The Pennsylvania State University
The Graduate School

ESSAYS ON MARKET DESIGN

A Dissertation in
Economics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2022

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Abstract

In chapter 1, In this paper, I introduce a profit-maximizing centralized marketplace into a decentralized market with search frictions. Agents choose between the centralized marketplace and the decentralized bilateral trade. I characterize the optimal marketplace in this market choice game using a mechanism design approach. In an equilibrium, the centralized marketplace and the decentralized trade coexist. The thickness of the centralized marketplace in this equilibrium does not depend on the search frictions. The profit of the marketplace decreases as the search frictions in the decentralized market are reduced. However, it is always higher than the half of the profit when the frictions are prohibitively high for decentralized trade. I derive conditions under which, this equilibrium results in higher welfare than either institution on its own.

In chapter 2, I provide two characterizations for the existence of stable matchings in this environment. Moreover, if ‘part-time’ contracts are allowed, I show that there is always a stable matching. Finally, I introduce a measure of instability in the market, measured as the amount of subsidy needed to ‘stabilize’ an efficient outcome.

In chapter 3, I show that the existence of a solution to a market design problem can be obtained as long as the designer’s and the agents’ preferences satisfy any sufficiently well-behaved abstract convexity, using ‘convex’ price orders that rank bundles instead of price vectors. Walrasian Equilibrium is obtained as a special case.

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Acknowledgments

I wish to thank my committee, Vijay Krishna, Kalyan Chatterjee, Nima Haghpanah, Ran Shorrer, and Chloe Tergiman. I have immensely benefited from each of their feedback and guidance.

I have been incredibly fortunate to have Vijay as my advisor. From his very first lecture to the end of the job market, every interaction has given me new ideas about both how to do research and how to communicate my research. Without Vijay's guidance, encouragement, and generosity, this work would not be possible. I can only hope I will be able to pay it forward to my students in the future.

Kalyan has always encouraged me to pursue my ideas and brought out many connections between my results and earlier work that I haven't been aware of. He was also kind enough to point out several weak spots *and* suggest remedies in early stages of my research. Nima made me realize different ways to think about some of my results and his detailed comments helped me get a better understanding of the contributions of the work. Ran has been incredibly helpful throughout, especially during the job market. He gave very useful comments about both the content and the presentation of the results; without him, my slides would have at most two colors.

Ed Green has been an amazing mentor to me. By a coincidence, he was the first Penn State faculty I have ever heard of; someone recommended me to read one of his lesser known papers ("On the Emergence of Parliamentary Government: The Role of Private Information") when I was still a sophomore. Years later, he was also the first Penn State faculty I talked to outside of the classroom, when he approached me at the Departmental Picnic and introduced himself. From then on, I used every opportunity to learn from him and he has been incredibly generous with his time and advice. His excellent taste extended to literature as well: a novel he recommended, "An Instance of the Fingerpost", turned out to be the best reading of my entire PhD period.

Nageeb Ali has been very generous with his time to discuss ideas, read drafts, give helpful feedback, and offer his advice on various topics. Henrique de Oliveira advised me during my third year; he encouraged me when I wanted to explore new topics and has been a supportive advisor. Our walks to Saint's to get coffee and sweets after our meetings have also been quite enjoyable. Rohit provided me with much needed nudges and support.

At Penn State, students who are interested in theory have been very fortunate to have many theory-related events throughout the years. On top of our excellent theory seminar

series with many crash courses, our reading groups, Kalyan's lunch workshop, Nima's "coffee hour" (where we gathered after seminars and discussed the work we have been presented), and two annual conferences, PETCO and TAPS (Pennsylvania Economic Theory Conference and Theory at Penn State) have been incredible opportunities to be exposed to the latest works by some amazing economists and to talk about economics in general. I am grateful to all of the organizers for the opportunities they provided.

I thank my parents, Gülsen and Zeki, and my brothers, Alp and Argun, for their unconditional love and support. They have always supported my studies even though it meant I had less time to spend with them. Needless to say, I wouldn't be here without their contributions and support throughout the years.

Without the company of my friends, the last six years would have been significantly less pleasant. Special thanks to Nita, Shomak Chakrabarti, Nino Doghonadze, Ersoy Dokumaci, Nilufer Gok, Esra Gul, Tanji Hwang, Murat Inan, Omer Karaduman, Gourika Khanna, Xiao Lin, Ali Polat, Gyu Rho, Garima Singal, Yuta Suzuki, Ece Teoman. Each of them tolerated more dad jokes from me than anyone can be reasonably expected. Tea parties, walks, hikes, trips, and actual parties with them have made it easy to relax after (but not necessarily immediately after) a day of hard work.

I may have spent more time in Nino's living room than my own and thus I owe her special thanks for being the best neighbor and host a friend can hope for[1] (and apologies for occupying her house). She is a founding member of our State College family and the facilitator of many fun memories.

Ece and Nino are the reason I got a driver's license at the tender age of 32. They forced me to get a learner's permit and then taught me how to drive by finding time in their busy schedules. If I can get a parking ticket today, it is all thanks to their efforts. (Seriously, thank you so much guys!)

I dedicate this work to Ece. She has been there for me at every stage of the way while she was going through the same path herself. She is the best coauthor I ever had; working with her taught me much more than the answers to the questions we have been trying to answer. She gave me everything I needed -and more- unprompted. She made my life better both emotionally and academically. I thank her for everything she has done for me over the years and apologize for standing in the way of her cat adoption (together with our landlady). Last but not least, I am grateful to her for not sending me to jail in Venezuela when I *allegedly* undercooked fish (but I freely admit to undercooking her burgers, often)[2].

I gratefully acknowledge the financial support from the National Science Foundation. The findings and conclusions do not necessarily reflect the view of the NSF.

[1]The fact that most of my Georgian vocabulary consists of food names and the nonemptiness of the said vocabulary are evidences that support my claims.

[2]This reminds me of another thing I should acknowledge. Ece and occasionally Nino have been the -mostly- voluntary subjects of my cooking, baking, and drinks mixing experiments, which were not even always meant to be successful. Although they drew the line at the roasted goose for a thanksgiving dinner, with their patience and honest feedback, I have been able to improve my skills in the kitchen over the years. So much so that a food critic known for her sincerity preferred my burgers to 'cheeseburger Happy Meal with extra fries and coke', and approved my carnita tacos; these are my proudest achievements.

Epigraph

"*In the beginning was simplicity.*"
Richard Dawkins, *The Selfish Gene*

"*We are going to die, and that makes us the lucky ones. Most people are never going to die because they are never going to be born. The potential people who could have been here in my place but who will in fact never see the light of day outnumber the sand grains of Arabia. Certainly those unborn ghosts include greater poets than Keats, scientists greater than Newton. We know this because the set of possible people allowed by our DNA so massively exceeds the set of actual people. In the teeth of these stupefying odds it is you and I, in our ordinariness, that are here. We privileged few, who won the lottery of birth against all odds, how dare we whine at our inevitable return to that prior state from which the vast majority have never stirred?*"
Richard Dawkins, *Unweawing the Rainbow*

"*Actually, we were at the other extreme from giants: we had become dwarfs! And I mean this quite literally. For, often, the right way to philosophize is to make yourself artificially stupid! Only by being 'stupid' can you break the barrier of the seemingly obvious.*"
Apostolos Doxiadis and Christos H. Papadimitriou, *Logicomix*

Chapter 1 | Coexistence of Centralized and Decentralized Markets

1.1 Introduction

Centralized marketplaces that bring buyers and sellers together have seen massive growth in the last decade. For instance, it has been estimated that Amazon generates half of all e-commerce sales in the US.[1] Another analysis estimates that usage of ride-share apps surpassed taxis in NYC as early as 2017.[2] Airbnb and Vrbo's market shares in vacation rentals reached half of the market.[3] All of these evidence suggest that the centralized marketplaces have significant positions in their respective markets.

These centralized marketplaces are successful because they reduce search and information frictions present in decentralized markets. In a decentralized market, an agent may not meet with a trading partner, and even when he meets with a partner, it may not be the right one. Centralized marketplaces reduce these frictions by attracting agents, collecting information from them, and making sure the realized matches are efficient enough to allow the marketplace to make some profit. In turn, the near certainty of trade on these platforms attracts many agents and gives the platforms significant market shares.

But the reduction in frictions is not without cost. Centralized marketplaces take commissions from the participants –buyers or sellers, or both. It is up to the individual traders to decide whether it is worthwhile to buy/sell in the centralized marketplace or to do so in a decentralized manner. The two institutions compete. And, the payoff

[1] Congress Majority Staff (2020).
[2] Schneider (2021).
[3] Hinote (2021).

from trading in the decentralized market is a function of the self-selected agents who also choose to trade there. Given the rising market shares of platforms like Amazon, Uber, etc., there are concerns about the possible monopolization of trade by such platforms (Khan, 2016). Indeed, a congressional panel investigating competition in the digital markets has asserted that Amazon has monopoly power as an intermediary in the US e-commerce market (Congress Majority Staff, 2020).

In this paper I ask: Would a centralized marketplace monopolize all trade or is there room for some forms of decentralized trade to coexist? Moreover, if coexistence is possible, what are the welfare and profit consequences of multiple trading modes? How is the centralized marketplace affected by search frictions?

The literature on competing platforms sheds some light on the competition when there are multiple profit-maximizing platforms. Search theory provides an understanding of the decentralized markets on their own. However, we know less about the impact of a centralized marketplace that competes with numerous other trading venues. Recent contributions to matching theory (Ashlagi and Roth, 2014; Ekmekci and Yenmez, 2019; Roth and Shorrer, 2021) have focused on how to make a marketplace attractive to agents, when there are other options. They show that even a benevolent marketplace can have difficulty in recruiting agents. It is natural to expect it to be even harder when the marketplace is a profit maximizer. To answer the questions above, I study a centralized marketplace introduced into a decentralized market with some frictions.

To illustrate the point, consider the example of someone who wants to buy or sell a used car. She can check the prices offered by Carvana, a platform specialized in the used car market. If the price offered by Carvana is acceptable, she can simply take this deal. However, if she thinks she can get a better deal by searching privately via newspaper ads, she can choose to do so. Even though Carvana may have a large market share in this sector, there are still endless possibilities for trading privately. When Carvana chooses what price to offer for each car, ignoring these possibilities would harm its profit. Moreover, it is hard to guess the impact of Carvana's response to these other options.

Financial markets provide another example. Many assets can be traded at the stock exchanges as well as over-the-counter. In the stock exchanges, there is essentially no uncertainty; agents can buy or sell at the posted prices. However, in the over-the-counter markets, trade is not as transparent; dealers often do not post prices in a public manner. Instead, they provide quotes when someone is interested in trading with them. The agents who trade with these dealers only observe the prices offered by a limited number of dealers before they trade. This paper provides a framework to think about the problems faced by a stock exchange that competes with over-the-counter trades.

In this paper, I develop a model of a centralized marketplace that competes for agents who also have the option to trade in a decentralized manner. I consider a setup with a single, indivisible good where each of a continuum of agents can buy or sell one unit of the good. The endowments are common knowledge whereas the valuations are the private information of the agents. A designer chooses an individually rational, incentive compatible mechanism to maximize revenue of the marketplace – say, the commissions charged for intermediation.

The decentralized market is modeled as in Diamond-Mortensen-Pissarides (Diamond, 1971; Mortensen, 1970; Pissarides, 1979); agents are randomly matched among those who choose to participate in the decentralized trade and then they engage in Nash Bargaining in each realized match.[4] I consider a market choice game where (i) the marketplace designer announces a mechanism, (ii) agents choose whether to join the mechanism or to search for a trading partner, (iii) outcomes are realized in both markets.

When every agent joins the centralized market, the outside option for each agent is not trading. Thus, of course the deviations are not profitable, given that the centralized market promises nonnegative payoffs. However, this is a very fragile equilibrium; I show that this equilibrium is always in dominated strategies. For agents with intermediate valuations, joining the centralized market is weakly dominated by joining the decentralized market. Therefore, I focus on equilibria in undominated strategies.

I first establish that in an equilibrium in undominated strategies, centralized marketplace and decentralized trade coexist. In this coexistence equilibrium, the agents with low and high values join the marketplace while the agents with intermediate values choose to search. To summarize, high surplus trades take place in the marketplace while low surplus trades happen privately.

The thickness of the centralized marketplace is independent of the search frictions. Even at the extremes, where the search frictions are absent or prohibitively high, the centralized marketplace targets and successfully attracts exactly the same agents to trade there. As the frictions decrease, the centralized marketplace has to "sweeten the deal" for traders to join there. Thus, with lower frictions in the decentralized market, the profit of the centralized marketplace from each trade is lower. It would be reasonable to expect the centralized marketplace to become more exclusive as a response. However, it is in fact optimal for the centralized marketplace to attract exactly the same agents.

[4]Later I establish robustness by showing that the results extend to trading with a double auction, instead of Nash Bargaining under some distributions.

One might expect that competition from the decentralized market will significantly decrease the profits of the marketplace. This is not the case. I demonstrate that the profit of the marketplace in the coexistence equilibrium is at least half of the profit that the marketplace would make if there had been no decentralized trade. Moreover, I show that the ratio of the reduction in profit of the marketplace as result of competition from the decentralized market is independent of the distribution of agents valuations. In fact, this ratio is *only* a function of the search friction – the probability of finding a trading partner – in the decentralized market. A decrease in the search frictions in the decentralized market decreases the profits of the marketplace. However, even if these frictions were absent, the profit of the marketplace is half of its profit when it operates on its own.

Next, I provide two types of welfare comparison. First, I focus on the traders' welfare in a Pareto sense. I show that decreasing the frictions in the decentralized market increases the payoff of each trader, no matter which market they choose to trade in. Then, I compare the total welfare -measured as gains from trade- created in this equilibrium to the welfare from either modes of trade operating on their own. Coexistence always improves the welfare over the centralized marketplace alone. Furthermore, I provide conditions under which the coexistence generates higher total welfare than the search market alone. Essentially, the decentralized market extends the extensive margin of trade (more agents trades in the coexistence equilibrium than in the baseline, single marketplace) while the marketplace extends the intensive margin (some agents trade with a higher probability in the coexistence equilibrium than when the decentralized market operates alone). Thus, the combination of these leads to increased efficiency

Finally, I discuss an alternative setup where there are multiple profit-maximizing marketplace designers competing with each other. First, I study the case where they are restricted to choose direct mechanisms. This environment is akin to Bertrand Competition and the insight from the Bertrand Equilibrium carries over: In equilibrium, the designers make zero profit and the outcome is equivalent to a Walrasian Equilibrium. Next, I allow the designers to choose more complex mechanisms and construct one particular equilibrium using strategies that include "price-matching guarantees." In standard Bertrand Competition, price-matching guarantees are known to allow monopoly pricing to be a Nash equilibrium. Here, each marketplace posts prices that are equivalent to the baseline marketplace where there is no decentralized market and the agents uniformly randomize over the marketplaces. In this case, the marketplaces share the baseline profit. This discontinuity between zero profit with direct mechanism and the collective baseline profit (similar to a cartel's monopoly profit) reflects the inadequacy of direct mechanisms when there are multiple designers. If agents could also join the decentralized market, then the marketplaces would each post prices equivalent to the coexistence equilibrium mechanism of the main model with price-matching guarantees, and the agents again randomize.

1.1.1 Literature Review

There are many studies that consider the problem of incentivizing participation to centralized marketplaces. However, many of them do not have a decentralized trade option, as in the literature on competing platforms. In the papers where there is a decentralized market as well, the centralized marketplace is often a benevolent one.

In the literature on competing platforms Rochet and Tirole (2003); Armstrong (2006), the questions mainly focus on the competition among platforms under numerous configurations of fee and price structures that could be employed by the platforms. The important distinction between this literature and my study is that in the context of competing platforms, the competition is between two profit-maximizing entities; they each react (or best respond) to the other's actions. However, here, the centralized marketplace is in competition with a fixed set of rules that cannot react to the marketplace's actions. This paper complements these studies by providing a insights into the nature of the competition in a decentralized market with a platform. More recently Hartline and Roughgarden (2014) also bridges the gap between mechanism design and two-sided markets with a model where sellers can choose to join a platform that sets a menu of selling procedures or to develop their own selling venue.

Following the financial crisis of 2007-2008, a series of papers initiated by Philippon and Skreta (2012); Tirole (2012) focused on the ability of the public interventions to increase the efficiency of the investments in the financial markets. In these papers, the designers of the centralized markets are concerned with social welfare. Specifically, they focus on markets with adverse selection (and moral hazard in the case of Tirole (2012)) in terms of the quality of the investments. Without the public intervention, the level of investments is below the socially optimal level. To reduce the adverse selection and increase the level of investment, the government introduces a program by overpaying for some assets and removing the weakest assets from the market. Fuchs and Skrzypacz (2015) studies a similar problem but considers the effects of the dynamic nature of the markets. By contrast, I focus on the profit-maximization problem of a centralized marketplace. Moreover, the nature of the frictions in the decentralized markets are different from the ones considered in this literature.

In another strand, some papers in matching theory Ashlagi and Roth (2014); Ekmekci and Yenmez (2019); Roth and Shorrer (2021) study the problem faced by a benevolent marketplace designer when the agents can choose between multiple venues. In kidney-exchange and school-choice settings, they show that it might be infeasible or undesirable to make sure everyone joins the centralized market, even if it aims to maximize the social welfare. This paper provides a natural counterpart where the centralized market is only concerned about is own profit.

Peivandi and Vohra (2021) study a model where agents are allowed to deviate from a market mechanism to trade among themselves according to any feasible trading protocol. Their main result states that almost every market mechanism is inherently unstable in the sense that there is always a positive measure of agents who would like to deviate from it. My findings provide a partial counterpart to their result: By restricting the possible deviations from the market mechanism, I am able to find a stable market structure where both the centralized marketplace and the outside trade are active.

Literature on the efficient dissolution of partnerships has important parallels with this study. Starting with Cramton et al. (1987) and with contributions by many others Mylovanov and Tröger (2014); Kittsteiner (2003); Fieseler et al. (2003); Loertscher and Wasser (2019); Figueroa and Skreta (2012), the setup in this literature includes a divisible good that is owned by many agents and a designer who wants to allocate the whole supply to one agent (or in some cases, to at least reduce the number of owners), thus dissolving the partnership. All of these papers have endogenous roles as buyers and sellers; each agent has some endowment which is less than the total endowment in the economy. Thus, like here, they obtain intermediate types who that are excluded from the trade and U-shaped utilities as a functions of agents types. Moreover, in extending the Nash bargaining, I used the efficient double auction which was introduced by Cramton et al. (1987) and was shown to have a unique equilibrium by Kittsteiner (2003).

Myerson and Satterthwaite (1983) studied the problem of choosing a trade mechanism to maximize the total welfare in the economy. Their main result shows that it is generically impossible to have an efficient trade mechanism -that allocates the good always to the agent who values it the most- without outside resources to finance it. In our model without a decentralized market, unsurprisingly, the welfare achieved is even less than what Myerson and Satterthwaite (1983) provides. However, introducing the option to search improves the efficiency of the market as a whole.

Miao (2006) also studies a similar environment with centralized and decentralized markets. When the search technology in the decentralized market is improved so that it can support the Walrasian Equilibrium, he shows that the centralized market serves a vanishingly small part of the population.

1.2 The Model

1.2.0.0.1 Setup I consider a market with a single, indivisible good. There is a continuum of agents on $[0, 1]$. Each agent has 1 unit of endowment of the good and has a demand for 2 units of it. As the good is indivisible, each agent can sell 1 unit, buy 1 unit, or neither buy nor sell any. Each agent has some valuation $\theta \in [0, 1]$ for a unit of the good. The valuations are drawn from some continuous distribution F with support $[0, 1]$, and they are agents' private information.

There is also a profit-maximizing mechanism designer, and they play the following game.

1.2.0.0.2 The Extensive-Form

1. Mechanism designer announces a mechanism and commits to it.

2. Agents observe the designer's marketplace and decide whether to join it or search for a trade partner in the decentralized market.

3. Trade happens in both markets simultaneously.

1.2.0.0.3 The Decentralized Market Following Diamond-Mortensen-Pissarides (Diamond, 1971; Mortensen, 1970; Pissarides, 1979), the decentralized trade is modeled as a search market. Agents who choose the decentralized trade are randomly matched to each other and then use Nash bargaining to divide the surplus created by their trade.

Formally, suppose Θ^d is the set of agents who join the search market in a strategy profile and μ is the measure with respect to the distribution F. Then, the measure of meetings in the search market will be given by a matching function $M(\mu(\Theta^d))$ as a function of the measure of agents in the search market, $\mu(\Theta^d)$.

In search theory, matching functions are commonly assumed to have constant returns to scale (CRS). This means that doubling the size of the market also doubles the number of meetings. Since I focus on a market where every agent has the same endowment and demand, this is a one-sided market. In this setup, CRS matching functions are simply linear in the size of the market: $M(\mu(\Theta^d) = m \times \mu(\Theta^d)$ where $m \in [0, 0.5]$ is the efficiency parameter of the search process. Then, probability that an agent finds a match in the search market, p is equal to $2m$ since the total measure of meetings is $M(\mu(\Theta^d)$, each agent is equally likely to be in any meeting, and there will be two agents in each meeting.

Notice that the probability of a match, p is independent of the set of agents who join the search market as well as the measure of the set. Since there is a one-to-one relationship between p and m, from now, an agents probability of finding a match is simply denoted by p. For the same reason, in this setup, p itself can be thought of as the primitive of the search market and the efficiency parameter of the matching process.

Given Θ^d and p, an agent with valuation θ computes his expected payoff from the decentralized market as follows: (i) He gets a match with the probability p. (ii) If he gets a match, it is a random draw, θ' from the distribution of types restricted to Θ^d. (iii) If he is matched to θ', his payoff from the Nash bargaining is half of the surplus created from their trade, $\frac{|\theta-\theta'|}{2}$. Thus, the expected payoff is given by

$$p\mathbb{E}\left[\frac{|\theta - \theta'|}{2}\middle|\theta' \in \Theta^d\right].$$

1.2.0.0.4 Designer's Strategies The designer can choose any deterministic mechanism for each set of participants, and commit to implementing it. That is, a strategy for designer is a collection of mechanisms $\{\mathcal{M}_\Theta | \Theta \subset [0, 1]\}$ where $\mathcal{M}_\Theta$ represents the mechanism implemented when the set of agents who choose the marketplace is Θ. For each Θ, M_Θ can be any direct or indirect mechanism with deterministic allocations and payments for each agent.

It is important to allow the designer to condition her marketplace on the set of participants for two reasons. First, without such conditioning, the mechanism can be infeasible for some sets of participants as market may fail to clear. Second, this provides a way to design the "off-the-equilibrium path" expected payoffs.

1.2.0.0.5 Revelation Principle and Individual Rationality Consider any equilibrium of the extensive-form game described above. Let Θ^* be the set of types that join the decentralized market in this equilibrium. By standard revelation principle arguments, the designer can instead offer a direct mechanism that implements the same outcome (allocation and payment) for each agent, given each set of possible participants. Since the agents with valuation in Θ^* joins the decentralized market in the original equilibrium, it must be the case that the centralized market is not strictly individually rational for them. Similarly, for other agents, the centralized market must be individually rational. Again, following the standard arguments, the direct mechanism given that the set of valuations of the participants is Θ^* will also be individually rational exactly for agents with valuations in Θ^*. Thus, from here on, I focus on direct mechanisms.

1.2.1 Bid-Ask Mechanisms are Without Loss

The mechanism implemented in the centralized marketplace has to be incentive compatible for every agents, whether their equilibrium behavior involves trading there or joining the decentralized market. Otherwise, there would be potential profitable deviations by changing the market and the report to the designer. Thus, in this subsection, I analyze the class of Bayesian Incentive Compatible mechanisms in this setting. The main finding of this section is that focusing on mechanisms with bid-ask prices is without loss of generality and profit, i.e., the designer can simply set a price for buying and a price for selling in the marketplace and let the agents choose whether they want to buy, sell or not trade. This is in line with the findings of Hagerty and Rogerson (1987) as it will become clear later.

As the mechanism has to be incentive compatible for every agent, for now, we can ignore the individual rationality constraints and simply study the consequences of incentive compatibility. Suppose a mechanism designer knows the distribution of valuations, F, and wants to design a Bayesian Incentive Compatible mechanism. Since there is a continuum of agents, there is no aggregate uncertainty. Thus, it is without loss to focus on direct, Ex-Post Incentive Compatible (or strategy-proof) mechanisms. Moreover, as agents are symmetric other than their valuations, I restrict attention to anonymous mechanisms; that is, the designer does not condition the mechanism on agents' 'names'.

The designer will choose a mechanism described by the quantities and the transfers as functions of agents' reported valuations. Specifically, the allocations and transfers are given by functions $q : [0,1] \to \mathbb{R}$ and $t : [0,1] \to \mathbb{R}$, respectively. Thus, an agent who reports his valuation is θ will get $q(\theta)$ units of the good and will pay $t(\theta)$. Note that both the allocation and the transfer can be either positive or negative, depending on whether the agent is buying or selling. Hence, the net utility of the agent with the valuation θ from this mechanism is

$$u(\theta) = \theta \min\{1, q(\theta)\} - t(\theta).$$

As agents have demands for two units, having more than 2 units of the good is the same as having 2 unit for the agent. Therefore, the utility from the traded quantity is capped at 1 unit. Of course, in an optimal mechanism, agents will never receive more goods than they demand as I shown in the Appendix B.

