimes \mathrm{id}"'] & M \otimes C \otimes C \arrow[r, "\pi_V \otimes \mathrm{id}_C \otimes \mathrm{id}_C"] & V \otimes C \otimes C\,,
\end{tikzcd}$$

commutes, the map $p_V : M \to V \otimes C$ is a morphism of comodules. A cogenerator π_V is said to be **free** if p_V is an isomorphism of comodules.

Proposition 2.4.12. *Let (M, ρ_M) and (N, ρ_N) be graded comodules over C, and $\pi_V : N \to V$ be a cogenerator of N. Then the pushforward map*

$$\pi_{V*} : \mathrm{coHom}_C(M, N) \to \mathrm{Hom}(M, V)$$

is an embedding of graded vector spaces. Moreover, if π_V is a free cogenerator, then π_{V*} is an isomorphism of graded vector spaces.

Proof. Let $\Psi_1, \Psi_2 \in \mathrm{coHom}_C(M, N)$ be comodule morphisms such that $\pi_{V*}(\Psi_1) = \pi_{V*}(\Psi_2)$. Then, by (2.6), one can show that

$$p_V \circ \Psi_1 = p_V \circ \Psi_2.$$

Since p_V is injective, we have $\Psi_1 = \Psi_2$.

Assume π_V is a free cogenerator. Since p_V is an isomorphism of comodules, it suffices to verify the case

$$N = V \otimes C, \qquad \pi_V = \mu_{V,\mathbb{K}} \circ (\mathrm{id}_V \otimes \epsilon) : V \otimes C \to V.$$

For each $f \in \mathrm{Hom}(M, V)$, it straightforward to show that the map

$$\Psi_f := (f \otimes \mathrm{id}_C) \circ \phi_M : M \to V \otimes C,$$

is a morphism of comodules such that $\pi_V \circ \Psi_f = f$. Thus, the proof is completed. $\square$

Example 2.4.13. Let V and W be a graded vector spaces. Then the pair $(V \otimes SW, \mathrm{id}_V \otimes \Delta)$ is a graded comodule over SW. The projection $\mathrm{pr} : V \otimes SW \twoheadrightarrow V \otimes S^0W \cong V$ is a free cogenerator, because the composition

$$V \otimes SW \xrightarrow{\mathrm{id}_{\mathcal{U}\mathfrak{g}} \otimes \Delta} V \otimes SW \otimes SW \xrightarrow{\mathrm{pr} \otimes \mathrm{id}_{SW}} V \otimes SW$$

is the identity map. Thus, by Proposition 2.4.12, the pushforward map

$$\mathrm{pr}_* : \mathrm{coHom}_{SW}(V \otimes SW, V \otimes SW) \to \mathrm{Hom}(V \otimes SW, V)$$

is an isomorphism of graded vector spaces. In fact, for $f \in \mathrm{Hom}(V \otimes SW, V)$, the composition

$$\Psi_f : V \otimes SW \xrightarrow{\mathrm{id}_V \otimes \Delta} V \otimes SW \otimes SW \xrightarrow{f \otimes \mathrm{id}_{SW}} V \otimes SW, \tag{2.7}$$

is the comodule morphism such that $\mathrm{pr}_*(\Psi_f) = f$.

2.5 Hochschild complexes and tensor coalgebras

We recall Getzler's construction of Hochschild complexes in [19] and show an isomorphism between the version here and our version in Section 4.1. For a dg algebra A, we consider Getzler's formulas as the natural formulas on $A[1]$, and ours are the formulas on A obtained by Getzler's formulas composed with proper degree-shifting maps.

2.5.1 Tensor coalgebras

Let V be a graded vector space. The **tensor coalgebra** $(TV, \Delta_T, \epsilon_T)$ over V is the graded vector space $TV = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ together with the counit $\epsilon_T = \mathrm{pr} : TV \twoheadrightarrow V^{\otimes 0} = \mathbb{K}$ and the coproduct $\Delta_T : TV \to TV \otimes TV$,

$$\Delta_T(v_1 \otimes \cdots \otimes v_n) = 1 \otimes (v_1 \otimes \cdots \otimes v_n) + (v_1 \otimes \cdots \otimes v_n) \otimes 1 \\ + \sum_{i=1}^{n-1} \big(v_1 \otimes \cdots \otimes v_i\big) \otimes \big(v_{i+1} \otimes \cdots \otimes v_n\big).$$

Let $\mathrm{pr}_V : TV \twoheadrightarrow V$ be the canonical projection. By imitating the techniques of cogenerators in Section 2.4.4, one can show that a coderivation $D \in \mathrm{coDer}(TV)$ is uniquely determined by $\mathrm{pr}_V \circ D \in \mathrm{Hom}(TV, V)$. In fact, for $q_k \in \mathrm{Hom}(V^{\otimes k}, V)$, the map $q = \sum_k q_k \in \mathrm{Hom}(TV, V)$ determines a coderivation D_q by the formulas

$$D_q|_{V^{\otimes n}} = \sum_{i+j+k=n} \mathrm{id}^{\otimes i} \otimes q_k \otimes \mathrm{id}^{\otimes j} : V^{\otimes n} \to V^{\otimes n-k+1}.$$

Since the space of coderivations with the graded commutator is a graded Lie algebra, the space $\mathrm{Hom}(TV, V)$ is also equipped with a Lie bracket

$$[f, g] := \mathrm{pr}_V \circ (D_f \circ D_g - (-1)^{|f||g|} D_g \circ D_f)$$

where $f, g \in \mathrm{Hom}(TV, V)$ are arbitrary homogeneous maps.

Tensor coalgebra over a shifted dg algebra

Let A be a dg algebra equipped with differential d_A and multiplication μ_A. Following [19], we denote

$$m_1(\mathfrak{s}a_1) := \mathfrak{s}d_A(a_1), \qquad m_2(\mathfrak{s}a_1, \mathfrak{s}a_2) := (-1)^{|a_1|}\mathfrak{s}\mu_A(a_1, a_2),$$

where $a_1, a_2 \in A$, and $\mathfrak{s} : A \to A[1]$ is the degree-shifting map of degree -1. Let $D_m \in \mathrm{coDer}(T(A[1]))$ be the coderivation generated by $m := m_1 + m_2 \in \mathrm{Hom}^1(T(A[1]), A[1])$. Since the dg algebra axioms

$$d_A^2 = 0, \qquad d_A \circ \mu_A = \mu_A \circ (d_A \otimes \mathrm{id} + \mathrm{id} \otimes d_A), \qquad \mu_A \circ (\mu_A \otimes \mathrm{id}) = \mu_A \circ (\mathrm{id} \otimes \mu_A)$$

are equivalent to

$$\mathrm{pr}_{A[1]} \circ [D_m, D_m]\big|_{A[1]} = 0, \quad \mathrm{pr}_{A[1]} \circ [D_m, D_m]\big|_{A[1]^{\otimes 2}} = 0, \quad \mathrm{pr}_{A[1]} \circ [D_m, D_m]\big|_{A[1]^{\otimes 3}} = 0,$$

respectively, one has the equation

$$[D_m, D_m] = 0.$$

Therefore, we have

Proposition 2.5.1. *The triple* $\big(\mathrm{Hom}(T(A[1]), A[1]), [m, -], [-, -]\big)$ *is a dg Lie algebra.*

Let $M_1 = [m, -]$, and $M_2 : \mathrm{Hom}(T(A[1]), A[1])^{\otimes 2} \to \mathrm{Hom}(T(A[1]), A[1])$ be the operation

$$M_2(D_1, D_2) := m_2 \circ (D_1 \otimes D_2) \circ \Delta_T$$

of degree one. One can show that

$$[D_M, D_M] = 0, \tag{2.8}$$

where $M = M_1 + M_2 \in \mathrm{Hom}\big(T\,\mathrm{Hom}(T(A[1]), A[1]), T\,\mathrm{Hom}(T(A[1]), A[1])\big)$. See [19, Proposition 1.7].

Let $\widehat{d_{\mathcal{H}}} : \mathrm{Hom}(T(A[1]), A) \to \mathrm{Hom}(T(A[1]), A)$ and $\widehat{\cup} : \mathrm{Hom}(T(A[1]), A)^{\otimes 2} \to \mathrm{Hom}(T(A[1]), A)$ be the unique maps satisfying the equations

$$M_1 \circ \mathfrak{s} = \mathfrak{s} \circ \widehat{d_{\mathcal{H}}}, \qquad M_2 \circ (\mathfrak{s} \otimes \mathfrak{s}) = \mathfrak{s} \circ \widehat{\cup},$$

where $\mathfrak{s} : \mathrm{Hom}(T(A[1]), A) \to \mathrm{Hom}(T(A[1]), A[1])$ is the degree-shifting map. By (2.8), we have the following

Proposition 2.5.2. *The triple* $\big(\mathrm{Hom}(T(A[1]), A), \widehat{d_{\mathcal{H}}}, \widehat{\cup}\big)$ *is a dg algebra.*

Remark 2.5.3. More generally, if A is an A_∞ algebra, then so is $\mathrm{Hom}(T(A[1]), A)$. The construction is closely related to the braces operations on $\mathrm{Hom}(T(A[1]), A[1])$. See [56, 20, 19].

2.5.2 Hochschild cochains and coderivations

Let (A, d_A, μ_A) be a dg algebra. Let

$$\mathsf{déc} : \mathrm{Hom}^r(A^{\otimes p}, A) \to \mathrm{Hom}^{p+r}(A[1]^{\otimes p}, A)$$

be the décalage map defined as

$$\mathsf{déc}(f)(\mathfrak{s}a_1 \otimes \cdots \otimes \mathfrak{s}a_p) = (-1)^{\sum_i (p-i)|a_i|} f(a_1 \otimes \cdots \otimes a_p)$$

for $f \in \mathrm{Hom}^r(A^{\otimes p}, A)$ and $a_1, \cdots, a_n \in A$. In other words, the map $\mathsf{déc}(f) \in \mathrm{Hom}^{p+r}(A[1]^{\otimes p}, A)$ is the unique map such that the diagram

$$\begin{tikzcd} A[1]^{\otimes p} \arrow[r, "\mathsf{déc}(f)"] & A \\ A^{\otimes p} \arrow[u, "\mathfrak{s}^{\otimes p}"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. Note that $m_1 = \mathfrak{s} \circ \mathsf{déc}(d_A)$ and $m_2 = \mathfrak{s} \circ \mathsf{déc}(\mu_A)$.

Proposition 2.5.4. *The map*

$$\mathfrak{s} \circ \mathsf{déc} \circ \mathfrak{s}^{-1} : (\mathrm{Hoch}^\bullet_\oplus(A)[1], [\![-,-]\!]) \to (\mathrm{Hom}(T(A[1]), A[1]), [-,-])$$

is an embedding of graded Lie algebras. In particular,

$$\mathsf{déc} \circ (d_{\mathcal{H}} + \partial_A) = \widehat{d_{\mathcal{H}}} \circ \mathsf{déc}.$$

Proof. Let $f \in \mathrm{Hoch}^{p_1,r_1}(A)$ and $g \in \mathrm{Hoch}^{p_2,r_2}(A)$. Since

$$\begin{aligned}
&(\mathfrak{s}\,\mathsf{déc}\,f) \circ \left(\mathrm{id}^{\otimes i-1} \otimes (\mathfrak{s}\,\mathsf{déc}\,g) \otimes \mathrm{id}^{\otimes p_1-i}\right) \circ \mathfrak{s}^{\otimes p_1+p_2-1} \\
&\quad = (-1)^{(i-1)(p_2+r_2-1)}(\mathfrak{s}\,\mathsf{déc}\,f) \circ \left(\mathfrak{s}^{\otimes i-1} \otimes (\mathfrak{s}\,\mathsf{déc}\,g \circ \mathfrak{s}^{\otimes p_2}) \otimes \mathfrak{s}^{\otimes p_1-i}\right) \\
&\quad = (-1)^{(i-1)(p_2+r_2-1)+(p_1-i)r_2}(\mathfrak{s}\,\mathsf{déc}\,f) \circ \mathfrak{s}^{\otimes p_1} \circ (\mathrm{id}^{\otimes i-1} \otimes g \otimes \mathrm{id}^{\otimes p_1-i}) \\
&\quad = (-1)^{(i-1)(p_2-1)+(p_1-1)r_2}\,\mathfrak{s} \circ f \circ (\mathrm{id}^{\otimes i-1} \otimes g \otimes \mathrm{id}^{\otimes p_1-i}),
\end{aligned}$$

it follows from (4.5) that

$$[\mathfrak{s}\,\mathsf{déc}\,f, \mathfrak{s}\,\mathsf{déc}\,g] \circ \mathfrak{s}^{\otimes p+s+1} = \mathfrak{s} \circ [\![f, g]\!],$$

which implies the assertion. $\square$

Proposition 2.5.5. *The map*

$$\mathsf{déc} : (\mathrm{Hoch}^{\bullet}_{\oplus}(A), d_{\mathcal{H}} + \partial_A, \cup\,) \to (\mathrm{Hom}(T(A[1]), A), \widehat{d_{\mathcal{H}}}, \widehat{\cup}\,)$$

is an embedding of dg algebras.

Proof. Let $f \in \mathrm{Hoch}^{p_1,r_1}(A)$ and $g \in \mathrm{Hoch}^{p_2,r_2}(A)$. Since

$$\begin{aligned}
\mathfrak{s}(\mathsf{déc}\,f \mathbin{\widehat{\cup}} \mathsf{déc}\,g) &= M_2 \circ (\mathfrak{s} \otimes \mathfrak{s})(\mathsf{déc}\,f \otimes \mathsf{déc}\,g) \\
&= m_2 \circ (\mathfrak{s} \otimes \mathfrak{s}) \circ (\mathsf{déc}\,f \otimes \mathsf{déc}\,g) \circ \Delta_T \\
&= \mathfrak{s} \circ \mu_A \circ (\mathsf{déc}\,f \otimes \mathsf{déc}\,g) \circ \Delta_T,
\end{aligned}$$

we have

$$\begin{aligned}
(\mathsf{déc}\,f \mathbin{\widehat{\cup}} \mathsf{déc}\,g) \circ (\mathfrak{s}^{\otimes p_1+p_2}) &= \mu_A \circ (\mathsf{déc}\,f \otimes \mathsf{déc}\,g) \circ \Delta_{TA[1]} \circ (\mathfrak{s}^{\otimes p_1+p_2}) \\
&= \mu_A \circ (\mathsf{déc}\,f \otimes \mathsf{déc}\,g) \circ (\mathfrak{s}^{\otimes p_1} \otimes \mathfrak{s}^{\otimes p_2}) \circ \Delta_{TA} \\
&= (-1)^{(p_2+r_2)p_1}\mu_A \circ (f \otimes g) \circ \Delta_{TA},
\end{aligned}$$

where $\Delta_{TA[1]}$ is the coproduct on $TA[1]$, and Δ_{TA} is the coproduct on TA. Thus, by comparing the above formula with (4.3), we conclude

$$(\mathsf{déc}\,f \mathbin{\widehat{\cup}} \mathsf{déc}\,g) \circ (\mathfrak{s}^{\otimes p_1+p_2}) = f \cup g,$$

which implies the assertion. $\square$

Chapter 3

Formal exponential map of dg manifolds

3.1 dg coalgebras associated with dg manifolds

Any dg manifold $(\mathcal{M}, Q)$ determines a pair of dg coalgebras over the dg ring $(C^\infty(\mathcal{M}), Q)$, namely $\mathcal{D}(\mathcal{M})$ and $\Gamma\big(S(T_\mathcal{M})\big)$. Below we will briefly describe these dg coalgebra structures. In the sequel, unless specified otherwise, we will always identify $(\mathcal{R}, d_\mathcal{R}) \cong (C^\infty(\mathcal{M}), Q)$.

First, let us consider the dg coalgebra structure on the left $\mathcal{R}$-module $\mathcal{D}(\mathcal{M})$ of differential operators on $\mathcal{M}$.

The comultiplication

$$\Delta : \mathcal{D}(\mathcal{M}) \to \mathcal{D}(\mathcal{M}) \otimes_\mathcal{R} \mathcal{D}(\mathcal{M}) \tag{3.1}$$

is defined by

$$(\Delta D)(f \otimes g) = D(f \cdot g),$$

where $f, g \in C^\infty(\mathcal{M})$ and $D \in \mathcal{D}(\mathcal{M})$.

The differential $\mathcal{L}_Q : \mathcal{D}(\mathcal{M}) \to \mathcal{D}(\mathcal{M})$ is defined as the commutator with Q, which is also the Lie derivative along the homological vector field Q:

$$\mathcal{L}_Q(D) = [\![Q, D]\!] = Q \cdot D - (-1)^{|D|} D \cdot Q \tag{3.2}$$

for any $D \in \mathcal{D}(\mathcal{M})$, where $[\![-, -]\!]$ denotes the commutator on $\mathcal{D}(\mathcal{M})$.

The induced differential on $\mathcal{D}(\mathcal{M}) \otimes_\mathcal{R} \mathcal{D}(\mathcal{M})$ is again the Lie derivative $\mathcal{L}_Q$, which coincides with $[\![Q, -]\!]$, with $[\![-, -]\!]$ being the Gerstenhaber bracket on polydifferential operators on $\mathcal{M}$.

The counit map
$$\epsilon : \mathcal{D}(\mathcal{M}) \to C^\infty(\mathcal{M}) \tag{3.3}$$
is the canonical projection, which evaluates a differential operator D on the constant function 1.

Note that $\mathcal{D}(\mathcal{M})$ admits a natural ascending filtration by the order of differential operators
$$C^\infty(\mathcal{M}) = \mathcal{D}^{\leq 0}(\mathcal{M}) \subset \cdots \subset \mathcal{D}^{\leq n}(\mathcal{M}) \subset \cdots$$
where $\mathcal{D}^{\leq n}(\mathcal{M})$ denotes the space of differential operators of order $\leq n$. The following proposition can be easily verified.

Proposition 3.1.1. *For any dg manifold $(\mathcal{M}, Q)$, the space of differential operators $\mathcal{D}(\mathcal{M})$ on $\mathcal{M}$, equipped with the comultiplication Δ, the differential $\mathcal{L}_Q$ and the counit ϵ as in* (3.1), (3.2) *and* (3.3)*, is a filtered dg cocommutative coalgebra over $(C^\infty(\mathcal{M}), Q)$.*

Next we describe the dg coalgebra structure on the left $\mathcal{R}$-module $\Gamma\big(S(T_\mathcal{M})\big)$.

The comultiplication
$$\Delta : \Gamma\big(S(T_\mathcal{M})\big) \to \Gamma\big(S(T_\mathcal{M})\big) \otimes_\mathcal{R} \Gamma\big(S(T_\mathcal{M})\big)$$
is given by
$$\begin{aligned}\Delta(X_1 \odot \cdots \odot X_n) = {} & 1 \otimes (X_1 \odot \cdots \odot X_n) + (X_1 \odot \cdots \odot X_n) \otimes 1 \\ & + \sum_{k=1}^{n-1} \sum_{\sigma \in \mathfrak{S}_k^{n-k}} \varepsilon \cdot (X_{\sigma(1)} \odot \cdots \odot X_{\sigma(k)}) \otimes (X_{\sigma(k+1)} \odot \cdots \odot X_{\sigma(n)}),\end{aligned} \tag{3.4}$$
where $X_1, \cdots, X_n \in \Gamma\big(T_\mathcal{M}\big)$. The symbol $\mathfrak{S}_k^{n-k}$ denotes the set of all $(k, n-k)$-shuffles and the symbol $\varepsilon := \varepsilon(X_1, X_2, \cdots, X_n)$ denotes the Koszul signs arising from the reordering of the homogeneous objects $X_1, X_2, \cdots, X_n$ in each term of the right hand side.

The differential
$$L_Q : \Gamma\big(S(T_\mathcal{M})\big) \to \Gamma\big(S(T_\mathcal{M})\big) \tag{3.5}$$
is the Lie derivative along the homological vector field Q. The induced differential on $\Gamma\big(S(T_\mathcal{M})\big) \otimes_\mathcal{R} \Gamma\big(S(T_\mathcal{M})\big) \cong \Gamma\big(S(T_\mathcal{M}) \otimes S(T_\mathcal{M})\big)$ is again the Lie derivative $\mathcal{L}_Q$.

The counit map
$$\epsilon : \Gamma\big(S(T_\mathcal{M})\big) \to C^\infty(\mathcal{M}) \tag{3.6}$$
is the canonical projection.

Note that $\Gamma\big(S(T_{\mathcal{M}})\big)$ admits a canonical ascending filtration

$$C^\infty(\mathcal{M}) = \Gamma\big(S^{\leq 0}(T_{\mathcal{M}})\big) \subset \cdots \subset \Gamma\big(S^{\leq n}(T_{\mathcal{M}})\big) \subset \cdots .$$

The following proposition is easily verified.

Proposition 3.1.2. *For any dg manifold $(\mathcal{M}, Q)$, the space $\Gamma\big(S(T_{\mathcal{M}})\big)$, equipped with the comultiplication Δ, the differential L_Q and the counit map ϵ as in* (3.4), (3.5) *and* (3.6)*, is a filtered dg cocommutative coalgebra over $(C^\infty(\mathcal{M}), Q)$.*

3.2 Formal exponential map of a dg manifold

Let $\mathcal{M}$ be a finite dimensional graded manifold and ∇ be an affine connection on $\mathcal{M}$. A purely algebraic description of the Poincaré–Birkhoff–Witt map has been extended to the context of $\mathbb{Z}$-*graded* manifolds by Liao–Stiénon [34]. As pointed out in the introduction, for an ordinary smooth manifold, the PBW map is a formal exponential map. In the same way, one can think of the PBW map of a $\mathbb{Z}$-graded manifold as an induced formal exponential map of 'the virtual exponential map'

$$\exp^\nabla : T_{\mathcal{M}} \to \mathcal{M} \times \mathcal{M} \tag{3.7}$$

by taking fiberwise ∞-jets.

Recall that the Poincaré–Birkhoff–Witt map

$$\operatorname{pbw}^\nabla : \Gamma\big(S(T_{\mathcal{M}})\big) \to \mathcal{D}(\mathcal{M}) \tag{3.8}$$

is defined by the inductive formula [34]:

$$\begin{aligned} \operatorname{pbw}^\nabla(f) &= f, \quad \forall f \in C^\infty(\mathcal{M}); \\ \operatorname{pbw}^\nabla(X) &= X, \quad \forall X \in \mathfrak{X}(\mathcal{M}); \end{aligned} \tag{3.9}$$

and

$$\operatorname{pbw}^\nabla(X_1 \odot \cdots \odot X_n) = \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \left(X_k \operatorname{pbw}^\nabla(\boldsymbol{X}^{\{k\}}) - \operatorname{pbw}^\nabla(\nabla_{X_k} \boldsymbol{X}^{\{k\}}) \right), \tag{3.10}$$

where $\boldsymbol{X} = X_1 \odot \cdots \odot X_n \in \Gamma\big(S^n(T_{\mathcal{M}})\big)$ for homogeneous vector fields $X_1, \cdots, X_n \in \mathfrak{X}(\mathcal{M})$ and $\varepsilon_k = (-1)^{|X_k|(|X_1| + \cdots + |X_{k-1}|)}$ is the Koszul sign.

Theorem 3.2.1 ([34])**.** *The map $\operatorname{pbw}^\nabla$ is an isomorphism of graded coalgebras from $\Gamma\big(S(T_{\mathcal{M}})\big)$ to $\mathcal{D}(\mathcal{M})$ over $C^\infty(\mathcal{M})$.*

Now, we assume there exists a homological vector field Q on $\mathcal{M}$ so that $(\mathcal{M}, Q)$ is a dg manifold. Then, both $\Gamma(S(T_{\mathcal{M}}))$ and $\mathcal{D}(\mathcal{M})$ in (3.8) are dg coalgebras over $(C^\infty(\mathcal{M}), Q)$, according to Propositions 3.1.1 and 3.1.2. We think of the elements of the dg coalgebra $\big(\Gamma\big(S(T_{\mathcal{M}})\big), L_Q\big)$ as fiberwise dg distributions on the dg vector bundle $\pi : T_{\mathcal{M}} \to \mathcal{M}$ with support the zero section — the homological vector field on $T_{\mathcal{M}}$ is the complete lift $\hat{Q}$ of the homological vector field Q on $\mathcal{M}$ [42, 52]. Likewise, we think of the elements of the dg coalgebra $\big(\mathcal{D}(\mathcal{M}), \mathcal{L}_Q\big)$ as fiberwise dg distributions on the dg fiber bundle $\mathrm{pr}_1 : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ with support the diagonal Δ — the homological vector field on $\mathcal{M} \times \mathcal{M}$ is (Q, Q). On the level of fiberwise ∞-jets, the fact that the virtual exponential map (3.7) is a map of dg manifolds is equivalent to the map $\mathrm{pbw}^\nabla : \big(\Gamma\big(S(T_{\mathcal{M}})\big), L_Q\big) \to \big(\mathcal{D}(\mathcal{M}), \mathcal{L}_Q\big)$ being an isomorphism of dg coalgebras over $(C^\infty(\mathcal{M}), Q)$. This consideration leads to the following

Theorem 3.2.2. *Let $(\mathcal{M}, Q)$ be a dg manifold. The Atiyah class $\alpha_{(\mathcal{M},Q)}$ vanishes if and only if there exists a torsion-free affine connection ∇ on $\mathcal{M}$ such that*

$$\mathrm{pbw}^\nabla : \big(\Gamma\big(S(T_{\mathcal{M}})\big), L_Q\big) \to \big(\mathcal{D}(\mathcal{M}), \mathcal{L}_Q\big)$$

is an isomorphism of dg coalgebras over $(C^\infty(\mathcal{M}), Q)$.

Remark 3.2.3. A similar theorem in the same spirit concerning the Atiyah class of Lie pairs was obtained in [31, Theorem 5.10]. It would be interesting to establish a result that encompasses both [31, Theorem 5.10] and Theorem 3.2.2 under a unified framework.

In order to prove Theorem 3.2.2, we first introduce a linear map

$$C^\nabla : \Gamma\big(S(T_{\mathcal{M}})\big) \to \mathcal{D}(\mathcal{M})$$

by

$$C^\nabla := \mathcal{L}_Q \circ \mathrm{pbw}^\nabla - \mathrm{pbw}^\nabla \circ L_Q. \tag{3.11}$$

One can easily check that C^∇ is a $C^\infty(\mathcal{M})$-linear map of degree $+1$. Moreover, for $n \geq 0$,

$$C^\nabla\big(\Gamma\big(S^{\leq n}(T_{\mathcal{M}})\big)\big) \subseteq \mathcal{D}^{\leq n-1}(\mathcal{M}).$$

The following proposition indicates that C^∇ can be completely determined by a recursive formula.

Proposition 3.2.4. *Let $(\mathcal{M}, Q)$ be a dg manifold, and ∇ a torsion-free affine connection on $\mathcal{M}$. Then the map C^∇ satisfies*

$$C^\nabla(f) = 0; \tag{3.12}$$

$$C^\nabla(X) = 0; \tag{3.13}$$

$$C^\nabla(X \odot Y) = -\operatorname{At}^\nabla_{(\mathcal{M},Q)}(X, Y), \tag{3.14}$$

for all $f \in C^\infty(\mathcal{M})$, $X, Y \in \mathfrak{X}(\mathcal{M})$, and, for $n \geq 3$, it satisfies the recursive formula

$$C^\nabla(\boldsymbol{X}) = \frac{1}{n}\sum_{k=1}^{n} \varepsilon_k \left[(-1)^{|X_k|} X_k \cdot C^\nabla(\boldsymbol{X}^{\{k\}}) - C^\nabla(\nabla_{X_k}\boldsymbol{X}^{\{k\}})\right] \\ - \frac{2}{n}\sum_{i<j} \varepsilon_i\varepsilon_j (-1)^{|X_i|\cdot|X_j|} \operatorname{pbw}^\nabla \left(\operatorname{At}^\nabla_{(\mathcal{M},Q)}(X_i, X_j) \odot \boldsymbol{X}^{\{i,j\}}\right), \quad (3.15)$$

where $\boldsymbol{X} = X_1 \odot \cdots \odot X_n \in \Gamma\big(S^n(T_\mathcal{M})\big)$ denotes the symmetric tensor product of n homogeneous vector fields $X_1, \cdots, X_n \in \mathfrak{X}(\mathcal{M})$; $\boldsymbol{X}^{\{k\}} = X_1 \odot \cdots \odot \widehat{X}_k \odot \cdots \odot X_n$ for any $1 \leq k \leq n$; $\boldsymbol{X}^{\{i,j\}} = X_1 \odot \cdots \odot \widehat{X}_i \odot \cdots \odot \widehat{X}_j \odot \cdots \odot X_n$ for any $1 \leq i < j \leq n$; and $\varepsilon_k = (-1)^{|X_k|(|X_1|+\cdots+|X_{k-1}|)}$ is the Koszul sign arising from the reordering $X_1, X_2, \cdots, X_n \mapsto X_k, X_1, X_2, \cdots, X_{k-1}, X_{k+1}, \cdots, X_n$.

We now prove Theorem 3.2.2 based on Proposition 3.2.4.

Proof of Theorem 3.2.2. Observe that, according to Proposition 2.2.4, $\alpha_{(\mathcal{M},Q)}$ vanishes if and only if there exists an affine connection ∇' for which $\operatorname{At}^{\nabla'}_{(\mathcal{M},Q)} = 0$. It follows from $\mathcal{L}_Q(\nabla') = \operatorname{At}^{\nabla'}_{(\mathcal{M},Q)} = 0$ that $\mathcal{L}_Q(T^{\nabla'}) = 0$. Therefore, if the Atiyah cocycle of the affine connection ∇' vanishes, then so does the Atiyah cocycle of the torsion-free connection $\nabla = \nabla' - \frac{1}{2}T^{\nabla'}$:

$$\operatorname{At}^\nabla_{(\mathcal{M},Q)} = \mathcal{L}_Q(\nabla) = \mathcal{L}_Q(\nabla' - \frac{1}{2}T^{\nabla'}) = \operatorname{At}^{\nabla'}_{(\mathcal{M},Q)} - \frac{1}{2}\mathcal{L}_Q(T^{\nabla'}) = 0.$$

Thus, it suffices to prove that $C^\nabla = 0$ if and only if $\operatorname{At}^\nabla_{(\mathcal{M},Q)} = 0$.

Assume that $C^\nabla = 0$. By Proposition 3.2.4, we have

$$\operatorname{At}^\nabla_{(\mathcal{M},Q)}(X, Y) = -C^\nabla(X \odot Y) = 0$$

for all $X, Y \in \mathfrak{X}(\mathcal{M})$.

Conversely, assume that $\operatorname{At}^\nabla_{(\mathcal{M},Q)} = 0$. Then we have $C^\nabla(X \odot Y) = 0$ by Proposition 3.2.4. Hence $C^\nabla(\boldsymbol{Y}) = 0$ for all $\boldsymbol{Y} \in \Gamma\big(S^{\leq 2}(T_\mathcal{M})\big)$. Moreover, Equation (3.15) can be written as

$$C^\nabla(\boldsymbol{X}) = \frac{1}{n}\sum_{k=1}^{n} \varepsilon_k \left[(-1)^{|X_k|} X_k \cdot C^\nabla(\boldsymbol{X}^{\{k\}}) - C^\nabla(\nabla_{X_k}\boldsymbol{X}^{\{k\}})\right]$$

for all $\boldsymbol{X} \in \big(\Gamma\big(S^{\geq 3}(T_\mathcal{M})\big)\big)$. Therefore, $C^\nabla = 0$ by the inductive argument. □

3.2.1 Proof of Proposition 3.2.4

Now we turn to the proof of Proposition 3.2.4. We will divide the proof into several lemmas.

Lemma 3.2.5. *Under the same hypothesis as in Proposition 3.2.4, Equations* (3.12), (3.13) *and* (3.14) *hold.*

Proof. Equations (3.12) and (3.13) follow immediately from Equation (3.9).

To prove Equation (3.14), let $X, Y \in \mathfrak{X}(\mathcal{M})$ be homogeneous vector fields. Since ∇ is torsion-free, we have

$$\nabla_X Y - (-1)^{|X|\cdot|Y|}\nabla_Y X = [X,Y] = XY - YX.$$

It then follows from Equation (3.10) that

$$\mathrm{pbw}^{\nabla}(X \odot Y) = XY - \nabla_X Y.$$

From there, we obtain

$$\mathcal{L}_Q \circ \mathrm{pbw}^{\nabla}(X \odot Y) = [Q,X]Y + (-1)^{|X|}X[Q,Y] - [Q, \nabla_X Y]$$

and

$$\begin{aligned}\mathrm{pbw}^{\nabla} \circ L_Q(X \odot Y) &= \mathrm{pbw}^{\nabla}\left([Q,X] \odot Y + (-1)^{|X|} X \odot [Q,Y]\right) \\ &= \left([Q,X]Y - \nabla_{[Q,X]}Y\right) + (-1)^{|X|}\left(X[Q,Y] - \nabla_X[Q,Y]\right).\end{aligned}$$

As a result, we have

$$\begin{aligned}C^{\nabla}(X \odot Y) &= (\mathcal{L}_Q \circ \mathrm{pbw}^{\nabla} - \mathrm{pbw}^{\nabla} \circ L_Q)(X \odot Y) \\ &= -\left([Q, \nabla_X Y] - \nabla_{[Q,X]}Y - (-1)^{|X|}\nabla_X[Q,Y]\right) \\ &= -\operatorname{At}^{\nabla}_{(\mathcal{M},Q)}(X,Y).\end{aligned}$$ □

In the sequel, we adopt the following notations. For any $\boldsymbol{X} = X_1 \odot \cdots \odot X_n \in \Gamma\big(S^n(T_{\mathcal{M}})\big)$, we write $\boldsymbol{X}^{\{k\}} = X_1 \odot \cdots \odot \widehat{X}_k \odot \cdots \odot X_n$; for $i \neq j$, we write $\boldsymbol{X}^{\{i,j\}} = X_1 \odot \cdots \odot \widehat{X}_i \odot \cdots \odot \widehat{X}_j \odot \cdots \odot X_n$, and for all $1 \leq i \leq n$, $\boldsymbol{X}^{\{i,i\}} = 0$.

