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INFORMATION FRESHNESS: FUNDAMENTAL LIMITS AND OPTIMIZATION METHODOLOGIES

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Abstract

In recent years, the pervasive availability of network connectivity and the proliferation of sensing devices have fundamentally transformed the operation and control of cyber-physical systems (CPSs), where continuous data streams are generated by the sensors, transferred through different communication pipelines, delivered to central controllers, and eventually processed and analyzed, facilitating real-time status monitoring and decision-making. In such systems, the timeliness of information is of paramount importance: Stale measurements may lead to state estimations that significantly deviate from the current system state, resulting in incorrect control commands and deteriorating the stability and resilience of the system. Therefore, there is a compelling need to quantify and study the "freshness" of information, to measure the impact of information "staleness" on system performance, and to develop information gathering and processing techniques to ensure information freshness in real-time systems. In order to measure and ensure the freshness of the information available to the monitor, a metric called *Age of Information (AoI)* has been introduced and analyzed in various status updating systems. Specifically, at time t, the AoI in the system is defined as $t - U(t)$, where $U(t)$ is the time stamp of the latest received update at the destination. Since AoI depends on data generation as well as queueing and transmission, it is fundamentally different from traditional network performance metrics, such as throughput and delay. The goal of this dissertation is to characterize the AoI in various status monitoring systems under practical system level constraints, and to design efficient sampling, coding and scheduling techniques to improve information freshness and optimize AoI by leveraging a diverse corpus of tools from queueing theory, optimization theory, combinatorial analysis and information theory.

We first provide an overview of the dissertation and a brief review of the related literature in Chapter 1.

In Chapter 2, we focus on the AoI in energy harvesting (EH) sensor networks where the channel between the sensor and the destination is noisy. Our goal is to design an optimal online status updating policy to minimize the long-term average AoI at the destination, subject to the energy causality constraint. Depending on whether there exists updating feedback to the source, we consider two possible scenarios: no updating feedback and perfect updating feedback. For both cases, we propose Best-effort Uniform updating (BU) policy and Best-effort Uniform updating with Retransmission (BUR) policy and prove their optimality under a broad class of online policies. In order to analyze the performance of the proposed policies, we come up with a novel virtual policy

based approach. Specifically, for both BU and BUR updating policies, we construct a sequence of virtual policies, which are strictly sub-optimal to their original counterparts, and eventually converge to them. Leveraging the virtual policies, we are able to decouple the effects of battery outage and updating errors in the performance analysis. We show that the long-term average AoI under virtual polices converges to the corresponding lower bound, which implies the optimality of the proposed policies.

In Chapter 3, we consider the MIMO broadcast setting where optimizing AoI through precoding and transmission scheduling is the focal point. Under the assumptions that the noise level is negligible in the channel and the instantaneous channel state information (CSI) is available to the transmitter at the beginning at each time slot, we propose novel coding and scheduling schemes under different setups that allow simultaneously transmissions of multiple users, and prove their optimalities. Our results indicate that the size of the updates plays a critical role in the design of the optimal updating schemes. In fact, when the update size is greater than one, wasting some transmission opportunities in order to deliver fresher updates may be better. The techniques developed to the show the optimality are novel and those techniques may be applicable for other AoI problems as well. The MIMO broadcast setting has been extensively studied for throughput optimization but existing study on AoI in broadcast channel rarely considers the impact of multiplexing gain on information freshness. This work bridges the gap between existing studies on MIMO broadcast channel and AoI.

In Chapter 4, we study AoI optimization for an symbol erasure channel with rateless codes. We formulate the problem as a Markov Decision Process (MDP) and identify the monotonic threshold structure of the optimal policy. We then consider a more practical scenario where limited feedback is available during the transmission of an update. We propose a type of threshold policies to approximate the multi-threshold structure of the optimal policy and derive the explicit expression of the time-average AoI under the threshold policies. We then numerically search for the optimal policy through structured dynamic programming, and compare its AoI performance with a threshold policy and other baseline policies. Numerical results corroborate the structural properties of the optimal solution and indicates that the performance under the special type of threshold policy is very close to the optimal policy.

In Chapter 5, we investigate the impact of coding on the AoI in a two-user broadcast symbol erasure channel with feedback. Assuming perfect feedback information at the source right after the transmission of each symbol, our objective is to design an adaptive coding scheme to achieve small AoI at both users. We propose a novel coding scheme to judiciously combine symbols from different updates together, and analyze the AoI at both users. Compared with a baseline greedy scheme, the proposed adaptive coding scheme improves the AoI at the weak user by orders of magnitude without compromising the AoI at the strong user.

In Chapter 6, we aim to establish the connection between AoI in network theory, information uncertainty in information theory, and detection delay in time series analysis. We consider a dynamic system whose state changes at discrete time points, and a state change will not be detected until an update generated after the change point is delivered to the destination for the first time. We introduce an information theoretic metric to measure the information freshness at the destination, and name it as generalized Age of Information (GAoI). We show that under any state-independent online updating policy, if the underlying state of the system evolves according to a stationary Markov chain, the GAoI is proportional to the AoI. Besides, the accumulative GAoI and AoI are proportional to the expected accumulative detection delay of all changes points over a period of time. Thus, any (G)AoI-optimal state-independent updating policy equivalently minimizes the corresponding expected change point detection delay, which validates the fundamental role of (G)AoI in real-time status monitoring. Besides, we also investigate a Bayesian change point detection scenario where the underlying state evolution is not stationary. Although AoI is no longer related to detection delay explicitly, we show that the accumulative GAoI is still an affine function of the expected detection delay, which indicates the versatility of GAoI in capturing information freshness in dynamic systems.

Finally, we conclude the dissertation in Chapter 7, with a brief discussion of potential future directions.

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Dedication

To my parents.

Chapter 1 | Introduction

In recent years, the pervasive availability of network connectivity and the proliferation of sensing devices have fundamentally transformed the operation and control of cyber-physical systems (CPSs), where continuous data streams are generated by the sensors, transferred through different communication pipelines, delivered to central controllers, and eventually processed and analyzed, facilitating real-time status monitoring and decision-making. In such systems, the timeliness of information is of paramount importance: Stale measurements may lead to state estimations that significantly deviate from the current system state, resulting in incorrect control commands and deteriorating the stability and resilience of the system. Therefore, there is a compelling need to quantify and study the "freshness" of information, to measure the impact of information "staleness" on system performance, and to develop information gathering and processing techniques to ensure information freshness in real-time systems. In order to measure and ensure the freshness of the information available to the monitor, a metric called *Age of Information (AoI)* has been introduced and analyzed in various status updating systems. Specifically, at time t, the AoI in the system is defined as $t - U(t)$, where $U(t)$ is the time stamp of the latest received update at the destination. Since AoI depends on data generation as well as queueing and transmission, it is fundamentally different from traditional network performance metrics, such as throughput and delay. The goal of this study is to characterize the AoI in various status monitoring systems under practical system level constraints, and to design efficient sampling, coding and scheduling techniques to improve information freshness and optimize AoI by leveraging a diverse corpus of tools from queueing theory, optimization theory, combinatorial analysis and information theory.

In Chapter 2, we focus on the AoI in energy harvesting (EH) sensor networks. We consider an EH sensor that continuously monitors a system and sends time-stamped status update to the destination. The sensor harvests energy from nature and uses it to power its updating operations. The destination keeps track of the system status through the successfully received updates. The objective is to design an optimal online status updating policy to minimize the long-term average AoI at the destination, subject to the energy causality constraint. Due to the noisy channel between the sensor and the destination, each transmitted update may be erased with a fixed probability and the AoI at the destination will be reset only when an update is successfully received. Depending on whether there exists updating feedback to the source, we consider two possible scenarios: no updating feedback and perfect updating feedback. In the former case, the source has no knowledge of whether an update is successful and it can only use the up-to-date energy arrival profile and updating decisions to decide the upcoming updating time points. In the latter case, the source receives an instantaneous feedback when an update is transmitted. Therefore, it can decide when to update next based on the feedback information, along with the information it uses for the no feedback case. For both cases, we propose Best-effort Uniform updating (BU) policy and Best-effort Uniform updating with Retransmission (BUR) policy and prove their optimality under a broad class of online policies. Although the proposed policies are intuitive, their optimality is quite challenging to establish, compared with existing work. The reason is that both battery outage and updating erasure will affect the AoI under the proposed policies. While the impact of either of those events can be analyzed relatively easily when isolated, it becomes challenging when both of them are involved. Besides, when there exists perfect updating feedback to the source, updating erasures under BUR will lead to subsequent retransmissions and energy consumption, thus affecting the battery outage probability in the future. Such complicated interplay between those two events makes the problem even more complicated. In order to overcome those difficulties, we propose a novel virtual policy based approach. Specifically, for both BU and BUR updating policies, we construct a sequence of virtual policies, which are strictly sub-optimal to their original counterparts, and eventually converge to them. Leveraging the virtual policies, we are able to decouple the effects of battery outage and updating errors in the performance analysis. We show that the long-term average AoI under virtual polices converges to the corresponding lower bound, which implies the optimality of the proposed policies.

In Chapter 3, we investigate novel coding and scheduling schemes that allow simultaneously transmissions of multiple users and achieve optimal summed average AoI. Specifically, we study a status monitoring system with K sources, each generating updates intended for one of the K users. The updates are transmitted to the monitors through a broadcast channel. Different from existing work, we consider block fading over the links between the transmitter and users, each of which is equipped with N antennas. Therefore, all users are able to receive an attenuated version of the transmitted signal. Then, under the assumption that the noise level is negligible in the channel, and the instantaneous channel state information is available to the transmitter at the beginning at each time slot, our objective is to investigate the optimal coding and transmission scheduling schemes for the minimization of the summed time average AoI over the users. We explicitly identify the optimal updating strategies for the MIMO broadcast channel under different setups and our results indicate that the size of the updates plays a critical role in the design of the optimal updating schemes. In fact, when the update size is greater than one, wasting some transmission opportunities in order to deliver fresher updates may be better. The techniques developed to the show the optimality are novel and we expect those techniques may be applicable for other AoI problems as well. The novel MIMO broadcast setting has been extensively studied for throughput optimization but existing study on AoI in broadcast channel rarely considers the impact of multiplexing gain on information freshness. This work bridges the gap between existing studies on MIMO broadcast channel and AoI.

In Chapter 4, we study AoI optimization for an erasure channel with rateless codes. We examine a status updating system where updates generated by the source are sent to the monitor through an erasure channel. Each update consists of k information symbols and encoded using rateless codes. We assume each transmitted encoded symbol will be erased according to an independent and identically distributed (i.i.d.) Bernoulli process and an update can be successfully decoded if k encoded symbols are received. Our objective is to optimize the information freshness at the monitor through adaptive transmissions of the encoded symbols, where we adopt the metric “Age of Information” (AoI) to measure the freshness of information. We begin with an ideal scenario where instant feedback is available to the source after the transmission of each symbol. Upon receiving feedback, the source has the choice to start transmitting a new update, or continue with the transmission of the previous update if it is not delivered yet. We formulate the problem as a Markov Decision Process (MDP) and identify the monotonic threshold structure of the optimal policy. We then consider a more practical scenario where limited feedback is available during the transmission of an update. We propose a type of threshold policies to approximate the multi-threshold structure of the optimal policy and derive the explicit expression of the time-average AoI under the threshold policies. We also prove performance guarantee for a special case. We then numerically search for the optimal policy through structured dynamic programming, and compare its AoI performance with a threshold policy and other baseline policies. Numerical results corroborate the structural properties of the optimal solution and indicates that the performance under the special type of threshold policy is very close to the optimal policy.

In Chapter 5, we investigate the impact of coding on the Age of Information (AoI) in a two-user broadcast symbol erasure channel with feedback. We assume each update consists of K symbols and the source is able to broadcast one symbol in each time slot. Due to random channel noise, the intended symbol at each user will be erased according to an independent and identically distributed (i.i.d.) Bernoulli process. A user is able to successfully decode an update if it accumulates sufficient information and successfully decodes the K symbols of the update. Assuming perfect feedback information at the source right after the transmission of each symbol, our objective is to design an adaptive coding scheme to achieve small AoI at both users. We propose a novel coding scheme to judiciously combine symbols from different updates together, and analyze the AoI at both users. Compared with a baseline greedy scheme, the proposed adaptive coding scheme improves the AoI at the weak user by orders of magnitude without compromising the AoI at the strong user.

In Chapter 6, we aim to establish the connection between Age of Information (AoI) in network theory, information uncertainty in information theory, and detection delay in time series analysis. We consider a dynamic system whose state changes at discrete time points, and a state change won't be detected until an update generated after the change point is delivered to the destination for the first time. We introduce an information theoretic metric to measure the information freshness at the destination, and name it as generalized Age of Information (GAoI). We show that under any state-independent online updating policy, if the underlying state of the system evolves according to a stationary Markov chain, the GAoI is proportional to the AoI. Besides, the accumulative GAoI and AoI are proportional to the expected accumulative detection delay of all changes points over a period of time. Thus, any (G)AoI-optimal state-independent updating policy equivalently minimizes the corresponding expected change point detection delay, which validates the fundamental role of (G)AoI in real-time status monitoring. Besides, we also investigate a Bayesian change point detection scenario where the underlying state evolution is not stationary. Although AoI is no longer related to detection delay explicitly, we show that the accumulative GAoI is still an affine function of the expected detection delay, which indicates the versatility of GAoI in capturing information freshness in dynamic systems.

Finally, we conclude the dissertation in Chapter 7, with a brief discussion of potential future directions.

1.1 Related Literature

1.1.1 AoI in Status Updating Systems

There are two main approaches in the study of AoI. The first approach is to characterize the AoI under given status updating policies [1–12]. The second approach is to design certain status updating policies to actively optimize AoI [13–15]. Modeling the status monitoring system as a queueing system, where updates are generated at the source according to a random process, the time average AoI has been analyzed in different queueing management settings. For systems with a single server, the corresponding AoI has been studied in single-source single-server queues [1], the $M/M/1$ Last-Come First-Served (LCFS) queue with preemption in service [2], the $M/M/1$ First-Come First-Served (FCFS) queue with multiple sources [3], the $M/M/1$ queue with multiple souces which only keeps the latest status packet of each source in the queue [4], the LCFS queue with gamma-distributed service time and Poisson update packet arrivals [5]. Moreover, in $M/M/1$ queue systems, packet deadlines are found to improve AoI performance in [6], and AoI in the presence of packet delivery errors is evaluated in [7]. The AoI in systems with multiple servers has been evaluated in [8, 9]. A related metric, Peak Age of Information (PAoI), is introduced and studied in [10, 11, 14]. For more complicated multi-hop networks, reference [12] introduces a novel stochastic hybrid system (SHS) approach to derive explicit age distributions. The optimality properties of a preemptive Last Generated First Served (LGFS) service discipline in a multi-hop network are identified in [13]. AoI optimization with the knowledge of the server state has been studied in [14]. The relationship between AoI and the MMSE in remote estimation of a Wiener process is investigated in [15].

1.1.2 AoI in State Estimation and Real-time Control of Stochastic Systems

Age of information has also demonstrated its fundamental role in the state estimation and real-time control of stochastic systems. In [16], the fundamental trade-off between the control performance and information staleness measured in AoI has been characterized, while in [17], it studies how the random AoI would alter the rate-cost tradeoff for a Gaussian linear control system, where the cost is measured in terms of the system-state mean-square stability. In [18], AoI has been adopted to solve the state estimation and control problem in a single-loop stochastic linear time-invariant (LTI) networked system. It shows that that minimizing the estimation error is equivalent to minimizing a non-negative and non-decreasing function of AoI. In [19], AoI has been utilized for the distributed estimation of the state of a discrete-time LTI process over a time-varying directed communication graph.

1.1.3 Coding for AoI Optimization

Recently, coding for AoI optimization has received increasing attention. In [20], two different coding strategies, i.e., rateless codes and maximum distance separable (MDS) codes, are studied for both single-user and multiple-user systems. They show that if the redundancy is carefully optimized in response to the channel erasure rate, the AoI performance of MDS coding can match that of rateless coding. In [21], the optimal transmission of rateless codes for AoI minimization in a single-user erasure channel has been characterized. Reference [22] proves that when the source alphabet and channel input alphabet have the same size, a Last Come First Served (LCFS) with no buffer policy is optimal. For an energy harvesting erasure channel, [23] shows that rateless coding with save-and-transmit scheme outperforms MDS based schemes. For streaming source coding system, the optimal prefix-free lossless coding scheme that minimizes the average peak AoI is proposed in [24]. The effect of codeword length on the average AoI is analyzed in [25, 26]. In [27], the benefits of network coding in a two-user broadcast packet erasure channel with updates from two streams are studied. It shows that coded randomized policies outperform their uncoded counterparts in terms of AoI.

Chapter 2
AoI Minimization for an Energy Harvesting Source with Updating Erasures

2.1 Introduction

The study of AoI with EH devices is motivated by the rapid development and deployment of these devices in internet of things (IoT) networks, such as the agricultural IoT deployment in rural areas where devices may be exclusively powered by natural sources such as wind or solar. A systematic investigation of AoI with EH devices is thus of practical importance to improve the resilience, adaptability, and efficiency of the overall system.

Due to the magnified tension between keeping information fresh and the stringent energy constraint, AoI in EH wireless networks has attracted increasing interests recently. An EH sensor harvests energy from the environment and uses it to power its sensing and communication operations. Due to the stochastic energy arrival process, all of the operations are subject to the so-called energy causality constraint. Under such constraint, various policies have been proposed to optimize different communication and sensing performance metrics [28–38]. Such sample path-wise constraint also makes the design and analysis of the status updating policy in EH systems extremely challenging. Under the assumption that the battery size is sufficiently large, [39] shows that updates should be scheduled only when the server is free to avoid queueing delay, and a *lazy* update policy that introduces inter-update delays outperforms the greedy policy. Reference [40] investigates AoI-optimal offline and online status updating policies, where the online problem is modeled as a Markov decision process and solved through dynamic programming. In [41–43], optimal online status updating policies under different assumptions on the battery size have been identified. Specifically, for the infinite battery case, [41] shows that the *best-effort uniform* updating policy, which updates at a constant rate when the source has sufficient energy, is optimal when the channel between source and destination is perfect. When the battery size is finite, the optimal policies are shown to have certain threshold structures [42, 43]. Offline policies to minimize AoI in EH channels have been studied in [44, 45]. Reference [23] analyzes the AoI performance of two channel coding schemes when channel erasures are present. Using the SHS tools proposed in [12], references [46] and [47] study the average AoI for a finite battery EH system, with and without preemption of packets in service allowed, respectively. Reference [48] considers a setting where extra information is carried by the timing of the update packets. A tradeoff between the average AoI and the average message rate is studied for several achievable schemes. The AoI in an EH cognitive radio network is studied in [49], and that in wireless powered networks has been analyzed in [50–52]. Other information freshness metrics, such as non-linear functions of AoI [53, 54], have also been investigated in the EH setting.

In this chapter, we take the imperfect updating channel into consideration and investigate the optimal updating policies of an EH system where updating erasures can happen. Assuming each update can be erased with a constant probability, the AoI at the destination will be reset only when an update is successfully received. Our objective is to design *online* status updating policies to minimize the average AoI at the destination. Depending on whether there exists updating feedback to the source, we consider two possible scenarios:

1) *No updating feedback.* In this case, the source has no knowledge of whether an update is successful. It can only use the update-to-date energy arrival profile and updating decisions as well as the statistical information, such as the energy arrival rate and the erasure probability of the channel, to decide the upcoming updating time points. We show that the *Best-effort Uniform updating* (BU) policy, which was shown to be optimal under the perfect channel setting in [41], is still optimal among a broad class of online policies.

2) *Perfect updating feedback.* In this case, the source receives an instantaneous feedback when an update is transmitted. Therefore, it can decide when to update next based on the feedback information, along with the information it uses for the no feedback case. For this case, we propose a *Best-effort Uniform updating with Retransmission* (BUR) policy and prove its optimality among a broad class of online policies.

We would like to point out that for the finite battery capacity setting with update erasures, the problem is in general much more complicated. Some results for the special case with a unit battery size can be found in subsequent works [55, 56]. Generally speaking, when the battery size is finite, the optimal updating policy would depend on the battery level, and usually exhibits certain threshold structure. Such finite battery setting is fundamentally different from the infinite battery setting considered in this work, and requires a completely different approach to tackle it.

We identify the optimal updating policies for each case. Although the proposed policies are quite intuitive, their optimality is quite challenging to establish, compared with [41]. This is because both battery outage and updating erasure will affect the AoI under the proposed policies. While the impact of either of those two events can be analyzed relatively easily when isolated, it becomes extremely challenging when both of them are involved. Besides, when there exists perfect updating feedback to the source, updating erasures under the BUR will lead to subsequent retransmissions and energy consumption, thus affecting the battery outage probability in the future. Such complicated interplay between those two events makes the problem even more complicated. In order to overcome such difficulties, we propose a novel *virtual policy* based approach. Specifically, for both BU and BUR updating policies, we construct a sequence of virtual policies, which are strictly suboptimal to their original counterparts, and eventually converge to them. Leveraging the virtual policies, we are able to decouple the effects of battery outage and updating errors in the performance analysis. We show that the long-term average AoI under virtual policies converges to the corresponding lower bound, which implies the optimality of the original policy.

2.2 System Model

Consider a scenario where an energy harvesting sensor continuously monitors a system and sends time-stamped status updates to a destination. The destination keeps track of system status through received updates. We use the metric Age of Information (AoI) to measure the "freshness" of the status information available at the destination.

We assume that the energy unit is normalized so that each status update requires one unit of energy. This energy unit represents the energy cost of both measuring and transmitting a status update. Assume energy arrives at the sensor according to a Poisson process with parameter λ. Hence, energy arrivals occur at discrete time instants $t_1, t_2, \ldots$. We assume $\lambda = 1$ for ease of exposition, since we can always scale the time

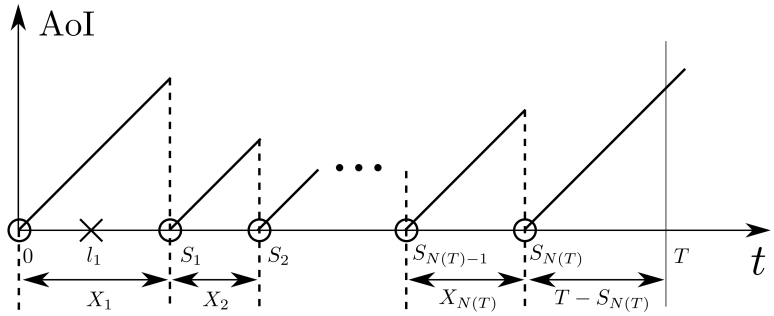


Figure 2.1: AoI as a function of t. Circles $\circ$ represent successful status updates, and crosses $\times$ represent failed status updates.

axis proportionally to make $\lambda = 1$ per unit time. The sensor is equipped with a battery to store harvested energy. In this chapter, we focus on the case where the battery size is infinite.

We assume that the time used to collect and transmit a status update is *negligible* compared with the time scale of the long-term average AoI in the system. Therefore, a status update can be generated and transmitted at any time as long as the energy level is greater than or equal to one. We assume that the channel between the source and the destination is time-invariant and noisy, thus with probability $1-p$, $0 < p \leq 1$, each update will be erased during transmission, independent of any other factors in the system. As shown in Fig. 2.1, the AoI at the destination will be reset to zero only when an update is successfully received. We consider two possible cases. For the no updating feedback case, the source has no information of the updating result. For the perfect updating feedback case, we assume there is a *perfect feedback* channel between the destination and the source, so that the source is notified about an updating failure once it happens.

A status update policy is denoted as $\pi := \{l_n\}_{n=1}^{\infty}$, where l_n is the nth update time at the *source*. However, due to random update erasures, only a subset of the update packets will be successfully delivered. Thus, the actual status update times at the *destination* are different from $\{l_n\}_{n=1}^{\infty}$ in general. Therefore, we use S_n to denote the nth actual update time at the *destination*. We assume $S_0 = l_0 = 0$, i.e., an update is successfully delivered right before time zero, and the system starts with an initial energy of E_0, $E_0 \geq 1$.

Define A_n as the total amount of energy harvested in $[l_{n-1}, l_n)$, and $E(l_n^-)$ as the energy level of the sensor right before the update time l_n. Then, under any feasible status update policy, the energy queue evolves as follows

$$E(l_1^-) = E_0 + A_1, \tag{2.1}$$

$$E(l_n^-) = E(l_{n-1}^-) - 1 + A_n, \quad n = 2, 3, \ldots. \tag{2.2}$$

Based on the Poisson arrival process assumption, A_n is an independent Poisson random variable with parameter $l_n - l_{n-1}$.

In order to ensure every update time is feasible, we must have the energy causality constraint satisfied all the time, i.e.,

$$E(l_n^-) \geq 1, \quad n = 1, 2, \ldots, \tag{2.3}$$

which indicates that the source will generate and transmit an update only when it has sufficient energy.

We use $M(T)$ and $N(T)$ to denote the number of status updates sent by the source and the number of status updates successfully received at the destination over $(0, T]$, respectively. Define $R(T)$ as the cumulative AoI at the destination over $[0, T]$. Denote the delay between two successful updates as $X_n := S_n - S_{n-1}$, for $n = 1, 2, \ldots$. Then,

$$R(T) = \frac{\sum_{i=1}^{N(T)} X_i^2 + (T - S_{N(T)})^2}{2}, \tag{2.4}$$

which corresponds to the area below the AoI curve over $[0, T]$, as shown in Fig. 2.1. The time-average AoI over the duration $[0, T]$ can then be expressed as $R(T)/T$.

Our objective is to determine the sequence of update times $l_1, l_2, \ldots$ at the *source*, so that the time average AoI at the *destination* is minimized, subject to the energy causality constraint. We focus on a set of *online* policies. Specifically, for the no updating feedback case, the information available for determining the updating point l_n includes the updating history $\{l_i\}_{i=0}^{n-1}$, the energy arrival profile over $[0, l_n)$, as well as the energy harvesting statistics (i.e., λ in this scenario) and the probability of updating success p. Denote the set of such online policies as Π_1. For the perfect updating feedback case, the source also utilizes up-to-date updating feedback to make its decisions. We denote the set of such online policies as Π_2. Then, the optimization problem can be formulated as

$$\begin{aligned} \min_{\pi \in \Pi} \quad & \limsup_{T \to +\infty} \mathbb{E}\left[\frac{R(T)}{T}\right] \\ \text{s.t.} \quad & (2.1) - (2.3), \end{aligned} \tag{2.5}$$

where Π equals Π_1 or Π_2, depending on the setting, and the expectation in the objective function is taken over all possible energy harvesting sample paths and update erasure patterns.

2.3 Status Updating Without Feedback

In this section, we will study the optimal status updating policy for the case where there is no update feedback available to the sensor. We show that the expected long-term average AoI has a lower bound for a broad class of online policies, which can be achieved by the BU updating policy.

2.3.1 A Lower Bound

Note that when battery size is infinite, no energy flow will happen, and the long-term average status updating rate is subject to the energy harvesting rate constraint. Specifically, we have the following lemma.

Lemma 1 (Lemma 1 in [37])**.** *Under any policy* $\pi \in \Pi_1$*, it must have* $\limsup_{T\to\infty} M(T)/T \leq 1$ *almost surely.*

We point out that Lemma 1 is also valid for all $\pi \in \Pi_2$, which will be discussed in Section 2.4.

Besides, we also have the following intuitive yet important observation.

Lemma 2. *For any* $\pi \in \Pi_1$ *that achieves a finite expected long-term average AoI, it must have* $\lim_{T\to\infty} M(T) = \infty$ *almost surely.*

Proof. We prove it by contradiction. Assume

$$\mathbb{P}\left[\lim_{T\to\infty} M(T) = \infty\right] < 1,$$

i.e., there exists $\epsilon > 0$ and $M_0 > 0$, such that

$$\mathbb{P}\left[\lim_{T\to\infty} M(T) < M_0\right] \geq \epsilon.$$

Define

$$p_n := (1-p)^{n-1}p, \tag{2.6}$$

i.e., the probability that l_n is the first successful update time after l_0. Then,

$$\begin{aligned}
&\limsup_{T\to\infty} \mathbb{E}\left[\frac{R(T)}{T}\right] \\
&\geq \lim_{T\to\infty} \frac{T^2}{2T} \cdot \mathbb{P}[\text{all } M(T) \text{ updates fail}, M(T) < M_0] \qquad (2.7)\\
&\geq \lim_{T\to\infty} \frac{T}{2}\left(1 - \sum_{i=1}^{M_0} p_i\right)\epsilon = \infty, \qquad (2.8)
\end{aligned}$$

which implies that the expected long-term average AoI cannot be finite. □

In order to obtain a valid lower bound, in the following, we only need to focus on the policies that achieve finite expected long-term average AoI. To facilitate the following analysis, we introduce a broad class of online policies defined as follows.

Definition 1 (Bounded Updating Policy)**.** *If under a policy $\pi \in \Pi_1$, the nth updating point at the source (i.e., l_n) satisfies $\mathbb{E}[l_n] < \infty$ for any fixed $n \in \{1, 2, \ldots\}$, π is called a bounded updating policy.*

Denote the set of bounded updating policies as Π_3. Then, $\Pi_3 \subset \Pi_1$. Intuitively, any practical status updating policy should be in Π_3, as it is undesirable to have any nth updating point (and the inter-update delay between any consecutive updating points before l_n) to become unbounded *in expectation*. We have the following lower bound for bounded updating policies.

Theorem 1 (Lower Bound for Channel without Feedback)**.** *For any policy $\pi \in \Pi_3$, the expected long-term average AoI is lower bounded by $\frac{2-p}{2p}$.*

The proof of Theorem 1 is provided in Appendix A.1. We note that a common approach to derive lower bounds on AoI is to first use a sample-path-based argument to obtain an almost sure lower bound, and then use Fatou's lemma to convert the almost sure bound to a bound in the expected AoI [57, 58]. However, both [57, 58] consider a multi-user scenario, and the derivation of the lower bounds focuses on how to balance the AoI of different users under the coupled scheduling constraints. In contrast, in this work, we mainly focus on a single-source setting, and the main challenge is the energy constraints imposed by the temporally evolving battery level, and the convoluted effects of battery outage and updating erasures on the AoI evolution. Correspondingly, the form and analysis of the lower bounds are quite different.

2.3.2 Optimal Online Status Updating

In this section, we propose online status updating policies to achieve the lower bound derived in Section 2.3.1. We will start with the BU updating policy introduced in [41]. Although we assume a noisy channel in this work, when there is no feedback available to the source, intuitively, it is still desirable for the source to update in a uniform fashion, so that the successfully received updates at the destination would be most uniformly distributed in time.

Definition 2 (BU Updating)**.** *The sensor is scheduled to update the status at* $s_n = n$, $n = 1, 2, \ldots$. *The sensor performs the task at* s_n *if* $E(s_n^-) \geq 1$*; Otherwise, the sensor keeps silent until the next scheduled status updating time point.*

Here we use s_n to denote the nth scheduled updating time point. It is in general different from the nth actual updating time l_n, since some scheduled updates may be infeasible due to battery outage.

BU updating ensures that the energy causality constraint is always satisfied. We expect that BU updating achieves the lower bound in Theorem 1. However, analyzing its AoI performance is very challenging. Although we are able to identify a renewal structure in the system status evolution under the BU updating policy (i.e., a renewal interval can begin right after the sensor successfully delivers an update and the battery state becomes $E_0 - 1$), the analysis of the expected average AoI over one renewal interval is still very complicated, mainly due to two reasons:

First, different from the perfect channel case [41], the actual update time at the destination S_n may deviate from the scheduled update time s_n due to two possible events: battery outage and update erasure. Although the average AoI can be characterized in systems where only one of such events can happen, it is hard to analyze the AoI when the effects of both events are involved.

Second, the expected length of such a renewal interval is unbounded. This is because the battery evolution under BU updating can be modeled as a Martingale process, and as we will show in the proof of Lemma 4, the expected time when it becomes empty for the first time (i.e., hitting time of zero) is infinity. Since with a non-zero probability the renewal interval contains such an interval, the expected length of each renewal interval is thus unbounded, and the corresponding expected average AoI becomes intractable.

To overcome such challenges, we will construct a sequence of *virtual* policies, and show that the expected time average AoI under those virtual policies approaches the lower bound in Theorem 1. Since such virtual policies are sub-optimal to the BU updating policy, the optimality of BU updating can thus be proved. In order to simplify the definition and analysis of the virtual policy, we assume $E_0 = 1$. The proof can be slightly modified to show that the optimality of the proposed policy is valid for any $E_0 \geq 0$.

Definition 3 (BU-ER$_{T_0}$)**.** *The sensor performs BU updating until the battery level after sending an updating becomes zero for the first time, or until time T_0^+, in which case the sensor depletes its battery; After that, when the battery level becomes higher than or equal to one after a successful update for the first time, the sensor reduces the battery level to one, and then repeats the process.*

Lemma 3. *For any $T_0 > 0$, BU-ER$_{T_0}$ updating policy is sub-optimal to the BU updating policy.*

Proof. We note that BU-ER$_{T_0}$ updating is identical to BU updating except the energy removal at time T_0 and when $E(s_n^+)$ becomes higher than one. Given the same energy harvesting sample path, the battery level under BU is always higher than that under BU-ER$_{T_0}$. Thus, BU-ER$_{T_0}$ incurs more infeasible status updates. With the same update erasure pattern, the instantaneous AoI under BU-ER$_{T_0}$ updating is always greater than or equal to that under BU updating sample path-wisely. Thus, the expected time-average AoI under BU-ER$_{T_0}$ is greater than or equal to that under BU, which proves the lemma. □

We note that the BU-ER$_{T_0}$ updating policy is a *renewal* type policy, i.e., the states of the system evolve according to a renewal process. To see this, we note that the updating process under BU-ER$_{T_0}$ works in cycles, where each cycle begins with the initial battery level to be one and the AoI to be zero, followed by i.i.d. battery and AoI evolution processes. Therefore, to analyze the expected long-term average AoI, it suffices to analyze the expected average AoI over one renewal interval. In the following, we will focus on the first renewal interval, and show that the corresponding expected average AoI converges to the lower bound in Theorem 1 as T_0 increases. As illustrated in Fig. 2.2, the renewal interval consists of two stages. The first stage starts at time zero and ends until the battery becomes empty for the first time, or until time T_0^+. We denote T_1 as the end of the first stage. We note that all scheduled status updating epochs over $(0, T_1]$ are feasible. The second stage starts at T_1 and ends when the battery level becomes higher than or equal to one after a successful update for the first time after T_1. We denote the duration of the second stage as T_2. The second stage thus ends at $T_1 + T_2$.

Lemma 4. *Under BU-ER$_{T_0}$ updating,* $\lim_{T_0 \to \infty} \mathbb{E}[T_1] = \infty$.

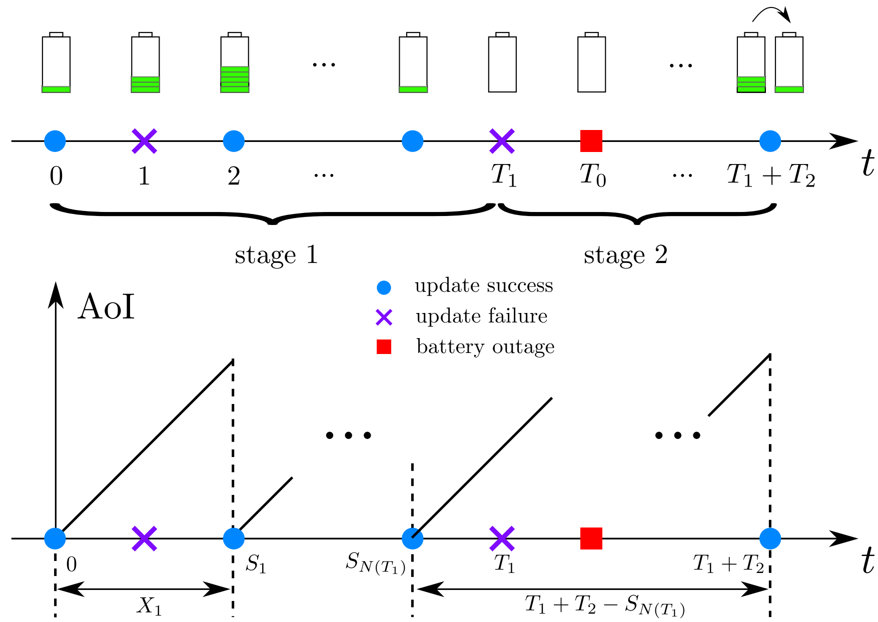


Figure 2.2: An illustration of the BU-ER$_{T_0}$ updating policy and the battery level right after each updating epoch. AoI will be reset to zero at the successul updating epochs.

Proof. Consider a "random walk" $\{\Omega_n\}_{n=0}^{\infty}$, which starts with 1 and increments with $A_n - 1$, where A_n is an i.i.d. Poisson random variable with parameter 1. Denote the first 0-hitting time for $\{\Omega_n\}_{n=0}^{\infty}$ as κ. Then $\Omega_0 = 1$ and $\Omega_\kappa = 0$. Note that when $T_0 \to \infty$, $\{\Omega_n\}_{n=0}^{\kappa}$ is identical to the battery level evolution process $\{E(s_n^+)\}_{n=0}^{\kappa}$ under the BU-ER$_{T_0}$ updating policy almost surely, and the corresponding $T_1 = \kappa$.

Define a Martingale process associated with $\{\Omega_n\}_{n=0}^{\infty}$ as $\{\exp(-\alpha\Omega_n - n\gamma(\alpha))\}_{n=0}^{\infty}$ with $\alpha > 0$ and $\gamma(\alpha) = e^{-\alpha} - (1-\alpha) > 0$. According to the proof of Theorem 1 in [37],

$$\exp(-\alpha\Omega_0) = \mathbb{E}[\exp(-\alpha\Omega_\kappa - \kappa\gamma(\alpha))]. \tag{2.9}$$

Taking the derivative of both sides of (2.9) with respect to α, we have

$$\Omega_0 \exp(-\alpha\Omega_0) = \mathbb{E}[(\Omega_\kappa + \kappa\gamma'(\alpha))\exp(-\alpha\Omega_\kappa - \kappa\gamma(\alpha))]. \tag{2.10}$$

Since $\Omega_0 = 1$ and $\Omega_\kappa = 0$, (2.10) can be reduced to

$$\exp(-\alpha) = \mathbb{E}[\kappa\gamma'(\alpha)\exp(-\kappa\gamma(\alpha))] \leq \mathbb{E}[\kappa\gamma'(\alpha)], \tag{2.11}$$

where the inequality follows from the fact that $\kappa\gamma(\alpha) \geq 0$.

Dividing both sides of (2.11) by $\gamma'(\alpha)$, we have

$$\mathbb{E}[\kappa] \geq \exp(-\alpha)/\gamma'(\alpha). \tag{2.12}$$

Note that

$$\lim_{\alpha\to 0} \gamma'(\alpha) = \lim_{\alpha\to 0}(-e^{-\alpha}+1) = 0^+. \tag{2.13}$$

Combining (2.12) and the fact that $T_1 = \kappa$ when $T_0 \to \infty$, we have

$$\lim_{T_0\to\infty} \mathbb{E}[T_1] \geq \lim_{\alpha\to 0} \exp(-\alpha)/\gamma'(\alpha) = \infty. \tag{2.14}$$

$\square$

Lemma 5. *Under BU-ER$_{T_0}$ updating,* $\mathbb{E}[T_2]$, $\mathbb{E}[T_2^2]$, $\mathbb{E}[T_1 - S_{N(T_1)}]$, $\mathbb{E}[(T_1 - S_{N(T_1)})^2]$ *are bounded.*

Proof. We consider another genie-aided virtual process starting at time T_1 as follows. The source performs BU-ER$_{T_0}$ after T_1, and keeps tracking the battery level and genie-informed update result. If a status update is erased and the battery level is above zero, the sensor depletes its battery and repeat the process. The process stops when the battery level after a successful update becomes one for the first time. Denote the duration of the second state as T_2'.

For each sample path, we can see that the battery level under the new virtual process is always less than or equal to that under BU-ER$_{T_0}$, due to the extra energy depletion after T_1 and before T_2'. Since the update erasure patterns are the same under both policies, we must have $T_2' > T_2$. We note that at each updating time point between T_1 and T_2', the battery level is above zero with probability $1-2e^{-1}$; and if the previous event happens, the update is successfully delivered with probability p. Therefore, T_2' under the new virtual policy is a geometric random variable with parameter $p(1-2e^{-1})$. Thus, its first and second moments are bounded. Therefore, $\mathbb{E}[T_2]$ and $\mathbb{E}[T_2^2]$ are bounded.

Next, we note that under the BU-ER$_{T_0}$ updating, the AoI over $[0, T_1]$ is a renewal reward process, which resets to zero at $\{S_i\}_{i=1}^{N(T_1)}$. According to Proposition 3.4.6 in [59], $\lim_{t\to\infty} \mathbb{E}[S_{N(t)} - t]$ is bounded. Therefore $\mathbb{E}[S_{N(T_1)} - T_1]$ is uniformly bounded for any T_1. Similarly, we can show that $\mathbb{E}[(S_{N(T_1)} - T_1)^2]$ is uniformly bounded. $\square$

Lemma 6. *As $T_0 \to \infty$, the expected long-term average AoI under BU-ER$_{T_0}$ is upper bounded by $\frac{2-p}{2p}$.*

Proof. First, we note that the

$$\begin{aligned}
&\lim_{T_0\to\infty} \frac{\mathbb{E}[(T_1+T_2-S_{N(T_1)})^2]}{2\mathbb{E}[T_1+T_2]}\\
&= \lim_{T_0\to\infty} \frac{\mathbb{E}[(T_1-S_{N(T_1)})^2]+\mathbb{E}[T_2^2]+2\mathbb{E}[T_1-S_{N(T_1)}]\mathbb{E}[T_2]}{2\mathbb{E}[T_1]}\\
&= 0, \qquad (2.15)
\end{aligned}$$

where the first equality follows from that the two events $T_1-S_{N(T_1)}$ and T_2 are independent, and the second equality follows from Lemma 4 and Lemma 5.

As illustrated in Fig. 2.2,

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{R(T)}{T}\right] \leq \frac{\sum_{i=1}^{N(T_1)} X_i^2 + (T_1+T_2-S_{N(T_1)})^2}{2\mathbb{E}[T_1+T_2]}.$$

Consider the channel state realization at the scheduled status updating epochs under BU (and BU-ER) updating. Let Y_i be the duration between the ith and $i-1$st epochs when the channel states are good and the corresponding update would be successful if it were sent. Then, $\{Y_i\}_{i=1}^{N(T_1)}$ is identical to $\{X_i\}_{i=1}^{N(T_1)}$. This is because there is no battery outage over $[0,T_1]$, and whether an update is successful or not only depends on the channel state. Combining with (2.15), we have

$$\begin{aligned}
\lim_{T_0\to\infty}\lim_{T\to\infty} \mathbb{E}\left[\frac{R(T)}{T}\right] &\leq \lim_{T_0\to\infty} \frac{\mathbb{E}[\sum_{i=1}^{N(T_1)} X_i^2]}{2\mathbb{E}[T_1+T_2]} && (2.16)\\
&\leq \lim_{T_0\to\infty} \frac{\mathbb{E}\left[\sum_{i=1}^{N(T_1)+1} Y_i^2\right]}{2\mathbb{E}\left[\sum_{i=1}^{N(T_1)+1} Y_i - (\sum_{i=1}^{N(T_1)+1} Y_i - T_1)\right]} && (2.17)\\
&= \lim_{T_0\to\infty} \frac{\mathbb{E}[N(T_1)+1]\mathbb{E}[Y_1^2]}{2\mathbb{E}[N(T_1)+1]\mathbb{E}[Y_1] - 2\mathbb{E}\left[\sum_{i=1}^{N(T_1)+1} Y_i - T_1\right]}, && (2.18)
\end{aligned}$$

where (2.18) follows from Wald's equality and the fact that $N(T_1)+1$ is a stopping time for $\{Y_i\}$ for any given T_1.