Proposition 1. *Bid-ask mechanisms are without loss of profit.*

In the Appendix A.1, I solve the designer's problem for the simpler case of prohibitive search frictions where the decentralized trade is impossible as a warm up exercise.

1.2.2 Solution Concept

In this work, I focus on subgame perfect equilibrium in stage-undominated strategies. What this means is that given the mechanism announced by the designer, agents should play undominated actions in the following subgame. The reason for focusing on this selection is simple: Everyone joining the same market is always an equilibrium. However, these equilibria are very fragile. In particular, I will show that it is a dominated action for all agents to join the same market. Next, I show that coexistence is an undominated equilibrium. Finally, for technical reasons, I assume that the designer targets a closed set of agents as the participants of the marketplace in the equilibrium path.

Next result shows that if both markets are active, then the segmentation must be given by an interval structure where the agents with the lowest and highest valuations trade in the centralized market while the agents with intermediate valuations trade bilaterally.

Proposition 2. *If there is an equilibrium in which both markets are active (a coexistence equilibrium), there must be some cutoffs* $\underline{\theta}$ *and* $\overline{\theta}$ *such that in the equilibrium path, agents with valuations in* $(\underline{\theta}, \overline{\theta})$ *join the decentralized market while the rest of the agents join the centralized market.*

1.2.3 Designer's Problem

From the Proposition 2, we know that in any coexistence equilibrium, the agents who join the mechanism will have valuations in $[0, \underline{\theta}] \cup [\overline{\theta}, 1]$ for some $\underline{\theta}$ and $\overline{\theta}$ with $0 \leq \underline{\theta} \leq \overline{\theta} \leq 1$. We can write their utilities as follows, using payoff equivalence.

$$u^m(\theta) = \begin{cases} u^m(\overline{\theta}) + \int_{\overline{\theta}}^{\theta} q(x)dx & \text{if } \theta \geq \overline{\theta} \\ u^m(0) + \int_{0}^{\theta} q(x)dx & \text{if } \theta \leq \underline{\theta}. \end{cases}$$

In this environment, there will be different binding individual rationality constraints for agents below $\underline{\theta}$ and for agents above $\overline{\theta}$. Thus, writing the utilities in this form is more convenient. Similarly, we can write the transfers:

$$t(\theta) = \begin{cases} \theta q(\theta) - u^m(\overline{\theta}) - \int_{\overline{\theta}}^{\theta} q(x)dx & \text{if } \theta \geq \overline{\theta}, \\ \theta q(\theta) - u^m(0) - \int_{0}^{\theta} q(x)dx. & \text{if } \theta \leq \underline{\theta}. \end{cases}$$

Now, we can study the profit from the optimal allocation given the cutoffs. The step-by-step derivation can be followed in the Appendix A.4 but here is the end-result:

$$\Pi_{\underline{\theta},\overline{\theta}} = \underbrace{\mathbb{P}[\theta \in [0,\underline{\theta}]]}_{\text{Measure of sellers}} \underbrace{\mathbb{E}[t(\theta)|\theta \in [0,\underline{\theta}]]}_{\text{Expected payment of a seller}} + \underbrace{\mathbb{P}[\theta \in [\overline{\theta},1]]}_{\text{Measure of buyers}} \underbrace{\mathbb{E}[t(\theta)|\theta \in [\overline{\theta},1]]}_{\text{Expected payment of a buyer}}$$

$$= \underbrace{-F(\underline{\theta})u^m(\underline{\theta}) - (1-F(\overline{\theta}))u^m(\overline{\theta})}_{\text{Compensations for joining the mechanism}}$$

$$+ \underbrace{\int_0^{\underline{\theta}} \left[\left(x + \frac{F(x)}{f(x)}\right) q(x)\right] f(x)dx + \int_{\overline{\theta}}^1 \left[\left(x - \frac{1-F(x)}{f(x)}\right) q(x)\right] f(x)dx.}_{\text{Profit if there were no search market}}$$

Here, the agents will get some compensations for joining the centralized marketplace instead of the decentralized market. This is because the agents are giving up the opportunity to trade in the decentralized market when they join the centralized marketplace. Thus, the individual rationality constraints are endogenously determined by the equilibrium segmentation. The centralized marketplace has to pay the agents this lost 'opportunity cost', on top of the standard information rents created by the informational asymmetry. The exact form of the compensation will become clear below when we consider the individual rationality constraints.

I defined the virtual cost, $\mathcal{C}$ and the virtual value $\mathcal{V}$ as:

$$\mathcal{C}(x) = x + \frac{F(x)}{f(x)} \text{ and } \mathcal{V}(x) = x - \frac{1 - F(x)}{f(x)}.$$

I continue to assume both of them are increasing and call such distributions regular.

1.2.3.0.1 The Constraints The support of the distribution of valuations is $[0, 1]$. Thus, the first restriction is $0 \leq \underline{\theta} \leq \overline{\theta} \leq 1$.

Second, we have a market-clearing constraint in the form of

$$\int_0^{\underline{\theta}} q(x)f(x)dx + \int_{\overline{\theta}}^1 q(x)f(x)dx \leq 0.$$

This simply says that the designer cannot sell more than she buys.

Third, since each agent has one unit of endowment, they cannot sell more than that. Thus, we need $-1 \leq q(\theta)$. In fact, since the good is indivisible, each agent must have $q(\theta) \in \{-1, 0, 1\}$ as their allocation.

Fourth, we know that for incentive compatibility in the mechanism, we need the allocation to be increasing. The rest of the requirements of the incentive compatibility are already embedded in the transfers. So, as long as the allocation is increasing, the mechanism will be incentive compatible.

Finally, we need to consider the implications of the individual rationality.

1.2.3.0.2 Individual Rationality Notice that in any profit maximizing mechanism, individual rationality (IR) constraint for at least one type of agent in each segment who joins the mechanism ($[0, \underline{\theta}]$ and $[\overline{\theta}, 1]$) must bind. If not, then uniformly increasing the payment of all agents in the particular segment without a binding IR until there is a binding constraint increases the profit.

Moreover, the binding IR constraints in the optimal mechanism must be $\underline{\theta}$ and $\overline{\theta}$. To see this, notice that under the Assumption 1, we want to construct an equilibrium such that agents in $[0, \underline{\theta}]$ and $[\overline{\theta}, 1]$ join the mechanism while the rest of the agents join the search market. In the equilibrium, each agent will choose the market that offer him a higher utility. Then, in the equilibrium, agents with valuations in $[0, \underline{\theta}]$ and $[\overline{\theta}, 1]$ must have a higher utility in the centralized marketplace and agents with valuations in $(\underline{\theta}, \overline{\theta})$

must have a higher expected utility in the decentralized market. Since the utilities from both markets are continuous in valuations, this means the utilities from the decentralized market, u^d, and the utilities from the centralized marketplaces, u^m, must cross each other at $\underline{\theta}$ and $\overline{\theta}$. Thus, agents with valuations $\underline{\theta}$ and $\overline{\theta}$ must be indifferent between the markets and these are the binding individual rationality constraints:

$$u^d(\underline{\theta}) = u^m(\underline{\theta}) = u^m(0) + \int_0^{\underline{\theta}} q(x)dx \text{ and } u^m(\overline{\theta}) = u^d(\overline{\theta}).$$

This allows us the make the following observation, proved in the Appendix A.5.

Lemma 1. *In a coexistence equilibrium with cutoffs $\underline{\theta}$, $\overline{\theta}$, allocations must be such that:*

$$q(\theta) = \begin{cases} -1 \;\; \textit{if}\; \theta \leq \underline{\theta}, \\ 1 \;\; \textit{if}\; \theta \geq \overline{\theta}. \end{cases}$$

This is a very useful observation. On intervals where the allocation is constant, transfer is constant as well. Thus, with this lemma, we know that the marketplace will be equivalent to posting some bid-ask prices. From here, we also know that the agents with valuations below $\underline{\theta}$ can only be sellers in any market and agents with valuations above $\overline{\theta}$ can only be buyers in any market.

1.2.4 Existence of Coexistence Equilibrium

Here, I show how the designer can find the optimal pair of $\underline{\theta}$ and $\overline{\theta}$. The Main Result of this section shows that, a coexistence equilibrium exists. In this equilibrium, the marketplace designer attracts the agents with very low and very high valuations. The rest of the agents are left to search. Moreover, cutoffs are such that the marginal cost of the highest seller in the marketplace is equal to the marginal revenue of the lowest buyer in the marketplace. Finally, the measures of buyers and sellers in the marketplace are equal to each other.

Theorem 1. *Suppose F is regular. Then, there exists $\underline{\theta}, \overline{\theta}$ such that for any $p \in [0, 1]$, in the coexistence equilibrium,*

- *agents in $[0, \underline{\theta}]$ and $[\overline{\theta}, 1]$ join the mechanism,*
- *agents in $(\underline{\theta}, \overline{\theta})$ join the search market,*

- $C(\underline{\theta}) = \mathcal{V}(\overline{\theta})$ *and* $F(\underline{\theta}) = 1 - F(\overline{\theta})$.

The proof is in the Appendix A.7.

To illustrate the utilities agents are offered in the mechanism and expect from the search market, suppose everyone gets a meeting in the decentralized market ($p = 1$) and agents are uniformly distributed over the unit interval. The optimal cutoffs for the are $\underline{\theta} = \frac{1}{4}$ and $\overline{\theta} = \frac{3}{4}$. Figure 1.1 shows the utilities agents can expect from either market *in the equilibrium.*

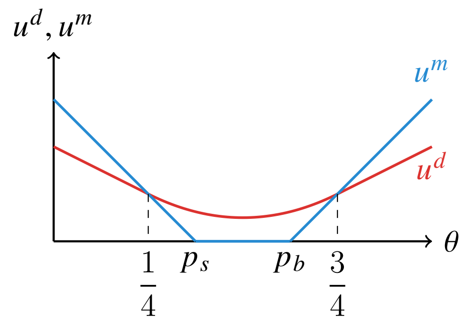

Figure 1.1. The utilities from the search market and the optimal mechanism under the coexistence equilibrium.

As it can be observed from the Figure 1.1, agents with intermediate types receive a lower utility in this equilibrium in the centralized marketplace. So they are happy to join the decentralized market. Moreover, the agents with low and high values are offered higher utilities in the centralized marketplace than they expect from the decentralized market. Thus, no agent has any unilateral profitable deviation.

The preceding theorem describes the structure of the coexistence equilibrium. However, it does not completely describe the mechanism that induces this equilibrium. Lemma 1 pins down the allocations and transfers for the agents who join the mechanism. What remains to be determined is what should be offered to agents in $[\underline{\theta}, \overline{\theta}]$.

In fact there are many mechanisms that would induce the same equilibrium that only differ in the off-path payoffs. Proposition 11 in the Appendix A.7.1 describes one such mechanism. As described before, it is equivalent to offering some bid-ask prices, i.e., prices for buying and selling. Essentially, I compute the transfers for the agents who join the mechanism and then extends the allocation and the transfer rules to the rest of the agents in a way that the allocations are increasing and we have $u^m(\theta) < u^d(\theta)$ for each $\theta \in (\underline{\theta}, \overline{\theta})$. The prices for buying and selling, p_b and p_s, implied by this proposition are the lowest and the highest valuations such that u^m is equal to 0, as can be seen in the Figure 1.1.

1.2.5 Profit of the Marketplace

Next, I establish the relationship between the profit in this equilibrium and the profit of the marketplace when there is no decentralized market. As the decentralized trade makes it costly to recruit agents to the centralized marketplace, it is not hard to guess that the decentralized trade decreases the profit of the centralized marketplace. Interestingly, the decrease in the profit only depends on the efficiency of the matching process in the decentralized market; it is independent of the distribution of the valuations.

Let Π^M denote the profit of the marketplace when there is no decentralized market.

Theorem 2. *Suppose F is regular. Then, the profit of the marketplace in coexistence is equal to $\left(1 - \frac{p}{2}\right)\Pi^M$.*

Moreover, the aggregate compensations agents receive in coexistence is equal to $\frac{p}{2}\Pi^M$.

It is surprising that the ratio of the profits is completely independent of the distribution of valuations. The ratio only depends on the probability that an agents finds a match in the search market, p. The intuition for this result is as follows. When there is no decentralized trade, the marketplace keeps all the surplus that remains after 'paying' the information rents. When agents can join the decentralized market, they find a match with probability p, and they get half of the surplus created by their meetings. I show that the expected surplus of a buyer and a seller is equal to the profit of the marketplace from each sale. Thus, in total, marketplace pays $\frac{p}{2}\Pi^M$ as compensation and the rest is kept as the profit of the marketplace.[5]

Since the ratio $1 - \frac{p}{2}$ is decreasing in p, the profit is also decreasing in p: As the matching process becomes more efficient, the decentralized market becomes more attractive. Then, the opportunity cost of each agent increases, meaning the compensations increase. Thus, the profit of the marketplace decreases.

Even if every agent finds a match in the decentralized market, meaning $p = 1$, the profit of the marketplace is half of the profit when it operates on its own. This provides a distribution-free lower bound for the profit as a function of Π^M. Here is why the marketplace can still make positive profit when $p = 1$. Due to the random matching process, even when each agent in the decentralized market gets a match with certainty, the matches may have small gains from trade. An agent might meet someone whose valuation is very close to her own. However, the marketplace solves this problem and makes profit by creating efficient matches. At the other extreme, $p = 0$, when there

[5] The result directly follows from the derivation of the coexistence profit in Appendix A.6.

is no match in the decentralized market, agents cannot trade bilaterally. Then, the marketplace makes the profit Π^M. Thus, the baseline with the marketplace alone is obtained as a special case.

Remark 1. The independence of compensation:profit ratio holds for off-equilibrium as well: Given any (potentially sub-optimal) cutoffs $\underline{\theta}$ and $\overline{\theta}$ for segmentation, the total compensations is equal to $\frac{p}{2}$ times the virtual surplus generated by this segmentation.

1.2.6 Thickness of the Marketplace

In this section, I compare the agents who trade on the centralized marketplace in the coexistence to those who get to trade in the marketplace when it operates on its own.

In the previous section, we have seen that the decrease in the profit of the marketplace is given by a ratio which is independent of the distribution. This means that the objective function of the marketplace, with or without the decentralized trade, is the same, up to a multiplication with a constant. Then, the solution -in terms of the allocation- is the same. The centralized marketplace wants to serve exactly the same types for each value of p including $p = 0$. Of course, for each value of p, the transfers are adjusted so that the cutoff types, $\underline{\theta}$ and $\overline{\theta}$, are always made indifferent between trading in the marketplace and the decentralized market.

Corollary 1. *The agents who trade on the marketplace are the same with or without the decentralized market. Thus, the thickness of the marketplace is unaffected by the decentralized trade.*

Although we can understand the corollary in light of the previous section, it is a counter-intuitive result on its own. When the agents have the option to trade in the decentralized market, they need to be compensated to join the centralized market. Thus, decentralized trade makes each trade in the marketplace costlier. Since the marketplace maximizes profit, it would be reasonable to expect the marketplace to become more exclusive to account for the higher costs. However, this is not optimal. Since the total compensations agents receive is a constant fraction of their virtual surplus, the optimal strategy is to maximize the virtual surplus. But Π^M is precisely the maximum attainable virtual surplus. Thus, marketplace keeps the same agents, no matter what happens in the decentralized market.

1.2.7 Welfare Comparisons

In this section, I first compare the consumers' welfare in a Pareto sense for different levels of p. Next, I compare the welfare generated in the coexistence to (i) welfare generated by the centralized market alone and (ii) welfare generated by the decentralized trade alone. For this analysis the welfare is measured as gains from trade. Thus, it includes the profit of the marketplace, when it exists. Omitted proofs are in the Appendix A.9.

1.2.7.1 Consumer Welfare

From Corollary 1, we know that the agents who trade in the decentralized market are the same. Then, conditional on meeting someone, agents' expected payoffs from the decentralized market are the same for each value of p. Thus, as p increases, agents' expected payoffs from the decentralized market increase. This directly leads to higher utilities for the intermediate types who join the decentralized market. It indirectly increases the payoff of each agent in the marketplace as well. When p increases, the cutoff types have a higher expected payoff from the decentralized market. Since the cutoff types remain the same, to make them indifferent under a higher p, centralized marketplace has to increase the compensations for each agent. Thus, a more efficient decentralized market increases the payoffs for all agents, regardless of their market choice.

Corollary 2. *Equilibrium payoff of each type is increasing in* p.

This provides a nice policy recommendation. Make the decentralized trade more efficient. It will make every agent strictly better off.

1.2.7.2 Total Welfare

Comparison between the coexistence and the centralized market alone is simple. Once again, by Corollary 1, the same agents trade on the marketplace, whether there is a decentralized market or not. Thus, the gains from trade generated on the marketplace is constant. However, in the coexistence, the agents with intermediate types also generate some trade. Thus, coexistence generates more gains under any regular distribution.

Corollary 3. *Coexistence leads to a higher welfare than the centralized market alone.*

For ease of exposition, I first focus on a concrete distribution for valuations to compare the welfare from the coexistence to the welfare from the decentralized trade alone. Later, I show that the result holds much more generally.

Proposition 3. *Suppose agents valuations are drawn from the uniform distribution over* $[0,1]$*. Then, the total welfare under the coexistence is greater than when either market operates on its own.*

Intuition behind this result is that the search market extends the extensive margin of trade by allowing more agents to trade while the centralized marketplace extends the intensive margin -by solving the search and matching frictions. Thus, when they operate together, everyone gets to trade with some probability and some agents trade for certain.

Although the proposition is stated for the uniform distribution, the result is true for many other distribution. Next, I provide a sufficient condition for distributions under which the coexistence is more efficient. This condition is satisfied by most commonly used distributions.

Assumption 1. Under F, the following inequality holds:

$$2\mathbb{E}[\theta F(\theta)|\theta \in [\underline{\theta}, \overline{\theta}]] + \mathbb{E}[\theta|\theta > \overline{\theta}] \geq \mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]] + \mathbb{E}[\theta F(\theta)|\theta \leq \underline{\theta}] + \mathbb{E}[\theta F(\theta)|\theta \geq \overline{\theta}],$$

where $\underline{\theta}$ and $\overline{\theta}$ are the optimal cutoffs for a coexistence equilibrium, given the distribution F.

Standard Normal Distribution, Logistic Distribution, Exponential Distribution, Standard Beta Distribution are among the distributions that has this property[6].

Proposition 4. *If F satisfies Assumption 1, then the total welfare under the coexistence equilibrium is greater than when either market operates on its own.*

1.2.8 Monopoly Equilibrium is in Dominated Strategies

If all agents join the centralized market, of course, unilateral deviations to the decentralized market are not profitable; you end up alone in a market. Thus, it is always an equilibrium for all agents to join the centralized market. Below I argue that this *monopoly equilibrium* is very fragile.

Proposition 5. *Each monopoly equilibrium is in dominated strategies.*

Proof. If all agents join the centralized market, outside option for each agent is not trading. Thus, for this subgame, the designer should announce a mechanism where

[6] Mathematica codes for computations that show these distributions satisfy this assumption are available upon request.

the individual rationality simply requires agents to get nonnegative payoffs. This is equivalent to the case of prohibitive search frictions where the probability of trade in the decentralized market is 0, so the analysis of the Appendix A.1 applies. Then, we learn that the designer must exclude agents with intermediate values, $(\underline{\theta}, \overline{\theta})$ from trade. This means these agents get 0 payoffs in the monopoly equilibrium. So in this case, their payoff is 0 in both markets.

Moreover, by construction of Proposition 11, the mechanism is always designed so that for each possible segmentation, the agents with intermediate valuations always receive a lower payoff than their expected payoff from search. Thus, for any segmentation other than the monopolization, they can expect a strictly higher payoff from the decentralized market.

Thus, joining the decentralized market weakly dominates joining the centralized market for agents with valuations in $(\underline{\theta}, \overline{\theta})$.

□

Here, the solution concept is actually 'stage-undominated subgame perfect equilibrium' because the domination depends on the designer choosing the optimal mechanisms for profit maximization. Of course, properly it should be called a 'stage-undominated perfect Bayesian equilibrium' as there is uncertainty about the agents' valuations ex-ante. However, as there is a continuum of agents and no aggregate uncertainty, it is natural to think about it as a subgame perfect equilibrium as well.

1.2.9 Coexistence Equilibrium with Double Auction

So far, I assumed that when agents are matched to each other in the decentralized market, they engage in Nash bargaining. This provides a nice benchmark since Nash bargaining is efficient: Even with an efficient trading protocol, the marketplace can attract the agents it targets and make positive profit. However, in Nash bargaining, the assumption is that the agents observe each others' valuations upon meeting. This creates an information asymmetry between markets since the designer cannot observe agents' valuations. Thus, we might be concerned that the results I have presented may not be robust to other models of decentralized trade. In this section, I consider an alternative bargaining protocol and I show that under some conditions, all of the main results hold true more generally.

Suppose the agents who join the decentralized market are randomly matched to each other and then they participate in a double auction. The random matching process is unchanged from the main model. The double auction works as follows. Each agent submits a bid, b_i. Agent with the higher bid buys the other's endowment and pays $\frac{b_i+b_j}{2}$, when bids are given by b_i and b_j; so they are trading a unit of the good at the midpoint of the bids.

This is a special case of the "simple trading rule" studied by Cramton et al. (1987). Assuming the valuations are drawn from the same, smooth distribution for each agent, they show that this game has a symmetric equilibrium where the bids are increasing in agents' valuations. Thus, the equilibrium is ex-post efficient in the sense that, the agent who values the good more ends up with the whole quantity. Moreover, Kittsteiner (2003) has shown that the symmetric equilibrium characterized by Cramton et al. (1987) is indeed the unique equilibrium.

Suppose the distribution of agents who participates in this double auction is given by some CDF, G. Suppose G is strictly increasing on its support, $[\underline{\theta}, \overline{\theta}]$, and is differentiable. Then, agents' bids in the unique equilibrium are given by:

$$b(\theta) = \theta - \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2}$$

It follows from the Proposition 5 of Cramton et al. (1987) that following this bidding strategy constitutes an equilibrium. Theorem 1 in Kittsteiner (2003) further shows that there is no other equilibrium.

Here, I focus on simple mechanisms that exclude an interval of agents: Suppose agents in $(\underline{\theta}, \overline{\theta})$ join the decentralized market and the rest join the centralized marketplace. Then, the endogenous distribution of agents in the decentralized market is simply F truncated from both below and above. Thus, $G(\theta) = \dfrac{F(x) - F(\underline{\theta})}{F(\overline{\theta}) - F(\underline{\theta})}$ on its support $(\underline{\theta}, \overline{\theta})$.[7]

In the Appendix A.10, I develop an analysis parallel to that on Nash bargaining for the double auction. I summarize my results in this section. Suppose the agents' valuations are drawn from the uniform distribution over $[0, 1]$. Then, essentially every result I obtained under the Nash bargaining holds true for the decentralized market with double auction as well:

Theorem 3. *There exists $\underline{\theta}, \overline{\theta}$ such that in the coexistence equilibrium,*

- *agents in $[0, \underline{\theta}]$ and $[\overline{\theta}, 1]$ join the mechanism,*

[7]The equilibrium of this double auction game when G has gaps in its support is not known.

- *agents in* $(\underline{\theta}, \overline{\theta})$ *join the decentralized market with double auction,*

- $C(\underline{\theta}) = \mathcal{V}(\overline{\theta})$ *and* $F(\underline{\theta}) = 1 - F(\overline{\theta})$.

Proposition 6. *The profit of the marketplace in coexistence is* $1 - \frac{5p}{6} = 1 - \frac{5m}{3}$ *times the profit the marketplace would make if there were no decentralized market with double auction.*

Proposition 7. *The total welfare under the coexistence equilibrium with the double auction is greater than when either market operates on its own.*

Proposition 8. *The agents who trade on the marketplace are the same with or without the decentralized market. Thus, the thickness of the marketplace is unaffected by the decentralized trade.*

Thus, exactly same agents trade in the centralized marketplace (i) for each level of friction in the decentralized market (including the extreme case of no decentralized trade with $p = 0$) and (ii) whether the decentralized trade happens according to Nash bargaining or a double auction.

1.3 Multiple Designers

A natural question to ask is how the equilibrium would change if instead of one centralized marketplace competing with a decentralized market, there were multiple centralized marketplaces competing with each other. This is closer to the models extensively studied by the competing platforms literature. However, it is still worthwhile to understand what the structure of the competition would look like within the framework of this paper.

Mechanism design problems with multiple designers are notoriously difficult to solve. The difficulty lies in the fact that when there are multiple designers, each one wants to condition her mechanism to the others' and none of them wants to post a mechanism that only depends on the agents' types. As a result, revelation principle does not hold in environments with multiple designers. Despite recent developments (Feng and Hartline, 2018), mechanism design without revelation principle is itself a notorious problem, even with a single designer.

One way the literature has dealt with this problem of infinite regress between competing mechanism is to explicitly assume that the designers each choose a direct mechanism. Although this is not without loss anymore, it at least provides a tractable way to study the problem. I will first follow this literature and then illustrate what else can be achieved with a non-direct mechanism.