Lemma 3.2.6. *Under the same hypothesis as in Proposition 3.2.4, for all* $\boldsymbol{X} = X_1 \odot \cdots \odot X_n \in \Gamma\big(S^n(T_{\mathcal{M}})\big)$ *with* $n \geq 3$*, we have*

$$\begin{aligned}\mathcal{L}_Q \circ \mathrm{pbw}^{\nabla}(\boldsymbol{X}) &= \frac{1}{n}\sum_{k=1}^{n} \varepsilon \cdot [Q, X_k] \cdot \mathrm{pbw}^{\nabla}(\boldsymbol{X}^{\{k\}}) \\ &\quad + \frac{1}{n}\sum_{k=1}^{n}\Big(\varepsilon \cdot X_k \cdot \mathcal{L}_Q\big(\mathrm{pbw}^{\nabla}(\boldsymbol{X}^{\{k\}})\big) - \varepsilon \cdot \mathcal{L}_Q\big(\mathrm{pbw}^{\nabla}(\nabla_{X_k}\boldsymbol{X}^{\{k\}})\big)\Big)\end{aligned}$$

and

$$\begin{aligned}&\mathrm{pbw}^\nabla \circ L_Q(\boldsymbol{X})\\&\quad=\frac{1}{n}\sum_{k=1}^{n}\Big(\varepsilon\cdot[Q,X_k]\cdot\mathrm{pbw}^\nabla(\boldsymbol{X}^{\{k\}})+\varepsilon\cdot X_k\cdot\mathrm{pbw}^\nabla\left(L_Q(\boldsymbol{X}^{\{k\}})\right)\\&\qquad-\varepsilon\cdot\mathrm{pbw}^\nabla\left(L_Q(\nabla_{X_k}\boldsymbol{X}^{\{k\}})\right)\Big)+\frac{1}{n}\sum_{i<j}\varepsilon\cdot\mathrm{pbw}^\nabla\Big(2\,\mathrm{At}^\nabla_{(\mathcal{M},Q)}(X_i,X_j)\odot\boldsymbol{X}^{\{i,j\}}\Big).\end{aligned}$$

In the two equations above and in the proof of the Lemma as well, the symbol $\varepsilon=\varepsilon(Q,X_1,\cdots,X_n)$ *denotes the Koszul signs arising from the reordering of the homogeneous objects* $Q,X_1,\cdots,X_n$ *in each term of the right hand sides.*

Proof. The formula for $\mathcal{L}_Q\circ\mathrm{pbw}^\nabla(\boldsymbol{X})$ is immediate from Equation (3.10).

Next, we will compute $\mathrm{pbw}^\nabla\circ L_Q(\boldsymbol{X})$. Since $L_Q(\boldsymbol{X})=\sum_{k=1}^n\varepsilon\cdot([Q,X_k]\odot\boldsymbol{X}^{\{k\}})$, applying Equation (3.10), we have

$$\mathrm{pbw}^\nabla\circ L_Q(\boldsymbol{X})=\frac{1}{n}\left(\mathcal{A}^1-\mathcal{A}^2+\mathcal{B}-\mathcal{C}\right),$$

where

$$\mathcal{A}^1:=\sum_{k=1}^{n}\varepsilon\cdot[Q,X_k]\cdot\mathrm{pbw}^\nabla(\boldsymbol{X}^{\{k\}}),\tag{3.16}$$

$$\mathcal{A}^2:=\sum_{k=1}^{n}\varepsilon\cdot\mathrm{pbw}^\nabla(\nabla_{[Q,X_k]}\boldsymbol{X}^{\{k\}}),$$

$$\mathcal{B}:=\sum_{k=1}^{n}\sum_{i=1}^{n}\varepsilon\cdot X_i\cdot\mathrm{pbw}^\nabla([Q,X_k]\odot\boldsymbol{X}^{\{i,k\}}),$$

$$\mathcal{C}:=\sum_{k=1}^{n}\sum_{i=1}^{n}\varepsilon\cdot\mathrm{pbw}^\nabla(\nabla_{X_i}([Q,X_k]\odot\boldsymbol{X}^{\{i,k\}}).$$

First, by changing the order of summation, we obtain

$$\begin{aligned}\mathcal{B}&=\sum_{i=1}^{n}\sum_{k=1}^{n}\varepsilon\cdot X_i\cdot\mathrm{pbw}^\nabla\Big([Q,X_k]\odot\boldsymbol{X}^{\{i,k\}}\Big)\\&=\sum_{i=1}^{n}\varepsilon\cdot X_i\cdot\mathrm{pbw}^\nabla\Big(L_Q(\boldsymbol{X}^{\{i\}})\Big).\end{aligned}\tag{3.17}$$

We also can write

$$\mathcal{A}^2 = \sum_{k=1}^{n}\sum_{i=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla}\left((\nabla_{[Q,X_k]} X_i) \odot \boldsymbol{X}^{\{k,i\}} \right)$$
$$= \sum_{k=1}^{n}\sum_{i=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla}\left((\nabla_{[Q,X_i]} X_k) \odot \boldsymbol{X}^{\{i,k\}} \right). \tag{3.18}$$

Now we also have

$$\sum_{k=1}^{n}\sum_{i=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla}\big([Q, X_k] \odot \nabla_{X_i} \boldsymbol{X}^{\{i,k\}} \big)$$
$$= \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla}\big([Q, X_k] \odot \nabla_{X_i} X_j \odot \boldsymbol{X}^{\{i,k,j\}} \big)$$
$$= \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla}\big(\nabla_{X_i} X_j \odot [Q, X_k] \odot \boldsymbol{X}^{\{i,k,j\}} \big)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla}\big(\nabla_{X_i} X_j \odot L_Q \boldsymbol{X}^{\{i,j\}} \big)$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla}\big(\nabla_{X_i} X_k \odot L_Q \boldsymbol{X}^{\{i,k\}} \big).$$

Therefore, it follows that

$$\sum_{k=1}^{n}\sum_{i=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla}\big([Q, \nabla_{X_i} X_k] \odot \boldsymbol{X}^{\{i,k\}} \big) + \sum_{k=1}^{n}\sum_{i=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla}\big([Q, X_k] \odot \nabla_{X_i} \boldsymbol{X}^{\{i,k\}} \big)$$
$$= \sum_{k=1}^{n}\sum_{i=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla}\big([Q, \nabla_{X_i} X_k] \odot \boldsymbol{X}^{\{i,k\}} \big) + \sum_{i=1}^{n}\sum_{k=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla}\big(\nabla_{X_i} X_k \odot L_Q \boldsymbol{X}^{\{i,k\}} \big)$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla} L_Q\big(\nabla_{X_i} X_k \odot \boldsymbol{X}^{\{i,k\}} \big)$$
$$= \sum_{i=1}^{n} \varepsilon \cdot \mathrm{pbw}^{\nabla}\big(L_Q(\nabla_{X_i} X^{\{i\}}) \big). \tag{3.19}$$

Moreover,

$$\mathcal{C} = \sum_{k=1}^{n}\sum_{i=1}^{n} \varepsilon\cdot\mathrm{pbw}^{\nabla}\big((\nabla_{X_i}[Q, X_k]) \odot \boldsymbol{X}^{\{i,k\}} \big) + \sum_{k=1}^{n}\sum_{i=1}^{n} \varepsilon\cdot\mathrm{pbw}^{\nabla}\big([Q, X_k] \odot \nabla_{X_i} \boldsymbol{X}^{\{i,k\}} \big). \tag{3.20}$$

Then by combining Equations (3.18), (3.19) and (3.20) and using the definition of Atiyah cocycles, we obtain

$$
\begin{aligned}
\mathcal{A}^2 + \mathcal{C} &= \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \operatorname{pbw}^\nabla \left(\left([Q, \nabla_{X_i} X_k] - \operatorname{At}^\nabla_{(\mathcal{M},Q)}(X_i, X_k) \right) \odot \boldsymbol{X}^{\{i,k\}} \right) \\
&\qquad + \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \operatorname{pbw}^\nabla \left([Q, X_k] \odot \nabla_{X_i} \boldsymbol{X}^{\{i,k\}} \right) \\
&= \sum_{i=1}^{n} \varepsilon \cdot \operatorname{pbw}^\nabla \left(L_Q(\nabla_{X_i} X^{\{i\}}) \right) - \sum_{i<j} \varepsilon \cdot \operatorname{pbw}^\nabla \left(2 \operatorname{At}^\nabla_{(\mathcal{M},Q)}(X_i, X_j) \odot \boldsymbol{X}^{\{i,j\}} \right).
\end{aligned}
\tag{3.21}
$$

The conclusion thus follows from Equations (3.16), (3.17), and (3.21). □

Proof of Proposition 3.2.4. Equations (3.12), (3.13) and (3.14) have been proved in Lemma 3.2.5. It remains to prove Equation (3.15). According to Lemma 3.2.6, we have

$$
\begin{aligned}
&\mathcal{L}_Q \circ \operatorname{pbw}^\nabla(\boldsymbol{X}) - \operatorname{pbw}^\nabla \circ L_Q(\boldsymbol{X}) \\
&\quad = \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k (-1)^{|X_k|} X_k \cdot (\mathcal{L}_Q \circ \operatorname{pbw}^\nabla - \operatorname{pbw}^\nabla \circ L_Q)(\boldsymbol{X}^{\{k\}}) \\
&\qquad - \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \big(\mathcal{L}_Q \circ \operatorname{pbw}^\nabla - \operatorname{pbw}^\nabla \circ L_Q \big)(\nabla_{X_k} \boldsymbol{X}^{\{k\}}) \\
&\qquad - \frac{1}{n} \sum_{i<j} \varepsilon_i \varepsilon_j (-1)^{|X_i| \cdot |X_j|} \operatorname{pbw}^\nabla \left(2 \operatorname{At}^\nabla_{(\mathcal{M},Q)}(X_i, X_j) \odot \boldsymbol{X}^{\{i,j\}} \right).
\end{aligned}
$$

This concludes the proof of Proposition 3.2.4. □

3.3 Atiyah class and homotopy Lie algebras

This section is devoted to the study of homotopy Lie algebras associated with the Atiyah class of dg manifolds.

3.3.1 Kapranov $L_\infty[1]$ algebras of dg manifolds

The Atiyah class of a holomorphic vector bundle is closely related to $L_\infty[1]$ algebras as shown by the pioneer work of Kapranov [23, 45, 46]. These $L_\infty[1]$

algebras play an important role in derived geometry [12, 39, 45] and construction of Rozansky–Witten invariants [23, 27, 46, 47, 55].

In this section, following Kapranov [23], we show that the Atiyah class of a dg manifold is related to $L_\infty[1]$ algebras in a similar fashion. We refer to [11, Sections 4 and 5] for the interpretation in terms of derived category.

Let $(\mathcal{M}, Q)$ be a dg manifold and let ∇ be an affine connection on $\mathcal{M}$. The Lie derivative $\mathcal{L}_Q$ along the homological vector field Q is a degree $+1$ coderivation of the dg coalgebra $\mathcal{D}(\mathcal{M})$ over $(C^\infty(\mathcal{M}), Q)$ according to Proposition 3.1.1.

Transferring $\mathcal{L}_Q$ from $\mathcal{D}(\mathcal{M})$ to $\Gamma\big(S(T_\mathcal{M})\big)$ by the graded coalgebra isomorphism pbw^∇ (3.8), we obtain a degree $+1$ coderivation δ^∇ of $\Gamma\big(S(T_\mathcal{M})\big)$:

$$\delta^\nabla := (\mathrm{pbw}^\nabla)^{-1} \circ \mathcal{L}_Q \circ \mathrm{pbw}^\nabla \,. \tag{3.22}$$

Therefore

$$\Big(\Gamma\big(S(T_\mathcal{M})\big), \delta^\nabla\Big) \tag{3.23}$$

is a dg coalgebra over the dg ring $(C^\infty(\mathcal{M}), Q)$.

Finally, dualizing δ^∇ over $(C^\infty(\mathcal{M}), Q)$, we obtain a degree $+1$ derivation:

$$D^\nabla : \Gamma\big(\widehat{S}(T^\vee_\mathcal{M})\big) \to \Gamma\big(\widehat{S}(T^\vee_\mathcal{M})\big) \tag{3.24}$$

Here we used the identification $\Gamma\big(\widehat{S}(T^\vee_\mathcal{M})\big) \cong \mathrm{Hom}_{C^\infty(\mathcal{M})}(\Gamma\big(S(T_\mathcal{M})\big), C^\infty(\mathcal{M}))$.

The following theorem was first announced in [42], but a proof was omitted. We will present a complete proof below.

Theorem 3.3.1. *Let $(\mathcal{M}, Q)$ be a dg manifold, and let ∇ be a torsion-free affine connection on $\mathcal{M}$.*

(i) The operator D^∇ is a derivation of degree $+1$ of the graded algebra $\Gamma\big(\widehat{S}(T^\vee_\mathcal{M})\big)$ satisfying $(D^\nabla)^2 = 0$. Thus $\Big(\Gamma\big(\widehat{S}(T^\vee_\mathcal{M})\big), D^\nabla\Big)$ is a dg algebra.

(ii) There exists a sequence of degree $+1$ sections $R_k \in \Gamma\big(S^k(T^\vee_\mathcal{M}) \otimes T_\mathcal{M}\big)$, $k \geqslant 2$ whose first term R_2 equals to $-\operatorname{At}^\nabla_{(\mathcal{M},Q)}$, such that

$$D^\nabla = \mathcal{L}_Q + \sum_{k=2}^\infty \widetilde{R_k},$$

where each $\widetilde{R_k} : \Gamma\big(\widehat{S}(T^\vee_\mathcal{M})\big) \to \Gamma\big(\widehat{S}(T^\vee_\mathcal{M})\big)$ denotes the $\mathcal{R}$-linear degree $+1$ derivation corresponding to R_k.

(iii) Different choices of torsion-free affine connections ∇ induce isomorphic dg algebras $\Big(\Gamma\big(\widehat{S}(T^\vee_\mathcal{M})\big), D^\nabla\Big)$.

Remark 3.3.2. The graded algebra $\Gamma\big(\widehat{S}(T^\vee_{\mathcal{M}})\big)$ can be thought of as the graded algebra of functions on a graded manifold $\widetilde{T}_{\mathcal{M}}$ with support M and D^∇ as a homological vector field on $\widetilde{T}_{\mathcal{M}}$. Note that $T_{\mathcal{M}}$ and $\widetilde{T}_{\mathcal{M}}$ are different graded manifolds: the support of $T_{\mathcal{M}}$ is T_M while the support of $\widetilde{T}_{\mathcal{M}}$ is M.

Before we prove this theorem, we need to recall some basic notations.

Recall that given a graded commutative algebra $\mathcal{R}$ and a graded $\mathcal{R}$-module V, the symmetric tensor algebra $(S_{\mathcal{R}}(V), \mu)$ over $\mathcal{R}$ admits a canonical graded coalgebra structure $\Delta : S_{\mathcal{R}}(V) \to S_{\mathcal{R}}(V) \otimes_{\mathcal{R}} S_{\mathcal{R}}(V)$ defined by [31]

$$\begin{aligned}\Delta(v_1 \odot \cdots \odot v_n) &= 1 \otimes (v_1 \odot \cdots \odot v_n) + (v_1 \odot \cdots \odot v_n) \otimes 1 \\ &\quad + \sum_{k=1}^{n-1} \sum_{\sigma \in \mathfrak{S}_k^{n-k}} \varepsilon \cdot \big(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}\big) \otimes \big(v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)}\big)\end{aligned}$$

for homogeneous elements $v_1, \cdots, v_n \in V$. Here the symbol $\varepsilon = \varepsilon(v_1, v_2, \cdots, v_n)$ denotes the Koszul signs arising from the reordering of the homogeneous objects $v_1, v_2, \cdots, v_n$ in each term of the right hand side.

The following lemma is standard— see, for example, [38, 31].

Lemma 3.3.3. *Let $\mathcal{R}$ be a graded commutative algebra and V be an $\mathcal{R}$-module. There is a natural isomorphism*

$$\mathrm{coDer}_{\mathcal{R}}(S_{\mathcal{R}}(V), S_{\mathcal{R}}(V)) \xrightarrow{\cong} \prod_{k=0}^{\infty} \mathrm{Hom}_{\mathcal{R}}(S^k_{\mathcal{R}}(V), V)$$

as $\mathcal{R}$-modules.

More explicitly, the correspondence between a sequence of maps $\{q_k\}_{k\geq 0}$ with $q_k \in \mathrm{Hom}_{\mathcal{R}}(S^k_{\mathcal{R}}(V), V)$ and a coderivation $Q \in \mathrm{coDer}_{\mathcal{R}}(S_{\mathcal{R}}(V), S_{\mathcal{R}}(V))$ is given by

$$\begin{aligned}Q(v_1 \odot \cdots \odot v_n) &= q_0(1) \odot v_1 \odot \cdots \odot v_n + q_n(v_1 \odot \cdots \odot v_n) \odot 1 \\ &\quad + \sum_{k=1}^{n-1} \sum_{\sigma \in \mathfrak{S}_k^{n-k}} \varepsilon \cdot q_k\big(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}\big) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)}\end{aligned} \tag{3.25}$$

for homogeneous vectors $v_1, \cdots, v_n \in V$.

For a given graded $\mathcal{R}$-coalgebra (C, Δ) and a graded $\mathcal{R}$-algebra (A, μ), the convolution product $\star$ on the graded vector space $\mathrm{Hom}_{\mathcal{R}}(C, A)$ is defined by

$$f \star g = \mu \circ (f \otimes g) \circ \Delta$$

$\forall f, g \in \mathrm{Hom}_{\mathcal{R}}(C, A)$. It is clear that $(\mathrm{Hom}_{\mathcal{R}}(C, A), \star)$ is a graded $\mathcal{R}$-algebra. In particular, since $S_{\mathcal{R}}(V)$ is both a graded coalgebra and a graded algebra, the space of $\mathcal{R}$-linear maps $\mathrm{Hom}_{\mathcal{R}}\left(S_{\mathcal{R}}(V), S_{\mathcal{R}}(V)\right)$ admits a convolution product:

$$(f \star g)(\mathbf{v}) = \sum_{(\mathbf{v})} (-1)^{|g| \cdot \left|\mathbf{v}_{(1)}\right|} f(\mathbf{v}_{(1)}) \odot g(\mathbf{v}_{(2)}), \tag{3.26}$$

where $\mathbf{v} \in S_{\mathcal{R}}(V)$ and $\Delta(\mathbf{v}) = \sum_{(\mathbf{v})} \mathbf{v}_{(1)} \otimes \mathbf{v}_{(2)}$.

Using the above notation (3.26), we may write Equation (3.25) as

$$Q = \sum_{k=0}^{\infty} \left(\bar{q}_k \star \mathrm{id}_{S_{\mathcal{R}}(V)}\right), \tag{3.27}$$

where the map $\bar{q}_k : S_{\mathcal{R}}(V) \to S_{\mathcal{R}}(V)$ is defined by the following commutative diagram:

$$\begin{tikzcd} S_{\mathcal{R}}(V) \arrow[r, "\bar{q}_k"] \arrow[d, two heads, "\mathrm{pr}_k"] & S_{\mathcal{R}}(V) \\ S^k_{\mathcal{R}}(V) \arrow[r, "q_k"] & S^1_{\mathcal{R}}(V). \arrow[u, hook] \end{tikzcd} \tag{3.28}$$

Here $\mathrm{pr}_k : S_{\mathcal{R}}(V) \to S^k_{\mathcal{R}}(V)$ denotes the canonical projection. We write id for $\mathrm{id}_{S_{\mathcal{R}}(V)}$ below if there is no confusion. We are now ready to give a detailed proof of Theorem 3.3.1.

Proof of Theorem 3.3.1. For (i), by construction, it is clear that the operator D^∇ in (3.24) is indeed a degree $+1$ derivation. Since Q is a homological vector field, from (3.22), it follows that $(\delta^\nabla)^2 = 0$. Therefore $(D^\nabla)^2 = 0$.

To prove (ii), consider the case when $\mathcal{R} = C^\infty(\mathcal{M})$ and $V = \Gamma\big(T_{\mathcal{M}}\big)$ in Lemma 3.3.3. Recall that C^∇ in (3.11) is $\mathcal{R}$-linear, and $\mathrm{pbw}^\nabla : \Gamma\big(S(T_{\mathcal{M}})\big) \to \mathcal{D}(\mathcal{M})$ is an isomorphism of graded coalgebras over $\mathcal{R}$. Since we have $L_Q \in \mathrm{coDer}_{\mathbb{K}}\left(\Gamma\big(S(T_{\mathcal{M}})\big)\right)$ and $\mathcal{L}_Q \in \mathrm{coDer}_{\mathbb{K}}(\mathcal{D}(\mathcal{M}))$, it thus follows that

$$(\mathrm{pbw}^\nabla)^{-1} \circ C^\nabla = (\mathrm{pbw}^\nabla)^{-1} \circ \mathcal{L}_Q \circ \mathrm{pbw}^\nabla - L_Q \in \mathrm{coDer}_{\mathcal{R}}(\Gamma\big(S(T_{\mathcal{M}})\big)).$$

Since both $\mathcal{L}_Q$ and L_Q are of degree $+1$ and pbw^∇ is of degree 0, it follows from Lemma 3.3.3 and Equation (3.27) that there exists a sequence of degree $+1$ sections $R_k \in \Gamma\big(S^k(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}}\big)$, $k \geqslant 0$, such that

$$(\mathrm{pbw}^\nabla)^{-1} \circ \mathcal{L}_Q \circ \mathrm{pbw}^\nabla - L_Q = \sum_{k=0}^{\infty} (\bar{R}_k \star \mathrm{id}). \tag{3.29}$$

Here we think of R_k as an $\mathcal{R}$-linear map $R_k : \Gamma\big(S^k(T_{\mathcal{M}})\big) \to \Gamma\big(T_{\mathcal{M}}\big)$ and $\bar{R}_k : \Gamma\big(S(T_{\mathcal{M}})\big) \to \Gamma\big(S(T_{\mathcal{M}})\big)$ defined as in Diagram (3.28).

From Equations (3.12), (3.13) and (3.14), it follows that

$$R_0 = 0, \quad R_1 = 0, \quad \text{and} \quad R_2 = -\operatorname{At}^{\nabla}_{(\mathcal{M},Q)}. \tag{3.30}$$

Thus the conclusion follows immediately from (3.29) by taking its $\mathcal{R}$-dual.

Finally, assume that ∇' is another torsion-free affine connection. Let $\phi := (\operatorname{pbw}^{\nabla'})^{-1} \circ \operatorname{pbw}^{\nabla}$. Then from Proposition 3.1.1, Proposition 3.1.2 and Theorem 3.2.1, it follows that

$$\phi : \big(\Gamma\big(S(T_{\mathcal{M}})\big), \delta^{\nabla}\big) \xrightarrow{\cong} \big(\Gamma\big(S(T_{\mathcal{M}})\big), \delta^{\nabla'}\big) \tag{3.31}$$

is an isomorphism of dg coalgebras over $(C^\infty(\mathcal{M}), Q)$. By dualizing it over the dg algebra $(C^\infty(\mathcal{M}), Q)$, we have that

$$\phi^T : \big(\Gamma\big(\widehat{S}(T^\vee_{\mathcal{M}})\big), D^{\nabla'}\big) \xrightarrow{\cong} \big(\Gamma\big(\widehat{S}(T^\vee_{\mathcal{M}})\big), D^{\nabla}\big)$$

is an isomorphism of dg algebras over $(C^\infty(\mathcal{M}), Q)$. This concludes the proof of the theorem. $\square$

Indeed, following Kapranov [23], one may consider $\big(\Gamma\big(\widehat{S}(T^\vee_{\mathcal{M}})\big), D^{\nabla}\big)$ as the 'dg algebra of functions' on the 'formal neighborhood' of the diagonal Δ of the product dg manifold $\big(\mathcal{M}\times\mathcal{M}, (Q,Q)\big)$: the PBW map $\operatorname{pbw}^{\nabla}$ is, by construction, a formal exponential map identifying a 'formal neighborhood' of the zero section of $T_{\mathcal{M}}$ to a 'formal neighborhood' of the diagonal of $\mathcal{M}\times\mathcal{M}$ as $\mathbb{Z}$-graded manifolds and Equation (3.22) asserts that D^{∇} is the homological vector field obtained on $T_{\mathcal{M}}$ by pullback of the vector field (Q,Q) on $\mathcal{M}\times\mathcal{M}$ through this formal exponential map. The readers are invited to compare Theorem 3.3.1 with [23, Theorem 2.8.2].

As an immediate consequence, we are ready to prove the main result of this section.

Theorem 3.3.4. *Let $(\mathcal{M}, Q)$ be a dg manifold. Each choice of an affine connection ∇ on $\mathcal{M}$ determines an $L_\infty[1]$ algebra structure on the space of vector fields $\mathfrak{X}(\mathcal{M})$. While the unary bracket $\lambda_1 : S^1\big(\mathfrak{X}(\mathcal{M})\big) \to \mathfrak{X}(\mathcal{M})$ is the Lie derivative $\mathcal{L}_Q$ along the homological vector field, the higher multibrackets $\lambda_k : S^k\big(\mathfrak{X}(\mathcal{M})\big) \to \mathfrak{X}(\mathcal{M})$, with $k \geq 2$, arise as the composition*

$$\lambda_k : S^k\big(\mathfrak{X}(\mathcal{M})\big) \to \Gamma\big(S^k(T_{\mathcal{M}})\big) \xrightarrow{R_k} \mathfrak{X}(\mathcal{M})$$

induced by a family of sections $\{R_k\}_{k\geq 2}$ of the vector bundles $S^k(T^\vee_{\mathcal{M}})\otimes T_{\mathcal{M}}$ starting with $R_2 = -\operatorname{At}^{\nabla}_{(\mathcal{M},Q)}$.

Furthermore, the $L_\infty[1]$ algebra structures on $\mathfrak{X}(\mathcal{M})$ arising from different choices of affine connections are all isomorphic.

For clarity, we point out that $S^k\big(\mathfrak{X}(\mathcal{M})\big)$ denotes the symmetric tensor product *over the field* $\mathbb{K}$ of k copies of $\mathfrak{X}(\mathcal{M})$. While λ_1 is merely a $\mathbb{K}$-linear endomorphism of $\mathfrak{X}(\mathcal{M})$, we note that, for all $k \geq 2$, the multibracket λ_k is $C^\infty(\mathcal{M})$-linear in each of its k arguments.

Proof. The first part follows immediately from the fact that $\big(\Gamma\big(S(T_\mathcal{M})\big), \delta^\nabla\big)$ as in (3.23) is a dg coalgebra over $(C^\infty(\mathcal{M}), Q)$.

The uniqueness is a direct consequence of Theorem 3.3.1 as well. Indeed, it is easier to derive it using the dg coalgebra $\big(\Gamma\big(S(T_\mathcal{M})\big), \delta^\nabla\big)$ as in (3.23). If ∇' is another torsion-free affine connection on $\mathcal{M}$, we know that $\phi : \big(\Gamma\big(S(T_\mathcal{M})\big), \delta^\nabla\big) \xrightarrow{\cong} \big(\Gamma\big(S(T_\mathcal{M})\big), \delta^{\nabla'}\big)$ as in (3.31) is an isomorphism of dg coalgebras over the dg ring $(C^\infty(\mathcal{M}), Q)$. Thus it follows that the sequence of maps $\{\phi_k\}_{k\geq 1}$ defined by the composition

$$\phi_k : S^k\left(\mathfrak{X}(\mathcal{M})\right) \to \Gamma\big(S^k(T_\mathcal{M})\big) \xrightarrow{\phi} \Gamma\big(S(T_\mathcal{M})\big) \xrightarrow{\mathrm{pr}_1} \Gamma\big(T_\mathcal{M}\big) = \mathfrak{X}(\mathcal{M})$$

is an isomorphism of $L_\infty[1]$ algebras. Indeed, it is simple to see from (3.9) that the linear term ϕ_1 is the identity map. □

Such an $L_\infty[1]$ algebra on $\mathfrak{X}(\mathcal{M})$ is called the **Kapranov $L_\infty[1]$ algebra** of the dg manifold $(\mathcal{M}, Q)$.

3.3.2 Recursive formula for multibrackets

It is clear that the Kapranov $L_\infty[1]$ algebra of a dg manifold in Theorem 3.3.4 is completely determined by the Atiyah 1-cocycle and

$$R_k \in \Gamma\big(S^k(T^\vee_\mathcal{M}) \otimes T_\mathcal{M}\big) \cong \Gamma\big(\operatorname{Hom}(S^k(T_\mathcal{M}), T_\mathcal{M})\big)$$

for $k \geq 3$.

Recall that, for the $L_\infty[1]$ algebra on the Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ associated to the Atiyah class of the tangent bundle T_X of a Kähler manifold X, Kapranov showed that the multibrackets can be described explicitly by a very simple formula [23]. For a general complex manifold, it was proved in [31] that they can be computed recursively as well. It is thus natural to ask if one can describe the multibrackets in Theorem 3.3.4 explicitly.

In what follows, we will give a characterization of these multibrackets, or equivalently all terms R_k, $k \geq 2$, by showing that they are completely determined by the Atiyah cocycle $\mathrm{At}^\nabla_{(\mathcal{M},Q)}$, the curvature R^∇, and their higher covariant derivatives, by a recursive formula.

We need to introduce some notations first.

By $\widetilde{d^\nabla} R_{n-1} \in \Gamma\big(S^n(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}}\big)$, we denote the symmetrized covariant derivative of R_{n-1}. That is, for any $\boldsymbol{X} \in \Gamma\big(S^n(T_{\mathcal{M}})\big)$,

$$\begin{aligned}\left(\widetilde{d^\nabla} R_{n-1}\right)(\boldsymbol{X}) &= \sum_{k=1}^{n} \varepsilon_k \big(\nabla_{X_k} R_{n-1}\big)(\boldsymbol{X}^{\{k\}}) \\ &= \sum_{k=1}^{n} \varepsilon_k \left((-1)^{|X_k|} \nabla_{X_k}\big(R_{n-1}(\boldsymbol{X}^{\{k\}})\big) - R_{n-1}\big(\nabla_{X_k} \boldsymbol{X}^{\{k\}}\big)\right).\end{aligned} \tag{3.32}$$

Here $\varepsilon_k = (-1)^{|X_k|(|X_1|+\cdots+|X_{k-1}|)}$ is the Koszul sign.

Let $B^\nabla : \Gamma\big(T_{\mathcal{M}} \otimes S(T_{\mathcal{M}})\big) \to \Gamma\big(S(T_{\mathcal{M}})\big)$ be the map defined by

$$B^\nabla(Y; \boldsymbol{X}) = (\mathrm{pbw}^\nabla)^{-1}\left(Y \cdot \mathrm{pbw}^\nabla(\boldsymbol{X})\right) - \nabla_Y \boldsymbol{X}, \tag{3.33}$$

$\forall Y \in \mathfrak{X}(\mathcal{M})$ and $\boldsymbol{X} \in \Gamma\big(S^n(T_{\mathcal{M}})\big)$. The following can be verified directly.

Lemma 3.3.5. *The map B^∇ is well defined and $\mathcal{R}$-linear. Hence B^∇ is indeed a bundle map*

$$B^\nabla : T_{\mathcal{M}} \otimes S(T_{\mathcal{M}}) \to S(T_{\mathcal{M}}).$$

As we will see below, the map B^∇ is completely determined by the curvature R^∇ and its higher covariant derivatives.

Let

$$\Gamma\big(\widehat{S}(T^\vee_{\mathcal{M}})\big) \otimes_{\mathcal{R}} \Gamma\big(S(T_{\mathcal{M}})\big) \xrightarrow{\langle - | - \rangle} \mathcal{R}$$

be the duality pairing defined by

$$\begin{aligned}&\langle \alpha_1 \odot \cdots \odot \alpha_q | X_1 \odot \cdots \odot X_p \rangle \\ &\qquad = \begin{cases} \sum_{\sigma \in S_p} \varepsilon \, \langle\, \alpha_1 \mid X_{\sigma(1)} \,\rangle \cdot \langle\, \alpha_2 \mid X_{\sigma(2)} \,\rangle \cdots \langle\, \alpha_p \mid X_{\sigma(p)} \,\rangle & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}\end{aligned}$$

for all homogeneous elements $\alpha_1, \ldots, \alpha_q \in \Gamma\big(T^\vee_{\mathcal{M}}\big)$ and $X_1, \ldots, X_p \in \Gamma\big(T_{\mathcal{M}}\big)$. The symbol $\varepsilon = \varepsilon(\alpha_1, \alpha_2, \cdots, \alpha_p, X_1, X_2, \cdots, X_p)$ denotes the Koszul signs arising from the reordering of the homogeneous objects $\alpha_1, \alpha_2, \cdots, \alpha_p, X_1, X_2, \cdots, X_p$ in each term of the right hand side.

The following is an immediate consequence of the Fedosov construction of graded manifolds [34, Theorem 5.6 and Proposition 5.2]. A short description on this topic can be found in Section 2.3.

Lemma 3.3.6.