Since $\mathbb{E}[N(T_1)+1]\mathbb{E}[Y_1] \geq \mathbb{E}[T_1]$, according to Lemma 4,

$$\lim_{T_0\to\infty} \mathbb{E}[N(T_1)+1]\mathbb{E}[Y_1] \geq \lim_{T_0\to\infty} \mathbb{E}[T_1] = \infty. \qquad (2.19)$$

Meanwhile, we have $\mathbb{E}\left[\sum_{i=1}^{N(T_1)+1} Y_i - T_1\right]$ uniformly bounded for any T_1 based on Proposition 3.4.6 in [59]. Therefore, (2.18) is equal to $\frac{\mathbb{E}[Y_1^2]}{2\mathbb{E}[Y_1]}$, i.e., $\frac{2-p}{2p}$. □

Theorem 1, Lemma 3 and Lemma 6 imply the optimality of the BU updating, as summarized in the following theorem.

Theorem 2 (Optimality of BU Updating)**.** *Among all policies in* Π_3*, the BU updating policy is optimal when updating feedback is unavailable, i.e.,*

$$\limsup_{T\to\infty} \mathbb{E}\left[\frac{R(T)}{T}\right] = \frac{2-p}{2p}.$$

2.4 Status Updating With Perfect Feedback

In this section, we consider the case where there exists perfect updating feedback to the sensor. With perfect updating feedback, the sensor has the choice to retransmit the update immediately or wait and update later, thus leading to optimal solutions different from the no feedback case. In order to facilitate the analysis, in the following, we focus on another class of online policies, termed as uniformly bounded policies.

2.4.1 A Lower Bound

Define K_i as the number of *attempted* updates (including the last successful one) between two successful updates at time S_{i-1} and S_i under any online policy in Π_2. Then, K_i could be any integer number greater than or equal to one.

Definition 4 (Uniformly bounded policy)**.** *Under a policy* $\pi \in \Pi_2$*, if: 1) there exists a function* $g(k)$ *such that when* $K_i = k$*,* $X_i \leq g(k)$*,* $\forall i$*, and* $\mathbb{E}[g^2(K_i)] < \infty$*, and 2)* $\mathbb{E}[M(t) - M(t-\Delta)] \leq C\Delta$ *for any* $\Delta > 0, t > 0$*, then,* π *is called a uniformly bounded policy.*

Roughly speaking, the first condition ensures that the source updates frequently so that the AoI at the destination does not grow unbounded in expectation; The second condition requires that the source does not update too frequently in any period of time. Such conditions are consistent with our intuition that the optimal policies should try to maintain a constant X_i as much as possible. We note that uniformly bounded policies do not have to be renewal or Markovian in general. Denote the set of uniformly bounded policies as Π_4, then $\Pi_4 \subset \Pi_2$. We have the following lemma.

Lemma 7. *For any* $\pi \in \Pi_4$*, it must have* $\lim_{T\to\infty} \frac{\mathbb{E}\left[X^2_{N(T)+1}\right]}{T} = 0$ *and* $\lim_{T\to\infty} \frac{\mathbb{E}\left[X_{N(T)+1}\right]}{T} = 0$*.*

The proof of this lemma is adapted from the proof of Theorem 3 in [41], and provided in Appendix A.2.

Besides, we also have the following observation.

Lemma 8. *Under any policy* $\pi \in \Pi_4$*, it must have* $\lim_{T\to\infty} \frac{\mathbb{E}[N(T)]}{T} \leq p$.

Proof. First, we observe that

$$\lim_{T\to\infty} \frac{\mathbb{E}[\sum_{i=1}^{N(T)+1} K_i]}{T} \leq \lim_{T\to\infty} \frac{E_0 + \mathbb{E}[\sum_{i=1}^{N(T)+1} A_i]}{T} \tag{2.20}$$

due to the energy causality constraint. We note that $A(t) - t$ is a continuous-time martingale, where $A(t)$ is a Poisson process with parameter one. Therefore, according to the optimal stopping time theorem [59], for any stopping time τ, we have $\mathbb{E}[A(\tau) - \tau] = \mathbb{E}[A(0) - 0] = 0$, i.e., $\mathbb{E}[A(\tau)] = \mathbb{E}[\tau]$. Since $S_{N(T)+1}$ is a stopping time associated with the past energy arrivals and update erasure patterns under any $\pi \in \Pi_4$, we have $\mathbb{E}[A(S_{N(T)+1})] = \mathbb{E}[S_{N(T)+1}]$. Plugging it into (2.20), we have

$$\lim_{T\to\infty} \frac{\mathbb{E}[\sum_{i=1}^{N(T)+1} K_i]}{T} \leq \lim_{T\to\infty} \frac{\mathbb{E}[S_{N(T)+1}]}{T} \tag{2.21}$$

$$= 1 + \lim_{T\to\infty} \frac{\mathbb{E}[X_{N(T)+1}]}{T} = 1, \tag{2.22}$$

where the last equality follows from Lemma 7.

Besides, we note that under any online policy $\pi \in \Pi_4$, K_i is an i.i.d. geometric random variable with parameter p. Therefore, applying Wald's equality, we have

$$\lim_{T\to\infty} \frac{\mathbb{E}[\sum_{i=1}^{N(T)+1} K_i]}{T} = \lim_{T\to\infty} \frac{\mathbb{E}[N(T)+1]\mathbb{E}[K_i]}{T} \tag{2.23}$$

$$= \lim_{T\to\infty} \frac{\mathbb{E}[N(T)+1]}{Tp}. \tag{2.24}$$

Combining with (2.22), we have $\lim_{T\to\infty} \frac{\mathbb{E}[N(T)+1]}{T} = \lim_{T\to\infty} \frac{\mathbb{E}[N(T)]}{T} \leq p$. □

In order to obtain a lower bound on the AoI for all $\pi \in \Pi_4$, we will first drop the energy causality constraint, and focus on those online policies that satisfy Lemma 8 and are also uniformly bounded. Denote the set of such policies as Π_5. Then, we have $\Pi_4 \subset \Pi_5$. Since not all policies in Π_5 would be feasible if the energy causality constraint is imposed, the minimum expected long-term AoI achieved by policies in Π_5 serves as a *lower bound* for policies in Π_4.

Theorem 3. *Any policy $\pi \in \Pi_5$ is suboptimal to a renewal policy, i.e., a policy under which the successful updating points $\{S_i\}_{i=1}^{\infty}$ form a renewal process. Besides, under the renewal policy, X_i only depends on K_i.*

A sketch of the proof is as follows: For any given policy $\pi \in \Pi_5$, we construct a renewal policy based on all possible sample paths under π. Specifically, our approach is to first average X_i over sample paths with the same K_i, so that all factors other than K_i that may affect X_i can be averaged out. Then, we form a linear combination of X_i, and use it as the inter-update delay under the new policy. Such a policy is a renewal policy, and each renewal interval only depends on K_i. Through rigorous stochastic analysis, we prove that the constructed renewal policy always outperforms the original policy. The detailed proof of Theorem 3 is provided in Section A.3.

In the following, we will focus on renewal policies in Π_2, and identify the AoI-optimal renewal policy.

Theorem 4. *Under the optimal renewal policy in Π_5, X_i equals a constant $\frac{1}{p}$ irrespective of K_i, and the corresponding long-term average AoI equals $\frac{1}{2p}$.*

Proof: Based on proof of Theorem 3, under the optimal renewal policy, X_i can only take values from a countable set of constants $\{x_1, x_2, \ldots\}$, depending on the realization of K_i. Specifically, X_i will equal x_k if $K_i = k$. Note that K_i is a geometric random variable with parameter p irrespective of the values of x_ks. Then, to minimize the expected long-term average AoI, it suffices to solve the following optimization problem:

$$\min_{\{x_k\}} \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]} \quad \text{s.t.} \quad \frac{1}{\mathbb{E}[X_i]} \leq p, \tag{2.25}$$

where the constraint follows from Lemma 8 and the property of renewal processes.

Applying the inequality that $\mathbb{E}[X^2] \geq \mathbb{E}^2[X]$ to the objective function and utilizing the constraint $\frac{1}{\mathbb{E}[X_i]} \leq p$, we have

$$\frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]} \geq \frac{\mathbb{E}[X_i]}{2} \geq \frac{1}{2p}, \tag{2.26}$$

where the equalities can be met if $X_i = \mathbb{E}[X_i] = \frac{1}{p}$.

■

Combining Theorem 3 and Theorem 4, we obtain a lower bound for all $\pi \in \Pi_4$ as follows.

Theorem 5. (Lower Bound for Channel with Perfect Feedback) *For any policy* $\pi \in \Pi_4$*, the expected long-term average AoI is lower bounded by* $\frac{1}{2p}$*.*

2.4.2 Optimal Online Status Updating

Motivated by the uniform structure of $\{X_i\}$ under the optimal renewal policy in Theorem 4, we define the *Best-effort Uniform updating with Retransmission* (BUR) policy as follows.

Definition 5 (BUR Updating)**.** *The sensor is scheduled to update the status at* $s_n = n/p$, $n = 1, 2, \ldots$. *The sensor keeps sending updates at* s_n *until an update is successful or until it runs out of battery; Otherwise, the sensor keeps silent until the next scheduled status update time.*

In order to prove that the BUR updating policy is optimal, we will first construct a sequence of policies which are sub-optimal to the BUR updating policy, and show that the limit of those suboptimal policies achieves the lower bound in Theorem 5.

Definition 6 (BUR with Energy Removal (BUR-ER$_{T_0}$))**.** *The sensor performs BUR updating policy until the battery level after sending an update becomes zero for the first time, or until time* T_0^+*, in which case the sensor depletes its battery after a successful update at* T_0*; After that, when the battery level becomes higher than or equal to one after a successful update for the first time, the sensor reduces the battery level to one, and then repeats the process.*

Lemma 9. *The BUR-ER*$_{T_0}$ *updating policy is suboptimal to the BUR updating policy.*

Proof. We note that the BUR-ER$_{T_0}$ updating policy is identical to the BUR updating policy up to the energy removal step. Given the same energy harvesting sample path, the battery level under BUR is always higher than that under BUR-ER$_{T_0}$. Thus, BUR-ER$_{T_0}$ incurs more infeasible status updating points. With the same update erasure pattern, the instantaneous AoI under BUR-ER$_{T_0}$ is always greater than or equal to that under BUR sample path-wisely. Thus, the expected time-average AoI under BUR-ER$_{T_0}$ is greater than or equal to that under BUR. □

Note that BUR-ER$_{T_0}$ updating is a renewal policy and Fig. 2.3 is an illustration of one renewal interval. In order to analyze the expected long-term average AoI, it suffices to analyze the expected average AoI over one renewal interval. Thus, we will focus on the first renewal interval, and show that the expected average AoI converges to the lower

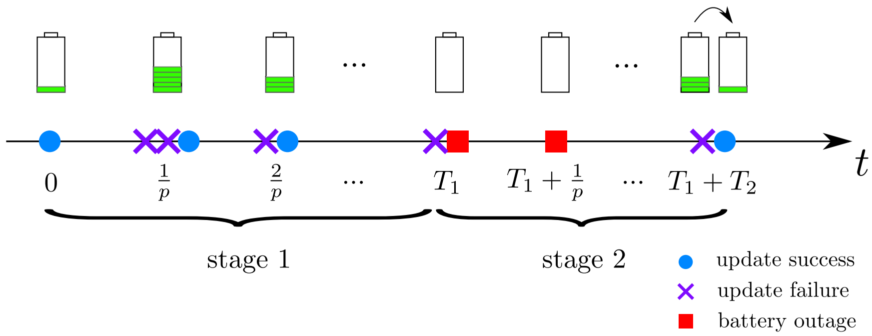


Figure 2.3: An illustration of the BUR-ER$_{T_0}$ updating policy and the battery level right after each updating epoch. AoI will be reset to zero at the successul updating epochs.

bound in Theorem 5. The renewal interval consists of two stages. The first stage starts at time zero and ends until the battery becomes empty for the first time, or until time T_0^+, denoted as T_1. We note that all scheduled updating points over $(0, T_1)$ are feasible. The second stage starts at T_1 and ends when the battery level after a successful update becomes higher than or equal to one for the first time after T_1, denoted as $T_1 + T_2$, where T_2 is the duration of the second stage.

Lemma 10. *Under BUR-ER$_{T_0}$ updating,* $\lim_{T_0 \to \infty} \mathbb{E}[T_1] = +\infty$.

Proof. Consider a "random walk" $\{\Omega_n\}_{n=0}^{\infty}$. It starts with 1 and the evolves as $\Omega_n = (\Omega_{n-1} + A_n - B_n)^+$, where A_n is an i.i.d. Poisson random variable with parameter $\frac{1}{p}$ and B_n is an i.i.d. geometric random variable with parameter p. Denote the first zero-hitting time for $\{\Omega_n\}_{n=0}^{\infty}$ as T_1. Then $\Omega_0 = 1$ and $\Omega_{T_1} = 0$. We note that when $T_0 = \infty$, $\{\Omega_n\}_{n=0}^{T_1}$ is identical to the battery level evolution process $\{E(s_n^+)\}_{n=0}^{T_1}$ under the BUR-ER$_{T_0}$ updating policy.

For ease of exposition, define $C_n := A_n - B_n$, and $\gamma(\alpha) := \log \mathbb{E}[e^{-\alpha C_n}]$ for $\alpha > 0$. Then, we have

$$\mathbb{E}[e^{-\alpha C_n - \gamma(\alpha)}] = 1. \tag{2.27}$$

Based on the definition of A_n, B_n and C_n, we have

$$\mathbb{E}[e^{-\alpha C_n}] = e^{\frac{1}{p}(e^{-\alpha}-1)} \frac{pe^{\alpha}}{1-(1-p)e^{\alpha}}. \tag{2.28}$$

Therefore,

$$\gamma(\alpha) = \log \mathbb{E}[e^{-\alpha C_n}] = \frac{1}{p}(e^{-\alpha} - 1) + \log \frac{pe^{\alpha}}{1-(1-p)e^{\alpha}}. \tag{2.29}$$

Taking derivative of (2.29), we get

$$\gamma'(\alpha) = -\frac{1}{p}e^{-\alpha} + \frac{1}{1-(1-p)e^{\alpha}}. \tag{2.30}$$

Next, we define a process associated with $\{\Omega_n\}_{n=0}^{\infty}$ as $\{e^{-\alpha\Omega_n - n\gamma(\alpha)}\}_{n=0}^{\infty}$. We note that

$$\begin{aligned}
&\mathbb{E}[e^{-\alpha\Omega_k-\gamma(\alpha)k}|\Omega_1,\ldots,\Omega_{k-1}] \\
&= \mathbb{E}[e^{-\alpha(\Omega_{k-1}+C_k)^+-\gamma(\alpha)k}|\Omega_1,\ldots,\Omega_{k-1}] \\
&\leq \mathbb{E}[e^{-\alpha(\Omega_{k-1}+C_k)-\gamma(\alpha)k}|\Omega_1,\ldots,\Omega_{k-1}] \\
&= e^{-\alpha\Omega_{k-1}-\gamma(\alpha)(k-1)}\mathbb{E}[e^{-\alpha C_k-\gamma(\alpha)}] \\
&= e^{-\alpha\Omega_{k-1}-\gamma(\alpha)(k-1)},
\end{aligned} \tag{2.31}$$

where (2.31) follows from (2.27). Therefore, $\{e^{-\alpha\Omega_n - n\gamma(\alpha)}\}_{n=0}^{\infty}$ is a super-martingale process, i.e.,

$$e^{-\alpha\Omega_0} \geq \mathbb{E}[e^{-\alpha\Omega_{T_1}-\gamma(\alpha)T_1}] \geq \mathbb{E}[1-(\alpha\Omega_{T_1}+T_1\gamma(\alpha))].$$

Since $\Omega_0 = 1$ and $\Omega_{T_1} = 0$, combining with (2.30), we have

$$\mathbb{E}[T_1] \geq \lim_{\alpha\to 0^+}\frac{1-e^{-\alpha\Omega_0}}{\gamma(\alpha)} = \lim_{\alpha\to 0^+}\frac{\Omega_0 e^{-\alpha\Omega_0}}{\gamma'(\alpha)} = \infty. \tag{2.32}$$

□

Lemma 11. *Under the BUR-ER$_{T_0}$ updating policy, $\mathbb{E}[T_2]$, $\mathbb{E}[T_2^2]$ are uniformly bounded.*

Proof. Under BUR-ER$_{T_0}$ updating policy, the number of energy arrivals over $[\frac{n}{p}, \frac{n+1}{p})$ (denoted as A_{n+1}) is a Poisson random variable with parameter $1/p$. If the source has sufficient energy, the total number of attempts at time $\frac{n+1}{p}$ (denoted as B_{n+1}) is an i.i.d. geometric random variable with parameter p. Therefore, if the battery is empty at time $\frac{n}{p}$, it will increase to one or above after a successful update at time $\frac{n+1}{p}$ only when $A_{n+1} - B_{n+1} \geq 1$, which will happen with a constant probability. Thus, pT_2 is a geometric random variable whose first and second moments are finite. □

Lemma 12. *As $T_0 \to \infty$, the expected long-term average AoI under BUR-ER$_{T_0}$ updating is upper bounded by $\frac{1}{2p}$.*

Proof. First, we note that

$$\lim_{T_0\to\infty} \frac{\mathbb{E}[(T_1+T_2-S_{N(T_1)})^2]}{2\mathbb{E}[T_1+T_2]} \leq \lim_{T_0\to\infty} \frac{\mathbb{E}[(T_2+\frac{1}{p})^2]}{2\mathbb{E}[T_1]} = 0, \tag{2.33}$$

where (2.33) follows from the fact that $T_1 - S_{N(T_1)}$ is upper bounded by $1/p$ under the BU-ER$_{T_0}$ policy, Lemma 10 and Lemma 11.

Next, we note that the BU-ER$_{T_0}$ updating policy is a renewal policy and the expected long-term average AoI is equal to the expected average AoI over one renewal interval. Therefore,

$$\begin{aligned}
&\lim_{T_0\to\infty}\lim_{T\to\infty} \mathbb{E}\left[\frac{R(T)}{T}\right] \\
&\leq \lim_{T_0\to\infty} \frac{\mathbb{E}[\sum_{i=1}^{N(T_1)} X_i^2 + (T_1+T_2-S_{N(T_1)})^2]}{2\mathbb{E}[T_1+T_2]} &(2.34)\\
&\leq \lim_{T_0\to\infty} \frac{\mathbb{E}[\sum_{i=1}^{N(T_1)} X_i^2]}{2\mathbb{E}[S_{N(T_1)}]} = \lim_{T_0\to\infty} \frac{\mathbb{E}[N(T_1)]\frac{1}{p^2}}{2\mathbb{E}[N(T_1)]\frac{1}{p}} = \frac{1}{2p}, &(2.35)
\end{aligned}$$

where (2.35) follows from (2.33) and the fact that $X_i = 1/p$ for $i \leq N(T_1)$ and $S_{N(T_1)} = N(T_1)/p$. □

Lemma 12 indicates that the expected time-average AoI under the BUR-ER$_{T_0}$ updating policy converges to the lower bound in Theorem 5 as T_0 goes to infinity. According to Lemma 9, BUR-ER$_{T_0}$ is suboptimal to BUR. Therefore, the BUR updating policy also achieves the lower bound, thus it is optimal. We summarize the optimality result in the next theorem.

Theorem 6 (Optimality of BUR Updating)**.** *Among all policies in* Π_4*, the BUR updating policy is optimal when transmission feedback is available, i.e.,*

$$\limsup_{T\to\infty} \mathbb{E}\left[\frac{R(T)}{T}\right] = \frac{1}{2p}.$$

We note that the main techniques we use to show the optimality of BU and BUR for the no-feedback and perfect feedback cases are quite similar: they both derive a lower bound and then use a series of virtual policies, which are sub-optimal to their original counterparts, to approach the lower bound. However, there exist subtle differences in the corresponding analysis, as summarized as follows: For the perfect feedback case, due to instantaneous feedback, the policy space is considerably larger than that under the no-feedback case. Therefore, the derivation of the lower bound is more sophisticated, where we first reduce the policy space to a subset of renewal policies, and then use the minimal AoI over the renewal policies to obtain the lower bound. On the other hand, due to the retransmission structure with perfect feedback, the constructed virtual policy BUR-ER is slightly simpler to analyze compared with BU-ER. As shown in Lemma 5 and Lemma 11, $\mathbb{E}[T_2]$, $\mathbb{E}[T_2^2]$ and $\mathbb{E}[T_1 - S_{N(T_1)}]$, $\mathbb{E}[(T_1 - S_{N(T_1)})^2]$ are easier to bound under BUR-ER compared with BU-ER.

2.5 Extension to Multiple-Source Systems

In previous sections, we focus on a single-source updating system, where the energy harvesting sensor monitors a single source and sends the updates to the destination. In this section, we consider a multiple-source extension of the original system, where the sensor continuously monitors M independent sources, and sends the updates to the corresponding destinations. The AoI at destination m, $1 \leq m \leq M$, will be reset to zero when an update from source m is successfully delivered. The generation and transmission of each update is subject to the energy availability of the sensor. Let $R_m(T)$ be the cumulative AoI at the m-th destination. Then, our objective is to minimize the total time-average AoI of the M destinations as follows:

$$\min_{\tilde{\pi} \in \tilde{\Pi}} \quad \limsup_{T \to +\infty} \sum_{m=1}^{M} \mathbb{E}\left[\frac{R_m(T)}{T}\right] \quad \text{s.t. } (2.1) - (2.3), \tag{2.36}$$

where a policy $\tilde{\pi} := \{(l_n, a_n)\}_{n=1}^{\infty}$ consists of the instant l_n the sensor performs updating as well as the corresponding destination a_n it updates at time l_n, and $\tilde{\Pi}$ is the corresponding set of online policies.

Following similar approaches as in Section 2.3 and Section 2.4, we can show that the objective function in (2.36) is lower bounded by $M^2 \frac{2-p}{2p}$ for the no-feedback case, and by $\frac{M^2}{2p}$ for the perfect feedback case, within the corresponding subset of bounded updating policies considered. In other words, the lower bounds are exactly M^2 times of their counterparts in the single-source system. Due to the symmetric setup, intuitively, the lower bounds would be achieved under certain symmetric updating policies.

We extend the BU and BUR updating policies for such multi-source scenario as follows: the scheduled updating instances s_n are determined in the same way as in the original policies, while the scheduled updating source at time s_n, denoted as d_n, is determined in a round-robin fashion, i.e., $d_n = (n - 1 \mod M) + 1$. The sensor would perform a scheduled updating if it has sufficient energy; Otherwise, it will keep silent until the next scheduled updating time. We term the extended policies as round-robin (RR) BU and round-robin BUR, respectively.

We note that the battery evolution and updating erasure under RR-BU and RR-BUR would be the same as that under their original counterparts. The only difference lies in the interleaving updating epochs for different destinations due to the rounb-robin updating fashion. However, with slight modification of the construction of the virtual policies, we can still enforce renewal updating structures in the system, and show that RR-BU and RR-BUR approach the corresponding lower bounds and are optimal for the no-feedback and feedback cases, respectively.

2.6 Simulation Results

In this section, we evaluate the performances for the proposed status updating policies through simulations. For each case, we generate sample paths for the Poisson energy harvesting process with $\lambda = 1$ and compute the sample average of the time average AoI over 1000 sample paths.

2.6.1 Status Updating Without Feedback

First, we evaluate the BU updating policy in Fig. 2.4. We vary $p = 0.2, 0.6, 1.0$, and plot both the time average AoI as a function of T and the corresponding lower bound in the figure. We observe that all time average AoI curves gradually approach the corresponding lower bound $\frac{2-p}{2p}$ as $T \to \infty$. The results show that the proposed BU updating policy is optimal. Note that the time average AoI is monotonically decreasing as p increases. This is intuitive since channel with better quality, i.e., larger p, will render smaller time average AoI.

Next, we evaluate the performances of virtual policies BU-ER$_{T_0}$ for different value of T_0 in Fig. 2.5. We fix $p = 0.6$ and plot the time average AoI under BU-ER$_{T_0}$ with $T_0 = 300, 600, 1800$. We also compare with a greedy updating policy and the BU updating policy. Under the greedy updating policy, the sensor updates instantly when one unit of energy arrives. As we observe in Fig. 2.5, the greedy policy results in the highest average AoI, and never approaches the lower bound. The time averaged AoI under the BU-ER$_{T_0}$ updating policy is monotonically decreasing as T_0 increases, and gradually approaches that under the BU updating policy. This is consistent with Lemma 3 and Lemma 6 that

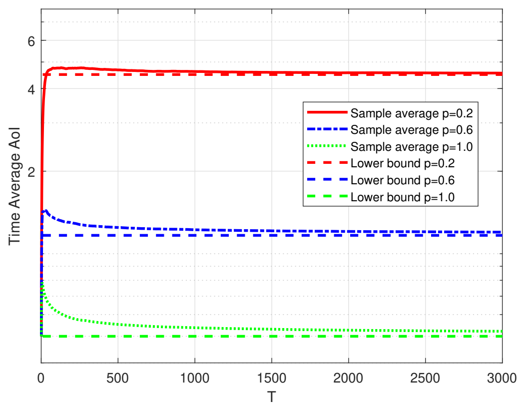


Figure 2.4: Performances of BU policy.

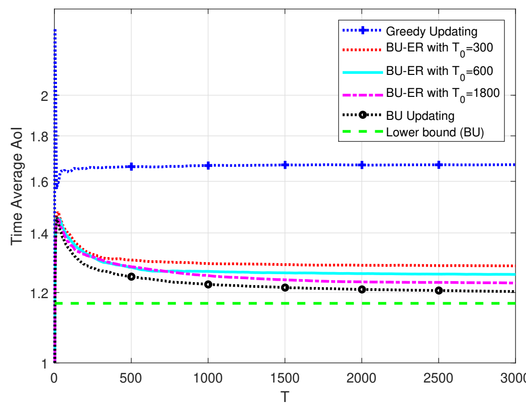


Figure 2.5: Performances of BU-ER policy.

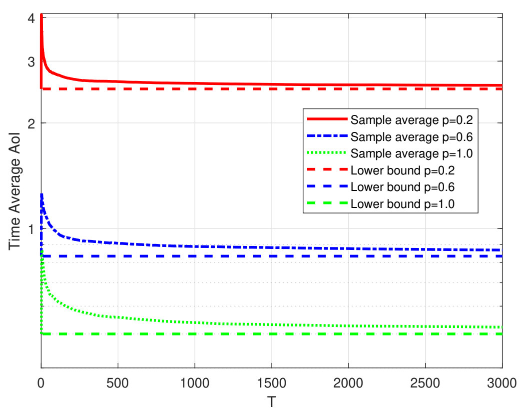


Figure 2.6: Performances of BUR policy.

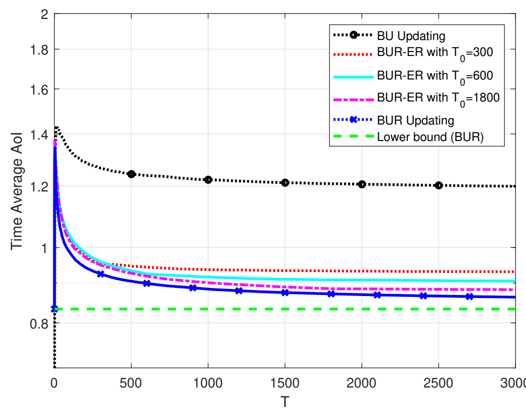


Figure 2.7: Performances of BUR-ER policy.

BU-ER$_{T_0}$ updating is sub-optimal to BU updating, and eventually converges to it when T_0 increases.

2.6.2 Status Updating With Perfect Feedback

Next, we evaluate the performances of the proposed online policies when perfect feedback is available to the sensor. In Fig. 2.6, under the BUR updating policy, we plot the time average AoI with $p = 0.2, 0.6, 1.0$ and the corresponding lower bound $\frac{1}{2p}$. We note that as $T \rightarrow \infty$, the time average AoI approaches the lower bound. Thus BUR updating is optimal. We then evaluate the performances of the BUR-ER$_{T_0}$ updating policy in Fig. 2.7. We fix $p = 0.6$, choose $T_0 = 300, 600, 1800$ and plot the time average AoI as a function of T. As a comparison, we also plot the time average AoI under the BU updating policy and the BUR updating policy in the figure. We note that the AoI under

BUR-ER$_{T_0}$ gradually decreases and approaches that under the BUR updating policy as T_0 increases, which is consistent with Lemma 9 and Lemma 12. The performance gap between the BU updating and the BUR updating indicates that exploiting updating feedback can significantly reduces time average AoI in the system.

2.6.3 Markovian Updating Erasure Process

In this subsection, we investigate the performances of the proposed policies when the i.i.d. packet erasure assumption is not satisfied. In particular, we consider a more practical scenario where the update erasure pattern evolves according to a two-state Markov chain, as illustrated in Fig. 2.8. Under this model, state 0 corresponds to an updating erasure, while state 1 corresponds to an updating success. Given the current update is erased (successful), the next update will be successful (erased) with probability q_1 (q_2). We set the parameters to satisfy $q_2 = \frac{(1-p)q_1}{p} \in [0, 1]$. We note that with the given transition probabilities, the stationary distribution of the updating states is $\pi_0 = 1 - p$, $\pi_1 = p$, which matches with the probabilities of update erasures and successes under the original i.i.d. erasure assumption. Besides, when $q_1 = p$, the Markov chain reduces to the i.i.d. Bernoulli erasure process. We aim to use this scenario to illustrate the impact of correlated updating erasures on the performances of BU and BUR. In the following experiments, we set $p = 0.5$, and vary the range of p_1 in $(0, 1)$. We consider different time horizon T, and take the sample average to get the corresponding time-average AoI.

In Fig. 2.9, we evaluate the performance of BU under the Markovian updating erasure model for the no-feedback case. We observe that the long-term average AoI monotonically decreases as q_1 increases, and when q_1 equals 0.5, it coincides with that under the i.i.d. Bernoulli erasure assumption. The monotonicity of AoI on q_1 can be explained in this way: First, we ignore possible battery outage during the updating process. We can explicitly express the stationary distribution of X_i as follows:

$$\mathbb{P}[X_i = k] = \begin{cases} 1 - q_2, & k = 1, \\ q_2(1 - q_1)^{k-1} q_1, & k \geq 2. \end{cases} \tag{2.37}$$

Thus, under the Markovian updating erasure model, the long-term average of the expected inter-update delay $\mathbb{E}[X_i] = \frac{1}{p}$, $\mathbb{E}[X_i^2] = \frac{1-p}{p}\left(\frac{2}{p_1} + \frac{1}{1-p}\right)$, which implies that the corresponding long-average AoI is monotonically increasing in p_1.

Then, we consider possible battery outage and the impact on the corresponding AoI. Based on our analysis in Section 2.3, we can see that battery outage happens

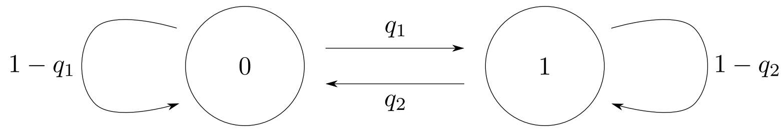


Figure 2.8: Markov chain of the Markovian packet drop.

rarely under the BU policy. Since battery evolution is independent with the underlying Markovian updating erasure process for the no-feedback base, we can conclude that the long-term average AoI would be the same as that under the no battery outage assumption. Therefore, the monotonicity in p_1 persists.

Next, we consider the perfect feedback case, and evaluate the performance of BUR updating under the Markovian updating erasure model. The long-term average AoI as a function of q_1 is plotted in Fig. 2.9. We note that within the given time horizon T, the time-average AoI approaches the lower bound $\frac{1}{2p}$ as q_1 increases. Besides, for the same q_1, when T increases, the corresponding time-average AoI is lower. This can be explained as follows: Under the BUR updating policy, the transmitter attempts to deliver a status update every $\frac{1}{p}$ unit time unless it depletes its battery. Thus, if battery outage never happens, the long-term average AoI would always be $\frac{1}{2p}$. Next, we consider the possible impact of battery outage. Different from the no-feedback case where battery outage is independent with update erasures, now, due to perfect feedback, the battery usage depends on the update erasure patterns. According to our analysis for the no-feedback case, the expected total number of attempts between two successful updates is $1/p$, while the corresponding variance is monotonically decreasing in q_1. Therefore, although we expect the long-term average AoI under the Markovian updating erasure model always converges to the lower bound as T increases, the corresponding convergence rate would depend on q_1: when q_1 increases, the converge rates is higher, due to the reduced variance of the total number of attempts between two successful updates.

2.6.4 Multiple-Source Updating Systems

Finally, we evaluate the performances of the RR-BU policy and the RR-BUR policy in multiple-source updating systems. We consider a system with $M = 3$ sources, and fix $p = 0.2, 0.6, 1.0$. We plot the summed time-average AoI as a function of T under RR-BU for the no-feedback case in Fig. 2.11 and that under RR-BUR for the perfect feedback case in Fig. 2.12. We also plot the lower bounds of the summed average AoI in the figures. We observe that the time-average AoI approaches the corresponding lower bound when

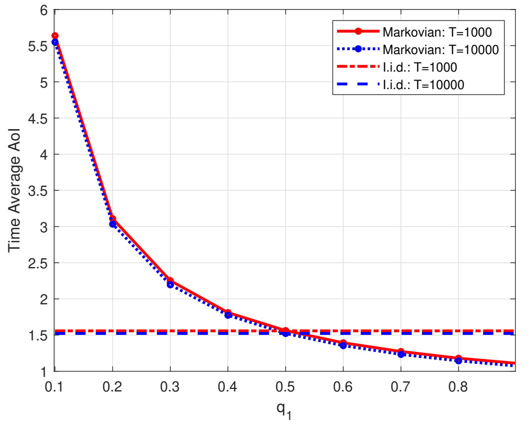


Figure 2.9: Markovian erasure without feedback.

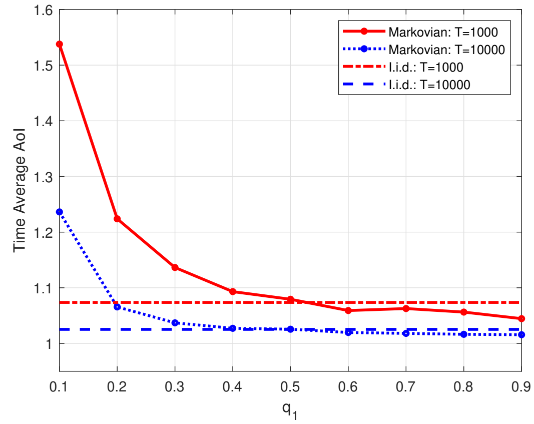


Figure 2.10: Markovian erasure with feedback.

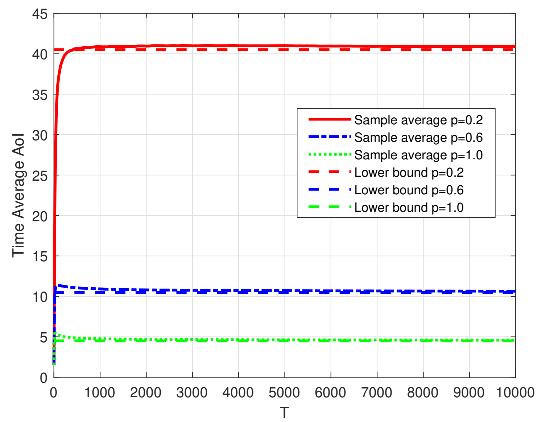


Figure 2.11: RR-BU for no-feedback scenario.

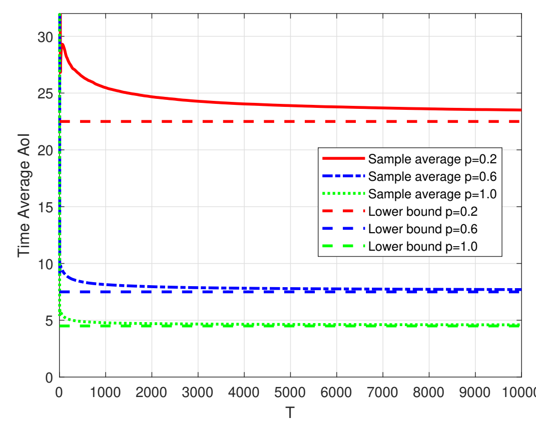


Figure 2.12: RR-BUR for perfect feedback scenario.

T is sufficiently large, which indicates that RR-BU and RR-BUR are optimal for the no-feedback and perfect feedback cases, respectively.

2.7 Conclusions

In this work, we considered the optimal online status update policies for an energy harvesting source in presence of updating erasures. We investigated both cases where no updating feedback or perfect feedback is available to the source. For each case, we first obtained a lower bound and then proved the proposed status updating policy can achieve the lower bound among a broadly defined class of policies. The optimality of proposed status update policies were proved through constructing a sequence of virtual status updating policies which are sub-optimal to the original policy and asymptotically achieve the lower bound. The performances of the proposed policies were evaluated through simulations. We point out that although we only showed the optimality of the proposed policies within a subset of online policies, we conjecture that their optimality can be extended for all online policies. How to generalize the results is one of our future steps. Another direction we would like to pursue is to investigate the impact of update erasures on the optimal updating policy for an EH source with finite battery.

Chapter 3 Precoding and Scheduling for AoI Minimization in MIMO Broadcast Channels

3.1 Introduction

One significant challenge in the timely delivery of information is the *scarce spectrum resources*. The proliferation of new use cases such as IoT, combined with the vastly crowded electromagnetic spectrum, calls for transformative information delivery schemes that are able to efficiently utilize limited spectrum resources and meet the quality of service requirements of multiple users. While

In systems where specific communication channels instead of abstract "servers" are considered, AoI optimization has also been extensively studied in [22, 60–63]. The minimum AoI scheduling problem with interfering links is studied in [60]. The AoI over multiple-access channels has been analyzed for both scheduled access with feedback and slotted ALOHA-like random access mechanisms [61]. Reference [62] investigates the minimization of the average AoI in status update systems with packet based transmissions over fading channels. The optimal achievable average AoI over an erasure channel has been studied in [22] for the cases when the source and channel-input alphabets have equal or different sizes. The optimal error toleration policy for AoI minimization during transmission of an update in an erasure channel with feedback has been investigated in [63].

This chapter investigates broadcast channels similar to those studied in [27, 57, 64–66]. Reference [57] studies the expected weighted sum AoI minimization problem of the single-hop broadcast network with minimum throughput constraints. It considers a system where the updates for users are generated periodically, and the transmission between the transmitter and each user can be erased with a constant probability. It shows that in a symmetric network, greedily updating the user with the highest instantaneous AoI is optimal. For general setups, it develops low-complexity scheduling policies with performance guarantees. In [64], it considers stochastic update arrivals while assuming no-buffer transmitter and reliable links between the transmitter and the users. It derives the Whittle's index in a closed-form and proposes a scheduling algorithm based on it. Reference [65] extends results in [57, 64] by jointly considering both unreliable links and stochastic update arrivals, and examines Whittle's index based scheduling policies. A common assumption in [57, 64, 65] is that *only one* user can be updated each time. Thus, the "broadcast" nature of wireless medium is not really exploited in those works.

Recently, a few works have taken some initial steps to explore the benefit of broadcasting on information freshness by relaxing the assumption that only one user can be updated each time [27, 66]. In [27], it considers a two-user broadcast *symbol* erasure channel with feedback, where a transmitted update can be successfully received by each of the users with certain probability. Based the instantaneous symbol delivery feedback, the transmitter is able to adaptively code the updates and improve the AoI performance of the corresponding uncoded policies. In [66], we consider a two-user broadcast *symbol* erasure channel, and propose an adaptive coding policy. We show that compared with a greedy transmission policy without coding, the AoI at the weak user can be improved by orders of magnitude without affecting that at the strong user. Both works in [27, 66] show the benefit of coding on AoI in those broadcast channels.

In this chapter, we investigate novel coding and scheduling schemes that allow simultaneously transmissions of multiple users and achieve optimal AoI. We consider a status monitoring system with K sources, each generating updates intended for one of the K recipients. The updates are transmitted to the monitors through a broadcast channel. Different from the models studied in existing works, we consider block fading over the links between the transmitter and receivers, each receiver equipped with N antennas. Therefore, all receivers are able to receive an attenuated version of the transmitted signal. Then, under the assumption that the noise level is negligible in the channel, and the instantaneous channel state information (CSI) is available to the transmitter at the beginning of each time slot, our objective is to investigate the optimal coding and transmission scheduling schemes for the minimization of the summed time-average AoI over the receivers.

3.1.1 Main Contributions

We summarize our main contributions in this chapter as follows.

First, we investigate a novel MIMO broadcast setting where optimizing AoI through precoding and transmission scheduling is the focal point. While precoding strategies for throughput optimization for such channel has been investigated extensively in the literature [67–69], maximizing information freshness is a very different aspect and requires unconventional treatment. On the other hand, existing study on AoI in broadcast channels rarely considers the impact of multiplexing gain on information freshness. The problem studied in this work bridges the gap between existing studies on MIMO broadcast channel and AoI, rendering novel precoding and transmission scheduling solutions.

Second, we explicitly identify the optimal updating strategies for the MIMO broadcast channel under different setups. Our result indicates that the size of updates plays a critical role on the design of the optimal updating schemes: When updates are of size one, the optimal schemes exhibit a round-robin structure. When updates are of size B, $B \geq 2$, round-robin updating may not be optimal. Rather, the transmitter may waste some transmission opportunities in order to deliver fresher updates. This is in contrast to conventional throughput-optimal transmission schemes in the literature. For the two-user case, we show that framed updating schemes are optimal.

Finally, the techniques we develop to show the optimality of the proposed updating schemes are novel. Due to combinatorial nature of the scheduling problem, establishing the optimality of an updating scheme is not straightforward. Toward that, we strive to obtain lower bounds that match with the summed long-term average AoI achieved under the proposed schemes. For the case when each user is equipped with one receiving antenna, we focus on consecutive time frames consisting of B time slots and identify a lower bound on the summed AoI over each frame based on a newly defined notion of Degree of Freedom (DoF). Such DoF characterizes the transmission and decoding capabilities of the system and determines the minimum possible AoI of the users. For the two-user case, we first investigate an updating scheme that always update users in an alternating fashion for the timely delivery of each update to the intended user. Such alternating updating schemes naturally partition the time axis into concatenating segments with different updating patterns. We then examine the updating patterns on those segments individually and obtain a lower bound on the corresponding AoI. Finally, we show that the lower bound on the summed long-term average AoI for the class of alternating policies remains valid for any policy. We believe those techniques are new in the study of AoI, and may be applicable for other problems in the area as well.