Suppose there are n marketplace designers competing with each other for agents' participation. Each of them is a profit-maximizer. Suppose they are restricted to choose direct mechanisms. The timeline is as follows: First, designers announce their mechanisms simultaneously. Then, agents choose which marketplace to join simultaneously. Finally, trades in each marketplace realize simultaneously.

From the analysis of the Appendix B, we know that this is equivalent to saying that each designer i chooses a pair of prices (bid-ask prices) p_b^i and p_s^i for buying and selling in the marketplace i, respectively.

This environment is essentially a model of Bertrand competition between the marketplaces where agents' roles as buyers and sellers are endogenously determined. The reasoning from the Bertrand Equilibrium still holds: If some designer i makes a positive profit, then another designer j can offer slightly smaller bid-ask spread, and increase her own profit discontinuously by recruiting the agents who were trading on the marketplace i. Thus, there cannot be an equilibrium where a designer makes a positive profit. Conversely, there cannot be an equilibrium where a designer makes a negative profit: It is always feasible to make 0 profit by setting a prohibitively high price to everyone.

Next, I argue that there are equilibria where all designers make 0 profit: Consider the strategy profiles where (i) at least one of the marketplaces set $p_b = p_s = m(F)$ where $m(F)$ is the median of the distribution of the valuations, (ii) the rest of the marketplaces either shut down or set prohibitively unfavorable prices, and (iii) all agents uniformly randomize over marketplaces where $p_b = p_s = m(F)$. Clearly, none of the designers or agents can improve their payoffs strictly and all such strategy profiles constitute equilibria.[8] Finally, there is no equilibrium where all marketplaces are inactive: If this happened, then each marketplace would simply deviate to announcing the baseline mechanism and make a positive profit.

Notice that $m(F)$ is the price that would clear the market if we had a competitive market: Setting the price at $m(F)$ ensures that the supply is equal to demand. Thus, the equilibria of this game replicate the Walrasian equilibrium due to the nature of the competition here.

We have seen that introducing another profit-maximizing marketplace reduced the profits to zero. In the previous section, we have observed that when we instead considered

[8]Notice that there are many other payoff-equivalent equilibria. For instance, for $n = 2$, suppose we have $p_b = p_s = m(F)$ on both marketplaces. Then, the agents are indifferent between the marketplaces. Any segmentation such that the market is cleared in each marketplace is also an equilibrium.

a competing decentralized market, the profit only decreased by a constant ratio, always less than the half. This stark difference is a result of the restriction to direct mechanisms when they are not without loss of generality. Next, I will briefly discuss the simplest way in which mechanisms can depend on each other and how it allows the designers to recoup the profits.

Now suppose the designers can announce any kind of mechanism. Thus, they can make their prices depend on the other marketplaces. One of the simplest ways to utilize this interdependence is by posting prices together with price-matching guarantees.

In an equilibrium with price-matching guarantees, each designer i announces prices p_b^i and p_s^i, and also promises to honor the prices $\min_j\{p_b^j : j = 1, \ldots, n\}$ and $\max_j\{p_s^j : j = 1, \ldots, n\}$ as well; so each agent can get the lowest price for buying and the highest price for selling from each marketplace. This is in line with introducing price-matching guarantees in a Bertrand competition setting and obtaining an equilibrium with monopoly pricing (Hess and Gerstner, 1991).

In the framework of this paper, it is see an equilibrium for all designers to announce the prices that are equivalent to the baseline mechanism and for all agents to uniformly randomize over all the marketplaces. In this case, the designers are splitting the baseline profit equally. Thus, they essentially operate as a cartel.

It is worth noting that this is not the unique equilibrium in this environment. Specifically, it is still an equilibrium for each designer to announce the median of the valuations as the posted price for both the buyers and sellers.[9]

Finally, in addition to choosing among these marketplaces, suppose agents also have the option to join the decentralized market, after observing the posted mechanisms. Then, the same argument works with a minor modification: The designers would each post prices equivalent to the coexistence mechanism with price-matching guarantees and the agents with extreme values would randomize over the marketplaces uniformly. Since no marketplace has any incentives to deviate, they collectively act as the centralized market from the main model. Given that the centralized market is the same (on the equilibrium-path), no agent has any incentives to deviate either. Thus, the same segmentation would constitute an equilibrium with the addition that the agents with extreme values now randomize over the marketplaces.

[9]Requiring coalition-proofness for the designers as well would make sure that they are collectively achieving the baseline profit in the equilibrium.

Chapter 2 Stable Matchings under Complements and Substitutes

2.1 Introduction

In this paper, I study the many-to-one matchings with transfers. Since Kelso and Crawford (1982), it has been known that a stable matching exists as long as the doctors are gross substitutes for each hospital. However, this assumption is clearly too strong to be a necessary condition: It requires each hospital to consider each set of doctors as substitutes. It is easy to generate examples where a stable matching exists even if some hospitals have preferences that don't satisfy substitutability. In general, any assumption imposed on preferences of every agents individually will be not necessary.

Here, I first prove an equivalence between the stable matchings in this environment and core of an associated game that I define. Given this, I use Bondareva-Shapley Theorem to obtain a necessary and sufficient condition for existence of a stable matching. Moreover, this equivalence also shows that if the doctors could work on 'part-time' contracts at different hospitals during different parts of the day, there would be a stable matching in any environment.

The condition implied by the Bondavera-Shapley Theorem itself is not minimal in the computational sense: It ignores the special structure of the coalitions here since in this setting the deviations are accomplished by a set of doctors and at most one hospital. I exploit this structure to further reduce the number of inequalities that needs to be checked to verify the existence of a stable matching.

Beyond these, I propose measuring the instability in an economy by the minimal amount of government subsidy needed to 'stabilize' the efficient matching.

2.2 Setup

Let H and D be finite sets of hospitals and doctors, respectively and $N = H \cup D$. We endow each hospital h with a revenue function $r_h(D')$ where $D' \subset D$ is the set of doctors working for it. Then, the preferences of hospitals are given by the profit function: $\pi_h = r_h(D') - \sum_{d \in D'} w_d$ where D' is the set of workers working for it at wages given by w_d. Each doctor d has a preference represented by $u_d(h, w) = u_d(h) + w$ where h is the hospital the doctor d works for and w is doctor d's wage.

In this model, each doctor can work for at most one hospital while each hospital can hire any set of doctors at any wage to satisfy their needs.

For a matching μ with wages vector $(w_{hd})_{hd \in \mu}$, let $\mu(h)$ stand for the set of doctors matched with the hospital h and $\mu(d)$ be the hospital that the doctor d is matched with or $\emptyset$, in case she is unemployed. Also, let w_d^μ stand for doctor d's wage in this matching.

Definition 1. A matching μ with wages vector $(w_{hd})_{\{hd\} \in \mu}$ is **blocked** if there exists a hospital $h \in H$, a set of doctors $D' \subset D$ and wages for the doctors in D', $(w_{hd})_{d \in D'}$ such that there exists a set of doctors D'' with $D'' \subset D' \cup \mu(h)$ and:

- $r_h(D'') - \sum_{d \in D''} w_{hd} \geq r_h(\mu(h)) - \sum_{d \in \mu(h)} w_{hd}$ (New profit of the hospital h is at least as high as before);
- For each doctor $d \in D'$, $u_d(h, w_{hd}) \geq u_d(\mu(d), w_d^\mu)$ (New utility of each doctor in D' is at least as high as before);
- For at least one agent in $D' \cup \{h\}$, the corresponding inequality above is strict.

Definition 2. A matching μ and wages vector $(w_{hd})_{hd \in \mu}$ is a **stable outcome** if

(i) It is not blocked;

(ii) For each $d \in D$, $u_d(\mu(h), w_d^\mu) \geq u_i(\emptyset)$;

(iii) For each $h \in H$, $r_h(\mu(h)) - \sum_{d \in \mu(h)} w_d \geq 0$.

2.3 Existence Through Coalitional Games

2.3.1 Efficiency

Next, we consider a way to characterize the stability. We introduce an efficiency axiom.

Definition 3. An outcome (μ, w) satisfies **efficiency** if

$$\forall H^* \subset H, \forall D^* \subset D, \sum_{i \in H^* \cup D^*} w_i \geq \max_{\substack{D_h \cap D_{h'} = \emptyset \\ \bigcup_{h \in H^*} D_h \subset D^*}} \{ \sum_{h \in H^*} r_h(D_h) + \sum_{h \in H^*} \sum_{d \in D_h} u_d(h) \}. \tag{2.1}$$

Lemma 2. *If an outcome (μ, w) is stable, then it satisfies efficiency.*

Proof. Suppose (μ, w) is stable. Then, for each $h \in H$, for each $D_h \subset D$, we have

$$\sum_{i \in \{h\} \cup D_h} w_i \geq r_h(D_h) + \sum_{d \in D_h} u_d(h). \tag{2.2}$$

Then, for any $H^* \subset H$, for any disjoint collection of sets $\{D_h | h \in H^*\}$, we have

$$\sum_{i \in H^* \cup D^*} w_i \geq \sum_{h \in H^*} r_h(D_h) + \sum_{h \in H^*} \sum_{d \in D_h} u_d(h). \tag{2.3}$$

So, the efficiency holds since above inequality holds for any allocation of any $D^* \subset D$, including the maximizer of the total value.

□

2.3.2 Existence

Let $\mathcal{C}$ be the set of nonempty coalitions that has at most one hospital in it:

$$\mathcal{C} = \{H' \cup D' | H' \subset H; D' \subset D; |H'| \leq 1\}. \tag{2.4}$$

Let $|\mathcal{C}| = k$; $|N| = n$.

For each coalition in $\mathcal{C}$, we will define an indicator vector $\delta_C \in \{0, 1\}^n$ that will help us list all coalitions systematically.

$$\forall i \in N, (\delta_C)_i = \begin{cases} 1 \text{ if } i \in C; \\ 0 \text{ if } i \notin S. \end{cases} \tag{2.5}$$

For each coalition C in $\mathcal{C}$, we define the value as follows:

$$\forall i \in N, v(C) = \begin{cases} r_h(C \setminus \{h\}) + \sum_{d \in C \cap D} u_d(h), & \text{if } \{h\} = C \cap H, \\ 0, & \text{if } C \cap H = \emptyset. \end{cases} \tag{2.6}$$

Now, we extend the definition of the value function to any set of agent in the following way: For any $N' \subset D \cup H$,

$$\forall i \in N, v(N') = \max_{D_h \text{ s. t. } D_h\text{'s, } h \in H \cap N' \text{ partition } D \cap N'} \sum_{h \in N' \cap H} v(\{h\} \cup D_h).$$

Now, this is a transferable utility game and we use the results for non-emptiness of the core to characterize existence of a stable matching.

Proposition 9. *A matching economy with wages has a stable outcome if and only if the associated transferable utility game v has a non-empty core.*

Proof. Suppose there is a stable outcome (μ, w). Then, for any $N' \subset H \cup D$, we have:

$$\sum_{i \in N'} w_i \geq v(N'). \tag{2.7}$$

Hence, w belongs to the core and the core is nonempty.

Now, assume that the core is nonempty and $w \in C(v)$. Then, for any h, any set of doctors $D' \subset D$, we have:

$$\sum_{i \in \{h\} \cup D'} w_i \geq v(\{h\} \cup D'). \tag{2.8}$$

But then, the outcome (μ, w) is individually rational not blocked where μ is any matching under which w is a feasible payoff vector. Hence, there exists a stable outcome. □

Of course, given this, clearly, Bondareva-Shapley theorem provides the necessary and sufficient conditions for the existence of a stable matching in this economy.

For any set of agents, N', let $\delta_{N'}$ be the indicator vector of membership. ($\delta_{N'}(i) = 1$ if $i \in N'$ and $= 0$ otherwise for $i \in H \cup D$.)

Corollary 4. *There exists a stable outcome (μ, w) if and only if for any set of balancing coefficients $(b_{N'})_{N' \subset N}$ (i.e., for any vector $(b_{N'})_{N' \subset N}$ such that $\sum_{N' \subset N} b_{N'} \delta_{N'} = 1$), we have that $v(N) \geq \sum_{N' \subset N} b_{N'} v(N')$.*

2.4 Part-time Contracts

Now, we are going to consider a more general definition of a matching which allows doctors to divide their time between some hospitals. We will show that in this case, a stable matching always exists. We are still in the same environment as above with a set of hospitals, $\boldsymbol{H}$, a set of doctors, $\boldsymbol{D}$ and their preferences as well as the set of admissible coalitions $\boldsymbol{C}$.

Definition 4. A time allocation vector $(p_C)_{C \in \boldsymbol{C}}$ is a (feasible) **matching with fractions** if

$$\sum_{C \in \boldsymbol{C}} p_C \delta_C = 1. \tag{2.9}$$

Notice that a time-allocation vector that is a matching with fractions is a balancing set of coefficients in the sense of Bondareva-Shapley. Moreover, it is also clear by the definition above that everyone has 1 unit of time to allocate.

The preferences will have a more complicated form here. Given a matching μ with fractions $(p_C)_{C \in \boldsymbol{C}}$ and wages $(w_{d,C})_{p_C > 0}$, the profit of hospital h is

$$\pi_h = \sum_{C \in \boldsymbol{C}} p_C (v(C) - \sum_{d \in C} w_{d,C}).$$

The utility of a doctor d is

$$u_d(\mu) = \sum_{C \in \boldsymbol{C}} p_C (u_d(C \setminus D) + w_{d,C}).$$

Definition 5. A matching with fractions $(p_C)_{C \in \boldsymbol{C}}$ is blocked if there is a coalition $C^* \in \boldsymbol{C}$ such that the members of C^* can all improve their payoff by increasing p_C and at least one member can strictly improve their payoff.

Definition 6. A matching with fractions $(p_C)_{C \in \boldsymbol{C}}$ and wages $(w_{d,C})_{p_C > 0}$ is stable if (i) it is not blocked; (ii) total revenue of each hospital is positive; and (iii) total utility of each doctor is positive.

Now, we provide a lemma that establishes the efficiency of a stable matching with fractions. The proof is parallel to Lemma 2 so we only sketch the idea.

Lemma 3. *Let $(p_C)_{C \in \boldsymbol{C}}$ and $(w_{d,C})_{p_C > 0}$ be stable. Then, for any $N' \subset \boldsymbol{H} \cup \boldsymbol{D}$, their total payoffs exceed anything they could attain by reshuffling their times among members of N'.*

Proof. If their total payoff was lower than total payoff they could manage to get by deviations in N', then there would be at least one coalition C such that its members can profitably increase p_C, contradicting stability. □

The next lemma (which is a Corollary to Bondareva-Shapley Theorem) shows that the efficiency is not only necessary but also sufficient for a matching to be stable.

Lemma 4. *Suppose a matching is $(p_C)_{C \in \mathcal{C}}$ is efficient. Then, it is stable.*

2.5 Linear Representation of Stability

2.5.1 Some Matrices

Let $\mathcal{C}$ be the set of nonempty coalitions that has at most one hospital in it:

$$\mathcal{C} = \{H' \cup D' | H' \subset H; D' \subset D; |H'| \leq 1\}. \tag{2.10}$$

Let $|\mathcal{C}| = k$; $|N| = n$.

For each coalition in $\mathcal{C}$, we will define an indicator vector $\delta_C \in \{0,1\}^n$ that will help us list all coalitions systematically.

$$\forall i \in N, (\delta_C)_i = \begin{cases} 1 \text{ if } i \in C; \\ 0 \text{ if } i \notin S. \end{cases} \tag{2.11}$$

For each coalition C in $\mathcal{C}$, we define the value as follows:

$$\forall i \in N, v(C) = \begin{cases} r_h(C \setminus \{h\}) + \sum_{d \in C \cap D} u_d(h), & \text{if } \{h\} = C \cap H, \\ 0, & \text{if } C \cap H = \emptyset. \end{cases} \tag{2.12}$$

We define the following matrices to help us represent the problem of finding stable matchings as a linear program.

$$A = \begin{bmatrix} \delta_{C_1} \\ \vdots \\ \delta_{C_k} \end{bmatrix}_{k \times n} \quad b = \begin{bmatrix} v(C_1) \\ \vdots \\ v(C_k) \end{bmatrix}_{k \times 1} \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}_{n \times 1}$$

Let A_i be the i^{th} row of A. Consider $A_i w = \sum_{j \in C_i} w_i$. If $A_i w \geq v(C_i) = b_i$, then they cannot benefit from deviating to the coalition C_i, since no matter how they share the value, they cannot make everyone weakly better off and someone strictly better off. Thus,

C_i cannot be a blocking set. We are going to use this idea to state stability as a linear inequality.

We will also need a matrix P of dimensions $p \times k$ such that it consists of 0's and 1's, and each row will represent a Partition of $H \cup D$ using the coalitions that has at most one hospital and p is the number of such partitions. This matrix will help us in stating the feasibility of a certain payoff vector w in an economy.

2.5.2 2 Hospitals, 2 Doctors Case

We illustrate these concepts with the two doctors, two hospital case.

In this case, the set nonempty coalitions with at most one hospital, C will look like follows:

$$C = \{\{h_1\}, \{h_2\}, \{d_1\}, \{d_2\}, \{d_1, d_2\}, \{h_1, d_1\}, \{h_1, d_2\}, \{h_1, d_1, d_2\}, \{h_2, d_1\}, \{h_2, d_2\}, \{h_2, d_1, d_2\}\}$$

Then, A will simply give the indicator for agents in a coalition in each of its rows; b will give the values of the coalitions and w is just a payoff vector for all agents.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}_{11\times 4} \quad b = \begin{bmatrix} v(\{h_1\}) \\ v(\{h_2\}) \\ v(\{d_1\}) \\ v(\{d_2\}) \\ v(\{d_1, d_2\}) \\ v(\{h_1, d_1\}) \\ v(\{h_1, d_2\}) \\ v(\{h_1, d_1, d_2\}) \\ v(\{h_2, d_1\}) \\ v(\{h_2, d_2\}) \\ v(\{h_2, d_1, d_2\}) \end{bmatrix}_{11\times 1} \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}_{4\times 1}$$

For example, A_{12} (12th row of A) represents the matching $h_2 - \{d_1, d_2\}$. We see, $A_{12}w = w_2 + w_3 + w_4$. If $A_{12}w \geq v(h_2 - \{d_1, d_2\})$, then $\{h_2, d_1, d_2\}$ cannot be a blocking set since they cannot be better off with this block as there is simply not enough value.

Next, we represents the partitions (or matchings) of the economy with the matrix P below. For example, the first row of P below represents the partition $\{\{h_1\}, \{h_2\}, \{d_1\}, \{d_2\}\}$ which is basically the "empty matching" that has no active contract and the last row represents $\{\{h_1 - d_2\}, \{h_2 - d_1\}\}$, so it is the matching where d_2 works for h_1 and d_1 works for h_2.

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}_{10 \times 11}$$

For instance, consider P_{12}. Notice that $P_{12}b = v(\{h_1 - d_2\}) + v(\{h_2 - d_1\})$. We will use this to evaluate whether a given payoff vector w is feasible under a matching or not in the next section.

2.5.3 Results

2.5.3.1 Existence

Lemma 5. *Let $w \in \mathbb{R}^n$ be such that $Aw \geq b$. Then, $PAw \geq Pb$.*

Proof. Let $w \in \mathbb{R}^n$ be such that $Aw \geq b$. Notice that PAw is just a vector with a length equal to the number of partitions in P, p and each entry in PAw is equal to sum of all payoffs: $(PAw)_i = \sum_{P_{ij}=1}^{p} \sum_{j \in C_i} w_j = \sum_{k=1}^{n} w_k$ where the last equality follows from the fact that each row of P is a partition of $H \cup D$.

Also, Pb is another vector again with a length equal to the number of partition in P and each of its entry j gives the total value of the corresponding partition's coalitions: $(Pb)_j = \sum_{Pji=1} v(C_i)$.

Now, $Aw \geq b$ implies that for each coalition C, $\sum_{i \in C} w_i \geq v(C)$.

Then, of course, for each row j, $(PAw)_j \geq (Pb)_j$. □

Claim 1. *There exists a stable matching if and only if there exists $w \in \mathbb{R}^n$ with $Aw \geq b$ and $PAw \ngeq Pb$.*

Proof. ($\Longrightarrow$ by Contrapositive) Suppose for each $w \in \mathbb{R}^n$, either $Aw \ngeq b$ or $PAw > Pb$. Let $w \in \mathbb{R}^n$.

If $PAw > Pb$, then, there is no partitioning of $H \cup D$, Q such that $\sum_{k=1}^{n} w_k = \sum_{C_j \in Q} v(C_j)$. Then, this w is not a feasible payoff vector in our environment and it cannot arise in any matching.

If $Aw \ngeq b$, then, there exists j such that $\sum_{i \in C_j} w_i = \sum_{i \in N} \delta_i w_i = (Aw)_j < b_j = v(C_j)$. To write this in economic terms, we have seen that for payoff vector w, there exists a coalition $C_j \in C$ such that $\sum_{i \in C_j} w_i < v(C_j)$. Then, either (μ, w) is not IR or C_j is a blocking set of agents. In either case, (μ, w) is not stable.

($\Longleftarrow$ by Contrapositive) Conversely, suppose there does not exist a stable matching. Then, for each matching μ and payoff vector w with $PAw \geq Pb$ and $PAw \ngtr Pb$, either (μ, w) is not IR for some agents or there exists some blocking set S. In either case, for some $C_j \in C$, we have $\sum_{i \in C_j} w_i < v(C_j)$. This means for each $w \in \mathbb{R}^n$, there exists $j \in N$ such that $b_j = v(C_j) > \sum_{i \in C_j} w_i = \sum_{i \in N} \delta_i w_i = (Aw)_j$, meaning $Aw \ngeq b$. □

2.6 Measuring Instability

Suppose we have an economy in which no stable outcome exists. Then, the next natural question is how far away is this economy from being stable. In order to answer this question, we consider the following problem.

$$
\begin{aligned}
w^* \in \arg\min_w \quad & \sum_{i \in N} w_i \\
s.t. \quad & \\
& Aw \geq b.
\end{aligned}
\tag{2.13}
$$

w^* is a minimal -in the sense of having the smallest sum- payoff vector that doesn't admit any blocking set. Then, $\sum_{i \in N} w_i^* - \max_j (PAb)_j$ is the amount of resources needed to stabilize the economy, since we can suggest the matching that corresponds to the j^{th} row of P and support it with difference above.

Chapter 3
Equilibria of Convex Economies

3.1 Introduction

Convexity has been one of the most common assumptions in many different lines of economic research, including showing the existence of equilibrium as well as in studying the efficiency, stability and other properties of the equilibria (Mas-Colell et al. (1995); Kreps (2012)). However, Euclidean convexity that we commonly refer to as convexity is not the only notion of betweenness. Even though the mathematical theory of abstract convexities has been developing for decades, economics has not benefited from these developments until recently. Although there were earlier attempts to incorporate these structures within economic frameworks (most notably Koshevoy (1999) and Nehring and Puppe (2007)), Richter and Rubinstein (2015) and Richter and Rubinstein (2017) introduced them to general equilibrium and decision theory contexts.

In the first part, I show that the existence of Nash equilibrium for generalized games can be extended to much more general convexities with appropriate modifications of the assumptions. Building on this result, I show the existence of competitive equilibrium under certain assumptions related to the interactions between the abstract convexity and the underlying topological space.

One important distinction to make is that in the equilibrium definition, price vector is replaced by a public ordering, as defined by Richter and Rubinstein (2015). The public ordering that clears the market orders the consumption bundles in the economy in a way that each individual maximizes her preferences when her budget is all the bundles rankled lower than her endowment. This generalizes the standard prices since in the standard economy, costs of consumption bundles define a public ordering such that the lower contour set of the endowment in this order is the budget set for each agent.

3.2 Abstract Convexities

3.2.1 Definitions and Familiar Results

We first introduce the concept of abstract convexity, then review and verify that many of the standard results about the standard convex preferences hold for abstract convex preferences as well with very similar proofs.[1] First, we want to define what is an abstract convexity and what it means to for a preference to be convex in this context. Preferences are complete and transitive binary relations in what follows unless otherwise stated.

We first give the general definition of convexity.

Definition 7. A family $\mathcal{C}$ of subsets of a set X is called a **convexity** on X if

(C-1) The empty set $\emptyset$ and the universal set X are in $\mathcal{C}$;

(C-2) $\mathcal{C}$ is stable for intersections;

(C-3) $\mathcal{C}$ is stable for nested unions.

The pair $(X, \mathcal{C})$ is called a **convex structure**.

Notice that all three of above axioms of convexity are satisfied by the standard convex sets. Next, we define the convex hull for a convexity in the same way we do for the standard convexity.

Definition 8. Given a convex structure $(X, \mathcal{C})$, for $A \subset X$, **convex hull** of A is defined as:

$$K(A) = \bigcap \{C | A \subset C \in \mathcal{C}\}. \tag{3.1}$$

Moreover, a set A is said to be convex if $A \in \mathcal{C}$ or equivalently $A = K(A)$.

Next, we define convex preferences more generally as was introduced by Richter and Rubinstein (2017).