(i) The bundle map $B^{\nabla} : T_{\mathcal{M}} \otimes S(T_{\mathcal{M}}) \to S(T_{\mathcal{M}})$ *in Lemma 3.3.5 is completely determined by the curvature* R^{∇} *and its higher covariant derivatives. More precisely, given any* $Y \in \mathfrak{X}(\mathcal{M})$*, provided that* $B^{\nabla}(Y; \boldsymbol{Y})$ *is known for all* $\boldsymbol{Y} \in \Gamma\big(S^{\leq n-1}(T_{\mathcal{M}})\big)$*, one can compute* $B^{\nabla}(Y; \boldsymbol{X})$ *for any* $\boldsymbol{X} \in \Gamma\big(S^{n}(T_{\mathcal{M}})\big)$*.*

(ii) Moreover, if $R^{\nabla} = 0$*, then* $B^{\nabla}(Y; \boldsymbol{X}) = Y \odot \boldsymbol{X}$*, for all* $Y \in \mathfrak{X}(\mathcal{M})$ *and* $\boldsymbol{X} \in \Gamma\big(S(T_{\mathcal{M}})\big)$*.*

Proof. (i). Let

$$\nabla^{\natural}_{Y} \boldsymbol{X} = (\mathrm{pbw}^{\nabla})^{-1}(Y \cdot \mathrm{pbw}^{\nabla}(\boldsymbol{X})).$$

Then by Equation (3.33),

$$B^{\nabla}(Y; \boldsymbol{X}) = \nabla^{\natural}_{Y} \boldsymbol{X} - \nabla_{Y} \boldsymbol{X}.$$

For the rest of the proof, we follow the notation from Section 2.3, in particular, Theorem 2.3.4. For all $\sigma \in \Gamma\big(\widehat{S}(T^{\vee}_{\mathcal{M}})\big)$, we have

$$\begin{aligned}
\langle\, \sigma \mid \nabla^{\natural}_{Y} \boldsymbol{X} - \nabla_{Y} \boldsymbol{X} \,\rangle &= (-1)^{|\sigma| \cdot |Y|} \langle\, \nabla_{Y} \sigma - \nabla^{\natural}_{Y} \sigma \mid \boldsymbol{X} \,\rangle \\
&= (-1)^{|\sigma| \cdot |Y|} \left\langle i_{Y}(d^{\nabla} - d^{\nabla^{\natural}})(\sigma) \middle| \boldsymbol{X} \right\rangle \\
&= (-1)^{|\sigma| \cdot |Y|} \left\langle i_{Y}(\delta - \widetilde{A^{\nabla}})(\sigma) \middle| \boldsymbol{X} \right\rangle \\
&= \langle\, \sigma \mid Y \odot \boldsymbol{X} \,\rangle - (-1)^{|\sigma| \cdot |Y|} \langle\, i_{Y} \widetilde{A^{\nabla}}(\sigma) \mid \boldsymbol{X} \,\rangle \\
&= \langle\, \sigma \mid Y \odot \boldsymbol{X} \,\rangle - \langle\, \sigma \mid (i_{Y} \widetilde{A^{\nabla}})^{T} \boldsymbol{X} \,\rangle.
\end{aligned}$$

Thus it follows that

$$B^{\nabla}(Y; \boldsymbol{X}) = Y \odot \boldsymbol{X} - (i_{Y} \widetilde{A^{\nabla}})^{T} \boldsymbol{X}.$$

The conclusion thus follows from Corollary 2.3.7.

(ii) Moreover, if $R^{\nabla} = 0$, then $A^{\nabla} = 0$ by Equation (2.2), and hence we obtain

$$B^{\nabla}(Y; \boldsymbol{X}) = Y \odot \boldsymbol{X}.$$

□

Theorem 3.3.7.

(i) The sections $R_n \in \Gamma\big(S^{n}(T^{\vee}_{\mathcal{M}}) \otimes T_{\mathcal{M}}\big)$*, with* $n \geq 3$*, are completely determined by the Atiyah 1-cocycle* $\mathrm{At}^{\nabla}_{(\mathcal{M},Q)}$*, the curvature* R^{∇}*, and their higher covariant derivatives, through the recursive formula*

$$R_n = \frac{1}{n} \sum_{k=2}^{n-1} \left[\overline{\big(\widetilde{d^{\nabla}} R_k \star \mathrm{id}\big)} + (1-k)(\bar{R}_k \star \mathrm{id}) - B^{\nabla} \circ (\bar{R}_k \otimes \mathrm{id}) \circ \Delta \right] + \frac{2}{n}(\bar{R}_2 \star \mathrm{id}). \quad (3.34)$$

(ii) In particular, if $R^\nabla = 0$, then $R_2 = -\operatorname{At}^\nabla_{(\mathcal{M},Q)}$ and $R_n = \frac{1}{n}\widetilde{d^\nabla} R_{n-1}$ for all $n \geq 3$.

In terms of Sweedler's notation $\Delta \boldsymbol{X} = \boldsymbol{X}_{(1)} \otimes \boldsymbol{X}_{(2)}$, one can rewrite Equation (3.34) as follows:

$$\begin{aligned} R_n(\boldsymbol{X}) = \frac{1}{n}\sum_{k=2}^{n-1} \Big[\Big(\widetilde{d^\nabla} R_k(\boldsymbol{X}_{(1)}) \odot \boldsymbol{X}_{(2)}\Big) + (1-k)\left(R_k(\boldsymbol{X}_{(1)}) \odot \boldsymbol{X}_{(2)}\right) \\ - B^\nabla\left(R_k(\boldsymbol{X}_{(1)}); \boldsymbol{X}_{(2)}\right)\Big] + \frac{2}{n}\left(R_2(\boldsymbol{X}_{(1)}) \odot \boldsymbol{X}_{(2)}\right). \end{aligned}$$

Now we proceed to prove Theorem 3.3.7. For any $\boldsymbol{X} \in \Gamma\big(S^n(T_\mathcal{M})\big)$, we can write

$$\begin{aligned} C^\nabla(\boldsymbol{X}) &= \operatorname{pbw}^\nabla \circ \Big((\operatorname{pbw}^\nabla)^{-1} \circ \mathcal{L}_Q \circ \operatorname{pbw}^\nabla - L_Q\Big)(\boldsymbol{X}) \\ &= \operatorname{pbw}^\nabla \left(\sum_{k=0}^{n} (\bar{R}_k \star \operatorname{id})(\boldsymbol{X})\right) && \text{by Eq. (3.29)} \\ &= \sum_{k=2}^{n} \operatorname{pbw}^\nabla \circ (\bar{R}_k \star \operatorname{id})(\boldsymbol{X}) && \text{by Eqs. (3.30).} \end{aligned} \tag{3.35}$$

In order to simplify the notation, we introduce a sequence of maps $B_k^\nabla : \Gamma\big(S(T_\mathcal{M})\big) \to \Gamma\big(S(T_\mathcal{M})\big)$, for $k \geq 2$, defined by

$$B_k^\nabla(\boldsymbol{X}) = B^\nabla \circ (\bar{R}_k \otimes \operatorname{id}) \circ \Delta(\boldsymbol{X}), \quad \forall \boldsymbol{X} \in \Gamma\big(S^n(T_\mathcal{M})\big).$$

Explicitly, in terms of Sweedler's notation $\Delta \boldsymbol{X} = \boldsymbol{X}_{(1)} \otimes \boldsymbol{X}_{(2)}$, we write

$$\begin{aligned} B_k^\nabla(\boldsymbol{X}) &= B^\nabla(R_k(\boldsymbol{X}_{(1)}); \boldsymbol{X}_{(2)}) \\ &= (\operatorname{pbw}^\nabla)^{-1}\left(R_k(\boldsymbol{X}_{(1)}) \cdot \operatorname{pbw}^\nabla(\boldsymbol{X}_{(2)})\right) - \nabla_{R_k(\boldsymbol{X}_{(1)})} \boldsymbol{X}_{(2)}. \end{aligned} \tag{3.36}$$

From Lemma 3.3.5, it follows that B_k^∇, with $k \geq 2$, is $\mathcal{R}$-linear. That is, B_k^∇, with $k \geq 2$, is indeed a bundle map $S(T_\mathcal{M}) \to S(T_\mathcal{M})$.

Proof of Theorem 3.3.7. (i) First, we will prove the recursive formula (3.34).

Pick any element $\boldsymbol{X} = X_1 \odot \cdots \odot X_n$ in $\Gamma\big(S^n(T_\mathcal{M})\big)$. Again, for the sake of simplicity, we use Sweedler's notation $\Delta \boldsymbol{X} = \boldsymbol{X}_{(1)} \otimes \boldsymbol{X}_{(2)}$ and the Koszul sign

$\varepsilon_k = (-1)^{|X_k|(|X_1|+\cdots+|X_{k-1}|)}$. For each l, by Equations (3.10) and (3.26), we have

$$
\begin{aligned}
&(n-l+1)\,\mathrm{pbw}^{\nabla}\circ(\bar{R}_l \star \mathrm{id})(\boldsymbol{X}) \\
&= (n-l+1)\,\mathrm{pbw}^{\nabla}(R_l(\boldsymbol{X}_{(1)}) \odot \boldsymbol{X}_{(2)}) \\
&= R_l(\boldsymbol{X}_{(1)})\cdot \mathrm{pbw}^{\nabla}(\boldsymbol{X}_{(2)}) - \mathrm{pbw}^{\nabla}\left(\nabla_{R_l(\boldsymbol{X}_{(1)})}\boldsymbol{X}_{(2)}\right) \\
&\quad + \sum_{k=1}^{n} \varepsilon_k (-1)^{|X_k|}\Big[X_k \cdot \mathrm{pbw}^{\nabla}\left(R_l(\boldsymbol{X}^{\{k\}}_{(1)}) \odot \boldsymbol{X}^{\{k\}}_{(2)}\right) \\
&\qquad\qquad\qquad - \mathrm{pbw}^{\nabla}\left(\nabla_{X_k}\left(R_l(\boldsymbol{X}^{\{k\}}_{(1)}) \odot \boldsymbol{X}^{\{k\}}_{(2)}\right)\right)\Big] \\
&= R_l(\boldsymbol{X}_{(1)})\cdot \mathrm{pbw}^{\nabla}(\boldsymbol{X}_{(2)}) - \mathrm{pbw}^{\nabla}\left(\nabla_{R_l(\boldsymbol{X}_{(1)})}\boldsymbol{X}_{(2)}\right) \\
&\quad + \sum_{k=1}^{n} \varepsilon_k (-1)^{|X_k|}\Big[X_k \cdot \mathrm{pbw}^{\nabla}\circ(\bar{R}_l \star \mathrm{id})(\boldsymbol{X}^{\{k\}}) \\
&\qquad\qquad\qquad - \mathrm{pbw}^{\nabla}\left(\nabla_{X_k}\left((\bar{R}_l \star \mathrm{id})(\boldsymbol{X}^{\{k\}})\right)\right)\Big].
\end{aligned}
$$

Combining it with Equation (3.36), we conclude that

$$
\begin{aligned}
&(n-l+1)\,\mathrm{pbw}^{\nabla}\circ(\bar{R}_l \star \mathrm{id})(\boldsymbol{X}) - \mathrm{pbw}^{\nabla}\circ B_l^{\nabla}(\boldsymbol{X}) \\
&= \sum_{k=1}^{n} \varepsilon_k (-1)^{|X_k|}\left[X_k \cdot \mathrm{pbw}^{\nabla}\circ(\bar{R}_l \star \mathrm{id})(\boldsymbol{X}^{\{k\}}) - \mathrm{pbw}^{\nabla}\left(\nabla_{X_k}\big((\bar{R}_l \star \mathrm{id})(\boldsymbol{X}^{\{k\}})\big)\right)\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(n-l+1)(\bar{R}_l \star \mathrm{id})(\boldsymbol{X}) - B_l^{\nabla}(\boldsymbol{X}) \\
&= \sum_{k=1}^{n} \varepsilon_k (-1)^{|X_k|}\Big[(\mathrm{pbw}^{\nabla})^{-1}\left(X_k \cdot \mathrm{pbw}^{\nabla}\circ(\bar{R}_l \star \mathrm{id})(\boldsymbol{X}^{\{k\}})\right) \\
&\qquad\qquad\qquad - \nabla_{X_k}\left((\bar{R}_l \star \mathrm{id})(\boldsymbol{X}^{\{k\}})\right)\Big].
\end{aligned} \tag{3.37}
$$

Also, for each l, by Equation (3.32), we have

$$
\begin{aligned}
&(\overline{\widetilde{d^{\nabla}R_l}} \star \mathrm{id})(\boldsymbol{X}) \\
&= \sum_{k=1}^{n} \varepsilon_k \left[(d^{\nabla}R_l)(X_k; \boldsymbol{X}^{\{k\}}_{(1)}) \odot \boldsymbol{X}^{\{k\}}_{(2)}\right] \\
&= \sum_{k=1}^{n} \varepsilon_k \left[(-1)^{|X_k|}\left(\left(\nabla_{X_k} R_l(\boldsymbol{X}^{\{k\}}_{(1)})\right) \odot \boldsymbol{X}^{\{k\}}_{(2)}\right) - \left(R_l\left(\nabla_{X_k}\boldsymbol{X}^{\{k\}}_{(1)}\right) \odot \boldsymbol{X}^{\{k\}}_{(2)}\right)\right]
\end{aligned}
$$

$$= \sum_{k=1}^{n} \varepsilon_k \Big[(-1)^{|X_k|} \Big(\Big(\nabla_{X_k} R_l(\boldsymbol{X}^{\{k\}}_{(1)}) \Big) \odot \boldsymbol{X}^{\{k\}}_{(2)} \Big) + (-1)^{|X_k| \cdot \left|\boldsymbol{X}^{\{k\}}_{(1)}\right|} \Big(R_l(\boldsymbol{X}^{\{k\}}_{(1)}) \odot \Big(\nabla_{X_k} \boldsymbol{X}^{\{k\}}_{(2)} \Big) \Big) \Big]$$

$$- \sum_{k=1}^{n} \varepsilon_k \Big[\Big(R_l \Big(\nabla_{X_k} \boldsymbol{X}^{\{k\}}_{(1)} \Big) \odot \boldsymbol{X}^{\{k\}}_{(2)} \Big) + (-1)^{|X_k| \cdot \left|\boldsymbol{X}^{\{k\}}_{(1)}\right|} \Big(R_l(\boldsymbol{X}^{\{k\}}_{(1)}) \odot \Big(\nabla_{X_k} \boldsymbol{X}^{\{k\}}_{(2)} \Big) \Big) \Big]$$

$$= \sum_{k=1}^{n} \varepsilon_k \left[(-1)^{|X_k|} \nabla_{X_k} \left((\bar{R}_l \star \mathrm{id})(\boldsymbol{X}^{\{k\}}) \right) - (\bar{R}_l \star \mathrm{id}) \left(\nabla_{X_k} \boldsymbol{X}^{\{k\}} \right) \right].$$

According to (3.30), we have $R_2 = -\operatorname{At}^{\nabla}_{(\mathcal{M},Q)}$. Hence

$$\mathrm{pbw}^{\nabla} \circ (\bar{R}_2 \star \mathrm{id})(\boldsymbol{X}) = -\sum_{i<j} \varepsilon_i \varepsilon_j (-1)^{|X_i| \cdot |X_j|} \, \mathrm{pbw}^{\nabla} \left(\operatorname{At}^{\nabla}_{(\mathcal{M},Q)}(X_i, X_j) \odot \boldsymbol{X}^{\{i,j\}} \right). \tag{3.38}$$

By Equations (3.15) and (3.38), we have

$$\begin{aligned}
&C^{\nabla}(\boldsymbol{X}) - \frac{2}{n} \mathrm{pbw}^{\nabla} \circ (\bar{R}_2 \star \mathrm{id})(\boldsymbol{X}) \\
&= \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \left[(-1)^{|X_k|} X_k \cdot C^{\nabla}(\boldsymbol{X}^{\{k\}}) - C^{\nabla}(\nabla_{X_k} \boldsymbol{X}^{\{k\}}) \right] \\
&= \frac{1}{n} \sum_{k=1}^{n} \sum_{l=2}^{n-1} \varepsilon_k \left[(-1)^{|X_k|} X_k \cdot \mathrm{pbw}^{\nabla} \circ (\bar{R}_l \star \mathrm{id})(\boldsymbol{X}^{\{k\}}) - \mathrm{pbw}^{\nabla} \circ (\bar{R}_l \star \mathrm{id})(\nabla_{X_k} \boldsymbol{X}^{\{k\}}) \right] \\
&= \frac{1}{n} \sum_{k=1}^{n} \sum_{l=2}^{n-1} \varepsilon_k \; (-1)^{|X_k|} \Big[X_k \cdot \mathrm{pbw}^{\nabla} \circ (\bar{R}_l \star \mathrm{id})(\boldsymbol{X}^{\{k\}}) \\
&\qquad\qquad - \mathrm{pbw}^{\nabla} \Big(\nabla_{X_k} \Big((\bar{R}_l \star \mathrm{id})(\boldsymbol{X}^{\{k\}}) \Big) \Big) \Big] \\
&\quad + \frac{1}{n} \sum_{k=1}^{n} \sum_{l=2}^{n-1} \varepsilon_k \Big[(-1)^{|X_k|} \mathrm{pbw}^{\nabla} \Big(\nabla_{X_k} \Big((\bar{R}_l \star \mathrm{id})(\boldsymbol{X}^{\{k\}}) \Big) \Big) \\
&\qquad\qquad - \mathrm{pbw}^{\nabla} \circ (\bar{R}_l \star \mathrm{id}) \Big(\nabla_{X_k} \boldsymbol{X}^{\{k\}} \Big) \Big],
\end{aligned}$$

where the second equality is obtained by applying Equation (3.35) to $C^{\nabla}(\boldsymbol{X}^{\{k\}})$ and $C^{\nabla}(\nabla_{X_k} \boldsymbol{X}^{\{k\}})$.

It thus follows that

$$(\mathrm{pbw}^{\nabla})^{-1} \circ C^{\nabla}(\boldsymbol{X}) - \frac{2}{n} (\bar{R}_2 \star \mathrm{id})(\boldsymbol{X}) = \alpha + \beta, \tag{3.39}$$

where

$$\alpha = \frac{1}{n}\sum_{k=1}^{n}\sum_{l=2}^{n-1}\varepsilon_k(-1)^{|X_k|}\Big[(\mathrm{pbw}^\nabla)^{-1}\left(X_k\cdot\mathrm{pbw}^\nabla\circ(\bar{R}_l\star\mathrm{id})(\boldsymbol{X}^{\{k\}})\right)$$

$$-\nabla_{X_k}\left((\bar{R}_l\star\mathrm{id})(\boldsymbol{X}^{\{k\}})\right)\Big],$$

and

$$\begin{aligned}\beta &= \frac{1}{n}\sum_{k=1}^{n}\sum_{l=2}^{n-1}\varepsilon_k\left[(-1)^{|X_k|}\left(\nabla_{X_k}\left((\bar{R}_l\star\mathrm{id})(\boldsymbol{X}^{\{k\}})\right)\right)-(\bar{R}_l\star\mathrm{id})\left(\nabla_{X_k}\boldsymbol{X}^{\{k\}}\right)\right]\\ &= \sum_{l=2}^{n-1}\frac{1}{n}\big(\overline{\widetilde{d^\nabla}R_l}\star\mathrm{id}\big)(\boldsymbol{X}).\end{aligned}\tag{3.40}$$

Now, according to (3.37),

$$\begin{aligned}&\alpha-\sum_{l=2}^{n-1}(\bar{R}_l\star\mathrm{id})(\boldsymbol{X})\\ &\quad=\sum_{l=2}^{n-1}\frac{1}{n}\left((n-l+1)(\bar{R}_l\star\mathrm{id})(\boldsymbol{X})-B_l^\nabla(\boldsymbol{X})\right)-\sum_{l=2}^{n-1}(\bar{R}_l\star\mathrm{id})(\boldsymbol{X})\\ &\quad=\frac{1}{n}\sum_{l=2}^{n-1}\left[(1-l)\left((\bar{R}_l\star\mathrm{id})(\boldsymbol{X})\right)-B_l^\nabla(\boldsymbol{X})\right].\end{aligned}\tag{3.41}$$

Equation (3.35) can be rewritten as

$$R_n(\boldsymbol{X})=(\mathrm{pbw}^\nabla)^{-1}\circ C^\nabla(\boldsymbol{X})-\sum_{k=2}^{n-1}(\bar{R}_k\star\mathrm{id})(\boldsymbol{X}).$$

Equations (3.39), (3.40) and (3.41) then yield Equation (3.34).

From (3.30), we know that $R_2=-\mathrm{At}^\nabla_{(\mathcal{M},Q)}$. According to Lemma 3.3.6, the bundle map B^∇ is completely determined by the curvature R^∇ and its higher covariant derivatives. It thus follows from the recursive formula (3.34) that, for any $n\geq 3$, R_n is determined by R_k with $k\leq n-1$, their covariant derivatives and the curvature. Thus, by inductive argument, R_n is completely determined by the Atiyah 1-cocycle, the curvature and their higher covariant derivatives.

(ii) Assume that $R^\nabla=0$. By Lemma 3.3.6, the bundle map $B^\nabla:T_\mathcal{M}\otimes S(T_\mathcal{M})\to S(T_\mathcal{M})$ is given by $B^\nabla(Y;\boldsymbol{X})=Y\odot\boldsymbol{X}$. Thus the formula $R_n(\boldsymbol{X})=\frac{1}{n}\widetilde{d^\nabla}R_{n-1}(\boldsymbol{X})$ can be obtained by induction argument, again using the recursive formula (3.34).

This concludes the proof of the theorem. $\square$

3.4 Examples of Kapranov L_∞ algebras

This section is devoted to the study of examples of Kapranov $L_\infty[1]$ algebras of some standard dg manifolds including those corresponding to $L_\infty[1]$ algebras, foliations and complex manifolds as in Examples 2.2.1 and 2.2.2.

3.4.1 dg manifolds associated with $L_\infty[1]$ algebras

Let $\mathfrak{g}$ be a finite dimensional L_∞ algebra with $d = \dim \mathfrak{g}$. Then $\mathfrak{g}[1]$ is an $L_\infty[1]$ algebra: the (canonical) symmetric coalgebra $(S(\mathfrak{g}[1]), \Delta)$ is equipped with a coderivation $\widetilde{Q} \in \mathrm{coDer}(S(\mathfrak{g}[1]))$ of degree $+1$ satisfying $\widetilde{Q} \circ \widetilde{Q} = 0$ and $\widetilde{Q}(1) = 0$. Indeed, $\widetilde{Q}$ is equivalent to a sequence of linear maps $q_k : S^k(\mathfrak{g}[1]) \to \mathfrak{g}[1],\ k \geq 1$, of degree $+1$ satisfying the generalized Jacobi identities. The map q_k is called the k-th multibracket.

Given an $L_\infty[1]$ algebra $\mathfrak{g}[1]$, we say a vector space $\mathfrak{M}$ is a $\mathfrak{g}[1]$-module if there exists a sequence of maps $\rho_k : S^k(\mathfrak{g}[1]) \otimes \mathfrak{M} \to \mathfrak{M}$ of degree $+1$, $\forall k \geq 0$, satisfying the standard compatibility condition [29]. If we write

$$\rho = \sum_{k\geq 0} \rho_k : S(\mathfrak{g}[1]) \otimes \mathfrak{M} \to \mathfrak{M}, \tag{3.42}$$

the compatibility condition is expressed explicitly as

$$\rho \circ \Big((\mathrm{id}_{S(\mathfrak{g}[1])} \otimes \rho) \circ (\Delta \otimes \mathrm{id}_{\mathfrak{M}}) + \widetilde{Q} \otimes \mathrm{id}_{\mathfrak{M}} \Big) = 0.$$

As an obvious example, we have the **trivial module** $\mathfrak{M} = \mathbb{K}$ together with the trivial action $\rho_k = 0$ for all $k \geq 0$. Another example is the **adjoint module** $\mathfrak{M} = \mathfrak{g}[1]$ with the adjoint action $\rho_k : S^k(\mathfrak{g}[1]) \otimes \mathfrak{g}[1] \to \mathfrak{g}[1]$ defined by

$$\rho_k(\boldsymbol{X} \otimes X) = q_{k+1}(\boldsymbol{X} \odot X),$$

where $\boldsymbol{X} \in S^k(\mathfrak{g}[1])$, $X \in \mathfrak{g}[1]$ and $q_{k+1} : S^{k+1}(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ is the multibracket of the $L_\infty[1]$ algebra $\mathfrak{g}[1]$. That is, $\{\rho_k\}_{k\geq 0}$ is defined by the following commutative diagram

$$\begin{tikzcd} S^k(\mathfrak{g}[1]) \otimes \mathfrak{g}[1] \arrow[rr, "\rho_k"] \arrow[dr, "\mathrm{sym}"'] & & \mathfrak{g}[1] \\ & S^{k+1}(\mathfrak{g}[1]) \arrow[ur, "q_{k+1}"'] & \end{tikzcd}$$

where $\mathrm{sym} : S^\bullet(\mathfrak{g}[1]) \otimes \mathfrak{g}[1] \to S^{\bullet+1}(\mathfrak{g}[1])$ is the canonical symmetrization map. By taking its dual, $(\mathfrak{g}[1])^\vee$ is also a $\mathfrak{g}[1]$-module, where the action is called the **coadjoint action**.

Throughout this section, we denote the degree of a homogeneous element $x \in \mathfrak{g}[1]$ by $|x|$. In particular, if $\mathfrak{g}$ is a Lie algebra concentrated in degree 0, then for any $x \in \mathfrak{g}[1]$, its degree is $|x| = -1$.

The associated Chevalley–Eilenberg cochain complex of a $\mathfrak{g}[1]$-module $\mathfrak{M}$ is

$$\mathcal{C}(\mathfrak{g}[1]; \mathfrak{M}) = \Big(\operatorname{Hom}\big(S(\mathfrak{g}[1]), \mathfrak{M}\big), d_{\mathrm{CE}}^{\mathfrak{M}} \Big),$$

where $d_{\mathrm{CE}}^{\mathfrak{M}}$ is defined by

$$d_{\mathrm{CE}}^{\mathfrak{M}}(F) = \rho \circ (\mathrm{id} \otimes F) \circ \Delta - (-1)^{|F|} F \circ \widetilde{Q},$$

for any homogeneous element $F \in \operatorname{Hom}\big(S(\mathfrak{g}[1]), \mathfrak{M}\big)$.

Observe that when $\mathfrak{M}$ is the trivial module $\mathbb{K}$, the associated Chevalley–Eilenberg cochain complex

$$\mathcal{C}(\mathfrak{g}[1]; \mathbb{K}) = \Big(\operatorname{Hom}\big(S(\mathfrak{g}[1]), \mathbb{K}\big), d_{\mathrm{CE}}^{\mathbb{K}} = d_{\mathrm{CE}} \Big)$$

is a dg algebra, equipped with the multiplication

$$f \odot g = \mu_{\mathbb{K}} \circ (f \otimes g) \circ \Delta : S(\mathfrak{g}[1]) \to \mathbb{K} \tag{3.43}$$

for any $f, g \in \operatorname{Hom}(S(\mathfrak{g}[1]), \mathbb{K})$. In other words, the dg algebra $(C^\infty(\mathfrak{g}[1]), Q)$ coincides with the Chevalley–Eilenberg cochain complex $\big(\mathcal{C}(\mathfrak{g}[1]; \mathbb{K}), d_{\mathrm{CE}}\big)$ of the trivial $\mathfrak{g}[1]$-module $\mathbb{K}$. That is, $\big(\mathcal{C}(\mathfrak{g}[1]; \mathbb{K}), d_{\mathrm{CE}}\big)$ is the dg algebra dual to the dg coalgebra $(S(\mathfrak{g}[1]), \widetilde{Q})$. Moreover, for any $\mathfrak{g}[1]$-module $\mathfrak{M}$, the Chevalley–Eilenberg cochain complex $\big(\mathcal{C}(\mathfrak{g}[1]; \mathfrak{M}), d_{\mathrm{CE}}^{\mathfrak{M}}\big)$ is a dg module over the dg algebra $(C^\infty(\mathfrak{g}[1]), Q)$, where the action, under the identification $\mu_0 : \mathbb{K} \otimes \mathfrak{M} \cong \mathfrak{M}$, is given by

$$f \cdot F = \mu_0 \circ (f \otimes F) \circ \Delta : S(\mathfrak{g}[1]) \to \mathfrak{M} \tag{3.44}$$

for any $f \in \operatorname{Hom}(S(\mathfrak{g}[1]), \mathbb{K})$ and $F \in \operatorname{Hom}(S(\mathfrak{g}[1]), \mathfrak{M})$. In particular, this means that it satisfies the compatibility condition

$$d_{\mathrm{CE}}^{\mathfrak{M}}(f \cdot F) = d_{\mathrm{CE}}(f) \cdot F + (-1)^{|f|} f \cdot d_{\mathrm{CE}}^{\mathfrak{M}}(F). \tag{3.45}$$

Therefore, the Chevalley–Eilenberg differential $d_{\mathrm{CE}}^{\mathfrak{M}}$ is completely determined by its image of elements in $\mathfrak{M}$, which is essentially induced by the action (3.42). More precisely, for any $x \in \mathfrak{M}$,

$$d_{\mathrm{CE}}^{\mathfrak{M}}(x) = \sum_{k \geq 0} \rho_k(-, x) \in \operatorname{Hom}(S(\mathfrak{g}[1]), \mathfrak{M}).$$

In particular, if $\mathfrak{M} = \mathfrak{g}[1]$ is the adjoint module of the finite dimensional $L_\infty[1]$ algebra $\mathfrak{g}[1]$ described above, the Chevalley–Eilenberg differential $d_{\mathrm{CE}}^{\mathfrak{g}[1]}$ (seen as an operator on $\widehat{S}(\mathfrak{g}[1])^\vee \otimes \mathfrak{g}[1]$) is determined by the relation

$$d_{\mathrm{CE}}^{\mathfrak{g}[1]}(x) = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \xi^{i_{k-1}} \odot \cdots \odot \xi^{i_1} \otimes q_k(e_{i_1} \odot \cdots \odot e_{i_{k-1}} \odot x), \quad \forall x \in \mathfrak{g}[1], \tag{3.46}$$

where $\{e_1, \cdots, e_d\}$ is a basis for $\mathfrak{g}[1]$ and $\{\xi^1, \cdots, \xi^d\}$ is the dual basis for $(\mathfrak{g}[1])^\vee$. In Equation (3.46) and in the remainder of the present section, we use the Einstein notation tacitly to avoid inserting summations over the indices $i_1, \ldots, i_{k-1}$ in many equations.

Remark 3.4.1. In terms of Sweedler's notation, we may write (3.43) as

$$(f \odot g)(\boldsymbol{X}) = \sum_{(\boldsymbol{X})} (-1)^{|g| \cdot \left|\boldsymbol{X}_{(1)}\right|} f(\boldsymbol{X}_{(1)}) g(\boldsymbol{X}_{(2)})$$

and (3.44) as

$$(f \cdot F)(\boldsymbol{X}) = \sum_{(\boldsymbol{X})} (-1)^{|F| \cdot \left|\boldsymbol{X}_{(1)}\right|} f(\boldsymbol{X}_{(1)}) F(\boldsymbol{X}_{(2)}),$$

where $f, g \in \mathrm{Hom}(S(\mathfrak{g}[1]), \mathbb{K})$, $F \in \mathrm{Hom}(S(\mathfrak{g}[1]), \mathfrak{M})$, $\boldsymbol{X} \in S(\mathfrak{g}[1])$ are homogeneous elements and $\Delta \boldsymbol{X} = \sum_{(\boldsymbol{X})} \boldsymbol{X}_{(1)} \otimes \boldsymbol{X}_{(2)}$.

We now proceed to describe the Kapranov $L_\infty[1]$ algebra of the dg manifold $(\mathfrak{g}[1], d_{\mathrm{CE}})$. Recall that $Q = d_{\mathrm{CE}}$ is defined by

$$Q(f) = d_{\mathrm{CE}}(f) = -(-1)^{|f|} f \circ \widetilde{Q} \tag{3.47}$$

for any homogeneous element $f \in \mathrm{Hom}(S(\mathfrak{g}[1]), \mathbb{K}) \cong C^\infty(\mathfrak{g}[1])$.

Let $\{e_1, \cdots, e_d\}$ be a basis of $\mathfrak{g}[1]$ and $\{x^1, \cdots, x^d\}$ its induced coordinate functions on $\mathfrak{g}[1]$ satisfying

$$x^i(e_j) = \langle\, x^i \mid e_j \,\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

We also use the notation

$$\frac{\partial}{\partial x^j} x^i := (-1)^{\left|x^i\right| \cdot \left|x^j\right|} \langle\, x^i \mid e_j \,\rangle. \tag{3.48}$$

Lemma 3.4.2. *Under the above notation, write the multibrackets as*

$$q_k(e_{i_1}, \cdots, e_{i_k}) = \sum_j c^j_{i_1\cdots i_k} e_j, \quad \forall k \geq 1.$$

Then the homological vector field $Q \in \mathfrak{X}(\mathfrak{g}[1])$ *can be written as*

$$Q = -\sum_j \sum_{k=1}^{\infty} \frac{1}{k!} c^j_{i_1\cdots i_k} x^{i_k} \odot \cdots \odot x^{i_1} \frac{\partial}{\partial x^j}.$$

Here, we are making tacit use of the Einstein summation convention for the indices $i_1, \ldots, i_k$.

Proof. As a vector field, Q can be written as $Q = \sum_j Q^j \frac{\partial}{\partial x^j}$ for some $Q^j \in C^\infty(\mathfrak{g}[1])$. Then, as a derivation of $C^\infty(\mathfrak{g}[1])$, Q satisfies $Q(x^j) = (-1)^{\left|x^j\right|} Q^j$ according to (3.48). On the other hand, according to (3.47), we have

$$\begin{aligned} \langle\, Q(x^j) \mid e_{i_1} \odot \cdots \odot e_{i_k} \,\rangle &= -(-1)^{\left|x^j\right|} \langle\, x^j \mid \widetilde{Q}(e_{i_1} \odot \cdots \odot e_{i_k}) \,\rangle \\ &= -(-1)^{\left|x^j\right|} c^j_{i_1\cdots i_k} \end{aligned}$$

for any $k \geq 1$.