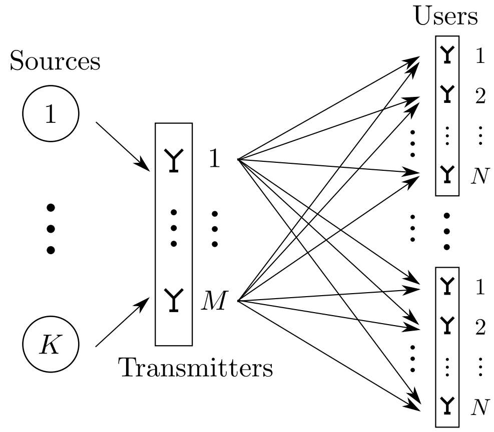


Figure 3.1: System model.

3.2 Problem Formulation

Consider a status updating system where there are K independent sources intended for K users. At the beginning of each time slot, an update of B symbols are generated at each source. The symbols are transmitted to the users through a MIMO broadcast channel, as illustrated in Fig. 3.1. We assume the transmitter is equipped with M transmitting antennas, and each receiver is equipped with N receiving antennas. Each user tracks the status of the source of interest based on the symbols it receives. We use (K, M, N, B) to denote the status monitoring system with K source-user pairs, M transmitting antennas, N antennas at each user, and update size B.

We refer the K updates generated at time slot t as $\mathbf{W}_t := (\mathbf{w}_t^{(1)}, \ldots, \mathbf{w}_t^{(K)})^\mathsf{T}$ where $\mathbf{w}_t^{(k)} := (w_t^{(k)}[1], \ldots, w_t^{(k)}[B])$ is the update intended for user k. We assume $w_t^{(k)}[b]$ is drawn from a finite field $\mathbb{F}_q$, and use $\mathbf{W}_t[:, b]$ to denote the b-th column of $\mathbf{W}_t$. The definitions of $\mathbf{W}_t$, $\mathbf{w}_t^{(k)}$ and $\mathbf{W}_t[:, b]$ are illustrated in Fig. 3.2. We assume at each time slot, the transmitter is able to transmit a symbol on each of its antennas and it takes one time slot to deliver the symbol. Let $\mathbf{x}_t := (x_t[1], \ldots, x_t[M])^\mathsf{T}$ be the symbols transmitted at time slot n. Throughout this chapter, we restrict to linear precoding schemes and assume $\mathbf{x}_t$ is a *linear* function of the previously generated symbols of updates $\{\mathbf{W}_\tau\}_{\tau=1}^t$.

Let $\mathbf{H}_t^{(k)} \in (\mathbb{F}_q)^{N \times M}$, $k \in [1 : K]$, be the channel state between the transmitter and user k, and denote $\mathbf{H}_t := ((\mathbf{H}_t^{(1)})^\mathsf{T}, \ldots, (\mathbf{H}_t^{(K)})^\mathsf{T})^\mathsf{T} \in (\mathbb{F}_q)^{KN \times M}$. The channel output at user k, denoted as $\mathbf{y}_t^{(k)} := (y_t^{(k)}[1], \ldots, y_t^{(k)}[N])^\mathsf{T}$, is modeled as

$$\mathbf{y}_t^{(k)} = \mathbf{H}_t^{(k)} \mathbf{x}_t, \tag{3.1}$$

$$\mathbf{W}_t = \begin{pmatrix} w_t^{(1)}[1] & w_t^{(1)}[2] & \cdots & w_t^{(1)}[B] \\ w_t^{(2)}[1] & w_t^{(2)}[2] & \cdots & w_t^{(2)}[B] \\ \vdots & \vdots & \ddots & \vdots \\ w_t^{(K)}[1] & w_t^{(K)}[2] & \cdots & w_t^{(K)}[B] \end{pmatrix} \begin{matrix} \mathbf{w}_t^{(1)} \\ \mathbf{w}_t^{(2)} \\ \vdots \\ \mathbf{w}_t^{(K)} \end{matrix}$$

$$\mathbf{W}_t[:,1] \quad \mathbf{W}_t[:,2] \quad \cdots \quad \mathbf{W}_t[:,B]$$

Figure 3.2: Update generated at time t. Update of user k generated at time t corresponds to the k-th row of $\mathbf{W}_t$. The b-th column of $\mathbf{W}_t$ is denoted as $\mathbf{W}_t[:,b]$.

where we assume the additive noise in the channel is negligible compared with the transmit signal and leave it out for ease of exposition.

We assume any submatrix of $\mathbf{H}_t$ is full rank almost surely, and $\mathbf{H}_t$ is available to the transmitter and the users at the beginning of each time slot. Then, the transmitter is able to design $\mathbf{x}_t$ based on the instantaneous channel state information (CSI) $\mathbf{H}_t$, the symbols in $\{\mathbf{W}_\tau\}_{\tau=1}^t$ and all previously transmitted symbols $\{\mathbf{x}_\tau\}_{\tau=1}^{t-1}$. Once $\mathbf{y}_t$ is received, each individual user k will try to recover updates from the corresponding source k based on received symbols $\{\mathbf{y}_\tau^{(k)}\}_{\tau=1}^t$ and historical CSI $\{\mathbf{H}_\tau^{(k)}\}_{\tau=1}^t$, i.e., update $\mathbf{w}_t^{(k)}$ of user k is recovered if user k is able to decode all symbols in update $\mathbf{w}_t^{(k)}$.

We adopt the metric *age of information* (AoI) to measure the freshness of the information at the users. Formally, the AoI at user k is the duration since the decoded freshest update was generated at the associated source k. Once an intended update is decoded at user k, its AoI is reset to the age of the update if it is fresher. If multiple updates from source k are decoded at the same time, the AoI is reset to the age of the freshest one. Let $\delta_t^{(k)}$ be the AoI of the k-th user at the end of time slot n. Then, the average AoI of user k is defined as

$$\Delta^{(k)} = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} \delta_t^{(k)}\right], \tag{3.2}$$

and the summed average AoI of K users is defined as $\Delta = \sum_{k=1}^K \Delta^{(k)}$. Our objective is to obtain an optimal precoding policy to determine $\{\mathbf{x}_t\}_t$, such that the summed average AoI Δ is minimized.

3.3 Main Results

Due to the combinatorial nature of the precoding and scheduling schemes in the MIMO broadcast channel, searching for the age-optimal updating policy is extremely complicated in general. In order to gain some insights to this general problem, in this chapter, we focus on two special scenario. In the first scenario, we restrict to the case when each user is equipped with one receiving antenna, while for the second scenario, we focus on systems with two users only. Our main results are summarized as follows.

Theorem 7. *For $(K, M, 1, B)$ systems, the following results hold:*

(i) If $K \leq M$, the minimum summed average AoI equals $\frac{1}{2}K(3B-1)$;

(ii) If $K = pM + q$, where $p \in \mathbb{N}$ and $q \in [0 : M-1]$, the minimum summed average AoI equals $\frac{1}{2}pM(pB + 2B - 1) + \frac{1}{2}q(2pB + 3B - 1)$.

Theorem 8. *For $(2, M, N, B)$ systems, the following results hold:*

(i) If $N \geq B$ and $\frac{M}{B} \geq 2$, the minimum summed average AoI equals 2;

(ii) If $N \geq B$ and $1 \leq \frac{M}{B} < 2$, the minimum summed average AoI equals 3;

(iii) If $N \geq M$, $0 < \frac{M}{B} < 1$, let $i = \lceil \frac{B}{M} \rceil - 1$ and $j = \lfloor \frac{1}{B/M - i} \rfloor$. Then, $\frac{j}{ij+1} \leq \frac{M}{B} < \frac{j+1}{(j+1)i+1}$, and the minimum summed average AoI equals $4i + 1 + \frac{2i+1}{ij+1}$ if $j \geq 2$, and equals $4i + 3$ if $j = 1$.

(iv) If $N \leq \frac{M}{2}$, $0 < \frac{N}{B} < 1$, let $i = \lceil \frac{B}{N} \rceil - 1$ and $j = \lfloor \frac{1}{B/N - i} \rfloor$. Then, $\frac{j}{ij+1} \leq \frac{N}{B} < \frac{j+1}{(j+1)i+1}$, and the minimum summed average AoI equals $3i + 1 + \frac{i+1}{ij+1}$.

We note that Theorem 8 explicitly characterizes the optimal AoI in all $(2, M, N, B)$ systems except for the case when $N < B$ and $N < M < 2N$. Although explicit identification of the optimal AoI for this case is extremely challenging and intractable, we are able to provide a lower bound on the summed average AoI, and obtain performance guarantee for a transmission policy as follows.

Theorem 9. *For $(2, M, N, B)$ systems, if $B = iN + j$, $i \in \mathbb{N}$, $j = B \pmod N$, and $N < M < 2N$, the minimum summed average AoI is lower bounded by $\Delta_{\mathsf{LB}} = 2i + \lceil \frac{2j}{N} \rceil$. Moreover, there exists a 2-optimal policy under which the summed average AoI is upper bounded by $2\Delta_{\mathsf{LB}}$.*

In the following, we first present updating schemes in Section 3.4 and Section 3.5 that achieve the summed time-average AoI in Theorem 7 and Theorem 8, and then provide the matching lower bounds in Section 3.6 and Section 3.7. In Section 3.8, for $(2, M, N, B)$ systems with $N < B$ and $N < M < 2N$, we investigate the lower bound and propose a 2-optimal policy. We conclude the chapter in Section 3.9 and defer some of the proofs to the Appendix.

3.4 Achievable Schemes for Theorem 7

In this section, we explicitly describe the optimal updating schemes that render the minimum summed average AoI stated in Theorem 7.

3.4.1 Achievable Scheme for the $(K, M, 1, B)$ System with $M \geq K$

First, we consider the case when $N = 1$, $M \geq K$. This corresponds to the case when each user is equipped with a single antenna, and the number of antennas at the transmitter is greater than the number of users. Since $N = 1$, each user can receive at most one linear combination of the transmitted symbols, implying that a B-symbol update takes at least B time slots to deliver. On the other hand, since $M \geq K$, the transmitter is able to send K independent symbols in each time slot. This motivates us to propose a simple synchronized updating scheme as follows:

Definition 7 (Synchronized updating)**.** *Partition the time axis into frames of length B starting at the beginning of time slot 1. Then, at the beginning of time slot $t = mB + b$, $m \in \mathbb{Z}_+$, $b \in [1 : B]$. The transmitter sends*

$$\mathbf{x}_t = \begin{pmatrix} \tilde{\mathbf{H}}_t^{-1} \mathbf{W}_{mB+1}[:, b] \\ \mathbf{0}_{(M-K)\times 1} \end{pmatrix}, \tag{3.3}$$

where $\tilde{\mathbf{H}}_t \in (\mathbb{F}_q)^{K\times K}$ is $\mathbf{H}_t$ knocked off the last $M - K$ columns.

We note that under the synchronized updating scheme, the b-th symbol of updates generated at the beginning of a time frame, i.e., $\{\mathbf{W}_{mB+1}^{(k)}\}_k$, is transmitted in the b-th time slot in the corresponding time frame simultaneously. By precoding the symbols according to (3.3), each user is able to cancel off the interference from other unintended updates and decode the designated update at the end of the time frame, i.e., at the end

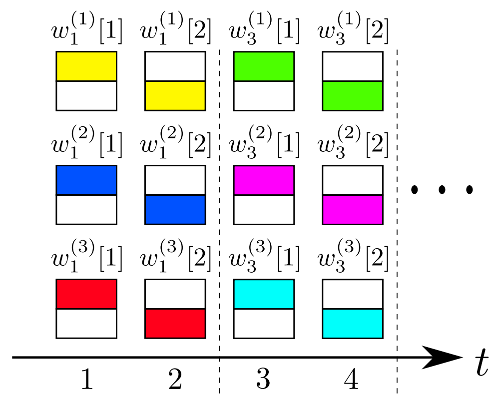


Figure 3.3: Synchronized updating scheme for the $(3, 4, 1, 2)$ system, where updates $\mathbf{W}_1, \mathbf{W}_3, \ldots$ are delivered at the end of time slots $2, 4, \ldots$.

of time slot $(m+1)B$. The synchronized updating scheme for the $(3, 4, 1, 2)$ system is shown in Fig. 3.3.

Tracking the AoI of each user $\delta_t^{(k)}$ in time frame consisting of time slots $[mB+1 : (m+1)B]$, $m \in \mathbb{N}$, we note that for general $B > 1$, it increases monotonically from $B+1$ to $2B-1$ until being reset to B at the end of the time frame. When $B = 1$, the AoI resets to 1 at the end of each time slot. Denote $\delta_{m:n}^{(k)} = \sum_{t=m}^{n} \delta_t^{(k)}$. Assume the initial AoI at time 0 is bounded for every user. Then,

$$\begin{aligned}\Delta &= \lim_{T\to\infty} \frac{K}{TB} \sum_{t=1}^{TB} \delta_t^{(k)} \\ &= \lim_{T\to\infty} \frac{K}{TB} \left(\delta_{1:B}^{(k)} + \sum_{m=1}^{T-1} \delta_{mB+1:(m+1)B}^{(k)} \right) \\ &= \frac{1}{2} K(3B-1). \end{aligned} \tag{3.4}$$

We note that the synchronized updating scheme is not the only updating scheme that achieves the AoI depicted in Theorem 7 for the $M \geq K$ case. Actually, instead of starting the transmission of new updates to all users synchronously, the transmitter can continuously update the users in an asynchronous way by introducing an offset $n_k \in [0 : B-1]$ to the time when the transmitter starts transmitting a new update to user k. Such an offset will only affect the updating time points of a user without changing the AoI evolution pattern between two updates. Thus, the long-term average AoI stays the same.

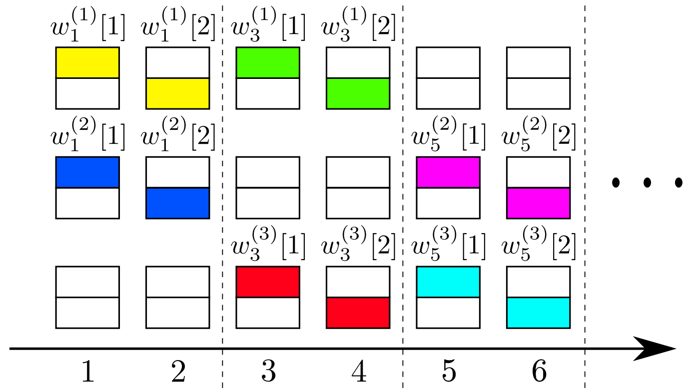


Figure 3.4: Round-robin synchronized updating for the $(3, 2, 1, 2)$ system.

3.4.2 Achievable Scheme for the $(K, M, 1, B)$ System with $M < K$

Next, we consider the case when $N = 1$, $M < K$. Compared with the scenario discussed in Sec. 3.4.1, we note that the number of transmitting antennas is now less than the number of users, which implies that not all users can be updated in a synchronized fashion. How to schedule the updating of each user to minimize the total AoI thus becomes non-trivial. We propose the following intuitive updating scheme and prove it is optimal afterwards.

Definition 8 (Round-robin synchronized updating)**.** *Partition the time axis into frames of length B starting at the beginning of time slot 1. Then, the transmitter selects M users to update in each frame in a round-robin fashion. Specifically, in the frame consisting of time slots $[mB + 1 : (m + 1)B]$, $m \in \mathbb{Z}_+$, the selected users to update are the M users $(mM + i - 1) \pmod K + 1$, $i = [1 : M]$. At the beginning of time slot $t = mB + b$, $b \in [1 : B]$, the transmitter sends $\mathbf{x}_t = \tilde{\mathbf{H}}_t^{-1}\tilde{\mathbf{W}}_{mB+1}[:, b]$, where $\tilde{\mathbf{H}}_t \in (\mathbb{F}_q)^{M\times M}$ and $\tilde{\mathbf{W}}_{mB+1} \in (\mathbb{F}_q)^{M\times 1}$ are $\mathbf{H}_t$ and $\mathbf{W}_{mB+1}$ knocked off the rows associated with the unselected $K - M$ users, respectively.*

Under the precoding and transmission scheme, the selected M users are able to decode the intended updates at the end of each time frame. The transmission strategy for the $(3, 2, 1, 2)$ system is given in Fig. 3.4 as an example.

In the following, we explicitly identify the summed time-average AoI under the round-robin updating scheme.

Lemma 13. *Let $K = pM + q$, where $q = K \pmod M$. Let k be the index of the i-th ranked user in the frame consisting of time slots $[mB + 1 : (m + 1)B]$, $m \in \mathbb{Z}_+$, i.e., $k := (mM + i - 1) \pmod K + 1$. Denote L_i as the frame index difference between the* *current frame and the next frame during which user k will be updated. Then, $L_i = p$ if $i + q \leq M$, and $L_i = p + 1$ if $i + q > M$. Besides, user k will be the $(\overline{i+q})$-th ranked user in the frame starting at $(m + L_i)B + 1$, where $\overline{x} := (x - 1) \pmod{M} + 1$.*

Proof. Under the round-robin updating policy, all the rest $K-1$ users should be updated exactly once between two consecutive updates of user k. Therefore, we must have $L_iM < i + K \leq (L_i + 1)M$. Since $K = pM + q$, the inequality becomes $L_iM < i + pM + q \leq (L_i + 1)M$. Therefore, if $i + q \leq M$, we must have $L_i = p$; Otherwise, $L_i = p+1$. Meanwhile, we note that the ranking of user k will be $(i+K-1) \pmod{M}+1$ in frame $m + L_i$, i.e., $\overline{i+q}$. □

We point out that under the round-robin synchronized updating scheme, each update takes exactly B time slots to transmit. Thus, the AoI at user k after each update is B, and it monotonically increases until the next updating time point.

Lemma 14. *Let $d := \gcd(q, M)$, $k := (mM + i - 1) \pmod{K} + 1$. Then, after the frame starting at $mB + 1$, the ranking of user k in the next M/d time frames during which user k is updated must be a permutation of $\mathcal{R}_i := \left\{\overline{i+d}, \overline{i+2d}, \ldots, \overline{i+\frac{M}{d}d}\right\}$.*

Proof. First, we note that $\overline{i+\frac{M}{d}d} = i$. Thus, for any $\ell \in \mathbb{N}$, $\overline{i+\ell d}$ must belong to $\mathcal{R}_i$.

Next, for any $\ell \in \mathbb{N}$, since $d = \gcd(q, M)$, $\frac{\ell q}{d} \in \mathbb{N}$, we must have $\overline{i+\ell q} = \overline{i+\frac{\ell q}{d}d}$. Thus, $\overline{i+\ell q}$ belong to $\mathcal{R}_i$, too.

Besides, for any $1 \leq \ell_1 < \ell_2 \leq M/d$, $\ell_1, \ell_2 \in \mathbb{N}$, we can show that $\overline{i+\ell_1 q} \neq \overline{i+\ell_2 q}$ through contradiction as follows: if $\overline{i+\ell_1 q} = \overline{i+\ell_2 q}$, we must have $(\ell_2 - \ell_1)q$ be an integer multiple of M, i.e., $(\ell_2 - \ell_1)q/d$ must be an integer multiple of M/d. Since $d = \gcd(q, M)$, it implies $\ell_2 - \ell_1$ must be an integer multiple of M/d, which contradicts with the assumption that $1 \leq \ell_1 < \ell_2 \leq M/d$.

Therefore, for $\overline{i+\ell q}$, $\ell = 1, \ldots, M/d$, they must equal M/d different values, which implies that the ranking of user k in M/d consecutive updating frames must be a permutation of $\mathcal{R}_i$. □

Remark 1. We note that for two users k_1, k_2, $k_1 \neq k_2$, if $\bar{k}_1 = \bar{k}_2$, they share the same set of rankings when updated. In total, there exist M different set of rankings $\{\mathcal{R}_i\}_{i=1}^M$.

Consider K/d consecutive frames. Since M users are updated in each frame, and the updating is performed in a round-robin fashion, each user is updated exactly M/d times. Therefore, the AoI evolution is periodic every K/d frames after the first update for each user. The long-term average AoI of any user is thus equal to the average AoI during any K/d consecutive frames after its first update.

Consider the AoI evolution of user k after its first update. We note that under the round-robin synchronized updating scheme, the ranking of user k when it is updated for the first time is $(k-1) \pmod{M} + 1$, i.e., $\overline{k}$. Consider the K/d consecutive frames starting at time $t = \frac{K}{d}B + 1$. According to Lemma 14, we have

$$\Delta^{(k)} = \frac{d}{KB} \sum_{t=\frac{K}{d}B+1}^{\frac{2K}{d}B} \delta_t^{(k)} = \frac{d}{KB} \sum_{j \in \mathcal{R}_{\overline{k}}} f(L_j), \tag{3.5}$$

where $f(L_j) := \frac{1}{2}[B + B(1+L_j) - 1]BL_j$ is the total AoI experienced by user i between two consecutive updates.

Thus,

$$\Delta = \sum_{k=1}^{K} \Delta^{(k)} = \frac{d}{KB} \sum_{k=1}^{K} \sum_{j \in \mathcal{R}_{\overline{k}}} f(L_j). \tag{3.6}$$

We note that $\{\mathcal{R}_{\bar{k}}\}_{k=1}^{K}$ actually corresponds to the rankings of the K users during any consecutive $\frac{K}{d}$ frames when they are updated. Since there are always M users selected in each frame, $\{\mathcal{R}_{\bar{k}}\}_{k=1}^{K}$ must contain M different elements from 1 to M, and each element appears exactly $\frac{K}{d}$ times. Applying this observation on Eqn. (3.6), we have

$$\Delta = \frac{d}{KB} \sum_{k=1}^{K} \sum_{j \in \mathcal{R}_{\overline{k}}} f(L_j) = \frac{d}{KB} \sum_{i=1}^{M} \frac{K}{d} f(L_i) \tag{3.7}$$

$$= \frac{1}{B} \left[(M-q) f(p) + q f(p+1) \right] \tag{3.8}$$

$$= \frac{Mp}{2}(Bp + 2B - 1) + \frac{q}{2}(2Bp + 3B - 1). \tag{3.9}$$

3.5 Achievable Schemes for Theorem 8

In this section, we investigate achievable schemes matching the minimum summed average AoI in Theorem 8. For those cases, we first focus on the $(2, M, B, B)$, $(2, M, M, B)$ and $(2, 2N, N, B)$ systems, respectively, and then show that the corresponding schemes can be applied to systems with general parameter setups.

3.5.1 Achievable Scheme for $(2, M, B, B)$ Systems with $M/B \geq 2$

Since $M/B \geq 2$, the transmitter is able to send at least $2B$ linear combinations of update symbols in each time slot. Therefore, at each time slot t, the transmitter chooses to transmit all $2B$ symbols of the newly generated updates $\mathbf{w}_t^{(1)}$ and $\mathbf{w}_t^{(2)}$. The precoding procedure is as follows: We knock off the last $M - 2B$ columns of $\mathbf{H}_t^{(1)}, \mathbf{H}_t^{(2)} \in (\mathbb{F}_q)^{B\times M}$ and let the remaining matrices be $\tilde{\mathbf{H}}_t^{(1)}, \tilde{\mathbf{H}}_t^{(2)} \in (\mathbb{F}_q)^{B\times 2B}$. Denote $\tilde{\mathbf{H}}_t := ((\tilde{\mathbf{H}}_t^{(1)})^\mathsf{T}, (\tilde{\mathbf{H}}_t^{(2)})^\mathsf{T})^\mathsf{T} \in (\mathbb{F}_q)^{2B\times 2B}$. At the beginning of time slot t, the transmitter selects

$$\mathbf{x}_t = \begin{pmatrix} \tilde{\mathbf{H}}_t^{-1} \begin{pmatrix} \mathbf{w}_t^{(1)} \\ \mathbf{w}_t^{(2)} \end{pmatrix} \\ \mathbf{0}_{(M-2B)\times 1} \end{pmatrix}. \tag{3.10}$$

Both users are able decode the intended update at the end of each time slot t, resetting the AoI to 1. Thus, the summed average AoI at the end of each time slot is 2.

3.5.2 Achievable Scheme for $(2, M, B, B)$ Systems with $1 \leq M/B < 2$

Next, we consider the scenario when $1 \leq M/B < 2$. Since $M < 2B$, the two newly generated updates can not be delivered in the same time slot simultaneously. On the other hand, since $M \geq B$, it indicates that at least one update can be delivered in each time slot. Thus, the question becomes whether the transmitter should utilize the remaining transmission capability to transmit another update partially. It turns out that a scheme that updates the two users alternately, one in each time slot, is optimal.

Specifically, at time slot t, the transmitter sends

$$\mathbf{x}_t = \begin{pmatrix} (\tilde{\mathbf{H}}_t^{(k)})^{-1}\mathbf{w}_t^{(k)} \\ \mathbf{0}_{(M-B)\times 1} \end{pmatrix}, \tag{3.11}$$

where $k = 1$ if t is odd, and $k = 2$ if t is even, and $\tilde{\mathbf{H}}_t \in (\mathbb{F}_q)^{B\times B}$ is $\mathbf{H}_t^{(k)}$ knocked off the last $M - B$ columns.

Then, at the end of time slot t, the transmitted update is decoded at the corresponding user. Since the AoI of each user resets to 1 every two time slots, the summed time-average AoI is 3.

3.5.3 Achievable Scheme for $(2, M, M, B)$ Systems with $\frac{j}{ij+1} \leq \frac{M}{B} < \frac{j+1}{(j+1)i+1}$, $i, j \in \mathbb{N}$

Consider the case when $\frac{M}{B} < 1$. We partition the range $(0, 1)$ into intervals in the form of $[\frac{j}{ij+1}, \frac{j+1}{(j+1)i+1})$ for $i = \lceil \frac{B}{M} \rceil - 1$ and $j = \lfloor \frac{1}{B/M-i} \rfloor$, and construct an achievable scheme for each possible interval that $\frac{M}{B}$ may lie in.

Definition 9 (Framed alternating updating)**.** *Partition the time axis into frames of length $ij+1$ starting at time slot 1. Then, the transmitter exhausts its transmission capability to update the two users alternatively until the end of the frame. Specifically, in the frame starting at $m(ij+1)+1$, let $t_n := m(ij+1)+ni+1$, $n \in [0 : j-1]$, i.e., the time slot during which a new update will be transmitted, and $k_n := (mj+n) \pmod 2 + 1$, i.e., the user that the new update is intended to. Then, when $t = t_n$, $n \in [1 : j-1]$,*

$$\mathbf{x}_t = \tilde{\mathbf{H}}_t^{-1} \begin{pmatrix} \mathbf{w}_{t_{n-1}}^{(k_{n-1})}[niM - (n-1)B + 1 : B] \\ \mathbf{w}_{t_n}^{(k_n)}[1 : (ni+1)M - nB] \end{pmatrix}, \tag{3.12}$$

where $\tilde{\mathbf{H}}_t \in (\mathbb{F}_q)^{M \times M}$ is the channel matrix between the M transmitting antennas and subsets of antennas at users k_{n-1} and k_n, respectively. When $t = t_n + l$, $n \in [0 : j-1]$, $l \in [1 : i-1]$, and $t = t_0$, $\mathbf{x}_t = \tilde{\mathbf{H}}_t^{-1}\mathbf{w}_{t_n}^{(k_n)}[(ni+l)M - nB + 1 : (ni+l+1)M - nB]$.

An example of the framed alternating updating policy for the $(2, 7, 7, 12)$ system is shown in Fig. 3.5.

Next, we track the AoI evolution under the framed alternating updating scheme. First, we note that under the proposed scheme, in each frame, the transmitter sends M symbols in each time slot until j updates are delivered to the two users alternately. Besides, since $\frac{nB}{M} \in [(i + \frac{1}{j+1})n, (i + \frac{1}{j})n)$, when $n \in [1 : j]$, the n-th updating time in the frame starting at time slot $(ij+1)m+1$ must be $t_n = (ij+1)m + ni + 1$. Moreover, since $nB < M(i+1)n$, there must be some transmission capability left after delivering the update in time slot t_n for $n < j$, which will be used to transmit a new one. Therefore, once the update is delivered at time $t_{n+1} = (ij+1)m + i(n+1) + 1$, the corresponding AoI is reset to $i+1$.

We track the AoI of both users under the updating scheme, and have the following observations.

1) j is even. For this case, the AoI evolution is periodic with period $ij+1$. Within each period, $\delta_t^{(1)}$ begins with $2i+2$ and resets to $i+1$ when $t = t_1, \ldots, t_{(j-2)/2}$ and monotonically increases in between; while $\delta_t^{(2)}$ begins with $i+2$ and resets to $i+1$ when

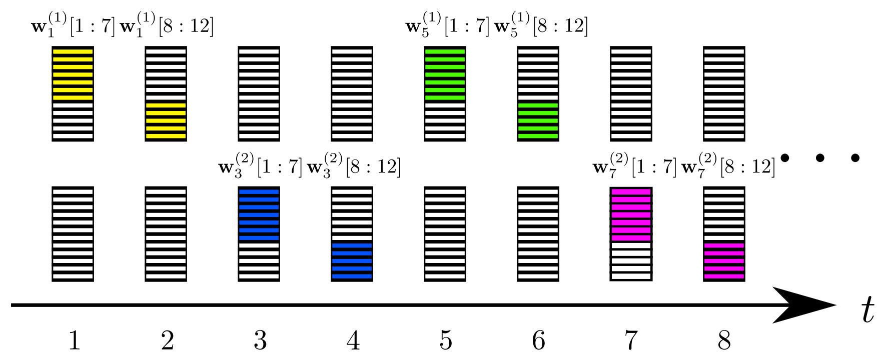


Figure 3.5: Framed alternating updating scheme for the $(2, 7, 7, 12)$ system. Since $\frac{M}{B} = \frac{7}{12} \in [\frac{1}{2}, \frac{2}{3})$, we have $i = 1, j = 1$, and the frame length is 2. Although the transmitter is able to deliver 7 linearly independent symbols in each time slot, it chooses to deliver 5 instead of 7 symbols in the last time slot of each frame.

$t = t_2, \ldots, t_{j/2}$ and monotonically increases in between. The summed long-term average AoI equals the summed AoI over any frame after the first one. Therefore,

$$\Delta^{(1)} = \frac{1}{ij+1}\left(\sum_{\ell=0}^{t_1-t_0}(2i+1+\ell) + \left(\frac{j}{2}-1\right)\sum_{\ell=1}^{2i}(i+\ell) + \sum_{\ell=1}^{i}(i+\ell)\right) \tag{3.13}$$

$$= \frac{1}{2}\left(4i+1+\frac{2i+1}{ij+1}\right). \tag{3.14}$$

Similarly, we can obtain $\Delta^{(2)}$, which equals $\Delta^{(1)}$. Combining them together, we have $\Delta = 4i + 1 + \frac{2i+1}{ij+1}$.

2) j is odd. For this case, under the framed alternating updating policy, one user will be updated $\frac{j-1}{2}$ times, while the other one will be updated $\frac{j+1}{2}$ times within each frame. Thus, the first user to update in next frame will be switched correspondingly. The AoI evolution is periodic with period $2(ij + 1)$. Following similar analysis as for the previous case, we can show that $\Delta = 4i + 1 + \frac{2i+1}{ij+1}$ if $j \geq 3$.

If $j = 1$,

$$\Delta^{(1)} = \Delta^{(2)} = \frac{1}{2(i+1)}\sum_{\ell=0}^{2i+1}(i+1+\ell) = \frac{4i+3}{2}. \tag{3.15}$$

Thus, $\Delta = 4i + 3$ if $j = 1$.

3.5.4 Achievable Scheme for $(2, 2N, N, B)$ Systems with $\frac{j}{ij+1} \leq \frac{N}{B} < \frac{j+1}{(j+1)i+1}$, $i, j \in \mathbb{N}$

Finally, we consider the case when $\frac{N}{B} < 1$ for $(2, 2N, N, B)$ Systems. For $\frac{N}{B} \in [\frac{j}{ij+1}, \frac{j+1}{(j+1)i+1})$, an achievable scheme can be constructed similar to that in Section 3.5.3 as follows.

Definition 10 (Framed synchronous updating)**.** *Partition the time axis into frames of length $ij+1$ starting at time slot 1. Then, the transmitter exhausts its transmission capability to update the two users simultaneously until the end of the frame. Specifically, in the frame starting at $m(ij+1)+1$, let $t_n := m(ij+1)+ni+1$, $n \in [0 : j-1]$, i.e., the time slot during which a new update will be transmitted. Then, when $t = t_n$, $n \in [1 : j-1]$,*

$$\mathbf{x}_t = \tilde{\mathbf{H}}_t^{-1} \begin{pmatrix} \mathbf{w}_{t_{n-1}}^{(1)}[niN-(n-1)B+1 : B] \\ \mathbf{w}_{t_{n-1}}^{(1)}[1 : (ni+1)N-nB] \\ \mathbf{w}_{t_n}^{(2)}[niN-(n-1)B+1 : B] \\ \mathbf{w}_{t_n}^{(2)}[1 : (ni+1)N-nB] \end{pmatrix}, \tag{3.16}$$

where $\tilde{\mathbf{H}}_t \in (\mathbb{F}_q)^{2N\times 2N}$ is the channel matrix between the M transmitting antennas and subsets of antennas at users k_{n-1} and k_n, respectively. When $t = t_n + l$, $n \in [0 : j-1]$, $l \in [1 : i-1]$, and $t = t_0$,

$$\mathbf{x}_t = \tilde{\mathbf{H}}_t^{-1} \begin{pmatrix} \mathbf{w}_{t_n}^{(1)}[(ni+l)N-nB+1 : (ni+l+1)N-nB] \\ \mathbf{w}_{t_n}^{(2)}[(ni+l)N-nB+1 : (ni+l+1)N-nB] \end{pmatrix}. \tag{3.17}$$

An example of the framed synchronous updating policy for the $(2, 14, 7, 12)$ system is shown in Fig. 3.6.

Different from the framed alternating updating scheme in Section 3.5.3, the AoI evolutions of two users are always the same under the framed synchronous updating scheme. Following a similar argument as in Section 3.5.3, we can show that the n-th updating time of each user in the frame starting at time slot $(ij+1)m+1$ must be $t_n = (ij+1)m+ni+1$ and the AoI of the delivered update at time $t_{n+1} = (ij+1)m+(n+1)i+1$ must be $i+1$.

Tracking the AoI of both users, the AoI evolution of each user is periodic with period $ij+1$. Within each period, both δ_t^1 and $\delta_t^{(2)}$ begins with $2i+2$ and resets to $i+1$ when $t = t_1, t_2, \ldots, t_{j-1}$ and monotonically increases in between. The summed long-term

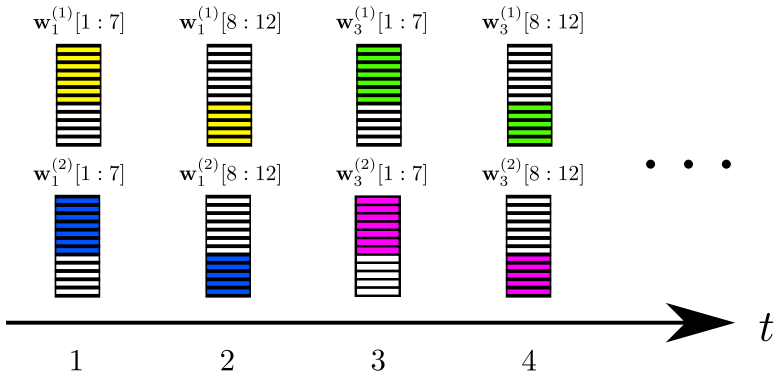


Figure 3.6: Framed synchronous updating scheme for the $(2, 14, 7, 12)$ system. Since $\frac{N}{B} = \frac{7}{12} \in [\frac{1}{2}, \frac{2}{3})$, we have $i = 1, j = 1$, and the frame length is 2. Although the transmitter is able to deliver 7 linearly independent symbols in each time slot, it chooses to deliver 5 instead of 7 symbols in the last time slot of each frame.

average AoI equals the summed AoI over any frame after the first one. Therefore,

$$\begin{aligned}
&\Delta^{(1)} = \Delta^{(2)} \\
&= \frac{1}{ij+1}\left(\sum_{\ell=0}^{t_1-t_0}(i+1+\ell) + (j-1)\sum_{\ell=1}^{i}(i+\ell)\right) \qquad (3.18)\\
&= \frac{3i+1}{2} + \frac{i+1}{2(ij+1)}. \qquad (3.19)
\end{aligned}$$

Thus, $\Delta = 3i + 1 + \frac{i+1}{ij+1}$.

3.5.5 Generalization to $(2, M, N, B)$ Systems

For $(2, M, N, B)$ systems with $N \geq B$, we note that all updating schemes described for the $(2, M, B, B)$ systems in Sections 3.5.1 and 3.5.2 are still applicable. This is equivalent to virtually removing $N - B$ antennas at each receiver, and the corresponding AoI evolution remains the same. Therefore, those updating schemes achieve the corresponding optimal summed average AoI specified in Theorem 8 (i)-(ii).

Likewise, updating schemes for $(2, M, M, B)$ systems described in Section 3.5.3 can be applied to $(2, M, N, B)$ systems with either $N \geq B > M$ or $B > N \geq M$, while the updating schemes for $(2, 2N, N, B)$ systems proposed in Section 3.5.4 can be applied to $(2, M, N, B)$ system with $M \geq 2N$. Therefore, the proposed updating schemes achieve the optimal summed average AoI specified in Theorem 8 (iii) and (iv), respectively.

3.6 Converse of Theorem 7

In this section, we prove the converse of Theorem 7, i.e., we will show that the summed long-term average AoI under any updating scheme cannot be lower than that specified in Theorem 7. Towards that, we first introduce the concept of degree of freedom (DoF) in this context, and then define a subset of schemes where the optimal scheme must lie in.

Definition 11 (Degree of Freedom (DoF))**.** *In a time slot, the degree of freedom (DoF) for a (K, M, N, B) system is the number of linearly independent equations that are delivered to users in the time slot, while the DoF allocated to a user is the number of linearly independent equations that the user receives in the time slot.*

The definition of DoF characterizes the transmission capability of the system: The total number of symbols decoded by a user cannot exceed the maximum number of linearly independent equations it can receive, as it needs n linearly independent equations to solve for the n unknown variables (symbols). For a (K, M, N, B) system, the maximum DoF for each user in any time slot is $\min\{M, N\}$, while the maximum DoF for the whole system is $\min\{M, NK\}$.

Lemma 15. *For any updating scheme, there always exists an equivalent updating scheme under which the precoding matrix is designed in such a way that the receivers' antennas receive raw symbols of the intended updates only and the DoF allocation remains the same.*

Proof. Without loss of generality, we assume the transmitter starts to update the users at time slot 1. Let n_k be the DoF allocated to user k under the original scheme at time slot 1. Then, we must have $n_k \leq N$, $\tilde{N} := \sum_{k=1}^{K} n_k \leq M$. Let $\mathbf{y}_1^{(k)} = (w_1^{(k)}[1], \ldots, w_1^{(k)}[n_k])^{\mathsf{T}}$ be the symbols received by user k at time 1 under the equivalent updating scheme. We can always design a precoding matrix in the form of

$$\tilde{\mathbf{H}}_1^{-1} := \begin{pmatrix} \tilde{H}_1^{(1)} \\ \tilde{H}_1^{(2)} \\ \vdots \\ \tilde{H}_1^{(K)} \end{pmatrix}^{-1}, \tag{3.20}$$

where $\tilde{H}_1^{(k)} \in (\mathbb{F}_q)^{n_k \times \tilde{N}}$ is a submatrix of $H_1^{(k)}$ corresponding to the CSI between the first $\tilde{N}$ transmitting antennas and the first n_k receiving antennas at user k. Under the assumption that any submatrix of $\mathbf{H}_t$ is full-rank almost surely, $\mathbf{y}_1^{(k)}$ will be delivered to user k in time slot 1. We note that the new updating scheme maintains the same DoF allocating under the original scheme. We then continue this process in time slot 2, during which raw symbols not included in $\{\mathbf{y}_1^{(k)}\}_{k=1}^K$ will be delivered. Since we always keep the DoF allocation the same under both schemes, under the newly constructed updating scheme, the intended updates will be decoded no later than that under the original scheme, rendering an equivalent or even better AoI performance. $\square$

Definition 12 (Set of efficient updating schemes Π_0)**.** *For the (K, M, N, B) system with any given initial state, denote Π_0 as a set of deterministic updating schemes that deliver raw packets to users only while satisfying the following properties:*

i) All transmitted updates will be decoded at the intended user and reset the corresponding AoI.

ii) Any delivered update $\mathbf{w}_t^{(k)}$ is transmitted starting from its generation time t.

iii) The transmitter will utilize the maximum DoF during the transmission of any update unless in the time slot when the update is delivered.

iv) Among the symbols delivered to the same user, symbols generated earlier are delivered no later than symbols generated later.

Theorem 10. *The updating scheme that achieves the minimum summed long-term average AoI lies in Π_0.*

Proof. First, we note that due to the deterministic system model, the optimal policy should be deterministic, as we can always execute the sample path that renders the minimum summed long-term average AoI under any randomized policy to outperform the original randomized policy.

Next, due to the deterministic setting, the system can foresee the AoI evolution under any deterministic updating scheme; Thus, it is unnecessary to transmit symbols that will not help to improve the AoI.

Property ii) can be shown by noticing that starting to transmit an older update instead of the newly generated one at time t leads to higher AoI when the update is delivered.

Property iii) is based on the following observation: Assume the DoF is not fully utilized during the transmission of an update $\mathbf{w}_t^{(k)}$ before it is delivered in time d, i.e., in a time slot t', $t \leq t' < d$, the DoF allocated to user k is less than N, and the total DoF allocated to all users is less than M. Then we can allocate at least one more DoF to the user k in time slot t' without affecting the DoF allocation to other users. Thus, one more symbol from $\mathbf{w}_t^{(k)}$ can be delivered in time t', potentially reducing the time used to deliver $\mathbf{w}_t^{(k)}$. Since an earlier delivery will strictly improve the AoI, the new policy performs better or at least the same as the original policy.

Property iv) can be proved through contradiction: assume under the optimal policy two updates $\mathbf{w}_{t_1}^{(k)}$ and $\mathbf{w}_{t_2}^{(k)}$, $t_1 < t_2$, are delivered to user k at time d_1 and d_2, $d_1 < d_2$, respectively. Assume $d_1 > t_2$. Since $\mathbf{w}_{t_2}^{(k)}$ must be transmitted at time t_2, at least one of its symbols is delivered to user k at time t_2. Meanwhile, since $\mathbf{w}_{t_1}^{(k)}$ is not delivered until d_1, we can always switch the transmission of one symbol from $\mathbf{w}_{t_2}^{(k)}$ with another symbol from $\mathbf{w}_{t_1}^{(k)}$ that is delivered at d_1 under the original scheme. This potentially shortens the delivery time for $\mathbf{w}_{t_1}^{(k)}$ without affecting the delivery time of $\mathbf{w}_{t_2}^{(k)}$, which improves the AoI. □

In the following, we will restrict to updating schemes in Π_0 only. Instead of considering the long-term average AoI, in the remaining of this section, we partition the time-axis into frames of length B, and investigate the minimum summed AoI in any frame. Since the summed long-term average AoI must be greater than the minimum time-average summed AoI in any frame, the latter serves as a lower bound for the former.