Definition 9. A preference $\succeq$ is **convex** w.r.t. $\mathcal{C}$ if for any alternative $a \in X$, the strict upper contour set $U(\succeq, a) = \{x | x \succ a\}$ is convex w.r.t. $\mathcal{C}$.

[1] Most of the material about convexities discussed in this section can be found in van de Vel (1993b).

Some simple but useful results has not been proven for these general convexities. So we prove them here to be prepared for the next section. The next definition introduces a measure of richness of the convexity by categorizing the convexities based on their polytopes. This will be useful in the next two lemmas.

Definition 10. A convex structure is of **arity** $\leq n$ provided its convex sets are precisely the sets C with the property that $co\{F\} \subset C$ for each subset F with $\#F \leq n$. (We will say of arity n (rather than of arity at most n) for ease of reading unless the distinction is important.)

Lemma 6. *Suppose the convexity on X is of arity n. If $\geq$ is convex, then the set of $\geq$-best points in a convex set is convex.*

Proof. Let $A \subset X$ and $B(\geq, A)$ be the set of $\geq$-best points in A. Assume that A is convex. Suppose there are n points $x_1, \ldots, x_n \in B(\geq, A)$. Then, $x_i \sim x_j$ for each $i, j = 1, \ldots, n$. Let $z \in K(\{x_1, \ldots, x_n\})$. Since $\geq$ is convex, the (weak) upper contour set is a convex set: $UC(\geq, x) = K(UC(\geq, x))$ and by monotonicity of the convex hull operator, $z \in UC(\geq, x)$ so that $z \geq x$. Also, since $x \in B(\geq, A)$, for each $t \in A$, we have $x \geq t$. Then, by transitivity, combining $z \geq x$ and $x \geq t$ yields $z \geq t$ for each $t \in A$. Hence, $z \in B(\geq, A)$ and $B(\geq, A)$ is convex. □

Definition 11. Suppose the convexity on X is of arity n. Let $f : X \to \mathbb{R}$. f is **quasi-concave** if, $f(z) \geq \min\{f(x_1), \ldots, f(x_n)\}$ for each z in $K(\{x_1, \ldots, x_n\})$.

Lemma 7. *Suppose the convexity on X is of arity n. Let $\geq$ be preference on X and a function $f : X \to \mathbb{R}$ represents it. $\geq$ is convex, if and only if, f is quasi-concave.*

Proof. If $\geq$ is convex, then the weak upper contour sets are convex: $UC(y, \geq) = K(UC(y, \geq))$ for each $y \in X$. Now, let $x_2, \ldots, x_n \in UC(y, \geq)$. By monotonicity of the convex hull operator, $K(y, x_2, \ldots, x_n) \subset UC(y, \geq)$. Then, for each $z \in K(y, x_2, \ldots, x_n)$, $z \geq y$. So, $u(z) \geq u(y) = \min\{u(y), u(x_2), \ldots, u(x_n)\}$ so that $u(\cdot)$ is quasi-concave.

Conversely, if $u(\cdot)$ is quasi-concave, take any $y, x_2, \ldots, x_n \in X$ such that $f(x_2), \ldots, f(x_n) \geq f(y)$. Then, by quasi-concavity, $f(z) \geq f(y)$ for each $z \in K(y, x_2, \ldots, x_n)$ and hence upper contour set of y is convex. □

3.2.1.1 A Specific Class of Convexities of Economic Interest

Richter and Rubinstein (2015, 2017) suggested a way to generate a convexity based on some *primitive orderings* in economic contexts:

Definition 12. Let X be a set of outcomes and Λ a set of (complete and transitive) binary relations that we refer to as **primitive orderings**. Then, convex hull operator of Λ-convexity on X is defined as follows: For each $A \subset X$

$$K(A) = \{x | \forall \geq_k \in \Lambda, \exists a_k \in A \text{ s.t. } x \geq_k a_k\}. \tag{3.2}$$

Remark 2. Richter and Rubinstein (2015) establishes that for every convex preference with respected to a Λ-convexity, the weak upper contour sets are also convex: $WU(\geq, a) = \{x | x \geq a\}$.

Then, this gives rise to the following equivalent definition of a convex preference for a convexity generated by primitive orderings, characterized in Proposition 6 of Richter and Rubinstein (2017).

Definition 13. A preference $\geq$ is **Λ-convex** if for any two alternatives $a, b \in X$; if for each $\geq_k \in \Lambda$, there is an alternative $y_k \in X$ with $y_k \neq b$ such that (i) $b \geq_k y_k$ and (ii) $y_k \geq a$, then $b \geq a$.

One motivation they provided is as follows: Suppose some professors are evaluating job market candidates based on research, teaching and charm. To convince them that candidate b should be chosen rather than a, you would find for each criterion an alternative c which is inferior to b according to that criteria but they still prefer c to a.

This approach encompasses the standard convexity as well. Let $X = \mathbb{R}^n$. Then, the orderings induced by all linear functionals in $\mathbb{R}^n$ would define the standard convexity. If we only consider the orderings induced by the strictly positive linear functionals, then this corresponds to a convexity such that if a preference is convex with respect to it, then it is strictly increasing.

3.2.2 Existence of a Nash Equilibrium

We are going to prove the existence of a competitive equilibrium by constructing a generalized game where a player's actions affect the other player's feasible actions. In this section, we prove the existence of a Nash equilibrium in a generalized game where the players have convex preferences and convex strategy spaces.

We present a generalization of Kakutani's fixed point theorem that replaces the standard convexity with an abstract one. Before stating the theorem, we give definitions of several concepts needed for this theorem:

- A Topological Convex Structure (tcs) is a set X with a convexity $\mathcal{C}$ and a topology $\mathcal{T}$ such that all polytopes of $\mathcal{C}$ are closed in $\mathcal{T}$.

- In a convexity, a *half-space* is a convex set whose complement is also convex.

- A tcs X is S_4 if for each pair of disjoint and non-empty convex sets C, D, there exists a half-space $H \subset X$ with $C \subset H$ and $D \subset X \setminus H$.

- A tcs X is *closure stable* if closure of each set is also convex.

- a tcs X is *properly locally convex* if each $x \in X$, has a *neighborhood base* of convex open sets. (I.e., a family of open and convex sets $\{N_\alpha\}$ such that every neighborhood N of x includes some member N_α of this family: $N_\alpha \subset N$.)

- A tcs X is FS_4 if for each pair of disjoint and non-empty convex closed sets C, D, there exists a continuous CP functional of X separating C and D.

Lemma 8 (Theorem 6.15 of Chapter 4.6, van de Vel (1993b))**.** *Let X be a compact Hausdorff tcs with connected convex sets. Let $F : X \rightrightarrows X$ be a nonempty-, convex-, closed-valued upper hemi-continuous correspondence. If either X is properly locally convex, closure stable and S_4 or X is FS_4, then F has a fixed point.*

Remark 3. To be able to use Berge's theorem, we need a metric space anyway and metric spaces are Hausdorff topological spaces. In what follows, we restrict ourselves to compact, properly locally convex, closure stable and S_4 metric spaces with connected convex sets. We could have considered FS_4 spaces as well but we do not take that route and focus on generalizations of Euclidean spaces here for now.

We will need the following lemma when we construct the "joint best responses" correspondence and apply Lemma 8.

Lemma 9. *Consider some tcs $\{X_1, \ldots, X_n\}$. If each X_i is a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets, the their product tcs $X = \Pi_{i=1}^{n}$ is also a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets.*

Proof. Consider the product tcs of some tcs $\{X_1, \ldots, X_n\}$ which are compact, properly locally convex, closure stable and S_4 metric spaces with connected convex sets. It is a standard result from topology that compactness and connectedness is preserved under products. Closure stability follows from theorem 1.10 of chapter 3.1 in van de Vel (1993b).

The product is S_4 by theorem 3.15 of chapter 1.3 in van de Vel (1993b). Finally, it is properly locally convex which can be seen by taking the product of neighborhood base in each dimension. □

We record the version of Berge's Theorem (or Theorem of Maximum) we will be using next.

Lemma 10. *Let X and Y be metric spaces and $f : X \times Y \Rightarrow \mathbb{R}$ be a continuous function. Suppose that $F : X \Rightarrow Y$ is a correspondence that is upper and lower hemi-continuous, and compact- and nonempty-valued. Then, the optimal choice correspondence $Z : X \Rightarrow Y$ defined by $Z(x) = \arg\max_{y \in F(x)} f(x, y)$ is upper hemi-continuous and the maximum value function $m : X \Rightarrow \mathbb{R}$ is continuous.*

Finally, we give formal definitions of a generalized game and a Nash equilibrium for such a game.

Definition 14. An n-player generalized game $G = (\{A_i, C_i, u_i\}_{i=1}^n)$ consists of, for each $i = 1, \ldots, n$,

(i) A set of actions A_i;

(ii) A constraint (feasibility) correspondence $C_i : A_{-i} \Rightarrow A_i$ (where $A_{-i} = \Pi_{j \in \{1,\ldots,n\} \setminus \{i\}} A_j$);

(iii) A utility function $u_i : \Pi_{j=1}^n A_j \Rightarrow \mathbb{R}$.

Definition 15. A Nash equilibrium for a generalized game $G = (\{A_i, C_i, u_i\}_{i=1}^n)$ is a strategy profile $(a_i^*)_{i=1}^n \in \Pi_{i=1}^n A_i$ such that, for each $i = 1, \ldots, n$,

(i) $a_i^* \in C_i(a_{-i}^*)$;

(ii) a_i^* maximizes $u(a_i, a*_{-i})$ over $a_i \in C_i(a_{-i}^*)$.

Now, we can prove the existence of Nash equilibrium in a generalized game.

Theorem 4. *Suppose that $G = (\{A_i, C_i, u_i\}_{i=1}^n)$ is a generalized game for which*

(i) Each A_i be a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets;

(ii) Each C_i is a continuous, non-empty-, closed- and convex-valued correspondence;

(iii) Each u_i is jointly continuous in $a \in A = \Pi_{j=1}^{n} A_j$ and represents a convex preference $\succeq_i$.

Then, G has a Nash equilibrium.

Proof. Consider the problem of maximizing the real-valued $u_i(a_i, a_{-i})$ with respect to $a_i \in C_i(a_{-i})$ for each $a_{-i} \in A_{-i}$ and for each $i \in N$. Then, applying Berge's theorem directly thanks to assumptions made above, we see that the best response correspondences $A_i^* : A_{-i} \rightrightarrows A_i$ are non-empty- and compact-valued, and upper hemi-continuous.

Take any $x \in \mathbb{R}$. Since we assumed that u_i represents a convex preference, its argmax on a convex set is convex by lemma 6. Hence, A^* is convex-valued for each player.

Define the joint best response correspondence $A^* : A \rightrightarrows A$ as follows: At each $a \in A$, we have $b \in A^*(a)$ if for each $i \in N$, $b_i \in A^*(b_{-i})$. It inherits the properties of individual best responses by lemma 9 and hence it is a non-empty-, convex-, closed-valued and upper hemi-continuous correspondence in a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets. Hence, by Kakutani's theorem 8, G has a fixed point and by construction, the fixed point is a Nash equilibrium. □

Corollary 5. *Suppose that $G = (\{A_i, C_i, u_i\}_{i=1}^{n})$ is a generalized game for which*

(i) Each A_i be a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets;

(ii) Best Response of each individual, $A_i^*(a_{-i})$ is an upper hemi-continuous, non-empty-, closed- and convex-valued correspondence.

Then, G has a Nash equilibrium.

Proof. The proof is same as the last step of the previous proposition: Define the joint best response correspondence $A^* : A \rightrightarrows A$ as follows: At each $a \in A$, we have $b \in A^*(a)$ if for each $i \in N$, $b_i \in A^*(b_{-i})$. It inherits the properties of individual best responses by lemma 9 and hence it is a non-empty-, convex-, closed-valued and upper hemi-continuous correspondence in a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets. Hence, by Kakutani's theorem 8, G has a fixed point and by construction, the fixed point is a Nash equilibrium. □

3.2.3 Topology and Convexity of Space of Public Orderings

As it is common in the general equilibrium theory (see for example Kreps (2012)), our existence result will employ an auctioneer who chooses a 'price' to clear the markets. However, unlike the standard framework where the set of strategies of the auctioneer can be reduced to the unit simplex of relevant dimension, it is a complicated object in this setting: The auctioneer chooses a public ordering. Notice that this is a generalization of the standard economy with prices. For example, with linear prices, possible price vectors are non-negative linear functionals, which are among primitive orderings of the standard economy. Moreover, linear prices also define an ordering on the set of allocations in the natural way: If $p \cdot x \geq p \cdot y$ then $x \succeq_P y$.

Showing that the set of strategies of the auctioneer is well-behaved enough to be used in Theorem 4 requires quite a lot of work and involves defining a convexity and topology on public orderings that are consistent with the individual's consumption spaces' convexities and topologies, then verifying that this convexity and topology inherits topological and convexity related properties of each individual's consumption space. Therefore, we reserve this for the appendix and summarize the results here.

Assumption 2. We assume that for an economy $\mathcal{E}$, the consumption sets of individuals are same: $X_i = X_j$ for each $i, j = 1, \ldots, n$.

This assumption ensures that we have a clear interpretation of the public ordering. If we don't make this assumption, the public ordering would be comparing the consumption bundles in spaces distinct from each other. Then, there would be situations in which an agent can afford a bundle according to the public ordering but this bundle is not even in his consumption space. We can deal with this mathematically but it would make little sense to do it that way.

An alternative, which is mathematically feasible is to define the public ordering as the union of public orderings of each individual's consumption space. I.e., $P \subset \cup_{i=1}^{n} X_i \times X_i$. Everything we are doing here can be done with this with appropriate adjustments but it would still be more difficult to interpret such an equilibrium with such a public ordering.

It is worth noting that we do not assume the consumption spaces of individuals to be endowed with the same convexity or topology. Even though the sets X_i are the same, we keep using the subscript to distinguish them from their product and to emphasize the individual convexities and topologies.

The following lemma which has been proven in the Appendix C shows that the set of public orderings of an auctioneer satisfies the assumptions of 4.

Lemma 11. *Let X_i be a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets. Then, the space of closed subsets of X_i^2 can be endowed with the Hausdorff metric, Vietoris topology and Vietoris convexity. Moreover, this tcs is also a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets.*

3.2.4 Feasible Trades and Admissible Public Orderings

We haven't yet specified the set of feasible trades. We do it implicitly by introducing a set of feasible allocations $\mathcal{F} \subset X = \Pi_{i=1}^n X_i$, given a profile of initial endowments. (Since the endowments will be fixed, we do not explicitly show the dependence of the set to the endowments in our notation.) We denote the set of admissible public orderings for an economy by $\mathcal{P}$ as defined in C.2.

We revise the definition of Walrasian equilibrium accordingly.

Definition 16. A Walrasian equilibrium for an economy $\mathcal{E} = \left\langle N, (X_i, \succeq_i, e^i)_{i=1}^n, \mathcal{F} \right\rangle$ is a public ordering $p \in \mathcal{P}$ and an allocation profile $x \in X$ such that

(i) For each consumer i, x^i solves the problem: maximize $u_i(y^i)$ subject to $y^i \in X_i$ and $e^i \succeq_p y^i$;

(ii) Markets clear: $x \in \mathcal{F}$.

Assumption 3. Given an economy $\mathcal{E} = \left\langle N, (X_i, \succeq_i, e^i)_{i=1}^n, \mathcal{F} \right\rangle$, there exists a preference $\succeq_A$ on $X \times \mathcal{P}$ such that

(i) It is represented by a continuous function $u_A(x, p)$ that is quasi-concave in p;

(ii) If $x \in X$ with $x \notin \mathcal{F}$ and $q \in \arg\max_{p \in \mathcal{P}} u(x, p)$, then there exists $i \in N$ such that $x^i \succ_q e^i$.

This assumption ensures that there is a well-behaved preference on the set of allocation and public ordering bundles such that, if some allocation is not feasible from some profile of initial endowments, then any public ordering that maximize this preference given this allocation has the property that there is at least one individual who cannot afford his allocation under this ordering.

Even though this higher order assumption looks obscure, considering the fact that the set of feasible allocation profiles is entirely arbitrary in this model, this is a necessary restriction. In the standard setting, the Walrasian auctioneer's preferences are continuous and quasi-concave in prices and the second part of the assumption is also satisfied thanks to linearity of the prices. Thus, this assumption is simply a generalization of what is implied by the linear price structure in the standard case.

In proving the existence of a Walrasian equilibrium under standard convexity, we consider an artificial auctioneer whose utility function is the value of the excess demand. In that case, it is of course possible for the auctioneer to find a price vector that would make some individual violate his budget, if the allocation profile is infeasible.[2] This assumption is an abstract counterpart to it. Thus, it serves a similar purpose and it will help us show the market clearance.

3.2.5 Existence of Competitive Equilibrium

The following lemma shows the continuity of the budget sets in public orderings and it has been proved in the Appendix C 22.

Lemma 12. *Consider an economy* $\mathcal{E} = \left\langle N, (X_i, \succeq_i, e^i)_{i=1}^n, \mathcal{F} \right\rangle$ *where* X_i *is a compact, properly locally convex, closure stable and* S_4 *metric space with connected convex sets and it is identical for each individual.*[3] *Let* p *be a continuous public ordering. Then,* $B_i(p) = \{x \in X_i | e^i \succeq_p x\}$ *is continuous.*

Now, we give an abstract version of previous Walrasian equilibrium existence theorem.

Proposition 10. *Consider an economy* $\mathcal{E} = \left\langle N, (X_i, \succeq_i, e^i)_{i=1}^n, \mathcal{F} \right\rangle$. *Assume:*

(i) For each $i, j \in N$, $X_i = X_j$ *and* X_i *is a compact, properly locally convex, closure stable and* S_4 *metric space with connected convex sets.*

(ii) $\succeq_i$ *is continuous and convex. (We take* u_i *to represent* $\succeq_i$ *for each individual* i *in* N.*)*

(iii) There exists a utility function u_A *on* $X \times \mathcal{P}$ *such that it is continuous in both arguments and quasi-concave in* $p \in \mathcal{P}$*; and if* $x \in X$ *with* $x \notin \mathcal{F}$ *and if* q *maximizes* $\succeq_A$ *given* x*, then there exists* $i \in N$ *such that* $x^i \succ_q e^i$.

[2]Indeed, the last step of the standard proof of existence of competitive equilibrium would also show this, using its contrapositive Kreps (2012).

[3]Notice that we require the set of possible allocations X_i to be identical for each agent but we do not require the associated topologies and convexities to be the same. Moreover, we do not rule out the possibility that some bundles are not available for some agents, it is implicitly embedded in the definition of feasible allocation profiles, $\mathcal{F}$.

Then, $\mathcal{E}$ has a Walrasian equilibrium.

Proof. Consider the following generalized game.

- The players are the individuals in N and an auctioneer.
- The strategy space $\mathcal{P}$ of the auctioneer is a non-empty convex compact subset of $c(X_i^2)$ such that each $p \in \mathcal{P}$ is a continuous, concave[4] and reflexive ordering on X_i.
- Each individual i's strategies are constrained by choice of the auctioneer: He must choose x^i from the set $\{x \in X_i | e^i \geq_p x\}$ where $\geq_p$ is the public ordering that the auctioneer chose. When i chooses x, his utility is $u_i(x)$.
- We endow the auctioneer with the preference $\geq_A$.

Each individual's strategy space has been assumed to be a non-empty tcs such that it is a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets, so the first condition of the previous proposition holds for them.

The auctioneer's strategy space is also given similarly as shown by lemma 11.

The auctioneer's feasibility correspondence is constant (hence continuous) and is equal to a compact, convex, non-empty set everywhere and hence it satisfies the second condition of the previous proposition.

The continuity and convexity of preferences of individuals is assumed.

The continuity of preferences of the auctioneer is also assumed.

Now, the feasibility correspondences of the individuals (as well as the auctioneer) are clearly convex (by concavity of the public ordering), compact (by continuity of the public ordering and the fact that X_i is compact) and nonempty (by reflexivity of the public ordering) everywhere. The trouble is continuity and it has been shown in the 22.

Then, Theorem 4 applies and there is a Nash equilibrium $(p, (x^i))$ of this generalized game. We want to show that this is a Walrasian equilibrium of this economy. Utility maximization is obvious by construction. We need to verify that markets clear.

Now, suppose $x \notin \mathcal{F}$. Then, we know that the best response of the auctioneer would require him to choose a public ordering such that at least one individual i cannot afford x^i given his endowment e^i under this public ordering. So, there cannot be a Nash equilibrium in which $x \notin \mathcal{F}$. Hence, the Nash equilibrium must be a Walrasian equilibrium of this economy.

□

[4]By a concave ordering, I mean one whose lower contour sets are convex.

Appendix A
Omitted Proofs from Chapter 1

A.1 Prohibitive Search Frictions

In the case of prohibitive search frictions, agents cannot trade outside the marketplace. Thus, the outside option is receiving 0 net utility from trade for each agent. The profit of the marketplace is the expected net payments. Thus, the designer seeks to maximize total payments, given incentive compatibility, individual rationality, and feasibility constraints.

$$
\begin{array}{lll}
\max\limits_{(q,t)} & \mathbb{E}_\theta\left[t(\theta)\right] & \\
\text{s. t.} & & \\
\text{(IC)} & \theta \min\{1, q(\theta)\} - t(\theta) & \geq \theta \min\{1, q(\theta', \theta)\} - t(\theta') \\
\text{(IR)} & \theta \min\{1, q(\theta)\} - t(\theta) & \geq 0 \\
\text{(Individual Feasibility)} & q(\theta) & \geq -1 \\
\text{(Aggregate Feasability)} & \mathbb{E}_\theta\left[q(\theta)\right] & \leq 0
\end{array}
$$

In an Online Appendix, I develop a series of lemmata to simplify this problem.[1] They allow the problem to be restated in terms of the virtual values and virtual costs, defined as follows.

Definition 17. An agent with reported valuation θ has virtual value, $\mathcal{V}(\theta)$, and virtual cost, $C(\theta)$, given by:

$$\mathcal{V}(\theta) = \theta - \frac{(1 - F(\theta))}{f(\theta)} \text{ and } C(\theta) = \theta + \frac{(F(\theta)}{f(\theta)}.$$

Virtual value and virtual cost can be thought of as the marginal revenue and marginal

[1] Online Appendix is available at this link.

cost. When an agent is a seller, that is, when an agent has a negative allocation, $q(\theta) < 0$, his deduction from the profit of the marketplace is the virtual cost. Similarly, when an agent is a buyer, $q(\theta) > 0$, his contribution to the profit is the virtual value. Then, the problem can be restated in these terms as follows:

$$\begin{aligned}
\max_{q(\cdot)} \quad & \mathbb{E}\left[q(\theta)\left(\mathbb{1}\{q(\theta) < 0\}\mathcal{C}(\theta) + \mathbb{1}\{q(\theta) > 0\}\mathcal{V}(\theta)\right)\right] \\
\text{s. t.} \quad & \\
& q(\theta) \text{ is increasing} \\
& q(\theta) \geq -1 \\
& \mathbb{E}_\theta\left[q(\theta)\right] = 0
\end{aligned}$$

Definition 18. The distribution of agents' valuations, F is **regular** if both $\mathcal{V}$ and $\mathcal{C}$ are increasing.

The regularity condition guarantees that the marketplace has a decreasing marginal revenue from having additional buyers and increasing marginal cost from having additional sellers. With this definition, we are ready to state the main result of this section, which characterizes the baseline optimal marketplace.

Theorem 5. *Suppose the distribution F is regular. Then, the optimal mechanism has the allocation rule*

$$q(\theta) = \begin{cases} -1 & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} < \theta < \overline{\theta} \\ 1 & \text{if } \theta \geq \overline{\theta} \end{cases}$$

and the transfer rule

$$t(\theta) = \begin{cases} -\underline{\theta} & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} < \theta < \overline{\theta} \\ \overline{\theta} & \text{if } \theta \geq \overline{\theta} \end{cases}$$

where $\underline{\theta}$ and $\overline{\theta}$ satisfies $\mathcal{C}(\underline{\theta}) = \mathcal{V}(\overline{\theta})$ and solves the problem

$$\max_{\underline{\theta},\overline{\theta}} \quad \left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right]$$

$$s.\ t.$$

$$F(\underline{\theta}) = 1 - F(\overline{\theta})$$

$$0 \le \underline{\theta} \le \overline{\theta} \le 1.$$

Notice that each agent below $\underline{\theta}$ sells 1 unit and gets paid $\underline{\theta}$, and each agent above $\overline{\theta}$ buys 1 unit and pays $\overline{\theta}$. If the designer posts $\underline{\theta}$ as the price for selling and $\overline{\theta}$ as the price for buying, and let agents choose what to do, the allocation above represents exactly what the agents would do. Thus, the designer can implement this mechanism in a very straightforward way by posting bid-ask prices.