Therefore, we may conclude that

$$Q^j = -\sum_{k=1}^{\infty} \frac{1}{k!} c^j_{i_1\cdots i_k} x^{i_k} \odot \cdots \odot x^{i_1}.$$

This completes the proof. $\square$

Note that we have a canonical trivialization of the tangent bundle

$$T_{\mathfrak{g}[1]} \cong \mathfrak{g}[1] \times \mathfrak{g}[1]. \tag{3.49}$$

Hence, we have the following identification

$$\begin{aligned} C^\infty(\mathfrak{g}[1]) \otimes \mathfrak{g}[1] \leftarrow &\mathfrak{X}(\mathfrak{g}[1]) \to \operatorname{Hom}\left(S(\mathfrak{g}[1]), \mathfrak{g}[1]\right) \\ f \otimes e_i \leftarrow\!\shortmid \;\; & f\frac{\partial}{\partial x^i} \;\; \mapsto \left(\boldsymbol{X} \mapsto (-1)^{|e_i|\cdot|\boldsymbol{X}|} \langle\, f \mid \boldsymbol{X} \,\rangle \cdot e_i\right), \end{aligned} \tag{3.50}$$

where $f \in \operatorname{Hom}(S(\mathfrak{g}[1]), \mathbb{K}) \cong C^\infty(\mathfrak{g}[1])$ is homogeneous and $\boldsymbol{X} \in S(\mathfrak{g}[1])$.

Lemma 3.4.3. *Under the identification* (3.50)*, the Lie derivative* $\mathcal{L}_Q = [Q, -] \in \operatorname{End}(\mathfrak{X}(\mathfrak{g}[1]))$ *corresponds to the Chevalley–Eilenberg differential* $d^{\mathfrak{g}[1]}_{\mathrm{CE}}$*, where* $\mathfrak{g}[1]$ *acts on* $\mathfrak{g}[1]$ *by the adjoint action.*

Proof. Recall that the Chevalley–Eilenberg differential $d_{\mathrm{CE}}^{\mathfrak{g}[1]}$ on $\mathfrak{g}[1]$ satisfies (3.45). On the other hand, we have

$$\mathcal{L}_Q(f\cdot F)=[Q,f\cdot F]=Q(f)\cdot F+(-1)^{|f|}f\cdot[Q,F]=Q(f)\cdot F+(-1)^{|f|}f\cdot\mathcal{L}_Q(F),$$

for any homogeneous element $f\in C^\infty(\mathfrak{g}[1])\cong\operatorname{Hom}(S(\mathfrak{g}[1]),\mathbb{K})$ and $F\in\mathfrak{X}(\mathfrak{g}[1])\cong\operatorname{Hom}(S(\mathfrak{g}[1]),\mathfrak{g}[1])$. Since $Q(f)=d_{\mathrm{CE}}(f)$ according to Equation (3.47), it suffices to prove the claim for each $\frac{\partial}{\partial x^i}$, $i=1,\ldots,d$.

We keep the notation $Q=\sum_j Q^j\frac{\partial}{\partial x^j}$. Now, by Lemma 3.4.2, we have

$$\begin{aligned}
\mathcal{L}_Q\left(\frac{\partial}{\partial x^i}\right)&=-(-1)^{\left|\frac{\partial}{\partial x^i}\right|}\sum_j\frac{\partial}{\partial x^i}(Q^j)\frac{\partial}{\partial x^j}\\
&=-(-1)^{\left|\frac{\partial}{\partial x^i}\right|}\left(-\sum_j\sum_{k=1}^{\infty}\frac{1}{k!}c^j_{i_1\cdots i_k}\frac{\partial(x^{i_k}\odot\cdots\odot x^{i_1})}{\partial x^i}\frac{\partial}{\partial x^j}\right)\\
&=(-1)^{\left|\frac{\partial}{\partial x^i}\right|+|x^i|}\sum_j\sum_{k=1}^{\infty}\frac{1}{(k-1)!}c^j_{i_1\cdots i_{k-1}i}x^{i_{k-1}}\odot\cdots\odot x^{i_1}\frac{\partial}{\partial x^j}\\
&=\sum_j\sum_{k=1}^{\infty}\frac{1}{(k-1)!}c^j_{i_1\cdots i_{k-1}i}x^{i_{k-1}}\odot\cdots\odot x^{i_1}\frac{\partial}{\partial x^j}.
\end{aligned}$$

The conclusion thus follows immediately by comparing the equation above with (3.46). □

The trivialization of the tangent bundle (3.49) induces an isomorphism

$$T^\vee_{\mathfrak{g}[1]}\otimes\operatorname{End}(T_{\mathfrak{g}[1]})\xrightarrow{\cong}\mathfrak{g}[1]\times\big((\mathfrak{g}[1])^\vee\otimes(\mathfrak{g}[1])^\vee\otimes\mathfrak{g}[1]\big)$$

of vector bundles. Lemma 3.4.3, comparing with (2.1), indicates that we have an isomorphism of cochain complexes:

$$\big(\Gamma\big(\mathfrak{g}[1];T^\vee_{\mathfrak{g}[1]}\otimes\operatorname{End}(T_{\mathfrak{g}[1]})\big)^\bullet,\mathcal{Q}\big)\xrightarrow{\cong}\big(\operatorname{Hom}^\bullet(S(\mathfrak{g}[1]),\mathfrak{M}),d_{\mathrm{CE}}^{\mathfrak{M}}\big),$$

where $\mathfrak{M}=(\mathfrak{g}[1])^\vee\otimes(\mathfrak{g}[1])^\vee\otimes\mathfrak{g}[1]$ is the tensor product of adjoint and coadjoint modules.

Thus we have the following

Corollary 3.4.4. *Let $(\mathcal{M},Q)=(\mathfrak{g}[1],d_{\mathrm{CE}})$ be the dg manifold corresponding to a finite-dimensional $L_\infty[1]$ algebra $\mathfrak{g}[1]$. There is a canonical isomorphism, for any $k\in\mathbb{Z}$,*

$$H^k\big(\Gamma(T^\vee_{\mathfrak{g}[1]}\otimes\operatorname{End}(T_{\mathfrak{g}[1]}))^\bullet,\mathcal{Q}\big)\cong H^k_{\mathrm{CE}}(\mathfrak{g}[1],(\mathfrak{g}[1])^\vee\otimes(\mathfrak{g}[1])^\vee\otimes\mathfrak{g}[1])$$

where the right hand side stands for the Chevalley–Eilenberg cohomology of the $L_\infty[1]$ algebra $\mathfrak{g}[1]$ with values in $(\mathfrak{g}[1])^\vee\otimes(\mathfrak{g}[1])^\vee\otimes\mathfrak{g}[1]$.

Remark 3.4.5. It is sometimes useful to use the Chevalley–Eilenberg cohomology of L_∞ algebra rather than $L_\infty[1]$ algebra. Then Corollary 3.4.4 can be rephrased as follows.

For any finite-dimensional L_∞ algebra $\mathfrak{g}$, there is a canonical isomorphism, for any $k \in \mathbb{Z}$,

$$H^k\big(\Gamma(T^\vee_{\mathfrak{g}[1]} \otimes \operatorname{End}(T_{\mathfrak{g}[1]}))^\bullet, \mathcal{Q}\big) \cong H^{k-1}_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \otimes \mathfrak{g}),$$

where the right hand side stands for the Chevalley–Eilenberg cohomology of the L_∞ algebra $\mathfrak{g}$ with values in $\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \otimes \mathfrak{g}$. Note that there is a degree shifting here.

We still keep the notation $d_{\mathrm{CE}} = Q = \sum_l Q^l \frac{\partial}{\partial x^l}$. Let $\nabla : \mathfrak{X}(\mathfrak{g}[1]) \otimes \mathfrak{X}(\mathfrak{g}[1]) \to \mathfrak{X}(\mathfrak{g}[1])$ be the trivial (torsion-free) connection: $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$. The corresponding Atiyah 1-cocycle $\operatorname{At}^\nabla_{(\mathfrak{g}[1], d_{\mathrm{CE}})} \in \Gamma\big(\operatorname{Hom}\big(S^2(T_{\mathfrak{g}[1]}), T_{\mathfrak{g}[1]}\big)\big)$ is completely determined by the relations

$$\begin{aligned}
&\operatorname{At}^\nabla_{(\mathfrak{g}[1], d_{\mathrm{CE}})}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\
&= -(-1)^{|x^i|} \nabla_{\frac{\partial}{\partial x^i}} \mathcal{L}_Q\left(\frac{\partial}{\partial x^j}\right) \\
&= \sum_l (-1)^{|x^i|+|x^j|} \frac{\partial^2}{\partial x^i \partial x^j}(Q^l) \frac{\partial}{\partial x^l} \qquad (3.51) \\
&= \sum_l (-1)^{|x^i|+|x^j|} \frac{\partial^2}{\partial x^i \partial x^j}\left(-\sum_{k=1}^\infty \frac{1}{k!} c^l_{i_1 \cdots i_k} x^{i_k} \odot \cdots \odot x^{i_1}\right) \frac{\partial}{\partial x^l} \\
&= -\sum_l \sum_{k=2}^\infty \frac{1}{(k-2)!} c^l_{i_1 \cdots i_{k-2} ij} x^{i_{k-2}} \odot \cdots \odot x^{i_1} \frac{\partial}{\partial x^l}, \qquad (3.52)
\end{aligned}$$

for all $i, j \in \{1, \ldots, d\}$.

Let $\widehat{\operatorname{At}^\nabla_{(\mathfrak{g}[1], d_{\mathrm{CE}})}}$ be the map defined by the following commutative diagram

$$\begin{tikzcd}
C^\infty(\mathfrak{g}[1]) \otimes S^2(\mathfrak{g}[1]) \arrow[r, "\simeq"] & \Gamma\big(S^2(T_{\mathfrak{g}[1]})\big) \arrow[r, "\operatorname{At}^\nabla_{(\mathfrak{g}[1], d_{\mathrm{CE}})}"] & \mathfrak{X}(\mathfrak{g}[1]) \arrow[d, "\simeq"] \\
S^2(\mathfrak{g}[1]) \arrow[u, hook] \arrow[rr, "\widehat{\operatorname{At}^\nabla_{(\mathfrak{g}[1], d_{\mathrm{CE}})}}"] & & \operatorname{Hom}(S(\mathfrak{g}[1]), \mathfrak{g}[1]).
\end{tikzcd}$$

Equation (3.52) implies that

$$\widehat{\operatorname{At}^\nabla_{(\mathfrak{g}[1], d_{\mathrm{CE}})}}(e_i, e_j): \quad e_{l_1} \odot \cdots \odot e_{l_k} \mapsto -q_{k+2}(e_i \odot e_j \odot e_{l_1} \odot \cdots \odot e_{l_k}).$$

Therefore, under the identification above, we have

$$\widehat{\mathrm{At}^{\nabla}_{(\mathfrak{g}[1],d_{\mathrm{CE}})}}(x,y): \quad \boldsymbol{X} \mapsto -q_{n+2}(x\odot y\odot \boldsymbol{X}),$$

for any $x,y\in\mathfrak{g}[1]$ and $\boldsymbol{X}\in S^n(\mathfrak{g}[1])$. Thus, by abuse of notation, we may write

$$\mathrm{At}^{\nabla}_{(\mathfrak{g}[1],d_{\mathrm{CE}})} = -\sum_{k\geq 2} q_k.$$

Proposition 3.4.6. *Let $\mathfrak{g}[1]$ be an $L_\infty[1]$ algebra with multibrackets $q_k : S^k(\mathfrak{g}[1]) \to \mathfrak{g}[1]$, $k\geq 1$. Then the Atiyah class $\alpha_{(\mathfrak{g}[1],d_{\mathrm{CE}})}$ of the dg manifold $(\mathfrak{g}[1],d_{\mathrm{CE}})$ is*

$$\begin{aligned}\alpha_{(\mathfrak{g}[1],d_{\mathrm{CE}})} = -\big[\sum_{k\geq 2} q_k\big] &\in H^1_{\mathrm{CE}}\big(\mathfrak{g}[1],(\mathfrak{g}[1])^\vee\otimes(\mathfrak{g}[1])^\vee\otimes\mathfrak{g}[1]\big)\\ &\cong H^1\big(\Gamma(T^\vee_{\mathfrak{g}[1]}\otimes \operatorname{End} T_{\mathfrak{g}[1]})^\bullet,\mathcal{Q}\big).\end{aligned}$$

Remark 3.4.7. We can rephrase Proposition 3.4.6 in terms of multibrackets of L_∞ algebra $\mathfrak{g}$ instead of $L_\infty[1]$ algebra $\mathfrak{g}[1]$. For a finite dimensional L_∞ algebra $\mathfrak{g}$ equipped with multibrackets $l_k:\Lambda^k\mathfrak{g}\to\mathfrak{g}$ of degree $2-k$ for $k\geq 1$, the Atiyah class $\alpha_{(\mathfrak{g}[1],d_{\mathrm{CE}})}$ of the dg manifold $(\mathfrak{g}[1],d_{\mathrm{CE}})$ is

$$\alpha_{(\mathfrak{g}[1],d_{\mathrm{CE}})} = \big[\sum_{k\geq 2} l_k\big] \in H^0_{\mathrm{CE}}\big(\mathfrak{g},\mathfrak{g}^\vee\otimes\mathfrak{g}^\vee\otimes\mathfrak{g}\big) \cong H^1\big(\Gamma(T^\vee_{\mathfrak{g}[1]}\otimes \operatorname{End} T_{\mathfrak{g}[1]})^\bullet,\mathcal{Q}\big),$$

where $H^0_{\mathrm{CE}}\big(\mathfrak{g},\mathfrak{g}^\vee\otimes\mathfrak{g}^\vee\otimes\mathfrak{g}\big)$ denotes the 0-th Chevalley–Eilenberg cohomology of the L_∞ algebra $\mathfrak{g}$ with values in the tensor product of adjoint and coadjoint modules $\mathfrak{g}^\vee\otimes\mathfrak{g}^\vee\otimes\mathfrak{g}$.

Since the trivial connection ∇ is flat, by the second part of Theorem 3.3.7, we know that

$$R_n = \frac{1}{n}\widetilde{d^\nabla} R_{n-1} \in \Gamma\big(\operatorname{Hom}(S^n(T_{\mathfrak{g}[1]}),T_{\mathfrak{g}[1]})\big)$$

for $n\geq 3$. As the connection ∇ is trivial, Equation (3.32) implies that

$$\begin{aligned}&\widetilde{d^\nabla} R_{n-1}\left(\frac{\partial}{\partial x^{i_1}}\odot\cdots\odot\frac{\partial}{\partial x^{i_n}}\right)\\ &\quad=\sum_{k=1}^n \varepsilon_k(-1)^{|x^{i_k}|}\nabla_{\frac{\partial}{\partial x^{i_k}}}\left(R_{n-1}\left(\frac{\partial}{\partial x^{i_1}}\odot\cdots\widehat{\frac{\partial}{\partial x^{i_k}}}\cdots\odot\frac{\partial}{\partial x^{i_n}}\right)\right)\\ &\quad=\sum_{k=1}^n \varepsilon_k(-1)^{|x^{i_k}|}\frac{\partial}{\partial x^{i_k}}\left(R_{n-1}\left(\frac{\partial}{\partial x^{i_1}}\odot\cdots\widehat{\frac{\partial}{\partial x^{i_k}}}\cdots\odot\frac{\partial}{\partial x^{i_n}}\right)\right)\end{aligned}$$

Here, $\epsilon_k = (-1)^{|x^{i_k}|(|x^{i_1}|\cdots+|x^{i_{k-1}}|)}$ is the Koszul sign. Starting from

$$R_2\left(\frac{\partial}{\partial x^{i_1}} \odot \frac{\partial}{\partial x^{i_2}}\right) = -(-1)^{|x^{i_1}|+|x^{i_2}|}\sum_j \frac{\partial^2 Q^j}{\partial x^{i_1}\partial x^{i_2}}\frac{\partial}{\partial x^j},$$

as in (3.51), we inductively obtain that

$$R_n\left(\frac{\partial}{\partial x^{i_1}} \odot \cdots \odot \frac{\partial}{\partial x^{i_n}}\right) = -(-1)^{|x^{i_1}|+\cdots+|x^{i_n}|}\sum_j \frac{\partial^n Q^j}{\partial x^{i_1}\cdots\partial x^{i_n}}\frac{\partial}{\partial x^j}.$$

According to Corollary 3.3.4, we obtain the following

Proposition 3.4.8. *Let $\mathfrak{g}[1]$ be a finite dimensional $L_\infty[1]$ algebra with multibrackets $q_k : S^k(\mathfrak{g}[1]) \to \mathfrak{g}[1]$, $k \geq 1$. Let $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ be its corresponding dg manifold. Choose the trivial connection. Then the multibrackets $\{\lambda_n\}_{n\geq 1}$ of the Kapranov $L_\infty[1]$ algebra structure on* $\mathrm{Hom}\,(S(\mathfrak{g}[1]), \mathfrak{g}[1]) \cong \widehat{S}(\mathfrak{g}[1])^\vee \otimes \mathfrak{g}[1]$*, being identified with $\mathfrak{X}(\mathfrak{g}[1])$ as in Equation* (3.50)*, are given as follows.*

1. *The unary bracket λ_1 coincides with the Chevalley–Eilenberg differential with values in the $L_\infty[1]$-adjoint module $\mathfrak{g}[1]$:*

$$\lambda_1 = d_{\mathrm{CE}}^{\mathfrak{g}[1]} : \widehat{S}(\mathfrak{g}[1])^\vee \otimes \mathfrak{g}[1] \to \widehat{S}(\mathfrak{g}[1])^\vee \otimes \mathfrak{g}[1]$$

2. *For any $n \geq 2$, λ_n is $\widehat{S}(\mathfrak{g}[1])^\vee$-linear in each of its n argument, and therefore can be considered as a linear map*

$$\lambda_n : S^n(\mathfrak{g}[1]) \to \widehat{S}(\mathfrak{g}[1])^\vee \otimes \mathfrak{g}[1]$$

   which is completely determined by

$$\lambda_n(\boldsymbol{X}) = \sum_{k=n}^{\infty} q_k(\boldsymbol{X} \odot -), \quad n \geq 2,$$

   where $\boldsymbol{X} \in S^n(\mathfrak{g}[1])$, and each $q_k(\boldsymbol{X} \odot -) : S^{k-n}(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ is defined by $\boldsymbol{Y} \mapsto q_k(\boldsymbol{X} \odot \boldsymbol{Y})$ for all $\boldsymbol{Y} \in S^{k-n}(\mathfrak{g}[1])$.

Example 3.4.9. If $\mathfrak{g}$ is a finite dimensional Lie algebra, then the Kapranov L_∞ algebra (i.e. (-1)-shifted Kapranov $L_\infty[1]$ algebra) of the dg manifold $(\mathfrak{g}[1], d_{\mathrm{CE}})$ is the dgla $\Lambda\mathfrak{g}^\vee \otimes \mathfrak{g}$, where the differential is the Chevalley–Eilenberg differential $d_{\mathrm{CE}}^{\mathfrak{g}}$ of the $\mathfrak{g}$-module $\mathfrak{g}$ (for the adjoint action), and the Lie bracket is $[\xi \otimes x, \eta \otimes y] = \xi \wedge \eta \otimes [x, y]$ for homogeneous $\xi, \eta \in \Lambda\mathfrak{g}^\vee$ and $x, y \in \mathfrak{g}$.

3.4.2 dg manifolds associated with complex manifolds and integrable distributions

Every complex manifold X determines a dg manifold $(T_X^{0,1}[1], \bar{\partial})$—see Example 2.2.2. This section is devoted to the description of the corresponding Kapranov $L_\infty[1]$ algebra. Recall that for a Kähler manifold X, Kapranov obtained an explicit description of an $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$, where the unary bracket is the Dolbeault operator $\bar{\partial}$ and the binary bracket is the Dolbeault cocycle of the Atiyah class of T_X [23, Theorem 2.6]. Kapranov proved the existence of an $L_\infty[1]$ algebra structure associated with the Atiyah class of the holomorphic tangent bundle of any complex manifold using formal geometry and PROP [23, Theorem 4.3]. See Theorem 3.4.11 below for the Dolbeault representations. Since $T_X^{0,1} \subset T_{\mathbb{C}}X$ is a complex integrable distribution, we will consider general integrable distributions over $\mathbb{K}$. Indeed such $L_\infty[1]$ algebra structures can be obtained in a more general perspective in terms of Lie pairs [31]. We recall its construction briefly below.

Let $F \subseteq T_{\mathbb{K}}M$ be an integrable distribution. Then $(F[1], d_F)$ is a dg manifold, whose algebra of smooth functions $C^\infty(F[1], \mathbb{K})$ is identified with $\Omega_F := \Gamma(\wedge F^\vee)$ and the homological vector field is the leafwise de Rham differential, i.e. the Chevalley–Eilenberg differential $d_F\colon \Omega_F^\bullet \to \Omega_F^{\bullet+1}$ of the Lie algebroid F. It is well known that the normal bundle $B := T_{\mathbb{K}}M/F$ is naturally an F-module, where the F-action is known as the **Bott connection** [9], defined by

$$\nabla_a^{\mathrm{Bott}} b = q([a, \tilde{b}]),$$

for all $a \in \Gamma(F)$, $b \in \Gamma(B)$ and $\tilde{b} \in \Gamma(T_{\mathbb{K}}M)$ such that $q(\tilde{b}) = b$. Here $q : T_{\mathbb{K}}M \to B$ denotes the canonical projection. Let $\mathcal{D}(M)$ be the space of $\mathbb{K}$-linear differential operators on M, and $R = C^\infty(M; \mathbb{K})$ be the space of $\mathbb{K}$-valued smooth functions on M. Then $\mathcal{D}(M)$ is an R-coalgebra equipped with the standard coproduct

$$\Delta\colon \mathcal{D}(M) \to \mathcal{D}(M) \otimes_R \mathcal{D}(M). \tag{3.53}$$

Let $\mathcal{D}(M)\Gamma(F) \subseteq \mathcal{D}(M)$ be the left ideal of $\mathcal{D}(M)$ generated by $\Gamma(F)$. Since

$$\Delta(\mathcal{D}(M)\Gamma(F)) \subseteq \mathcal{D}(M) \otimes_R \mathcal{D}(M)\Gamma(F) + \mathcal{D}(M)\Gamma(F) \otimes_R \mathcal{D}(M),$$

the coproduct (3.53) descends to a well-defined coproduct over R

$$\Delta : \mathcal{D}(B) \to \mathcal{D}(B) \otimes_R \mathcal{D}(B), \tag{3.54}$$

on the quotient space $\mathcal{D}(B) := \frac{\mathcal{D}(M)}{\mathcal{D}(M)\Gamma(F)}$. Hence $\mathcal{D}(B)$ is an R-coalgebra as well, called the R-coalgebra of *differential operators transverse to* F [54].

It is well known that $\mathcal{D}(B)$ is an F-module [31, 30], where the F-action is given by

$$a \cdot \overline{u} = \overline{a \circ u}, \tag{3.55}$$

for any $a \in \Gamma(F)$ and $u \in \mathcal{D}(M)$ — the symbol $\overline{x}$ denotes the image of x under the quotient map $\mathcal{D}(M) \to \mathcal{D}(B)$. Here $\circ$ denotes the composition of differential operators. Moreover, F acts on $\mathcal{D}(B)$ by coderivations. Indeed, the associated Chevalley–Eilenberg differential

$$d_F^{\mathcal{U}}: \quad \Omega_F^\bullet(\mathcal{D}(B)) \to \Omega_F^{\bullet+1}(\mathcal{D}(B))$$

is a coderivation of the Ω_F-linear coproduct

$$\Delta : \Omega_F(\mathcal{D}(B)) \to \Omega_F(\mathcal{D}(B)) \otimes_{\Omega_F} \Omega_F(\mathcal{D}(B))$$

extending the coproduct (3.54) on $\mathcal{D}(B)$. Thus $(\Omega_F(\mathcal{D}(B)), d_F^{\mathcal{U}}, \Delta)$ is a dg coalgebra over (Ω_F, d_F).

Let $j\colon B \to T_{\mathbb{K}}M$ be a splitting of the short exact sequence of vector bundles over M:

$$0 \to F \xrightarrow{i} T_{\mathbb{K}}M \xrightarrow{q} B \to 0. \tag{3.56}$$

Choose a torsion-free linear connection ∇^B of the vector bundle B, i.e. a $T_{\mathbb{K}}M$-connection on B satisfying the condition:

$$\nabla_X^B\big(q(Y)\big) - \nabla_Y^B\big(q(X)\big) - q\big([X,Y]\big) = 0,$$

for any $X, Y \in \Gamma\big(T_{\mathbb{K}}M\big)$. It is known [31, Lemma 5.2] that a torsion-free linear connection ∇^B automatically extends the Bott representation of F on B, that is, $\nabla_a^B \bar{X} = \nabla_a^{\mathrm{Bott}} \bar{X}$, $\forall a \in \Gamma\big(F\big)$ and $\bar{X} \in \Gamma\big(B\big)$.

According to [31, 30], the pair (j, ∇^B) determines an isomorphism of R-coalgebras

$$\overline{\mathrm{pbw}}\colon \Gamma(S(B)) \to \mathcal{D}(B),$$

called the PBW isomorphism for the Lie pair $(T_{\mathbb{K}}M, F)$, which is defined recursively by the relations

$$\begin{aligned} &\overline{\mathrm{pbw}}(f) = f, \quad \forall f \in R, \\ &\overline{\mathrm{pbw}}(b) = \overline{j(b)}, \quad \forall b \in \Gamma(B), \end{aligned}$$

and

$$\overline{\mathrm{pbw}}(b_1 \odot \cdots \odot b_n) = \frac{1}{n} \sum_{k=1}^{n} \left\{ j(b_k) \cdot \overline{\mathrm{pbw}}(b^{\{k\}}) - \overline{\mathrm{pbw}}(\nabla^B_{j(b_k)}(b^{\{k\}})) \right\},$$

where we keep the notation from (3.55) and $b^{\{k\}} = b_1 \odot \cdots \odot b_{k-1} \odot b_{k+1} \odot \cdots \odot b_n$. Extending this isomorphism of R-coalgebras Ω_F-linearly, we obtain an isomorphism of Ω_F-coalgebras

$$\overline{\mathrm{pbw}} \colon \Omega_F(S(B)) \to \Omega_F(\mathcal{D}(B)). \tag{3.57}$$

Transferring the coderivation $d_F^{\mathcal{U}}$ of $\Omega_F(\mathcal{D}(B))$ to $\Omega_F(S(B))$ via the isomorphism (3.57), we obtain a degree $+1$ coderivation $\overline{\delta}$ of $\Omega_F(S(B))$:

$$\overline{\delta} := (\overline{\mathrm{pbw}})^{-1} \circ d_F^{\mathcal{U}} \circ \overline{\mathrm{pbw}} : \Omega_F^\bullet(S(B)) \to \Omega_F^{\bullet+1}(S(B)).$$

Thus

$$\big(\Omega_F(S(B)), \overline{\delta}, \Delta\big)$$

is a dg coalgebra over (Ω_F, d_F).

By dualizing $\overline{\delta}$ over the dg algebra (Ω_F, d_F), we obtain a degree $+1$ derivation

$$D : \Omega_F^\bullet(\widehat{S}(B^\vee)) \to \Omega_F^{\bullet+1}(\widehat{S}(B^\vee)). \tag{3.58}$$

According to [31, Theorem 5.7], D in (3.58) can be expressed as

$$D = d_F^{\nabla^{\mathrm{Bott}}} + \sum_{k=2}^{\infty} \widetilde{\mathcal{R}}_k,$$

where

1. $d_F^{\nabla^{\mathrm{Bott}}}$ is the Chevalley–Eilenberg differential corresponding to the Bott connection of F on $\widehat{S}\big(B^\vee\big)$;

2. for any $k \geq 2$, $\widetilde{\mathcal{R}}_k : \Omega_F^\bullet(\widehat{S}(B^\vee)) \to \Omega_F^{\bullet+1}(\widehat{S}(B^\vee))$ is the $\Omega_F^\bullet$-linear degree $+1$ derivation acting by contraction induced from a section $\mathcal{R}_k \in \Omega_F^1(S^k(B^\vee) \otimes B)$;

3. $\mathcal{R}_2 \in \Omega_F^1(S^2(B^\vee) \otimes B)$ is the Atiyah 1-cocycle $\mathrm{At}_{T_{\mathbb{K}}M/F}^{\nabla^{\mathrm{Bott}}}$ associated with the connection ∇^B defined by

   $$\mathcal{R}_2(a, \bar{X}) = \nabla_a^B \nabla_X^B - \nabla_X^B \nabla_a^B - \nabla_{[a,X]}^B,$$

   for all $a \in \Gamma(F)$ and $X \in \Gamma(T_{\mathbb{K}}M)$, where $\bar{X} \in \Gamma\big(B\big)$ denotes the image of X under the quotient map $T_{\mathbb{K}}M \to T_{\mathbb{K}}M/F$.

A priori, $\mathcal{R}_2 \in \Omega_F^1(B^\vee \otimes \mathrm{End}(B))$, but the torsion-free assumption guarantees that it is indeed an element in $\Omega_F^1(S^2(B^\vee) \otimes B)$. Its cohomology class $\alpha_{T_{\mathbb{K}}M/F} \in \mathbb{H}_{\mathrm{CE}}^1(F, B^\vee \otimes \mathrm{End}(B))$ is independent of the choice of ∇^B and is called the Atiyah class of the Lie pair $(T_{\mathbb{K}}M, F)$ [9]. Note that $\Omega_F(\widehat{S}(B^\vee))$ is the algebra of functions on $F[1] \oplus B$. Thus $(F[1] \oplus B, D)$ is a dg manifold with support M, called a **Kapranov dg manifold** associated with the Lie pair $(T_{\mathbb{K}}M, F)$ [31]. One can prove that the various Kapranov dg manifold structures on $F[1] \oplus B$ resulting from all possible choices of splitting and connection are all isomorphic.

Theorem 3.4.10 ([31, Theorem 5.7])**.** *Let $F \subseteq T_{\mathbb{K}}M$ be an integrable distribution. The choice of a splitting $j\colon B \to T_{\mathbb{K}}M$ of the short exact sequence* (3.56) *and a torsion-free linear connection ∇^B of the vector bundle B determines an $L_\infty[1]$ algebra structure on the graded vector space $\Omega_F^\bullet(B)$ defined by a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of multibrackets such that each λ_k, with $k \geqslant 2$, is Ω_F-multilinear, and*

- *the unary bracket λ_1 is the Chevalley–Eilenberg differential $d_F^{\nabla^{\mathrm{Bott}}}$ associated with the Bott connection ∇^{Bott} of F on B;*
- *the binary bracket λ_2 is the map*
$$\lambda_2 : \Omega_F^{j_1}(B) \otimes \Omega_F^{j_2}(B) \to \Omega_F^{j_1+j_2+1}(B)$$
induced by the Atiyah 1*-cocycle $\mathcal{R}_2 \in \Omega_F^1(S^2(B^\vee) \otimes B)$;*
- *for every $k \geqslant 3$, the k-th multibracket λ_k is the composition of the wedge product*
$$\Omega_F^{j_1}(B) \otimes \cdots \otimes \Omega_F^{j_k}(B) \to \Omega_F^{j_1+\cdots+j_k}\big(B^{\otimes k}\big)$$
with the map
$$\Omega_F^{j_1+\cdots+j_k}\big(B^{\otimes k}\big) \to \Omega_F^{j_1+\cdots+j_k+1}(B)$$
induced by an element $R_k \in \Omega_F^1\big(S^k(B^\vee) \otimes B\big) \subset \Omega_F^1\big((B^\vee)^{\otimes k} \otimes B\big)$.

Moreover, the $L_\infty[1]$ algebra structure on $\Omega_F^\bullet(B)$ is unique up to isomorphisms in the sense that those resulting from all possible choices of splitting and connection are all isomorphic.

Any such $L_\infty[1]$ algebra structure on $\Omega_F^\bullet(B)$ is called a **Kapranov $L_\infty[1]$ algebra** of the integrable distribution F.

As a special case, consider a complex manifold X. The subbundle $F = T_X^{0,1} \subset T_{\mathbb{C}}X$ is an integrable distribution, and the normal bundle $B := T_{\mathbb{C}}X/T_X^{0,1}$ is naturally identified with $T_X^{1,0}$. Moreover, the Chevalley–Eilenberg differential associated with the Bott F-connection on $T_X^{1,0}$ becomes the Dolbeault operator
$$\bar{\partial}\colon \Omega_X^{0,\bullet}(T_X^{1,0}) \to \Omega_X^{0,\bullet+1}(T_X^{1,0}).$$

The following is an immediate consequence of Theorem 3.4.10, which extends Kapranov's construction for Kähler manifolds [23, Theorem 2.6] to all complex manifolds.