Lemma 16. *For the $(K, M, 1, B)$ system, under any policy $\pi \in \Pi_0$, during any consecutive B time slots, at most $\min\{K, M\}$ updates are delivered, each to a different user.*

Proof. First, we note that the maximum DoF for the $(K, M, 1, B)$ system in any time slot is $\min\{K, M\}$. Thus, the maximum number of linearly independent equations delivered in each frame is $B\min\{K, M\}$, which implies that at most $\min\{K, M\}$ updates can be decoded in any frame. Next, we note that the maximum DoF for each user is 1 since it only has one receiving antenna. Thus, at most one update can be decoded for each user in any frame. Therefore, in any time frame, at most $\min\{K, M\}$ updates are delivered, each for a different user. □

Theorem 11. *For the $(K, M, 1, B)$ system with $K \leq M$, the summed AoI in frame consisting of time slots $[mB + 1 : (m + 1)B]$, $m \in \mathbb{Z}_{\geq 2}$, is lower bounded by $\frac{1}{2}K(3B - 1)$.*

Proof. We consider the updating scheme that minimizes the summed AoI in the given time frame and ignore the AoI evolution outside the time window. The summed AoI in the frame is determined by the last update before time $mB + 1$ and the update within the frame for each user. Denote the last updating time for the K users prior to time $mB+1$ as $t_1 \leq t_2 \leq \ldots t_K \leq mB$. Then, we have the following observations.

First, if $t_1 < (m-1)B+1$, we can always construct an alternative updating scheme under which another update is delivered to the same user at time t_1+B without violating Lemma 16 and reduce its summed AoI in the frame considered. Thus, to obtain a lower bound on the summed AoI in the frame, we restrict to the scenario $t_1 \geq (m-1)B+1$.

Next, we note that the summed AoI in the frame is minimum when the reset AoI at $t_1, t_2, \ldots, t_K$ are equal to B, as each update takes at least B time slots to deliver.

Finally, we point out that the summed AoI in the frame can be minimized if the next updating happens exactly B time slots after the previous updating for each user, i.e., at time t_1+B, t_2+B, ..., t_K+B.

Calculating the cumulative AoI of each user during frame m, we have

$$\begin{aligned}\delta^{(k)}_{mB+1:(m+1)B} &\geq \sum_{\ell=(m-1)B+1}^{t_k+B-1} (\ell - t_k + B) + \sum_{\ell=t_k+B}^{(m+1)B} (\ell - t_k) \\ &= \sum_{\ell=B}^{2B-1} \ell = \frac{1}{2}B(3B-1), \end{aligned} \tag{3.21}$$

and the summed AoI in the frame is lower bounded by $\frac{1}{2}KB(3B-1)$. □

Theorem 12. *For the $(K, M, 1, B)$ system with $K = pM+q$, where $p \in \mathbb{N}$, $q \in [0 : M-1]$, the summed AoI in the frame consisting of time slots $[mB+1 : (m+1)B]$, $m \in \mathbb{Z}_{\geq p+1}$, is lower bounded by $\frac{1}{2}pM(pM+2B-1) + \frac{1}{2}q(2pB+3B-1)$.*

Proof. Similar to the $K \leq M$ case, the summed AoI in the frame is determined by the last update before time $mB+1$ and the update in the frame for each user. Denote the last updating time for the K users prior to time $mB+1$ as $t_1 \leq t_2 \leq \ldots t_K \leq mB$. Then, we have the following observations.

Since $K > M$, according to Lemma 16, at most M users can be updated in each frame. Then, to obtain a lower bound on the summed AoI in the frame, we assume $t_{K-\ell M+1}, \ldots, t_{K-(\ell-1)M}$ lie in the frame starting at $(m-\ell)B+1$, $1 \leq \ell \leq p$, and $t_1, \ldots t_q$ lie in the frame starting at $(m-p-1)B+1$. This is because if the update times are not in the corresponding frames, we can always reschedule the transmission of updates without violating Lemma 16 and reduce the corresponding AoI contribution from those updates in the frame starting at $mB+1$.

Then, following the same argument as for the $K \leq M$ case, the summed AoI in the frame starting at $mB+1$ is minimum when the reset AoI at $t_1, t_2, \ldots, t_K$ are equal to B.

Besides, to minimize the summed AoI in the frame, the transmitter should update the M users with the highest AoIs during the frame. Due to the constraints imposed by Lemma 16, the updates should happen at time $t_{K-M+1}+B$, $t_{K-M+2}+B$, ..., t_K+B.

Calculating the summed AoI of all users during the frame, we have the lower bound hold. □

3.7 Converse of Theorem 8

In the following, we let $\bar{\delta}_t$ be the vector consisting of $\{\delta_t^{(k)}\}_k$ arranged in the increasing order, and name it the *AoI pattern* at time t. We note that the summed AoI in any time slot can be determined by the AoI pattern without considering the specific AoI at each user. We name the AoI pattern that renders the minimum summed AoI in any time slot as the *minimum AoI pattern*. We note that the summed long-term AoI is lower bounded by the sum of the AoIs in the minimum AoI pattern.

For the first two cases in Theorem 8, we can obtain lower bounds as follows.

For the case when $N \geq B$ and $\frac{M}{B} \geq 2$, the AoI at each user is lower bounded by one due to the transmission delay, i.e., the minimum AoI pattern at any time slot is $(1,1)$. Therefore, the summed average AoI is lower bounded by 2.

For the case when $N \geq B$ and $1 \leq \frac{M}{B} < 2$, at most one update generated at the beginning of time slot t can be delivered. Thus, at the end of time slot t, at most one user can be updated with AoI reset as 1, while the other user is either not updated, or updated with AoI reset as 2. The minimum AoI pattern is thus $(1,2)$ and the summed AoI is lower bounded by 3. Those two lower bounds match with the AoI obtained under the updating schemes described in Section 3.5.1 and Section 3.5.2, indicating the optimality of the updating schemes.

In the following, we provide a matching lower bound for case (iii) in Theorem 8, i.e., when $N \geq M$ and $\frac{j}{ij+1} \leq \frac{M}{B} < \frac{j+1}{(j+1)i+1}$, for some $i, j \in \mathbb{N}$. The lower bound for Theorem 8 (iv) can be derived similarly and deferred to Appendix B.2.

For a $(2, M, N, B)$ system with $N \geq M$, we note that the maximum DoF of the whole system is M, while the maximum DoF allocated to individual users is also M. The transmitter needs to decide how to split its DoF between the two users in each time slot.

In the following, we first obtain a lower bound for a subset of policies named as alternating updating schemes, and then show that the lower bound applies to any policy lying in Π_0.

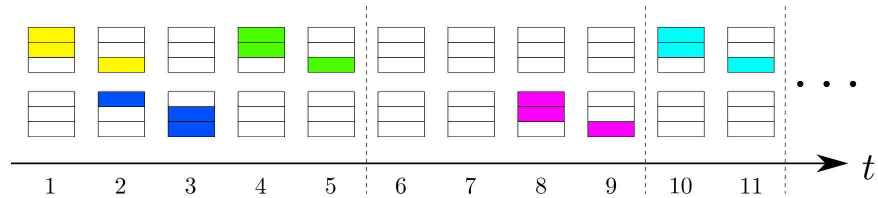


Figure 3.7: An alternating updating scheme for the $(2, 2, 3, 3)$ system, which can be equivalently represented as $(B_3, B_{2,1}, B_1, \cdots)$.

Definition 13 (Set of alternating updating schemes Π_1)**.** *Under an alternating updating scheme $\pi \in \Pi_1 \subset \Pi_0$, in each time slot, the transmitter utilizes all of its DoF on a single user unless an update is decoded. Besides, the two users are updated alternately.*

Remark 2. We note that under the alternating policy, the user to be updated next is always the user with higher AoI.

For any policy $\pi \in \Pi_1$, it can be represented as a sequence of blocks, where each block $B_{u,v}$ consists of v idling time slots followed by $l_u := \lceil uB/M \rceil$ time slots, during which the transmitter exhausts its DoF to send u updates to the two users alternately. When $v = 0$, we simply express $B_{u,0}$ as B_u. An updating scheme for the $(2, 2, 3, 3)$ system is shown in Fig. 3.7, which can be represented by $(B_3, B_{2,1}, B_1, \cdots)$ as illustrated.

Lemma 17. *For the $(2, M, N, B)$ system with $N \geq M$ and $\frac{j}{ij+1} \leq \frac{M}{B} < \frac{j+1}{(j+1)i+1}$, $i, j \in \mathbb{N}$, the minimum AoI pattern in any time slot is $(i + 1, 2i + \lceil \frac{2}{j} \rceil)$.*

Proof. In order to decode an update, the transmitter needs to deliver at least B linearly independent equations to the user. Due to the DoF constraint, it requires at least $\lceil B/M \rceil$ time slots. Since $i + \frac{1}{j} \geq \frac{B}{M} > i + \frac{1}{j+1}$, $\lceil \frac{B}{M} \rceil = i + 1$ for any $i, j \in \mathbb{N}$. Thus, the minimum AoI for any user in any time slot is $i + 1$.

In order to update both users, it requires to deliver at least $2B$ linearly independent equations, which needs $2i + \lceil \frac{2}{j} \rceil$ time slots. Thus, the minimum AoI pattern is $(i + 1, 2i + \lceil \frac{2}{j} \rceil)$. □

Remark 3. According to Lemma 17, we can see that if $j = 1$, the minimum AoI pattern is $(i + 1, 2i + 2)$; if $j \geq 2$, the minimum AoI pattern becomes $(i + 1, 2i + 1)$.

In the following, we will first study a work-conserving updating scheme $\pi_1 \in \Pi_1$, under which the transmitter exhausts its DoF at each time slot and update the two users continuously. By establishing the relationship between policy π_1 and block B_u, we will identify a lower bound on the summed average AoI over a block B_u, based on which we are able to obtain a lower bound on the summed average AoI for any policy in Π_1.

Without loss of generality, under policy π_1, we assume the initial AoI pattern at time 0 is the minimum AoI pattern $(i+1, 2i+\lceil \frac{2}{j} \rceil)$.

Lemma 18. *For the $(2, M, N, B)$ system with $N \geq M$ and $\frac{j}{ij+1} \leq \frac{M}{B} < \frac{j+1}{(j+1)i+1}$, $i, j \in \mathbb{N}$, under policy π_1, the duration between two consecutive delivered updates is either i or $i+1$ time slots.*

Proof. Assume under policy π_1, the m-th update is delivered at time slot t_m. We will show that the $(m+1)$-th update is delivered either at time $t_m + i$ or at time $t_m + i + 1$.

1) If the m-th update takes up all DoFs in time slot t_m, then, the $(m+1)$-th update is generated at time slot $t_m + 1$, which must be delivered at time $t_m + i + 1$ as it takes exactly $i+1$ time slots to deliver.

2) If the m-th update takes D_m DoFs of time slot t_m where $D_m \in [1 : M-1]$, then, the $(m+1)$-th update will be generated at time t_m under policy π_1 and take up the remaining $M - D_m$ DoFs. It will be delivered at time $t_m + \lceil \frac{B-M+D_m}{M} \rceil$. Since $\lceil \frac{B}{M} \rceil = i+1$, we have

$$\left\lceil \frac{B-M+D_m}{M} \right\rceil \geq \left\lceil \frac{B+1}{M} \right\rceil - 1 \geq \left\lceil \frac{B}{M} \right\rceil - 1 = i, \tag{3.22}$$

$$\left\lceil \frac{B-M+D_m}{M} \right\rceil \leq \left\lceil \frac{B-1}{M} \right\rceil \leq \left\lceil \frac{B}{M} \right\rceil = i+1. \tag{3.23}$$

Hence, it will be delivered either at time $t_m + i$ or at time $t_m + i + 1$. □

Label the delivered updates starting at time 1 in the order of their delivery time. Let U_m be the index of the m-th update whose delivery time is $i+1$ time slots after the previous delivered update. Since the first update generated at time slot 1 is delivered at the end of time slot $i+1$, we have $U_1 = 1$. We can see that the delivery times of the following $j-1$ delivered updates are exactly i time slots after the previous delivery time while the $(j+1)$-th updating time is $i+1$ time slots after the previous update, hence $U_2 = j+1$. In general, for $m \in \mathbb{N}$, the duration between the delivery times of updates U_m and $U_m - 1$ equals $i+1$ and the durations between any other two consecutive updates are i. Thus, the delivery time for $U_m - 1$ is at the end of time slot $(U_m - 1)i + m - 1$.

By the DoF constraint, we have

$$[i(U_m - 1) + m - 1]M \geq (U_m - 1)B, \tag{3.24}$$

i.e., the maximum DoF over $[1 : (U_m - 1)i + m - 1]$ must be greater than the DoF required to deliver $U_m - 1$ updates.

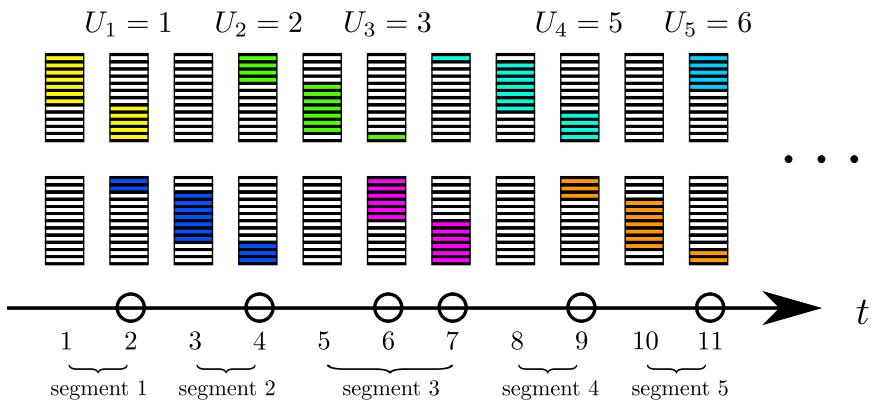


Figure 3.8: Updating pattern in the $(2, 7, 12, 12)$ system under π_1. We have $i = 1$, $j = 1$. Circles represent delivery times of updates. Since $i = 1$, the segments can be obtained by tracking the updates whose delivery time is 2 time slots after the previous delivery time. We note that the length of each segment is either 2 (i.e., $ij + 1$) or 3 (i.e., $(j + 1)i + 1$).

Similarly, update U_m is delivered at time $U_m i + m$. Thus,

$$[iU_m + m - 1]M < U_m B, \tag{3.25}$$

i.e., the maximum DoF over $[1 : U_m i + m - 1]$ must be less than the DoF required to deliver U_m updates.

Eqn. (3.24) and Eqn. (3.25) imply that

$$\frac{(m-1)M/B}{1 - iM/B} < U_m \leq \frac{(m-1)M/B}{1 - iM/B} + 1. \tag{3.26}$$

Since $\frac{j}{ij+1} \leq \frac{M}{B} < \frac{j+1}{(j+1)i+1}$, we have

$$j-1 < \frac{M/B}{1-iM/B} - 1 < U_{m+1} - U_m < \frac{M/B}{1-iM/B} + 1 < j+2. \tag{3.27}$$

Under the constraint that U_m and U_{m+1} are integers, we must have

$$U_{m+1} - U_m = j \quad \text{or} \quad j + 1. \tag{3.28}$$

We now partition the time axis into segments by the delivery time of updates $\{U_m - 1\}_{m=2}^{\infty}$, i.e., $[1 : ij + 1]$, ..., $[(U_m - 1)i + m : (U_{m+1} - 1)i + m]$, ... According to Eqn. (3.28), the segment length is either $ij + 1$ or $(j + 1)i + 1$. An example of the definition of U_m and the segments for the $(2, 7, 12, 12)$ system is illustrated in Fig. 3.8.

Table 3.1: Minimum AoI pattern for segments of length $ij+1$, $j \geq 2$, $\ell \in [2:j-1]$. An update is delivered at the end of the time slots in the last column.

Time slot	$(U_m-1)i+m$	$\cdots$	U_mi+m-1	U_mi+m
Minimum AoI pattern	$(i+2,2i+2)$	$\cdots$	$(2i+1,3i+1)$	$(i+1,2i+2)$
Time slot	U_mi+m+1	$\cdots$	$(U_m+1)i+m-1$	$(U_m+1)i+m$
Minimum AoI pattern	$(i+2,2i+3)$	$\cdots$	$(2i,3i+1)$	$(i+1,2i+1)$
Time slot	$(U_m+\ell-1)i+m+1$	$\cdots$	$(U_m+\ell)i+m-1$	$(U_m+\ell)i+m$
Minimum AoI pattern	$(i+2,2i+2)$	$\cdots$	$(2i,3i)$	$(i+1,2i+1)$

Table 3.2: Minimum AoI pattern for segments of length $(j+1)i+1$, $j \geq 2$, $\ell \in [3:j]$. An update is delivered at the end of the time slots in the last column.

Time slot	$(U_m-1)i+m$	$\cdots$	U_mi+m-1	U_mi+m
Minimum AoI pattern	$(i+2,2i+2)$	$\cdots$	$(2i+1,3i+1)$	$(i+2,2i+2)$
Time slot	U_mi+m+1	$\cdots$	$(U_m+1)i+m-1$	$(U_m+1)i+m$
Minimum AoI pattern	$(i+3,2i+3)$	$\cdots$	$(2i+1,3i+1)$	$(i+1,2i+2)$
Time slot	$(U_m+1)i+m+1$	$\cdots$	$(U_m+2)i+m-1$	$(U_m+2)i+m$
Minimum AoI pattern	$(i+2,2i+3)$	$\cdots$	$(2i,3i+1)$	$(i+1,2i+1)$
Time slot	$(U_m+\ell-1)i+m+1$	$\cdots$	$(U_m+\ell)i+m-1$	$(U_m+\ell)i+m$
Minimum AoI pattern	$(i+2,2i+2)$	$\cdots$	$(2i,3i)$	$(i+1,2i+1)$

Lemma 19. *For the $(2,M,N,B)$ system with $N \geq B$ and $\frac{j}{ij+1} \leq \frac{M}{B} < \frac{j+1}{(j+1)i+1}$, $i,j \in \mathbb{N}$, under policy π_1, the summed average AoI over segment $[(U_m-1)i+m : (U_{m+1}-1)i+m]$ is lower bounded by $\Delta_{\min} = 4i+1+\frac{2i+1}{ij+1}$ if $j \in \mathbb{Z}_{\geq 2}$, and by $\Delta_{\min} = 4i+3$ if $j=1$.*

Proof. 1) $j \geq 2$. We start with the case when the segment length is $ij+1$, i.e., $U_{m+1}-U_m = j$. According to Remark 3, the minimum AoI pattern at the end of time slot $(U_m-1)i+m-1$ is $(i+1,2i+1)$. Hence, the minimum AoI pattern at the first time slot of the segment starting at $(U_m-1)i+m$ is $(i+2,2i+2)$. We note that under π_1, update U_m is delivered at time U_mi+m, with minimum age $i+1$. This would happen if U_m is generated at time $(U_m-1)i+m$. After that, updates $U_m+1,\ldots,U_m+j-1$ are delivered sequentially after i time slots since the previous delivery. Thus, the minimum age of those updates when delivered is i. Due to the alternating updating structure, the user with higher AoI will always be updated next under π_1. Thus, the minimum AoI pattern over the segment can thus be specified (cf. Table 3.1), and the corresponding minimum summed average AoI over the duration is $4i+1+\frac{2i+1}{ij+1}$.

Next, we consider the case when $U_{m+1}-U_m = j+1$ and the corresponding segment length is $(j+1)i+1$. We will show that the minimum AoI pattern when U_m is delivered, i.e., at the end of time slot U_mi+m, is $(i+2,2i+2)$ instead of $(i+1,2i+2)$, i.e., update

U_m must be generated at time slot $(U_m - 1)i + m - 1$ instead of $(U_m - 1)i + m$. We prove it by contradiction.

Assume update U_m is generated at time slot $(U_m - 1)i + m$. Since update $U_m - 1$ is delivered at time $(U_m - 1)i + m - 1$, update U_m would consume all DoF at time slot $(U_m - 1)i + m$ under policy π_1. Thus, under policy π_1, the DoF allocation for updates $U_m, \ldots, U_{m+1} - 1$ would be the same as that for updates $U_1, \ldots, U_2 - 1$. Therefore, the length of segment $[(U_m - 1)i + m : (U_{m+1} - 1)i + m]$ would be identical to that of segment $[1, ij + 1]$, i.e., $ij + 1$. This contradicts with the assumption that the segment is of length $(j + 1)i + 1$, which indicates that update U_m must be generated at time slot $(U_m - 1)i + m - 1$, and reset the AoI of the corresponding user to $i + 2$ instead of $i + 1$ when delivered.

With the minimum AoI pattern at the end of time slot $U_m i + m$ by $(i + 2, 2i + 2)$, the minimum AoI pattern can be identified (cf. Table 3.2). The minimum summed average AoI over the segment can thus be calculated, which is equal to $4i + 1 + \frac{4i+1}{(j+1)i+1}$.

Combining those two cases, we can see that the summed average AoI over any segment is lower bounded by $4i + 1 + \frac{2i+1}{ij+1}$.

2) $j = 1$. For this case, the segment length is either $i + 1$ or $2i + 1$. According to Remark 3, the minimum AoI pattern is $(i + 1, 2i + 2)$. If the segment length is $i + 1$, there is only one update at the end of the segment, which resets the AoI to $i + 1$. The corresponding summed average AoI over the segment can be calculated, which is equal to $4i + 3$.

When the segment length is equal to $2i + 1$, two updates are delivered over the segment, one is at time i and the other is at time $2i + 1$. With the minimum AoI pattern $(i + 1, 2i + 2)$, we can show that the summed average AoI is still lower bounded by $4i + 3$. □

Remark 4. We note that for all $i, j \in \mathbb{N}$, the minimum summed average AoI over the first $\ell i + 1$ time slots in each segment $[(U_m - 1)i + m : (U_{m+1} - 1)i + m]$ is monotonically decreasing in ℓ for $\ell \in [1 : U_{m+1} - U_m]$.

Next, we relate the AoI pattern under π_1 with block B_u under any alternative updating policy in Π_1. Recall that l_u is the number of time slots required to deliver u updates in a block B_u. Then, the updating scheme π_1 over $[1, l_u]$ is identical to a block B_u except that some DoF at time slot l_u under B_u may not be exhausted.

We note that B_u can be partitioned into segments $[1 : ij+1]$, ..., $[(U_{m_u-1}-1)i+m_u-1 : (U_{m_u} - 1)i + m_u - 1]$ and a residue $[(U_{m_u} - 1)i + m_u : l_u]$, where $m_u = \max\{m : U_m < l_u\}$. According to Remark 4, the summed average AoI of the residue is lower bound by $\Delta_{\min}$.

Lemma 20. *If $x, y, z, w, t \in \mathbb{R}_{>0}$ satisfy inequalities $\frac{x}{y} \geq t$ and $\frac{z}{w} \geq t$, then $\frac{x+z}{y+w} \geq t$.*

Since the summed average AoI over each segment is lower bounded by the quantity in Lemma 19, then, based on Lemma 20, the summed average AoI over any block B_u is lower bounded by the quantity as well.

Next, we will show that the lower bound for block B_u is also a valid lower bound for blocks $B_{u,v}$, $\forall v > 0$.

Lemma 21. *For the $(2, M, N, B)$ system with $N \geq M$ and $\frac{j}{ij+1} \leq \frac{M}{B} < \frac{j+1}{(j+1)i+1}$, $i, j \in \mathbb{N}$, the summed average AoI over $B_{u,v}$ is lower bounded by $\Delta_{\min} = 4i + 1 + \frac{2i+1}{ij+1}$ if $j \in \mathbb{Z}_{\geq 2}$, and by $\Delta_{\min} = 4i + 3$ if $j = 1$.*

Proof. Recall that in a block $B_{u,v}$, there are v idle time slots before B_u. Let $\delta_0^{(1)}$, $\delta_0^{(2)}$ be the AoI at time zero. Without loss of generality, we assume $B_{u,v}$ starts at time slot 1. Since there is no updating over the first v time slots, the AoI of user k, $k = 1, 2$, will monotonically increase until the first successful update at time $v + t_k$. Thus, the existence of idling time slots affects the AoI evolution until the first update for each user occurs. Let $v + l_u$ be the end of block $B_{u,v}$.

Let $A_1^{(k)}$ be the AoI increment induced by the idling time slots at user k, as illustrated by the shaded area in Fig. 3.9. Meanwhile, denote $A_2^{(k)}$ as the summed AoI over B_u when no idling time slot is present, corresponding to the unshaded area in Fig. 3.9.

Then, we have

$$A_1^{(k)} = \sum_{\ell=1}^{v} (\delta_0^{(k)} + \ell) + v \cdot t_k, \quad k = 1, 2. \tag{3.29}$$

Let $\Delta_{B_{u,v}}$ be the summed average AoI over $B_{u,v}$. Then, $\Delta_{B_{u,v}} = \frac{\sum_{k=1}^{2} (A_1^{(k)} + A_2^{(k)})}{v + l_u}$. We note that

$$\begin{aligned} \frac{\sum_{k=1}^{2} A_1^{(k)}}{v} &= \delta_0^{(1)} + \delta_0^{(2)} + (1 + v) + t_1 + t_2 \\ &\geq 2\left(3i + 1 + \left\lceil \frac{2}{j} \right\rceil\right) + (1 + v), \end{aligned} \tag{3.30}$$

where the last inequality follows from Lemma 17.

Note that $6i + 2\lceil \frac{2}{j} \rceil + v + 1 > 4i + 1 + \frac{2i+1}{ij+1}$ if $j > 2$ and $6i + 2\lceil \frac{2}{j} \rceil + v + 1 > 4i + 3$ if $j = 1$. By applying Lemma 20, we have the lower bounds $\Delta_{\min}$ hold for $B_{u,v}$ as well. □

Since every policy in Π_1 can be decomposed as a sequence of blocks in the form of $B_{u,v}$, the lower bound on each block applies to the long-term average. Thus, $\Delta_{\min}$ is a

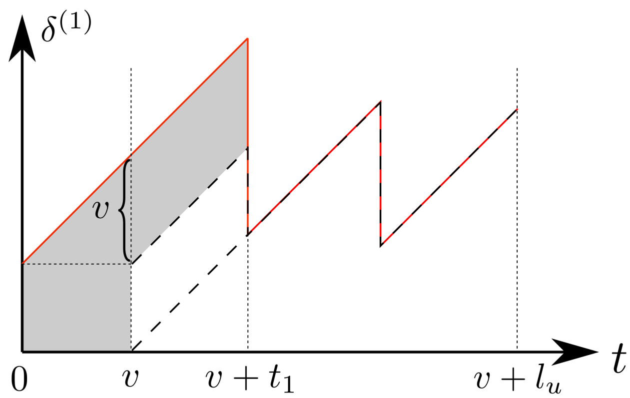


Figure 3.9: AoI evolution of user 1 over an extended block. The dashed line is the AoI evolution if idling period does not exist.

lower bound on the summed average AoI for all policies in Π_1. To prove that the lower bound applies to all policies, it suffices to show that no other policy in Π_0 can achieve AoI lower than $\Delta_{\min}$.

Theorem 13. *For the* $(2, M, N, B)$ *system with* $N \geq M$ *and* $M/B \in [\frac{j}{ij+1}, \frac{j+1}{(j+1)i+1})$*,* $i, j \in \mathbb{N}$*, the summed average AoI under any policy* $\pi \in \Pi_0$ *is lower bounded by* $\Delta_{\min} = 4i + 1 + \frac{2i+1}{ij+1}$ *if* $j \in \mathbb{Z}_{\geq 2}$*, and by* $\Delta_{\min} = 4i + 3$ *if* $j = 1$*.*

The proof of Theorem 13 is provided in Appendix B.1.

3.8 Proof of Theorem 9

For the two-user system with $N < B$ and $N < M < 2N$, it is extremely challenging to identify the exact minimum average AoI due to the combinatorial nature of the problem. In this section, we strive to obtain a lower bound on the summed average AoI and propose a 2-optimal policy.

3.8.1 Lower Bound

We provide a lower bound on the summed average AoI, as summarized in the following lemma.

Lemma 22. *For the* $(2, M, N, B)$ *system with* $N < M < 2N$*,* $B = iN + j$*,* $i \in \mathbb{N}$*,* $j \in [0 : N-1]$*, the summed average AoI is lower bounded by* $\Delta_{\mathsf{LB}} := 2i + \lceil \frac{2j}{N} \rceil$*.*

Proof. Denote d_1 and d_2 are the AoI at user 1 and user 2 at an time slot k, respectively. Without loss of generality, assume $d_1 \leq d_2$. Let $k - d_1$ and $k - d_2$ as the generation times of the freshest update received at user 1 and user 2, respectively, and u_1 and u_2 as the corresponding updates. Define x_ℓ and y_ℓ as the DoFs allocated for the transmission of updates u_1 and u_2 at time slot $k - \ell$, respectively. Then, we must have the following conditions satisfied:

$$\sum_{\ell=1}^{d_1} x_\ell = B, \quad \sum_{\ell=1}^{d_2} y_\ell = B, \tag{3.31}$$

$$0 \leq x_\ell + y_\ell \leq M, \quad \ell = 1, \ldots, d_2, \tag{3.32}$$

$$0 \leq x_\ell, y_\ell \leq N, \quad \ell = 1, \ldots, d_2. \tag{3.33}$$

Thus,

$$2B = \sum_{\ell=1}^{d_1} (x_\ell + y_\ell) + \sum_{\ell=d_1+1}^{d_2} y_\ell \leq d_1 M + (d_2 - d_1) N. \tag{3.34}$$

Since $B = iN + j$ and $M < 2N$, the above inequality becomes

$$d_1 + d_2 \geq 2i + \frac{2j}{N} + \frac{d_1(2N - M)}{N} > 2i + \frac{2j}{N}. \tag{3.35}$$

Besides, since $d_1 + d_2$ is an integer, we must have $d_1 + d_2 \geq 2i + \lceil \frac{2j}{N} \rceil$, which provides a valid lower bound on the summed average AoI at any time slot k. □

3.8.2 Framed Alternating Updating

We propose a framed alternating updating scheme and provide its performance guarantee subsequently.

Definition 14 (Framed alternating updating)**.** *Partition the time axis into frames of length* $2i$ *if* $j = 0$*,* $2i + 1$ *if* $0 < 2j < M$*, or* $2i + 2$ *otherwise.*

1) If $j = 0$*, or* $2j > M$*, within each frame, the transmitter first utilizes its transmission capability to update the user with higher AoI until its AoI resets. Then, at the beginning of the next time slot, the transmitter starts to transmit a new update to the other user until the end of the frame.*

2) If $0 < 2j < M$*, in each frame, the transmitter first utilizes its transmission capability to update the user with higher AoI until its AoI resets at the* i*-th time slot.*

Within the same time slot, the transmitter exhausts the remaining transmission capability to start transmitting an new update for the other user until the end of the frame.

Remark 5. For the $(2, M, N, B)$ system with $N < B$ and $N < M < 2N$, in each time slot, it is important to decide whether or not to exploit the remaining $M - N$ DoFs when N DoFs have been used to update one user. Exploiting the remaining DoFs may lead to earlier updating of the other user, while wasting them may shorten the transmission time of an update and reduces its age when delivered. When M gets close to N, the benefit of utilizing the $M - N$ remaining DoFs is offset by the elongated age of the update. As a result, we expect that the framed alternating updating scheme performs close to optimal when M approaches N.

In order to characterize the AoI performance, we track the AoI evolution under the framed alternating updating scheme. Note that when $j = 0$, each user is updated every $2i$ time slots, and when it is updated, its AoI is reset as i and starts increasing until next update. The summed average AoI thus equals $\Delta = 4i - 1$. When $2j > M$, similar analysis shows that the summed average AoI $\Delta = 4i + 3$. When $0 < 2j < M$, each user is updated every $2i + 1$ time slots, and when it is updated, its AoI is reset as $i + 1$. Therefore, the summed average AoI is $\Delta = 4i + 1$.

Note that the lower bound in Theorem 9 becomes $\Delta_{\mathsf{LB}} = 2i$ if $j = 0$, $\Delta_{\mathsf{LB}} = 2i + 1$ if $0 < 2j < M$, and $\Delta_{\mathsf{LB}} = 2i + 2$. Combining the summed average AoI of the framed alternating updating scheme and the lower bound, we have

$$\frac{\Delta}{\Delta_{\mathsf{LB}}} \leq \max\left\{\frac{4i-1}{2i}, \frac{4i+1}{2i+1}, \frac{4i+3}{2i+2}\right\} < 2, \tag{3.36}$$

i.e., in $(2, M, N, B)$ system with $N < B$ and $N < M < 2N$, the summed average AoI under the framed alternating updating scheme is at most twice the minimum summed average AoI and the proposed policy is 2-optimal.

3.9 Conclusions and discussions

In this chapter, we investigated the AoI optimization problem in MIMO broadcast channels with various numbers of users, transmitting and receiving antennas and update sizes. Due to the combinatorial nature of the problem and the complex AoI evolution in a dynamic system, identifying the optimal updating scheme becomes challenging.

We considered two specific scenarios, where in the first scenario, each receiver has one antenna, and in the second scenario, it only has two users. We developed different updating schemes for those cases and showed their optimality through rigorous analysis. Although the optimal schemes seem intuitive, establishing their optimality is non-trivial. Toward that, we developed some novel approaches. We think those approaches will be useful for the AoI-optimal updating schemes in noise-free MIMO broadcast channels with other parameters. Besides, we expect that those techniques can be extended to handle more practical noisy channels by leveraging the deterministic channel models proposed in [70]. Due to the coupled dynamics of AoI evolution, the general AoI-tradeoff among multiple users are intractable. However, we expect that the approaches developed in this chapter can be adopted to identify certain Pareto optimal points on the AoI of multiple users. We leave this as one of our future steps.

Chapter 4
Timely Updates With Rateless Codes Over A Symbol Erasure Channel

4.1 Introduction

The difference between information freshness metrics, such as AoI, and traditional network performance metrics, such as throughput, is manifested in a setting where the average transmission time of an update is long compared with the average inter-arrival time of the updates. This may lead to a scenario where new updates arrive at the transmitter during the transmission of an "old" update. While preempting the transmission of the old update and starting to transmit the new one degrades the throughput performance, it may actually improve the information freshness at the destination. Whether the source should preempt the current transmission in general requires delicate analysis.

AoI of coded status updating for erasure channels have been studied recently [20, 23, 27, 66]. The long-term average AoI under two different coding strategies, i.e., rateless codes and maximum distance separable (MDS) codes, are characterized for single and multiple monitor systems in [20]. It is shown that MDS coding can match the AoI performance of rateless codes if the redundancy is carefully optimized in response to the channel erasure rate. In [23], it considers an energy harvesting erasure channel and shows that rateless coding with save-and-transmit scheme outperforms MDS based schemes. For two-user broadcast *symbol* erasure channel with feedback, it is shown in [27] that adaptively code the updates is beneficial for AoI performance compared with uncoded policies, and in [66], an adaptive coding scheme is proposed under which the AoI at the weak user is enhanced by orders of magnitude without affecting that at the strong user.

In this chapter, we focus on AoI optimization for an erasure channel with rateless codes. While the AoI performance for rateless codes has been analyzed in [20, 23], in this work, our objective is to design online transmission scheduling policies for the rateless codes so that the long-term average AoI is minimized. Specifically, during the transmission of an update, the source has the choice to preempt the current transmission and switch to a new update, or to continue transmitting the current update until it is successfully decoded, based on the instantaneous feedback from the destination.

4.1.1 Main Contributions

The main contribution of this chapter is three-fold.

First, we investigate the AoI minimization problem over a symbol erasure channel. Assume each update contains k symbols. Then, the update is decoded at the destination if and only if at least k relevant symbols are successfully delivered. While the age performance under the IIR and FR regimes are studied in [20], the problem studied in this work focus on the optimal scheduling policy that actively decide whether or not to discard current update and transmit a new one.

Second, we explicitly characterize the structure of the age-optimal policy. Our result indicates that the optimal policy exhibits multiple thresholds structure. Then, we consider a limited feedback scenario and propose a type of threshold policies. The explicit expression of the time-average AoI under the threshold policies are derived and a performance guarantee for a special case is also provided.

Finally, towards the structure of the optimal policy, we leverage the powerful tool Markov Decision Process (MDP). The key step is to reduce the average cost problem as the α-discounted problem and determine the structural properties through value iteration. For the more practical limited feedback scenario, the time-average analysis is based on renewal theory. Moreover, numerical results suggests that the special type of threshold policy performs surprisingly well compared to the optimal policy.

4.2 System Model and Problem Formulation

We consider a status monitoring system where a source and a monitor is connected by a symbol erasure channel. The source continuously transmits updates to the monitor, where each update consists of k information symbols. To combat symbol erasures, we assume each update is encoded by a rateless code such that when k coded symbols are correctly received by the monitor, the update is successfully decoded. We assume the source transmits one coded symbol in each time slot, which can be successfully delivered with probability p. We assume instantaneous transmission feedback is available during the transmission of an update. We begin with an ideal scenario where feedback is available at the end of each time slot, indicating whether the symbol transmitted in the time slot has been erased or not. Upon receiving the feedback, the source has to decide whether to start transmitting a new update, or to continue transmitting the previous one if it has not been successfully decoded yet.

Denote the decision of the source at the beginning of time slot t as $a_t \in \{0, 1\}$, where $a_t = 1$ represents transmitting a new update and $a_t = 0$ represents transmitting the unfinished update (if there is one) or being idle. As illustrated in Fig.4.1, whenever $a_t = 1$, the source starts sending a new update, whose age will continuously grow (as indicated by the red dotted curve), and the AoI at the destination will grow simultaneously as well (as indicated by the blue curve). Once the update is successfully decoded, the AoI at the destination will be reset to the age of the delivered update. Due to the random erasures happening during the transmission, each update may take different time to get delivered.

Define $R(T)$ as the total age of information experienced by the system over $[0, T]$. We focus on a set of *online* policies Π, in which the information available for determining a_t includes the decision history $\{a_i\}_{i=1}^{t-1}$, the up-to-date feedback information, as well as the system parameters (i.e., k, p in this scenario). The optimization problem can be formulated as

$$\min_{\theta \in \Pi} \quad \limsup_{T \to \infty} \mathbb{E}\left[\frac{R(T)}{T}\right] \tag{4.1}$$

where the expectation in the objective function is taken over all possible erasure patterns.

4.3 Perfect Feedback Scenario

In this section, we consider the perfect feedback case where the source is able to receive the instantaneous feedback after each transmission in each time slot. Thus, the source can track the number of delivered symbols as well as the AoI at the destination.

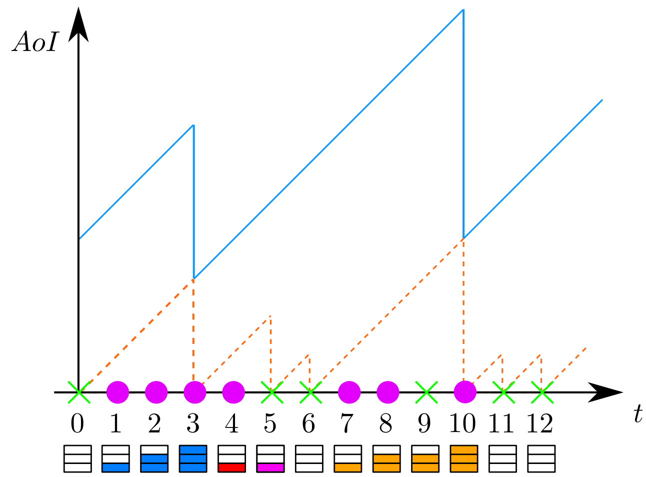


Figure 4.1: AoI evolution when $k = 3$. Circles and crosses represent successful transmissions and erasures, respectively. The stacked rectangles represent the number of delivered symbols.

4.3.1 MDP Formulation

The optimization problem in (4.1) is very challenging to solve in general, due to the random erasures and the temporal dependency in the AoI evolution. In order to make the problem analytically tractable, in the following, we will focus on stationary Markovian policies where the decision a_t only depends on the state tuple $\mathbf{s}_t := (\delta_t, d_t, l_t)$. Here δ_t and d_t are the AoI at the destination and the age of unfinished update at the beginning of time slot t, respectively. If $d_t = 0$, it indicates that an update has just been decoded. l_t is the number of successfully delivered symbols of the unfinished update at the beginning of time slot t. When $d_t = 0$, we have $l_t = 0$ as well. We note that $l_t \in \{0, 1, \ldots, k-1\}$.

Assume at time slot t, the state $\mathbf{s}$ has form $\mathbf{s}_t = (\delta, d, l)$. Then, under decision $a_t = 1$, the system will transit from state $\mathbf{s}_t$ to $\mathbf{s}_{t+1}$ according to following equation:

$$\mathbf{s}_{t+1} = \begin{cases} (\delta+1, 1, 1), & \text{with prob. } p, \\ (\delta+1, 1, 0), & \text{with prob. } 1-p. \end{cases} \tag{4.2}$$

If $a_t = 0$ and $l < k-1$,

$$\mathbf{s}_{t+1} = \begin{cases} (\delta+1, d+1, l+1), & \text{with prob. } p, \\ (\delta+1, d+1, l), & \text{with prob. } 1-p, \end{cases} \tag{4.3}$$

and if $a_t = 0$ and $l = k - 1$,

$$\mathbf{s}_{t+1} = \begin{cases} (d+1, 0, 0), & \text{with prob. } p, \\ (\delta+1, d+1, k-1), & \text{with prob. } 1-p. \end{cases} \tag{4.4}$$

Let $C(\mathbf{s}_t)$ be the instantaneous AoI at the beginning of time slot t under state $\mathbf{s}_t$. Then, $C(\mathbf{s}_t) = \delta_t$.

Let Π' be the set of stationary Markovian policies where a_t only depends on $\mathbf{s}_t$. Then, the long-term average AoI under scheduling policy $\theta \in \Pi'$ is given by

$$V(\theta) = \limsup_{T\to\infty} \frac{1}{T+1}\mathbb{E}_\theta \left[\sum_{t=0}^{T} C(\mathbf{s}_t)|\mathbf{s}_0\right], \tag{4.5}$$

where $\mathbb{E}_\theta$ represents the expectation when scheduling policy θ is employed.