Proof of Theorem 5. Since the allocation needs to be increasing, if $q(\theta) < 0$ for some θ, we would have $q(\theta') < 0$ for each $\theta' \le \theta$. Similarly, if $q(\theta) > 0$ for some θ, we would have $q(\theta') > 0$ for each $\theta' \ge \theta$. So, let $0 \le \underline{\theta} \le \overline{\theta} \le 1$ such that $\underline{\theta}$ is the supremum of values with negative allocation and $\overline{\theta}$ is the infimum of the values with positive allocation. Then, we can write the objective function as follows:

$$\begin{aligned}\Pi^M &= \mathbb{P}[\theta \in [0,\underline{\theta}]]\mathbb{E}[\mathcal{C}(\theta)q(\theta)|\theta \in [0,\underline{\theta}]] + \mathbb{P}[\theta \in [\overline{\theta},1]]\mathbb{E}[\mathcal{V}(\theta)q(\theta)|\theta \in [\overline{\theta},1]] \\ &= \int_0^{\underline{\theta}} \mathcal{C}(x)q(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)q(x)f(x)dx\end{aligned}$$

Note that for $\theta \le \underline{\theta}$, we must have $q(\theta) = -1$ as -1 is the only possible negative allocation with the indivisible good and similarly, $q(\theta) = 1$ for $\theta \ge \overline{\theta}$. Thus, the optimal allocation will have the following form and the next step is to choose the cutoffs, $\underline{\theta}$ and $\overline{\theta}$ optimally.

$$q(\theta) = \begin{cases} -1 & \text{if } \theta \le \underline{\theta} \\ 0 & \text{if } \underline{\theta} \le \theta \le \overline{\theta} \\ 1 & \text{if } \theta \ge \overline{\theta} \end{cases}$$

Then, the problem can be restated as follows:

$$\max_{\underline{\theta},\overline{\theta}} \quad \left[-\int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)f(x)dx\right]$$

s. t.

$$F(\underline{\theta}) = 1 - F(\overline{\theta})$$
$$0 \leq \underline{\theta} \leq \overline{\theta} \leq 1$$

Integrating out the total virtual value and the virtual cost using integration by parts shows that the objective function is equal to:

$$-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta})) = F(\underline{\theta})(\overline{\theta} - \underline{\theta})$$

where the equality is obtained by using the feasibility condition $F(\underline{\theta}) = 1 - F(\overline{\theta})$. Then, there exists a solution to this problem. Moreover, the solution is interior: If $\underline{\theta} = 0$ or $\underline{\theta} = \overline{\theta}$, the profit is 0. However, positive profit is feasible by any feasible interior solution as is clear from the objective function.

Moreover, by using the formula for the transfer rule and the optimal allocation from above, we can compute the transfers in the mechanism to be

$$t(\theta) = \begin{cases} -\underline{\theta} & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} \leq \theta \leq \overline{\theta} \\ \overline{\theta} & \text{if } \theta \geq \overline{\theta} \end{cases}$$

Notice that each 'seller' gets the same payment while each 'buyer' pays the same amount. Thus, this mechanism is equivalent to offering bid-ask prices that the transfer rule above suggest, that is a price for buying and a price for selling, and letting agents choose whether they want to buy or sell or not trade.

Finally, notice that the solution must have $C(\underline{\theta}) = \mathcal{V}(\overline{\theta})$: If we had $C(\underline{\theta}) > \mathcal{V}(\overline{\theta})$, decreasing $\underline{\theta}$ and adjusting $\overline{\theta}$ accordingly for feasibility would increase the profit since buying from $\underline{\theta}$ is costlier than what selling to $\overline{\theta}$ pays off. Similarly, if we had $C(\underline{\theta}) < \mathcal{V}(\overline{\theta})$, then increasing $\underline{\theta}$ and adjusting $\overline{\theta}$ so that the feasibility binds would again increase the profit since there are more agents whose trade is profitable. □

A.1.1 Illustrative Example with Uniform Distribution

Suppose θ is distributed uniformly over $[0, 1]$ with c.d.f. $F(\theta) = \theta$. Then, the virtual values and costs are given by

$$\mathcal{V}(\theta) = 2\theta - 1 \text{ and } C(\theta) = 2\theta\,.$$

It is easy to show that the objective function becomes $\underline{\theta}(1 - 2\underline{\theta})$ after substituting for $\overline{\theta} = 1 - \underline{\theta}$ (feasibility). So, the optimal cutoffs are $\underline{\theta} = \frac{1}{4}$ and $\overline{\theta} = \frac{3}{4}$.

Let us consider two agents with valuations θ_1 and θ_2. Then, 1 will buy from 2 in the optimal marketplace when it operates on its own if and only if $\theta_1 \geq 0.75$ and $\theta_2 \leq 0.75$, and 2 will buy from 1 if and only if $\theta_2 \geq 0.75$ and $\theta_1 \leq 0.75$. The Figure A.1 depicts the space of (θ_1, θ_2) where the shaded areas represent these trading regions.

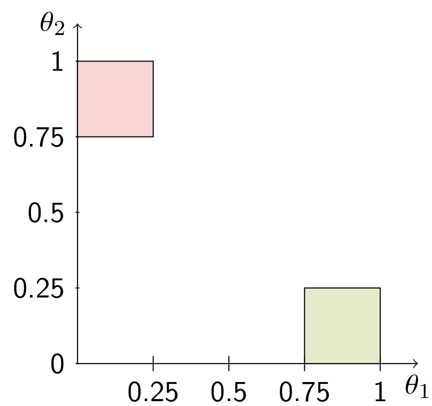

Figure A.1. x-axis represents $\theta_1 \in [0, 1]$ and y-axis represents $\theta_2 \in [0, 1]$. Green and red areas show the type profiles at which agent 1 and 2 is the buyer, respectively.

Using the payment formula from the theorem above, we can compute the payments as below:

$$t(\theta) = \begin{cases} -\dfrac{1}{4}, & \text{if } \theta \leq \dfrac{1}{4}\,, \\ 0, & \text{if } \dfrac{1}{4} \leq \theta \leq \dfrac{3}{4}\,, \\ \dfrac{1}{4}, & \text{if } \theta \geq \dfrac{3}{4}\,. \end{cases}$$

The large area of the type space where there is no trade motivates our consideration for the decentralized market. The marketplace excludes these type profiles from trade because their is not profitable; this is akin to a monopolist or a monopsonist excluding some agents from trade. Indeed, the marketplace acts as both a monopolist and a monopsonist.

However, unlike the buyers a monopolist excludes, the agents the marketplace excludes actually have a surplus that they can create, if they were allowed to trade. Thus, assumption that they will remain in the marketplace even though they are not trading is not very realistic. Next, I am going to allow them to choose between the marketplace and a decentralized market.

A.2 Slope of Utilities from Search Market

Lemma 13. *Under any equilibrium,* $-\frac{p}{2} \leq \frac{\partial u^d(\theta)}{\partial \theta} \leq \frac{p}{2}$ *for each agent.*

Proof. Suppose Θ^d is the set of agents who join the decentralized market, $\mu(\Theta^d) = \mathbb{P}[x \in \Theta^d]$ their measure, and let $\theta \in Cov(\Theta^d)$. Then,

$$\begin{aligned}
u^d(\theta) &= \frac{p}{2}\mathbb{P}[x > \theta | x \in \Theta^d] \left[\mathbb{E}[x|x > \theta, x \in \Theta^d] - \theta\right] \\
&+ \frac{p}{2}\mathbb{P}[x < \theta | x \in \Theta^d] \left[\theta - \mathbb{E}[x|x < \theta, x \in \Theta^d]\right] \\
&= \frac{p}{2}\frac{\mathbb{P}[x > \theta, x \in \Theta^d]}{\mu(\Theta^d)} \left[\mathbb{E}[x|x > \theta, x \in \Theta^d] - \theta\right] \\
&+ \frac{p}{2}\frac{\mathbb{P}[x < \theta, x \in \Theta^d]}{\mu(\Theta^d)} \left[\theta - \mathbb{E}[x|x < \theta, x \in \Theta^d]\right] \\
&= \frac{p}{2\mu(\Theta^d)}\left[\int_{\{x \in \Theta^d : x > \theta\}} x f(x) dx - \theta \mathbb{P}[x > \theta, x \in \Theta^d]\right] \\
&+ \frac{p}{2\mu(\Theta^d)}\left[\theta \mathbb{P}[x < \theta, x \in \Theta^d] - \int_{\{x \in \Theta^d : x < \theta\}} x f(x) dx\right] \\
\frac{\partial u^d(\theta)}{\partial \theta} &= \frac{p}{2\mu(\Theta^d)}\left[\mathbb{P}[x < \theta, x \in \Theta^d] - \mathbb{P}[x > \theta, x \in \Theta^d]\right].
\end{aligned}$$

If $\theta \leq \theta'$ for each $\theta' \in \Theta^d$, then

$$u^d(\theta) = \frac{p}{2}\mathbb{P}[x > \theta | x \in \Theta^d] \left[\mathbb{E}[x|x \in \Theta^d] - \theta\right]$$

$$= \frac{p}{2}\left[\int_{\{x\in\Theta^d\}} \frac{xf(x)dx}{\mu(\Theta^d)} - \theta]\right]$$

$$\frac{\partial u^d(\theta)}{\partial\theta} = -\frac{p}{2}.$$

Finally, if $\theta \geq \theta'$ for each $\theta' \in \Theta^d$, then

$$u^d(\theta) = \frac{p}{2}\mathbb{P}[x < \theta | x \in \Theta^d]\left[\theta - \mathbb{E}[x|x \in \Theta^d]\right]$$

$$= \frac{p}{2}\left[\theta - \int_{\{x\in\Theta^d\}} \frac{xf(x)dx}{\mu(\Theta^d)}]\right]$$

$$\frac{\partial u^d(\theta)}{\partial\theta} = \frac{p}{2}.$$

□

A.3 Segmentation in Coexistence Equilibrium

Proof of Proposition 2. We start with a simple observation: For the mechanism to make a positive profit, there has to be both agents who buy and sell at the marketplace. This means for a positive measure of agents, $u^m(\theta) \geq u^d(\theta)$ on both regions with $\frac{\partial u^m(\theta)}{\partial\theta} = 1$ and $\frac{\partial u^m(\theta)}{\partial\theta} = -1$.

We have shown in Appendix A.2 that for an arbitrary segmentation of agents, the expected utility from search has a slope between -0.5 and 0.5.

Notice that if for an agent $\frac{\partial u^m(\theta)}{\partial\theta} = 1$, then for each $\theta' > \theta$, $q(\theta') = 1$ by the envelope condition and monotonicity of the allocation for an IC mechanism. Similarly, if $\frac{\partial u^m(\theta)}{\partial\theta} = -1$, then for each $\theta' < \theta$, $q(\theta') = -1$.

Given this, if for θ, $u^m(\theta) \geq u^d(\theta)$ and $\frac{\partial u^m(\theta)}{\partial\theta} = 1$, then for each $\theta' > \theta$, $u^m(\theta') \geq u^d(\theta')$ and similarly for the sellers. Thus, let $\underline{\theta}$ be the highest value such that $u^m(\theta) \geq u^d(\theta)$ and $q(\theta) = -1$ in the equilibrium. Similarly, let $\overline{\theta}$ be the lowest value such that $u^m(\theta) \geq u^d(\theta)$ and $q(\theta) = 1$.

There must be at least one type such that $u^m(\theta) = u^d(\theta)$. If not, either the utilities from search are above the utilities from the mechanism everywhere so that no one comes to the marketplace and the profit of the marketplace is zero or the utilities from the mechanism is strictly higher everywhere so everyone is in the mechanism and the mechanism can reduce the utilities until some IR constraint binds to strictly increase the profit.

Next, we argue that it cannot be the case that u^m and u^d are only tangent at $\underline{\theta}$ and $\overline{\theta}$, the utilities have to cross each other at these cutoffs: If they are only tangent but does not cross each other, then $\underline{\theta}$ or $\overline{\theta}$ would have to be the point of a kink on u^m. Then, only one of the cutoffs is at a kink, and an interval of agents near the other cutoff join the search market. Suppose there is a kink at $\overline{\theta}$. Then, if the search market is active, u^d has to be increasing at $\overline{\theta}$ since everyone above it is in the mechanism so that an agent with the value $\overline{\theta}$ can only be a buyer in the search market. For there to be agents in the search market, u^d should cross u^m at a point θ such that u^d is decreasing since it cannot cross u^m on the part it is constant or has a slope of 1 below $\overline{\theta}$. But if there is such a point, then u^d would be increasing at θ, since all agents in the search market will be below it as well, which shows this is impossible. Thus, u^m and u^d cannot be tangent at $\underline{\theta}$ and $\overline{\theta}$, they have to cross each other at these points.

Then, due to the shape of the feasible utility functions (u^m can have slopes -1, 0, and 1 in this order and u^d is first decreasing and then increasing -with a potentially constant 0 slope in the middle- with a slope that remains between $-\frac{1}{2}$ and $\frac{1}{2}$), either all agents with values in $[\underline{\theta}, \overline{\theta}]$ join the search market or the flat part of the u^m crosses u^d twice again, in which case agents with values in $[\underline{\theta}, a]$ and $[b, \overline{\theta}]$ join the search market for some $\underline{\theta} < a < b < \overline{\theta}$ and agents with values in a, b join the mechanism as well. Moreover, in the latter case, we need $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$; otherwise the u^d would be either strictly decreasing or strictly increasing for agents with values in $[a, b]$, in which case u^d and u^m would not cross at both a and b, as this case requires. When $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$, u^d would be flat, as it can be seen from the slope we computed above. So, we can write the profit as follows where the case with $a = b$ corresponds to the situation where the flat part of u^m does not cross u^d.

$$\Pi = -F(\underline{\theta})u^d(\underline{\theta}) - (1 - F(\overline{\theta}))u^d(\overline{\theta}) - (F(b) - F(a))u^d(a) - \int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\overline{\theta}}^1 V(x)f(x)dx$$

Moreover, the constraints are $0 \le \underline{\theta} \le a \le b \le \overline{\theta} \le 1$, $F(\underline{\theta}) \ge 1 - F(\overline{\theta})$ and $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$.

Let $\Theta^d = [\underline{\theta}, a] \cup [b, \overline{\theta}]$. Then, by noting that $(1 - F(\overline{\theta}) - F(\underline{\theta})) \le 0$ by feasibility, it is easy to show that Π is decreasing in $\underline{\theta}$ in the feasible space. We will use this to show that the feasibility binds.

Next, we consider the Lagrangian problem to study the KKT conditions. Here, we will initially relax the problem by relaxing the equality constraint $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$ to $F(a) + F(b) \geq F(\underline{\theta}) + F(\overline{\theta})$ but focus on solutions where it binds. This is how we learn that the feasibility constraint must bind. We will then use feasibility to observe that $a = b$ should hold in the equilibrium, which means that any coexistence equilibrium has an interval of types in the decentralized market.

$$
\begin{aligned}
\mathcal{L}(\underline{\theta}, a, b, \overline{\theta}, \lambda) = \Pi &+ \lambda_1(F(\underline{\theta}) + F(\overline{\theta}) - 1) + \lambda_2(1 - \overline{\theta}) + \lambda_3(\overline{\theta} - b) + \lambda_4(b - a) + \lambda_5(a - \underline{\theta}) + \lambda_6\underline{\theta} \\
&+\lambda_7(F(a) + F(b) - F(\underline{\theta}) - F(\overline{\theta})) \\
&\frac{\partial \mathcal{L}}{\partial \underline{\theta}} = \frac{\partial \Pi}{\partial \underline{\theta}} + \lambda_1 f(\underline{\theta}) - \lambda_5 + \lambda_6 - \lambda_7 f(\underline{\theta}) = 0 \\
&\frac{\partial \mathcal{L}}{\partial a} = \frac{\partial \Pi}{\partial a} - \lambda_4 + \lambda_5 + \lambda_7 f(a) = 0 \\
&\frac{\partial \mathcal{L}}{\partial b} = \frac{\partial \Pi}{\partial b} - \lambda_3 + \lambda_4 + \lambda_7 f(b) = 0 \\
&\frac{\partial \mathcal{L}}{\partial \overline{\theta}} = \frac{\partial \Pi}{\partial \overline{\theta}} + \lambda_1 f(\overline{\theta}) - \lambda_3 + \lambda_4 - \lambda_7 f(\overline{\theta}) = 0 \\
&\lambda_i \geq 0 \\
&\lambda_1(F(\underline{\theta}) + F(\overline{\theta}) - 1) = 0 \\
&\lambda_2(1 - \overline{\theta}) = 0 \\
&\lambda_3(\overline{\theta} - b) = 0 \\
&\lambda_4(b - a) = 0 \\
&\lambda_5(a - \underline{\theta}) = 0 \\
&\lambda_6\underline{\theta} = 0 \\
&\lambda_7(F(a) + F(b) - F(\underline{\theta}) - F(\overline{\theta})) = 0.
\end{aligned}
$$

First, we note that for $\Pi > 0$, we need $1 > \overline{\theta}$. Moreover, for $1 > \overline{\theta}$, we need $\underline{\theta} > 0$ by feasibility. Then, we have $\lambda_2 = \lambda_6 = 0$ by complementary slackness conditions.

Remember that for $\underline{\theta} > 0$, we have $\frac{\partial \Pi}{\partial \underline{\theta}} < 0$. Then, since $\lambda_6 = 0$ and $\lambda_5, \lambda_7 \geq 0$, for $\frac{\partial \mathcal{L}}{\partial \underline{\theta}} = 0$, we need $\lambda_1 > 0$. By complementary slackness, this implies the feasibility constraint must bind.

Then, the profit, Π becomes:

$$-\frac{p(F(b)-F(a))}{2\left[F(\overline{\theta})-F(b)+F(a)-F(\underline{\theta})\right]}\left[\int_{b}^{\overline{\theta}} xf(x)dx-\int_{\underline{\theta}}^{a} xf(x)dx\right]+\frac{2-p}{2}\left[-\underline{\theta}F(\underline{\theta})-\overline{\theta}F(\overline{\theta})+\overline{\theta}\right].$$

Next we are going to argue that in any solution to the above problem with a positive profit, we must have $a=b$. Suppose $(\underline{\theta},a,b,\overline{\theta})$ maximizes Π and $b>a$. Remember that we reject any solution that does not satisfy $F(a)-F(\underline{\theta})=F(\overline{\theta})-F(b)$, since this is an equilibrium requirement. Then, the differences of integrals in the above equation is nonnegative. Moreover, it is strictly positive if $\Pi>0$:

Notice that for $\Pi>0$, we need $\overline{\theta}>\underline{\theta}$. If $\overline{\theta}=\underline{\theta}$, then it must be the case that $\overline{\theta}=a=b=\underline{\theta}=F^{-1}(0.5)$ and then we can verify that $\Pi=0$. $\overline{\theta}>\underline{\theta}$ implies $\overline{\theta}>b$ and $a>\underline{\theta}$ because (i) we need $F(a)-F(\underline{\theta})=F(\overline{\theta})-F(b)$ in the equilibrium and (ii) $\overline{\theta}=b>a=\underline{\theta}$ cannot happen in the equilibrium as shown before stating the Lagrangian problem. But when we have $\overline{\theta}>b\geq a>\underline{\theta}$ and $F(a)-F(\underline{\theta})=F(\overline{\theta})-F(b)$, we have:

$$\begin{aligned}
&\left[\int_{b}^{\overline{\theta}} xf(x)dx-\int_{\underline{\theta}}^{a} xf(x)dx\right]\\
=&\left[(F(\overline{\theta})-F(b))\int_{b}^{\overline{\theta}}\frac{xf(x)dx}{(F(\overline{\theta})-F(b))}-(F(a)-F(\underline{\theta}))\int_{\underline{\theta}}^{a}\frac{xf(x)dx}{(F(a)-F(\underline{\theta}))}\right]\\
=&(F(\overline{\theta})-F(b))\mathbb{E}[x|x\in[b,\overline{\theta}]]-(F(a)-F(\underline{\theta}))\mathbb{E}[x|x\in[\underline{\theta},a]]\\
=&(F(\overline{\theta})-F(b))\left[\mathbb{E}[x|x\in[b,\overline{\theta}]]-\mathbb{E}[x|x\in[\underline{\theta},a]]\right]>0.
\end{aligned}$$

Then, we must have $a=b$, since this does not effect the virtual surplus but minimizes the cost. Moreover, it must be the case that $F(a)=F(b)=F^{-1}(\frac{1}{2})$ since we need $F(a)-F(\underline{\theta})=F(\overline{\theta})-F(b)$ and the feasibility binds. Thus, any coexistence equilibrium must be such that in the equilibrium segmentation, an interval of intermediate types join the decentralized market. □

A.4 Simplifying the Profit Function

Here we simplify the profit function.

$$\Pi_{\underline{\theta},\overline{\theta}} = \int_{0}^{\underline{\theta}} t(\theta) f(\theta) d\theta + \int_{\overline{\theta}}^{1} t(\theta) f(\theta) d\theta.$$

We will study each integral separately. We start with the first one.

$$\int_{0}^{\underline{\theta}} t(\theta) f(\theta) d\theta \tag{A.1}$$

$$= \int_{0}^{\underline{\theta}} \left[\theta q(\theta) - u^m(0) - \int_{0}^{\theta} q(x) dx \right] f(\theta) d\theta \tag{A.2}$$

$$= \int_{0}^{\underline{\theta}} \left[\theta q(\theta) - u^m(0) \right] f(\theta) d\theta - \int_{0}^{\underline{\theta}} \int_{0}^{\theta} q(x) f(\theta) dx d\theta \tag{A.3}$$

$$= \int_{0}^{\underline{\theta}} \left[\theta q(\theta) - u^m(0) \right] f(\theta) d\theta - \int_{0}^{\underline{\theta}} \int_{x}^{\underline{\theta}} q(x) f(\theta) d\theta dx \tag{A.4}$$

$$= \int_{0}^{\underline{\theta}} \left[\theta q(\theta) - u^m(0) \right] f(\theta) d\theta - \int_{0}^{\underline{\theta}} q(x) \int_{x}^{\underline{\theta}} f(\theta) d\theta dx \tag{A.5}$$

$$= \int_{0}^{\underline{\theta}} \left[\theta q(\theta) - u^m(0) \right] f(\theta) d\theta - \int_{0}^{\underline{\theta}} q(x) (F(\underline{\theta}) - F(x)) dx \tag{A.6}$$

$$= \int_{0}^{\underline{\theta}} \left[-u^m(0) + \left(x - \frac{F(\underline{\theta}) - F(x)}{f(x)} \right) q(x) \right] f(x) dx \tag{A.7}$$

$$= \int_{0}^{\underline{\theta}} \left[-u^m(\underline{\theta}) + \int_{0}^{\underline{\theta}} q(y) dy + \left(x - \frac{F(\underline{\theta}) - F(x)}{f(x)} \right) q(x) \right] f(x) dx \tag{A.8}$$

$$= \int_{0}^{\underline{\theta}} F(\underline{\theta}) q(y) \frac{f(y)}{f(y)} dy + \int_{0}^{\underline{\theta}} \left[-u^m(\underline{\theta}) + \left(x - \frac{F(\underline{\theta}) - F(x)}{f(x)} \right) q(x) \right] f(x) dx \tag{A.9}$$

$$= \int_{0}^{\underline{\theta}} \left[-u^m(\underline{\theta}) + \left(x + \frac{F(x)}{f(x)} \right) q(x) \right] f(x) dx \tag{A.10}$$

$$= -F(\underline{\theta})u^m(\underline{\theta}) + \int_0^{\underline{\theta}} \left[\left(x + \frac{F(x)}{f(x)}\right) q(x)\right] f(x)dx \tag{A.11}$$

In line 4, we change the order of integration; in line 5, we isolate the inner integral by extracting the allocations out; in line 6, we replace the value of the inner integral; in line 7, we merge the sum back; in line 8, we replace the value of the utility of the lowest type; in line 9, we integrate out the information rent for these types; in line 10, we cancel the new double integral with the $-\underline{\theta}q(x)$ as that integral turns out to be just the integral of $\underline{\theta}q(x)$ by changing the order of integration as above. We follow the similar steps for the transfers from $[\overline{\theta}, 1]$.

$$\begin{aligned}
&\int_{\overline{\theta}}^{1} t(\theta)f(\theta)d\theta \\
&= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta}) - \int_{\overline{\theta}}^{\theta} q(x)dx\right] f(\theta)d\theta \\
&= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta})\right] f(\theta)d\theta - \int_{\overline{\theta}}^{1}\int_{\overline{\theta}}^{\theta} q(x)f(\theta)dxd\theta \\
&= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta})\right] f(\theta)d\theta - \int_{\overline{\theta}}^{1}\int_{x}^{1} q(x)f(\theta)d\theta dx \\
&= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta})\right] f(\theta)d\theta - \int_{\overline{\theta}}^{1} q(x)\int_{x}^{1} f(\theta)d\theta dx \\
&= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta})\right] f(\theta)d\theta - \int_{\overline{\theta}}^{1} q(x)(1-F(x))dx \\
&= \int_{\overline{\theta}}^{1} \left[-u^m(\overline{\theta}) + \left(x - \frac{1-F(x)}{f(x)}\right) q(x)\right] f(x)dx \\
&= -(1-F(\overline{\theta}))u^m(\overline{\theta}) + \int_{\overline{\theta}}^{1} \left[\left(x - \frac{1-F(x)}{f(x)}\right) q(x)\right] f(x)dx
\end{aligned}$$