Theorem 3.4.11 ([31, Theorem 5.24]). *For a given complex manifold X, any torsion-free $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$ determines an $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ such that*

- *the unary bracket λ_1 is the Dolbeault operator*
$$\overline{\partial} : \Omega^{0,j}(T_X^{1,0}) \to \Omega^{0,j+1}(T_X^{1,0});$$
- *the binary bracket λ_2 is the map*
$$\lambda_2 : \Omega^{0,j_1}(T_X^{1,0}) \otimes \Omega^{0,j_2}(T_X^{1,0}) \to \Omega^{0,j_1+j_2+1}(T_X^{1,0})$$
induced by the Dolbeault representative of the Atiyah 1-cocycle $R_2 \in \Omega^{0,1}\big(S^2(T_X^{1,0})^\vee \otimes T_X^{1,0}\big)$;
- *for every $k \geqslant 3$, the k-th multibracket λ_k is the composition of the wedge product*
$$\Omega^{0,j_1}(T_X^{1,0}) \otimes \cdots \otimes \Omega^{0,j_k}(T_X^{1,0}) \to \Omega^{0,j_1+\cdots+j_k}\big((T_X^{1,0})^{\otimes k}\big)$$
with the map
$$\Omega^{0,j_1+\cdots+j_k}\big((T_X^{1,0})^{\otimes k}\big) \to \Omega^{0,j_1+\cdots+j_k+1}(T_X^{1,0})$$
induced by an element R_k of the subspace $\Omega^{0,1}\big(S^k\big((T_X^{1,0})^\vee\big) \otimes T_X^{1,0}\big)$ of $\Omega^{0,1}\big(((T_X^{1,0})^\vee)^{\otimes k} \otimes T_X^{1,0}\big)$, completely determined by the Atiyah 1-cocycle R_2, the curvature of $\nabla^{1,0}$, and their higher covariant derivatives.

Moreover, the $L_\infty[1]$ algebra structure on $\Omega^{0,\bullet}(T_X^{1,0})$ is unique up to isomorphisms.

Now we are ready to consider the Kapranov $L_\infty[1]$ algebra of the dg manifold $(F[1], d_F)$. Let
$$\tilde{\Phi} \colon \mathcal{D}(F[1]) \to \Omega_F(\mathcal{D}(B))$$
be the map defined by $\tilde{\Phi}(D) = \overline{\pi_*(D)}$, where $\pi_* \colon \mathcal{D}(F[1]) \to \Omega_F \otimes_R \mathcal{D}(M)$ is the pushforward map
$$\pi_*(D)(f) = D(\pi^* f), \qquad \forall D \in \mathcal{D}(F[1]), \forall f \in R$$
and $\overline{\pi_*(D)} \in \Omega_F(\mathcal{D}(B))$ denotes the class of $\pi_*(D)$ in $\Omega_F \otimes_R \frac{\mathcal{D}(M)}{\mathcal{D}(M)\Gamma(F)} \cong \Omega_F(\mathcal{D}(B))$.

Theorem 3.4.12 ([54, 10]). *There exists a contraction of dg Ω_F-modules*
$$\begin{tikzcd} (\mathcal{D}(F[1]), \mathcal{L}_Q) \arrow[loop left, "\tilde{H}"] \arrow[r, shift left, "\tilde{\Phi}"] & (\Omega_F(\mathcal{D}(B)), d_F^{\mathcal{U}}), \arrow[l, shift left, "\tilde{\Psi}"] \end{tikzcd} \tag{3.59}$$
where the projection $\tilde{\Phi}$ is a morphism of Ω_F-coalgebras.

Choose a torsion-free affine connection ∇ on $F[1]$. We write

$$\mathrm{pbw} : \Gamma\big(S(T_{F[1]})\big) \to \mathcal{D}(F[1])$$

for the corresponding Poincaré–Birkhoff–Witt map as in (3.8).

By conjugating the PBW maps pbw and $\overline{\mathrm{pbw}}$, respectively, on the left hand side and on the right hand side of (3.59), we obtain

Corollary 3.4.13. *There exists a contraction of dg Ω_F-modules*

$$H \circlearrowright \big(\Gamma\big(S(T_{F[1]})\big), \mathrm{pbw}^{-1} \circ \mathcal{L}_Q \circ \mathrm{pbw}\big) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \big(\Omega_F(S(B)), \overline{\mathrm{pbw}}^{-1} \circ d_F^{\mathcal{U}} \circ \overline{\mathrm{pbw}}\big),$$

where the projection $\Phi := \overline{\mathrm{pbw}}^{-1} \circ \tilde{\Phi} \circ \mathrm{pbw}$ is a morphism of Ω_F-coalgebras.

The projection Φ determines a sequence of maps $\{\phi_k\}_{k\geq 1}$ making the diagrams

$$\begin{tikzcd}
S^k_{\mathbb{K}}\big(\mathfrak{X}(F[1])\big) \arrow[r, "\phi_k"] \arrow[d] & \Omega_F(B) \\
\Gamma\big(S(T_{F[1]})\big) \arrow[r, "\Phi"] & \Omega_F\big(S(B)\big) \arrow[u]
\end{tikzcd} \tag{3.60}$$

commutative. Note that $\phi_1 : \mathfrak{X}(F[1]) \to \Omega_F(B)$ is the composition

$$\mathfrak{X}(F[1]) \xrightarrow{\pi_*} \Omega_F(T_{\mathbb{K}}M) \xrightarrow{q} \Omega_F(B).$$

Theorem 3.4.14. *Let $F \subseteq T_{\mathbb{K}}M$ be an integrable distribution. Then the sequence of Ω_F-multilinear maps $\{\phi_k\}_{k\geq 1}$ defined by the commutative diagrams (3.60) constitutes a quasi-isomorphism from the Kapranov $L_\infty[1]$ algebra $\mathfrak{X}(F[1])$ arising from the dg manifold $(F[1], d_F)$ to the Kapranov $L_\infty[1]$ algebra $\Omega_F^\bullet(B)$ arising (as in Theorem 3.4.10) from the integrable distribution F.*

As an immediate consequence, we have

Corollary 3.4.15. *For any complex manifold X, consider its corresponding dg manifold $(T_X^{0,1}[1], \bar{\partial})$ as in Example 2.2.2. The Kapranov $L_\infty[1]$ algebra $\mathfrak{X}(T_X^{0,1}[1])$ is quasi-isomorphic to the $L_\infty[1]$ algebra $\Omega^{0,\bullet}(T_X^{1,0})$—see Theorem 3.4.11. The quasi-isomorphism $\{\phi_k\}_{k\geq 1}$, in which each map ϕ_k is $\Omega_X^{0,\bullet}$-multilinear, is given by (3.60) (with $F = T_X^{0,1}$ and $B = T_X^{1,0}$), and in particular, the linear part $\phi_1 : \mathfrak{X}(T_X^{0,1}[1]) \to \Omega^{0,\bullet}(T_X^{1,0})$ is given by the composition*

$$\mathfrak{X}(T_X^{0,1}[1]) \xrightarrow{\pi_*} \Omega^{0,\bullet}(T_X^{\mathbb{C}}) \xrightarrow{\mathrm{pr}} \Omega^{0,\bullet}(T_X^{1,0}).$$

Chapter 4

Keller admissible triples and Duflo theorem

4.1 Hochschild complexes of differential graded algebras

In this section, we recall the explicit formulas of the Gerstenhaber bracket and the cup product on the Hochschild cochain complex of a differential graded algebra. These structures were first introduced by Gerstenhaber [18] for an ungraded ring, and were generalized to dg algebras by various authors in slightly different ways. The formulas in this section are obtained by composing the formulas in [19] with proper degree-shifting maps. More details can be found in Section 2.5.

Let (A, d_A) and (B, d_B) be dg algebras. Recall that a *dg A-B-bimodule* (X, d_X) is a graded A-B-bimodule X together with a differential d_X such that

$$d_X(axb) = d_A(a)xb + (-1)^{|a|} a d_X(x) b + (-1)^{|a|+|x|} ax d_B(b),$$

for $a \in A,\ x \in X,\ b \in B$.

Let (M, d_M) be a dg A-A-bimodule. A **Hochschild cochain** of degree (p, r) of A with values in M is an element in

$$\mathrm{Hoch}^{p,r}(A, M) := \mathrm{Hom}^r\left(A^{\otimes p}, M\right).$$

The **Hochschild cochain complex** of A with values in M is the space

$$\mathrm{Hoch}^{\bullet}_{\oplus}(A, M) := \bigoplus_{p+r=\bullet} \mathrm{Hoch}^{p,r}(A, M)$$

together with the differential $d_{\mathcal{H}} + \partial : \mathrm{Hoch}^{\bullet}_{\oplus}(A, M) \to \mathrm{Hoch}^{\bullet+1}_{\oplus}(A, M)$, where

$$\begin{aligned} d_{\mathcal{H}}(f)(a_0, \cdots, a_p) := & (-1)^{(p+r-1)+r|a_0|} a_0 f(a_1, \cdots, a_p) \\ & + \sum_{i=0}^{p-1} (-1)^{p+r+i} f(a_0, \cdots, a_i a_{i+1}, \cdots, a_p) \\ & + (-1)^r f(a_0, \cdots, a_{p-1}) a_p \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} & \partial(f)(a_1, \cdots, a_p) \\ & := d_M f(a_1, \cdots, a_p) - (-1)^r \sum_{i=1}^{p} (-1)^{|a_1|+\cdots+|a_{i-1}|} f(a_1, \cdots, d_A a_i, \cdots, a_p), \end{aligned} \tag{4.2}$$

for $f \in \mathrm{Hoch}^{p,r}(A, M)$, $a_0, \cdots, a_p \in A$. The cohomology of $\big(\mathrm{Hoch}^{\bullet}_{\oplus}(A, M), d_{\mathcal{H}} + \partial\big)$ is called the **Hochschild cohomology** of (A, d_A) with values in (M, d_M), denoted by $\mathrm{HH}^{\bullet}_{\oplus}(A, M)$. In the case $(M, d_M) = (A, d_A)$, we denote $\partial_A = \partial$, $\mathrm{Hoch}^{\bullet}_{\oplus}(A, d_A) = \mathrm{Hoch}^{\bullet}_{\oplus}(A, A)$ and $\mathrm{HH}^{\bullet}_{\oplus}(A, d_A) = \mathrm{HH}^{\bullet}_{\oplus}(A, A)$. We will omit d_A in the notations if the differential is clear from the context.

Remark 4.1.1. In the literature, the Hochschild cohomology of a dg algebra (A, d_A) is defined by derived functors which can be computed by the *product-total complex* of the double complex $\big(\mathrm{Hoch}(A), d_{\mathcal{H}}, \partial_A\big)$,

$$\mathrm{Hoch}^{\bullet}_{\Pi}(A) := \prod_{p+r=\bullet} \mathrm{Hoch}^{p,r}(A)$$

whose cohomology $\mathrm{HH}^{\bullet}_{\Pi}(A)$ is different from $\mathrm{HH}^{\bullet}_{\oplus}(A)$ in general.

The sum Hochschild cohomology $\mathrm{HH}^{\bullet}_{\oplus}(A)$ is sometimes referred as the compactly supported Hochschild cohomology [44].

Example 4.1.2. Let (A, d_A) be the dg algebra $(S(\mathbb{K}[1])^{\vee}, 0) \cong (\mathbb{K}[x]/(x^2), 0) \cong (\mathbb{K} \oplus \mathbb{K}x, 0)$, where x is a formal variable of degree one. It is straightforward to show that

$$\mathrm{HH}^0_{\oplus}(A) \cong \bigoplus_{n=0}^{\infty} \mathbb{K} \neq \prod_{n=0}^{\infty} \mathbb{K} \cong \mathrm{HH}^0_{\Pi}(A).$$

Let $f \in \mathrm{Hoch}^{p_1,r_1}(A)$ and $g \in \mathrm{Hoch}^{p_2,r_2}(A)$. The *cup product* $f \cup g \in \mathrm{Hoch}^{p_1+p_2,r_1+r_2}(A)$ is defined by the formula

$$\begin{aligned} & (f \cup g)(a_1, \cdots, a_{p_1+p_2}) \\ & := (-1)^{p_1 p_2 + r_2(|a_1|+\cdots+|a_{p_1}|+p_1)} f(a_1, \cdots, a_{p_1}) \cdot g(a_{p_1+1}, \cdots, a_{p_1+p_2}), \end{aligned} \tag{4.3}$$

for $a_1, \cdots, a_{p_1+p_2} \in A$.

Let $f \circ_i g \in \mathrm{Hoch}^{p_1+p_2-1, r_1+r_2}(A)$ be the *i-th composition*

$$f \circ_i g := f \circ (\mathrm{id}^{\otimes i-1} \otimes g \otimes \mathrm{id}^{\otimes p_1 - i}). \tag{4.4}$$

The *Gerstenhaber bracket* $[\![f, g]\!] \in \mathrm{Hoch}^{p_1+p_2-1, r_1+r_2}(A)$ of f and g is

$$\begin{aligned}[\![f, g]\!] := \sum_{i=1}^{p_1} (-1)^{(p_1-1)r_2+(i-1)(p_2-1)} f \circ_i g \\ - (-1)^{(p_1+r_1-1)(p_2+r_2-1)} \sum_{j=1}^{p_2} (-1)^{(p_2-1)r_1+(j-1)(p_1-1)} g \circ_j f.\end{aligned} \tag{4.5}$$

One can show that

$$d_{\mathcal{H}} = [\![\mu_A, -]\!], \qquad \partial_A = [\![d_A, -]\!],$$

where the multiplication $\mu_A : a \otimes b \mapsto a \cdot b$ in A is considered as a Hochschild cochain in $\mathrm{Hoch}^{2,0}(A)$.

The following proposition can be shown by a direct computation as in [18].

Proposition 4.1.3. *Let (A, d_A) be a dg algebra.*

(i) The shifted Hochschild cochain complex $\big(\mathrm{Hoch}^\bullet_\oplus(A)[1], d_{\mathcal{H}} + \partial_A, [\![-,-]\!]\big)$ together with the Gerstenhaber bracket $[\![-,-]\!]$ is a differential graded Lie algebra.

(ii) The Hochschild cochain complex $\big(\mathrm{Hoch}^\bullet_\oplus(A), d_{\mathcal{H}} + \partial_A, \cup\big)$ together with the cup product $\cup$ is a differential graded algebra.

Furthermore, the Hochschild cohomology $\mathrm{HH}^\bullet_\oplus(A)$ is a Gerstenhaber algebra.

4.2 Hochschild complexes of differential graded bimodules

Let (A, d_A) and (B, d_B) be dg algebras, and (X, d_X) be a dg A-B-bimodule. In [24], Keller constructed a dg category of two objects from (X, d_X) and studied its Hochschild cohomology which can be computed by a product-total complex $\mathrm{Hoch}^\bullet_\Pi(X)$. In this section, we consider the dg algebra $A \ltimes X \rtimes B$ and introduce the (sum) Hochschild complex $\mathrm{Hoch}^\bullet_\oplus(X)$ of X as a subcomplex of the Hochschild complex $\mathrm{Hoch}^\bullet_\oplus(A \ltimes X \rtimes B)$. This is a sum analogue of the Keller's complex $\mathrm{Hoch}^\bullet_\Pi(X)$.

Let $A \ltimes X \rtimes B$ be the dg algebra whose underlying graded vector space is the direct sum $A \oplus X \oplus B$, and whose multiplication and differential are defined, respectively, by the formulas

$$\begin{aligned}(a_1 + x_1 + b_1) \cdot (a_2 + x_2 + b_2) &:= a_1a_2 + (a_1x_2 + x_1b_2) + b_1b_2,\\ d(a_1 + x_1 + b_1) &:= d_A(a_1) + d_X(x_1) + d_B(b_1),\end{aligned}$$

for $a_1, a_2 \in A$, $x_1, x_2 \in X$, $b_1, b_2 \in B$. It is straightforward to show that $A \ltimes X \rtimes B$ is a dg algebra.

We will use the notations

$$\begin{aligned}\mathrm{Hoch}^{p,q,r}(A, X, B) &:= \mathrm{Hom}^r(A^{\otimes p} \otimes X \otimes B^{\otimes q}, X),\\ \mathrm{Hoch}^\bullet_\oplus(A, X, B) &:= \bigoplus_{p+q+r+1=\bullet} \mathrm{Hoch}^{p,q,r}(A, X, B),\\ \mathrm{Hoch}^\bullet_\oplus(X) &:= \mathrm{Hoch}^\bullet_\oplus(A) \oplus \mathrm{Hoch}^\bullet_\oplus(A, X, B) \oplus \mathrm{Hoch}^\bullet_\oplus(B).\end{aligned}$$

The space $\mathrm{Hoch}^\bullet_\oplus(X)$ can be embedded into $\mathrm{Hoch}^\bullet_\oplus(A \ltimes X \rtimes B)$ as follows:

$$\begin{aligned}\mathrm{Hoch}^p_\oplus(A) \hookrightarrow \mathrm{Hoch}^p_\oplus(A \ltimes X \rtimes B) &: f_A \mapsto i_A \circ f_A \circ \mathrm{pr}_A^{\otimes p},\\ \mathrm{Hoch}^q_\oplus(B) \hookrightarrow \mathrm{Hoch}^q_\oplus(A \ltimes X \rtimes B) &: f_B \mapsto i_B \circ f_B \circ \mathrm{pr}_B^{\otimes q},\end{aligned}$$

and for $n = p + q + r + 1$,

$$\mathrm{Hoch}^{p,q,r}(A, X, B) \hookrightarrow \mathrm{Hoch}^n_\oplus(A \ltimes X \rtimes B) : f_X \mapsto i_X \circ f_X \circ (\mathrm{pr}_A^{\otimes p} \otimes \mathrm{pr}_X \otimes \mathrm{pr}_B^{\otimes q}),$$

where i_A, i_B, i_X are the inclusions from A, B, X into $A \ltimes X \rtimes B$, respectively, and $\mathrm{pr}_A, \mathrm{pr}_B, \mathrm{pr}_X$ are the projections from $A \ltimes X \rtimes B$ onto A, B, X, respectively. We will omit i and pr by abuse of notation. With this embedding, one can show that the subspace $\mathrm{Hoch}^\bullet_\oplus(X)$ is closed under the differential, Gerstenhaber bracket and cup product in $\mathrm{Hoch}^\bullet_\oplus(A \ltimes X \rtimes B)$.

Proposition 4.2.1. *Let X be a dg A-B-bimodule. The subspace $\mathrm{Hoch}^\bullet_\oplus(X)$ is closed under the differential $d_{\mathcal{H}} + \partial$, Gerstenhaber bracket $[\![-,-]\!]$ and cup product $\cup$ in $\mathrm{Hoch}^\bullet_\oplus(A \ltimes X \rtimes B)$.*

In the rest of this section, we describe the differential, Gerstenhaber bracket and cup product on $\mathrm{Hoch}^\bullet_\oplus(X)$.

Let $d_{\mathcal{H}} + \partial$ be the restriction of the differential of $\mathrm{Hoch}^\bullet_\oplus(A \ltimes X \rtimes B)$ to $\mathrm{Hoch}^\bullet_\oplus(X)$. Here, $d_{\mathcal{H}}$ denotes the Hochschild differential, and ∂ denotes the differential induced by the dg structure of A, B and X. The Hochschild differential $d_{\mathcal{H}}$ can be decomposed as

$$d_{\mathcal{H}} = d_{\mathcal{H}}^{A} + d_{\mathcal{H}}^{AX} + d_{\mathcal{H}L}^{X} + d_{\mathcal{H}R}^{X} + d_{\mathcal{H}}^{XB} + d_{\mathcal{H}}^{B}$$

which acts on $\mathrm{Hoch}_{\oplus}^{\bullet}(X)$ as in the following diagram:

$$\begin{tikzcd}
 & \mathrm{Hoch}_{\oplus}^{\bullet}(A,X,B) \arrow[loop above, "d_{\mathcal{H}}^{X}=d_{\mathcal{H}L}^{X}+d_{\mathcal{H}R}^{X}"] & \\
\mathrm{Hoch}_{\oplus}^{\bullet}(A) \arrow[ur, "d_{\mathcal{H}}^{AX}"] \arrow[loop left, "d_{\mathcal{H}}^{A}"] & & \mathrm{Hoch}_{\oplus}^{\bullet}(B) \arrow[ul, "d_{\mathcal{H}}^{XB}"'] \arrow[loop right, "d_{\mathcal{H}}^{B}"]
\end{tikzcd}$$

where $d_{\mathcal{H}}^{A}$ and $d_{\mathcal{H}}^{B}$ are the Hochschild differentials of A and B, respectively, and the other components $d_{\mathcal{H}}^{AX}, d_{\mathcal{H}L}^{X}, d_{\mathcal{H}R}^{X}, d_{\mathcal{H}}^{XB}$ are described as follows.

Let $f_A \in \mathrm{Hoch}^{p,r}(A), f_B \in \mathrm{Hoch}^{q,r}(B), f_X \in \mathrm{Hoch}^{p,q,r}(A,X,B)$, $a_i \in A$, $x \in X$, and $b_j \in B$. We have

$$\begin{aligned}
d_{\mathcal{H}}^{AX}(f_A) &\in \mathrm{Hoch}^{p,0,r}(A,X,B), & d_{\mathcal{H}}^{XB}(f_B) &\in \mathrm{Hoch}^{0,q,r}(A,X,B),\\
d_{\mathcal{H}L}^{X}(f_X) &\in \mathrm{Hoch}^{p+1,q,r}(A,X,B), & d_{\mathcal{H}R}^{X}(f_X) &\in \mathrm{Hoch}^{p,q+1,r}(A,X,B)
\end{aligned}$$

which are defined by

$$\begin{aligned}
d_{\mathcal{H}}^{AX}(f_A)\big(a_1,\cdots,a_p;x\big) &:= (-1)^{r} f_A(a_1,\cdots,a_p)\cdot x,\\
d_{\mathcal{H}}^{XB}(f_B)\big(x;b_1,\cdots,b_q\big) &:= (-1)^{q+r-1+r|x|} x\cdot f_B(b_1,\cdots,b_q),
\end{aligned}$$

$$\begin{aligned}
&d_{\mathcal{H}L}^{X}(f_X)\big(a_0,\cdots,a_p;x\,;b_1,\cdots,b_q\big)\\
&\qquad := (-1)^{p+q+r+r|a_0|} a_0\cdot f_X(a_1,\cdots,a_p;x\,;b_1,\cdots,b_q)\\
&\qquad\quad + \sum_{i=0}^{p-1}(-1)^{p+q+r+i+1}\ f_X(a_0,\cdots,a_i a_{i+1},\cdots,a_p;x\,;b_1,\cdots,b_q)\\
&\qquad\quad + (-1)^{q+r+1}\ f_X(a_0,\cdots,a_{p-1};a_p\cdot x\,;b_1,\cdots,b_q),\\
&d_{\mathcal{H}R}^{X}(f_X)\big(a_1,\cdots,a_p;x\,;b_0,\cdots,b_q\big)\\
&\qquad := (-1)^{q+r+1} f_X(a_1,\cdots,a_p;x\cdot b_0\,;b_1,\cdots,b_q)\\
&\qquad\quad + \sum_{j=0}^{q-1}(-1)^{q+r+j}\ f_X(a_1,\cdots,a_p;x\,;b_0,\cdots,b_j b_{j+1},\cdots,b_q)\\
&\qquad\quad + (-1)^{r}\ f_X(a_1,\cdots,a_p;x\,;b_0,\cdots,b_{q-1})\cdot b_q.
\end{aligned}$$

For the component ∂, we have

$$\partial = \partial_A + \partial_B + \partial_X,$$

where ∂_A and ∂_B are the differentials defined by (4.2) on A and B, respectively, and the Hochschild cochain $\partial_X(f_X) \in \mathrm{Hoch}^{p,q,r+1}(A, X, B)$ is defined by

$$\begin{aligned}&\partial_X(f_X) := d_X \circ f_X \\ -&(-1)^r f_X \circ \Big(\sum_{i=0}^{p-1} \mathrm{id}^{\otimes i} \otimes d_A \otimes \mathrm{id}^{\otimes p+q-i} + \mathrm{id}^{\otimes p} \otimes d_X \otimes \mathrm{id}^{\otimes q} + \sum_{j=1}^{q} \mathrm{id}^{\otimes p+j} \otimes d_B \otimes \mathrm{id}^{\otimes q-j} \Big).\end{aligned}$$

The cochain complex $\big(\mathrm{Hoch}^\bullet_\oplus(X), d_{\mathcal{H}} + \partial\big)$ will be referred as the **Hochschild cochain complex** of the dg A-B-bimodule X.

Let $\circ_i$ be the i-th composition in $\mathrm{Hoch}^\bullet_\oplus(A \ltimes X \rtimes B)$, which is defined as in (4.4). We have

$$f_A \circ_i f_X = f_A \circ_i f_B = f_B \circ_i f_A = f_B \circ_i f_X = 0,$$

for any i, and

$$f_X \circ_j f_A = 0, \quad f_X \circ_k f'_X = 0, \quad f_X \circ_l f_B = 0,$$

if $j > p$, $k \neq p+1$, $l < p+2$. Here, f'_X is a Hochschild cochain in $\mathrm{Hoch}^\bullet_\oplus(A, X, B) \subset \mathrm{Hoch}^\bullet_\oplus(X)$.

Furthermore, it can be shown by (4.3) that

$$f_A \cup f_B = 0, \quad f_X \cup f_A = 0, \quad f_X \cup f'_X = 0, \quad f_B \cup f_A = 0, \quad f_B \cup f_X = 0,$$

and

$$f_A \cup f_X,\ f_X \cup f_B \in \mathrm{Hoch}^\bullet_\oplus(A, X, B).$$

As a consequence, we have the following

Proposition 4.2.2. *Let $\pi_A : \mathrm{Hoch}^\bullet_\oplus(X) \twoheadrightarrow \mathrm{Hoch}^\bullet_\oplus(A)$, $\pi_B : \mathrm{Hoch}^\bullet_\oplus(X) \twoheadrightarrow \mathrm{Hoch}^\bullet_\oplus(B)$ be the natural projections, and let $\iota_A : \mathrm{Hoch}^\bullet_\oplus(A) \hookrightarrow \mathrm{Hoch}^\bullet_\oplus(X)$, $\iota_B : \mathrm{Hoch}^\bullet_\oplus(B) \hookrightarrow \mathrm{Hoch}^\bullet_\oplus(X)$ be the natural inclusions.*

(i) The projections π_A and π_B are cochain maps.

(ii) The projections π_A, π_B and the inclusions ι_A, ι_B respect the compositions $\circ_i$ and thus preserve the Gerstenhaber brackets.

(iii) The projections π_A, π_B and the inclusions ι_A, ι_B preserve the cup products.

Note that the inclusions ι_A and ι_B are *not* cochain maps.

4.3 Keller admissible triples

Let A and B be dg algebras, and X be a dg A-B-bimodule. Let ρ_A and ρ_B be the maps

$$\begin{aligned} \rho_A &: A \to \mathrm{Hom}_{B^{\mathrm{op}}}(X,X), \quad \rho_A(a)(x) := a \cdot x, \\ \rho_B &: B^{\mathrm{op}} \to \mathrm{Hom}_A(X,X), \quad \rho_B(b)(x) := (-1)^{|x||b|} x \cdot b. \end{aligned}$$

The spaces $\mathrm{Hom}_{B^{\mathrm{op}}}(X,X)$ and $\mathrm{Hom}_A(X,X)$ are equipped with the differential $\partial_X = [d_X, -]$, where d_X is the given differential on X, and $[-,-]$ is the bracket induced by graded commutators. Note that the maps ρ_A and ρ_B are morphisms of dg algebras.

Since *sum* Hochschild cohomologies are not well-behaved under quasi-isomorphisms, we will require the action maps ρ_A and ρ_B satisfy a technical condition, called *weak cone-nilpotency* in this paper, so that they induce quasi-isomorphisms between Hochschild cohomologies. In order to define weak cone-nilpotency, we first introduce a condition that a sum Hochschild cohomology vanishes.

4.3.1 A vanishing condition of Hochschild cohomology

Unlike the *product* Hochschild cohomology $\mathrm{HH}_{\prod}(A,M)$ of A with values in M, which vanishes for an acyclic A-A-bimodule M, the *sum* Hochschild cohomology $\mathrm{HH}_{\oplus}(A,M)$ does not necessarily vanish for an acyclic A-A-bimodule M. In the following, we describe a condition on an acyclic dg A-A-bimodule M that the sum Hochschild cohomology $\mathrm{HH}_{\oplus}(A,M)$ vanishes.

Let (A, d_A) be a dg algebra and (M, d_M) be an *acyclic* dg A-A-bimodule. Observe that, since the underlying space of M is an acyclic dg vector space over $\mathbb{K}$, there is a homotopy operator $h : M \to M$ of degree -1 such that

$$d_M \circ h + h \circ d_M = \mathrm{id}_M .$$

We will refer such a homotopy operator $h : M \to M$ as a *contracting homotopy* for (M, d_M).

For each fixed n, we have the induced map

$$H := h_* : \mathrm{Hom}^{\bullet}(A^{\otimes n}, M) \to \mathrm{Hom}^{\bullet -1}(A^{\otimes n}, M)$$

of degree -1, defined by $H(f) = h \circ f$.

Lemma 4.3.1. *For each n, the induced operator H satisfies the homotopy equation*

$$\partial \circ H + H \circ \partial = \mathrm{id}_{\mathrm{Hom}(A^{\otimes n}, M)}$$

on $\big(\mathrm{Hom}^\bullet(A^{\otimes n}, M), \partial\big)$, where ∂ is the differential induced by the dg structures.

Proof. Let $f \in \mathrm{Hom}^r(A^{\otimes n}, M)$.

$$\begin{aligned}\partial \circ H(f)(a_1 \otimes \cdots \otimes a_n) &= d_M \circ H(f)(a_1 \otimes \cdots \otimes a_n)\\ &\quad - (-1)^{r-1} \sum_{i=1}^{n} (-1)^{|a_1| + \cdots + |a_{i-1}|} h \circ f(\cdots \otimes a_{i-1} \otimes d_A(a_i) \otimes a_{i+1} \otimes \cdots)\end{aligned}$$

$$\begin{aligned}H \circ \partial(f)(a_1 \otimes \cdots \otimes a_n) &= h \circ d_M(f)(a_1 \otimes \cdots \otimes a_n)\\ &\quad - (-1)^{r} \sum_{i=1}^{n} (-1)^{|a_1| + \cdots + |a_{i-1}|} h \circ f(\cdots \otimes a_{i-1} \otimes d_A(a_i) \otimes a_{i+1} \otimes \cdots)\end{aligned}$$

Thus we have $(\partial \circ H + H \circ \partial)(f) = (d_M \circ h + h \circ d_M) \circ (f) = f$. □

As a result, we have an operator $H : \mathrm{Hoch}_\oplus^\bullet(A, M) \to \mathrm{Hoch}_\oplus^{\bullet - 1}(A, M)$ of degree -1 on the Hochschild cochain complex $\big(\mathrm{Hoch}_\oplus^\bullet(A, M), d_{\mathcal{H}} + \partial\big)$. However, it is *not* a contracting homotopy. Indeed, we have

$$H \circ (\partial + d_{\mathcal{H}}) + (\partial + d_{\mathcal{H}}) \circ H - \mathrm{id}_{\mathrm{Hoch}_\oplus(A,M)} = H \circ d_{\mathcal{H}} + d_{\mathcal{H}} \circ H$$

which does *not* vanish in general.

For each n, we define a sequence of maps

$$\mathfrak{h}_k : \mathrm{Hom}^\bullet(A^{\otimes n}, M) \to \mathrm{Hom}^{\bullet - k - 1}(A^{\otimes n+k}, M), \qquad k \geq 0, \tag{4.6}$$

by

$$\mathfrak{h}_k := (H \circ d_{\mathcal{H}})^k \circ H = H \circ (d_{\mathcal{H}} \circ H)^k.$$

In particular, $\mathfrak{h}_0 = H = h_*$. Furthermore, we have

$$\mathfrak{h} := \sum_{k=0}^{\infty} (-1)^k \mathfrak{h}_k : \mathrm{Hom}^\bullet(A^{\otimes n}, M) \to \prod_{k=0}^{\infty} \mathrm{Hom}^{\bullet - k - 1}(A^{\otimes n+k}, M),$$

which defines a map $\mathfrak{h} : \mathrm{Hoch}_\oplus^\bullet(A, M) \to \mathrm{Hoch}_\Pi^{\bullet - 1}(A, M)$ of degree -1. Note that this map $\mathfrak{h}$ depends on a choice of contracting homotopy h for M.

Lemma 4.3.2. *The map* $\mathfrak{h} : \mathrm{Hoch}^{\bullet}_{\oplus}(A, M) \to \mathrm{Hoch}^{\bullet-1}_{\Pi}(A, M)$ *satisfies the equation*

$$\mathfrak{h} \circ (\partial + d_{\mathcal{H}})(f) + (\partial + d_{\mathcal{H}}) \circ \mathfrak{h}(f) = f$$

for $f \in \mathrm{Hoch}^{\bullet}_{\oplus}(A, M)$.

Proof. It suffices to show that $\partial\mathfrak{h}_k + \mathfrak{h}_k\partial = d_{\mathcal{H}}\mathfrak{h}_{k-1} + \mathfrak{h}_{k-1}d_{\mathcal{H}}$. Recall that we have

$$\begin{gathered}\partial H + H\partial = \mathrm{id} \\ \partial d_{\mathcal{H}} + d_{\mathcal{H}}\partial = 0 \\ d_{\mathcal{H}}^2 = 0.\end{gathered}$$

We prove the assertion by induction on k. For $k = 1$, we consider $\mathfrak{h}_0 = H$ and $\mathfrak{h}_1 = Hd_{\mathcal{H}}H$. Since

$$\partial(Hd_{\mathcal{H}}H) = (\mathrm{id} - H\partial)d_{\mathcal{H}}H = d_{\mathcal{H}}H + Hd_{\mathcal{H}}\partial H = d_{\mathcal{H}}H + Hd_{\mathcal{H}}(\mathrm{id} - H\partial),$$

we have

$$\partial\mathfrak{h}_1 + \mathfrak{h}_1\partial = d_{\mathcal{H}}\mathfrak{h}_0 + \mathfrak{h}_0 d_{\mathcal{H}}.$$

Suppose that the equation $\partial\mathfrak{h}_n + \mathfrak{h}_n\partial = d_{\mathcal{H}}\mathfrak{h}_{n-1} + \mathfrak{h}_{n-1}d_{\mathcal{H}}$ holds for $n = k$. Then we have

$$\begin{aligned}\partial\mathfrak{h}_{k+1} = \partial H d_{\mathcal{H}}\mathfrak{h}_k = (\mathrm{id} - H\partial)d_{\mathcal{H}}\mathfrak{h}_k = d_{\mathcal{H}}\mathfrak{h}_k + Hd_{\mathcal{H}}\partial\mathfrak{h}_k = \\ = d_{\mathcal{H}}\mathfrak{h}_k + Hd_{\mathcal{H}}(d_{\mathcal{H}}\mathfrak{h}_{k-1} + \mathfrak{h}_{k-1}d_{\mathcal{H}} - \mathfrak{h}_k\partial) = d_{\mathcal{H}}\mathfrak{h}_k + \mathfrak{h}_k d_{\mathcal{H}} - \mathfrak{h}_{k+1}\partial.\end{aligned}$$

This proves the lemma. □

As a result, the map $\mathfrak{h} : \mathrm{Hoch}^{\bullet}_{\oplus}(A, M) \to \mathrm{Hoch}^{\bullet-1}_{\Pi}(A, M)$ defines a contracting homotopy for the sum Hochschild complex $\big(\mathrm{Hoch}^{\bullet}_{\oplus}(A, M), d_{\mathcal{H}} + \partial\big)$ if and only if for each $f \in \mathrm{Hoch}^{\bullet}_{\oplus}(A, M)$, there exists an integer $N = N(f)$ such that $\mathfrak{h}_k(f) = 0$ if $k > N$.