In order to obtain the optimal solution to (4.5), we first define the corresponding discounted cost problem with a discount factor $\alpha \in (0,1)$, and obtain the dynamic programming formulation

$$V_{n+1}^\alpha(\mathbf{s}) = \min_{a\in\{0,1\}} C(\mathbf{s}) + \alpha\mathbb{E}[V_n^\alpha(\mathbf{s}')|\mathbf{s}, a]. \tag{4.6}$$

The expectation in (4.6) is taken over all possible state $\mathbf{s}'$ transiting from state $\mathbf{s}$ with action a, and $V_0^\alpha(\mathbf{s}) = 0$ for all $\mathbf{s}$. As $n \to \infty$, $V_n^\alpha(\mathbf{s}) \to V^\alpha(\mathbf{s})$, which is the unique solution of the following optimal equation

$$V^\alpha(\mathbf{s}) = \min_{a\in\{0,1\}} C(\mathbf{s}) + \alpha\mathbb{E}[V^\alpha(\mathbf{s}')|\mathbf{s}, a]. \tag{4.7}$$

This is a three-dimensional MDP, which is difficult to solve in general. In the following, we first determine some structural properties of the optimal policy for the α-discounted problem. We can then prove that the structure still exists when $\alpha \to 1$, i.e., for the average cost problem.

4.3.2 Structure of the Optimal Policy

Before we proceed, we introduce the following state-action value functions

$$Q_n^\alpha(\mathbf{s}; a) := C(\mathbf{s}) + \alpha\mathbb{E}[V_n^\alpha(\mathbf{s}')|\mathbf{s}, a],$$
$$Q^\alpha(\mathbf{s}; a) := C(\mathbf{s}) + \alpha\mathbb{E}[V^\alpha(\mathbf{s}')|\mathbf{s}, a].$$

Then the optimality conditions in (4.6) and (4.7) are equivalent to

$$V_{n+1}^{\alpha}(\mathbf{s}) = \min_{a\in\{0,1\}} Q_n^{\alpha}(\mathbf{s}; a), \tag{4.8}$$

$$V^{\alpha}(\mathbf{s}) = \min_{a\in\{0,1\}} Q^{\alpha}(\mathbf{s}; a). \tag{4.9}$$

We note that for any valid state $\mathbf{s} := (\delta, d, l)$, we must have $\delta \geq d + k$, $d \geq l$ and $l < k$. Define the set of valid states as $\mathcal{S}$.

Lemma 23. *For any* $\mathbf{s} \in \mathcal{S}$, $V_n^{\alpha}(\delta, d, l)$ *is monotonically increasing in* δ *at every iteration* n. *Therefore,* $V^{\alpha}(\delta, d, l)$ *monotonically increases in* δ.

Proof. We prove it by induction. It is obviously true when $n = 0$. We assume it is true for an $n \geq 0$. Then, we will prove it holds for the $(n+1)$-th iteration as well.

i) If $l < k - 1$, we have

$$Q_n^{\alpha}(\delta, d, l; 0) = \delta + p\alpha V_n^{\alpha}(\delta + 1, d + 1, l + 1) + (1 - p)\alpha V_n^{\alpha}(\delta + 1, d + 1, l), \tag{4.10}$$

$$Q_n^{\alpha}(\delta, d, l; 1) = \delta + p\alpha V_n^{\alpha}(\delta + 1, 1, 1) + (1 - p)\alpha V_n^{\alpha}(\delta + 1, 1, 0). \tag{4.11}$$

Based on the assumption, $Q_n^{\alpha}(\delta, d, l; 0)$ and $Q_n^{\alpha}(\delta, d, l; 1)$ are both non-decreasing in δ. Thus, after taking the minimum of them, $V_{n+1}^{\alpha}(\delta, d, l)$ is increasing in d as well.

ii) If $l = k - 1$, we have

$$Q_n^{\alpha}(\delta, d, k - 1; 0) = \delta + p\alpha V_n^{\alpha}(d + 1, 0, 0) + (1 - p)\alpha V_n^{\alpha}(\delta + 1, d + 1, k - 1), \tag{4.12}$$

$$Q_n^{\alpha}(\delta, d, k - 1; 1) = \delta + p\alpha V_n^{\alpha}(d + 1, 1, 1) + (1 - p)\alpha V_n^{\alpha}(d + 1, 1, 0). \tag{4.13}$$

By the same reasoning, we conclude $V_{n+1}^{\alpha}(\delta, d, k - 1)$ is increasing in δ.

The proof is completed after we combine both cases. □

Lemma 24. *For any* $\mathbf{s} \in \mathcal{S}$, $V_n^{\alpha}(\delta, d, l)$ *is monotonically increasing in* d *at every iteration* n. *Therefore,* $V^{\alpha}(\delta, d, l)$ *monotonically increases in* d.

Proof. The proof of Lemma 24 is similar to the proof of Lemma 23 and is provided here for the completeness of this chapter.

We prove it by induction. Clearly, it is true when $n = 0$. Assume it is true for an $n \geq 0$. We will prove it holds for the $(n+1)$-th iteration.

i) If $d = 0$, we must have $l = 0$. Observe that

$$V_{n+1}^{\alpha}(\delta, 0, 0) = \delta + p\alpha V_n^{\alpha}(\delta + 1, 1, 1) + (1 - p)\alpha V_n^{\alpha}(\delta + 1, 1, 0), \tag{4.14}$$

$$Q_n^\alpha(\delta,1,0;0) = \delta + p\alpha V_n^\alpha(\delta+1,2,1) + (1-p)\alpha V_n^\alpha(\delta+1,2,1), \tag{4.15}$$
$$Q_n^\alpha(\delta,1,0;1) = \delta + p\alpha V_n^\alpha(\delta+1,1,1) + (1-p)\alpha V_n^\alpha(\delta+1,1,0). \tag{4.16}$$

By the induction assumption, $Q_n^\alpha(\delta,1,0;0) \geq V_{n+1}^\alpha(\delta,0,0)$. Meanwhile, note that $Q_n^\alpha(\delta,1,0;1) = V_{n+1}^\alpha(\delta,0,0)$ and we have

$$\begin{aligned} V_{n+1}^\alpha(\delta,1,0) &= \min\{Q_n^\alpha(\delta,1,0;0), Q_n^\alpha(\delta,1,0;1)\} \\ &\geq V_{n+1}^\alpha(\delta,0,0). \end{aligned} \tag{4.17}$$

ii) If $d \neq 0$ and $l < k-1$, we have

$$Q_n^\alpha(\delta,d,l;0) = \delta + p\alpha V_n^\alpha(\delta+1,d+1,l+1) + (1-p)\alpha V_n^\alpha(\delta+1,d+1,l), \tag{4.18}$$
$$Q_n^\alpha(\delta,d,l;1) = \delta + p\alpha V_n^\alpha(\delta+1,1,1) + (1-p)\alpha V_n^\alpha(\delta+1,1,0). \tag{4.19}$$

Based on the assumption, $Q_n^\alpha(\delta,d,l;0)$ is increasing in d. Note that $Q_n^\alpha(\delta,d,l;1)$ is independent of d. Therefore $V_{n+1}^\alpha(\delta,d,l)$ which takes the minimum is also increasing in d.

iii) If $l = k-1$, observe that

$$Q_n^\alpha(\delta,d,k-1;0) = \delta + p\alpha V_n^\alpha(d+1,0,0) + (1-p)\alpha V_n^\alpha(\delta+1,d+1,k-1), \tag{4.20}$$
$$Q_n^\alpha(\delta,d,k-1;1) = \delta + p\alpha V_n^\alpha(\delta+1,1,1) + (1-p)\alpha V_n^\alpha(\delta+1,1,0). \tag{4.21}$$

By Lemma 23 and induction assumption, $Q_n^\alpha(\delta,d,k-1;0)$ is increasing in d. Combining with the fact that $Q_n^\alpha(\delta,d,k-1;1)$ is independent of d, we conclude that $V_{n+1}^\alpha(\delta,d,k-1)$ which takes the minimum is also an increasing function in s.

Combining three cases together, we complete the proof. □

We have established the monotonicity of $V^\alpha(\delta,d,l)$ in δ and d in Lemma 23 and Lemma 24, respectively. In the following, we aim to show the monotonicity of $V^\alpha(\delta,d,l)$ in l as well. In order to prove it in Lemma 27, we will first introduce Lemma 25 and Lemma 26.

Lemma 25. *For any* $\mathbf{s} \in \mathcal{S}$*,* $V_n^\alpha(\delta,d,l) \leq V_n^\alpha(\delta,0,0)$ *at every iteration* n*. Therefore,* $V^\alpha(\delta,d,l) \leq V^\alpha(\delta,0,0)$*.*

Proof. Proof is by induction. It holds when $n=0$. Assume it is true for an $n \geq 0$. We aim to show that it holds for $n+1$ as well.

According to Eqn. (4.8), we have

$$\begin{aligned}
V^\alpha_{n+1}(\delta,0,0) &= \min\{Q^\alpha_n(\delta,0,0;0), Q^\alpha_n(\delta,0,0;1)\}\\
&= \delta + \alpha\min\{V^\alpha_n(\delta+1,0,0), pV^\alpha_n(\delta+1,1,1) + (1-p)V^\alpha_n(\delta+1,1,0)\}\\
&\geq Q^\alpha_n(\delta,0,0;1),
\end{aligned}$$

where the last inequality is based on the assumption that Lemma 25 holds for an $n \geq 0$.

Besides, based on the definition of $Q^\alpha_n(\mathbf{s};a)$, we have

$$V^\alpha_{n+1}(\delta,d,l) \leq Q^\alpha_n(\delta,d,l;1) = Q^\alpha_n(\delta,0,0;1).$$

Therefore, $V^\alpha_{n+1}(\delta,d,l) \leq V^\alpha_{n+1}(\delta,0,0)$. □

Roughly speaking, Lemma 25 implies that $(\delta,0,0)$ is the "*worst*" state among the set of states of form (δ,d,l). Based on Lemma 25, we also have the following corollary.

Corollary 1. *Whenever the source successfully delivers an update, it should start transmitting a new update immediately.*

Corollary 1 indicates that the zero-wait policy in this scenario is optimal, which is in contrast to the result in [14]. This is because under our setup, the source is allowed to preempt the transmission of an update at any time, while in [14], the server will finish the transmission once it starts.

Combining Lemma 24 and Lemma 25, we also have the following corollary.

Corollary 2. *Whenever the source starts transmitting a new update and the first symbol is erased, it should discard it and start transmitting a new update immediately. Besides, for any* $(\delta,d,0) \in \mathcal{S}$, $V^\alpha_n(\delta,d,0) = V^\alpha_n(\delta,0,0)$ *at every iteration* n.

The next lemma characterizes the relationship between the value function at state $(\delta,d,k-1)$ and state $(d,0,0)$.

Lemma 26. *For any* $(\delta,d,k-1) \in \mathcal{S}$, $V^\alpha_n(\delta,d,k-1) \geq V^\alpha_n(d,0,0)$ *at every iteration* n. *Therefore,* $V^\alpha(\delta,d,k-1) \geq V^\alpha(d,0,0)$.

Proof. The statement is true for $n=0$. Assume for an $n \geq 0$, $V^\alpha_n(\delta,d,k-1) \geq V^\alpha_n(d,0,0)$ for any state $(\delta,d,k-1)$ in $\mathcal{S}$. We aim to show the inequality holds for $n+1$ as well.

We note that

$$V^\alpha_{n+1}(d,0,0) = Q^\alpha_n(d,0,0;1) \tag{4.22}$$

$$= d + p\alpha V_n^\alpha(d+1,1,1) + (1-p)\alpha V_n^\alpha(d+1,1,0) \quad (4.23)$$

$$= d + p\alpha V_n^\alpha(d+1,1,1) + (1-p)\alpha V_n^\alpha(d+1,0,0), \quad (4.24)$$

where (4.22) is due to Corollary 1, and (4.24) is based on Corollary 2.

Meanwhile,

$$\begin{aligned} &Q_n^\alpha(\delta, d, k-1; 1) \\ &= \delta + p\alpha V_n^\alpha(\delta+1,1,1) + (1-p)\alpha V_n^\alpha(\delta+1,1,0) \\ &= \delta + p\alpha V_n^\alpha(\delta+1,1,1) + (1-p)\alpha V_n^\alpha(\delta+1,0,0). \end{aligned}$$

Therefore, based on Lemma 23 and the fact that $\delta \geq d+k$, we have

$$Q_n^\alpha(\delta, d, k-1; 1) \geq V_{n+1}^\alpha(d, 0, 0). \quad (4.25)$$

Besides,

$$\begin{aligned} &Q_n^\alpha(\delta, d, k-1; 0) \\ &= \delta + p\alpha V_n^\alpha(d+1,0,0) + (1-p)\alpha V_n^\alpha(\delta+1, d+1, k-1) \\ &\geq d + p\alpha V_n^\alpha(d+1,1,1) + (1-p)\alpha V_n^\alpha(d+1,0,0) \quad (4.26) \\ &= V_{n+1}^\alpha(d, 0, 0), \quad (4.27) \end{aligned}$$

where (4.26) is based on Lemma 25 and the assumption that $V_n^\alpha(\delta+1, d+1, k-1) \geq V_n^\alpha(d+1, 0, 0)$ for an $n \geq 0$.

Combining (4.25) and (4.27), we have $V_{n+1}^\alpha(\delta, d, k-1) \geq V_{n+1}^\alpha(d, 0, 0)$. □

Now, We are ready to establish the monotonicity structure of l on state value function $V^\alpha(\delta, d, l)$.

Lemma 27. *For any $\mathbf{s} \in \mathcal{S}$, $V_n^\alpha(\delta, d, l)$ is monotonically decreasing in l at every iteration n. Therefore, $V^\alpha(\delta, d, l)$ monotonically decreases in l.*

Proof. We prove it through induction. Similar to previous proofs, we note that it is true when $n = 0$ and we assume it holds for an $n \geq 0$. Then, we show that it holds for the $(n+1)$-th iteration as well.

We note that $Q_n^\alpha(\delta, d, l+1; 1) = Q_n^\alpha(\delta, d, l; 1)$. Besides, for any valid state $(\delta, d, l) \in \mathcal{S}$, $l < k-2$, we have

$$Q_n^\alpha(\delta, d, l; 0)$$

$$
\begin{aligned}
&= \delta + p\alpha V_n^\alpha(\delta+1, d+1, l+1) + (1-p)\alpha V_n^\alpha(\delta+1, d+1, l) \\
&\geq \delta + p\alpha V_n^\alpha(\delta+1, d+1, l+2) + (1-p)\alpha V_n^\alpha(\delta+1, d+1, l+1) \\
&= Q_n^\alpha(\delta, d, l+1; 0),
\end{aligned}
$$

where the inequality is based on the assumption that $V_n^\alpha(\delta, d, l)$ decreases in l.

When $l = k-2$, we have

$$
\begin{aligned}
&Q_n^\alpha(\delta, d, k-2; 0) \\
&= \delta + \alpha p V_n^\alpha(\delta+1, d+1, k-1) + \alpha(1-p) V_n^\alpha(\delta+1, d+1, k-2) \\
&\geq \delta + p\alpha V_n^\alpha(d+1, 0, 0) + (1-p)\alpha V_n^\alpha(\delta+1, d+1, k-1) \\
&= Q_n^\alpha(\delta, d, k-1),
\end{aligned}
$$

where the inequality follows from Lemma 26 and the assumption that $V_n^\alpha(\delta+1, d+1, l)$ decreases in l .

Since $V_{n+1}^\alpha(\mathbf{s})$ takes the minimum of $Q_n^\alpha(\mathbf{s}; 0)$ and $Q_n^\alpha(\mathbf{s}; 1)$, after combining both cases, we have $V_{n+1}^\alpha(d, s, l) \leq V_{n+1}^\alpha(d, s, l+1)$. □

In the following, we will prove two threshold structures of the optimal policy based on Lemma 23 through Lemma 27. The first one is the threshold structure with respect to the age of unfinished update, as described in Theorem 14, while the second one is the threshold structure on the number of successfully delivered symbols of the update being transmitted, as detailed in Theorem 15.

Theorem 14. *Under the optimal policy in Π', if the source decides to discard the unfinished update and start transmitting a new update at state (δ, d, l), it should make the same decision at state $(\delta, d+1, l)$.*

Proof. To show Theorem 14, it suffices to show that

$$
\Delta Q_n^\alpha(\delta, d, l) := Q_n^\alpha(\delta, d, l; 0) - Q_n^\alpha(\delta, d, l; 1) \tag{4.28}
$$

is monotonically increasing in d for every n.

We note that $Q_n^\alpha(\delta, d, l; 1) = Q_n^\alpha(\delta, d', l'; 1)$ for any valid state $(\delta, d', l') \in \mathcal{S}$. Therefore, it is equivalent to show that $Q_n^\alpha(\delta, d, l; 0)$ is non-decreasing in d.

When $l < k-1$, we have

$$
Q_n^\alpha(\delta, d, l; 0) = \delta + p\alpha V_n^\alpha(\delta+1, d+1, l+1) + (1-p)\alpha V_n^\alpha(\delta+1, d+1, l), \tag{4.29}
$$

which is non-decreasing in d according to Lemma 24.

When $l = k-1$, we have

$$Q_n^\alpha(\delta, d, k-1; 0) = \delta + p\alpha V_n^\alpha(d+1, 0, 0) + (1-p)\alpha V_n^\alpha(\delta+1, d+1, k-1), \tag{4.30}$$

which is monotonically increasing in d according to Lemma 23 and Lemma 24.

Combining both cases, we have $Q_n^\alpha(\delta, d, l; 0)$ and $\Delta Q_n^\alpha(\delta, d, l)$ monotonically increases in d for all n. □

Theorem 15. *Under the optimal policy in Π', if the source decides to keep sending the unfinished update at state (δ, d, l), it should make the same decision at state $(\delta, d, l+1)$.*

Proof. To show Theorem 15, it suffices to show that $\Delta Q_n^\alpha(\delta, d, l)$ is monotonically decreasing in l for every n. We prove this through contradiction. Assume $\Delta Q_n^\alpha(\delta, d, l) < 0$, however, $\Delta Q_n^\alpha(\delta, d, l+1) \geq 0$. Then, we must have

$$Q_n^\alpha(\delta, d, l+1; 0) \geq Q_n^\alpha(\delta, d, l; 0). \tag{4.31}$$

Hence,

$$\begin{aligned} &V_n^\alpha(\delta, d, l+1) \\ &= \min\{Q_n^\alpha(\delta, d, l+1; 0), Q_n^\alpha(\delta, d, l+1; 1)\} &(4.32)\\ &\geq \min\{Q_n^\alpha(\delta, d, l; 0), Q_n^\alpha(\delta, d, l; 1)\} &(4.33)\\ &= V_n^\alpha(\delta, d, l), &(4.34) \end{aligned}$$

which contradicts the fact that $V_n^\alpha(\delta, d, l+1)$ is decreasing in l, as shown in Lemma 27. Therefore $\Delta Q_n^\alpha(\delta, d, l)$ is monotonically decreasing in l. □

Theorem 16. *Under the optimal policy in Π', if the source decides to keep sending the unfinished update at state (δ, d, l), it should make the same decision at state $(\delta+1, d+1, l+1)$.*

Proof. To prove Theorem 16, we will prove its contrapositive instead, i.e., if $\Delta Q^\alpha(\delta+1, d+1, l+1) > 0$, we must have $\Delta Q^\alpha(\delta, d, l) \geq 0$. We focus on the states with $l < k-1$.

First, we note that

$$\begin{aligned} &\Delta Q^\alpha(\delta, d, l) \\ &= p\alpha[V^\alpha(\delta+1, d+1, l+1) - V^\alpha(\delta+1, 1, 1)] \end{aligned}$$

$$+ (1-p)\alpha[V^{\alpha}(\delta+1, d+1, l) - V^{\alpha}(\delta+1, 0, 0)]. \tag{4.35}$$

If $\Delta Q^{\alpha}(\delta+1, d+1, l+1) > 0$,

$$V^{\alpha}(\delta+1, d+1, l+1) = Q^{\alpha}(\delta+1, d+1, l+1; 1). \tag{4.36}$$

Besides, according to Corollary 1, we have

$$V^{\alpha}(\delta+1, 0, 0) = Q^{\alpha}(\delta+1, 0, 0; 1) = Q^{\alpha}(\delta+1, d+1, d+1; 1). \tag{4.37}$$

Meanwhile, according to Lemma 25, we have

$$V^{\alpha}(\delta+1, 0, 0) \geq V^{\alpha}(\delta+1, 1, 1). \tag{4.38}$$

Combining (4.36)(4.37) and (4.38), we have

$$V^{\alpha}(\delta+1, d+1, l+1) - V^{\alpha}(\delta+1, 1, 1) \leq 0. \tag{4.39}$$

On the other hand, Theorem 15 indicates that $\Delta Q^{\alpha}(\delta+1, d+1, l) \geq \Delta Q^{\alpha}(\delta+1, d+1, l+1) > 0$. Therefore,

$$V^{\alpha}(\delta+1, d+1, l) = Q^{\alpha}(\delta+1, d+1, l; 1) \tag{4.40}$$

$$= Q^{\alpha}(\delta+1, 0, 0; 1) = V^{\alpha}(\delta+1, 0, 0). \tag{4.41}$$

Combining (4.39)(4.41) with (4.35), we have $\Delta Q^{\alpha}(\delta, d, l) \geq 0$. □

Applying Theorem 16 recursively and combining with Corollary 2, Theorem 14, Theorem 15, we characterize the structure of the optimal policy in the following theorem.

Theorem 17. *Consider the optimal policy in* Π'. *Starting at a state* $(\delta_0, 0, 0)$, *there exists a threshold* τ_{d,δ_0} *for every* $d \geq 0$, *such that if the source attempts to transmit an update for* d *time slots, and* $l < \tau_{d,\delta_0}$ *coded symbols are successfully delivered, the source should quit the unfinished update and start transmitting a new update at state* $(\delta_0 + d, d, l)$; *Otherwise, it will keep transmitting until the update is successfully decoded. Besides,* τ_{d,δ_0} *monotonically increases in* d.

We point out that all of the structural properties of the optimal policy derived for the α-discounted problem hold when $\alpha \to 1$ [71]. Thus, the optimal policy for the time-average problem also exhibits similar threshold structure.

4.4 Limit Feedback Scenario

In Section 4.3, we assume instantaneous feedback is available to the transmitter after each transmission and derive the corresponding optimal transmission policy through dynamic programming. However, due to limited resources in practice, it is inefficient for the monitor to send feedback information in each time slot. It is thus desirable to design more practical updating schemes with less feedback information without deteriorating the AoI performance significantly. In the sequel, we study the case where at most one additional feedback information is sent to the source throughout the transmission of the rateless codes. Specifically, we consider a special type of policies, termed as $\pi_{m,n}$ policy. Compared with conventional transmission of rateless codes where an ACK is sent to the source when the update is decoded, the updating schemes proposed here require at most one more feedback and significantly improve the AoI performance potentially.

Specifically, under the $\pi_{m,n}$ policy, the monitor tracks the delivery of encoded symbols and sends a NACK to the source if less than m symbols are successfully delivered after n transmissions of the current update. If the source does not receive the NACK after the n-th transmission, it will continue the transmission until the update is delivered. Otherwise, it will discard the current update and start to transmit a new one. We note that there are two different scenario depending on whether whether n is greater than the update size k. We point out that the $\pi_{m,n}$ policy is equivalent to the following policy: The monitor tracks the delivery of symbols and sends back an ACK once m symbols are received within n transmissions. If the source does not receive the ACK after the n-th transmission, it will discard the current update and start to transmit a new one. Observe that for any sample path realization of the symbol erasure pattern, those two policies perform exactly the same.

Let X_i be the i-th inter-update time and Y_i be the AoI after reset. If $\{(X_i, Y_i)\}_{i=1}^{\infty}$ are independent and identically distributed, the long-term average AoI can be expressed as [20]

$$\Delta = \mathbb{E}[Y_i] + \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}. \tag{4.42}$$

We can show that under the $\pi_{m,n}$ policy, the inter-update delay and the AoI after each update are i.i.d. in time, thus, Eqn. (4.42) applies.

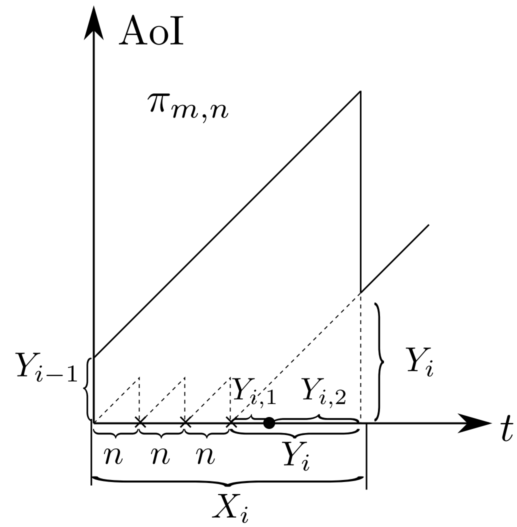


Figure 4.2: The i-th inter-update interval under the $\pi_{m,n}$ policy.

4.4.1 AoI under The $\pi_{m,n}$ Policy

For ease of exposition, denote $F_m(n)$ as the CDF of the negative binomial random variable with parameters m, p, i.e.,

$$F_m(n) = \sum_{\ell=m}^{n} \binom{\ell-1}{m-1} p^m (1-p)^{\ell-m}, \quad n \in \mathbb{Z}_{\geq m}. \tag{4.43}$$

Fig. 4.2 shows the AoI evolution of the i-th inter-update interval. Note that $m \leq k$ and $m \leq n$. Decompose the length of the i-th inter-update interval X_i as $X_i = nZ_i + Y_i$ where Z_i is the number of re-transmissions. Let $\gamma := F_m(n)$. Then Z_i is a geometric distribution with parameter γ, i.e., $\mathbb{P}[Z_i = z] = (1-\gamma)^z \gamma$, $z \in \mathbb{Z}_{\geq 0}$, whose first and second moments of Z_i can be expressed as

$$\mathbb{E}[Z_i] = \frac{1-\gamma}{\gamma}, \tag{4.44}$$

$$\mathbb{E}[Z_i^2] = \frac{\gamma^2 - 3\gamma + 2}{\gamma^2}. \tag{4.45}$$

Furthermore, Y_i can be decomposed as $Y_i = Y_{i,1} + Y_{i,2}$ where $Y_{i,1}$ is the number of time slots to deliver the m symbols of the delivered update and $Y_{i,2}$ is the number of time slots to deliver the remaining $k - m$ symbols. Clearly, $Y_{i,1}$ and $Y_{i,2}$ are independent with each other. The PMFs of $Y_{i,1}$ and $Y_{i,2}$ are

$$\mathbb{P}[Y_{i,1} = y_1] = \frac{1}{\gamma}\binom{y_1}{m-1} p^r (1-p)^{y_1-m}, \quad y_1 \in [m:n], \tag{4.46}$$

$$\mathbb{P}[Y_{i,2} = y_2] = \binom{y_2 - 1}{k-m-1} p^{k-m} (1-p)^{y_2-k+m}, \quad y_2 \in \mathbb{Z}_{\geq k-m}. \tag{4.47}$$

Note that $\sum_{y_1=m}^{n} \mathbb{P}[Y_{i,2} = y_2] = \sum_{y_2=k-m}^{\infty} \mathbb{P}[Y_{i,1} = y_1] = 1$. The first moment of $Y_{i,1}$ is

$$\mathbb{E}[Y_{i,1}] = \frac{1}{\gamma} \sum_{y_1=m}^{n} y_1 \binom{y_1 - 1}{m-1} p^m (1-p)^{y_1-m} \tag{4.48}$$

$$= \frac{m}{p\gamma} \sum_{y_1'=m+1}^{n+1} \binom{y_1' - 1}{m_1' - 1} p^{m'} (1-p)^{y_1'-m_1'} \tag{4.49}$$

$$= \frac{m}{p\gamma} F_{m+1}(n+1). \tag{4.50}$$

Similarly, we have

$$\mathbb{E}[Y_{i,1}(Y_{i,1}+1)] = \frac{m(m+1)}{p^2\gamma} F_{m+2}(n+2). \tag{4.51}$$

Hence, the second moment of $Y_{i,1}$ is

$$\mathbb{E}[Y_{i,1}^2] = \frac{m(m+1)}{p^2\gamma} F_{m+2}(n+2) - \frac{m}{p\gamma} F_{m+1}(n+1). \tag{4.52}$$

We note that $Y_{i,2} - k + m \in \mathbb{Z}_{\geq 0}$ is a negative binomial distribution and it is the sum of $k-m$ independent and identically distributed geometric random variables with parameter p. Thus, the moment generating function of $Y_{i,2} - k + m$, denoted as $M_{Y_{i,2}-k+m}$, equals the $(k-m)$-th power of that of the geometric random variable with parameter p, i.e.,

$$M_{Y_{i,2}-k+m}(r) = \left[\frac{p}{1-(1-p)e^r}\right]^{k-m}. \tag{4.53}$$

Then, the first and second moments of $Y_{i,2}$ can be derived as

$$\mathbb{E}[Y_{i,2}] = \frac{k-m}{p}, \tag{4.54}$$

$$\mathbb{E}[Y_{i,2}^2] = \frac{(k-m)(k-m+1-p)}{p^2}. \tag{4.55}$$

Therefore,

$$\mathbb{E}[Y_i] = \frac{m}{p\gamma}F_{m+1}(n+1) + \frac{k-m}{p}, \tag{4.56}$$

$$\mathbb{E}[Y_i^2] = \frac{m(m+1)}{p^2\gamma}F_{m+2}(n+2) + \frac{m(2k-2m-p)}{p^2\gamma}F_{m+1}(n+1) + \frac{(k-m)(k-m+1-p)}{p^2}. \tag{4.57}$$

Combining the previous results, the long-term average AoI under $\pi_{m,n}$ can be expressed in terms of Z_i and Y_i as follows

$$\Delta_{\pi_{m,n}} = \mathbb{E}[Y_i] + \frac{n^2\mathbb{E}[Z_i^2] + 2n\mathbb{E}[Z_i]\mathbb{E}[Y_i] + \mathbb{E}[Y_i^2]}{2\left(n\mathbb{E}[Z_i] + \mathbb{E}[Y_i]\right)} \tag{4.58}$$

$$= \frac{mF_{m+1}(n+1)}{pF_m(n)} + \frac{k-m}{p} + \frac{1}{2}[mpF_{m+1}(n+1) + p(k-m-np)F_m(n) + np^2]^{-1}\mathcal{D} \tag{4.59}$$

where

$$\mathcal{D} = m(m+1)F_{m+2}(n+2) + m\left(2np\frac{1-F_m(n)}{F_m(n)} + 2k - 2m - p\right)F_{m+1}(n+1) + n^2p^2F_m^2(n) + [-np(3np+2) + (k-m)(k-m+1-p)]F_m(n) + 2np(np+k-m). \tag{4.60}$$

Remark: When $m = n = 1$, we have $\gamma = p$ and $\eta_1 = \eta_2 = 1$. The corresponding long-term average AoI equals

$$\Delta_{\pi_{1,1}} = \frac{3k+p-1}{2p}. \tag{4.61}$$

We concluding this section by comparing policy $\pi_{1,1}$ and the non-preemptive policy. It has been noted in [20] that the long-term average AoI under this non-preemptive policy is

$$\bar{\Delta} = \frac{3k+1-p}{2p}. \tag{4.62}$$

The difference of the long-term average AoI is

$$\bar{\Delta} - \Delta_{\pi_{1,1}} = \frac{1-p}{p} \geq 0, \tag{4.63}$$

which indicates that $\pi_{1,1}$ always outperforms than the non-preemptive policy.

4.4.2 Performance Comparison

In this section, we compare the performance of the $\pi_{1,1}$ policy with the non-preemptive updating policy where the source keeps sending encoded symbols until the corresponding update has been successfully decoded. We will also analytically characterize the performance gap between the $\pi_{1,1}$ policy and the optimal policy characterize in Section III.

Next, we compare the $\pi_{1,1}$ policy with the optimal policy and provide performance guarantees for it.

To facilitate our analysis, we will focus on the set of policies Π^* where the random process incurred by policy $\theta \in \Pi^*$ is stationary and ergodic in the first and second moments of inter-update time X_i.

Note that an update is delivered to the monitor after k symbols are received and decoded at the monitor and at least k time slots are required to transmit k symbols. Thus, we have the following lemma.

Lemma 28. *Under any updating policy, the instantaneous AoI at any time must be greater than k, i.e., $\delta_t \geq k$.*

Denote $M(T)$ as the number of successful updates at the monitor over $[1 : T]$. The following lemma characterizes the long-term average updating rate constraint.

Lemma 29. *Under any updating policy θ, it must have $\lim_{T\to\infty} M(T)/T \leq p/k$ almost surely.*

Proof: We differentiate two types of delivered symbols: one is the delivered symbols that are useful for decoding an update; the other one is the delivered symbols that are discarded, i.e., the corresponding updates are discarded. Let l_T be the number of all delivered symbols up to time T and $\tilde{l}_T$ be the first type of symbols up to time T. Note that the ratio between l_T and T is p, independent of the decision sequence as the delivery probability of each symbol is a constant p. Moreover, since an decoded update consists of exactly k symbols, it must have $kM(T) \leq \tilde{l}_T$. Then we have

$$\lim_{T\to\infty} \frac{M(T)}{T} \leq \lim_{T\to\infty} \frac{\tilde{l}_T/k}{T} \leq \lim_{T\to\infty} \frac{l_T/k}{T} = \frac{p}{k}. \tag{4.64}$$

■

In the sequel, we will focus on the update rate constraint instead of symbol delivery rate constraint. Meanwhile, in Lemma 30 we also ignore the fact that $\delta \geq k$ by Lemma 28 pick it up in Theorem 18.

Consider the set of virtual policies Π_v ($\supset \Pi'$) under which we are able to control the channel state, i.e., the update transmission is feasible as long as the designed process is consistent with the long-term update rate constraint in Lemma 29. We will then design a virtual policy $\theta_v \in \Pi_v$ and prove the average AoI under policy θ_v is a lower bound of any policy $\theta \in \Pi_v$. The key idea is to arrange successful update delivery time as uniformly as possible. The next lemma characterizes this phenomenon.

Lemma 30. *Assume an update is delivered at time* 0 *and time* T*, and there are* $n-1$ *updates delivery at integer times between* 0 *and* T*. Let* $T = m_1 \left\lfloor \frac{T}{n} \right\rfloor + m_2 \left\lceil \frac{T}{n} \right\rceil$, $m_1, m_2 \in \mathbb{N}$*. Then, the average AoI is lower bounded by* $\frac{1}{2T}\left(m_1 \left\lfloor \frac{T}{n} \right\rfloor^2 + m_2 \left\lceil \frac{T}{n} \right\rceil^2\right)$*, in which the update happens at each time point* $\frac{kT}{n}$ *for* $k = 1, 2, \ldots, n-1$*.*

Proof: We will show by contradiction that under the policy minimizing the AoI over $[0, T]$, the inter-update delivery time X_n has to be either $\left\lfloor \frac{T}{n} \right\rfloor$ or $\left\lceil \frac{T}{n} \right\rceil$.

Assume there are two different inter-update delivery time X_i and X_j where $X_i \leq X_j - 2$. We construct a new scheme under which $X_i' = X_i + 1$ and $X_j' = X_j - 1$ and $X_k' = X_k$ for $k \neq i, j$. Then, $\sum_{i=1}^{n} X_i^2 > \sum_{i=1}^{n} (X_i')^2$, i.e., the AoI under the newly constructed scheme performs better. Therefore, under the policy minimizing the AoI over $[0, T]$, for any inter-update delivery time X_i and X_j where $X_i \leq X_j$, it must have either $X_i = X_j$ or $X_i = X_j + 1$. It turns out that the only possible assignment is m_1 number of inter-update time $\left\lfloor \frac{T}{n} \right\rfloor$ and m_2 number of inter-update time $\left\lceil \frac{T}{n} \right\rceil$ and the corresponding average AoI over $[0, T]$ is $\frac{1}{2T}\left(m_1 \left\lfloor \frac{T}{n} \right\rfloor^2 + m_2 \left\lceil \frac{T}{n} \right\rceil^2\right)$. ■

We remark that if the delivery of update is allowed at non-integer time, then the average AoI over $[0, T]$ is lower bounded by $\frac{T}{2n}$ where each update happens at time $\frac{kT}{n}$ for $k = [1 : n-1]$.

Now we are ready to derive a universal lower bound for policy $\theta \in \Pi_v$ by constructing a policy $\theta_v \in \Pi_v$. As mentioned earlier, we will consider Lemma 28 in the following theorem.

Theorem 18. *For any policy* $\theta \in \Pi^*$*, the expected long-term average AoI is lower bounded by* $\Delta_* = \frac{k}{p}\left(\frac{1}{2} + p\right)$*.*

Proof: For any policy $\theta \in \Pi^*$, consider T large enough, by Lemma 29 the number of successful update delivery is upper bounded by $\frac{pT}{k}$. By rearranging the successful update time suggested in Lemma 30, we have the following lower bound

$$\Delta_\theta = k + \lim_{T\to\infty} \frac{T}{2(M(T)-1)} = k + \lim_{T\to\infty} \frac{T}{2M(T)} \geq k + \frac{k}{2p}, \tag{4.65}$$

where k appeared in the first summation comes from Lemma 28.

We remark that in the above equation, we use sample mean instead ensemble mean, which is valid by the ergodicity assumption on the first and second moments of inter-update delivery time X_i. ■

The following corollary follows from Eqn. (4.61) and Theorem 18.

Corollary 3. *The difference between the average AoI under the policy* $\pi_{1,1}$ *and that under the optimal policy is upper bounded by* $\frac{1-p}{p}\left(k-\frac{1}{2}\right)$.

Remark: For fixed update size k, as the delivery probability p approaches 1, the performance gap approaches 0. I.e., the approximate policy $\pi_{1,1}$ tends to be the optimal policy as p goes to 1. Moreover, as p approaches 1, the non-preemptive policy also approaches to the optimal policy by Eqn. (4.63).

4.5 Numerical Results

In this section, we first numerically search for the optimal policy and identify the threshold structure. Then, we investigate the general $\pi_{m,n}$ policy through simulation. Finally, we evaluate the performances of the policy $\pi_{1,1}$, the non-preemptivepolicy as well as the optimal policy.

4.5.1 Structural Properties of the Optimal Policy

We propose a structured value iteration algorithm to obtain the thresholds, as detailed in Algorithm 1. In order to reduce the computational complexity, we define an approximate MDP as follows: We define δ_m as the boundary AoI, and truncate the state space of the original MDP as $\mathcal{S}_m = \{\mathbf{s} \in \mathcal{S} : \delta \leq \delta_m, d \leq \delta_m - k\}$. If under the action a_t, $\mathbf{s}_{t+1}$ becomes outside $\mathcal{S}_m$, we will let it be the corresponding capped boundary state. We can show that the approximate MDP is identical to the original MDP when $\delta_m \to \infty$. Besides, we also leverage the structure of the optimal policy to reduce the computational complexity.

Fig. 4.7 shows the structure of the optimal policy. In Fig. 4.7(a), we observe the threshold structure on d for fixed δ, l. Fig. 4.7(b) indicates the threshold structure on l. Finally, Fig. 4.7(b)-(d) exhibits the properties described in Theorem 16 and Theorem 17.

Algorithm 1 Structured Value Iteration.

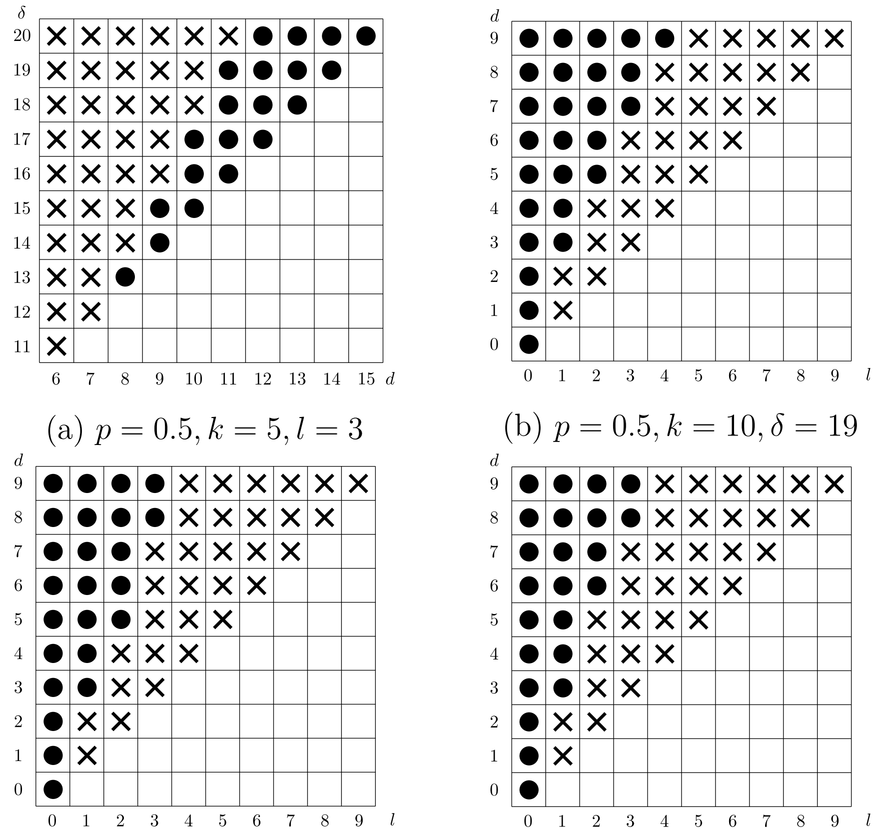
Initialize: $V_0(\mathbf{s}) = 0, \forall \mathbf{s} \in \mathcal{S}_m$.
for $i = 0 : n$ **do**
  for $\forall(\delta, d, l) \in \mathcal{S}_m$ **do**
    $Q_i(\mathbf{s}; a) = C(\mathbf{s}) + \mathbb{E}[V_i(\mathbf{s}')|\mathbf{s}, a]$
    if $l = 0$ **then** $a^*(\mathbf{s}) = 0$;
    else if $\exists l' < l, a^*(\delta, d, l') = 0$ **then** $a^*(\mathbf{s}) = 0$
    else if $\exists d' < d, a^*(\delta, d', l) = 1$ **then** $a^*(\mathbf{s}) = 1$
    else $a^*(\mathbf{s}) = \arg\min_{a \in \{0,1\}} Q_i(\mathbf{s}; a)$
    $V_{i+1}(\mathbf{s}) = Q_i(\mathbf{s}; a^*(\mathbf{s})) - V_i(\mathbf{s}_0)$

m \ n	1	2	3	4	5	6	7	8	9	10
1	8.3571	8.5879	8.7025	8.7530	8.7735	8.7813	8.7842	8.7852	8.7855	8.7857
2	/	9.0270	8.6429	8.6697	8.7217	8.7552	8.7724	8.7802	8.7836	8.7849
3	/	/	12.0159	9.4154	8.9127	8.7963	8.7751	8.7761	8.7800	8.7828
4	/	/	/	18.6579	11.4103	9.7076	9.1355	8.9199	8.8362	8.8041

Table 4.1: Performance of the $\pi_{m,n}$ policy for $k = 4$ and $p = 0.7$.