A.5 Individual Rationality and the Allocations

Proof of Lemma 1. Notice that just below $\underline{\theta}$, the utility from the mechanism should have a left derivative below $-\frac{p}{2}$:

$$\begin{aligned}
u^m(\theta) = u^m(0) + \int_0^{\theta} q(x)dx \geq u^d(\theta) &\iff \\
u^d(\underline{\theta}) - \int_0^{\underline{\theta}} q(x)dx + \int_0^{\theta} q(x)dx \geq u^d(\theta) &\iff \\
u^d(\underline{\theta}) - \int_{\theta}^{\underline{\theta}} q(x)dx \geq u^d(\theta) &\iff \\
p\left[\frac{\int_{\underline{\theta}}^{\overline{\theta}} xf(x)dx}{2\left(F(\overline{\theta}) - F(\underline{\theta})\right)} - \frac{\underline{\theta}}{2}\right] - \int_{\theta}^{\underline{\theta}} q(x)dx \geq p\left[\frac{\int_{\underline{\theta}}^{\overline{\theta}} xf(x)dx}{2\left(F(\overline{\theta}) - F(\underline{\theta})\right)} - \frac{\theta}{2}\right] &\iff \\
p\frac{\theta - \underline{\theta}}{2} \geq \int_{\theta}^{\underline{\theta}} q(x)dx &\iff \\
p\frac{\theta - \underline{\theta}}{2} \geq u^m(\underline{\theta}) - u^m(\theta).&
\end{aligned}$$

Since only possible allocations are −1, 0, and 1, this means we must have $q(\underline{\theta}) = -1$. Moreover, due to monotonicity of the allocation, for each $\theta \in [0, \underline{\theta}]$, this implies $q(\theta) = -1$. Following the same steps around $\overline{\theta}$ also shows that $q(\theta) = 1$ for each $\theta \in [\overline{\theta}, 1]$. We note the observations we have made here in the following lemma. □

A.6 Profit from the coexistence equilibrium

Using the Lemma 1,

$$\int_0^{\underline{\theta}} C(x)q(x)f(x)dx + \int_{\overline{\theta}}^{1} V(y)q(y)f(y)dy = \int_{\overline{\theta}}^{1} V(y)f(y)dy - \int_0^{\underline{\theta}} C(x)f(x)dx$$

From the Appendix A.1, we have

$$-\int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\overline{\theta}}^1 V(y)f(y)dy = -\underline{\theta}F(\underline{\theta}) - \overline{\theta}F(\overline{\theta}) + \overline{\theta}$$

Moreover, under a CRS matching function M for the decentralized market, p is independent of the segmentation of the market. Then, using the binding IR constraints, the compensations paid to the agents will be given by:

$$\begin{aligned}
&F(\underline{\theta})u^d(\underline{\theta}) + (1-F(\overline{\theta}))u^d(\overline{\theta}) \\
&= p\frac{F(\underline{\theta})}{2}\left[\frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta)d\theta}{F(\overline{\theta})-F(\underline{\theta})} - \underline{\theta}\right] + p\frac{1-F(\overline{\theta})}{2}\left[\overline{\theta} - \frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta)d\theta}{F(\overline{\theta})-F(\underline{\theta})}\right] \\
&= p\frac{1}{2}\left(F(\underline{\theta})\left[\frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta)d\theta}{F(\overline{\theta})-F(\underline{\theta})} - \underline{\theta}\right] + (1-F(\overline{\theta}))\left[\overline{\theta} - \frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta)d\theta}{F(\overline{\theta})-F(\underline{\theta})}\right]\right) \\
&= \frac{p}{2}\left(F(\underline{\theta})E[\theta|\underline{\theta} \le \theta \le \overline{\theta}] - \underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta})) - (1-F(\overline{\theta}))E[\theta|\underline{\theta} \le \theta \le \overline{\theta}]\right) \\
&= \frac{p}{2}\left(\left(F(\underline{\theta}) - (1-F(\overline{\theta}))\right)E[\theta|\underline{\theta} \le \theta \le \overline{\theta}] - \underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Pi \\
&= -\frac{p}{2}\left(\left(F(\underline{\theta}) - (1-F(\overline{\theta}))\right)E[\theta|\underline{\theta} \le \theta \le \overline{\theta}] - \underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right) + \left[-\underline{\theta}F(\underline{\theta}) - \overline{\theta}F(\overline{\theta}) + \overline{\theta}\right] \\
&= \frac{1}{2}\left((2-p)\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right] - p\left[F(\underline{\theta}) - (1-F(\overline{\theta}))\right]E[\theta|\underline{\theta} \le \theta \le \overline{\theta}]\right)
\end{aligned}$$

Notice that

$$\mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]] = \frac{\int_{\underline{\theta}}^{\overline{\theta}} xf(x)dx}{F(\overline{\theta}) - F(\underline{\theta})}$$

$$\frac{\partial \mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]]}{\partial \underline{\theta}} = \frac{-\underline{\theta} f(\underline{\theta})[F(\overline{\theta}) - F(\underline{\theta})] + f(\underline{\theta}) \left[\int_{\underline{\theta}}^{\overline{\theta}} x f(x) dx\right]}{[F(\overline{\theta}) - F(\underline{\theta})]^2}$$
$$= f(\underline{\theta}) \frac{\int_{\underline{\theta}}^{\overline{\theta}} x f(x) dx - \underline{\theta}[F(\overline{\theta}) - F(\underline{\theta})]}{[F(\overline{\theta}) - F(\underline{\theta})]^2}$$
$$= f(\underline{\theta}) \frac{\mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]] - \underline{\theta}}{[F(\overline{\theta}) - F(\underline{\theta})]} > 0.$$

Then, using this, $2\frac{\partial \Pi}{\partial \underline{\theta}} =$

$$-(2-p)F(\underline{\theta}) - (2-p)\underline{\theta} f(\underline{\theta}) - p[f(\underline{\theta})]E[\theta|\underline{\theta} \leq \theta \leq \overline{\theta}] - p\left[F(\underline{\theta}) - (1 - F(\overline{\theta}))\right] \frac{\partial \mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]]}{\partial \underline{\theta}}$$

For a feasible mechanism, we need $F(\underline{\theta}) - (1 - F(\overline{\theta})) \geq 0$. Thus, each term above is negative, and Π is decreasing in $\underline{\theta}$. Given that Π is decreasing in $\underline{\theta}$, $F(\underline{\theta}) - (1 - F(\overline{\theta})) \geq 0$ will bind in any equilibrium. Therefore,

$$\Pi = \frac{1}{2}(2-p)\left[-\underline{\theta} F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right]$$
$$= \frac{2-p}{2}\left[-\underline{\theta} F(\underline{\theta}) + \overline{\theta}(F(\underline{\theta}))\right]$$
$$= \frac{2-p}{2}\left[F(\underline{\theta})[\overline{\theta} - \underline{\theta}]\right]$$

Notice that for interior values with $\overline{\theta} > \underline{\theta}$, $\Pi > 0$. Thus, positive profit is feasible and will be achieved in the equilibrium.

A.7 Existence of a Coexistence Equilibrium

Proof of Theorem 1. In the Appendix A.6, I show that the profit from any pair of thresholds, $\underline{\theta}$ and $\overline{\theta}$, can be written as follows.

$$\Pi = \frac{1}{2}\left([2-p]\left[-\underline{\theta} F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right] - p\left[F(\underline{\theta}) - (1 - F(\overline{\theta}))\right] E[\theta|\underline{\theta} \leq \theta \leq \overline{\theta}]\right)$$

Our constraints are $0 \leq \underline{\theta} \leq \overline{\theta} \leq 1$ and $F(\underline{\theta}) \geq 1 - F(\overline{\theta})$. First, we cannot have $\overline{\theta} < m(F)$ where $m(F)$ is the median of F since that would require $0.5 > F(\overline{\theta}) \geq F(\underline{\theta}) \geq$

$1 - F(\overline{\theta}) \geq 0.5$. Second, Appendix A.6 also shows that Π is strictly decreasing in $\underline{\theta}$. Then, $F(\underline{\theta}) \geq 1 - F(\overline{\theta})$ should bind. Third, $\underline{\theta} \leq m(F) \leq \overline{\theta}$ as a result of previous two observations. Thus, for a strictly increasing F, this is essentially a single parameter problem with a continuous objective and a compact domain. Hence, it has a solution by Weierstrass Theorem. Moreover, the solution is interior in the sense that the constraints $\underline{\theta} \leq m(F) \leq \overline{\theta}$ do not bind. If they did, then the profit would be 0 while it is possible to achieve a positive profit when they do not bind. (Appendix A.6 shows this in more detail as well.) When the feasibility condition binds, the expectation term disappears from the profit. Then, the profit in this equilibrium is equal to the profit when there was no search market, times a constant, $\frac{2-p}{2}$. Thus, the solution must still have $\mathcal{C}(\underline{\theta}) = \mathcal{V}(\overline{\theta})$, as shown in the Theorem 5. The mechanism is constructed so that no agent has any profitable deviation either in market choice or the message to the designer.

□

A.7.1 The Mechanism in the Coexistence Equilibrium

Proposition 11. *In the coexistence equilibrium, the mechanism the designer offers has the following allocation and transfer rules:*

$$q(\theta) = \begin{cases} -1 & \textit{if } \theta \leq \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\underline{\theta}}{2} \\ 0 & \textit{if } \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\underline{\theta}}{2} \leq \theta \leq \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\overline{\theta}}{2} \\ 1 & \textit{if } \theta \geq \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\overline{\theta}}{2} \end{cases}$$

$$t(\theta) = \begin{cases} -\dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\underline{\theta}}{2} & \textit{if } \theta \leq \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\underline{\theta}}{2} \\ 0 & \textit{if } \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\underline{\theta}}{2} \leq \theta \leq \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\overline{\theta}}{2} \\ \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\overline{\theta}}{2} & \textit{if } \theta \geq \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\overline{\theta}}{2} \end{cases}$$

Proof of Proposition 11. First, we compute the transfers for the agents who join the mechanism using the binding IR constraints for $\underline{\theta}$ and $\overline{\theta}$.

In the coexistence equilibrium we construct, for an agent with $\theta \in [0, \underline{\theta}]$,

$$
\begin{aligned}
t(\theta) &= \theta q(\theta) - u^m(0) - \int_0^{\theta} q(x)dx \\
&= \theta(-1) - u^m(0) - \theta(-1) \\
&= -u^m(0) = -u^d(\underline{\theta}) + \int_0^{\underline{\theta}} q(x)dx = -u^d(\underline{\theta}) + \int_0^{\underline{\theta}} (-1)dx \\
&= -\frac{p}{2}\left[E[x|x \in [\underline{\theta}, \overline{\theta}]] - \underline{\theta}\right] - \underline{\theta} \\
&= -\frac{p}{2}E[x|x \in [\underline{\theta}, \overline{\theta}]] - \underline{\theta}\frac{2-p}{2}
\end{aligned}
$$

Similarly, we can compute the transfer of agents with $\theta \in [\overline{\theta}, 1]$,

$$
\begin{aligned}
t(\theta) &= \theta q(\theta) - u^m(\overline{\theta}) - \int_{\overline{\theta}}^{\theta} q(x)dx \\
&= \theta(1) - u^m(\overline{\theta}) - 1(\theta - \overline{\theta}) \\
&= \overline{\theta} - u^m(\overline{\theta}) = \overline{\theta} - \frac{p}{2}\left[\overline{\theta} - E[x|x \in [\underline{\theta}, \overline{\theta}]]\right] \\
&= \frac{p}{2}E[x|x \in [\underline{\theta}, \overline{\theta}]] + \overline{\theta}\frac{2-p}{2}
\end{aligned}
$$

Knowing these, the designer can offer -1 allocation to all agents whose valuations are below $\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\underline{\theta}}{2}$. Similarly, for agents with valuations above $\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\overline{\theta}}{2}$, 1 unit of allocation can be offered. In between, they are not offered any trade. This allocation is clearly increasing. Moreover, accompanied by the transfers $-\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\underline{\theta}}{2}$ for agents with negative allocations and $\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\overline{\theta}}{2}$ for agents with positive allocations, agents with values in $(\underline{\theta}, \overline{\theta})$ would strictly prefer the search market. To see this, note that these agents have utilities with slopes -1 until u^m hits 0, then it is constant at 0 and then it has the slope 1, after $\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\overline{\theta}}{2}$. Moreover, u^m and u^d are equal at $\underline{\theta}$ and $\overline{\theta}$. Since the slope of u^d is bounded between $-\frac{p}{2}$ and $\frac{p}{2}$, and u^d is positive, u^m and u^d cannot cross each other at any point other than $\underline{\theta}$ and $\overline{\theta}$. Thus, u^d remains below u^m for values in $(\underline{\theta}, \overline{\theta})$.

□

A.8 Simple Economics of Optimal Marketplaces

Here is how Figure A.2 works. The aggregate compensations for buyers is equal to $\frac{p}{2}q^* \times (\underline{\theta} - \mathbb{E}_d)$ where $\mathbb{E}_d$ the expected valuation in the decentralized market, i.e., $\mathbb{E}_d = \mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]]$. Similarly, the aggregate compensations for sellers is equal to $\frac{p}{2}q^* \times (\mathbb{E}_d - \underline{\theta})$. Since $\mathbb{E}_d$ is a point between $\underline{\theta}$ and $\overline{\theta}$, this means total compensations will be equal to $\dfrac{p}{2}$ times the area of the rectangle between $\underline{\theta}$ and $\overline{\theta}$ on the vertical axis and between 0 and q_C^*; $(\overline{\theta} - \underline{\theta}) \times q_C^*$.

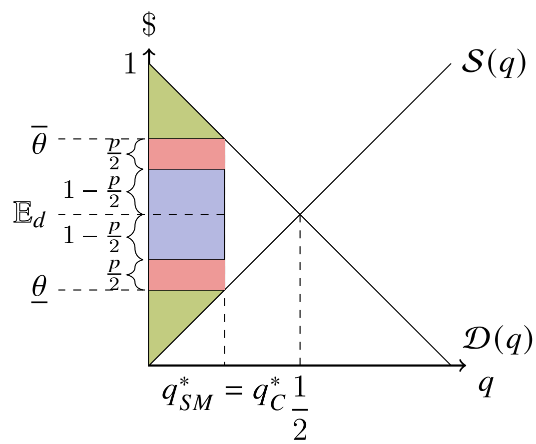

Figure A.2. The ratio of the compensations only depend on p and not the distribution.

A.9 Efficiency of Coexistence

A.9.1 Under Uniform Distribution

Proof of Proposition 3. Suppose everyone is in the search market. Then, the total welfare can be computed as follows:

$$\begin{aligned}\mathbb{E}[u^d(\theta)] &= p\int_0^1 [\theta\theta - (1-\theta)\theta]d\theta \\ &= p\int_0^1 [2\theta^2 - \theta]d\theta \\ &= p\left[\frac{2\theta^3}{3} - \frac{\theta^2}{2}\right]_0^1 = \frac{p}{6}.\end{aligned}$$

Next, we compute the welfare created by the marketplace alone in the coexistence equilibrium. The welfare marketplace generates will be more than enough to exceed the total welfare of the pure search market, so we do not need to compute the welfare created by the search market in the coexistence.

The profit function from the coexistence equilibrium under the uniform distribution is a constant times $\underline{\theta}(\overline{\theta} - \underline{\theta}) = \underline{\theta}(1 - 2\underline{\theta})$ using the fact that the feasibility binds so that $\underline{\theta} = 1 - \overline{\theta}$. This is maximized at $\underline{\theta} = \frac{1}{4}$ and $\overline{\theta} = \frac{3}{4}$. Then, the welfare generated by the marketplace is

$$\int_{0.75}^{1} \theta_b d\theta_b - \int_{0}^{0.25} \theta_s d\theta_s = \left[\frac{\theta^2}{2}\right]_{0.75}^{1} - \left[\frac{\theta^2}{2}\right]_{0}^{0.25} = \frac{3}{16}.$$

The total welfare from the search market is $\frac{p}{6} \leq \frac{1}{6}$ for any matching function since the probability a meeting will be less than or equal to 1. Moreover, $\frac{3}{16} > \frac{1}{6}$. Thus, for any matching function, the coexistence equilibrium creates a welfare higher than the pure search market. □

A.9.2 Under General Distribution

Proof of Proposition 4. For the pure search market, the total welfare created is given by

$$\begin{aligned} &\int_{0}^{1} \left[pF(\theta)\theta - p(1 - F(\theta))\theta\right] f(\theta)d\theta \\ =&p \int_{0}^{1} \theta \left[2F(\theta) - 1\right] f(\theta)d\theta \end{aligned}$$

In the first line above, $pF(\theta)$ is the probability that the agent with the value θ meets with an agent with a value less than θ, so she gets θ in the trade and $p(1 - F(\theta))$ is the probability that she meets with an agent whose value is higher so she loses θ. As with the uniform case, we ignore the transfers in the utilities as the transfers will cancel in the search market.

In the coexistence equilibrium, the total welfare created in the search market is

$$\int_{0}^{1} \left[p\left[\frac{F(\theta) - F(\underline{\theta})}{F(\overline{\theta}) - F(\underline{\theta})}\right]\theta - p\left[\frac{F(\overline{\theta}) - F(\theta)}{F(\overline{\theta}) - F(\underline{\theta})}\right]\theta\right] f(\theta)d\theta$$

$$=p \int_{\underline{\theta}}^{\overline{\theta}} \theta \left[\frac{2F(\theta) - F(\underline{\theta}) - F(\overline{\theta})}{F(\overline{\theta}) - F(\underline{\theta})} \right] f(\theta) d\theta$$

$$=p \int_{\underline{\theta}}^{\overline{\theta}} \theta \left[\frac{2F(\theta) - 1}{F(\overline{\theta}) - F(\underline{\theta})} \right] f(\theta) d\theta$$

The welfare generated by the marketplace in the coexistence is

$$\int_{\overline{\theta}}^{1} x f(x) dx - \int_{0}^{\underline{\theta}} x f(x) dx$$

$$p \int_{\underline{\theta}}^{\overline{\theta}} \theta \left[\frac{2F(\theta) - 1}{F(\overline{\theta}) - F(\underline{\theta})} \right] f(\theta) d\theta + \int_{\overline{\theta}}^{1} x f(x) dx - \int_{0}^{\underline{\theta}} x f(x) dx \geq p \int_{0}^{1} \theta \left[2F(\theta) - 1 \right] f(\theta) d\theta$$

$$\iff p \int_{\underline{\theta}}^{\overline{\theta}} \theta \left[\frac{2F(\theta)}{F(\overline{\theta}) - F(\underline{\theta})} \right] f(\theta) d\theta + \int_{\overline{\theta}}^{1} x f(x) dx + p \int_{0}^{1} \theta f(\theta) d\theta$$

$$\geq p \int_{\underline{\theta}}^{\overline{\theta}} \left[\frac{\theta f(\theta) d\theta}{F(\overline{\theta}) - F(\underline{\theta})} \right] + p \int_{0}^{1} \theta \left[2F(\theta) \right] f(\theta) d\theta + \int_{0}^{\underline{\theta}} x f(x) dx$$

$$\iff p \int_{\underline{\theta}}^{\overline{\theta}} \theta \left[\frac{2F(\theta)}{F(\overline{\theta}) - F(\underline{\theta})} \right] f(\theta) d\theta + (1 + p) \int_{\overline{\theta}}^{1} \theta f(\theta) d\theta$$

$$\geq p \left[\frac{2F(\underline{\theta})}{1 - 2F(\underline{\theta})} \right] \int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta) d\theta + 2p \int_{0}^{1} \theta \left[F(\theta) \right] f(\theta) d\theta + (1 - p) \int_{0}^{\underline{\theta}} x f(x) dx$$

In the first line above, either we have

$$\int_{\underline{\theta}}^{\overline{\theta}} \theta \left[\frac{2F(\theta) - 1}{F(\overline{\theta}) - F(\underline{\theta})} \right] f(\theta) d\theta \geq \int_{0}^{1} \theta \left[2F(\theta) - 1 \right] f(\theta) d\theta$$

in which case the inequality is satisfied for any p, since $\int_{\overline{\theta}}^{1} x f(x) dx - \int_{0}^{\underline{\theta}} x f(x) dx \geq 0$ when $F(\underline{\theta}) = 1 - F(\overline{\theta})$ or

$$\int_{\underline{\theta}}^{\overline{\theta}} \theta \left[\frac{2F(\theta) - 1}{F(\overline{\theta}) - F(\underline{\theta})} \right] f(\theta) d\theta < \int_{0}^{1} \theta \left[2F(\theta) - 1 \right] f(\theta) d\theta.$$

Thus, to show that for any p, the coexistence is more efficient, it is enough to show it with $p = 1$.

$\square$

A.10 Double Auction

To compute all agents' expected payoffs from the decentralized market, we need to know the optimal bids of agents whose valuations lie outside $(\underline{\theta}, \overline{\theta})$. For agents with values below $\theta \leq \underline{\theta}$, if they join the decentralized market and get matched to someone, the best response is to bid $b(\underline{\theta})$ and for agents with values above $\overline{\theta}$, the best response is to bid $b(\overline{\theta})$. I show this in two steps. First, I show that for any agent, the optimal bid must lie in $[b(\underline{\theta}), b(\overline{\theta})]$. Then, I show that for agents with values less than $\underline{\theta}$ the optimal bid is $b(\underline{\theta})$ while for agents with values above $\overline{\theta}$, the optimal bid is $b(\overline{\theta})$. This is stated in the next lemma and proved in the Appendix A.10.1.

Lemma 14. *For each $\theta \in [0, \underline{\theta}]$, $b(\theta) = b(\underline{\theta})$ and for each $\theta \in [\overline{\theta}, 1]$, $b(\theta) = b(\overline{\theta})$.*

Agents' utilities from the decentralized market:

Suppose $\theta \in [\underline{\theta}, \overline{\theta}]$. If the agent has the lower value, she has the lower bid, since the bidding function is monotone. Then, the agent gives up her endowment but gets paid. If the agent has the higher value, then, she has the higher bid so she gets the other's endowment and pays for it. Thus, expected payoff is

$$u^{da}(\theta) = pG(\theta)\left[\theta - \int_{\underline{\theta}}^{\theta} \frac{\frac{1}{2}[b(x) + b(\theta)]g(x)dx}{G(\theta)}\right] + p(1 - G(\theta))\left[\int_{\theta}^{\overline{\theta}} \frac{\frac{1}{2}[b(x) + b(\theta)]g(x)dx}{1 - G(\theta)} - \theta\right]$$

$$= p\theta[2G(\theta) - 1] + \frac{p}{2}\int_{\theta}^{\overline{\theta}} [b(x) + b(\theta)]g(x)dx - \frac{p}{2}\int_{\underline{\theta}}^{\theta} [b(x) + b(\theta)]g(x)dx$$

$$= p\theta[2G(\theta) - 1] + \frac{p}{2}(1 - 2G(\theta))b(\theta) + \frac{p}{2}\int_{\theta}^{\overline{\theta}} b(x)g(x)dx - \frac{p}{2}\int_{\underline{\theta}}^{\theta} b(x)g(x)dx$$

If $\theta \leq \underline{\theta}$, then, as above argument shows, the optimal bid is $b(\underline{\theta})$ and the expected payoff from the decentralized market is:

$$u^{da}(\theta) = p\left[\int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{2}[b(\underline{\theta}) + b(x)]g(x)dx - \theta\right]$$

Similarly, agents with $\theta \geq \overline{\theta}$ bid $b(\overline{\theta})$ and get the expected payoff:

$$u^{da}(\theta) = p\left[\theta - \int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{2}[b(\overline{\theta}) + b(x)]g(x)dx\right]$$

Next, we compute $u^{da}(\underline{\theta})$ and $u^{da}(\overline{\theta})$ by using the formulas for $b(\cdot)$, as the IR constraints of $\underline{\theta}$ and $\overline{\theta}$ will again play an important role in the equilibrium. The details can be found in the Appendix A.10.2 but here is the end result:

$$u^{da}(\underline{\theta}) = \frac{p}{2}\left[-\underline{\theta} + 4\int_{\underline{\theta}}^{G^{-1}(\frac{1}{2})} \left[G(x) - \frac{1}{2}\right]^2 dx + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]$$

$$u^{da}(\overline{\theta}) = \frac{p}{2}\left[\overline{\theta} + 4\int_{G^{-1}(\frac{1}{2})}^{\overline{\theta}} \left[G(x) - \frac{1}{2}\right]^2 dx - \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]$$

In the coexistence equilibrium, we are going to make the IR constraints of $\underline{\theta}$ and $\overline{\theta}$ bind, as otherwise decreasing the payment until they bind increases the profit. For these cutoffs to work, we need the slope of the utility from the decentralized market, $\frac{\partial u^{da}(\theta)}{\partial \theta}$ for types below $\underline{\theta}$ to be greater than -1, since -1 is the allocation they will be offered in the mechanism and we want u^{da} to be less than u^m on this region. Moreover, we need the slope of u^{da} to be greater than -1 around $\underline{\theta}$ and the slope should be increasing (thus the utility function should be convex). Finally, we need the slope of the utility from the decentralized market to be less than 1 for agents with values above $\overline{\theta}$.