Definition 4.3.3. We say an acyclic dg A-A-bimodule M is **pointwisely nilpotent** if there exists a contracting operator h for (M, d_M) such that the induced sequence $\{\mathfrak{h}_k\}_{k=0}^{\infty}$ of maps defined in (4.6) satisfies the following: for each $f \in \mathrm{Hoch}^{\bullet}_{\oplus}(A, M)$, there exists an integer $N = N(f)$ such that $\mathfrak{h}_k(f) = 0$ for all $k > N$.

Proposition 4.3.4. *Let* A *be a dg algebra, and* M *be an acyclic dg* A*-*A*-bimodule. Then* $\mathrm{HH}^{\bullet}_{\oplus}(A, M) = 0$ *if* M *is pointwisely nilpotent.*

If the dg algebra A is finite dimensional, the pointwise nilpotency of an acyclic dg A-A-bimodule M is obtained by an ascending filtration of M compatible with a contracting homotopy for M.

Lemma 4.3.5. *Let A be a finite dimensional dg algebra. An acyclic dg A-A-bimodule M is pointwisely nilpotent if there exists an ascending filtration*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M$$

of A-A-subbimodules of M (not necessarily closed under the dg structure) and a contracting homotopy h for M such that $\bigcup_{q\geq 0} M_q = M$ and

$$h(M_{q+1}) \subset M_q$$

for all $q \geq 0$.

Proof. Let $f \in \mathrm{Hom}^r(A^{\otimes p}, M)$. Observe that if $f(A^{\otimes p}) \subset M_q$, then $(d_{\mathcal{H}} \circ H)(f) \in \mathrm{Hom}^{r-1}(A^{\otimes p+1}, M)$ and

$$(d_{\mathcal{H}} \circ H)(f)(A^{\otimes p+1}) \subset A \cdot h(M_q) + h(M_q) \cdot A \subset M_{q-1}. \tag{4.7}$$

Since A is finite dimensional, there is $k \geq 0$ such that $f(A^{\otimes p}) \subset M_k$. By applying (4.7) repeatedly, we have

$$\mathfrak{h}_k(f)(A^{\otimes p+k}) = H \circ (d_{\mathcal{H}} \circ H)^k(f)(A^{\otimes p+k}) \subset h(M_0) = \{0\}$$

which proves the lemma. □

4.3.2 Weak cone-nilpotency and Keller admissible triples

Definition 4.3.6. Let A be a dg algebra, and M and N be dg A-A-bimodules. A quasi-isomorphism $\phi : M \to N$ of A-A-bimodules is said to be **weakly cone-nilpotent** if one of the following conditions is satisfied:

(i) The mapping cone of ϕ is pointwisely nilpotent.

(ii) The map ϕ has a right inverse whose mapping cone is pointwisely nilpotent.

(iii) The map ϕ has a left inverse whose mapping cone is pointwisely nilpotent.

Let $\phi : \mathcal{C}^\bullet \to \mathcal{D}^\bullet$ be a map of cochain complexes $(\mathcal{C}^\bullet, d_{\mathcal{C}})$ and $(\mathcal{D}^\bullet, d_{\mathcal{D}})$. In the present paper, among various conventions, the mapping cone of ϕ is a cochain complex $\mathrm{Cone}^\bullet(\phi) = (\mathcal{C}^{\bullet+1} \oplus \mathcal{D}^\bullet, d_{\mathrm{Cone}(\phi)})$ where the differential is defined by

$$d_{\mathrm{Cone}(\phi)}(c, d) = \big(- d_{\mathcal{C}}(c), \phi(c) + d_{\mathcal{D}}(d)\big)$$

for $c \in \mathcal{C}$ and $d \in \mathcal{D}$. Note that if $(\mathcal{C}, d_{\mathcal{C}})$ and $(\mathcal{D}, d_{\mathcal{D}})$ are dg A-A-bimodules, then $\mathrm{Cone}(\phi)$ also carries the dg A-A-bimodule structures.

Lemma 4.3.7. *Let $\phi : M \to N$ be a quasi-isomorphism of A-A-bimodules. If ϕ is weakly cone-nilpotent, then the induced map*

$$\phi_* : \mathrm{Hoch}_\oplus(A, M) \to \mathrm{Hoch}_\oplus(A, N)$$

is a quasi-isomorphism.

Proof. If the mapping cone $\mathrm{Cone}(\phi)$ of ϕ is pointwisely nilpotent, then the lemma follows from Proposition 4.3.4 and the fact that

$$\mathrm{Cone}(\phi_*) \cong \mathrm{Hoch}_\oplus(A, \mathrm{Cone}(\phi))$$

where $\mathrm{Cone}(\phi_*)$ is the mapping cone of the pushforward map $\phi_* : \mathrm{Hoch}_\oplus(A, M) \to \mathrm{Hoch}_\oplus(A, N)$.

Let τ be a right/left inverse of ϕ. By the same reason, if the mapping cone of τ is pointwisely nilpotent, then the pushforward map $\tau_* : \mathrm{Hoch}_\oplus(A, N) \to \mathrm{Hoch}_\oplus(A, M)$ is a quasi-isomorphism. Since τ_* is a right/left inverse of ϕ_*, the map ϕ_* is also a quasi-isomorphism. $\square$

Definition 4.3.8. A triple (A, X, B) is called a **Keller admissible triple** if (i) the action maps

$$\begin{aligned} \rho_A &: (A, d_A) \to \big(\mathrm{Hom}_{B^{\mathrm{op}}}(X, X), \partial_X\big), \\ \rho_B &: (B^{\mathrm{op}}, d_B) \to \big(\mathrm{Hom}_A(X, X), \partial_X\big) \end{aligned}$$

are weakly cone-nilpotent quasi-isomorphisms, and (ii) the sequences

$$0 \to \mathrm{Hom}_A(X, X) \hookrightarrow \mathrm{Hom}(X, X) \xrightarrow{d^A_{\mathcal{H}}} \cdots \xrightarrow{d^A_{\mathcal{H}}} \mathrm{Hom}(A^{\otimes n} \otimes X, X) \xrightarrow{d^A_{\mathcal{H}}} \cdots \quad (4.8)$$

$$0 \to \mathrm{Hom}_{B^{\mathrm{op}}}(X, X) \hookrightarrow \mathrm{Hom}(X, X) \xrightarrow{d^B_{\mathcal{H}}} \cdots \xrightarrow{d^B_{\mathcal{H}}} \mathrm{Hom}(X \otimes B^{\otimes n}, X) \xrightarrow{d^B_{\mathcal{H}}} \cdots \quad (4.9)$$

are exact.

The main theorem of this paper is the following

Theorem 4.3.9. *Let (A, X, B) be a Keller admissible triple. The two projections*

$$\begin{tikzcd} & \mathrm{Hoch}^\bullet_\oplus(X) \arrow[dl, two heads, "\pi_A"'] \arrow[dr, two heads, "\pi_B"] & \\ \mathrm{Hoch}^\bullet_\oplus(A) & & \mathrm{Hoch}^\bullet_\oplus(B) \end{tikzcd}$$

induce isomorphisms of Gerstenhaber algebras on cohomologies.

The rest of the section is devoted to proving Theorem 4.3.9.

4.3.3 Hochschild complexes with values in $\mathrm{Hom}_{B^{\mathrm{op}}}(X,X)$ and $\mathrm{Hom}_A(X,X)$

Since $\mathrm{Hom}_{B^{\mathrm{op}}}(X,X)$ is equipped with the A-A-bimodule structure

$$(a\cdot f)(x) := a\cdot f(x), \qquad (f\cdot a)(x) := f(a\cdot x),$$

we have the Hochschild complex $\mathrm{Hoch}_{\oplus}^{\bullet}\big(A, \mathrm{Hom}_{B^{\mathrm{op}}}(X,X)\big)$ of A with values in $\mathrm{Hom}_{B^{\mathrm{op}}}(X,X)$. Similarly, with the B-B-bimodule structure

$$(b\cdot f)(x) := (-1)^{|b|(|f|+|x|)} f(x\cdot b), \qquad (f\cdot b)(x) := (-1)^{|b||x|} f(x)\cdot b,$$

on $\mathrm{Hom}_A(X,X)$, we have the Hochschild complex $\mathrm{Hoch}_{\oplus}^{\bullet}\big(B, \mathrm{Hom}_A(X,X)\big)$ with values in $\mathrm{Hom}_A(X,X)$.

Suppose X is a dg A-B-bimodule such that the sequence (4.8) is exact. For each p, r, since the functor $\mathrm{Hom}_{\mathbb{K}}(A^{\otimes p}, -)$ is exact, we have an exact sequence of vector spaces:

$$\begin{aligned} 0 \to \mathrm{Hom}^r\left(A^{\otimes p}, \mathrm{Hom}_{B^{\mathrm{op}}}(X,X)\right) &\hookrightarrow \mathrm{Hom}^r\left(A^{\otimes p}, \mathrm{Hom}(X,X)\right) \to \\ \to \mathrm{Hom}^r\left(A^{\otimes p}, \mathrm{Hom}(X\otimes B, X)\right) &\to \mathrm{Hom}^r\left(A^{\otimes p}, \mathrm{Hom}(X\otimes B^{\otimes 2}, X)\right) \to \cdots \end{aligned} \tag{4.10}$$

Under the isomorphism

$$\mathrm{Hom}^r(A^{\otimes p}\otimes X\otimes B^{\otimes q}, X) \cong \mathrm{Hom}^r\left(A^{\otimes p}, \mathrm{Hom}(X\otimes B^{\otimes q}, X)\right),$$

the sequence (4.10) can be rephrased as the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}^r\left(A^{\otimes p}, \mathrm{Hom}_{B^{\mathrm{op}}}(X,X)\right) \xhookrightarrow{\Phi^{p,r}} \mathrm{Hoch}^{p,0,r}(A,X,B) \xrightarrow{d_{\mathcal{H}R}^X} \\ \xrightarrow{d_{\mathcal{H}R}^X} \mathrm{Hoch}^{p,1,r}(A,X,B) \xrightarrow{d_{\mathcal{H}R}^X} \mathrm{Hoch}^{p,2,r}(A,X,B) \xrightarrow{d_{\mathcal{H}R}^X} \cdots \end{aligned}$$

for each p, r.

We define $\Phi = \sum_{p,r} (-1)^r\ \Phi^{p,r}$. Then we have a double complex

$$\begin{tikzcd} & \vdots & \vdots & \vdots & \\ 0 \arrow[r] & \mathcal{C}^{-1,n+1} \arrow[u] \arrow[r, "\Phi"] & \mathcal{C}^{0,n+1} \arrow[u] \arrow[r, "d_{\mathcal{H}R}^X"] & \mathcal{C}^{1,n+1} \arrow[u] \arrow[r, "d_{\mathcal{H}R}^X"] & \cdots \\ 0 \arrow[r] & \mathcal{C}^{-1,n} \arrow[u, "d_{\mathcal{H}}+\partial"] \arrow[r, "\Phi"] & \mathcal{C}^{0,n} \arrow[u, "d_{\mathcal{H}L}^X+\partial_X"'] \arrow[r, "d_{\mathcal{H}R}^X"] & \mathcal{C}^{1,n} \arrow[u, "d_{\mathcal{H}L}^X+\partial_X"'] \arrow[r, "d_{\mathcal{H}R}^X"] & \cdots \\ & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \end{tikzcd} \tag{4.11}$$

where

$$\mathcal{C}^{-1,n} = \bigoplus_{p+r=n-1} \operatorname{Hom}^r\big(A^{\otimes p}, \operatorname{Hom}_{B^{\mathrm{op}}}(X,X)\big),$$
$$\mathcal{C}^{q,n} = \bigoplus_{p+r+1=n} \operatorname{Hoch}^{p,q,r}(A,X,B), \quad q \geq 0.$$

Here, the complex $\big(\mathcal{C}^{-1,\bullet+1},\ d_{\mathcal{H}}+\partial\big)$ is the Hochschild cochain complex of A with values in $\operatorname{Hom}_{B^{\mathrm{op}}}(X,X)$, and the total complex $\big(\bigoplus_{\substack{q\geq 0,\\ q+n=\bullet}} \mathcal{C}^{q,n},\ d^X_{\mathcal{H}L}+\partial_X+d^X_{\mathcal{H}R}\big)$ is a subcomplex of $\big(\operatorname{Hoch}^\bullet_\oplus(X),\ d^X_{\mathcal{H}}+\partial_X\big)$.

Similarly, one has the map $\Psi : \operatorname{Hoch}^\bullet_\oplus(B, \operatorname{Hom}_A(X,X)) \to \operatorname{Hoch}^{\bullet+1}_\oplus(A,X,B)$,

$$\Psi(f)(x\,; b_1,\cdots,b_q) := (-1)^{q+r-1}(-1)^{|x|(|b_1|+\cdots+|b_q|)} f(b_1\otimes\cdots\otimes b_q)(x),$$

for $f \in \operatorname{Hom}^r(B^{\otimes q}, \operatorname{Hom}_A(X,X))$ and has a double complex analogous to (4.11).

We need the following lemma in Homological algebra [57, Lemma 2.7.3]:

Lemma 4.3.10 (Acyclic Assembly Lemma)**.** *Let $\mathcal{B}^{p,q}$ be a double complex, equipped with differentials*

$$d_1 : \mathcal{B}^{p,q} \to \mathcal{B}^{p+1,q}$$
$$d_2 : \mathcal{B}^{p,q} \to \mathcal{B}^{p,q+1}.$$

Suppose that the 2nd quadrant of $\mathcal{B}$ vanishes (i.e. $\mathcal{B}^{p,q} = 0$ if $p<0$ and $q>0$). If every row of $\mathcal{B}$ is exact, then the direct sum total complex $\operatorname{tot}^\oplus(\mathcal{B})$ is acyclic.

Applying Lemma 4.3.10 to (4.11), we get

Lemma 4.3.11. *Let A and B be dg algebras, and X be a dg A-B-bimodule. If the sequence* (4.8) *is exact, then the map*

$$\Phi : \big(\operatorname{Hoch}^\bullet_\oplus(A, \operatorname{Hom}_{B^{\mathrm{op}}}(X,X))[-1],\ -(d_{\mathcal{H}}+\partial)\big) \to \big(\operatorname{Hoch}^\bullet_\oplus(A,X,B),\ d^X_{\mathcal{H}}+\partial_X\big),$$

is a quasi-isomorphism.

Similarly, if the sequence (4.9) *is exact, then the map*

$$\Psi : \big(\operatorname{Hoch}^\bullet_\oplus(B, \operatorname{Hom}_A(X,X))[-1],\ -(d_{\mathcal{H}}+\partial)\big) \to \big(\operatorname{Hoch}^\bullet_\oplus(A,X,B),\ d^X_{\mathcal{H}}+\partial_X\big),$$

is a quasi-isomorphism.

Proof. We show the assertion for the first part, and the second part follows from a similar argument.

It is clear that the map Φ is a cochain map. Observe that the mapping cone $\operatorname{Cone}(\Phi)$ of Φ is the direct sum total complex $\operatorname{tot}^\oplus(\mathcal{C})$ of (4.11). By Lemma 4.3.10, the total complex $\operatorname{tot}^\oplus(\mathcal{C}) = \operatorname{Cone}(\Phi)$ is acyclic, and therefore, we conclude that Φ is a quasi-isomorphism. $\square$

4.3.4 Proof Theorem 4.3.9

Let (A, X, B) be a Keller admissible triple. Observe that the action map $\rho_A : (A, d_A) \to \big(\mathrm{Hom}_{B^{\mathrm{op}}}(X, X), \partial_X\big)$ is a dg A-A-bimodule map, and $\rho_B : (B^{\mathrm{op}}, d_B) \to \big(\mathrm{Hom}_A(X, X), \partial_X\big)$ is a dg B-B-bimodule map. Thus, by Lemma 4.3.7, the induced maps

$$\rho_{A*} : \Big(\mathrm{Hoch}^\bullet_\oplus(A),\ -(d^A_{\mathcal{H}} + \partial_A)\Big) \to \Big(\mathrm{Hoch}^\bullet_\oplus\big(A, \mathrm{Hom}_{B^{\mathrm{op}}}(X, X)\big),\ -(d_{\mathcal{H}} + \partial)\Big),$$
$$\rho_{B*} : \Big(\mathrm{Hoch}^\bullet_\oplus(B),\ -(d^B_{\mathcal{H}} + \partial_B)\Big) \to \Big(\mathrm{Hoch}^\bullet_\oplus\big(B, \mathrm{Hom}_A(X, X)\big),\ -(d_{\mathcal{H}} + \partial)\Big)$$

are quasi-isomorphisms.

To prove Theorem 4.3.9, we need the following

Lemma 4.3.12. *If the map*

$$\Phi \circ \rho_{A*} : \big(\mathrm{Hoch}^\bullet_\oplus(A)[-1],\ -(d^A_{\mathcal{H}} + \partial_A)\ \big) \to \big(\mathrm{Hoch}^\bullet_\oplus(A, X, B),\ d^X_{\mathcal{H}} + \partial_X\ \big)$$

is a quasi-isomorphism of cochain complexes, then so is the projection $\pi_B : \mathrm{Hoch}^\bullet_\oplus(X) \twoheadrightarrow \mathrm{Hoch}^\bullet_\oplus(B)$. *Similarly, if*

$$\Psi \circ \rho_{B*} : \big(\mathrm{Hoch}^\bullet_\oplus(B)[-1], -(d^B_{\mathcal{H}} + \partial_B)\big) \to \big(\mathrm{Hoch}^\bullet_\oplus(A, X, B), d^X_{\mathcal{H}} + \partial_X\big)$$

is a quasi-isomorphism, then so is the projection $\pi_A : \mathrm{Hoch}^\bullet_\oplus(X) \twoheadrightarrow \mathrm{Hoch}^\bullet_\oplus(A)$.

Proof. We only prove the first statement. The second one follows from a similar argument.

Since the projection $\pi_B : \big(\mathrm{Hoch}^\bullet_\oplus(X), d_{\mathcal{H}} + \partial\big) \twoheadrightarrow \big(\mathrm{Hoch}^\bullet_\oplus(B), d^B_{\mathcal{H}} + \partial_B\big)$ is surjective, it is a quasi-isomorphism if and only if the kernel

$$\ker(\pi_B) = \big(\mathrm{Hoch}^\bullet_\oplus(A) \oplus \mathrm{Hoch}^\bullet_\oplus(A, X, B),\ d^{AX}_{\mathcal{H}} + d^A_{\mathcal{H}} + d^X_{\mathcal{H}} + \partial_X + \partial_A\ \big)$$

is acyclic. Since

$$\begin{aligned}\Phi \circ \rho_{A*}(f)(a_1, \cdots, a_p;\, x) &= (-1)^r f(a_1, \cdots, a_p) \cdot x \\ &= d^{AX}_{\mathcal{H}}(f)(a_1, \cdots, a_p;\, x),\end{aligned}$$

for $f \in \mathrm{Hoch}^{p,r}(A)$, the mapping cone

$$\mathrm{Cone}(\Phi \circ \rho_{A*}) = \big(\mathrm{Hoch}^\bullet_\oplus(A) \oplus \mathrm{Hoch}^\bullet_\oplus(A, X, B),\ \Phi \circ \rho_{A*} + d^A_{\mathcal{H}} + d^X_{\mathcal{H}} + \partial_X + \partial_A\ \big)$$

of $\Phi \circ \rho_{A*}$ coincides with the kernel $\ker(\pi_B)$. Since $\Phi \circ \rho_{A*}$ is a quasi-isomorphism, the mapping cone $\mathrm{Cone}(\Phi \circ \rho_{A*}) = \ker(\pi_B)$ is acyclic. Therefore, π_B is a quasi-isomorphism. $\square$

Proof of Theorem 4.3.9. Let (A, X, B) be a Keller admissible triple. By Lemma 4.3.11, the embedding maps Φ and Ψ are quasi-isomorphisms. Since the pushforward maps ρ_{A*} and ρ_{B*} are also quasi-isomorphisms, it follows from Lemma 4.3.12 that π_A and π_B are quasi-isomorphisms. $\square$

Remark 4.3.13. Theorem 4.3.9 is a sum analogue of Keller's theorem in [24] which respects the *product* Hochschild cohomology. In fact, in [24, 51, 21], analogous theorems were proved for bigraded algebras where a Hochschild complex is endowed with three types of degrees — one from Hochschild construction, two from the given bigrading. Although a direct sum is taken for bigraded components in [51], the total complex is still different from our sum Hochschild complexes. Thus, there is no clear relation between these theorems and Theorem 4.3.9.

4.4 Keller admissible triples associated with Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. In this section, we prove the triple

$$\begin{aligned}
(A, d_A) &= \big(\mathcal{U}\mathfrak{g}, 0\big), \\
(B, d_B) &= \big(\operatorname{Hom}(S(\mathfrak{g}[1]), \mathbb{K}), d_{\mathfrak{g}}\big) \cong \big(S(\mathfrak{g}[1])^\vee, d_{\mathfrak{g}}\big), \\
(X, d_X) &= (\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), d_X)
\end{aligned}$$

is a Keller admissible triple, where $d_{\mathfrak{g}} : S^\bullet(\mathfrak{g}[1])^\vee \to S^{\bullet+1}(\mathfrak{g}[1])^\vee$ is the Chevalley–Eilenberg differential defined as in (2.4), $\mathcal{U}\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$, and $d_X : \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]) \to \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])$ is defined by

$$\begin{aligned}
d_X(u \otimes x_1 \odot \cdots \odot x_n) := &\sum_{i=1}^{n} (-1)^{i+1}\; u \cdot \mathrm{s}x_i \otimes x_1 \odot \cdots \widehat{x_i} \cdots \odot x_n \\
&+ \sum_{i<j} (-1)^{i+j}\; u \otimes \mathrm{s}^{-1}[\mathrm{s}x_i, \mathrm{s}x_j]_{\mathfrak{g}} \odot x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \odot x_n.
\end{aligned}$$

Here, $\mathcal{U}\mathfrak{g}$ is considered as a dg algebra concentrated at degree zero, $\mathrm{s} : \mathfrak{g}[1] \to \mathfrak{g}$ is the degree-shifting map of degree $+1$, and $u \in \mathcal{U}\mathfrak{g}$, $x_1, \cdots, x_n \in \mathfrak{g}[1]$. This triple is adapted from [25, Example 6.5].

It is well known that the complex $\big(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), d_X\big)$ is a free resolution of $\mathbb{K}$ in the category of $\mathcal{U}\mathfrak{g}$-modules.

Remark 4.4.1. In our grading setting, the degrees are chosen to be compatible with the Koszul sign convention, and a few classical formulations need to be modified correspondingly. In fact, in the literature, the expressions $(\Lambda^\bullet \mathfrak{g}^\vee, d_{\mathfrak{g}})$

and $(\mathcal{U}\mathfrak{g} \otimes \Lambda^\bullet \mathfrak{g},\ d_X)$ are more common than the expressions of graded symmetric tensors in this paper. The two types of expressions are isomorphic as complexes. Nevertheless, in the category of graded vector spaces, an element in $\Lambda^\bullet \mathfrak{g}$ should be considered to be of degree zero which is *not* the expected degree. Thus, in order to avoid confusion and to keep the consistency of degree counting, we choose the expressions $S(\mathfrak{g}[1])^\vee$ and $\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])$. See Section 2.4.2 for more details.

We describe the right $S(\mathfrak{g}[1])^\vee$-action $\llcorner$ on $S(\mathfrak{g}[1])$ induced by (graded) contraction: For any $x, x_i \in \mathfrak{g}[1]$, $\xi \in (\mathfrak{g}[1])^\vee$, we define

$$\begin{aligned} 1 \llcorner \xi &:= 0, \\ (x_1 \odot \cdots \odot x_n) \llcorner \xi &:= \sum_{i=1}^{n} (-1)^{n-i} \langle\, x_i \mid \xi \,\rangle (x_1 \odot \cdots \widehat{x_i} \cdots \odot x_n), \end{aligned}$$

and extend it by the module axiom. Similarly, we also have the right $S(\mathfrak{g}[1])$-action $\lrcorner$ on $S(\mathfrak{g}[1])^\vee$. Note that, in this way, we have

$$\begin{aligned} x \llcorner \xi &= \langle\, x \mid \xi \,\rangle = -\xi(x), \\ \xi \lrcorner x &= \langle\, \xi \mid x \,\rangle = \xi(x). \end{aligned}$$

The space $X = \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])$ carries the left $\mathcal{U}\mathfrak{g}$-action induced by the multiplication in $\mathcal{U}\mathfrak{g}$ and the right $S(\mathfrak{g}[1])^\vee$-action induced by $\llcorner$.

Lemma 4.4.2. *The dg vector space $\big(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]),\ d_X\big)$ is a dg $\mathcal{U}\mathfrak{g}$–$S(\mathfrak{g}[1])^\vee$-bimodule.*

Proof. This lemma follows from Example 2.4.10. □

4.4.1 The action maps

We first prove the action maps are weakly cone-nilpotent quasi-isomorphisms.

Lemma 4.4.3. *The action map*

$$\begin{aligned} \rho_A : (\mathcal{U}\mathfrak{g}, 0) &\to \Big(\mathrm{Hom}_{(S(\mathfrak{g}[1])^\vee)^{\mathrm{op}}} \big(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])\big), \partial_X \Big), \\ \rho_A(v)(u \otimes \mathbf{x}) &= (vu) \otimes \mathbf{x}, \end{aligned}$$

is a weakly cone-nilpotent quasi-isomorphism.

Proof. Since $B = S(\mathfrak{g}[1])^\vee$ is a graded commutative algebra, a right $S(\mathfrak{g}[1])^\vee$-module structure is equivalent to a left $S(\mathfrak{g}[1])^\vee$-module structure. Thus, by Proposition 2.4.9 and Example 2.4.10, we have

$$\mathrm{Hom}^\bullet_{(S(\mathfrak{g}[1])^\vee)^{\mathrm{op}}}(X, X) = \mathrm{coHom}^\bullet_{S(\mathfrak{g}[1])}(X, X),$$

where $X = \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])$. Furthermore, since the natural projection

$$\mathrm{pr} : \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]) \twoheadrightarrow \mathcal{U}\mathfrak{g} \otimes S^0(\mathfrak{g}[1]) \cong \mathcal{U}\mathfrak{g}$$

is a free cogenerator of the graded comodule $X = \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])$, it follows from Proposition 2.4.12 that the induced map

$$\mathrm{pr}_* : \mathrm{Hom}^\bullet_{B^{\mathrm{op}}}(X, X) \to \mathrm{Hom}^\bullet_{\mathbb{K}}(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g}) \cong \mathrm{Hom}^\bullet_{\mathbb{K}}\big(S(\mathfrak{g}[1]), \mathrm{Hom}_{\mathbb{K}}(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})\big)$$

is an isomorphism of graded vector spaces. See Section 2.4 for details.

Let $\tilde{\partial}_X$ be the differential on $\mathrm{Hom}\big(S(\mathfrak{g}[1]), \mathrm{Hom}(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})\big)$ induced by ∂_X under the isomorphism pr_*. By (2.7), one can show that

$$\begin{aligned} &\tilde{\partial}_X(f)(x_0 \odot \cdots \odot x_n)(u) \\ &\quad = \sum_{i=0}^{n} (-1)^{n-i} \big(f(x_0 \odot \cdots \widehat{x_i} \cdots \odot x_n)(u)\, \mathrm{s}x_i - f(x_0 \odot \cdots \widehat{x_i} \cdots \odot x_n)(u \mathrm{s}x_i) \big) \\ &\qquad - (-1)^n \sum_{i<j} (-1)^{i+j} f(\mathrm{s}^{-1}[\mathrm{s}x_i, \mathrm{s}x_j]_{\mathfrak{g}} \odot x_0 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \odot x_n)(u) \end{aligned}$$

for $f \in \mathrm{Hom}\big(S^n(\mathfrak{g}[1]), \mathrm{Hom}(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})\big)$, $x_i \in \mathfrak{g}[1]$ and $u \in \mathcal{U}\mathfrak{g}$. Let $\blacklozenge : \mathfrak{g} \times \mathrm{Hom}(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g}) \to \mathrm{Hom}(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})$ be the Lie algebra action

$$(y \blacklozenge g)(u) = g(uy) - g(u)y$$

for $g \in \mathrm{Hom}(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})$ and $y \in \mathfrak{g}$. Then the differential $\tilde{\partial}_X$ coincides with the Chevalley–Eilenberg differential $d^{\blacklozenge}_{\mathrm{CE}}$ associated with the action $\blacklozenge$.

Recall that one has the isomorphism (see, for example, [57, Exercise 7.3.5])

$$H^\bullet_{\mathrm{CE}}\big(\mathfrak{g}, \mathrm{Hom}(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})\big) \cong \mathrm{Ext}^\bullet_{\mathcal{U}\mathfrak{g}^{\mathrm{op}}}(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g}) \cong \mathcal{U}\mathfrak{g}.$$

Since the action map ρ_A induces a right inverse of the isomorphism $\mathrm{pr}_* : H\big(\mathrm{Hom}^\bullet_{B^{\mathrm{op}}}(X, X), \partial_X\big) \to \mathcal{U}\mathfrak{g}$, the lemma follows.

For weak cone-nilpotency, we observe that the dg algebra $\mathcal{U}\mathfrak{g}$ is concentrated in degree 0 and the mapping cone $M := \mathrm{Cone}(\rho_A)$ of ρ_A is bounded below. Thus, there exists $N > 0$ such that $\mathrm{Hom}^r(A^{\otimes n}, M) = 0$ if $r < -N$. Therefore, for any choice of homotopy operator on M and for any $f \in \mathrm{Hoch}^s_{\oplus}(A, M)$, the induced maps $\{\mathfrak{h}_k\}$ satisfy $\mathfrak{h}_k(f) = 0$ for $k > s + N$. This implies that the mapping cone $M = \mathrm{Cone}(\rho_A)$ is pointwisely nilpotent. □

The quasi-isomorphism property for the other action ρ_B follows from the proof of [25, Lemma 6.5 (a)]. In fact, the augmentation map $\varepsilon : \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]) \twoheadrightarrow \mathcal{U}\mathfrak{g} \twoheadrightarrow \mathbb{K}$ induces a quasi-isomorphism

$$\begin{aligned} \varepsilon_* : \big(\mathrm{Hom}_{\mathcal{U}\mathfrak{g}}(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])), \partial_X\big) &\to \big(\mathrm{Hom}_{\mathcal{U}\mathfrak{g}}\big(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), \mathbb{K}\big), \partial_{X,\mathbb{K}}\big) \\ &\cong \big(S(\mathfrak{g}[1])^\vee, d_{\mathfrak{g}}\big) \end{aligned}$$

which defines a left inverse for ρ_B.

Lemma 4.4.4. *The mapping cone of*

$$\varepsilon_* : \big(\operatorname{Hom}_{\mathcal{U}\mathfrak{g}}(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])), \partial_X\big) \to \big((S(\mathfrak{g}[1])^\vee)^{\mathrm{op}}, d_{\mathfrak{g}}\big)$$

is pointwisely nilpotent. Therefore, the action map

$$\rho_B : \big(\big(S(\mathfrak{g}[1])^\vee\big)^{\mathrm{op}}, d_{\mathfrak{g}}\big) \to \big(\operatorname{Hom}_{\mathcal{U}\mathfrak{g}}\big(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])\big), \partial_X\big),$$
$$\rho_B(b)(u \otimes \mathbf{x}) = (-1)^{|\mathbf{x}||b|} u \otimes (\mathbf{x} \llcorner b),$$

is a weakly cone-nilpotent quasi-isomorphism.

Proof. The mapping cone of ε_* is

$$\operatorname{Cone}(\varepsilon_*) \cong \operatorname{Hom}_{\mathcal{U}\mathfrak{g}}\Big(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), \big(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])\big)[1] \oplus \mathbb{K}\Big)$$

equipped with the differential $\partial : \operatorname{Cone}(\varepsilon_*) \to \operatorname{Cone}(\varepsilon_*)$,

$$\partial(f) := d_\varepsilon \circ f - (-1)^{|f|} f \circ d_X$$

where d_ε is the differential on the mapping cone $\operatorname{Cone}(\varepsilon)$ of $\varepsilon : \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]) \to \mathbb{K}$.