4.5.2 Performance of the $\pi_{m,n}$ Policy

We evaluate the performances of the approximate $\pi_{m,n}$ policy in Table 4.1. We fix system parameters $k = 4$ and $p = 0.7$ and evaluate policies for different m and n. Note that for fixed m, the age performance of the $\pi_{m,n}$ policy is a convex function of parameter n. We observe three possibilities: 1) For $n = 1$, the time average AoI is increasing and the optimal one is the policy $\pi_{1,1}$; 2) For $n = 2$ and $n = 3$, the time average AoI is decreasing first and then increasing. However, the optimal ones underperform the $\pi_{1,1}$ policy; 3) For $n \geq 4$, the long-term average AoI in terms of m is decreasing a function. Moreover, the time average AoI approaches 8.7857 as n becomes larger. This is intuitive as we expect the threshold has little impact on the AoI if n is comparably larger than the update size k. In fact, the baseline non-preemptive policy has the same AoI performance 8.7857, as expected. We also perform exhaust search for the optimal approximate policy $\pi_{m,n}$ with various k and p ranging from $k \in [1 : 50]$, $p \in [0.1 : 0.1 : 0.9]$. It turns out that the $\pi_{1,1}$ policy is always the optimal policy and it outperforms the baseline non-preemptive policy by a fair margin, as shown in the next section.



(a) $p = 0.5, k = 5, l = 3$ (b) $p = 0.5, k = 10, \delta = 19$

(c) $p = 0.5, k = 10, \delta = 20$ (d) $p = 0.5, k = 10, \delta = 21$

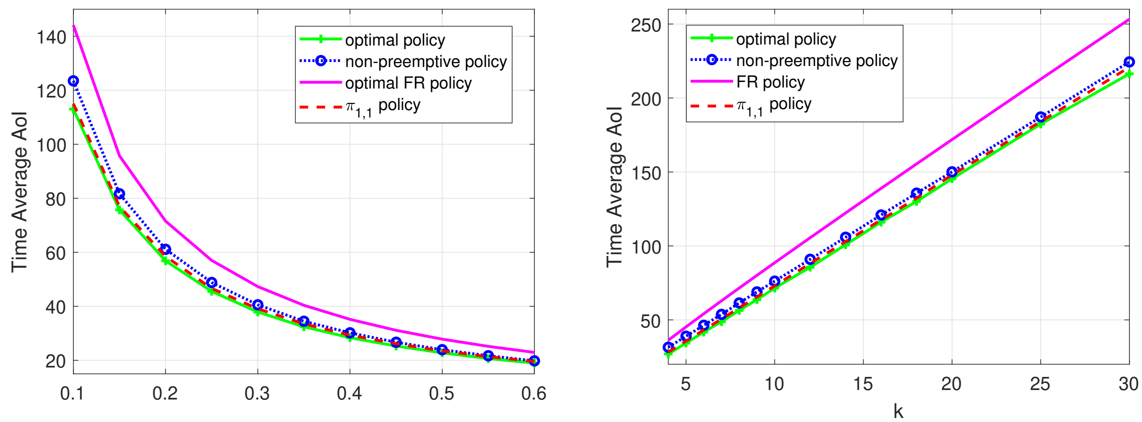
Figure 4.7: Threshold structure. Dots represent new transmission and crosses represent continuing unfinished transmission.

4.5.3 Performance Comparison

In this section, we evaluate the performances of the proposed approximate threshold policy $\pi_{1,1}$, the non-preemptive policy, the optimal FR policy and the optimal policy. Under fixed redundancy (FR) strategy, each update is encoded as n coded symbols and the update is successfully decoded if and only if at least k symbols are delivered [20]. In the simulation, for each p, k, we numerically search the optimal n which minimizes the AoI and term it as the optimal FR policy.

We fix update size $k = 8$ and vary p in $(0.1, 0.6)$ and plot the long-term average AoI of the approximate threshold policy $\pi_{1,1}$ in Fig. 4.8(a). While in Fig. 4.8(b), we fix $p = 0.2$ and plot the time-average AoI as a function of K within range $[4, 30]$. For comparison, the optimal policy identified in Section 4.5.1 and the baseline non-preemptive policy and the optimal FR policy are evaluated[1]. We observe that the approximate threshold

[1]The optimal n of the FR regime is $[109, 73, 54, 43, 36, 31, 27, 23, 21, 19, 17]$ for scenario (a) while the optimal n is $[26, 33, 40, 47, 54, 61, 68, 81, 94, 107, 119, 132, 163, 193]$ for scenario (b).



(a) AoI as a function of p when $K = 8$.

(b) AoI as a function of k when $p = 0.2$.

Figure 4.8: Performance comparison between $\pi_{1,1}$, the optimal policy and the non-preemptive policy.

policy outperforms the non-preemptive policy, which is consistent with our analysis in Section 4.4. Moreover, the FR regime perform the worse, which is due to the fact that it does not exploit any feedback information. Surprisingly, the performance gap between the approximate threshold policy and the optimal policy is fairly small, which indicates that slightly deviating from the non-preemptive policy can greatly reduce the time average AoI.

4.6 Conclusions

In this chapter, we considered the optimal transmission scheduling of rateless codes in an erasure channel for AoI optimization. Theoretical analysis indicates that the optimal policy has a monotonic threshold structure. An approximate threshold policy $\pi_{1,1}$ which slightly deviates from the non-preemptive policy is proposed. The closed-form expressions of both policy $\pi_{1,1}$ and the non-preemptive policy are derived. We also identify the performance guarantee on policy $\pi_{1,1}$ and the non-preemptive policy. Numerical results indicates that a small deviation of the non-preemptive policy can reduce the AoI significantly.

Chapter 5 Adaptive Coding for Information Freshness in a Two-user Broadcast Erasure Channel

5.1 Introduction

In this chapter, we consider a two-user broadcast symbol erasure channel. The source continuously broadcasts encoded symbols of status updates to two users. Due to random erasures over the link from the source to each user, the users may not be able to decode the intended update at the same time. Assuming perfect feedback information at the source so it knows exactly which encoded symbols have been delivered to which user(s), our objective is to design an adaptive coding scheme to judiciously generate the encoded symbol each time, so that both users can successfully decode updates from the source promptly. Intuitively, there exists a tension between the AoI of the two users. To see this, consider the scenario where the channel between the source and one user (termed as the strong user) is statistically better than the other (termed as the weak user). Assume the source prioritizes one of the users and will immediately switch to a new update once the previous update has been successfully decoded by the user with priority. Depending on whether the strong or weak user has the priority, there exist two different situations: If the strong user has the priority, then, with higher probability, the weak user will not be able to decode the same update when the strong user decodes. Thus, its AoI will keep growing until it eventually decodes one update successfully, leading to a higher AoI. On the other hand, if the weak user has the priority, then, with higher probability, the strong user will decode before the weak user decodes. The source will continue transmitting the same update until the weak one decodes. Compared with the first case, the AoI at the weak user will be lower, at the price of increasing the AoI at the strong user due to waiting. In analogy to the capacity region of broadcast channels, in this status updating setting, all achievable AoI pairs at both users form a region. While a complete characterization of such an *AoI region* seems too ambitious at this stage, as a first step, we will investigate certain "achievable points within the AoI region under specific coding and transmission schemes.

Specifically, we will consider two updating schemes: a greedy scheme that always prioritizes the strong user, and another adaptive coding scheme that prioritizes the strong user but also takes the information freshness at the weak user into consideration. We aim to show that, compared with the greedy scheme, the adaptive coding scheme strictly improves the AoI at the weak user without compromising the AoI at the strong user. Such improvement becomes more prominent when the size of updates increases.

5.2 System Model

We consider a broadcast symbol erasure channel consisting of one source and two users. The source keeps generating updates of K information symbols, and is able to broadcast one *encoded* symbol to the users in each time slot. Assume the link between the source and user i, $i = 1, 2$ is noisy and each symbol can be erased over the link independently according to an i.i.d. Bernoulli process. Let x_t be the broadcast symbol at time slot t, and $y_{1,t}$ and $y_{2,t}$ be the corresponding received symbol at user 1 and user 2, respectively. Then, $y_{i,t}$ equals x_t with probability p_i and equals $\varnothing$ if x_t is erased. Without loss of generality, we assume $p_1 > p_2$, i.e., user 1 is the strong user while user 2 is the weak user. We also assume perfect feedback information at the source right after each transmission. Therefore, the source knows which symbols have been received at each user at any time.

Denote the update generated at the source at time t as $\mathbf{u}_t := \{u_t(k)\}_{k=1}^K$, where $u_t(k)$ is the k-th information symbol. Let $\mathbf{u}^t := \{\mathbf{u}_\tau\}_{\tau=1}^t$, $\mathbf{y}_i^t := \{y_{i,\tau}\}_{\tau=1}^t$ for $i = 1, 2$. Then, in general, $x_t = f_t(\mathbf{u}^t, \mathbf{y}_1^{t-1}, \mathbf{y}_2^{t-1})$, where f_t is the encoding function at time t.

At the end of time slot t, user i tries to decode an update based on $\mathbf{y}_i^t$. An update is successfully decoded if the K information symbols of the update are successfully decoded. If $\mathbf{u}_\tau$ is decoded at time t at user i, the instantaneous AoI at user i, denoted as $\delta_i(t)$, will reset to $t - \tau$. If multiple updates are decoded simultaneously, $\delta_i(t)$ will be reset to the smallest age of the decoded updates.

Let

$$\Delta_i := \limsup_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T \delta_i(t)dt\right] \tag{5.1}$$

be the expected long-term average AoI at user i. Our ultimate goal is to characterize the maximum (Δ_1, Δ_2) region over all possible coding schemes. While this is an extremely challenging problem in general, in this chapter, we focus on specific coding schemes and characterize the corresponding achievable AoI pairs (Δ_1, Δ_2). Specifically, we aim to show that by adaptively combining symbols from different updates into an encoded symbol, the AoI at the weak user can be significantly improved.

5.3 Greedy Scheme

In this section, we introduce a baseline greedy scheme. Assume that the source adopts the infinite incremental redundancy (IIR) strategy in [20]. Under the IIR strategy, each K-symbol update is encoded by a rateless code, such as a fountain code. The source keeps broadcasting the encoded symbols of an update to both users. Under the greedy scheme, the source prioritizes the strong user (i.e., user 1) and aims to minimize time average AoI Δ_1. Thus, as soon as user 1 decodes an update, the source will switch to a new update and start broadcasting it.

Following the analysis for the single user case in [20], we can show that the long-term average AoI of user 1 is equal to

$$\Delta_1 = \frac{K}{p_1}\left(\frac{3}{2} + \frac{1-p_1}{K}\right). \tag{5.2}$$

Next, we will characterize Δ_2. In order to compare with the inter-update coding scheme proposed later, we focus on the lower bound on Δ_2 in this section.

Let $Y_{1,i}$, $Y_{2,i}$, $i = 1, 2, \ldots$ be independent random variables with the following distributions

$$\mathbb{P}[Y_{1,i} = n] = \binom{n-1}{k-1} p_1^k (1-p_1)^{n-k} \tag{5.3}$$

$$\mathbb{P}[Y_{2,i} = n] = \binom{n-1}{k-1} p_2^k (1-p_2)^{n-k} \tag{5.4}$$

Under the greedy policy I, $Y_{1,i}$ is the i-th inter-update delay for the first user, while $Y_{2,i}$

can be interpreted as the *required* transmission time for the user to decode the same update. If $Y_{1,i} < Y_{2,i}$, then user 2 won't be able to decode the update, as the source will start transmitting a new update immediately. If $Y_{2,i} \leq Y_{1,i}$, user 2 will be able to decode the update no later than user 1, and its age will be updated to $Y_{2,i}$. Since the source will greedily update user 1, it will continue transmitting the update until user 1 also decodes it successfully.

Label the updates that user 1 successfully decodes as $1, 2, \ldots$. Define N_j, $j = 1, 2, \ldots$ as the index of the updates user 2 successfully decodes. Then,

$$N_j = \min\{i \mid Y_{1,i} \geq Y_{2,i}, i > N_{j-1} \ldots\} \tag{5.5}$$

with $N_0 = 0$.

Define

$$S_j = \sum_{i=N_{j-1}+1}^{N_j} Y_{1,i}, \quad D_j = \sum_{i=N_{j-1}+1}^{N_j - 1} Y_{1,i} + Y_{2,N_j}. \tag{5.6}$$

Then, we can show that under the greedy policy, $\Delta_2 \geq \frac{\mathbb{E}[D_j^2]}{2\mathbb{E}[S_j]}$.

Due to the complicated distributions of D_j and S_j, a closed-form expression of Δ_2 is intractable. Therefore, in the following, we focus on the asymptotic regime when K is sufficiently large, and derive an approximation of Δ_2.

Theorem 19. *Under the greedy policy, $\Delta_2 = \Omega(K(1/\bar{q})^K)$, where $\bar{q} := \frac{p_1 p_2}{(1-\sqrt{(1-p_1)(1-p_2)})^2} < 1$.*

Proof: Due to the i.i.d. distributions of $\{S_j\}$ and $\{D_j\}$, for any j, we have

$$\mathbb{E}[S_j] = \mathbb{E}[N_1] \cdot \mathbb{E}[Y_{1,i}] \tag{5.7}$$

$$\mathbb{E}[D_j] = \mathbb{E}[N_1] \cdot \mathbb{E}[\min\{Y_{1,i}, Y_{2,i}\}] \tag{5.8}$$

where N_1 is a geometric random variable with parameter

$$p_g := \mathbb{P}[Y_{1,i} \geq Y_{2,i}]. \tag{5.9}$$

Let X_k be the difference between two independent geometric random variables with parameters p_1 and p_2, respectively. Then,

$$\mathbb{P}[Y_{1,i} \geq Y_{2,i}] = \mathbb{P}\left[\sum_{k=1}^{K} X_k \geq 0\right]. \tag{5.10}$$

Since

$$\mathbb{E}[\exp(\alpha \sum_{k=1}^{K} X_k)] = \left[\frac{p_1 e^{\alpha}}{1-(1-p_1)e^{\alpha}} \frac{p_2 e^{-\alpha}}{1-(1-p_2)e^{-\alpha}}\right]^K,$$

let $\alpha := \frac{1}{2}\ln\frac{1-p_2}{1-p_1}$. Then, based on Markov's inequality, we have

$$p_g = \mathbb{P}\left[\sum_{k=1}^{K} X_k \geq 0\right] \leq \bar{q}^K. \tag{5.11}$$

where $\bar{q}$ is defined in Theorem 19.

On the other hand, since $Y_{1,i}$ is the summation of K i.i.d. geometric random variables with parameter p_1, for any $\alpha < 0$, we have

$$\mathbb{P}\left[Y_{1,i} \leq \frac{K}{2p_1}\right] \leq \frac{\mathbb{E}[\exp(\alpha Y_{1,i})]}{\exp\left(\frac{\alpha K}{2p_1}\right)} \tag{5.12}$$

$$= \left(\frac{p_1}{e^{-\alpha}-(1-p_1)} e^{-\frac{\alpha}{2p_1}}\right)^K \tag{5.13}$$

$$\leq \left(\frac{p_1\left(1-\alpha/(2p_1)+\alpha^2/(8p_1^2)\right)}{p_1-\alpha}\right)^K \tag{5.14}$$

By setting $\alpha = -p_1$, we have

$$\mathbb{P}\left[Y_{1,i} \leq \frac{K}{2p_1}\right] \leq \left(\frac{13}{16}\right)^K \tag{5.15}$$

Thus,

$$\mathbb{P}\left[\{Y_{1,i} < Y_{2,i}\} \cap \left\{Y_{1,i} > \frac{K}{2p_1}\right\}\right] \tag{5.16}$$

$$= 1 - \mathbb{P}\left[\{Y_{1,i} \geq Y_{2,i}\} \cup \left\{Y_{1,i} \leq \frac{K}{2p_1}\right\}\right] \tag{5.17}$$

$$\geq 1 - \mathbb{P}[Y_{1,i} \geq Y_{2,i}] - \mathbb{P}\left[Y_{1,i} \leq \frac{K}{2p_1}\right] \tag{5.18}$$

$$\geq 1 - \bar{q}^K - \left(\frac{13}{16}\right)^K \tag{5.19}$$

which implies that

$$\mathbb{E}[\min\{Y_{1,i}, Y_{2,i}\}] \geq \frac{K}{2p_1}\left(1 - \bar{q}^K - \left(\frac{13}{16}\right)^K\right) \tag{5.20}$$

Putting pieces together, we have

$$\Delta_2 \geq \frac{\mathbb{E}[D_j^2]}{2\mathbb{E}[S_j]} \geq \frac{(\mathbb{E}[D_j])^2}{2\mathbb{E}[S_j]} \tag{5.21}$$

$$= \frac{\mathbb{E}[N_1] \cdot (\mathbb{E}[\min\{Y_{1,i}, Y_{2,i}\}])^2}{2\mathbb{E}[Y_{1,i}]} \tag{5.22}$$

$$\geq \frac{1}{\bar{q}^K}\left(\frac{K}{2p_1}\right)^2\left(1 - \bar{q}^K - \left(\frac{13}{16}\right)^K\right)^2 \frac{p_1}{2K} \tag{5.23}$$

$$= \Omega(K(1/\bar{q})^K) \tag{5.24}$$

■

5.4 Adaptive Coding Scheme

Next, we present a novel adaptive coding scheme to strictly improve the greedy scheme. Our intuition is that by adaptively combining information symbols from different updates, the AoI of user 1 won't be affected while the AoI at user 2 will be significantly reduced.

The adaptive coding and updating scheme works in cycles, where each cycle begins with a phase 1, possibly followed by a phase 2. We use $\mathbf{w} := \{w_1, w_2\}$ to indicate the updates that the users intend to decode at current time slot. If $w_1 = w_2$, they aim to decode the same update, and the system works in phase 1; otherwise, the system operates in phase 2. Initially, we set $\mathbf{w} = (\mathbf{u}_1, \mathbf{u}_1)$ at $t = 1$. We also use $k_i, i = 1, 2$ to track the total number of random linear equations involving w_i that have been received by user i. let $\mathbf{v}_t := (v_{1,t}, v_{2,t})$ be the transmission status in time slot t. If $v_{i,t} = 1$, it indicates the symbol broadcast at time t has been successfully received at user i; otherwise, it is erased. At the end of each time slot, the encoder will update $\mathbf{w}$ and k_1, k_2 based on the received feedback $\mathbf{v}_t$, and decide the coding strategy for $t + 1$.

The coding scheme is elaborated as follows.

- **Phase 1:** In Phase 1, the source adopts rateless codes to encode w_1 and transmits encoded symbols continuously until user 1 receives K encoded symbols and successfully decodes w_1 at the end of a time slot t. Then, w_1 will be reset to $\mathbf{u}_{t+1}$. Depending on whether user 2 has decoded w_2 at the end of t or not, there are two different scenarios.

(a) User 2 has decoded w_2 at time t. Then, w_2 will be reset to $\mathbf{u}_{t+1}$. The system enters phase 1 of the next coding cycle at $t+1$.

(b) User 2 has not decoded w_2 yet. The system then enters phase 2 at $t+1$.

We note that during phase 1, k_i will keep increasing according to $\mathbf{v}_{i,t}$ until it reaches K; it will then be reset to zero if w_i is changed to a new update.

- **Phase 2:** During phase 2, the source will broadcast two different types of symbols: 1) the (k_1+1)th uncoded information symbol of w_1, denoted as $w_1(k_1+1)$, or 2) a random linear combination of $w_1(k_1+1)$ and the symbols of w_2. Denote $c_t \in \{1,2\}$ as the type of symbols broadcast at time t. At the beginning of phase 2, $c_t = 1$. Then, the selection of c_{t+1}, as well as the updating of $\mathbf{w}$, k_1 and k_2, depends on c_t and $\mathbf{v}_t$, and is described as follows.

  (a) $c_t = 1$**.** First, we note that since the transmitted symbol is from w_1 only, k_2 will stay the same. We further divide this case into two subcases: 1) $v_{1,t} = 1$. We increase k_1 by one, and then compare it with K. If it equals K, it indicates that update w_1 is delivered to user 1. We will then update w_1 to the new update generated at next time slot, i.e., $\mathbf{u}_{t+1}$, and reset k_1 to 0. At the same time, if $k_2 = K$, we will update w_2 to $\mathbf{u}_{t+1}$, and the system enters phase 1 of the new updating cycle; otherwise, the system stays in phase 2. If $k_1 < K$, we keep w_1, w_2 unchanged, and let $c_{t+1} = 1$. 2) $v_{1,t} = 0$. Then, k_1, w_1, w_2 will stay the same. If $v_{2,t} = 1$, set $c_{t+1} = 2$; otherwise, $c_{t+1} = 1$.

  (b) $c_t = 2$**.** For user i, if $v_{i,t} = 1$, we will increase k_i by one, and then compare with K. We consider the following subcases: 1) $v_{1,t} = 1$. Similar to the case $c_t = 1$, if $k_1 = k_2 = K$, we update w_1 and w_2 to $\mathbf{u}_{t+1}$, and the system enters phase 1 of the new updating cycle; Otherwise, it stays in phase 2 of the current cycle with $c_{t+1} = 1$. If $k_1 = K$, we update w_1 to $\mathbf{u}_{t+1}$, reset $k_1 = 0$. 2) $v_{1,t} = 0$. We keep w_1, w_2 the same, and let $c_{t+1} = 2$.

The procedure is summarized in Algorithm 2. The adaptive coding scheme has two important features: First, we note that the source transmits three types of symbols: a

Algorithm 2 Adaptive Coding and Updating

```
Initialization: t = 1, w = (u_1, u_1), k_1 = k_2 = 0.
while t do
    if w_1 = w_2 then                                   ▷ Phase 1
        Send an encoded symbol of w_1 and receive v_t;
        k_i = min{k_i + v_{i,t}, K}, i = 1, 2;
        if k_1 = K then
            w_1 = u_{t+1}, k_1 = 0;
            if k_2 = K then
                w_2 = u_{t+1}, k_2 = 0;
            else
                c_{t+1} = 1;
    else                                                ▷ Phase 2
        if c_t = 1 then                                 ▷ Uncoded symbol
            Send w_1(k_1 + 1) and receive v_t;
            k_1 = k_1 + v_{i,t};
            if k_1 = K then
                w_1 = u_{t+1}, k_1 = 0;
                if k_2 = K then
                    w_2 = u_{t+1}, k_2 = 0;
            if v_{1,t} = 1 or v = (0, 0) then
                c_{t+1} = 1;
            else
                c_{t+1} = 2.
        else                                            ▷ Inter-update coding
            Encode w_1(k_1 + 1) and w_2 and transmit.
            k_i = min{k_i + v_{i,t}, K}, i = 1, 2;
            if k_1 = K then
                w_1 = u_{t+1}, k_1 = 0;
                if k_2 = K then
                    w_2 = u_{t+1}, k_2 = 0;
            if v_{1,t} = 1 then
                c_{t+1} = 1;
            else
                c_{t+1} = 2;
    t = t + 1;
```

coded symbol of w_1 (which equals w_2) in phase 1, an uncoded symbol $w_1(k_1+1)$, or a mixture of $w_1(k_1+1)$ and w_2 in phase 2. Since w_2 is already decoded by user 1 at the end of phase 1, once user 1 successfully receives any type of such symbols, it accumulates one more novel equation regarding w_1. Besides, once it successfully accumulates K equations of w_1 and decodes it, the source will switch to a new update immediately. Thus, the adaptive coding scheme works in a greedy fashion for user 1.

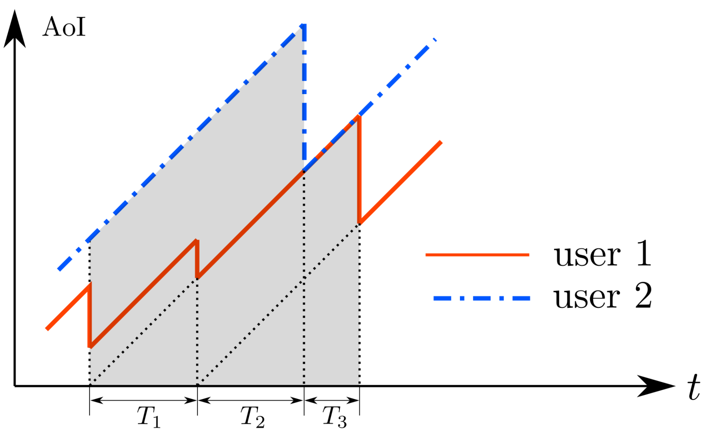


Figure 5.1: An illustration of the updating cycle.

Second, in each updating cycle, user 2 only decodes one update w_2. It leverages the diversity of channel conditions to accumulate novel information regarding w_2 in phase 2. Specifically, in phase 2, the source would only broadcast a random mixture of $w_1(k_1+1)$ and w_2 after user 2 successfully receives $w_1(k_1+1)$ and user 1 has not received it yet. Thus, once user 2 receives such an encoded symbol, it can stripe $w_1(k_1+1)$ away from the mixture, and obtain another novel equation regarding w_2. By judiciously selecting the broadcast symbols, we ensure that the information received by user 2 does not involve too many unknown variables, avoiding unnecessary decoding delay.

5.5 Analysis of the AoI at both Users

First, we note that under the adaptive coding scheme, the source always broadcasts a new equation involving the update that user 1 demands in each time slot. Therefore, the AoI of user 1 evolves exactly the same as under the greedy scheme. Thus, the long-term average AoI of user 1 remains unchanged, which is equal to $\Delta_1 = \frac{K}{p_1}\left(\frac{3}{2} + \frac{1-p}{K}\right)$.

Next, we will analyze the AoI of user 2 under the adaptive coding scheme. We note that the resulted updating cycles form a renewal process, where each renewal interval begins when both users demand the same updates (i.e., $w_1 = w_2$), and ends when user 1 successfully decodes an update after user 2 decodes w_2. As shown in Fig. 5.1, we further decompose each renewal interval into three different stages: phase 1, phase 2a, which begins when the system enters phase 2 and ends when user 2 decodes w_2, and phase 2b, the duration user 1 takes to complete the current update w_1 after user 2 decodes w_2. We denote the lengths of those stages as T_1, T_2 and T_3, respectively. In the following, we will analyze each of them individually, and then obtain an upper bound on Δ_2.

5.5.1 Analysis of T_1

Since T_1 follows a negative binomial distribution with parameter K, p_1, we have

$$\mathbb{E}[T_1] = \frac{K}{p_1}, \quad \mathbb{E}[T_1^2] = \frac{K^2}{p_1^2} + \frac{(1-p_1)K}{p_1}. \tag{5.25}$$

5.5.2 Analysis of T_2

Let $\bar{k}_2$ be the number of symbols delivered to user 2 during phase 1. Then, $\bar{k}_2 = \min\left\{\sum_{t=1}^{T_1} v_{2,t}, K\right\}$. The system will enter phase 2 if $\bar{k}_2 < K$.

In order to analyze T_2, we first introduce a Markov chain associated with the coding and updating process in phase 2, as shown in Fig. 5.2. The Markov chain has two states named "uncoded and "encoded", corresponding to the encoding decisions $c_t = 1$ and $c_t = 2$, respectively. The evolution of the Markov chain depends on the transmission results $\mathbf{v}_t$, similar to the coding scheme in phase 2.

In order to track the number of equations user 2 receives regarding w_2 in phase 2, we associate a reward with each transition of the Markov chain. The reward denotes the increment of k_2 after each transmission. As depicted in Section 5.4, k_2 will increase by one at the end of time slot t if $c_t = 2$ and $v_{2,t} = 1$. This is because x_t is a linear combination of $w_1(k_1+1)$ and w_2, and $w_1(k_1+1)$ has been successfully received by user 2 previously. Thus, after the successful transmission of x_t, user 2 can strip $w_1(k_1+1)$ away from x_t, and obtain a new linear equation about w_2. The transition probabilities and the associated rewards are shown in Fig. 5.2.

Phase 2a) begins at state "uncoded", and ends when user 2 accumulates K equations regarding w_2 and successfully decodes it. Although the Markovian structure admits a closed-form stationary distribution, the non-asymptotic analysis of T_2 is not straightforward. To make it tractable, we will consider a renewal reward process embedded in the Markov structure, and leverage tools such as stopping time theory to analyze it.

Define a renewal process where each renewal interval corresponds to the duration between two consecutive visits to state "uncoded" under the Markov chain. Let $\{Z_j\}_j$ be the lengths of the renewal intervals, and $\{W_j\}_j$ be the total rewards (i.e., total increments of k_2) over individual renewal intervals. Then, we have the following observations.

Proposition 1. *Let V_j be a geometric random variable with parameter p_1, and R_t be i.i.d. Bernoulli random variables with parameter p_2. Then, (Z_j, W_j) are i.i.d. random*

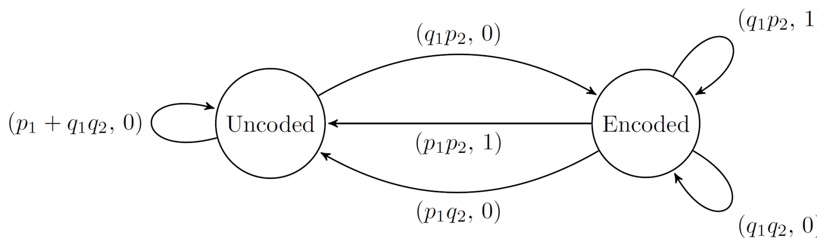


Figure 5.2: Associated Markov chain in phase 2.

pairs with

$$(Z_j, W_j) = \begin{cases} (1,0), & w.p. \quad p_1 + q_1q_2, \\ (1+V_j, \sum_{t=1}^{V_j} R_t) & w.p. \quad 1-(p_1+q_1q_2), \end{cases}$$

where $q_i := 1 - p_i$, $i = 1, 2$.

Proof: Under the Markov chain, $Z_j = 1$ if $v_{1,t} = 1$ or $\mathbf{v}_t = (0,0)$, which happens with probability $p_1 + q_1q_2$. Otherwise, the system enters state "encoded" and stays there until user 1 successfully receives an encoded symbol (i.e., $v_{1,t} = 1$). The total duration the system stays in state "encoded", denoted as V_j, is thus a geometric random variable with parameter p_1. The reward obtained over Z_j is thus equal to the total number of successful transmissions when the system stays in the state "encoded", which is the summation of V_j i.i.d. Bernoulli random variables with probability p_2. ■

Proposition 2. *T_2 is upper bounded by $\sum_{j=1}^{N_1} Z_j$, where N_1 is a stopping time determined by $N_1 = \min\left\{n \middle| \sum_{j=1}^{n} W_j \geq K\right\}$.*

Proof: Under the original coding and updating process, when the system enters phase 2a, we have $0 \leq \bar{k}_2 < K$. Phase 2a ends as soon as the cumulative number of equations received by user 2 regarding w_2 reaches K. Thus, T_2 will be upper bounded by $\sum_{j=1}^{N_1} Z_j$ under each sample path. Since N_1 only depends on observed W_js, it is a stopping time. ■

We point out that T_2 essentially depends on phase 1 through $\bar{k}_2$. The upper bound of T_2 removes such dependency by relaxing $\bar{k}_2$ to zero.

Lemma 31. *The first and second moments of Z_j equal*

$$\mathbb{E}[Z_j] = 1 + \frac{q_1p_2}{p_1}, \quad \mathbb{E}[Z_j^2] = 1 - p_2 + \frac{(2-p_1)p_2}{p_1^2}.$$

Proof: Note that the first and second moments of Z_j is independent on j, and we denote the interval as Z, the duration between consecutive visits to state "uncoded" under the Markov chain (Fig 5.2). Let F be the duration from visit from state "encoded" state "uncoded". Then, we have

$$\mathbb{E}[Z] = (1 - q_1 p_2) + q_1 p_2 (1 + \mathbb{E}[F]),$$
$$\mathbb{E}[F] = p_1 + q_1 (1 + \mathbb{E}[F]),$$
$$\mathbb{E}[Z^2] = (1 - q_1 p_2) + q_1 p_2 \mathbb{E}[(1 + F)^2],$$
$$\mathbb{E}[F^2] = p_1 + q_1 \mathbb{E}[(1 + F)^2].$$

Solving the above equations, we have

$$\mathbb{E}[Z] = 1 + \frac{q_1 p_2}{p_1},$$
$$\mathbb{E}[Z^2] = 1 - p_2 + \frac{(2 - p_1) p_2}{p_1^2},$$
$$\mathbb{E}[F] = \frac{1}{p_1},$$
$$\mathbb{E}[F^2] = \frac{2 - p_1}{p_1^2}.$$

■

Next, we will derive proper bounds for $\mathbb{E}[T_2]$ and $\mathbb{E}[T_2^2]$. In order to simplify the analysis, we consider *capped* rewards $\bar{W}_j$ instead of W_j. Specifically, let $\bar{W}_j = \min\{W_j, 1\}$. and define another stopping time $\bar{N}_1$ as $\bar{N}_1 = \min\left\{n \middle| \sum_{j=1}^{n} \bar{W}_j = K\right\}$. We have the following observation.

Lemma 32. *The capped reward process* $\{\bar{W}_j\}$ *is an i.i.d. Bernoulli process with parameter* $r := p_1 + q_1 q_2 + \frac{p_1 p_2 q_1 q_2}{1 - q_1 q_2}$. *Besides,*

$$\mathbb{E}[\bar{N}_1] = \frac{K}{1 - r}, \quad \mathbb{E}[\bar{N}_1^2] = \frac{K(K + r)}{(1 - r)^2}.$$

Proof: Based on the definitions of W_j and $\bar{W}_j$, we have

$$\mathbb{P}[\bar{W}_j = 0] = \mathbb{P}[W_j = 0]$$
$$= p_1 + q_1 q_2 + q_1 p_2 \sum_{l=0}^{\infty} (q_1 q_2)^l p_1 q_2 := r.$$

Since $\bar{N}_1$ is essentially a negative binomial random variable with parameters K and r, its first and second moments can thus be easily derived. ■

Before we proceed to bound the first and second moments of $\sum_{j=1}^{\bar{N}_1} Z_j$, we introduce the following lemma.

Lemma 33 (Sharp Moment Inequality from [72]). *Let $\{X_t\}$ be a sequence of independent non-negative random variables, τ be a stopping time, and τ' be a copy of τ independent with $\{X_t\}$. Then,*

$$\mathbb{E}\left(\sum_{i=1}^{\tau} X_i\right)^p \leq 2^{p-1}\mathbb{E}\left(\sum_{i=1}^{\tau'} X_i\right)^p, \quad 1 \leq p < \infty.$$

Lemma 33 enables us to decouple the dependency between $\bar{N}_1$ and $\{Z_j\}$ to obtain the corresponding upper bounds as follows.

Lemma 34. *Based on the definitions of Z_j, $\bar{N}_1$, we have*

$$\begin{aligned}\mathbb{E}\left[\sum_{j=1}^{\bar{N}_1} Z_j\right] &= \frac{K}{1-r}\left(1+\frac{q_1 p_2}{p_1}\right),\\ \mathbb{E}\left(\sum_{j=1}^{\bar{N}_1} Z_j\right)^2 &\leq \frac{2K(K+2r-1)}{(1-r)^2}\left(1+\frac{q_1 p_2}{p_1}\right)^2\\ &\quad + \frac{2K}{1-r}\left(1-p_2+\frac{(2-p_1)p_2}{p_1^2}\right).\end{aligned}$$

The first equality in Lemma 34 can be proved based on Wald's identity. The bound on the second moment is based on Lemma 33 by setting $p = 2$, and the results from Lemma 31 and Lemma 32.

Remark: Since $\bar{W}_j$ is a capped version of W_j, $\bar{N}_1 \geq N_1$ under every sample path. Thus, the results in Lemma 34 serve as upper bounds for $\mathbb{E}\left[\sum_{j=1}^{N_1} Z_j\right]$ and $\mathbb{E}\left(\sum_{j=1}^{\bar{N}_1} Z_j\right)^2$, which also upper bound $\mathbb{E}[T_1]$ and $\mathbb{E}[T_1^2]$ according to Proposition 2. Therefore, we have $\mathbb{E}[T_1] = O(K)$ and $\mathbb{E}[T_1^2] = O(K^2)$.

5.5.3 Analysis of T_3

Since T_3 is the remaining time that user 1 takes to complete current update after user 2 decodes w_2, the remaining symbol user 1 demand is upper bounded by K. Therefore, T_3

is bounded by the summation of K i.i.d. geometric random variables with parameter p_1. $\mathbb{E}[T_3]$ and $\mathbb{E}[T_3^2]$ can thus be bounded in the same way as T_1.

5.5.4 Bound Δ_2

As illustrated in Fig. 5.1, under the adaptive coding scheme, we have

$$\Delta_2 \leq \mathbb{E}[T_1 + T_2 + T_3] + \frac{\mathbb{E}[(T_1 + T_2 + T_3)^2]}{2\mathbb{E}[T_1 + T_2 + T_3]}.$$

Combining the results on T_1, T_2 and T_3, we obtain the following theorem.

Theorem 20. *Under the adaptive coding scheme, we have* $\Delta_1 = \frac{K}{p_1}\left(\frac{3}{2} + \frac{1-p}{K}\right)$, $\Delta_2 = O(K)$.

Theorem 20 indicates that, compared with the greedy scheme in Sec. 5.3, the adaptive coding scheme improves the AoI at user 2 from $\Omega((1/q)^K K)$ to $O(K)$ without affecting the AoI at user 1.

5.6 Numerical Results

In this section, we evaluate the proposed coding schemes through simulation. We plot the sample average of AoI over 50 sample paths.

First, we fix $K = 10$, $p_2 = 0.2$ and vary $p_1 \in (0.2, 0.5]$. The corresponding AoI under the greedy scheme and the adaptive coding scheme is depicted in Fig. 5.3. We note that the AoI at user 1 under the adaptive coding scheme matches that under the greedy scheme. Besides, the average AoI at user 2 increases in p_1 in a super-linear fashion under the greedy scheme, while it only increases approximately linearly under the adaptive coding scheme. This indicates that the coding gain is more prominent when the channel qualities of both users differ more significantly.

Next, we fix $p_1 = 0.7$ and $p_2 = 0.4$ and evaluate the sample average AoI with different sizes of the update K. We observe similar results in Fig. 5.4. We note that the AoI at user 2 increases super-linearly in K under the greedy scheme, and only scales linearly in K under the adaptive coding scheme, which corroborates the theoretical results in Theorem 19 and Theorem 20. We also observe that the greedy scheme outperforms the adaptive coding scheme when K is small. This is bacause for small K, the probability that user 2 decodes an update earlier than user 1 is considerable. Thus, greedily switching

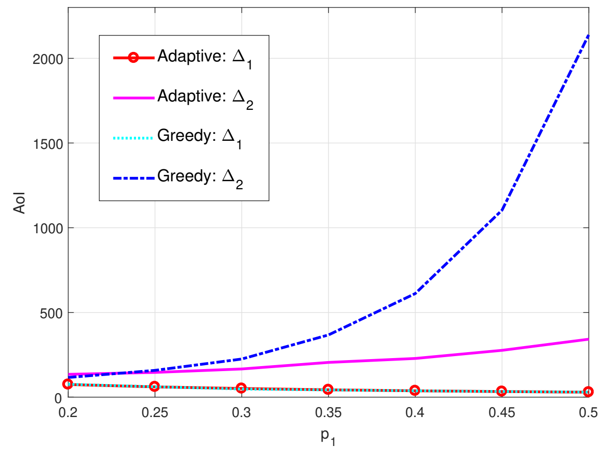


Figure 5.3: AoI as a function of p_1 when $p_2 = 0.2, K = 10$.

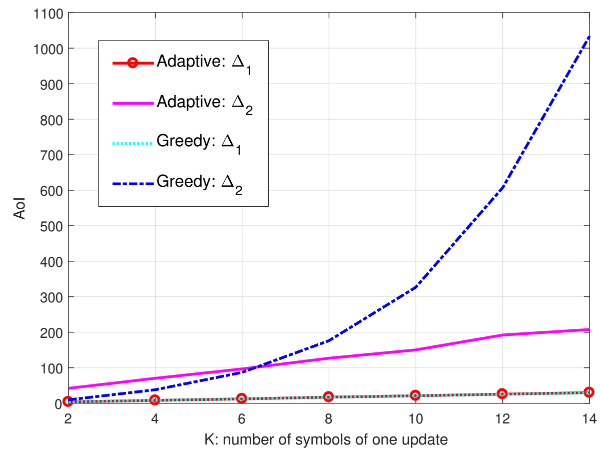


Figure 5.4: AoI as a function of K when $p_1 = 0.7, p_2 = 0.4$.

to a new update if user 1 decodes an update earlier than user 2 may even lead to earlier decoding of the new update and lower AoI at user 2.

Chapter 6 Information Freshness for Timely Detection of Status Changes

6.1 Introduction

Even though AoI can be used as a measure of "staleness" of the status information in a system, it does not take the dynamics of the underlying status evolution into consideration. Recently, there are a few attempts to take the properties of the monitored signal into the definition of information freshness. A metric called "Age of Synchronization" (AoS) is proposed in [73]. It refers to the duration since the destination became *desynchronized* with the source. In the same spirit, another metric called "Age of Incorrect Information" (AoII) is proposed in [74]. AoII takes both the time that the monitor is unaware of the correct status of the system and the difference between the current estimate at the monitor and the actual state of system into the definition. With particular penalty functions, AoII reduces to AoS. Reference [75] proposes to use the mutual information between the real-time source value and the delivered samples at the receiver to quantify the freshness of the information contained in the delivered samples. It shows that for a time-homogeneous Markov chain, the mutual information can be expressed as a non-negative and non-increasing function of the age. In [76], a metric called "Value of Information" (VoI) is introduced to facilitate packet scheduling for low-error Kalman filter based estimation. VoI depends on the age of the packet, as well as the mutual information between the packet content and the system status, which is equivalent to the variance of the noise associated with the measurement. Generally speaking, how to define a universal information freshness metric that accounts for dynamically evolving system states remains open.

In this chapter, we propose an information theoretic measure of information freshness by taking the dynamics of the monitored system into account, where we introduce a two-dimensional discrete-time Markov chain to model the underlying state changes.

6.1.1 Main Contributions

We summarize our main contributions in this chapter as follows.

First, the introduced information theoretic measure generalizes the definition of AoI. It takes the age of updates and the dynamics of the monitored system into the definition, and suggests a unified approach to define proper age penalty functions [77] for various dynamic systems.

Second, we establish fundamental relationship between AoI and the expected detection delay of status changes under any state-independent online updating policies. Such relationship validates the critical role of AoI in real-time status monitoring when the system evolution is stationary. It enables us to safely decouple the properties of the underlying system from the design of age-optimal sampling, scheduling and coding policies, without compromising the effectiveness of the delivered updates on tracking status changes of the system.

Third, we show that the generalized AoI is an affine function of the expected detection delay in an adapted Bayesian change point detection setting. This observation suggests that GAoI is a proper measure of information freshness even when the underlying system evolution is not stationary.