Lemma 15.

$$\frac{\partial u^{da}(\theta)}{\partial \theta} = \begin{cases} -p & \textit{if } \theta \leq \underline{\theta} \\ p(G(\theta) - 1) & \textit{if } \underline{\theta} \leq \theta \leq \overline{\theta} \\ p & \textit{if } \theta \geq \overline{\theta} \end{cases}$$

Proof can be found in the Appendix A.10.3.

Since this is between -1 and 1, for agents with values in $[\underline{\theta}, \overline{\theta}]$, the designer can indeed offer lower utilities to these agents. One way of doing this would be offering the allocation -1 for agents between $\underline{\theta}$ and $G^{-1}(\frac{1}{2})$ and allocation 1 for agents between $G^{-1}(\frac{1}{2})$ and $\overline{\theta}$. This would make sure these agents are offered utilities lower than their expected payoff from the decentralized market since $u^m(\underline{\theta}) = u^{da}(\underline{\theta})$ and $u^m(\overline{\theta}) = u^{da}(\overline{\theta})$. (This may offer utilities below zero for some agents. The designer may not be concerned about this, since these agents are not wanted anyway. However, if the designer wishes the mechanism to offer nonnegative utilities, this can be achieved by flattening the utility when it reaches zero, as the Proposition 11 does.)

Now, we look at the profit function. As before, it is equal to

$$\begin{aligned}
\Pi_{\underline{\theta},\overline{\theta}} &= \mathbb{P}[\theta \in [0,\underline{\theta}]]\mathbb{E}[t(\theta)|\theta \in [0,\underline{\theta}]] + \mathbb{P}[\theta \in [\overline{\theta},1]]\mathbb{E}[t(\theta)|\theta \in [\overline{\theta},1]] \\
&= \int_0^{\underline{\theta}} t(\theta)f(\theta)d\theta + \int_{\overline{\theta}}^1 t(\theta)f(\theta)d\theta \\
&= -F(\underline{\theta})u^d(\underline{\theta}) - (1-F(\overline{\theta}))u^d(\overline{\theta}) + \int_0^{\underline{\theta}} \mathcal{C}(x)q(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)q(x)f(x)dx \\
&= -F(\underline{\theta})u^d(\underline{\theta}) - (1-F(\overline{\theta}))u^d(\overline{\theta}) - \int_0^{\underline{\theta}} \mathcal{C}(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)f(x)dx
\end{aligned}$$

First two terms are the compensations for agents to join the centralized marketplace, while the last two terms are the total virtual surplus. We know expression for the total surplus from before because that part is unchanged:

$$-\int_0^{\underline{\theta}} \mathcal{C}(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)f(x)dx = -\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta})).$$

Next, we study the compensations, since now they will be different from what we had for the Nash bargaining.

$$F(\underline{\theta})u^d(\underline{\theta}) + (1-F(\overline{\theta}))u^d(\overline{\theta})$$

$$
\begin{aligned}
=&F(\underline{\theta})p\left[\frac{1}{2}\left[b(\underline{\theta})+\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]-\underline{\theta}\right]+(1-F(\overline{\theta}))p\left[\overline{\theta}-\frac{1}{2}\left[b(\overline{\theta})+\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]\right] \\
=&\frac{p}{2}\left[F(\underline{\theta})\left[b(\underline{\theta})+\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx-2\underline{\theta}\right]+(1-F(\overline{\theta}))\left[2\overline{\theta}-b(\overline{\theta})-\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]\right] \\
=&p\left[-\underline{\theta}F(\underline{\theta})+\overline{\theta}(1-F(\overline{\theta}))\right] \\
+&\frac{p}{2}\left[\left[F(\underline{\theta})+F(\overline{\theta})-1\right]\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx+F(\underline{\theta})b(\underline{\theta})-(1-F(\overline{\theta}))b(\overline{\theta})\right]
\end{aligned}
$$

Remember that the optimal bidding strategy is given by

$$
b(\theta)=\theta-\frac{\int_{G^{-1}(\frac{1}{2})}^{\theta}[G(x)-\frac{1}{2}]^2dx}{[G(\theta)-\frac{1}{2}]^2}
$$

with $G(\theta)=\dfrac{F(x)-F(\underline{\theta})}{F(\overline{\theta})-F(\underline{\theta})}$ on $[\underline{\theta},\overline{\theta}]$ and $G^{-1}(\frac{1}{2})=F^{-1}\left(\frac{F(\underline{\theta})+F(\overline{\theta})}{2}\right)$. Although it is relatively easy to show that a coexistence equilibrium exists with arbitrary distributions, the general solution to the profit maximization problem is too complicated to provide some useful comparative statics. Hence, I focus on the uniform distribution, $U[0,1]$ from here on to be able to find a closed form solution to the problem above.

Using the uniform distribution, with some algebra (see Appendix A.10.4), we can show that

$$
b(\theta)=\frac{\overline{\theta}+\underline{\theta}+4\theta}{6},
$$

$$
\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx=\frac{\overline{\theta}+\underline{\theta}}{2}.
$$

Now, we can plug these back into the expression for the profit to see that it is decreasing in $\underline{\theta}$. Thus, the feasibility must bind, which means $\underline{\theta}=1-\overline{\theta}$. Using this, we simplify the profit further and obtain the following simple expression for the profit. (The derivations can be followed in Appendix A.10.5.)

$$\Pi_{\underline{\theta},\overline{\theta}} = \frac{6-5p}{6}\left[\underline{\theta}(\overline{\theta}-\underline{\theta})\right] = \frac{6-5p}{6}\Pi^M$$

Clearly, this problem has an interior solution, which is the same as the solution of the problem of the marketplace when it operated on its own: The profits in two cases are equal up to a constant multiplier. Thus, almost everything we have seen under the Nash bargaining hold here with the uniform distribution, with the exception of the change ratio of the profit.

A.10.1 Optimal Bids for Agents in the Marketplace

Proof of Lemma 14. Step 1: Bidding below $b(\underline{\theta})$ can never be optimal: For any bid $b \leq b(\underline{\theta})$, the agent sells her endowment with certainty but get paid less than she would get if she bid $b(\underline{\theta})$. Mathematically, the expected utility of an agent who bids $b \leq b(\underline{\theta})$ is given by:

$$\frac{1}{2}\int_{\underline{\theta}}^{\overline{\theta}}[b+b(x)]g(x)dx - \theta$$

Thus, bids strictly below $b(\underline{\theta})$ cannot be optimal.

Similarly, bids strictly above $b(\overline{\theta})$ cannot be optimal either. In that case, the agent's expected payoff would be

$$\theta - \frac{1}{2}\int_{\underline{\theta}}^{\overline{\theta}}[b+b(x)]g(x)dx$$

Hence, for each $\theta \in [0,\underline{\theta}] \cup [\overline{\theta},1]$, the optimal bid must be the bid of some type joins the decentralized market, that is: $b(\theta) \in [b(\underline{\theta}), b(\overline{\theta})]$.

Step 2: Let us first define the following notation: If an agent bids $b(\theta')$, then the expected price for selling is $p_s(\theta') = \frac{1}{2}\int_{\underline{\theta}}^{\theta'}\frac{[b(\theta')+b(x)]g(x)dx}{1-G(\theta')}$ and the expected price for buying is $p_b(\theta') = \frac{1}{2}\int_{\theta'}^{\overline{\theta}}\frac{[b(\theta')+b(x)]g(x)dx}{G(\theta')}$.

Since the best response of agent with value $\underline{\theta}$ is $b(\underline{\theta})$, her expected payoff from this bid should be higher than any other $b(\theta')$ by revealed preference. Then,

$$\begin{aligned} p_s(\underline{\theta}) - \underline{\theta} &\geq (1 - G(\theta')) \left[p_s(\theta') - \underline{\theta}\right] + G(\theta') \left[\underline{\theta} - p_b(\theta')\right] \\ &= \underline{\theta}[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\ \iff p_s(\underline{\theta}) - (1 - G(\theta'))p_s(\theta') + G(\theta')p_b(\theta') &\geq 2\underline{\theta}G(\theta') \end{aligned}$$

Suppose $\theta \leq \underline{\theta}$. We want to show that bidding $b(\underline{\theta})$ gives a higher payoff than any other type's bid $b(\theta')$:

$$\begin{aligned} p_s(\underline{\theta}) - \theta &\geq (1 - G(\theta')) \left[p_s(\theta') - \theta\right] + G(\theta') \left[\underline{\theta} - p_b(\theta')\right] \\ &= \theta[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\ p_s(\underline{\theta}) - (1 - G(\theta'))p_s(\theta') + G(\theta')p_b(\theta') &\geq 2\theta G(\theta') \end{aligned}$$

But this is true since $p_s(\underline{\theta}) - (1 - G(\theta'))p_s(\theta') + G(\theta')p_b(\theta') \geq 2\underline{\theta}G(\theta') \geq 2\theta G(\theta')$ where the first inequality follows from the revealed preference argument above and the second one follows from $\underline{\theta} \geq \theta$ and $G(\theta') \geq 0$.

Similarly, the best response of an agent with value $\underline{\theta}$ is $b(\overline{\theta})$. Thus,

$$\begin{aligned} \overline{\theta} - p_b(\overline{\theta}) &\geq (1 - G(\theta')) \left[p_s(\theta') - \overline{\theta}\right] + G(\theta') \left[\overline{\theta} - p_b(\theta')\right] \\ &= \overline{\theta}[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\ \iff 2\overline{\theta}(1 - G(\theta')) &\geq p_b(\overline{\theta}) + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \end{aligned}$$

Suppose $\theta \geq \overline{\theta}$. In this case, we want to show that bidding $b(\overline{\theta})$ gives a higher payoff than any other type's bid $b(\theta')$:

$$\begin{aligned} \theta - p_b(\overline{\theta}) &\geq (1 - G(\theta')) \left[p_s(\theta') - \theta\right] + G(\theta') \left[\theta - p_b(\theta')\right] \\ &= \theta[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\ \iff 2\theta(1 - G(\theta')) &\geq p_b(\overline{\theta}) + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \end{aligned}$$

Again, this is true since $2\theta(1 - G(\theta')) \geq 2\overline{\theta}(1 - G(\theta')) \geq p_b(\overline{\theta}) + (1 - G(\theta'))p_s(\theta') -$

$G(\theta')p_b(\theta')$ where the first inequality again follows from the revealed preference argument above and the second one follows from $\theta \geq \overline{\theta}$ and $1 - G(\theta') \geq 0$. □

A.10.2 Binding IR constraints

$$
\begin{aligned}
u^{da}(\underline{\theta}) &= p\left[\int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{2}[b(\underline{\theta}) + b(x)]g(x)dx - \underline{\theta}\right] \\
&= p\left[\frac{1}{2}\int_{\underline{\theta}}^{\overline{\theta}} [b(\underline{\theta}) + b(x)]g(x)dx - \underline{\theta}\right] \\
&= p\left[\frac{1}{2}\left[b(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right] - \underline{\theta}\right] \\
&= p\left[\frac{1}{2}\left[\underline{\theta} - \frac{\int_{G^{-1}(\frac{1}{2})}^{\underline{\theta}} \left[G(x) - \frac{1}{2}\right]^2 dx}{\left[G(\underline{\theta}) - \frac{1}{2}\right]^2} + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right] - \underline{\theta}\right] \\
&= \frac{p}{2}\left[-\underline{\theta} + 4\int_{\underline{\theta}}^{G^{-1}(\frac{1}{2})} \left[G(x) - \frac{1}{2}\right]^2 dx + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]
\end{aligned}
$$

Similarly, we obtain,

$$
\begin{aligned}
u^{da}(\overline{\theta}) &= p\left[\overline{\theta} - \int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{2}[b(\overline{\theta}) + b(x)]g(x)dx\right] \\
&= p\left[\overline{\theta} - \frac{1}{2}\left[b(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]\right]
\end{aligned}
$$

$$= p\left[\overline{\theta} - \frac{1}{2}\left[\overline{\theta} - \frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\overline{\theta}} \left[G(x) - \frac{1}{2}\right]^2 dx}{\left[G(\overline{\theta}) - \frac{1}{2}\right]^2} + \int\limits_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]\right]$$

$$= \frac{p}{2}\left[\overline{\theta} + 4\int\limits_{G^{-1}(\frac{1}{2})}^{\overline{\theta}} \left[G(x) - \frac{1}{2}\right]^2 dx - \int\limits_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]$$

A.10.3 Slope of Utilities from the Double Auction

Proof of Lemma 15. It is easy to verify that for agents with values less than $\underline{\theta}$, $\frac{\partial u^{da}(\theta)}{\partial \theta} = -p \geq -1$ and for agents with values above $\overline{\theta}$, $\frac{\partial u^{da}(\theta)}{\partial \theta} = p \leq 1$. Next we show that $\frac{\partial u^{da}(\theta)}{\partial \theta}$ is greater than -1 for $\underline{\theta}$ and less than 1 for $\overline{\theta}$.

First, we need the derivative of the bidding function:

$$b(\theta) = \theta - \frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2}$$

$$b'(\theta) = 1 - \frac{[G(\theta) - \frac{1}{2}]^4 - 2g(\theta)(G(\theta) - \frac{1}{2}) \int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^4}$$

$$= \frac{2g(\theta)(G(\theta) - \frac{1}{2}) \int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^4}$$

$$= 2g(\theta)\frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^3}$$

$$u^{da}(\theta) = p\theta[2G(\theta) - 1] + \frac{p}{2}(1 - 2G(\theta))b(\theta) + \frac{p}{2}\int_{\theta}^{\overline{\theta}} b(x)g(x)dx - \frac{p}{2}\int_{\underline{\theta}}^{\theta} b(x)g(x)dx$$

$$\begin{aligned}
\frac{\partial u^{da}(\theta)}{\partial\theta} &= p\left[(2G(\theta) - 1) + 2\theta g(\theta)\right] + \frac{p}{2}\left[-2g(\theta)b(\theta) + (1 - 2G(\theta))b'(\theta)\right] - \frac{p}{2}2b(\theta)g(\theta) \\
&= p(2G(\theta) - 1) + 2pg(\theta)(\theta - b(\theta)) + \frac{p}{2}\left[(1 - 2G(\theta))b'(\theta)\right] \\
&= p\left[G(\theta) - 1 + 2g(\theta)(\theta - b(\theta)) + \frac{1}{2}\left[(1 - 2G(\theta))b'(\theta)\right]\right] \\
&= p\left[G(\theta) - 1 + 2g(\theta)\left[\theta - \theta + \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta}[G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2}\right] + \frac{1}{2}(1 - 2G(\theta))b'(\theta)\right] \\
&= p\left[G(\theta) - 1 + 2g(\theta)\frac{\int_{G^{-1}(\frac{1}{2})}^{\theta}[G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2} + \frac{1}{2}(1 - 2G(\theta))\left[2g(\theta)\frac{\int_{G^{-1}(\frac{1}{2})}^{\theta}[G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^3}\right]\right] \\
&= p\left[G(\theta) - 1 + 2g(\theta)\frac{\int_{G^{-1}(\frac{1}{2})}^{\theta}[G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2} - 2g(\theta)\frac{\int_{G^{-1}(\frac{1}{2})}^{\theta}[G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2}\right] \\
&= p\left[G(\theta) - 1\right]
\end{aligned}$$

□

A.10.4 Bids with Uniform Distribution

$$b(\theta) = \theta - \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2} = \theta - \frac{\int_{\frac{\underline{\theta}+\overline{\theta}}{2}}^{\theta} \left[\frac{x-\underline{\theta}}{\overline{\theta}-\underline{\theta}} - \frac{1}{2}\right]^2 dx}{\left[\frac{\theta-\underline{\theta}}{\overline{\theta}-\underline{\theta}} - \frac{1}{2}\right]^2}$$

$$= \theta - \frac{\int_{\frac{\underline{\theta}+\overline{\theta}}{2}}^{\theta} \left[\frac{2x-\underline{\theta}-\overline{\theta}}{2(\overline{\theta}-\underline{\theta})}\right]^2 dx}{\left[\frac{2\theta-\underline{\theta}-\overline{\theta}}{2(\overline{\theta}-\underline{\theta})}\right]^2} = \theta - \frac{\int_{\frac{\underline{\theta}+\overline{\theta}}{2}}^{\theta} \left[2x-\underline{\theta}-\overline{\theta}\right]^2 dx}{\left[2\theta-\underline{\theta}-\overline{\theta}\right]^2}$$

$$= \theta - \frac{\left[\frac{1}{2\times 3}\left[2x-\underline{\theta}-\overline{\theta}\right]^3\right]_{\frac{\underline{\theta}+\overline{\theta}}{2}}^{\theta}}{\left[2\theta-\underline{\theta}-\overline{\theta}\right]^2} = \theta + \frac{1}{6}\frac{\left[2\theta-\underline{\theta}-\overline{\theta}\right]^3}{\left[2\theta-\underline{\theta}-\overline{\theta}\right]^2}$$

$$= \theta - \frac{2\theta-\underline{\theta}-\overline{\theta}}{6} = \frac{4\theta+\underline{\theta}+\overline{\theta}}{6}$$

Next, we compute another expression from the profit function:

$$\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx = \int_{\underline{\theta}}^{\overline{\theta}} \frac{4\theta+\underline{\theta}+\overline{\theta}}{6}\frac{1}{\overline{\theta}-\underline{\theta}}dx = \frac{1}{6(\overline{\theta}-\underline{\theta})}\int_{\underline{\theta}}^{\overline{\theta}} (4\theta+\underline{\theta}+\overline{\theta})dx$$

$$= \frac{1}{6(\overline{\theta}-\underline{\theta})}\left[2\overline{\theta}^2 - 2\underline{\theta}^2 + (\overline{\theta}+\underline{\theta})(\overline{\theta}-\underline{\theta})\right] = \frac{1}{6(\overline{\theta}-\underline{\theta})}\left[3(\overline{\theta}+\underline{\theta})(\overline{\theta}-\underline{\theta})\right] = \frac{\overline{\theta}+\underline{\theta}}{2}$$

A.10.5 Profit from Coexistence Equilibrium under Double Auction

$$\Pi_{\underline{\theta},\overline{\theta}} = -\int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)f(x)dx - F(\underline{\theta})u^d(\underline{\theta}) - (1-F(\overline{\theta}))u^d(\overline{\theta})$$

$$= \left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right] - p\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right]$$

$$
-\frac{p}{2}\left[\left[F(\underline{\theta})+F(\overline{\theta})-1\right]\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx+F(\underline{\theta})b(\underline{\theta})-(1-F(\overline{\theta}))b(\overline{\theta})\right]
$$

$$
=(1-p)\left[-\underline{\theta}F(\underline{\theta})+\overline{\theta}(1-F(\overline{\theta}))\right]
$$

$$
-\frac{p}{2}\left[\left[F(\underline{\theta})+F(\overline{\theta})-1\right]\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx+F(\underline{\theta})b(\underline{\theta})-(1-F(\overline{\theta}))b(\overline{\theta})\right]
$$

$$
=(1-p)\left[-\underline{\theta}^2+\overline{\theta}(1-\overline{\theta})\right]
$$

$$
-\frac{p}{2}\left[\left[\underline{\theta}+\overline{\theta}-1\right]\frac{\overline{\theta}+\underline{\theta}}{2}+\underline{\theta}\frac{5\underline{\theta}+\overline{\theta}}{6}-(1-\overline{\theta})\frac{5\overline{\theta}+\underline{\theta}}{6}\right]
$$

The profit is decreasing in $\underline{\theta}$:

$$
\frac{\partial \Pi_{\underline{\theta},\overline{\theta}}}{\partial \underline{\theta}}=(1-p)\left[-2\underline{\theta}\right]-\frac{p}{2}\left[\frac{\overline{\theta}+\underline{\theta}}{2}+\frac{1}{2}\left[\underline{\theta}+\overline{\theta}-1\right]+\frac{5\underline{\theta}+\overline{\theta}}{6}+\underline{\theta}\frac{5}{6}-(1-\overline{\theta})\frac{1}{6}\right]
$$

$$
=(1-p)\left[-2\underline{\theta}\right]-\frac{p}{2}\left[\frac{2}{3}(4\underline{\theta}+2\overline{\theta}-1)\right]\leq 0
$$

Notice that the first summand is negative and inside the brackets of the second summand is positive since $\underline{\theta}\geq 1-\overline{\theta}$ by feasibility. Thus, the profit is decreasing in $\underline{\theta}$. Hence, the feasibility binds and we have $\underline{\theta}=1-\overline{\theta}$, otherwise decreasing $\underline{\theta}$ until the feasibility binds strictly increases the profit. Then, we have

$$
\Pi_{\underline{\theta},\overline{\theta}}=(1-p)\left[-\underline{\theta}^2+\overline{\theta}(1-\overline{\theta})\right]-\frac{p}{2}\left[\underline{\theta}\frac{5\underline{\theta}+\overline{\theta}}{6}-(1-\overline{\theta})\frac{5\overline{\theta}+\underline{\theta}}{6}\right]
$$

$$
=(1-p)\left[-\underline{\theta}^2+\overline{\theta}(1-\overline{\theta})\right]-\frac{p}{2}\left[\underline{\theta}\frac{5\underline{\theta}+\overline{\theta}-5\overline{\theta}-\underline{\theta}}{6}\right]
$$

$$
=(1-p)\left[-\underline{\theta}^2+(1-\underline{\theta})\underline{\theta}\right]-\frac{p}{2}\left[\underline{\theta}\frac{5\underline{\theta}+(1-\underline{\theta})-5(1-\underline{\theta})-\underline{\theta}}{6}\right]
$$

$$
=(1-p)\left[\underline{\theta}(1-2\underline{\theta})\right]-\frac{p}{2}\left[\underline{\theta}\frac{8\underline{\theta}-4}{6}\right]=(1-p)\left[\underline{\theta}(1-2\underline{\theta})\right]+\frac{p}{6}\left[\underline{\theta}(1-2\underline{\theta})\right]
$$

$$
\frac{6-5p}{6}\left[\underline{\theta}(1-2\underline{\theta})\right]=\frac{6-5p}{6}\Pi^M
$$

Appendix B
Some Results used in Appendix A

The results below have first been obtained in Idem (2021) for an environment with finitely many agents, divisible goods and arbitrary endowments. Here I restate them for the environment I study in the Coexistence of Centralized and Decentralized Markets, with the proofs adjusted accordingly. Moreover, here I focus on the case where the decentralized trade is not possible due to prohibitive frictions. This simplifies the individual rationality constraints to the usual form where the mechanism needs to guarantee nonnegative utility. The main text deals with the complications arising from endogenous outside option created by the search market participants.

B.1 The Environment

I describe the setup and the initial statement of the mechanism design problem here for convenience.

- Good: There is a single, indivisible good in the market.
- Agents: There is a continuum of agents on $[0, 1]$.
- Endowments: Each agent has 1 unit of endowment of the good.
- Demands: Each agent demands up to 2 units of the good. Since the good is indivisible, this means, they can consume 0, 1, or 2 units, depending on whether they buy or sell, or neither buy nor sell.

- Valuations: Each agent has some valuation $\theta \in [0,1]$ for a unit of the good. The valuations are drawn from some distribution F with support $[0,1]$. Agents' valuations are their private information.

- Marketplace: A mechanism designer wants to design a mechanism to maximize its profit. She knows the distribution of valuations, F.

By revelation principle, I focus on direct mechanisms. Moreover, as agents are symmetric other than their valuations, I focus on anonymous mechanisms, which is without loss. Then, the designer will choose a mechanism that allocates $q : \theta \to \mathbb{R}$ units of good to each agent with valuation θ and asks her to pay $t : \theta \to \mathbb{R}$. Hence, the net utility of the agent with the valuation θ from the monagorastic mechanism is

$$u(\theta) = \theta \min\{1, q(\theta)\} - t(\theta).$$

As agents have demands for two units, having more than 2 unit of the good is same as having 2 unit. As such, the expression for the utility above caps the maximum net trade that increases the utility at 1, since the agent already has 1 unit of endowment.

The profit of the marketplace is the net payments. Thus, the designer seeks to maximize total payment, given the incentive compatibility, individual rationality, and feasibility constraints.

$$\max_{(q,t)} \int_{[0,1]} t(\theta) f(\theta) d\theta$$

s. t.

$$\begin{array}{lll}
\text{(IC)} & \theta \min\{1, q(\theta)\} - t(\theta) & \geq \theta \min\{1, q(\theta')\} - t(\theta') \\
\text{(IR)} & \theta \min\{1, q(\theta)\} - t(\theta) & \geq 0 \\
\text{(Individual Feasibility)} & q(\theta) & \geq -1 \\
\text{(Aggregate Feasability)} & \int_{[0,1]} q(\theta) f(\theta) d\theta & \leq 0
\end{array}$$

B.2 Simplifying The Designer's Problem

We first develop a series of lemmata that help us state the maximization problem above as a concave program.

Lemma 16 (Monotonicity)**.** *Suppose (q, t) is a direct, IC mechanism. Then,*

1. *If* $q(\theta) < 1$ *for some* $\theta \in [0,1]$*, then* $q(\theta)$ *is increasing at* (θ)*.*

2. *If* $q(\theta) \geq 1$ *for some* $\theta \in [0,1]$*, then* $q(\theta') \geq 1$ *for each* $\theta' \geq \theta$*.*

The proof is standard, except for taking care of the capacities so it can be found in the Appendix B.4.

The next lemma presents the derivative of the utility of an agent in an IC mechanism.