Let $\{e_1, \cdots, e_d\}$ be a basis for $\mathfrak{g}[1]$, $\mathrm{s} : \mathfrak{g}[1] \to \mathfrak{g}$ be the degree-shifting map of degree $+1$, and $\mathfrak{s} : \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]) \to \big(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])\big)[1]$ be the degree-shifting map of degree -1. Note that the mapping cone $\operatorname{Cone}(\varepsilon)$ is a filtered complex with respect to the filtration

$$F^{-p} := \Big(\bigoplus_{k=0}^{p} \mathcal{U}\mathfrak{g}^{\leq p-k} \otimes S^k(\mathfrak{g}[1])\Big)[1] \oplus \mathbb{K}$$

whose basis is the set

$$\{(0,1)\} \cup \{(\mathfrak{s}(\mathrm{s}e_{i_1} \cdots \mathrm{s}e_{i_l} \otimes e_{j_1} \odot \cdots \odot e_{j_k}), 0) \mid i_1 \leq \cdots \leq i_l,\ j_1 < \cdots < j_k,\ k+l \leq p\}$$

for $p \geq 0$, and $F^{-p} = 0$ if $p < 0$. It is known [22, Lemma VII.4.1] that the quotient complex F^{-p}/F^{-p+1} is exact. As a result, the inclusion map $F^{-p+1} \hookrightarrow F^{-p}$ is a quasi-isomorphism, and thus F^{-p} is exact for each p. See [22, Section VII.4] or [5, Section XIII.7] for details.

We will inductively construct a sequence of homotopy operators $h_p : F^{-p} \to F^{-p}$ of degree -1 such that (i) $h_p|_{F^{-p+1}} = h_{p-1}$, and (ii) the equation $d_\varepsilon \circ h_p + h_p \circ d_\varepsilon = \mathrm{id}_{F^{-p}}$ holds in F^{-p}. Since $\cup_p F^{-p} = \operatorname{Cone}(\varepsilon)$, such a sequence defines a contracting homotopy $h : \operatorname{Cone}(\varepsilon) \to \operatorname{Cone}(\varepsilon)$ for $\operatorname{Cone}(\varepsilon)$ with the property $h(F^{-p}) \subset F^{-p}$.

In the case $p = 0$, one can choose h_0 to be the isomorphism $\mathbb{K} \xrightarrow{\cong} \mathcal{U}\mathfrak{g}^{\leq 0} \otimes S^0(\mathfrak{g}[1])$. Assume we have a homotopy operator $h_p : F^{-p} \to F^{-p}$ with properties (i) and (ii). Then since F^{-p-1} is an exact sequence of vector spaces, one can assign the value of h_{p+1} at

$$(\mathfrak{s}(\mathrm{s}e_{i_1} \cdots \mathrm{s}e_{i_l} \otimes e_{j_1} \odot \cdots \odot e_{j_k}), 0), \quad i_1 \leq \cdots \leq i_l,\ j_1 < \cdots < j_k,\ k + l = p + 1,$$

inductively on k so that the equation $d_\varepsilon \circ h_{p+1} + h_{p+1} \circ d_\varepsilon = \mathrm{id}_{F^{-p-1}}$ holds. In this way, one can obtain a homotopy operators h_p with properties (i) and (ii) for each $p \geq 0$.

Recall that, by the Poincaré–Birkhoff–Witt theorem,

$$\{(0,1)\} \cup \{(\mathfrak{s}(\mathrm{s}e_{i_1} \cdots \mathrm{s}e_{i_l} \otimes e_{j_1} \odot \cdots \odot e_{j_k}), 0) \mid i_1 \leq \cdots \leq i_l,\ j_1 < \cdots < j_k\}$$

is a basis of $\big(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])\big)[1] \oplus \mathbb{K}$. By the construction, $h((0,1)) = (\mathfrak{s}(1 \otimes 1), 0)$. Also, each basis vector of the form $\mathbf{v} = (\mathfrak{s}(\mathrm{s}e_{i_1} \cdots \mathrm{s}e_{i_l} \otimes e_{j_1} \odot \cdots \odot e_{j_k}), 0)$ belongs to $F^{-(k+l)}$, and thus

$$h(\mathbf{v}) \in F^{-(k+l)} \cap \big(\mathcal{U}\mathfrak{g} \otimes S^{k+1}(\mathfrak{g}[1])\big)[1] \oplus \{0\} = \big(\mathcal{U}\mathfrak{g}^{\leq l-1} \otimes S^{k+1}(\mathfrak{g}[1])\big)[1] \oplus \{0\}.$$

Therefore, we have

$$h\Big(\big(\mathcal{U}\mathfrak{g}^{\leq q} \otimes S(\mathfrak{g}[1])\big)[1] \oplus \mathbb{K}\Big) \subset \big(\mathcal{U}\mathfrak{g}^{\leq q-1} \otimes S(\mathfrak{g}[1])\big)[1] \oplus \{0\}. \tag{4.12}$$

Now, we define $S(\mathfrak{g}[1])^\vee$-$S(\mathfrak{g}[1])^\vee$-subbimodules of $\mathrm{Cone}(\varepsilon_*)$ by

$$M_0 = \mathrm{Hom}_{\mathcal{U}\mathfrak{g}}\Big(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), \big(\mathcal{U}\mathfrak{g}^{\leq 0} \otimes S(\mathfrak{g}[1])\big)[1] \oplus \{0\}\Big)$$

and

$$M_q = \mathrm{Hom}_{\mathcal{U}\mathfrak{g}}\Big(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), \big(\mathcal{U}\mathfrak{g}^{\leq q} \otimes S(\mathfrak{g}[1])\big)[1] \oplus \mathbb{K}\Big)$$

for each $q > 0$. Then $\{M_q\}_{q \geq 0}$ forms an ascending filtration of $\mathrm{Cone}(\varepsilon_*)$ satisfying $\bigcup_q M_q = \mathrm{Cone}(\varepsilon_*)$. Moreover, (4.12) implies that the induced contracting homotopy $\tilde{h} = h_*$ for $\mathrm{Cone}(\varepsilon_*)$ satisfies $\tilde{h}(M_q) \subset M_{q-1}$. Thus, by Lemma 4.3.5, the mapping cone $\mathrm{Cone}(\varepsilon_*)$ is pointwisely nilpotent, and thus ρ_B is weakly cone-nilpotent. $\square$

4.4.2 Exactness of the sequences

Now we show the sequences (4.8) and (4.9) are exact for the triple $(A, X, B) = (\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), S(\mathfrak{g}[1])^\vee)$.

Let V be a vector space concentrated at degree zero, $\{e_1, \cdots, e_d\}$ be a basis for $V[1]$, and $\{\epsilon^1, \cdots, \epsilon^d\}$ be the dual basis for $(V[1])^\vee$ such that $\langle\, \epsilon^j \mid e_i \,\rangle = \delta_i^j$. We denote

$$\omega := e_1 \odot \cdots \odot e_d \in S^d(V[1]), \quad \tau := \epsilon^d \odot \cdots \odot \epsilon^1 \in S^d(V[1])^\vee.$$

To prove the exactness of (4.8), we need the following technical lemma.

Lemma 4.4.5. *For $\mathbf{x} \in S(V[1])$, $b \in S(V[1])^\vee$, we have*

(i) $\omega \llcorner (\tau \lrcorner \mathbf{x}) = (-1)^{d-|\mathbf{x}|}\ \mathbf{x}$,

(ii) $\tau \lrcorner (\mathbf{x} \llcorner b) = (-1)^{|b|}\ (\tau \lrcorner \mathbf{x}) \odot b$.

Proof. In the following computation, we use multi-index notations and Einstein summation convention. Let E_i be the d-tuple with 1 at the i-th component and 0 elsewhere. We denote $e_I = e_{i_1} \odot \cdots \odot e_{i_k}$ and $\epsilon^J = \epsilon^{j_1} \odot \cdots \odot \epsilon^{j_l}$, where $I = E_{i_1} + \cdots + E_{i_k}$, $i_1 < \cdots < i_k$, and $J = E_{j_1} + \cdots + E_{j_l}$, $j_1 < \cdots < j_l$. We say I is smaller than J, denoted $I \leq J$, if $\{i_1, \cdots, i_k\} \subset \{j_1, \cdots, j_l\}$. If $I \leq J$, we denote $J - I$ to be the tuple associated with $\{j_1, \cdots, j_l\} \setminus \{i_1, \cdots, i_k\}$.

Let $T = E_1 + \cdots + E_d$, and $T - I = E_{\tilde{\imath}_1} + \cdots + E_{\tilde{\imath}_{d-k}}$, $\tilde{\imath}_1 < \cdots < \tilde{\imath}_{d-k}$. We have

$$\tau \lrcorner e_I = (\epsilon^d \cdots \epsilon^1) \lrcorner e_{i_1} \cdots \lrcorner e_{i_k} = (-1)^{i_1 + \cdots + i_k - \frac{k(k+1)}{2}}\ \epsilon_{\mathrm{rev}}^{T-I},$$

where $\epsilon_{\mathrm{rev}}^{T-I} = \epsilon^{\tilde{\imath}_{d-k}} \odot \cdots \odot \epsilon^{\tilde{\imath}_1}$. Thus,

$$\begin{aligned} \omega \llcorner (\tau \lrcorner e_I) &= (-1)^{i_1 + \cdots + i_k - \frac{k(k+1)}{2}}\ (e_1 \odot \cdots \odot e_d) \llcorner \epsilon^{\tilde{\imath}_{d-k}} \cdots \llcorner \epsilon^{\tilde{\imath}_1} \\ &= (-1)^{i_1 + \cdots + i_k - \frac{k(k+1)}{2}} (-1)^{(d - \tilde{\imath}_1 - d + k) + \cdots + (d - \tilde{\imath}_{d-k} - 1)}\ e_I \\ &= (-1)^{d-k}\ e_I. \end{aligned}$$

Since $\{e_I\}_I$ is a basis for $S(V[1])$, the first equation follows.

For the second equation, it suffices to verify it for any $\mathbf{x} = e_I$ and $b = \epsilon^J$. If $J \not\leq I$, it is clear that

$$(\tau \lrcorner e_I) \odot \epsilon^J = 0 = \tau \lrcorner (e_I \llcorner \epsilon^J).$$

Thus, we can assume $J \leq I$. Let $j_p = i_{\jmath_p}$ and $I - J = E_{i_{\tilde{\jmath}_1}} + \cdots + E_{i_{\tilde{\jmath}_{k-l}}}$. Then we have

$$\begin{aligned} e_I \llcorner \epsilon^J &= (-1)^l (-1)^{(k - \jmath_1) + \cdots + (k - \jmath_l)} e_{I-J}, \\ \tau \lrcorner (e_I \llcorner \epsilon^J) &= (-1)^{i_{\tilde{\jmath}_1} + \cdots + i_{\tilde{\jmath}_{k-l}}} (-1)^{\jmath_1 + \cdots + \jmath_l + \frac{k^2 + k + l^2 + l}{2}}\ \epsilon_{\mathrm{rev}}^{T-I+J}. \end{aligned}$$

Furthermore,

$$
\begin{aligned}
\tau \lrcorner\, e_I &= (-1)^{(i_1-1)+\cdots+(i_k-k)} \epsilon_{\text{rev}}^{T-I}, \\
(\tau \lrcorner\, e_I) \odot \epsilon^J &= (-1)^{(i_1-1)+\cdots+(i_k-k)} (-1)^{(i_{j_1}-j_1)+\cdots+(i_{j_l}-j_l+l-1)} \epsilon_{\text{rev}}^{T-I+J} \\
&= (-1)^l \ \tau \lrcorner\, (e_I \llcorner\, \epsilon^J).
\end{aligned}
$$

Thus the proof is complete. $\square$

Lemma 4.4.6. *For* $(A, X, B) = (\mathcal{U}\mathfrak{g},\ \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]),\ S(\mathfrak{g}[1])^\vee)$, *the sequence* (4.8) *is exact.*

Proof. Define an operator

$$
h_R : \mathrm{Hom}^r(X \otimes B^{\otimes q+1}, X) \to \mathrm{Hom}^r(X \otimes B^{\otimes q}, X),
$$

for each $q \geq 0$, by

$$
h_R(f)\big((u \otimes \mathbf{x})\,; b_1, \cdots, b_q\big) := (-1)^{r+1}(-1)^{d-|\mathbf{x}|}\ f\big((u \otimes \omega)\,; (\tau \lrcorner\, \mathbf{x}), b_1, \cdots, b_q\big),
$$

where $u \in \mathcal{U}\mathfrak{g}$, $\mathbf{x} \in S(\mathfrak{g}[1])$, $b_1, \cdots, b_q \in S(\mathfrak{g}[1])^\vee$. By Lemma 4.4.5, one can show that

$$
\begin{aligned}
&(h_R d^X_{\mathcal{H}R} f)\big((u \otimes \mathbf{x})\,; b_1, \cdots, b_q\big) \\
&\qquad = (-1)^{d-|\mathbf{x}|} f\big((u \otimes \omega) \llcorner\, (\tau \lrcorner\, \mathbf{x})\,; b_1, \cdots, b_q\big) \\
&\qquad\quad + (-1)^{d-|\mathbf{x}|+1} f\big((u \otimes \omega)\,; ((\tau \lrcorner\, \mathbf{x}) \odot b_1), \cdots, b_q\big) \\
&\qquad\quad + \sum_{j=1}^{q-1} (-1)^{d-|\mathbf{x}|+j+1} f\big((u \otimes \omega)\,; (\tau \lrcorner\, \mathbf{x}), \cdots, b_j b_{j+1}, \cdots, b_q\big) \\
&\qquad\quad + (-1)^{d-|\mathbf{x}|+q+1} f\big((u \otimes \omega)\,; (\tau \lrcorner\, \mathbf{x}), \cdots, b_{q-1}\big) \llcorner\, b_q,
\end{aligned}
$$

and

$$
\begin{aligned}
&(d^X_{\mathcal{H}R} h_R f)\big((u \otimes \mathbf{x})\,; b_1, \cdots, b_q\big) \\
&\qquad = (-1)^{d-|\mathbf{x}|} f\big((u \otimes \omega)\,; (\tau \lrcorner\, \mathbf{x}) \odot b_1,\ b_2, \cdots, b_q\big) \\
&\qquad\quad + \sum_{j=1}^{q-1} (-1)^{j+d-|\mathbf{x}|} f\big((u \otimes \omega)\,; (\tau \lrcorner\, \mathbf{x}), b_1, \cdots b_j b_{j+1} \cdots, b_q\big) \\
&\qquad\quad + (-1)^{q+d-|\mathbf{x}|} f\big((u \otimes \omega)\,; (\tau \lrcorner\, \mathbf{x}), b_1, \cdots, b_{q-1}\big) \llcorner\, b_q
\end{aligned}
$$

Thus, we have

$$
d^X_{\mathcal{H}R} h_R + h_R d^X_{\mathcal{H}R} = \mathrm{id},
$$

and the cohomologies vanish except the zeroth cohomology. Since the zeroth cohomology is

$$\ker\Big(\operatorname{Hom}^r(X,X)\xrightarrow{d^X_{\mathcal{H}R}}\operatorname{Hom}^r(X\otimes B,X)\Big)=\operatorname{Hom}^r_{B^{\mathrm{op}}}(X,X),$$

the proof is completed. $\square$

Finally, we prove the exactness of the sequence (4.9).

Lemma 4.4.7. *For* $(A,X,B)=(\mathcal{U}\mathfrak{g},\ \mathcal{U}\mathfrak{g}\otimes S(\mathfrak{g}[1]),\ S(\mathfrak{g}[1])^\vee)$*, the sequence* (4.9) *is exact.*

Proof. In order to prove this assertion, we define an operator

$$h_L:\operatorname{Hom}^r(A^{\otimes p+1}\otimes X,X)\to\operatorname{Hom}^r(A^{\otimes p}\otimes X,X),$$

for each $p\geq 0$, by

$$h_L(f)\big(a_1,\cdots,a_p;(u\otimes\mathbf{x})\big):=(-1)^r f\big(a_1,\cdots,a_p,u;(1\otimes\mathbf{x})\big),$$

where $a_1,\cdots,a_p,u\in\mathcal{U}\mathfrak{g}$ and $\mathbf{x}\in S(\mathfrak{g}[1])$. It is straightforward to show that

$$d^X_{\mathcal{H}L}h_L+h_Ld^X_{\mathcal{H}L}=\mathrm{id}\,.$$

Thus,

$$H^n\big(\operatorname{Hom}^r(A^{\otimes\bullet}\otimes X,X),d^X_{\mathcal{H}L}\big)=0,$$

for any $n>0$ and any $r\in\mathbb{Z}$. Furthermore, since the zeroth cohomology is

$$\ker\Big(\operatorname{Hom}^r(X,X)\xrightarrow{d^X_{\mathcal{H}L}}\operatorname{Hom}^r(A\otimes X,X)\Big)=\operatorname{Hom}^r_A(X,X),$$

the proof is completed. $\square$

Therefore, we have the following

Theorem 4.4.8. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra. The triple* $\big(\mathcal{U}\mathfrak{g},\ \mathcal{U}\mathfrak{g}\otimes S(\mathfrak{g}[1]),\ S(\mathfrak{g}[1])^\vee\big)$ *is a Keller admissible triple. Therefore, the projections*

$$\begin{tikzcd}[column sep=small] & \operatorname{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g}\otimes S(\mathfrak{g}[1])) \arrow[dl, two heads, "\pi_A"'] \arrow[dr, two heads, "\pi_B"] & \\ \operatorname{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g}) & & \operatorname{Hoch}^\bullet_\oplus(S(\mathfrak{g}[1])^\vee)\end{tikzcd}$$

induce isomorphisms of Gerstenhaber algebras on cohomologies.

Example 4.4.9. For the 1-dimensional Lie algebra $\mathfrak{g} = \mathbb{K}$, it follows that $\mathcal{U}\mathfrak{g} = S\mathfrak{g} \cong \bigoplus_{n=0}^{\infty} \mathbb{K}$ and $(S(\mathfrak{g}[1])^\vee, 0) = (\mathbb{K}[x]/(x^2), 0)$ as in Example 4.1.2. We have

$$\mathrm{HH}^0_{\oplus}(\mathcal{U}\mathfrak{g}) \cong \bigoplus_{n=0}^{\infty} \mathbb{K} \cong \mathrm{HH}^0_{\oplus}(S(\mathfrak{g}[1])^\vee, 0).$$

However, the 0-th *product* Hochschild cohomology of $S(\mathfrak{g}[1])^\vee$ is

$$\mathrm{HH}^0_{\Pi}(S(\mathfrak{g}[1])^\vee, 0) \cong \prod_{n=0}^{\infty} \mathbb{K}$$

which is *not* isomorphic to $\mathrm{HH}^0_{\Pi}(\mathcal{U}\mathfrak{g}) = \mathrm{HH}^0_{\oplus}(\mathcal{U}\mathfrak{g})$.

4.5 Application to Duflo theorem

A dg manifold $(\mathcal{M}, Q)$ is a graded manifold $\mathcal{M}$ together with a homological vector field Q. Such a structure plays an important role in various fields of mathematics. See, for example, [1, 6, 40, 2]. In the present paper, we consider the dg manifold $(\mathfrak{g}[1], d_{\mathfrak{g}})$ associated with a finite-dimensional Lie algebra $\mathfrak{g}$ whose function algebra is the Chevalley–Eilenberg dg algebra $(S(\mathfrak{g}[1])^\vee, d_{\mathfrak{g}})$.

On the dg manifold $(\mathfrak{g}[1], d_{\mathfrak{g}})$, one has the dg algebra $\big({}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathfrak{g}[1]), [d_{\mathfrak{g}}, -]\big)$ of polyvector fields and the dg algebra $\big({}_{\oplus}\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathfrak{g}[1]), d_{\mathcal{H}} + [\![d_{\mathfrak{g}}, -]\!]\big)$ of polydifferential operators on $(\mathfrak{g}[1], d_{\mathfrak{g}})$. According to [35, Theorem 4.3], one has the Kontsevich–Duflo-type map

$$\mathrm{hkr} \circ \mathrm{td}^{1/2}_{\mathfrak{g}[1]} : \big({}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathfrak{g}[1]), d_T\big) \to \big({}_{\oplus}\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathfrak{g}[1]), d_D\big),$$

which induces a graded algebra isomorphism on their cohomologies. Here,

$$\begin{aligned} d_T &:= [d_{\mathfrak{g}}, -], \\ d_D &:= d_{\mathcal{H}} + [\![d_{\mathfrak{g}}, -]\!], \end{aligned}$$

and $\mathrm{td}_{\mathfrak{g}[1]} \in \prod_{k\geq 0} \big(\Gamma(\Lambda^k T^\vee_{\mathfrak{g}[1]})\big)^k$ denotes the Todd cocycle associated with the trivial connection on $\mathfrak{g}[1]$. We refer the reader to [35] for an introduction to Kontsevich–Duflo-type theorem for dg manifolds.

Let $J \in \widehat{S}(\mathfrak{g}^\vee)$ be the *Duflo element* which is the formal power series associated with $J(x) = \det\left(\frac{1-e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}\right)$, $x \in \mathfrak{g}$. Its square root $J^{1/2}$ acts on $S\mathfrak{g}$ as a formal differential operator, and this map induces an operator on the Chevalley–Eilenberg cohomology $H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S\mathfrak{g})$. In this section, we construct isomorphisms of graded algebras

$$\Phi_T : \mathbb{H}^\bullet({}_\oplus\mathcal{T}_{\mathrm{poly}}(\mathfrak{g}[1]), d_T) \xrightarrow{\cong} H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S\mathfrak{g}),$$
$$\Phi_D : \mathbb{H}^\bullet({}_\oplus\mathcal{D}_{\mathrm{poly}}(\mathfrak{g}[1]), d_D) \xrightarrow{\cong} H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}\mathfrak{g}),$$

and prove the following

Theorem 4.5.1. *The diagram*

$$\begin{tikzcd}
\mathbb{H}^\bullet({}_\oplus\mathcal{T}_{\mathrm{poly}}(\mathfrak{g}[1]), d_T) \arrow[r, "\mathrm{Td}^{1/2}_{\mathfrak{g}[1]}"] \arrow[d, "\Phi_T"] & \mathbb{H}^\bullet({}_\oplus\mathcal{T}_{\mathrm{poly}}(\mathfrak{g}[1]), d_T) \arrow[r, "\mathrm{hkr}"] \arrow[d, "\Phi_T"] & \mathbb{H}^\bullet({}_\oplus\mathcal{D}_{\mathrm{poly}}(\mathfrak{g}[1]), d_D) \arrow[d, "\Phi_D"] \\
H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S\mathfrak{g}) \arrow[r, "J^{1/2}"] & H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S\mathfrak{g}) \arrow[r, "\mathrm{pbw}"] & H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}\mathfrak{g})
\end{tikzcd} \tag{4.13}$$

commutes.

In this way, we obtain a precise relation between the Kontsevich–Duflo-type isomorphism [35] for the dg manifold $(\mathfrak{g}[1], d_{\mathfrak{g}})$ and the Kontsevich–Duflo isomorphism [15, 28, 43] for the Lie algebra $\mathfrak{g}$. In particular, we recover the Kontsevich–Duflo theorem:

Corollary 4.5.2 (Kontsevich)**.** *The map*

$$\mathrm{pbw} \circ J^{1/2} : H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S\mathfrak{g}) \to H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}\mathfrak{g})$$

is an isomorphism of graded algebras.

The diagram (4.13) consists of two parts: left half and right half. The commutativity of the left half diagram is established in Section 4.5.3, and the commutativity of the right half diagram is proved in Proposition 4.5.5.

4.5.1 Polyvector fields on $\mathfrak{g}[1]$

The dg algebra ${}_\oplus\mathcal{T}^\bullet_{\mathrm{poly}}(\mathfrak{g}[1])$ of polyvector fields on $(\mathfrak{g}[1], d_{\mathfrak{g}})$ consists of the graded vector space

$${}_\oplus\mathcal{T}^\bullet_{\mathrm{poly}}(\mathfrak{g}[1]) = \Gamma\big(\mathfrak{g}[1]; S(T_{\mathfrak{g}[1]}[-1])\big) \cong S(\mathfrak{g}[1])^\vee \otimes S\mathfrak{g}$$

equipped with the differential $d_T = [d_{\mathfrak{g}}, -]$ and the natural multiplication

$$(\mathfrak{f}_1 \otimes \tilde{\mathbf{x}}_1) \cdot (\mathfrak{f}_2 \otimes \tilde{\mathbf{x}}_2) = (\mathfrak{f}_1 \odot \mathfrak{f}_2) \otimes (\tilde{\mathbf{x}}_1 \odot \tilde{\mathbf{x}}_2),$$

where $[-,-]$ is the Schouten bracket, $\mathfrak{f}_1, \mathfrak{f}_2 \in S(\mathfrak{g}[1])^\vee$ and $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2 \in S\mathfrak{g}$.

Let $\tilde{\Phi}_T : {}_\oplus\mathcal{T}^\bullet_{\mathrm{poly}}(\mathfrak{g}[1]) \to \mathrm{Hom}(S^\bullet(\mathfrak{g}[1]), S\mathfrak{g})$ be the map

$$\tilde{\Phi}_T(\mathfrak{f} \otimes \mathrm{s}x_1 \odot \cdots \odot \mathrm{s}x_q) : \mathbf{y} \mapsto \langle\, \mathfrak{f} \mid \mathbf{y}\, \rangle \cdot \mathrm{s}x_1 \odot \cdots \odot \mathrm{s}x_q$$

where $\mathfrak{f} \in S(\mathfrak{g}[1])^\vee$, $x_i \in \mathfrak{g}[1]$, $\mathbf{y} \in S(\mathfrak{g}[1])$ are homogeneous, and $\mathrm{s} : \mathfrak{g}[1] \to \mathfrak{g}$ is the degree-shifting map of degree $+1$. Here, the dg algebra $\mathrm{Hom}(S^\bullet(\mathfrak{g}[1]), S\mathfrak{g})$ is equipped with the convolution product $\star$ and the Chevalley–Eilenberg differential induced by the adjoint action. See Section 2.4 for the precise definitions. The following lemma follows from a direct computation.

Lemma 4.5.3. *The map $\tilde{\Phi}_T : {}_\oplus\mathcal{T}^\bullet_{\mathrm{poly}}(\mathfrak{g}[1]) \to \mathrm{Hom}(S^\bullet(\mathfrak{g}[1]), S\mathfrak{g})$ is an isomorphism of dg algebras. In particular, the induced map*

$$\Phi_T : \mathbb{H}^\bullet({}_\oplus\mathcal{T}_{\mathrm{poly}}(\mathfrak{g}[1]), d_T) \to H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S\mathfrak{g})$$

is an isomorphism of graded algebra.

4.5.2 Polydifferential operators on $\mathfrak{g}[1]$

The dg algebra ${}_\oplus\mathcal{D}^\bullet_{\mathrm{poly}}(\mathfrak{g}[1])$ of polydifferential operators on $(\mathfrak{g}[1], d_\mathfrak{g})$ is the graded vector space

$$\begin{aligned} {}_\oplus\mathcal{D}^\bullet_{\mathrm{poly}}(\mathfrak{g}[1]) &\cong \bigoplus_{p+r=\bullet} \Big(S(\mathfrak{g}[1])^\vee \otimes \big(S(\mathfrak{g}[1])\big)^{\otimes p}\Big)^r \\ &\qquad\qquad\qquad \cong \bigoplus_{p+r=\bullet} \mathrm{Hom}^r\left((S(\mathfrak{g}[1])^\vee)^{\otimes p}, S(\mathfrak{g}[1])^\vee \right), \end{aligned}$$

equipped with the differential $d_D = d_\mathcal{H} + [\![d_\mathfrak{g}, -]\!]$ and the cup product. In other words, the dg algebra ${}_\oplus\mathcal{D}^\bullet_{\mathrm{poly}}(\mathfrak{g}[1])$ is isomorphic to the dg algebra $\mathrm{Hoch}^\bullet_\oplus(S(\mathfrak{g}[1])^\vee, d_\mathfrak{g})$ of Hochschild cochains of the dg algebra $(S(\mathfrak{g}[1])^\vee, d_\mathfrak{g})$, and thus,

$$\mathbb{H}^\bullet({}_\oplus\mathcal{D}_{\mathrm{poly}}(\mathfrak{g}[1]), d_D) \cong \mathrm{HH}^\bullet_\oplus(S(\mathfrak{g}[1])^\vee, d_\mathfrak{g}).$$

We will omit $d_\mathfrak{g}$ in the Hochschild cohomology/complex for simplicity.

By Theorem 4.4.8, there is an isomorphism $\Phi_1 : \mathrm{HH}^\bullet_\oplus(S(\mathfrak{g}[1])^\vee) \to \mathrm{HH}^\bullet_\oplus(\mathcal{U}\mathfrak{g})$ of graded algebras. Furthermore, it is well known that $\mathrm{HH}^\bullet_\oplus(\mathcal{U}\mathfrak{g})$ is isomorphic to $H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}\mathfrak{g})$. In fact, this isomorphism is represented by the cochain map $\tilde{\Phi}_2 : \mathrm{Hom}(\mathcal{U}\mathfrak{g}^{\otimes \bullet}, \mathcal{U}\mathfrak{g}) \to \mathrm{Hom}(S^\bullet(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g})$,

$$\tilde{\Phi}_2(f) : x_1 \odot \cdots \odot x_p \mapsto \sum_{\sigma \in S_p} (-1)^\sigma f(\mathrm{s}x_{\sigma(1)} \otimes \cdots \otimes \mathrm{s}x_{\sigma(p)}), \tag{4.14}$$

for $x_1, \cdots, x_p \in \mathfrak{g}[1]$ and $f \in \mathrm{Hom}(\mathcal{U}\mathfrak{g}^{\otimes p}, \mathcal{U}\mathfrak{g})$. The dg algebra $\mathrm{Hom}(S^\bullet(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g})$ is equipped with the convolution product $\star$ and the Chevalley–Eilenberg differential $d_{\mathrm{CE}}^{\mathcal{U}\mathfrak{g}}$ of adjoint action. See (2.3) and (2.5) for the precise definitions of $\star$ and $d_{\mathrm{CE}}^{\mathcal{U}\mathfrak{g}}$. The following lemma is standard [5].

Lemma 4.5.4 (Cartan–Eilenberg). *The map $\tilde{\Phi}_2$ is a quasi-isomorphism of dg algebras. In particular, the induced map*

$$\Phi_2 : \mathrm{HH}_\oplus^\bullet(\mathcal{U}\mathfrak{g}) \to H_{\mathrm{CE}}^\bullet(\mathfrak{g}, \mathcal{U}\mathfrak{g})$$

is an isomorphism of graded algebras.

Therefore, the map

$$\Phi_D := \Phi_2 \circ \Phi_1 : \mathbb{H}^\bullet({}_\oplus\mathcal{D}_{\mathrm{poly}}(\mathfrak{g}[1]), d_D) \cong \mathrm{HH}_\oplus^\bullet(S(\mathfrak{g}[1])^\vee) \to H_{\mathrm{CE}}^\bullet(\mathfrak{g}, \mathcal{U}\mathfrak{g}),$$

is an isomorphism of graded algebras.

4.5.3 Todd class and Duflo element

The Todd class of a dg manifold $(\mathcal{M}, Q)$ can be defined via the Atiyah class which measures the obstruction to the existence of Q-invariant connections. In the following, we present an Atiyah cocycle and a Todd cocycle of the dg manifold $(\mathfrak{g}[1], d_\mathfrak{g})$, and we compare this Todd cocycle with the Duflo element of $\mathfrak{g}$. We refer the reader to [42] for a general introduction to the Atiyah and Todd classes of a dg manifold.

Let ∇^0 be the trivial connection on $\mathfrak{g}[1]$. The *Atiyah cocycle*

$$\mathrm{at}_\mathfrak{g} \in S(\mathfrak{g}[1])^\vee \otimes (\mathfrak{g}[1])^\vee \otimes \mathrm{End}(\mathfrak{g}[1]) \cong \mathrm{Hom}(\mathfrak{g}[1] \otimes \mathfrak{g}[1],\ S(\mathfrak{g}[1])^\vee \otimes \mathfrak{g}[1])$$

associated with ∇^0 is characterized by

$$\mathrm{at}_\mathfrak{g}(x, y) = \mathrm{s}^{-1}[\mathrm{s}x, \mathrm{s}y]_\mathfrak{g}$$

for $x, y \in \mathfrak{g}[1]$. Since $\mathrm{at}_\mathfrak{g}(x, -) : \mathfrak{g}[1] \to \mathfrak{g}[1]$ maps the odd component to the odd component, the *Todd cocycle* $\mathrm{td}_{\mathfrak{g}[1]} \in \prod_{k \geq 0} \left(\Gamma(\Lambda^k T_{\mathfrak{g}[1]}^\vee)\right)^k \cong \widehat{S}\mathfrak{g}^\vee$ associated with ∇^0 is

$$\mathrm{td}_{\mathfrak{g}[1]}(x) = \mathrm{Ber}\left(\frac{\mathrm{at}_\mathfrak{g}(x, -)}{1 - e^{-\mathrm{at}_\mathfrak{g}(x,-)}}\right) = \det\left(\frac{1 - e^{-\mathrm{ad}_{\mathrm{s}x}}}{\mathrm{ad}_{\mathrm{s}x}}\right) = J(\mathrm{s}x).$$

In other words, the Todd cocycle $\mathrm{td}_{\mathfrak{g}[1]}$ is identified with the *Duflo element* $J \in \widehat{S}\mathfrak{g}^\vee$ under the isomorphism

$$\prod_{k \geq 0} \left(\Gamma(\Lambda^k T_{\mathfrak{g}[1]}^\vee)\right)^k \cong \prod_{k \geq 0} \left(S(\mathfrak{g}[1])^\vee \otimes \Lambda^k(\mathfrak{g}[1])^\vee\right)^k \cong \prod_{k \geq 0} S^k\mathfrak{g}^\vee = \widehat{S}\mathfrak{g}^\vee. \tag{4.15}$$

A Todd cocycle is closed under the coboundary operator $L_{d_{\mathfrak{g}}}$. In our case, this means the Todd cocycle $\operatorname{td}_{\mathfrak{g}[1]}$ is $\mathfrak{g}$-invariant. Thus, the square root $\operatorname{td}_{\mathfrak{g}[1]}^{1/2}$ is also $\mathfrak{g}$-invariant and acts on $H^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, S\mathfrak{g})$ by contraction.