6.2 System Model

Consider a single-node status monitoring scenario where a sensor monitors the status of an underlying system and sends updates to a destination through a communication channel. In order to capture the dynamics of the monitored system, we consider a discrete-time model, where the status of the system in each time slot n is denoted as X_n. We assume X_n takes values from a *finite* alphabet $\mathcal{X}$. At the beginning of time slot n, X_n may stay the same as X_{n-1}, or change to a different value. We let $T_n \in \{0, 1, 2, \ldots\}$ denote the number of time slots the system has stayed in the current status. Then,

$$T_n = \begin{cases} T_{n-1} + 1, & \text{if } X_n = X_{n-1}, \\ 0, & \text{if } X_n \neq X_{n-1}. \end{cases}$$

We assume the evolution of $\{T_n\}$ depends on the current status X_n in general. Specifically, we denote $q_i(x) := \mathbb{P}[T_{n+1} = 0|T_n = i, X_n = x]$, and denote the corresponding transition matrix as $P_T(x)$. Besides, if $T_n = 0$, i.e., a status change happens at the beginning of time slot n, X_n evolves according to a Markov chain with transition matrix P_X, as illustrated in Fig. 6.1. Denote the status of the system at time n as $U_n := (X_n, T_n)$. We can show that U_n evolves according to a two-dimensional Markov chain. Essentially, such a model accommodates the scenario where $\{X_n\}$ is Markovian, as well as certain scenarios where $\{X_n\}$ cannot be simply modeled as a Markov chain, thus is more general in modeling dynamic status changes.

Assume each state update sampled at time n is a time-stamped tuple $U_n := (X_n, T_n)$, i.e., it does not just contain the time-stamped status X_n, but also the duration that the system has stayed in the current status. The receiver tracks the system status based on received updates. The objective of this work is to investigate the fundamental impact of information freshness on the timely detection of status changes. As a first step, we focus on state-independent online policies defined as follows.

Definition 15 (State-independent online updating policies)**.** *Let $s_1, s_2, \ldots \in \mathbb{N}$ be the sampling time points and $d_1, d_2, \ldots \in \mathbb{N}$ be the corresponding delivery times of the updates U_{s_1}, U_{s_2}, ... under an updating policy, where we let $d_i = \infty$ if the ith update is never delivered. If s_i only depends on previous sampling points $\{s_j\}_{j=1}^{i-1}$ and up-to-date delivery times $\{d_j : d_j \leq s_i\}$, and d_is are independent with $\{U_n\}$, the policy is called a state-independent online updating policy.*

We also formally define detection delay as follows.

Definition 16 (Detection delay of status changes)**.** *If the system status changes at the beginning of time slot n, i.e., $T_n = 0$, the detection delay of this status change equals* $\min\{d_i : s_i \geq T_n\} - T_n$.

The definition of detection delay is intuitive in the sense that a status change won't be detected until a status update collected after the change is delivered to the destination for the first time. From an information theoretic perspective, if multiple status changes happen between the sampling points of two consecutively delivered updates, as shown in Fig. 6.2, the destination won't be able to *locate* all of them except the last one based on $U_{\delta(n)}$ only. However, we still consider them to be *detected* in order to properly measure the detection delay performance of the updating schemes.

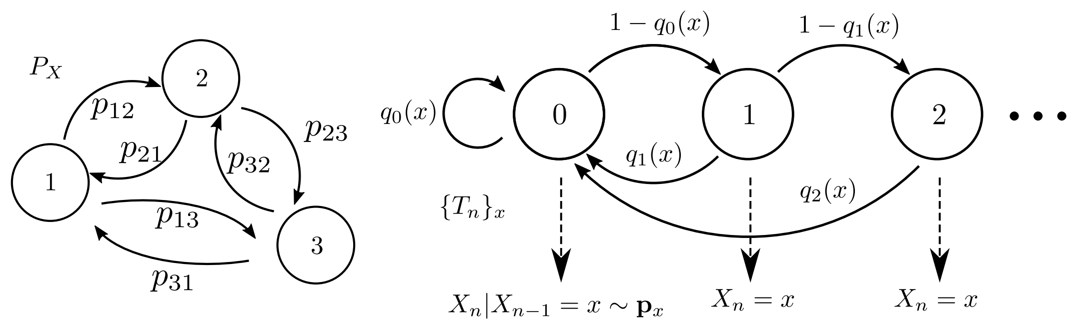


Figure 6.1: An example of the discrete-time state transition model where $\mathcal{X} = \{1, 2, 3\}$.

6.3 Information Theoretic Characterization of Information Freshness

Assume the latest status update at the destination at time n was sampled at time $\delta(n)$. Then, the instantaneous AoI at the destination is $n - \delta(n)$. We propose the following metric as an information staleness measure at the destination at time n:

$$\Phi(n) := H(U^{\delta(n)+1:n}|U_{\delta(n)}), \tag{6.1}$$

where $U^{n:n+k} := \{U_n, U_{n+1}, \dots, U_{n+k}\}$. Intuitively, $\Phi(n)$ captures the "novelty" in random process $\{U_n\}$ since time $\delta(n)$, which also indicates the uncertainty level at the destination regarding the monitored system status over duration $[\delta(n)+1, n]$. We term it as *generalized Age of Information* (GAoI) in this chapter.

The GAoI defined in (6.1) essentially measures information freshness from an ensemble's perspective, i.e., it averages over all possible realizations of $\{U_n\}$. Therefore it is *state-independent*, i.e., it is independent with the information content of the received updates (i.e., $U_{\delta(n)}$); Rather, it only relies on stochastic information of the state evolution process (i.e., $\{P_T(x)\}$ and P_X), as well as the time stamps of received updates.

Intuitively, the information contained in $U_{\delta(n)}$ may indicate how fast the system status may change from the previous status, and how much uncertainty/novelty is generated since the latest update was generated. This motivates the definition of *state-dependent* GAoI as follows.

Define

$$\Phi(n|U_{\delta(n)} = u) = H(U^{\delta(n)+1:n}|U_{\delta(n)} = u), \tag{6.2}$$

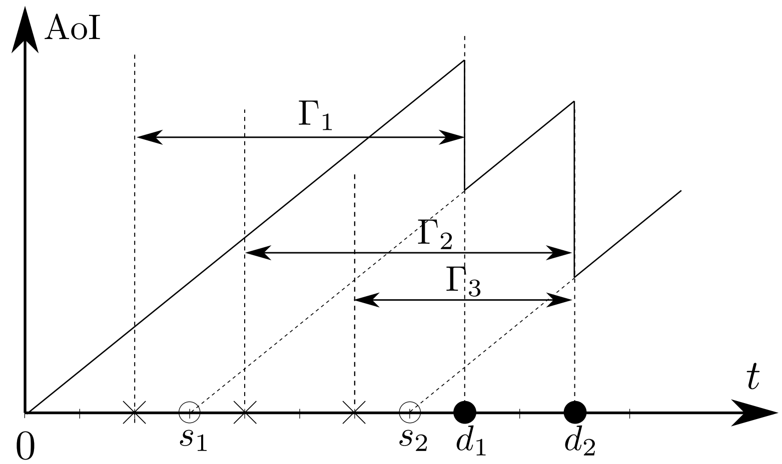


Figure 6.2: AoI evolution with given status change points. × represents status changes, ○ represents sampling points, and solid circle represents delivery time of an update. Γ_i is the detection delay for the ith status change.

i.e., the uncertainty in system states over $[\delta(n)+1, n]$ given the latest observation $U_{\delta(n)} = u$.

Then, based on the definition of conditional entropy, we have

$$\Phi(n) = \sum_{u} \Phi(n|U_{\delta(n)} = u) \cdot \mathbb{P}[U_{\delta(n)} = u]. \tag{6.3}$$

The discrete-time status evolution model enables us to explicitly discuss the relationship between GAoI, AoI and the detection delay of status changes. In the following, we investigate two scenarios, and establish their relationships.

6.4 Stationary Markovian State Evolution

In this section, we focus on stationary Markovian state evolution models.

Proposition 3. *Assume (X_n, T_n) evolves according to a stationary Markov chain $\mathcal{C}$ defined by transition matrices P_X and $\{P_T(x)\}_{x\in\mathcal{X}}$. Then,*

$$\Phi(n) = (n - \delta(n)) \cdot H(P_X, \{P_T(x)\}_{x\in\mathcal{X}}),$$

where $H(P_X, \{P_T(x)\}_{x\in\mathcal{X}})$ is the entropy rate of $\mathcal{C}$.

Proof. By the definition of generalized AoI, we have

$$\Phi(n) = H(U^{\delta(n)+1:n}|U_{\delta(n)})$$

$$
\begin{aligned}
&= \sum_{i=\delta(n)+1}^{n} H(U_i|U^{\delta(n):i-1}) \\
&= \sum_{i=\delta(n)+1}^{n} H(U_i|U_{i-1}) \quad (6.4)\\
&= (n-\delta(n)) \cdot H(P_X, \{P_T(x)\}_{x\in\mathcal{X}}) \quad (6.5)
\end{aligned}
$$

where Eqn. (6.5) follows from the fact that U_n evolves according to a two dimensional Markov chain characterized by $(P_X, \{P_T(x)\}_{x\in\mathcal{X}})$. □

Proposition 3 can be proved based on the property of stationary Markov chains. It indicates that the information freshness measure $\Phi(n)$ is proportional to the instantaneous age $n-\delta(n)$ if $H(P_X, \{P_T(x)\}_{x\in\mathcal{X}}) \neq 0$, which validates the effectiveness of AoI in capturing uncertainty in the system status from the destination's perspective.

Besides, compared with AoI which only depends on time, $\Phi(n)$ also captures the dynamics of the underlying status evolution. For systems with frequent state changes, or with very dynamic state transitions, $\Phi(n)$ grows quickly as age increases, which implies that the information gets stale quickly. On the other hand, in systems with more deterministic state changes, information will "age" at a slower rate.

One extreme case is when the status of the system evolves in a periodic pattern, i.e., given an initial status X_0, the system evolves according to a cycle over the states in $\mathcal{X}$. Thus, $\Phi(n) = 0$, i.e., $U_{\delta(n)}$ will never get expired, since it can be used to accurately predict the state of the system in any upcoming time. Therefore, AoI itself cannot be used for a measure of information freshness for this scenario. This special case illustrates the importance of taking the dynamics of status evolution into the definition of information freshness.

Proposition 4. *Assume the stationary distribution of* (X_n, T_n) *exists and denote it as* $\{\mu_{x,i}\}_{x\in\mathcal{X}, i\in\mathbb{Z}_+}$*, where* $\mu_{x,i} = \mathbb{P}[(X_n,T_n) = (x,i)]$*. Then,* $H(P_X, \{P_T(x)\}_{x\in\mathcal{X}})$ *equals*

$$
\sum_{x\in\mathcal{X}} \mu_{x,0} \sum_{i=0}^{\infty} \prod_{j=0}^{i-1} (1-q_j(x))[H(q_i(x), 1-q_i(x)) + q_i(x) H(\mathbf{p}_x)],
$$

where $\mathbf{p}_x$ *is the row associated with status* x *in* P_X*, and* $H(\pi)$ *is the entropy of distribution* π*.*

Corollary 4. *If* $P_T(x)$ *is homogeneous, i.e.,* $P_T(x) = P_T$ *for all state* $x \in \mathcal{X}$*, then, the* *entropy rate of the Markov chain $\mathcal{C}$ equals*

$$H(P_X, P_T) = H(P_T) + H(P_X)\mathbb{P}(T_n = 0). \tag{6.6}$$

Our main result is summarized as follows.

Theorem 21. *Assume the monitor is informed of the initial system state at time 0. Then, under any state-independent online updating policy,*

$$\frac{\bar{\Phi}(T)}{H(P_X, \{P_T(x)\})} = \bar{\Delta}(T) = \frac{\bar{\Gamma}(T)}{\mathbb{P}(T_n = 0)},$$

where $\bar{\Phi}(T) := \sum_{n=1}^{T} \Phi(n)$, $\bar{\Gamma}(T)$ is the expected total detection delay of the state changes over $[1, T]$, and $\bar{\Delta}(T)$ is the expected total AoI experienced at the destination over $[1, T]$.

Proof: Consider the sampling times and update delivery times under a state-independent online updating policy. For ease of exposition, let s_i and d_i be the ith sampling time and the corresponding delivery time. Consider the first T time slots $[1, T]$ during which K updates are delivered. We collectively denote the delivery times as $\{d_i\}_{i=1}^{K}$ and the corresponding sampling times as $\{s_i\}_{i=1}^{K}$. We also define $d_0 = s_0 = 0$ and $d_{K+1} = s_{K+1} = T$. Without loss of generality, we can assume $d_i < d_j$ whenever $s_i < s_j$. In fact, if there exists $s_j \geq s_i$ such that $d_j \leq d_i$, the information delivered at time d_i is stale compared to the previously received information U_{s_j}. Thus, we can exclude such updates without affecting the (G)AoI evolution or detection delay. Define the detection delay of a status change at time n ($n \leq T$) restricted to $[1, T]$ as $\Gamma = \min\{d_i : s_i \geq n, i \leq K+1\} - n$. The expected total detection delay over $[1, T]$ can be expressed as

$$\bar{\Gamma}(T) = \mathbb{P}(T_n = 0) \sum_{i=0}^{K} \sum_{j=s_i+1}^{s_{i+1}} (d_{i+1} - j), \tag{6.7}$$

where the equality is based on the fact that $\mathbb{P}(T_n = 0)$ is a constant for all n under the assumption that $\{U_n\}$ is a stationary Markov chain. Then, the expected total detection delay over $[1, T]$ scaled by $1/\mathbb{P}(T_n = 0)$ is

$$\frac{\bar{\Gamma}(T)}{\mathbb{P}(T_n = 0)} = \frac{1}{2}T^2 - \frac{1}{2}T - \sum_{i=1}^{K} s_i(d_{i+1} - d_i). \tag{6.8}$$

The expected AoI experienced over $[1, T]$ is

$$\bar{\Delta}(T) = \sum_{j=0}^{K+1} \sum_{i=0}^{d_{j+1}-d_j-1} (d_j - s_j + i) \tag{6.9}$$

$$= \frac{1}{2}T^2 - \frac{1}{2}T - \sum_{i=1}^{K} s_i(d_{i+1} - d_i). \tag{6.10}$$

Hence, $\bar{\Gamma}(T) = \bar{\Delta}(T) \cdot \mathbb{P}(T_n = 0)$. We note that this equation holds for any possible realizations of s_is and d_is under a state-independent online updating policy. Thus, it still holds if we take expectations with respect to s_is and d_is. The first equality then follows from Proposition 3. ■

Theorem 21 validates the fundamental role of information freshness on timely detection of status changes. It indicates that minimizing AoI through state-independent sampling, transmission scheduling or coding is equivalent to minimizing the expected detection delay of the status changes in the system.

6.5 Non-stationary Markovian State Evolution

The result in Section 6.4 relies on the stationary Markovian state evolution assumption. It is also intuitive that under the stationary Markovian assumption, $\Phi(n)$ scale proportionally to the instantaneous age at the destination. However, such assumptions are quite restrictive in practice. This motivates us to investigate a broader class of status change models where the stationary assumption may not hold. While this seems challenging for a general setting, as a first step, we consider the following status change model, which is adapted from the classical Bayesian change point detection model in the literature.

We assume the system status starts with an initial state 0 at $n = 0$. At an unknown change point $\theta \in \mathbb{N}$, the system status changes to 1. Under the Bayesian setting, it assumes that θ is a random variable following a geometric distribution, i.e., $\mathbb{P}[\theta = \tau] = p(1-p)^{\tau-1}$, $\tau = 1, 2, \ldots$.

We can show that the system status $\{X_n\}$ can be modeled as a Markov chain shown in Fig. 6.3. Since it has an absorbing state 1, and the initial state $X_0 = 0$, the Markov chain is no longer stationary.

Since the state evolution can be characterized by a one-dimensional Markov chain $\{X_n\}$, the information theoretic definitions of information freshness in Eqns. (6.1) and

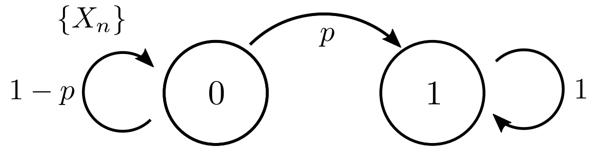


Figure 6.3: Equivalent Markov chain for the Bayesian change point model.

(6.2) are equivalent to the following definitions, respectively:

$$\Phi(n) = H(X^{\delta(n)+1:n} | X_{\delta(n)}), \tag{6.11}$$

$$\Phi(n | X_{\delta(n)} = x) = H(X^{\delta(n)+1:n} | X_{\delta(n)} = x). \tag{6.12}$$

We have the following observations.

Proposition 5. *Under the Bayesian change point model,*

$$\Phi(n | X_{\delta(n)} = 0) = 0,$$
$$\Phi(n | X_{\delta(n)} = 1) = H(p, \cdots, (1-p)^{n-\delta(n)-1} p, (1-p)^{n-\delta(n)}),$$

where $H(p_1, p_2, \cdots, p_n) = -\sum_i p_i \log p_i$.

Theorem 22. *Under any state-independent online updating policy,*

$$\bar{\Phi}(T) = \frac{H(p, 1-p)}{p} \bar{\Gamma}(T) + C(T),$$

where $\bar{\Phi}(T) := \sum_{n=1}^{T} \Phi(n)$, $\bar{\Gamma}(T)$ *is the expected detection delay of the state change over* $[1, T]$, *and* $C(T)$ *is a constant for fixed* T.

Proof: Denote $h(x)$ as

$$h(x) := H(p, \cdots, (1-p)^{x-1} p, (1-p)^x), \forall x \in \mathbb{N}. \tag{6.13}$$

Based on the definition of entropy, we have the following recursive formula:

$$h(x+1) = h(x) + (1-p)^x h(p). \tag{6.14}$$

Denote $h(1) := H(p, 1-p)$. Then,

$$h(x) = \frac{1 - (1-p)^x}{p} h(1). \tag{6.15}$$

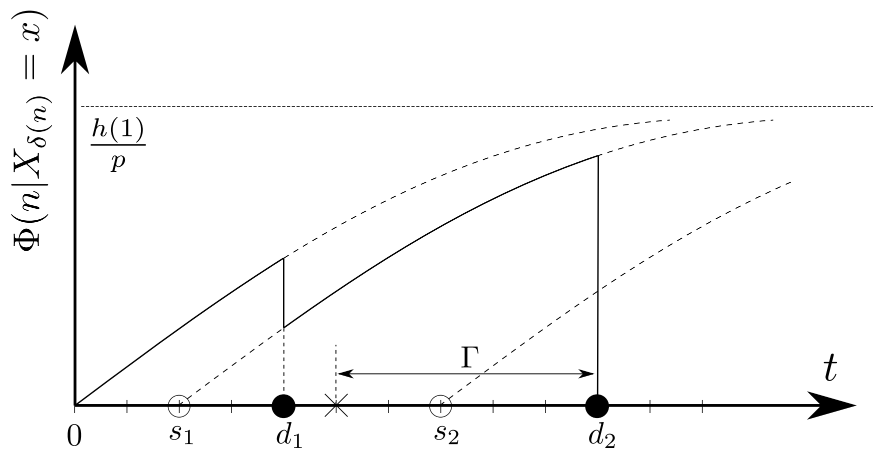


Figure 6.4: Generalized AoI evolution with given sampling times s_is and delivery times d_i. $\times$ represents the change point, and Γ represents the detection delay.

Plugging (6.15) into the expression in Proposition 5, we have

$$\Phi(n|X_{\delta(n)}=0)=\frac{1-(1-p)^{n-\delta(n)}}{p}h(1). \tag{6.16}$$

Then, a uniform expression for the state-dependent GAoI is as follows:

$$\begin{aligned}&\Phi(n|X_{\delta(n)}=x)\\&=\frac{1-(1-p)^{n-\delta(n)}}{p}h(1)\mathbf{1}_{X_{\delta(n)}=0},\quad x\in\{0,1\}.\end{aligned} \tag{6.17}$$

Similar to the proof of Theorem 21, we consider the sampling points $s_1, s_2, \ldots, s_K$ and corresponding delivery points $d_1, d_2, \ldots, d_K$ of the K updates delivered over $[1, T]$ under a given state-independent updating policy, as shown in Fig. 6.4. Without loss of generality, we assume $0 < s_1 < s_2 < \ldots < s_K < T$ and $0 < d_1 < d_2 < \ldots < d_K < T$. We define $s_0 = d_0 = 0$ and $s_{K+1} = d_{K+1} = T$ for ease of exposition.

The accumulative generalized GAoI can be expressed as

$$\sum_{n=1}^{T}\Phi(n|X_{\delta(n)}=x)=\sum_{i=0}^{K}\sum_{j=d_i+1}^{d_{i+1}}\frac{1-(1-p)^{j-s_i}}{p}h(1)\mathbf{1}_{X_{s_i}=0}.$$

Taking expectation with respect to x, we have

$$\begin{aligned}&\bar{\Phi}(T)\\&=\sum_{i=0}^{K}\sum_{j=d_i+1}^{d_{i+1}}\frac{1-(1-p)^{j-s_i}}{p}h(1)(1-p)^{s_i}\end{aligned}$$

$$= \frac{h(1)}{p}\left(\frac{(1-p)[(1-p)^T-1]}{p} + \sum_{i=0}^{K}(d_{i+1}-d_i)(1-p)^{s_i}\right). \tag{6.18}$$

On the other hand, the expected detection delay can be calculated as follows:

$$\begin{aligned}
&\bar{\Gamma}(T) \\
&= \sum_{i=0}^{K}\sum_{j=s_i+1}^{s_{i+1}}(1-p)^{j-1}p(d_{i+1}-j) && (6.19)\\
&= -\sum_{k=1}^{T}k(1-p)^{k-1}p + \sum_{i=0}^{K}\sum_{j=s_i+1}^{s_{i+1}}(1-p)^{j-1}pd_{i+1} && (6.20)\\
&= -\sum_{k=1}^{T}k(1-p)^{k-1}p + \sum_{i=0}^{K}d_{i+1}[(1-p)^{s_i}-(1-p)^{s_{i+1}}] \\
&= -T(1-p)^T - \sum_{k=1}^{T}k(1-p)^{k-1}p + \sum_{i=0}^{K}(d_{i+1}-d_i)(1-p)^{s_i}. && (6.21)
\end{aligned}$$

Since the updating policy is state-independent, taking expectation of (6.18) and (6.21) with respect to s_is, we have the proof complete. ■

Theorem 22 indicates that GAoI is an affine function of the expected detection delay for the Bayesian change point model under any state-independent updating policy. We can verify that such relationship no longer holds between the AoI and the expected detection delay. Such result suggests that GAoI is a more accommodating measure of information freshness compared with AoI.

6.6 Simulation Results

In this section, we evaluate our results through simulations. We evaluate the AoI, GAoI and the status change detection delay under two state evolution models: a stationary two-state symmetric Markov chain and a Bayesian change point model.

6.6.1 Stationary Two-state Symmetric Markov chain

First, we consider a two-state symmetric Markov chain where the probability of status change is p, as illustrated in Fig. 6.5. We adopt two different updating schemes: The first one is uniform sampling with instant delivery where the sampling occurs at $n = 50, 100, \ldots$. The other one is greedy sampling policy with random delivery time, where the sampling happens whenever an update is delivered to the destination. We assume the delivery

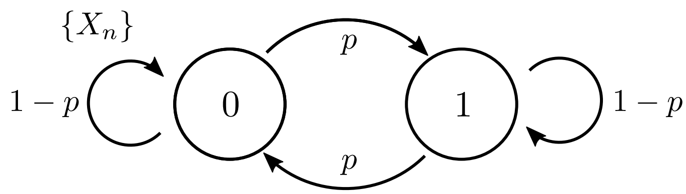


Figure 6.5: Generalized AoI evolution with given sampling points.

time of each update is uniformly distributed in $[20, 80]$. We note that the sampling rates of the two schemes are actually the same, and both policies are state-independent. In this simulation, we fix $p = 0.6$. For each updating policy, we generate 100 sample paths, where the initial state is randomly selected according to the stationary distribution. For each sample path, we track the status changes and obtain the total detection delay, which is then averaged over T. We also track the AoI evolution and calculate its time average. As shown in Fig. 6.6, after scaling AoI by a factor of $\mathbb{P}(T_n = 0)$, the ensemble average matches the ensemble average of the detection delay under both updating policies. This is consistent with the theoretical results in Theorem 21.

6.6.2 Bayesian Change Point Model

Next, we evaluate the relationship between the GAoI and the detection delay under a Bayesian change point model. We set $p = 0.04$ and track the detection delay and the evolution of the state-dependent GAoI for each sample path under the two updating policies described in Sec. 6.6.1. We fix the sampling rate for the uniform sampling as once every five time slots. For the other policy, we let the random delivery time be uniformly distributed in $[2, 8]$. We generate 2000 sample paths over duration $[1, 100]$. The ensemble average of the cumulative GAoI and the ensemble average of the detection delay are compared and plotted in Fig. 6.7. The GAoI curves are scaled by a factor of $\frac{p}{H(p,1-p)}$, as suggested in Theorem 22. It is noteworthy that the difference between the scaled GAoI and the detection delay is a constant under both updating policies, which corroborates the theoretical result in Theorem 22.

6.7 Conclusions

In this chapter, we introduce an information theoretic measure of information freshness and investigate its relationship between AoI and detection delay of status changes in a status monitoring system. Our results validates the fundamental role of AoI in timely change detection when the underlying system states evolve according to a stationary

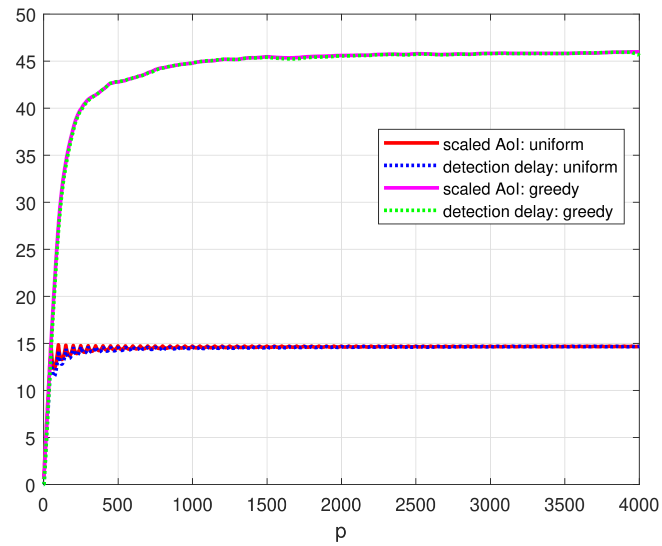


Figure 6.6: AoI and detection delay for a stationary Markovian status evolution model.

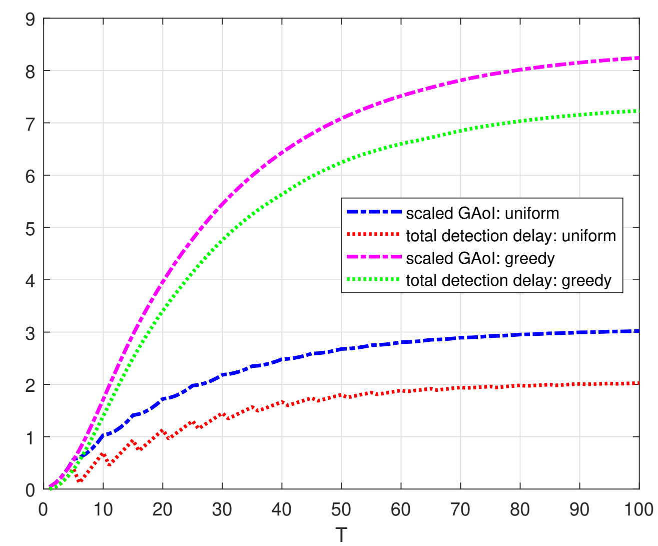


Figure 6.7: GAoI and detection delay for a Bayesian change point model.

Markov chain. It also indicates that for a special non-stationary status change model, GAoI is a more proper measure. In the future, we will investigate the relationships between those metrics under state-dependent updating policies.

Chapter 7
Summary and Outlook

7.1 Dissertation Summary

In this dissertation, we investigated Age of Information (AoI) in various status monitoring systems under practical system level constraints, and designed efficient sampling, coding and scheduling techniques to improve information freshness and optimize AoI. We leveraged a diverse corpus of tools from queueing theory, optimization theory, combinatorial analysis and information theory.

In Chapter 2, we studied the the optimal online scheduling policies for minimizing long-term average AoI in an energy harvesting sensor networks where the channel between the sensor and the destination is noisy. For both the no updating feedback scenario and the perfect feedback scenario, we proposed Best-effort Uniform updating (BU) policy and Best-effort Uniform updating with Retransmission (BUR) policy respectively and prove their optimality under a broad class of online policies.

In Chapter 3, we considered the impact of multiplexing gain on information freshness and bridged the gap between existing studies on MIMO broadcast channel and AoI. We investigated novel coding and scheduling schemes for the MIMO broadcast channel that allow simultaneously transmissions of multiple users and achieve optimal summed average AoI. Our results indicate that when the update size is greater than one, wasting some transmission opportunities in order to deliver fresher updates may be better.

In Chapter 4, we studied AoI optimization for a symbol erasure channel with rateless codes. We considered both the instant feedback scenario and the limited feedback scenario. Our results showcase the threshold structure of the optimal policies in both scenarios. Numerical results corroborate the structural properties of the optimal solution and indicates that the performance under the special type of threshold policy is very close to the optimal policy.

In Chapter 5, we investigated the impact of coding on the AoI in a two-user broadcast symbol erasure channel with feedback. We proposed a novel coding scheme to judiciously combine symbols from different updates together, and analyzed the AoI at both users. Compared with a baseline greedy scheme, the proposed adaptive coding scheme improves the AoI at the weak user by orders of magnitude without compromising the AoI at the strong user.

In Chapter 6, we introduced an information theoretic metric generalized Age of Information (GAoI) to measure the information freshness. Under stationary Markov chain scenario, we showed that any (G)AoI-optimal state-independent updating policy equivalently minimizes the corresponding expected change point detection delay, which validates the fundamental role of (G)AoI in real-time status monitoring. Under Bayesian change point detection, although AoI is no longer related to detection delay explicitly, we showed that the accumulative GAoI is still an affine function of the expected detection delay, which indicates the versatility of GAoI in capturing information freshness in dynamic systems.

7.2 Research Outlook

As a novel notion of timeliness, AoI leads to an immense wealth of possible problems in different areas. The importance of AoI as a performance metric is mainly due to the desire to maintain information as fresh as possible at the destination. As a result, minimizing AoI is related to the value of information and it is naturally related to information theoretic metrics.

In addition, there are plenty of applications where AoI may play a critical role for system performance, such as in caching and communication networks. One important and promising area is to investigate the AoI in dynamic control systems, especially in autonomous vehicles and control networks with multiple sources. In such systems, machine learning may be adopted a useful tool to implement actions based on the relative values of AoI of received signal.

7.2.1 Connection with Information Theory

It is of importance to connect Age of Information with information theoretic metrics such as entropy and mutual information. An interesting study is to investigate the impact of AoI on the predictability of a source. Depending on transmission schemes selected based on feedback or knowledge of the source and destination, the amount of transmitted data can be decreased without losing any information.

In general, the need to deliver updates in a timely manner prevents the use of long codewords. The application of coding strategies with low latency is critical. In unreliable wireless channels, long codewords can lead to lower failure probabilities but longer latency. An possible research direction is to design optimal codeword length for minimizing average AoI.

7.2.2 AoI and Caching

Caching has attracted increasing attention in content delivery systems. Due to the demanding online services, caching of messages has become important. In such scenarios, caching and content placement is an interesting area for future research. There have been several works on caching policies including minimizing cache misses by the request rates and popularity on content age [78], and updating dynamic content to minimize the average age of cached items [79].

7.2.3 AoI and Reinforcement Learning

Recently, there are few works starting to utilize reinforcement learning (RL) to minimize AoI. Formulating the AoI minimization problem as a constrained Markov Decision Process, [80] solved it by value iteration and SARSA algorithms on discrete state space in a point-to-point status updating system and [81] extended it to multi-user setting. In [82], a deep RL based algorithm was proposed to solve an AoI minimization problem in continuous state space.

In practice, it is difficult to obtain the statistics of network models as there are many interacting phenomena in the networked system. Learning algorithms, however, can be used to minimize the AoI with no prior assumptions of the network. Thus, evaluating existing RL methods is important to show the learning-based methods can be applied to keep the data fresh in networks.

Appendix A
Proofs for Chapter 2

A.1 Proof of Theorem 1

Define $S_i^T := \min\{S_i, T\}$, $l_n^T := \min\{l_n, T\}$, and $p_n := (1-p)^{n-1}p$. Then, under any $\pi \in \Pi_3$, the expected average AoI over $[0, T]$ can be expressed as

$$\mathbb{E}\left[\frac{R(T)}{T}\right] = \frac{1}{T}\mathbb{E}\left[\sum_{i=0}^{N(T)} \frac{(S_{i+1}^T - S_i)^2}{2}\right] \tag{A.1}$$

$$= \frac{1}{2T}\mathbb{E}\left[\sum_{n=1}^{M(T)} p_n l_n^2 + \left(1 - \sum_{n=1}^{M(T)} p_n\right) T^2 + \sum_{n=1}^{M(T)} \sum_{j=1}^{\infty} (l_{n+j}^T - l_n)^2 p p_j\right], \tag{A.2}$$

where the first two terms inside the expectation in (A.2) correspond to the AoI contribution over $[0, S_1^T]$, and the last term correspond to the AoI contribution over any other $[S_i, S_{i+1}^T]$. This can be explained as follows. With fixed updating epochs $\{l_n\}$, depending on the realization of the channel state, the interval $[0, T]$ can be decomposed into segments, separated by successful updates. The probability to have $[l_n, l_{n+j}^T]$, $1 \leq n \leq M(T), j \geq 1$, as one of such segment equals pp_j, which corresponds to the event that update at l_n succeeds, and the next successful update is at l_{n+j}. The corresponding AoI contribution over $[l_n, l_{n+j}^T]$ thus needs to be weighted by pp_j when the expected AoI is calculated. Since the AoI contribution over $[0, S_1^T]$ is always positive, in the following, we will drop it to obtain a lower bound, i.e.,

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{R(T)}{T}\right]$$

$$\geq \lim_{T\to\infty} \frac{1}{2T}\mathbb{E}\left[p\sum_{j=1}^{\infty} p_j \sum_{n=1}^{M(T)} (l_{n+j}^T - l_n)^2\right] \tag{A.3}$$

$$\geq \lim_{T\to\infty} \frac{1}{2T}\mathbb{E}\left[p\sum_{j=1}^{\infty} p_j \frac{1}{M(T)} \left(\sum_{n=1}^{M(T)} (l_{n+j}^T - l_n)\right)^2\right] \tag{A.4}$$

$$= \lim_{T\to\infty} \frac{1}{2T}\mathbb{E}\left[p\sum_{j=1}^{\infty} p_j \frac{1}{M(T)} \left(jT - \sum_{n=1}^{j} l_n^T\right)^2\right] \tag{A.5}$$

$$= \lim_{T\to\infty} \frac{1}{2}p\sum_{j=1}^{\infty} p_j j^2 \mathbb{E}\left[\frac{(T-\bar{l}_j^T)^2}{M(T)T}\right], \tag{A.6}$$

where (A.4) is based on a consequence of Jensen's inequality that $\frac{1}{n}\sum_{i=1}^n x_i^2 \geq \left(\frac{1}{n}\sum_{i=1}^n x_i\right)^2$ for any $x_i \in \mathbb{R}$ and (A.5) is obtained after rearranging the items in the summation in (A.4) and considering the cases $j \leq M(T)$ and $j > M(T)$ separately. After extracting a factor j^2 from the squared summation in (A.5) and pushing the factor $\frac{1}{T}$ and the expectation operator into the summation, we obtain (A.6), where $\bar{l}_j^T := \sum_{n=1}^j l_n^T/j$.

Since each term in the summation in (A.6) is positive, we can switch the order of limit and summation. We note that for any given j, $\mathbb{E}[\bar{l}_j^T] \leq \mathbb{E}[l_j] < \infty$ according to the definition of bounded policy. Besides, for any policy that renders a finite expected average AoI, we must have $\lim_{T\to\infty} M(T) = \infty$ almost surely according to Lemma 2. Therefore, according to the bounded convergence theorem [83], we have

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{\bar{l}_j^T}{M(T)}\right] = 0, \quad \lim_{T\to\infty} \mathbb{E}\left[\frac{(\bar{l}_j^T)^2}{M(T)T}\right] = 0. \tag{A.7}$$

Combining with (A.6), we have

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{R(T)}{T}\right] \geq \frac{1}{2}p\sum_{j=1}^{\infty} p_j j^2 \lim_{T\to\infty} \mathbb{E}\left[\frac{T}{M(T)}\right] \tag{A.8}$$

$$\geq \frac{1}{2}p\sum_{j=1}^{\infty} j^2(1-p)^{j-1}p = \frac{2-p}{2p}, \tag{A.9}$$

where the first inequality follows from Lemma 1.

A.2 Proof of Lemma 7

We first prove $\lim_{T\to\infty} \frac{\mathbb{E}\left[X_{N(T)+1}^2\right]}{T} = 0$.

Denote $F_n(t)$ as the cumulative distribution function of S_n under a uniform bounded policy, i.e., $F_n(t) = \mathbb{P}[S_n \leq t]$. Recall that $N(t)$ is the number of status updates successfully received at the destination over $(0, t]$. We have

$$\mathbb{E}[N(t)] = \sum_{n=0}^{\infty} F_n(t). \tag{A.10}$$

We note that

$$\begin{aligned}
&\mathbb{E}[X_{n+1}^2 \mathbf{1}_{S_{n+1}>T} | S_n = t] \\
&= \mathbb{E}[X_{n+1}^2 \mathbf{1}_{X_{n+1}>T-t} | S_n = t] && \text{(A.11)}\\
&\leq \mathbb{E}_k[g^2(k) \mathbf{1}_{g(k)>T-t} | S_n = t, K_{n+1} = k] && \text{(A.12)}\\
&= \mathbb{E}_k[g^2(k) \mathbf{1}_{g(k)>T-t} | K_{n+1} = k] && \text{(A.13)}\\
&:= G(T-t), && \text{(A.14)}
\end{aligned}$$

where (A.12) follows from the definition of uniformly bounded policy and (A.13) follows from the fact that $g(k)$ is independent of other parameters. We note that

$$\lim_{\Delta \to \infty} G(\Delta) = 0. \tag{A.15}$$

Besides,

$$\begin{aligned}
&\mathbb{E}[X_{N(T)+1}^2] \\
&= \sum_{n=0}^{\infty} \int_0^T \mathbb{E}[X_{n+1}^2 \mathbf{1}_{S_{n+1}>T} | S_n = t] dF_n(t) && \text{(A.16)}\\
&\leq \int_0^T G(T-t) d\left(\sum_{n=0}^{\infty} F_n(t)\right) && \text{(A.17)}\\
&= \int_0^T G(T-t) d\mathbb{E}[N(t)], && \text{(A.18)}
\end{aligned}$$

where (A.17) follows from (A.14), and (A.18) follows from (A.10).

For any fixed Δ satisfying $0 \leq \Delta \leq T$, we have

$$\begin{aligned}
&\frac{1}{T} \int_0^T G(T-t) d\mathbb{E}[N(t)] \\
&= \frac{1}{T} \int_0^{T-\Delta} G(T-t) d\mathbb{E}[N(t)] + \frac{1}{T} \int_{T-\Delta}^{T} G(T-t) d\mathbb{E}[N(t)] && \text{(A.19)}\\
&\leq G(\Delta) \frac{\mathbb{E}[N(T-\Delta)]}{T} + G(0) \frac{\mathbb{E}[N(T)] - \mathbb{E}[N(T-\Delta)]}{T}, && \text{(A.20)}
\end{aligned}$$

where (A.20) follows from that fact that $G(t)$ is a non-increasing function.

Recall that $M(t)$ is defined as the total number of attempted status updates over $(0,t]$, which is upper bounded by the total number of energy arrivals $A(t)+E_0$ due to the energy causality constraint. We observe that

$$\begin{aligned}
&\lim_{T\to\infty} G(\Delta)\frac{\mathbb{E}[N(T-\Delta)]}{T} \\
&= \lim_{T\to\infty} G(\Delta)\frac{\mathbb{E}[pM(T-\Delta)]}{T} && \text{(A.21)}\\
&\le \lim_{T\to\infty} G(\Delta)\frac{p\mathbb{E}[A(T-\Delta)+E_0]}{T} && \text{(A.22)}\\
&= \lim_{T\to\infty} G(\Delta)\frac{p(T-\Delta+E_0)}{T} = pG(\Delta). && \text{(A.23)}
\end{aligned}$$

Based on the definition of uniformly bounded policy in Definition 4, we have

$$\begin{aligned}
&\lim_{T\to\infty} \frac{\mathbb{E}[N(T)]-\mathbb{E}[N(T-\Delta)]}{T} \\
&= \lim_{T\to\infty} \frac{p\mathbb{E}[M(T)]-p\mathbb{E}[M(T-\Delta)]}{T} && \text{(A.24)}\\
&\le \lim_{T\to\infty} \frac{pC\Delta}{T} = 0. && \text{(A.25)}
\end{aligned}$$

Combining (A.20), (A.23) and (A.25), we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T G(T-t)d\mathbb{E}[N(T)] = pG(\Delta) \tag{A.26}$$

for any $\Delta \ge 0$. Therefore, by letting $\Delta\to\infty$ we have $\lim_{T\to\infty}\frac{\mathbb{E}\left[X^2_{N(T)+1}\right]}{T} = \lim_{\Delta\to\infty} pG(\Delta) = 0$, where the last equality follows from (A.15).

Since $\mathbb{E}^2[X_{N(T)+1}] \le \mathbb{E}[X^2_{N(T)+1}]$, we have $\lim_{T\to\infty}\frac{\mathbb{E}\left[X_{N(T)+1}\right]}{T} = 0$ as well.

A.3 Proof of Theorem 3

The proof is adapted from the proof of Theorem 3 in [41]. For completeness, we provide the detailed proof here.

We define

$$\hat{X}_{i+1}(k) := \mathbb{E}[X_{i+1}|i \le N(T), K_{i+1}=k] \tag{A.27}$$

$$= \frac{\mathbb{E}[X_{i+1}\mathbf{1}_{i\leq N(T)}|K_{i+1}=k]}{\mathbb{E}[\mathbf{1}_{i\leq N(T)}|K_{i+1}=k]} \tag{A.28}$$

$$= \frac{\mathbb{E}[X_{i+1}\mathbf{1}_{i\leq N(T)}|K_{i+1}=k]}{\mathbb{E}[\mathbf{1}_{i\leq N(T)}]}, \tag{A.29}$$

where the last equality follows from the fact that the two events $i \leq N(T)$ and $K_{i+1} = k$ are independent of each other under any online policy in Π_5.