Lemma 17 (Envelope Condition)**.** *If* (q,t) *is a direct, IC mechanism, then for each* $\theta \in [0,1]$

$$\frac{\partial u(\theta)}{\partial \theta} = \begin{cases} q(\theta), & \textit{if } q(\theta) < 1, \\ 1, & \textit{otherwise.} \end{cases}$$

Again, the proof is similar to standard arguments and can be found in Appendix B.5.

Notation: For any direct mechanism (q,t), let

$$q^*(\theta) = \begin{cases} q(\theta), & \text{if } q(\theta) < 1, \\ 1, & \text{otherwise.} \end{cases}$$

Note that for a direct, IC mechanism, $q^*(\theta)$ is also weakly increasing.

The next lemma gives the representation of the utility of each type as the integral of the allocation rule, using the previous lemma.

Lemma 18 (Payoff Equivalence)**.** *If* (q,t) *is a direct, IC mechanism, then*

$$u(\theta) = u(0) + \int_0^\theta q^*(x)dx,$$

for each $\theta \in [0.1]$*.*

Proof. Since $u(\theta)$ is convex θ on both regions where $q(\theta) > 1$ and $q(\theta) \leq 1$ separately, it is absolutely continuous in θ. Then, it is the integral of its derivative. □

Next, we pin down the transfer rule in an IC mechanism.

Lemma 19 (Revenue Equivalence)**.** *If* (q,t) *is a direct, IC mechanism, then*

$$t(\theta) = -u(0) + \theta q^*(\theta) - \int_0^\theta q^*(x)dx,$$

for each $\theta \in [0,1]$*.*

Proof. From the definition of $u(\theta)$ and the previous lemma. □

Now we show that the necessary conditions above for incentive compatibility of a mechanism are also sufficient to establish the incentive compatibility of a mechanism.

Lemma 20. *Let (q,t) be a direct mechanism. The mechanism is incentive compatible if and only if,*

1. *$q^*(\theta)$ is increasing at θ;*
2. *$t(\theta) = -u(0) + \theta q^*(\theta) - \int_0^\theta q^*(x)dx$.*

Proof can be found in Appendix B.6.

The next result shows that deterministic incentive compatible mechanisms are bid-ask price mechanisms.

Proposition 12. *If (q,t) is a direct, deterministic and incentive compatible mechanism, then there are two types $\underline{\theta}$ and $\overline{\theta}$ and prices p_s, p_0, and p_b such that*

$$q(\theta) = \begin{cases} -1 & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} < \theta < \overline{\theta} \\ 1 & \text{if } \theta \geq \overline{\theta} \end{cases}$$

and the transfer rule

$$t(\theta) = \begin{cases} -p_s & \text{if } \theta \leq \underline{\theta} \\ p_0 & \text{if } \underline{\theta} < \theta < \overline{\theta} \\ p_b & \text{if } \theta \geq \overline{\theta} \end{cases}$$

Proof. The form of the allocation follows from the fact that the good is indivisible, the mechanism is deterministic, and that the incentive compatible mechanisms have monotone allocations. Given this, integrating the revenue equivalence formula shows that for each allocation outcome, there is a unique price. □

The next proposition provides the characterization of the IR mechanisms by establishing the types with the lowest utilities. The reason this is an issue in this model is that in an auction, the lowest allocation an agent could receive is 0. Hence, the utility is always increasing in agent's type, as can be seen from the envelope condition. Of course, this means the lowest type has the lowest utility. However, here, an agent with a relatively low type can be a seller, which means he would get a negative allocation. Therefore, the utility of the lowest type is not the lowest utility, which can again by seen from the envelope condition.

Proposition 13. *Let (q, t) be a direct IC mechanism. Then, it is IR if and only if,*

$$\theta^* q^*(\theta^*) \geq t(\theta^*),$$

where θ^* is defined as

1. $\theta^* = 0$ *if* $q^*(0) \geq 0$,
2. $\theta^* = 1$ *if* $q^*(1) < 0$,
3. *a solution to* $q^*(\theta^*) = 0$ *if such a type exists,*
4. θ *such that for each* $\theta' < \theta$, $q(\theta') < 0$ *and for each* $\theta' > \theta$, $q(\theta') > 0$.

Proof. Case 1: Suppose $q^*(0) \geq 0$. Then, by Lemma 18, incentive compatibility of a mechanism implies that the associated ex-post utilities $u(\theta)$ are increasing in θ. Hence, if $u(0) \geq 0$, we have $u(\theta) \geq 0$ for each $\theta \in [0, 1]$.

Case 2: Suppose $q^*(1) < 0$. Then, by Lemma 18, $u(\theta)$ are decreasing and hence, $u(1)$ is the lowest payoff. Hence, if it is nonnegative, all other types' payoffs are nonnegative as above.

Cases 3 and 4: Suppose θ^* is defined as in the Case 3 or Case 4. Then, by Lemma 18, $u(\theta)$ is decreasing up to θ^* and increasing after that point. Hence, type θ^* has the lowest payoff. So, if $u(\theta^*) \geq 0$, each type's IR condition must also hold.

□

Lemma 21. *If an IC and IR mechanism maximizes the expected revenue of the designer, then,*

$$t(\theta^*) = \theta^* q(\theta^*)$$

where θ^* is defined as

1. $\theta^* = 0$ *if* $q^*(0) \geq 0$,

2. $\theta^* = 1$ *if* $q^*(1) < 0$,

3. *the solution to* $q^*(\theta^*) = 0$ *if such a type exists,*

4. θ *such that for each* $\theta' < \theta$, $q(\theta') < 0$ *and for each* $\theta' > \theta$, $q(\theta') > 0$.

Proof. The previous proposition shows that IC and IR mechanisms must have $\theta^* q(\theta^*)$ greater than $t(\theta^*)$. However, if $\theta^* q(\theta^*) > t(\theta^*)$, then the seller can increase the expected revenue by increasing $t(0)$ and keeping the allocation rule the same. This would increase all types' payments and the revenue strictly, contradicting revenue maximization.

□

Using the condition about θ^* from Lemma 21 and the previous lemmata, we have

$$
\begin{aligned}
\theta^* q^*(\theta^*) \quad &= t(\theta^*) \\
= \quad & -u(0) + \theta^* q^*(\theta^*) - \int_0^{\theta^*} q^*(x)dx \\
\iff \quad & u(0) = -\int_0^{\theta^*} q^*(x)dx \\
\iff \quad & t(\theta) = \int_0^{\theta^*} q^*(x)dx + \theta q^*(\theta) - \int_0^{\theta} q^*(x)dx
\end{aligned}
$$

Now we are ready to show that the allocation rule in a revenue-maximizing mechanism is not 'wasteful'.

Proposition 14. *Let* (q, t) *be a direct mechanism that maximizes the revenue of the designer. Then,* $q(\theta) \leq 1$ *with probability 1 and the aggregate feasibility holds with equality:* $\int_{[0,1]} q(\theta) f(\theta) d\theta = 0$.

Proof. First, suppose that in the optimal mechanism, there exists a set $\Theta \subset [0, 1]$ with a positive measure such that for each $\theta \in \Theta$, $q(\theta) > 1$. Notice that decreasing the allocation to 1 unit has no effect on the agent's payoff. Hence, it doesn't effect any IC or IR constraints.

Next, let us examine the transfer rule in a direct, IC mechanism:

$$
t(\theta) = \int_0^{\theta^*} q^*(x)dx + \theta q^*(\theta) - \int_0^{\theta} q^*(x)dx.
$$

If we have $q(\theta) > 1$ for a positive measure of types, then we must have $q(\theta) < 0$ for a corresponding positive measure of types by the aggregate feasibility constraint.

Hence, if we reduced $q(\theta) = 1$ for $\theta \in \Theta$, this wouldn't affect any constraints but instead strictly increase profit as it allows us to increase $q(\theta) < 0$ for a positive measure of types, contradicting the optimality of the mechanism.

By the same argument, having $\int_{[0,1]} q(\theta)f(\theta)d\theta < 0$ cannot be optimal: Either buying less from types or selling more to some types would increase their payments, strictly increasing the profit. □

Now, by fixing $t(\theta)$ to the characterization we have from above, we can restate the problem as follows.

$$
\begin{aligned}
\max_{q} \quad & \int_{[0,1]} \left[\int_{\{y|q(y)\leq 0\}} q(x)dx + \left(\theta q(\theta) - \int_{0}^{\theta} q(x)dx \right) \right] f(\theta)d\theta \\
\text{s. t.} \quad & \\
& q(\theta) \text{ is increasing} \\
& q(\theta) \geq -1 \\
& \int_{[0,1]} q(\theta)f(\theta)d\theta = 0
\end{aligned}
$$

After some transformations[1], the problem above can be rewritten as follows:

$$
\begin{aligned}
\max_{q} \quad & \left[\int_{[0,1]} q(\theta) \left[\frac{\mathbb{1}\{q(\theta) \leq 0\}}{f(\theta)} + \left(\theta - \frac{(1-F(\theta))}{f(\theta)} \right) \right] f(\theta)d\theta \right] \\
\text{s. t.} \quad & \\
& q(\theta) \text{ is increasing} \\
& q(\theta) \geq -1 \\
& \int_{[0,1]} q(\theta)f(\theta)d\theta = 0
\end{aligned}
$$

B.3 Omitted Proofs

B.4 Proof of Lemma 1

Proof. Let $\theta, \theta' \in [0,1]$. Then, by incentive compatibility

$$\theta \min\{1, q(\theta)\} - t(\theta) \geq \theta \min\{1, q(\theta')\} - t(\theta')$$

[1]The details can be followed in Appendix B.7.

and

$$\theta' \min\{1, q(\theta)\} - t(\theta) \leq \theta' \min\{1, q(\theta')\} - t(\theta').$$

Subtracting the second inequality from the first one leads to:

$$(\theta - \theta') \min\{1, q(\theta)\} \geq (\theta - \theta') \min\{1, q(\theta')\}$$

Suppose $q(\theta) < 1$ and $\theta > \theta'$. Then, we have

$$\begin{aligned} \min\{1, q(\theta)\} &\geq \min\{1, q(\theta')\} \iff \\ 1 > q(\theta) &\geq \min\{1, q(\theta')\} \iff \\ q(\theta) &\geq q(\theta') \end{aligned}$$

Now suppose $q(\theta) \geq 1$ and $\theta' \geq \theta$. Then,

$$\begin{aligned} \min\{1, q(\theta')\} &\geq \min\{1, q(\theta)\} \iff \\ \min\{1, q(\theta')\} &\geq 1 \iff \\ q(\theta') &\geq 1. \end{aligned}$$

□

B.5 Proof of Lemma 2

Proof. First, suppose $q(\theta) < 1$. Then, IC implies that for type θ agent:

$$\begin{aligned} u(\theta) &= \max_{\theta' \in [0,1]} \min\{1, q(\theta')\}\theta - t(\theta') \\ &= \max_{\theta' \in [0,1]} q(\theta')\theta - t(\theta'). \end{aligned}$$

Notice that the RHS is the maximum of affine functions of θ, so $u(\theta)$ is convex in θ on this region. Hence, $u(\theta)$ is differentiable almost everywhere in θ on this region. For any θ at which it is differentiable, for $\delta > 0$, IC implies that

$$\begin{aligned} & \lim_{\delta \to 0} \frac{u(\theta + \delta) - u(\theta)}{\delta} \\ \geq & \lim_{\delta \to 0} \frac{(q(\theta)(\theta + \delta) - t(\theta)) - (q(\theta)\theta - t(\theta))}{\delta} = q(\theta). \\ & \lim_{\delta \to 0} \frac{u(\theta) - u(\theta - \delta)}{\delta} \\ \leq & \lim_{\delta \to 0} \frac{(q(\theta)\theta - t(\theta)) - (q(\theta)(\theta - \delta) - t(\theta))}{\delta} = q(\theta). \end{aligned}$$

Then, two inequalities together imply that

$$\frac{\partial u(\theta)}{\partial \theta} = q(\theta).$$

Now suppose $q(\theta) \geq 1$. Then,

$$u(\theta) = \min\{1, q(\theta)\}\theta - t(\theta) = \theta - t(\theta).$$

Notice that $t(\theta)$ must be constant in θ on the region with $q(\theta) \geq 1$: Since agent's effective allocation is constant, otherwise, i would simply choose the type with the least cost. Then, of course,

$$\frac{\partial u(\theta)}{\partial \theta} = 1.$$

□

B.6 Proof of Proposition 1

Proof. We want to show that for each $\theta, \theta' \in [0, 1]$, we have

$$\begin{aligned}
& u(\theta) \geq \theta \min\{1, q(\theta')\} - t(\theta') \\
\iff \; & u(\theta) \geq \theta \min\{1, q(\theta')\} + \theta' \min\{1, q(\theta')\} \\
& -\theta' \min\{1, q(\theta')\} - t(\theta') \\
\iff \; & u(\theta) \geq \theta \min\{1, q(\theta')\} - \theta' \min\{1, q(\theta')\} \\
& +u(\theta') \\
\iff \; & u(\theta) - u(\theta') \geq (\theta - \theta') \min\{1, q(\theta')\} \\
\iff \; & \int_{\theta'}^{\theta} q^*(x)dx \geq \int_{\theta'}^{\theta} q^*(\theta')dx
\end{aligned}$$

Suppose $\theta > \theta'$. Since $q^*(\cdot)$ is increasing, $q^*(x) \geq q^*(\theta')$ for each $x \in [\theta', \theta]$. Then, the last inequality above holds. Similar analysis holds for the case of $\theta < \theta'$.

□

B.7 Transformations of the Designer's Problem

We start with the problem in Equation B.2 and make the following transformation:

$$\begin{aligned}
& \int\limits_{[0,1]} \int\limits_{0}^{\theta} q(x)dx f(\theta)d\theta \\
= & \int\limits_{[0,1]} \int\limits_{x}^{\overline{\theta}} f(\theta)d\theta q(x)dx \\
= & \int\limits_{[0,1]} q(x)(1-F(x))dx \\
= & \int\limits_{[0,1]} q(\theta)\left(\frac{(1-F(\theta))}{f(\theta)}\right) f(\theta)d\theta
\end{aligned}$$

So, the second part of the objective function becomes:

$$\begin{aligned}
& \int\limits_{[0,1]} \theta q(\theta)f(\theta)d\theta - \int\limits_{[0,1]} q(\theta)\left(\frac{(1-F(\theta))}{f(\theta)}\right) f(\theta)d\theta \\
= & \int\limits_{[0,1]} \left(\theta q(\theta) - q(\theta)\frac{(1-F(\theta))}{f(\theta)}\right) f(\theta)d\theta \\
= & \int\limits_{[0,1]} q(\theta)\left(\theta - \frac{(1-F(\theta))}{f(\theta)}\right) f(\theta)d\theta
\end{aligned}$$

Next we look at the first summand in the objective function above. Notice that inside is actually a constant, so it can be expressed as below:

$$\begin{aligned}
& \int\limits_{[0,1]} \left[\int\limits_{\{y|q(y)\leq 0\}} q(x)dx\right] f(\theta)d\theta \\
= & \int\limits_{\{y|q(y)\leq 0\}} q(x)dx \\
= & \int\limits_{[0,1]} q(x)\mathbb{1}\{q(x)\leq 0\}dx \\
= & \int\limits_{[0,1]} q(\theta)\frac{\mathbb{1}\{q(\theta)\leq 0\}}{f(\theta)} f(\theta)d\theta
\end{aligned}$$

Finally, the objective function can be written as:

$$\begin{aligned}
& \int\limits_{[0,1]} q(\theta)\frac{\mathbb{1}\{q(\theta)\leq 0\}}{f(\theta)} f(\theta)d\theta + \int\limits_{[0,1]} q(\theta)\left(\theta - \frac{(1-F(\theta))}{f(\theta)}\right) f(\theta)d\theta \\
= & \int\limits_{[0,1]} q(\theta)\left[\frac{\mathbb{1}\{q(\theta)\leq 0\}}{f(\theta)} + \left(\theta - \frac{(1-F(\theta))}{f(\theta)}\right)\right] f(\theta)d\theta
\end{aligned}$$

Hence, the revenue maximization problem can be expressed as

$$\max_{(q,t)} \int_{\Theta} q(\theta)\left[\frac{\mathbb{1}\{q(\theta) \leq 0\}}{f(\theta)} + \left(\theta - \frac{(1-F(\theta))}{f(\theta)}\right)\right] f(\theta) d\theta$$

s. t.

$$q(\theta) \text{ is increasing in } \theta$$

$$q(\theta) \geq -1$$

$$0 \geq \int_{[0,1]} q(\theta) d\theta$$

Appendix C
Omitted Proofs from Chapter 3

C.1 Continuity of Budget Sets

Consider an economy $\mathcal{E} = \left\langle N, (X_i, \succeq_i, e^i)_{i=1}^n, \mathcal{F} \right\rangle$ where X_i is a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets; and consumption spaces are identical for individuals.

We define the budget sets $B_i(p) = \{x \in X_i | e^i \succeq_p x\}$. In this section, we want to show the continuity of these sets in the public ordering, p. We first give the relevant definitions and results that we are going to use.

A closed-valued correspondence (between metric spaces A, B) $F : A \rightrightarrows B$ is *upper hemi-continuous*, if at each $a \in A$, for each $a_n \in A^\infty$ that converges to a and for each $b_n \in B^\infty$ with $b_n \in F(a_n), \forall n = 1, \ldots$ that converges to b, $b \in F(a)$.

A correspondence (between metric spaces A, B) $F : A \rightrightarrows B$ is *lower hemi-continuous*, if at each $a \in A$, for each $a_n \in A^\infty$ that converges to a and for each $b \in F(a)$, there exists a subsequence a_{n_k} and a sequence $b_k \in B^\infty$ that converges to b with $b_k \in F(a_{n_k})$, for each k.

A correspondence is *continuous* if it is both upper hemi-continuous and lower hemi-continuous.

Now, we define the (Kuratowski) convergence of a sequence of sets. Let Z be a topological space. Let $\{Z_n\}$ be a sequence of subsets of Z.

$\limsup Z_n$ is the set of elements z such that there is a sequence of elements z_k and a subsequence $\{Z_{n_k}\}$ such that $z_k \in Z_{n_k}$ and z_k converges to z.

$\liminf Z_n$ is the set of elements z such that there is a sequence of elements z_n with $z_k \in Z_n$ and z_n converges to z.

$Z' = \lim Z_n$ if $\liminf Z_n = Z_n \limsup Z_n$. If this is the case, Z_n converges to Z' in Kuratowski sense.

Consider $(c(X_i^2), T_V)$. This is a compact Hausdorff space, since it is a compact metric space, with the Hausdorff metric. Then, by Theorem 4.7 in Illanes and Nadler (1999), convergence in Kuratowski sense and Convergence in T_V are equivalent on $c(X_i^2)$. Now, we can prove the following lemma.

Lemma 22. *Consider an economy* $\mathcal{E} = \left\langle N, (X_i, \geq_i, e^i)_{i=1}^n, \mathcal{F} \right\rangle$ *where* X_i *is a compact, properly locally convex, closure stable and* S_4 *metric space with connected convex sets and identical for each individual. Let* p *be a continuous public ordering. Then,* $B_i(p) = \{x \in X_i | e^i \geq_p x\}$ *is continuous.*

Proof. Upper hemi-continuity: Suppose $\{p_m\}$ is a sequence of public orderings in P approaching p and x_m^i is feasible at p_m for each m and $\lim_m x_m^i = x^i$. By feasibility, $x_m^i \in X_i$ for each m, so by compactness, $x^i \in X_i$. Again, by feasibility, $e^i \geq_{p_m} x_m^i$. Now, it is known that in compact metric spaces, Kuratowski convergence and convergence in Hausdorff metric coincide. Then, since $\{p_m\}$ converges to p, (the inner limit) $\liminf p_m$ is equal to p as a set, since p is the limit and hence is equal to inner and outer limit by definition. Now, since $e^i \geq_{p_m} x_m^i$, we have $(x_m^i, e^i) \in p$ with (x^i, e^i) being its limit. Then, (x^i, e^i) is in the $\liminf p_m$ and hence in p. Hence, $e^i \geq_p x^i$, yielding upper hemi-continuity.

Lower hemi-continuity: Let p_m be a sequence of public orderings that converges to some public ordering p and let x^i be a feasible allocation for individual i with the endowment e^i under p. (So, $e^i \geq_p x^i$ or equivalently, $(x^i, e^i) \in p$.) We want to show that there is a subsequence p_{m_k} of p_m and a sequence of allocations x_k^i such that it converges to x^i and for each k, $e^i \geq_{p_k} x_k^i$. Now, using the fact that $(x^i, e^i) \in p$ where p is the limit of p_m and hence is equal to the outer limit, by definition of the outer limit, we have a sequence of points (x_k^i) and a subsequence of sets p_{m_k} of p_m such that $(x_k^i, e) \in P_{m_k}$ with (x_k^i, e^i) converging to (x^i, e^i). The subsequence of public orderings p_{m_k} and the sequence of allocations x_k^i with the stated qualities was precisely what we needed for lower hemi-continuity. □

C.2 Space of Public Orderings

Assume throughout this section that X_i is a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets.

Notation: $X = \Pi_{i=1}^n X_i$ and $X_i^2 = X_i \times X_i$. We call $x_i \in X_i$ an *allocation* and $x \in X$ an *allocation profile*.

Given the metrics on the individual's consumption spaces, X_i, we endow X_i^2 with the *Manhattan metric*: Let d_i be the metric on X_i. Then, we define:

$$d(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i), \forall x, y \in X_i^2. \tag{C.1}$$

Let $c(X_i^2)$ be the set of (nonempty) compact subsets of X_i^2. We define the *Hausdorff metric* on $c(X_i^2)$ as follows:

$$d_H(A,B) = \max\{\max_{z\in A}\min_{t\in B}\{d(z,t)\}, \max_{z\in B}\min_{t\in A}\{d(z,t)\}\}, \forall A, B \in c(X_i^2). \tag{C.2}$$

It is routine to check that both of these metrics are well-defined and indeed, metrics on their respective spaces.

Let (Y, T) be an arbitrary topological space and let $cl(Y)$ be the set of (nonempty) closed sets in Y. Then, the Vietoris topology on $cl(Y)$ is the smallest topology T_V such that

(i) If $U \in T$, then $\{A \in CL(Y) | A \subset U\} \in T_V$.

(ii) If B is T-closed, then $\{A \in cl(Y) | A \subset B\}$ is T_V-closed.

Let the topology generated by d_H be denoted by T_H. Also, notice that $cl(X_i^2) = c(X_i^2)$ since closed subsets of a compact space is compact. Now, by theorem 3.1 in Illanes and Nadler (1999), $T_V = T_H$. Moreover, by theorem 3.5 there, $(c(X_i^2), T_V)$ is compact.

Next, we define a convexity on $c(X_i^2)$. To do this, we first take the product convexity on X_i as the convexity on X_i^2. We need a couple of definitions.

Let $\mathcal{C}$ be a collection of subset of a set Z. We say that $(Z, \mathcal{C})$ is a *closure structure*, if (i) $\mathcal{C}$ includes the empty set and Z, (ii) $\mathcal{C}$ is closed under (arbitrary) intersections.

We say that a collection $\mathcal{S}$ of subset of a set Z is a *subbase* for a convex structure $(Z, \mathcal{C})$, if $\mathcal{S} \subset \mathcal{C}$ and $\mathcal{C}$ is the coarsest convexity that includes $\mathcal{S}$.

Let Z be a set and $A_1, \ldots, A_n \subset Z$. Define $\langle A_1, \ldots, A_n\rangle = \{B \subset 2^Z | B \subset \cup_{i=1}^n A_i; \forall i = 1, \ldots, n, B \cap A_i \neq \emptyset\}$.

Let $(Z, \mathcal{D})$ be a closure structure; $\mathcal{D}_* \equiv \mathcal{D} \setminus \{\emptyset\}$. Then, the sets $\langle D\rangle \cap \mathcal{D}_*$ and $\langle D, Z\rangle \cap \mathcal{D}_*$ for $D \in \mathcal{D}$ generate the *Vietoris Convexity* on the set $\mathcal{D}_*$.

Let $\mathcal{H}_*$ be the set of all (nonempty) closed, convex subset of X_i^2. Then, by definition, it defines a Vietoris convexity on X_i^2, $\mathcal{C}_V$.

We can indeed do this as by Proposition 3.10.4 of Chapter 3.3 in van de Vel (1993b), Vietoris metric and Vietoris convexity are compatible.[1]

[1] This follows since when the space is compact, the convex closure of union of two compact convex set is also compact, as argued by van de Vel (1984) (Remark 1.7).

By 3.7 in van de Vel (1993a), this space is S_4 and has connected convex sets, inheriting the properties of X_i^2, which inherits properties of X_i. Theorem 2.6 in van de Vel (1993a) implies that this space is properly locally convex and Theorem 2.4 there implies that it is closure stable. Hence, the space of (nonempty) closed subsets of X_i^2 is a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets. We summarize these results in the following lemma for future reference.

Lemma 23. *Let X_i be a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets. Then, the space of closed subsets of X_i^2 can be endowed with the Hausdorff metric, Vietoris topology and Vietoris convexity. Moreover, this tcs is also a compact, properly locally convex, closure stable and S_4 metric space with connected convex sets.*

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