With the isomorphism (4.15), one can show that the left half of the diagram (4.13) commutes.

4.5.4 Hochschild–Kostant–Rosenberg map and Poincaré–Birkhoff–Witt isomorphism

The last step of proving Theorem 4.5.1 is to show that the two well-known isomorphisms — *Hochschild–Kostant–Rosenberg isomorphism* and *Poincaré–Birkhoff–Witt isomorphism* — are isomorphic via Φ_T and Φ_D. We first recall the definitions of these two isomorphisms.

Recall that for $\mathbf{x} \in S^k(\mathfrak{g}[1])$, the interior product $\iota_{\mathbf{x}} : S(\mathfrak{g}[1])^\vee \to S(\mathfrak{g}[1])^\vee$ is characterized by

$$\iota_{\mathbf{x}}(\mathfrak{f}) = (-1)^{|\mathbf{x}|\cdot|\mathfrak{f}|}\,\mathfrak{f}(\mathbf{x}\odot -) : S^{n-k}(\mathfrak{g}[1]) \to \mathbb{K}, \quad \forall\, \mathfrak{f} \in \operatorname{Hom}(S^n(\mathfrak{g}[1]), \mathbb{K}) \cong S^n(\mathfrak{g}[1])^\vee,$$

if $n \geq k$, and $\iota_{\mathbf{x}}(\mathfrak{f}) = 0$ if $n < k$. The *Hochschild–Kostant–Rosenberg map* [35, Section 4] on the graded manifold $\mathfrak{g}[1]$ is the map

$$\begin{gathered} \operatorname{hkr} : {}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathfrak{g}[1]) \to {}_{\oplus}\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathfrak{g}[1]) \cong \operatorname{Hoch}^{\bullet}_{\oplus}(S(\mathfrak{g}[1])^\vee), \\ \operatorname{hkr}\big(\mathfrak{f} \otimes (\mathrm{s}x_1 \odot \cdots \odot \mathrm{s}x_q)\big) = \frac{1}{q!} \sum_{\sigma \in S_q} \mathfrak{f} \odot \iota_{x_{\sigma(1)}} \otimes \cdots \otimes \iota_{x_{\sigma(q)}}, \end{gathered}$$

for $\mathfrak{f} \otimes (\mathrm{s}x_1 \odot \cdots \odot \mathrm{s}x_q) \in S(\mathfrak{g}[1])^\vee \otimes S^q\mathfrak{g} \subset {}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathfrak{g}[1])$. Here, the Hochschild cochain $\mathfrak{f} \odot \iota_{x_1} \otimes \cdots \otimes \iota_{x_q} : (S(\mathfrak{g}[1])^\vee)^{\otimes q} \to S(\mathfrak{g}[1])^\vee$ is defined by

$$\mathfrak{f} \odot \iota_{x_1} \otimes \cdots \otimes \iota_{x_q} : (b_1 \otimes \cdots \otimes b_q) \mapsto (-1)^{\sum_{i=1}^{q}(q-i)|b_i|}\, \mathfrak{f} \odot \iota_{x_1} b_1 \odot \cdots \odot \iota_{x_q} b_q$$

for $b_1, \cdots, b_q \in S(\mathfrak{g}[1])^\vee$. This map induces an isomorphism of vector spaces $\operatorname{hkr} : \mathbb{H}^{\bullet}({}_{\oplus}\mathcal{T}_{\mathrm{poly}}(\mathfrak{g}[1]), d_T) \to \mathrm{HH}^{\bullet}_{\oplus}(S(\mathfrak{g}[1])^\vee)$ on their cohomologies.

The *Poincaré–Birkhoff–Witt isomorphism* is an isomorphism $\operatorname{pbw} : S\mathfrak{g} \to \mathcal{U}\mathfrak{g}$,

$$\operatorname{pbw}(\mathrm{s}x_1 \odot \cdots \odot \mathrm{s}x_q) = \sum_{\sigma \in S_n} \frac{1}{n!} \mathrm{s}x_{\sigma(1)} \cdots \mathrm{s}x_{\sigma(q)}, \tag{4.16}$$

of $\mathfrak{g}$-modules which induces an isomorphism of the Chevalley–Eilenberg complexes

$$\operatorname{pbw} : \big(\operatorname{Hom}(S(\mathfrak{g}[1]), S\mathfrak{g}), d^{S\mathfrak{g}}_{\mathrm{CE}}\big) \to \big(\operatorname{Hom}(S(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g}), d^{\mathcal{U}\mathfrak{g}}_{\mathrm{CE}}\big).$$

Proposition 4.5.5. *The diagram*

$$\begin{tikzcd} \mathbb{H}^\bullet({}_\oplus\mathcal{T}_{\mathrm{poly}}(\mathfrak{g}[1]), d_T) \arrow[r, "\mathrm{hkr}"] \arrow[d, "\Phi_T"'] & \mathrm{HH}^\bullet_\oplus(S(\mathfrak{g}[1])^\vee) \arrow[d, "\Phi_D=\Phi_2\circ\Phi_1"] \\ H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S\mathfrak{g}) \arrow[r, "\mathrm{pbw}"'] & H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}\mathfrak{g}) \end{tikzcd}$$

commutes, where $\mathrm{pbw} : H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S\mathfrak{g}) \to H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}\mathfrak{g})$ *is the map induced by* (4.16).

The rest of the section is devoted to proving Proposition 4.5.5 which is the last step of proving Theorem 4.5.1.

The isomorphism $\Phi_1 = \pi_{A*} \circ (\pi_{B*})^{-1} : \mathrm{HH}^\bullet_\oplus(S(\mathfrak{g}[1])^\vee) \xrightarrow{\cong} \mathrm{HH}^\bullet_\oplus(\mathcal{U}\mathfrak{g})$, according to Theorem 4.3.9, is induced by the surjective quasi-isomorphisms

$$\mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g}) \overset{\pi_A}{\twoheadleftarrow} \mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])) \overset{\pi_B}{\twoheadrightarrow} \mathrm{Hoch}^\bullet_\oplus(S(\mathfrak{g}[1])^\vee).$$

Note that the inclusion $\iota_B : \mathrm{Hoch}^\bullet_\oplus(S(\mathfrak{g}[1])^\vee) \hookrightarrow \mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]))$ is *not* a cochain map, and there is no obvious representation of $(\pi_{B*})^{-1}$ on the cochain level. Thus, it is difficult to verify $\mathrm{pbw} \circ \Phi_T = \Phi_D \circ \mathrm{hkr} = \Phi_2 \circ \pi_{A*} \circ (\pi_{B*})^{-1} \circ \mathrm{hkr}$ directly. We solve this issue by lifting π_B and hkr to a pullback complex.

Pullback complex

Let $\mathrm{hkr}^* \mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]))$ be the pullback complex

$$\begin{tikzcd} \mathrm{hkr}^* \mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])) \arrow[r, dashed, "\widetilde{\mathrm{hkr}}"] \arrow[d, dashed, "\tilde{\pi}_B"'] & \mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])) \arrow[d, "\pi_B"] \\ {}_\oplus\mathcal{T}^\bullet_{\mathrm{poly}}(\mathfrak{g}[1]) \arrow[r, "\mathrm{hkr}"'] & \mathrm{Hoch}^\bullet_\oplus(S(\mathfrak{g}[1])^\vee). \end{tikzcd}$$

More precisely,

$$\begin{aligned} &\mathrm{hkr}^* \mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])) \\ &\qquad = \mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g}) \oplus \mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), S(\mathfrak{g}[1])^\vee) \oplus {}_\oplus\mathcal{T}^\bullet_{\mathrm{poly}}(\mathfrak{g}[1]) \end{aligned}$$

together with the differential

$$D = d^A_{\mathcal{H}} + d^{AX}_{\mathcal{H}} + d^X_{\mathcal{H}} + \partial_X + (d^{XB}_{\mathcal{H}} \circ \mathrm{hkr}) + [d_\mathfrak{g}, -].$$

Since $\pi_B : \mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])) \to \mathrm{Hoch}^\bullet_\oplus(S(\mathfrak{g}[1])^\vee)$ is a surjective quasi-isomorphism, the kernel $\ker(\pi_B)$ is acyclic. Also note that $\ker(\pi_B) = \ker(\tilde{\pi}_B)$ by construction. Thus, the projection

$$\tilde{\pi}_B : \mathrm{hkr}^* \mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])) \to {}_\oplus\mathcal{T}^\bullet_{\mathrm{poly}}(\mathfrak{g}[1])$$

is also a surjective quasi-isomorphism. Therefore, to prove Proposition 4.5.5, it suffices to show that the diagram

$$\begin{tikzcd} \mathrm{hkr}^* \operatorname{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])) \arrow[r, "\widetilde{\mathrm{hkr}}"] \arrow[d, "\tilde{\Phi}_T \circ \tilde{\pi}_B"'] & \operatorname{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])) \arrow[r, "\pi_A"] & \operatorname{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g}) \arrow[d, "\tilde{\Phi}_2"] \\ \operatorname{Hom}(S(\mathfrak{g}[1]), S\mathfrak{g}) \arrow[rr, "\mathrm{pbw}"'] & & \operatorname{Hom}(S(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g}) \end{tikzcd}$$

commutes up to homotopy. We prove it by constructing an explicit homotopy operator.

Homotopy operator

Let

$$\psi_1 = \tilde{\Phi}_2 \circ \pi_A \circ \widetilde{\mathrm{hkr}},$$
$$\psi_2 = \mathrm{pbw} \circ \tilde{\Phi}_T \circ \tilde{\pi}_B.$$

Now, our task is to construct a homotopy operator $h : \mathrm{hkr}^* \operatorname{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])) \to \operatorname{Hom}(S(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g})$ satisfying the equation

$$\psi_1 - \psi_2 = h \circ D + d^{\mathcal{U}\mathfrak{g}}_{\mathrm{CE}} \circ h. \tag{4.17}$$

Notations In the rest of this section, we denote $A = \mathcal{U}\mathfrak{g}$, $B = S(\mathfrak{g}[1])^\vee$ and $X = \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])$. A basis of $\mathfrak{g}[1]$ is denoted by $\{e_1, \cdots, e_d\}$, and its dual basis is denoted by $\{\epsilon^1, \cdots, \epsilon^d\}$. The symbol x_i is an element of $\mathfrak{g}[1]$ for each i, $\mathrm{s} : \mathfrak{g}[1] \to \mathfrak{g}$ is the degree-shifting map, and $\blacklozenge$ denotes the adjoint action. We need the technical maps

(i) $\widetilde{\mathrm{sym}} : S^p(\mathfrak{g}[1]) \to \mathcal{U}\mathfrak{g}^{\otimes p}$,

$$\widetilde{\mathrm{sym}}(x_1 \odot \cdots \odot x_p) = \sum_{\sigma \in S_p} (-1)^\sigma \mathrm{s}x_{\sigma(1)} \otimes \cdots \otimes \mathrm{s}x_{\sigma(p)},$$

(ii) $\mathcal{J}_{q,r} : S^{q+r}(\mathfrak{g}[1]) \to \mathcal{U}\mathfrak{g} \otimes S^{q+r}(\mathfrak{g}[1]) \otimes (\mathfrak{g}[1]^\vee)^{\otimes q} \otimes \mathfrak{g}[1]^{\otimes q}$,

$$\mathcal{J}_{q,r}(\mathbf{x}) = 1 \otimes \mathbf{x} \otimes \sum_{i_1, \cdots, i_q} \big((\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_q}) \otimes (e_{i_q} \otimes \cdots \otimes e_{i_1}) \big),$$

for $\mathbf{x} \in S^{q+r}(\mathfrak{g}[1])$,

(iii) $\phi : \mathfrak{g}[1]^{\otimes q} \to \mathcal{U}\mathfrak{g}$,

$$\phi(x_1 \otimes \cdots \otimes x_q) = \mathrm{s}x_1 \cdots \mathrm{s}x_q,$$

(iv) the projection $\mathrm{pr}_{\mathcal{U}\mathfrak{g}} : \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]) \twoheadrightarrow \mathcal{U}\mathfrak{g} \otimes S^0(\mathfrak{g}[1]) \cong \mathcal{U}\mathfrak{g}$.

For $\mathbf{x} = x_1 \odot \cdots \odot x_n \in S^n(\mathfrak{g}[1])$, we denote $\mathbf{x}^{\{i\}} := x_1 \odot \cdots \widehat{x_i} \cdots \odot x_n$. Following Sweedler notation, we write the formula of the comultiplication $\Delta : S(\mathfrak{g}[1]) \to S(\mathfrak{g}[1]) \otimes S(\mathfrak{g}[1])$ as

$$\Delta(\mathbf{x}) = \sum_k \mathbf{x}_{(1),k} \otimes \mathbf{x}_{(2),n-k} = \mathbf{x}_{(1),k} \otimes \mathbf{x}_{(2),n-k},$$

where $\mathbf{x}_{(1),k} \in S^k(\mathfrak{g}[1])$ and $\mathbf{x}_{(2),n-k} \in S^{n-k}(\mathfrak{g}[1])$.

For $f \in \mathrm{Hom}^r(A^{\otimes p} \otimes X \otimes B^{\otimes q}, X)$, we define

$$h_{p,q,r} : \mathrm{Hom}^r(A^{\otimes p} \otimes X \otimes B^{\otimes q}, X) \to \mathrm{Hom}(S^{p+q+r}(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g})$$
$$h_{p,q,r}(f) = \mathrm{sgn}(p,q,r)\ \mu_{\mathcal{U}\mathfrak{g}} \circ (\mathrm{pr}_{\mathcal{U}\mathfrak{g}} \otimes \phi) \circ (f \otimes 1) \circ (\widetilde{\mathrm{sym}} \otimes \mathcal{J}_{q,r}) \circ \Delta_{p,q+r}$$

where $\mathrm{sgn}(p,q,r) = (-1)^{qr+r+q(q+1)/2}$. Using Sweedler notation, one can write

$$\begin{aligned} &h_{p,q,r}(f)(\mathbf{x}) \\ &= \mathrm{sgn}(p,q,r) \sum_{i_1,\cdots,i_q} \mathrm{pr}_{\mathcal{U}\mathfrak{g}} \circ f\left(\widetilde{\mathrm{sym}}(\mathbf{x}_{(1),p}) \otimes (1 \otimes \mathbf{x}_{(2),q+r}) \otimes (\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_q})\right) \cdot (\mathrm{s}e_{i_q} \cdots \mathrm{s}e_{i_1}) \end{aligned}$$

for $\mathbf{x} \in S^{p+q+r}(\mathfrak{g}[1])$. Now, we define the homotopy operator

$$h : \mathrm{hkr}^* \, \mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])) \to \mathrm{Hom}(S(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g})$$

to be the operator extending $\sum h_{p,q,r}$ by zero, i.e. the operator h is equal to the composition

$$\begin{aligned} \mathrm{hkr}^* \, \mathrm{Hoch}^\bullet_\oplus(\mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1])) &\xrightarrow{\mathrm{pr}} \mathrm{Hoch}_\oplus(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g} \otimes S(\mathfrak{g}[1]), S(\mathfrak{g}[1])^\vee) \\ &\xrightarrow{\sum h_{p,q,r}} \mathrm{Hom}(S(\mathfrak{g}[1]), \mathcal{U}\mathfrak{g}). \end{aligned}$$

A-side Since $A = \mathcal{U}\mathfrak{g}$ is concentrated at degree zero, we have

$$(d^{AX}_{\mathcal{H}} f_A)(a_1 \otimes \cdots \otimes a_p \otimes x) = f_A(a_1 \otimes \cdots \otimes a_p) \cdot x$$

for $f_A \in \mathrm{Hom}(\mathcal{U}\mathfrak{g}^{\otimes p}, \mathcal{U}\mathfrak{g}) \subset \mathrm{Hoch}^\bullet_\oplus(X)$, $x \in X$ and $a_i \in \mathcal{U}\mathfrak{g}$. Thus, for $\mathbf{x} = x_1 \odot \cdots \odot x_p \in S(\mathfrak{g}[1])$,

$$\begin{aligned} h(d^{AX}_{\mathcal{H}} f_A)(\mathbf{x}) &= \mathrm{sgn}(p,0,0)\ \mathrm{pr}_{\mathcal{U}\mathfrak{g}} \circ (d^{AX}_{\mathcal{H}} f_A)(\widetilde{\mathrm{sym}}(\mathbf{x}) \otimes (1 \otimes 1)) \\ &= \sum_{\sigma \in S_p} (-1)^\sigma f_A(\mathrm{s}x_{\sigma(1)} \otimes \cdots \otimes \mathrm{s}x_{\sigma(p)}). \end{aligned}$$

Furthermore, since $\psi_2(f_A) = \mathrm{pbw} \circ \widetilde{\Phi}_T \circ \tilde{\pi}_B(f_A) = 0$ and

$$\psi_1(f_A)(\mathbf{x}) = (\tilde{\Phi}_2 \circ \pi_A \circ \widetilde{\mathrm{hkr}})(f_A)(\mathbf{x}) = \sum_{\sigma \in S_p} (-1)^{\sigma} f_A(\mathrm{s}x_{\sigma(1)} \otimes \cdots \otimes \mathrm{s}x_{\sigma(p)}),$$

we have

$$(h \circ D + d_{\mathrm{CE}}^{\mathcal{U}\mathfrak{g}} \circ h)(f_A) = h(d_{\mathcal{H}}^{AX} f_A) = (\psi_1 - \psi_2)(f_A).$$

B**-side** For $f_B = \mathfrak{f} \otimes \mathrm{s}e_{j_1} \odot \cdots \odot \mathrm{s}e_{j_q} \in S^{q+r}(\mathfrak{g}[1])^\vee \otimes S^q\mathfrak{g} \subset {}_{\oplus}\mathcal{T}_{\mathrm{poly}}(\mathfrak{g}[1])$ and $\mathbf{x} \in S^{q+r}(\mathfrak{g}[1])$, we have

$$\begin{aligned}
&\mathrm{sgn}(0,q,r)h(d_{\mathcal{H}}^{XB} \circ \mathrm{hkr}(f_B))(\mathbf{x}) \\
&\quad = \sum_{i_1,\cdots,i_q} \mathrm{pr}_{\mathcal{U}\mathfrak{g}} \circ (d_{\mathcal{H}}^{XB} \circ \mathrm{hkr}(f_B))\Big((1 \otimes \mathbf{x}) \otimes (\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_q})\Big) \cdot (\mathrm{s}e_{i_q} \cdots \mathrm{s}e_{i_1}) \\
&\quad = (-1)^{q+r-1+r(q+r)} \sum_{i_1,\cdots,i_q} \mathrm{pr}_{\mathcal{U}\mathfrak{g}} \Big((1 \otimes \mathbf{x}) \llcorner \mathrm{hkr}(f_B)(\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_q})\Big) \cdot (\mathrm{s}e_{i_q} \cdots \mathrm{s}e_{i_1}) \\
&\quad = (-1)^{q+r-1+r(q+r)+q(q+1)/2} (\mathbf{x} \llcorner \mathfrak{f}) \frac{1}{q!} \sum_{\sigma \in S_q} \mathrm{s}e_{j_{\sigma(q)}} \cdots \mathrm{s}e_{j_{\sigma(1)}} \\
&\quad = (-1)^{qr+q-1+q(q+1)/2} (-1)^{q+r} \mathfrak{f}(\mathbf{x}) \cdot \mathrm{pbw}(\mathrm{s}e_{j_1} \odot \cdots \odot \mathrm{s}e_{j_q}) \\
&\quad = -\,\mathrm{sgn}(0,q,r)\mathfrak{f}(\mathbf{x}) \cdot \mathrm{pbw}(\mathrm{s}e_{j_1} \odot \cdots \odot \mathrm{s}e_{j_q}).
\end{aligned}$$

Since $\psi_1(f_B) = \tilde{\Phi}_2 \circ \pi_A \circ \widetilde{\mathrm{hkr}}(f_B) = 0$ and

$$\psi_2(f_B) = \mathrm{pbw} \circ \tilde{\Phi}_T(\mathfrak{f} \otimes \mathrm{s}e_1 \odot \cdots \odot \mathrm{s}e_q) : \mathbf{x} \mapsto \mathfrak{f}(\mathbf{x}) \cdot \mathrm{pbw}(\mathrm{s}e_1 \odot \cdots \odot \mathrm{s}e_q),$$

we conclude that

$$(h \circ D + d_{\mathrm{CE}}^{\mathcal{U}\mathfrak{g}} \circ h)(f_B) = h \circ d_{\mathcal{H}}^{XB} \circ \mathrm{hkr}(f_B) = (\psi_1 - \psi_2)(f_B).$$

X**-side** Let $f \in \mathrm{Hom}^r(A^{\otimes p} \otimes X \otimes B^{\otimes q}, X)$ and $\mathbf{x} = x_1 \odot \cdots \odot x_{p+q+r+1} \in S^{p+q+r+1}(\mathfrak{g}[1])$. To verify the homotopy equation, we need to compute the following 4 terms.

Term I:

$$\begin{aligned}
d_{\mathrm{CE}}^{\mathcal{U}\mathfrak{g}}(hf)(\mathbf{x}) &= \left(\sum_{i=1}^{p+q+r+1} (-1)^{i+p+q+r} (\mathrm{s}x_i) \bullet (hf)(\mathbf{x}^{\{i\}}) \right) - (-1)^{p+q+r} (hf)(\partial_{\mathfrak{g}} \mathbf{x}) \\
&= \left(\sum_{i=1}^{p+q+r+1} (-1)^{i+p+q+r} \,\mathrm{sgn}(p,q,r)(\mathfrak{D}_1^i - \mathfrak{D}_2^i) \right) \\
&\qquad\qquad - (-1)^{p+q+r} \,\mathrm{sgn}(p,q,r)(\mathfrak{F}_1 + \mathfrak{F}_2),
\end{aligned}$$

where

$$\mathfrak{D}_1^i = \sum_{i_1,\cdots,i_q} \mathrm{s}x_i \cdot \Big(\mathrm{pr}_{\mathcal{U}\mathfrak{g}} \circ f\Big(\widetilde{\mathrm{sym}}(\mathbf{x}^{\{i\}}_{(1),p}) \otimes (1 \otimes \mathbf{x}^{\{i\}}_{(2),q+r}) \otimes (\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_q})\Big)\Big) \cdot (\mathrm{s}e_{i_q} \cdots \mathrm{s}e_{i_1}),$$

$$\mathfrak{D}_2^i = \sum_{i_1,\cdots,i_q} \Big(\mathrm{pr}_{\mathcal{U}\mathfrak{g}} \circ f\Big(\widetilde{\mathrm{sym}}(\mathbf{x}^{\{i\}}_{(1),p}) \otimes (1 \otimes \mathbf{x}^{\{i\}}_{(2),q+r}) \otimes (\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_q})\Big)\Big) \cdot (\mathrm{s}e_{i_q} \cdots \mathrm{s}e_{i_1}) \cdot \mathrm{s}x_i,$$

$$\mathfrak{F}_1 = \sum_{i_1,\cdots,i_q} \mathrm{pr}_{\mathcal{U}\mathfrak{g}} \circ f\Big(\widetilde{\mathrm{sym}}(\partial_{\mathfrak{g}}\mathbf{x}_{(1),p+1}) \otimes (1 \otimes \mathbf{x}_{(2),q+r}) \otimes (\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_q})\Big) \cdot (\mathrm{s}e_{i_q} \cdots \mathrm{s}e_{i_1}),$$

$$\mathfrak{F}_2 = (-1)^p \sum_{i_1,\cdots,i_q} \mathrm{pr}_{\mathcal{U}\mathfrak{g}} \circ f\Big(\widetilde{\mathrm{sym}}(\mathbf{x}_{(1),p}) \otimes (1 \otimes \partial_{\mathfrak{g}}\mathbf{x}_{(2),q+r+1}) \otimes (\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_q})\Big) \cdot (\mathrm{s}e_{i_q} \cdots \mathrm{s}e_{i_1}).$$

Term II:

$$h(d^X_{\mathcal{H}L} f)(\mathbf{x}) = \mathrm{sgn}(p+1,q,r)(-1)^{p+q+r} \cdot \left\{ \left(\sum_{i=1}^{p+q+1} (-1)^{i+1} \mathfrak{A}_1^i \right) + \mathfrak{A}_{\mathrm{mid}} + \left((-1)^{p+1} \sum_{i=1}^{p+q+r+1} (-1)^{i+1+p} \mathfrak{A}_2^i \right) \right\},$$

where

$$\mathfrak{A}_1^i = \sum_{i_1,\cdots,i_q} \mathrm{s}x_i \cdot \Big(\mathrm{pr}_{\mathcal{U}\mathfrak{g}} \circ f\Big(\widetilde{\mathrm{sym}}(\mathbf{x}^{\{i\}}_{(1),p}) \otimes (1 \otimes \mathbf{x}^{\{i\}}_{(2),q+r}) \otimes (\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_q})\Big)\Big) \cdot (\mathrm{s}e_{i_q} \cdots \mathrm{s}e_{i_1}),$$

$$\mathfrak{A}_2^i = \sum_{i_1,\cdots,i_q} \mathrm{pr}_{\mathcal{U}\mathfrak{g}} \circ f\Big(\widetilde{\mathrm{sym}}(\mathbf{x}^{\{i\}}_{(1),p}) \otimes (\mathrm{s}x_i \otimes \mathbf{x}^{\{i\}}_{(2),q+r}) \otimes (\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_q})\Big) \cdot (\mathrm{s}e_{i_q} \cdots \mathrm{s}e_{i_1}),$$

$$\mathfrak{A}_{\mathrm{mid}} = \sum_{i_1,\cdots,i_q} \sum_{j=1}^{p} (-1)^j \cdot \mathrm{pr}_{\mathcal{U}\mathfrak{g}} \circ f\Big(\big(\mu_j \circ \widetilde{\mathrm{sym}}(\mathbf{x}_{(1),p+1})\big) \otimes (1 \otimes \mathbf{x}_{(2),q+r}) \otimes (\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_q})\Big) \cdot (\mathrm{s}e_{i_q} \cdots \mathrm{s}e_{i_1}),$$

and the map $\mu_j : \mathcal{U}\mathfrak{g}^{\otimes p+1} \to \mathcal{U}\mathfrak{g}^{\otimes p}$, $j = 1, \cdots, p$, is defined by

$$\mu_j(u_1 \otimes \cdots \otimes u_{p+1}) = u_1 \otimes \cdots \otimes u_j u_{j+1} \otimes \cdots \otimes u_{p+1}.$$

Term III:

$$h(d^X_{\mathcal{H}R} f)(\mathbf{x}) = \mathrm{sgn}(p,q+1,r)(-1)^{p+q+r}(\mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3),$$

where

$$\mathfrak{B}_1 = -\sum_{i_1,\cdots,i_q}\sum_{i=1}^{p+q+r+1}(-1)^{q+r-i}\cdot$$
$$\mathrm{pr}_{\mathcal{U}\mathfrak{g}}\circ f\Big(\widetilde{\mathrm{sym}}(\mathbf{x}_{(1),p}^{\{i\}})\otimes(1\otimes\mathbf{x}_{(2),q+r}^{\{i\}})\otimes(\epsilon^{i_1}\otimes\cdots\otimes\epsilon^{i_q})\Big)\cdot(\mathrm{se}_{i_q}\cdots\mathrm{se}_{i_1}\mathrm{s}x_i)$$

$$\mathfrak{B}_2 = \sum_{i_1,\cdots,i_{q+1}}\sum_{k=1}^{q}(-1)^{p+1+k}\cdot$$
$$\mathrm{pr}_{\mathcal{U}\mathfrak{g}}\circ f\Big(\widetilde{\mathrm{sym}}(\mathbf{x}_{(1),p})\otimes(1\otimes\mathbf{x}_{(2),q+r+1})\otimes(\epsilon^{i_1}\otimes\cdots\epsilon^{i_k}\odot\epsilon^{i_{k+1}}\cdots\otimes\epsilon^{i_{q+1}})\Big)$$
$$\cdot(\mathrm{se}_{i_{q+1}}\cdots\mathrm{se}_{i_1})$$

$$\mathfrak{B}_3 = \sum_{i_1,\cdots,i_q,j}(-1)^{p+q}\cdot$$
$$\mathrm{pr}_{\mathcal{U}\mathfrak{g}}\Big\{f\Big(\widetilde{\mathrm{sym}}(\mathbf{x}_{(1),p})\otimes(1\otimes\mathbf{x}_{(2),q+r+1})\otimes(\epsilon^{i_1}\otimes\cdots\otimes\epsilon^{i_q})\Big)\llcorner\,\epsilon^j\Big\}\cdot(\mathrm{se}_j\mathrm{se}_{i_q}\cdots\mathrm{se}_{i_1})$$

Term IV:

$$h(\partial_X f)(\mathbf{x}) = \mathrm{sgn}(p,q,r+1)(\mathfrak{I}_1-(-1)^r(\mathfrak{I}_2+\mathfrak{I}_3)),$$

where

$$\mathfrak{I}_1 = \sum_{i_1,\cdots,i_q}\mathrm{pr}_{\mathcal{U}\mathfrak{g}}\circ d_X\circ f\Big(\widetilde{\mathrm{sym}}(\mathbf{x}_{(1),p})\otimes(1\otimes\mathbf{x}_{(2),q+r+1})\otimes(\epsilon^{i_1}\otimes\cdots\otimes\epsilon^{i_q})\Big)\cdot(\mathrm{se}_{i_q}\cdots\mathrm{se}_{i_1})$$

$$\mathfrak{I}_2 = \sum_{i_1,\cdots,i_q}\mathrm{pr}_{\mathcal{U}\mathfrak{g}}\circ f\Big(\widetilde{\mathrm{sym}}(\mathbf{x}_{(1),p})\otimes d_X(1\otimes\mathbf{x}_{(2),q+r+1})\otimes(\epsilon^{i_1}\otimes\cdots\otimes\epsilon^{i_q})\Big)\cdot(\mathrm{se}_{i_q}\cdots\mathrm{se}_{i_1})$$

$$\mathfrak{I}_3 = \sum_{i_1,\cdots,i_q}\sum_{k=1}^{q}(-1)^{q+r+k}\cdot$$
$$\mathrm{pr}_{\mathcal{U}\mathfrak{g}}\circ f\Big(\widetilde{\mathrm{sym}}(\mathbf{x}_{(1),p})\otimes(1\otimes\mathbf{x}_{(2),q+r+1})\otimes(\epsilon^{i_1}\otimes\cdots\otimes d_{\mathfrak{g}}\epsilon^{i_k}\otimes\cdots\otimes\epsilon^{i_q})\Big)$$
$$\cdot(\mathrm{se}_{i_q}\cdots\mathrm{se}_{i_1}).$$

Note that we have the following equations

$$
\begin{gathered}
\mathfrak{D}_1^i = \mathfrak{A}_1^i \\
\sum_i (-1)^{q+r+i+1} \mathfrak{D}_2^i = \mathfrak{B}_1 \\
\mathfrak{F}_1 = \mathfrak{A}_{\mathrm{mid}} \\
\mathfrak{I}_2 = (-1)^p \mathfrak{F}_2 + \sum_{i=1}^{p+q+r+1} (-1)^{i+p+1} \mathfrak{A}_2^i \\
\mathfrak{B}_2 = (-1)^{p+q+r} \mathfrak{I}_3 \\
\mathfrak{B}_3 = (-1)^{p+q+1} \mathfrak{I}_1
\end{gathered}
$$

for any $\mathbf{x} \in S^{p+q+r+1}(\mathfrak{g}[1])$. Here, the last second equation is obtained by

$$
(d_{\mathfrak{g}} \epsilon^i) \otimes \mathrm{s}e_i = \sum_{a<b} \epsilon^a \odot \epsilon^b \otimes [\mathrm{s}e_a, \mathrm{s}e_b] = \sum_{a<b} \left(\epsilon^a \odot \epsilon^b \otimes \mathrm{s}e_a \mathrm{s}e_b + \epsilon^b \odot \epsilon^a \otimes \mathrm{s}e_b \mathrm{s}e_a \right)
$$

which is induced by

$$
[\mathrm{s}e_a, \mathrm{s}e_b] = c_{ab}^i \mathrm{s}e_i, \quad d_{\mathfrak{g}} \epsilon^i = c_{ab}^i \epsilon^a \odot \epsilon^b,
$$

and the last equation is obtained by

$$
d_X(u \otimes e_i) = u(\mathrm{s}e_i) \otimes 1 = -(u(\mathrm{s}e_i) \otimes e_i) \llcorner \epsilon^i.
$$

Since $(\psi_1 - \psi_2)(f) = 0$, the homotopy equation (4.17) is equivalent to the system

$$
\begin{gathered}
\operatorname{sgn}(p+1, q, r) = \operatorname{sgn}(p, q, r), \\
\operatorname{sgn}(p, q+1, r) = (-1)^{q+r+1} \operatorname{sgn}(p, q, r), \\
\operatorname{sgn}(p, q, r+1) = (-1)^{q+1} \operatorname{sgn}(p, q, r),
\end{gathered}
$$

and the sign function

$$
\operatorname{sgn}(p, q, r) = (-1)^{qr + r + q(q+1)/2}
$$

is a solution of this system. Therefore, we conclude that

$$
(h \circ D + d_{\mathrm{CE}}^{\mathcal{U}\mathfrak{g}} \circ h)(f) = 0 = (\psi_1 - \psi_2)(f).
$$

This completes the proof of Proposition 4.5.5, and thus the proof of Theorem 4.5.1 is also complete.

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