Taking expectation on both sides of $(A.29)$ with respect to k, we have

$$\mathbb{E}_k[\hat{X}_{i+1}(k)] \cdot \mathbb{E}[\mathbf{1}_{i\leq N(T)}] = \mathbb{E}[X_{i+1}\mathbf{1}_{i\leq N(T)}]. \tag{A.30}$$

Meanwhile, we note that

$$\left(\hat{X}_{i+1}(k)\mathbb{E}[\mathbf{1}_{i\leq N(T)}]\right)^2 = \left(\mathbb{E}\left[X_{i+1}\mathbf{1}_{i\leq N(T)}\middle|K_{i+1}=k\right]\right)^2 \tag{A.31}$$

$$\leq \mathbb{E}\left[X_{i+1}^2\mathbf{1}_{i\leq N(T)}\middle|K_{i+1}=k\right]\mathbb{E}[\mathbf{1}_{i\leq N(T)}|K_{i+1}=k] \tag{A.32}$$

$$= \mathbb{E}\left[X_{i+1}^2\mathbf{1}_{i\leq N(T)}\middle|K_{i+1}=k\right]\mathbb{E}[\mathbf{1}_{i\leq N(T)}], \tag{A.33}$$

where (A.32) follows from the Cauchy-Schwartz inequality. Dividing both sides of (A.33) by $\mathbb{E}[\mathbf{1}_{i\leq N(T)}]$ and taking expectation with respect to k, we have

$$\mathbb{E}_k\left[\hat{X}_{i+1}^2(k)\right]\mathbb{E}[\mathbf{1}_{i\leq N(T)}] \leq \mathbb{E}\left[X_{i+1}^2\mathbf{1}_{i\leq N(T)}\right]. \tag{A.34}$$

Next, based on Lemma 7, we have

$$\lim_{T\to\infty}\frac{\mathbb{E}\left[R(T)\right]}{T} = \lim_{T\to\infty}\frac{\mathbb{E}\left[\sum_{i=1}^{N(T)+1}X_i^2\right]}{2T} \tag{A.35}$$

$$= \lim_{T\to\infty}\frac{\sum_{i=0}^{\infty}\mathbb{E}\left[X_{i+1}^2\mathbf{1}_{i\leq N(T)}\right]}{2T} \tag{A.36}$$

$$\geq \lim_{T\to\infty}\frac{\mathbb{E}\left[\sum_{i=0}^{\infty}X_{i+1}^2\mathbf{1}_{i\leq N(T)}\right]}{2\mathbb{E}[\sum_{i=0}^{\infty}X_{i+1}\mathbf{1}_{i\leq N(T)}]} \tag{A.37}$$

$$\geq \lim_{T\to\infty}\frac{\sum_{i=0}^{\infty}\mathbb{E}_k[\hat{X}_{i+1}^2(k)]\cdot\mathbb{E}[\mathbf{1}_{i\leq N(T)}]}{2\sum_{i=0}^{\infty}\mathbb{E}_k[\hat{X}_{i+1}(k)]\cdot\mathbb{E}[\mathbf{1}_{i\leq N(T)}]}, \tag{A.38}$$

where in (A.38) the first inequality follows from the fact that $T \leq \sum_{i=0}^{\infty}X_{i+1}\mathbf{1}_{i\leq N(T)}$ for every sample path, and the second inequality follows from (A.30) and (A.34).

Define

$$\rho_{i+1} := \frac{\mathbb{E}_k[\hat{X}_{i+1}(k)] \cdot \mathbb{E}[\mathbf{1}_{i \leq N(T)}]}{\sum_{i=0}^{\infty} \mathbb{E}_k[\hat{X}_{i+1}(k)] \cdot \mathbb{E}[\mathbf{1}_{i \leq N(T)}]}. \tag{A.39}$$

We note that $\{\rho_{i+1}\}_{i=0}^{\infty}$ is a valid distribution. Therefore, based on Cauchy-Schwartz inequality, we have

$$\begin{aligned} &\left(\sum_{i=0}^{\infty} \frac{\hat{X}_{i+1}^2(k)}{\mathbb{E}[\hat{X}_{i+1}(k)]}\rho_{i+1}\right)\left(\sum_{i=0}^{\infty} \mathbb{E}[\hat{X}_{i+1}(k)]\rho_{i+1}\right) \\ &\geq \left(\sum_{i=0}^{\infty} \hat{X}_{i+1}(k)\rho_{i+1}\right)^2 := \left(\bar{X}(k)\right)^2, \end{aligned} \tag{A.40}$$

where $\bar{X}(k) := \sum_{i=0}^{\infty} \hat{X}_{i+1}(k)\rho_{i+1}$. This is equivalent to

$$\sum_{i=0}^{\infty} \frac{\hat{X}_{i+1}^2(k)}{\mathbb{E}[\hat{X}_{i+1}(k)]}\rho_{i+1} \geq \frac{\left(\bar{X}(k)\right)^2}{\sum_{i=0}^{\infty} \mathbb{E}[\hat{X}_{i+1}(k)]\rho_{i+1}} = \frac{\left(\bar{X}(k)\right)^2}{\mathbb{E}[\bar{X}(k)]}. \tag{A.41}$$

We note that (A.38) equals $\lim_{T\to\infty} \sum_{i=0}^{\infty} \frac{\mathbb{E}_k[\hat{X}_{i+1}^2(k)]}{2\mathbb{E}_k[\hat{X}_{i+1}(k)]}\rho_{i+1}$, which is lower bounded by $\lim_{T\to\infty} \frac{\mathbb{E}[\bar{X}^2(k)]}{2\mathbb{E}[\bar{X}(k)]}$ according to (A.41).

Before we proceed to define the renewal policy, we will first show that $\bar{X}(k+1) - \bar{X}(k) \geq 0$ for $k = 1, 2, \ldots$. Consider the $(i+1)$st inter-update delay X_{i+1} where $i \leq N(T)$. Group all sample paths that share the same history up to the kth attempt together. Depending on whether the kth attempt is successful, we can further divide them into two subgroups. Then, all those who fail at the kth attempt will experience longer inter-update delay than those who succeed at the kth attempt. Since each attempt is successful with probability p independently, after taking expectation over all such sample paths we must have

$$\hat{X}_{i+1}(k+1) \geq \hat{X}_{i+1}(k). \tag{A.42}$$

From (A.42) and the definition of $\bar{X}(k)$, we then have $\bar{X}(k+1) - \bar{X}(k) \geq 0$ for $k = 1, 2, \ldots$.

Then, we define the a renewal policy as follows: Starting at $t = 0$, the sensor will first update at time $\bar{X}(1)$ and observe the feedback. If the update is successful, the sensor will wait for $\bar{X}(0)$ and update again; Otherwise, it will update again after waiting for $\bar{X}(2) - \bar{X}(1)$. The process continues after waiting for $\bar{X}(k+1) - \bar{X}(k)$, where $k - 1$ is the number of *failed* updated since the last successful update.

Define $q_i := \frac{\mathbb{E}[\mathbf{1}_{i \leq N(T)}]}{\mathbb{E}[N(T)+1]}$. We note that $\sum_{i=0}^{\infty} q_i = 1$, thus $\{q_i\}_{i=0}^{\infty}$ is a valid distribution.

Based on the definitions of $\bar{X}(k)$ and ρ_{i+1}, we have

$$\begin{aligned}
&\mathbb{E}_k[\bar{X}(k)] \\
&= \mathbb{E}_k\left[\sum_{i=0}^{\infty} \hat{X}_{i+1}(k)\frac{\mathbb{E}_k[\hat{X}_{i+1}(k)]\cdot\mathbb{E}[\mathbf{1}_{i\le N(T)}]}{\sum_{i=0}^{\infty}\mathbb{E}_k[\hat{X}_{i+1}(k)]\cdot\mathbb{E}[\mathbf{1}_{i\le N(T)}]}\right] \qquad \text{(A.43)}\\
&= \frac{\sum_{i=0}^{\infty}\mathbb{E}_k^2[\hat{X}_{i+1}(k)]\cdot\mathbb{E}[\mathbf{1}_{i\le N(T)}]}{\sum_{i=0}^{\infty}\mathbb{E}_k[\hat{X}_{i+1}(k)]\cdot\mathbb{E}[\mathbf{1}_{i\le N(T)}]} \qquad \text{(A.44)}\\
&= \frac{\sum_{i=0}^{\infty} q_i\mathbb{E}^2[\hat{X}_{i+1}]}{\sum_{i=0}^{\infty} q_i\mathbb{E}[\hat{X}_{i+1}]} \ge \frac{\left(\sum_{i=0}^{\infty} q_i\mathbb{E}[\hat{X}_{i+1}]\right)^2}{\sum_{i=0}^{\infty} q_i\mathbb{E}[\hat{X}_{i+1}]} \qquad \text{(A.45)}\\
&= \sum_{i=0}^{\infty} q_i\mathbb{E}[\hat{X}_{i+1}] = \sum_{i=0}^{\infty}\mathbb{E}[\hat{X}_{i+1}]\frac{\mathbb{E}[\mathbf{1}_{i\le N(T)}]}{\mathbb{E}[N(T)+1]} \qquad \text{(A.46)}\\
&= \frac{\sum_{i=0}^{\infty}\mathbb{E}[X_{i+1}\mathbf{1}_{i\le N(T)}]}{\mathbb{E}[N(T)+1]} \ge \frac{T}{\mathbb{E}[N(T)+1]} \qquad \text{(A.47)}
\end{aligned}$$

where (A.45) follows from Jensen's inequality and (A.47) follows from (A.30).

Let $\bar{N}(T)$ denote the number of completed renewal intervals under policy $\{\bar{X}(k)\}$ by time T. Then, according to the elementary renewal theorem [59],

$$\lim_{T\to\infty}\frac{\mathbb{E}[\bar{N}(T)]}{T} = \lim_{T\to\infty}\frac{1}{\mathbb{E}_k[\bar{X}(k)]} \le \lim_{T\to\infty}\frac{\mathbb{E}[N(T)+1]}{T} \le p.$$

Therefore, for any $\pi \in \Pi_5$, we can always construct a renewal policy that is also in Π_5, and achieves a shorter long-term average AoI.

Appendix B
Proofs for Chapter 3

B.1 Proof of Theorem 13

First, we point it out that for the $(2, M, N, B)$ systems, under any policy in Π_0, at any time t, there exist at most two updates that are partially transmitted in any time slot. This is due to property iv) in Definition 12, i.e., the transmitter will not start transmitting a new update to a user until the previous one has been delivered to the same user. Therefore, at any time t, there exists at most one partially transmitted update for each user.

Next, for $(2, M, N, B)$ systems with $B > M$, we have the following observation.

Lemma 35. *For the $(2, M, N, B)$ system with $B > M$, consider two consecutive successful deliveries of updates from the transmitter under the optimal policy in Π_0. Denote their delivery times as d_1, d_2, respectively, $d_1 \leq d_2$, and the corresponding generation times as t_1, t_2. With a little abuse of notation, we name those two updates $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively. Then, either of the following two scenarios must be true: 1) $t_1 < d_1 \leq t_2 < d_2$. 2) $t_2 < t_1 < d_1 \leq d_2$, and over time $t_1 \leq t < d_1$, the transmitter utilizes all of its DoF to transmit $\mathbf{w}_1$.*

Proof. Recall that for all policies in Π_0, the transmitter only uses its DoF to deliver updates that eventually reset the AoI. In the following, we show that any policy $\pi \in \Pi_0$ that violates the structures can be strictly improved to reduce the AoI. We consider the following cases:

i) $t_1 \leq t_2 < d_1 \leq d_2$. Recall that for any policy in Π_0, all delivered symbols are transmitted starting from their generation times. Thus, the transmitter must begin to transmit $\mathbf{w}_2$ at time t_2. We then consider an alternative policy under which the transmitter will utilize the DoF that was allocated to transmit $\mathbf{w}_2$ under the original policy at time slot t_2 and afterwards for $\mathbf{w}_1$ until $\mathbf{w}_1$ is delivered. Apparently, $\mathbf{w}_1$ will be delivered no later than d_1, which potentially improves the AoI of the corresponding user. After that, the transmitter will reallocate the DoF that was allocated for $\mathbf{w}_1$ and $\mathbf{w}_2$ to transmit a new update $\mathbf{w}'_2$, where $\mathbf{w}'_2$ and $\mathbf{w}_2$ are intended for the same user. Since the total allocated DoF remains the same under both policies, it ensures that $\mathbf{w}'_2$ will be delivered at t_2, which will reset the corresponding AoI with a smaller age. Thus, the overall AoI will be strictly improved under the alternative policy, indicating that this case cannot exist under the optimal policy.

ii) $t_2 < t_1 < d_1 \leq d_2$, and there exists at least one time slot t, $t_1 \leq t < d_1$, during which the transmitter does not exhaust its DoF to transmit $\mathbf{w}_1$. Following the similar argument as in case i), we can construct an alternative policy under which the transmitter exhausts its DoF to transmit $\mathbf{w}_1$ until its delivered, and utilizes the remaining DoF to deliver $\mathbf{w}_2$. This will improve the AoI of the user that decodes $\mathbf{w}_1$, without impacting the AoI of the other user. Thus, we can safely exclude this case for the optimal policies in Π_0 without compromising the optimality. $\square$

Then, in order to show that the lower bound in Theorem 13 applies to all policies in Π_0 for the $(2, M, N, B)$ system with $B > M$, the optimal policy in Π_0 must exhibit the following structural properties.

Lemma 36. *For the $(2, M, N, B)$ system with $B > M$, under any optimal policy in Π_0, a successful updating always updates the user with higher AoI.*

Proof. We consider an updating policy starting at a time slot t_0. Assume at the beginning of t_0, the AoI at users 1 and 2 are $\delta_0^{(1)}$, $\delta_0^{(2)}$, respectively, where $\delta_0^{(1)} > \delta_0^{(2)}$.

Denote the first two delivered updates after time t_0 as $\mathbf{w}_1$ and $\mathbf{w}_2$. We assume their transmission pattern complies with Lemma 35. We aim to show that these two updates always update user 1 and then user 2. We consider the two possible transmission structures separately.

i) $t_2 < t_1 < d_1 \leq d_2$. First, we note that those two updates are intended for different users. This is because if both $\mathbf{w}_1$ and $\mathbf{w}_2$ are intended for the same user, then the delivery of $\mathbf{w}_2$ will not reset the AoI at the corresponding user, as $\mathbf{w}_2$ is more stale than $\mathbf{w}_1$. This violates the assumption that under the optimal policy in Π_0, all delivered updates reset the corresponding AoI.

Next, we assume $\mathbf{w}_1$ and $\mathbf{w}_2$ are intended for user 2 and user 1, respectively. We note that under this scheme, user 2 will be updated at d_1 while user 1 is updated at d_1. We aim to show that this is strictly sub-optimal. For that, we consider an alternative policy

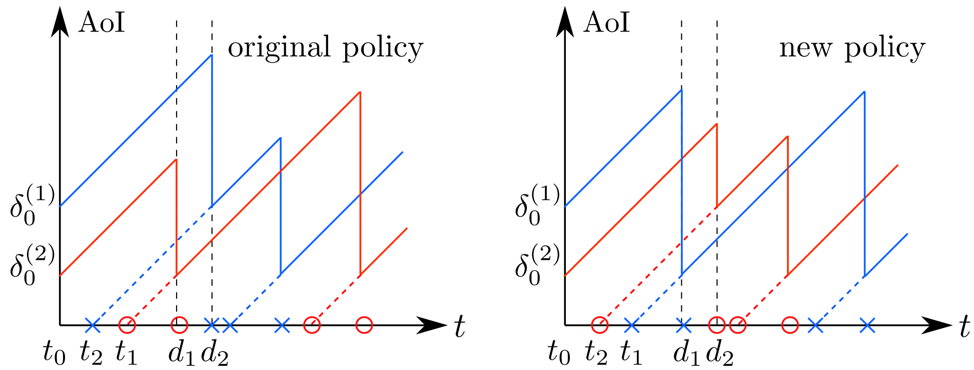


Figure B.1: Comparison between the new policy and the original policy.

where the transmitter replaces each update delivered after t_0 with an update generated at the same time but intended for the other user. We note that the AoI evolution at both users remains the same under both policies up to d_1, and are switched after d_2, as illustrated in Fig. B.1. Between d_1 and d_2, since $\delta_0^{(1)} > \delta_0^{(2)}$, resetting user 1 at d_1 instead of d_2 leads to reduced summed AoI. Therefore, $\mathbf{w}_1$ and $\mathbf{w}_2$ should be intended for user 1 and user 2, respectively, under the optimal policy.

ii) $t_1 < d_1 \leq t_2 < d_2$. We now consider the following cases:

ii-a) $\mathbf{w}_1$ and $\mathbf{w}_2$ are both intended for user 2. For this case, we construct a new policy by replacing $\mathbf{w}_1$ with another update $\mathbf{w}_1'$ generated at the same time but intended for user 1. Then, under the new policy, the AoI of user 1 will be reset at time d_1, while the AoI of user 2 will be reset at d_2 only. Therefore, after d_2, the AoI of user 2 remains the same under both policies, while the AoI of user 1 will be strictly improved. Besides, between d_1 and d_2, the summed AoI of both users is strictly improved under the new policy, as resetting user with higher age (user 1) leads to lower summed AoI. Therefore, $\mathbf{w}_1$ and $\mathbf{w}_2$ cannot be intended for user 2 under the optimal policy.

ii-b) $\mathbf{w}_1$ and $\mathbf{w}_2$ are intended for user 2 and user 1, respectively. For this case, we construct a new policy by replacing each update transmitted after t_0 by the update generated at the same time but intended for the other user. Then, under the new policy, after d_2, the AoI evolution of user 1 and user 2 will be switched. Between d_1 and d_2, the summed AoI of both users is strictly improved under the new policy, as resetting user with higher age (user 1) leads to lower summed AoI.

Combining cases ii-a) and ii-b), we can see that $\mathbf{w}_1$ must be intended for user 1 under the optimal policy.

Therefore, for the two possible updating structures, the next update must be intended for the user with higher AoI. Since $\mathbf{w}_1$ and $\mathbf{w}_2$ must intend for user 1 and user 2, respectively, for the first structure, we repeat this argument for updates after d_2. For the second structure, we only showed that $\mathbf{w}_1$ must intend for user 1, while $\mathbf{w}_2$ may intend for either user, depending on the updating structure after d_1. We then repeat the argument after d_1 for the second structure. Then, we can conclude that each delivered update should update the user with higher AoI. □

Remark 6. Note that the delivery structure described in Lemma 36 does not necessarily imply the alternating transmission structure described in Definition 13, due to the second possible transmission pattern depicted in Lemma 35.

For ease of exposition, let $\tilde{\Pi}_0 \subset \Pi_0$ be the set of policies satisfying Lemmas 35-36. According to Theorem 3, no free DoF is available during the transmission of an update, which naturally leads to the definition of generalized blocks as follows.

Definition 17 (Generalized Block $\tilde{B}_{u,v}$)**.** *Block $\tilde{B}_{u,v}$ consists of v idling time slots followed by $\lceil uB/M \rceil$ time slots during which the transmitter exhausts its DoF to send u* useful *updates to the two users. When $v = 0$, we simply express $\tilde{B}_{u,0}$ as $\tilde{B}_u$.*

Compared with the block $B_{u,v}$ defined in Section 3.7, in generalized blocks $\tilde{B}_{u,v}$, we do not impose the alternating updating structure. Any policy $\pi \in \tilde{\Pi}_0$ can be represented as a sequence of generalized blocks.

Next, we consider the DoF allocation within each generalized block. We introduce the definition of resource chunk as follows.

Definition 18 (Resource chunk)**.** *A resource chunk in block $\tilde{B}_{u,v}$ is the smallest subset of the utilized uB DoFs in the block satisfying the following conditions: 1) At least one update is delivered using the DoF in each chuck; 2) During the transmission of the update(s) satisfying 1), there does not exist any other partially transmitted update in the system.*

Lemma 37. *There are two types of resource chunks in each block $\tilde{B}_{u,v}$: Type-1: A chunk consisting of B DoFs allocated to the transmission of a single update. Type-2: A chunk consisting of $2B$ DoFs allocated to the transmission of two updates (denoted as $\mathbf{w}_1$ and $\mathbf{w}_2$) with $t_2 < t_1 < d_1 \leq d_2$. Besides, the transmission time of $\mathbf{w}_1$ is always equal to $i + 1$.*

Lemma 37 can be shown based on the structure depicted in Lemma 35.

Definition 19 (Type-2 resource chunk re-allocation)**.** *A re-allocated Type-2 resource chunk will allocate the DoFs in the original resource chunk to users in the order of their original updating times. Moreover, all future updates delivered after this chunk are replaced by updates with the same generation time but intended for the other user.*

Remark 7. Based on Lemma 36, the first delivered update using a Type-2 resource chunk, i.e., the update delivered at time slot d_1, is always intended for the user with higher AoI. After re-allocation, such user should receive the first update as well.

We note that when all Type-2 resource chunks are re-allocated, the corresponding updating schemes becomes an alternating updating policy in Π_1. Thus, to show that the summed average AoI under any policy in $\tilde{\Pi}_0$ is lower bounded by the same quantity $\Delta_{\min}$ suggested in Theorem 19, it suffices to show that the re-allocation of Type-2 resource chunks always improve the summed average AoI.

As we have shown in the proof of Lemma 17, we first note that the minimum possible transmission time for one update is $i+1$ time slots under any updating policy in Π_0.

Lemma 38. *If after the re-allocation of a Type-2 resource chunk, the transmission time of the first delivered update is $i+1$, then the re-allocation always improves the AoI under the original resource chunk allocation.*

Proof. Let $\delta_0^{(1)}$ and $\delta_0^{(2)}$ be the initial AoI of user 1 and user 2 at the beginning of this Type-2 chunk. Without loss of generality, we assume $\delta_0^{(2)} > \delta_0^{(1)}$. Then, user 2 will be updated first. Assume the generation times and delivery times of the two updates under the original resource chunk allocation are $t_1 < t_2 < d_2 \leq d_1$. Then, users 1 and 2 will be updated at times d_1 and d_2, respectively. Besides, based on Lemma 37, we have $d_2 - t_2 = i+1$. After the re-allocation, the DoF will be utilized to update user 2 first, starting at time t_1. Denote the corresponding delivery time as d_1'. Then, by the assumption that the transmission time of the first delivered update after reallocation is $i+1$, we have $d_1' - t_1 = i+1$. Let t_2' be the generation time of the second delivered update after re-allocation. Then, $t_1 < t_2 \leq d_1' \leq t_2' \leq d_2 < d_1$.

For clarity, we define $\alpha := t_2 - t_1 \leq i+1$ and $\beta := d_1 - d_1 \leq d_2' - t_2' \leq i+2$. As illustrated in Fig. B.2, the AoI evolution before d_1' stays the same after re-allocation. Besides, at time d_1, under the original allocation, the AoIs at users 1 and 2 are $d_1 - t_1$, $d_1 - t_2$ respectively. After re-allocation, the AoIs become $d_1 - t_2'$, $d_1 - t_1$, respectively. Since the DoF allocation after this resource chunk will be switched between the two users, user 2 will be updated next after the re-allocation. Since $d_1 - t_2' \leq d_1 - t_2$. the AoI evolution after d_2 will be improved after the re-allocation. It remains to show that the AoI over time slots $[d_1' : d_1 - 1]$ after the re-allocation is strictly improved.

Since the AoI evolution of user 1 stays the same before d_1 and the difference solely depends on the AoI of user 2, the age difference between that under the Type-2 resource

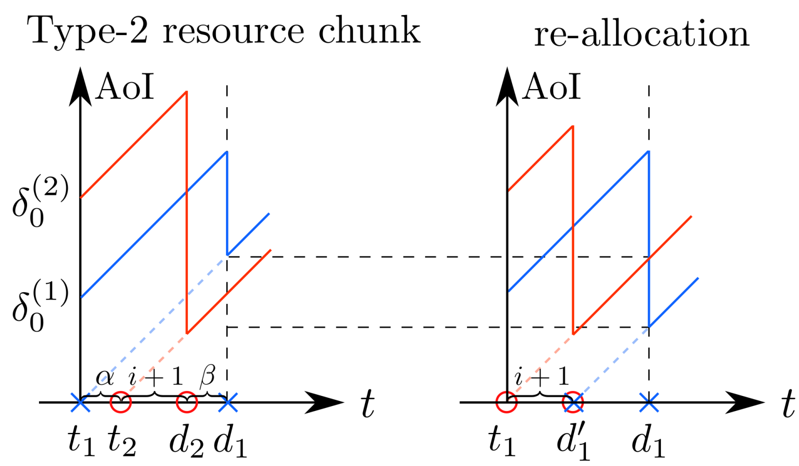


Figure B.2: Re-allocation of Type-2 resource chunk.

chunk allocation and the re-allocation is

$$\left(\sum_{\ell=1}^{\alpha}(\delta_0^{(2)}+i+\ell)+\sum_{\ell=1}^{\beta-1}(i+\ell)\right)-\sum_{\ell=1}^{\alpha+\beta-1}(i+\ell)$$
$$=\sum_{\ell=1}^{\alpha}\delta_0^{(2)}-\sum_{\ell=1}^{\beta-1}\alpha=\alpha(\delta_0^{(2)}-\beta+1)\geq 0,$$

where the last inequality is based on the fact that the minimum possible AoI of the user with higher AoI is $i+1+\lceil\frac{2}{j}\rceil$ by Lemma 17, and the fact that $\beta\leq i+2$. □

For a generalized block $\tilde{B}_{u,v}$, after all Type-2 resource chunks are re-allocated, it becomes a block $B_{u,v}$, which can be partitioned into segments as in Sec. 3.7. Recall that each segment consists of either $ij+1$ or $(j+1)i+1$ time slots. Besides, for any policy in Π_1, the transmission time of any update is either $i+1$ or $i+2$.

Note that the transmission times of all updates in the segments of length $ij+1$ in $B_{u,v}$ are $i+1$. Then, according to Lemma 38, under the original optimal updating scheme, it must not contain any Type-2 resource chunk before the first updating time slot in the next segment. The only possible segments that contain Type-2 resource chunks under the original optimal policy are segments of length $(j+1)i+1$ where the transmission time of the first update is $i+2$ and that of any other update is exactly $i+1$. Thus, under the original optimal policy, the Type-2 resource chunk can only be used to transmit the first two updates in the segment. In the following, we will show that the lower bound $\Delta_{\min}$ suggested in Lemma 19 still holds for such segments.

Lemma 39. *For segments of length $(j+1)i+1$ where the first two updates are transmitted*

Table B.1: Minimum AoI pattern of the first two updates for segments of length $(j+1)i+1$, $j \geq 2$. First row is the time slot and the second row is the minimum AoI pattern. The first delivered updates is generated at the end of time slot $(U_m - 1)i + m + \gamma$, and the two updates are delivered at $U_m i + m + \gamma$, $(U_m + 1)i + m$, respectively.

$(U_m-1)i+m$	$\cdots$	$(U_m-1)i+m+\gamma-1$	$(U_m-1)i+m+\gamma$
$(i+2, 2i+2)$	$\cdots$	$(i+\gamma+1, 2i+\gamma+1)$	$(i+\gamma+2, 2i+\gamma+2)$
$(U_m-1)i+m+\gamma+1$	$\cdots$	$U_m i+m+\gamma-1$	$U_m i+m+\gamma$
$(i+\gamma+3, 2i+\gamma+3)$	$\cdots$	$(2i+\gamma+1, 3i+\gamma+1)$	$(i+1, 2i+\gamma+2)$
$U_m i+m+\gamma+1$	$\cdots$	$(U_m+1)i+m-1$	$(U_m+1)i+m$
$(i+2, 2i+\gamma+3)$	$\cdots$	$(2i-\gamma, 3i+1)$	$(2i-\gamma+1, 2i+2)$

Table B.2: Minimum AoI pattern of for segments of length $2i + 1$. First row is the time slot and the second row is the minimum AoI pattern. The first delivered update is generated at the end of time slot $(U_m - 1)i + m + \gamma$, and the two updates are delivered at $U_m i + m + \gamma$, $(U_m + 1)i + m$, respectively.

$(U_m-1)i+m$	$\cdots$	$(U_m-1)i+m+\gamma-1$	$(U_m-1)i+m+\gamma$
$(i+2, 2i+3)$	$\cdots$	$(i+\gamma+1, 2i+\gamma+2)$	$(i+\gamma+2, 2i+\gamma+3)$
$(U_m-1)i+m+\gamma+1$	$\cdots$	$U_m i+m+\gamma-1$	$U_m i+m+\gamma$
$(i+\gamma+3, 2i+\gamma+4)$	$\cdots$	$(2i+\gamma+1, 3i+\gamma+2)$	$(i+1, 2i+\gamma+2)$
$U_m i+m+\gamma+1$	$\cdots$	$(U_m+1)i+m-1$	$(U_m+1)i+m$
$(i+2, 2i+\gamma+3)$	$\cdots$	$(2i-\gamma, 3i+1)$	$(2i-\gamma+1, 2i+2)$

using a Type-2 resource chunk under the original optimal policy, the summed average AoI is lower bounded by $\Delta_{\min} = 4i + 1 + \frac{2i+1}{ij+1}$ *if* $j \in \mathbb{Z}_{\geq 2}$, *and by* $\Delta_{\min} = 4i + 3$ *if* $j = 1$.

Proof. 1) $j \geq 2$. Consider a segment of length $(j + 1)i + 1$ by U_m that consists of time slots $[(U_m - 1)i + m : (U_m + j)i + m]$. Following similar arguments as in the proof of Lemma 19, we can show that update U_m is generated at time $(U_m - 1)i + m - 1$ instead of $(U_m - 1)i + m$. Note that under the original resource allocation, within the Type-2 resource chunk, update U_m is delivered after update $U_m + 1$. Assume $U_m + 1$ is generated at time $(U_m - 1)i + m + \gamma$, $\gamma \in [0 : i]$. It will consume all DoF until it is delivered at time $(U_m - 1)i + m + \gamma + i$. Then the minimum AoI pattern of the first two updates over the segment can be specified (cf. Table B.1). At the end of time slot $(U_m + 1)i + m$, the minimum AoI pattern is $(2i - \gamma + 1, 2i + 2)$ under the original resource allocation and $(i + 1, 2i + 2)$ under the re-allocation (cf. Table 3.2). Since $2i - \gamma + 1 \geq i + 1$ and the remaining updating follows the alternating updating rules, it suffices to show that the summed AoI over $[(U_m - 1)i + m : (U_m + 1)i + m]$ under the original allocation is greater than that under the re-allocation. In fact, the summed AoI over $[(U_m - 1)i + m : (U_m + 1)i + m]$ under the original allocation is $8i^2 + 9i + 3 + \gamma^2 + 3$

(cf. Table B.1) and that under the re-allocation (cf. Table 3.2) is $8i^2 + 9i + 3$, which completes the proof.

2) $j = 1$. The Type-2 resource chunk has length $2i + 1$. According to Remark 3, the minimum AoI pattern is $(i + 1, 2i + 2)$ instead of $(i + 2, 2i + 1)$. Therefore, the minimum AoI pattern at the first time slot of the segment is $(i + 2, 2i + 3)$ instead of $(i+2, 2i+2)$ (cf. B.2). Similar to 1), we assume that update $U_m + 1$ is generated at time $(U_m - 1)i + m + \gamma$, $\gamma \in [0 : i]$, and the AoI pattern with the original Type-2 resource allocation can be specified (cf. Table B.2). The corresponding summed average AoI is $4i + 3 + \frac{\gamma^2 + (i+1)\gamma}{2i+1}$, which is greater than or equal to $4i + 3$, the summed average AoI under the re-allocation. □

In summary, the summed average AoI of any segment of the generalized block $\tilde{B}_{u,v}$ is lower bounded by $\Delta_{\min}$. Together with Lemma 21, we can show that the summed average AoI of the generalized block $\tilde{B}_{u,v}$ is also bounded by $\Delta_{\min}$. Therefore, the summed average AoI under any policy in $\tilde{\Pi}_0$ is lower bounded by $\Delta_{\min}$.

B.2 Converse of Theorem 8 (iv)

In this section, we provide a proof of the converse of Theorem 8 (iv). I.e., for the case when $N \le \frac{M}{2}$, $\frac{j}{ij+1} \le \frac{N}{B} < \frac{j+1}{(j+1)i+1}$ where $i = \lceil \frac{B}{N} \rceil - 1$ and $j = \lfloor \frac{1}{B/N - i} \rfloor$, we aim to show that the minimum summed average AoI is lower bounded by $3i + 1 + \frac{i+1}{ij+1}$ if $j \ge 1$, and by $3i - 1$ if $j = 0$. We adopt a similar approach as in the proof of the converse of Theorem 8 (iii). For the sake of completeness and brevity, we provide key steps and omit proofs of lemmas if they are essentially the same as their counterparts in Section 3.7.

Since the number of transmit antennas M is more than the total number of antennas at the two users, each user is able to decode N symbols simultaneously and the minimum summed average AoI is twice the minimum average AoI of the single-user system $(1, N, N, B)$. Thus, in the following, we will focus on the minimum average AoI of the single user system $(1, N, N, B)$.

Since the optimal scheme is guaranteed to lie within Π_0 by Theorem 10, any policy $\pi \in \Pi_0$ in $(1, N, N, B)$ systems can be represented as a sequence of blocks, where each block $B_{u,v}$ consists of v idling time slots followed by $l_u := \lceil uB/N \rceil$ time slots, during which the transmitter exhausts all DoF to send u updates. When $v = 0$, we express $B_{u,0}$ as B_u.

We first study a work-conserving updating scheme $\pi \in \Pi_0$, under which the transmitter exhausts its DoF at each time slot.

Lemma 40. *For the* $(1, N, N, B)$ *system with* $\frac{j}{ij+1} \leq \frac{N}{B} < \frac{j+1}{(j+1)i+1}$, $i, j \in \mathbb{N}$, *we have*

(i) the minimum AoI pattern in any time slot is $i + 1$;

(ii) under policy π, *the duration between two consecutive delivered updates is either* i *or* $i + 1$ *time slots.*

The proof of Lemma 40 is similar to that in Lemma 17 and Lemma 18 and omitted.

Label the delivered updates starting at time 1 in the order of their delivery time. Let U_m be the index of the m-th update whose delivery time is $i + 1$ time slots after the previous delivered update. By the DoF constraint, we have

$$[i(U_m - 1) + m - 1]N \geq (U_m - 1)B, \tag{B.1}$$

$$[iU_m + m - 1]N \geq U_m B. \tag{B.2}$$

Since $\frac{j}{ij+1} \leq \frac{N}{B} < \frac{j+1}{(j+1)i+1}$, $i, j \in \mathbb{N}$, solving the above inequalities gives

$$j-1 < \frac{N/B}{1-iN/B} - 1 < U_{m+1} - U_m < \frac{N/B}{1-iN/B} + 1 < j+2. \tag{B.3}$$

Since U_m and U_{m+1} are integers, we have

$$U_{m+1} - U_m = j \quad \text{or} \quad j + 1. \tag{B.4}$$

Partition the time axis into segments by the delivery time of updates $\{U_m - 1\}_{m=2}^{\infty}$, i.e., $[1 : ij + 1]$, $\ldots$, $[(U_m - 1)i + m : (U_{m+1} - 1)i + m]$, $\ldots$. By Eqn. (B.4), the segment length is either $ij + 1$ or $(j + 1)i + 1$. An example of the definition of U_m and the segments for the $(1, 7, 7, 12)$ system is shown in Fig. B.3. We point it out that all updates in Fig. B.3 are intended for one user only while in Fig. 3.8, updates in the first row are intended for the first user and updates in the second row are intended for the second user.

Lemma 41. *For the* $(1, N, N, B)$ *system with* $\frac{j}{ij+1} \leq \frac{N}{B} < \frac{j+1}{(j+1)i+1}$, $i, j \in \mathbb{N}$, *under policy* π, *the summed average AoI over segment* $[(U_m - 1)i + m : (U_{m+1} - 1)i + m]$ *is lower bounded by* $\Delta_{\min} = \frac{3i+1}{2} + \frac{i+1}{2(ij+1)}$.

Proof. 1) The segment length is $ij + 1$, i.e., $U_{m+1} - U_m = j$. The minimum AoI pattern over the segment can be specified in Table B.3 and the corresponding minimum average AoI is $\frac{3i+1}{2} + \frac{i+1}{2(ij+1)}$.

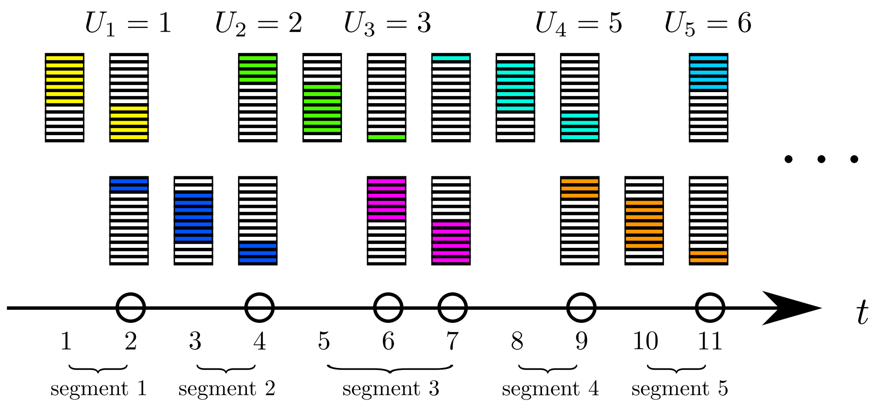


Figure B.3: Updating pattern in the $(1, 7, 7, 12)$ system under π_1. We have $i = 1$, $j = 1$. Circles represent delivery times of updates. Since $i = 1$, the segments can be obtained by tracking the updates whose delivery time is 2 time slots after the previous delivery time. We note that the length of each segment is either 2 (i.e., $ij + 1$) or 3 (i.e., $(j + 1)i + 1$).

Table B.3: Minimum AoI pattern for segments of length $ij + 1$, $j \in \mathbb{N}$, $\ell \in [1 : j - 1]$. An update is delivered at the end of the time slots in the last column.

Time slot	$(U_m - 1)i + m$	$\cdots$	$U_m i + m - 1$	$U_m i + m$
Minimum AoI pattern	$i + 2$	$\cdots$	$2i + 1$	$i + 1$
Time slot	$(U_m + \ell - 1)i + m + 1$	$\cdots$	$(U_m + \ell)i + m - 1$	$(U_m + \ell)i + m$
Minimum AoI pattern	$i + 2$	$\cdots$	$2i$	$i + 1$

2) The segment length is $(j + 1)i + 1$, i.e., $U_{m+1} - U_m = j + 1$. By a similar argument as in Lemma 19, we can show that the minimum AoI pattern when U_m is delivered, i.e., at the end of time slot $U_m i + m$, is $i + 2$ instead of $i + 1$. As a result, the minimum AoI pattern over segment of length $(j+1)i+1$ can be specified in Table B.4 and the minimum average AoI over the segment is $\frac{3i+1}{2} + \frac{3i+1}{2[(j+1)i+1]}$, which is greater than $\frac{3i+1}{2} + \frac{i+1}{2(ij+1)}$.

Combining the two cases, the average AoI over any segment is lower bounded by $\Delta_{\min}$. □

Table B.4: Minimum AoI pattern for segments of length $(j + 1)i + 1$, $j \in \mathbb{N}$, $\ell \in [2 : j]$. An update is delivered at the end of the time slots in the last column.

Time slot	$(U_m - 1)i + m$	$\cdots$	$U_m i + m - 1$	$U_m i + m$
Minimum AoI pattern	$i + 2$	$\cdots$	$2i + 1$	$i + 2$
Time slot	$U_m i + m + 1$	$\cdots$	$(U_m + 1)i + m - 1$	$(U_m + 1)i + m$
Minimum AoI pattern	$i + 3$	$\cdots$	$2i + 1$	$i + 1$
Time slot	$(U_m + \ell - 1)i + m + 1$	$\cdots$	$(U_m + \ell)i + m - 1$	$(U_m + \ell)i + m$
Minimum AoI pattern	$i + 2$	$\cdots$	$2i$	$i + 1$

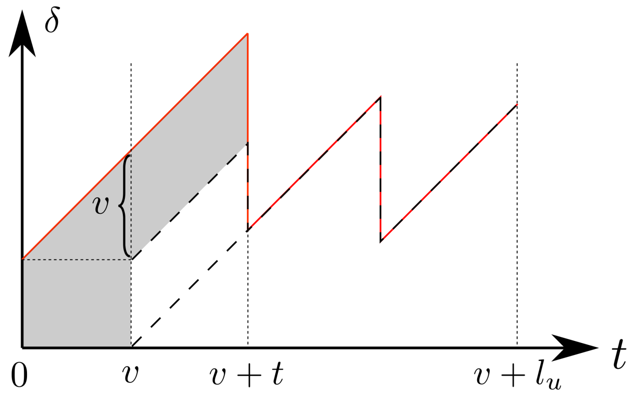


Figure B.4: AoI evolution over an extended block. The dashed line is the AoI evolution if idling period does not exist.

Remark 8. Note that for all $i, j \in \mathbb{N}$, the minimum average AoI over the first $\ell i + 1$ time slots in each segment $[(U_m - 1)i + m : (U_{m+1} - 1)i + m]$ is monotonically decreasing in ℓ for $\ell \in [1 : U_{m+1} - U_m]$.

Next, we relate the AoI pattern under π with block B_u. The updating policy π over $[1, l_u]$ is identical to a block B_u except that the DoF at time slot l_u may not be exhausted. Partition B_u into segments $[1 : ij + 1]$, ..., $[(U_{m_u-1} - 1)i + m_u - 1 : (U_{m_u} - 1)i + m_u - 1]$ and a residue $[(U_{m_u-1} - 1)i + m_u : l_u]$, where $m_u = \max\{m : U_m < l_u\}$. According to Remark 8, the average AoI of the residue is lower bounded by $\Delta_{\min}$.

For block $B_{u,v}$, the lower bound $\Delta_{\min}$ still holds as summarized in the following lemma.

Lemma 42. *For the $(1, N, N, B)$ system with $\frac{j}{ij+1} \le \frac{N}{B} < \frac{j+1}{(j+1)i+1}$, $i, j \in \mathbb{N}$, the summed average AoI over $B_{u,v}$ is lower bounded by $\Delta_{\min} = \frac{3i+1}{2} + \frac{i+1}{2(ij+1)}$.*

Proof. Assume $B_{u,v}$ starts at time slot 1. Let δ_0 be the AoI at time zero, $v + t$ be the first update delivery time and $v + l_u$ be the end of block $B_{u,v}$. Then, the existence of idling time slots affects the AoI evolution until $v + t$. By Lemma 40, $\delta_0 \ge i + 1$ and $t \ge i + 1$.

Let A_1 be the AoI increment induced by the idling time slots, as shown by the shaded area in Fig. B.4. Denote A_2 be the summed AoI over B_u when no idling time slot is present, corresponding to unshaded aread in Fig. B.4. We have

$$\begin{aligned} \frac{A_1}{v} &= \frac{1}{v}\left(\sum_{\ell=1}^{v}(\delta_0 + \ell) + v \cdot t\right) = 2i + 2 + \frac{v+1}{2} \\ &\ge \frac{3i+1}{2} + \frac{i+1}{2(ij+1)}. \end{aligned} \tag{B.5}$$

Let $\Delta_{B_{u,v}}$ be the summed average AoI over $B_{u,v}$. Then

$$\Delta_{B_{u,v}} = \frac{A_1 + A_2}{v + l_u} \geq \Delta_{\min}, \tag{B.6}$$

where the last inequality follows from Lemma 20. $\square$

We summarize the converse result in the following theorem.

Theorem 23. *For the* $(2, M, N, B)$ *system with* $N \geq \frac{M}{2}$ *and* $M/B \in [\frac{j}{ij+1}, \frac{j+1}{(j+1)i+1})$, $i, j \in \mathbb{N}$, *the summed average AoI under any policy* $\pi \in \Pi_0$ *is lower bounded by* $\Delta_{\min} = 3i + 1 + \frac{i+1}{ij+1}$.

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